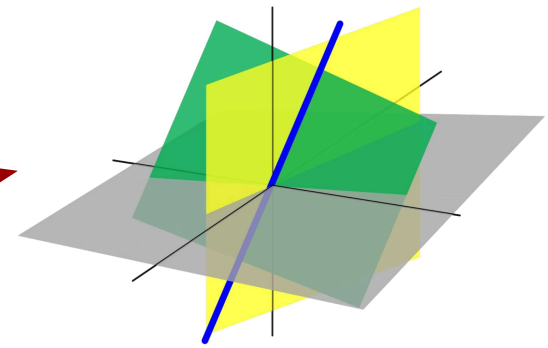
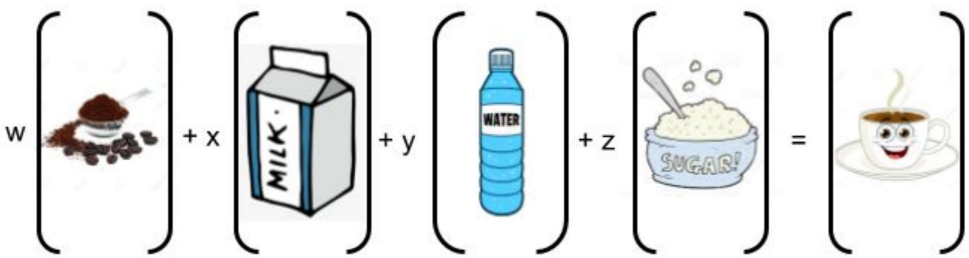
From Coffee Space to Vector Space: Foundations of Linear Algebra

Machine Learning
BITS F464



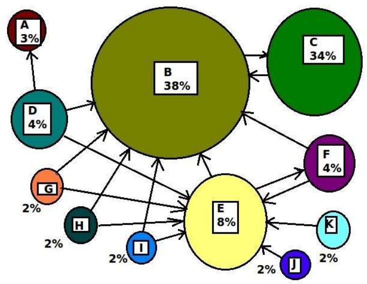
Coffee Space- Vector Space

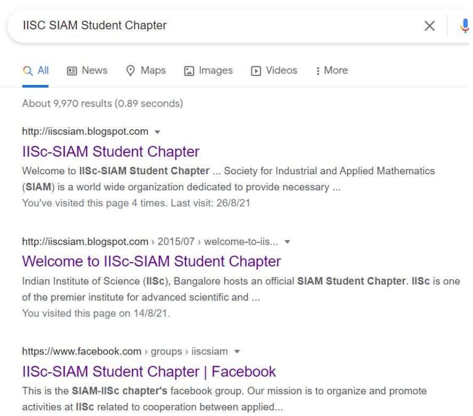


Harikrishnan N B
15 Sept 2022

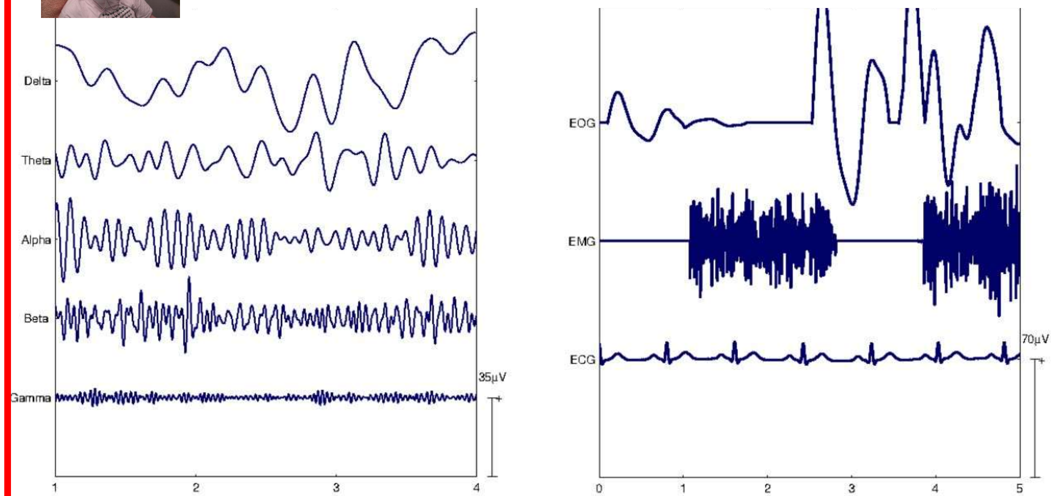
Why should I learn Linear Algebra?

Mathematics of Google Search - Page Rank





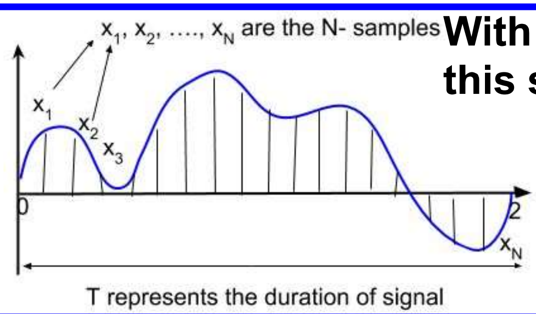
Common EEG Artifacts



(a) Brain Rhythms

(b) Artifacts

Figure 1. (a) *Five normal brain rhythms, from low to high frequencies.* Delta, Theta, Alpha, Beta and Gamma rhythms comprise the background EEG spectrum. (b) *Three different types of artifacts.* Ocular, muscular and cardiac artifacts are the most frequent physiological contaminants in the literature on EEG artifact removal.



With what all frequency this signal is made up of ?

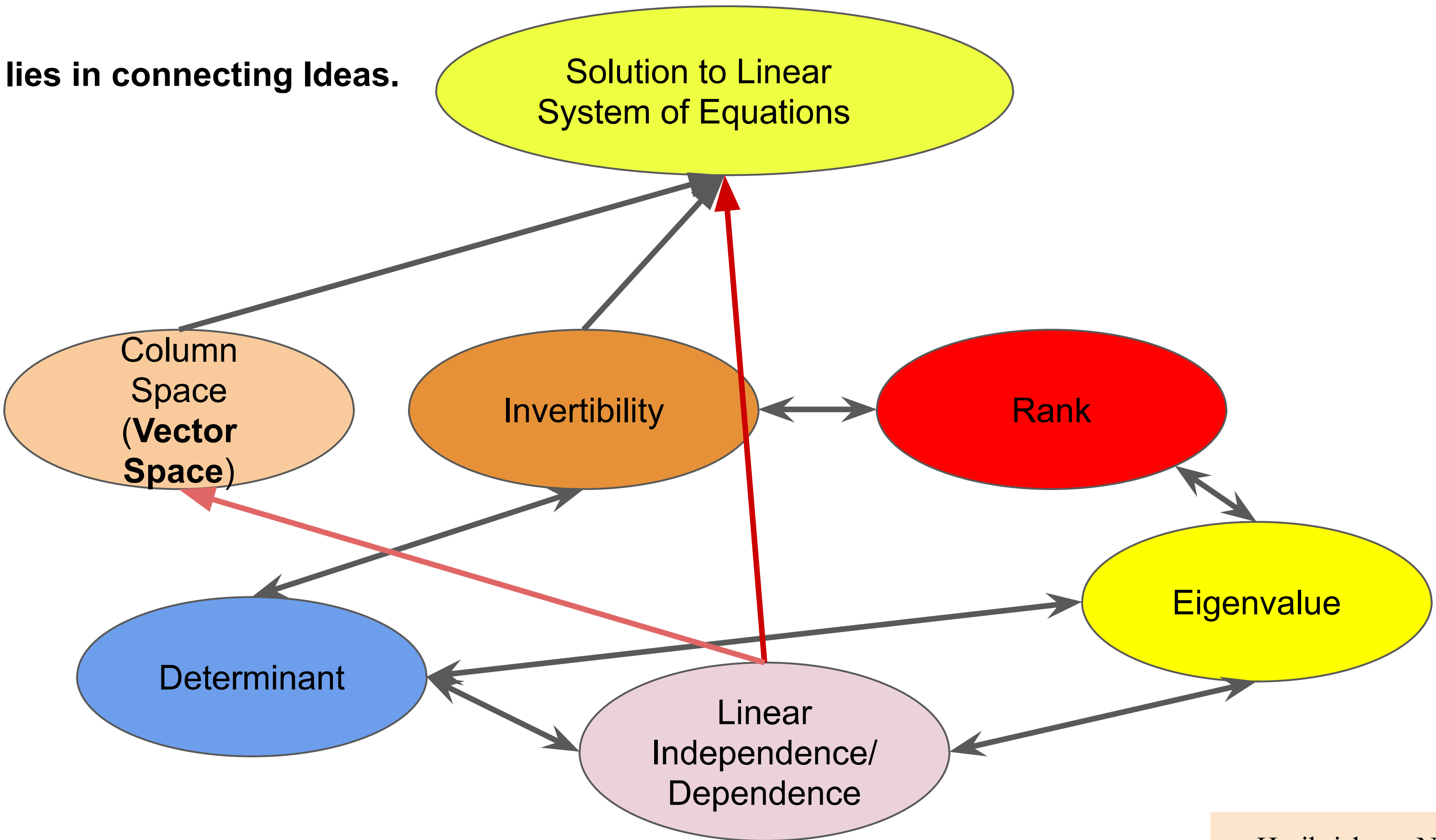
1. http://pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/index.html (Page Rank)
2. Reference: Urigüen, J. A., & Garcia-Zapirain, B. (2015). EEG artifact removal—state-of-the-art and guidelines. *Journal of neural engineering*, *12*(3), 031001.

Goal of Linear Algebra

- The central problem of Linear Algebra is to **understand** a system of linear equations.
- **Understanding involves**
  - Insights about row picture and column picture.
  - Explore the existence of solution to the system of linear equations.
  - Insights about column space, row space, right null space, left null space.
  - **What new can we say about the system?**

OUR APPROACH

Beauty lies in connecting Ideas.



Two Equations and Two Unknowns-
Algebraic Interpretation

$$-x + y = 0$$
$$2x + y = 3$$

What is the value of the **unknown variables *x* and *y*** that satisfies this system of linear equations?

Elimination

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & 1 & 3 \end{array}\right] R_1 \rightarrow 2R_1 \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & 1 & 3 \end{array}\right] \quad R_2 \rightarrow R_1 + R_2 \quad \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 3 & 3 \end{array}\right]$$

$$3y = 3$$
$$y = 1$$

Sub. *y* = *1* in Equation 1, we get: ***x* = *1***

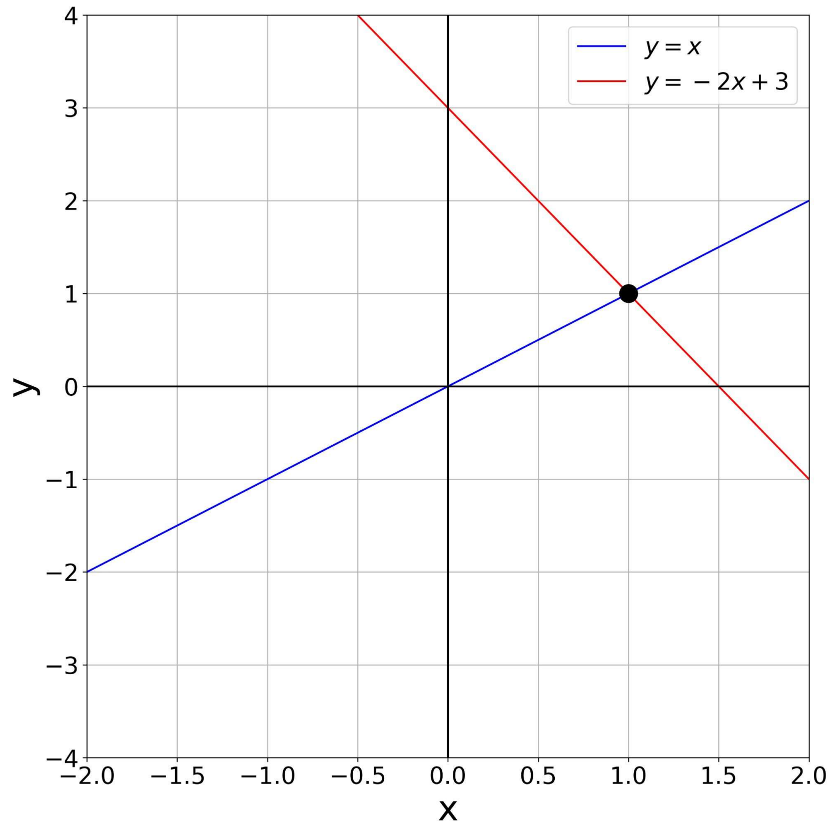
$$\text{Solution: } x = 1, \text{and } y = 1$$

Two Equations and Two Unknowns-
Geometric Interpretation

$$-x + y = 0$$
$$2x + y = 3$$

Row Picture

$$y = x$$
$$y = -2x + 3$$



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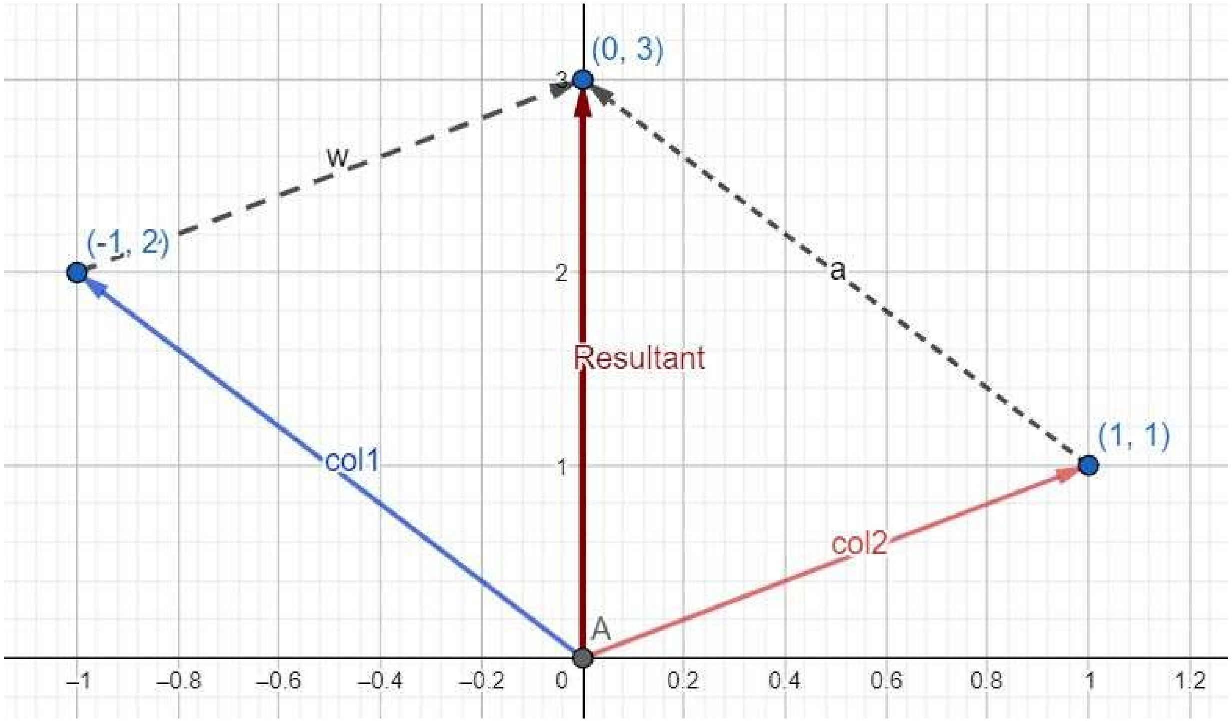
Two Equations and Two Unknowns-
Geometric Interpretation

$$-x + y = 0$$
$$2x + y = 3$$

Column Picture

$$x \begin{bmatrix} -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



Two Equations and Two Unknowns- Some Observations

$$-x + y = 0$$
$$2x + y = 3$$

$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Leftrightarrow \boxed{A\vec{x} = b}$$

$$\Updownarrow$$

$$x \begin{bmatrix} -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

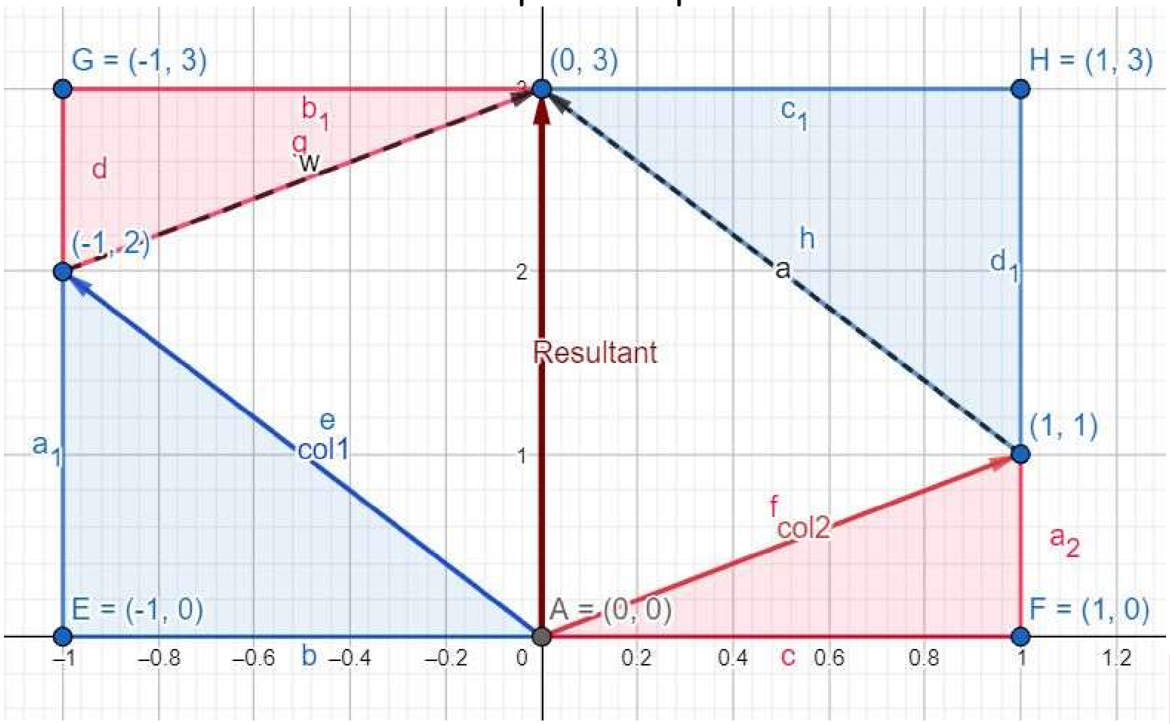
$A\vec{x} = b$ is the weighted linear combinations of columns of A

Two Equations and Two Unknowns- Determinant

$$-x + y = 0$$
$$2x + y = 3$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$|A| = -3$$



Two Equations and Two Unknowns-
Invertibility

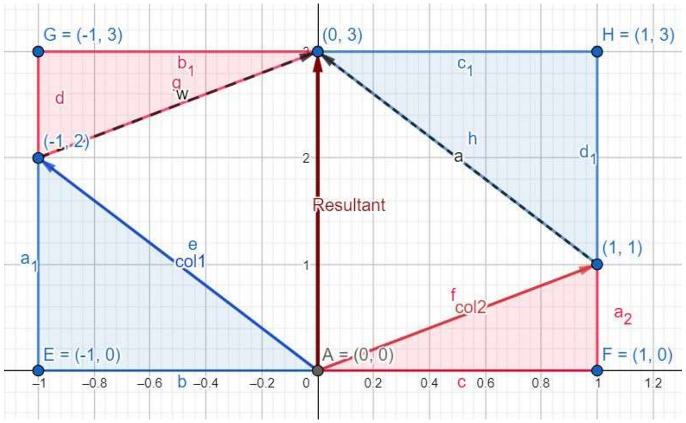
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$-x + y = 0$$
$$2x + y = 3$$

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$|A| = -3$$



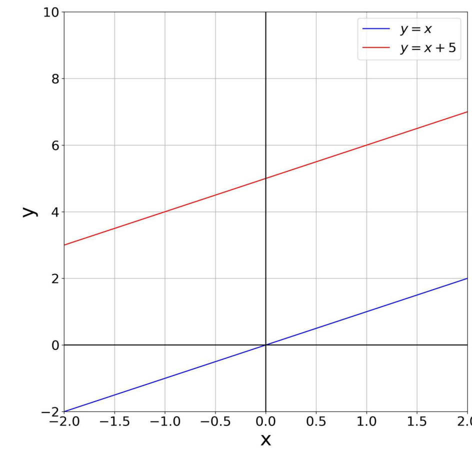
Two Equations and Two Unknowns- Contd..

$$-x + y = 0$$
$$-x + y = 5$$

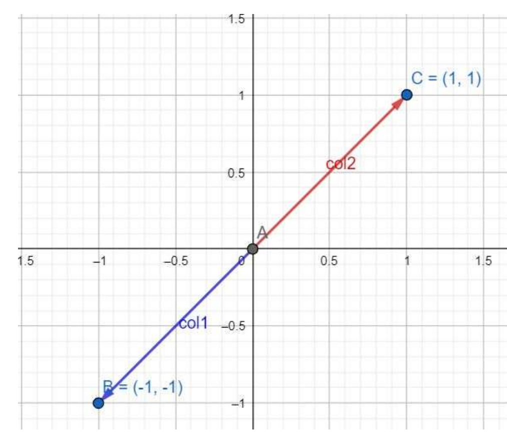
Permanent Breakdown of Elimination

$$-x + y = 0$$
$$-x + y = 5$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$



$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 5 \end{array}\right] \quad R_2 \rightarrow R_2 - R_1 \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 5 \end{array}\right]$$



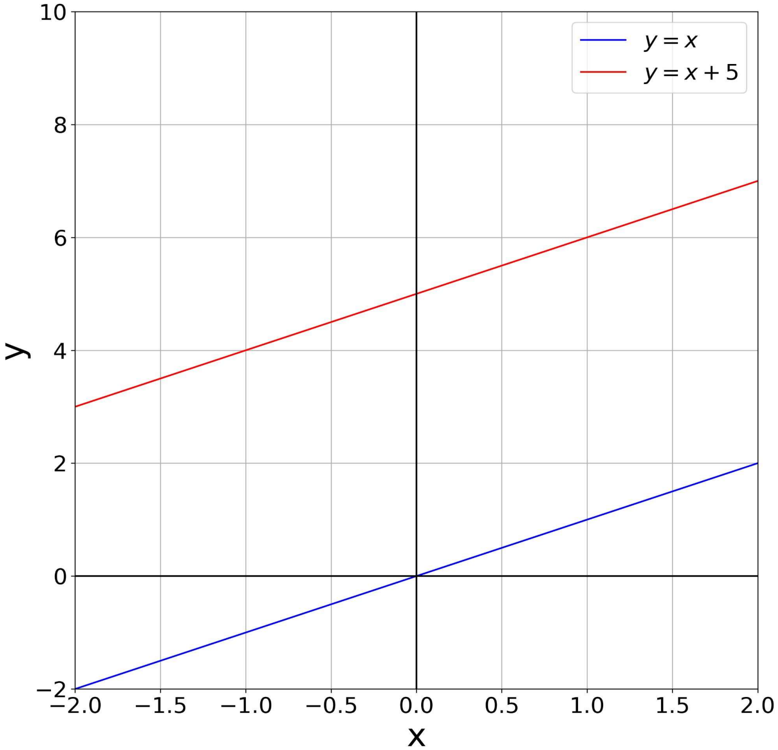
$$0y = 5$$

Permanent Breakdown of elimination (NO SOLUTION)

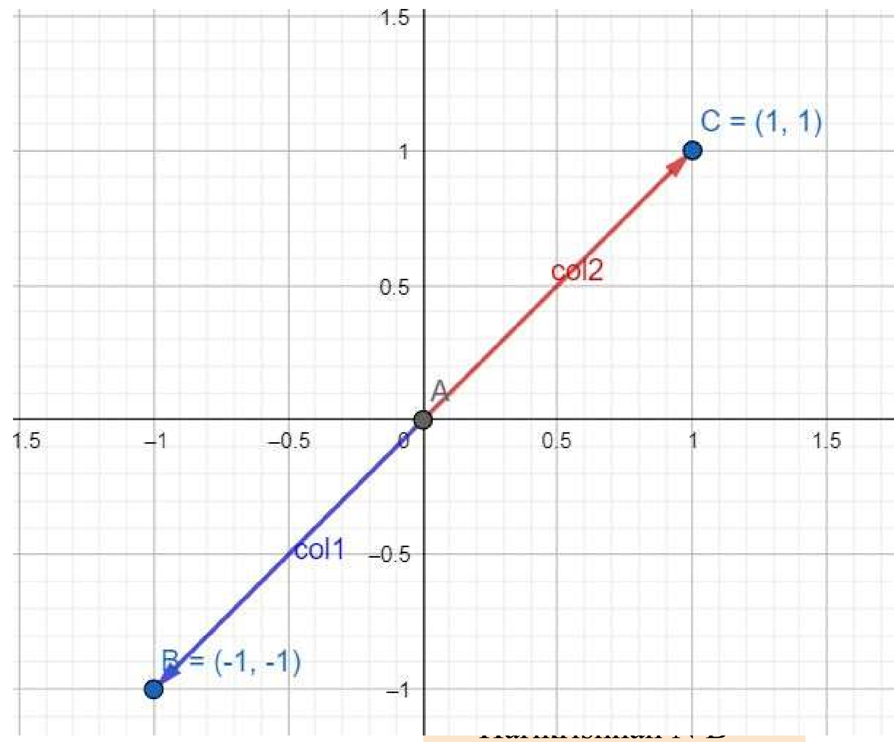
Two Equations and Two Unknowns- Contd..

$$-x + y = 0$$
$$-x + y = 5$$

Row Picture



Column Picture



Two Equations and Two Unknowns- Invertibility

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

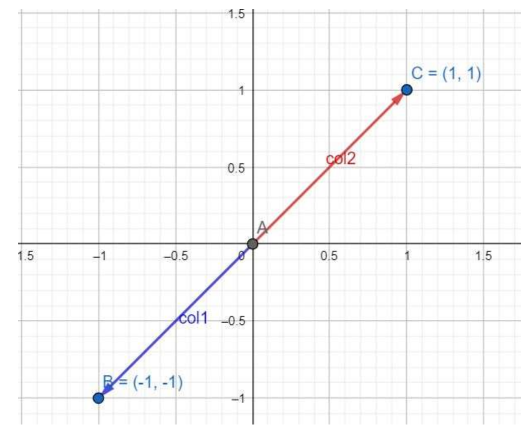
$$-x + y = 0$$
$$-x + y = 5$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

NOT INVERTIBLE

$$|A| = 0$$

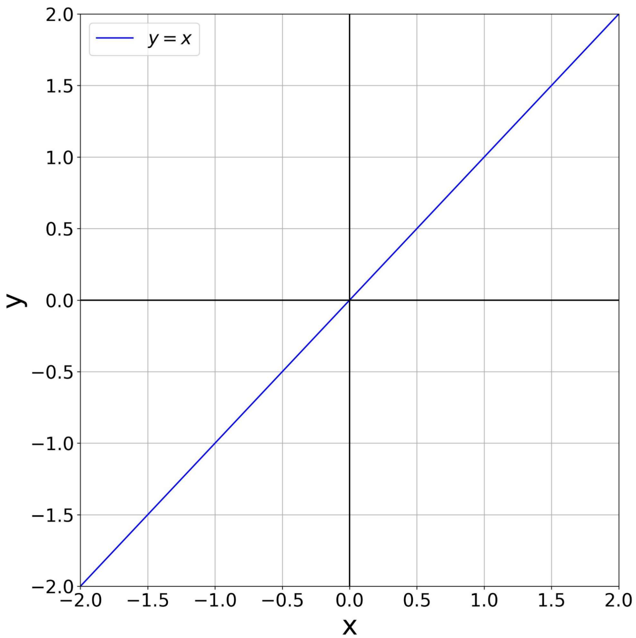


Two Equations and Two Unknowns-
Contd..

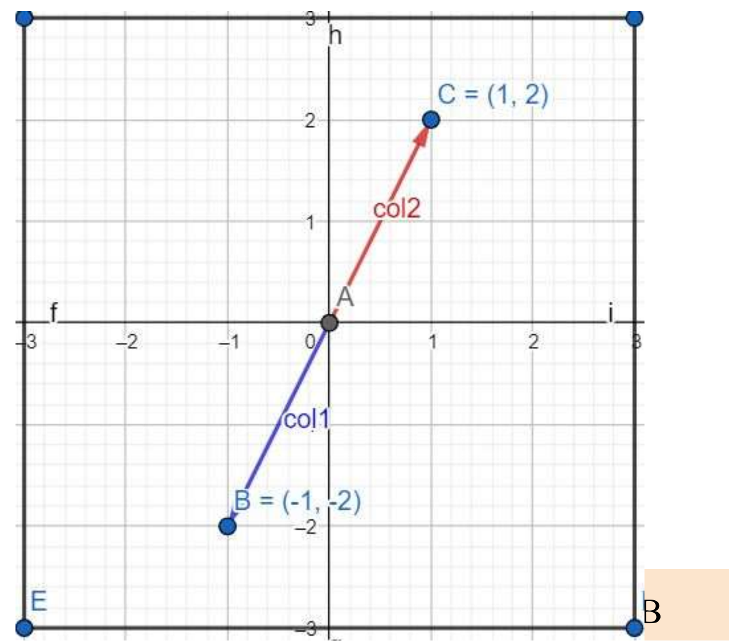
$$-x + y = 0$$
$$-2x + 2y = 0$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row Picture



Column Picture



Two Equations and Two Unknowns- Contd..

$$
\begin{aligned}
-x + y &= 0 \\
-2x + 2y &= 0
\end{aligned}
\qquad
\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}
\begin{bmatrix} x \\ y \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

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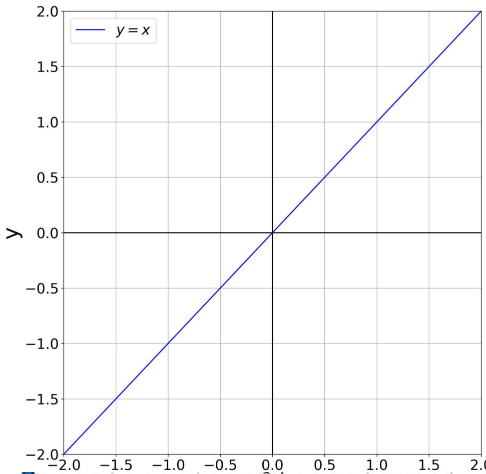
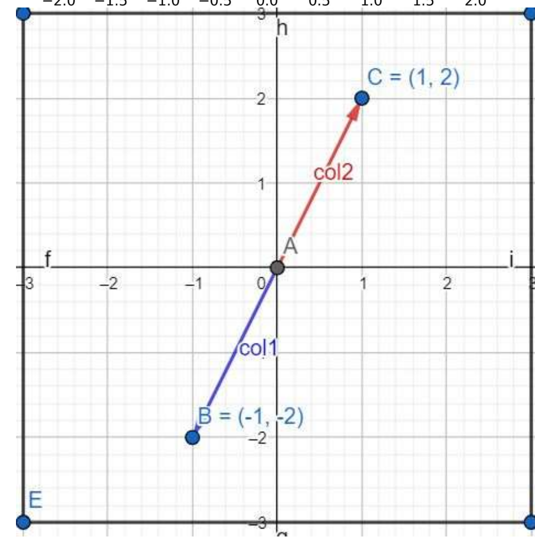
Temporary Breakdown of Elimination

$$-x + y = 0$$
$$-\ 2x + 2y = 0$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] R_2 \to R_2 - R_1 \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$



$$0y = 0$$

***y* can take any value (Infinitely many solutions)**

Two Equations and Two Unknowns- Invertibility

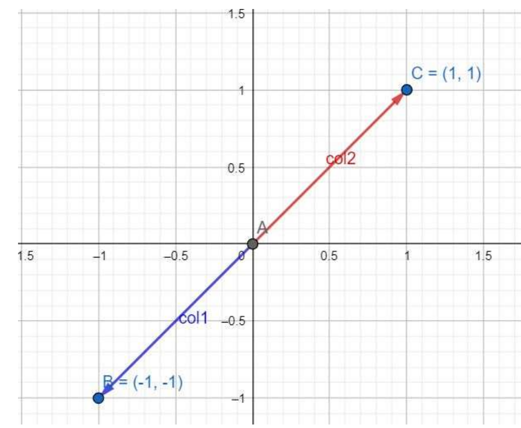
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-x + y = 0$$
$$-2x + 2y = 0$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

NOT INVERTIBLE!!!



$$|A| = 0$$

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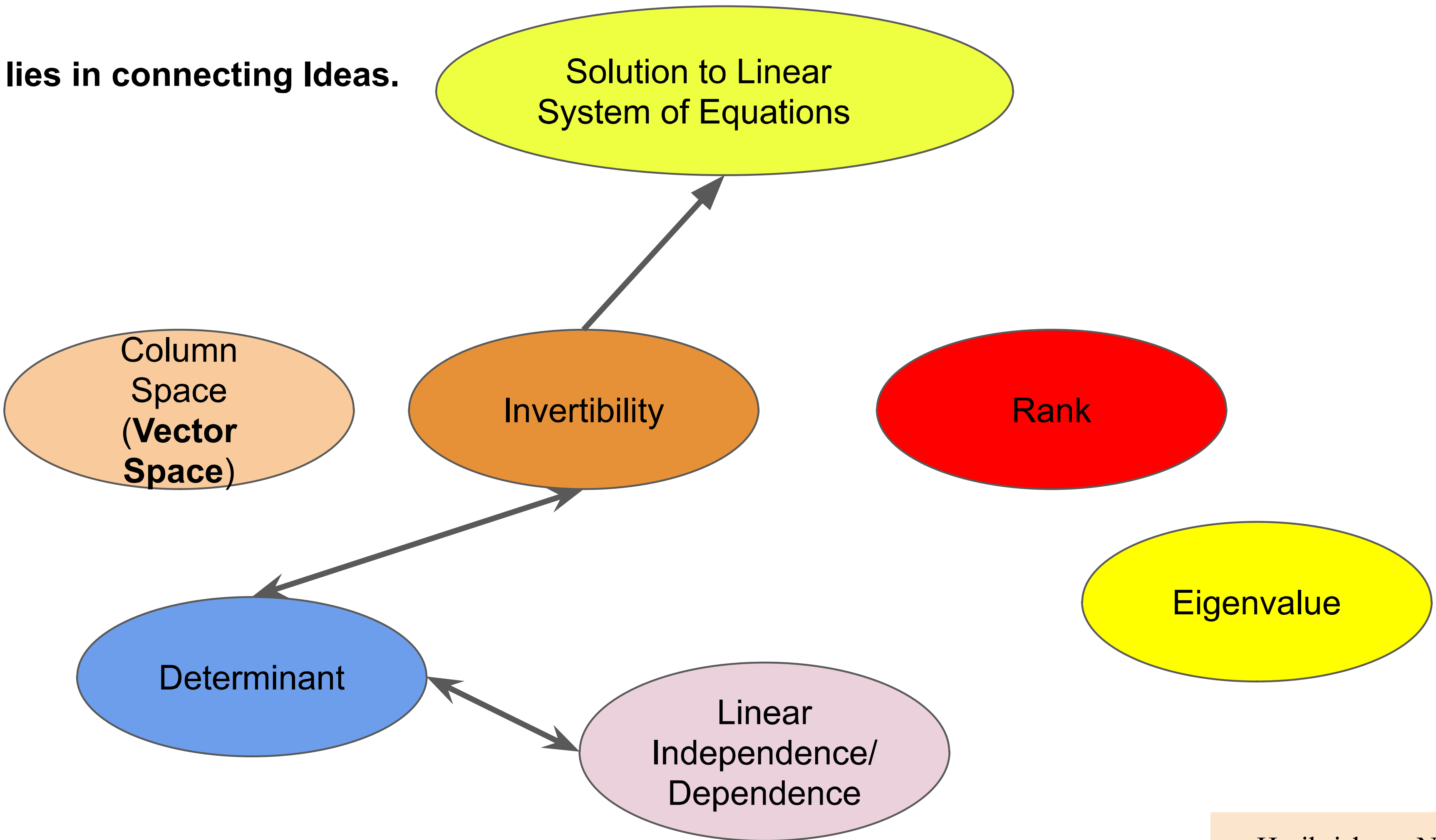
Beauty lies in connecting Ideas.



Dot Product/ Matrix Multiplication/ Inverse

$$\vec{x}.\vec{y} = ||x||||y|| \cos\theta = x^T y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} row_1.col \\ row_2.col \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

Orthogonal and Orthonormal vectors

Orthogonal vectors

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

L2 - norm = $\sqrt{2}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Orthonormal vectors

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

L2 - norm = 1

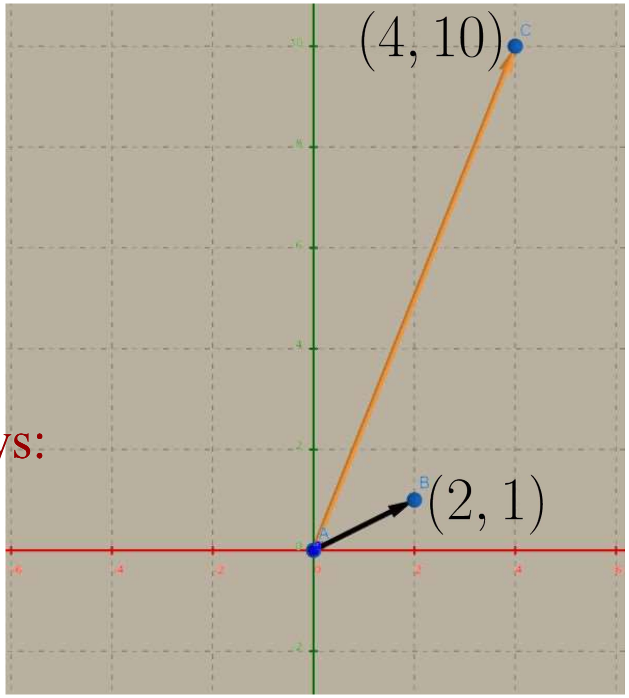
Orthogonal Matrix or Orthonormal Matrix

$$XX^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Vector Multiplication as a Transformation

Intuition for Matrix vector multiplication for Square Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$



Matrix(Square Matrix) vector multiplication can be seen as follows:

- Rotation
- Stretching or Shrinking

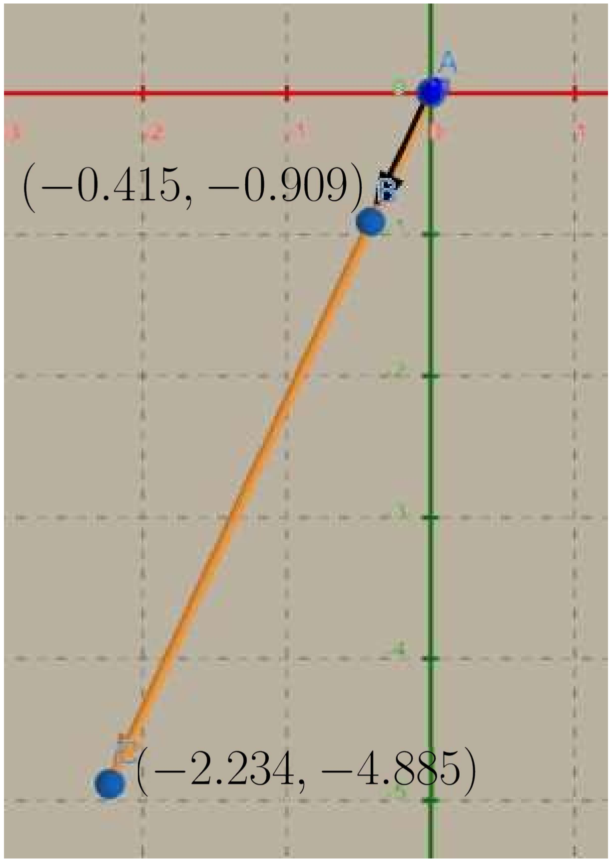
Special Vectors

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-0.415\\ -0.909\end{bmatrix} = \begin{bmatrix}-2.234\\ -4.885\end{bmatrix} = 5.372\begin{bmatrix}-0.415\\ -0.909\end{bmatrix}$$

$$A\vec{x} \qquad \lambda\vec{x}$$

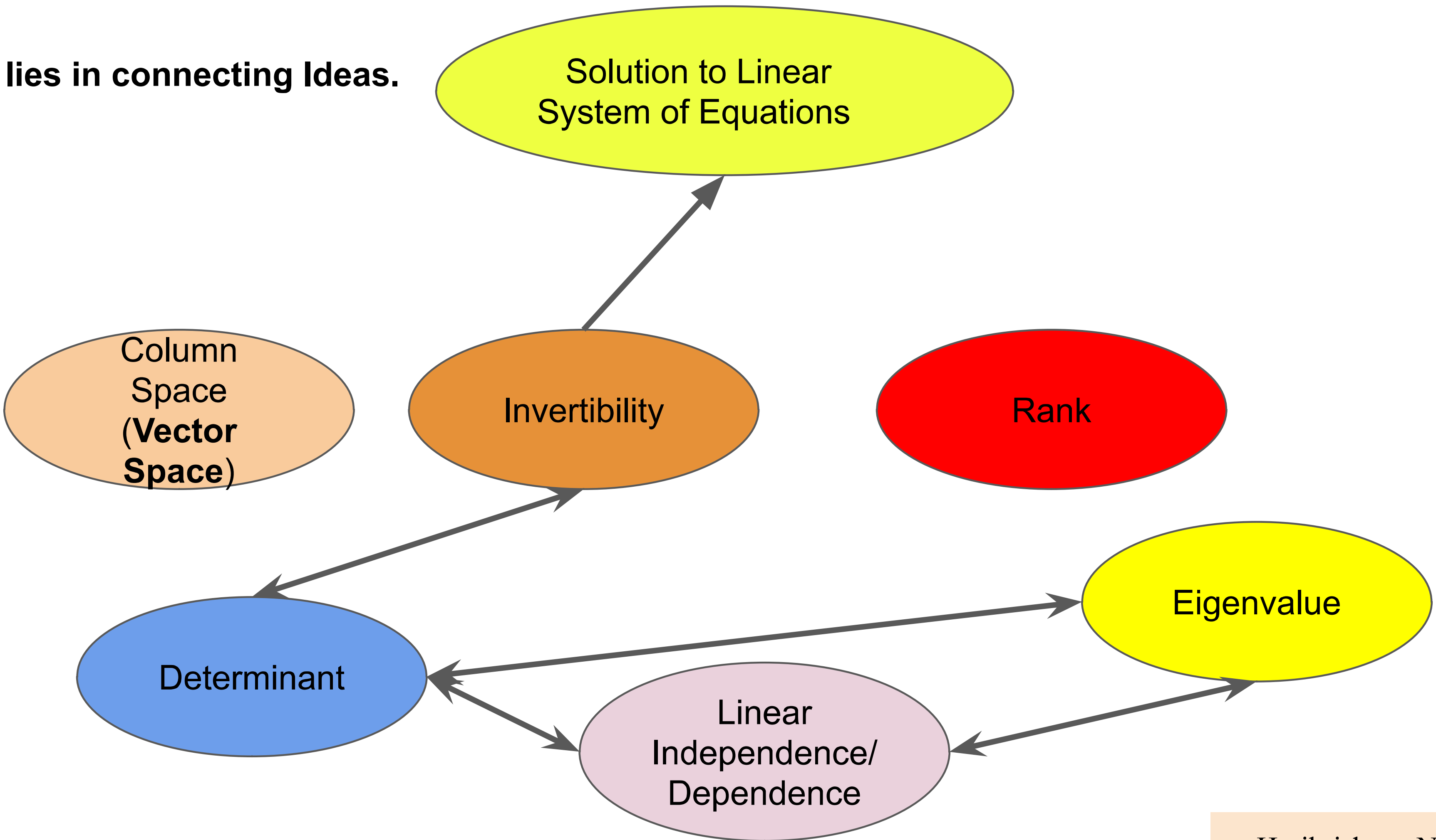
$$A\vec{x} = \lambda\vec{x}$$

1. Direction of $\vec{x}$ is unchanged. (No rotation)
2. Only the magnitude is scaled by a factor λ
3. $\vec{x}$ - **eigenvector of matrix A**
4. λ - **eigenvalue of matrix A**



$$A\vec{x} = \lambda\vec{x}$$

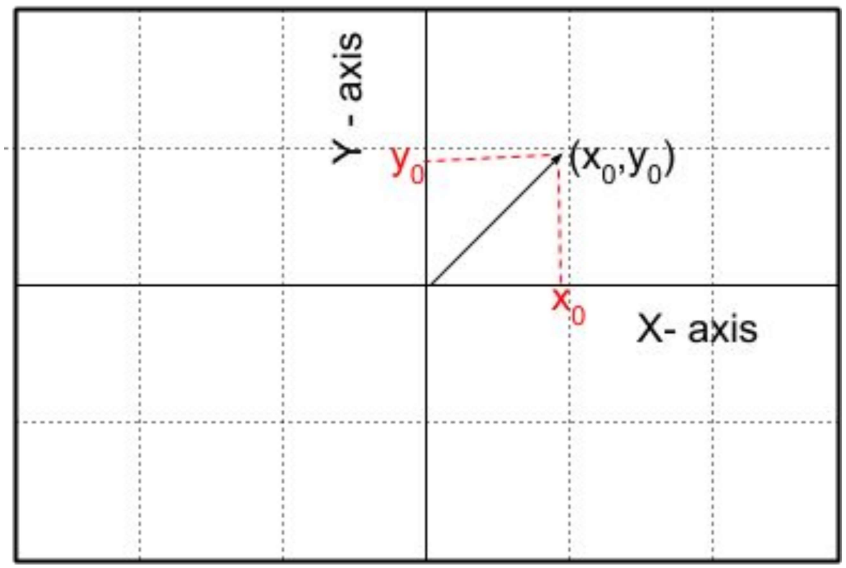
Beauty lies in connecting Ideas.



Second Iteration

Vectors - Different Understanding

Physicists



Computer Scientist

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Mathematicians

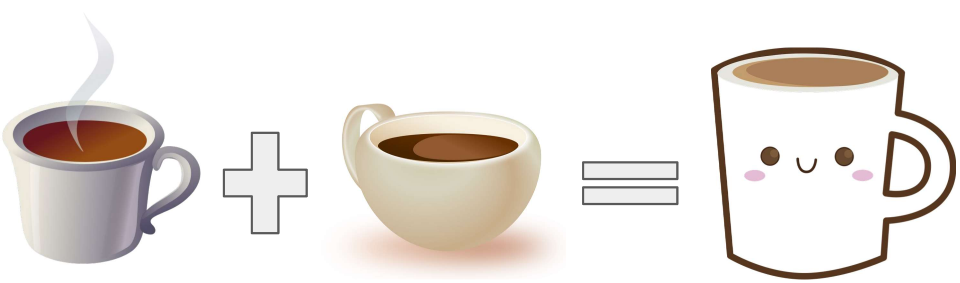
Vector space is a **collection of objects**(it can be anything) called vectors which satisfies mainly two important properties:

1. **closed under vector addition**
2. **closed under scalar multiplication**.

Vector Space - Coffee Space

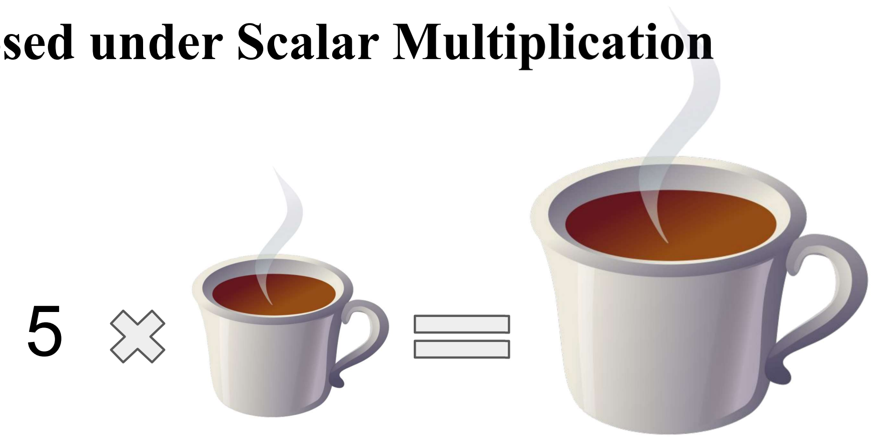
Coffee Space - In Coffee space we have different kinds of coffee with varying strength. Now we will understand the vector space properties with this metaphor.

Closed under Vector Addition



Adding two coffee’s will give you another coffee which is in the coffee space

Closed under Scalar Multiplication



Scaling a coffee will give a coffee which is in the coffee space

Vector Space

- A real vector space is a set/collection of "*vectors*" together with the rules for vector addition and multiplication by real numbers.*

*Strang, Gilbert. *Linear Algebra and Its Applications*. Cengage Learning, 2017.

Dimension and Basis of a Vector Space

Dimension of a Vector space - Every vector space has a dimension. Dimension is the number of basis vectors required to span the vector space.

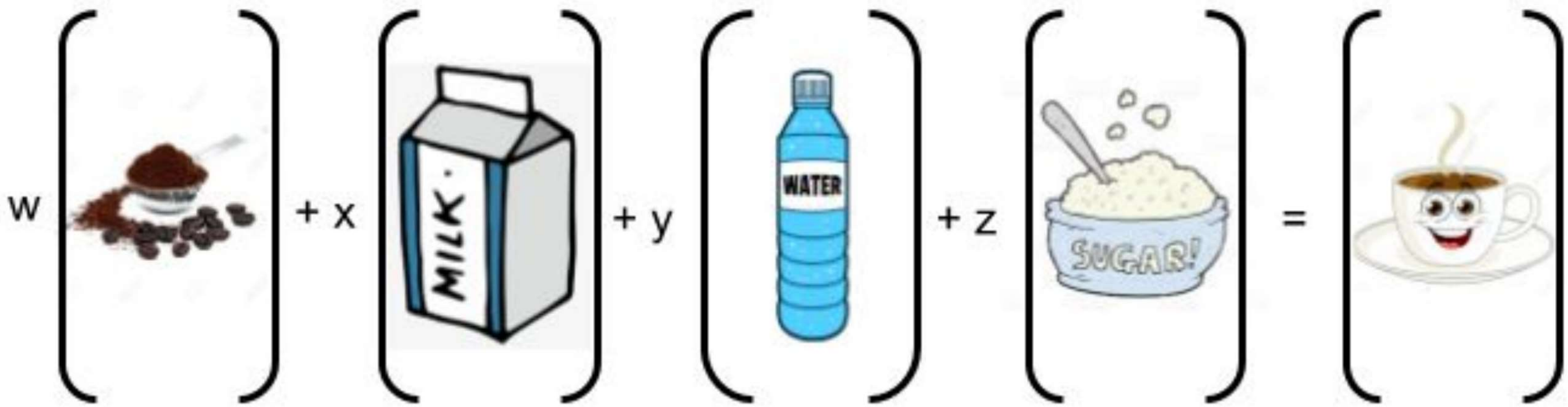
Properties of Basis Vectors -

- Basis vectors has to be linearly independent.
- Basis vectors should span the vector space.

Dimension and Basis of a Coffee Space

- **Linear Independence**
- **Span the space**

Coffee Space- Vector Space



Coffee powder, milk, water and sugar are the basis vectors. Since there are only 4 basis vectors then coffee space has a dimension of 4.

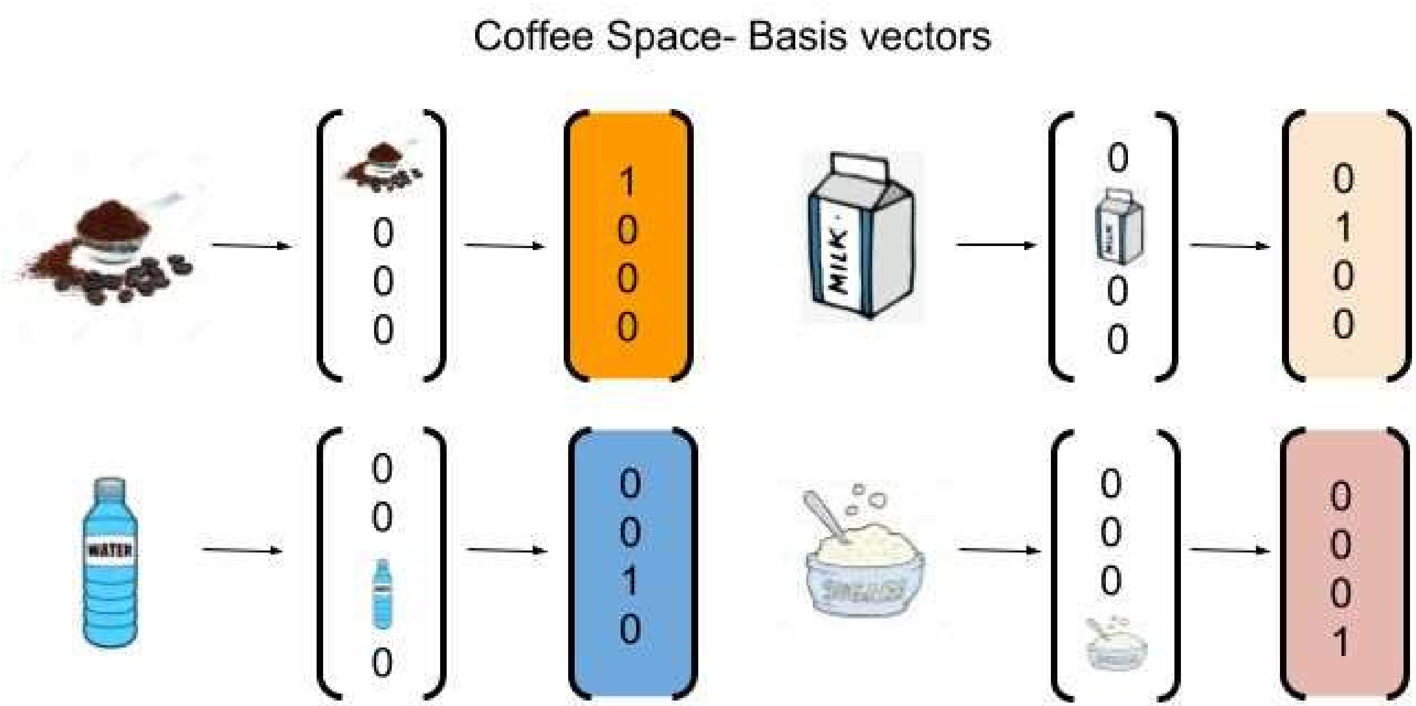
My Friend's Horrible Coffee

My Friend’s Horrible Coffee

My Friend's Horrible Coffee

$$2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\4\\3\end{bmatrix}$$

Visualizing Coffee Space Basis Vectors



Matrix Multiplication - Visualization