Q1 - Endsem - CS6100 - TDAA

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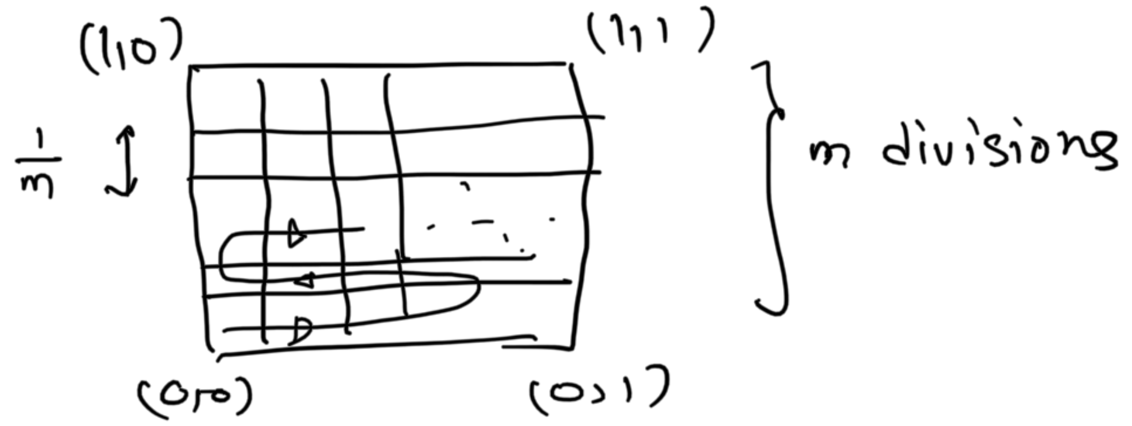
a)



Given that available algorithm for steiner tree is of complexity 2^n ($|F| = n$)
So we need $\log_2 n$ points per division to get polynomial complexity ($n^{O(1)}$)

Expected no. of points per square ($\frac{1}{m} \times \frac{1}{m}$)

$$\Rightarrow \frac{n}{m^2} = \log n$$

$$m = \left(\frac{n}{\log n}\right)^{1/2}$$

Algorithm:

Step 1: Divide $[0,1]^2$ in m^2 small squares.
Each square will give G_i, F_i
(partitions of G, F present in the square)
Apply our 2^n algorithm to G_i, F_i.

Step 2: (Combining problems)
we have got ST in each small square, now we start from left bottom square

and join any leaf in it to any leaf in the square right to it. we will continue like this in snake fashion (shown in diagram) until top right/left square. Since we are joining leaves tree property remains intact. this combining process may not be optimal but the cost incurred will be well with in the bounds.

$$\text{cost of combining} \leq m^2 \, o(\text{diameter of square})$$
$$= o(m^2 \times \frac{1}{m})$$
$$= o(m) \quad (\text{Same as TSP})$$

$$\text{Total cost} = \sum_{i=1}^{m^2} ST(F_i, G_i) + \text{cost of combining}$$

$$= \sum_{i=1}^{m^2} ST(F_i, G_i) + o(m) \quad \text{—①}$$

b) Expected running time:

$$\Rightarrow m^2 \underbrace{o(2^{\frac{n}{m^2}})}_{\text{expected time for each square}} + o(m^2)$$

$$\Rightarrow \frac{n}{\log n} 2^{\frac{n}{n}\log n} + \frac{n}{\log n}$$

$$\Rightarrow o\left(\frac{n^2+n}{\log n}\right) \simeq o\left(\frac{n^2}{\log n}\right)$$

Cost computation:

* Growth bound for $ST(F,G)$ $[G \subseteq F]$

→ Since $ST(F,G)$ is eucledian and Sub additive it satisfies growth bound

$$ST(F,G) \leq O\left(n^{\frac{d-1}{d}}\right) \leq O(\sqrt{n}) \quad \text{—} (2)$$

in expectation

* Bound on $ST_B(F,G)$

By definition

$$ST_B(F,G) \leq ST(F,G)$$

from (2)

$$ST_B(F,G) \leq O(\sqrt{n}) \quad \text{—} (3)$$

Take (1)

$$\Rightarrow PT(x) \leq \sum_{i=1}^{m^2} ST(F_i,G_i) + O(m)$$

→ By point wise closeness which is implication of subadditivity, smoothness and boundary functional.

$$ST(F_i,G_i) \leq ST_B(F_i,G_i) + \frac{1}{m} c\, |F_i|^{\frac{0}{1}}$$

→ factor $1/m$ is due to reduce square dimensions

$$\Rightarrow PT(x) \leq \sum_{1}^{m^2} (ST_B(F_i,G_i) + \sum_{1}^{m^2} \frac{c}{m} + O(m)$$

$$\Rightarrow PT(x) \leq \sum_{1}^{m^2} ST_B(F_i,G_i) + \sum_{1}^{m^2} \frac{c}{m} + O(m)$$

→ w.k.T boundry functionals are super additive

$$\Rightarrow PT(n) \le ST_B(F,G) + \sum_{1}^{m^2} \frac{c}{m} + o(m)$$

→ from (3)

$$\Rightarrow PT(n) \le o(\sqrt{n}) + \sum_{1}^{m^2} \frac{c}{m} + o(m)$$

$$\Rightarrow PT(n) \le o(\sqrt{n}) + m^{\cancel{2}} \frac{c}{\cancel{m}} + o(m)$$

$$\Rightarrow PT(n) \le o(\sqrt{n}) + o(m)$$

→ Substituting $m = \sqrt{n/\log n}$

$$\Rightarrow PT(n) \le o(\sqrt{n} + \sqrt{n/\log n})$$ in expectation

$$\boxed{E(PT(x)) \le o(\sqrt{n} + \sqrt{n/\log n})} \longrightarrow (4)$$

* Given that $ST(F,G)$ satisfies limit theorem

$$\lim_{n\to\infty} \left(\frac{ST(F,G)}{n^{1/2}} \right) = \alpha$$ almost surely $Pr = 1 - O(1)$

So we can say for sufficiently large n

$$E(ST(F,G)) \ge o(n^{1/2}) \quad — (5)$$

from (4) & (5)

$$\Rightarrow \quad \frac{E(PT(X))}{E(ST(X))} \leq o\left(\frac{\sqrt{n}}{\sqrt{n}} + \frac{\sqrt{n/\log n}}{\sqrt{n}}\right)$$

$$\Rightarrow \quad \boxed{\frac{E(PT(X))}{E(ST(X))} \leq o\left(1 + \frac{1}{\sqrt{\log n}}\right)}$$

c) Time complexity $= o(\underbrace{\sum_{i=1}^{m^2} 2^{n_i}}_{\text{each square}} + \underbrace{m^2}_{\text{combing}})$

$$\Rightarrow E(TC) = E\left[\sum_{i=1}^{m^2} 2^{n_i} + m^2\right]$$

$\rightarrow$ $E[X+Y] = E[X] + E[Y]$

$$\Rightarrow \sum_{i=1}^{m^2} E[2^{n_i}] + E[m^2] = E(TC)$$

$\rightarrow$ m value doent change with pertorbation

$$\Rightarrow m^2 + \sum_{i=1}^{m^2} E[2^{n_i}] = E(TC)$$

$\rightarrow$ let $Y = [0,1]^2$ and S is small square after division

$f(Y) \leq \phi \rightarrow f(S) \leq \phi/m^2$

w.k.t $\quad f(S) = Pr[x \in S] = \phi/m^2 \rightarrow$ 6

$\rightarrow$ from 6 $\quad E[n_i] = \sum_{1}^{n} P[n \in S_i]$

$$E[n_i] = n\phi/m^2 \rightarrow 7$$

→ 2^{n} is convex hence we can say $E[2^{n}] \geq 2^{E[n]}$ but this is not useful as ϕ is bounded ($\phi<1$) we can safely assume that

$$E[2^{n_i}] = O(2^{E(n_i)})$$

(for some large constant)

$$\Rightarrow E(TC) \leq m^2 + O\left(\sum_{1}^{m^2} 2^{E(n_i)}\right)$$

$$\leq m^2 + \sum_{1}^{m^2} 2^{n\phi/m^2}$$

$$\leq m^2 + m^2\, 2^{n\phi/m^2}$$

→ Substituting $m = \sqrt{n/\log n}$

$$E(TC) = O\left(\frac{n}{\log n} + \frac{n}{\log n}\, 2^{\phi \times \log n}\right)$$

$$= O\left(\frac{n}{\log n} + \frac{n \cdot n^{\phi}}{\log n}\right)$$

$$\boxed{E(TC) \simeq \frac{n^{(\phi+1)}}{\log n}}$$