Hofstadter Hamiltonian

1 Introduction

The Hofstadter Hamiltonian deals with the two dimensional tight binding model in the presence of an external magnetic field. The solution of this Hamiltonian gives two surprising results - the first being the self similar nature of spectrum obtained and the second, the difference in the behaviour of the spectrum spectrum depending on rationality or irrationality of a parameter pertaining to the magnetic flux. Intuition tells us that a small change in a rational parameter would make it irrational and vice versa. This would result in the experimentally measurable spectrum being discontinuous everywhere, which is unreasonable. We will see along this set of notes how this behaviour of differentiating rational and irrational numbers emerge. Further, we will show how this apparent paradox of measurable quantities being discontinuous everywhere can be explained away.

2 Derivation of the Difference Equation

For studying the effects of the magnetic field, we will be considering the tight binding two dimensional square lattice with spacing a. A uniform magnetic field $\vec{H}$ is applied perpendicular to it. Further, we will be restricting ourself to a single Bloch band to study the effects of the applied magnetic field.

For the two dimensional tight binding model, the Bloch energy function is given by,

$$W(\vec{k}) = 2E_0 \left(\cos(k_x a) + \cos(k_y a)\right) \tag{1}$$

To account for the effect of the magnetic field, we replace $\hbar k$ with $\vec{p} - \frac{e\vec{A}}{c}$, where $\vec{A}$ is the vector potential. This is known as the "Peierls Substitution". We would have then created an operator out of $W(\vec{k})$, corresponding to the energy function for the tight binding model in the presence of a magnetic field.

We further re-write $W(\vec{k})$ as,

$$W(\vec{k}) = 2E_0 \left[\cos\left(\frac{(\vec{p}_x - e\vec{A}_x/c)a}{\hbar}\right) + \cos\left(\frac{(\vec{p}_y - e\vec{A}_y/c)a}{\hbar}\right)\right] \tag{2}$$

Using the available gauge freedom, we choose the Landau gauge, $\vec{A} = H(0, x, 0)$. Further, writing $W(\vec{k})$ in terms of exponential functions,

$$W(\vec{k}) = E_0 \left[\exp\left(\frac{ia\vec{p}_x}{\hbar}\right) + \exp\left(\frac{-ia\vec{p}_x}{\hbar}\right) + \exp\left(\frac{i(\vec{p}_y - eHx/c)a}{\hbar}\right) + \exp\left(\frac{-i(\vec{p}_y - eHx/c)a}{\hbar}\right)\right] \tag{3}$$

Recall that $\exp\left(\frac{-iap_x}{\hbar}\right)$ and $\exp\left(\frac{-iap_y}{\hbar}\right)$ corresponds to translation operator in the x and y direction respectively.

Now operating $W(\vec{k})$ on the wave function $\psi(x, y)$ we get,

$$E_0 \left[\psi(x+a, y) + \psi(x-a, y) + \exp\left(\frac{-ieHax}{\hbar c}\right)\psi(x, y+a) + \exp\left(\frac{ieHax}{\hbar c}\right)\psi(x, y-a)\right] = E\psi(x, y) \tag{4}$$

We make the following substitutions,

$$x = ma \qquad y = na \qquad \frac{E}{E_0} = \epsilon \tag{5}$$

Since the coefficients in Eq(4) only depends on x, it is reasonable to assume plane wave solution in the y direction.

$$\psi(ma, na) = e^{i\nu n} g(m) \tag{6}$$

Finally we introduce the parameter which would be responsible for all the interesting effects,

$$\alpha = \frac{a^2 H}{2\pi(\hbar c/e)} \tag{7}$$

Note that α is dimensionless. Physically, it corresponds to the ratio of flux through a lattice cell to one flux quantum. A value of $\alpha = 1$ corresponds to 10^9 gauss for a lattice spacing $a = 2$ A°.

Making all the substations in Eq(4), the Schrödinger equation turns into a one dimensional difference equation,

$$g(m+1) + g(m-1) + 2\cos\left(2\pi m\alpha - \nu\right) g(m) = \epsilon g(m) \tag{8}$$

Eq(8) is sometimes refered to as Harper's equation.

3 Calculation of the Spectrum and the Rationality Conditions

Note that most of the results in this section have not been proved. They have merely been mentioned. For proofs of the results, refer to D. Hofstadter's thesis work, "The Energy Levels of Bloch Electrons in a Magnetic Field"

Eq(8) can be re written in a matrix form as,

$$\begin{pmatrix} g(m+1) \\ g(m) \end{pmatrix} = \begin{pmatrix} \epsilon - 2\cos(2\pi m\alpha - \nu) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g(m) \\ g(m-1) \end{pmatrix} \tag{9}$$

The 2×2 matrix in Eq(9) is called as $A(m)$. Using the product of m matrices, Eq(9) can be rewritten as,

$$\begin{pmatrix} g(m+1) \\ g(m) \end{pmatrix} = A(m)A(m-1)\cdots A(2)A(1) \begin{pmatrix} g(1) \\ g(0) \end{pmatrix} \tag{10}$$

Since the A matrix has a cosine function, we expect that the A matrices are periodic. This would mean that long products of the A matrices would repeat with the same period. Now let us assume that the A matrices are periodic in m, with a period q. From this, we get the condition for periodicity as,

$$2\pi\alpha(m+q) = 2\pi\alpha m - \nu + 2\pi m \tag{11}$$

$$\implies \alpha = \frac{p}{q} \tag{12}$$

This is means that the A matrix will be periodic, provided that α is rational.

We also define the product of q consecutive A matrices as Q.

$$Q = A(q)A(q-1)\cdots A(2)A(1) \tag{13}$$

We are also required to impose a physical constraint of boundedness on the wave-function, $g(m)$ for all m. This can be restated as a constraint on the matrix Q. For the functions $g(m)$ to be bounded, we require that the eigenvalues of matrix Q to be of unit magnitude. This can be further rewritten as a constraint on the trace of matrix Q as follows,

$$|TrQ(\epsilon;\nu)| \leq 2 \tag{14}$$

D. Hofstadter has shown that, ν merely shifts the graph up or down, the shape of the graph is unaffected by it. The spectrum we are interested in is the union over all values of ν, such

that Eq(14) is satisfied.

Further, as proved in D. Hofstadter's thesis work, we can rewrite the trace as,

$$TrQ(\epsilon;\nu) = TrQ(\epsilon) + 2f(\nu) \tag{15}$$

where $f(\nu)$ is a periodic function of unit amplitude. Whenever we write $Q(\epsilon)$, without explicit reference to ν, we mean, $Q(\epsilon;\frac{\pi}{2q})$[1](The justification for such substitution is provided in D. Hofstadter's thesis work). Using the triangular inequality, Eq(14) can be rewritten as,

$$|TrQ(\epsilon)| \leq 4 \tag{16}$$

Eq(16) has made our lives simpler as it has shifted the condition of union over all values of ν to a simple inequality for $\nu = \frac{\pi}{2q}$. Remember that TrQ refers to $TrQ(\epsilon;\nu = \frac{\pi}{2q})$. This is particularly useful in numerical calculations as it reduces the number of calculations required for calculating the spectrum for each value of α significantly.

Eq(16) would always give a polynomial of degree q in ϵ. Hence, we would expect that there are q regions where the inequality given in Eq(16) is satisfied. This is the basis for the interesting behaviour of the spectrum looking discontinuous everywhere. For $\alpha = \frac{p}{q}$, a small change in α could cause a large change in the value of q. This would mean that a small fluctuation in α could cause the previous line to split into many, many different parts! We will resolve this apparent anomaly in the upcoming sections.

A few results of the spectrum belonging to a particular value of α are mentioned,

(i) Spectrum(α) and spectrum(α+N) are identical.

(ii) Spectrum(α) and spectrum(-α) are identical.

(iii) ϵ only belongs to the spectrum(α) if and only if $-\epsilon$ belongs to the spectrum(α).

(iv) If ϵ belongs to the spectrum(α) for any α, then $-4 \leq \epsilon \leq 4$.

Using properties (i) and (iv), we define a unit cell, which is restricted in ϵ between -4 and 4, and restricted in α along an interval of unit length. We choose the unit interval [0,1]. Further, as the consequences of these properties, we see that the unit cell has two axes of reflection, $\epsilon = 0$ and $\alpha = \frac{1}{2}$.

So to obtain the spectrum, we are finding values of ϵ where Eq(16) is valid for a given value of α. Then we would increment the values of α and repeat the process again for all discretized values of ϵ. This procedure is repeated till we exhaust all the values of α we are interested

[1]In the original article by D. Hofstadter's, "Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields", there is a typographic error where $Q(\epsilon)$ is defined as $Q(\epsilon;\frac{1}{2q})$. The error was rectified after referring to the thesis work of D. Hodstadter, where it is defined correctly as we have defined.

in. The resulting graph, Fig 1, computed numerically[2], is sometimes called as Hofstadter's butterfly.

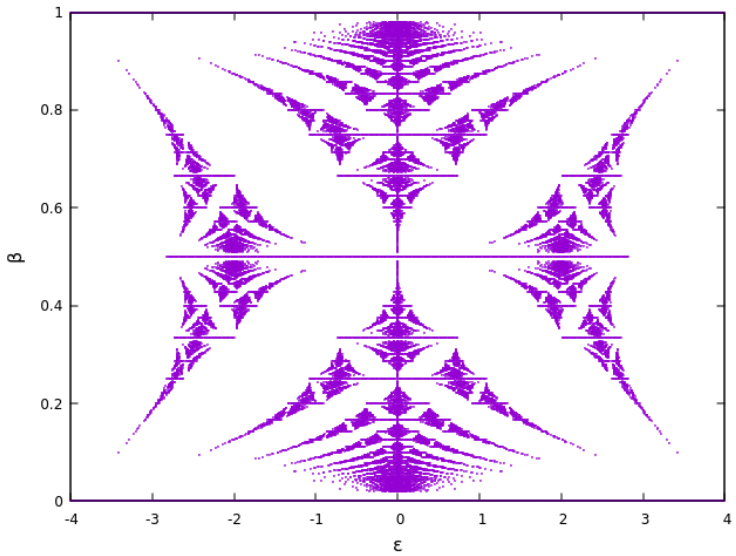


Figure 1: Spectrum inside a unit cell. Also referred to as Hofstadter's butterfly

4 Recursive Structure of the Graph

All the statements made in this section are based on the results obtained from numerical calculations by D. Hofstadter. All the results mentioned are hence to be taken as "empirically proved".

The graph obtained has some unusual properties. This graph is sometimes referred to as Hofstadter's butterfly because of gaps forming a shape similar to a butterfly. Further, these butterflies are stacked in a self similar pattern. In this section, we would try to explain the rules of formation of these hierarchically organized patterns.

First, let us define the notion of a unit cell in this graph. A unit cell is any portion of the graph located between two successive integral values of α. We refer to the unit cell between N and $N+1$ as the N^{th} unit cell. Every unit cell has a local variable β, which runs from 0 to 1. β by definition is the fractional part of α, denoted by $\{\alpha\}$. At $\beta = 0$ and $\beta = 1$, we have band stretching across the full width of the unit cell. These band are referred to as the cell walls os the unit cell.

Certain rational values of the unit cell play an important role in the description of the structure of the unit cell. These are the **pure cases**, given by,

[2]The program used to compute the graph can be found at

$$\frac{1}{N} \qquad \text{and} \qquad 1-\frac{1}{N} \qquad (N \geq 2) \tag{17}$$

and the **special cases**, given by,

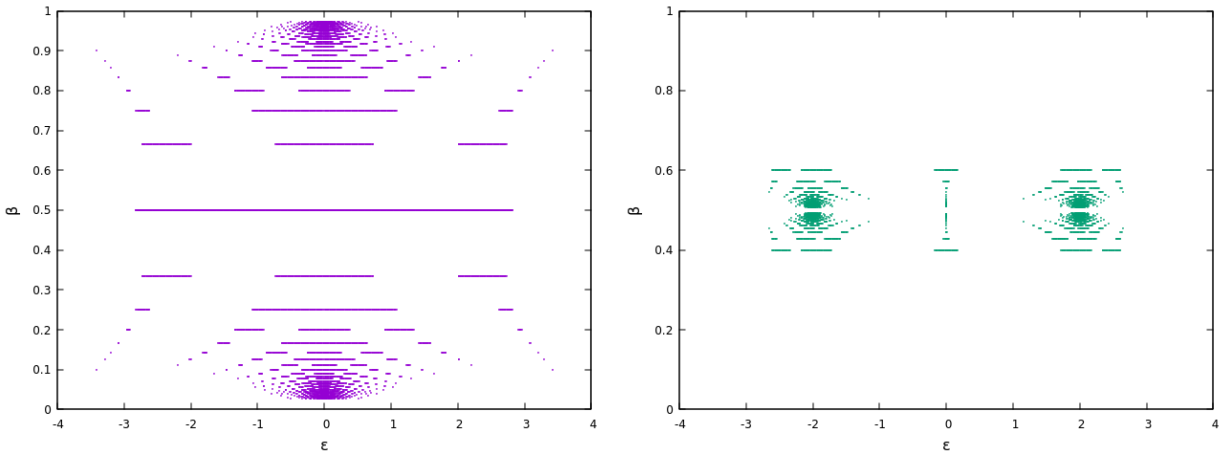
$$\frac{N}{2N+1} \qquad \text{and} \qquad \frac{N+1}{2N+1} \qquad (N \geq 2) \tag{18}$$

The spectra belonging to these rational values defines a "skeleton" on which rest of the graph is hung. By using a recursive process, the rest of the graph can be built. This recursive process involves compressing the original skeleton to a fraction of it's original size, then distorting the vertical and horizontal scale, and finally reinserting it in the region between the two "wings". Continuing this process indefinitely would result in the complete graph. To turn this process into precise definition, we require these three "empirically proved" statements,

Statement I

At any height in the unit cell, where β equals the "pure cases", i.e, either $\frac{1}{N}$ or $1-\frac{1}{N}$, there are N bands between the left and the right of the cell. When N is even, it might seem like there are only $N-1$ bands, but we consider the bands touching at $\epsilon = 0$ as two distinct bands, and refer to them as "degenerate" bands. When N goes to infinity, the ratio of band size to gap size goes to zero. We see that the different values of N allowed for the pure cases enables it to span across the whole of the unit cell ($\beta = \frac{1}{2}$ when $N = 2$. $\beta \to 0$ or 1 when $N \to \infty$). A representation of these pure cases can be seen in Fig 2a.

When β equals the "special cases", i.e, either $\frac{N}{2N+1}$ or $\frac{N+1}{2N+1}$, we get a similar set of bands.We are only interested in the centremost set of bands for the special cases. The width of these centremost bands fo to zero as N increases. We see that $\beta = \frac{2}{5}$ or $\frac{3}{5}$ when $N = 2$ and $\beta \to \frac{1}{2}$ as $N \to \infty$. We can see a representation of these special cases in Fig 2b. We are only interested in the set of bands at the centre.



(a) Pure Cases for $N \leq 37$ (b) Special Cases for $N \leq 37$

Figure 2

Superimposing images from Fig 2a and the central bands from Fig 2b, we get Fig 3. The procedure of constructing the subcells will be performed on this figure.

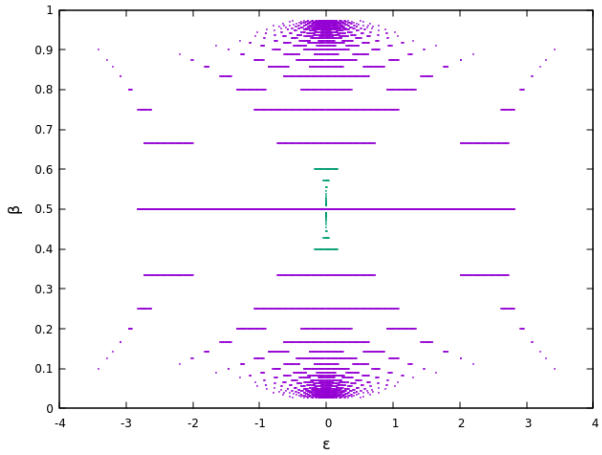


Figure 3: Superimposed image of pure case and central bands of special case

Rules for Subcell Construction

As the next two statements depend on the definition of what a subcell is, we shall provide the rules of it's construction. The definitions of the subcell is assumed to be clear from the procedure of construction. It is best to refer Fig 4 while going to the procedure so that the idea is clear.

The subcells are defined into three categories, the L chain, the R chain and the C chain. The L and R chains have the same procedure for construction, while the C chain differs from it. The rules are as follows.

To construct L and R chain, connect the edges of the outermost bands of neighbouring pure cases with straight lines. The trapezoidal boxes hence formed on the left and right sides corresponds to the L and R chain respectively. This is seen as the blue lines in the outer ends and the red lines connecting the inner ends in Fig 4.

To construct the C chain, first connect all the middle bands of pure cases with $N > 2$ with straight lines with the $N = 2$ special case. Now connect the remaining centremost bands of the special cases with straight lines. The regions formed from this procedure is referred to as the C chain. This is seen as the green lines in Fig 4. Note that C_0 and C_{-1} connects the first special case with all the pure cases, where as the other C subcells are created by connecting the special cases amongst each other.

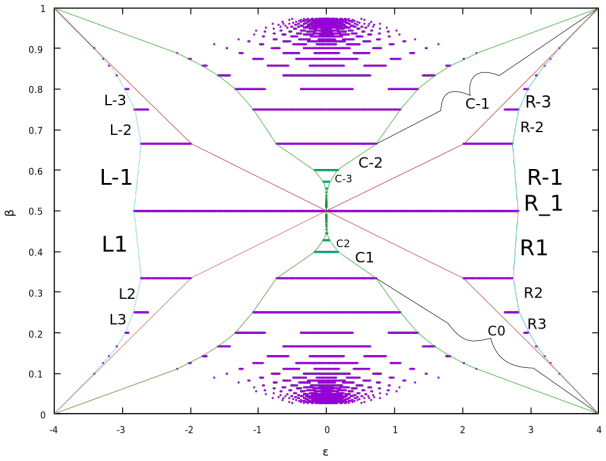


Figure 4: Constructed subcells with labelling scheme

Statement II

The regions of a cell outside its subcells are gaps. (This region which looks like the wings of a butterfly with no bands are called gaps).

Statement III

Each subcell of any cell can be given a local variable, which is defined in terms of the local variable of the parent cell. Each subcell, when indexed by its own local variable is a cell in its own right, i.e, it satisfies all the statements I, II and III.

We will now try to define the subcell's local variable, β' in terms of the outer cell's local variable, β. Let us assume that $\beta \leq \frac{1}{2}$. Then let N' be defined by,

$$N' = \left[\frac{1}{\beta}\right] \tag{19}$$

where $[x]$ denotes the greatest integer less than or equal to x.

For L or R type subcells, we relate β' and β by

$$\beta = (N' + \beta')^{-1} \tag{20}$$

This can be justified as follows because we know β lies between $\frac{1}{N}$ and $\frac{1}{N+1}$ (for pure cases of the first kind). Comparing it with Eq(20), we can see that β' lies between 0 and 1. For pure cases of the second kind, a similar argument can be show. Let us define $\Lambda(\beta)$ as the function that relates β to β'. The closed form of $\Lambda(\beta)$ can be written as,

$$\beta' = \Lambda(\beta) \equiv \frac{1 - N'\beta}{\beta} \tag{21}$$

For C type subcells, the relation is given by,

$$\beta = \left(2 + \frac{1}{\alpha'}\right)^{-1} \tag{22}$$

$$\beta' = \{\alpha'\} \tag{23}$$

where, $\{x\}$ denotes the fractional part of x.

Again this can be justified as follows for the first kind of special cases. β lies between $\frac{N}{2N+1}$ and $\frac{N+1}{2N+3}$ which is the same as $\frac{1}{2+\frac{1}{N}}$ and $\frac{1}{2+\frac{1}{N+1}}$. Comparing with, Eq(22), we see that β' lies between 0 and 1. Again a similar argument can be constructed for the special cases of second kind. We denote the function that relates β to β' as $\Gamma(\beta)$. The closed form for $\Gamma(\beta)$ can be written as,

$$\beta' = \Gamma(\beta) \equiv \left\{\frac{\beta}{1 - 2\beta}\right\} \tag{24}$$

Now, if $\frac{1}{2} < \beta < 1$, we define $\beta' = 1 - \beta$. All these definitions together gives a unique value between 0 and 1 to the inner local variable β'. Hence we have shown that each of the subcells have similar properties to that of the unit cell.

One property that the subcell is missing is that of degeneracy for even values of N. The bands corresponding to even N does not join in the middle, rather, there is a small gap between the two bands.

According to Azbel, as mentioned by the author, the spectrum is completely determined by the continued fraction of α. Now, if we use Eq(20), and iterate, that is, replace β by β' and β' by β'', and continue this process ad infinitum, we get the representation,

$$\beta = \cfrac{1}{N_1 + \cfrac{1}{N_2 + \cfrac{1}{N_3 + \cdots}}} \tag{25}$$

The representation given by Eq(25) is unique and will terminate for any rational value of α. Azbel predicts that the spectrum(α) will consist of N_1 bands, each of which breaks into N_2 sub-bands, each of which breaks into N_3 sub-bands, and so on. No information regarding the C subcells is obtained from this expansion.

When N_2,N_3,$\cdots$ are very large, we can compare Eq(25) with statement I. We see that for pure cases, the band breaks into N_1 different bands.

5 How the Bands are Clustered

Before understanding the pattern of clustering, let us define a notation to choose a subcell from the hierarchical structure of the subcells. Let us do this by an example. We will define a subcell $U_7L_{-2}C_0R_3L_1$. The notation is to be understood as follows, U_7 stands for the 7^{th} unit cell. L_{-2} stands for the -2^{th} subcell in L subcells of the 7^th unit cell. C_0 stands for the 0^{th} subcell in the C subcells nested within the previous two, and so on and so forth.

Suppose we are interested in describing the pattern of the bands formed at $\alpha = \frac{p}{q}$. β is the local variable defined as $\{\alpha\}$. Let use denote the pattern found at the value β by $\Pi(\beta)$. The recursive structure we had defined in the previous section shows that each band is roughly divided into three regions, one each in the L, R and C subcells. That is, we can denote the pattern formed in the unit cell, $\Pi(\beta)$, as the combination of three different, $\Pi(\beta')$, where each β' represents the value of the new local variable defined in each of the subcells. From Eq(20) and Eq(22), we know how β and β' are related. So we can define the structure formed recursively as follows,

$$\Pi(\beta) = \Pi(\Lambda(\beta))\Pi(\Gamma(\beta))\Pi(\Lambda(\beta)) \tag{26}$$

Note that the the expression in Eq(26) should not be seen as multiplication, rather it should be seen as placing the patterns formed side by side.

Let us take the example of $\alpha = \frac{5}{17}$. It's spectrum looks as follows,

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We can also represent it as follows,

(2-1-2)-(2-3-2)-(2-1-2)

We can use Eq(26) to explain why the breakdown is 5-7-5. Using the equations Eq(21) and Eq(24) ($\Lambda\left(\frac{5}{17}\right) = \frac{2}{5}$ and $\Gamma\left(\frac{5}{17}\right) = \frac{5}{7}$), we can write the pattern as,

$$\Pi\left(\frac{5}{17}\right) = \Pi\left(\frac{2}{5}\right)\Pi\left(\frac{5}{7}\right)\Pi\left(\frac{2}{5}\right) \tag{27}$$

Now remember that for $\alpha = \frac{p}{q}$, the band is broken into q different segments. Hence we see that, Eq(27) is broken into segments, 5-7-5.

We can carry out this process again to get the next nested structure,

$$\Pi\left(\frac{2}{5}\right) = \Pi\left(\frac{1}{2}\right)\Pi\left(0\right)\Pi\left(\frac{1}{2}\right) \tag{28}$$

$$\Pi\left(\frac{5}{7}\right) = \Pi\left(\frac{1}{2}\right)\Pi\left(\frac{1}{3}\right)\Pi\left(\frac{1}{2}\right) \tag{29}$$

This corresponds to the structure,

$$\Pi\left(\frac{2}{5}\right) = \text{2-1-2} \tag{30}$$

$$\Pi\left(\frac{5}{7}\right) = \text{2-3-2} \tag{31}$$

Using this information, we see that the 5-7-5 structure is further broken down into (2-1-2)-(2-3-2)-(2-1-2), that is,

$$\Pi\left(\frac{5}{17}\right) = \text{(2-1-2)-(2-3-2)-(2-1-2)} \tag{32}$$

Using the recursive algorithm, we get the information that either the line is not composed of smaller lines, that is, there is no substructure to the given line. We are guaranteed to get this result because the iterative process terminates for any rational values of α. The number of times this operation must be carried out for the process is terminate is however hard to predict.

6 Spectra Belonging to Irrational Fields

So far what ever analysis we have done, we have assumed that α is a rational value. Now what happens if α is irrational?

Using the functions Λ and Γ on an irrational value of α, the newly defined value will also turn out to be irrational. Repeating this process over and over again, we see that this process never terminates! Visually we would get a band no matter how much "closer" we try to look, there is always a finer structure visible! Using some theorems of topology, it can be shown that this spectrum is homeomorphic to a Cantor set. This set has the bizarre property of being uncountable yet having a zero measure. Intuitively this means that there are as many points in the spectrum as there are real numbers, that is uncountably infinite, yet you cannot find two points that are very close to each other.

7 Magnetic Field Fluctuations create a "Blurred Graph"

After working out so much of theoretical jargon, now it is the right time to ask the question of its physical significance in real life experiment. Though from the previous sections, we have shown that the rational values of $\alpha = \frac{p}{q}$ is highly dependant on the value of q, and a small change in α might deviate value of q to a significant extend, looking at the spectrum we see some kind of continuity in the graph.

D. Hofstadter proved in his thesis, the following theorem,

Theorem. *For any value of α' approaching α, the points on the Spectrum(α') approaches points formed by Spectrum of α.*

This is on the line of observation of vertical displacement of points on the Physical Spectrum. The points seem not only to br smudged along vertical, but there also seems to be a notion of continuity among those points, though they have discontinuous gap in between. This was confirmed by Hofstadter by defining $M(\alpha)$ as a Lebesgue Measure of the band at given value of α. For all rational values of α, $M(\alpha)$ is taken to be positive on the basis that every rational points on the graph would have a finite length of neighbouring points. But on the same terms, $M(\alpha)$ of irrational points would be 0. In layman's terms, M(α) can be considered as some sort of "sum of length" of all the points which are arbitrarily close to each other for a given value of α. Surrounding a rational number, we expect an irrational number, arbitrary close to the rational number. Hence, $M(\alpha)$ for rational numbers would be discontinuous. In simpler terms if one tries to draw a line connecting values of $M(\alpha)$ starting from a rational number, they would bump into a irrational number with $M(\alpha) = 0$, and the line would be discontinuous. But contrary to one's expectation, M(α) is continuous at irrational values.

This issue of continuity and discontinuity has a simple resolution in the physical level. It is almost impossible to create a very sharp magnetic field without any fluctuation to a high degree of precision. On the same line, it is in no way possible to create an irrational field magnetic field. Creation of such a field would mean precision up to infinite number of decimal places. Each decimal place would have to be filled with a particular definite value. So in short it can be said that a real magnetic field has an inbuilt uncertainty of $\Delta\vec{H}$, which can be rewritten in terms of α as $\Delta\alpha$. Now it is assumed that the spectrum obtained by introducing error in α (Called the physical spectrum), may be obtained by a union of all the spectra obtained from the values $\left[\alpha - \frac{\Delta\alpha}{2}, \alpha + \frac{\Delta\alpha}{2}\right]$. Symbolically,

$$\text{Physical Spectrum}(\alpha) = \bigcup_{\alpha' \in \Delta} \text{Spectrum}(\alpha') \tag{33}$$

with Δ being the interval $\left[\alpha - \frac{\Delta\alpha}{2}, \alpha + \frac{\Delta\alpha}{2}\right]$.

The smeared graph can be reproduced computationally by considering a random small fluctuation in the value of α confined within a height of $\Delta\alpha$. For our case it is $\frac{1}{100}$ and hence a

height of $\frac{\Delta\alpha}{2} = \frac{1}{200}$.

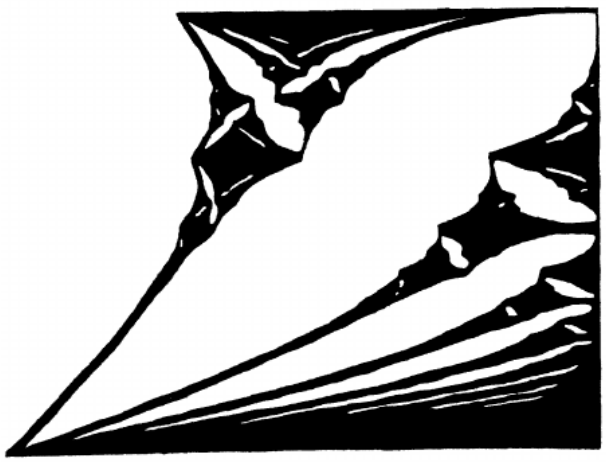

Figure 5: The smeared spectrum of Hofstadter's butterfly for a quadrant (Taken from the original article).

As can be seen from the graph it is smeared out but overall nature is preserved.

With the decrease in the value of $\Delta\alpha$, in principal, we expect to recover the finer details of the graph, but in true physical scenario, we can never expect it to be 0, but only expect to reduce it to an acceptable value.

8 Possible Experimental Test

After going through all the rigorous derivation and extracting some physical meaning that one might expect from real life experiment, it might be a good time to explore various possible experimental test that can be undertaken. It is near to impossible to perform any such experiment on real crystal as the magnetic field needed would exceed 10^9 G for a typical crystal with a lattice spacing a = 2 Å and $\alpha = 1$. One way to reduce such a huge magnetic field would be to increase crystal spacing. This can be achieved using lead-less field effect transistor with a high electric field applied across it. On application of electric field, the holes and the electrons would be driven to one side of the device and concentrated in a thin layer there. In this situation, the electrons behave like a two dimensional gas of charged particles. Now if the device is prepared in advance with a dielectric layer which is non-uniform, and periodic in two dimensions, then the two dimensional gas would be moving in a periodic potential. This periodic potential can be manufactured to our requirement. In particular, we can construct a tight binding model with increased lattice spacing to study the effect of the magnetic field on it.

With the increased spacing, our magnetic field would approximately be reduced from 10^9 G to 10^5 G for a spacing of 200 Å. Now with the application of uniform magnetic field perpendicular to the plane of the gas, electromagnetic radiation of various wavelengths can

be applied to study band structure. From such an experiment, we expect to obtain the Hofstadter's butterfly spectrum.

References

- Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields - D. Hofstadter - 1976
- The Energy Levels of Bloch Electrons in a Magnetic Field - D. Hofstadter - 1974 (Thesis work)