APPLICATION OF QUADRATURE METHOD FOR SOLVING INTEGRALS AND INTEGRAL EQUATION

A Project Report Submitted for

INTERNSHIP

in

MATHEMATICS

by

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1 Introduction

Here, we derive and analyze Quadrature methods for evaluating definite integral of the form

$$I = \int_a^b f(x)\, dx \tag{1.1}$$

in the interval $[a, b]$. Most of such integrals cannot be evaluated analytically, and with many others, it is faster to integrate numerically then finding the complex antiderivative of $f(x)$ and then evaluate it.After deriving the Quadrature Methods, we make use of it to find the value of some integrals and compare it with exact solution by finding out the error.

After this, we solved Fredholm Integral Equation of Second Kind using the Quadrature Method, that is we consider the problem of solving approximately the integral equation of the form

$$\phi(x) + \int_\alpha^\beta K(x,t)\, \phi(t)\, dt = f(x) \qquad (\alpha < x < \beta) \tag{1.2}$$

$$\implies L\phi(x) = f(x) \tag{1.3}$$

where α, β are constant, $f(x)$, $K(x,t)$ are known while $\phi(x)$ is unknown function and L is integral operator

2 Quadrature Methods

Here our main task is to find out approximate value of Integral

$$I = \int w(x)f(x)\, dx \tag{2.1}$$

where $w(x) > 0$ in $[a, b]$ is the weight function.
Here we assume that $w(x)$ and $w(x)f(x)$ are integrable, the integral is approximated by finite linear combination of values of $f(x)$ in the form

$$I = \int w(x)f(x)\, dx = \sum w_k f(x_k) + R_n \tag{2.2}$$

where x_k are the nodes and w_k are weight of Quadrature Formula and R_n is the error in approximation.

$$R_n = \int w(x)f(x)\, dx - \sum w_k f(x_k) \tag{2.3}$$

2.1 Methods based on Interpolation

Given n sub-interval in $[a, b]$ with $n+1$ nodes and the corresponding values of $f(x_k)$, the lagrange interpolating polynomial fitting the data $(x_k, f(x_k))$ where k varies over 1,2,3,...,n is given by

$$f(x) = P_n(x) = \sum_{k=0}^{n} l_k(x) f_k + \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi) \qquad x_0 < \xi < x_n \tag{2.4}$$

where $l_k(x)$ are Lagrange Interpolating Polynomial

$$l_k(x) = \frac{\pi(x)}{(x - x_k)\pi'(x)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \tag{2.5}$$

$$\pi(x) = \prod_{i=0}^{n} (x - x_i) \tag{2.6}$$

Now using interpolating polynomial (2.4) and integrate it in $[a, b]$

$$I = \int_a^b w(x)f(x)\, dx \tag{2.7}$$

$$= \sum_{k=0}^{n} \left[\int_a^b w(x) l_k(x)\, dx \right] f_k + \int_a^b \frac{w(x)\pi(x)}{(n+1)!} f^{n+1}(\xi)\, dx \tag{2.8}$$

$$= \sum_{k=0}^{n} w_k f_k + R_n \tag{2.9}$$

where $w_k = \int_a^b w(x) l_k(x)\, dx$
Since the method is exact for polynomials of degree $\leq n, R_n = 0$ when $f(x) = x^i, i = 1, 2, 3, ..., n$ and $R_n = 0$ when $f(x) = x^{n+1}$. Thus we can write error term as

$$R_n = \frac{c}{(n+1)!} f^{n+1}(\xi), \qquad \xi \in (a, b) \tag{2.10}$$

$$\text{where } c = \int_a^b w(x) x^{n+1}\, dx - \sum_{k=0}^{n} w_k x_k^{n+1} \tag{2.11}$$

2.1.1 Simpson's $\frac{1}{3}$rd Rule

This method is based on interpolation method to evaluate the definite integral. This Rule gives approximately exact value for a polynomial f upto degree 2. For applying this rule the number of subinterval must be even.

Derivation: Let $a = x_0$ and $b = x_2$ are the end point of f(x) and $h = \frac{b-a}{n}$ and weight function w(x)=1.

Standard Simpson's $\frac{1}{3}$rd Rule(n=2)
By Quadratic Lagrange Interpolation

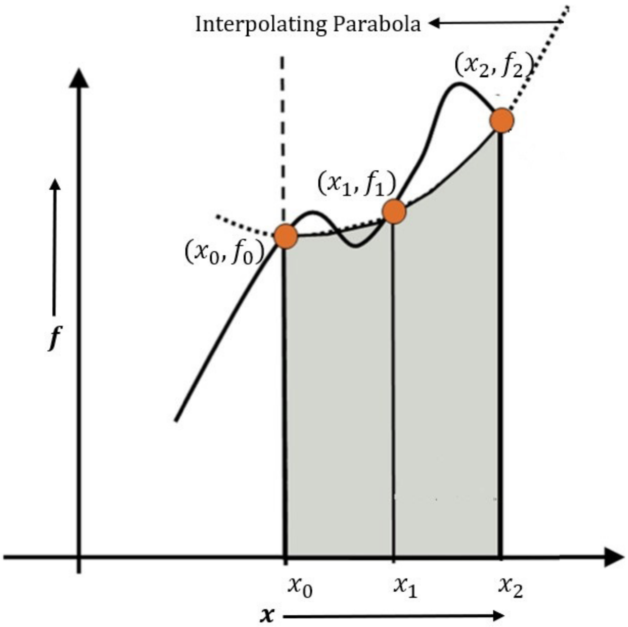

Figure 1: Standard Simpson's $\frac{1}{3}$rd Rule

$$f(x) \approx P_2(x) = L_0 f_0 + L_1 f_1 + L_2 f_2 \tag{2.12}$$

$$\implies f(x) \approx \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2 \tag{2.13}$$

$$\implies \int_{x_0}^{x_2} f(x)\, dx \approx \int_{x_0}^{x_2} \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0\, dx + \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_1\, dx + \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2\, dx \tag{2.14}$$

Substituting $x - x_1 = s$ we get

$$\implies \int_{x_0}^{x_2} f(x)\, dx \approx \frac{f_0}{2h^2} \int_{-h}^{h} s(s - h)\, ds + \frac{f_1}{-h^2} \int_{-h}^{h} (s + h)(s - h)\, ds + \frac{f_2}{2h^2} \int_{-h}^{h} (s + h) s\, ds \tag{2.15}$$

$$\implies \int_{x_0}^{x_2} f(x)\, dx \approx \frac{h}{3} [f_0 + 4f_1 + f_2] \tag{2.16}$$

Error Analysis

Standard Simpson's $\frac{1}{3}$rd Rule integrate exactly for polynomials of degree ≤ 2, so we have

$$c = \int_a^b x^3\,dx - \frac{b-a}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right] = \frac{b^4-a^4}{4} - \frac{b-a}{4}[a^3+b^3+a^2b+ab^2] = 0 \tag{2.17}$$

which show that Simpson's $\frac{1}{3}$rd Rule is exact for polynomial of degree 3 also. So the error terms becomes

$$R_2 = \frac{c}{4!}f^4(\xi) \qquad \xi \in (0,2) \tag{2.18}$$

where

$$c = \int_a^b x^4\,dx - \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] = -\frac{(b-a)^5}{120} \tag{2.19}$$

$$R_2 = \frac{(b-a)^5}{120(4!)}f^4(\xi) = -\frac{h^5}{90}f^4(\xi), \qquad \xi \in (0,2) \tag{2.20}$$

2.2 Composite Simpson's $\frac{1}{3}$rd Rule($n \geq 2$)

Using Standard Simpson's Rule we have

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] \tag{2.21}$$

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] \tag{2.22}$$

....

$$\int_{x_{n-2}}^{x_n} f(x)\,dx \approx \frac{h}{3}\left[f_{n-2} + 4f_{n-1} + f_n\right] \tag{2.23}$$

Adding (2.21)-(2.23) we get

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + ... + f_{n-1}) + 2(f_2 + f_4 + ... + f_{n-2}) + f_n\right] \tag{2.24}$$

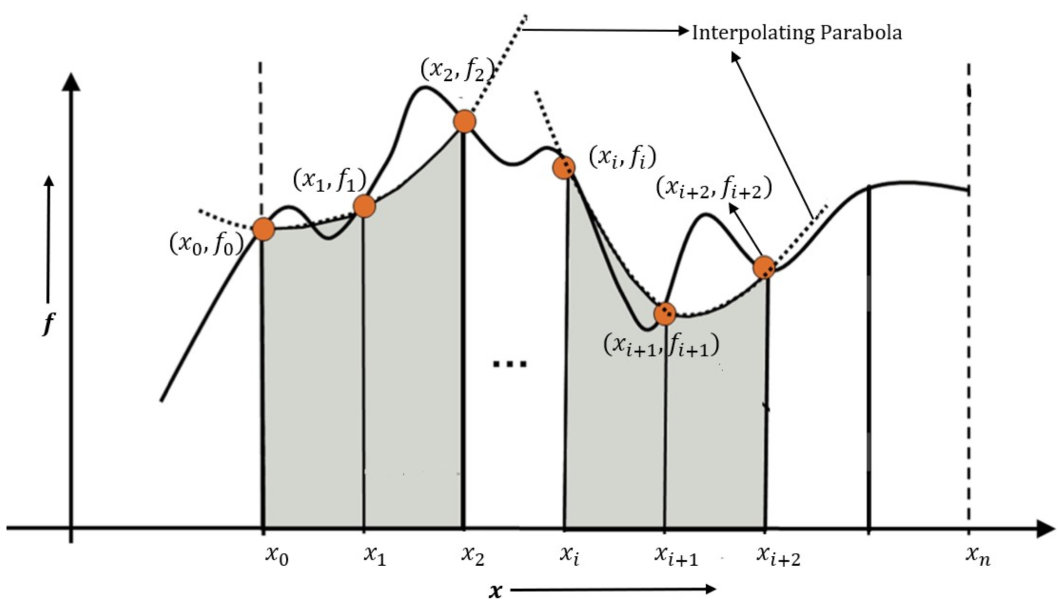

Figure 2: Composite Simpson's $\frac{1}{3}$rd Rule

Error Analysis
The error using Composite Simpson's $\frac{1}{3}$rd Rule is

$$R(f,x) = \frac{-h^5}{90}[f^4(\xi_1) + f^4(\xi_2) + ... + f^4(\xi_{n/2})] \tag{2.25}$$

where $x_0 < \xi_1 < x_2$, $x_2 < \xi_2 < x_4$,......, $x_{n-2} < \xi_{n/2} < x_n$
The bound on the error is given by

$$|R(f,x)| \leq \frac{h^5}{90}[|f^4(\xi_1)| + |f^4(\xi_2)| + ... + |f^4(\xi_{n/2})|] = \frac{(b-a)h^4 M_4}{180} \tag{2.26}$$

where $M_4 = \max_{a\leq\xi\leq b}|f^4(\xi)|$

2.3 Gaussian Integration Method

In the integration method (2.2), the nodes x_k and weight w_k, k=0,1,2,....,n can be obtained by making the formula exact for polynomial of degree upto m. When the nodes are also to be determined, we have $m = 2n + 1$ and the methods are called Gaussian Integration Method. Since any finite interval $[a, b]$ can be transformed to $[-1, 1]$ using the transformation

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \tag{2.27}$$

We consider the integral of form

$$\int_{-1}^{1} w(x)f(x)\,dx = \sum_{k=0}^{n} w_k f_k + R_n \tag{2.28}$$

2.3.1 Gauss-Chebyshev Quadrature Formula

For $w(x) = \frac{1}{\sqrt{1-x^2}}$, the method of the form

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx \sum_{k=0}^{n} w_k f_k \tag{2.29}$$

are called Gauss-Chebyshev integration method.

Determination of Nodes and weights through Orthogonal Polynomials

Theorem 2.1. *If $x_k' s$ are selected as zeros of an orthogonal polynomial, orthogonal with respect to the weight function $w(x)$ over$[a, b]$, then the formula (2.29) has a precision $2n + 1$.*

Proof. Let us consider a polynomial $f(x)$ such that $deg(f(x)) \leq 2n + 1$.
Let $q_n(x)$ be interpolating Lagrange polynomial of degree $\leq$ n, interpolating (x_i, f_i) for i=0,1,2,...,n

$$q_n(x) = \sum_{k=0}^{n} l_k(x)\,f(x_k) \qquad \text{where} \quad l_k(x) = \frac{\pi(x)}{(x-x_k)\pi'(x_k)} \tag{2.30}$$

Since the polynomial $[f(x) - q_n(x)]$ has zeros at x_0, x_1, ..., x_n we have

$$f(x) - q_n(x) = P_{n+1}(x)\,r_n(x) \tag{2.31}$$

where $P_{n+1}(x_i) = 0$ for $i = 0, 1, ..., n$ and $r_n(x)$ is a polynomial with $deg(r_n(x)) \leq n$. Integrating (2.31) with the weight function $w(x)$ we get

$$\int_a^b w(x)(f(x) - q_n(x))\,dx = \int_a^b w(x)\,P_{n+1}(x)\,r_n(x)\,dx \tag{2.32}$$

$$\implies \int_a^b w(x)f(x)\,dx = \int_a^b w(x)q_n(x)\,dx + \int_a^b w(x)\,P_{n+1}(x)\,r_n(x)\,dx \tag{2.33}$$

If $P_{n+1}(x)$ is a orthogonal polynomial, orthogonal with respect to the weight function $w(x)$ to all the polynomial of degree less than or equal to n, So $int_a^b w(x)\,P_{n+1}(x)\,r_n(x)\,dx = 0$, so we get

$$\int_a^b w(x)f(x)\,dx = \int_a^b w(x)\,q_n(x)\,dx = \sum_{k=0}^{n} w_k\,f_k \tag{2.34}$$

where $w_k = \int_a^b w(x)\,l_k(x)\,dx$ This proves that $I = \int_a^b w(x)f(x)\,dx = \sum_{k=0}^{n} w_k\,f_k$ has a precision of $2n+1$. $\square$

The nodes $x_k's$ are of $n+1$ point formula are the roots of Chebyshev polynomial.

$$T_{n+1}(x) = \cos\left((n+1)\arccos x\right) = 0 \tag{2.35}$$

The nodes $x_k's$ are given by the equation

$$x_k = \cos\left(\frac{(2k+1)\pi}{2n+2}\right) \qquad k = 0, 1, 2, ..., n \tag{2.36}$$

and the weights are given by

$$w_k = \int_a^b w(x) l_k(x)\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_k)\pi'(x_k)}\, dx \qquad k = 0,\ 1,\ 2, ..., n \tag{2.37}$$

Derivation of $n+1$ point Formula

Taking $n=0$, we have

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx w_0 f_0 \tag{2.38}$$

where $x_0 = \cos\frac{(2(0)+1)\pi}{2} = 0$ and

$$w_0 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{x}{x(1)}\, dx = \pi \tag{2.39}$$

Taking $n=1$, we have

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx w_0 f_0 + w_1 f_1 \tag{2.40}$$

where $x_0 = \cos\frac{(2(0)+1)\pi}{4} = \frac{1}{\sqrt{2}}$, $x_1 = \cos\frac{(2(1)+1)\pi}{4} = -\frac{1}{\sqrt{2}}$ and

$$w_0 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\left(x+\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)}\, dx = \frac{\pi}{2} \tag{2.41}$$

$$w_1 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\left(x-\frac{1}{\sqrt{2}}\right)}{\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)}\, dx = \frac{\pi}{2} \tag{2.42}$$

Taking $n=2$, we have

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx w_0 f_0 + w_1 f_1 + w_2 f_2 \tag{2.43}$$

where $x_0 = \cos\frac{(2(0)+1)\pi}{6} = \frac{\sqrt{3}}{2}$, $x_1 = \cos\frac{(2(1)+1)\pi}{6} = 0$, $x_2 = \cos\frac{(2(2)+1)\pi}{6} = -\frac{\sqrt{3}}{2}$ and

$$w_0 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{(x-0)\left(x+\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}-0\right)\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)}\, dx = \frac{\pi}{3} \tag{2.44}$$

$$w_1 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\left(x-\frac{\sqrt{3}}{2}\right)\left(x+\frac{\sqrt{3}}{2}\right)}{\left(0-\frac{\sqrt{3}}{2}\right)\left(0+\frac{\sqrt{3}}{2}\right)}\, dx = \frac{\pi}{3} \tag{2.45}$$

$$w_2 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\pi(x)}{(x-x_0)\pi'(x_0)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\left(x-\frac{\sqrt{3}}{2}\right)(x-0)}{\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}-0\right)}\, dx = \frac{\pi}{3} \tag{2.46}$$

Generalising in this ways, the weight of $n+1$ point formula is given by

$$w_k = \frac{\pi}{(n+1)} \qquad k = 0, 1, 2, \ldots, n \tag{2.47}$$

So (2.29) becomes

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx \frac{\pi}{n+1} \sum_{k=0}^{n} f_k \tag{2.48}$$

3 Example of Integrals

Example 1: $\mathrm{I} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: The exact solution of integral is given by

$$I_e = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}\left[\arcsin(x)\right]_0^b = \frac{\pi}{2} \tag{3.1}$$

The approximate solution using Simpson's Rule is given by

$$I_s = \frac{h}{3}\left[f_0 + 4\sum_{\substack{k=1\\ \text{(k odd)}}} f_k + 2\sum_{\substack{k=2\\ \text{(k even)}}} f_k + f_n\right] \tag{3.2}$$

Using transformation (2.27) and Gauss-Chebyshev Quadrature Formula the approximate solution is given by

$$I_c = \frac{\pi}{n+1}\sum_{k=0}^{n} f_k \tag{3.3}$$

For different value of n the approximate solution and error is obtained in Table 1.

Table 1: Exact and Approximate solution of integral in Example 1

Number of sub-interval (n)	Exact Solution (I_e)	Approx. Sol. using Simpson's Rule (I_s)	Approx. Sol. using Gauss-Chebyshev (I_c)	Absolute Error $E_s = \lvert I_e - I_s\rvert$	Absolute Error $E_c = \lvert I_e - I_c\rvert$
2	1.5708	2.0942	1.6274	0.5234	0.0566
4	1.5708	1.6905	1.8128	0.1197	0.2420
6	1.5708	1.5734	1.7063	0.0026	0.1355
8	1.5708	1.5212	1.7215	0.0496	0.1507
10	1.5708	1.4929	1.6582	0.0779	0.0874

Example 2: $\mathrm{I} = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: The exact solution of integral is given by

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{b\to 1}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{b\to 1}\left[\arcsin(x)\right]_0^b = \pi \tag{3.4}$$

The approximate solution using Simpson's Rule is given by

$$I_s = \frac{h}{3}\left[f_0 + 4\sum_{\substack{k=1\\ \text{(k odd)}}} f_k + 2\sum_{\substack{k=2\\ \text{(k even)}}} f_k + f_n\right] \tag{3.5}$$

Using transformation (2.27) and Gauss-Chebyshev Quadrature Formula the approximate solution is given by

$$I_c = \frac{\pi}{n+1}\sum_{k=0}^{n} f_k \tag{3.6}$$

For different value of n the approximate solution and error is obtained in Table 2.

Table 2: Exact and Approximate solution of integral in Example 2

Number of sub-interval (n)	Exact Solution (I_e)	Approx. Sol. using Simpson's Rule (I_s)	Approx. Sol. using Gauss-Chebyshev (I_c)	Absolute Error $E_s = \lvert I_e - I_s\rvert$	Absolute Error $E_c = \lvert I_e - I_c\rvert$
2	3.1416	5.9986	3.1416	2.8570	0
4	3.1416	4.1885	3.1416	1.0469	0
6	3.1416	3.6370	3.1416	0.4954	0
8	3.1416	3.3809	3.1416	0.2393	0
10	3.1416	3.2371	3.1416	0.0955	0

4 Quadrature Method for Solving Fredholm Equation of Second Kind

Fredholm Equation occurs in problems of Science and Engineering. By replacing the Fredholm Integral equation of second kind (1.2) by quadrature method, we get

$$\phi(x) + \sum_{k=0}^{n} w_k \, \phi(t_k) \, K(x, t_k) = f(x) \tag{4.1}$$

where w_k are the weights and $t_k's$ are interpolating points. Since t_k are nodes, so $t_0 = \alpha$, $t_k = \alpha + kh$ and $t_n = \beta$. At $x = t_i$

$$\phi_i + \sum_{k=0}^{n} w_k \, \phi_k \, K_{ik} = f_i \qquad i = 0, 1, 2, ..., n \tag{4.2}$$

where $f_i = f(t_i)$, $\phi_i = \phi(t_i)$ and $K_{ik} = K(t_i, t_k)$

4.1 Simpson's $\frac{1}{3}$rd Rule

$$\phi_i + \frac{h}{3} \left[K_{i0}\phi_0 + 4 \sum_{\substack{k=1 \\ \text{(k odd)}}}^{n-1} K_{ik}\phi_k + 2 \sum_{\substack{k=2 \\ \text{(k even)}}}^{n-2} K_{ik}\phi_k + K_{in}\phi_n \right] = f_i \tag{4.3}$$

Solving the system of Equations (4.3) we will get the value of ϕ_0, ϕ_1 , ..., ϕ_n.

4.2 Gauss-Chebyshev Quadrature Method

$$\phi_i + \frac{\pi}{n+1} \sum_{k=0}^{n} \frac{\phi_k \, K_{ik}}{w(t_k)} = f_i \tag{4.4}$$

Solving the system of Equations (4.4) with transformation (2.27) we will get the value of ϕ_0, ϕ_1 , ..., ϕ_n.

Example 3: $K(x, t) = -(xt + x^2t^2)$, $f(x) = 1$, $\alpha = -1$, $\beta = 1$

Solution: The exact solution of the corresponding Fredholm integral equation (cf. [2]) is given by

$$\phi_e(x) = 1 + \frac{10}{9}x^2 \tag{4.5}$$

The approximate and exact solution of Fredholm integral Equation are nearly equal for n=6 as shown in Figure 3. The approximate solution and absolute errors are obtained for some discretized point in Table 3.

Table 3: Exact and Approximate solution of Fredholm integral equation given in Example 3 using n=6

x	-1	-0.5	0	0.5	1
$\phi_e(x)$	2.1111	1.2778	1	1.2778	2.1111
$\phi_s(x)$	2.1172	1.2793	1	1.2793	2.1172
$\phi_c(x)$	2.176	1.294	1	1.294	2.176
$E_s = \|\phi_e(x) - \phi_s(x)\|$	0.0061	0.0015	0	0.0015	0.0061
$E_c = \|\phi_e(x) - \phi_c(x)\|$	0.0649	0.0162	0	0.0162	0.0649

where,
$\phi_e(x)$ is the Exact solution.
$\phi_s(x)$ is the Approximate solution by Simpson's Rule.
$\phi_c(x)$ is the Approximate solution by Gauss-Chebyshev Method.
E_s is the Absolute error using Simpson's Rule.
E_c is the Absolute error using Gauss-Chebyshev Method.

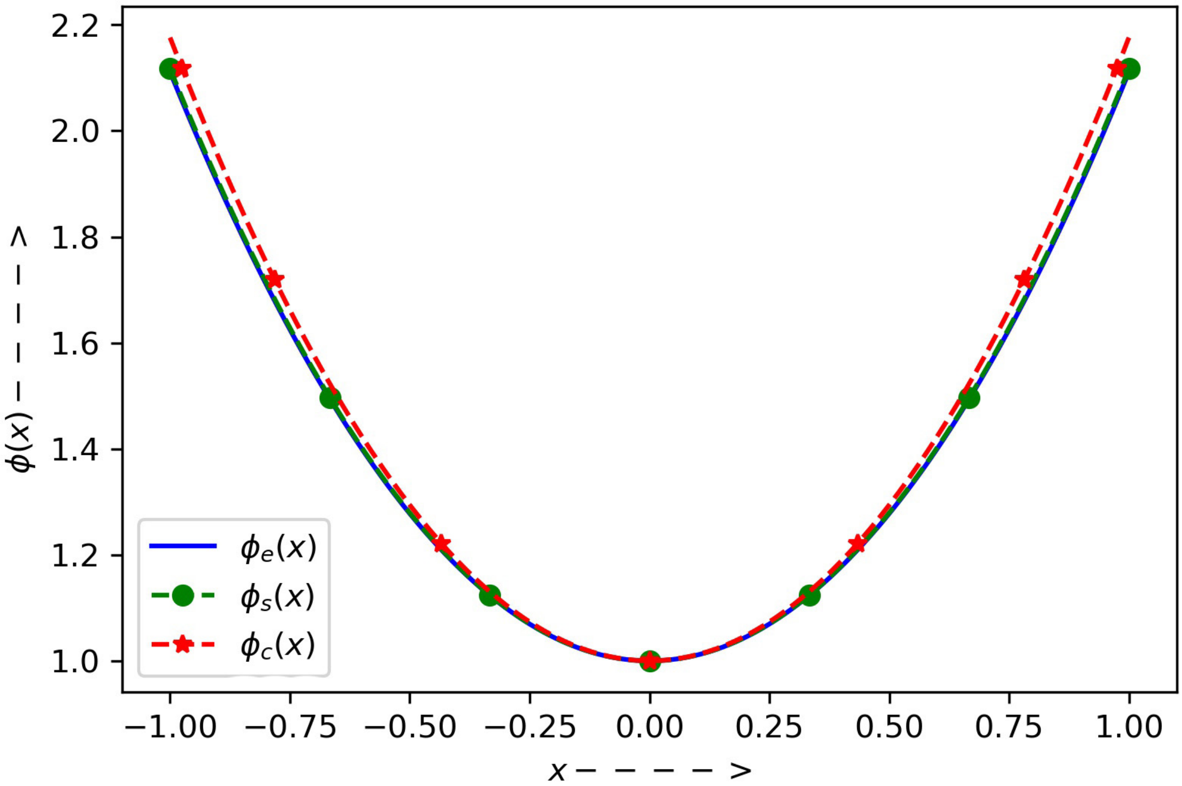

Figure 3: Exact and Approximate solution of Fredholm Integral equation given in Example 3 with n=6

Example 4: $K(x,t) = -(x^2 + t^2)$, $f(x) = x^2$, $\alpha = 0$, $\beta = 1$

Solution: The exact solution of the corresponding Fredholm integral equation (cf. [2]) is given by

$$\phi_e(x) = \frac{30}{11}x^2 + \frac{9}{11} \tag{4.6}$$

The approximate and exact solution of Fredholm integral Equation are nearly equal for n=6 as shown in Figure 4. The approximate solution and absolute errors are obtained for some discretized point in Table 4.

Table 4: Exact and Approximate solution of Fredholm integral equation given in Example 4 using n=4

x	0	0.25	0.5	0.75	1
$\phi_e(x)$	0.8182	0.9886	1.5	2.3523	3.5455
$\phi_s(x)$	0.8221	0.9929	1.5053	2.3594	3.5552
$\phi_c(x)$	0.9475	1.134	1.6934	2.6258	3.9311
$E_s = \|\phi_e(x) - \phi_s(x)\|$	0.0039	0.0042	0.0053	0.0072	0.0097
$E_c = \|\phi_e(x) - \phi_c(x)\|$	0.1293	0.1454	0.1934	0.2735	0.3857

where,
$\phi_e(x)$ is the Exact solution.
$\phi_s(x)$ is the Approximate solution by Simpson's Rule.
$\phi_c(x)$ is the Approximate solution by Gauss-Chebyshev Method.
E_s is the Absolute error using Simpson's Rule.
E_c is the Absolute error using Gauss-Chebyshev Method.

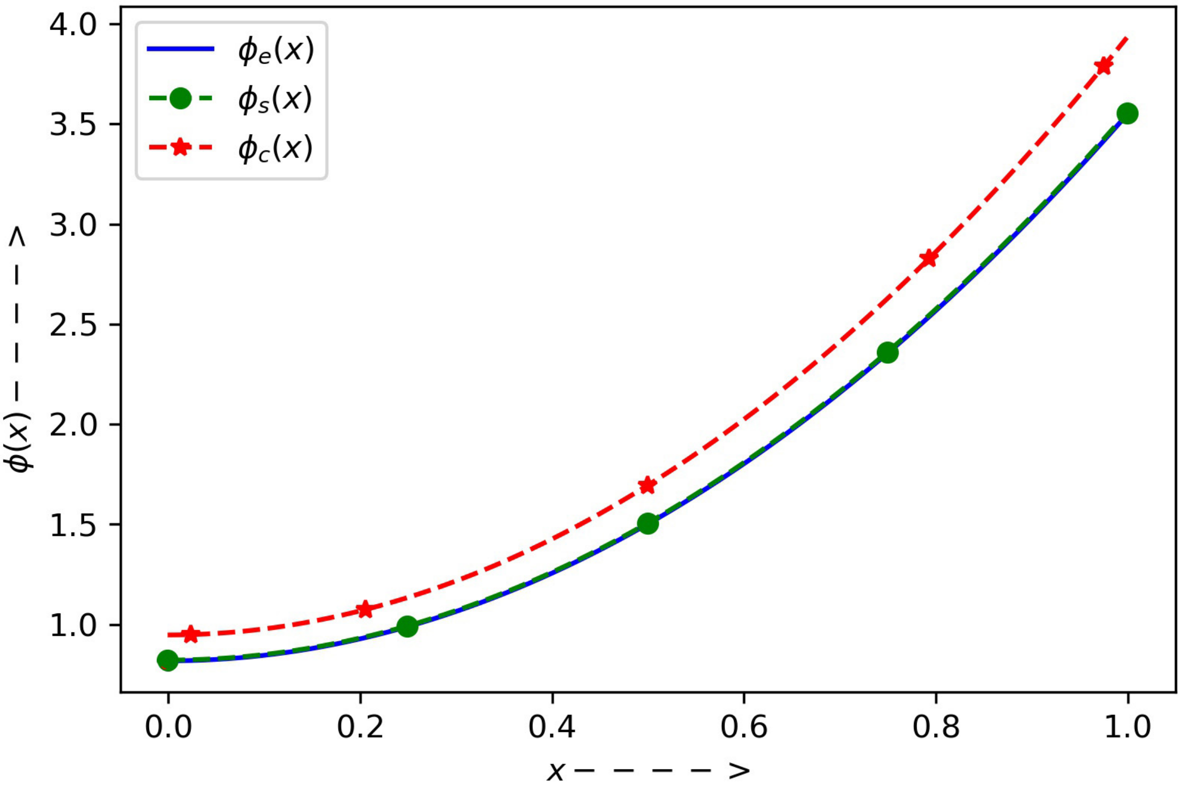

Figure 4: Exact and Approximate solution of Fredholm Integral equation given in Example 4 with n=4

Example 5: $K(x,t) = -(\sqrt{x} + \sqrt{t})$, $f(x) = 1 + x$, $\alpha = 0$, $\beta = 1$

Solution: The exact solution of the corresponding Fredholm integral equation (cf. [1]) is given by

$$\phi_e(x) = \frac{-129}{70} - \frac{141}{35}\sqrt{x} + x \tag{4.7}$$

The approximate and exact solution of Fredholm integral Equation are nearly equal for n=6 as shown in Figure 5. The approximate solution and absolute errors are obtained for some discretized point in Table 5.

Table 5: Exact and Approximate solution of Fredholm integral equation given in Example 5 using n=4

x	0	0.25	0.5	0.75	1
$\phi_e(x)$	-1.8429	-3.6071	-4.1915	-4.5817	-4.8714
$\phi_s(x)$	-1.9135	-3.7212	-4.3235	-4.7275	-5.0289
$\phi_c(x)$	-2.1347	-3.4486	-3.988	-4.3492	-4.6163
$E_s = \lvert\phi_e(x) - \phi_s(x)\rvert$	0.0707	0.114	0.132	0.1458	0.1574
$E_c = \lvert\phi_e(x) - \phi_c(x)\rvert$	0.2919	0.1586	0.2035	0.2325	0.2552

where,
$\phi_e(x)$ is the Exact solution.
$\phi_s(x)$ is the Approximate solution by Simpson's Rule.
$\phi_c(x)$ is the Approximate solution by Gauss-Chebyshev Method.
E_s is the Absolute error using Simpson's Rule.
E_c is the Absolute error using Gauss-Chebyshev Method.

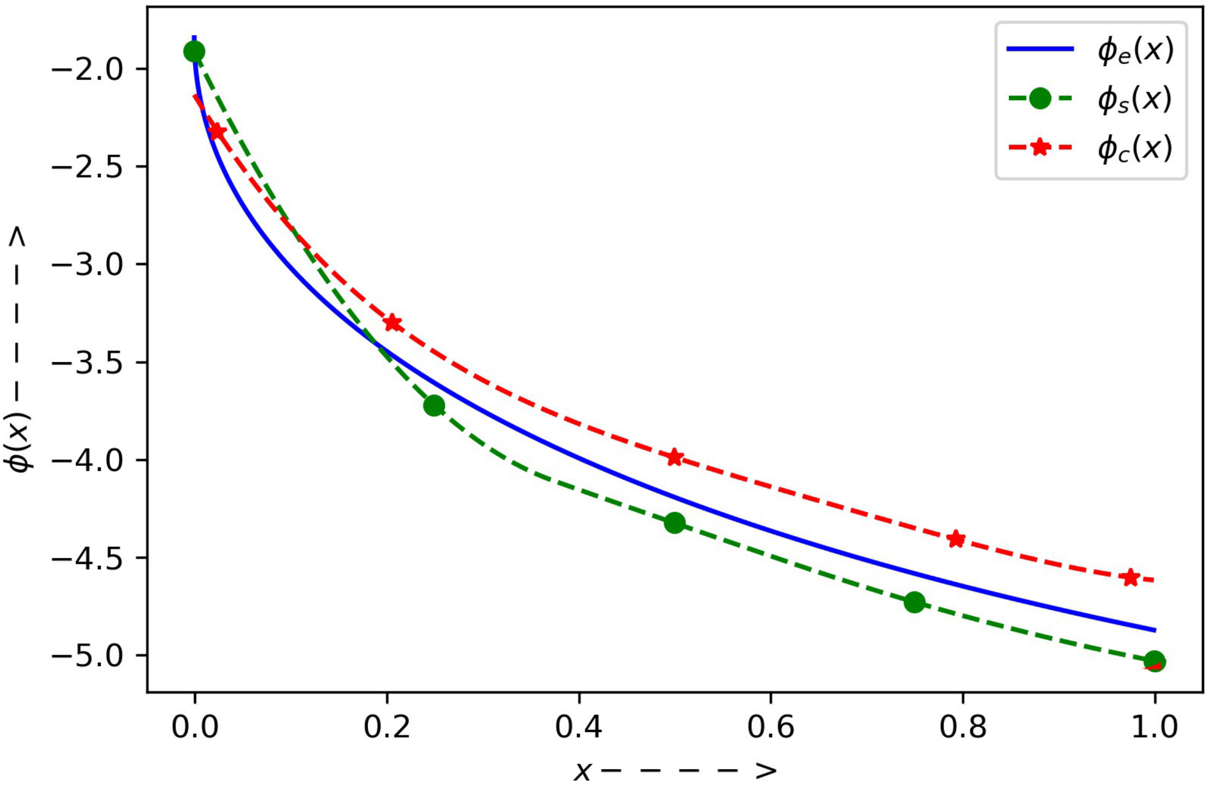

Figure 5: Exact and Approximate solution of Fredholm Integral equation given in Example 5 with n=4

Example 6: $K(x,t) = -(\cos x + \cos t)$, $f(x) = \sin x$, $\alpha = 0$, $\beta = \pi$

Solution: The exact solution of the corresponding Fredholm integral equation (cf. [1]) is given by

$$\phi_e(x) = \sin x + \frac{4}{2-\pi^2}\cos x + \frac{2\pi}{2-\pi^2} \tag{4.8}$$

The approximate and exact solution of Fredholm integral Equation are nearly equal for n=6 as shown in Figure 6. The approximate solution and absolute errors are obtained for some discretized point in Table 6.

Table 6: Exact and Approximate solution of Fredholm integral equation given in Example 4 using n=4

x	0	0.25	0.5	0.75	1
$\phi_e(x)$	0.8182	0.9886	1.5	2.3523	3.5455
$\phi_s(x)$	0.8221	0.9929	1.5053	2.3594	3.5552
$\phi_c(x)$	0.9475	1.134	1.6934	2.6258	3.9311
$E_s = \lvert\phi_e(x) - \phi_s(x)\rvert$	0.0039	0.0042	0.0053	0.0072	0.0097
$E_c = \lvert\phi_e(x) - \phi_c(x)\rvert$	0.1293	0.1454	0.1934	0.2735	0.3857

where,
$\phi_e(x)$ is the Exact solution.
$\phi_s(x)$ is the Approximate solution by Simpson's Rule.
$\phi_c(x)$ is the Approximate solution by Gauss-Chebyshev Method.
E_s is the Absolute error using Simpson's Rule.
E_c is the Absolute error using Gauss-Chebyshev Method.

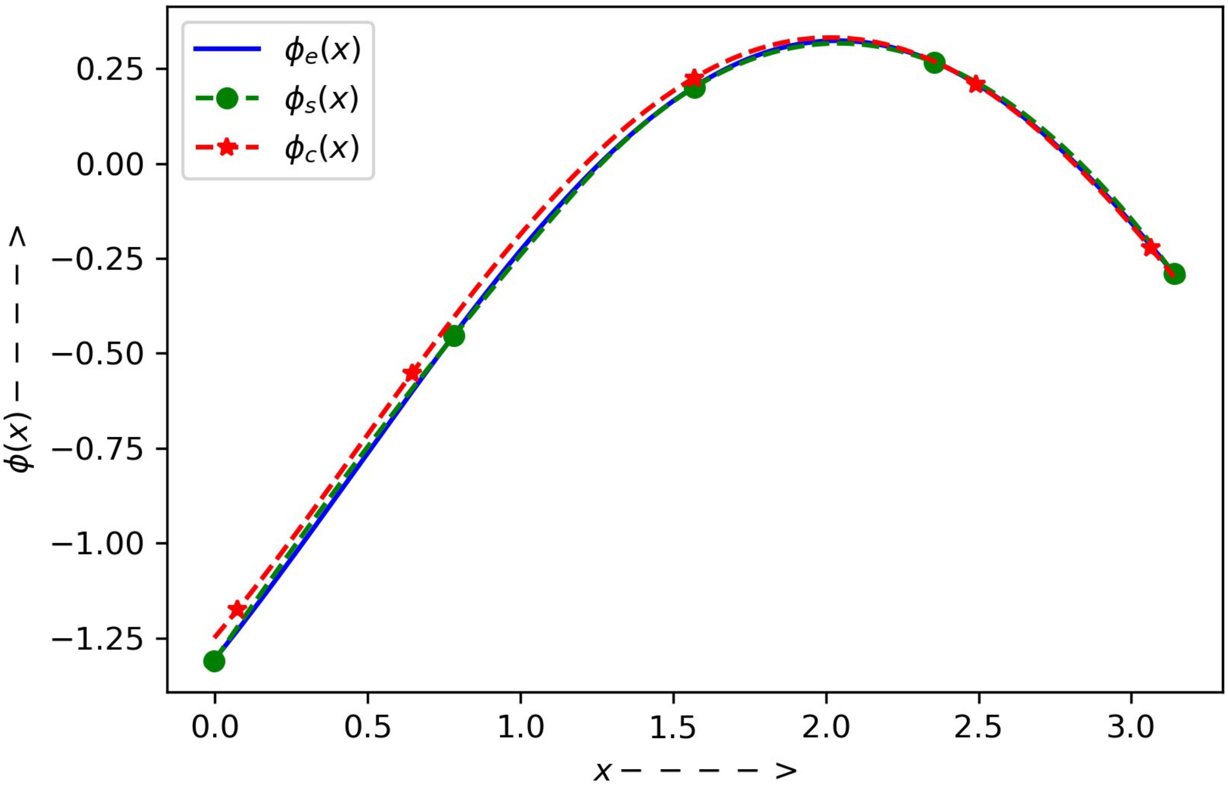

Figure 6: Exact and Approximate solution of Fredholm Integral equation given in Example 6 with n=4

5 Conclusion

Quadrature Method is applied to solve the Integral and Fredholm Integral Equation of Second Kind. In each case approximate solution is obtained using Simpson's $\frac{1}{3}$rd Rule and Gauss-Chebyshev Quadrature Formula and the absolute error is determined in each case.

References

[1] A. Chakrabarti, S.C. Martha, Approximate solutions of Fredholm integral equations of the second kind, Appl. Math. Comput. 211 (2009) 459-466

[2] R.P. Kanwal, Linear Integral Equations, Academic Press, Orlando, 1973, pp. 12.

[3] S.R.K. Iyengar, R.K. Jain, Numerical Methods, New Age International (P) Limited, 2009

[4] Richard L. Burden, J. Douglas Faires, Numerical Analysis, Cengage Learning, 2011