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# UNIT-3

# MULTIPLE INTEGRALS

## Double Integrals:

### Evaluation of Double Integration -

Double Integral of a func $f(x,y)$ Over a Region R can be evaluated by 2 Successive integrations. There are 2 diff methods to evaluate a double Integral.

**Method 1 -**

Let the Region 'R' i.e, PQRS be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ & The Lines $x = a$, $x = b$.

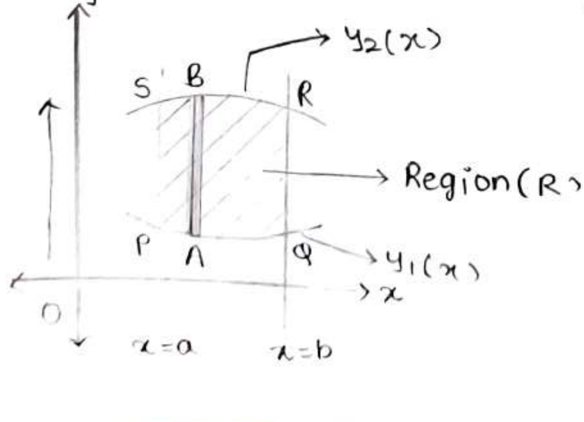
In the Region PQRS draw a vertical strip AB, Along the strip AB, $y$ varies from $y_1$ to $y_2$ & $x$ is fixed. Therefore, the double Integral is Integrated w.r.t 'y' b/w the Limits $y_1, y_2$, treating 'x' as const. Now, move the strip AB horizontally from $x=a$ to $x=b$ to cover the entire region PQRS.

Thus, the Result of the 1st Integral is integrated w.r.t $x$ b/w the Limits A & B. Hence,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \int_{y_1(x)}^{y_2(x)} f(x,y)\,dx\,dy$$

$$= \int_{x=a}^{x=b} \left[ \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right] dx$$

↳ Treating $x$ as const.

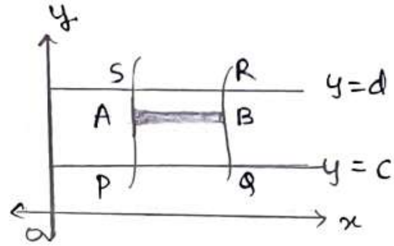


Method-2.

Let the Region 'R' i.e, PQRS be bounded by the curves $x = x_1(y)$ & $x = x_2(y)$ and the Lines $y=c$ to $y=d$.

In the Region PQRS draw a horizontal strip AB, Along the strip AB, $x$ varies from $x_1$ to $x_2$ & $y$ is fixed. Therefore, the double integral is Integrated 1st w.r.t '$x$' b/w the Limits $x_1, x_2$, treating '$y$' as const. Now, move the strip AB vertically from $y=c$ to $y=d$ to cover the entire region PQRS. Thus, the result of the 1st Integral is integrated w.r.t '$y$' b/w the Limits C & D.

Hence, $$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\left[\underbrace{\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx}_{\text{Treat } y \text{ as const}}\right]dy$$




Note:

If the Limits of both $x$ & $y$ are const then the function $f(x,y)$ can be integrated with any variable first, then the double integral over 'R' is

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx = \int_{y=c}^{y=d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

→ If $f(x,y)$ is given as explicit form, Let it be $f(x,y) = f_1(x)\, f_2(y)$

then, $$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=c}^{d} f_1(x) f_2(y)\,dx\,dy = \left[\int_{x=a}^{b} f_1(x)\,dx\right]\left[\int_{y=c}^{d} f_2(y)\,dy\right]$$

Working Rule for Evaluation of double integration over a given region:-

1] If the Region is bounded by more than 1 curve then find the points of Intersection of all the curves.

2] Draw all the curves & mark their P.O.I.

3] Identify the Region of Integration.

4] Draw a vertical or a horizontal strip in the Region, whichever makes the integration easier.

5] The vertical strip starts from the lowest part of the region & terminates on the highest part of the region.

6] for Vertical Strip →

(a) The Lower Limit of $y$ is obtained from the curve where the Vertical Strip Starts & the upper limit of $y$ is obtained from the curve where it terminates.

(b) The Lower Limit of $x$ is the $x$-coordinate of the left most point of the region & the upper limit of $x$ is the $x$-coordinate of the right most point of the Region.

7] The horizontal strip starts from the left part of the region & terminates on the Right part of the region.

8] For Horizontal strip →

(a) The Lower Limit of $x$ is obtained from the curve where the horizontal strip starts & the upper Limit of $x$ is obtained from the curve where it terminates.

(b) The lower Limit of $y$ is $y$-coordinate of the lowest point part of the region & the upper limit of $y$ is $y$-coordinate of the highest point of the Region.

9] If variation along the strip changes within the Region, then the Region is divided into 2 parts

1] Evaluate double integral $\int_2^a \int_2^b \frac{dx\,dy}{xy}$

$$= \left[\int_2^b \frac{1}{x}\,dx\right] \times \left[\int_2^a \frac{1}{y}\,dy\right]$$

Limits of $x, y$ are constant

$f(x,y) = \frac{1}{xy}$ explicit fun$^c$.

$$= [\ln x]_2^b \, [\ln y]_2^a$$

$$= [\ln b - \ln 2][\ln a - \ln 2] = \ln\left(\frac{b}{2}\right)\ln\left(\frac{a}{2}\right)$$

$$\cancel{= (\ln a)^2 + (\ln 2)^2 - 2\ln a \ln 2}$$

2] Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$a = \sqrt{1-y^2}$

$a^2 = 1-y^2$

$$= \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\right] dy = \int_0^1 \left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{\sqrt{\frac{1-y^2}{2}}}{\sqrt{1-y^2}}\right) - \sin^{-1}(0)\right] dy = \int_0^1 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)\;dy$$

$$= \int_0^1 \left(\frac{\pi}{4} - 0\right) dy = \frac{\pi}{4}\int_0^1 1\,dy = \frac{\pi}{4}(y)_0^1 = \frac{\pi}{4}$$

3] $\int_0^{2a} \int_0^{y=\sqrt{2ax-x^2}} xy\,dy\,dx$

$$= \int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} xy\,dy\right] dx = \int_0^{2a} \left[\frac{xy^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx = \int_0^{2a} \left[\frac{x(2ax-x^2)}{2} - 0\right] dx$$

$$= \int_0^{2a} \left[\frac{2ax^2}{2} - \frac{x^3}{2}\right] dx = \int_0^{2a} \left[ax^2 - \frac{x^3}{2}\right] dx = \left[\frac{ax^3}{3} - \frac{x^4}{8}\right]_0^{2a}$$

$$= \frac{8a^4}{3} - \frac{16a^4}{8} - 0 + 0 = \frac{8a^4}{3} - \frac{16a^4}{8} = a^4\left(\frac{8}{3} - 2\right) = \frac{2a^4}{3}$$

4] $\int_{1}^{2}\int_{x=-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx\,dy$

$f(x)\int g(x)dx - \int \frac{d}{dx} f(x)$

$g(x)$

$\hookrightarrow y^2$

$f(x) = (2-y)$

$$= \int_1^2 \left[ \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx \right] dy = \int_1^2 \left[ 2y^2 \frac{x^3}{3} \right]_{-\sqrt{2-y}}^{\sqrt{2-y}} dy.$$

$$= \int_1^2 \left[ \frac{2y^2(2-y)^{3/2}}{3} + \frac{2y^2(2-y)^{3/2}}{3} \right] dy = \int_1^2 \frac{4y^2}{3}(2-y)^{3/2}\,dy$$

$$= \frac{4}{3}\int_1^2 y^2(2-y)^{3/2}\,dy = \frac{4}{3}\left[ \int_1^0 (2-t^2)^2\, t^3 \cdot -2t\,dt \right]$$

$2-y = t^2$

$-\frac{dy}{dx} = 2t\frac{dt}{dx}$

$dy = -2t\,dt$

$$= \frac{4}{3}\left[ \int_0^1 2(2-t^2)^2 t^4\,dt \right] = \frac{4}{3}\left[ 2\int_0^1 (4+t^4-4t^2)\,t^4\,dt \right]$$

L.L $y = 2-t^2$

$1 = 2-t^2$

$t^2 = 1$

$t = 1$

$$= \frac{4}{3}\left[ 2\int_0^1 4t^4 + t^8 - 4t^6\,dt \right]$$

U.L $t = 0$

$$= \frac{8}{3}\left[ \frac{4t^5}{5} + \frac{t^9}{9} - \frac{4t^7}{7} \right]_0^1 = \frac{8}{3}\left[ \frac{4}{5} + \frac{1}{9} - \frac{4}{7} \right]$$

$$= \frac{8}{3}\left[ \frac{107}{315} \right] = \frac{856}{945}$$

5] $\int_0^1\int_0^x e^{x+y}\,dx\,dy = \int_0^1\int_0^x e^x e^y\,dx\,dy = \int_0^1\int_0^x e^x e^y\,dy\,dx$

$$= \int_0^1 \left[ \int_0^x e^x e^y\,dy \right] dx = \int_0^1 e^x\,[e^y]_0^x\,dx \quad \int_0^1 e^{2x} - e^x\,dx$$

$$= \left[ \frac{e^{2x}}{2} \right]_0^1 - [e^x]_0^1 = \left[ \frac{e^2}{2} - \frac{1}{2} \right] - e = \frac{1}{2}[e^2-1] - e + 1 = \frac{[e^2+1-2e]}{2}$$

$$= \frac{(e-1)^2}{2}$$

6] $\int_0^1\int_0^{1/y} y\,e^{xy}\,dx\,dy = \int_0^1 \left[ \int_0^{1/y} y \cdot e^{xy}\,dx \right] dy = \int_0^1 y \cdot \left[ \frac{e^{xy}}{y} \right]_1^{1/y} dy$

$$= \int_0^1 y \cdot \left[ \frac{e}{\frac{1}{y}} - e^y \right] dy = \int_0^1 ey^2 - e^y y\,dy$$

$$\left[\frac{y^3}{3}\right]_0^1 - \left[ye^y - e^y\right]_0^1 \quad e(0) - \frac{(-1\times 1)}{+1}$$

$$(y-1)e^y$$

$$= e\left[\frac{1}{3} - 0\right] - \left[e - e - [1-1]\right] \quad \frac{e}{3} + 1$$

$$\int xe^x = xe^x - \int e^x dx$$

$f(x)\; g(x) \quad xe^x - e^x$

$$xe^x + e^x - e^x$$

=

7] $$\int_1^{\log 8}\int_{x=0}^{x=\log y} e^{x+y}\,dx\,dy \quad \int_1^{\log 8}\left[\int_0^{\log y} e^x e^y\,dx\right]dy = \int_1^{\log 8} e^y\left[e^{\log y} - e^0\right]dy$$

$$\int_1^{\log 8}\left[e^y[y-1]\,dy\right] = \int_1^{\log 8}\left[e^y y - e^y\right]dy = \left((y-1)e^y - e^y\right]_1^{\log 8}$$

$$= (\log 8 - 1)e^{\log 8} - e^{\log 8}\Big] - \left[0 - e^1\right]$$

$$= (\log 8 - 1)8 \underbrace{- 8 + e}_{} = 8(\log 8 - 1)$$

$$\int_0^1\int_{y^2}^{y}(1+xy^2)\,dx\,dy = \int_0^1\left[\int_{y^2}^{y}(1+xy^2)\,dx\right]dy = \int_0^1\left[x + \frac{x^2y^2}{2}\right]_{y^2}^{y} dy$$

$$= \int_0^1\left[y + \frac{y^4}{2} - \left[y^2 + \frac{y^6}{2}\right]\right]dy = \int_0^1\left[y + \frac{y^4}{2} - y^2 + \frac{y^6}{2}\right]dy$$

$$= \left[y^2 + \frac{y^5}{10} - \frac{y^3}{3} + \frac{y^7}{14}\right]_0^1 dy \quad \left[1 + \frac{1}{10} - \frac{1}{3} + \frac{1}{14}\right]$$

=

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Change the order of integration & evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

→ [given order of integration – $dy\,dx$
we have to change it to $dx\,dy$]

Given limits : $y = x^2 \to$ Parabola symmetric about $y$-axis & passing through $(0,0)$

& $y = 2-x \to$ St line cuts $x$-axis at $(2,0)$ & $y$-axis at $(0,2)$

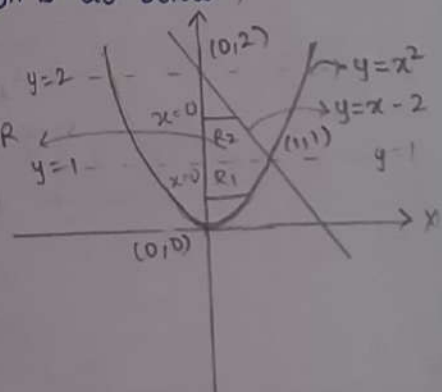
The intersection points of the curve $y = x^2$, $y = 2-x$ are given by

$$x^2 = 2-x \Rightarrow x^2+x-2=0$$
$$x = 1, -2$$

when $x=1, y=1$

$\therefore$ The intersection point of the 2 curves is $(1,1)$ and $x$ limits are from 0 to 1.

Hence the region of Integration is as below




Take Horizontal strips in the region $R_1$ & $R_2$

For $R_1$ : $x$ varies from $x=0$ to $x=\sqrt{y}$

& $y$ varies from $y=0$ to $y=1$

For $R_2$ : $x$ varies from $x=0$ to $x=2-y$

$y$ varies from $y=1$ and $y=2$

$$\therefore \int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{y=2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2y}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2y}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{(2-y)^2 y}{2}\,dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \int_1^2 \frac{(4+y^2-4y)y}{2}\,dy$$

$$= \frac{1}{6} + \int_1^2 \frac{4y+y^3-4y^2}{2}\,dy = \frac{1}{6} + \left[y^2 + \frac{y^4}{8} - \frac{2y^3}{3}\right]_1^2$$

Change o

$\iint_R f($

1] Draw
line

2] Mark

3] Take th

4] Evaluate

Note : A sim

$$= \frac{1}{6} + \left[4 + 2 - \frac{16}{3} - \left[1 + \frac{1}{8} - \frac{2}{3}\right]\right] = \frac{1}{6} + \left[6 - \frac{16}{3} - 1 - \frac{1}{8} + \frac{2}{3}\right]$$

$$= \frac{1}{6} + \left[5 - \frac{14}{3} - \frac{1}{8}\right]$$

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \frac{3}{8}$$

2) $\int_0^1 \int_1^{2-x} xy\,dx\,dy$

Given order → dydx

Change it to → dxdy

Given Limits, $y = 1$ → Line

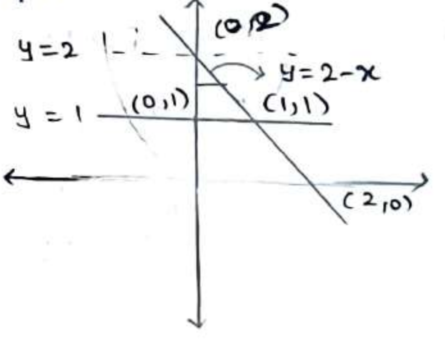
$y = 2 - x$ → Line cuts at $(2,0)$ & $(0,2)$

x axis, y axis

P.O.I → $y = 1$ & $y = 2 - x$

$1 = 2 - x$

$x = 1$ → $(1,1)$ Intersection pt.




Take Horizontal Strip in $R_1$

For $R_1$: $x$ varies from $0$ to $2-y$

$y$ varies from $1$ to $2$

$$\int_0^1 \int_1^{2-x} xy\,dy\,dx = \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2 y}{2}\right]_0^{2-y} dy = \int_1^2 \left[\frac{4y + y^3 - 4y^2}{2}\right] dy = \left[y^2 + \frac{y^4}{8} - \frac{2y^3}{3}\right]_1^2$$

$$= \frac{5}{24}$$

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Evaluate $\iint_R y\,dx\,dy$. where R is the Region bounded by the parabolas $y^2=4x$ and $x^2=4y$

Sol 1] Find the Intersecting points of the curves

2] Draw the region bounded by the curves and mark all the intersecting Points.

3] Take either Horizontal (or) vertical strip in the Region & find the Limits x & y and then using these Limits we will evaluate the given double Integral.

To find the Intersecting pts of the curves $y^2=4x \text{ --- } (1)$, $x^2=4y \Rightarrow \underbrace{y=\frac{x^2}{4}}_{(2)}$

Sub (2) in (1), $\left(\frac{x^2}{4}\right)^2=4x \Rightarrow x^4=4^3x$

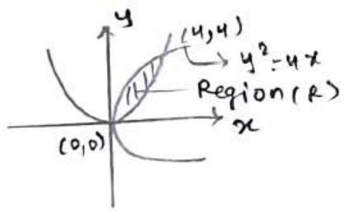
$$\Rightarrow x(x^3-4^3)=0$$

$$\Rightarrow x=0,4$$

when $x=0 \Rightarrow y=0$ $\left[\because \text{from } (1)\right]$

when $x=4, y=4$

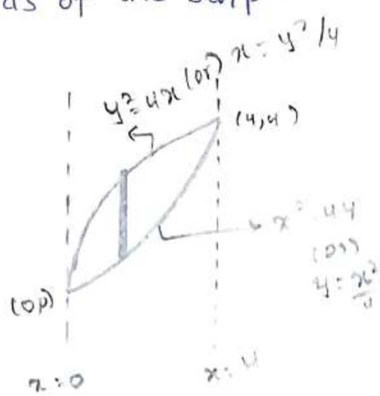
$\therefore$ The Intersecting points of the given curves are (0,0) (4,4). The region bounded by the curve is as below




Taking vertical strip in the Region Then the ends of the strip are on the curves $x^2=4y$ & $y^2=4x$.

Hence, the Limits of y are $\frac{x^2}{4}$ to $2\sqrt{x}$

Moving the vertical strip horizontally to cover the entire region. Then the limits of x are 0 to 4.




$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx$$

$$= \int_{x=0}^{4}\left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right]dx$$

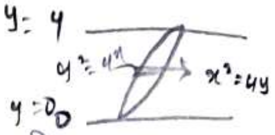
$$= \frac{1}{2}\int_0^4 \left[(2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2\right] dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$\frac{1}{2}\left[2(16) - \frac{4^5}{80}\right] = \frac{1}{2}\left[32 - \frac{1024}{80}\right]$$

$$= 16 - \frac{32}{5} = \frac{48}{5}$$

<u>Method 2:</u>



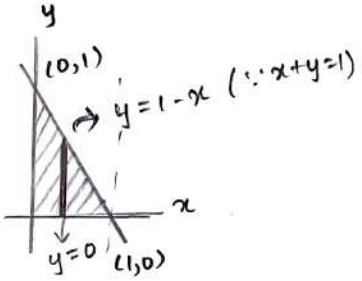

$$\int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} y\, dx\, dy$$

$y\, dx\, dy$

2) Evaluate $\int x^2+y^2 dx dy$ in the +ve Quandrant for which $x+y \leq 1$

R (bounded by $y=0$, $x=0$, $x+y=1$)

∵ The Intersecting points of the given curves are $(0,1)$ & $(1,0)$. The region bounded by the curve is as show in fig.




Taking vertical strip in the region Then the ends of strip are on the curves $y=0$ to $y=1-x$

$y \to 0$ to $1-x$
$x \to 0$ to $1$

The Limits of $y$ are $0$ to $1-x$

$x \to 0$ to $1$

$$\int_0^1\int_0^{1-x} (x^2+y^2)\, dy\, dx = \int_0^1 \left[\cancel{y} + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[1-x + \frac{(1-x)^3}{3}\right] dx$$

$$\frac{(1-x)^2}{-2}\Big|_0^1 \left[\frac{(1-x)^4}{-3}\right]_0^1 = -\left[\frac{(1-1)^4}{-3} + \frac{(1-0)^4}{-3}\right] + \frac{(1-1)^2}{2} + \frac{(1-0)^2}{2}$$

$$= 0 + \frac{-1}{3} + \frac{1}{2}$$

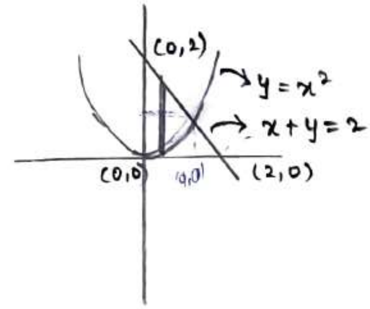
$$= \frac{1}{6}$$

Horizontal →

$x \to 0$ to $1-y$

$y \to 0$ to $1$

$\iint_R y\,dx\,dy$ where $R$ is domain bounded by $y$ - axis, the curve $y = x^2$ & the line $x+y=2$ in first quadrant




$x+y=2.$

$y = 2-x$

$\Rightarrow x^2+x+2=0$

$x = 1, -2.$

$x=1, y=1$ ($1^{st}$ Quadrant)

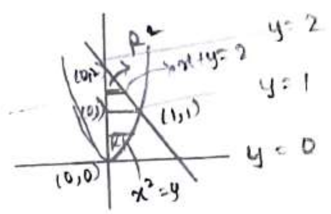
$x=-2, y=0$

$$\int_0^1\int_{x^2}^{2-x} y\,dx\,dy = \int_0^1\int_{x^2}^{2-x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx = \left[\frac{(2-x)^3}{6} - \frac{x^5}{10}\right]_0^1$$

$$= \frac{2}{6} - \frac{1}{10} = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$$

(or) $$\iint_R y\,dx\,dy = \int_0^1\int_0^{\sqrt{y}} y\,dx\,dy + \int_1^2\int_0^{2-y} y\,dx\,dy$$

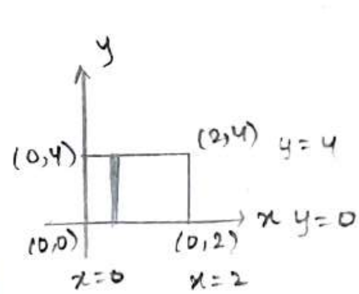



4) $\iint_R f(x,y)\,dx\,dy$ where $f(x,y) = 2xy$ & R is Rectangle with vertices

$(0,0)$ $(2,0)$ & $(0,4)$ $(2,4)$

Limits $\rightarrow$ $y = 0$ to $4$

$x = 0$ to $2$




$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dx\,dy = \left[\int_{x=a}^{b} f(x)\,dx\right]\left[\int_{y=c}^{d} f(y)\,dy\right]$$

$$= \int_0^2 2x\,dx \times \int_0^4 y\,dy = \left[2\times\frac{x^2}{2}\right]_0^2 \times \left[\frac{y^2}{2}\right]_0^4$$

$$= (4-0) \times 8 = 32.$$

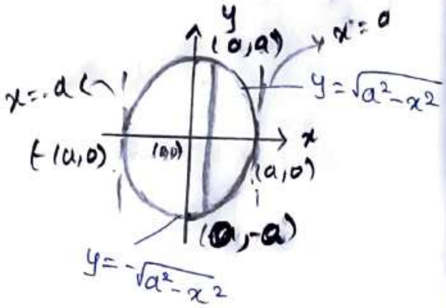
$= \frac{16}{3}$

5] Evaluate $\iint_R dx\,dy$ where R is the Region bounded by the circle.

$x^2+y^2=a^2$

$y^2 = a^2-x^2$

$y = \pm\sqrt{a^2-x^2}$




$x \to -a$ to $a$

$y \to -\sqrt{a^2-x^2}$ to $\sqrt{a^2+x^2}$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}} dy\,dx \quad \int_{-a}^{a}[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}}\,dx = \int_{-a}^{a}\left[\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right]dx$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx.$$

$$= 2\left[\frac{x}{a}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{a}$$

$$= 2\left[\frac{a}{a}\sqrt{0}+\frac{a^2}{2}\sin^{-1}(1)-\left[0+\frac{a^2}{2}\sin^{-1}(-1)\right]\right]$$

$$= 2\left[\frac{a^2}{2}\frac{\pi}{2}+\frac{a^2}{2}\frac{\pi}{2}\right] = 2\left[\frac{2a^2\pi}{4}\right]$$

$$= a^2(\pi)$$

---

Change of order of Integration.

$$\iint_R f(x,y)\,dy\,dx = \int_{x=a}^{b}\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx$$

1] Draw region of Integration by drawing curves $y=f_1(x)$, $y=f_2(x)$ and line $x=a$ & $x=b$.

2] Mark all the Intersection pts.

3] Take the Horizontal strip in the region & find the Limits for both x & y

4] Evaluate the double integral with new limits.

Note : A similar procedure to evaluate $\int_{y=c}^{y=d}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$

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## Change of variables in double integral

Sometimes, the evaluation of double or triple integral with its present form may not be simple to evaluate. By choice of an appropriate coordinate system, given integral can be transformed to a simpler integral involving the new variables.

### 1) Transformation of coordinates

Let $x = g(u,v)$ $y = h(u,v)$ be the relations b/w the old variables $x, y$ with the new variables $u, v$ of ~~with~~ the new coordinate system, then,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(g,h)\,|J|\,du\,dv$$

where, $J$ is Jacobian of $x, y$ w.r.t $u, v$ and is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

### 2) Cartesian to polar coordinates

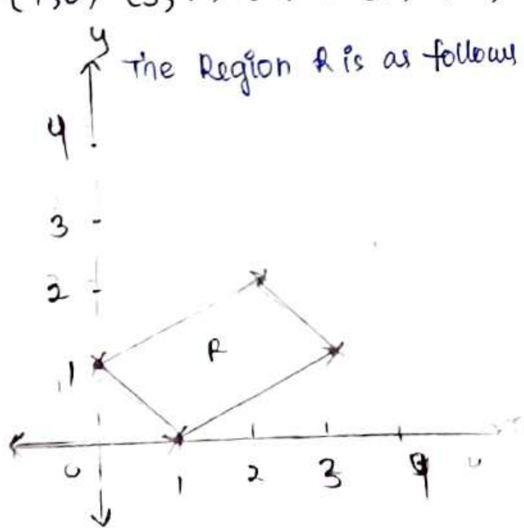
Let $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

---

1] Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is llgram in $xy$ plane with vertices $(1,0)$ $(3,1)$ $(2,2)$ $(0,1)$, $u = x+y$, $v = x-2y$ (using Transformation)

The Region R is as follow




Given transformations are $u = x+y - (1)$
$v = x-2y - (2)$

$2(1)+2 \Rightarrow 2u+v = 3x \Rightarrow x = \frac{1}{3}(2u+v)$

$(1)-(2) \Rightarrow u-v = 3y \Rightarrow y = \frac{1}{3}(u-v)$

Now, $\frac{\partial x}{\partial u} = \frac{2}{3}$, $\frac{\partial x}{\partial v} = \frac{1}{3}$, $\frac{\partial y}{\partial u} = \frac{1}{3}$, $\frac{\partial y}{\partial v} = \frac{-1}{3}$

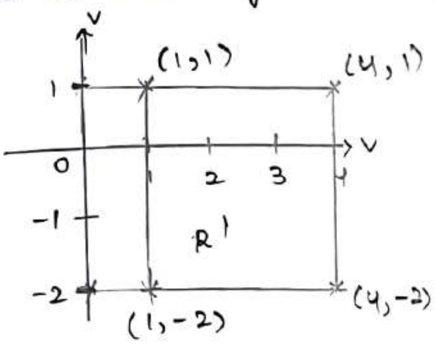
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-1}{3}$$

Find vertices of new region to find new limits

| $(x,y) \to$ old | $(u,v) \to$ new |
|---|---|
| $(1,0)$ | $(1,1)$ |
| $(0,1)$ | $(1,-2)$ |
| $(3,1)$ | $(4,1)$ |
| $(2,2)$ | $(4,-2)$ |

$u = x+y$, $v = x-2y$

Hence the new region is as follows




The Limits of u,v are 1 to 4 and -2 to +1

$$\therefore \iint_R (x+y)^2 \, dx\,dy = \iint_{R'} u^2 |J| \, du\,dv$$

$$= \int_{v=-2}^{1} \int_{u=1}^{4} u^2 \left| \frac{-1}{3} \right| du\,dv = \frac{1}{3} \int_{-2}^{1} \int_{1}^{4} u^2 \, du\,dv$$

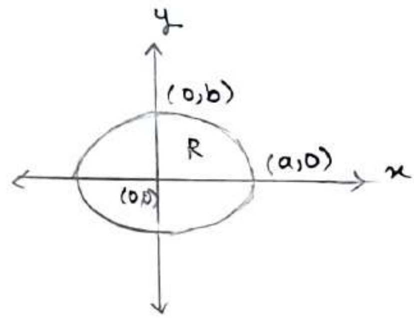
$$\frac{1}{3} \int_{-2}^{1} \left( \frac{u^3}{3} \right)^4 dv$$

$$= \frac{63}{3} = 21$$

---

Evaluate $\iint_R (x^2+y^2) \, dx\,dy$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the $Q_1$ ($1^{st}$ Quadrant) by using transformation, $x = au, y = bv$

$(0,0)$ $(a,0)$ $(0,b)$




Given, $x = au$, $y = bv$ $\quad u = \frac{x}{a}, v = \frac{y}{b}$

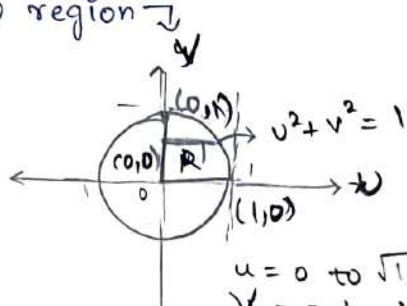
$$\frac{\partial x}{\partial u} = a, \frac{\partial y}{\partial u} = 0, \frac{\partial x}{\partial v} = 0, \frac{\partial y}{\partial v} = b$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

New Vertices $\to$ New limits

| $(x,y) \to$ old | $(u,v) \to$ new |
|---|---|
| $(0,0)$ | $(0,0)$ |
| $(a,0)$ | $(1,0)$ |
| $(0,b)$ | $(0,1)$ |

New region $\to$




$u = 0$ to $\sqrt{1-v^2}$
$v = 0$ to $1$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \iint_{R'} (a^2u^2+b^2v^2)\,ab\,du\,dv$$

$$= \iint_{R'} (a^3bu^2 + ab^3v^2)\,du\,dv = \int_0^1 \left[\int_0^{u=\sqrt{1-v^2}} (a^3bu^2 + ab^3v^2)\,du\right] dv$$

$$= \int_0^1 \left[\frac{a^3bu^3}{3} + 0\right]_0^{\sqrt{1-v^2}} dv = \frac{a^3b}{3}\int_0^1 \left[(1-v^2)\sqrt{1-v^2}\right] dv$$

$v = \sin\theta$ or $\cos\theta$

$$= \frac{a^3b}{3}\int_0^1 (1-v^2)^{3/2}\,dv$$

$$\int_0^1 \left(\frac{a^3bu^3}{3} + ab^3v^2 \cdot u\right)_0^{\sqrt{1-v^2}} = \frac{a^3b\sqrt{1-v^2}(1-v^2)}{3} + ab^3v^2\sqrt{1-v^2}$$

$$ab(a^2+b^2)\sqrt{1-v^2}\left(\frac{1-v^2}{3} + v^2\right)$$

$$ab(a^2+b^2)$$

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# Triple Integrals

This is an extension of double Integral. If $f(x,y,z)$ is a 3. variable func, then triple Integrals are denoted by $\iiint_V f(x,y,z)\,dx\,dy\,dz$

$$= \int_{x=a}^{b} \int_{y=h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dx\,dy\,dz$$

Note: If the limits of $x, y, z$ are constants, then triple Integrals can be evaluated in any order as per the given limits.

---

1] Evaluate $\int_0^1 \int_1^2 \int_2^3 xyz\,dx\,dy\,dz$

$$= \int_0^1 \int_1^2 yz\left(\int_2^3 x\,dx\right) dy\,dz = \int_0^1 \int_1^2 yz\left(\frac{x^2}{2}\right)_2^3 dy\,dz = \frac{5}{2}\int_0^1 z\left\{\int_1^2 y\,dy\right\} dz$$

$$= \frac{5}{2}\int_0^1 z\left[\frac{y^2}{2}\right]_1^2 dz = \frac{5}{2}\cdot\frac{3}{2}\int_0^1 z\,dz = \frac{15}{4}\left[\frac{z^2}{2}\right]_0^1 = \frac{15}{8}$$

(or)

$$= \int_0^1 z\,dz \times \int_1^2 y\,dy \times \int_2^3 x\,dx = \frac{1}{2}\times\frac{3}{2}\times\frac{5}{2} = \frac{15}{8}$$

---

2] Evaluate $\int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\,dx\,dy\,dz = \int_0^a \int_0^x \int_0^{x+y} \underset{e^x e^y e^z}{e^{x+y+z}}\,dz\,dy\,dx$

$$= \int_0^a \int_0^x e^x e^y \left(e^z\right)_0^{x+y} dy\,dx = \int_0^a \int_0^x e^x e^y \left(e^{x+y} - 1\right) dy\,dx$$

$e^x e^y (e^x e^y - 1)$

$(e^x e^y)^2 - e^x e^y$

$$= \int_0^a e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x - e^x\left(e^y\right)_0^x dx = \int_0^a \frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\,dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3}{2}\times\frac{e^{2x}}{2} + e^x\right]_0^a = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} - e^a - \left[\frac{1}{8} - \frac{3}{4} + 1\right]$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$$

3] Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$z \quad x \quad y$

$$= \int_{-1}^{1}\int_{0}^{z}(x+z)+\left[\frac{y^2}{2}\right]_{x-z}^{x+z}dx\,dz = \int_{-1}^{1}\int_{0}^{z}(x+z)+\frac{(x+z)^2}{4}-\frac{(x-z)^2}{4}\,dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}(x+z)+xz\,dx\,dz = \int_{-1}^{1}\int_{0}^{z}\cancel{x^2z+xz^2}\,(x+z+xz)\,dx\,dz$$

$$= \int_{-1}^{1}(1+z+z)_0^z\,dz = \int_{-1}^{1}(1+2z)\,dz = (z+z^2)_{-1}^{1} = 2-(-1+1) = 0$$

---

4] $\int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x}\log z\,.\,dz\,dx\,dy = \int_{1}^{e}\int_{1}^{\log y}(z\log z - z)_1^{e^x}\,dx\,dy$

$$= \int_{1}^{e}\int_{1}^{\log y}(e^x.x-e^x)-(0-1)\,dx\,dy = \int_{1}^{e}\int_{1}^{\log y}(xe^x-e^x)+1\,dx\,dy$$

$$= \int_{1}^{e}\left[xe^x-e^x-e^x+x\right]_1^{\log y}dy \qquad xe^x-\int 1e^x.1$$

$$\log y\frac{y^2}{2}-\int\frac{1}{y}\frac{y^2}{2} = \frac{y^2\log y}{2}$$

$$= \int_{1}^{e}\left[xe^x-2e^x+x\right]_1^{\log y}dy = \int_{1}^{e}\left[\log y\,.\,y-2y+\log y-(e-2e+1)\right]dy$$

$$= \int_{1}^{e}\left[y\log y+\log y-2y+e-1\right]dy = \left[\frac{y^2\log y}{2}-\frac{y^2}{4}+y\log y-y-\frac{2y^2}{2}+(e-1)y\right]_1^e$$

$$= \left[\frac{e^2}{2}-\frac{e^2}{4}+\cancel{e}-\cancel{e}-\underbrace{\frac{2e^2}{2}}_{-\cancel{e^2}}+\underbrace{(e-1)e}_{\cancel{e^2}-e}\right]-\left[0-\frac{1}{4}+0-1-1+e-1\right]$$

$$= \left[\frac{e^2}{2}-\frac{e^2}{4}-e\right]-\left[\frac{-13}{4}+e\right]$$

$$= +\frac{1}{4}e^2-e+\frac{13}{4}-e = \frac{e^2}{4}-2e+\frac{13}{4} = \frac{1}{4}(e^2-8e+13)$$

$$\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,dx\,dy\,dz = \int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x}e^{y}e^{z}\,dz\,dy\,dx$$

$x \quad y \quad z$

$$= \int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x}e^{y}\cdot e^{z}\,dz\,dy\,dx = \int_{0}^{\log 2}\int_{0}^{x} e^{x}e^{y}\left(e^{z}\right)_{0}^{x+\log y}\,dy\,dx$$

$$= \int_{0}^{\log 2}\int_{0}^{x} e^{x}e^{y}\left(e^{x}\cdot e^{\log y}-1\right)dy\,dx = \int_{0}^{\log 2}\int_{0}^{x} e^{2x}\cdot e^{y+\log y} - e^{x}e^{y}\,dy\,dx$$

$$= \int_{0}^{\log 2}\int_{0}^{x} \cancel{\left[e^{2x}\cdot\frac{e^{y+\log y}}{1+\frac{1}{y}}\right.} \; e^{2x}\cdot e^{y}\cdot y - e^{x}e^{y}\,dy\,dx$$

$$= \int_{0}^{\log 2}\left[e^{2x}\cdot\left[y e^{y}-e^{y}\right]-e^{x}e^{y}\right]_{0}^{x}dx = \int_{0}^{\log 2}\left[e^{2x}\cdot\left[x e^{x}-e^{x}\right]-e^{2x}\right] - \left[e^{2x}\cdot[0-1]-e^{x}\right]dx$$

$$= \int_{0}^{\log 2}\left[e^{2x}\left[x e^{x}-e^{x}-1\right]\right]-\left[-e^{2x}-e^{x}\right]dx$$

$$= \int_{0}^{\log 2}\left[x e^{3x}-e^{3x}-\cancel{e^{2x}}\right]+\cancel{e^{2x}}+e^{x}\,dx = \int_{0}^{\log 2}\left[x e^{3x}-e^{3x}+e^{x}\right]dx$$

$$= \left[\frac{x\cdot e^{3x}}{3}-\frac{e^{3x}}{9}-\frac{e^{3x}}{3}+e^{x}\right]_{0}^{\log 2} = \log 2\cdot\frac{8}{3}-\frac{8}{9}-\frac{8}{3}+2 - \left[0-\frac{1}{9}-\frac{1}{3}+1\right]$$

$$= \frac{8}{3}\log 2-\frac{14}{9}-\frac{5}{9} = \frac{8}{3}\log 2-\frac{19}{9}.$$

---

$$\int_{0}^{2\pi}\int_{0}^{b}\int_{-h}^{h}(z^{2}+r^{2}\sin^{2}\theta)\,dz\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{b}\left[\frac{z^{3}}{3}+r^{2}\sin^{2}\theta\cdot z\right]_{-h}^{h}dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{b}\left[\frac{h^{3}}{3}+hr^{2}\sin^{2}\theta+\frac{h^{3}}{3}+hr^{2}\sin^{2}\theta\right]dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{b}\frac{2h^{3}}{3}+2hr^{2}\sin^{2}\theta\,dr\,d\theta$$

3]

$$\int_0^{2\pi}\int_0^{b} \frac{2h^3}{3} + 2hr^2\sin^2\theta \, dr\, d\theta = \int_0^{2\pi} \left[\left[\frac{2h^3}{3}\cdot r\right] + (2h\sin^2\theta)\frac{r^3}{3}\right]_0^b d\theta$$

$$= \int_0^{2\pi} \frac{b2h^3}{3} + (2h\sin^2\theta)\frac{b^3}{3} - 0 \, d\theta = \left[\frac{2bh^3}{3}\theta + \frac{2b^3h}{3}\left[\frac{1}{2} - \frac{\sin 2\theta}{4}\right]\right]_0^{2\pi}$$

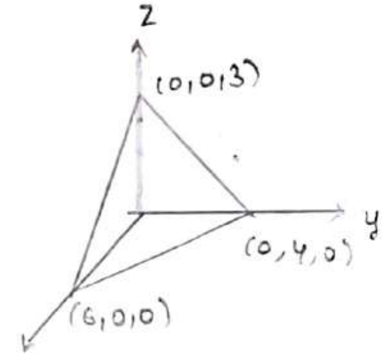
$$\int \sin^2\theta = \int \frac{1-\cos 2\theta}{2} = \frac{1}{2} - \frac{\sin 2\theta}{4}$$

$$= \left[\frac{2bh^3}{3}\cdot 2\pi + \frac{2b^3h}{3}\left[\frac{1}{2} - 0\right]\right] - \left[0 + 0\right]$$

$$\Rightarrow \quad = \frac{2}{3}\pi bh\,(2h^2 + b^2)$$

---

Evaluate $\iiint_V dx\,dy\,dz$ where $v$ is a finite region of space formed by the planes $x=0, y=0, z=0$, $2x+3y+4z=12$



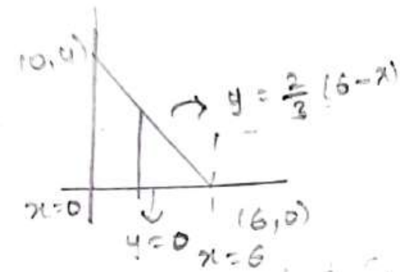

For the given solid, the limits are

$z = 0$ to $\frac{1}{4}(12-2x-3y)$

$y = 0$ to $\frac{2}{3}(6-x)$

[∵ For the region R in $xy$ plane in the eqn of the line joining $(6,0), (0,4)$ is]

xy-plane




$$\frac{x-6}{0-6} = \frac{y-0}{4-0} \Rightarrow y = \frac{2}{3}(6-x)$$

and for $x$ the limits are 0 to 6

$$\therefore \iiint_V dx\,dy\,dz = \int_{x=0}^{6}\int_{y=0}^{\frac{2}{3}(6-x)}\int_{z=0}^{\frac{1}{4}(12-2x-3y)} dz\,dy\,dx.$$

$$= \int_0^6 \int^{\frac{2}{3}(6-x)} \frac{1}{4}(12-2x-3y)\,dy\,dx$$

$$= \int_0^6 \left( \frac{1}{4}\left(12y - 2xy - \frac{3y^2}{2}\right)_0^{2/3(6-x)} dx = \frac{1}{4}\int_0^6 \left( 8(6-x) - \frac{4x}{3}(6-x) - \frac{2}{3}(6-x)^2 \right) dx$$

$$= \frac{1}{4}\int_0^6 48 - 8x - \frac{24x}{3} + \frac{4x^2}{3} - \frac{2}{3}(6-x)^2\, dx = \frac{1}{4}\left[48x - \frac{8x^2}{2} - \frac{24x^2}{2\times 3} + \frac{4x^3}{9} + \frac{2}{9}(6-x)^3\right]$$

$$288 - 288 \qquad 6^3 \times \frac{2}{9}$$

$$= \frac{1}{4}\left[48(6) - 4(36) - 4(36) + \frac{4}{9}\times 36\times 6 - \frac{2}{9}\times 6^3\right] = \frac{1}{4}\times 6\times 6\times 6\times \frac{2}{9}$$

---

$\iiint xy^2 z\,dx\,dy\,dz$ taken positive ~~Operand~~ Octane of the sphere $x^2+y^2+z^2=a^2$

$z = 0$ to $\sqrt{a^2-x^2-y^2}$

$z=0$    $y = 0$ to $\sqrt{a^2-x^2}$

$x^2+y^2=a^2$    $x = 0$ to $a$

$x^2=a^2$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\,dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}} \left(xy^2\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{xy^2(a^2-x^2-y^2)}{2}\,dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{xy^2a^2 - x^3y^2 - xy^4}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy^2a^2 - x^3y^2 - xy^4\,dy\,dx = \frac{1}{2}\int_0^a \left(xa^2\frac{y^3}{3} - \frac{x^3y^3}{3} - \frac{xy^5}{5}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \frac{xa^2(a^2-x^2)^{3/2}}{3} - \frac{x^3(a^2-x^2)^{3/2}}{3} - \frac{x(a^2-x^2)^{5/2}}{5}\,dx$$

$$= \frac{1}{2}$$

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3] 1) Evaluate $\iiint_E (3-4x)\,dv$ where $E$ is the region below $z=4-xy$ & above the region in the $xy$ plane $\underset{\hookrightarrow z=0}{}$ defined $0\le x\le 2,\ 0\le y\le 1$

$$\iiint_E (3-4x)\,dv = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{4-xy} (3-4x)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^1 \left[3z-4xz\right]_0^{4-xy} dy\,dx = \int_0^2\int_0^1 [12-3xy-4x(4-xy)]-0\,dy\,dx$$

$$= \int_0^2\int_0^1 12-3xy-16x+4x^2y\,dy\,dx = \int_0^2 \left[12y-3x\frac{y^2}{2}-16xy+\frac{4x^2y^2}{2}\right]_0^1 dx$$

$$= \int_0^2 \left[12-\frac{3x}{2}-16x+\frac{4x^2}{2}\right]-0\,dx$$

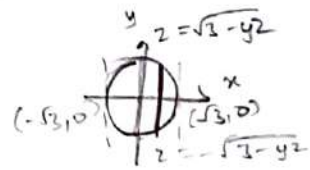
$$= \left[12x-\frac{3x^2}{4}-\frac{16x^2}{2}+\frac{\overset{2}{4}x^3}{\underset{3}{6}}\right]_0^2 = 24-3-32+\frac{16}{3} = -11+\frac{16}{3} = \frac{-17}{3}$$

2) Evaluate $\iiint_E yz\,dv$, where $E$ is the region bounded by $x=2y^2+2z^2-5$ and the plane $x=1$

$x=2y^2+2z^2-5$ } POI

$2y^2+2z^2-5=1 \Leftarrow x=1$

$y^2+z^2=3$ ( POI)




It is a circle with centre $(0,0)$ & radius $\sqrt{3}$

$z=-\sqrt{3-y^2}$ to $\sqrt{3-y^2}$

$y=-\sqrt{3}$ to $\sqrt{3}$

$$= \int_{y=-\sqrt{3}}^{\sqrt{3}}\int_{z=-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{x=2y^2+2z^2-5}^{1} yz\,dx\,dz\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \left[xyz\right]_{2y^2+2z^2-5}^{1} dz\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \left[yz-(2y^2+2z^2-5)yz\right] dz\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} (yz - 2y^3z - 2yz^3 + 5yz)\,dz\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left( y\frac{z^2}{2} - y^3z^2 - 2y\frac{z^4}{4} + 5y\frac{z^2}{2} \right)_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left( \frac{3y-y^3}{2} - \overset{y^3(3-y^2)}{\cancel{(3-y^2)^{3/2} z^2}} - \frac{2y(3-y)^2}{4} + \frac{5y(3-y^2)}{2} \right) - \left( \frac{3y-y^3}{2} + \overset{y^3(3-y^2)}{\cancel{(3-y^2)^{3/2} z^2}} - \downarrow \quad \downarrow \right) dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \cancel{2(3-y^2)^{3/2} z^2\,dy} = \int_{-\sqrt{3}}^{3} -2y^3(3-y^2)\,dy$$

$f(y) = -2y^3(3-y^2)$
$f(-y) = 2y^3(3-y^2)$ } odd

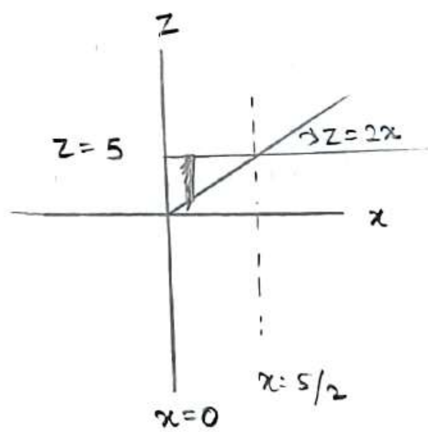
$= 0$ //

5) $\iiint (12y-8x)\,dV$, $y = 10-2z$ in front of region in $xz$ plane ↓ $y=0$

$z = 2x$, $z = 5$, $x = 0$

$z : 2x$ to $5$ (or) $x = 0$ to $z/2$

$x : 0$ to $5/2$ $\quad z = 0$ to $5$




$$= \int_{x=0}^{5/2} \int_{2x}^{5} \int_{y=0}^{10-2z} (12y-8x)\,dy\,dz\,dx$$

$$= \int_{0}^{5/2} \int_{2x}^{5} (6y^2 - 8xy)_0^{10-2z}\,dz\,dx$$

$$= \int_0^{5/2} \int_{2x}^{5} 6(10-2z)^2 - 8x(10-2z)\,dz\,dx = \int_0^{5/2} \int_{2x}^{5} 6(100+4z^2-40z) - (+80x-16xz)\,dz\,dx$$

$$= \int_0^{5/2} \left[ 600z + 24z^3/3 - 120z^2 - 80xz + 8xz^2 \right]_{2x}^{5}$$

$$\int_0^{5/2} \cancel{3060} + 1000 - \cancel{3060} - 400x + 200x - \left[1200x + 64x^3 - 480x^2 - 160x^2 + 32x^3\right] dx$$

$$= \left[1000x - 200\frac{x^2}{2} - 600x^2 - \overset{16}{\cancel{64}}\frac{x^4}{\cancel{4}} - \overset{160}{\cancel{480}}\frac{x^3}{\cancel{3}} - 160\frac{x^3}{3} + \overset{8}{\cancel{32}}\frac{x^4}{\cancel{4}}\right]_0^{5/2}$$

$$= 2500 - 100\times\frac{25}{4} - 600\times\frac{25}{4} - \overset{4}{\cancel{16}}\times\frac{25}{\cancel{4}}\times\frac{25}{\cancel{4}} - 160\times\frac{125}{8} - 160\times\frac{125}{8\times 3} + \overset{2}{\cancel{8}}\times\frac{25}{4}\times\frac{25}{\cancel{4}_2}$$

$$= \cancel{2500} - 625 - 3750 - \cancel{2500} - 5000 - \frac{2500}{3} + \frac{625}{2}$$

$$= \frac{3125}{6}$$

---

## VOLUME

→ Volume of a Solid under the surface $z = f(x,y)$ is given by

$V = \iint_R z\,dx\,dy$. where R is the projection of surface $z = f(x,y)$ on $xy$ plane.

Q→ Using double Integral, find the volume of tetrahedron bounded by the coordinate planes & the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$V = \iint_R z\,dx\,dy = \iint_R c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\,dy$$

$$= \int_0^a \int_{y=0}^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx$$

$$= \int_0^a c\left(\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dx$$

$$= \int_0^a c\left[\left(1 - \frac{x}{a}\right)b\left(1 - \frac{x}{a}\right) - \frac{b^2\left(1 - \frac{x}{a}\right)^2}{4b^2}\right] dx$$

$$= \int_0^a c\left[b\left(1-\frac{x}{a}\right)^2 - \frac{1}{4}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= c\int_0^a \left[\frac{b}{a^2}(a-x)^2 - \frac{1}{4a^2}(a-x)^2\right] dx = \frac{c}{a^2}\int_0^a (a-x)^2\left(b-\frac{1}{4}\right) dx$$

$$= \frac{c\left(b-\frac{1}{4}\right)}{a^2}\left(\frac{(a-x)^3}{-1}\right)_0^a = \frac{c(4b-1)}{4a^2}\left[0+a^3\right]$$

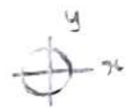
$$= \frac{ac(4b-1)}{4}$$

---

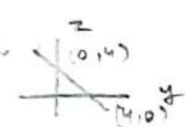
30/5/24

Find the volume bounded by the regions $x^2+y^2=4$, $y+z=4$ & $z=0$ then using $\iint$ evaluate volume

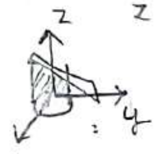
$x^2+y^2=4 \rightarrow$




$y+z=4 \rightarrow$




$z=0 \rightarrow$ on xy plane




$$V = \iint_R z\,dx\,dy$$

$$= \iint_R (4-y)\,dx\,dy$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[4\sqrt{4-x^2} - \frac{4-x^2}{2} + 4\sqrt{4-x^2} + \frac{4-x^2}{2}\right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx$$

$$= 2\int_0^2 8\sqrt{4-x^2}\,dx = 16\pi$$

$dx = 2\cos\theta\, d\theta$

$x = 2\sin\theta$

$2 = 2\sin\theta$

$\sin\theta = 1$

$\theta = \pi/2$

$0 = 0$

$\frac{\pi}{4}$

$\frac{1-\cos 2\theta}{2}$

$\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)_0^{\pi/2}$

$\frac{\pi}{4}\times 8$

**Note:**

Volume bd by $x^2+y^2=4$, $y+z=4$, $z=0$ using Triple Integrals is given by

$$V = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-y} dz\,dy\,dx$$

$4\cos^2\theta\, d\theta$

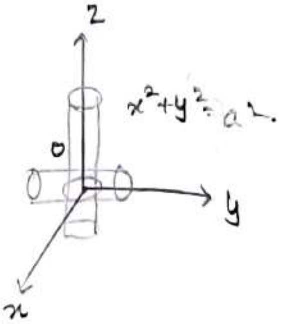
$$= \int_0^{\pi/2}\sqrt{4(1-\sin^2\theta)}\, d\theta$$

$4(1-0)$

$$= 16\int_0^{\pi/2} 2\cos\theta\, d\theta\; 2\cos\theta\, d\theta$$

$$= 4(\sin\theta)_0^{\pi/2}$$

2) Find the volume bounded by $x^2+y^2=a^2$; $x^2+z^2=a^2$




$$V = \iint z\,dx\,dy$$

$$V = \iint z\,dx\,dy$$

$$= 2 \iint_{x^2+y^2=a^2} \sqrt{a^2-x^2}\,dx\,dy$$

$$= 2 \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx$$

$$= 2 \int_{-a}^{a} 2 \cdot \int_{0}^{\sqrt{a^2-x^2}} \underline{\sqrt{a^2-x^2}}\,dy\,dx$$

$$V = \iiint dx\,dy\,dz$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 4 \int_{-a}^{a} \left[\sqrt{a^2-x^2} \cdot y\right]_{0}^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{-a}^{a} (a^2-x^2)\,dx$$

$$= 8 \int_{0}^{a} (a^2-x^2)\,dx$$

$$= 8\left[a^2x - \frac{x^3}{3}\right]_0^a = 8\left[a^3 - \frac{a^3}{3}\right]$$

$$= 8\left[\frac{2a^3}{3}\right] = \frac{16a^3}{3}$$

---

3) Find the volume bounded by $x^2+y^2=1$, $2x+3y+4z=12$, $xy$ plane

$$\downarrow$$

$$z = \frac{1}{4}(12-2x-3y)$$

$$V = \iint z\,dx\,dy$$

$$= \iint \frac{1}{4}(12-2x-3y)\,dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4}(12-2x-3y)\,dy\,dx$$

$$= \int_{-1}^{1} \int 3 - \frac{x}{2} - \frac{3}{4}y\,dy\,dx = \int_{-1}^{1} \left[3y - \frac{xy}{2} - \frac{3y^2}{8}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left[ 3\sqrt{1-x^2} - \frac{x\sqrt{1-x^2}}{2} - \frac{3(1-x^2)}{8} - \left[ -3\sqrt{1-x^2} + \frac{x\sqrt{1-x^2}}{2} + \frac{3(1-x^2)}{8} \right] \right] dx$$

$$\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_{-1}^{1} 6\sqrt{1-x^2} - x\sqrt{1-x^2}\, dx$$

$$\sqrt{1-x^2} - \int \frac{(-2x)}{2\sqrt{1-x^2}} = \sqrt{1-x^2} + \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}} \quad \frac{f'(x)}{f(x)}$$

$$= 6\left[\frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{-1}^{1} - \left[\sqrt{1-x^2} + \frac{1}{2}\log\sqrt{1-x^2}\right]_{-1}^{1} \qquad \log f(x)$$

$$= 6\left[\frac{1}{2}\sqrt{0} + \frac{1}{2}\sin^{-1}(1) + 0 - \frac{1}{2}\sin^{-1}(-1)\right] - [0+0]$$

$$= 6\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{6\pi}{2}$$

$$= 3\pi$$

---

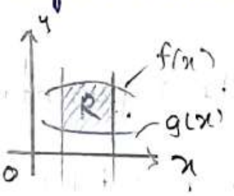
3/8/24

**Area enclosed by a closed curve:**

Consider the area enclosed by the curves $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ in $xy$ plane

The area of the region are bounded by the given curves is given by $\iint_R dy\,dx$

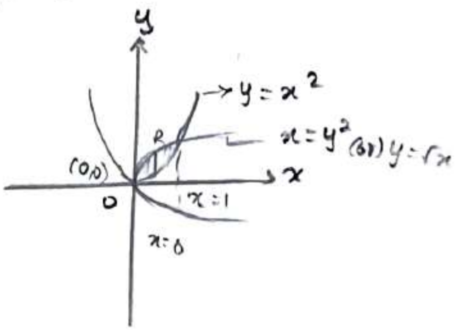
$$\boxed{\iint_R dy\,dx = \int_a^b \int_{g(x)}^{f(x)} dy\,dx}$$




1] Find the area enclosed by the parabola's $x^2 = y$ & $y^2 = x$

Given curves are $x^2 = y$, $y^2 = x$

The region bounded by the curves is as follows




The intersection points of the 2 curves are given by, $x^2 = \sqrt{x} \Rightarrow x^4 = x$

$$\Rightarrow x(x^3 - 1) = 0$$

$$\Rightarrow x = 0, x = 1$$

for $x=0, y=0$ & for $x=1, y=1$

$\therefore$ The Intersection points are $(0,0)$ $(1,1)$

$\therefore$ The are enclosed by the given corves is given by

$$\iint_R dy\,dx = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} dy\,dx$$

$$= \int_0^1 \sqrt{x} - x^2\,dx = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

---

2) $y = 4x - x^2$, $y = x$

| $x$ | $y$ |
|---|---|
| 1 | 3 |
| 2 | 4 |
| 3 | 3 |
| 4 | 0 |

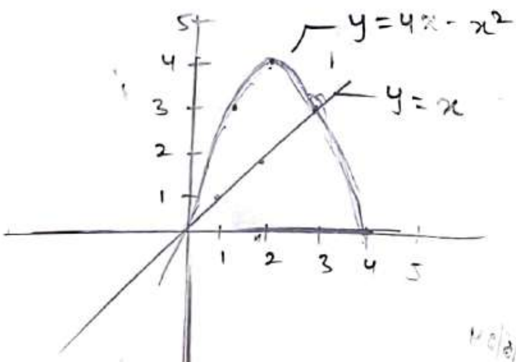
$4x - x^2 = x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, x = 3$

$y = 0, y = 3$




$$\iint_R dy\,dx = \int_0^3\int_x^{4x-x^2} dy\,dx = \int_0^3 4x - x^2 - x\,dx = \int_0^3 +3x - x^2\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{9}{2}$$

---

1) Find the area of the region bounded by the curves $y^2 = 4ax$, $x+y=3a$ & $y=0$

2) Find by $\iint$ Area enclosed by the corves $y = 2-x$ & $y^2 = 2(2-x)$

2) find the area of the plane in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the $1^{st}$ quadrant.

## Change of Variable in Triple Integrals.

Let $x = \phi_1(u,v,w)$ $y = \phi_2(u,v,w)$ $z = \phi_3(u,v,w)$ be the transformations from cartesian coordinates $u,v,w$ then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(\phi_1,\phi_2,\phi_3)\,|J|\,du\,dv\,dw$$

J - Jacobian of $x,y,z$ w.r.t $u,v,w$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Change of variables from Cartesian to Spherical coordinates

we use spherical coordinates when the given curves or eq^ns are symmetric about origin like spheres, cones.

The Relation b/w spherical coordinates & cartesian coordinates are -

$x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$ $\quad r,\theta,\phi$ & $x,y,z$

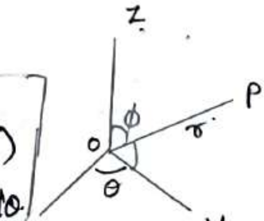
$$J = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$= \cos\phi\left(-r^2\sin^2\theta\sin\phi\cos\phi - r^2\sin\phi\cos\phi\cos^2\theta\right) - r\sin\phi\left(r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta\right)$$

$$= \cos\phi\left(-r^2(\sin\phi\cos\phi)\right) - r\sin\phi\left(r\sin^2\phi\right) = -r^2\sin\phi\cos^2\phi - r^2\sin^3\phi$$

$$= -r^2\sin\phi$$

$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2\sin\phi\,dr\,d\phi\,d\theta}$$



Note:

If the region of the Integration is a sphere $x^2+y^2+z^2=a^2$ with centre at $(0,0,0)$ at radius $a$, then the limits of $r, \theta, \phi$ are as follows:

i) For positive octect of the sphere
$r : 0$ to $a$
$\theta = 0$ to $\pi/2$
$\phi : 0$ to $\pi/2$

ii) For Hemisphere
$r : 0$ to $a$
$\theta = 0$ to $2\pi$
$\phi = 0$ to $\pi/2$

iii) For complete sphere.
$r :$ 0 to $a$
$\theta :$ 0 to $2\pi$
$\phi :$ 0 to $\pi$

1] Evaluate $\iiint x^2+y^2+z^2 \, dx\,dy\,dz$ over the volume enclosed by the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates.

Let the spherical polar coordinates $r, \theta, \phi$ & $x, y, z$

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi$$

Then $J = r^2\sin\phi$

for a complete sphere : $r : 0$ to $1$
$\theta : 0$ to $2\pi$
$\phi : 0$ to $\pi$

$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \iiint r^2 |J| \, dr\,d\phi\,d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^2\sin\phi \, r^2 \, dr\,d\phi\,d\theta$$

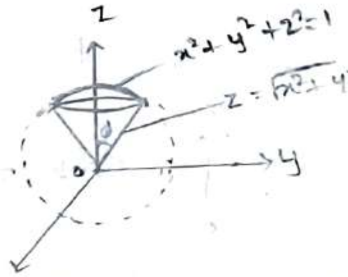
$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{r^5}{5}\sin\phi\right)_0^{1} d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi}{5}\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{5}(-\cos\phi)_0^{\pi}\,d\theta = \int_0^{2\pi}\frac{1}{5}(-\cos\pi+\cos 0)\,d\theta = \frac{2}{5}\left[2\pi - 0\right]$$

$$= \frac{4\pi}{5}$$

2] Evaluate $\iiint\limits_E 3z\,dv$ over region $x^2+y^2+z^2=1$, $z=\sqrt{x^2+y^2}$

$\downarrow$ sphere $\qquad \downarrow$ cone




$r$: 0 to 1

$\theta$: 0 to $2\pi$

$\phi$: 0 to $\pi/4$

$z = \sqrt{r^2\sin\phi^2}$

$$\iiint 3r\cos\phi \, r^2\sin\phi \, dr\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 3r^3\sin\phi\cos\phi \, dr\,d\phi\,d\theta$$

for the cone: $z=\sqrt{x^2+y^2}$

$z=\sqrt{r^2\sin^2\phi}$

$r\cos\phi = r\sin\phi$

$\tan\phi = 1$

$\phi = \pi/4$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left(\frac{3r^4}{4}\sin^2\phi\cancel{\cos\phi}\right)_0^1 d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{3}{4}\sin^2\phi\cancel{\cos\phi}\,d\phi\,d\theta$$

$$= \cancel{\int_0^{2\pi}\frac{3}{4}\left[-\sin^2\phi+\cos^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi}\frac{3}{4}\left[-\frac{1}{2}+\frac{1}{2}-(0+1)\right] d\theta}$$

$$= \cancel{-\frac{3}{4}(2\pi-0) = -\frac{3\pi}{2}}$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\frac{3}{4}\sin^2\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{3}{4}\left(\frac{1-\cos2\phi}{2}\right)d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\cancel{\frac{3}{8}-\frac{3\cos2\phi}{8}\,d\phi\,d\theta} = \int_0^{2\pi}\left(\frac{3\phi}{8}+\frac{3\sin2\phi}{16}\right)_0^{\pi/4} d\theta$$

$$= \int_0^{2\pi}\left(\frac{3\pi}{32}+\frac{3}{16}\right)-(0)\,d\theta = \frac{3\pi}{32}(2\pi-0)+\frac{3}{16}(2\pi-0) = \frac{6\pi^2}{32}+\frac{3\pi}{8}$$

$$= \frac{3\pi^2}{16}+\frac{3\pi}{8}$$

$$\frac{3}{8}\iint \sin2\phi\,d\phi\,d\theta = -\frac{3}{8}\int\left(\frac{\cos2\phi}{2}\right)_0^{\pi/4} = -\frac{3}{16}\int_0^{2\pi}(0-1)\,d\theta$$

$$= -\frac{3}{16}(2\pi-0) = \frac{3\pi}{8}$$

3) Evaluate $\iiint_E (x^2+y^2)\, dx\,dy\,dz$ where E is the region $-x^2+y^2+z^2=4$ with $y \geq 0$ is

$r: 0$ to $2$.

$\theta: 0$ to $\pi$

$\phi: 0$ to $\pi$

$x^2+y^2 = r^2\sin^2\phi$ $\quad r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \dfrac{\sin 3\theta - 3\sin\theta}{4}$

$$\int_0^\pi\int_0^\pi\int_0^2 r^2\sin^2\phi\,(r^2\sin\phi)\,dr\,d\phi\,d\theta = \int_0^\pi\int_0^\pi\int_0^{2} r^4\sin^3\phi\,dr\,d\phi\,d\theta$$

$$= \int_0^\pi\int_0^\pi \left[\frac{r^5}{5}\sin^3\phi\right]_0^1 d\phi\,d\theta = \int_0^\pi\int_0^\pi \frac{32}{5}\sin^3\phi\,d\phi\,d\theta$$

$$= -\frac{32}{5}\int_0^\pi\int_0^\pi \frac{\sin 3\theta - 3\sin\theta}{4}\,d\phi\,d\theta = \frac{-32}{20}\int_0^\pi \left[\frac{-\cos 3\theta}{3} + 3\cos\theta\right]_0^\pi d\theta$$

$$= -\frac{32}{20}\int_0^\pi \left[-\frac{\cos 3\pi}{3} + 3\cos\pi\right] - \left[\frac{-\cos 0}{3} + 3\cos 0\right] d\theta$$

$$= \frac{32}{20}\int_0^\pi \left[\frac{1}{3} - 3\right] - \left[\frac{-1}{3} + 3\right] d\theta = \frac{-1}{\cancel{20}_{\cancel{10}\,5}} \times \frac{-\cancel{16}^{\cancel{8}\,4}}{3}(\pi - 0)$$

$$= +\frac{4}{15}\pi \times 32 =$$

$$\int_0^\pi\int_0^\pi\int_0^2 r^2\sin^2\phi\, r^2\sin\phi\,dr\,d\phi\,d\theta = \int_0^\pi\int_0^\pi\int_0^2 r^4\sin^3\phi\,dr\,d\phi\,d\theta$$

$$\int_0^\pi\int_0^\pi \left(\frac{r^5}{5}\sin^3\phi\right)_0^2 d\phi\,d\theta = \frac{32}{5}\int_0^\pi\int_0^\pi \frac{\sin 3\theta - 3\sin\theta}{4}\,d\phi\,d\theta = \frac{32}{5\times 4}\int_0^\pi \left[\frac{-\cos 3\theta}{3} + 3\cos\theta\right]_0^\pi$$

$$= \frac{8}{5}\int_0^\pi \left[\frac{-\cos 3\pi}{3} + 3\cos\pi\right] - \left[\frac{-\cos 0}{3} + 3\cos 0\right] d\theta = \frac{8}{5}\int_0^\pi \left[\frac{1}{3} - 3\right] - \left[\frac{-1}{3} + 3\right] d\theta$$

$$= \frac{8}{5}\int_0^\pi \frac{-19}{3}\,d\theta = -\frac{8\times 19}{15}\pi$$

HW

1) $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\quad 9a^2+x^2-6ax = 4ax$ $\quad x+y=3a$ & $y^2=4ax$

$\frac{10a \pm \sqrt{100a^2-36a^2}}{2}$ $\quad x^2-10ax+9a^2=0$ $\quad a=1$ $\quad y=3a-x$ $\quad a=1$

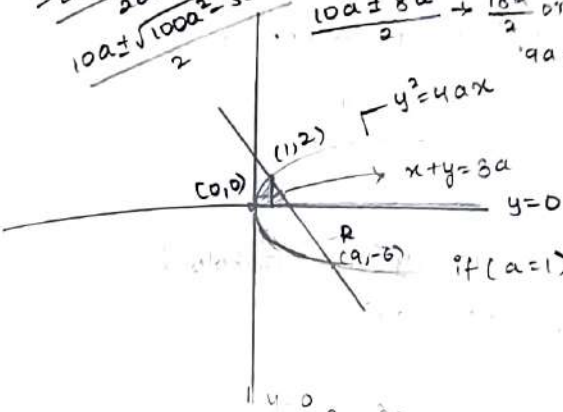
$\frac{10a \pm 8a}{2} \to \frac{18a}{2}$ or $\frac{2a}{2}$ $\quad y^2=12a$ $\quad a=3$

$9a$ or $a$ $\quad y^2=4x$ $\quad x=3a \quad y=\sqrt{12a}$

$\frac{x+y=3a}{y=0}$ $\quad$ Ex $\quad a=1$

$x=3$

$x+y=3$ $\quad y=\sqrt{12}$




if $(a=1)$

$y^2=4ax$ to $x+y=3a$

$x=y^2/4a \quad x=3a-y$

$$\text{Area} = \int_0^a \int_{y^2/4a}^{3a-y} dy\,dx = \int_0^a 3a-y-\frac{y^2}{4a}\,dy = \left(3ay-\frac{y^2}{2}-\frac{y^3}{12a}\right)_0^a$$

$$= 3a^2-\frac{a^2}{2}-\frac{a^2}{12} = \frac{29a^2}{12}$$

2) $y=2-x$ & $y^2=2(2-x)=4-2x$

$(2-x)(2-x)=2(2-x)$

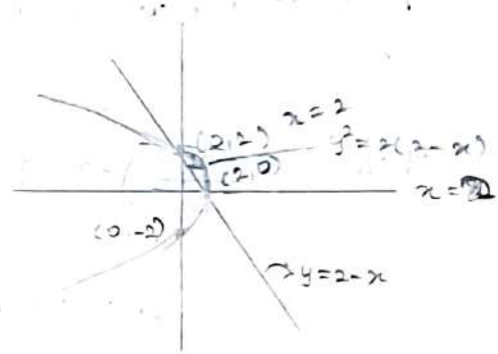
$2-x=2 \quad (2-x)[2-x-2]=0$

$x=0 \quad (2-x)(-x)=0$

$2-x=0$

$x=2-y$ to $x=2-\frac{y^2}{2}$ $\quad x=2$ or $x=0$

$y=0$ to $y=2$ $\quad y=0 \quad y=2$



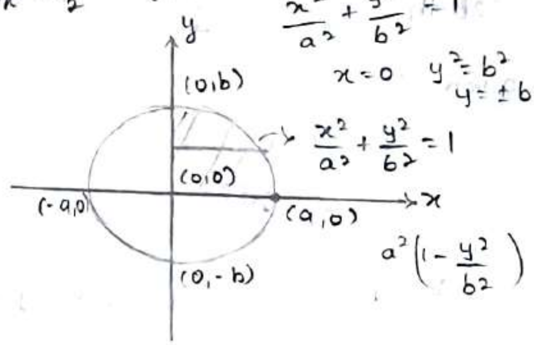

$$\int_0^2 \int_{2-y}^{2-y^2/2} dx\,dy = \int_0^2 2-\frac{y^2}{2}-2+y\,dy = \left[-\frac{y^3}{6}+\frac{y^2}{2}\right]_0^2 = -\frac{8}{6}+2 = -\frac{3}{2}+2 = -\frac{1}{2}$$

3) $x=0$ to $\sqrt{a^2\left(1-\frac{y^2}{b^2}\right)}$ $\quad \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ $\quad y=0 \quad x=\pm a$

$y=0$ to $b$ $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x=0 \quad y^2=b^2$

$y=\pm b$




$a^2\left(1-\frac{y^2}{b^2}\right)$

$$\int_0^b \sqrt{a^2\left(1-\frac{y^2}{b^2}\right)}\,dy = \frac{a}{b}\int_0^b \sqrt{b^2-y^2}$$

$$= \frac{a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b$$

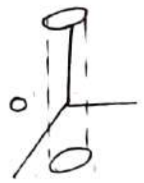
$$= \frac{a}{b}\left[\left(\frac{b}{2}\sqrt{0}+\frac{b^2}{2}\sin^{-1}(1)\right)-0\right] = \frac{a}{b}\times\frac{b^2}{2}\frac{\pi}{2} = \frac{\pi ab}{4}$$

5|6|24

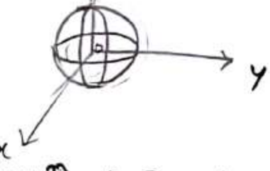
1] circular cylinder

sym abt z axis

$x^2+y^2=a^2 \rightarrow$



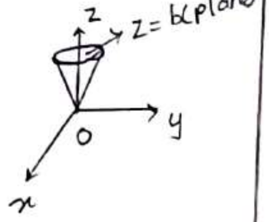

3) $x^2+y^2+z^2=a^2 \rightarrow$ sphere




2] $z^2=c^2(x^2+y^2)$

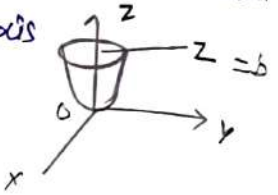
cor cone

$z = c\sqrt{x^2+y^2}$




4] $z = c(x^2+y^2) \rightarrow$ Paraboloid

sym abt z axis




Change of variables from cartesian coordinates to cylindrical coordinates.

The Relation b/w cartesian $x, y, z$ & $r, \theta, z$ cylindrical $\boxed{x = r\cos\theta,\ y = r\sin\theta,\ z = z}$

Then, $J\left(\frac{x,y,z}{r,\theta,z}\right) = r$ & $\iiint f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r,\theta,z)\, r\,dz\,dr\,d\theta$

We use cylindrical coordinates when the region of Integration bounded by cylinders along the z axis, planes through z axis, planes $\perp^r$ to z axis.

1] Using cylindrical coordinates find the volume of cylinder with base radius 'a' & height 'h'

Let the cylindrical coordinates be $x = r\cos\theta, y = r\sin\theta, z = z$ then $dx\,dy\,dz = r\,dz\,d\ldots$

given, $z : 0$ to $h$

$r : 0$ to $a$

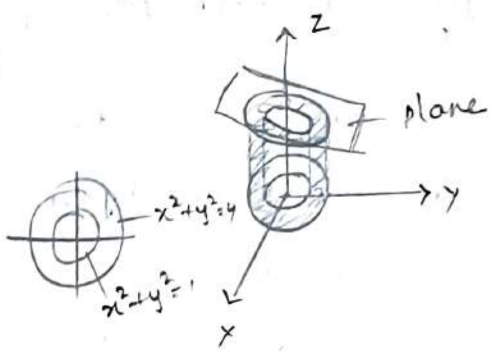
$\theta : 0$ to $2\pi$

Volume of cylinder is $V = \iiint dv = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} dz\,(r\,dr\,d\theta)$

$$= \int_0^{2\pi}\int_0^a rh\,dr\,d\theta = \int_0^{2\pi}\left(\frac{hr^2}{2}\right)_0^a = \int_0^{2\pi}\left(\frac{a^2h}{2}\right) = \frac{2\pi a^2 h}{2}.$$

$$= \pi a^2 h$$

2) Evaluate $\iiint_E y\,dv$, $z = x+2$ above $xy$ plane & b/w the cylinder $x^2+y^2=1$ & $x^2+y^2=4$




$z : 0$ to $x+2 = 0$ to $r\cos\theta+2$

$r : 1$ to $2$

$\theta : 0$ to $2\pi$

$$\iiint y\,dv = \int_{0\,(\theta)}^{2\pi}\int_{1\,(r)}^{2}\int_{0\,(z)}^{r\cos\theta+2} r\sin\theta\,(r\,dz\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta(r\cos\theta+2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^3\sin\theta\cos\theta + 2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^4}{4}\sin\theta\cos\theta + \frac{2r^3}{3}\sin\theta\right)_1^2 d\theta$$

$$= \int_0^{2\pi}\left(4\sin\theta\cos\theta + \frac{16}{3}\sin\theta\right) - \left(\frac{1}{4}\sin\theta\cos\theta + \frac{2}{3}\sin\theta\right) d\theta$$

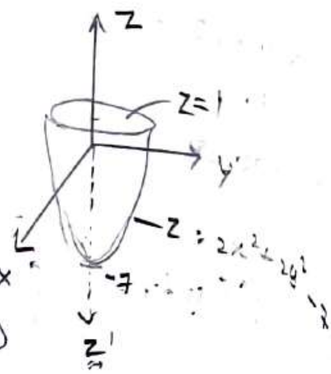
$$= \int_0^{2\pi} \frac{15}{4}\sin\theta\cos\theta + \frac{14}{3}\sin\theta\,d\theta$$

$$= \frac{15}{8}\int_0^{2\pi}\sin 2\theta\,d\theta + \frac{14}{3}\int_0^{2\pi}\sin\theta\,d\theta = \frac{15}{8}\left(\frac{-\cos 2\theta}{2}\right)_0^{2\pi} + \frac{14}{3}(-\cos\theta)_0^{2\pi}$$

$$= \frac{15}{8}\underbrace{\left(\frac{-1}{2}+\frac{1}{2}\right)}_{0} + \frac{14}{3}(-1+1) = 0$$

3) Evaluate $\iiint_E 4xy\,dv$ where $E$ is region bounded by $z = 2x^2+2y^2-7$, $z = 1$

$z : 2x^2+2y^2-7$ to $1$ = $2r^2-7$ to $1$




The region in 2D plane is given by eqns ;

$$z = 2x^2+2y^2-7 = 1$$

$$2x^2+2y^2 = 8$$

$$x^2+y^2 = 4 \text{ ( circle with centre } (0,0), r=2)$$

$r : 0$ to $2$

$\theta : 0$ to $2\pi$

$$\iiint_E 4xy\,dv = \iiint 4r\sin\theta r\cos\theta\, dz\,dr\,d\theta\, r$$

$$= \int_0^{2\pi}\int_0^{2}\int_{2r^2-7}^{1} \underbrace{4r^3\sin\theta\cos\theta}_{\sin 2\theta \,=\, 2r^3\sin 2\theta}\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left[2r^3\sin 2\theta(1) - 2r^3\sin 2\theta(2r^2-7)\right] dr\,d\theta$$

$$= \iint 2r^3\sin 2\theta - 4r^5\sin 2\theta + 14r^3\sin 2\theta \, dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{2}\sin 2\theta - \frac{4r^6}{6}\sin 2\theta + \frac{14r^4}{4}\sin 2\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[8\sin 2\theta - \frac{128}{3}\sin 2\theta + 56\sin 2\theta\right] d\theta$$

$$= \left(-8\cos 2\theta\right)_0^{2\pi} + \frac{128}{3}\left(\cos 2\theta\right)_0^{2\pi} + 56\left(\cos 2\theta\right)_0^{2\pi}$$

$$= -8(+1-1) + \frac{128}{3}(1-1) + 56(1-1)$$

$$= 0$$

4) $\iiint_E z\,dv$, E is region b/w. planes $x+y+z=2$ & $x=0$ & inside the cylinder $y^2+z^2=1$ (Along x axis)

$y = r\cos\theta,\ z = r\sin\theta,\ x=0$ $\qquad$ $x+y+z=2$

$dx\,dy\,dz = dx\,(r\,d\theta\,dr)$ $\qquad$ $\Rightarrow x = 2-y-z = 2-r\cos\theta - r\sin\theta$

$r$ : 0 to 1

$\theta$ : 0 to $2\pi$

$$\iiint_E z\,dv = \int_0^{2\pi}\int_0^1\int_0^{2-y-z} z\cdot r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^{2-y-z=(2-r\cos\theta-r\sin\theta)} r^2\sin\theta\,dx\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\sin\theta\,(2-r\cos\theta-r\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^2\sin\theta - \frac{r^3\overbrace{2\sin\theta\cos\theta}^{\sin 2\theta}}{2} - r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2r^3}{3}\sin\theta - \frac{r^4}{8}\sin 2\theta - \frac{r^4}{4}\sin^2\theta\right]_0^1 d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\frac{1}{4}(\ ) = \frac{1}{8} - \frac{1}{8}\cos 2\theta$

$$= \int_0^{2\pi} \frac{2}{3}\sin\theta - \frac{1}{8}\sin 2\theta \underbrace{- \frac{1}{4}\sin^2\theta}_{-\frac{1}{8}+\frac{1}{8}\cos 2\theta}\ d\theta$$

$$= \left(\frac{2}{3}\cos\theta + \frac{1}{16}\cos 2\theta - \frac{1}{8}\theta + \frac{1}{16}\sin 2\theta\right)_0^{2\pi}$$

$$= \frac{2}{3}(1-1) + \frac{1}{16}(1-1) - \frac{1}{8}(2\pi) + \frac{1}{16}(0)$$

$$= \frac{-\pi}{4}$$

5] $\iiint_E z\,dv$, $x+y+z=2$ & $z=0$ & $x^2+y^2=1$

$z$ : 0 to $2-r\cos\theta-r\sin\theta$

$r$ : 0 to 1

$\theta$ : 0 to $2\pi$

$$\iiint_E z\,dv = \int_0^{2\pi}\int_0^1\int_0^{2-r\cos\theta-r\sin\theta} z\,dz\,r\,dr\,d\theta \quad \int_0^{2\pi}\int_0^1 \left(\frac{z^2}{2}\right)_0^{2-r\cos\theta-r\sin\theta} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (2-r\cos\theta-r\sin\theta)^2\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (4+r^2\cos^2\theta+r^2\sin^2\theta-4r\cos\theta-4r\sin\theta+2r^2\sin\theta\cos\theta)\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (4+r^2-4r(\cos\theta+\sin\theta)+r^2\sin 2\theta)\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 4r+r^3-4r^2(\cos\theta+\sin\theta)+r^3\sin 2\theta\, dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[2r^2+\frac{r^4}{4}-\frac{4r^3}{3}(\cos\theta+\sin\theta)+\frac{r^4}{4}\sin 2\theta\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[2+\frac{1}{4}-\frac{4}{3}\cos\theta-\frac{4}{3}\sin\theta+\frac{1}{4}\sin 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[\frac{9}{4}\theta-\frac{4}{3}\sin\theta+\frac{4}{3}\cos\theta-\frac{1}{8}\cos 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{9}{2}\pi-\frac{4}{3}(0)+\frac{4}{3}(0)-\frac{1}{8}(0)\right]$$

$$= \frac{9\pi}{4}$$

---

6) $\iiint (x^2+y^2)\,dx\,dy\,dz$ taken over the volume bounded by the $xy$ plane & the Paraboloid $z=9-x^2-y^2$

$z$ : 0 to $9-r^2\sin^2\theta-r^2\cos^2\theta = 9-r^2$

$z=0$

$x^2+y^2=9 \Rightarrow r=3$

$r$ : 0 to 3

$\theta$ : 0 to $2\pi$

$$\iiint_E x^2+y^2\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2.\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{3} r^3(9-r^2)\,dr\,d\theta$$

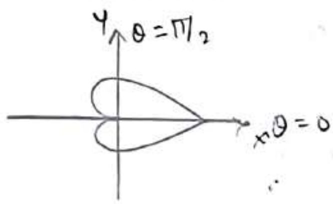
$$= \int_0^{2\pi}\int_0^{3} 9r^3 - r^5\,dr\,d\theta = \int_0^{2\pi}\left(\frac{9r^4}{4}-\frac{r^6}{6}\right)_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{\cancel{8}\times 729}{4} - \frac{243}{2}\right) d\theta = \int_0^{2\pi}\left(\frac{\overset{729}{\cancel{6561}}}{4} - \frac{243}{2}\right) d\theta$$

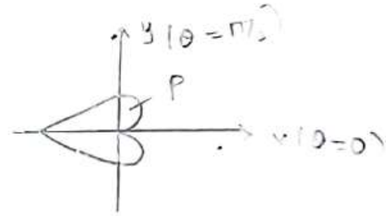
$$= \frac{\overset{243}{\cancel{6075}}}{4}\times 2\pi = \frac{\cancel{6075}\pi}{\cancel{2}} = \frac{243}{2}\pi$$

---

Cardioid $\rightarrow$ $r = a(1+\cos\theta)$

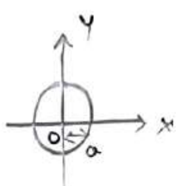



$r = a(1-\cos\theta)$




circle $\rightarrow$ $r = a$

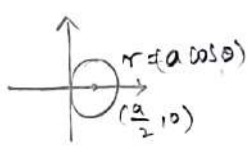
$r^2 = a^2 \Rightarrow x^2+y^2 = a^2$



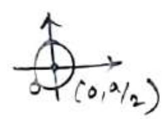

$r = a\cos\theta$

$r^2 = a(r\cos\theta)$

$x^2+y^2 = ax$

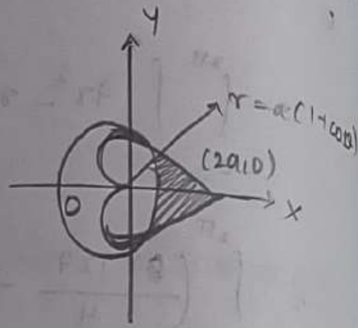



$r = a\sin\theta$




$\frac{3\pi a^2}{2}$

2) Find the area which is inside the cardiod $r = a(1+\cos\theta)$ & Outside the circle $r = a$




$r$ : $a$ to $a(1+\cos\theta)$

$\theta$ : $0$ to $\pi$

$$A = \iint_R dx\,dy = 2\int_{\theta=0}^{\pi}\int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\, d\theta = \int_0^{\pi} a^2(1+\cos^2\theta + 2\cos\theta)\, d\theta$$

$$= \int_0^{\pi} a^2\left(1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right) d\theta$$

$$= \left(a^2\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + 2\sin\theta\right)_0^{\pi}$$

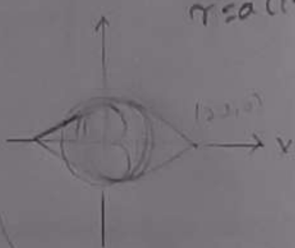
$$= a^2\pi + \frac{\pi}{2} + 0 + 0$$

$$= \frac{\pi(2a^2+1)}{2} \qquad \frac{a^2}{4}(\pi+8)$$

3) Inside $\odot^{le}$ outside cardiod $r = a(1-\cos\theta)$

$r = a\sin\theta$

$r = a(1-\cos\theta)$




$a^2(1-\pi/4)$

Find all of a loop $r = a(1+\cos\theta)$