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UNIT-4

Vector differentiation & Line Integration

Field : If a function is defined in any region of space for every point of the region, then this region is known as field.

Scalar point function: A function $\phi(x,y,z)$ is called a scalar point func defined in the region R, if it associates a scalar quantity with every point in the region R of space.

Ex : Temperature distribution in a heated body

Vector point function: A function $\bar{f}(x,y,z)$ is called a vector point func defined in the Region R, if it associates a vector quantity with every point in the Region R of space.

Ex : The velocity of a moving field, gravitational force, etc..

Vector differential operator: It is denoted by del (∇) and is defined as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

Gradient: The Gradient of a Scalar point function $\phi(x,y,z)$ is denoted by gradϕ or $\nabla\phi$ and is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

Note* → WKT, the total derivative of ϕ is given by

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

$$= \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right) \cdot (dx i + dy j + dz k)$$

$$= (\nabla\phi) \cdot (d\bar{r})$$

$$\boxed{d\phi = |\nabla\phi||d\bar{r}|\cos\theta}$$

→ If $\theta = 0$ then $d\bar{r}, \nabla\phi$ are in the same direction & the max value of ($\because \cos\theta$ is max → $\theta = 0°$) $d\phi$ is $|\nabla\phi||d\bar{r}|$

Divergence of a vector point function:

Let $\bar{f}$ be any continuously differential vector point func, then the divergence of $\bar{f}$ is denoted by $\underline{div\bar{f} \text{ or } \nabla\cdot\bar{f}}$ and is defined as

$$div\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{where } \bar{f} = f_1 i + f_2 j + f_3 k$$

Curl of a vector point function:

Let $\bar{f}$ be any continuously differential vector point func, and Let $\bar{f} = f_1 i + f_2 j + f_3 k$ then curl is denoted by $\underline{curl\,\bar{f} \text{ (or) } \nabla\times\bar{f}}$ and is defined as

$$curl\,\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Solenoidal vector:

A vector point function $\bar{f}$ is said to be Solenoidal if $\boxed{div\bar{f} = 0}$

Irrotational of a vector:

A vector point function $\bar{f}$ is said to be irrotational if $\boxed{Curl\,\bar{f} = 0}$

Position vector:

$\bar{r} = xi + yj + zk$, then, Length of position vector

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ and $d\bar{r} = dx\,i + dy\,j + dz\,k$

1] Find the gradient of the func, $\phi = x^2yz + xy^2z$ at $(1,1,2)$

Sol given, $\phi = x^2yz + xy^2z$

gradient of ϕ is given by

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$= \frac{\partial(x^2yz+xy^2z)}{\partial x}i + \frac{\partial(x^2yz+xy^2z)}{\partial y}j + \frac{\partial(x^2yz+xy^2z)}{\partial z}k$$

$$\nabla\phi = (2xyz+y^2z)i+(x^2z+2xyz)j+(x^2y+xy^2)k$$

At $(1,1,2)$, $\nabla\phi = 6i+6j+2k$

2) If $\bar{f} = xy^2 i + 2x^2yz j - 3yz^2 k$ then find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

$$= f_1 i + f_2 j + f_3 k \text{ (say)}$$

$$f_1 = xy^2,\ f_2 = 2x^2yz,\ f_3 = -3yz^2$$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$$

$$\operatorname{div}\bar{f} = y^2+2x^2z-6yz$$

At $(1,-1,1)$, $\operatorname{div}\bar{f} = 1+2+6=9$

3) Find $\operatorname{div}\bar{f}$, if $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

$$\operatorname{grad}(x^3+y^3+z^3-3xyz) = \nabla(x^3+y^3+z^3-3xyz)$$

$$= \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)i+\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)j+\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)k$$

$$\bar{f} = (3x^2-3yz)i+(3y^2-3xz)j+(3z^2-3xy)k$$

$$\operatorname{div}\bar{f} = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$\operatorname{div}\bar{f} = 6x+6y+6z$$

4) Find $\operatorname{div}\bar{f}$, if $\bar{f} = r^n\bar{r}$ find n if it is Solenoidal

we have $\bar{r} = xi+yj+zk$, then $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ then $r^n = (x^2+y^2+z^2)^{n/2}$

Here, $r^2 = x^2+y^2+z^2$

diff w.r.t x partially, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ $|||^{ly}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\bar{f} = r^n\bar{r} = r^n x i + r^n y j + r^n z k = f_1 i + f_2 j + f_3 k \text{ (say)}$$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(r^n x)+\frac{\partial}{\partial y}(r^n y)+\frac{\partial}{\partial z}(r^n z)$$

$$= \left(nr^{n-1}\frac{\partial r}{\partial x}\right)x + r^n + \left(nr^{n-1}\frac{\partial r}{\partial y}\right)y + r^n + \left(nr^{n-1}\frac{\partial r}{\partial z}\right)z + r^n$$

$$= \left(nr^{n-1}\cdot\frac{x}{r}\right)x + 3r^n + \left(nr^{n-1}\frac{y}{r}\right)y + \left(nr^{n-1}\frac{z}{r}\right)z$$

$$\text{div}\,\bar{f} = 3r^n + n r^{n-2}(x^2+y^2+z^2) \qquad -3r^n = n r^{n-2}(x^2+y^2+z^2)$$

$$\text{solenoidal} \Rightarrow \text{div}\,\bar{f} = 0 \Rightarrow r^n\left(3 + n r^{-2}(x^2+y^2+z^2)\right) = 0 \qquad 3/r^2 = x^2+y^2+z^2$$

$$\boxed{n = -3}$$

5] $\bar{f} = (x+3y)i + (y-2z)j + (x+pz)k$, $p = ?$

$$\text{div}.\bar{f} = 1 + 1 + p$$

solenoidal $\boxed{p = -2}$

6] P.T $\text{div}\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$ where $r = xi + yj + zk$

$$\frac{\bar{r}}{r} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

$$\text{div}\left(\frac{\bar{r}}{r}\right) = \frac{1}{\sqrt{x^2+y^2+z^2}}\Big(1 \cdot \quad (\text{or}) \quad \text{div}\left(\frac{xi+yj+zk}{r}\right) = \text{div}\left(\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \left(\frac{1}{r} + \frac{1}{r} + \frac{1}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r}\right) \cdot x + \frac{1}{r} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right) y + \frac{1}{r} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right) z + \frac{1}{r}$$

$$\frac{-1}{r^2}\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}} \cdot x + \frac{3}{r} + \frac{-1}{r^2}\frac{\partial r}{\partial y} + \frac{-1}{r^2}\frac{\partial r}{\partial z}$$

$$-\frac{x^2}{r^3} + \frac{3}{r} - \frac{y^2}{r^3} - \frac{z^2}{r^3} = \frac{3}{r} - \frac{(x^2+y^2+z^2)}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

7) Evaluate $\nabla \cdot \frac{\bar{r}}{r^3}$

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\bar{f} = \frac{\bar{r}}{r^3} = \frac{xi+yj+zk}{r^3} \qquad \frac{x}{r^3}\ \frac{y}{r^3}\ \frac{z}{r^3}$$

$$\nabla\bar{f} = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = x \cdot \frac{\partial}{\partial x}(r^{-3}) + \frac{3}{r^3} + y \cdot \frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) + z\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)$$

$$= \frac{-3}{r^4}\left(\frac{x}{r}\right)x + y\frac{-3}{r^4}\frac{y}{r} + z\frac{-3}{r^4}\frac{z}{r} + \frac{3}{r^3} = -3\left(\frac{x^2+y^2+z^2}{r^5}\right) + \frac{3}{r^3}$$

$$= \frac{-3(r^2)}{r^5} + \frac{3}{r^3} = 0$$

8) Given, $\bar{f} = xy^2 i + 2x^2yz j - 3yz^2 k = f_1 i + f_2 j + f_3 k$ (say) curl at $(1,-1,1)$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} = i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]$$

$$-j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

curl at $(1,-1,1)$

$$= i\left[-3z^2 - 2x^2y\right] - j\left[0-0\right] = k\left[4xyz - 2xy\right]$$

$$\text{curl } \bar{f}_{(1,-1,1)} = -i - 2k$$

9] curl $(r^n \bar{r}) = 0$ (or) S.T $r^n \bar{r}$ is irrotational

$\bar{f} = r^n \bar{r}$ $\qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\bar{f} = r^n x i + r^n y j + r^n z k$$

$$\text{curl}(r^n \bar{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} = r^n \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= r^n (i0 + j0 + k0) = 0$$

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10) Find the constants a,b,c if the vector $\bar{f} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ is irrotational

$$\text{curl}(\bar{f}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = i(c-3) - j(2-a) + k(b-3)$$

$$\text{curl } \bar{f} = \bar{0} \Rightarrow c-3=0,\ 2-a=0,\ b-3=0$$

$$a=2,\ b=3,\ c=3$$

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1) Find the constants a, b, c so that the vector $\bar{A} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

*** is irrotational & find ϕ such that $\bar{A} = \nabla\phi$

$\Rightarrow$ curl $(\bar{A}) = \bar{0} \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$0i+0j+0k = i(c+1) - j(4-a) + k(b-2)$

$c=-1, \ a=4, \ b=2$

$\Rightarrow \bar{A} = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k$

$\bar{A} = \nabla\phi$

$\Rightarrow (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

WKT, Total derivative of ϕ is $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

Integrating on both sides

$$\int d\phi = \int_{y,z \text{ const}} \frac{\partial\phi}{\partial x}dx + \int_{x,z \text{ const}} \frac{\partial\phi}{\partial y}dy + \int_{x,y \text{ const}} \frac{\partial\phi}{\partial z}dz + c$$

$$\phi = \int_{y,z \text{ const}} (x+2y+4z)dx + \int_{x,z \text{ const}} (2x-3y-z)dy + \int_{x,y \text{ const}} (4x-y+2z)dz + c$$

$$\phi = \frac{x^2}{2} + \underbrace{2xy + 4xz + 2xy}_{\text{repetition not allowed}} - \frac{3y^2}{2} - yz + 4xz - yz + z^2 + c$$

[consider only once]

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + c$$

2] S.T $\operatorname{div}(\operatorname{grad} r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Laplace Transform

F $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\operatorname{grad}(r^m) = \operatorname{grad}(\sqrt{x^2+y^2+z^2}) = \frac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2})\bar{i} + \frac{\partial}{\partial y}(\sqrt{x^2+y^2+z^2})\bar{j} + \frac{\partial}{\partial z}(\sqrt{x^2+y^2+z^2})\bar{k}$$

$$\operatorname{div}(\operatorname{grad}(r^m)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\right)$$

(or)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}r^n\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}r^n\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}r^n\right)$$

$$= \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial z}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(nr^{n-1}\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(nr^{n-1}\frac{z}{r}\right)$$

$$= \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2}y) + \frac{\partial}{\partial z}(nr^{n-2}z)$$

$$= n\left[(n-2)r^{n-3}x\frac{\partial r}{\partial x} + r^{n-2}\right] + n\left[(n-2)r^{n-3}y\frac{\partial r}{\partial y} + r^{n-2}\right] + n\left[(n-2)r^{n-3}z\frac{\partial r}{\partial z} + r^{n-2}\right]$$

$$= n\left[(n-2)r^{n-3}x\frac{x}{r} + r^{n-2}\right] + n\left[(n-2)r^{n-3}y\cdot\frac{y}{r} + r^{n-2}\right] + n\left[(n-2)r^{n-3}z\cdot\frac{z}{r} + r^{n-2}\right]$$

$$= n\left[(n-2)r^{n-4}x^2 + r^{n-2} + (n-2)r^{n-4}y^2 + r^{n-2} + (n-2)r^{n-4}z^2 + r^{n-2}\right]$$

$$= n\left[(n-2)r^{n-4}(x^2+y^2+z^2) + 3r^{n-2}\right] = n\left[(n-2)r^{n-4}r^2 + 3r^{n-2}\right]$$

$$n(n+1)r^{n-2} \leftarrow nr^{n-2}[(n-2+3)] \leftarrow = n\left[(n-2)r^{n-2} + 3r^{n-2}\right]$$

→ $\phi(x,y,z)=c$ is called a level surface

→ Normal to the surface : If $\phi(x,y,z)=c$ is a level surface, then its normal is given by $\nabla\phi$

→ Angle b/w the surfaces: Let $f(x,y,z)=c_1$, $g(x,y,z)=c_2$ be two surfaces, then the angle b/w the surfaces is given by the angle b/w the normals. Hence, the angle b/w surfaces is $\theta=\cos^{-1}\left(\frac{\nabla f\cdot\nabla g}{|\nabla f||\nabla g|}\right)$

1] Find a unit normal vector to the given surface x^2y+2xz at point $(2,-2,3)$

$$\nabla f = \frac{\partial}{\partial x}(x^2y+2xz)i+\frac{\partial}{\partial y}(x^2y+2xz)j+\frac{\partial}{\partial z}(x^2y+2xz)k$$

$$= (2xy+2z)i+(x^2)j+(2x)k$$

$$(\nabla f)_{(2,-2,3)} = (-8+6)i+4j+4k = -2i+4j+4k$$

$$(|\nabla f|_p) = \sqrt{4+16+16} = 6$$

unit Normal vector is $\frac{(\nabla f)_P}{(|\nabla f_P|)} = -\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k = \frac{-i+2j+2k}{3}$

2] Find the angle b/w the normals to the surface $xy=z^2$ at the points $(4,1,2)$ & $(3,3,-3)$

$$xy-z^2=0$$
$$f=c$$

$$\nabla f = \frac{\partial}{\partial x}(xy-z^2)i+\frac{\partial}{\partial y}(xy-z^2)j+\frac{\partial}{\partial z}(xy-z^2)k$$

$$\nabla f = yi+xj-2zk$$

$(\nabla f)_{P(4,1,2)} = i+4j-4k$; $(\nabla f)_{(3,3,-3)} = 3i+3j+6k$

$(|\nabla f|)_P = \sqrt{1+16+16}=\sqrt{33}$; $(|\nabla f|)_{P_2} = \sqrt{9+9+36} = \sqrt{54}$

$$\theta = \cos^{-1}\left(\frac{\nabla f_{P_1}\cdot\nabla f_{P_2}}{|\nabla f_{P_1}||\nabla f_{P_2}|}\right) = \cos^{-1}\left(\frac{3+12-24}{\sqrt{33}\sqrt{54}}\right) = \cos^{-1}\left(\frac{-9}{9\sqrt{22}}\right) = 102.30^\circ$$

3] Find the angle of intersection of spheres $\underbrace{x^2+y^2+z^2=29}_{f_1}$ & $\underbrace{x^2+y^2+z^2+4x-6y-8z-47=0}_{f_2}$ at $P(4,-3,2)$.

$$\nabla f_1 = \frac{\partial}{\partial x}(x^2+y^2+z^2-29)i + \frac{\partial}{\partial y}(x^2+y^2+z^2-29)j + \frac{\partial}{\partial z}(x^2+y^2+z^2-29)k$$

$$\nabla f_1 = 2xi+2yj+2zk$$

$$(\nabla f_1)_p = 8i-6j+4k \ , \ |(\nabla f_1)_p| = \sqrt{8^2+6^2+4^2} = 2\sqrt{29}$$

$$\nabla f_2 = \frac{\partial}{\partial x}(x^2+y^2+z^2+4x-6y-8z-47)i + \frac{\partial}{\partial y}(x^2+y^2+z^2+4x-6y-8z-47)j$$
$$+ \frac{\partial}{\partial z}(x^2+y^2+z^2+4x-6y-8z-47)k$$

$$\nabla f_2 = (2x+4)i + (2y-6)j + (2z-8)k$$

$$(\nabla f_2)_p = 12i - 12j - 4k$$

$$|(\nabla f_2)_p| = 4\sqrt{19}$$

$$\theta = \cos^{-1}\left(\frac{(\nabla f_1)_p(\nabla f_2)_p}{|\nabla f_{1p}||\nabla f_{2p}|}\right) = \cos^{-1}\left(\frac{96+72-16}{4\sqrt{19}\cdot 2\sqrt{29}}\right) = \cos^{-1}\left(\frac{152}{8\sqrt{551}}\right)$$

$$\theta = 36.03^\circ$$

4] Find the values of a & b so that the surfaces $\underbrace{ax^2-byz=(a+2)x}_{f}$ & $\underbrace{4x^2y+z^3=4}_{g}$ may intersect orthogonally at $P(1,-1,2)$

$ax^2-byz-(a+2)x$

$$\nabla f = (2ax-(a+2))i + (-bz)j + (-by)k$$

$a \frac{5}{2}, b\ 1.$

$$(\nabla f)_p = (2a-a-2)i - 2bj + bk$$

$$\nabla g = (8xy)i + (4x^2)j + (3z^2)k$$

$$(\nabla g)_p = -8i + 4j + 12k$$

$$\theta = 90^\circ \Rightarrow (\nabla f)_p \cdot (\nabla g)_p = 0 \Rightarrow -8(a-2) + 4(-2b) + 12b = 0$$
$$\Rightarrow -8a + 16 - 8b + 12b = 0$$
$$\Rightarrow -8a + 4b + 16 = 0 \quad -(1)$$

sub $(1,-1,2)$ in f $\Rightarrow$ ~~a+bu~~ $a+2b = a+2 \Rightarrow \boxed{b=1}$

sub $b=1$ in (1) $\Rightarrow$ $4b = 8a-16 = \cancel{4}$ $a = 20/8$

$\boxed{\cancel{b} = \frac{-1}{2}}$ x $\boxed{a = \frac{5}{2}}$

~~Directional derivative~~

$\rightarrow$ The directional derivative of a scalar point function of a point P in the direction of vector

$\bar{a}$ = $(\nabla\phi)_P \frac{\bar{a}}{|\bar{a}|}$ (or) $(\nabla\phi)_P \cdot \hat{a}$

i) $\bar{a} = a_1 i + a_2 j + a_3 k$

ii) $\bar{a}$ = line joining 2 points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$= \overline{PQ} = \overline{OQ} - \overline{OP} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

$= (x_2 - x_1) i + (y_2 - y_1) j + (z_2 - z_1) k$

iii] $\bar{a}$ = Normal to the surface $f = c$ at a point A

$= (\nabla f)_A$

$\rightarrow$ The directional derivative is maximum in the direction of $\nabla\phi$ & the greatest value of directional derivative of ϕ at a point P is equal to $|\nabla\phi|_P$

1] Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $i + 2j + 2k$ at point $(1, 2, 0)$.

$\nabla f = (y+z) i + (x+z) j + (x+y) k$

$(\nabla f)_{(1,2,0)} = 2i + j + 3k$

$|\bar{a}| = \sqrt{1+4+4} = 3$

directional derivative $= (\nabla\phi)_P \frac{\bar{a}}{|\bar{a}|} = \frac{2+2+6}{3} = \frac{10}{3}$

2) Find the directional derivative of xyz^2+xz at $P(1,1,1)$ in direction of normal to surface $3xy^2+y=z$ at $(0,1,1)$

let $f = xyz^2+xz$, $P(1,1,1)$

$g = 3xy^2+y-z$, $Q(0,1,1)$

normal to surface (g) is $(\nabla g)_Q$

$\bar{a} = (\nabla g)_Q =$

$\nabla g = 3y^2 i + (6xy+1)j + (-1)K \Rightarrow (\nabla g)_Q = 3i+j-K$

$\nabla f = (yz^2+z)i + (xz^2)j + (2xyz+x)K \Rightarrow (\nabla f)_P = 2i+j+3K$

Directional derivative $= (\nabla f)_P \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{(2i+j+3K)(3i+j-K)}{\sqrt{11}}$

$$= \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Find the direction derivative of function $f = x^2-y^2+2z^2$ at point $P(1,2,3)$ in the direction of line PQ. where $Q(5,0,4)$

$PQ = (5-1, 0-2, 4-3) = 4i-2j+K = \bar{a}$

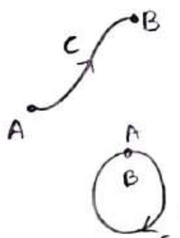
$\nabla f = 2xi-2yj+4zK$; $(\nabla f)_{(1,2,3)} = 2i-4j+12K$

Direction derivative $= (\nabla f)_P \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{8+8+12}{\sqrt{16+4+1}} = \dfrac{28}{\sqrt{21}} = \dfrac{4\sqrt{21}}{3}$

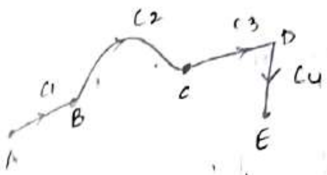
19/6/24

Vector Integration:

1) closed curve: Let c be a curve in space, Let A be the initial point and B be the terminal point of the curve C. when, the direction along c oriented from A to B is positive then the direction from B to A is -ve. If the 2 points A & B coincide then the curve is called closed curve.



2. Smooth curve: A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable. A curve C is said to piecewise smooth if it is the union of finite no. of smooth curve.



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

3) Line Integrals: Let $\bar{f}$ be the vector point function defined & continuous along the curve c then the line integral of $\bar{f}$ along the curve c is given by $\int_C \bar{f} \cdot d\bar{r}$

Note: other types of line integrals

$\int_C \bar{f} \times d\bar{r}$, $\int_C \phi \, d\bar{r}$ ϕ – Scalar point function

Circulation: If $\bar{F}$ represents the velocity of a fluid particle and c is a closed curve then the integral symbol is must $\oint_C \bar{f} \cdot d\bar{r}$ is called the circulation of $\bar{f}$ along the curve c.

→ If $\int_C \bar{F} \cdot d\bar{r} = 0$ then the field $\bar{f}$ is said to be conservative i.e, no work is done & the energy is conserved.

→ If $\oint_C \bar{f} \cdot d\bar{r} = 0$ then, $\bar{F}$ is irrotational.

Workdone by a force: If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB then, the total workdone by $\bar{F}$ during the displacement from A to B is given by the line integral $\int_A^B \bar{F} \cdot d\bar{r}$

→ If the force $\bar{F}$ is conservative i.e, $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice versa. In this case, $\text{Curl } \bar{F} = \text{Curl}(\nabla\phi) = 0$

ϕ – Scalar differential/Potential function.

Note: If $\bar{F}$ is conservative (if and only if) $\Leftrightarrow$ $\boxed{\nabla \times \bar{f} = 0 \Leftrightarrow \bar{F} \text{ is irrotational}}$

In 2D, $\bar{F} = F_1 \hat{i} + F_2 \hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C F_1 \, dx + F_2 \, dy$$

In 3D, $\bar{F} = F_1 i + F_2 j + F_3 k$, $\bar{r} = xi + yj + zk$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \underbrace{F_1 dx + F_2 dy + F_3 dz}_{\text{single}}$$

evaluate

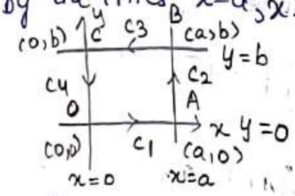
1) Evaluate $\oint_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2+y^2)i - 2xy\,j$ & the curve c is the rectangle in the xy-plane bounded by $x=0, x=a, y=0, y=b$

Sol Given, $\bar{F} = (x^2+y^2)i - 2xy\,j$

we have $\bar{r} = xi + yj$ then $d\bar{r} = dx\,i + dy\,j$

Now, $\bar{F} \cdot d\bar{r} = [(x^2+y^2)i + 2xyj] \cdot [dx\,i + dy\,j] = (x^2+y^2)dx + 2xy\,dy$

Here the closed curve c is a rectangle bounded by the lines $x=a, x=a$, $y=0, y=b$



Thus $C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r} + \int_{C_3} \bar{F} \cdot d\bar{r} + \int_{C_4} \bar{F} \cdot d\bar{r} \quad \text{—(1)}$$

Along C_1: $y=0, dy=0$

$$\therefore \int_{C_1} \bar{F} \cdot d\bar{r} = \int_{C_1} (x^2+y^2)dx - 2xy\,dy = \int_0^a x^2 dx - 0 = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along C_2 : we have $x=a \Rightarrow dx=0$ & y varies from 0 to b

$$\int_{C_2} \bar{F} \cdot d\bar{r} = \int_{C_2} (x^2+y^2)dx - 2xy\,dy = \int_0^b 0 - 2ay\,dy = (-ay^2)_0^b = -ab^2$$

Along C_3: ~~$x=0, dx=0$~~ $y=b, dy=0$, x varies from a to 0

$$\int_{C_3} \bar{F} \cdot d\bar{r} = \int_{C_3} (x^2+y^2)dx - 2xy\,dy = \int_{C_3} (x^2+y^2)\,dx. = \int_a^0 (x^2+b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0 = -\frac{a^3}{3} - ab^2$$

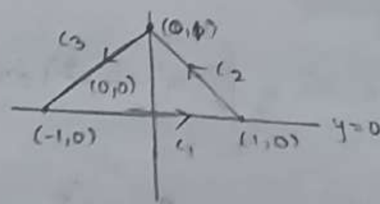
Along C_4: $x=0$, $y = b$ to 0
$\downarrow$ $dx=0$

$$\int_{C_4} \bar{F} \cdot d\bar{r} = \int_b^0 0\,dy = 0$$

$\therefore$ From (1) $\rightarrow \oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0 = -2ab^2$

2) Evaluate $\oint_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y^2\hat{i} - x^2\hat{j}$ & c is $\triangle$le whose vertices are (1,0), (0,1), (-1,0) in xy-plane.

line eqn $\rightarrow \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{(y_2-y_1)}$



(0,1) (-1,0) $\Rightarrow \left(\frac{x-1}{-1} = \frac{y-0}{0-0}\right)$

$c_3 \rightarrow \frac{x-0}{-1} = \frac{y-1}{0-1} \Rightarrow -x = -(y-1) \Rightarrow x = y-1 \Rightarrow y = 1+x$

$c_2 \rightarrow$ (0,1) (1,0) $\Rightarrow \frac{x-0}{1-0} = \frac{y-1}{0-1} = x = -y+1 \Rightarrow y = 1-x$

$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r} + \int_{C_3} \bar{F}\cdot d\bar{r}$ — (1)

Along c_1: $y=0, dy=0$, $x \rightarrow 1$ to 0

$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} y^2 dx - x^2 dy = 0$

Along c_2: $y = 1-x$, $dy = -dx$ $x \rightarrow 1$ to 0

$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{C_2} y^2 dx - x^2 dy = \int_1^0 (1-x)^2 dx + x^2 dx$

$= -\frac{(1-x)^3}{3} + \frac{x^3}{3} = -\frac{(1-1)^3}{3} - \frac{1}{3} + \frac{(1-0)^3}{3} + 0 = \frac{-2}{3}$

$\frac{(1-x)^3}{-3} \quad -\frac{1}{3} - \frac{1}{3}$

Along c_3: $y = 1+x$, $x = 0$ to -1

$dy = -1$

$\int_0 (1+x)^2 dx - x^2 dx = \left[\frac{(1+x)^3}{3} - \frac{x^3}{3}\right]_0^{-1}$

$= \frac{0}{3} + \frac{1}{3} - \frac{0}{3} - 0 = 0$

8) Evaluate $F = (5xy - 6x^2 y)\,i + (2y - 4x)\,j$ then $\oint_C F \cdot d\bar{r}$ along the curve C in xy plane $y = x^3$ from (1,1) to (2,8) Ans 35

$\frac{dy}{dx} = 3x^2$

F $F = (5x^4 - 6x^5)\,i + (2x^3 - 4x)\,j$

$dy = 3x^2$

$\int \bar{F} \cdot d\bar{r} = \int_1^2 (5x^4 - 6x^5)\,dx + (2x^3 - 4x)\,3x^2\,dx$

$= \int_1^2 (5x^4 - 6x^5 + 6x^5 - 12x^3)\,dx = \left[x^5 - x^6 + x^6 - 3x^4\right]_1^2$

$= \cancel{32 - 48 + 1 - 3}$ $[32 - 16 + 64 - 48] - [1 - 2 + 1 - 3]$

$= 32 - [-8]$

$= 35\,4$

3) If $\bar{f} = 3xy\,i - 5z\,j + 10x\,k$ then Evaluate $\int_C \bar{F} \cdot d\bar{r}$ over C along curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from t = 1 to 2.

$F = \underbrace{6(t^2+1)t^2}_{6t^4+6t^2}\,i - 5t^3\,j + (10t^2 + 10)\,k$

$\bar{F} = \underbrace{3(t^2+1)(2t^2)}_{2t^4+2t^2}\,i - 5t^3\,j + (10t^2 + 10)\,k$

$dx = 2t\,dt$, $dy = 4t\,dt$, $dz = 3t^2\,dt$

$\int_C \bar{F} \cdot d\bar{r} = \int (6t^4 + 6t^2)(2t\,dt) - 5t^3(4t)\,dt + (10t^2 + 10)(3t^2)\,dt$

$\int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$

$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = \left[\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right]$

$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$

$= [2^7 + 2^6 + 48 + 80] - [2 + 2 + 3 + 10]$

$= [\;320\;] - [17] = 303$

4) If $\bar{F} = y\hat{i} + z\hat{j} + x\hat{k}$ then find circulation of $\bar{F}$ along the curve C where C is circle where $x^2+y^2=1$, $z=0$ $\begin{matrix} r\cos\theta \\ r\sin\theta \end{matrix}$ $\theta \to 0$ to 2π

$dz = 0$ $x = r\cos\theta = \cos\theta$

$-\pi$ $y = r\sin\theta = \sin\theta$

sol

$$\int \bar{F}\cdot d\bar{r} = \int y\,dx + \underset{0}{z}\,dy + x\,dz$$ $dx = -\sin\theta\,d\theta$

$dy = \cos\theta\,d\theta$

$$= \int_0^{2\pi} \sin\theta\,dx + \cancel{\cos\theta\,dy}$$

$$= \int_0^{2\pi} -\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\,d\theta + \cancel{\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\,d\theta}$$ $$\int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

$$\cancel{= \int_0^{2\pi} \cos 2\theta\,d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = 0}$$

$$= -\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)_0^{2\pi} = -\pi$$

5) Find the workdone by the force $\bar{F} = (3x^2-6yz)\hat{i} + (2y+3xz)\hat{j} + (1-4xyz^2)\hat{k}$ in moving particle from $(0,0,0)$ to $(1,1,1)$ along the curve C $x=t$, $y=t^2$, $z=t^3$

t: 0 to 1

$$\text{workdone} = \int \bar{F}\cdot d\bar{r} = \int_C (3x^2-6yz)\,dx + (2y+3xz)\,dy + (1-4xyz^2)\,dz$$

$$= \int_C (3t^2-6t^5)\,dt + (2t^2+3t^4)\,2t\,dt + (1-4t^9)\,3t^2\,dt$$

$$= \int_0^1 (3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{4})\,dt$$

$$= \left[t^3 - t^6 + t^3 + t^5 + t^2 - \frac{12t^{12}}{12}\right]_0^1$$

$$= 1 - 1 + 1 + 1 + 1 - \frac{\cancel{3}\,1\times\cancel{4}}{\cancel{12}\,\cancel{4}} = 3 - \frac{1}{4} = \frac{11}{4} = 2$$

20/6/24

1) find the workdone by force $\bar{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ by moving a Particle from $(1,-1,2)$ to $(3,2,-1)$ in the field, P.T $\bar{F}$ is conservative

F **Sol** $\bar{F}$ conservative $\Leftrightarrow \nabla\times\bar{F}=0 \Leftrightarrow \bar{F}=\nabla\phi$ $\quad d\phi \to \phi$?

$$W = \int_C \bar{F}\cdot d\bar{r} = \int_A^B d\phi = [\phi(x,y,z)]_A^B = \phi(B)-\phi(A) = \phi(3,2,-1)-\phi(1,-1,2)$$

$$= 2(3)(2)(-1) \quad d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$\int d\phi = \int_{y,z\ \text{const}} \frac{\partial\phi}{\partial x}dx + \int_{x,z\ \text{const}} \frac{\partial\phi}{\partial y}dy + \int_{x,y\ \text{const}} \frac{\partial\phi}{\partial z}dz + C$$

$$\phi = \int_{y,z\ \text{const}} 2xyz^3\,dx + \int x^2z^3\,dy + \int 3x^2yz^2\,dz + C \quad = i($$

$$= yz^3\cdot x^2 + x^2yz^3 + x^2yz^3 + C$$

$$\phi = x^2yz^3 + C$$

$$\phi(3,2,1) = (9)(2)(-1) + C \qquad \phi(1,-1,2) = (1)(-1)(8) = -8$$

$$= -18 + C$$

$$\phi(3,2,1) - \phi(1,-1,2) = -10 \qquad \boxed{-10}$$

2) Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x-3y)i + (y-2x)j$ & C is closed curve in xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to 2π $\qquad 6\pi$

$dx = -2\sin t\,dt$, $dy = 3\cos t\,dt$

$$\int \bar{F}\cdot d\bar{r} = \int F_1\,dx + F_2\,dy$$

$$= \int (x-3y)\,dx + (y-2x)\,dy$$

$$= \int (2\cos t - 9\sin t)(-2\sin t)\,dt + (3\sin t - 4\cos t)3\cos t\,dt$$

$$= \int -4\sin t\cos t + 18\sin^2 t + 9\sin t\cos t - 12\cos^2 t\,dt$$

$$= \int -2\sin 2t + 18\left(\frac{1-\cos 2t}{2}\right) + \frac{9}{2}\sin 2t - 12\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= \left[\frac{2\cos 2t}{2} + 9\left(t - \frac{\sin 2t}{2}\right) - \frac{9}{4}\cos 2t - 6t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \cos 4\pi + 9\left(2\pi - \frac{\sin 4\pi}{2}\right) - \frac{9}{4}\cos 4\pi - 12\pi - \frac{\sin 4\pi}{2}$$

$$-\left[\cos 0 + 9(0-0) - \frac{9}{4}\cos 0 - 6(0) - 0\right]$$

$$= 1 + 18\pi - \frac{9}{4} - 12\pi - 0 - 1 + \frac{9}{4}$$

$$= 6\pi$$

3) If $\bar{F} = (x^2 - 27)\,i - 6yz\,j + 8xz^2\,k$ then Evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$; $(1,0,0)$ to $(1,1,0)$ & $(1,1,0)$ to $(1,1,1)$

$$\frac{z_2 - z_1}{z_2 - z_1} = \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - 1}$$

$$\frac{88}{3}$$

$(0,0,0)$ to $(1,0,0)$ $x = t \Rightarrow dx = dt$

$\rightarrow (0,0,0)$ to $(t,0,0)$ $x=0 \rightarrow t=0$ - LL, $x=1 \rightarrow t=1$ - UL

$y, z = 0$

$$\int \bar{F}\cdot d\bar{r} = \int (x^2 - 27)\,dx = \int_0^1 t^2 - 27\,dt = \left[\frac{t^3}{3} - 27t\right]_0^1 = -\frac{80}{3}$$

$(1,0,0)$ to $(1,1,0)$ $y = t \Rightarrow dy = dt$

$z = 0$, $x = 1$, $dx = 0$, $y = 0$ to 1

$$\int \bar{F}\cdot d\bar{r} = \int_0^1 (1-27)\cancel{dx}\ 6(0)\cancel{dt} + 0\,dz = 0$$

$(1,1,0)$ to $(1,1,1)$ $x=1$ $dx=0$

$z \rightarrow 0$ to 1 $y=1$, $dy=0$

$$\int_0^1 8t^2\,dt = \left(\frac{8t^3}{3}\right)_0^1 = \frac{8}{3}$$

$$\int_C F\cdot d\bar{r} = -\frac{80}{3} + 0 + \frac{8}{3} = -\frac{72}{3} = -24$$

4) Find the workdone by the force $\bar{F} = zi + xj + yk$ when it moves a particle along the arc of the curve $\bar{r} = \cos t\, i + \sin t\, j - t k$ from $t = 0$ to 2π

$x\,i + y\,j + z\,k$

write in terms of t

$$\boxed{-\pi}$$

$\bar{F} = -t i + \cos t j + \sin t k$, $\bar{r} = \cos t i + \sin t j - t k$

$\bar{F}\cdot d\bar{r} = -t\cos t + \sin t \cos t - t\sin t$

$\int t\cos t = t(+\sin t) - \int 1(+\sin t)dt$

$= + t\sin t + \cos t$

$\int t\sin t = t(-\cos t) - \int 1(-\cos t)dt$

$= -t\cos t + \int \cos t\,dt$

$= -t\cos t + \sin t$

$$\int \bar{F}\cdot d\bar{r} = \int -t\cos t + \frac{2\sin t\cos t}{2} - t\sin t$$

$$= \int -t\cos t + \frac{\sin 2t}{2} - t\sin t$$

$$= \int_0^{2\pi} -t\cos t + \frac{\sin 2t}{2} - t\sin t = \left[-t\sin t - \cos t - \frac{\cos 2t}{4} + t\cos t - \sin t\right]_0^{2\pi}$$

$$= -2\pi\sin 2\pi - \cos 2\pi - \frac{\cos 2\pi}{4} + 2\pi\cos 2\pi - \sin 2\pi - \left[0 - \cos 0 - \frac{\cos 0}{4} + 0 - 0\right]$$

$$= 0 - 1 - \frac{1}{4} + 2\pi - 0 + 1 + \frac{1}{4}$$

$$= 2\pi$$

5] P.T $\bar{F} = (x^2+xy^2)i + (y^2+x^2y)j$ is conservative & Find the Scalar Potential function.

$\bar{F} \to$ conservative $\Leftrightarrow \nabla \times \bar{F} = 0 \Leftrightarrow \bar{F} = \nabla\phi$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\int d\phi = \int_{y,z\ \text{const}} \frac{\partial\phi}{\partial x}dx + \int_{x,z\ \text{const}} \frac{\partial\phi}{\partial y}dy + \int_{x,y\ \text{const}} \frac{\partial\phi}{\partial z}dz$$

$$= k(2xy - 2xy)$$

$$= 0$$

$$= \int (x^2+xy^2)dx + \int (y^2+x^2y)dy$$

$$\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \underbrace{\frac{y^3}{3} + \frac{x^2y^2}{2}}_{\text{repeated}} + C$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2}$$

6] S.T $\bar{F} = (2xy+z^3)i + (x^2)j + (3xz^2)k$ is a conservative force field. Find Scalar potential & workdone by $\bar{F}$ in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= i(0) - j(3z^2 - 3z^2) + k(2x - 2x)$$

$$= 0$$

$$\phi = \int_{y,z} (2xy+z^3)\,dx + \int_{x,z} x^2\,dy + \int_{x,y} 3xz^2\,dz$$

$$\phi = x^2y + xz^3 + \underbrace{x^2y + xz^3}_{\text{repeated}} + C = x^2y + xz^3$$

$$\phi(3,1,4) = 9(1) + 3(64) = 9 + 192 = 201 + C$$

$$\phi(1,-2,1) = (1)(-2) + 1(1) = -1 + C$$

$$F = 201 + 1 = 202$$

7] $\bar{F} = (e^x z - 2xy)i - (x^2-1)j + (e^x + z)k$ (same as 6th). ~~Hence~~ $\int \bar{F} \cdot d\bar{r}$ from $(0,1,-1)$ to $(2,3,0)$

$$\phi = \int_{y,z} e^x z - 2xy \, dx + \int_{x,z} -(x^2-1) dy + \int_{x,y} e^x + z \, dz$$

$$= e^x z - x^2 y - x^2 y + y + e^x z + \frac{z^2}{2}$$

$$\phi = e^x z - x^2 y + y + \frac{z^2}{2}$$

$$\phi(2,3,0) - \phi(0,1,-1)$$

$$= e^2(0) - 4(3) + 3 + 0 - \left[(-1) - 0 + 1 + \frac{1}{2}\right]$$

$$= -9 - \frac{1}{2}$$

$$= \frac{-19}{2}.$$