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Chapter - 5

Function Approximation Tools in Engineering

Curved - fitting: least sq's curve fitting procedures:-

with an experimetal data, the data is plotted on a graph phaper and a straight line is drawn to the plotted points. This is the useal method to fit a mathematical equation to Experimental data. The method of least squares is the most systematic procedure to fit a unique curve to the given data points. its application is wide in practical computations

let the set of data points be (x_i, y_i), $i = 1, 2, \ldots m$

suppose the curve $y = f(x)$ is fitted to this data. let the observed value at $x = x_i$ is y_i and the corresponding value on the curve is $f(x_i)$. let e_i is the error of approximation at $x = x_i$ then we have

$$\boxed{e_i = y_i - f(x_i)} \quad \text{—} (1)$$

Consider $S = [y_1 - f(x_1)]^2 + (y_2 - f(x_2))^2 + \ldots [y_m - f(x_m)]^2$

$$= e_1^2 + e_2^2 + \ldots + e_m^2 \quad \text{—} (2)$$

The method of least sq. consists of minimising S.

Fitting a straight line:

let $y = a_0 + a_1 x$ is a straight line to be fitted to the given data. Then

$$S = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \ldots + [y_m - a_0 + a_1 x_m]^2 \quad \text{—} (3)$$

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Chapter - 5

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with an experimental data, the data is plotted on a graph paper and a straight line is drawn to the plotted points. this is the usual method to fit a mathematical equation to Experimental data. The method of least squares is the most systematic procedure to fit a unique curve to the given data points. its application is wide in practical computations

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$$\boxed{e_i = y_i - f(x_i)}$$

Consider $S = [y_1 - f(x_1)]^2 + (y_2 - f(x_2))^2 + \cdots [y_m - f(x_m)]^2$ — (2)

$= e_1^2 + e_2^2 + \cdots + e_m^2$

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Fitting a straight line:-

let $y = a_0 + a_1 x$ is a straight line to be fitted to the given data. Then

$$S = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \cdots + [y_m - a_0 + a_1 x_m]^2 \quad (3)$$

→ The normal equations are

$\Sigma y = ma_0 + a_1 \Sigma x$

$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2$

solving the above equations we get required st. line equation.

Q) Fit a straight line for the following data

x	6	7	7	8	8	8	9	9	10
y	5	5	4	5	4	3	4	3	3

x	y	xy	x^2
6	5	30	36
7	5	35	49
7	4	28	49
8	5	40	64
8	4	32	64
8	3	24	64
9	4	36	81
9	3	27	81
10	3	30	100

$\Sigma x = 72$ $\Sigma y = 36$ $\Sigma x^2 = 588$ $\Sigma xy = 282$

let the required line equation be $y = a_0 + a_1 x$

The normal equation of st. line are

$\Sigma y = ma_0 + a_1 \Sigma x$

trick to remember

$y = a_0 + a_1 x$

Apply Σ

$\Sigma y = \Sigma a_0 + \Sigma a_1 x$

$\boxed{\Sigma y = ma_0 + a_1 \Sigma x}$ — ①

$y = a_0 + a_1 x$

multiply by x

$xy = a_0 x + a_1 x^2$

Apply Σ

$\boxed{\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2}$ ②

$36 = 9a_0 + a_1 72$ — ③

$282 = a_0 72 + a_1 (588)$ — ④

$\therefore a_0 = 8$ $a_1 = \frac{-1}{2}$

$\boxed{y = 8 - (0.5)x}$

Non-linear Curve fitting:

Fitting a Parabola by least squares method:

let the Parabola be $y = a_0 + a_1 x + a_2 x^2$ — ①

The Normal equations are

$\Sigma y = ma_0 + a_1 \Sigma x + a_2 \Sigma x^2$ — ②

$xy = a_0 x + a_1 x^2 + a_2 x^3$

$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2 + a_2 \Sigma x^3$ — ③

$x^2 y = a_0 x^2 + a_1 x^3 + a_2 x^4$

$\boxed{\Sigma x^2 y = a_0 \Sigma x^2 + a_1 \Sigma x^3 + a_2 \Sigma x^4}$ — ④

solving ②, ③ & ④ we get a_0, a_1, a_2

put a_0, a_1, a_2 values in ① which give required Parabola

Fit a second degree polynomial to a following data by a method of least squares.

x	10	12	15	23	20
y	14	17	23	25	21

Sol let the Parabola be $y = a_0 + a_1 x + a_2 x^2$

The normal equations are

$\Sigma y = ma_0 + a_1 \Sigma x + a_2 \Sigma x^2$

$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2 + a_2 \Sigma x^3$

$\Sigma x^2 y = a_0 \Sigma x^2 + a_1 \Sigma x^3 + a_2 \Sigma x^4$

x	y	x^2	x^3	x^4	xy	x^2y
10	14	100	1000	10000	140	1400
12	17	144	1728	20736	204	2448
15	23	225	3375	50625	345	5175
23	25	529	12167	279841	575	13225
20	21	400	8000	160000	420	8400
$\Sigma x=80$	$\Sigma y=100$	1398	26270	521202	1684	30648

$100 = 5a_0 + a_1 80 + a_2(1398)$

$1684 = a_0(80) + a_1(1398) + a_2(26270)$

$30648 = a_0(1398) + a_1(26270) + a_2(521202)$

$a_0 = -3.727$

$a_1 = 3.009$

$a_2 = -0.0694$

$\therefore$ Parabola eqn is

$y = (-3.727) + (3.009)x - (0.0694)x^2$

Fitting of exponential curve by the method of least Squares:

let (x_i, y_i), $i = 1,2,3 \ldots n$ be the set of n values and let, relations b/w x and y be $\boxed{y = ab^x}$ — (1)

Taking logarithm on both sides of equation

$\boxed{\log_e y = \log_e a + x \log_e b}$ — (2)

put $Y = \log_e y$, $A = \log_e a$, $B = \log_e b$, $x = X$

$\boxed{\cancel{Y = A + XB}}$

$\boxed{Y = A + BX}$ — (3) $\rightarrow$ is a linear curve

Normal equations are

$\Sigma Y = nA + B\Sigma X$ — (4)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ — (5)

solving (4), (5) we get A, B

$A = \log_e a$ $\quad B = \log_e b$

$\boxed{a = e^A}$ $\quad \boxed{b = e^B}$

Q) Fit a curve of the form $y = ab^x$ to the following data

x	y	$x = X$	$Y = \log_e y$	X^2	XY
1	1	1	0	1	0
2	1.2	2	0.1823	4	0.3646
3	1.8	3	0.5877	9	1.7634
4	2.5	4	0.9163	16	3.6652
5	3.6	5	1.2809	25	6.4045
6	4.7	6	1.5476	36	9.2856
7	6.6	7	1.8871	49	13.2097
8	9.1	8	2.2083	64	17.6664
		36	8.6103	204	52.3594

soln: let the curve be $y = ab^x$

$\log_e y = \log_e a + x \log_e b$

put $Y = \log_e y$, $A = \log_e a$, $B = \log_e b$, $X = x$ $\Rightarrow$ $\boxed{Y = A + BX}$

The Normal equations are

$\Sigma Y = nA + B\Sigma X$

$\Sigma XY = A\Sigma X + B\Sigma X^2$

$8.6103 = 8A + B(36)$

$52.3594 = A\,36 + B(204)$

$A = -0.3822 \Rightarrow \log_e a = -0.3822$

$a = e^{-0.3822}$

$a = 0.6823$

$B = 0.3241$

$b = e^{0.3241} = 1.3827$

$\therefore$ The eq. Curve is

$\boxed{y = (0.6823)(1.3827)^x}$

Fitting a power curve by least sq's method :

let $\boxed{y = a x^b}$ — (1)

$\boxed{\log_e y = \log_e a + b \log_e x}$

Put $Y = \log_e y$, $A = \log_e a$, $X = \log_e x$, $b = B$

Now $\boxed{Y = A + BX}$ — (2)

Normal eq's are

$\Sigma Y = nA + B\Sigma X$ — (3)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ — (4)

solving (3) and (4) we get A and B

calculate a by $A = \log_e a$

$\boxed{a = e^A}$

Q) Fit a curve of the form $y = ax^b$ to the following data

x	y	$X = \log_e x$	$Y = \log_e y$	X^2	XY
20	22	2.9957	3.0910	8.9742	9.2597
16	41	2.7725	3.7135	7.6873	10.2963
10	120	2.3025	4.7874	5.3019	11.0237
11	89	2.3973	4.4886	5.7499	10.7632
14	56	2.6390	4.0253	6.9648	10.6234
		13			

Normal eq are

$\Sigma Y = nA + B\Sigma X$ $\qquad$ $20.1061 = 5A + B(13.1079)$

$\Sigma XY = A\Sigma X + B\Sigma X^2$ $\qquad$ $51.9663 = A(13.1079) + B(34.6781)$

$A = 10.2146$

$b = B = -2.3624$

$a = e^A = e^{10.2146}$

$= 27298.8539$

The req curve is

$y = (27298.8539)x^{-2.3624}$

Mean squared error :

Mean squared error $= \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

where n = no. of data points

y_i = observed value

$\hat{y}$ = expected value

Q) Fit a st line of the form $y = a_0 + a_1 x$ from the following data and also mean sq. error.

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
15	204	55	748

Soln: let the req curve be $y = a_0 + a_1 x$.

The normal eq's are

$\Sigma y = n a_0 + a_1 \Sigma x$

$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2$

$204 = 5a_0 + a_1(15)$

$748 = a_0(15) + a_1(55)$

$\boxed{a_0 = 0}$ $\boxed{a_1 = 13.6}$

∴ The st. line eq. is $y = 0 + 13.6(x)$

$\boxed{y = (13.6)(x)}$

Expected values

$x=1, \hat{y} = 13.6$

$x=2, \hat{y}_2 = 27.2$

$x=3, \hat{y}_3 = 40.8$

$x=4, \hat{y}_4 = 54.4$

$x=5, \hat{y}_5 = 68$

$= \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

mean sqrd error $= \frac{1}{5}\left((0.4)^2 + (0.2)^2 + (-0.8)^2 + (0.6)^2 + 0^2\right)$

$= \frac{1}{5}(0.16 + 0.04 + 0.64 + 0.36)$

$= \frac{1.2}{5} = 0.24$

Chebychev's polynomial:-

The differential eqn $(1-x^2)y'' - xy' + n^2y = 0$ is called chebychev's ~~poly~~ differential eqn. The solutions of chebychev's differential eqn are called chebychevs polynomial

The chebychevs polynomial $\longrightarrow$ of first kind $T_n(x)$ and second kind $U_n(x)$ are defined by

$\boxed{T_n(x) = \cos(n\cos^{-1}x)}$

$U_n(x) = \sin(n\cos^{-1}x)$

$n = 0, 1, 2, 3 \ldots$

we should take $\boxed{x = \cos\theta}$

Note:- $\boxed{T_n(1) = 1}$

we have $T_n(x) = \cos(n\cos^{-1}x)$

~~$T_n(x) = \cos$~~

$T_n(1) = \cos(0)$

$\boxed{T_n(1) = 1}$

$U_n(1) = \sin(n\cos^{-1}(1))$

$= \sin(0)$

$\boxed{U_n(1) = 0}$

Chebychev's polynomials:-

$T_n(x) = \cos(n\cos^{-1}x)$

$\boxed{T_n(x) = \cos(n\theta)}$ $\boxed{\because x = \cos\theta}$

Put $n = 0$, $\boxed{T_0(x) = 1}$

$T_1(x) = \cos\theta$

$\boxed{T_1(x) = x}$

$n = 2$, $T_2(x) = \cos 2\theta$

$= 2\cos^2\theta - 1$

$= 2x^2 - 1$

$n = 3$, $T_3(x) = \cos 3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$= 4x^3 - 3x$

$n = 4$, $T_4(x) = \cos 4\theta$

$= \cos 2(2\theta)$

$= 2\cos^2(2\theta) - 1$

$= 2(2\cos^2\theta - 1)^2 - 1$

$= 2(2x^2 - 1)^2 - 1$

$= 2(4x^4 + 1 - 4x^2) - 1 = 8x^4 - 8x^2 + 1$

$T_5(x) = \cos 5\theta = 16x^5 - 8x^2 + 1$

$=$ ~~$\cos(3\theta + 2\theta)$~~

$=$ ~~$\cos 3\theta \cdot \cos 2\theta - \sin 3\theta \cdot \sin 2\theta$~~

~~orthogonal projection~~

orthogonal Property of chebychev polynomial:-

Prove that $\int_{-1}^{+1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$

Put $\boxed{x = \cos\theta}$

$\frac{dx}{d\theta} = -\sin\theta$, $\quad -1 = \cos\theta \Rightarrow \theta = \pi$

$\boxed{dx = -\sin\theta \, d\theta}$ $\quad 1 = \cos\theta \Rightarrow \theta = 0$

$T_n(x) = \cos n\theta$

$T_m(x) = \cos m\theta$

L.H.S

$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} \, dx = \int_{\pi}^{0} \frac{\cos m\theta \cos n\theta}{\sqrt{1-\cos^2\theta}} (-\sin\theta) \, d\theta$

Due to minus we can change the limits

$= \int_0^{\pi} \cos m\theta \cos n\theta$

Case I:- $m \neq n$

$= \frac{1}{2} \int_0^{\pi} 2\cos m\theta \cos n\theta$

$= \frac{1}{2} \int_0^{\pi} (\cos(m+n)\theta + \cos(m-n)\theta) \, d\theta$

$= \frac{1}{2} \left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n} \right]_0^{\pi}$

$= \frac{1}{2}[0 + 0] = 0$

Case II : $m = n$

$L.H.S = \int_0^{\pi} \cos m\theta \cos n\theta \, d\theta$

$= \int_0^{\pi} \cos^2 n\theta \, d\theta$

$= \int_0^{\pi} \left(\frac{\cos 2n\theta + 1}{2}\right) d\theta$

$= \frac{1}{2}\left(\int_0^{\pi} (\cos 2n\theta + 1) d\theta\right)$

$= \frac{1}{2}\left(\underbrace{\left(\frac{\sin 2n\theta}{2n}\right)_0^{\pi}}_{0} + (\theta)_0^{\pi}\right)$

$= \frac{1}{2}(\pi) = \frac{\pi}{2}$

Case III :- $m = n = 0$

$L.H.S = \int_0^{\pi} (1)(1) \, d\theta = \int_0^{\pi} 1 \, d\theta = (\theta)_0^{\pi}$

$= \pi - 0 = \pi$

Recurrence relation :-

(1) Prove that $T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x) = 0$

Put $x = \cos\theta$

$T_n(x) = \cos n\theta$

L.H.S

$\cos(A+B) + \cos(A-B) = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos(n+1)\theta - 2\cos\theta\cos n\theta + \cos(n-1)\theta$

$= \cancel{2\cos\frac{(n+1)+(n-1)}{2}\theta\cos\frac{(n+1)-(n-1)}{2} - 2\cos\theta\cos n\theta}$

$= 2\cos\theta\cos n\theta - 2\cos\theta\cos n\theta$

$= 0$

$= R.H.S$

(2) $(1-x^2) T_n'(x) = -nx T_n(x) + n(T_{n-1}(x))$

L.H.S

$T_n(x) = \cos(n\cos^{-1}x)$

$T_n'(x) = -\sin(n\cos^{-1}x) \, n \frac{-1}{\sqrt{1-x^2}}$

$= n\frac{\sin(n\cos^{-1}x)}{\sqrt{1-x^2}}$

$L.H.S = (1-x^2) T_n'(x) = \frac{\overset{\sqrt{1-x^2}}{\cancel{(1-x^2)}} \sin(n\cos^{-1}x)}{\cancel{\sqrt{1-x^2}}} \times n$

$= n \sin\theta \sin(n\theta)$

$\boxed{x = \cos\theta}$

$\boxed{\theta = \cos^{-1}x}$

$R.H.S = -n\cos\theta\cos n\theta + n(\cos(n-1)\theta)$

$= -n\cos\theta\cos n\theta + n(\cos(n\theta - \theta))$

$= -n\cos\theta\cos(n\theta) + n[\cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta]$

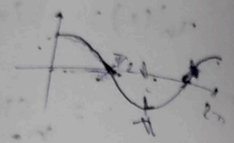
$= n\sin\theta\sin(n\theta)$

$= L.H.S$

Prove that

i) $T_n(-1) = (-1)^n$ ii) $T_{2n}(0) = (-1)^n$ (iii) $T_{2n+1}(0) = 0$

i) $T_n(x) = \cos(n\cos^{-1}x)$



$T_n(-1) = \cos -n\pi$

$= \cos n\pi$

$= (-1)^n$

ii) $T_{2n}(0) = (-1)^n$ iii) $T_{2n+1}(0) = 0$

$\boxed{T_n(x) = \cos(n\cos^{-1}x)}$ $T_n(x) = \cos(n\cos^{-1}x)$

$T_{2n}(0) = \cos 2n\frac{\pi}{2}$ $T_{2n+1}(0) = \cos((2n+1)\cos^{-1}(0))$

$= \cos n\pi$ $T_{2n+1}(0) = \cos\left((2n+1)\frac{\pi}{2}\right)$

$= (-1)^n$

$= \cos\left(n\pi + \frac{\pi}{2}\right)$

$= \underbrace{\cos n\pi \cos\frac{\pi}{2}}_{0} - \underbrace{\sin n\pi \sin\frac{\pi}{2}}_{0}$

$= 0 - 0 = 0$

Q) Express the polynomial $x^3 + 2x^2 - x + 2(1)$ in terms of Chebychev's polynomial

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1 \Rightarrow x^2 = \frac{T_2(x) + T_0(x)}{2}$

$T_3(x) = 4x^3 - 3x \Rightarrow x^3 = \frac{T_3(x) + 3(T_1(x))}{4}$

$x^3 + 2x^2 - x + 2 = \frac{1}{4}\left[T_3(x) + 3T_1(x)\right] + 2\cdot\frac{1}{2}\left[\frac{T_2(x) + T_0(x)}{2}\right]$

$- T_1(x) + 2T_0(x)$

$= \frac{T_3(x)}{4} + \frac{3}{4}T_1(x) + T_0(x) + T_2(x) - T_1(x) + 2T_0(x)$

$= \frac{1}{4}T_3(x) - \frac{1}{4}T_1(x) + T_2(x) + 3T_0(x)$

Q) Prove that $\int_{-1}^{1} x^6(1-x^2)^{-1/2} T_8(x)\,dx = 0$

Put $x = \cos\theta$ $x = -1$: $\boxed{\theta = \pi}$ $x = 1$: $\boxed{\theta = 0}$

$\frac{dx}{d\theta} = -\sin\theta$

$dx = -\sin\theta\,d\theta$

L.H.S: $\int_{\pi}^{0} \cos^6\theta\,(1-\cos^2\theta)^{-1/2} \cos(8\cos^{-1}\cos\theta)(-\sin\theta)\,d\theta$

$\int_0^{\pi} \cos^6\theta \cos 8\theta\,d\theta = \int_0^{\pi} (\cos^3\theta)^2 \cos 8\theta\,d\theta$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\boxed{\frac{\cos 3A + 3\cos A}{4} = \cos^3 A}$

$= \int_0^{\pi} \frac{1}{4}\left[\cos 3\theta + 3\cos\theta\right] \frac{1}{4}(\cos 3\theta + 3\cos\theta) \cos 8\theta\,d\theta$

$= \frac{1}{16}\left(\int_0^{\pi} (\cos^2 3\theta + 9\cos^2\theta + 6\cos 3\theta\cos\theta)\cos 8\theta\,d\theta\right)$

$~~= \frac{1}{16}\int_0^{\pi}\cos^2 3\theta\cos 8\theta\,d\theta~~$

$= \frac{1}{16}\left(\int_0^{\pi}\left[\frac{1+\cos 6\theta}{2} + 3[\cos 4\theta + \cos 2\theta] + \frac{9}{2}(1+\cos 2\theta)\right]\cos 8\theta\,d\theta\right)$

$= \frac{1}{16}\left(\int_0^{\pi}\frac{1}{2}\cos 8\theta\,d\theta + \frac{1}{2}\underbrace{\int_0^{\pi}\cos 8\theta\cos 6\theta\,d\theta}_{0} + 3\int_0^{\pi}\cos 4\theta\cos 8\theta\,d\theta + 3\int_0^{\pi}\cos 2\theta\cos\theta\right. +$

$\left.\frac{9}{2}\int_0^{\pi}\cos 8\theta\,d\theta + \frac{9}{2}\int_0^{\pi}\cos 2\theta\cos 8\theta\,d\theta\right)$

$\boxed{\int_0^{\pi}\cos mx\cos nx = 0 \quad \boxed{m \neq n}}$

$= \frac{1}{16}\left(\frac{1}{2}\times\frac{1}{8}\underbrace{\left[\sin 8\theta\right]_0^{\pi}}_{0} + \frac{9}{2}\frac{1}{8}\times\left[\sin 8\theta\right]_0^{\pi}\right)$

$= \frac{1}{16}\left[0 + 0\right] = 0 = \text{R.H.S}$

Chebychev's Series :-

let i) $f(x)$ be a continous function

ii) $f(x)$ have continous derivates in $[-1, 1]$

Then $f(x)$ can be expressed as the infinite series

known as chebychev's series

$$f(x) = a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \dots$$

where $a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx$

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{T_n(x) f(x)}{\sqrt{1-x^2}} \, dx$$

Q) Approximate the func. with a chebychev series reduce the result at $x = 1$

$$f(x) = \sin^{-1} x$$

$$\int_{-n}^{n} \text{odd fn} = 0$$

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx$$

$$= \frac{1}{\pi} \int_{-1}^{1} \underbrace{\frac{\sin^{-1} x}{\sqrt{1-x^2}}}_{\text{odd function}} \, dx$$

$\underbrace{\text{Therefore}}$ $\int_{-1}^{1} \text{odd funct} = 0$

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{T_n(x) f(x)}{\sqrt{1-x^2}}$$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

Here all even coefficents are zeros

$a_2 = a_4 = a_6 = a_8 \dots = 0$

put $n = 2m+1$

$$a_{2m+1} = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_{2m+1}(x)}{\sqrt{1-x^2}} \, dx$$

Put $x = \cos\theta$

$$= \frac{2}{\pi} \int_{\pi}^{0} \frac{\sin^{-1}(\cos\theta) \cos(2m+1)\theta (-\sin\theta) \, d\theta}{\sqrt{1-\cos^2\theta}}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin^{-1} \sin\left(\frac{\pi}{2} - \theta\right) \cos(2m+1)\theta \, d\theta$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \left(\frac{\pi}{2} - \theta\right) \cos(2m+1)\theta \, d\theta$$

$$= \int_{0}^{\pi} \cos(2m+1)\theta \, d\theta - \frac{2}{\pi} \int_{0}^{\pi} \theta \cos(2m+1)\theta \, d\theta$$

$$= \underbrace{\left(\frac{\sin(2m+1)}{(2m+1)}\right)_0^{\pi}}_{0} - \frac{2}{\pi} \left[\theta \int_{0}^{\pi} \cos(2m+1)\theta \, d\theta - \int_{0}^{\pi} \frac{d\theta}{d\theta} \left(\int \cos(2m+1)\theta \, d\theta \right) d\theta \right]$$

$$= -\frac{2}{\pi} \left[\theta(0) - \int_{0}^{\pi} \frac{\sin(2m+1)}{2m+1} \right]$$

$$= \frac{-2}{\pi} \left[\frac{\cos(2m+1)\theta}{(2m+1)^2} \right]_0^{\pi}$$

$$= \frac{-2}{\pi} \left[\frac{-1-1}{(2m+1)^2} \right] = \frac{4}{\pi (2m+1)^2}$$

$$a_{2m+1} = \frac{4}{\pi (2m+1)^2}$$

$m = 0$, $a_1 = \frac{4}{\pi}$

$m = 1$, $a_3 = \frac{4}{\pi}\left[\frac{1}{3^2}\right]$

$m = 2$, $a_5 = \frac{4}{\pi}\left[\frac{1}{5^2}\right]$

The chebychev series

$$f(x) = \frac{4}{\pi} T_1(x) + \frac{4}{\pi}\left[\frac{1}{3^2}\right] T_3(x) + \frac{4}{\pi}\left[\frac{1}{5^2}\right] T_5(x) + \dots$$

$$\underbrace{\sin^{-1} x}_{\sin^{-1}(1)} = \frac{4}{\pi} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$= \frac{\pi}{2}$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Generation function of $T_n(x)$:

Prove that $\frac{1-xt}{1-2tx+t^2} = \sum_{n=0}^{\infty} t^n T_n(x)$

L.H.S: put $x = \cos\theta$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$= \frac{e^{i\theta}+e^{-i\theta}}{2}$

$$\frac{1-xt}{1-2tx+t^2} = \frac{1-t\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}{1-2t\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+t^2}$$

$$= \frac{2-te^{i\theta}-te^{-i\theta}}{2\left(1-te^{i\theta}-te^{-i\theta}+t^2\right)} = \frac{2-t(e^{i\theta}+e^{-i\theta})}{2\left(1-t(e^{i\theta}+e^{-i\theta})+t^2\right)}$$

$$= \frac{2-t(e^{i\theta}+e^{-i\theta})}{2\left((1-te^{i\theta})(1-te^{-i\theta})\right)}$$

$$= \frac{1}{2}\left[\frac{1}{(1-te^{i\theta})} + \frac{1}{(1-te^{-i\theta})}\right]$$

$(1-x)^{-1} = 1+x+x^2+x^3+\dots = \sum_{n=0} x^n$

$$= \frac{1}{2}\left[(1-te^{i\theta})^{-1} + (1-te^{-i\theta})^{-1}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(te^{i\theta})^n + \sum_{n=0}^{\infty}(te^{-i\theta})^n\right]$$

$e^{i(n\theta)} = \cos n\theta + i\sin(n\theta)$

$e^{-in\theta} = \cos n\theta - i\sin(n\theta)$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} t^n e^{i(n\theta)} + \sum_{n=0}^{\infty} t^n e^{-i(n\theta)}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(t^n)\left(e^{i(n\theta)}+e^{-i(n\theta)}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} t^n\, 2\cos n\theta\right] = \sum_{n=0}^{\infty} t^n \cos n\theta = \sum_{n=0}^{\infty} t^n T_n(x)$$

Taylor's polynomials (function approx. by Taylor's polynomial)

Taylor Polynomial

$$T_n(x) = f(c) + \frac{(x-c)f'(c)}{1!} + \frac{(x-c)^2 f''(c)}{2!} + \dots + \frac{(x-c)^n f^n(c)}{n!}$$

is called the ~~tay~~ n^{th} degree Taylor polynomial for $f(x)$ at c

Note: If $c=0$ then it is called n^{th} degree ~~maclaurin's~~ Maclaurin's polynomial.

Q) determine the 5^{th} degree ~~maclaurins~~ Taylor's polynomial centered at $x=0$ for given function $f(x) = \cos x$.

Sol:

$$T_5(x) = f(c) + \frac{(x-c)f'(c)}{1!} + \frac{(x-c)^2 f''(c)}{2!} + \frac{(x-c)^3 f'''(c)}{3!} + \frac{(x-c)^4 f^{iv}(c)}{4!} + \frac{(x-c)^5 f^{v}(c)}{5!}$$

Given centered at $x=0$ that is $\boxed{c=0}$

$f(c) = f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$

$f^{v}(x) = -\sin x \Rightarrow f^{v}(0) = 0$

$$T_5(x) = 1 + \frac{(x-0)\,0}{1} + \frac{x^2(-1)}{2!} + \frac{x^3(0)}{3!} + \frac{(x-0)^4\,1}{4!} + \frac{(x-1)^5(0)}{5!}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}}$$

Q) Determine 4th degree Taylor's polynomial centered at $x = 1$ for given function $\ln x$

$$T_n(x) = f(c) + \frac{(x-c)f'(c)}{1!} + \frac{(x-c)^2 f''(c)}{2!} + \frac{(x-c)^3 f'''(c)}{3!} + \frac{(x-c)^4 f^{iv}(c)}{4!}$$

$f(x) = \ln(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -\frac{6}{x^4} \Rightarrow f^{iv}(1) = -6$

UNIT V

Taylor's series for function of two variables.

$\boxed{f_x = \frac{\partial f}{\partial x}}$

The Taylor series is

~~$f(x,y) = f(a,b) + \left[(x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)}\right.$~~

$$f(x,y) = f(a,b) + \left[(x-a) f_{x(a,b)} + (y-b) f_{y(a,b)}\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx(a,b)} + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx(a,b)} + 3(x-a)^2(y-b) f_{xxy(a,b)} + 3(x-a)(y-b)^2 f_{xyy(a,b)} + (y-b)^3 f_{yyy}(a,b)\right]$$
$+ \dots$

is known as Taylor series (or) Taylors expansion (or) Taylor series expansion of $f(x,y)$ about the point (a,b)

Maclaurin's series expansion is a special expansion of Taylor series when the expansion is above origin i.e $(0,0)$.

Q) Use Taylor ~~series~~ theorem to expand $f(x,y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$

Sol: The Taylors expansion of $f(x,y)$ at point $1,2$ is

$$f(x,y) = f(1,2) + \left[(x-1) f_{x(1,2)} + (y-2) f_y(1,2)\right]\frac{1}{1!} + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)\right] + \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,2) + (y-2)^3 f_{yyy}(1,2) + 3 f_{xxy}(1,2)(x-1)^2(y-2) + 3(x-1)(y-2)^2 f_{xyy}(1,2)\right]$$

$f(x,y) = x^2 + xy + y^2 \Rightarrow f(1,2) = 1 + 2 + 4 = 7$

$f_x = \frac{\partial f}{\partial x} = 2x + y \Rightarrow f_x(1,2) = 4$

$f_y = \frac{\partial f}{\partial y} = 2y + x \Rightarrow f_y(1,2) = 5$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x+y) = 2 \Rightarrow f_{xx}(1,2) = 2$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y+x) = 1 \Rightarrow f_{xy}(1,2) = 1$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial (2y+x)}{\partial y} = 2 \Rightarrow f_{yy}(1,2) = 2$$

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}(2) = 0$$ of power more than 2

$$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}(2) = 0$$ the dow is 0

∴ The Taylor series of

$$\underbrace{f(x,y)}_{x^2+xy+y^2} = 7 + \frac{1}{1!}\left[(x-1)4 + (y-2)5\right] + \frac{1}{2}\left[(x-1)^2(2) + (y-2)^2 2 + 2(x-1)(y-2)(1)\right] + 0$$

$$\boxed{x^2 + \cancel{y} + y^2 = \text{'' '' '' '' '' '' ''}}$$