28/10/24 (Monday)

UNIT-IV

Generating functions

- → G.f
- → G.f sequence
- → G.f counting sequence
- → G.f coefficient
- → G.f counting sequence coefficient
- → R.R = reccurence relation
- → first order linear homogeneous R.R
- → Second " " " "
- → Third " " " "

* Generating functions (G.f):-

- Let $a_0, a_1, a_2, a_3 - - - a_n$ be a series $\langle a_r \rangle, r=0, 1, - - - n$ & $x^0, x^1, x^2, - - - x^n$ be the coefficients of above series then G.f $G(x) = a_0 x^0 + a_1 x^1 + - - a_n x^n$

$$= \sum_{r=0}^{n} a_r x^r$$

$$\boxed{G(x) = \sum_{r=0}^{n} a_r x^r}$$ here $r = 0, 1, 2 - - - n$

*** Problem ①:-** Find the G.f of $(3+x)^3$

Sol:- $(3+x)^3 = 3^3\left(1+\frac{x}{3}\right)^3$

$$= 3^3\left({}^3C_0 + {}^3C_1\left(\frac{x}{3}\right)^1 + {}^3C_2\left(\frac{x}{3}\right)^2 + {}^3C_3\left(\frac{x}{3}\right)^3\right)$$

$$= 3^3\left(1 + x + \frac{x^2}{3} + \frac{x^3}{27}\right)$$

$$= 27 + 27x + 9x^2 + x^3$$

$$= 27x^0 + 27 \cdot x^1 + 9 \cdot x^2 + 1 \cdot x^3$$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^n$$

* Problem ②: $(1-x)^{-1}+2x^2$. Find the generating function of given expression.

Sol:- $(1-x)^{-1}+2x^2$

$= (1+x+x^2+x^3+ \cdots +x^n)+2x^2$

$= 1+x+3x^2+x^3+ \cdots +x^n$

$= 1\cdot x^0+1\cdot x^1+3\cdot x^2+ \cdots +1\cdot x^n$

* Problem ③:- find the g.f of $e^{2x}+(1-x)^{-2}$

Sol:- $e^{2x}+(1-x)^{-2}$

$= 1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+ \cdots \frac{(2x)^n}{n!}+(1-x)^{-2}$

$= 1+2x+2x^2+\frac{4}{3}x^3+ \cdots \frac{(2x)^n}{n!}+(1+2x+3x^2+ \cdots)$

$= 2+4x+5x^2+\frac{16}{3}x^3+ \cdots$

$= 2\cdot x^0+4\cdot x^1+5\cdot x^2+\frac{16}{3}\cdot x^3+ \cdots$

* Problem ④:- Find the g.f of $6(1+x)^{-2}+x^3-2x^5$

Sol:- $= 6-12x+18x^2-24x^3+30x^4-36x^5+x^3-2x^5$

$= 6-12x+18x^2-23x^3+30x^4-38x^5$

$= 6-12x+18x^2-23x^3+30x^4-38x^5$

$= 6\cdot x^0-12\cdot x^1+18\cdot x^2-23\cdot x^3+30\cdot x^4-38\cdot x^5$

* Problem ⑤:- Find the below sequences of g.f

① 1, 1, 1, 1 - - - ② 1, -1, 1, -1, 1, -1 - - -

③ 1, 2, 3, 4 - - - ④ 1, -2, 3, -4 - - -

⑤ 1, 1, 0, 1, 0, 1, 1 - - - ⑥ 0, 0, 1, 1, 1 - - -

⑦ 0, 0, 1, -1, 1, -1 - - - ⑧ 6, 0, 6, 0 - - -

⑨ 0, 0, 3, -4, 5, -6 - - - ⑩ 1, 2, 0, 0, 5 - - -

sol:- (1.) $1, 1, 1, 1 \ldots$

$G(x) = a_0 x^0 + a_1 x^1 + \ldots$

$a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1$

$= 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3$

$= 1 + x + x^2 + x^3 = (1-x)^{-1}$

(2.) $1, -1, 1, -1, \ldots$

$G(x) = 1 \cdot x^0 - 1 \cdot x^1 + 1 \cdot x^2 - 1 \cdot x^3$

$= 1 - x + x^2 - x^3 = (1+x)^{-1}$

(3.) $1, 2, 3, 4 \ldots$

$G(x) = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + \ldots$

$= 1 + 2x + 3x^2 + 4x^3 = (1-x)^{-2}$

(4.) $1, -2, 3, -4 \ldots$

$G(x) = 1 \cdot x^0 - 2 \cdot x^1 + 3 \cdot x^2 - 4 \cdot x^3 + \ldots$

$= 1 - 2x + 3x^2 - 4x^3 = (1+x)^{-2}$

(5.) $1, 1, 0, 1, 0, 1, 1 \ldots$

$G(x) = 1 \cdot x^0 + 1 \cdot x^1 + 0 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 1 \cdot x^5$

$= 1 + x + x^3 + x^5 + x^6 - [x^2 + x^4] + [x^2 + x^4]$

$= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 - [x^2 + x^4]$

$= (1-x)^{-1} - x^2 - x^4$

(6.) $0, 0, 1, 1, 1 \ldots$

$G(x) = 0 \cdot x^0 + 0 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + 1 \cdot x^4$

$= x^2 + x^3 + x^4 - [1 + x] + [1 + x]$

$= (1-x)^{-1} - 1 - x$

(7.) $0, 0, 1, -1, 1, -1$

$G(x) = 0 \cdot x^0 + 0 \cdot x^1 + 1 \cdot x^2 + (-1) x^3 + 1 \cdot x^4 + (-1) x^5$

$= x^2 - x^3 + x^4 - x^5$

$= x^2 - x^3 + x^4 - x^5 - [1 - x] + [1 - x] = (1+x)^{-1} - 1 + x$

8.) 6, 0, 6, 0 - - -

$G(x) = 6 \cdot x^0 + 0 \cdot x^1 + 6 \cdot x^2 + 0 \cdot x^3$

$= 6 + 6x^2$

$= 6(1 + x^2) - [6x + 6x^3] + [6x + 6x^3]$

$= 6[(1-x)^{-1} - x - x^3]$

9.) 0, 0, 3, -4, 5, -6

$G(x) = 0 \cdot x^0 + 0 \cdot x^1 + 3 \cdot x^2 - 4 \cdot x^3$

$= 3x^2 - 4x^3 + 5x^4 - 6x^5 - [1 - 2x] + [1 - 2x]$

$= (1+x)^{-2} - 1 + 2x$

10.) 1, 2, 0, 0, 5 - - -

$G(x) = 1 \cdot x^0 + 2 \cdot x^1 + 0 \cdot x^2 + 0 \cdot x^3 + 5 \cdot x^4 + - - -$

$= 1 + 2x + 5x^4 - [3x^2 + 4x^3] + [3x^2 + 4x^3]$

$= (1-x)^{-2} - 3x^2 - 4x^3$

7/11/24 Thursday

* Problem (1): Find the below sequences of generating functions

1.) 0, 1, 1, 1, - - - -

2.) 0, 1, 2, 3, 4 - - - -

3.) 0, 1, -2, 3, -4 - - - -

4.) 0, 0, 6, 6, 6 - - -

sol: 1.) 0, 1, 1, 1 - - - -

$= 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + - - -$

$= (1 + x + x^2 + - - -) - 1$

$= (1-x)^{-1} - 1$

(or) $= x(1 + x + x^2 + - -)$

$= x(1-x)^{-1}$

2.) 0, 1, 2, 3, 4 - - - -

$= 0 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + 3 \cdot x^3 + 4 \cdot x^4 - - -$

$= x + 2x^2 + 3x^3 + 4x^4$

$= x(1 + 2x + 3x^2 + 4x^3)$

$= x(1-x)^{-2}$

3.) $0, 1, -2, 3, -4$

$= 0 \cdot x^0 + 1 \cdot x^1 + (-2)x^2 + 3x^3 + (-4)x^4$

$= x - 2x^2 + 3x^3 - 4x^4$

$= x(1 - 2x + 3x^2 - 4x^3)$

$= x(1+x)^{-2}$

4.) $0, 0, 6, 6, 6 - - -$

$= 0 \cdot x^0 + 0 \cdot x^1 + 6 \cdot x^2 + 6 \cdot x^3 + 6x^4$

$= 6x^2 + 6x^3 + 6x^4$

$= 6x^2(1 + x + x^2)$

$= 6x^2(1-x)^{-1}$

*Problem ②:- Find the below sequence series of G.f

1.) $1^2, 2^2, 3^2, 4^2 - - -$

Sol:- S-1:- $1^2, 2^2, 3^2, 4^2 - - -$

S-2:- $1, 2, 3, 4 - - - = (1-x)^{-2}$

S-3:- $1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + - - = (1-x)^{-2}$

S-4:- Both sides multiply 'x'

S-5:- $x(1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + - - -) = x(1-x)^{-2}$

S-6:- $1 \cdot x^1 + 2 \cdot x^2 + 3 \cdot x^3 + 4 \cdot x^4 + - - = x(1-x)^{-2}$

S-7: Both sides apply di[ff] ... e below sequence

S-8: $\frac{\partial}{\partial x}(1\cdot x^1+2\cdot x^2+3\cdot x^3$

S-9: $1^2\cdot x^0+2^2 x+3^2 x^2$

S-10: $1^2\cdot x^0+2^2\cdot x+3^2\cdot x^2+4$ $\quad x=0,1,2,3$

Problem (3): find the below sequence series of G.f

i) $0^2, 1^2, 2^2, 3^2, \ldots$ $\quad 0^2\cdot x^0+1^2\cdot x^1+2^2\cdot x^2+3^2\cdot x^3+\ldots=\frac{1+x}{(1-x)^3}$

Sol: S-1: $x(1+2^2\cdot x+3^2\cdot x^2+\ldots)=\frac{x(1+x)}{(1-x)^3}$

S-2:

Problem (4): Find the below sequence series of G.f

i) $1^3, 2^3, 3^3, 4^3, \ldots$

Sol: $1^2, 2^3, 3^3, 4^3, \ldots$

S-1: $1, 2, 3, 4, \ldots = (1-x)^{-2}$

S-2: $1\cdot x^0+2\cdot x^1+3\cdot x^2+4\cdot x^3+\ldots = (1-x)^{-2}$

S-3: Both sides multiply with 'x'

S-4: $x(1\cdot x^0+2\cdot x^1+3\cdot x^2)=x(1-x)^{-2}$

S-5: $1\cdot x^1+2\cdot x^2+3\cdot x^3+\ldots=x(1-x)^{-2}$

S-6: Differentiate on both sides

S-7:

S-8: $\frac{\partial}{\partial x}(1\cdot x^1+2\cdot x^2+3\cdot x^3+\ldots)=\frac{\partial}{\partial x}(x(1-x)^{-2})$

S-9: $1^2 x^0+2^2\cdot x+3^2\cdot x^2+4^2\cdot x^3=(1-x)^{-2}-2x(1-x)^{-3}(-1)$

$= x + 2x^2 + 3x^3 + 4x^4 + \dots = \frac{1+x}{(1-x)^3}$

$= x(1+2x$ multiply with x

$x(1 + \dots x^2 + \dots) = \frac{x(1+x)}{(1-x)^3}$

3) $0, 1, -2, 3, -4$

$= 1 \cdot x + 2 \cdot x^2 + 3^2 \cdot x^3 + \dots = \frac{x(1+x)}{1-x}$

S-14:- Bothsides differentiate with x

S-15:- $\frac{\partial}{\partial x}(1^2 \cdot x^1 + 2^2 \cdot x^2 + 3^2 \cdot x^3 + \dots) = \frac{\partial}{\partial x} \frac{(x+x^2)}{(1-x)^3}$

S-16:- $1^3 \cdot x^0 + 2^3 \cdot x + 3^3 \cdot x^2 + \dots) = \frac{1+4x+x^2}{(1-x)^4}$

*Problem(5):- Find the below sequence series of g.f

1) $0^3, 1^3, 2^3 \dots$

Sol:-

S-17:- $0^3 \cdot x^0 + 1^3 x^1 + 2^3 \cdot x \dots = \frac{1+4x+x^2}{(1-x)^4}$

S-18:- $x(1^3 x^0 + 2^3 \cdot x^1 + \dots) = \frac{x(1+x^2+4x)}{(1-x)^4}$

*Problem(6):- Find the below sequence series of G.f

1) $0, 2, 6, 12 \dots$

Sol:- S-1:- $0, 2, 6, 12 \dots$

S-2:- $(0+0^2), (1+1^2), (2+2^2), (3+3^2) \dots$

S-3:- $(r+r^2)$. $r = 0, 1, 2, 3, \dots$

A: $x(1-x)^{-2} + \frac{x(1+x)}{(1-x)^3}$

* Problem (7): Find the below sequence series of G.f

8, 26, 54 ...

Sol: $< 3(r+1) + 5(r+1)^2 >$, $r = 0, 1, 2, 3$

$< 3(1, 2, 3 \ldots) + 5(1^2, 2^2, 3^2 \ldots) >$

$= \boxed{< 3(1-x)^{-2} + 5\frac{(1+x)}{(1-x)^3} >}$

Monday

11/11/24

* Generating function Counting Sequence:-

Let $e_1, e_2, e_3, e_4 \ldots e_n$ different types of boxes & "r" similar balls then "e_1 box counting sequence $f(e_1) = x^0 + x^1 + x^2 \ldots x^r$

"e_2 box counting sequence $f(e_2) = x^0 + x^1 + x^2 \ldots x^r$

"e_n box counting sequence $f(e_n) = x^0 + x^1 + x^2 \ldots x^r$

$f(x) = f(e_1) \cdot f(e_2) \cdot f(e_3) \ldots f(e_n) = (x^0 + x^1 + \ldots x^r)(x^0 + x^1 + \ldots x^r) \ldots (x^0 + x^1 + \ldots x^r)$

* Problem (1):- Let $x_1 + x_2 + x_3 + x_4 + x_5 = 25$, where x_1 is even, x_2 is odd, x_3 is prime, $x_4 \geq 5$ & $10 \leq x_5 \leq 20$. Find g.f counting sequence.

Sol: S-1: Let $x_1 + x_2 + x_3 + x_4 + x_5 = 25$,

x_1 is even, x_2 is odd, x_3 is prime, $x_4 \geq 5$ & $10 \leq x_5 \leq 20$

S-2: $\boxed{x_1} + \boxed{x_2} + \boxed{x_3} + \boxed{x_4} + \boxed{x_5} = 25$

S-3: Counting seqaence of x_1 box

$f_1(x) = x^0 + x^2 + x^4 + \cdots + x^{24}$

counting sequence of x_2 box

$f_2(x) = x^1 + x^3 + \cdots + x^{25}$

counting sequence of x_3 box

$f_3(x) = x^2 + x^3 + x^5 + x^7 + x^{11} + x^{15} + x^{19} + x^{23} + x^{17}$

counting sequence of x_4 box

$f_4(x) = x^5 + x^6 + x^7 + \cdots + x^{25}$

counting sequence of x_5 box

$f_5(x) = x^{10} + x^{11} + \cdots x^{20}$

S-4: $G(x) = f_1(x) f_2(x) f_3(x) f_4(x) f_5(x)$

$G(x) = (x^0 + \cdots + x^{24})(x^1 + \cdots + x^{25})(x^2 + \cdots + x^{23})(x^5 + \cdots x^{25})$
$(x^{10} + \cdots x^{20})$

*Problem (2): A total 15 mangoes distributing into A, B, C persons A must get atmost 5 mangoes, B get atleast 7 mangoes, c must get atmost 10 mangoes atleast 5

sol: St-1: 15 mangoes distributing into the persons A, B & C

when $A \leq 5$
$B \geq 7$
& $5 \leq C \leq 10$

S-2: $A + B + C = 15$

S-3:

Person 'A' counting sequence $f(A) = x^0 + \cdots x^5$
Person 'B' counting sequence $f(B) = x^7 + \cdots + x^{15}$
Person 'C' counting sequence $f(C) = x^5 + \cdots + x^{10}$

4- $G(x) = (x^0 + \cdots + x^5)(x^7 + \cdots + x^{15})(x^5 + \cdots + x^{10})$

Problem(3):- Find the G.f.C's of 5 a's, 4 b's, 3 c's

{5 a's, 4 b's, 3 c's}

ol:

{5 a's, 4 b's, 3 c's}

-1:- {5 a's, 4 b's, 3 c's}

2:- Counting sequence of 5 a's $= x^0 + \cdots + x^5$
Counting sequence of 4 b's $= x^0 + \cdots + x^4$
Counting sequence of 3 c's $= x^0 + \cdots + x^3$

-3: $G(x) = (x^0 + \cdots + x^5)(x^0 + \cdots x^4)(x^0 + \cdots + x^3)$

Problem(4):- Find the G.f.C.s of 5 digit positive number

d:-
S-1:- five digit positive number

2:- _ _ _ _ _

3:- Counting sequence of 1st digit number
$= x^1 + x^2 + \cdots + x^9$
Counting sequence of 2nd digit number
$= x^0 + x^1 + x^2 + \cdots + x^9$
Counting sequence of 3rd digit number
$= x^0 + \cdots + x^9$
Counting sequence of 4th digit number
$= x^0 + \cdots x^9$
Counting sequence of 5th digit number $= x^0 + \cdots + x^9$

S-4 :- $G(x) = (x^1 + \cdots + x^9)(x^0 + \cdots x^9)^4$

* Generating function coefficients:-

- $\frac{1}{(1-x)} = (1-x)^{-1} = \sum_{r=0}^{n} x^r$
- $\frac{1}{(1+x)} = (1+x)^{-1} = \sum_{r=0}^{n} (-x)^r = (-1)^n \sum_{r=0}^{n} x^r$
- $\frac{1}{(1-ax)} = (1-ax)^{-1} = \sum_{r=0}^{n} (ax)^r = a^r \sum_{r=0}^{n} x^r$
- $\frac{1}{(1+ax)} = (1+ax)^{-1} = \sum_{r=0}^{n} (-ax)^r = (-a)^r \sum_{r=0}^{n} x^r$
- $\frac{1}{(1-x)^n} = {}^{(n-1+r)}C_r \sum_{r=0}^{n} x^r$
- $\frac{1}{(1+x)^n} = {}^{(n-1+r)}C_r (-1)^n \sum_{r=0}^{n} x^r$
- $\frac{1}{(1-ax)^n} = {}^{(n-1+r)}C_r \, a^r \sum_{r=0}^{n} x^r$
- $\frac{1}{(1+ax)^n} = {}^{(n-1+r)}C_r \, (-a)^r \sum_{r=0}^{n} x^r$

13/11/24 Wednesday

*** Problem ①:-** Find the g.f coefficients of below questions

① $\frac{1}{(2+3x)}$ ② $\frac{1}{(3-\frac{5}{2}x)}$ ③ $\frac{1}{(2+3x)^3}$ ④ $\frac{1}{(5+\frac{4}{7}x)^7}$

⑤ $\frac{1}{(2-3x)^8}$ ⑥ $\frac{1}{(3-415x)^9}$

Sol: (1) $\frac{1}{(2+3x)} = \frac{1}{2}\left(1+\frac{3}{2}x\right)^{-1} = \frac{1}{2}$

$= \boxed{\frac{1}{2}\left(-\frac{3}{2}\right)^r \sum_{r=0}^{n} x^r}$

(2) $\frac{1}{(3-\frac{5}{2}x)} = \frac{1}{3}\left(\frac{1}{1-\frac{5}{6}x}\right) = \frac{1}{3}\left(1-\frac{5}{6}x\right)^{-1}$

$= \boxed{\frac{1}{3}\left(\frac{5}{6}\right)^r \sum_{r=0}^{n} x^r}$

(3) $\frac{1}{(2+3x)^3} = \frac{1}{8}\left(1+\frac{3}{2}x\right)^{-3}$

$= \boxed{\frac{1}{8}\,{}^{2+r}C_r\left(-\frac{3}{2}\right)^r \sum_{r=0}^{n} x^r}$

(4.) $\frac{1}{(5+\frac{4}{7}x)^7} = \frac{1}{5^7}\left(1+\frac{4}{35}x\right)^{-7}$

$= \boxed{\frac{1}{5^7}\,{}^{6+r}C_r\left(-\frac{4}{35}\right)^r \sum_{r=0}^{n} x^r}$

(5.) $\frac{1}{(2-3x)^8} = \frac{1}{2^8}\left(1-\frac{3}{2}x\right)^{-8}$

$= \boxed{\frac{1}{2^8} \times {}^{7+r}C_r\left(\frac{3}{2}\right)^r \sum_{r=0}^{n} x^r}$

(6.) $\frac{1}{(3-\frac{4}{5}x)^9} = \frac{1}{3^9}\left(1-\frac{4}{15}x\right)^{-9}$

$= \boxed{\frac{1}{3^9} \times {}^{8+r}C_r\left(\frac{4}{15}\right)^r \sum_{r=0}^{n} x^r}$

*Note:- $(1+x+x^2+x^3+\cdots+x^n) = \frac{1-x^{n+1}}{(1-x)}$

[Generating function coefficient theorem]

*Problem ②:- Find the coefficient of x^{10} in $(x^2+x^3+x^4 - - - x^7)^3$

Sol:-

S-1: $(x^2+x^3+x^4+ - - + x^7)^3 = x^6(1+x+ - - x^5)^3$

$= x^6\left[\frac{(1-x^6)}{(1-x)}\right]^3 = x^6(1-x^6)^3(1-x)^{-3}$

$= x^6(1-x^6)^3 \; {}^{2+r}C_r \sum_{r=0}^{n} x^r$

$= x^6\left[1 - {}^3C_1 x^6 + {}^3C_2 x^{12} - {}^3C_3 x^{18}\right] {}^{2+r}C_r \sum_{r=0}^{n} x^r$

$= {}^6C_4$ (coefficient of x^{10})

*Problem ③:- Find the above g.f counting sequence of coefficient of x^{16} in

$(x^2+ - - x^5)(x+x^2+ - - x^7)(1+ - - x^9)$

Sol:-

S-1:- $(x^2+ - - x^5)(x+ - - x^7)(1+ - - - x^9)$

$(1+ - - x^3)\,x(1+ - x^6)\,(1+ - x^9)$

S-2:- $x^2(1+ - - x^3)(1+ - - x^6)(1+ - - x^9)$

S-3:- $x^3(1+ - - x^3)(1+ - - x^6)(1+ - - x^9)$

$= x^3\frac{(1-x^4)}{1-x}\frac{(1-x^7)}{1-x}\frac{(1-x^{10})}{1-x}$

$= x^3\frac{(1-x^4)(1-x^7)(1-x^{10})}{(1-x)^3}$

S-4:- $x^3(1-x^4)(1-x^7)(1-x^{10}) \; {}^{2+r}C_r \sum_{r=0}^{n} x^r$

S-5:- ${}^{15}C_{13} - {}^5C_3 - {}^8C_6 - {}^{11}C_9 + {}^4C_2$

*Problem ④:- Find the coefficient of x^0 in

$\left(3x^2 - \frac{2}{x}\right)^{15}$

Sol:- $\left(3x^2 - \frac{2}{x}\right)^{15} = (3x^2)^{15}\left(1 - \frac{2}{3x^3}\right)^{15}$

$= 3^{15} x^{30}\left[{}^{15}C_0 - {}^{15}C_1 \times \left(\frac{3}{3x^3}\right)^1 - \cdots + {}^{15}C_{10}\left(\frac{2}{3x^3}\right)^{10}\right.$

x^0 coefficient is:- $3^{15} \times {}^{15}C_{10} \times \frac{2^{10}}{3^{10}}$

$= \boxed{3^5 \times 2^{10} \times {}^{15}C_{10}}$

*Problem ⑤:- Let $x_1 + x_2 + x_3 = r$

① Find the a_{10}, where $x_i \le 4$

② Find the a_{50}, where $5 \le x_1$, $x_2 \ge 10$ & $10 \le x_3 \le 45$

Sol:- (1)

$= \frac{(1-x^5)^3}{(1-x)^3}$

$(1-x^5)^3 (1-x)^{-3}$

$= (1-x^5)^3 \; {}^{2+r}C_r \sum_{r=0}^{\infty} x^r$

$= [{}^3C_0 - {}^3C_1 x^5 + {}^3C_2 x^{10} - {}^3C_3 x^{15}] \; {}^{2+r}C_r \sum_{r=0}^{\infty} x^r$

$\boxed{{}^{12}C_{10} - {}^3C_1 \times {}^7C_5 + {}^3C_2}$

② $x_1 + x_2 + x_3 = r$

S-2: Counting sequence of $X_1 = x^5 + \cdots + x^{50}$

Counting sequence of $X_2 = x^{10} + \cdots x^{50}$

Counting sequence of $X_3 = (x^{10} + \cdots x^{45})$

$$G(x) = (x^5 + \cdots + x^{50})(x^{10} + \cdots x^{45})(x^{10} + \cdots + x^{50})$$

$$= x^{25}(1 + \cdots + x^{45})(1 + \cdots + x^{35})(1 + \cdots + x^{40})$$

$$= x^{25} \frac{(1-x^{46})}{(1-x)} \frac{(1-x^{36})}{(1-x)} \frac{(1-x^{41})}{1-x}$$

$$= x^{25}(1-x^{46})(1-x^{36})(1-x^{41})(1-x)^{-3}$$

$$= x^{25}(1-x^{46})(1-x^{36})(1-x^{41})(2+r)C_r \sum_{r=0}^{n} x^r$$

$$= {}^{27}C_{25}$$

*Problem (6):- Find the coefficient of x^9 in $(1+x^3+x^8)^{10}$

Sol:-

$$(1+x^3+x^8)^{10} = (x^0+x^3+x^8)^{10} \Rightarrow n = 10$$

$$x^9 = (x^0)^{n_1}(x^3)^{n_2}(x^8)^{n_3}$$

$n = n_1 + n_2 + n_3$

$$x^9 \Rightarrow \frac{n!}{n_1!\,n_2!\,n_3!}$$

$$x^9 = (x^0)^7 (x^3)^3 (x^8)^0$$

$$n_1 = 7,\ n_2 = 3,\ n_3 = 0$$

coefficient of $x^9 = \frac{10!}{7!\,3!\,0!}$

Problem⑦:- Find the coefficient of x^{16} in

$(1+x^4+x^8)^{10}$

Sol:- $(1+x^4+x^8)^{10} \Rightarrow n=10$

$$x^{16} = (1)^{n_1}(x^4)^{n_2}(x^8)^{n_3} \quad \Big\} \; n = n_1+n_2+n_3 = 10$$

i.) $x^{16} = (x^0)^6 (x^4)^4 (x^8)^0$

$n_1 = 6, n_2 = 4, n_3 = 0$

coefficient of x^{16} is $= \frac{10!}{6!4!0!}$

ii.) $x^{16} = (x^0)^8 (x^4)^0 (x^8)^2$

$\Rightarrow n_1 = 8, n_2 = 0, n_3 = 2$

$x^{16} = \frac{10!}{8!0!2!}$

iii.) $x^{16} = (x^0)^7 (x^4)^2 (x^8)^1$

$\Rightarrow n_1 = 7, n_2 = 2, n_3 = 1$

$x^{16} = \frac{10!}{7!2!1!}$

coefficient of x^{16} is $= \frac{10!}{6!4!0!} + \frac{10!}{8!0!2!} + \frac{10!}{7!2!1!}$

Problem⑧:- Find the coefficient of x^{23} & x^{32} in $(1+x^5+x^8)^{10}$

Sol:-

(14/11/23)

Thursday

* Reccurence Relations:-

- A relation connecting general term with one or more of the sequencing terms such a relation is called recurrence relation.

Ex:- ① $a_n = 5 \cdot a_{n-1}$. find the R.R

Sol:- S-1:- $a_n = 5a_{n-1}$

S-2:- Put $n=1 \Rightarrow a_1 = 5 \cdot a_0$, $\frac{a_1}{a_0} = 5$

$n=2 \Rightarrow a_2 = 5 \cdot a_1$, $\frac{a_2}{a_1} = 5$

$n=3 \Rightarrow a_3 = 5a_2$, $\frac{a_3}{a_2} = 5$

$$\boxed{\therefore \frac{a_n}{a_{n-1}} = 5}$$

② Let $a_n = 3n+7$. Find the R.R

Sol:- S-1:- $a_n = 3n+7$

S-2:- $n=0 \Rightarrow a_0 = 7$, $a_1 - a_0 = 3$

$n=1 \Rightarrow a_1 = 10$, $a_2 - a_1 = 3$

$n=2 \Rightarrow a_2 = 13$, $a_3 - a_2 = 3$

$n=3 \Rightarrow a_3 = 16$

$\therefore a_n - a_{n-1} = 3$

$$\Rightarrow \boxed{\therefore a_n = 3 + a_{n-1}}$$

③ Let $a_n = n^2$. Find the R.R

Sol:- S-1:- $a_n = n^2$

S-2:- $n=0 \Rightarrow a_0 = 0$, $a_1 - a_0 = 1$

$n=1 \Rightarrow a_1 = 1$, $a_2 - a_1 = 3$

$n=2 \Rightarrow a_2 = 4$, $a_3 - a_2 = 5$

$n=3 \Rightarrow a_3 = 9$

$a_n - a_{n-1} = (2n-1)$

$$\boxed{\therefore a_n = (2n-1) + a_{n-1}}$$

7. Let $a_n = n(n+2)$. Find R.R

Sol:- $a_n = n(n+2)$

$n=0 \Rightarrow a_0 = 0$ } $(2n+1)$

$n=1 \Rightarrow a_1 = 3$ } $(2n+1)$

$n=2 \Rightarrow a_2 = 8$ } $(2n+1)$

$n=3 \Rightarrow a_3 = 15$ } $(2n+1)$

$n=4 \Rightarrow a_4 = 24$

$a_n - a_{n-1} = 2n+1$

$$\boxed{\therefore a_n = (2n+1) + a_{n-1}}$$

8. The fibonacci function $f(n) = F(n)$ here $f(0)=0$, $f(1) = 1$. Find $f(9)$ & $f(10)$

Sol:- $F(n+2) = F(n+1) + F(n)$

$F(n) = F(n-1) + F(n-2)$

Put $n=0 \Rightarrow F(1) + F(0)$

$= 1 + 0 = 1$

$$\boxed{F(2) = 1}$$

Put $n=1 \Rightarrow F(3) = F(2) + F(1)$

$= 1 + 1$

$$\boxed{F(3) = 2}$$

Put $n=2 \Rightarrow F(4) = F(3) + F(2)$

$= 2 + 1$

$$\boxed{F(4) = 3}$$

Put $n=3 \Rightarrow F(5) = F(4) + F(3)$

$= 3 + 2$

$$\boxed{F(5) = 5}$$

Put $n=4 \Rightarrow F(6) = F(5) + F(4)$

$= 5 + 3$

$= 8$

put $n=5 \Rightarrow F(7)=F(6)+F(5)$
$=8+5$
$\boxed{F(7)=13}$

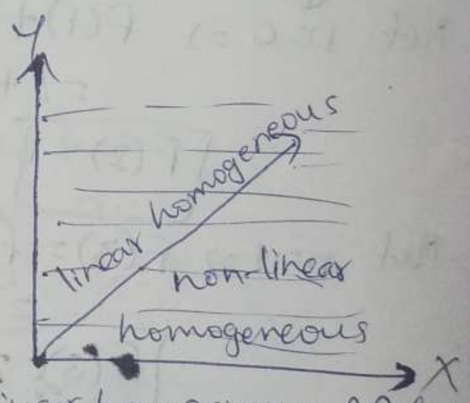
put $n=6 \Rightarrow F(8)=F(7)+F(6)$
$=13+8$
$\boxed{F(8)=21}$

put $n=7 \Rightarrow F(9)=F(8)+F(7)$
$=21+13$
$\boxed{F(9)=34}$

put $n=8 \Rightarrow F(10)=F(9)+F(8)$
$=34+21$
$\boxed{F(10)=55}$

* First order linear homogeneous Recurrence Relations:-

· The general form of linear homogeneous R.R



linear homogeneous R.R $\Rightarrow f(n)=0$
non " " " $\Rightarrow f(n)\neq 0$

S-1:- $a_n = c \cdot a_{n-1} + f(n-1)$

S-2:- put $n=n+1$

S-3:- $a_{n+1} = c \cdot a_n + f(n)$

S-4:- put $n=0 \Rightarrow a_1 = c \cdot a_0 + f(0)$

put $n=1 \Rightarrow a_2 = c \cdot a_1 + f(1)$
$= c\{c \cdot a_0 + f(0)\} + f(1)$
$= c^2 \cdot a_0 + c\, f(0) + f(1)$

put $(n=2) \Rightarrow a_3 = c \cdot a_2 + f(2)$
$= c\{c^2 \cdot a_0 + c \cdot f(0) + f(1)\} + f(2)$
$= c^3 a_0 + c^2 f(0) + c \cdot f(1) + f(2)$

put $r = n-1 \Rightarrow a_n = c^n \cdot a_0 + c^{n-1} \cdot f(0) + c^{n-2} f(1) + c^{n-3} f(2) - - -$

apply linear homogeneous R.R

$f(n) = 0$

$\boxed{a_n = c^n a_0}$

Problem ①: Let $a_n = 4a_{n-1}, a_0 = 3$. Find the R.R

Sol: S-1: $a_n = 4a_{n-1}, a_0 = 3$

S-2: put $n=1 \Rightarrow a_1 = 4a_0$

$a_1 = 12$ $\Big\} \frac{a_1}{a_0} = 4$

put $n=2 \Rightarrow a_2 = 4 \cdot a_1$ $\Big\} \frac{a_2}{a_1} = 4$

$= 4 \times 12$

$= 48$ $\Big\} \frac{a_3}{a_2} = 4$

put $n=3 \Rightarrow a_3 = 4 \cdot a_0 = 192$

$\therefore a_n = c^n a_0$

$\Rightarrow a_n = 4^n \cdot a_0$

$= 4^n \cdot 3$

$\boxed{a_n = 3 \cdot 4^n}$

* Problem ②: Let $a_n = 7a_{n-1}, a_2 = 98$. Find the R.R

Sol:

S-1: $a_n = 7a_{n-1}$

S-2: Put $n = 3 \Rightarrow a_3 = 7 \cdot a_2$

$= 7 \cdot 98$

$= 686$

$\frac{a_3}{a_2} = 7$

$\frac{a_2}{a_1} = 7$

$a_1 = \frac{98}{7}$

$\boxed{a_1 = 14}$

$\frac{a_1}{a_0} = 7$

$a_0 = \frac{14}{7}$

$a_0 = 2$

Problem (3): Find the below sequences of R.R

1) 0, 2, 6, 12 - - -   2) 2, 10, 50, 250 - - -

Sol:- 1) 0, 2, 6, 12 - - -

S-1:- 0, 2, 6, 12 - - -

S-2:-

$a_0 = 0$, $a_1 = 2$: $a_1 - a_0 = 2$ $(2n)$ $n=1$

$a_2 = 6$: $a_2 - a_1 = 4$ $(2n)$ $n=2$

$a_3 = 12$: $a_3 - a_2 = 6$ $(2n)$ $n=3$

$\vdots$

$a_n - a_{n-1} = 2n$ $n=n$

$$a_n - a_0 = \cancel{2}\left(\frac{n(n+1)}{\cancel{2}}\right)$$

$$\boxed{a_n = a_0 + n(n+1)}$$

2) 2, 10, 50, 250

S-1:- 2, 10, 50, 250

$a_0 = 2$, $a_1 = 10$: $\frac{a_1}{a_0} = 5$

$a_2 = 50$: $\frac{a_2}{a_1} = 5$

$a_3 = 250$: $\frac{a_3}{a_2} = 5$

$$\boxed{\therefore c = 5}$$

$$a_n = 5^n \cdot 2$$

$$\boxed{\therefore a_n = 2 \cdot 5^n}$$

* Second order linear homogeneous R.R:-

The general term of second order linear homogeneous R.R

$$\boxed{C_n \cdot K_n + C_{n-1} \cdot K_{n-1} + C_{n-2} K_{n-2} = 0}$$

$$\Rightarrow K_{n-2}[C_n k^2 + C_{n-1} k + C_{n-2}] = 0$$

$$\boxed{C_n k^2 + C_{n-1} k + C_{n-2} = 0}$$

here C_n, C_{n-1}, C_{n-2} are always constant values

above equation equal to $k^2 + k + 1 = 0$

case (i):- The roots k_1, k_2 both are real and different then

$$\boxed{a_n = A \cdot k_1^n + B \cdot k_2^n}$$

case (ii):- The roots k_1, k_2 both are real and equal

$$\boxed{a_n = (A + Bn)k^n}$$

case (iii):- The roots k_1, k_2, k_3 are real and different then

$a_n = A\,k_1^n + Bn\,k_2^n + Cn^2 k_3^n$

Case (iv): The roots k_1, k_2, k_3 are real and equal then

$\boxed{a_n = (A + Bn + Cn^2)\,k^n}$

Case (v): The roots k_1, k_2 are complex roots then

$a_n = r^n (A\cos n\theta + B\sin n\theta)$

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

Saturday

16/11/24

*Problem (1): find the R.R of $a_n + a_{n-1} - 6a_{n-2} = 0$

$a_0 = -1,\ a_1 = 8$

Sol: S-1: above equation is second order linear homogeneous R.R

S-2: $C_n \cdot a_n + C_{n-1} \cdot a_{n-1} + C_{n-2} \cdot a_{n-2} = 0$

S-3: $1 \cdot a_n + 1 \cdot a_{n-1} - 6 \cdot a_{n-2} = 0$

here
$$\left.\begin{array}{l} C_n = 1 \\ C_{n-1} = 1 \\ C_{n-2} = -6 \end{array}\right\} \text{constant values}$$

S-4: above equation convert it into second order linear homogeneous R.R

S-5: $1 \cdot k^2 + 1 \cdot k - 6 = 0$

S-6: $k^2 + k - 6 = 0$

$(k-2)(k+3) = 0$

$k = 2, 3$

S-7: k_1 & k_2 both are different so, $a_n = A \cdot k_1^n + B \cdot k_2^n$

$\boxed{\therefore a_n = A \cdot 2^n + B(-3)^n}$

S-8: Put $n=0$ $a_0 \Rightarrow -1 = A \cdot 2^0 + B(-3)^0$

$A + B = -1 \longrightarrow (1)$

Put $n=1$ $\Rightarrow a_1 \Rightarrow 8 = A \cdot 2^1 + B(-3)^1$

$2A - 3B = 8 \longrightarrow (2)$

solve (1) & (2)

$3A + 3B = -3$
$2A - 3B = 8$
$5A = 5$ $\Rightarrow$ $A = 1$, $B = -2$

S-9: $\boxed{\therefore a_n = 2^n - 2(-3)^n}$

* Third order linear homogeneous R.R:

* Problem(2): Find the R.R of $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0$

$a_0 = 1, a_1 = 5, a_2 = 1$

Sol: S-1: $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0$, $a_0 = 1, a_1 = 5, a_2 = 1$

S-2: above equation is third order linear homogeneous R.R

S-3: $C_n \cdot a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + C_{n-3} \cdot a_{n-3} = 0$

S-4: $1 \cdot a_n + 1 \cdot a_{n-1} - 8 \cdot a_{n-2} - 12 \cdot a_{n-3} = 0$

here
$$\left.\begin{array}{l} C_n = 1 \\ C_{n-1} = 1 \\ C_{n-2} = -8 \\ C_{n-3} = -12 \end{array}\right\} \text{constant values}$$

S-5: Above equation convert it into third order linear homogeneous R.R

S-6: $1 \cdot K^3 + 1 \cdot K^2 - 8 \cdot K - 12 = 0$

S-7: $K^3 + K^2 - 8K - 12 = 0$

$K = 3, -2, -2$

S-8: Here K_1 is different & K_2, K_3 are equal

$a_n = (A + Bn) K^n + C \cdot K_3^n$

$\boxed{a_n = (A + Bn)(-2)^n + C \cdot 3^n}$

S-9: put $n = 0 \Rightarrow a_0 \Rightarrow 1 = (A + B(0))(-2)^0 + C \cdot 3^0$

$\boxed{A + C = 1}$

put $n = 1 \Rightarrow a_1 \Rightarrow 5 = (A + B)(-2)^1 + C \cdot 3^1$

$\boxed{5 = -2A - 2B + 3C}$

put $n = 2 \Rightarrow a_2 \Rightarrow 1 = (A + 2B)4 + C \cdot 3^2$

$\boxed{4A + 8B + 9C = 1}$

$\boxed{A = 0, B = -1, C = 1}$

S-10: $\boxed{\therefore a_n = (-n)(-2)^n + 3^n}$