UNIT-1

Ordinary differential Equations of the first order

* Linear differential Eqn of first order :-
  1) Solution of Exact.
  2) Linear and Bernoulli Equation.
  3) Newtons Law of cooling
  4) Growth and decay
  5) Modeling an R-L circuit

* Non-Linear differential Eqns of first order :-
  1) Equations solving solvable for P
  2) Equations solvable for x $\cancel{x}$
  3) Equations solvable for y y

* **Differential Equation :-**

An Equation containing the derivatives of one or more dependent variables with respect to one or more indep variables is said to be differential Equation

Eg :- ① $\frac{dy}{dx} + 10y = e^x$ (First order)

$y \rightarrow$ Dependent variable ; $x \rightarrow$ indep var

② $\frac{d^2y}{dx^2} - \frac{dy}{dyx} + 9y = 0$ (sec. order)

③ $\frac{dx}{dt} + \frac{dy}{dt} = 2x+y$ D.V $\rightarrow x, y$ I.V $\rightarrow t$

④ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ D.V $\rightarrow u$ I.V $\rightarrow x, y$

* **Types of Differential Eqns :-**

There are two main types of differential Eqns

1) Ordinary D.E 2) Partial D.E

① **Ordinary Diff. Eqns :- (ODE)**

If a D.E contains only ordinary derivatives of one or more dep. var w.r.t single Indep. var it is said to be ordinary D.E

Eg :- ① $\frac{dy}{dx} + 10y = e^x$ ③ $\frac{dx}{dt} + \frac{dy}{dt} = 2x+y$

② $\frac{d^2y}{dx} - \frac{dy}{dx} + 9y = 0$

(2) Partial Diff. Eqn :- (PDE)

A D.E involving Partial Derivatives of one or more dep. var w.r.t two or more than two indep. var is called partial D.E.

Eg:- ① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

② $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$

③ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\left(\begin{matrix} D.V \to 2\ (u,v) \\ I.V \to x,y \end{matrix}\right)$

* Order of the Diff. Eqns :-

The order of the D.E (either ODE or PDE) is the order of the highest derivative present in the Equation.

Exponant of derivative / degree

Eg:- ① $\frac{d^3y}{dx^3} + 25\cdot\left(\frac{d^2y}{dx^2}\right)^{4} - 10\cdot\frac{dy}{dx} - 4y = \log x$

Order of given D.E is 3

② $\frac{\partial u}{\partial x} = \left(\frac{\partial^2 u}{\partial t^2}\right)^3 - 2\left(\frac{\partial u}{\partial t}\right)^4$

Order = 2

* Degree of the Diff Eqns :-

(I)f a D.E (either ODE or PDE) can be written polynomial in the D.E
The Power of highest order derivative
(free from radicals and fractions)

Eg:- ① $\frac{\partial u}{\partial x} = \left(\frac{\partial^2 u}{\partial t^2}\right)^3 - 2\left(\frac{\partial u}{\partial t}\right)^4$ deg = 3

② $\left(\frac{d^3y}{dx^3}\right) + 25\left(\frac{d^2y}{dx^2}\right)^4 - 10\cdot\frac{dy}{dx} - 4y = \log x$ deg = 1

How to Solve

An Equation of the form $\frac{dy}{dx} = f(x,y)$ is called diff. Eqn of first order and first degree

Methods to solve

1) Variable Seperable Method

2) Homogeneous and Non-Homo

3) Exact Eqns

Those which can be make exact by use of integration factors

4) Linear Eqns and Bernoulis Equation

① Variable Seperable Method :-

If the D.E $\frac{dy}{dx} = f(x,y)$ can be Expressed in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (or) $g(y).dy - f(x).dx = 0$ (or)

$$g(y)dy = f(x).dx \quad \text{—①}$$

where f, g are continous fn's of single variable then it is said to be of the form variable Seperable

Integrating ① then

General sol^n is $\int f(x).dx - \int g(y)dy = c$

where c is any arbitary constant

* Example Problems :-

Q1. Solve the D.E $\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$

Sol: Given D.E is $\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}} \Rightarrow \sqrt{1+x^2}$

$$\frac{dy}{\sqrt{1+y^2}} + \frac{dx}{\sqrt{1+x^2}} = 0$$

Integrating on both sides

G. Soln is $\sinh^{-1}x + \sinh^{-1}y = c$

Q2. Solve the D.E $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

Sol: Given D.E is $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

Seperating the variables

$$\frac{3e^x}{(1-e^x)} \cdot dx + \frac{\sec^2 y}{\tan y} dy = 0$$

Integ on both sides

$$\int \frac{3e^x}{(1-e^x)} \cdot dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$-3\log(1-e^x) + \log(\tan y) = c$$

$$\Rightarrow \log\left(\frac{\tan y}{(1-e^x)^3}\right) = \log c$$

$$\Rightarrow \tan y = (1-e^x)^3 c$$

$\therefore$ G. Soln of given D.E is $\tan y = (1-e^x)^3 c$

Q3. Solve the D.E $\frac{dy}{dx} = (4x+y+1)^2$

Sol: Given D.E is $\frac{dy}{dx} = (4x+y+1)^2$

let $4x+y+1 = v$

$4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$

$\Rightarrow \frac{dv}{dx} = v^2+4$

$\Rightarrow \frac{dv}{v^2+4} = dx$

Integrating on both side

$\int \frac{1}{v^2+4} \cdot dv = \int dx$ $\qquad \left(\int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c\right)$

$= \frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + c$

$\Rightarrow$ G.S $= \frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

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(2) Homogeneous Differential Equations :-

A fn $f(x,y)$ is said to be Homogeneous function in x, y of degree n, if $f(kx, ky) = k^n f(x,y)$ $\forall$ values of k where n is constant

Eg:- (1) Let $f(x,y) = \frac{x^2+y^2}{x^3+y^3}$

$f(kx, ky) = \frac{(kx)^2+(ky)^2}{(kx)^3+(ky)^3} = \frac{k^2}{k^3}\left(\frac{x^2+y^2}{x^3+y^3}\right) = k^{-1} f(x,y)$

$\Rightarrow f(kx, ky) = k^{-1} f(x,y)$

$\therefore f(x,y)$ is homo fn of degree -1

② $f(x,y) = \dfrac{y^2 + x^2 e^{x/y}}{x+y}$

$$f(kx, ky) = \frac{k^2y^2 + k^2x^2 e^{x/y}}{k(x+y)} = k\, f(x,y)$$

$\therefore$ $f(x,y)$ is homo fn of degree 1

* A differential Equation $\frac{dy}{dx} = f(x,y)$ of first order and first degree is called Homogeneous in x and y if the fn $f(x,y)$ is a homogeneous fn of degree zero in x and y.

⇒ General Soln of Homo. D.E :-

Let the Given D.E be $\frac{dy}{dx} = f(x,y)$ — ①

$\because$ $f(x,y)$ is a homo. fn of deg zero.

we can write $f(x,y) = \phi\left(\frac{y}{x}\right)$ — ②

from eq ① and ②

$$\Rightarrow \frac{dy}{dx} = \phi\left(\frac{y}{x}\right) \longrightarrow ③$$

Put $v = \frac{y}{x}$ $\Rightarrow$ $y = vx$ — ④ $\Rightarrow$ $\frac{dy}{dx} = \frac{dv}{dx}x + v(1)$

$$\Rightarrow \frac{dy}{dx} = v + x\cdot\frac{dv}{dx} \longrightarrow ④$$

from ③ and ④.

$$v + x\cdot\frac{dv}{dx} = \phi(v)$$

$$x\cdot\frac{dv}{dx} = \phi(v) - v$$

Seperating the variable $\frac{dv}{\phi(v) - v} = \frac{dx}{x}$

Integ on b/s :- $\int \frac{dv}{\phi(v) - v} = \int \frac{dx}{x} \longrightarrow$ G.S

Eg :- Solve $(x^2+y^2)\,dx = 2xy\cdot dy$

Sol: Given D.E is $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

This is a homo. D.E of deg 0

$\because f(kx,ky) = \frac{k^2(x^2+y^2)}{k^2(2xy)} = k^0 f(x,y)$

Put $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx}$

$\Rightarrow v + x.\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$

$\Rightarrow v + x.\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\Rightarrow x.\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$

$\Rightarrow x.\frac{dv}{dx} = \frac{1-v^2}{2v}$

Seperating the variables

$\Rightarrow \frac{2v}{1-v^2}\,dv = \frac{dx}{x}$

Integ on b/s

$\int \frac{2v}{1-v^2}\,dv = \int \frac{dx}{x}$

$\rightarrow -\log(1-v^2) = \log x + \log c$

$\Rightarrow \log x + \log(1-v^2) = \log c$

$\Rightarrow x(1-v^2) = c \quad (\because v = \frac{y}{x})$

$\Rightarrow x\left(1 - \frac{y^2}{x^2}\right) = c \longrightarrow$ G. soln

Q) Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$ $\quad x^2y\,dx = (x^3+y^3)\,dy$

Given D.E is $\frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)}$

$$f(kx, ky) = \frac{k^3(x^2y)}{k^3(x^3+y^3)} = f(x,y)$$

This is homo D.E of deg 0

Put $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

$$\Rightarrow v + x\cdot\frac{dv}{dx} = \frac{x^2(vx)}{x^3+v^3x^3} = \frac{v}{1+v^3}$$

$$\Rightarrow x\cdot\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow x\cdot\frac{dv}{dx} = \frac{v - v - v^4}{1+v^3} = \frac{-v^4}{1+v^3}$$

$$-\frac{1+v^3}{v^4}\,dv = \frac{dx}{x}$$

$v^{-4} = \frac{v^{-3}}{-3}$

Integ on b/s

$v^4 - 4v^3$

$v^4 = t$

$$\Rightarrow -\int \frac{1+v^3}{v^4}\,dv = \int \frac{dx}{x}$$

$4v^3\,dv = dt$

$v^3\,dv$

$$\Rightarrow -\int \frac{1}{v^4}\,dv + \int \frac{v^3}{v^4}\,dv = \log x + \log c$$

$$\Rightarrow \frac{1}{3}\,\frac{1}{v^3} + \log v = \log x + \log c$$

$$\Rightarrow \frac{1}{3v^3} + \log v - \log x = \log c \Rightarrow \frac{1}{3v^3} - \log vx = \log c$$

$$\Rightarrow \frac{1}{3\left(\frac{y}{x}\right)^3} + \log\left(\frac{y}{x}\right) = \log c \qquad \Rightarrow \frac{x^3}{3y^3} - \log y = \log c$$

$$\Rightarrow \frac{x^3}{3y^3} = \log\left(\frac{cx}{y}\right)$$

(20) solve $x^3 \cdot \frac{dy}{dx} = y^3 + y^2\sqrt{y^2-x^2}$

$$\frac{dy}{dx} = \frac{y^3+y^2\sqrt{y^2-x^2}}{x^3}$$

Homo. D.E of deg Zero

$$y = vx \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}(x)$$

$$\Rightarrow v + x\cdot\frac{dv}{dx} = \frac{v^3x^3+v^2x^2\sqrt{v^2x^2-x^2}}{x^3}$$

$$= \frac{v^3x^3+v^2x^3\sqrt{v^2-1}}{x^3}$$

$$= v^3+v^2\sqrt{v^2-1}$$

$$\Rightarrow x\cdot\frac{dv}{dx} = v^3+v^2\sqrt{v^2-1} - v$$

$$\Rightarrow \cancel{v}\,\frac{1}{v^3+v^2\sqrt{v^2-1}-v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \int\frac{1}{v^3+v^2\sqrt{v^2-1}-v}\cdot dv = \log x + \log c$$

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta = 1+\tan^2$

$v = \tan\theta$

$$\cancel{v = \tan\theta \Rightarrow dv = \sec^2\theta \cdot d\theta}$$

$$\cancel{\Rightarrow \int\frac{1}{\tan^3\theta + \tan^2\theta(}}$$

$$\Rightarrow \int\frac{dv}{v^3+v^2\sqrt{v^2-1}-v} = \log x + \log c$$

$v^3 - v$

$v(v^2-1) + v^2\sqrt{v^2-1}$

$$\Rightarrow \int\frac{dv}{v^2(v-1)+v^2\sqrt{v^2-1}} = \log x + \log c$$

$v(\sqrt{v^2-1})(\sqrt{v^2-1}$

$$\int\frac{dv}{v^2\sqrt{v^2-1}+\sqrt{v^2-1}\,(v\sqrt{v^2-1})} = \log x + \log c$$

$$\Rightarrow \frac{dv}{v\sqrt{v^2-1}\,(v+\sqrt{v^2-1})} \times \frac{v-\sqrt{v^2-1}}{v-\sqrt{v^2-1}} =$$

$$\Rightarrow \frac{dv}{v\sqrt{v^2-1}\,(v+\sqrt{v^2-1})} , \frac{dv}{v\sqrt{v^2-1}\,(\cancel{v^2}-\cancel{v^2}+1)} \times \frac{\cancel{v-\sqrt{v^2-1}}}{v}$$

$$\Rightarrow \frac{dv\,(v-\sqrt{v^2-1})}{v\sqrt{v^2-1}}$$

$$\int \frac{\cancel{v}\,dv}{\cancel{v}\sqrt{v^2-1}} - \int \frac{\sqrt{v^2-1}\,dv}{v\sqrt{v^2-1}}$$

$$\log(v+\sqrt{v^2-1}) - \log v = \log x + \log c$$

$$\log\left(\frac{v+\sqrt{v^2-1}}{v}\right) = \log xc \quad = \frac{\frac{y}{x}+\sqrt{\frac{y^2}{x^2}-1}}{\frac{y}{x}} = xc$$

$$\Rightarrow \quad y+\sqrt{y^2-x^2} = cxy$$

(3.) $[1+e^{x/y}].\,dx + e^{x/y}\left[1-\frac{x}{y}\right]dy = 0$

Ans: $x + e^{x/y}.\,y = c$

Sol: Given :- $[1+e^{x/y}].\,dx = -e^{x/y}\left[1-\frac{x}{y}\right]dy$

$$\Rightarrow \frac{dy}{dx} = \frac{[1+e^{x/y}]}{-e^{x/y}\left[1-\frac{x}{y}\right]}$$

Homo. ~~deo~~ d.E of deg zero

$$y = vx \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}(x)$$

$$\Rightarrow v + x.\frac{dv}{dx} = \frac{1+e^{x/y}}{-e^{x/y}\left[1-\frac{x}{y}\right]} = \frac{1+e^{\frac{1}{v}}}{-e^{\frac{1}{v}}\left[1-\frac{1}{v}\right]}$$

$$\Rightarrow x.\frac{dv}{dx} = \frac{1+e^{1/v}}{-e^{1/v}\left(1-\frac{1}{v}\right)} - v$$

$$\Rightarrow \frac{dv}{\frac{1+e^{\frac{1}{v}}}{-e^{\frac{1}{v}}\left[1-\frac{1}{v}\right]} - v} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{-\left[\frac{1+e^{1/v}}{e^{1/v}\left(1-\frac{1}{v}\right)} + v\right]} = \log x + c$$

$$\Rightarrow \int \frac{dv}{-\left[\frac{1+e^{1/v} + v.e^{\frac{1}{v}} - e^{\frac{1}{v}}}{e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}\right]} = \log x + c$$

$$\Rightarrow \int \frac{dv}{-\left(\frac{1+v.e^{\frac{1}{v}}}{e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}\right)}$$

or

$$\frac{dx}{dy} = -\frac{e^{x/y}\left[1-\frac{x}{y}\right]}{1+e^{x/y}}$$

$$\frac{x}{y} = v \Rightarrow x = vy \Rightarrow \frac{dx}{dy} = v + y.\frac{dv}{dy}$$

$$\frac{dx}{dy} = v + y.\frac{dv}{dy} = -\frac{e^{v}\left[1-v\right]}{1+e^{v}}$$

$$\Rightarrow y.\frac{dv}{dy} = \frac{-e^{v} + v.e^{v}}{1+e^{v}} - v$$

$$\Rightarrow \frac{dv}{\left(\frac{-e^{v} + ve^{v} - v - ve^{v}}{1+e^{v}}\right)} = \frac{dy}{y}$$

$$\int \frac{1+e^{v}}{-(e^{v}+v)} \cdot dv = \int \frac{dy}{y}$$

$$-\int \frac{1+e^v}{(e^v+v)}\,dv = \log y + \log c$$

$$-\log(e^v+v) = \log y + \log c$$

$$-\left(\log(e^v+v) + \log y\right) = \log c$$

$$\Rightarrow \quad y(e^v+v) = c$$

$$\Rightarrow y\left(e^{x/y} + \frac{x}{y}\right) = c$$

$$\Rightarrow \quad x + y.e^{x/y} = c$$

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③ Exact Differential Equation :-

A D.E which can be obtained directly by differentiating of their Premitive (solⁿ) without eliminating or transformation is called E.D.E

Eg:- $x^3 + y^3 = c$

diff on both sides

$$3x^2.dx + 3y^2 dy = 0$$

$$x^2 dx + y^2 dy = 0$$

$$M\,dx + N\,dy = 0 \longrightarrow \text{form}$$

→ Exact D.E

OR

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a first order and first degree D.E, where M, N are real valued fn's for some x, y then the Equation

$Mdx + Ndy = 0$ is said to be an Exact D.E

if $\exists$ a fn. f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

Eg:- $xdy + ydx = 0$ is obtained by differentiating

$xy = c$ [$\because d(xy) = xdy + ydx$]

* Necessary condition for Exact D.E :-

$\Rightarrow$ if $M(x,y)$ and $N(x,y)$ are 2 real valued fns

which have continuous partial derivatives then a

necessary and sufficient condition for the D.E

$Mdx + Ndy = 0$ to be a Exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ The General soln of given Exact D.E $Mdx + Ndy = 0$ is

$\int M dx + \int N dy = C$

(y const) (free from x)

only those terms which don't contain x

where C is any constant

$Mdx + Ndy = 0$

M N

$\frac{\partial M}{\partial y} = ?$ $\frac{\partial N}{\partial x} = ?$

Q) Solve the D.E $2xydx + (1+x^2)dy = 0$

Sol:- The Given D.E is $2xy\,dx + (1+x^2)dy = 0$ —— (1)

by comparing (1) with $Mdx + Ndy = 0$;

$M = 2xy$; $N = 1+x^2$

$\Rightarrow \frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The Given D.E is Exact D.E

The G.S of ① $= \int\limits_{y\text{ const}} M\,dx + \int\limits_{\text{free from } x} N\,dy = C$

$$\int 2xy\,dx + \int 1.\,dy = C$$

$$y.\frac{2x^2}{2} + y = C \Rightarrow x^2y + y = C \Rightarrow \underline{\underline{(x^2+1)y = C}}$$

Q) solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$ is Exact ?

Sol: The Given D.E is $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$\Rightarrow M = x^2 - 4xy - 2y^2$; $N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4x - 4y$ $\quad \frac{\partial N}{\partial x} = -4y - 4x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The Given D.E is Exact D.E

G.S of ① $= \int\limits_{y\text{ cons}} (x^2 - 4xy - 2y^2)\,dx + \int\limits_{\text{free from } x} (y^2 - 4xy - 2x^2)\,dy = C$

$$\Rightarrow \frac{x^3}{3} - 4y.\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = C$$

$$\Rightarrow \frac{x^3+y^3}{3} - 2x^2y - 2y^2x = C$$

$$\Rightarrow x^3 + y^3 - 6xy(x+y) = C$$

Q) $(hx+by+f)\,dy + (ax+hy+g)\,dx = 0$

Ans $ax^2+2hxy+2gx + 2fy + by^2 = C$

$M\,dx + N\,dy = 0$

$M = hx+by+f \quad N$

$M = ax+hy+g \quad N = hx+by+f$

$\frac{\partial M}{\partial y} = 0+h \quad$ & $\frac{\partial N}{\partial x} = h$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ E.D.E

$\Rightarrow \int (ax+hy+g)\,dx + \int (h\cancel{x}+by+f)\,dy = C$

y const   free from x

$\Rightarrow a\cdot\frac{x^2}{2} + hyx + gx + b\cdot\frac{y^2}{2} + fy = C$

$\Rightarrow ax^2 + 2hyx + 2gx + by^2 + 2fy = C$

Q) Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Ans $(e^y+1)\sin x = C$

Sol: $M\,dx + N\,dy = 0$

$M = (e^y+1)\cos x \quad N = e^y \sin x$

$\frac{\partial M}{\partial y} = \cancel{\cos x} \quad \frac{\partial N}{\partial x} = e^y(\cos x)$

$e^y\cos x + \cos x$

$\frac{\partial M}{\partial y} = \cos x \cdot e^y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ E.D.E

$$\int_{y\ const} (e^y+1)\cos x\,dx + \int e^y \sin x\,dy = C$$

free for x

$$\int e\ (e^y+1)\int \cos x.dx + e^y = C$$

$$\Rightarrow (e^y+1)(+\sin x) + e^y = C$$

Q) Solve $(1+e^{x/y})\,dx + e^{x/y}\left[1+\frac{x}{y}\right]dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = (1+e^{x/y}) \qquad N = e^{x/y}\left[1-\frac{x}{y}\right]$$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left(-\frac{1}{y^2}\right)x \qquad \frac{\partial N}{\partial x} = e^{x/y}\left(\frac{1}{y}\right) - e^{x/y} - e^{x/y}\left(\frac{x}{y}\right)$$

$$= e^{x/y}\left(\frac{1}{y}\right) - \left[e^{x/y}\left(\frac{x}{y}\right) + e^{x/y}\left(\frac{1}{y}\right)\right]$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{x}{y^2}\right) \qquad \Rightarrow e^{x/y}\left[1-\cancel{\frac{x}{y}}-\cancel{\frac{1}{y}}\right]$$

$\Rightarrow$ E.D.E

$$\int_{y\ const}(1+e^{x/y})\,dx + \int 0.dy = C$$

$$\Rightarrow x + \frac{e^{x/y}}{\frac{1}{y}} = C \qquad \Rightarrow x + ye^{x/y} = C$$

Q) $\left(xe^{xy}+2y\right)\frac{dy}{dx}+ye^{xy}=0$

Sol $\left(xe^{xy}+2y\right)dy+ye^{xy}.dx=0$

$M\,dx+N\,dy=0$

$M=ye^{xy}$ $\quad N=xe^{xy}+2y$

$\frac{\partial M}{\partial y}=ye^{xy}.x+e^{xy}$ $\quad \frac{\partial N}{\partial x}=x.e^{xy}.y+e^{xy}+0$

$=e^{xy}(xy+1)$ $\quad =e^{xy}(xy+1)$

$\Rightarrow$ E.D.E

$\underset{y\text{ const}}{\int}ye^{xy}.dx+\underset{\text{free from }x}{\int}(\cancel{xe^{xy}}+2y)\,dy=\cancel{0}C$

$y\int e^{xy}dx+\int 2y\,dy=\cancel{0}C$

$\cancel{y}.\frac{e^{xy}}{\cancel{y}}+\cancel{2}.\frac{y^2}{\cancel{2}}=\cancel{0}C$

$\Rightarrow e^{xy}+y^2=C$

Q) solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log|x|-x\sin y\right)dy=0$

Ans:- $y(x+\log|x|+\cos y(x))=C$

Sol: $M\,dx+N\,dy=0$

$M=y\left(1+\frac{1}{x}\right)+\cos y$ $\quad N=x+\log|x|-x\sin y$

$=y+\frac{y}{x}+\cos y$

$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$ $\quad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$

E.D.E

$$\Rightarrow \int_{y\ \text{const}} y + \frac{y}{x} + \cos y \cdot dx + \int 0 \cdot dy = 0$$

$$\Rightarrow y \cdot x + y \cdot \log x + \cos y(x) = 0$$

$$\Rightarrow y(x + \log x) + x\cos y = 0$$

Q) solve $[r + \sin\theta - \cos\theta]\,dr + r[\sin\theta + \cos\theta]\,d\theta = 0$

Ans:- $r^2 + 2r(\sin\theta - \cos\theta) = 0$

$$M\,dr + N\,d\theta = 0$$

$r = x \;;\; \theta = y$

$M = r + \sin\theta - \cos\theta$ $\qquad N = (\sin\theta + \cos\theta)r$

$\frac{\partial M}{\partial \theta} = \cos\theta + \sin\theta$ $\qquad \frac{\partial N}{\partial r} = \sin\theta + \cos\theta$

E.D.E

$$\Rightarrow \int_{\theta\ \text{const}} (r + \sin\theta - \cos\theta)\,dr + \int \cancel{r(\sin\theta + \cos\theta)\,d\theta} = C$$

free from r

$$\Rightarrow \frac{r^2}{2} + \sin\theta \cdot r - \cos\theta \cdot r = C$$

$$\Rightarrow r^2 + 2r(\sin\theta - \cos\theta) = C$$

*NON EXACT D.E :-

* Methods to find Integrating Factor :-

method-1 :- To find I.F of $Mdx + Ndy = 0$.

I.F's can be found by inspection.

For this purpose, the following differentials should be kept in kep mind.

* $d(xy) = x\,dy + y\,dx$
* $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
* $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$
* $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$
* $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$
* $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$
* $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{x^2+y^2}$
* $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$
* $d(\log(xy)) = \frac{y\,dx + x\,dy}{xy}$
* $d\left(\log(x^2+y^2)\right) = 2\frac{(x\,dx + y\,dy)}{x^2+y^2}$
* $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

Q) Solve $(1+xy)\,x\,dy + y(1-yx)\,dx = 0$

Sol:- Given Equation is $(1+xy)\,x\,dy + (1-yx)\,y\,dy = 0$

$x\,dy + y\,dy + xy\,(x\,dy - y\,dx) = 0$

Dividing with x^2y^2,

$$\frac{x\,dy + y\,dx}{x^2 + y^2} + \frac{x\,dy - y\,dx}{xy} = 0$$

$$\Rightarrow \frac{d(xy)}{(xy)^2} + \left(-\frac{1}{x}\,dx + \frac{1}{y}\,dy\right) = 0 \qquad \left(\because \int \frac{d}{x^2}\,dx = \frac{-1}{x}\right)$$

Integrating :- $\frac{-1}{xy} - \log x + \log y = \log c$

G. Soln :- $\frac{-1}{xy} - \log x + \log y = \log c$

Q) solve $x\,dx + y\,dy = \frac{x\,dy - y\,dx}{x^2+y^2}$

Sol:- Given Eq^n is $x\,dx + y\,dy = \frac{x\,dy - y\,dx}{x^2+y^2}$

$$\Rightarrow d\left(\frac{x^2+y^2}{2}\right) = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Integrating

$$\frac{x^2+y^2}{2} = \tan^{-1}\left(\frac{y}{x}\right) + c$$

General sol^n is $\frac{x^2+y^2}{2} = \tan^{-1}\left(\frac{y}{x}\right) + C$

Q) Solve $x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$

Sol:- Given Eqn is $x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$ → ①

① can be written as

$$d\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = a^2 \cdot d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Integrating :- $\frac{x^2}{2} + \frac{y^2}{2} = a^2 \tan^{-1}\left(\frac{y}{x}\right) + c$

∴ General solⁿ is $\frac{x^2+y^2}{2} = a^2 \tan^{-1}\left(\frac{y}{x}\right) + c$

Q) Solve $\frac{y\,dx - x\,dy}{x^2} + e^y dy = 0$

Sol:- The Given Eqⁿ is $\frac{y\,dx - x\,dy}{x^2} + e^y dy = 0$ — ①

① can be written as

$$e^y dy = \frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

Integrating

$$e^y = \frac{y}{x} + c$$

General solution ⇒ $e^y = \frac{y}{x} + c$

Q) Solve $\frac{y\,dx - x\,dy}{xy} + 2x\sin x^2 dx = 0$

Sol:- Given Eqⁿ is $\frac{y\,dx - x\,dy}{xy} + 2x\sin x^2 dx$ → ①

① ⇒ $d\left(\log\left(\frac{x}{y}\right)\right) + 2x\sin x^2 dx$

Integrating

$$\log\left(\frac{x}{y}\right) + \int 2x\sin x^2\,dx = c$$

I L A T E

$$\log\left(\frac{x}{y}\right) + 2\int x\sin x^2\,dx = c$$

$\int uv = u\int v - \int u'\int v$

$$\Rightarrow \log\left(\frac{x}{y}\right) + 2\left[x\int \sin x^2 \cdot dx - \int\left(\int \sin x^2 dx\right) dx\right]$$

$$\Rightarrow \log\left(\frac{x}{y}\right) + 2\left[x\int \sin x \cdot \sin x \cdot dx - \int\left(\int \sin x \sin x\, dx\right) dx\right]$$

$$\Rightarrow \int \sin x \sin x \cdot dx = \sin x\int \sin x\,dx - \int \cos x\int \sin x\,dx \cdot dx$$

$$= \sin x\left(\frac{-\cos x}{1}\right) - \int \cos x(-\cos x)\cdot dx$$

$$\Rightarrow -\sin x\cos x + \int \cos x \cdot \cos x \cdot dx$$

$$\int \sin^2 x \cdot dx = -\frac{\sin x\cos x}{2} + \frac{1}{2}x$$

$$\log\left(\frac{x}{y}\right) + 2\left[x\left(\frac{-\sin x\cos x + x}{2}\right)\right] -$$

$$\Rightarrow \log\left(\frac{x}{y}\right) + 2\int x\sin x^2\,dx = c$$

let $x^2 = t$

$2x\,dx = dt$

$$\Rightarrow \log\left(\frac{x}{y}\right) + \int \sin t\,dt = c$$

$$\Rightarrow \log\left(\frac{x}{y}\right) - \cos t = c$$

$$\Rightarrow \log\frac{x}{y} - \cos x^2 = c$$

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Method - 2 :- (Homo)

To find an integrating factor of $M\,dx + N\,dy = 0$ is a homogeneous

if $M(x,y).dx + N(x,y)\,dy = 0$ is a homogeneous D.E and $Mx + Ny \neq 0$ then

$\frac{1}{Mx+Ny}$ is an I.F of $M\,dx + N\,dy = 0$

Q> Solve $x^2y\,dx - (x^3+y^3)\,dxy = 0$

Sol: Given D.E is $x^2y\,dx - (x^3+y^3)\,dy = 0$ —— (1)

$(M\,dx - N\,dy) = 0$

$M = x^2y$ $\qquad$ $N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2$ $\qquad$ $\frac{\partial N}{\partial x} = -3x^2$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given D.E is non-exact D.E

$$\frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)}$$

$$f(kx, ky) = k^0 f(x,y)$$

The Eqn is homogeneous D.E

$$Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$$

$$\Rightarrow I.F = \frac{1}{Mx+Ny} = -\frac{1}{y^4}$$

Multiply (1) with I.F

$$\left(-\frac{x^2y}{y^4}\right)dx + \frac{(x^3+y^3)}{y^4}\,dy = 0 \quad \text{—— (2)}$$

$M_1 dx + N_1 dy = 0$

$M_1 = -\frac{x^2}{y^3}$ $\quad$ $N_1 = \frac{x^3+y^3}{y^4}$

$\frac{\partial M_1}{\partial y} = -x^2(-3)y^{-4}$ $\quad$ $\frac{\partial N_1}{\partial x} = \frac{1}{y^4}(3x^2)$

$= \frac{3x^2}{y^4}$ $\quad$ $= \frac{3x^2}{y^4}$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore$ (2) is Exact D.E

The General Sol^n of (2) is

$$\underset{y\ \text{const}}{\int} M_1 dx + \underset{\text{free from } x}{\int} N_1 dy = C$$

$$-\frac{1}{y^3}\int x^2 dx + \int \frac{1}{y} dy = C$$

$$\Rightarrow -\frac{1}{y^3} \times \frac{x^3}{3} + \log|y| = C$$

$$\Rightarrow \frac{-x^3}{3y^3} + \log|y| = C$$

$\therefore$ G. sol of Given D.E is $\frac{-x^3}{3y^3} + \log|y| = C$

Q) Solve $y^2 dx + (x^2 - xy - y^2) dy = 0$

$M dx + N dy = 0$

$M = y^2$ $\quad$ $N = x^2 - xy - y^2$

$\frac{\partial M}{\partial y} = 2y$ $\quad$ $\frac{\partial N}{\partial x} = 2x - y$

non exact

$$y^2dx = -(x^2-xy-y^2)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{-(x^2-xy-y^2)} \qquad \frac{k^2y^2}{-(k^2x^2-k^2xy-k^2y^2)}$$

$$f(kx,ky) = k^0 f(x,y)$$

Homo. D.E

$$Mx+Ny = y^2x + x^2y - xy^2 - y^3$$

$$= \cancel{2xy^2} + x^2y - y^3 = y(x^2-y^2) \neq 0$$

$$I.F = \frac{1}{Mx+Ny}$$

(1) × I.F

$$\Rightarrow \frac{y}{x^2-y^2}dx + \frac{x^2-xy-y^2}{y(x^2-y^2)}dy = 0$$

$$M_1 = \frac{y}{x^2-y^2} \qquad N_1 = \frac{x^2-xy-y^2}{y(x^2-y^2)}$$

$$\frac{\partial M_1}{\partial y} = \frac{x^2+y^2}{(x^2-y^2)^2} \qquad \frac{\partial N_1}{\partial x} = \frac{x^2+xy^2}{(x^2-y^2)^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Exact

$$\int_{y\text{-const}} M_1\,dx + \int_{\text{free from } x} N_1\,dy = C$$

$$\int \frac{y}{x^2-y^2}dx + \int \frac{1}{y}dy = C$$

$\Rightarrow \frac{1}{2}\log\left(\frac{x-y}{x+y}\right) + \log|y| = c$

$\Rightarrow \left(\frac{x-y}{x+y}\right)^{1/2} y = c$

$\Rightarrow (x-y)y^2 = c(x+y) \longrightarrow$ General solⁿ

Method - 3 : (non homo)

To find I.F of $Mdx + Ndy = 0$

If the Eqⁿ $Mdx + Ndy = 0$ is of the form

$y f(x,y)\,dx + x\,g(x,y)\,dy = 0$ and $Mx - Ny \neq 0$

Then $\frac{1}{Mx - Ny}$ is an I.F of $Mdx + Ndy = 0$

(Q) Solve $y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$

Sol: Given D.E is $y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$ —— (1)

$M = y(x^2y^2+2)$ $\quad N = x(2-2x^2y^2)$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2$ $\quad \frac{\partial N}{\partial x} = 2 - 6xy^2$

Non Exact D.E

$Mx - Ny = x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3$

$Mx - Ny = 3x^3y^3 \neq 0$

$I.F = \frac{1}{3x^3y^3}$

(1) × I.F

$$\Rightarrow \frac{y(x^2y^2+2)}{3x^3y^3}dx + \frac{x(2-2x^2y^2)}{3x^3y^3}dy = 0$$

$$M_1 = \frac{y(x^2y^2+2)}{3x^3y^3} \qquad N_1 = \frac{2x - 2x^3y^2}{3x^3y^3}$$

$$= \frac{1}{3x} + \frac{2}{3x^3y^2} \qquad = \frac{2}{3x^2y^3} - \frac{2}{3y}$$

$$\frac{\partial M_1}{\partial y} = \frac{2}{3x^3}(-2y^{-3}) \qquad \frac{\partial N_1}{\partial x} = \frac{2}{3y^3}(-2x^{-3})$$

$$= \frac{-4}{3x^3y^3} \qquad = \frac{-4}{3x^3y^3}$$

Exact D.E

$$\Rightarrow \underset{y \text{ const}}{\int} M_1\,dx + \underset{\text{free from } x}{\int} N_1\,dy = C$$

$$\int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int -\frac{2}{3y}\,dy = C$$

$$\frac{1}{3}\log|x| + \frac{2}{3y^2}\frac{x^{-2}}{-2} - \frac{2}{3}\log|y| = C$$

$$\Rightarrow \frac{1}{3}\log|x| - \frac{1}{3x^2y^2} - \frac{2}{3}\log|y| = C$$

$$\log(x)^{1/3} - \log(y)^{2/3} - \frac{1}{3x^2y^2} = C$$

$$\frac{x^{1/3}}{y^{2/3}} - \frac{1}{3x^2y^2} = C$$

$$\Rightarrow \left(\frac{x}{y^2}\right)^{1/3} - \frac{1}{3x^2y^2} = C \longrightarrow \text{General sol}^n$$

Q) Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$ — ①

Given D.E $M dx + N dy = 0$

$M = (xy\sin xy + \cos xy)y \qquad N = x^2y\sin xy - x\cos xy$

$\frac{\partial M}{\partial y} = xy^2\sin xy + y\cos xy$

$\frac{\partial M}{\partial y} = x\left[y^2 x\cos xy + \sin xy \cdot 2y\right] + \left[y(-\sin xy)x + \cos xy\right]$

$\Rightarrow x^2y^2\cos xy + 2xy\sin xy - xy\sin xy + \cos xy$

$\frac{\partial N}{\partial x} = y\cdot\left[x^2(\cos xy)y + \sin xy\,(2x)\right] - \left[x(-\sin xy)y + \cos xy\right]$

$= x^2y^2\cos xy + 2xy\sin xy + xy\sin xy - \cos xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

non exact

$Mx - Ny = \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$

$\neq 0$

$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$

① × I.F

$\frac{(xy^2\sin xy + y\cos xy)dx + (x^2y\sin xy - x\cos xy)dy}{2xy\cos xy} = 0$

$= \left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$

$M_1 = \frac{y}{2}\tan xy + \frac{1}{2x}$ $\quad N_1 = \frac{x}{2}\tan xy - \frac{1}{2y}$

$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[y.\sec^2 xy\,(x) + \tan xy\right]$ $\quad \frac{\partial N_1}{\partial x} = \frac{1}{2}\left[x\sec^2 xy\,(y) + \tan xy\right]$

$\Rightarrow$ Exact D.E

$\Rightarrow \int M_1\, dx + \int N_1\, dy = C$

$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int\left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = c$

$\frac{y}{2}\frac{\log|\sec xy|}{y} + \frac{1}{2}\log|x| + \frac{1}{2}\log|y| = C$

$\frac{1}{2}\left[\log\left(\frac{x\sec xy}{y}\right)\right] = C$

$\Rightarrow \left(\frac{x\sec xy}{y}\right)^{1/2} = C$

Method-4 :-

To find an I.F of $Mdx + Ndy = 0$

if $\exists$ a continuous single variable fn $f(x)$

such that $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then

$e^{\int f(x)\,dx}$ is an I.F of $Mdx + Ndy = 0$

Q) Solve $2xy\,(dy) - (x^2+y^2+1)\,dx = 0$ —— (1)

$$M\,dx + N\,dy = 0$$

$$N = 2xy \;;\; M = -x^2-y^2-1$$

$$\frac{\partial M}{\partial y} = -2y \;;\; \frac{\partial N}{\partial x} = 2y$$

non-exact

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}\left(-2y-2y\right) = -\frac{2}{x} = f(x)$$

$$\therefore \text{I.F} = e^{\int f(x)\,dx} = e^{-2\int \frac{1}{x}\,dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}} = \frac{1}{x^2}$$

(1) $\times \frac{1}{x^2} \Rightarrow \frac{2y}{x}\,dy - \left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx$ —— (2)

$$M_1\,dx + N_1\,dy = 0$$

$$M_1 = -1-\frac{y^2}{x^2} \;;\; N_1 = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{-2y}{x^2} \;;\; \frac{\partial N_1}{\partial x} = \frac{-2y}{x^2}$$

Exact D.F

$$\int\limits_{y\text{-const}} M_1\,dx + \int\limits_{\text{free from } x} N_1\,dy = C$$

$$\int\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx + \int 0\,dy = C$$

$$\Rightarrow -x + \frac{y^2}{x} + \frac{1}{x} = C \longrightarrow \text{General sol}^n$$

(Q) $(x^4e^x - 2mxy^2)dx + 2mx^2y\,dy = 0$ — (1)

Ans: $e^x + \frac{my^2}{x^2} = c$

$M\,dx + N\,dy = 0$

$M_0 = x^4e^x - 2mxy^2$

$\frac{\partial M}{\partial y} = -2mx(2y) = -4mxy$

$N = 2mx^2y \Rightarrow \frac{\partial N}{\partial x} = 2my(2x) = 4mxy$

non exact.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2mx^2y}\left(-4mxy - 4mxy\right)$

$= \frac{-8mxy}{2mx^2y} = \frac{-4}{x} = f(x)$

$I.F = e^{\int f(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{\log x^{-4}} = \frac{1}{x^4}$

(1) $\times \frac{1}{x^4}$

$\Rightarrow \left(e^x - \frac{2my^2}{x^3}\right)dx + \frac{2my}{x^2}\,dy = 0$

$M_1 = e^x - \frac{2my^2}{x^3}$ $\quad N_1 = \frac{2my}{x^2}$

$\int_{y\ \text{cons}} M_1\,dx + \int N_1\,dy = c$

$\Rightarrow \int\left(e^x - \frac{2my^2}{x^3}\right)dx + \int \frac{2my}{x^2}\,dy = c$ (0)

$e^x - 2my^2\frac{x^{-3+1}}{-2} = c$

$\Rightarrow e^x + \frac{my^2}{x^2} = c$

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Method 5 :-

To find integrating factor of $Mdx + Ndy = 0$ if there exists a continuous and differentiable single variable function $g(y)$ such that $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ then $e^{\int g(y).dy}$ is integrating factor of $Mdx + Ndy = 0$

Q) Solve $y(2xy + e^x)dx - e^x dy = 0$

$$Mdx + Ndy = 0$$

$M = y(2xy + e^x)$; $N = -e^x$

$\frac{\partial M}{\partial y} = 2x(2y) + e^x$ $\qquad$ $\frac{\partial N}{\partial x} = -e^x$

non-exact

$$\frac{1}{y(2xy+e^x)}\left(-e^x - 4xy - e^x\right) = \frac{-2(e^x + \cancel{2xy})}{y(2xy + e^x)} = -\frac{2}{y}$$

$$e^{\int -\frac{2}{y}dy} \Rightarrow \frac{1}{y^2}$$

$$\Rightarrow \left(\frac{2y^2}{x} + \frac{ye^x}{x^2}\right)dx - \frac{e^x}{x^2}dy = 0$$

$\frac{\delta M_1}{\partial y} = \frac{2}{x}(2y) + \frac{e^x}{x^2}$; $\frac{\delta N_1}{\partial x} = \frac{e^x}{x^2}$ ILATE

$2y^2\log x + y\int \frac{e^x}{x^2}dx - 0 = C$

$\Rightarrow 2y^2\log x + y[$

$\int e^x x^{-2}$

$e^x\left(\frac{x^{-1}}{-1}\right) + \int$

$\int x^{-2}e^x$

$x^{-2}e^x + \int -2x^{-3}.e^x.dx$

$$\left(2x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2} dy = 0 + C$$

$$2\frac{x^2}{2} + \frac{1}{y} e^x - 0 = 0 + C$$

$$x^2 + \frac{e^x}{y} = C$$

Q) $y(xy + e^x)dx - e^x dy = 0$

$M\,dx + N\,dy = 0$

$$\frac{\partial M}{\partial y} = x(2y) + e^x \qquad \frac{\partial N}{\partial x} = -e^x$$

$$\frac{1}{y(xy+e^x)} \left(-e^x - 2xy - e^x\right) \Rightarrow \frac{-2}{y}$$

$I.F = \frac{1}{y^2}$

$$\Rightarrow \left(x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2} dy = 0$$

$$\frac{\partial M}{\partial y} = e^x\left(\frac{-1}{y^2}\right), \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2} e^x$$

$$\int x + \frac{e^x}{y} \cdot dx - 0 = C$$

$$\Rightarrow \frac{x^2}{2} + \frac{1}{y} e^x = C \quad \Rightarrow \quad \frac{x^2}{2} + \frac{e^x}{y} = C$$

(2) $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3)dy = 0$

$Mdx + Ndy = 0$

$\frac{\partial M}{\partial y} = x(2y) - 0 \quad ; \quad \frac{\partial N}{\partial x} = y^2(6x) + y(2x) - 6x^2$

non- exat

$\frac{1}{M} \quad \frac{1}{xy^2 - x^2}\left(6xy^2 + 2xy - 6x^2 - 2xy\right)$

$= \frac{6xy^2 - 6x^2 4}{xy^2 - x^2} = 6$

$I.F = e^{\int 6 \cdot dy} = \not{f} \; e^{6y}$

$e^{6y}(xy^2 - x^2)dx + e^{6y}(3x^2y^2 + x^2y - 2x^3)dy = 0$

$\cancel{6} \; e^{6y}y^2\frac{x^2}{2} - e^{6y}\cdot\frac{x^3}{3} + 0 \cdot dy = c$

$\Rightarrow \frac{x^2y^2e^{6y}}{2} - e^{6y}\frac{x^3}{3} = c$

$\Rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) = c$

(3) $2xy\,dy - (x^2 + y^2 + 1)dx = 0$

A $-x + \frac{y^2}{x} + \frac{1}{x} = C$

④ $(x^2-y^2)dx = 2xy\,dy$

$(x^2-y^2)\,dx - 2xy\,dy = 0$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = -2y$

exact

$\int x^2-y^2\,dx - \int \cancel{2xy\,dy}^{\,0} = c$

$\frac{x^3}{3} - y^2x = c$

$\Rightarrow \quad x^3 - 3xy^2 = c$

⑤ $e^x \sin y\,dx + e^x \cos y\,dy = y \sin xy\,dx + x \sin xy\,dy$

$(e^x \sin y - y \sin xy)\,dx = (x \sin xy - e^x \cos y)\,dy$

$(e^x \sin y - y \sin xy)\,dx - (x \sin xy - e^x \cos y)\,dy = 0$

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = e^x \cos y - [y \cos xy\,(x) + \sin xy]$

$= e^x \cos y - xy \cos xy - \sin xy$

$\frac{\partial N}{\partial x} = -[x \cos xy\,(y) + \sin xy] + \cos y\, e^x$

exact

$\sin y \int e^x - y \int \sin xy\,dx = c$

$\sin y\, e^x + y\,\frac{\cos xy}{y} = c$

$\Rightarrow \quad e^x \sin y + \cos xy = c$

⑥ $y(2x^2y+e^x)dx = (e^x+y^3)dy$

$y(2x^2y+e^x)dx - (e^x+y^3)dy = 0$

$M = 2x^2y^2+e^xy \qquad N = -e^x-y^3$

$\frac{\partial M}{\partial y} = 2x^2(2y)+e^x \qquad \frac{\partial N}{\partial x} = -e^x$

$= 4x^2y+e^x$

non exact

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{-(e^x+y^3)}\left[4x^2y+e^x+e^x\right]$

$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{2x^2y^2+e^xy}\left[-e^x-4x^2y-e^x\right]$

$= \frac{1}{y(2x^2y+e^x)}\left[-2(e^x+2x^2y)\right]$

$= \frac{-2}{y}$

$\Rightarrow I.F = e^{\int g(y).d.} = e^{-2\int \frac{1}{y}dy} = \frac{1}{y^2}$

$\Rightarrow$ ① × I.F

$\Rightarrow \left(2x^2+\frac{e^x}{y}\right)dx - \left(\frac{e^x}{y^2}+\frac{1}{y}y\right)dy = 0$

$\frac{\partial M}{\partial y} = e^x\left(\frac{-1}{y^2}\right) \qquad \frac{\partial N}{\partial x} = -\frac{1}{y^2}e^x$

exact

$2\frac{x^3}{3}+\frac{1}{y}e^x - \cancel{\log y}\ y^2/2 = c$

$\Rightarrow \frac{2}{3}x^3+\frac{e^x}{y}-\frac{y^2}{2} = c$

(7) $(x^2 - ay)dx = (ax - y^2)dy$

$(x^2 - ay)dx - (ax - y^2)dy = 0$

$M dx + N dy = 0$

$\frac{\partial M}{\partial y} = -a \quad \frac{\partial N}{\partial x} = -a$

exact

$\int x^2 - ay \, dx + \int y^2 dy = c$

$\frac{x^3}{3} - ayx + \frac{y^3}{3} = C$

(8) $(y - x^2) dx + (x^2 \cot y - x) dy = 0$

$M dx + N dy = 0$

$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = \cot y (2x) - 1$

non exact

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y - x^2}\left[\cot y (2x) - 1 - 1\right]$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 \cot y - x}\left[+1 - \cot y (2x) + 1\right]$

$= \frac{2(1 - x\cot y)}{x(x\cot y - 1)} = -\frac{2}{x}$

$I.F = \frac{1}{x^2}$

$\left(\frac{y}{x^2} - 1\right)dx + \left(\cot y - \frac{1}{x}\right)dy = 0$

exact.

$y\left[\frac{x^{-1}}{-1}\right] - x + \log|\sin xy| = 0$

$\Rightarrow -\frac{y}{x} - x + \log|\sin xy| = 0$

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* Linear differential Equation of first order and first degree.

1. Def: A D.E is of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x is called a linear D.E in y

General solⁿ:-

⇒ write the given Eqⁿ in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{— (1)}$$

⇒ Calculating the integrating factor $e^{\int P.dx}$

⇒ General solⁿ of (1) is $(I.F)y = \int (I.F)Q.dx + C$

2. Def: A D.E is of the form $\frac{dx}{dy} + P(y)x = Q(y)$ is called Linear D.E in x

General solⁿ:

⇒ write the given Eqⁿ in the form

$$\frac{dx}{dy} + P(y)x = Q(y) \quad \text{— (1)}$$

⇒ $I.F = e^{\int P.dy}$

⇒ G. solⁿ of (1) is $(I.F)x = \int (I.F)Q\,dy + C$

Q) Solve $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$

$(x^2+1)\frac{dy}{dx} = \frac{1}{x^2+1} - 4xy$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(x^2+1)^2} - \frac{4xy}{(x^2+1)}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{4x}{x^2+1}\right)y = \frac{1}{(x^2+1)^2}$

$P(x) = \frac{4x}{x^2+1}$; $Q(x) = \frac{1}{(x^2+1)^2}$

$\Rightarrow I.F = e^{\int \frac{4x}{x^2+1} \cdot dx}$ $\quad 2\int \frac{2x}{x^2+1} dx$

$= (x^2+1)^2$ $\quad 2.\log(x^2+1)$

$(I.F)y = \int Q(x).I.F.dx$

$(x^2+1)^2 y = \int \frac{1}{(x^2+1)^2} \cdot (x^2+1)^2 \cdot dx$

$\Rightarrow (x^2+1)^2 y = x + C \longrightarrow$ G. soln of given D.E

Q) $x \cdot \frac{dy}{dx} + 2y - x^2 \log x = 0$

$\frac{dy}{dx} + \frac{2}{x}y = x\log x$

$\frac{dy}{dx} + P(x)y = Q(x)$

$e^{\int \frac{2}{x} \cdot dx} = e^{2.\log x} = x^2$

$x^2 y = \int x\log x \cdot x^2 \cdot dx$

ILATE

$\Rightarrow x^2 y = \int x^3 \log x \cdot dx$

$= \int \underset{u}{\log x} \cdot \underset{v}{x^3} \cdot dx$

$\log x \int x^3 dx - \int \frac{1}{x} \int x^3 dx$

$\Rightarrow \log x \cdot \frac{x^4}{4} - \int \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{4}3}}{4} \cdot dx$

$\log x \cdot \frac{x^4}{4} - \frac{1}{4} \cdot \frac{x^4}{4}$

$\Rightarrow \frac{x^{\cancel{4}2}}{4} (\log x - 1) = \cancel{x^2} y$

$\Rightarrow 4y = x^2 (\log x - 1)$

Ⓠ $x \cdot \cos x \cdot \frac{dy}{dx} + (x \sin x + \cos x) y = 1$

$\frac{dy}{dx} + (\tan x) y + \frac{1}{x} y = \frac{1}{x \cos x}$

$\Rightarrow \frac{dy}{dx} + y \left(\tan x + \frac{1}{x} \right) = \frac{1}{x \cos x}$

$e^{\int (\tan x + \frac{1}{x}) dx} = e^{\int \tan x + \int \frac{1}{x} dx}$

$= e^{-\log |\cos x| + \log x}$

$\Rightarrow \cancel{x \cos x}, \frac{x}{\cos x}$

$\cos^{-2} x$,

$\frac{x}{\cos x} y = \int \frac{\cancel{x}}{\cos x} \cdot \frac{1}{\cancel{x} \cos x} \cdot dx$

$= \int \frac{1}{\cos^2 x} \cdot dx = \int \frac{\sec x}{\cos x} \cdot dx$

$= \int \frac{\sec x \tan x}{\sin x} dx = \int \sec^2 x \, dx = \tan x + c$

$\Rightarrow x y \sec x = \tan x + c$

Ⓠ $x\log x \cdot \frac{dy}{dx} + y = 2\log x$

$$\frac{dy}{dx} + \frac{1}{x\log x} y = \frac{2}{x}$$

$\cancel{\log x = t}$

$\frac{1}{x} = dt$

$$e^{\int \frac{1}{x\log x} \cdot dx} \Rightarrow e^{\int \frac{dt}{t}}$$

$\log x = t$

$\frac{1}{x} dx = dt$

$$= t \Rightarrow \log x$$

$$(\log x) y = \int \log x \cdot \frac{2}{x} \cdot dx$$

$$y \log x = 2\int t\, dt$$

$$= 2 \cdot \frac{t^2}{2} = (\log x)^2 + C$$

$$y \log x = (\log x)^2 + C$$

Ⓠ solve $(1+y^2)dx = (\tan^{-1} y - x) dy$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1} y$$

$$\frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{\tan^{-1} y}{1+y^2}$$

Linear D.E in x

$$I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

G. soln $= (I.F)x = \int g(y) \cdot IF \cdot dy + C$

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{(1+y^2)} \cdot e^{\tan^{-1} y} \cdot dy + C$$

$\tan^{-1} y = t$

$\frac{1}{1+y^2} dy = dt$

$$= \int t \cdot e^t \cdot dt$$

$= (t-1)e^{t} + c$ $\quad te^{t} - \int e^{t}$

$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

Q) $(x+2y^3)\frac{dy}{dx} = y$

$\frac{dx}{dy}\cdot y = x + 2y^3$

$\frac{dx}{dy}\cdot(y) - x = 2y^3$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$

$P(y) = -\frac{1}{y}$: $Q(y) = 2y^2$

$e^{\int P dy} = e^{-\log y} = \frac{1}{y}$

$\frac{1}{y}x = \int 2y^{\cancel{2}}\left(\frac{1}{\cancel{y}}\right)\cdot dy = \frac{2y^2}{2} + c$

$\frac{x}{y} = y^2 + C$

Q) $dx + (2x\cot\theta + \sin 2\theta)d\theta = 0$

$\frac{dx}{d\theta} = -2x\cot\theta - \sin 2\theta$

$\frac{dx}{d\theta} + (2\cot\theta)x = -\sin 2\theta$

$I.F = e^{\int 2\cot\theta \cdot d\theta} = e^{2\int \cot\theta\cdot d\theta} = e^{2\log|\sin\theta|}$

$I.F = \sin^2\theta$

$\sin^2\theta = t$

$2\sin\theta\cos\theta \cdot d\theta = dt$

$\sin 2\theta \cdot d\theta = dt$

$\Rightarrow (\sin^2\theta)x = \int \sin^2\theta\cdot(-\sin 2\theta)d\theta$

$= -\int \sin^2\theta - \sin 2\theta . d\theta$

$= -\int t\,dt = -\frac{t^2}{2} = -\frac{\sin^4\theta}{2} + C$

$$\Rightarrow x\sin^2\theta + \frac{\sin^4\theta}{2} = C$$

(4) $$\frac{dy}{dx} - \frac{2y}{x} = \frac{5x^2}{(2+x)(3-2x)}$$

$$P(x) = -\frac{2}{x} \; ; \; Q(x) = \frac{5x^2}{(2+x)(3-2x)}$$

$$e^{\int P(x)dx} = e^{-2(\log x)} = \frac{1}{x^2}$$

$$\frac{1}{x^2} y = \int \frac{1}{x^2} \cdot \frac{5x^2}{(2+x)(3-2x)} \cdot dx$$

$$x^2 + 2\cdot\left(\frac{x}{4}\right) + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3$$

$$= 5\int \frac{1}{6-4x+3x-2x^2} dx$$

$$= -5\int \frac{1}{2x^2+x-6} dx = -\frac{5}{2}\int \frac{1}{\left(x^2+\frac{x}{2}-3\right)}$$

$$x^2 + \frac{1}{16} + \frac{x}{2} - \frac{1}{16}$$

$$= \frac{-5}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2 - \frac{1}{16} - 3}$$

$$-\frac{1}{16} - 3$$

$$-\left(\frac{1}{16}+3\right)$$

$$= -\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2 - \frac{49}{16}}$$

$$-\frac{49}{16}$$

$$= -\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2 - \left(\frac{7}{4}\right)^2} \cdot dx \qquad \frac{1}{x^2-a^2}$$

$$= -\frac{5}{2} \frac{1}{2\left(\frac{7}{4}\right)} \log\left|\frac{x+\frac{1}{4}-\frac{7}{4}}{x+\frac{1}{4}+\frac{7}{4}}\right|$$

$$\frac{y}{x^2} = \frac{-5}{7} \log\left|\frac{x-\frac{6}{4}}{x+\frac{8}{4}}\right| + C$$

$$y = \frac{5x^2}{7} \log\left|\frac{x+2}{x-\frac{3}{2}}\right| + C$$

(Q) $\cosh x \cdot \frac{dy}{dx} + y\sinh x = 2\cosh^2 x \sinh x$

$$\frac{dy}{dx} + \tanh x \cdot y = 2\sinh x \cosh x$$

$$e^{\int \tanh x \, dx} = \cosh x$$

$$(\cosh x)y = \int \cosh x \cdot (2\sinh x \cosh x)\,dx + C$$

$\cosh^2 x = t^2$

$\cancel{\sinh x\, dx = dt}$ $\frac{2t\,dt}{2t}$

$2\cosh x \sinh x\, dx = dt$

$$= \int t^2 (2t\,dt)$$

$$= 2\int t^2 \, dt = 2\frac{t^3}{3} + C$$

$$y\cosh x = \frac{2}{3}\cosh^3 x + C$$

* Bernoullis Equation :-

An Equation is of the form $\frac{dy}{dx} + Py = Qy^n$ — ①
is called Bernoullis Eqn if P and Q are consts or function of x alone and n is a real const

case-1 :- if $n = 1$ then ① can be written as

$n=0 \rightarrow$ linear

$n=1 \rightarrow$ variable seperable

$$\frac{dy}{dx} + (P-Q)y = 0 \quad \text{—— ②}$$

Here the variables are seperable.

General Solⁿ is $\int \frac{dy}{y} + \int (P-Q)\,dx = C$

case-2 :- If $n \neq 1$, multiply ① with $\frac{1}{y^n}$ we get

$$\frac{dy}{dx} \cdot y^{-n} + Py^{1-n} = Q \quad \text{—— ③}$$

Let $y^{1-n} = u$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

i.e $y^{-n}\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{du}{dx}$ —— ④

from ③ and ④ we get $\frac{1}{1-n}\frac{dy}{dx} + Pu = Q$

i.e $\frac{du}{dx} + (1-n)Pu = (1-n)Q \longrightarrow ⑤$

This is a linear eqn of 1st order in u and this can be solved as described earlier. At the end we substitute for u and get the req soln

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Q) Solve $x.\frac{dy}{dx} + y = x^3y^6$

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Given D.E is $x\frac{dy}{dx} + y = x^3y^6$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right)y = x^2y^6 \quad —①$$

↳ This is Bernoullis D.E

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\times\frac{1}{y^5} = x^2$$

$$\frac{1}{y^5} = u$$

$$\frac{du}{dx} = -5y^{-6}.\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{-5}{y^6}\frac{dy}{dx}$$

Substi in ①

$$\Rightarrow -\frac{1}{5}\frac{du}{dx} + \frac{1}{x}(u) = x^2$$

$$\Rightarrow \frac{du}{dx} - \frac{5}{x}u = -5x^2$$

This is a Linear D.E in u

$$I.F = e^{\int P dx}$$

$$= e^{\int -\frac{5}{x} dx} = \frac{1}{x^5}$$

$$u(I.F) = \int Q . I.F . dx$$

$$u . \left(\frac{1}{x^5}\right) = \int -5 \cancel{x^2} \times \frac{1}{\cancel{x^5}_{3}} . dx + C$$

$$\frac{1}{(xy)^5} = -5 \int \frac{1}{x^3} dx + C$$

$$= -5 \left(\frac{x^{-2}}{-2}\right) + C \Rightarrow \frac{5}{2x^2} + C$$

$$\Rightarrow \frac{1}{(xy)^5} = \frac{5}{2x^2} + C \longrightarrow \text{General sol}^n \text{ of given D.E}$$

(Q) $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$

$$\frac{1}{y^2} . \frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x \sin x$$

$$\frac{1}{y} = u \Rightarrow -1 y^{-2} \frac{dy}{dx} = \frac{du}{dx}$$

$$-\frac{du}{dx} + \frac{1}{x} u = x \sin x$$

$$\frac{du}{dx} - \frac{1}{x} u = -x \sin x$$

$$I.F = e^{\int -\frac{1}{x} dx} = \frac{1}{x}$$

$$\frac{1}{x} u = -\int \frac{1}{\cancel{x}} . \cancel{x} \sin x \, dx + C$$

$$\frac{1}{xy} = + \cos x + C$$

(2) $\frac{dy}{dx}(x^2y^3+xy)=1$

$\Rightarrow \frac{dx}{dy}-xy=x^2y^3$

$\frac{dx}{dy}+P(y)x=Q(y)x^n$

$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}xy=y^3$

$-\frac{1}{x}=u \Rightarrow -\left[-1x^{-2}\frac{dx}{dy}\right]=\frac{du}{dy}$

$\Rightarrow \frac{du}{dy}+u(y)=y^3$

$I.F=e^{\int y\,dy}=e^{\frac{y^2}{2}}$

$y^3=t$
$3y^2=dt$

$e^{y^2/2}u=\int e^{y^2/2}y^3.dy+C$ I LATE

$e^{y^2/2}\left(-\frac{1}{x}\right)=y^3\int e^{y^2/2}dy-\int 3y^2\int e^{y^2/2}.dy$

$=y^3.\frac{e^{y^2/2}}{2y/2}-3\int y^2\times\frac{e^{y^2/2}}{y}$

$=y^2.y^2\left[e^{y^2/2}\right]-3\left[y.\frac{e^{y^2/2}}{y}-\int\frac{e^{y^2/2}}{y}.dy\right]$

$e^{y^2/2}\left(\frac{1}{x}\right)=(2-y^2)e^{y^2/2}+C$

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H.W

(Q) $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$

$\Rightarrow (x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2)$

$\frac{x}{-(1+y^2)} + \frac{e^{\tan^{-1}y}}{(1+y^2)} = \frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{e^{\tan^{-1}y}}{1+y^2}$

$I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$

$e^{\tan^{-1}y} . x = \int e^{\tan^{-1}y} . \frac{e^{\tan^{-1}y}}{1+y^2} dy = \int \frac{e^{2\tan^{-1}y}}{1+y^2} . dy$

$\tan^{-1}y = t$

$\frac{1}{1+y^2} dy = dt$

$\Rightarrow \int \frac{e^{2t}}{1} dt = \frac{e^{2t}}{2} + C$

$e^{\tan^{-1}y} , x = \frac{e^{2\tan^{-1}y}}{2} + C$

$\Rightarrow$

$x . e^{-y} = -e^{-y}(y+2) + C$

(Q) $(x+y+1)\frac{dy}{dx} = 1$ Sol^n :-

$\Rightarrow \frac{dx}{dy} = x + y + 1 \Rightarrow \frac{dx}{dy} - (1)x = y + 1$

$e^{\int -1 dy} = e^{-y}$

$e^{-y}(x) = \int e^{-y} . (y+1) dy = (y+1)\frac{e^{-y}}{-1} + \int \cancel{-e^{-y}} . e^{-y} dy$

$= -(y+1)e^{-y} - e^{-y} = -e^{-y}(y+2) + C$

Q solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

$$e^{\int \frac{1}{1+y^2}\cdot dy} = e^{\tan^{-1}y}$$

$$e^{\tan^{-1}y}\cdot x = \int e^{\tan^{-1}y}\cdot \frac{\tan^{-1}y}{1+y^2}\cdot dy$$

$$= \int e^t \cdot t\, dt \qquad t = \tan^{-1}y$$

$$= te^t - \int e^t dt$$

$$= e^t(t-1) = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

$$\Rightarrow x = \tan^{-1}y - 1 + C$$

$$\Rightarrow x = \tan^{-1}y + C$$

$y' + y = e^{e^x}$

Q $\frac{dy}{dx} + y = e^{e^x}$

$$\frac{dy}{dx} + (1)y = e^{e^x}$$

$e^x = t$

$e^x dx = dt$

$$e^{\int 1 dx} = e^x$$

$$e^x \cdot y = \int e^x \cdot e^{e^x}\cdot dx = \int e^{x+e^x}\cdot dx$$

$$= \int e^t \cdot dt = e^t \Rightarrow e^{e^x} + C$$

Sol^n :- $ye^x = e^{e^x} + C$ //

Q) Find the equation of the curve satisfying the D.E $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$ and passing through origin.

Sol: $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$$e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1+x^2$$

$$(1+x^2)y = \int \cancel{(1+x^2)} \frac{4x^2}{\cancel{(1+x^2)}} dx$$

$$= 4 \cdot \frac{x^3}{3} + C$$

$$\Rightarrow y(1+x^2) = \frac{4}{3}x^3 + C$$

Q) $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$

$$-\frac{dr}{d\theta} = 2r\cot\theta + \sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta$$

$$e^{2\int \cot\theta \cdot d\theta} = e^{2\log|\sin\theta|} = \sin^2\theta$$

$\sin^2\theta = t$

$2\sin\theta\cos\theta \, d\theta = dt$

$$(\sin^2\theta)r = -\int \sin^2\theta \cdot \sin 2\theta \cdot d\theta$$

$$-\int t\,dt = -\frac{t^2}{2} + C = -\frac{\sin^4\theta}{2} + C$$

Solⁿ: $r\sin^2\theta = -\frac{\sin^4\theta}{2} + C$

(Q) $\frac{dy}{dx} - \frac{2y}{x} = \frac{5x^2}{(2+x)(3-2x)}$

$$\frac{dy}{dx} + \left(\frac{-2}{x}\right)y = \frac{5x^2}{(2+x)(3-2x)}$$

$$e^{-2\int \frac{1}{x}dx} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{1}{x^2}y = \int \frac{1}{\cancel{x^2}} \cdot \frac{5\cancel{x^2}}{(2+x)(3-2x)} \cdot dx$$

$$\frac{x}{2} = 2x\left(\frac{1}{4}\right)$$

$$= 5\int \frac{1}{6-4x+3x-2x^2} dx$$

$$= -5\int \frac{1}{2x^2+x-6} dx \qquad x^2 + 2\left(\frac{x}{4}\right) \quad \frac{x}{2}$$

$$= -\frac{5}{2}\int \frac{1}{x^2+\frac{x}{2}-3} dx$$

$$= -\frac{5}{2}\int \frac{1}{x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2-3} dx \qquad \frac{1}{16}+3$$

$$= -\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{7}{4}\right)^2} dx \qquad \frac{49}{16}$$

$$= -\frac{5}{2} \cdot \frac{1}{2\left(\frac{7}{4}\right)} \log\left|\frac{x+\frac{1}{4}-\frac{7}{4}}{x+\frac{1}{4}+\frac{7}{4}}\right|$$

$$\frac{y}{x^2} = -\frac{5}{7}\log\left|\frac{x-\frac{3}{2}}{x+2}\right| + c$$

$$y = +\frac{5x^2}{7}\log\left|\frac{x+2}{x-\frac{3}{2}}\right| + C$$

(Q) $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x} \cdot y = \frac{\sin 2x}{\log x}$ $\qquad \log x = t$

$\frac{1}{x} dx = dt$

$\Rightarrow e^{\int \frac{1}{x\log x} dx} = e^{\int \frac{1}{t} dt} = t = \log x$

$\Rightarrow (\log x) y = \int (\log x) \frac{\sin 2x}{\log x} dx + c$

$y(\log x) = -\frac{\cos 2x}{2} + c$

B
(Q) Solve $3\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$.

$\Rightarrow \frac{dy}{dx} - \left(\frac{y}{3}\right) co$

$\frac{dy}{dx} + \left(-\frac{\cos x}{3}\right) y = \left(\frac{\sin 2x - \cos x}{3}\right) y^4$

$\frac{1}{y^4} \cdot \frac{dy}{dx} + \left(-\frac{\cos x}{3}\right)\frac{1}{y^3} = \frac{\sin 2x - \cos x}{3}$

$\frac{1}{y^3} = u \Rightarrow -3y^{-4} \cdot \frac{dy}{dx} = \frac{du}{dx} = -\frac{1}{3}\frac{du}{dx}$

$\Rightarrow -\frac{1}{3}\frac{du}{dx} - \frac{\cos x}{3} u = \frac{\sin 2x - \cos x}{3}$

$\frac{du}{dx} + (\cos x) u = \cos x - \sin 2x$

$e^{\int \cos x \, dx} = e^{\sin x}$

$\Rightarrow e^{\sin x} \cdot \frac{1}{y^3} = \int e^{\sin x} (\cos x - \sin 2x) dx + c$

$$e^{\sin x} \cdot \frac{1}{y^3} = \int e^{\sin x} (\cos x - 2\sin x \cos x)\, dx + C$$

$$= \int e^{\sin x} \cos x (1 - 2\sin x)\, dx + C$$

$\sin x = t$

$\cos x\, dx = dt$

$$= \int e^t (1-2t)\, dt + C$$

$$= \int e^t . dt - 2\int e^t . t\, dt$$

$$= e^t - 2\left[t . e^t - \int e^t . dt\right]$$

$$= e^t - 2\left[t e^t - e^t\right]$$

$$= e^t - 2e^t[t-1] = e^t[1 - 2t + 2]$$

$$= e^t(3-2t) + C$$

$$\Rightarrow e^{\sin x} \cdot \frac{1}{y^3} = e^{\sin x}(3 - 2\sin x) + C$$

(2) $\frac{dy}{dx} + yx = y^2 . e^{x^2/2} \sin x$

$$\frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{y} x = e^{x^2/2} \sin x$$

$$\frac{1}{y} = v \Rightarrow -y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow -\frac{dv}{dx} + x(v) = e^{x^2/2} \sin x$$

$$\Rightarrow \frac{dv}{dx} - x(v) = -e^{x^2/2} \sin x$$

$$e^{\int -x\,dx} = \cancel{\tfrac{1}{x}} \quad e^{-\int x\,dx} = e^{-\frac{x^2}{2}}$$

$$\cancel{\tfrac{1}{x}}\,\cancel{(v)} = \int \cancel{\tfrac{1}{x}}\, e^{x^2/2} \sin x\,dx$$

$$\cancel{\tfrac{1}{xy}} = \quad e^{-\frac{x^2}{2}} \cdot \frac{1}{y} = -\int e^{-\frac{x^2}{2}} \cdot e^{x^2/2} \sin x\,dx$$

$$\Rightarrow e^{-x^2/2} \cdot \frac{1}{y} = +\cos x + c$$

$$\Rightarrow e^{-x^2/y} \times \frac{1}{y} = \cos x + c$$

Q solve $\dfrac{dy}{dx} - y\tan x = \dfrac{\sin x \cos^2 x}{y^2}$

$$y^2 \frac{dy}{dx} - y^3 \cdot \tan x = \sin x \cos^2 x$$

$$-y^3 = v \Rightarrow -3y^2 \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow -\frac{1}{3}\frac{dv}{dx} + (\tan x)v = \sin x \cos^2 x$$

$$\Rightarrow \frac{dv}{dx} - 3\tan x\, v = -3\sin x \cos^2 x$$

$$e^{\int -3\tan x\,dx} = e^{-3\int \tan x\,dx} = e^{+3\log|\cos x|}$$

$$= \cos^3 x$$

$$\cos^3 x \cdot (-y^3) = -3\int \cos^3 x \cdot \sin x \cdot \cos^2 x \cdot dx$$

$\cos x = t$

$-\sin x\,dx = dt$

$$= +3\int t^5\,dt = -\frac{t^6}{6} + c$$

$$\Rightarrow y^3 \cos^3 x = -\frac{\cos^6 x}{2} + c$$

23/5/22

* Newtons Law of cooling :-

Statement : The rate of change of temperature of a body is proportional to the difference of the temp of body and that of the surrounding medium.

Let θ be the temp of the body at time t and θ_0 be the temp of the surrounding medium (usually air). By the newton's Law of cooling, we have $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

i.e $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\therefore \frac{d\theta}{\theta - \theta_0} = -k\,dt$ (variable seperable)

Integ on b/s

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$$\log(\theta - \theta_0) = -kt + C \quad \text{—— (1)}$$

if initially $\theta = \theta_1$ is temp of body at $t=0$

then (1) $\Rightarrow$ $c = \log(\theta_1 - \theta_0)$

Sub (2) in (1) we get

$$\log(\theta - \theta_0) = -kt + \log(\theta_1 - \theta_0)$$

$$\log\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = -kt$$

$$\Rightarrow \frac{\theta-\theta_0}{\theta_1-\theta_0} = e^{-kt}$$

$$\theta-\theta_0 = (\theta_1-\theta_0)e^{-kt}$$

$$\therefore \theta = \theta_0 + (\theta_1-\theta_0)e^{-kt}$$

which gives the temperature of the body at time t

* working rule :-

1. write given θ_0, θ_1, t_0, t.
2. at ~~t=0~~ $t=t_0$ we get θ_0 C
3. find K value
4. find the unknown θ at t
5.

Q) A body is originally at $80°c$ and cool down to $60°c$ in 20min. If the temp of air is $40°c$ find the temp of body after 40 min

Sol: ~~$\theta_0 = 80°$~~, ~~$\theta_1 = 60°c$~~ ~~$t_1 = 20$min~~

$\theta = 80°c$ when $t = 0$ min
$\downarrow$
$\theta = 60°c$ $t = 20$ min

sur temp $= 40°c$

~~$C = \log(\theta_1 - \theta_0) = \log(80 -$~~

By newton's Law of cooling

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\theta = \theta_0 + e^{-kt}(c) \quad \text{—— (1)}$$

Step-1: find c value

$\theta_0 = 40°c \qquad \theta = 80°c$

$$80 = 40 + c e^{-k(0)}$$

$$40 = c$$

sub in (1)

$$60 = 40 + c e^{-k(20)}$$

$$20 = 40 e^{-k(20)}$$

$$\frac{1}{2} = e^{-20k}$$

$$-20k = \log \frac{1}{2}$$

$$k = \frac{1}{20} \log 2$$

$$\theta = \theta_0 + c e^{-kt}$$

$$= 40 + 40 e^{-\frac{40}{20}\log 2}$$

$$= 40 + 40\left(\frac{1}{4}\right)$$

$$\boxed{\theta = 50°c}$$

(Q) An obj whose temp is 75°c cools in an atm of const temp 25°c at the rate of $k\theta$. θ being the excess temp of body over that of the temp. If after 10 min, the temp of the obj falls to 65°c. Find its temp after 20 min. Also Find the time required to cool to 55°c at $t=0$ $\quad \theta = 75°c$

$t=10$ $\quad \theta - 65°c$ $\quad \theta_0 = 25°c$

1) $t=20$ $\quad \theta = ?$

2) $t=?$ $\quad \theta = 55°c$

Given data $\quad t=0 \quad \theta = 75°c \quad \theta_0 = 25°c$

By N.L.C

$$\boxed{\theta = \theta_0 + c e^{-kt}} \longrightarrow (1)$$

Step-1 :- Find c

$$75 = 25 + c\, e^{-k(0)}$$

$$\Rightarrow c = 50$$

sub in (1)

$$65 = 25 + 50\, e^{-k(10)}$$

$$\frac{40}{50} = e^{-10k} \Rightarrow \log\frac{5}{4} = 10k$$

$$k = \frac{1}{10}\log\frac{5}{4}$$

$t=20$; $\theta = ?$ $\quad c=50$; $\quad k = -\frac{1}{10}\log\left(\frac{4}{5}\right)$

(1) $\theta = 25 + 50\, e^{-\frac{1}{10}\log\frac{5}{4}\times 20^2}$

$$= 25 + 50\, e^{-2\log 5/4}$$

$$= 25 + 50\left(\frac{4}{5}\right)^2 = 25 + \cancel{50}^2\left(\frac{16}{25}\right)$$

$$= 25 + 32 = 57°c$$

(2) $\theta = 55°$; $t = ?$

$$55 = 25 + 50\, e^{-t\times\frac{1}{10}\log\frac{5}{4}}$$

$$30 = 50\, e^{\log\left(\frac{4}{5}\right)^{t/10}}$$

$$\frac{3}{5} = \left(\frac{4}{5}\right)^{t/10} \Rightarrow \log\frac{3}{5} = \frac{t}{10}\log\frac{4}{5}$$

$$\frac{t}{10} = \frac{\log(3/5)}{\log(4/5)} \Rightarrow t = 22.9\,\text{min}$$

24/5/22

* Law of Natural Growth and Decay

Let $x(t)$ be amount of a substance at time t and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount of $x(t)$ of chemically changing substance is proportional to the amount of substance at that time

i.e. $\frac{dx}{dt} \propto x$ +ve $\rightarrow$ growth

-ve $\rightarrow$ decay

$$\frac{dx}{dt} = \pm kx$$

where k is constant of proportionality. This D.E can also described in a simple way. The population growth radio active decay.

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\,dt$$

$$\Rightarrow \int \frac{dx}{dx} = \int k\,dt + c$$

$$\Rightarrow \log x = kt + \log c$$

$$\log \frac{x}{c} = kt$$

$$\Rightarrow \frac{x}{c} = e^{kt}$$

$$\boxed{x = c.e^{kt}}$$

decay :-

$$\frac{dx}{dt} = -kx$$

$$\frac{dx}{x} = -k\,dt$$

$$\log\left(\frac{x}{c}\right) = -kt$$

$$\frac{x}{c} = e^{-kt}$$

$$\boxed{x = c.e^{-kt}}$$

(Q) A bacteria culture, growing exponentially increases from 100 to 400 grams in 10 hours. How much was present after 3 hours.

Sol:- at $t=0$ hours $\quad N=100$

$t=10$ hours $\quad N=400$

$t=3$ hours $\quad N=?$

By the Law of natural growth

$$N = Ce^{kt} \longrightarrow (1)$$

(i) $t=0$; $N=100$

$$100 = Ce^{k(0)}$$

$$C=100$$

(ii) $t=10$, $N=400$, $C=100$

$$400 = 100\,e^{k(10)}$$

$$4 = e^{10k} \Rightarrow \log 4 = 10k$$

$$k = \frac{1}{10}\log 4$$

(iii) $t=3$; $C=100$, $k=\frac{1}{10}\log 4$; $N=?$

$$N = 100\,e^{\frac{3}{10}\log 4}$$

$$= 100\,e^{\log(4)^{3/10}}$$

$$= 100 \times 4^{3/10}$$

$$N = 151.$$

(Q) The number N of bacteria in a cultural grow at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1½ hours?

Sol: Given. at $t=0$ $N=100$

at $t=1$ $N=332$

at $t=1\frac{1}{2}$ $N=?$

$=3/2$

By Law of natural growth.

$$N = c e^{kt} \longrightarrow (1)$$

(i) $t=0$; $N=100$

$$100 = c e^{k(0)} \Rightarrow c=100$$

(ii) $t = 1hr = 60\,min$ $N=332$, $c=100$

$$332 = 100 e^{60k}$$

$$k = \frac{1}{60}\log\left(\frac{332}{100}\right)$$

(iii) $t=90$; $c=100$; $k = \frac{1}{60}\log\left(\frac{332}{100}\right)$ $N=?$

$$N = ce^{kt}$$

$$N = 100 \cdot e^{\frac{90}{60}\log\left(\frac{332}{100}\right)}$$

$$= 100 \times \left(\frac{332}{100}\right)^{3/2}$$

$$N = 605$$

26/5/22

* L-R CIRCUITS :-

Basic Relations :-

* $i = \frac{dq}{dt}$ or $q = \int i\,dt$

→ voltage drop across resistance $R = Ri$

→ Voltage drop across inductance $L = L \cdot \frac{di}{dt}$

→ voltage drop across capacitance $C = \frac{q}{V}$

$V = \frac{q}{C}$

Kirchoff's voltage Law :-

The algebraic sum of the voltage drops in any closed circuit is equal to the resultant electromolive force in the circuit.

Differential Eqns of Electric Circuits.

There are two types of electric circuits

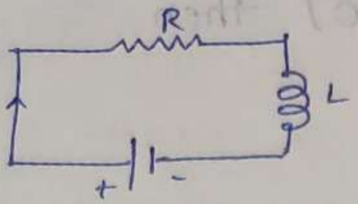
1. R-L-E circuits
2. R-C-E circuits

[with the help of above KVL we can formulate D.E of given electric circuit]

* Resistance (R)
* Inductance (L)
* Capacitence (C)
* EMF (E) or Battery

R-L circuit :-

Consider a circuit containing Resistance "R" and inductance L in series with a voltage source "E" as shown in diagram.



* Let 'i' be the current flowing in the circuit at any time t. Then we know

voltage drop across resistance $R = Ri$

voltage drop across inductance $L = L.\frac{di}{dt}$

applied voltage is E

$\Rightarrow$ By Kirchoff's Law

$$Ri + L.\frac{di}{dt} = E$$

$$\Rightarrow L.\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \left(\frac{R}{L}\right) i = \frac{E}{L}$$

which is D.E of R-L circuit

[and it is linear equation in i w.r.t dt]

$$I.F = e^{\int P dt} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

General Soln :-

where $P = \frac{R}{L}$; $Q = \frac{E}{L}$

$$i \times I.F = \int Q.\,I.F.\,dt + C$$

$$i \times e^{\frac{Rt}{L}} = \int \frac{E}{L}.e^{\frac{Rt}{L}}.dt + C$$

$$i.e^{\frac{Rt}{L}} = \frac{E}{L}.\frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + C$$

$$\Rightarrow i.e^{\frac{Rt}{L}} = \frac{E}{\cancel{L}}.\frac{L}{R}\,e^{\frac{Rt}{L}} + C$$

$$\boxed{i.e^{\frac{Rt}{L}} = \frac{E}{R}\,e^{\frac{Rt}{L}} + C}$$ $\rightarrow$ final soln of R-L circuit

Find i initially :- $[i=0, t=0]$ then

$$0 = \frac{E}{R} e^{0} + c$$

$$\boxed{c = -\frac{E}{R}}$$

Substitute c value in

$$i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + c$$

$$\Rightarrow i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} - \frac{E}{R}$$

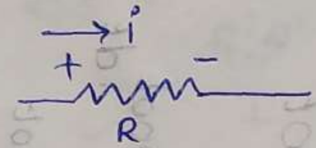
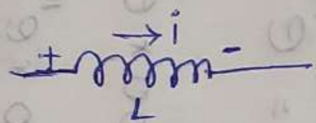
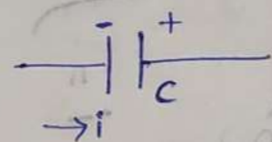
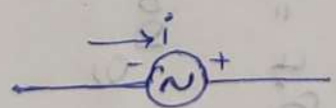
$$\Rightarrow i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} \left[e^{\frac{Rt}{L}} - 1 \right]$$

divide by $e^{\frac{Rt}{L}}$

$$\Rightarrow i = \frac{E}{R} - \frac{E}{R} e^{-\frac{Rt}{L}}$$

$$\Rightarrow \boxed{i = \frac{E}{R} \left[1 - e^{-\frac{Rt}{L}} \right]}$$

* Symbols *

Element	Symbol	unit
Quantity of electricity	q	coulumb
Current (= time rate flow of electricity)	i	ampere (A)
Resistance, R		ohm (Ω)
Inductance, L		henry (H)
Capacitance, C		Farad (F)
Electromotive Force or voltage (E)	 E = variable	volt (V)

Q. A Resistance of 100 Ω and an inductance of 0.1H are connected in series with battery of 20V. Find i in circuit at any time t.

Sol: Given. $L \cdot \frac{di}{dt} + Ri = E$

$R = 100\,\Omega$ & $L = 0.1\,H$, $E = 20V$

$$0.1 \times \frac{di}{dt} + 100\,i = 20$$

$$\Rightarrow \frac{di}{dt} + \left(\frac{100}{0.1}\right) i = \frac{20}{0.1}$$

$$\Rightarrow \frac{di}{dt} + 1000\,i = 200$$

It is of the form $\frac{di}{dt} + Pi = Q$

$$P = 1000 \,;\, Q = 200$$

$$I.F = e^{\int 1000\,dt} = e^{1000t}$$

$$\Rightarrow e^{1000t} \times i = \int 200 \times e^{1000t}\,dt + C$$

$$= 200 \times \frac{e^{1000t}}{1000} + C$$

$$i \times e^{1000t} = \frac{e^{1000t}}{5} + C$$

$$\Rightarrow i = \frac{1}{5} + \frac{C}{e^{1000t}}$$

For $C \Rightarrow t = 0 \,;\, i = 0$

$$0 = 0.2 + C \cdot e^{-1000(0)} \Rightarrow C = -0.2$$

$$i = 0.2 + C \cdot e^{-1000t}$$

$$\boxed{i = 0.2\left[1 - e^{-1000t}\right]}$$

(c) 2 9-volt batteries are connected to a series in which the Inductance is $\frac{1}{4}H$ and resistance of 8Ω. Determine the current $f(t)$, if the initial current is zero

Sol: $L.\frac{di}{dt} + Ri = E$

$$\Rightarrow \frac{1}{4}\cdot\frac{di}{dt} + 8i = 18$$

$$\Rightarrow \frac{di}{dt} + 32i = 72$$

$$I.F = e^{32t}$$

$$G.\ sol^n = e^{32t} \times i = \int e^{32t}\cdot 72.\,dt$$

$$= 72 \times \frac{e^{32t}}{32}$$

$$\Rightarrow i \times e^{32t} = \frac{72}{32}e^{32t} + c$$

$$\Rightarrow i = \frac{9}{4} + ce^{-32t}$$

$i=0$ for $t=0$

$$\Rightarrow c = -\frac{9}{4}$$

$$\Rightarrow i = \frac{9}{4}\left[1 - e^{-32t}\right]$$

As $t \to \infty$ $i(t) = \frac{9}{4}$ because as $t \to \infty$ $-\frac{9}{4}e^{-32t} \to 0$

$\therefore -\frac{9}{4}e^{-32t}$ is called transient term

$\frac{9}{4}$ is called steady term.

$\frac{9}{4}$ amp is steady state current.

Q) An inductance of L henry and a resistance $10\,\Omega$ are connected in series with EMF of 100V. If i is initially zero, and is equal to 9 amp after 1 second. Find L and find current after 0.5s

Sol: $L \cdot \frac{di}{dt} + Ri = E$

$R = 10 ; E = 100$

$L \cdot \frac{di}{dt} + 10i = 100$

$\Rightarrow \frac{di}{dt} + \left(\frac{10}{L}\right) i = \frac{100}{L}$

$I.F = e^{\int \frac{10}{L} dt} = e^{\frac{10}{L}t}$

G.sol :- $i \times e^{\frac{10}{L}t} = \int e^{\frac{10}{L}t} \times \frac{100}{L} \cdot dt + c$

$= \frac{100}{L} \cdot \frac{e^{\frac{10}{L}t}}{\frac{10}{L}} + c$

$i \cdot e^{\frac{10}{L}t} = 10\, e^{\frac{10}{L}t} + c$

$\Rightarrow i = 10 + c\, e^{-\frac{10}{L}t}$

$i = 0 ; t = 0$

$c = -10$

$\Rightarrow i = 10\left[1 - e^{-\frac{10}{L}t}\right]$

$i = 9 \quad t = 1$

$9 = 10\left[1 - e^{-\frac{10}{L}(1)}\right]$

$\Rightarrow 9 = 10 - 10\, e^{-\frac{10}{L}}$

$\Rightarrow 9e^{\frac{10}{L}} = 10e^{\frac{10}{L}} - 10$

$\Rightarrow e^{\frac{10}{L}} = 10$

$\Rightarrow \log e^{\frac{10}{L}} = \log 10$

$\Rightarrow \frac{10}{L} = \log 10$

$\Rightarrow L = \frac{10}{\log 10} = 4.34$ Henry

$\Rightarrow i = 10 + c.\, e^{-\frac{10}{L}t}$

$L = 4.34\ ; \quad t = 0.5$

$i = 10 - 10 e^{-\frac{10}{4.34}(0.5)}$

$i = 6.84$ amperes

$i = 6.84A$

(Q) A const EMF E volts is applied to a circuit containing a constant R ohms in series and constant inductance L henries. If the initial current is zero, show that current builds upto half its theoretical maximum in $\frac{L \log 2}{R}$ secs.

Sol: $\frac{di}{dt} + \left(\frac{R}{L}\right) i = \frac{E}{L}$

G. soln:- $i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + c \longrightarrow (1)$

$i = 0,\ t = 0$ then $c = -\frac{E}{R}$

$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \longrightarrow (2)$

Current in circuit will be max when t takes takes large values i.e current will be max at $t \to \infty$

$$\Rightarrow i_{max} = \frac{E}{R}\left(1-e^{-\infty}\right) = \frac{E}{R}(1-0)$$

$$i_{max} = \frac{E}{R}$$

let $t=T$ be time at which $i = \frac{1}{2}max$

Eqn (2) becomes

$$\frac{1}{2} i_{max} = \frac{E}{R}\left(1-e^{-\frac{RT}{L}}\right)$$

$$\frac{1}{2} \times \frac{E}{R} = \frac{E}{R}\left(1-e^{-\frac{RT}{L}}\right)$$

$$e^{-\frac{RT}{L}} = 1-\frac{1}{2} = \frac{1}{2}$$

$$\frac{-RT}{L} = \log\left(\frac{1}{2}\right)$$

$$= \log 1 - \log 2$$

$$\Rightarrow \frac{RT}{L} = \log 2$$

$$\Rightarrow T = \frac{L}{R}\log 2 \text{ sec}$$

(Q) In a series RL circuit $L = 4H$; $R = 100\,\Omega$ and $E = 200V$. Find the values of current as a function of time. Assume that the initial current is zero. Find the current with $t = 2$ sec.

$$L.\frac{di}{dt} + iR = E$$

$L = 4$
$R = 100$
$E = 200$

$$\Rightarrow 4 \cdot \frac{di}{dt} + 100i = 200$$

$$\frac{di}{dt} + 25i = 50$$

$$I.F = e^{\int P.dt} = e^{25t}$$

$$i \times e^{25t} = \int 50 e^{25t} \cdot dt + c$$

$$\Rightarrow i \times e^{25t} = 2 \cdot e^{25t} + C$$

$$i = 2 + C \cdot e^{-25t}$$

$i = 0$, $t = 0$ then

$$0 = 2 + c e^{-25(0)} \Rightarrow C = -2$$

$$i \cdot e^{25t} = 2 \cdot e^{25t} - 2$$

$$i = 2\left[1 - e^{-25t}\right]$$

$t = 2sec$ then $i = 2\left[1 - e^{-25(2)}\right]$

$$= 2\left[1 - e^{-50}\right]$$

$$i = 2 \text{ Amp}$$

(Q) An inductance of 1 Henry and a resistance of a $2\,\Omega$ are connected in a Series with a constant emf E volts. If the current is initially zero and is 10A after 5 sec find E.

Sol:- $L \cdot \frac{di}{dt} + iR = E$

$$1 \cdot \frac{di}{dt} + 2i = E \Rightarrow \frac{di}{dt} = E - 2i$$

variable seperable

$$\frac{di}{E-2i} = dt \Rightarrow \int \frac{di}{E-2i} = \int dt$$

$t=0 ; i=0$
$t=5 \quad i=10$

$$\int_0^{10} \frac{1}{E-2i}\, di = \int_0^5 dt$$

t limits are $0 \to 5$
i limits are $0 \to 10$

$U = E - 2i$

$du = -2\,di$

$-\frac{1}{2} du = di$

$$\Rightarrow \int_0^{10} \frac{-\frac{1}{2} du}{U} = \int_0^5 dt = [t]_0^5$$

$$\Rightarrow -\frac{1}{2}\left[\log(E-2i)\right]_0^{10} = 5$$

$$\Rightarrow -\frac{1}{2}\left[\log(E-20) - \log E\right] = 5$$

$$\Rightarrow \log\left(\frac{E-20}{E}\right) = -10$$

$$\Rightarrow \frac{E-20}{E} = e^{-10} \Rightarrow E - Ee^{-10} = 20$$

$$\Rightarrow E(1-e^{-10}) = 20$$

$$\Rightarrow E = \frac{20}{1-e^{-10}} = 20V$$

$$\boxed{\therefore E = 20V}$$