# UNIT-III

## Matrix decomposition and least Squares of algebraic systems.

### LU Decomposition method

This method is based on the fact that a sq. matrix A can be factorized into the form LU where L is the lower traingular matrix. U is the upper triangular matrix;

Note: To write the matrix A as LU the given matrix should satisfy the following condition

1) The matrix A is a sq. matrix
2) The leading principle minors of all orders are non-zero.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

leading principle minor order $1 = |1| = 1 \neq 0$

leading principle minor of order $2 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \neq 0$

leading principle minor of order $3 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 = 0$

$\therefore$ LU decomposition is not possible for the given matrix A.

By using LU decomposition we can find the solution of system of linear equations.

The matrix form of the given system of Linear equation is

$AX = B$ — (1)

$A = LU$ — (2)

Put eq (2) in eq (1)

$LUX = B$ — (3)

let $UX = Y$ — (4)

$LY = B$ — (5)

first solve $LY = B$

and then $UX = B$

Note: In LU decomposition method there are 2 methods

i) do little method ~~in this method we consider A =~~

ii) crout's method

i) do little method:

In this methode we consider $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

ii) crouts method:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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Q) Solve the eqn $2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

by Doolittle's method (LU decomposition)

The matrix form of given system of linear equations is $AX = B$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

leading Principal Minor of order $1 = |2| = 2 (\neq 0)$

leading Principal Minor of order $2 = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 (\neq 0)$

leading principal Minor of order $3 = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 18 (\neq 0)$

All leading Principal minor's of A are non-zero. So, we can write matrix A as LU.

$A = LU$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$\boxed{u_{11}=2}$ $\boxed{u_{12}=3}$ $\boxed{u_{13}=1}$

$l_{21}u_{11}=1$ $\quad l_{21}u_{12}+u_{22}=2$ $\quad l_{21}u_{13}+u_{23}=3$

$\boxed{l_{21}=\frac{1}{2}}$ $\quad \frac{1}{2}(3)+u_{22}=2$ $\quad \frac{1}{2}+u_{23}=3$

$\boxed{u_{22}=\frac{1}{2}}$ $\quad \boxed{u_{23}=\frac{5}{2}}$

$l_{31}u_{11}+l_{32}=3$ $\quad l_{31}u_{12}+l_{32}u_{22}=1$ $\quad l_{31}u_{13}+l_{32}u_{23}+u_{33}=2$

$l_{31}(2)+0=3$ $\quad \frac{3}{2}(3)+l_{32}(\frac{1}{2})=1$ $\quad \frac{3}{2}(1)+(-7)(\frac{5}{2})+u_{33}=2$

$\boxed{l_{31}=\frac{3}{2}}$ $\quad l_{32}=2-9$ $\quad u_{33}=2+\frac{35}{2}-\frac{3}{2}$

$\boxed{l_{32}=-7}$ $\quad \boxed{u_{33}=18}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{pmatrix}$$

$AX = B$ — ①

$A = LU$ — ②

$LUX = B$ — ③

let $UX = Y$ ④ then $LY = B$ ⑤

$LY = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 3/2 & -7 & 1 \end{pmatrix}_{3\times3} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3\times1} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} y_1 \\ \frac{y_1}{2}+y_2 \\ \frac{3y_1}{2}-7y_2+y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\boxed{y_1=9}$, $\frac{9}{2}+y_2=6$ $\quad \boxed{y_2=\frac{3}{2}}$

$\frac{3y_1}{2}-7y_2+y_3=8$

$3(\frac{9}{2})-7(\frac{3}{2})+y_3=8$

$y_3 = 8+\frac{21}{2}-\frac{27}{2} = 8-3=5$

$UX = Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ \frac{y}{2}+\frac{5z}{2} \\ 18z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$18z=5$ $\quad \boxed{z=\frac{5}{18}}$

$\frac{y}{2}+\frac{5z}{2}=\frac{3}{2}$

$\boxed{y=\frac{29}{18}}$

$2x+3y+z=9$

$x = \frac{70}{18} = \frac{35}{9}$

$\boxed{x=\frac{35}{9}}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 35/9 \\ 29/18 \\ 5/18 \end{bmatrix}$$

Q) Solve the following equation using Crout's method

$x_1+x_2+x_3=1$

$4x_1+3x_2-x_3=6$

$3x_1+5x_2+3x_3=4$

The Matrix form of the given sys of linear eq's is $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

leading Principle minor order 1 $= |1| = 1 \neq 0$

leading Principle minor order 2 $= \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -1 \neq 0$

leading Principle minor order 3 $= \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{vmatrix} = 10 \neq 0$

All leading Principal minor orders of A to so we can write A as LU.

$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix} = \begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}l_{11} & l_{11}u_{12} & l_{11}u_{13}\\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23}\\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{bmatrix}$$

$\boxed{l_{11}=1}$ $u_{12}=\frac{1}{1}=1$ $\boxed{u_{12}=1}$ $l_{11}u_{13}=1$ $\boxed{u_{13}=1}$

$\boxed{l_{21}=4}$ $4(1)+l_{22}=3$ $\boxed{l_{22}=-1}$ $4(1)+(-1)u_{23}=-1$ $\boxed{u_{23}=5}$

$\boxed{l_{31}=3}$ $3(1)+l_{32}=5$ $\boxed{l_{32}=2}$ $3(1)+(2)(5)+l_{33}=3$ $\boxed{l_{33}=-10}$

$$L=\begin{bmatrix}1&0&0\\4&-1&0\\3&2&-10\end{bmatrix} \quad U=\begin{bmatrix}1&1&1\\0&1&5\\0&0&1\end{bmatrix}$$

$AX=B$ — (1)

$A=LU$ — (2)

$LUX=B$ — (3)

let $UX=Y$ — (4) then $LY=B$ — (5)

$LY=B$

$$\begin{bmatrix}1&0&0\\4&-1&0\\3&2&-10\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\6\\4\end{bmatrix}$$

$\boxed{y_1=1}$

$4y_1-y_2=6$

$\boxed{y_2=-2}$

$3y_1+2y_2-10y_3=4 \Rightarrow 3-4-10y_3=4$ $\boxed{y_3=\frac{-1}{2}}$

$UX=Y$

$$\begin{bmatrix}1&1&1\\0&1&5\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\-2\\\frac{-1}{2}\end{bmatrix}$$

$x_3=\frac{-1}{2}$

$x_2+5x_3=-2$

$x_2=-2+\frac{5}{2}$

$\boxed{x_2=\frac{1}{2}}$

$x_1+x_2+x_3=1$

$\boxed{x_1=1}$

$\therefore$ The req. solution $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1/2\\-1/2\end{bmatrix}$

## Cholesky decomposition method:-

If the matrix A is symmetric and it is +ve definite (or) all leading principle minors are +ve then we can write matrix A as $L.L^T$, where L is lower traingular matrix by using cholesky decomposition, we find the solution of sys. of linear eq's

The matrix form of given syst. of linear eq's is $AX=B$ — (1)

$A=L\cdot L^T$ — (2)

Put (2) in (1):

$LL^TX=B$ — (3)

let $L^TX=Y$ — (4)

then $LY=B$ — (5)

First solve $LY=B$ and then $L^TX=Y$

This gives the req. solution to the given system of linear equations

Notes-

All leading principal minor orders of A ≠ 0 so we can write A as LU.

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$\boxed{l_{11}=1}$ $\quad u_{12}=\frac{1}{1}=1$ $\quad \boxed{u_{12}=1}$ $\quad l_{11}u_{13}=1$ $\quad \boxed{u_{13}=1}$

$\boxed{l_{21}=4}$ $\quad 4(1)+l_{22}=3$ $\quad \boxed{l_{22}=-1}$ $\quad 4(1)+(-1)u_{23}=-1$ $\quad \boxed{u_{23}=5}$

$\boxed{l_{31}=3}$ $\quad 3(1)+l_{32}=5$ $\quad \boxed{l_{32}=2}$ $\quad 3(1)+2(5)+l_{33}=3$ $\quad \boxed{l_{33}=-10}$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad U=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$AX=B$ — (1)

$A=LU$ — (2)

$LUX=B$ — (3)

let $UX=Y$ — (4) then $LY=B$ — (5)

$LY=B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$\boxed{y_1=1}$

$4y_1-y_2=6$

$\boxed{y_2=-2}$

$3y_1+2y_2-10y_3=4 \Rightarrow 3-4-10y_3=4$ $\quad \boxed{y_3=\frac{-1}{2}}$

$UX=Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ \frac{-1}{2} \end{bmatrix}$$

$x_3=\frac{-1}{2}$

$x_2+5x_3=-2$

$x_2=-2+\frac{5}{2}$

$\boxed{x_2=\frac{1}{2}}$

$x_1+x_2+x_3=1$

$\boxed{x_1=1}$

∴ The req. solution $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$

**Cholesky decomposition method:-**

all leading principle minors are +ve

If the matrix A is symmetric and it is +ve definite then we can write matrix A as $L.L^T$, where L is lower triangular matrix. By using cholesky decomposition we find the solution of sys. of linear eq's

The matrix form of given syst. of linear eq's is $AX=B$ — (1)

$A=L\cdot L^T$ — (2)

Put (2) in (1) :

$LL^TX=B$ — (3)

let $L^TX=Y$ — (4)

then $LY=B$ — (5)

First solve $LY=B$ and then $L^TX=Y$

This gives the req. solution to the given system of linear equations

**Notes:-**

Q) Solve the equations $25x + 15y - 5z = 35$
$15x + 18y = 33$
$-5x + 0 + 11z = 6$ using cholesky decomposition.

Sol: The matrix form of every given sys. of linear eq's is $\boxed{AX = B}$

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

i) A is symmetric.

leading principle minor of order 1 $= |25| = 25$ +ve

2 $= \begin{vmatrix} 25 & 15 \\ 15 & 18 \end{vmatrix} = 450 - 225 = 225$ +ve

3 $= \begin{vmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{vmatrix} = 2025$ +ve

All leading principle minors of A are +ve so the matrix A has +ve definite

So, $A = LL^T$ where L is lower traingular matrix

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$l_{11}^2 = 25$
$\boxed{l_{11} = 5}$

$l_{21}l_{11} = 15$
$\boxed{l_{21} = 3}$

$l_{31}l_{11} = -5$
$\boxed{l_{31} = -1}$

$l_{21}^2 + l_{22}^2 = 18$
$l_{22}^2 = 9$
$\boxed{l_{22} = 3}$

$l_{21}l_{31} + l_{22}l_{32} = 0$
$3(-1) + 3(l_{32}) = 0$
$\boxed{l_{32} = 1}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 11$
$l_{33}^2 = 11 - 1 - 1$
$l_{33}^2 = 9$
$\boxed{l_{33} = 3}$

$$\therefore L = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$AX = B$ — ①   $\boxed{A = LL^T}$ — ②

From ① & ② $LL^TX = B$
let $L^TX = Y$
then $\boxed{LY = B}$

Solve $LY = B$

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

$5y_1 = 35$
$\boxed{y_1 = 7}$

$3y_1 + 3y_2 = 33$
$7 + y_2 = 11$
$\boxed{y_2 = 4}$

$-y_1 + y_2 + 3y_3 = 6$
$-7 + 4 + 3y_3 = 6$
$3y_3 = 9$
$\boxed{y_3 = 3}$

Now $L^TX = Y$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$5x + 3y - z = 7$
$3y + z = 4$
$3z = 3$
$\boxed{z = 1}$
$3y = 3$
$\boxed{y = 1}$
$5x + 3 - 1 = 7$
$5x = 5$
$\boxed{x = 1}$

$\therefore$ The solution $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ The solution is $x = 1, y = 1, z = 1$

Q) Solve the eqn's

$$6x + 15y + 55z = 76$$
$$15x + 55y + 225z = 295$$
$$55x + 225y + 979z = 1259$$

The matrix form of given sys. of linear eq's is $AX = B$

$$A = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 76 \\ 295 \\ 1259 \end{bmatrix}$$

i) A is symmetric

ii) leading principle ~~order~~ minor of order 1 : $|6|$ +ve

$$2 : \begin{vmatrix} 6 & 15 \\ 15 & 55 \end{vmatrix} = 105 \text{ +ve}$$

$$3 : \begin{vmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{vmatrix} = 3920 \text{ +ve}$$

$\therefore A = LL^T$

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$l_{11}^2 = 6$

$\boxed{l_{11} = \sqrt{6}}$

$l_{11} l_{21} = 15$

$\boxed{l_{21} = \frac{15}{\sqrt{6}}}$

$l_{11} l_{31} = 55$

$\boxed{l_{31} = \frac{55}{\sqrt{6}}}$

$l_{21}^2 + l_{22}^2 = 55$

$l_{22}^2 = 55 - \frac{225}{6} = \frac{105}{6} \Rightarrow l_{22} = \frac{\sqrt{105}}{\sqrt{6}}$

$l_{21} l_{31} + l_{22} l_{32} = 225$

$\frac{15}{\sqrt{6}} \times \frac{55}{\sqrt{6}} + \frac{\sqrt{105}}{\sqrt{6}} l_{32} = 225$

~~$l_{32} = 225 \times 6$~~

$\sqrt{\frac{105}{6}} l_{32} = 225 - \frac{825}{6}$

$l_{32} = \frac{525}{6} \frac{\sqrt{6}}{\sqrt{105}}$

$= \frac{525}{\sqrt{6}\sqrt{105}}$

$= \frac{1}{\sqrt{6}} \sqrt{\frac{275625}{105}}$

$l_{32} = \frac{\sqrt{2625}}{\sqrt{6}}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 979$

$\frac{3025}{6} + \frac{\cancel{275625}\ 2625}{6 \times 105} + l_{33}^2 = 979$

$l_{33}^2 = 979 - \frac{2625 + 3025}{6}$

$= \frac{224}{6}$

$\boxed{l_{33} = \frac{\sqrt{224}}{\sqrt{6}}}$

$AX = B$

$LL^T X = B$

$L^T X = Y$

$\boxed{LY = B}$

$$\begin{bmatrix} \sqrt{6} & 0 & 0 \\ \frac{15}{\sqrt{6}} & \sqrt{\frac{105}{6}} & 0 \\ \frac{55}{\sqrt{6}} & \frac{\sqrt{2625}}{\sqrt{6}} & \frac{\sqrt{224}}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 76 \\ 295 \\ 1259 \end{bmatrix}$$

$\boxed{y_1 = \frac{76}{\sqrt{6}}}$

$\frac{15}{\sqrt{6}} \times \frac{76}{\sqrt{6}} \times \sqrt{\frac{105}{6}} y_2 = 295$

$\sqrt{\frac{105}{6}} y_2 = 295 - \frac{1140}{6}$

$= \frac{1770 - 1140}{6}$

$= \frac{630}{6} = 105$

$y_2 = \sqrt{105} \times \frac{\sqrt{6}}{\sqrt{105}}$

$\boxed{y_2 = \sqrt{630}}$

$\frac{55}{\sqrt{6}} y_1 + \frac{\sqrt{2625}}{\sqrt{6}} y_2 + \frac{\sqrt{224}}{\sqrt{6}} y_3 = 1259$

$\frac{55}{\sqrt{6}} \frac{76}{\sqrt{6}} + \frac{\sqrt{2625}}{\sqrt{6}} \times \sqrt{630} + \frac{\sqrt{224}}{\sqrt{6}} y_3 = 1259$

$\frac{\sqrt{1653750}}{\sqrt{6}} + \frac{\sqrt{224}}{\sqrt{6}} y_3 = 1259 - \frac{4180}{6}$

$\frac{\sqrt{224}}{\sqrt{6}} y_3 = \frac{3374}{6} - 525$

$y_3 = \frac{224}{6} \times \frac{\sqrt{6}}{\sqrt{224}}$

$\boxed{y_3 = \frac{\sqrt{224}}{\sqrt{6}}}$

$L^T x = y$

$$\begin{bmatrix} \sqrt{6} & 15/\sqrt{6} & 55/\sqrt{6} \\ 0 & \sqrt{105}/\sqrt{6} & \frac{\sqrt{2625}}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{224}{6}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 76/\sqrt{6} \\ \sqrt{630} \\ \sqrt{\frac{224}{6}} \end{bmatrix}$$

$$\boxed{z = 1}$$

$$\frac{\sqrt{105}}{\sqrt{6}} y + \frac{\sqrt{2625}}{\sqrt{6}}(1) = \sqrt{630}$$

$$\frac{\sqrt{105}}{\sqrt{6}} y = \sqrt{630} - \frac{\sqrt{2625}}{\sqrt{6}}$$

$$\cancel{\frac{1155}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{105}}} = \frac{\sqrt{3780} - \sqrt{2625}}{\sqrt{105}}$$

$$\cancel{y = \sqrt{12705}} \qquad = \sqrt{36} - \sqrt{25}$$

$$= 6 - 5 = 1$$

$$\cancel{\sqrt{6}x + \frac{15}{\sqrt{6}}\sqrt{12705} + \frac{55}{\sqrt{6}} = \frac{76}{\sqrt{6}}}$$

$$\cancel{6x + 15\sqrt{12705} + 55 = 76}$$

$$6x = 76 - 55 -$$

$$\sqrt{6}x + \frac{15}{\sqrt{6}}(1) + \frac{55}{\sqrt{6}}(1) = \frac{76}{\sqrt{6}}$$

$$\sqrt{6}x + \frac{70}{\sqrt{6}} = \frac{76}{\sqrt{6}}$$

$$\sqrt{6}x = \frac{6}{\sqrt{6}}$$

$$\boxed{x = 1}$$

$$\therefore X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

## Gram-Schmdit orthogonalization process

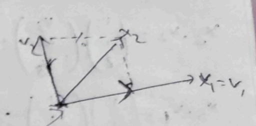
Let $\{x_1, x_2, x_3\}$ is L.I, by using gram-schmdit process we construct orthonormal set of vectors

**Step 1**:- $v_1 = x_1$

**Step 2**:- $v_2 = x_2 - \text{proj}_{v_1}^{x_2}$

$$= x_2 - \left(\frac{x_2 \cdot v_1}{|v_1|}\right)\frac{v_1}{|v_1|}$$

$$\boxed{v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1}$$




**Step 3**:- $v_3 = x_3 - \text{proj}_{v_1}^{x_3} - \text{proj}_{v_2}^{x_3}$

$$= x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

Now, the new set $\{v_1, v_2, v_3\}$ is orthogonal

The orthogonal set $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

Q) Construct an orthonormal set ~~of~~ by using the vectors

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{3\times 1}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$ $\quad |A| = 1 \neq 0$

$R_2 \to R_2 + R_1$ $\quad$ Rank $= 3 =$ no. of vectors

$\{x_1, x_2, x_3\}$ is L.I

**Step 1**:- $v_1 = x_1$

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$x_2 \cdot v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2$$

$$v_1 \cdot v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3$$

**Step 2**:- $v_2 = x_2 - \text{Proj}_{v_1}^{x_2}$

$$= x_2 - \left(\frac{(x_2 \cdot v_1)}{(v_1 \cdot v_1)}\right) v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$v_3 = x_3 - \text{proj}_{v_2}^{x_3} - \text{proj}_{v_1}^{x_3}$

$= x_3 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$x_3 \cdot v_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} = 1/3 + 2/3 + \frac{2}{3} = \frac{5}{3}$

$x_3 \cdot v_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix} = 1 - 1 + 2 = 2$

$v_2 \cdot v_2 = \begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} = \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

$\therefore v_3 = \begin{pmatrix}1\\1\\2\end{pmatrix} - \left(\frac{5/3}{2/3}\right)\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} - \left(\frac{2}{3}\right)\begin{pmatrix}1\\-1\\1\end{pmatrix}$

$\cancel{= \frac{5}{2}}$

$= \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}5/6\\5/3\\5/6\end{pmatrix} - \begin{pmatrix}2/3\\-2/3\\2/3\end{pmatrix}$

$= \begin{pmatrix}1 - \frac{5}{6} - \frac{2}{3}\\ 1 - 5/3 + 2/3 \\ 2 - 5/6 - 2/3\end{pmatrix} = \begin{bmatrix}\frac{6-5-4}{6}\\ \frac{3-5+2}{3}\\ \frac{12-5-4}{6}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\0\\1/2\end{bmatrix}$

$v_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$ $v_2 = \begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$ $v_3 = \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$

$v_1 \perp v_2$; $v_2 \perp v_3$, $v_3 \perp v_1$

$||v_1|| = \sqrt{3}$

$||v_2|| = \sqrt{\frac{6}{9}} = \sqrt{2/3}$

$||v_3|| = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

$u_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$

$u_2 = \frac{v_2}{||v_2||} = \frac{\sqrt{3}}{\sqrt{2}}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} = \begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$

$u_3 = \frac{v_3}{||v_3||} = \frac{1}{\frac{1}{\sqrt{2}}}\begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$

$\{u_1, u_2, u_3\}$ is orthonormal set

$Q = [u_1\ u_2\ u_3] = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2}\\ -1/\sqrt{3} & 2/\sqrt{6} & 0\\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2}\end{bmatrix}$

**QR decomposition:**

If $A$ is an $m \times n$ matrix with the linearly independent column vectors then $A$ can be factored as $A = QR$

$(A_{m\times n} = Q_{m\times n} R_{n\times n})$

$(A_{n\times n} = Q_{n\times n} R_{n\times n})$

where the columns of $Q$ are orthonormal set of vector and $R$ is an uppertriangle matrix.

Note: $A$ is sq. matrix of order $n$ and the columns of $A$ are L.I then

$A = QR$

$Q \rightarrow$ orthogonal matrix

$R \rightarrow$ upper triangular matrix

Q) Find the QR factorization of $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 3 & 3 & 4 \end{bmatrix}$

Rank of $A$: $|A| = 1(4-9)+1(-9)+2(-3)$

$= -5-9-6$

$= -20$

Rank $= 3 =$ no of column vectors

$\therefore$ The column vectors of $A$ are L.I

Let $x_1 = \begin{pmatrix}1\\0\\3\end{pmatrix}$, $x_2 = \begin{pmatrix}-1\\1\\3\end{pmatrix}$, $x_3 = \begin{pmatrix}2\\3\\4\end{pmatrix}$.

By using Gram-Schmidt process

$v_1 = x_1 = \begin{pmatrix}1\\0\\3\end{pmatrix}$

$v_2 = x_2 - \text{proj}_{v_1}^{x_2}$

$= x_2 - \left(\dfrac{x_2 \cdot v_1}{v_1 \cdot v_1}\right)(v_1)$

$x_2 \cdot v_1 = \begin{pmatrix}-1\\1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\0\\3\end{pmatrix}$

$= -1+9 = 8$

$v_1 \cdot v_1 = \begin{pmatrix}1\\0\\3\end{pmatrix} \cdot \begin{pmatrix}1\\0\\3\end{pmatrix} = 10$

$v_2 = \begin{pmatrix}-1\\1\\3\end{pmatrix} - \dfrac{84}{105}\begin{pmatrix}1\\0\\3\end{pmatrix}$

$= \begin{pmatrix}-1-\frac{4}{5}\\ 1-0 \\ 3-\frac{4(3)}{5}\end{pmatrix} = \begin{bmatrix}-9/5\\1\\ \frac{3}{5}\end{bmatrix}$

$v_3 = x_3 - \text{proj}_{v_1}^{x_3} - \text{proj}_{v_2}^{x_3}$

$= x_3 - \left(\dfrac{(x_3 \cdot v_1)}{(v_1 \cdot v_1)}\right) v_1 - \left(\dfrac{x_3 \cdot v_2}{v_2 \cdot v_2}\right)(v_2)$

$x_3 \cdot v_1 = \begin{pmatrix}2\\3\\4\end{pmatrix} \cdot \begin{pmatrix}1\\0\\3\end{pmatrix}$

$= 2+12 = 14$

$v_2 \cdot v_2 = \begin{bmatrix}-9/5\\1\\3/5\end{bmatrix} \begin{bmatrix}-9/5\\1\\3/5\end{bmatrix} = \frac{81}{25} + \frac{25}{25} + \frac{9}{25} = \frac{115}{25} = \frac{23}{5}$

$x_3 \cdot v_2 = \begin{bmatrix}2\\3\\4\end{bmatrix} \begin{bmatrix}-9/5\\1\\3/5\end{bmatrix}$

$= -\frac{18}{5} + \frac{15}{5} + \frac{12}{5}$

$= -\frac{18}{5} + \frac{27}{5} = \frac{9}{5}$

$v_3 = \begin{bmatrix}2\\3\\4\end{bmatrix} - \frac{14}{10}\begin{pmatrix}1\\0\\3\end{pmatrix} - \left(\dfrac{9/5}{23/5}\right)\begin{pmatrix}-9/5\\1\\3/5\end{pmatrix}$

$= \begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}7/5\\0\\21/5\end{bmatrix} - \begin{bmatrix}81/115\\9/23\\27/115\end{bmatrix}$

$= \begin{bmatrix}\frac{230-161-81}{115}\\ \frac{60-9}{23} \\ \frac{460-483-27}{115}\end{bmatrix}$

$= \begin{bmatrix}\frac{-12}{115}\\ \frac{51}{23} \\ \frac{-50}{115}\end{bmatrix}$

23 × 5 = 115, × 2 = 230

23 × 7 = 161

$3 - \frac{9}{23}$

$\frac{69-9}{23} = \frac{60}{23}$

117 × 4 = 460; 21 × 23

$Q = [v_1 \; v_2 \; v_3]$

$A = QR$

$Q^T A = \underbrace{Q^T Q}_{I} R$

$\boxed{Q^T A = R}$ By this way u can find R

Q) Find a QR decomposition of matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

To find Rank A

let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ For lower triangular matrix diagonal elements are eigenvalues and product of Eigenvalues is $|A|$.

$|A| = 1$

$P(A) = 3$

$\{v_1, v_2, v_3\}$ are L.I

By Gram-Schmidt process

Step 1: $u_1 = v_1$

$\boxed{u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

Step 2: $u_2 = v_2 - \text{Proj}_{u_1} v_2$

$= v_2 - \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right)(u_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v_2 \cdot u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0+1+1 = 2$

$\boxed{u_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$

$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$

Check if $u_1 \perp u_2$ $\quad u_1 \cdot u_2 = 0 \quad \therefore u_1 \perp u_2$

Step 3: $u_3 = v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3$

$= v_3 - \frac{(v_3 \cdot u_1)}{(u_1 \cdot u_1)}(u_1) - \left(\frac{v_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2$

$v_3 \cdot u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$

$v_3 \cdot u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 1/3$

$u_2 \cdot u_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1/3}{2/3}\right)\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$0 - 1/3 - \frac{3}{2} \times \frac{2}{3}$

$= \begin{bmatrix} 0 - \frac{1}{3} + \frac{1}{3} \\ 0 - \frac{1}{3} - \frac{1}{6} \\ 0 - \frac{1}{3} - \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ $u_3 = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$

Here $u_1 \perp u_2$, $u_2 \perp u_3$, $u_3 \perp u_1$

$\|u_1\| = \sqrt{3}$

$\|u_2\| = \frac{\sqrt{2}}{\sqrt{3}}$

$\|u_3\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -\frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} & 0 \\ 1/\sqrt{3} & \frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \\ 1/\sqrt{3} & \frac{1}{3}\frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad \frac{\sqrt{3}}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}$$

$$= \begin{bmatrix} 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & \frac{-1}{\sqrt{2}} \\ 1/\sqrt{3} & 1/\sqrt{6} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$A = QR$

$Q^T A = Q^T Q R$

$\boxed{Q^T A = R}$

$$R = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -\sqrt{2}/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ \frac{-\sqrt{2}}{\sqrt{3}} + \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$A = QR$

$$A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Q) Find the QR decomposition of $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 3 = no. of vectors

$\therefore \{v_1, v_2, v_3\}$ is L.I

By gram-schmidt process.

Step 1 :- $u_1 = v_1$

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

Step 2 :- $u_2 = v_2 - \text{Proj}_{u_1} v_2$

$$= v_2 - \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right) u_1$$

$$v_2 \cdot u_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 6$$

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4$$

$$u_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \boxed{\begin{bmatrix} -\frac{5}{2} \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}}$$

$-1 - \frac{3}{2}$

$\frac{-5}{2}$

$4 - \frac{3}{2}$

$\frac{8-3}{2} = \frac{5}{2}$

$-1 - \frac{3}{2}$

$\frac{-2-3}{2} = \frac{-5}{2}$

$\frac{4-3}{2} = \frac{8-3}{2}$

$u_3 = v_3 - \text{Proj}_{u_1}^{v_3} - \text{Proj}_{u_2}^{v_3}$

$= v_3 - \left(\frac{v_3 \cdot u_1}{u_1 \cdot u_1}\right)(u_1) - \left(\frac{v_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2$

$v_3 \cdot u_1 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4$

$v_3 \cdot u_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \frac{5}{2}(-4 - 2 + 2) = -10$

$u_2 \cdot u_2 = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} \cdot \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = 4\left(\frac{25}{4}\right) = 25$

$u_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-\cancel{10}^{\,2}}{\cancel{25}_{\,5}}\right) \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$

$u_3 = \begin{pmatrix} 4 - 1 - 1 \\ -2 - 1 + 1 \\ 2 - 1 + 1 \\ 0 - 1 - 1 \end{pmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$, $u_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$

here $u_1 \perp u_2$, $u_2 \perp u_3$, $u_3 \perp u_1$

$Q = \left[ \frac{u_1}{\|u_1\|} \quad \frac{u_2}{\|u_2\|} \quad \frac{u_3}{\|u_3\|} \right]$

$\|u_1\| = \sqrt{4} = 2$

$\|u_2\| = \sqrt{4\left(\frac{25}{4}\right)} = 5$

$\|u_3\| = \sqrt{4(4)} = 4$

$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$

$A = QR$

$Q^T A = Q^T Q R$

$\boxed{R = Q^T A}$

$R = \underset{3\times 4}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}}$

$= \begin{bmatrix} 1 & \overset{3}{-\frac{1}{2} + 2 + 2} & 2 \\ 0 & 1 + 2 + 2 & -2 \\ 0 & 0 & 2 + 2 \end{bmatrix}$ $\quad -\frac{1}{2} + 2 + 2 - \frac{1}{2}$

$R = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

## Singular value decomposition (SVD)

The SVD of any matrix of order $m \times n$ is defined as

$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^T_{n\times n}$

where U and V are orthogonal matrices of order m and n respectively, $\Sigma$ is a matrix of order $m \times n$ which having singular values of A in the main diagonal and other elements are zero's.

### Singular values

If A is any matrix of order $m \times n$ and $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ are the eigen values of $A^TA$ then

$\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$ ..... $\sigma_n = \sqrt{\lambda_n}$

are called singular values of A.

Note

→ If A is any matrix of order $m \times n$ then $A^TA$ and $AA^T$ are symmetric matrices of diff. orders or same but both matrices have same eigen values and the bigger matrix has "zero" as an additional eigen value.

if $AA^T_{2\times 2} \to 1, 2$

$A^TA_{3\times 3} \to 1, 2, \underline{0}$

→ The columns of V are orthonormal eigen vectors of $A^TA$.

→ The columns of U are orthonormal eigen vectors of $AA^T$.

Find SVD of matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ ... $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$= U \Sigma V^T$

$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Char. eq is

$\lambda^3 - (3)\lambda^2 + (1+1)\lambda - 0 = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda(\lambda^2 - 2\lambda - \lambda + 2) = 0$

$\lambda(\lambda(\lambda-2) - (\lambda-2)) = 0$

$\lambda[(\lambda-1)(\lambda-2)] = 0$

$\boxed{\lambda = 0, 1, 2}$

(write the Evalues in descending order)

The eigen values are 2, 1, 0

The eigen vector corr. E.value $\lambda = 2$

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an E.vec corr. E.value $\lambda = 2$.

$(A^TA)x = 2x$

$(A^TA - 2I)x = 0$

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + R_1$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$R_3 \leftrightarrow R_2$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ \dots \end{bmatrix}$

$r = 3 - 2 = 1$

$\boxed{x_3 = 0}$

$-x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = x_1}$

let $x_1 = k$

$x_2 = k$

∴ E.vec corr. to E.value $\lambda = 2$

is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the L.I.E.V corr. E.val $\boxed{\lambda = 2}$

The E.vec corr. E.val. $\lambda = 1$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Echelon

$\boxed{x_2 = 0}$

$\boxed{x_1 = 0}$

$\boxed{x_3 = k}$

∴ E.vec. corr. to E.val $\lambda = 1$

is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

L.I.O.E.vec corr. E.value $\boxed{\lambda = 1}$

U

The E.vec corr. to E.value ($\lambda=0$) is

$$\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}1&1&0\\0&0&0\\0&0&1\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$x_1+x_2=0$ $\quad$ $\boxed{n-r=1}$

$x_3=0$

let $x_2=k$ :

$\boxed{x_1=-k}$

$\therefore$ The E.vec corr. to E.val $\lambda=0$ is $k\begin{bmatrix}-1\\+1\\0\end{bmatrix}$

L.I.E.vec corr. to E.val $\boxed{\lambda=0}$

~~The orthogonal matrix~~

The orth. matrix U is

$$U=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$x_1=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $\quad x_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad x_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$

Clearly $x_1 \perp x_2$, $x_2 \perp x_3$, $x_3 \perp x_1$.

$\|x_1\|=\sqrt{2}$ $\quad \|x_2\|=1$ $\quad \|x_3\|=\sqrt{2}$

$\therefore$ The orthogonal matrix:

$$V=\left[\frac{x_1}{\|x_1\|}\ \frac{x_2}{\|x_2\|}\ \frac{x_3}{\|x_3\|}\right]$$

$$V=\begin{bmatrix}\frac{1}{\sqrt{2}}&0&-1/\sqrt{2}\\1/\sqrt{2}&0&1/\sqrt{2}\\0&1&0\end{bmatrix}$$

$$AA^T=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}2&0\\0&1\end{bmatrix}$$

E.values are 2, 1

E.vec corr. to E.val, $\lambda=2$ is

$$\begin{bmatrix}0&0\\0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$-x_2=0$

$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$ $\quad \boxed{x_2=0}$ $\quad \boxed{x_1=k}$ $\quad$ E.vec $=k\begin{bmatrix}1\\0\end{bmatrix}$

E.vec corr. to E.val. $\lambda=1$ is

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\boxed{x_1=0}$

$\boxed{x_2=k}$ $\quad$ E.vec $=k\begin{bmatrix}0\\1\end{bmatrix}$

$$A=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$$

The eigen value of $A^TA$ are 2, 1, 0

$\therefore$ singular values $\sigma_1=\sqrt{2}$ $\quad \sigma_2=1$ $\quad \sigma_3=0$

$$\Sigma_{2\times3}=\begin{bmatrix}\sqrt{2}&0&0\\0&1&0\end{bmatrix}_{2\times3}$$

$A=U\Sigma V^T$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\sqrt{2}&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\\-1/\sqrt{2}&1/\sqrt{2}&0\end{bmatrix}=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$$

$\cong A$

Q) Find the SVD of Matrix $A=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$ $\quad A=U\Sigma V^T$

$$A^TA=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

$\lambda^2-(4)\lambda+3=0$

$\lambda^2-3\lambda-\lambda+3=0$

$\lambda(\lambda-3)-(\lambda-3)=0$

$(\lambda-1)(\lambda-3)=0$

$\boxed{\lambda=1,3}$

Eigen values are 3, 1.

E.vec of E.val. $\lambda=3$

$$\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$R_2 \to R_2+R_1$

$$\begin{bmatrix}-1&1\\0&0\end{bmatrix}$$

$-x_1+x_2=0$

$\boxed{x_1=x_2}$

let $x_1=k$

$x_2=k$

E.v is $k\begin{bmatrix}1\\1\end{bmatrix}$

E.vec of E.val. $\lambda=1$

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$R_2 \to R_2-R_1$

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}$$

$x_1+x_2=0$

$\boxed{x_1=-x_2}$

let $x_1=k$

$\boxed{x_2=-k}$

E.v is $k\begin{bmatrix}1\\-1\end{bmatrix}$

The orthogonal matrix $V=\left[\frac{x_1}{\|x_1\|}\ \frac{x_2}{\|x_2\|}\right]$

$$V=\begin{bmatrix}\frac{1}{\sqrt{2}}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}$$

$$AA^T=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}_{3\times2}\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}_{2\times3}$$

$$=\begin{bmatrix}2&1&1\\1&1&0\\1&0&1\end{bmatrix}$$

E.Val are 3, 1, 0.

E.vec corres. Eval $\lambda=3$ is

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = x_2}$ $\quad -x_1 + x_2 + x_3 = 0$ $\quad \boxed{x_1 = 2K}$

let $x_2 = K$
$\boxed{x_3 = K}$

E.vec is $K\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ corr. E.val. $\lambda=3$

Optimal methods to find $U$

$A = U\Sigma V^T$

$AV = U\Sigma \underbrace{V^T V}_{I}$

$\boxed{AV = U\Sigma}$

$\Rightarrow AV_i = U_i \sigma_i$

$\boxed{U_i = \frac{1}{\sigma_i} AV_i}$

The singular values are $\sqrt{3}, \sqrt{1}$

$\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$

$U_1 = \frac{1}{\sigma_1} AV_1$

$= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$U_2 = \frac{1}{\sigma_2} AV_2$

$= \frac{1}{1}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

we got 2 coloumns of $U$ matrix calc. 3rd coloumn by E.vector $\boxed{\lambda=0}$

$(AA^T - 0I)x = 0$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to 2R_2 - R_1$
$R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$\boxed{x_2 = x_3}$

let $x_2 = K$

$2x_1 + 2K = 0$

$\boxed{x_1 = -K}$

E.V is $K\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ corr. E.val $\boxed{\lambda=0}$

$x_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\frac{x_3}{\|x_3\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = U_3$

matrix $U = (U_1 \ U_2 \ U_3)$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$ order of $A$

$U\Sigma V^T$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}_{3\times 3}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}_{2\times 2}$$

Q. Find the SVD of the matrix $A = \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix}$

Ans.

$A^T A = \begin{pmatrix} -4 & 1 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix}$

$= \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$

$\lambda^2 - (82)\lambda + 81 = 0$

$\lambda^2 - 82\lambda + 81 = 0$

$\lambda^2 - 81\lambda - \lambda + 81 = 0$

$\lambda(\lambda - 81) - (\lambda - 81) = 0$

$(\lambda - 1)(\lambda - 81) = 0$

$\lambda = 81, 1$

Eigen values are 81, 1

$[A^T A - 81 I] X = 0$

$\begin{bmatrix} -64 & 32 \\ 32 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_2 \to 2R_2 + R_1$

$\begin{pmatrix} -64 & 32 \\ 0 & 0 \end{pmatrix}$

Rank = 1

$-64x_1 + 32x_2 = 0$

$32x_2 = 64x_1$

$x_2 = 2x_1$

let $x_1 = k$

$x_2 = 2k$

∴ E.vector corr. E.value $\lambda = 81$ is $k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

E.vec corr. E.val $\lambda = 1$

$(A^T A - I) X = 0$

$\begin{bmatrix} 16 & 32 \\ 32 & 64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\begin{bmatrix} 16 & 32 \\ 0 & 0 \end{bmatrix}$

$16x_1 + 32x_2 = 0$

$x_1 + 2x_2 = 0$

let $x_2 = k$

$\boxed{x_1 = -2k}$

∴ E.vec corr. to E.val $\lambda = 1$ is $k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$\sigma_1 = \sqrt{81} = 9$

$\sigma_2 = \sqrt{1} = 1$

$u_i = \frac{1}{\sigma_i} A v_i$

$u_1 = \frac{1}{9} \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} -18/\sqrt{5} \\ 9/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$u_2 = \frac{1}{1} [\ldots] [\ldots 1/\sqrt{5}]$

$= \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$U = [u_1 \; u_2]$

$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$U \Sigma V^T$

$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$= \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} = A$

2/12/23

## Generalised inverse of a matrix

### Moore-Penrose Pseudo inverse of a matrix

**Over Determined system:**
The number of equations are more than the number of unknows in the system of equations than that system is called over determined system.

Example: $x+5y=-2, 3x-y=4, 2x+2y=1$

**Under determined System:**
The no. of unknowns are more than the number of equations in the system of equations, then that system is under under determined system.

Ex: $x+y+z=1, x+2y+z=1$

**Full Row Rank (Row Rank):** Rank of matrix of order $m\times n$ is equal to the no. of rows in a matrix, then matrix has full row rank.

(or)

Number of l.i. rows are equal to rank of matrix, then the matrix has full row rank.

**Full Column Rank (~~Row~~ column Rank):** Rank of matrix of order $m\times n$ is equal to the no. of columns in the matrix, then matrix has full column Rank.

(or)

No. of linearly independent columns are equal to rank of the matrix, then matrix has full coloumn Rank.

**Rank deficient**
Rank of a matrix of order $m\times n$ is less than the no. of rows and columns in the matrix, the matrix is Rank deficient.

**Pseudoinverses:** in mathematics, and in particular linear algebra a pseudo inverse $A^+$ of a matrix $A$ is a generalization of the inverse matrix. The most widely known type of matrix pseudoinverse is the Moore-Penrose pseudoinverse. A common use of the pseudoinverse is to compute a 'best-fit' (least squares) solution to a system of linear equations that lacks a unique solution. The pseudoinverse is defined and unique for all matrices whose entries are real or complex numbers. Matrix inverse exists for square matrices only. Real world data is not always square. Furthermore real world data is not always consistent and might contain repetitions. To deal with real world data generalized inverse for rectangular matrix is needed. It can be computed using the singular value decomposition also.

Pseudoinverse of a matrix of order $m\times n$ is defined as

1. If the colouns of a matrix $A$ are linearly independent, so $A^TA$ is invertible and we obtain the pseudo inverse with the following formula (Full coloumn Rank)

$$A^+ = (A^T.A)^{-1}.A^T$$

2. However, if the rows of the matrix are l.ind, we obtain the pseudo inverse with the formula (full row Rank)

$$A^+ = A^T(A.A^T)^{-1}$$

3. If $A$ has rank deficient, then the pseudo inverse of $A$ is defined as $A^+ = (U\Sigma V^T)^{-1} = \cancel{(V\Sigma V^T)} = V\Sigma^{-1}U^T$

$(V^T)^{-1}\Sigma^{-1}U^{-1}$
$(V^T)^{-1}\Sigma^{-1}U^T$

If $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}_{m\times n}$ then $\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \end{bmatrix}_{n\times m}$

changes

**Example**
Find the pseudo inverse of $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}_{2\times 4}$

$\det \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3-8 = -5 \neq 0$

$\therefore$ Rank $(A) = 2$

$\rightarrow$ The matrix $A$ has full row rank.

The pseudo inverse of A is $A^+ = A^T(A \cdot A^T)^{-1}$ symmetric matrix

$$AA^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}_{2\times4} \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}_{4\times2} = \begin{matrix} 1+4+1+9 & 4+6+2+3 \\ 4+6+2+3 & 16+9+4+1 \end{matrix} = \begin{bmatrix} 15 & 15 \\ 15 & 30 \end{bmatrix}$$

$|AA^T| = 15(30) - 15(15)$

$= 15(30-15)$

$= 225$

$$(AA^T)^{-1} = \frac{1}{225}\begin{bmatrix} 30 & -15 \\ -15 & 15 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}_{4\times2} \frac{1}{225}\begin{bmatrix} 30 & -15 \\ -15 & 15 \end{bmatrix}_{2\times2}$$

$$= \frac{1}{225}\begin{bmatrix} 30-60 & -15+60 \\ 60-45 & 15 \\ 0 & 15 \\ 90-15 & -30 \end{bmatrix}$$

$$= \frac{1}{225}\begin{bmatrix} -30 & 45 \\ 15 & 15 \\ 0 & 15 \\ 75 & -30 \end{bmatrix} = \begin{bmatrix} \frac{-2}{15} & \frac{1}{5} \\ \frac{1}{15} & \frac{1}{15} \\ 0 & \frac{1}{15} \\ \frac{1}{3} & \frac{-2}{15} \end{bmatrix}$$

$$= \begin{bmatrix} 0.13 & 0.2 \\ 0.06 & 0.06 \\ 0 & 0.06 \\ 0.33 & -0.13 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}} \underset{A^+}{\begin{bmatrix} \frac{-2}{15} & \frac{1}{5} \\ \frac{1}{15} & \frac{1}{15} \\ 0 & \frac{1}{15} \\ \frac{1}{3} & \frac{-2}{15} \end{bmatrix}} = \frac{-2}{15} + \frac{2}{15} + 1 \quad \frac{1}{5} + \frac{2}{15} + \frac{1}{15} - \frac{6}{15}$$

$\frac{3}{15} - \frac{3}{15}$

$\frac{-8+3}{15} + \frac{1}{3} \quad \frac{4}{5} + \frac{3+2}{15} - \frac{2}{15}$

$\frac{4}{3} - \frac{1}{3} + \frac{2}{15}$

$\frac{4}{5} + \frac{1}{3} - \frac{2}{15}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^+A = \begin{bmatrix} \frac{-2}{15} & \frac{1}{5} \\ \frac{1}{15} & \frac{1}{15} \\ 0 & \frac{1}{15} \\ \frac{1}{3} & -\frac{2}{15} \end{bmatrix}_{4\times2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}_{2\times4}$$

$$= \begin{bmatrix} \frac{-2}{15} + \frac{4\times3}{5\times3} & \frac{-4}{15} + \frac{3\times3}{5\times3} & \frac{-2}{15} + \frac{2\times3}{5\times3} & \frac{-6}{15} + \frac{1}{5} \\ \frac{5}{15} & \frac{5}{15} & \frac{3}{15} & \frac{4}{15} \\ \frac{4}{15} & 3/15 & 2/15 & 1/15 \\ \frac{5\cdot1}{3\times5} - \frac{8}{15} & \frac{5\times2}{5\times3} - \frac{2\times3}{5\times3} & \frac{1}{3} - \frac{4}{15} & 1 - \frac{2}{15} \end{bmatrix}$$

$$= \begin{matrix} \frac{10}{15} & \frac{5}{15} & \frac{4}{15} & \frac{-1}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{5} & \frac{4}{15} \\ \frac{4}{6} & \frac{8-3}{15} & \frac{2}{15} & 1/15 \\ \frac{-3}{15} \end{matrix}$$

Q) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

rank = 2

coloum

$A^+ = (A^T A)^{-1} A^T$

$$(A^T \cdot A)^{-1} = \left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\right)^{-1} = \frac{1}{35}\begin{bmatrix} 6 & -7 \\ -7 & 14 \end{bmatrix}$$

$$A^+ = (A^T \cdot A)^{-1} \cdot A^T = \frac{1}{35}\begin{bmatrix} 6 & -7 \\ -7 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} -8 & -1 & 11 \\ 21 & 0 & -7 \end{bmatrix}$$

Q) Find pseudo inverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Rank = 1 ≠ no. of rows & no. of colums

so we find through SVD for pseudo inverse.

$$A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

characteristic eq.

$\lambda^2 - 4\lambda = 0$

$\boxed{\lambda = 0}$ $\boxed{\lambda = 4}$

2 Singular values of matrix A are $\sqrt{4}, 0$ i.e $2, 0$.

E.vec corr. to E.val $\lambda = 4$

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector

$(A^TA)x = 4x$

$(A^TA - 4I)x = 0$

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_2 \to R_2 + R_1$

$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$ Rank = 1, $n - r = 2 - 1 = 1$

$-2x_1 + 2x_2 = 0$

$\boxed{x_2 = x_1}$

let $x_1 = k$

$\therefore x_2 = k$

E.vec is $k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ L.I.E.V

corres. to e.value $\lambda = 0$

E.vec corr. E.val $\boxed{\lambda = 0}$

$(A^TA - 0I)x = 0$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_2 \to R_2 - R_1$

$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ rank = 1, $n - r = 1$.

$2x_1 + 2x_2 = 0$

$\boxed{x_1 = -x_2}$

~~let $x_1 = k$~~

let $x_2 = +k$

$x_1 = -k$

E.vec is $k\begin{bmatrix} -1 \\ +1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ clearly $x_1 \perp x_2$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\|x_1\| = \sqrt{2}$, $\|x_2\| = \sqrt{2}$

$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$AA^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\|x_1\| = \sqrt{2}$ $\|x_2\| = \sqrt{2}$

$\therefore U = \begin{bmatrix} +1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}$

$\because \frac{1}{0}$ doesn't exist

$\therefore A^+ = V\Sigma^{-1}U^T$

$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

## Least sq. sol$^n$ of an overdetermined system

over determined System:

The no. of eq.'s is more than no. of unknowns in the sys. of equation then that system is called over determined system.

Note:

An over determined system is almost inconsistant

→ To solve the overdetermined problem instead of solving $AX = B$ we solve $(A^TA)\hat{x} = A^Tb$

(or) $\hat{x} = (A^TA)^{-1}A^Tb$

This solution we call it as least sq's solution.

→ Solve $x + y = 1$, $x + 2y = 2$, $x + 3y = 2$

The a $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$
$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Rank of $A = 2$
Rank of $[A:b] = 3$

$2 \neq 3$

∴ no solution but we can find approx. solution

To find least sq. soln

$$(A^TA)\hat{x} = (A^Tb)$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}_{2\times3} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{3\times2}$$

$$A^TA = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Possible ways: $A\hat{x} = b$, $\hat{x} = A^{-1}b$

$\hat{x} = (A^TA)^{-1}(A^Tb)$

The least sq soln $\hat{x} = (A^TA)^{-1}(A^Tb)$

$$= \begin{pmatrix} 7/3 & -1 \\ -1 & 1/2 \end{pmatrix}\begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{bmatrix} \frac{35}{3} - 11 \\ -5 + \frac{11}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}$$

∴ The least sq soln is

$\hat{x}_1 = \frac{2}{3}$ $\hat{x}_2 = \frac{1}{2}$

Q) Solve the overdetermined sys.

$x = 3$
$x + y = 4$
$x + 2y = 1$

$(A^TA)\hat{x} = (A^Tb)$
$\hat{x} = (A^TA)^{-1}(A^Tb)$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{40}{6} - 3 \\ -4 + 3 \end{bmatrix} = \begin{bmatrix} \frac{22}{6} \\ -1 \end{bmatrix}$$

The least sq's of solution is $\hat{x}_1 = \frac{22}{6}$ $\hat{x}_2 = -1$

least square soltn of over determined system of equations by QR factorization:

The least sq soltn of overdetermined system of eq's is

$$(A^TA)\hat{x} = A^Tb$$

(or) $\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$

In QR decomposition $A = QR$

$$\hat{x} = ((QR)^TQR)^{-1}(QR)^Tb$$

$$= (R^TQ^TQR)^{-1}(R^TQ^Tb)$$

(R is invertible upper traingular matrix.)

$$= (R^TR)^{-1}(R^TQ^Tb)$$

$$= R^{-1}(R^T)^{-1}R^TQ^Tb$$

$$\boxed{\hat{x} = R^{-1}Q^Tb}$$

Q) Find the least sq. solution by QR decomposition where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ -1 & -1 & -4 \\ 0 & 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 4 \\ -7 \\ -3 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ -1 & -1 & -4 \\ 0 & 1 & -4 \end{bmatrix} \quad \text{let } x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \; x_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} \; x_3 = \begin{bmatrix} 2 \\ 3 \\ -4 \\ -4 \end{bmatrix}$$

$[x_1\ x_2\ x_3] \to$ L.I not so find rank

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$
$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 3 = no. of column vectors
$\{x_1, x_2, x_3\}$ is L.I
$\therefore$ QR decomposition exists

Gram schmidt Process

step 1: $v_1 = x_1$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$v_2 = x_2 - \text{proj}_{x_1} x_2 = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1}(x_1)$$

$$x_2 \cdot x_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 3 + 2 + 1 = 6$$

$$x_1 \cdot x_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 3$$

$$v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 1 \end{pmatrix} - \frac{6}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_3 = x_3 - \text{proj}_{v_1} x_3 - \text{proj}_{v_2} x_3$$

$$= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1}(v_1) - \frac{x_3 \cdot v_2}{v_2 \cdot v_2}(v_2)$$

$$x_3 \cdot v_1 = \begin{pmatrix} 2 \\ 3 \\ -4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 2 + 3 + 4 = 9$$

$$x_3 \cdot v_2 = \begin{pmatrix} 2 \\ 3 \\ -4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 2 - 4 - 4 = -6$$

$$v_2 \cdot v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 3$$

$$v_3 = \begin{pmatrix} 2 \\ 3 \\ -4 \\ -4 \end{pmatrix} - \frac{9}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{(+6)}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 2 - 3 + 2 \\ 3 - 3 + 0 \\ -4 + 3 + 2 \\ -4 + 0 + 2 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$A = QR$
$R = Q^{-1}A$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \; v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \; v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} & \frac{v_3}{\|v_3\|} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

$$R = Q^{-1}A = Q^T A$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 0 & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}_{3\times4} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ -1 & -1 & -4 \\ 0 & 1 & -4 \end{bmatrix}_{4\times3}$$

$$= \begin{bmatrix} 3/\sqrt{3} & \frac{5}{\sqrt{3}} + \frac{1}{\sqrt{3}} & 9/\sqrt{3} \\ 0 & 3/\sqrt{3} & -6/\sqrt{3} \\ 0 & 0 & \frac{6}{\sqrt{6}} \end{bmatrix}$$

$\frac{5+2+2}{\sqrt{3}}$ $\quad 2+4-4$

$\frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}}$

$\frac{2}{\sqrt{6}} - \frac{4}{6} + \frac{8}{\sqrt{6}}$

$$\begin{bmatrix} \sqrt{3} & 2\sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{3} & -2\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

$$\hat{x} = R^{-1} Q^T b$$

$$= \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2\sqrt{3} & \sqrt{3} & 0 \\ 3\sqrt{3} & -2\sqrt{3} & \sqrt{6} \end{bmatrix} \Big[ \quad \Big]$$

$$\therefore R = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} & 3\sqrt{3} \\ 0 & \sqrt{3} & -2\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} 0.5773 & -1.154 & -2.857 \\ 0 & 0.5773 & 0.8164 \\ 0 & 0 & 0.4082 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0.5773 & -1.154 & -2.857 \\ 0 & 0.5773 & 0.8164 \\ 0 & 0 & 0.4082 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 0 & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}_{3\times 4} \begin{bmatrix} 7 \\ 4 \\ -7 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7+4+7}{\sqrt{3}} \\ \frac{7-7-3}{\sqrt{3}} \\ \frac{7-7+6}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 6\sqrt{3} \\ -\sqrt{3} \\ \sqrt{6} \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 \\ 0.9998 \\ 0.9998 \end{bmatrix}$$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{bmatrix} 1 \\ 0.9998 \\ 0.9998 \end{bmatrix}$$

→ The eigen decomposition of a symmetric matrix :-

we know that every real symmetric matrix is always diagonalizable moreover it is diagonalizing through an orthogonal matrix. i.e $P^{-1}AP = D$ ($P$ is orthogonal matrix)

pre-multiply both with $P$ and post multiply with $P^{-1}$

$$P P^{-1} A P P^{-1} = P D P^{-1}$$

$$\boxed{A = PDP^{-1}}$$

$$\boxed{A = PDP^T} \quad \because P \text{ is an orthogonal } \boxed{P^{-1} = P^T}$$

Find the eigen decomposition of $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}_{2\times 2}$

$$\lambda^2 - (6)\lambda + 5 = 0$$

$$\lambda^2 - (6\lambda) + 5 = 0$$

$$\lambda^2 - 5\lambda - \lambda + 5 = 0$$

$$\lambda(\lambda - 5) - (\lambda - 5) = 0$$

$$\lambda = 1, 5$$

$(3-\lambda)^2 - 4 = 0$

$\lambda^2 + 9 - 6\lambda - 4 -$

$\lambda^2 - 6\lambda + 5 = 0$

E.vec corr. to E.val $\lambda = 5$

let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an E.vec. corr. to E.val $\lambda = 5$

$$[A - 5I]X = 0$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$$

rank = 1

$n - r = 1$

$-x + y = 0$

$\boxed{y = x}$

let $x = k \therefore y = k$

$X = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is E.vec.

E.vec corr. to E.val $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

Rank = 1 $\quad x + y = 0$

$n - r = 1 \quad x = -y$

let $y = k$

$x = -k$

$\therefore X_2 = k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is E.vec

$x_1 = \begin{bmatrix}1\\1\end{bmatrix}$ $x_2 = \begin{bmatrix}-1\\1\end{bmatrix}$ $x_1 \perp x_2 \rightarrow$ pairwise orthogonal

orthogonal matrix $P = \left[\frac{x_1}{||x_1||} \quad \frac{x_2}{||x_2||}\right]$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{eigen values}$$

The eigen decomposition of $A = PDP^T$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} & 5/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5+1}{2} & \frac{5}{2} - \frac{1}{2} \\ \frac{5}{2} - \frac{1}{2} & \frac{5}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

**Note**:- The matrix A is diagonolisable matrix then eigen decomposition of $A = PDP^{-1}$