18/03/24

(2) Higher Order Differential Equation

→ $D = \frac{d}{dx}$ is called differential operator so that $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots \frac{d^n}{dx^n}$ are denoted by $D, D^2, D^3, \ldots D^n$.

⇒ An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n(x)y = Q(x)$ where, $P_1(x), P_2(x) \ldots P_n(x)$ are all continuous real valued functions of x is called linear differential equation of order n.

Linear differential equation with constant coefficients: an

⇒ L.D.E with constant coefficients

eq^n of the form $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots P_n y = Q(x)$ —— (1)

where $P_1, P_2 \ldots P_n$ are constants and $Q(x)$ is a continuous funtion is called L.D.E with constant coefficients.

→ using the differential operator (1) can be written as $D^n y + P_1 D^{n-1} y + \ldots + P_n y = Q(x)$

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n)y = Q(x)$$

$$\Rightarrow f(D)y = Q(x) \quad \text{—— (2)}$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$ is a polynomial in D.

→ The solution of ② is given by

$$y = y_c + y_p \quad (y_c - y_{C.F},\ y_p = y_{PI})$$

where, → y_c is complementary function
which is the G.S of $f(D)y = 0$

→ y_p is particular integral. which is
obtained from $y_P = \frac{1}{f(D)} Q(x)$

[LDE in second and higher order].

TO Find the complementary function
(the G.S of $f(D)y = 0$)

I Consider an auxillary equation $f(m) = 0$

Note: If $y_1, y_2 \ldots y_n$ are independent
solutions of $f(D)y = 0$,
then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is the G.S
where $c_1, c_2 \ldots c_n$ are arbitary constants

Since, $f(D)$ is a polymonial of degree n
than $f(n) = 0$ is an algebraic eq^n of degree n
so, there will be n roots.
let them be, $m_1, m_2, \ldots m_n$.
Based on the nature of the roots the
complimentary function will be written as
follows.

Nature of the root	Complimentary function
1) If all roots are real & distinct i.e, $m_1 \neq m_2 \neq \dots \neq m_n$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
2) If one real root is repeated and rest of the real roots are distinct i.e. $m_1 = m_2 \neq m_3 \neq \dots \neq m_n$	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
3) If one real root is repeated 3 times and rest of the roots are real & distinct. i.e., $m_1 = m_2 = m_3 \neq m_4 \dots \neq m_n$	$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + c_n e^{m_n x}$
4) If two roots are complex and rest of the roots are real distinct. i.e., $m_1 = a+ib$, $m_2 = a-ib$, $m_3 \neq 3_4 \neq \dots \neq m_n$	$y = e^{ax}[c_1 \cos bx + c_2 \sin bx] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
5) If two complex roots are repeated twice & rest of the roots are real and distinct i.e, $m_1 = a+ib$, $m_2 = a+ib$, $m_3 = a-ib$, $m_4 = a-ib$, $m_5 \neq m_6 \neq \dots \neq m_n$	$y = e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$

Q1) solve $y'' - y = 0$

G.D.E $\Rightarrow y'' - y = 0$

$\Rightarrow D^2 y - y = 0$

$(D^2 - 1)y = 0$

this is of the form $f(D)y = 0$

where $f(D) = D^2 - 1$

consider an Auxillary Eq^n $f(m) = 0$

$m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

let $m_1 = 1$, $m_2 = -1$ (real & distinct)

$\therefore$ sol^n is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{x} + c_2 e^{-x}$ //

I (2) $(D^3 - 1)y = 0$

~~G.D.E~~ = $(D^3 - 1)y = 0$

this is of the form $f(D)y = 0$

$f(D) = D^3 - 1$

A.E $f(m) = 0$

$m^3 - 1 = 0$

$m = \omega, \omega^2, 1$

$m = 1, \dfrac{-1 + \sqrt{3}i}{2}, \dfrac{-1 - \sqrt{3}i}{2}$

$\therefore$ Sol^n is $y = c_1 e^{x} + e^{-1/2 x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

$= c_1 e^{x} + e^{-1/2 x}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$

(3) $(D^2 + 6D + 9$

$f(D$

$f($

A.E

$\therefore$

(4) $(D^2$

(5) y

(3) $(D^2+6D+9)y=0$

$f(D)y=0$

$f(D)=D^2+6D+9$

A.E $f(m)=0$

$m^2+6m+9=0$

$m=-3,-3$ (real, repeated twice)

$\therefore y=(c_1+c_2x)e^{-3x}$

(4) $(D^2+4)^2y=0$

$f(D)y=0$

$f(D)=(D^2+4)^2$

A.E $f(m)=(m^2+4)^2=0$

$m^4+8m^2+16=0$

$m=2i,-2i,2i,-2i$

$=\pm 2i, \pm 2i$

$y=e^{0x}[(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x]$

$=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x.$

(5) $y'''+16y'=0$, $y(0)=1$, $y'(0)=2$, $y'''(0)=-1$

$y'''+16y'=0$

$D^3y+16Dy=0$

$(D^3+16D)y=0$

$f(D)y=0$

A.E $f(m)=m^3+16m=0$

$m=0,4i,-4i$

$\therefore y=c_1e^{0x}+e^{0x}[c_2\cos 4x+c_3\sin 4x]$

$y=c_1+[c_2\cos 4x+c_3\sin 4x]$

$y(0) = 1 \Rightarrow$

(8) Consider the equation $f(D)y = Q(x)$ then Particular Integral P.I is $y_P = \frac{1}{f(a)} e^{ax}$

Consider the Eq^n $f(D)y = Q(x)$

$\Rightarrow$ PI is $y_P = \frac{1}{f(D)} Q(x)$

Case 1: If $Q(x) = e^{ax}$ then $y_P = \begin{cases} \frac{1}{f(a)} e^{ax} & \text{if } f(a) \neq 0 \\ x^K \left[\frac{1}{f^{(K)}(a)} e^{ax}\right] & \text{if } f(a) = f'(a) = \dots = f^{(K-1)}(a) = 0,\ f^{(K)}(a) \neq 0 \end{cases}$

$D \to a$

Q) solve $(D^2-1)y = e^x + 2e^{3x}$

Sol: G.D.E $(D^2-1)y = e^x + 2e^{3x}$

this is of the form $f(D)y = Q(x)$

where $f(D) = D^2-1$, $Q(x) = e^x + 2e^{3x}$

The soln is $y = y_c + y_P$

To find y_c: consider $(D^2-1)y = 0$

A.E $f(m) = 0 \Rightarrow m^2-1=0 \Rightarrow m = \pm 1$ (real & distinct)

$y_c = c_1 e^{1x} + c_2 e^{-x}$

To find y_P: $y_P = \frac{1}{D^2-1}\left[e^x + 2e^{3x}\right]$

$= \frac{1}{D^2-1} e^{x} + 2\left[\frac{1}{D^2-1} e^{3x}\right]$

$D \to a = 1$ $\qquad D \to a = 3$

$= x\left[\frac{1}{2D} e^{x}\right] + 2\left[\frac{1}{3^2-1} e^{3x}\right]$

$= \frac{x}{2}\left[e^{x}\right] + 2\left[\frac{1}{8}\right] e^{3x}$

$= \frac{xe^{x}}{2} + \frac{1}{4} e^{3x}$

the sol is $y = y_c + y_p$

$y = c_1 e^{x} + c_2 e^{-x} + \frac{xe^{x}}{2} + \frac{e^{3x}}{4}$.

Q) $y'' - y' - 2y = 3e^{2x}$, $y(0) = 0$, $y'(0) = -2$

$D^2 y - Dy - 2y = 3e^{2x}$

$(D^2 - D - 2)y = 3e^{2x}$

$f(D)y = Q(x)$

the sol^n is $y = y_c + y_p$

$y_c = (D^2 - D - 2)y = 0$

A.E $f(m) = m^2 - m - 2 = 0 \Rightarrow m = -1, 2$

$y_c = c_1 e^{-x} + c_2 e^{2x}$

$y_p = \frac{1}{D^2 - D - 2}(3e^{2x})$ $\qquad$??-!

$D \to 2$

$y_p = x\left[\frac{1}{f'(D)}(3e^{2x})\right]$ $\quad = x\left(\frac{1}{2D-1} 3e^{2x}\right)$

$$= x\left(\frac{1}{2\cdot 2-1}\, 3e^{2x}\right) = xe^{2x}.$$

$$\therefore\ y = c_1e^{-x} + c_2e^{2x} + xe^{2x} \text{ --- } (1)$$

$$y' = -c_1e^{-x} + 2c_2e^{2x} + e^{2x} + 2xe^{2x} \text{ --- } (2)$$

$y(0)=0 \Rightarrow x=0, y=0$

sub in (1) $\Rightarrow$ $0 = c_1 + c_2$ --- (3)

$y'(0) = -2$, sub in (2)

$$-2 = -c_1 + 2c_2 + 1$$

$$c_1 - 2c_2 = 3 \text{ --- } (4)$$

Solving (3) & (4) $\Rightarrow c_2 = -1,\ c_1 = 1$

Sol^n is $y = e^{-x} - e^{2x} + xe^{2x}$ //.

Q3) Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

$$f(D)y = Q(x)$$

Sol^n: $y = y_c + y_p$

A.E $f(m) \Rightarrow m^3 - 5m^2 + 7m - 3 = 0$

$$m = 3, 1, 1.$$

$$y_c = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x}$$

$$y_c = (c_1 + c_2x)e^{x} + c_3e^{3x}$$

$$y_p = \frac{1}{D^3 - 5D^2 + 7D - 3}\left[e^{2x}\left[\frac{e^x + e^{-x}}{2}\right]\right]$$

$$\left(\because \cosh x = \frac{e^x + e^{-x}}{2}\right)$$

$$y_P = \frac{1}{D^3-5D^2+7D-3}\left[\frac{e^{3x}}{2}+\frac{e^x}{2}\right]$$

27 −45 + 21 − 3

48

$$\Rightarrow y_P = \frac{1}{D^3-5D^2+7D-3}\left(\frac{e^{3x}}{2}\right) + \frac{1}{D^3-5D^2+7D-3}\left[\frac{e^x}{2}\right]$$

$D \to 3$ $\quad$ $D \to 1$

$$y_P = x\left[\frac{1}{f'(D)}\left[\frac{e^{3x}}{2}\right]\right] + x^2\left[\frac{1}{f''(D)}\left[\frac{e^x}{2}\right]\right]$$

$$y_P = \frac{x}{2}\left[\frac{1}{3D^2-10D+7}e^{3x}\right] + \frac{x^2}{2}\left[\frac{1}{6D-10}e^x\right]$$

$$= \frac{x}{2}\left[\frac{1}{4}e^{3x}\right] + \frac{x}{2}\left[\frac{-1}{4}e^x\right]$$

$$= \frac{x}{8}e^{3x} - \frac{x}{8}e^x$$

$$y = y_P + y_c = (c_1 + c_2x)e^x + c_3e^{3x} + \frac{x}{8}e^{3x} - \frac{x}{8}e^x.$$

Q4) $(D^2+6D+9)^2 y = 2e^{-3x}$

2(2D+6)

2(D²+6D+9)(2D+6)

12( )

24

$f(D)y = Q(x)$

$y = y_c + y_p$

A.E $\Rightarrow f(m) = 0 \Rightarrow (m^2+6m+9)^2 = 0$

$$(m+3)^4 = 0$$

$$m = -3, -3, -3, -3$$

$$y_c = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{-3x}$$

$$y_p = \frac{1}{(D+3)^4}2e^{-3x}$$

$D \to -3$

$$y_p = x^4\left[\frac{1}{f''''(D)}2e^{-3x}\right] = x^4\left[\frac{1}{\cancel{24}_{12}}\cancel{2}e^{-3x}\right]$$

$$= \frac{x^4}{12}e^{-3x}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{-3x} + \frac{x^4}{12} e^{-3x}$$

Case 2: if $Q(x) = \sin bx$ (or $\cos bx$)

Then $y_p = \frac{1}{f(D)} \sin bx$ (or $\cos bx$)

Let $f(D) = \phi(D^2)$ $\quad D^2 \rightarrow -b^2$

$\rightarrow$ if $\phi(-b^2) \neq 0$, then $y_p = \frac{1}{\phi(-b^2)} \sin bx$ (or $\cos bx$)

$\rightarrow$ if $\phi(-b^2) = 0$ & $\phi'(-b^2) \neq 0$ then

$$y_p = x \left[\frac{1}{\phi'(-b^2)} \sin bx \text{ (or } \cos bx) \right]$$

Q1) Solve $(D^2 - 3D + 2)y = \cos 2x$

Sol: $(f(D))y = Q(x)$

A.E $\Rightarrow f(m) = 0 \Rightarrow m^2 - 3m + 2 = 0$

$m = 1, 2$

$y = y_c + y_p$

$y_c = c_1 e^x + c_2 e^{2x}$

$$y_p = \frac{1}{(D^2 - 3D + 2)} \cos 2x$$

Let $f(D) = \phi(D^2)$

$D^2 \rightarrow -b^2$

$D^2 \rightarrow -2^2 = -4$

$$y_p = \frac{1}{(-4 - 3D + 2)} \cos 2x$$

$$y_P = \frac{-1}{2+3D}\cos 2x$$

$$= \frac{-(2-3D)}{(2+3D)(2-3D)}\cos 2x$$

$$= \frac{-(2-3D)}{4-9D^2}\cos 2x \qquad D^2 \to -2^2$$

$$= \frac{-(2-3D)}{40}\cos 2x = \frac{-1}{40}\left[2\cos 2x - 3D\cos 2x\right]$$

$$= \frac{-1}{40}\left[2\cos 2x - 3(-2\sin 2x)\right]$$

$$= \frac{-1}{20}\cos 2x - \frac{3}{20}\sin 2x$$

$$\Rightarrow y = y_c + y_p = c_1e^{x} + c_2e^{2x} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$$

Q) Solve $(D^2-1)^2 y = 2\sin^2 x$

It is of the form $(f(D))y = Q(x)$

$y = y_c + y_p$.

A.E $\Rightarrow f(m) = 0 \Rightarrow (m^2-1)^2 = 0$

$m = \pm 1, \pm 1$

$$y_c = (c_1 + c_2x)e^{x} + (c_3 + c_4x)e^{-x}$$

$$y_p = \frac{1}{(D^2-1)^2} 2\sin^2 x = \frac{1}{(D^2-1)^2}\cancel{2}\left[\frac{1-\cos 2x}{\cancel{2}}\right]$$

$$= \frac{1}{(D^2-1)^2}\left[1-\cos 2x\right]$$

$$= \frac{1}{(D^2-1)^2} - \frac{1}{(D^2-1)^2}\cos 2x$$

$$= 1 - \frac{1}{25}\cos 2x \,//.$$

Q_3) Solve $(D^2-4D+3)y = \sin x \sin 3x + e^{3x}$

$f(D)y = Q(x)$.

$y = y_c + y_p$

A.E $\Rightarrow f(m)=0 \Rightarrow m^2-4m+3=0$

$m = 1, 3$

$y_c = c_1 e^{x} + c_2 e^{3x}$

$$y_p = \frac{1}{(D^2-4D+3)} \frac{2\sin x \sin 3x}{2} + \frac{1}{(D^2-4D+3)} e^{3x}$$

$$= \frac{1}{2(D^2-4D+3)} [\cos(2x) - \cos(4x)] + \frac{1}{(D^2-4D+3)} e^{3x}$$

$$= \frac{1}{2(D^2-4D+3)} (\cos(2x)) - \frac{1}{2(D^2-4D+3)} [\cos(4x)] + \frac{1}{(D^2-4D+3)} e^{3x}$$

$D^2 \to -2^2 \qquad D^2 \to -4^2 \qquad D \to 3$

$$= \frac{-1}{2[1+4D]} (\cos(2x)) + \frac{1}{2(13+4D)} [\cos(4x)] + x\left[\frac{1}{f'(D)} e^{3x}\right]$$

$$= \frac{-(1-4D)}{2[1-16D^2]} \cos 2x + \frac{(13-4D)}{2(169-16D^2)} \cos 4x + x\left[\frac{1}{2} e^{3x}\right]$$

$$= \frac{-(1-4D)}{2(65)} \cos 2x + \frac{(13-4D)}{2(425)} \cos 4x + \frac{x}{2} e^{3x}$$

$$= \frac{-1}{130} \cos 2x + \frac{4}{130} D\cos 2x + \frac{13}{850} \cos 4x - \frac{2}{425} D\cos 4x$$

$$+ \frac{x}{2} e^{3x}$$

$$= \frac{-1}{130}\cos 2x - \frac{8}{130}\sin 2x + \frac{13}{850}\cos 4x + \frac{8}{425}\sin 4x$$

$$y = c_1 e^{x} + c_2 e^{3x} - \frac{1}{130}\cos 2x - \frac{8}{130}\sin 2x + \frac{13}{850}\cos 4x + \frac{8}{425}\sin 4x + \frac{x}{2}e^{3x}$$

(4) $(D^2+9)^2 y = \cos 3x$

It is of the form $f(D)y = Q(x)$

Sol^n: $y = y_c + y_p$

A.E $\Rightarrow$ y_c: $f(m)=0 \Rightarrow (m^2+9)^2 = 0$

$m = \pm 3i, \pm 3i$

$$y_c = e^{0x}\left[(c_1+c_2x)\cos 3x + (c_3+c_4x)\sin 3x\right]$$

$$y_c = (c_1+c_2x)\cos 3x + (c_3+c_4x)\sin 3x.$$

$$y_p = \frac{1}{(D^2+9)^2}\cos 3x$$

$(b=3)$ [if $(D^2 \to -3^2) \Rightarrow f(D) = 0$]

$$= x^2 \frac{1}{f''(D)}\cos 3x$$

$2(D^2+9)$

$D^4 + 81 + 18D^2$

$4D^3 + 36D$

$12D^2 + 36$

$-108 + 36$

-72

$$= x^2 \frac{1}{12D^2+36}\cos 3x \qquad D^2 \to -3^2 = -9$$

$$= \frac{-x^2}{72}\cos 3x$$

$$y = (c_1+c_2x)\cos 3x + (c_3+c_4x)\sin 3x - \frac{x^2}{72}\cos 3x.$$

(5) $y'' + 4y' + 4y = 4\cos x + 3\sin x$, $y(0)=0$, $y'(0)=0$.

$D^2y + 4yD + 4y = 4\cos x + 3\sin x$

$(D^2+4D+4)y = 4\cos x + 3\sin x$

$f(D)y = Q(x)$

Sol^n: $y = y_c + y_p$

(6)

$A\cdot E \Rightarrow y(m)=0 \Rightarrow m^2+4m+4=0$

$m=-2,-2$

$y_c = (c_1+c_2x)e^{-2x}$

$y_p = \frac{1}{D^2+4D+4}[4\cos x+3\sin x]$

$= \frac{4}{D^2+4D+4}\cos x + \frac{3}{D^2+4D+4}\sin x$

$D^2 \to -1^2 \qquad D^2 \to -1^2$

$= \frac{4}{4D+3}\cos x + \frac{3}{4D+3}\sin x$

$= \frac{4(4D-3)}{(4D+3)(4D-3)}\cos x + \frac{3(4D-3)}{(4D+3)(4D-3)}\sin x$

$= \frac{4(4D-3)}{16D^2-9}\cos x + \frac{3(4D-3)}{16D^2-9}\sin x$

$D^2 \to -1^2 \qquad D^2 \to -1^2$

$= -\frac{4(4D-3)}{25}\cos x - \frac{3(4D-3)}{25}\sin x$

$= \frac{-4}{25}[4D\cos x - 3\cos x] - \frac{3}{25}[4D\sin x - 3\sin x]$

$= \frac{-4}{25}[-4\sin x - 3\cos x] - \frac{3}{25}[4\cos x - 3\sin x]$

$= \frac{16}{25}\sin x + \cancel{\frac{12}{25}\cos x} - \cancel{\frac{12}{25}\cos x} + \frac{9}{25}\sin x$

$= \sin x$

$y(0)=0 \Rightarrow c_1=0$

$y'(0)=0 \Rightarrow c_2=-1$

$y = (c_1+c_2x)e^{-2x} + \sin x$

$y = (-x)e^{-2x} + \sin x$

(6) $y'' + 4y' + 20y = 23\sin t - 15\cos t$, $y(0) = 0$, $y'(0) = -1$

$D^2y + 4Dy + 20y = 23\sin t - 15\cos t$

$(D^2 + 4D + 20)y = 23\sin t - 15\cos t$

$(f(D))y = Q(x)$

Sol^n: $y = y_c + y_p$

A.E $y(m) = 0 \Rightarrow m^2 + 4m + 20 = 0$

$m = -2+4i, -2-4i$

$y_c = e^{-2x}[c_1\cos 4x + c_2\sin 4x]$

$$y_p = \frac{23}{D^2+4D+20}\sin t - \frac{15}{D^2+4D+20}\cos t$$

$D^2 \to -1$ $\quad D^2 \to -1$

$$y_p = \frac{23}{4D+19}\sin t - \frac{15}{4D+19}\cos t$$

$$= \frac{23(4D-19)}{16D^2-361}\sin t - \frac{15(4D-19)}{16D^2-361}\cos t$$

$D^2 \to -1$ $\quad D^2 \to -1$

$$= -\frac{23}{377}(4D-19)\sin t + \frac{15}{377}(4D-19)\cos t$$

$$= \frac{-23}{377}[4D\sin t - 19\sin t] + \frac{15}{377}[4D\cos t - 19\cos t]$$

$$= \frac{-23}{377}(4\cos t) + \frac{437}{377}\sin t + \frac{15}{377}(4\sin t) - \frac{285}{377}\cos t$$

$$= \frac{-92}{377}\cos t + \frac{437}{377}\sin t - \frac{60}{377}\sin t - \frac{285}{377}\cos t$$

$= -\cos t + \sin t$

$y_p = \sin t - \cos t$

$$y = e^{-2x}[c_1 \cos 4x + c_2 \sin 4x] + \sin x - \cos x$$

$y(0) = 0$

$\Rightarrow 0 = c_1 - 1 \Rightarrow c_1 = 1$

$y'(0) = -1$

$$y' = -2e^{-2x}[c_1 \cos 4x + c_2 \sin 4x] + e^{-2x}[-c_1 4 \sin 4x + 4c_2 \cos 4x] + \cos x + \sin x$$

$$-1 = -2[c_1] + [4c_2] + 1$$

$$-1 = -2 + 4c_2 + 1 \Rightarrow c_2 = 0$$

$$\therefore y = e^{-2x}[\cos 4x] + \sin x - \cos x$$

<u>case 3</u>: If $Q(x) = x^k$ then $y_p = \frac{1}{f(D)} x^k$

$\Rightarrow$ Express $f(D)$ in the form of $1 \pm \phi(D)$ by taking lower degree term common from $f(D)$. Then,

$$y_p = \frac{1}{1 \pm \phi(D)} x^k$$ (or a polynomial of deg k)

$$= [1 \pm \phi(D)]^{-1} x^k$$

Expand ~~$(1 \pm x)$~~

$[1 \pm \phi(D)]^{-1}$ using binomial expansion upto D^k

[$\because x^k$ has k non-zero derivatives]

And then operate each term in the expansion on to the function x^k so that y_P is obtained.

Q1) solve $(D^2+D+1)y = x^3$

$$f(D)y = Q(x)$$

$$y = y_c + y_P$$

A.E $f(m)=0 \Rightarrow m^2+m+1=0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore y_c = e^{-1/2 x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$y_P = \frac{1}{D^2+D+1}\cdot x^3$$

$$y_P = \frac{1}{1+(D+D^2)}x^3 = \left[1+(D+D^2)\right]^{-1}x^3$$

$$y_P = \left[1-(D+D^2)+(D+D^2)^2-(D+D^2)^3\right]x^3$$

$$= \left[1 - D - \cancel{D^2} + \cancel{D^2} + 2D^3 - D^3\right]x^3$$

$$= \left[1 - D + D^3\right]x^3$$

$$= x^3 - Dx^3 + D^3x^3$$

$$= x^3 - 3x^2 + 6$$

[$\because D^4$ & higher powers of D are vomitted since 4^{th} & higher order derivatives of x^3 are zero]

$\therefore$ the sol^n of G.D.E is

$$y = y_c + y_P$$

$$= e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 6$$

(2) $(D^3+2D^2+D)y = e^{2x}+x^2+x+\sin 2x$.

Sol: G.D.E $\Rightarrow (D^3+2D^2+D)y = e^{2x}+x^2+x+\sin 2x$

$f(D)y = Q(x)$

Soln: $y = y_c + y_p$

A.E $\Rightarrow y(m)=0 \Rightarrow m^3+2m^2+m=0$

$m = -1, 0, -1$

$\therefore y_c = [c_1+c_2x]e^{-x} + c_3e^{0x}$

$y_c = [c_1+c_2x]e^{-x} + c_3$

$$y_p = \frac{1}{D^3+2D^2+D}\left[e^{2x}+x^2+x+\sin 2x\right]$$

$$= \frac{1}{D^3+2D^2+D}e^{2x} + \frac{1}{D^3+2D^2+D}(x^2+x) + \frac{1}{D^3+2D^2+D}\sin 2x$$

$$y_{p_1} = \frac{1}{D^3+2D^2+D}e^{2x}$$

$a=2$

$$= \frac{1}{8+8+2}e^{2x} = \frac{1}{18}e^{2x}$$

$$y_{p_2} = \frac{1}{D^3+2D^2+D}(x^2+x)$$

$$= \frac{1}{D(1+D^2+2D)}(x^2+x)$$

$$= \frac{1}{D}\left[[1+D^2+2D]^{-1}(x^2+x)\right]$$

$$= \frac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2\right](x^2+x)$$

$$= \frac{1}{D}\left[1 - D^2 - 2D + 4D^2\right](x^2+x)$$

$2x+1$

2

$$= \frac{1}{D}\left[1 - 2D + 3D^2\right](x^2+x)$$

$$= \frac{1}{D}\left[x^2+x - 2D(x^2+x) + 3D^2(x^2+x)\right]$$

$$= \frac{1}{D}\left[x^2+x - 2(2x+1) + 3(2)\right]$$

$$= \frac{1}{D}\left[x^2+x-4x-2+6\right] = \frac{1}{D}\left\{x^2-3x+4\right\}$$

$$= \int (x^2-3x+4)\,dx = \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$$y_{P_3} = \frac{1}{D^3+2D^2+D}\sin 2x$$

$b = 2$

$D^2 \to -4$

$$= \frac{1}{-4D-8+D}\sin 2x = \frac{-1}{3D+8}\sin 2x$$

$$= -\frac{(3D-8)}{9D^2-64}\sin 2x$$

$D^2 \to -4$

$$= \frac{3D-8}{100}\sin 2x$$

$$= \frac{1}{100}\left[3D\sin 2x - 8\sin 2x\right]$$

$$= \frac{3}{100}\left[2\cos 2x\right] - \frac{8}{100}\sin 2x)$$

$$= \frac{1}{50}\left(3\cos 2x - \frac{4}{50}\sin 2x\right)$$

$$y_p = \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{1}{50}\left(3\cos 2x - 4\sin 2x\right)$$

$y = y_p + y_c$

$y = c_3 + (c_1 + c_2 x)e^{-x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x +$

$\frac{1}{50}[3\cos 2x - 4\sin 2x]$

③ $(D^2+3D+2)y = 2\cos(2x+3) + 2e^x + x^2$

$f(D)y = Q(x)$

Sol^n: $y = y_c + y_p$

A.E $\Rightarrow$ $f(m) = 0 \Rightarrow m^2 + 3m + 2 = 0$

$m = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_{P_1} = \frac{(2)}{(D^2+3D+2)} \cos(2x+3)$

$b = 2$

$D^2 \rightarrow -4$

$= \frac{2}{3D-2} \cos(2x+3) = \frac{2(3D+2)}{9D^2-4} \cos(2x+3)$

$D^2 \rightarrow -4$

$= \frac{2(3D+2)}{-40} \cos(2x+3)$

$= \frac{-1}{20}[3D\cos(2x+3) + 2\cos(2x+3)]$

$= \frac{-1}{20}[3(-2\sin(2x+3)) + 2\cos(2x+3)]$

$= \cancel{\frac{3}{10}[\sin(2x+3) + \cos(2x+3)]}$

$= -\frac{1}{10}[\cos(2x+3) - 3\sin(2x+3)]$

$$y_{p_2} = \frac{2}{(D^2+3D+2)} e^x$$

$a=1$

$D \to a = 1$

$$y_{p_2} = \frac{2}{6} e^x = \frac{1}{3} e^x$$

$$y_{p_3} = \frac{1}{D^2+3D+2} x^2$$

$$= \frac{1}{2\left[1+\frac{(D^2+3D)}{2}\right]} x^2$$

$$= \frac{1}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2+3D}{2}+\frac{9D^2}{4}\right] x^2$$

$$= \frac{1}{2}\left[\frac{4-2D^2-6D+9D^2}{4}\right] x^2$$

$$= \frac{1}{2}\left[\frac{7D^2-6D+4}{4}\right] x^2$$

$$= \frac{1}{8}\left[7D^2x^2-6Dx^2+4x^2\right]$$

$$= \frac{1}{8}\left[7(2)-6(2x)+4x^2\right]$$

$$= \frac{1}{8}\left[14-12x+4x^2\right]$$

$$y_{p_3} = \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

$$y_p = -\frac{1}{10}\left[\cos(2x+3)-3\sin(2x+3)\right]+\frac{e^x}{3}+ \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

$y = y_c + y_p$

$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{10}\left[\cos(2x+3) - 3\sin(2x+3)\right]$
$\quad + \frac{e^x}{3} + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$

Case 4: If $Q(x) = e^{ax} V$ where $V = \sin bx / \cos bx / x^k$

then $y_p = \frac{1}{f(D)} e^{ax} V = e^{ax}\left(\frac{1}{f(D+a)} V\right)$

$\hookrightarrow$ apply above 3 cases

① $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x$

$f(D)y = Q(x)$

$f(m) = 0 \Rightarrow m^2 - 4m + 3 = 0$

$m = 1, 3$

$y_c = c_1 e^x + c_2 e^{3x}$

$y_p = \frac{1}{D^2 - 4D + 3}\left[e^x \cos 2x + \cos 3x\right]$

$y_{p_1} = \frac{1}{D^2 - 4D + 3}\left(e^x \cos 2x\right)$

$= e^x \left[\frac{1}{D^2 - 4D + 3} \cos 2x\right] \quad D \to (D+1)$

$= e^x \left\{\frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x\right\}$

$$= e^x\left[\frac{1}{D^2-2D}\cos 2x\right]$$

case 2

$D^2 \to -2^2$

$$= -e^x\left[\frac{1}{4+2D}\cos 2x\right]$$

$$= -e^x\left[\frac{4-2D}{16-4D^2}\cos 2x\right]$$

$D^2 \to -4$

$$= \frac{-e^x}{32}\left[4\cos 2x - 2D\cos 2x\right]$$

$$= \frac{-e^x}{32}\left[4\cos 2x + 4\sin 2x\right] = -\frac{e^x}{8}\left[\cos 2x + \sin 2x\right]$$

$$y_{P_2} = \frac{1}{D^2-4D+3}\cos 3x$$

$D^2 \to -9$

$$= \frac{-1}{4D+6}\cos 3x$$

$$= \frac{-(4D-6)}{16D^2-36}\cos 3x$$

$D^2 \to -9$

$$= \frac{4D-6}{180}\cos 3x$$

$$= \frac{1}{180}\left[4D\cos 3x - 6\cos 3x\right]$$

$$= \frac{1}{480}\left[4(-3\sin 3x - 6\cos 3x\right]$$

$$= \frac{-1}{80}\left[\cos 3x + 2\sin 3x\right]$$

Q) solve $(D^2+2)y = x^2 e^{3x} + e^x \cos 2x$

Sol: Sol^n $y = y_c + y_p$

$f(m)=0 \Rightarrow m^2+2=0$

$m = \sqrt{2}i, -\sqrt{2}i$

$y_c = e^{0x}[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x]$

$= c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x]$

$y_p = \frac{1}{D^2+2}[x^2 e^{3x} + e^x \cos 2x]$

$= \underbrace{\frac{1}{D^2+2} x^2 e^{3x}}_{y_{p_1}} + \underbrace{\frac{1}{D^2+2} e^x \cos 2x}_{y_{p_2}}$

$y_{p_1} = \frac{1}{D^2+2} x^2 e^{3x}$ $\quad D \to D+a = D+3$

$= e^{3x}\left[\frac{1}{(D+3)^2+2} x^2\right]$

$= e^{3x}\left[\frac{1}{D^2+6D+11} x^2\right]$

$= \frac{e^{3x}}{11}\left[\frac{1}{1+\frac{D^2+6D}{11}} x^2\right]$

$= \frac{e^{3x}}{11}\left[1+\frac{D^2+6D}{11}\right]^{-1} x^2$

$= \frac{e^{3x}}{11}\left[1-\frac{D^2+6D}{11}+\left(\frac{D^2+6D}{11}\right)^2\right] x^2$

$$= \frac{e^{3x}}{11}\left[1 - \frac{D^2}{11} - \frac{6D}{11} + \frac{36D^2}{121}\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121}\right]x^2$$

$$= \frac{e^{3x}}{11}\left[x^2 - \frac{6}{11}(2x) + \frac{25}{121}(2)\right]$$

$$= \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$$

$$y_{P_2} = \frac{1}{D^2+2} e^x \cos 2x \quad [e^{ax}V \to a=1,\ V=\cos 2x]$$

$$= e^x\left[\frac{1}{D^2+2}\cos 2x\right]$$

$D \to D+1$

$$= e^x\left[\frac{1}{(D+1)^2+2}\cos 2x\right]$$

$$= e^x\left[\frac{1}{D^2+2D+3}\cos 2x\right] \quad (\text{case } 2)$$

$D^2 \to -4$

$$= e^x\left[\frac{1}{2D-1}\cos 2x\right]$$

$$= e^x\left[\frac{2D-1}{4D^2-1}\cos 2x\right] \quad D^2 \to -4$$

$$= e^x\left[\frac{2D+1}{-17}\cos 2x\right]$$

$$= -\frac{e^x}{17}\left[2D\cos 2x + \cos 2x\right] = -\frac{e^x}{17}\left[-4\sin 2x + \cos 2x\right]$$

$$= \frac{e^x}{17}\left[4\sin 2x - \cos 2x\right]$$

$\therefore y_p = \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] + \frac{e^x}{17}\left[4\sin 2x - \cos 2x\right]$

Complete Sol^n is

$$y = y_c + y_p$$

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$$
$$+ \frac{e^x}{17}\left[4\sin 2x - \cos 2x\right].$$

<u>Case 5</u>: If $Q(x) = xV$ where $V = e^{ax}(\sin bx / \cos bx)$

then $y_p = \frac{1}{f(D)}(xV) = x\left[\frac{1}{f(D)}V\right] - \left[\frac{f'(D)}{(f(D))^2}V\right]$

$$(\text{or}) \quad \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V.$$

Case 6: If $Q(x) = x^m V$ where $V = \sin bx$ or $\cos bx$

then $y_p = \frac{1}{f(D)}x^m V$.

if $V = \sin bx$ then $y_p = \frac{1}{f(D)}x^m \sin bx = \text{I.P of}$
$$\left[\frac{1}{f(D)}x^m e^{ibx}\right]$$
case 4

if $V = \cos bx$ then

$$y_p = \frac{1}{f(D)}x^m \cos bx = \text{R.P of}\left[\frac{1}{f(D)}x^m e^{ibx}\right]$$
case 4

Q1) solve $(D^2-2D+1)y = xe^x \sin x$

Sol:- $f(D)y = Q(x)$

$y = y_c + y_p$

$f(m)=0 \Rightarrow m^2-2m+1=0$

$m = 1, 1$

$y_c = (c_1+c_2x)e^x$

$y_p = \frac{1}{D^2-2D+1} xe^x \sin x$

$y_p = \frac{1}{(D-1)^2} e^x (x\sin x)$ $(e^{ax}v \rightarrow a=1, v = x\sin x)$

$y_p = e^x \left[\frac{1}{(D-1)^2} x\sin x \right]$

$D \rightarrow D+1$

$= e^x \left[\frac{1}{(D+1-1)^2} x\sin x \right] = e^x \left[\frac{1}{D^2} x \sin x \right]$

$(xv \rightarrow v = \sin x)$

$= e^x \left[x\left[\frac{1}{D^2}\sin x\right] - \frac{2D}{D^4}\sin x \right]$ $D^2 \rightarrow -1$

$= e^x \left[x(-\sin x) - \frac{2D}{(-1)^2}\sin x \right]$

$= e^x [-x\sin x - 2\cos x]$

$= -e^x [x\sin x + 2\cos x]$

$y = (c_1+c_2x)e^x \cancel{+ \frac{e^{x}}{D^2-2D+1}} - e^x[x\sin x + 2\cos x]$

Q2) $(D^2+4)y = x\cos^2 x$

$f(D)y = Q(x)$

$y = y_c + y_p$

$f(m)=0 \Rightarrow m^2+4=0$

$m = 2i, -2i$

$y_c = e^{0x}[c_1\cos 2x + c_2 \sin 2x]$

8) $(D^2+4)y = x\sin^2 x$

$f = f_c + f_p$

$f(m) = 0 \Rightarrow m^2+4=0$

$m = \pm 2i$

$y_c = e^{0x}[c_1\cos 2x + c_2\sin 2x]$

$y_c = [c_1\cos 2x + c_2\sin 2x]$

$y_p = \frac{1}{D^2+4} x\sin^2 x$

$1-\sin^2 x = \frac{1+\cos 2x}{2}$

$1 - \frac{(1+\cos 2x)}{2}$

$\frac{1+\cos 2x}{2}$

$= \frac{1}{D^2+4} x\left[\frac{1-\cos 2x}{2}\right]$

$= \frac{1}{2}\left[\frac{1}{D^2+4}x - \frac{1}{D^2+4}x\cos 2x\right]$

$y_{p_1} = \frac{1}{2}\left(\frac{1}{D^2+4}x\right)$

$= \frac{1}{8}\left[\frac{1}{\left(1+\frac{D^2}{4}\right)}x\right]$

$= \frac{1}{8}\left[1+\frac{D^2}{4}\right]^{-1}x$

$= \frac{1}{8}\left[1-\frac{D^2}{4}\right]x$

$= \frac{1}{8}\left[x - \frac{D^2}{4}x\right]$

$= \frac{x}{8}$

$2(D^2+4)(2D)$

$4(D^2+4)+4D(\ldots)$

$$y_{P_2} = \frac{-1}{2}\left(\frac{1}{D^2+4}x\cos 2x\right)$$

$$= \frac{-1}{2}\left[x\left[\frac{1}{D^2+4}\cos 2x\right] - \left[\frac{2D}{(D^2+4)^2}\cos 2x\right]\right]$$

$$= -\frac{1}{2}\left[x^2\left[\frac{1}{2D}\cos 2x\right] - 2D\left[\frac{x^2}{\phi''(D)}\cos 2x\right]\right]$$

$$= \frac{-1}{2}\left[x^2\left[\frac{1}{2D}\cos 2x\right] - 2D\left(\frac{x^2}{12D^2+16}\cos 2x\right]\right]$$

$$= \frac{-1}{2}\left\{\left[\frac{x^2}{2}\int \cos 2x\,dx\right] - 2D\left[\frac{x^2}{-32}\cos 2x\right]\right]$$

-48
16
32

$$= -\frac{1}{2}\left\{\left[\frac{x^2}{2}\cdot\frac{\sin 2x}{2}\right] + \frac{1}{16}D\left(x^2\cos 2x\right)\right]$$

$$= -\frac{1}{2}\left[\frac{x^2\sin 2x}{4} + \frac{1}{16}\left(-x^2 2\sin 2x\right) + \cos 2x\,(2x)\right)$$

$$= -\frac{1}{8}x^2\sin 2x + \frac{x^2}{16}\sin 2x - \frac{2x}{32}\cos 2x$$

$$y_p = \frac{x}{8} - \frac{x^2\sin 2x}{16} - \frac{x}{16}\cos 2x$$

$$\Downarrow$$

case 6 : $y_p = \frac{x}{8} - \frac{x^2\sin 2x}{16} - \frac{x}{32}\cos 2x$ ✓

~~Case 6~~.

Q) $(D^2+9)y = xe^{2x}\cos x$

$f(m)=0 \Rightarrow m^2+9=0$

$-\frac{1}{4}\left(-\left(\frac{}{3} \quad 8\right)\right)$

$y = y_c + y_p$

$''$

(2) $(D^2+1)x = t\cos 2t$, $x=0, \frac{dx}{dt} = 0$ at $t=0$

$f(D)x = Q(t)$

$m^2+1 = 0$

$m = \pm i$

$x_c = c_1 \cos t + c_2 \sin t$

$$x_p = \frac{1}{D^2+1} t\cos 2t$$

$$= t\left[\frac{1}{D^2+1}\cos 2t\right] - \left[\frac{2D}{(D^2+1)^2}\cos 2t\right]$$

$D^2 \to -4$

$$= t\left[\frac{1}{-3}\cos 2t\right] - \left[\frac{2D}{D^4+1+2D^2}\cos 2t\right]$$

$$= -\frac{t}{3}\cos 2t - 2D\left[\frac{1}{16+1-8}\cos 2t\right]$$

$$= -\frac{t}{3}\cos 2t - 2D\left[\frac{1}{9}\cos 2t\right]$$

$$= -\frac{t}{3}\cos 2t - \frac{2}{9}(-2\sin 2t)$$

$$x_p = -\frac{t}{3}\cos 2t + \frac{4}{9}\sin 2t$$

$$x = c_1\cos t + c_2 \sin t - \frac{t}{3}\cos 2t + \frac{4}{9}\sin 2t$$

$$\frac{dx}{dt} = -\frac{1}{3}\cos 2t + \frac{2t}{3}\sin 2t + \frac{8}{9}\cos 2t$$
$$- c_1 \sin t + c_2 \cos t$$

$$-\frac{1}{3} + \frac{8}{9} + c_2 = 0$$

$$c_2 = -\frac{5}{9}$$

$$x = 0 \Rightarrow \quad c_1 \cancel{-t} = 0$$

$$\therefore x = -\frac{5}{9}\sin t - \frac{t}{3}\cos 2t + \frac{4}{9}\sin 2t$$

(3) $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

Sol: $m^2 - 4m + 4 = 0$

$m = 2, 2$

$$y_c = (c_1 + c_2) e^{2x}$$

$$y_P = \frac{1}{D^2 - 4D + 4}(8x^2 e^{2x} \sin 2x).$$

$$= e^{2x}\left[\frac{1}{(D+2)^2 - 4(D+2) + 4} 8x^2 \sin 2x\right]$$

$$= 8e^{2x}\left[\frac{1}{D^2} \cancel{8}x^2 \sin 2x\right]$$

$$= 8e^{2x}\left[\text{I.P of } \frac{1}{D^2} x^2 e^{i2x}\right]$$

$$= 8e^{2x}. \text{ I.P of } e^{i2x}\left[\frac{1}{(D+2i)^2} x^2\right]$$

$$= 8e^{2x}. \text{ I.P of } e^{i2x}\left(\frac{1}{D^2 + 4Di - 4} x^2\right)$$

$$= 8e^{2x} \cdot I.P \text{ of } -\frac{e^{i2x}}{4}\left[\frac{1}{1-\frac{D^2+4Di}{4}} x^2\right]$$

$$= 8e^{2x} \; I.P \text{ of } -\frac{e^{i2x}}{4}\left[1-\frac{D^2+4Di}{4}\right]^{-1} x^2$$

$$= 8e^{2x} \; I.P \text{ of } -\frac{e^{i2x}}{4}\left[1+\frac{D^2+4Di}{4}+\frac{16D^2i^2}{16}\right] x^2$$

$$= 8e^{2x} \; I.P \text{ of } -\frac{e^{i2x}}{4}\left[x^2+\frac{1}{2}+i(2x)-2\right]$$

$$= 8e^{2x} \cdot I.P \text{ of } -\frac{1}{4}(\cos 2x + i\sin 2x)$$

$$\left[x^2+i2x-\frac{3}{2}\right]$$

$$= 8e^{2x}\left[-\frac{x\cos 2x}{2}-\frac{\sin 2x \cdot x^2}{4}+\frac{3\sin 2x}{8}\right]$$

$$= -2e^{2x}\left[2x\cos 2x+\sin 2x \cdot x^2-\frac{3\sin 2x}{2}\right]$$

$$= -2e^{2x}\left[2x\cos 2x+\sin 2x\left[x^2-\frac{3}{2}\right]\right]$$

$$y = (c_1+c_2x)e^{x} - 2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

Linear equations of second and higher order with variable coefficients.

1) Cauchy's (or) Cauchy-Euler linear Eqn:

An Eq^n of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} \cdots + a_n y = Q(x)$$

where, $a_0, a_1, a_2, \cdots a_n$ are constansts.

we will solve this Eq^n by reducing to linear equation with constant coefficients as follows:

Step 1: Let $x = e^z \Rightarrow z = \log x$ and $\frac{dz}{dx} = \frac{1}{x}$

Step 2: Replace $xD, x^2D^2, x^3D^3 \ldots x^nD^n$ by $\theta, \theta(\theta-1), \theta(\theta-1)(\theta-2) \ldots \theta(\theta-1)(\theta-2)\cdots(\theta-(n-1))$ in Cauchy's equation so that the given Eq^n will be reduced to linear eqn with constant coefficients. of the form

$$f(\theta)y = Q(z) \quad \text{—(I)}$$

the complete sol^n of (I) is $y = y_c + y_p$ (in terms of z)

Step 3: Finally replace z by $\ln x$ in the sol of (I)

Q) Solve $(x^2D^2 - 3xD + 1)y = \frac{\log x \sin(\log x) + 1}{x}$ —①

Sol: Let $x = e^z$ then $z = \log x$ and $\frac{dz}{dx} = \frac{1}{x}$

Now Replace xD, x^2D^2 by $\theta, \theta(\theta-1)$ in (1)

where $D=\frac{d}{dx}, \theta=\frac{d}{dz}$

then (1) $\Rightarrow \left(\theta(\theta-1)-3\theta+1\right)y = \frac{z\sin z+1}{e^z}$

$$(\theta^2-4\theta+1)y = e^{-z} z\sin z + e^{-z} \quad \text{--- (2)}$$

this is of the form $f(\theta)y = Q(z)$

where, $f(\theta) = \theta^2-4\theta+1$

It's solⁿ is $y = y_c + y_p$

To find y_c: consider an A.E $f(m)=0$

$m^2-4m+1=0 \rightarrow m = 2\pm\sqrt{3}$

$$y_c = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z}$$

To find y_p: $y_p = \frac{1}{\theta^2-4\theta+1}\left[e^{-z} z\sin z + e^{-z}\right]$

$$y_{p_1} = \frac{e^{-z} z\sin z}{\theta^2-4\theta+1}$$

$$= e^{-z}\left[\frac{1}{(\theta-1)^2-4(\theta-1)+1} z\sin z\right]$$

$$= e^{-z}\left[\frac{1}{\theta^2-6\theta+6} z\sin z\right]$$

$$= e^{-z}\left(z\left[\frac{1}{\theta^2-6\theta+6}\sin z\right) - \frac{2\theta-6}{(\theta^2-6\theta+6)^2}\sin z\right]$$

$$= e^{-z}\left[z\left(\frac{1}{-1-6\theta+6}\sin z\right) - \frac{2\theta-6}{(-1-6\theta+6)^2}\sin z\right]$$

$$= e^{-z}\left[z\left(\frac{1}{5-6\theta}\sin z\right) - \frac{2\theta-6}{(5-6\theta)^2}\sin z\right]$$

$$= e^{-z}\left[z\left(\frac{5+6\theta}{25-36\theta^2}\sin z\right) - \frac{(2\theta-6)\cancel{(5+6\theta)}}{25+36\theta^2-60\theta}\sin z\right)$$

$$= e^{-z}\left[z\left(\frac{5+6\theta}{61}\sin z\right) - \frac{2\theta-6}{-11-60\theta}\sin z\right.$$

$$= e^{-z}\left(\frac{z}{61}\left[5\sin z + 6\cos z\right] + \frac{(2\theta-6)(11-60\theta)}{(11+60\theta)(11-60\theta)}\sin z\right.$$

$$= e^{-z}\left[\frac{z}{61}\left[5\sin z + 6\cos z\right] + \frac{54\sin z + 382\cos z}{61^2}\right]$$

$$= e^{-z}\left[\frac{z}{61}\left[5\sin z + 6\cos z\right] + \frac{382\cos z + 54\sin z}{3721}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{\theta^2-4\theta+1}e^{-z}$$

$$= \frac{1}{(-1)^2-4(-1)+1}e^{-z} = \frac{1}{6}e^{-z}$$

20 - 4200 - 66 + 3600

3600 / 121

3870

$$\therefore y_p = y_{p_1} + y_{p_2}$$

$$= e^{-z}\left[\frac{z}{61}\left[5\sin z + 6\cos z\right] + \frac{382\cos z + 54\sin z}{3721}\right] + \frac{1}{6}e^{-z}$$

$$y = y_c + y_p$$

$$y = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{382\cos z + 54\sin z}{3721} + \frac{1}{6}\right]$$

$z \to \log x$

$$y = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + \frac{1}{x}\left[\frac{\log x}{61}(5\sin(\log x) + 6\cos(\log x)) + \frac{382\cos(\log x) + 54\sin(\log x)}{3721} + \frac{1}{6}\right]$$

Legendre's linear Eq^n:

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = Q(x) \quad \text{--- (1)}$$

where $a_0, a_1, \dots a_n$ are constants.

$Q(x)$ is a function of x.

Here, we will solve this Eq^n by reducing to linear equation with constant coefficients.

Steps to solve:

1. Let $ax+b = e^z$ then $z = \ln(ax+b)$, $\frac{dz}{dx} = \frac{a}{ax+b}$
2. Replace $(ax+b)D, (ax+b)^2D^2, (ax+b)^3D^3 \dots (ax+b)^n D^n$ by $a\theta, a^2\theta(\theta-1), a^3\theta(\theta-1)(\theta-2) \dots a^n\theta(\theta-1)\dots(\theta-(n-1))$ in eq (1) where,

$D = \frac{d}{dx}$ & $\theta = \frac{d}{dz}$

so that ① will be reduced to linear Eq^n with constant coefficients in form of

$f(\theta)y = Q(z)$ — ②

we will solve ② by using suitable formulas

3. Finally, Replace z by $\ln(ax+b)$ in sol of ②

Q) $\left[(x+1)^2 D^2 + (x+1)D\right] y = (2x+3)(2x+4)$ ①

Sol^n: this is legendre's linear Eq^n

with $a=1, b=1$

let $(x+1) = e^z$ then $z = \log(x+1)$

Replace $(x+1)D, (x+1)^2 D^2$ by $1.\theta, 1^2\theta(\theta-1)$

in ① where $D = \frac{d}{dx}, \theta = \frac{d}{dz}$

i.e, $\theta, \theta(\theta-1)$

$[\theta(\theta-1)+\theta)y = (2(e^z-1)+3)(2(e^z-1)+4)$

$\theta^2 y = (2e^z+1)(2e^z+2)$

$= 4e^{2z} + 6e^z + 2$ — ②

this is of the form $f(\theta)y = Q(z)$

It's sol^n is, $y = y_c + y_p$

To find y_c: consider an A.E

$m^2 = 0$

$m = 0, 0$

$y_c = c_1 + c_2 z$

$$y_p = \frac{1}{\theta^2}\left[4e^{2z} + 6e^{z} + 2\right]$$

$$= \frac{1}{\theta^2}4e^{2z} + \frac{1}{\theta^2}6e^{z} + \frac{1}{\theta^2}2$$

$$= \frac{4}{\theta}\int e^{2z}dz + \frac{6}{\theta}\int e^{z}dz + \frac{1}{\theta}\int 2\,dz$$

$$= \frac{4}{2}\int e^{2z}dz + 6\int e^{z}dz + \int 2z\,dz$$

$$= e^{2z} + 6e^{z} + z^2$$

$$y = c_1 + c_2 z + e^{2z} + 6e^{z} + z^2$$

$\therefore$ the complete sol^n of (1) is

$$y = c_1 + c_2\log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$$

(2) $(1-2x)^3\frac{d^3y}{dx^3} + (1-2x)\frac{dy}{dx} - 2y = x$

$(1-2x)^3 D^3y + (1-2x)Dy - 2y = x$ — (1)

$a = -2,\ b = 1$

$(1-2x) = e^{z}$ then $z = \log(1-2x)$

Replace $(1-2x)^3\frac{d^3y}{dx^3} + (1-2x)\frac{dy}{dx}$ by $-8\theta(\theta-1)(\theta-2)$

$\cancel{4\theta}$ -2θ in (1)

$$[-8\theta(\theta-1)(\theta-2) - 2\theta - 2]y = \frac{1-e^{z}}{2}$$

$$[-8\theta^3 + 24\theta^2 - 18\theta - 2]y = \frac{1-e^{z}}{2}$$

It is of the form $f(\theta)y = Q(z)$

Sol^n is, $y = y_c + y_p$

$A \cdot E \Rightarrow -8m^3 + 24m^2 - 18m - 2 = 0$

$$m = -0.09,\ 1.5489 \pm 0.3925\,i$$

$$y_c = c_1 e^{-0.09z} + e^{1.5489z}\left[c_2 \cos 0.3925z + c_3 \sin 0.3925z\right]$$

$$y_p = \frac{1}{-8\theta^3 + 24\theta^2 - 18\theta - 2}\left[\frac{1 - e^z}{2}\right]$$

$$= \frac{1 \cdot e^0}{2[-8\theta^3 + 24\theta^2 - 18\theta - 2]} - \frac{1 \cdot e^z}{2[-8\theta^3 + 24\theta^2 - 18\theta - 2]}$$

$(\theta \to 1)$

$$= \frac{-1}{4} - \frac{1}{2(-4)} e^z$$

$$= \frac{-1}{4} + \frac{1}{8} e^z$$

$$y = c_1 e^{-0.09z} + e^{1.5489z}\left[c_2 \cos 0.3925z + c_3 \sin 0.3925z\right] - \frac{1}{4} + \frac{1}{8} e^z.$$

$z \to \log(1-2x)$.

$$y = c_1 (1-2x)^{-0.09} + (1-2x)^{1.5489}\left[c_2 \cos 0.3925 \log(1-2x) + c_3 \sin 0.3925 \log(1-2x)\right] - \frac{1}{4} + \frac{1}{8}(1-2x)$$

(9) $(x+1)^2 \dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 4y = x^2 + x + 1.$

solve.

Method of Variation of Parameters:

An Eq^n of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ —— ①

where $P(x), Q(x), R(x)$ are functions of x is called the linear Eq^n of the second order with variable coefficients.

We will solve this Eq^n as follows:

i) Find the complementary function of ①.

let it be $y_c = c_1 u(x) + c_2 v(x)$

ii) Find the wronskian of $u(x), v(x)$

i.e. $w(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$

$$w(x) = uv' - u'v$$

iii) We will assume the particular solution

$$y_p = A(x)u(x) + B(x)v(x)$$

iv) Find $A(x), B(x)$ using the formulas

$$A(x) = -\int \frac{vR}{w}dx, \quad B(x) = \int \frac{uR}{w}dx$$

v) Sub $A(x), B(x)$ in in ②

vi) write complete sol^n of ①

i.e., $y = y_c + y_p$.

8) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol:- $D^2y - y = \frac{2}{1+e^x}$

$$(D^2-1)y = \frac{2}{1+e^x} \quad -(1)$$

This is of the form. $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} +$

$$Q(x)y = R(x)$$

where $P(x)=0$, $Q(x)=-1$, $R(x) = \frac{2}{1+e^x}$

It's sol is $y = y_c + y_p$

To find y_c : consider an A.E $f(m)=0$

$$m^2-1=0 \Rightarrow m = \pm 1$$

$$\therefore y_c = c_1 e^x + c_2 e^{-x} = c_1 u(x) + c_2 (v(x))$$

$$u(x) = e^x, \; v(x) = e^{-x}$$

the wronskian of $u(x), v(x)$ is $w(x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

$$w(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2$$

let y_p be, $y_p = A(x)u(x) + B(x)v(x)$

where, $A(x) = -\int \frac{vR}{w} dx$

$$= -\int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{\cancel{1+e^x} \; -2} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{e^{-x}}{e^x(1+e^{-x})} dx$$

$= \int \frac{e^{-2x}}{1+e^{-x}} dx$ let $e^{-x} = t$

$A(x) = \int \frac{t}{1+t}(-dt) = -\int \frac{t+1-1}{1+t}(dt)$

$= -[t - \ln(1+t)]$

$= \ln(1+e^{-x}) - e^{-x}$

$B(x) = \int \frac{uR}{W} dx = \int \frac{e^{x} \cdot \frac{2}{1+e^{x}}}{-2} dx$

$= -\int \frac{e^{x}}{1+e^{x}} dx = -\log(1+e^{x})$

$y_P = [\ln(1+e^{-x}) - e^{-x}]e^{x} + [-\log(1+e^{x})]e^{-x}$

$y = y_c + y_{p} //.$

Q) $(D^2 - 2D)y = e^{x}\sin x$

A.E $\Rightarrow m^2 - 2m = 0$

$m = 0, +2$

$y_c = c_1 + c_2 e^{2x}.$

$u(x) = 1, \; v(x) = e^{2x}.$

$w(x) = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$

$y_p = A(x)u(x) + B(x)v(x)$

$\Rightarrow A(x) = -\int \frac{vR}{W} dx$

$= -\int \frac{e^{2x} \cdot e^{x}\sin x}{2e^{2x}} dx = -\frac{1}{2}\int e^{x}\sin x \, dx$

$$= \frac{-1}{2}\left[\frac{e^x}{2}[\sin x - \cos x]\right]$$

$$= -\frac{e^x}{4}[\sin x - \cos x]$$

$$B(x) = \int \frac{UR}{W} dw = \int \frac{e^x \sin x}{2e^{2x}} dx$$

$$= \frac{1}{2}$$

Q) solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

sol:- $x^2D^2y - 3xDy + 4y = (1+x)^2 \Rightarrow \cancel{D^2y - \frac{3}{x}Dy + \frac{4}{x^2}y = \frac{(1+x)^2}{x^2}}$

$\cancel{\left(D^2 - \frac{3}{x}D + \frac{4}{x^2}\right)y = \frac{(1+x)^2}{x^2}}$

let $x = e^z$ then $z = \log x$

$$(\theta(\theta-1) - 3\theta + 4)y = (1+e^z)^2$$

$$y_c \Rightarrow m^2 - 4m + 4 = 0$$

$$m = 2, +2$$

$$y_c = (c_1 + c_2x)e^{2x}$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 4}[1+e^z]^2$$

$$= \frac{1}{\theta^2-4\theta+4}[1+e^{2z}+2e^z]$$

$$= \frac{1 \cdot e^0}{\theta^2-4\theta+4} + \frac{1}{\theta^2-4\theta+4}e^{2z} + \frac{1}{\theta^2-4\theta+4}2e^z$$

$$= \frac{1}{4} + \frac{z^2}{f''(\theta)}e^{2z} + 2e^z$$

$$= \frac{1}{4} + \frac{z^2}{2}e^{2z} + 2e^z$$

$$y = (c_1 + c_2 x) e^{2z} + \frac{1}{4} + \frac{z^2 e^{2z}}{2} + 2e^z$$

$z = \log x$

$$y = (c_1 + c_2 x) e^{2\log x} + \frac{1}{4} + \frac{z^2 e^{2\log x}}{2} + 2e^{\log x}$$

$$y = (c_1 + c_2 x) x^2 + \frac{1}{4} + \frac{(\log x)^2 \cdot x^2}{2} + 2x$$

(2) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

$x = e^z, \ z = \log x$

Sol: $(x^3 D^3 + 2x^2 D^2 + 2) y = 10\left(x + \frac{1}{x}\right)$

$\theta^2 - 3\theta + 2$

$$(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2) y = 10(e^z + e^{-z})$$

$$y_c \Rightarrow (\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} + 2) y = 10(e^z + e^{-z})$$

$$(\theta^3 - \theta^2 + 2) y = 10(e^z + e^{-z})$$

A.E $\Rightarrow m^3 - m^2 + 2 = 0$.

$m = -1, 1+i, 1-i$

$$y_c = c_1 e^{-z} + e^{z}[c_2 \cos x + c_3 \sin x].$$

$3\theta^2 - 2\theta$

$$y_p = \frac{10}{(\theta^3 - \theta^2 + 2)} (e^z + e^{-z})$$

$$= 10\left[\frac{1}{\theta^3 - \theta^2 + 2} e^z + \frac{1}{\theta^3 - \theta^2 + 2} e^{-z}\right]$$

$$= 10\left[\frac{1\cdot e^{z}}{2} + \frac{z}{f'(0)} e^{-z}\right]$$

$$= 10\left[\frac{e^{z}}{2} + \frac{z}{5} e^{-z}\right]$$

$$= 5e^{z} + 2ze^{-z} \qquad z \to \log x$$

$$y = c_1 e^{-z} + e^{z}(c_2 \cos x + c_3 \sin x) + 5x + 2\log x \cdot \frac{1}{x}.$$

(3) $(x-1)^3 \frac{d^3y}{dx^3} + 2(x-1)^2 \frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1)$

Sol:- this is legendre's linear eq^n with

$a = 1,\ b = -1$

let $(x-1) = e^{z} \Rightarrow z = \log(x-1) \Rightarrow x = e^{z}+1$

$$(x-1)^3 D^3y + 2(x-1)^2 D^2y - 4(x-1)Dy + 4y = 4\log(x-1)$$

$$\left([\theta(\theta-1)(\theta-2)] + 2\theta(\theta-1) - 4\theta + 4\right)y = 4\log(e^{z}+1)$$

$$\Rightarrow (\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} - 4\theta + 4)y = 4\log(e^{z}+1)$$

$$(\theta^3 - \theta^2 - 4\theta + 4)y = 4\log(e^{z}+1)$$

A.E $\Rightarrow m^3 - m^2 - 4m + 4 = 0$

$m = -2, 2, 1$

$$y_c = c_1 e^{-2z} + c_2 e^{2z} + c_3 e^{z}.$$

$$y_P = \frac{1}{\theta^3-\theta^2-4\theta+4} 4\log(e^z+1)$$

$$= 4\left[\frac{1}{\theta^3-\theta^2-4\theta+4}(z)\right]$$

$$= 4\left[\frac{1}{1+\frac{\theta^3-\theta^2-4\theta}{4}} z\right]$$

$$= \left[1+\frac{\theta^3-\theta^2-4\theta}{4}\right]^{-1} z$$

$$= \left[1-\frac{\theta^3-\theta^2-4\theta}{4}\right] z$$

$$= z + \theta(z) = z+1$$

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^z + \log(x-1) + 1$$

(4) $(3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2+x+1$

Sol: $a=3, b=2.$

$(3x+2) = e^z, \; z = \log(3x+2), \; x = \frac{e^z-2}{3}$

$((3x+2)^2 D^2 + 5(3x+2)D - 3)y = x^2+x+1$

$(9\theta(\theta-1) + 15\theta - 3)y = x^2+x+1$

$(9\theta^2+6\theta-3)y = \left(\frac{e^z-2}{3}\right)^2 + \left(\frac{e^z-2}{3}\right) + 1$

$A.E \Rightarrow \quad 9m^2+6m-3=0$

$m = +1/3, -1$

$y_c = c_1 e^{1/3 x} + c_2 e^{-x}$

$$y_p = \frac{1}{9\theta^2+6\theta-3}\left[\left(\frac{e^z-2}{3}\right)^2+\left(\frac{e^z-2}{3}\right)+1\right]$$

$$y_p = \frac{1}{9\theta^2+6\theta-3}\left[\frac{e^{2z}+4-4e^z+e^z-2+3}{3}\right]$$

$$= \frac{1}{9\theta^2+6\theta-3}\left[\frac{e^{2z}-3e^z+5}{3}\right]$$

$$= \frac{1}{3}\left[\frac{1}{9\theta^2+6\theta-3}e^{2z} - \frac{3}{9\theta^2+6\theta-3}e^z + \frac{5\cdot e^0}{9\theta^2+6\theta-3}\right]$$

$$= \frac{1}{3}\left[\frac{1}{45}e^{2z} - \frac{3}{12}e^z - \frac{5}{3}\right]$$

$$y = c_1 e^{1/3 z} + c_2 e^{-z} + \frac{1}{3}\left[\frac{1}{45}e^{2z} - \frac{3}{12}e^z - \frac{5}{3}\right]$$

$z = \log(3x+2)$

$$y = c_1(3x+2)^{1/3} + c_2\cdot\frac{1}{3x+2} + \frac{1}{3}\left[\frac{1}{45}(3x+2)^2 - \frac{1}{4}(3x+2) - \frac{5}{3}\right]$$