Unit-4 Multivariable defferential calclus & function optimization

Partial differentiation:

In mathematics sometimes the function depends on 2 (or) more variables In this case the concept of Partial diff. arises. Generally, partial derivates are used in vector calculus and differential geometry

Function of two variables:

There are three variable say x, y, z and value of z depends upon the values of x, y then z is called a function of two variables x and y. it is denoted by $z = f(x, y)$

Here z is dependent variable & x, y are the independent variables such a function can be visualized as a surface in 3D.

Ex: volume of a cylindrical cone of radius r and height h is given by $V = \frac{\pi r^2 h}{3}$ here V is function of 2 variable r, h. V is depend. variable while r, h are indi. var.s

Partial derivates of 1st order:

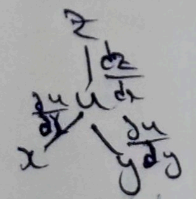
let $z = f(x, y)$ the $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ (both exisist) are called 1st order partial derivates of z.

→ The second order partial derivates of $z = f(x, y)$ are

$$\frac{\partial^2 z}{\partial x^2} \text{ and, } \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$$

Chain rule:

let $z = f(u)$, where u is a function of x, y.

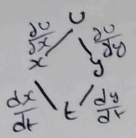


$$\boxed{\frac{\partial z}{\partial x} = \frac{dz}{dx}\,\frac{\partial u}{\partial x}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{dz}{dx}\cdot\frac{\partial u}{\partial y}}$$

Composite function of one variable:

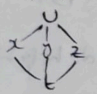
If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$ then u is a function of t and is called the composite funct. of a single variable t.



$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$$

Here $\frac{du}{dt}$ is total derivative of 'u' w.r.t 't'

Note:-
let $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$ then total derivative of 't' is



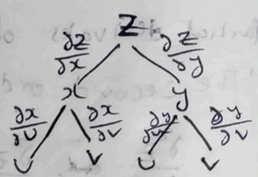
$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

Composite function of two variables:

If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is a function of u, v and is called the composite function of two variables u and v.

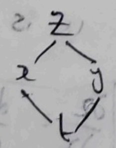
$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}}$$



z is func. of u, v so we can find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

Q) If $z = e^{xy}$, $x = t\cos t$, $y = t\sin t$ find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$



A) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= (e^{xy}y)\cdot(t(-\sin t) + \cos t) + (e^{xy}\cdot x)(\sin t + t\cos t)$

$= e^{xy}(y\cos t + x\sin t + xt\cos t - yt\sin t)$

$= e^{xy}(y(\cos t - t\sin t) + x(\sin t + t\cos t))$

at $t = \frac{\pi}{2}$

$x = \frac{\pi}{2}\cos\frac{\pi}{2}$ $\quad$ $y = \frac{\pi}{2}\sin\frac{\pi}{2}$

$\boxed{x = 0}$ $\quad$ $\boxed{y = \frac{\pi}{2}}$

$= e^{(t\cos t)(t\sin t)}(t\sin t(\cos t - t\sin t) + t\cos t(\sin t + t\cos t))$

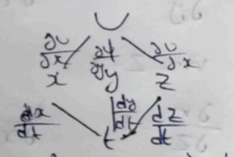
$= (e^0)\left(\frac{\pi}{2}\left(0 - \frac{\pi}{2}(1)\right) + 0(1 + 0)\right)$

$= \frac{\pi}{2}\left(-\frac{\pi}{2}\right)$

$= -\frac{\pi^2}{4}$

Q) If $u = x^2 + y^2 + z^2$ where $x = e^t$, $y = e^t\sin t$, $z = e^t\cos t$ find $\frac{du}{dt}$

Sol: $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$



u is func. of 't'.

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial y} = 2z$

$\frac{dx}{dt} = e^t$ $\quad$ $\frac{dy}{dt} = e^t\cos t + e^t\sin t$ $\quad$ $\frac{dz}{dt} = e^t\cos t - e^t\sin t$

$\frac{du}{dt} = 2xe^t + 2y(e^t\cos t + e^t\sin t) + 2z(e^t\cos t - e^t\sin t)$

$= 2(e^t)^2 + 2(e^t)^2(\sin t\cos t + \sin^2 t) + 2(e^t)^2(\cos^2 t - \sin t\cos t)$

$= 2(e^{2t}) + 2e^{2t}(\underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{\sin t\cos t} - \cancel{\sin t\cos t})$

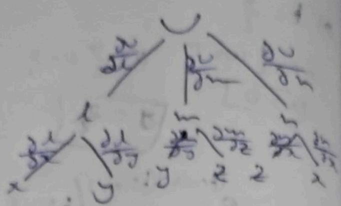
$= 4e^{2t}$

If $U = F(x-y, y-z, z-x)$ then S.T $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$

Sol: let $x - y = l$ $\quad U = F(l, m, n)$

$y - z = m$

$z - x = n$



$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial U}{\partial n} \cdot \frac{\partial n}{\partial x}$

$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial l} \frac{\partial l}{\partial y} + \frac{\partial U}{\partial m} \frac{\partial m}{\partial y}$

U is function of x, y, z

$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial m} \frac{\partial m}{\partial z} + \frac{\partial U}{\partial n} \cdot \frac{\partial n}{\partial z}$

$l = x - y$

$\frac{\partial l}{\partial x} = 1 \quad \frac{\partial l}{\partial y} = -1$

$m = y - z$

$\frac{\partial m}{\partial y} = 1 \quad \frac{\partial m}{\partial z} = -1$

$n = z - x$

$\frac{\partial n}{\partial z} = 1 \quad \frac{\partial n}{\partial x} = -1$

$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial l}(1) + \frac{\partial U}{\partial n}(-1)$

$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial l}(-1) + \frac{\partial U}{\partial m}(1)$

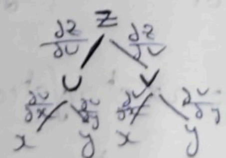
$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial m}(-1) + \frac{\partial U}{\partial n}(1)$

$\boxed{\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0}$

Hence proved.

2) If $z = F(u, v)$ where $u = x^2 + y^2$, $v = 2xy$ then S.T $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 - v^2}\left(\frac{\partial z}{\partial u}\right)$

Sol: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$



$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2(0 + y) = 2y$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 2(0 + x) = 2x$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y)$

$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(2x^2) + \frac{\partial z}{\partial v}(2xy)$ — ①

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot (2y) + \frac{\partial z}{\partial v}(2x)$

$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(2y^2) + \frac{\partial z}{\partial v}(2xy)$ — ②

① − ②

$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = (2x^2 - 2y^2)\left(\frac{\partial z}{\partial u}\right)$

$= 2(x^2 - y^2)\left(\frac{\partial z}{\partial u}\right)$

$\cancel{= 2(x^2 + y^2 + 2xy - y^2 - 2xy)}$

$\cancel{= 2((x^2 + y^2 + 2xy)(x^2 + y^2 - 2xy))}$

$u^2 - v^2 = (x^2 + y^2)^2 - (2xy)^2$

$= (x^2)^2 + (y^2)^2 + 2x^2y^2 - 4x^2y^2$

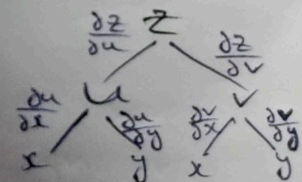
$= (x^2)^2 + (y^2)^2 - 2x^2y^2$

$= (x^2 - y^2)^2 \Rightarrow (x^2 - y^2) = \sqrt{u^2 - v^2}$

$\therefore x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2)\left(\frac{\partial z}{\partial u}\right)$

Q) If $z = F(u, v)$ where $u = \log(x^2 + y^2)$, $v = \frac{y}{x}$ then S.T

$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$



$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$= \frac{\partial z}{\partial u} \frac{(2y)}{x^2 + y^2} + \frac{\partial z}{\partial v}\left(\frac{1}{x}\right)$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$= \frac{\partial z}{\partial u}\left(\frac{2x}{x^2 + y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{2xy}{x^2+y^2} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{2xy}{x^2+y^2} \sim \frac{\partial z}{\partial u}$$

$$= \frac{\partial z}{\partial v} + \frac{\partial z}{\partial v}\left(\frac{y^2}{x^2}\right)$$

$$\boxed{x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}}$$

Hence proved

Jacobian:- guarantee

If $u(x,y)$ and $v(x,y)$ are two continuous functions of the independent variables x,y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}; \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous then the Jacobian of u,v w.r.t x,y is denoted by $J\left(\frac{u,v}{x,y}\right)$ (or) $\frac{\partial(u,v)}{\partial(x,y)}$ is defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Note:

If $u = u(x,y,z)$; $v = v(x,y,z)$; $w = w(x,y,z)$ w.r.t x,y,z is then the Jacobian of u,v,w given by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note:-

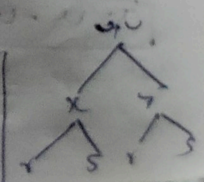
The imp. application of Jacobian is connection with the change of variables in multiple integrals also one can test the dependence of functional relations using the concept of Jacobian.

Properties of Jacobian:-

1) J_1 is Jacobian of u,v w.r.t x,y and J_2 is Jacobian of x,y w.r.t u,v then $\boxed{J_1 \times J_2 = 1}$

2) If u,v are functions of x,y and x,y are functions of r,s then

$$\boxed{\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}}$$



Q) If ~~u= x+y+z~~ $x = u(1+v)$, $y = v(1+u)$ then Prove that $\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = (1+v)$ $\quad \frac{\partial x}{\partial v} = u(1) = u$

$\frac{\partial y}{\partial u} = v(1)$ $\quad \frac{\partial y}{\partial v} = (1+u)$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+vu+v+u-vu$$
$$= 1+v+u$$

Hence proved.

Q) If $u = x+y+z$, $uv = y+z$, $uvw = z$. The S.T $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$z = uvw$

$y = uv - uvw = uv(1-w)$

$x = u + \cancel{uvw} - uv - \cancel{uvw}$

$\boxed{x = u - uv}$

$\frac{\partial x}{\partial u} = 1-v$ $\quad \frac{\partial y}{\partial u} = v - vw$

$\frac{\partial x}{\partial v} = -u$ $\quad \frac{\partial y}{\partial v} = u - uw$

$\frac{\partial x}{\partial w} = 0$ $\quad \frac{\partial y}{\partial w} = -uv$

$\frac{\partial z}{\partial u} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)(u^2v - \cancel{u^2vw} + \cancel{u^2vw})$
$+ u(uv^2 - \cancel{uv^2w} + \cancel{uv^2w})$
$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$

$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v}$$

Hence Proved

Q) If $x = r\cos\theta$, $y = r\sin\theta$ find the Jacobian $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ and $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$ and hence S.T $JJ' = 1$

Ans) $x = r\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{x}{r}\right)$

$y = r\sin\theta$ $\quad \theta = \sin^{-1}\left(\frac{y}{r}\right)$

$\tan\theta = \frac{y/r}{x/r} = \frac{y}{x}$

$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

$\boxed{x^2 + y^2 = r^2}$ (1)

$\boxed{r = \sqrt{x^2+y^2}}$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$\frac{\partial x}{\partial r} = (1)\cos\theta$ $\quad \frac{\partial x}{\partial \theta} = r(-\sin\theta) = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$ $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r(1) = r$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$\frac{\partial r}{\partial x} = \frac{\frac{\partial}{\partial x}(x^2+y^2)}{2\sqrt{x^2+y^2}}$

$= \frac{2x}{2\sqrt{x^2+y^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$\boxed{J = r}$ $\boxed{J' = \frac{1}{r}}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{y}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-1}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{(x^2+y^2)x}$$

$$= \frac{x}{x^2+y^2}$$

$$J \times J' = r \times \frac{1}{r} = 1$$

$\boxed{JJ' = 1}$

Hence proved

Q) If $x = uv$, $y = \frac{u}{v}$ S.T $JJ' = 1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = v$ $\frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v}$ $\frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v}$$

$$= \frac{-2u}{v}$$

$$= -2y$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$xy = uv \cdot \frac{u}{v}$

$\boxed{xy = u^2}$ $\boxed{u = \sqrt{xy}}$

$\frac{x}{y} = \frac{uv}{u} \cdot \frac{v}{1}$ $xv = v^2$ $\boxed{v = \sqrt{\frac{x}{y}}}$

$u = \sqrt{xy}$ $\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{xy}}(y)$

$v = \sqrt{\frac{x}{y}}$ $\frac{\partial u}{\partial y} = \frac{(x)}{2\sqrt{xy}}$

$$\frac{\partial v}{\partial x} = \frac{1}{2\sqrt{\frac{x}{y}}}\left(\frac{1}{y}\right)$$

$$\frac{\partial v}{\partial y} = \frac{1}{2\sqrt{\frac{x}{y}}}\left(x\left(\frac{-1}{y^2}\right)\right)$$

$$J' = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{\sqrt{y}}{2\sqrt{x}\,y} & \frac{-x\sqrt{y}}{2\sqrt{x}\,y^2} \end{vmatrix}$$

$$= \frac{-xy\sqrt{y}}{4x\sqrt{y}\,y^2} - \frac{x\sqrt{y} \times y}{4x\,y\sqrt{y}}$$

$$= \frac{-2xy\sqrt{y}}{4x\sqrt{y}\,y^2} = -\frac{1}{2y}$$

$\boxed{J' = -\frac{1}{2y}}$

$\boxed{J = -2y}$

$$J \times J' = (-2y)\left(\frac{-1}{2y}\right)$$

$\boxed{JJ' = 1}$

Hence proved.

Functionally dependent :-

If the functions u and v of the independent variables x, y are functionally dependent then the Jacobian is $\frac{\partial(u,v)}{\partial(x,y)}$ is 0.

Note: If the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ the u, v are said to be functionally independant

Q. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$ hence prove that u and v are functionally dependent and find the relation b/w them.

Sol:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{x^2+1} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$\boxed{J = 0}$$

$\therefore$ u and v are functionally dependent.

$v = \tan^{-1}x + \tan^{-1}y$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan v = u$

$\boxed{v = \tan^{-1}u}$ The relation b/w u & v is

If $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$ verify whether u, v are functionally dependent. If so, find relation b/w them.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(x-y) - (x+y)}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2}$$

$$= \frac{x-y+(x+y)}{(x-y)^2}$$

$$= \frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x-y)^2(y) - 2(x-y)(xy)}{((x-y)^2)^2}$$

$$= \frac{(x-y)\left((x-y)y - 2xy\right)}{(x-y)^4}$$

$$= \frac{-y^2 - xy}{(x-y)^3}$$

$$= \frac{-y(x+y)}{(x-y)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)^2(x) - 2(x-y)(-1)xy}{((x-y)^2)^2}$$

$$= \frac{(x-y)\left(x(x-y) + 2xy\right)}{(x-y)^3}$$

$$= \frac{x^2+xy}{(x-y)^3}$$

$$= \frac{x(x+y)}{(x-y)^3}$$

$$J = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ -\frac{y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = \frac{-2xy(x+y)}{(x-y)^5} + \frac{2xy(x+y)}{(x-y)^5}$$

$$= 0$$

$\boxed{J = 0}$ $\therefore$ u and v

$u = \frac{x+y}{x-y}$ $\qquad v = \frac{xy}{(x-y)^2}$

S.O.B.S

$$u^2 = \frac{(x+y)^2}{(x-y)^2}$$

$$u^2 = \frac{x^2+y^2}{(x-y)^2} + 2v$$

$$u^2 = \frac{(x-y)^2 + 4xy}{(x-y)^2}$$

$$= 1 + 4v$$

$$\boxed{u^2 = 1 + 4v}$$

Q) determine whether the following funct's are functionally dependent (or) not, if functionally dependent find the relation among them.

$u = xy + yz + zx$

$v = x^2 + y^2 + z^2$

$w = x + y + z$

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = y + z \quad \frac{\partial u}{\partial y} = x + z \quad \frac{\partial u}{\partial z} = x + y$

$\frac{\partial v}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2y \quad \frac{\partial v}{\partial z} = 2z$

$\frac{\partial w}{\partial x} = 1 \quad \frac{\partial w}{\partial y} = 1 \quad \frac{\partial w}{\partial y} = 1$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= (y+z)(2y-2z) - (x+z)(2x-2z) + (x+y)(2x-2y)$

$= 2\left[y^2 - yz + zy - z^2 - x^2 + xz - xz + z^2 + x^2 - y^2 + xy - xy\right]$

$= 2[0]$

$\boxed{J = 0}$

$\therefore$ u, v and w are functionaly dependent.

$w^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$\boxed{w^2 = v + 2u}$

Q) Determine whether the followins functions are functionally dependent (or), if dependent find the relation among them. not

$u = x^2 e^{-y} \cosh z$

$v = x^2 e^{-y} \sinh z$

$w = 3x^4 e^{-2y}$

$u^2 - v^2 = x^4 e^{-2y} (\cosh^2 z - \sinh^2 z)$

$= (x^4 \cdot e^{-2y})(1)$

$= x^2 e^{-2y}$

$\boxed{u^2 - v^2 = \frac{w}{3}}$

$\frac{\partial u}{\partial x} = 2x e^{-y} \cosh z \quad \frac{\partial u}{\partial y} = 8x^2$

~~Unconstrained~~ Unconstrained optimization of functions using Hessian Matrix

Maxima-Minima ~~of two~~ functions of two variables:

Definition: A function $f(x,y)$ of two variables is said be maximum at (a,b) if $f(a+h, b+k) - f(a,b) < 0$ for sufficiently small +ve (or) -ve integer value of h and k, and minimum if $f(a+h, b+k) - f(a,b) > 0$

The points at which maxima and minima occur are also known as extrema of critical points and the maximum and minimum values taken together are extreme values of the functions.

observation:

1. A function $f(x,y)$ may also ~~attain~~ attain its extreme values of the boundary.

2. The maxima-minima so defined are local relative maxima or local relative minima. Thus, a maximum value may not be the greatest and minimum may not be the least of all the values of the function in any finite region.