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1)

UNIT-5

Surface Integration & vector Integral theorems

Green's Theorem in a plane:

(Transformation b/w Line Integral & surface Integral)

If R is a closed region in xy plane bounded by a simple closed curve and if M, N are continuous functions of x & y having continuous derivatives in ~~are~~ R, then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ where c is traversed in the +ve direction i.e, anticlockwise direction.

Plane region → obtained from

1] Verify Green's theorem in a plane for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where c is the region bounded by $y=\sqrt{x}$ & $y=x^2$

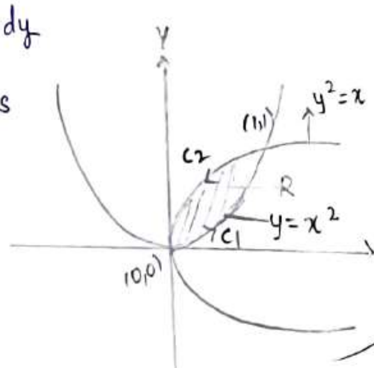
Sol Given, Integral $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$.

compare this with $\oint_C M\,dx + N\,dy$.

Then, $M = 3x^2 - 8y^2$; $N = 4y - 6xy$

By Green's Theorem, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

The region bounded by the two curves $y=x^2, y^2=x$ is as shown in the fig.



The intersection points of the two curves are obtained by, $y=x^2 \Leftrightarrow y=(y^2)^2 \Rightarrow y^4-y=0 \Rightarrow y=0,1$

Thus $x=0, x=1$, Here the intersection points are $(0,0)$ $(1,1)$

Evaluating $\oint_C M\,dx + N\,dy$

Since $c = C_1 \cup C_2$ then $\oint_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$

Along C_1: we have $y=x^2 \Rightarrow dy = 2x\,dx$ and varies from 0 to 1

$\therefore \int_{C_1} M\,dx + N\,dy = \int_{C_1} (3x^2-8y^2)dx + (4y-6xy)dy$

$= \int_0^1 (3x^2-8x^4)dx + (4x^2-6x^3)2x\,dx$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx = \left[x^3 - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\right]_0^1 = 1 - \frac{8}{5} + 2 - \frac{12}{5}$$

$$= 3 - \frac{20}{5} = -1$$

Along C_2 : we have $y^2 = x$ i.e, $x = y^2 \Rightarrow dx = 2y\,dy$ and y varies from 1 to 0

$$\therefore \int_{C_2} Mdx + Ndy = \int_{C_2} (3x^2 - 8y^2)(2y\,dy) + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy = \left[y^6 - 4y^4 + 2y^2 - \frac{3y^4}{2}\right]_1^0$$

$$= 0 - \left[1 - 4 + 2 - \frac{3}{2}\right] = 0 - \left[-1 - \frac{3}{2}\right] = \frac{5}{2}$$

$$\therefore \oint_C Mdx + Ndy = \oint_C (3x^2 - 5y^2)\,dx + (4y - 6xy)\,dy = -1 + \frac{5}{2} = \frac{3}{2}$$

Evaluating $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

we have $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

Now, $\frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx = 10 \int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} = 10 \int_0^1 \left(\frac{x}{2} - \frac{x^4}{2}\right)$$

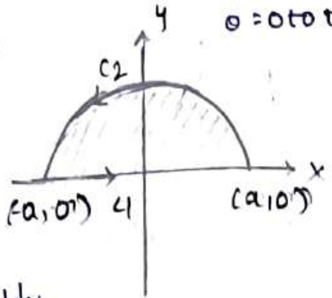
$$= \frac{10}{2}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{3}{5 \times 2}\right] = \frac{3}{2}$$

$$\therefore \oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence, Green's theorem is verified.

2] Evaluate $\oint_C (2x^2-y^2)\,dx+(x^2+y^2)\,dy$ where c is boundary of area enclosed by x axis & upper half of the circle $x^2+y^2=a^2$ using Green's Theorem

$r: 0 \text{ to } a$
$\theta = 0 \text{ to } \pi$



compare, we get $M=2x^2-y^2,\ N=x^2+y^2$

The intersection pts are $(-a,0)$ & $(a,0)$

$$C = C_1 \cup C_2$$

$$\oint_C M\,dx+N\,dy = \oint_{C_1} M\,dx+N\,dy + \oint_{C_2} M\,dx+N\,dy$$

Along C_1: we have $y=0$, $dy=0$ & x varies from $-a$ to a

$$\therefore \int_{C_1} M\,dx+N\,dy = \int_{C_1}(2x^2-y^2)\,dx+(x^2+y^2)\,dy = \int_{-a}^{a} 2x^2\,dx$$

No Need

$$= 2\left[\frac{x^3}{3}\right]_{-a}^{a} = 2\left[\frac{-a^3}{3}-\frac{a^3}{3}\right] = \frac{-4a^3}{3}$$

C_2:

(or)

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \qquad \frac{\partial N}{\partial x}=2x,\ \frac{\partial M}{\partial y}=-2y$$

cartesian

$$\iint_R (2x+2y)\,dx\,dy = 2\iint_R (x+y)\,dx\,dy = 2\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a}\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = 2\int_{-a}^{a} x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\,dx$$

(or)

$$x=r\cos\theta,\ y=r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$$

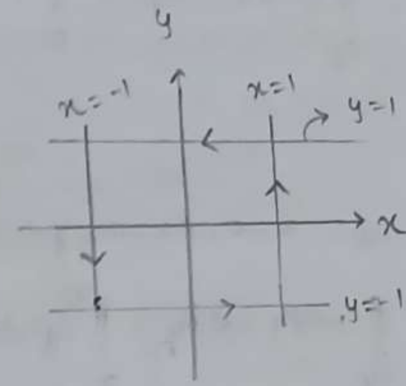
Polar.

$$2\int_0^{\pi}\int_0^{a}(r\cos\theta+r\sin\theta)(r\,dr\,d\theta) = 2\int_0^{\pi}\int_0^{a}(r^2\cos\theta+r^2\sin\theta)\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^3}{3}(\cos\theta+\sin\theta)\right]_0^a d\theta = 2\int_0^{\pi}\frac{a^3}{3}(\cos\theta+\sin\theta)\,d\theta = \frac{2a^3}{3}\left[+\sin\theta-\cos\theta\right]_0^{\pi}$$

$$= \frac{2a^3}{3}\left[0-(-1)-[0-1]\right] = \frac{4a^3}{3}$$

3] Use green's theorem $\oint_c x^2(1+y)dx + (y^3+x^3)dy$ where c is the square bounded by $y=\pm 1, x=\pm 1$



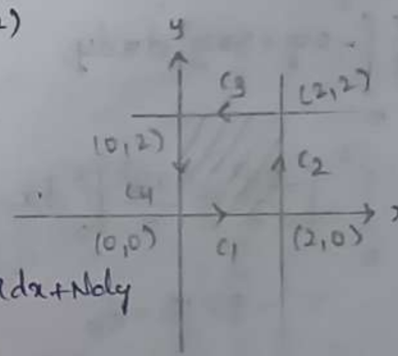
Here, $M = x^2(1+y) = x^2 + x^2 y$, $N = y^3 + x^3$

$$\frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = 3x^2$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3x^2 - x^2)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2x^2\,dx\,dy = 2\int_{-1}^{1}\left[\frac{x^3}{3}\right]_{-1}^{1} dy = 2\int_{-1}^{1} \frac{2}{3}\,dy = \frac{4}{3}\left[y\right]_{-1}^{1} = \frac{8}{3}.$$

4] Verify green's theorem in the plane for $\oint_c (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where c is the square with vertices (0,0) & (2,0) & (2,2) & (0,2)



$M = x^2 - xy^3$, $N = y^2 - 2xy$.

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\oint_c M dx + N dy = \oint_{C_1} M dx + N dy + \oint_{C_2} M dx + N dy + \oint_{C_3} M dx + N dy + \oint_{C_4} M dx + N dy$$

Along C_1 : we have $y=0$, $dy=0$, x varies from 0 to 2

$$\int_{C_1} M dx + N dy = \int_{C_1} (x^2 - xy^3)dx + (y^2 - 2xy)dy = \int_{C_1} x^2 dx = \int_0^2 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Along C_2 : $x=2$, $dx=0$, $y \to 0$ to 2

$$\int_{C_2} M dx + N dy = \int_0^2 0 + (y^2 - 4y)dy = \left[\frac{y^3}{3} - y^2 \cdot 2\right]_0^2 = \frac{8}{3} - 8 = \frac{-16}{3}$$

Along c_3: $y=2, dy=0 \quad x \to 2 \text{ to } 0$

$$\oint_{c_3} (Mdx+Ndy) = \int_2^0 (x^2 - x(8))\,dx = \left[\frac{x^3}{3} - 4x^2\right]_2^0$$

$$= 0 - \left[\frac{8}{3} - 16\right] = \frac{40}{3}$$

Along c_4: $x=0, dx=0, y \to (2 \text{ to } 0)$

$$\oint_{c_4} Mdx + Ndy = \int_2^0 y^2\,dy = \left[\frac{y^3}{3}\right]_2^0 = 0 - \frac{8}{3}$$

$$c = c_1 + c_2 + c_3 + c_4 = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} = -3xy^2$$

$$= \int_0^2\int_0^2 -2y + 3xy^2 \,dx\,dy$$

$$= \int_0^2 \left[\frac{-2y^2}{2} + \frac{3xy^3}{3}\right]_0^2 dx = \int_0^2 [-4 + 8x]\,dx = (-4x + 4x^2)_0^2$$

$$= -8 + 16 = 8$$

$$\therefore \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

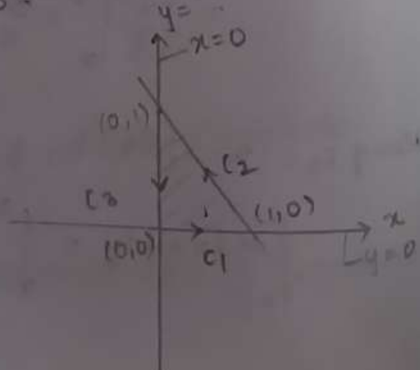
Hence, green's theorem is verified

5] verify green's therom for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is the region bounded by $x=0, y=0, x+y=1$

$\to (1,0)$

$(0,1)$ $\qquad 5/3$

$M = 3x^2 - 8y^2, \; N = 4y - 6xy \qquad \frac{\partial N}{\partial x} = -6y \quad \frac{\partial M}{\partial y} = -16y$



c_1: $y=0, dy=0, \; x \to 0 \text{ to } 1$

$$\oint_{c_1} Mdx + Ndy = \int_0^1 3x^2\,dx = (x^3)_0^1 = 1$$

c_2 : $x+y=1$

$\Rightarrow x = y-1 \Rightarrow y = 1-x$.

$-dx = dy$

$$4-4x-6x(1-x)$$

$$\oint_{c_2} Mdx+Ndy = \int_1^0 3x^2 - 8(x+1)^2\,dx + (4(x+1)-6x(x+1))dx$$

$$4-4x-6x+6x^2$$

$$(1-x)^2$$

$$3x^2-8(1+x^2-2x)\,dx$$

$$= \int_1^0 (3x^2 - 8x^2 - 8 - 16x + 4x + 4 - 6x^2 - 6x)\,dx$$

$$3x^2 - 8 - 8x^2 + 16x + 4 - 10x + 6x^2$$

$$-5x^2+6x-4$$

$$= \int_1^0 -11x^2 - 18x - 4\,dx = \left[-11\frac{x^3}{3} - 9x^2 - 4x\right]_1^0$$

$$-\frac{11}{3}(-1) - 9(-1) - 4(-1)$$

$$= 0 - \left[\frac{-11}{3} - 9 - 4\right] = \frac{8}{3} \quad \frac{11}{3} - 9 - 4 \quad \left[\frac{5x^3}{3} + 3x^2 - 4x\right]_1^0$$

$$= 0 - \left[\frac{-5}{3} - 1\right]$$

c_3 : $x=0$, $dx=0$, $y \to 1$ to 0

$$\oint_{c_3} Mdx+Ndy = \int_1^0 4y\,dy = 4\left[\frac{y^2}{2}\right]_1^0 = 4\left[0-\frac{1}{2}\right] = -2$$

$$C = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1\int_0^{1-x} 10y\,dy\,dx = 10\int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 1\,dx = 5.$$

$$= 10\int_0^1 \frac{(1-x)^2}{2} - 0\,dx$$

$$= 10\int_0^1 \frac{1}{2} + \frac{x^2}{2} - \frac{2x}{2}\,dx$$

$$= 10\left[\frac{1}{2}x + \frac{x^3}{6} - \frac{x^2}{2}\right]_0^1$$

$$= 10\left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2}\right]$$

$$= \frac{5}{3}$$

6] Show that the area bounded by a simple closed curve c is given by $\frac{1}{2}\oint_c x\,dy - y\,dx$ & hence find the area of (i) ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$A=\iint dx\,dy$

ii) circle $\rightarrow x^2+y^2=a^2$

consider $\oint_c x\,dy - y\,dx = \oint M\,dx + N\,dy$

Then $M=-y,\ N=x$

$$\frac{\partial M}{\partial y}=-1,\ \frac{\partial N}{\partial x}=1$$

By Green's theorem $\oint_c M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

$$\Rightarrow \oint_c x\,dy - y\,dx = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

A – Area of Region

$$\therefore A = \frac{1}{2}\oint_c x\,dy - y\,dx.$$

(i) ellipse $\rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad x=a\cos\theta,\ y=b\sin\theta \quad \theta: 0$ to 2π

$$A=\frac{1}{2}\oint_c x\,dy-y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos\theta\,(b\cos\theta)\,d\theta - b\sin\theta\,(-a\sin\theta)\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} ab\cos^2\theta + ab\sin^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} ab(1)\,d\theta = \frac{1}{2}ab(2\pi)$$

$$=\pi ab$$

(ii) circle $\rightarrow x^2+y^2=a^2$

$x=a\cos\theta,\ y=a\sin\theta \quad \theta: 0$ to 2π

$$A=\frac{1}{2}\oint_c x\,dy-y\,dx = \frac{1}{2}\int_0^{2\pi}\left(a\cos\theta\,(a\cos\theta) - a\sin\theta\,(a\cos\theta)\right)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \frac{a^2}{2}(2\pi) = \pi a^2$$

Surface Integrals.

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The surface integral over a curved surface s is generalisation of $\iint$ of plane region R. Let $\bar{F} = F_1 i + F_2 j + F_3 k$ be a continuous vector point function defined over a surface s. Then, surface integral of $\bar{F}$ over a surface s is given by $\int_S \bar{F}.\bar{n}\, ds$ or $\oiint_S \bar{F}.\bar{n}\, ds$

where s is the surface, ds is Area of surfaces, $\bar{n}$ is unit normal vector at a point on the surface s in the direction of outward normal to s.

Evaluation of Surface Integral

A surface integral is evaluated by expressing it as a double integral over region R. Region R is projection of surface s on the coordinate planes (xy / yz / zx plane). Let R be the projection of s on xy plane & $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\bar{n}$ then,

$$\hat{n} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$$

Now, $dx\,dy$ = projection of ds on xy-plane $= \cos\gamma\, ds$

$$\Rightarrow ds = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

Hence, $$\int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$||^{ly}$ if R is projection of s on yz-plane then, $ds = \dfrac{dy\,dz}{|\bar{n}.\bar{i}|}$

$$\& \int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dy\,dz}{|\bar{n}.\bar{i}|}$$

If R is projection of s on xz-plane then, $ds = \dfrac{dx\,dz}{|\bar{n}.\bar{j}|}$

$$\& \int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dx\,dz}{|\bar{n}.\bar{j}|}$$

Cartesian form

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_S (F_1 i+F_2 j+F_3 k)(\cos\alpha\, i+\cos\beta\, j+\cos\gamma\, k)\,ds$$

$$= \int_S F_1\cos\alpha\, ds + F_2\cos\beta\, ds + F_3\cos\gamma\, ds$$

$$= \int_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

The flux through a surface is a measure of rate at which the fluid is flowing through it.

Flux can be computed by using a surface integral $\int_S \bar{F}\cdot\bar{n}\,ds$ velocity of fluid Area

i] Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ then $\bar{F} = 18zi - 12j + 3yk$ where s is the part of surface of the plane $2x+3y+6z=12$ located in the 1st Octant.

Sol $\bar{F} = 18z\,i - 12j + 3yk$

Let $\phi = 2x+3y+6z-12$

The normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{dx} + j\frac{\partial\phi}{\partial y} + k\frac{d\phi}{dz}$,

$$\Rightarrow \nabla\phi = 2i+3j+6k$$

$\therefore$ The unit normal vector to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\Rightarrow \bar{n} = \frac{2i+3j+6k}{\sqrt{4+9+36}} = \frac{2i+3j+6k}{7} \quad \text{then } \bar{n}\cdot\bar{k} = \frac{6}{7}$$

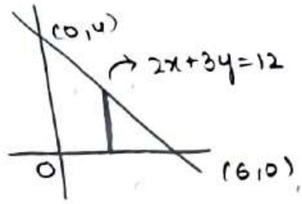
Let R be the Orientation of the surface ϕ in xy plane

$$\text{Then, } ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{6/7} = \frac{7}{6}dx\,dy$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R (18z\,i - 12j + 3yk)\cdot\left(\frac{2i+3j+6k}{7}\right)\left(\frac{7}{6}dx\,dy\right)$$

$$= \frac{1}{6}\iint_R (36z - 36 + 18y)\,dx\,dy$$

$$= \frac{1}{6}\int_{x=0}^{6}\int_{y=0}^{\frac{1}{3}(12-2x)} \left[6(12-2x-3y) - 36+18y\right] dy\,dx .$$



$$= \frac{1}{6}\int_{0}^{6}\int_{0}^{\frac{1}{3}(12-2x)} \underbrace{72-12x-\cancel{18y}-36+\cancel{18y}}\ dy\,dx$$

$$(-12x - \quad +36)\,dy\,dx = -12(x + \quad -3)\,dy\,dx$$

$$= -2\int_{0}^{6}\int_{0}^{\frac{1}{3}(12-2x)} x + \quad -3\ dy\,dx = -2\int_{0}^{6}\left[xy + \quad - 3y\right]_{0}^{\frac{1}{3}(12-2x)} dx$$

$$= -2\int_{0}^{6} \frac{x}{3}(12-2x) + \frac{\cancel{(12-2x)^2}}{\cancel{18}} - 3\left(\frac{1}{3}(12-2x)\right) dx$$

$$= -2\int_{0}^{6} 4x - \frac{2x^2}{3} + \frac{\cancel{144+4x^2-48x}}{\cancel{18}} - 12 + 2x\ dx$$

$$= -2\int_{0}^{6} 2x - \frac{2x^2}{3} + \frac{2x^2}{9} - \frac{8x}{3} + 8 - 12\,dx$$

$$= -2\int_{0}^{6} -\frac{2x}{3} - \frac{4x^2}{9} - 4\,dx$$

$$= -2\left[-\frac{2x^2}{6} - \frac{4x^3}{27} - 4x\right]_{0}^{6} = -2\left[-\frac{2(36)}{6} - \frac{4\times 36\times 6}{27} - 24\right]$$

$$-12 - 32 - 24$$

$$= 24$$

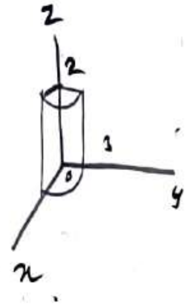
2] Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ if $\bar{F} = yz\,i + 2y^2 j + xz^2 k$ & S is the surface of cylinder $x^2+y^2=9$ contained in 1st octant b/w planes $z=0$, $z=2$ (234)

$x^2+y^2=9 \rightarrow \phi \qquad \bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = 2x\,i + 2y\,j \qquad |\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 6$

$\bar{n} = \frac{xi+yj}{3}$

$$ds = \frac{dy\,dz}{|\bar{n}\cdot j|} = \frac{dy\,dz}{\frac{x}{6}} = \frac{\cancel{6}^{3}}{x}dy\,dz$$



$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R (yz i + 2y^2 j + xz^2 k)\cdot\left(\frac{xi+yj}{3}\right)\frac{\cancel{6}3}{x}\,dy\,dz$$

$$= \int_0^{2}\int_0^{y=3} xyz + 2y^3\,dy\,dz$$

$$= \int_0^2\int_0^3 (9-y^2)yz + 2y^3\,dy\,dz = \int_0^2\int_0^3 9yz - y^3z + 2y^3\,dy\,dz$$

$$= \int_0^2 \left[9z\frac{y^2}{2} - \frac{y^4}{4}z + \frac{y^4}{2}\right]_0^3 dz = \int_0^2\left[\frac{27z}{2} - \frac{81z}{4} + \frac{81}{2}\right]dz$$

$$= \left[\frac{27z^2}{4} - \frac{81z^2}{8} + \frac{81z}{2}\right] = 27 - \frac{81}{2} + 81$$

Let v be volume bounded by s then volume bounded by s then volume integral of $\bar{F}$ in region is denoted by $\int_V \bar{F}\cdot dv = \iiint_V F_1 i + F_2 j + F_3 k\; dx\,dy\,dz$.

Guass divergence theorem:

Transformation between surface & volume integrals. Let s be a closed surface enclosing v.. If. $\bar{F}$ is continuously differentiable vector point function

$$\int_V \operatorname{div}\bar{F}\cdot dv = \int_S \bar{F}\cdot\bar{n}\,ds$$

$$\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \int_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

1) compute $\int ax^2 + by^2 + cz^2\; ds$

Then $\bar{F}\cdot\bar{n} = ax^2 + by^2 + cz^2$ on sphere

$$x^2+y^2+z^2 = 1$$

$$\phi = x^2+y^2+z^2-1$$

Normal to the $\phi = \nabla\phi = 2xi + 2yj + 2zk$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + 2yj + 2zk}{\sqrt{4(x^2+y^2+z^2)}} = xi + yj + zk$$

$$\bar{f}\cdot\bar{n} = \bar{f}\cdot(xi + yj + zk) = ax^2 + by^2 + cz^2$$

Thus $\bar{f} = axi + byj + czk = f_1 i + f_2 j + f_3 k$

Now $\operatorname{div} \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = a + b + c$

By Guass the

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_V \operatorname{div}\cdot\bar{F}\, dv$$

$$= \int_V (a+b+c)\, dv$$

$$= (a+b+c)(v)$$

$$= (a+b+c)\, \pi r^3\, 4/3$$

$$= \frac{4\pi}{3}(a+b+c)$$

27/6/23

2) $\iint_S \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = 4xi - 2y^2 j + z^2 k$

use divergence theorem & s is surface bounded by the region $x^2+y^2=4$, $z=0$, $z=3$ only closed surface

$$\operatorname{div}\bar{F} = \nabla\bar{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(4xi - 2y^2 j + z^2 k)$$

$$\operatorname{div}\bar{F} = (4 - 4y + 2z)$$

$$\iiint (4 - 4y + 2z)\, dx\, dy\, dz.$$

cylindrical coordinates

$x = r\cos\theta$ $\quad r = 0$ to 2

$y = r\sin\theta$ $\quad \theta = 0$ to 2π

$z = z$ $\quad z = 0$ to 3

$$\int_0^{2\pi}\int_0^2\int_0^3 (4-4r\sin\theta+2z)\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\left[4z-4rz\sin\theta+z^2\right]_0^3 dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left[12-12r\sin\theta+9\right] dr\, d\theta = \int_0^{2\pi}\int_0^2 \left[12r-12r^2\sin\theta+9r\right] dr\, d\theta$$

$$= \int_0^{2\pi}\left[6r^2-4r^3\sin\theta+\frac{9r^2}{2}\right]_0^2 d\theta = \int_0^{2\pi}\left[24-32\sin\theta+18\right] d\theta$$

$$= \left[24\theta+32\cos\theta+18\theta\right]_0^{2\pi} = 48\pi+36\pi \cancel{+} 32+32$$

$$= 84\pi$$

2) Verify guass divergence theorem for $\bar{F}=(x^3-yz)i-2x^2yj+zK$ taken over surface of cube bounded by the planes $x=y=z=a$ & the coordinate planes.

Ans By Guass divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$$

$$\bar{F} = (x^3-yz)i-2x^2yj+zK$$

$$= F_1 i+F_2 j+F_3 K$$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$= 3x^2-2x^2+1$$

$$\operatorname{div}\bar{F} = x^2+1$$

Now, $$\iiint_V \operatorname{div}\bar{F}\, dv = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2+1)\,dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{x^3}{3}+x\right]_0^a dy\,dz$$

$$= \int_0^a\int_0^a \frac{a^3}{3}+a\; dy\,dz = \int_0^a \frac{a^4}{3}+a^2$$

$$= \frac{a^5}{3}+a^3$$

For the cube we have 6 surfaces $y=0, -j$

S_1 (OABC), S_2(FGDE), S_3(OAFE), S_4(CBGD), S_5 = (OCDE), S_6(ABGF)

on S_1: $z=0, \bar{n}=-\bar{k}, ds=\frac{dxdy}{|\bar{n}\cdot\bar{k}|}=dxdy$

$$\iint_{S_1}\bar{F}\cdot\bar{n}\,ds=\iint_{S_1}\left[(x^3-yz)\bar{i}-2x^2y\bar{j}+z\bar{k}\right]\cdot(-\bar{k})\,dxdy$$

$$=\iint_{S_1}-z\,dxdy=0 \quad (\because \text{xy plane} \Rightarrow z=0)$$

on S_2 $z=a, \bar{n}=\bar{k}, ds=dxdy$

$$\iint_{S_2}\bar{F}\cdot\bar{n}\,ds=\iint_{S_2}z\,dxdy=\iint_{S_2}a\,dxdy=\int_0^a\int_0^a a\,dxdy=a^3$$

on S_3 $y=0, \bar{n}=-\bar{j}, ds=\frac{dxdz}{|\bar{n}\cdot\bar{j}|}=dxdz$

$$=\iint_{S_3}2x^2y\,dxdz=0$$

on S_4 $y=a$, $\bar{n}=\bar{j}$

$$=\iint_{S_3}-2x^2y\,dxdz=-2a\int_0^a\int_0^a x^2\,dxdz=\frac{-2a^5}{3}$$

on S_5 $x=0, \bar{n}=-\bar{i}, ds=\frac{dydz}{|\bar{n}\cdot\bar{i}|}=dydz$

$$\int_0^a\int_0^a yz\,dydz=\frac{a^4}{4}$$

on S_6 $x=a, \bar{n}=\bar{i}$

$$\int_0^a\int_0^a(a^3-yz)\,dydz=a^5-\frac{a^4}{4}$$

$$\int\bar{F}\cdot\bar{n}\,ds=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}+\iint_{S_5}+\iint_{S_6}=a^3+\frac{a^5}{3}$$

01/07/24

1] Evaluate $\int_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ using divergence theorem $x^2+y^2+z^2=a^2$

By divergence theorem, $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$

Given, $\int_S \bar{F}\cdot\bar{n}\,ds = \int_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \int_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$

Then $F_1 = x$, $F_2 = y$, $F_3 = z$

Now, $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

$\therefore$ From ①, $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V 3\,dv$

$\Rightarrow \int_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = 3\int_V dv = 3$(volume of sphere $x^2+y^2+z^2=a^2$)

$= 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$

2] Use divergence theorem to evaluate $\iint_S \bar{F}\cdot ds$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$, $z=3$

Ans $\boxed{84\pi}$

By divergence theorem, $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
r : 0 to 2
θ : 0 to 2π
z : 0 to 3

$= \int_S \bar{F}\cdot\bar{n}\,ds$

$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} =$

$\nabla\cdot\bar{F}$ $\quad \operatorname{div}\bar{F} = 4 - 4y + 2z$

$\int_S \operatorname{div}\bar{F} = \int_S (4-4y+2z)\,dv = \iiint (4-4y+2z)\,dv$

$= \int_0^{2\pi}\int_0^{2}\int_0^{3} (4-4y+2z)\,dz\,dr\,d\theta\, r = \int_0^{2\pi}\int_0^{2} r\left[4z-4yz+z^2\right]_0^3\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{2} r\left[12-12y+9\right]dr\,d\theta = \int_0^{2\pi}(12-12y+9)\left(\frac{r^2}{2}\right)_0^2 d\theta$

$= \int_0^{2\pi}(12-12y+9)(2)\,d\theta$

$$= \int_0^{2\pi}\int_0^2 r\left[12 - 12r\sin\theta + 9\right] dr\,d\theta = \int_0 \left[6r^2 - 4r^3\sin\theta + \frac{9r^2}{2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[24 - 32\sin\theta + 18\right] d\theta = \left[24\theta + 32\cos\theta + 18\theta\right]_0^{2\pi} = 48\pi + 32 + 36\pi$$

$$= 84\pi$$

3) Verify divergence theorem for $2x^2y\,i - y^2 j + 4xz^2 k$ taken over the region of 1st octant of cylinder, $y^2+z^2=9$ & $x=2$

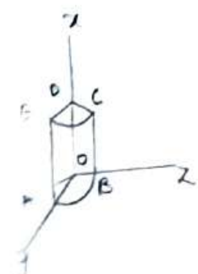
$S_1(OAB) \rightarrow \bar{n} = -\bar{i}$ $\quad S_3(OBCD) \rightarrow \bar{n} = -\bar{j}$

$S_2(CDE) \Rightarrow \bar{n} = \bar{i}$ $\quad S_4(AODE) \rightarrow \bar{n} = \bar{j}$

$S_5(ABCE) \rightarrow \bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\downarrow$

surface of cylinder.



$$\operatorname{div}\bar{F} = 4x - 2y + 8xz$$

$x \rightarrow 0$ to 2

$x \rightarrow x$

$y \rightarrow r\sin\theta$

$z \rightarrow 3\cos\theta$

$r\,dr\,d\theta$

$3\,dr\,d\theta$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv = \int_V 4xy - 2y + 8xz\,dv$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{x=0}^{2} \underbrace{4xy - 2y + 8xz}_{4xy - 2r\cos\theta - 8xr\sin\theta\,dx\,dr\,d\theta}\,dx\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}\left[2x^2y - 2xy + 4x^2z\right]_0^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}\left[8y - 4y + 16z\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^{3}\left[8y - 4r\cos\theta + 16r\sin\theta\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi/2}\left[8r^3\cos\theta/3 - \frac{4r^3}{3}\cos\theta + \frac{16r^3}{3}\sin\theta\right]_0^3 d\theta$$

$$= \int_0^{\pi/2}\left[\frac{4r^3/3\cos\theta}{\cos\theta} - 36\cos\theta + 144\sin\theta\right] d\theta$$

$$= \left[\underbrace{36\theta + 36\sin\theta}_{+36\sin\theta} - 144\cos\theta\right]_0^{\pi/2} = 18\pi + 180$$

on S_1: $x = 0$

$$\Rightarrow \int_{S_1} \bar{F}\cdot\bar{n}\,ds = \underline{0}$$

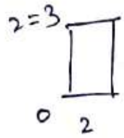
on S_2: $x = 2,\ \bar{n} = \bar{i},\ ds = \dfrac{dy\,dz}{|\bar{n}\cdot i|} = dy\,dz$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint 2x^2y\,dy.dz = \int_0^3\int_0^3 8y.dy\,dz = \underline{72}$$

on S_3: (xz plane) $\bar{n} = -\bar{j}$, $ds = \dfrac{dx\,dz}{|\bar{n}\cdot j|} = dx\,dz$

$\downarrow$ $y = 0$

$$\int_{S_3} \bar{F}\cdot\bar{n}\,ds = \underline{0}$$



$x \to 0$ to 2
$z \to 0$ to 3

on S_4: xy plane $\bar{n} = -\bar{k}$, $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = dx\,dy$

$\downarrow$

$z = 0$

$$\int_{S_4} \bar{F}\cdot\bar{n}\,ds = \underline{0}$$

on S_5: Let $\phi = y^2 + z^2 - 9$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K = 2yj + 2zK \qquad ds = \frac{dx\,dz}{|\bar{n}\cdot j|} \to xz\text{ plane}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2yj + 2zK}{\sqrt{4(y^2+z^2)}} = \frac{2yj + 2zK}{6} = \frac{yj + zK}{3} \quad \cancel{\bar{n}\cdot ds}$$

$$= \int \qquad \bar{F}\cdot\bar{n} = \frac{-y^3 + 4xz^3}{3},\ \bar{n}\cdot k = \frac{z}{3} \quad \frac{1}{3}\sqrt{9-y^2}$$

$$\iint_R \frac{4xz^3 - y^3}{\sqrt{9-y^2}}\,dx\,dy = \int_0^2\int_0^3 \left[4x(9-y^2) - y^3(9-y^2)^{-1/2}\right] dy\,dx$$

$$= \int_0^2 12x\,dx - 18\int_0^2 dx$$

~~$x \to 0$ to 2~~
$z \to 0$ to 3

$$= 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2$$

$$= 144 - 36$$
$$= 108.$$

4] verify divergence theorem $\bar{F} = x^2 i + y^2 j + z^2 k$ over surface S of the solid cut off by the plane $x+y+z=a$ in 1^{st} octant. $\frac{a^4}{4}$

5] Use divergence theorem to evaluate $\iint_S yz^2 i + zx^2 j + 2z^2 k \cdot d\bar{s}$ $\iiint$ where S is the closed surface bounded by xy plane & upper half of sphere $x^2+y^2+z^2=a^2$ above this plane. spherical coordinates (At last)

4] By Guass divergence, $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$

$\phi = x+y+z-a$

$\nabla\phi = i+j+k$ $\quad\quad$ $\bar{n} = \frac{j+i+k}{\sqrt{1+1+1}} = \frac{i+j+k}{\sqrt{3}}$

Let projection of S is on xy plane $\Rightarrow z=0$ $\quad\quad$ $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{1/\sqrt{3}}$

$$\int \bar{F}\cdot\bar{n}\,ds = \int x^2 i + y^2 j + z^2 k \cdot \left(\frac{i+j+k}{\cancel{\sqrt{3}}}\right)\cdot\cancel{\sqrt{3}}\cdot dx\,dy.$$

$x+y=a$
$y=0$
$x=a$.

$$= \int_0^a\int_0^{a-x} x^2+y^2+z^2\,dx\,dy$$

$$= \int_0^a\int_0^{a-x} x^2+y^2+\underbrace{(a-x-y)^2}_{x^2+y^2+a^2-2ax-2ay+2xy}\,dx\,dy \quad [\because x+y+z=a]$$

$$= \int_0^a\int_0^{a-x} 2x^2+2y^2-2ax-2ay+2xy+a^2\,dx\,dy$$

$$= \int_0^a \left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x}$$

$$= \int_0^a \left[2x^2(a-x) + \frac{2(a-x)^3}{3} + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)\right]$$

$$= \int_0^a -\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2a^3}{3}\,dx = \left[-\frac{5}{3}\frac{x^4}{4} + \frac{3ax^3}{3} - \frac{2a^2x^2}{2} + \frac{2a^4}{12}\right]_0^a$$

$$\int \bar{F}\cdot\bar{n}\,ds = \frac{a^4}{4}.$$

$\operatorname{div}\bar{F} = 2x+2y+2z$

$$\iiint \operatorname{div} \bar{F}\, dv = \int_0^a \int_0^{a-x} \int_0^{a-x-y} 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^a \int_0^{a-x} \left[xz + yz + \frac{z^2}{2}\right]_0^{a-x-y} dx\, dy = 2\int_0^a \int_0^{a-x} x(a-x-y) + y(a-x-y) + \frac{(a-x-y)^2}{2}\, dy\, dx$$

$$= 2\int_0^a \int_0^{a-x} (a-x-y)\left[x + y + \frac{a-x-y}{2}\right] dx\, dy = 2\iint \frac{(a-x-y)(a+x+y)}{2}\, dy\, dx$$

$$= \int_0^a \int_0^{a-x} (a^2 - x^2 - y^2 - 2xy)\, dx\, dy = \int_0^a \left[a^2 y - x^2 y - \frac{y^3}{3} - xy^2\right]_0^{a-x} dx$$

$$= \int_0^a (a-x)(2a^2 - x^2 - ax)\, dx = \frac{a^4}{4}$$

3] Stokes theorem:

Transformation b/w line integral & surface integral

Let s be a open surface bounded by a closed non intersective curve,

or if $\bar{F}$ be any differentiable vector point function,

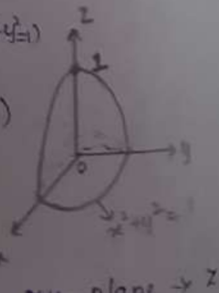
$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds$$

where c is take in +ve direction & $\bar{n} \rightarrow$ unit normal vector at any Point of the Surface S which is drawn Outwards

1] Use stoke's theorem to evaluate $\int_S \operatorname{curl} \bar{F} \cdot ds$ over the surface of paraboloid $z + x^2 + y^2 = 1$, $z \geq 0$ where $\bar{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

C - boundary of circle $(x^2+y^2=1)$

$(\because z = 1 - x^2 - y^2 \geq 0$ gives $x^2+y^2=1)$



By stoke's theorem $\bar{F} \cdot d\bar{r} = y\,dx + z\,dy + x\,dz$

$\oint_C \bar{F} \cdot d\bar{r} = \int_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds$ $\rightarrow y\,dx$ (1)

$x = \cos\theta$ $dx = -\sin\theta\, d\theta$

$y = \sin\theta$

xy - plane $\rightarrow z = 0$, $dz = 0$

$$\int \bar{F} \cdot d\bar{r} = \int_0^{2\pi} -\sin^2\theta\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{-1 + \cos 2\theta}{2}\right] d\theta = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \pi$$

$\therefore$ from (1), $\int_S \text{curl}\,\bar{F}\cdot ds = \pi$ //

2] Verify stoke's theorem for $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$ over the upper half surface of Sphere $x^2+y^2+z^2=1$ bounded by the projection on xy plane.

The curve c is $x^2+y^2=1$, $z=0$ (xy plane)
$\quad \hookrightarrow dz=0$

Now, $\int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx - yz^2 dy - y^2 z\,dz$

$$= \int_C (2x-y)\,dx \quad [\because z=0, dz=0] \qquad \begin{array}{l} x=\cos\theta \\ y=\sin\theta \\ \theta: 0 \text{ to } 2\pi \end{array}$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)\,d\theta \qquad dx = -\sin\theta$$

$$= \cancel{(2\sin\theta + \cos\theta)_0^{2\pi}} = \cancel{0 + [0-1] = 0}$$

$$= \int_0^{2\pi} -2\sin\theta\cos\theta + \sin^2\theta\, d\theta$$

$$= \int_0^{2\pi} -\sin 2\theta + \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \left(\frac{1}{2} + \pi - 0\right) - \left[\frac{1}{2} + 0 - 0\right] = \pi \,//$$

Now, $\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix}$ $\qquad ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$= i\left(-2yz + 2yz\right) - j(0-0) + k(0+1)$$

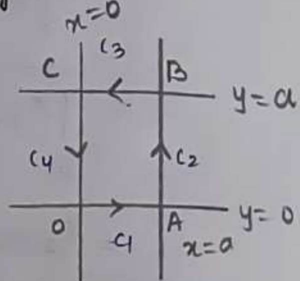
$$= \cancel{(-2yz + yz^2)i} + k$$

$\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2i - z^2 j - y^2 k}{\sqrt{(\quad)^2}}$ $\qquad \bar{n}\cdot\bar{k} = \dfrac{-y^2}{(\quad)} \times \dfrac{(\quad)}{-y^2}\,dx\,dy$

$x = \cos\theta$
$y = \sin\theta$

$\text{curl}\,\bar{F}\cdot \dfrac{2i - z^2 j - y^2 k}{(\quad)} \times \dfrac{dx\,dy}{-y^2} \times \dfrac{(\quad)}{} = \dfrac{-y^2}{-y^2}\,dx\,dy$ $\quad \frac{1}{2}\int 2\pi \cos\theta$ $\left(\frac{\sin\theta}{2}\right)$

3] Verify stoke's theorem $\bar{F} = x^2 i + xy j$ integrated along the square in plane $z=0$ $\downarrow$ xy plane whose sides are along the lines $x=0, y=0, x=a, y=a$



$c : c_1 \cup c_2 \cup c_3 \cup c_4$

By stokes theorem →

$$\oint_c \bar{F} \cdot d\bar{r} = \int_s \text{curl}\, \bar{F} \cdot \bar{n}\, ds$$

$$\text{curl}\, \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = i(0) - j(0) + k(y-0) = yk$$

$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

~~$\bar{n} = \nabla\phi$~~ $\bar{n} = \bar{k}$ (xy plane);

$ds = dx\,dy$

$$\int_s yk \cdot k\, dx\,dy = \int_0^a\int_0^a y\, dx\,dy = \left[\frac{a^2}{2}x\right]_0^a = \frac{a^3}{2} = \text{RHS}$$

$$\text{LHS} = \oint_c \bar{F}\cdot d\bar{r} = c_1 \cup c_2 \cup c_3 \cup c_4$$

on c_1 : $x : 0$ to a, $y = 0 \Rightarrow dy = 0$

$$\int \bar{F}\cdot d\bar{r} = \int x^2 dx + xy\,dy = \int_0^a x^2 dx = \frac{a^3}{3}$$

on c_2 : $y : 0$ to a, $x = a$, $dx = 0$

$$\int \bar{F}\cdot d\bar{r} = \int x^2 dx + (a) y\, dy = \int ay\,dy = a\left(\frac{y^2}{2}\right)_0^a = \frac{a^3}{2}$$

on c_3 : $y = a$, $dy = 0$, $x = a$ to 0

$$\int \bar{F}\cdot d\bar{r} = \int x^2 dx + xy\,dy = \int_a^0 x^2 dx = \left(\frac{x^3}{3}\right)_a^0 = \frac{-a^3}{3}$$

on c_4 : $x = 0$, $dx = 0$, $y = a$ to a

$$\int \bar{F}\cdot d\bar{r} = \int x^2 dx + xy\,dy = 0$$

$$\oint_c \bar{F}\cdot d\bar{r} = \frac{a^3}{2}$$

Stoke's theorem is verified

LHS = RHS.

$F = x^2 i - yz j + k$ integrated around the square

4] ve

$x=0, y=0, z=0, x=1, y=1, z=1$ 5] Evaluate by stokes theorem

$\oint_c e^x dx + 2y\,dy - dz$ where c is curve $x^2+y^2=9$ & $z=2$

0

5] $\bar{r} = xi + yj + zk$

$\bar{F}\cdot d\bar{r} = e^x dx + 2y\,dy - dz$ $\quad \bar{F} = e^x i + 2y j - \bar{k}$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = i(0-0) - j(0-0) + k(0-0) = 0_{//}$$

By stokes's theorem

$$\int_s \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 0$$

$$\oint_c \bar{F}\cdot d\bar{r} = \int_s \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$\therefore \oint_c \bar{F}\cdot d\bar{r} = \int_s \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 0_{//}$$

5] Divergence theorem

$$\int_s \bar{F}\cdot ds = \iiint \text{div}\,\bar{F}\,dv$$

xy plane

$$\nabla\bar{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\iint \bar{F}\cdot ds = \iiint_v 4z\,dx\,dy\,dz$$

$$= 4\int_{r=0}^{a}\int_{\phi=0}^{\pi}\int_{\theta}^{2\pi} (r\cos\theta)(r^2\sin\theta)\,dr\,d\theta\,d\phi \quad (\because 4 \text{ parts})$$

$\frac{r^3}{2}\sin 2\theta$

$$= 4\int_0^a\int_{\phi}^{\pi} r^3 \sin\theta\cos\theta\,[\theta]_0^{2\pi}\,dr\,d\phi$$

$\left(\frac{r^4}{8}\right)\sin\theta$

$2\pi\ \frac{a^4}{8}$ $\sin 2\theta$ $-\cos 2\theta$

$$= 4\int_0^a\int_0^{\pi} r^3 \underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}}(2\pi)\,dr\,d\theta$$

πa^4

$$= 4\pi\int_0^a r^3\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi} dr$$

$$= -2\pi\int_0^a r^3[+1-1] = 0$$