SUBHAPREET PATRO
2211CSO10547
GROUP-7A

DISCRETE MATHEMATICS UNIT-I ASSIGNMENT

1) (a) If $A=\{a,b,c,d\}$ and $B=\{x,y,z\}$. Let R be the following relation from A to B. $R=\{(a,x),(a,z),(d,y),(c,x),(b,z),(d,x)\}$.

(i) Determine the matrix of the relation

(ii) Draw the arrow diagram of R

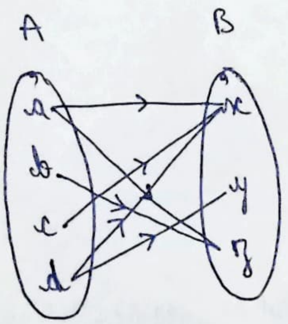
(iii) Find the inverse relation R^{-1} of R

Soln

(i)

$$\begin{array}{c|ccc} & x & y & z \\ \hline a & & & \\ b & & & \\ c & & & \\ d & & & \end{array} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

(ii)



(iii)

$R^{-1}=\{(x,a),(z,a),(y,d),(x,c),(z,b),(x,d)\}$

(b) Prove that for any positive integer m, the relation congruence modulo m is an equivalence relation on the integers.

Soln Given $R=\{(a,b)\,/\,a\equiv b \pmod m;\ a,b,m\in Z\}$

Reflexive: $\forall x\in Z,\ (x,x)\in R$

$\Rightarrow a\equiv a \pmod m$

$\Rightarrow \dfrac{a-a}{m}=\dfrac{0}{m}=0$

$\therefore (a,a)\in R$

Symmetric: $\forall\ a, b \in Z \qquad a \equiv b \pmod m$

$$\Rightarrow \frac{a-b}{m} = k$$

$$\Rightarrow -\frac{(b-a)}{m} = k$$

$$\Rightarrow \frac{b-a}{m} = -k$$

$$\Rightarrow (b, a) \in R$$

Transitive: $\forall\ a, b, c \in Z \qquad a \equiv b \pmod m$ and $b \equiv c \pmod m$

$$\Rightarrow \frac{a-b}{m} = k_1, \quad \frac{b-c}{m} = k_2$$

Now, $\dfrac{a-b+b-c}{m} = k_1 + k_2$

$$\Rightarrow \frac{a-c}{m} = k_1 + k_2$$

$$\Rightarrow a \equiv c \pmod m$$

$$\Rightarrow (a, c) \in R$$

$\therefore$ R is an equivalence relation since R is reflexive, symmetric and transitive.

2) (a) Let R is a relation on set of real numbers and it is defined as $(a, b) \in R$ iff $x - y$ is an integer. Then show that R is an equivalence relation.

Sol: Given $R = \{(a, b) \,/\, (a-b) \in Z\}$

Reflexive: $\forall\ a \in R$, $(a, a) \in R$

$$\Rightarrow a - a = 0 \in Z$$

$$\therefore (a, a) \in R$$

$\therefore$ R is reflexive.

Symmetric: $\forall a, b \in R$, $(a,b) \in R$

$\Rightarrow a - b = k$

$\Rightarrow -(b-a) = k$

$\Rightarrow b - a = -k \in Z$

$\therefore (b,a) \in R$

$\therefore$ R is symmetric

Transitive: $\forall a, b, c \in R$; $(a,b) \in R$, $(b,c) \in R$

$\Rightarrow a - b = k_1$

$b - c = k_2$

$\Rightarrow a - b + b - c = k_1 + k_2$

$\Rightarrow a - c = k \in Z$

$\therefore (a,c) \in R$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation

(b) Suppose $(a, b) \in R$ iff the price of book a is greater than or equal to the price of book b and the number of pages of book a greater than or equal to the number of pages in b. Show that R is partially ordered relation.

Sol:

Given $R = \{(a,b) / a \geq b ; a, b \in Z\}$

Reflexive: Let $a \in Z$

Always $a \geq a$

$\Rightarrow (a,a) \in R$

$\therefore$ R is reflexive.

Antisymmetric:- Let $a, b \in Z$ & $(a,b) \in R$, ~~$(b,a) \in R$~~

$\Rightarrow a \geq b,\ b \geq a$

$\Rightarrow a = b$ (only possible case) $\Rightarrow (b,a) \notin R$

$\therefore R$ is antisymmetric

Transitive:- Let $a, b, c \in Z$ & $(a,b) \in R$, $(b,c) \in R$

$\Rightarrow a \geq b;\ b \geq c$

$\Rightarrow a \geq c$

$\Rightarrow (a,c) \in R$

$\therefore R$ is transitive

$\therefore R$ is partially ordering.

3) (a) Prove that $f(x) = x^3$ is a one-one function from $R \to R$ where R is the set of real numbers. Also, prove that $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ for $f, g : Q \to Q$ such that $f(x) = 4x$ and $g(x) = x+5$.

Sol:- Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ [Here $f(x) = x^3$].

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1^3 - x_2^3 = 0$ $\left[\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)\right]$

$\Rightarrow (x_1 - x_2)^3 + 3x_1x_2(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)\left[(x_1 - x_2)^2 + 3x_1x_2\right] = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_2^2 + x_1x_2) = 0$

$\Rightarrow x_1 - x_2 = 0$ | $\Rightarrow x_1^2 + x_2^2 + x_1x_2 = 0$

$\Rightarrow x_1 = x_2$ | Not possible

$\therefore$ It is a one-one function.

$f, g : Q \to Q \quad f(x) = 4x \quad , \quad g(x) = x+5$

$f(x) = 4x = y$

$\Rightarrow x = \frac{y}{4}$

$\Rightarrow f^{-1}(y) = \frac{y}{4}$

$\Rightarrow \boxed{f^{-1}(x) = \frac{x}{4}}$

$g(x) = x+5$

$\Rightarrow x+5 = y$

$\Rightarrow x = y-5$

$\Rightarrow g^{-1}(y) = y-5$

$\Rightarrow \boxed{g^{-1}(x) = x-5}$

~~fog~~ Now $g \circ f = g(f(x))$

$= g(4x)$

$= 4x+5$

$\Rightarrow g \circ f = 4x+5 = y$

$\Rightarrow 4x = y-5$

$\Rightarrow x = \frac{y-5}{4}$

$\Rightarrow (g \circ f)^{-1}(y) = \frac{y-5}{4}$

$\Rightarrow \boxed{(g \circ f^{-1})(x) = \cancel{y}\ \frac{x-5}{4}} \rightarrow (1)$

Now $f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(x))$

$= f^{-1}(x-5)$

$\Rightarrow \boxed{f^{-1} \circ g^{-1} = \frac{x-5}{4}} \rightarrow (2)$

Since $(1) = (2)$

$\therefore f^{-1} \circ g^{-1} = (g \circ f)^{-1}$

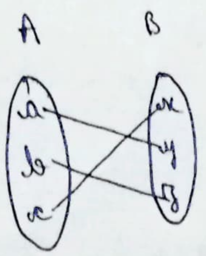
Hence, Proved

(b) Define one-one and onto functions and explain the composition of functions with diagram. Let f and g are two functions from $R \to R$ where R is the set of real numbers. Find $(f \circ f)(x)$ if $f(x) = 3x^2$ and $(f \circ g)(x)$, $(g \circ f)(x)$ if $f(x) = x^2 - 2$ and $g(x) = x + 4$.

<u>Sol</u>: • <u>One-One function (Injection)</u>:

A function $f : A \to B$ is ~~call~~ said to be a one-one function if all elements in set A have different images in set B.

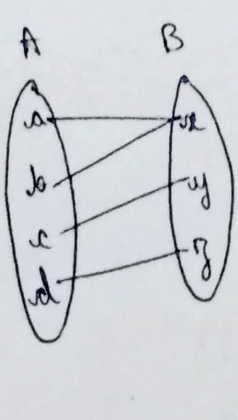
<u>Note</u>: $f : A \to B$ is a one-one function if $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ then $\boxed{x_1 = x_2}$



• <u>Onto function (Surjective function)</u>:

A function $f : A \to B$ is said to be an onto function if every element of B has a pre-image in A under f.

f is onto function if the range of f is equal to its codomain.



4) Define POSET. Let R is a relation on set of integers (Z) and defined as $R = \{(x,y) \mid x/y\}$ then prove that Z is POSET.

Sol: Given set is Z and $R = \{(x,y) \mid x/y \,;\, x,y \in Z\}$

Reflexive: $x \in Z$

Always x/x

$\Rightarrow (x,x) \in R$

$\therefore$ R is reflexive

Antisymmetric:- Let $x, y \in Z$ & $(x,y) \in R, (y,x) \in R$

$\Rightarrow x/y \,,\, y/x$

$\Rightarrow x = y$ (only possible case) $\Rightarrow (y,x) \notin R$

$\therefore$ R is antisymmetric

Transitive: Let $x, y, z \in Z$

$(x,y) \in R, (y,z) \in R$

$\Rightarrow x/y \,,\, y/z$

$\Rightarrow x/z$

$\Rightarrow (x,z) \in R$

$\therefore$ R is transitive

$\therefore$ R is partially ordering.

- Partial Ordering Relation:

→ Let 'A' be any set and 'R' is a relation defined on 'A' Then R is said to be Partial Ordering relation if R is reflexive, antisymmetric and transitive.

→ The set A with a partial order R defined on it is called Partially ordered set (or) POSET.

$f, g : R \to R$

$f(x) = 3x^2$

Now, $(f \circ f)(x) = f(f(x))$

$= f(3x^2)$

$= 3(3x^2)^2$

$= 3(9x^4)$

$= 27x^4$

$f(x) = x^2 - 2$, $g(x) = x + 4$

Now, $(f \circ g)(x) = f(g(x))$

$= f(x+4)$

$= (x+4)^2 - 2$

$= x^2 + 16 + 8x - 2$

$= x^2 + 8x + 14$

Now, $(g \circ f)(x) = g(f(x))$

$= g(x^2 - 2)$

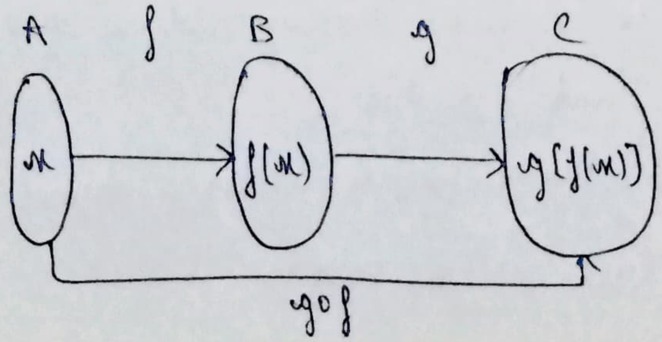
$= x^2 - 2 + 4$

$= x^2 + 2$

- <u>Composition of functions</u>:

Let $f : A \to B$, $g : B \to C$ be two functions then the composition of f and g is denoted by $g \circ f$ and is defined by $g \circ f : A \to C$

i.e $(g \circ f)(x) = g[f(x)]$

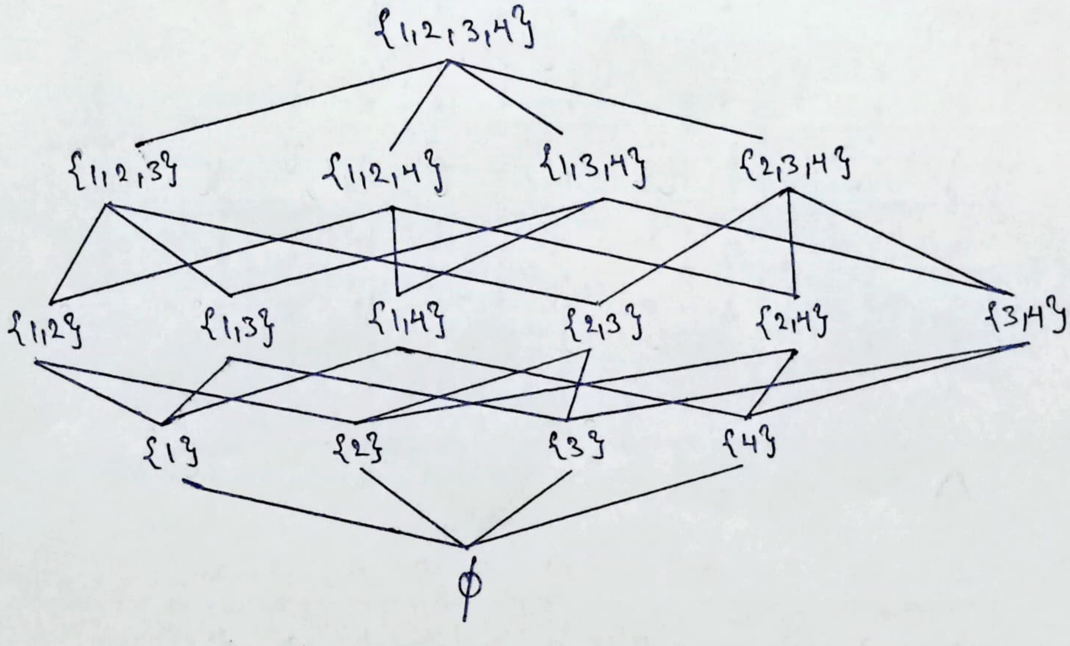


5) (a) Draw the Hasse diagram for the POSET $(P(S), \subseteq)$, where $S = \{1,2,3,4\}$.

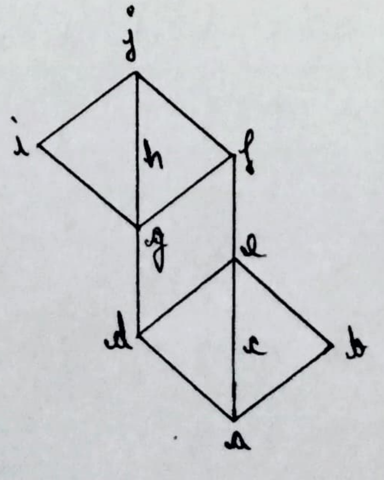
Soln Given $S = \{1,2,3,4\}$.

$R = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

Hasse Diagram



(b) Prove that the given Hasse diagram is a lattice and also verify, if it is a distributive lattice or not.



Soln

$\vee$	a	b	c	d	e	f	g	h	i	j
a	a	b	c	d	e	f	g	h	i	j
b	b	b	e	e	e	f	f	j	j	j
c	c	e	c	e	e	f	f	j	j	j
d	d	e	e	d	e	f	g	h	i	j
e	e	e	e	e	e	f	f	j	j	j
f	f	f	f	f	f	f	f	j	j	j
g	g	f	f	g	e	f	g	h	i	j
h	h	j	j	h	j	j	h	h	j	j
i	i	j	j	i	j	j	i	j	i	j
j	j	j	j	j	j	j	j	j	j	j

$\wedge$	a	b	c	d	e	f	g	h	i	j
a	a	a	a	a	a	a	a	a	a	a
b	a	b	a	a	b	b	a	a	a	a
c	a	a	c	a	c	c	a	a	a	a
d	a	a	a	d	d	d	d	d	d	d
e	a	b	c	d	e	e	d	d	d	d
f	a	b	c	d	e	f	g	g	g	f
g	a	a	a	d	d	g	g	g	g	g
h	a	a	a	d	d	g	g	h	g	h
i	a	a	a	d	d	g	g	g	i	i
j	a	b	c	d	e	f	g	h	i	j

$\therefore$ The given Hasse diagram is a lattice.

(i)

LHS	RHS	LHS	RHS
$a \vee (b \wedge c)$	$(a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c)$	$(a \wedge b) \vee (a \wedge c)$
$\Rightarrow a \vee a$	$\Rightarrow b \wedge c$	$\Rightarrow a \wedge e$	$\Rightarrow a \vee a$
$\Rightarrow a$	$\Rightarrow a$	$\Rightarrow a$	$\Rightarrow a$

(ii)

LHS	RHS	LHS	RHS
$d \vee (e \wedge f)$	$(d \vee e) \wedge (d \wedge f)$	$d \wedge (e \vee f)$	$(d \wedge e) \vee (d \vee f)$
$\Rightarrow d \vee e$	$\Rightarrow e \wedge f$	$\Rightarrow d \wedge f$	$\Rightarrow d \vee d$
$\Rightarrow e$	$\Rightarrow e$	$\Rightarrow d$	$\Rightarrow d$

(iii)

LHS	RHS	LHS	RHS
$g \vee (h \wedge i)$	$(g \vee h) \wedge (g \vee i)$	$g \wedge (h \vee i)$	$(g \wedge h) \vee (g \wedge i)$
$\Rightarrow g \vee g$	$\Rightarrow h \wedge i$	$\Rightarrow g \wedge j$	$\Rightarrow g \vee g$
$\Rightarrow g$	$\Rightarrow g$	$\Rightarrow g$	$\Rightarrow g$

$\because$ LHS = RHS

$\therefore$ It is a distributive lattice.