Probability & Statistics

Introduction to Probability, Conditional probability, Baye's theorem, Discrete and Continuous random variable, Binomial and Poisson Distribution.

Introduction to Probability:

On tossing a coin, the outcome will be either a head or a tail. The result is easily predictable. But what if you toss two coins at the same time?. The result can be a combination of head and tail. In the latter case, the correct answer cannot be obtained. So only one can predict the possibility of a result. This predication is known as probability. The probab explains something to do with a chance, that means it may happen or might not

Applications of probability:

① Probability is used in all sections

ex: Sports, Weather reports, blood samples.

② Probability is also used in insurance, Games, mutual funds, stocks etc.

Probability: To know that probability first we need to know the following

Random Experiment: An experiment have more ~~outco~~ outcomes if the result is not certain and is any one of the several possible outcomes then the experiment is called Random experiment

ex: ① Tossing a coin

② Throwing a die

Event: Outcome of random Experiment is called Event

ex: $E_1 = \{e_1, e_3, e_5\}$, $E_2 = \{e_2, e_4, e_6\}$

Mutually Exclusive Events: No common element in the events

ex: $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

Sample Space: The set of all possible outcomes of a Random experiment is called the sample space.

ex: Two coins are tossed at a time possible outcomes are HH, TT, TH, TT. If a coin is tossed 'n' times or 'n' coins tossed at a time the no. of possible outcomes is $\boxed{2^n}$

Probability: If an experiment is performed, 'n' is the total number of outcomes and m is the favourable cases of an event E. Then the probability of E is:

$$P(E) = \frac{\text{no. of favourable cases}}{\text{Total no. of outcomes}}$$

ex: ① Tossing a coin total no. of outcomes $S = \{H, T\}$
If we want probability of getting H. Then

$$P(H) = \frac{1}{2} = \frac{\text{favourable case}}{\text{Total no. of outcomes}}$$

② 11^{ly} throwing a die

Probability of getting 1 is $P(1) = 1/6$

" " " 2 " $P(2) = 1/6$

⋮

" " " 6 is $P(6) = 1/6$

Conditional Probability

The conditional probability is the probability of an event of the given one

ex: Suppose we want the probability of red card having number 4.

W.K.T in a pack of cards 26 are red, in that only two cards are numbered 4

$\therefore$ $P(\text{four card}/\text{red}) = \frac{2}{26} = \frac{1}{13}$

P(red card numbered 4) = $1/13$

Generally, we define conditional probability:

The probability of A given B i.e $P(A/B)$ where $P(B) > 0$ $P(A/B) = \frac{P(A \cap B)}{P(B)}$, $P(B/A) = \frac{P(A \cap B)}{P(A)}$

Addition Theorem of probability:

If A and B are two events

① $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

② $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

If A, B & C are mutually exclusive events then

① $P(A \cup B) = P(A) + P(B)$

② $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Note: ① $0 \leq P(E) \leq 1$ The probability of an event lies b/w 0 & 1 only

② Total probability is equal to 1

① In a class there are 10 boys and 5 girls. A committee of 4 students is to be selected from the class. Find the probability for the committee to contain atleast 3 girls

Soln: There are 10 boys and 5 girls

We need to find probability of committee of 4 members containing atleast 3 girls.

Let take E be the Event forming committee with atleast 3 girls

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{\text{favourable cases}}{\text{total no. of cases}}$

n(E) : ① 4 girls, no boys

② 3 girls, 1 boy

$\therefore n(E) = {}^5C_4 \times {}^{10}C_0 + {}^5C_3 \times {}^{10}C_1 \rightarrow (i)$

and $n(S) = {}^{15}C_4$

10+5 = 15 Total no. of students

4 → no. of students to be selected.

W.K.T $\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$

$n(E) = \frac{5!}{(5-4)!\,4!} \times 1 + \frac{5!}{(5-3)!\,3!} \times 10 \quad (\because {}^nC_0 = 1,\ {}^nC_1 = n)$

$n(E) = \frac{5!}{4!} + \frac{5!}{2!\,3!} \times 10$

$= \frac{5 \times \cancel{4!}}{\cancel{4!}} + \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}} \times 10$

$= 5 + (10 \times 10)$

$= 100 + 5$

$= 105$

$$n(s) = {}^{15}c_4 = \frac{15!}{(15-4)!\,4!} = \frac{15!}{11!\times 4!} = \frac{15\times \cancel{14}^{7}\times 13\times \cancel{12}\times \cancel{11!}}{\cancel{11!}\times 1\times \cancel{2}\times \cancel{3}\times \cancel{4}}$$

$$= \frac{15\times 7\times 13}{1} = 1365$$

sub in (I)

$$P(E) = \frac{n(E)}{n(S)} = \frac{105}{1365} \simeq 0.076$$

(2) In a group there are 3 men and 2 women. Three persons are selected at random from this group. Find the probability that one man and two women or two men and one women are selected.

soln. The group contains 3 men and 2 women = 5 people

The total no. of people selected from group are 3.

i.e $^5C_3 \Rightarrow n(s) = {}^5C_3 = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!} = 10$

Let E be the favourable case

$$n(E) = ({}^3C_1 \times {}^2C_2) + ({}^3C_2 \times {}^2C_1)$$

$$= (3\times 1) + (3\times 2)$$

$$= 3+6 = 9$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{9}{10} = 0.9$$

(3) A class consist of 6 girls and 10 boys. If a committee of 3 is choosen at random from the class. Find the probability that

(i) 3 boys are selected

(ii) Exactly 2 girls are selected

Sol: Given 6 girls and 10 boys = 16

$\therefore$ 3 members are choosen for committee

$$n(S) = {}^{16}C_3 = \frac{16!}{(16-3)!\,3!} = \frac{16!}{13!\,3!}$$

Let E be the favouriable event;

(i) 3 boys are selected: $\Rightarrow n(E) = {}^{10}C_3 = \frac{10!}{7!\,3!}$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{10!}{7!\,\cancel{3!}} \times \frac{\cancel{13!} \times \cancel{3!}}{16!}$$

$$= \frac{\cancel{10} \times 9 \times \cancel{8} \times \cancel{7!}}{\cancel{7!}} \times \frac{\cancel{13!}}{\cancel{16} \times \cancel{15} \times 14 \times \cancel{13!}}$$

$$P(E) = \frac{3}{14} = 0.214$$

(ii) Exactly 2 girls are selected:

$$n(E) = {}^{6}C_2 \times {}^{10}C_1 = \frac{6!}{4!\,2!} \times 10 = \frac{5 \times \cancel{6}}{\cancel{2}} \times 10 = 15$$

$$n(S) = {}^{16}C_3 = \frac{16!}{13!\,3!} = \frac{\cancel{16} \times \cancel{15} \times 14 \times \cancel{13!}}{\cancel{13!} \times 1 \times \cancel{2} \times \cancel{3}}$$

$$= 8 \times 5 \times 14 = 560$$

$$\therefore P(E) = \frac{15\cancel{0}}{56\cancel{0}} = \frac{15}{56} = 0.267$$

Problems on Addition Theorem :

① From a city three newspapers A, B, C are published A is read by 20%, B is read by 16%, C is read by 14%, Both A and B read by 8%; Both B and C read by 4%, A and C read by 5%. And all the 3 read by 2%. What is percentage of the population read atleast one paper.

Sol: Probability of population read A paper = 20%.

i.e $P(A) = \frac{20}{100} = 0.2$

Prob of population read B paper = 16% i.e $P(B) = 0.16$

" " " C " = 14% i.e $P(C) = 0.14$

||ly $P(A \cap B) = 8\% = \frac{8}{100} = 0.08$

$P(B \cap C) = 4\% = \frac{4}{100} = 0.04$

$P(A \cap C) = 5\% = \frac{5}{100} = 0.05$

$P(A \cap B \cap C) = 2\% = \frac{2}{100} = 0.02$

The the prob of population read atleast one paper is ~~P(AUBUC)~~ P(A or B or C)

i.e $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= 0.2 + 0.16 + 0.14 - 0.08 - 0.04 - 0.05 + 0.02$

$= 0.35 = 35\%$

(2) Given that $P(A) = 0.35$, $P(B) = 0.73$, $P(A \cap B) = 0.1$
Then find the probabilities $P(A \cup B)$, $P(\bar{A} \cup \bar{B})$, $P(\bar{A} \cap B)$

Soln: By addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.35 + 0.73 - 0.14$$

$$P(A \cup B) = 0.94$$

$$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.14 = 0$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.73 - 0.14 = 0.59$$

(3) A sample space contains of seven events nam E_i. Where $i = 1, 2, 3, 4, 5, 6, 7$ which assignment of probability 0.05, 0.2, 0.2, 0.25, 0.15, 0.1, 0.05 respectively. Let $A = \{E_1, E_4, E_6\}$, $B = \{E_2, E_4, E_7\}$ and $C = \{E_2, E_3, E_5, E_7\}$. Then find the following $P(A)$, $P(B)$, $P(C)$, $P(A \cup B)$

Soln: We know by Addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \longrightarrow \text{I}$$

When $P(A) = P(E_1) + P(E_4) + P(E_6) = 0.05 + 0.25 + 0.1$

$$= 0.4 \longrightarrow 1$$

$P(B) = P(E_2) + P(E_4) + P(E_7) = 0.2 + 0.25 + 0.05 = 0.5$ $\to 2$

$P(A \cap B)$ is the probability of $A \cap B$.

Where $A \cap B = \{E_4\}$

$$\therefore P(A \cap B) = P(E_4) = 0.25 \longrightarrow 3$$

Then subs ①, ② & ③ in ①

$\therefore P(A\cup B) = 0.4 + 0.5 - 0.25 = \underline{0.65}$

Problems on Conditional Probability:

① Let A and B be two events such that $P(A) = \frac{1}{3}$; $P(B) = 3/4$ and $P(A\cup B) = 11/12$, then obtain the conditional probabilities of the event.

Soln. Given that $P(A) = 1/3$, $P(B) = 3/4$,

$P(A\cup B) = 11/12$

We need to find $P(A/B)$ & $P(B/A)$

by conditional probability

$P(A/B) = \frac{P(A\cap B)}{P(B)}$; $P(B/A) = \frac{P(A\cap B)}{P(A)}$

So we need to find $P(A\cap B)$ first

by addition theorem w.k.T

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow P(A\cap B) = P(A) + P(B) - P(A\cup B)$

$= \frac{1}{3} + \frac{3}{4} - \frac{11}{12} = \frac{4+9-11}{12} = \frac{2}{12} = \frac{1}{6}$

$\therefore P(A/B) = \frac{1/6}{3/4}$; $P(B/A) = \frac{1/6}{1/3}$

$= \frac{1}{\cancel{6}_3} \times \frac{\cancel{4}^2}{3}$ $= \frac{1}{\cancel{6}_2} \times \frac{\cancel{3}}{1}$

$= 2/9$ $= 1/2$

$\boxed{\therefore P(A/B) = 2/9}$ $\boxed{P(B/A) = 1/2}$

② Ten numbered cards are there from 1 to 15 and two cards chosen at random such that the sum of the numbers on both the cards is even. Find the probability that the chosen cards are odd-numbered.

Sol: Given : ① Ten cards 1 to 15

② Two cards chosen randomly

③ Sum of two cards = even

④ The two cards are odd.

The numbers 1 to 15

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

The number cards which are even

$\{2, 4, 6, 8, 10, 12, 14\} = 7$

The number cards which are odd

$\{1, 3, 5, 7, 9, 11, 13, 15\} = 8$

and we know that

even + even = even

Odd + odd = Odd

Let A be the event of selecting odd numbers $n(A) = {}^8C_2 \longrightarrow ①$

Let B be the event of selecting two cards whose sum is even $n(B) = {}^7C_2 + {}^8C_2 \rightarrow$ 2

and $n(A \cap B) = {}^8C_2 \longrightarrow$ 3

Then $P(A|B) =$ Prob of selecting two odd cards whose sum is even

by conditional Probability

$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} \longrightarrow$ I

W.K.T $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{\text{favourable cases}}{\text{Total no. of cases}} \rightarrow$ 4

total we have 15 cards

n(S): selecting two cards from 15

$n(S) = {}^{15}C_2 \rightarrow$ 5

Subs 3 & 5 in 4

$P(A \cap B) = \frac{{}^8C_2}{{}^{15}C_2}$ and also $P(B) = \frac{n(B)}{n(S)} = \frac{{}^7C_2 + {}^8C_2}{{}^{15}C_2}$

sub 6 & 7 in I

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{{}^8C_2}{\cancel{{}^{15}C_2}} \times \frac{\cancel{{}^{15}C_2}}{{}^7C_2 + {}^8C_2}$$

$$= \frac{8!}{(8-2)!\,2!} \times \frac{1}{\frac{7!}{5!\,2!} + \frac{8!}{6!\,2!}}$$

$$= \frac{7 \times \cancel{8}^{4}}{\cancel{2}} \times \frac{1}{\frac{\cancel{6}^{3} \times 7}{\cancel{2}} + \frac{7 \times 8}{2}} = 28 \times \frac{1}{21+28}$$

$$= \frac{28}{49} = \frac{4}{7}$$

Prob

② Let E and F are events of experiment such that $P(E) = \frac{3}{10}$, $P(F) = \frac{1}{2}$ and $P(F/E) = \frac{2}{3}$
Find the value of (i) $P(E \cap F)$ (ii) $P(E/F)$ iv) $P(E \cup F)$

Soln:

Given $P(E) = \frac{3}{10}$, $P(F/E) = 2/3$, $P(F) = 1/2$

We need to find:

(i) $P(E \cap F)$

W.K.T by conditional Probability

$$P(F/E) = \frac{P(E \cap F)}{P(E)}$$

$$\Rightarrow P(E \cap F) = P(F/E) \cdot P(E)$$

$$= \frac{2}{3} \cdot \frac{3}{10} = 1/5$$

(ii) $P(E/F) = \frac{P(E \cap F)}{P(F)}$

$$= \frac{1/5}{1/2} = \frac{1}{5} \times 2 = \frac{2}{5}$$

iii) $P(E \cup F) = ?$

by addition thm:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{3}{10} + \frac{1}{2} - \frac{1}{5} = \frac{3+5-2}{10} = \frac{6}{10} = \frac{3}{5}$$

Prob

③ The probability of a student passing in science is 4/5 and the student passing in both science and maths is ½. What is the probability of that student passing in maths knowing that he passed in science?

soln: Given $P(S) = 4/5$ S: student passing in Science

$P(S \cap M) = 1/2$ M: " " in Maths

We need to find $P(M|S) = ?$

By Conditional Probability

$$P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{1/2}{4/5} = \frac{1}{2} \times \frac{5}{4} = \frac{5}{8}$$

∴ student passing in Maths, knowing he passed in science is 5/8

Independent Events: The outcome of one event doesnot affect the outcome of the another event. Then both Events are called independent Events.

Ex: Throwing two coins at a time. The results of one doesn't affect the result of another.

Note: If A and B are independent iff

$$P(A \cap B) = P(A) \cdot P(B)$$

① $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A)$ for $P(B) > 0$

② $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B)}{\cancel{P(A)}} = P(B)$, for $P(A) > 0$

Multiplication Theorem:

Let E_1, E_2 be the two events in a random experiments then

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1)$$
$$= P(E_2) \cdot P(E_1/E_2)$$

Bayes theorem: If $E_1, E_2, E_3 \ldots E_n$ are mutu- Exclusive Events, such that $P(E_i) > 0$ $(i = 1, 2, \ldots n)$ in a sample space S and A is any other Event in S such that $P(A) > 0$ (A can only occur in combination of any one of the event of $E_1, E_2 \ldots E_n$) Where $P(E_1), P(E_2), P(E_3) \ldots P(E_n)$ & $P(A/E_1)$, $P(A/E_2) \ldots P(A/E_n)$ are known values. Then

$$P(E_k/A) = \frac{P(E_k) \cdot P(A/E_k)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots + P(E_n) P(A/E_n)}$$

① In a bolt factory, Machines A, B, C manufactures respectively 20%, 30% and 50% of the total output of their outputs 5, 4 and 2 percent are known to be defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by (i) Machine A

(ii) Machine B

Soln: Given A, B, C Machines

Probability of Machine A manufactures bolts $= 20\%$

i.e $P(A) = \frac{20}{100}$

$|||^{ly}$ Probability Machine B manufactures bolts

i.e $P(B) = \frac{30}{100}$

$||^{rly}$ $P(C) = \frac{50}{100}$

Let D be the defective bolts

Probability of defective bolts from A = 5%

i.e. $P(D/A) = \frac{5}{100}$

Probability of defective bolts from B = 4%

$P(D/B) = \frac{4}{100}$

$||^{rly}$ $P(D/C) = \frac{2}{100}$

Then probability of taking bolt from product which is expected from A which is already defective is

$$P(A/D) = \frac{P(A) \cdot P(D/A)}{P(A) \cdot P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{\frac{200}{100} \times \frac{5}{100}}{\frac{20}{100} \times \frac{5}{100} + \frac{30}{100} \times \frac{4}{100} + \frac{50}{100} \times \frac{2}{100}} = \frac{100}{100+120+100} = \frac{100}{320} = \frac{5}{16}$$

$||^{rly}$

$$P(B/D) = \frac{P(B) \cdot P(D/B)}{P(A) \cdot P(D/A) + P(B)P(D/B) + P(C) \cdot P(D/C)}$$

$$= \frac{\frac{30}{100} \times \frac{4}{100}}{320} = \frac{12}{32} = \frac{3}{8}$$

$$P(C/D) = \frac{P(C) \cdot P(D/C)}{P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)}$$

$$= \frac{50 \times 2}{320} = \frac{10\cancel{0}}{32\cancel{0}} = \frac{5}{16}$$

(2) A company lauches an advertising compaign of new product on TV, Radio and in print media in an area where 30% watch TV, 50% listen to Radio and rest rely on newspapers for all information. It is estimated that a person who sees the advertisement on TV will buy the product with probability of 0.6, a person who heard it on Radio is expected to buy the product with probability 0.3 and seeing the advertisement in print will convience a person to buy the product with probability 0.1. A consumer chosen at random, is found to have purchased the product. What is the probability that he heard about the product on Radio?

Soln: An advertising Compaign of newproduct

who watch on TV is $P(A) = 30\%$

listen on Radio is $P(B) = 50\%$

Read Newspaper is $P(C) = 20\%$

Probability of buying newproduct (D)

by watching TV $P(D|A) = 0.6$

listening Radio $P(D|B) = 0.3$

Read Newspaper $P(D|C) = 0.1$

Then probu of buying product by listening radio is

$$P(B/D) = \frac{P(B) \cdot P(D|B)}{P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C)}$$

$$= \frac{\frac{50}{100} \times \frac{3}{10}}{\frac{30}{100} \times \frac{6}{10} + \frac{50}{100} \times \frac{3}{10} + \frac{20}{100} \times \frac{1}{10}} = \frac{150}{350} = \frac{3}{7}$$

(3) In a certain assembly plan 3 machines B_1, B_2, B_3 makes products 30%, 45%, and 25% respectively. It is known from the past experience that 2%, 3%, and 2% of the products made by each machine respectively are defective. What is the probability that the product is defective? If the product was chosen randomly and found to be defective, what is the probability that it was made by (i) Machine B_1, (ii) Machine B_2.

Soln. Probability of products made by B_1 is $P(B_1) = 30\%$

" " " B_2 is $P(B_2) = 45\%$

" " " B_3 is $P(B_3) = 25\%$

The defective products made by B_1 is $P(D/B_1) = 2\%$

" " " B_2 is $P(D/B_2) = 3\%$

" " " B_3 is $P(D/B_3) = 2\%$

Then the probability product which is choosen randomly from B_1 is

$$P(B_1/D) = \frac{P(B_1) \cdot P(D/B_1)}{P(B_1) P(D/B_1) + P(B_2) \cdot P(D/B_2) + P(B_3) \cdot P(D/B_3)}$$

$$= \frac{\frac{30}{100} \times \frac{2}{100}}{\frac{30}{100} \times \frac{2}{100} + \frac{45}{100} \times \frac{3}{100} + \frac{25}{100} \times \frac{2}{100}} = \frac{60}{245} = \frac{12}{49} \simeq 0.245$$

IIly $$P(B_2/D) = \frac{P(B_2) P(D/B_2)}{P(B_1) P(D/B_1) + P(B_2) P(D/B_2) + P(B_3) P(D/B_3)} = \frac{135}{245} = 0.551$$