24/2/23:-

Cartesian product :- Let A, B are 2 non empty sets then the cartesian product of A and B is denoted by $A \times B$, and it is defined as set of all possible ordered pairs where ordered pairs (x, y) the first element is from first set A and the second element is from second set B.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

Note :-

i) If $A_1, A_2 \ldots A_n$ are n sets then the cartesian product of $A_1 \times A_2 \times \ldots A_n$:

$$= \{(x_1, x_2, \ldots x_n) \mid x_i \in A_i \text{ for } i = n\}$$

ii) $A \times B \neq B \times A$ since for any 2 distinct $(x, y) \neq (y, x)$

iii) $A \times B = B \times A \iff A = B$

iv) if $n(A) = m$, $n(B) = n$ then $n(A \times B) = m \times n$.

Ex:- $A = \{1,2\}$, $B = \{a,b\}$

$n(A) = 2$ $n(B) = 2$

$A \times B = \{(1,a), (1,b), (2,a)(2,b)\}$

$n(A \times B) = 4$.

2) $A = \{1,2\}$ $B = \{a,b\}$ $C = \{x,y\}$

$A \times B \times C = \{(1,a,x), (1,a,y), (1,b,x),$
$(1,b,y), (2,a,x), (2,a,y)$
$(2,b,x), (2,b,y)\}$

$n(A \times B \times C) = 8$

Binary Relation:- A Relation R from set $A \to B$ is defined as a subset of cartesian product of A and B.

$$\boxed{R \subseteq A \times B}$$

If R is a Relation from $A \to B$ and if $(x,y) \in R$ then we can also write as $x R y \simeq (x,y) \in R$

Unary Relation:- A Relation R from $A \to A$ is called Unary Relation.

n-ary relation:- A Relation R from n sets ie, $A_1, A_2, \ldots A_n$

i.e, $R \subseteq A_1 \times A_2 \times \cdots A_n$

Domain and Range of a Relation:-

Example for Binary Relation:-

$A = \{(x, y)\}$, $B = \{1, 2, 3\}$

$(1, 2)$

$A \times B = \{(1,1), (1,2) (1,3), (2,1) (2,2) (2,3)\}$

$R_1 = \{(x, y) \mid (x-y) \text{ is divisible by } 2\}$

$= \{(1,1) (1,3), (2,2)\}$

$R_2 = \{(x, y) \mid x \mid y\}$

* Domain: Set of all first elements in a Relation R where R is from $A \to B$

* Range: Set of all second elements in R is called Range

Domain $= \{ x \mid$ for some $(x,y) \in R$ and $y \in B \}$

Range $= \{ y \mid$ for some $x \in A \}$

$A = \{1, 2, 3\}$, $B = \{a, b\}$

$R = \{(1,b), (2,a), (3,a)(3,b)\}$

$D = \{1,2,3\}$, Range $= \{a, b\}$

<u>Operations on Relation:-</u> Let R, S be any 2 Relations. The Union of R and S

i) $R \cup S = \{ (x,y) \mid (x,y) \in R$ or $(x,y) \in S$ or both $\}$

ii) $R \cap S = \{ (x,y) \mid (x,y) \in R$ and $(x,y) \in S \}$

iii) $R - S = \{ (x,y) \mid (x,y) \in R$ and $(x,y) \notin S \}$

iv) $S - R = \{ (x,y) \mid (x,y) \in S$ and $(x,y) \notin R \}$

v) $R \Delta S = (R - S) \cup (S - R)$

$A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$

$R = \{(1, a), (3, b), (4, c)\}$

$S = \{(2, b), (3, a), (4, c)\}$

i) $R \cup S = \{(1, a), (3, b), (4, c), (2, b), (3, a), (4, c)\}$

ii) $R \cap S = \{(4, c)\}$

iii) $R - S = \{(1, a), (3, b)\}$

iv) $S - R = \{(2, b), (3, a)\}$

v) $R \Delta S = \{(1, a), (3, b), (2, b), (3, a)\}$

Types of Relations:-

i) **Empty Relation:-** A Relation R from set A to set B is called empty Relation, if $R = \phi$.

ii) **Universal Relation:** If $R = A \times B$, then (R) it is said to be Universal Relation. It is denoted by 'U' and it is the cartesian product of $A \times B$.

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3) **Identity Relation**:- A Relation R on set A is said to be Identity Relation if $R = \{(x,x) \mid x \in A\}$ and it is denoted by I

$$I = \{(x,x) \mid x \in A\}$$

Ex: $A = \{1,2,3\}$

$$I = \{(1,1),(2,2),(3,3)\}$$

4) **Inverse Relation**: let A Relation R on set A to set B then the inverse of R is denoted by R^{-1} and it is defined as

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Ex:- $A = \{1,2,3\}$

$B = \{x,y\}$

$R = \{(1,x),(2,y),(3,x),(3,y)\}$

$R^{-1} = \{(x,1),(y,2),(x,3),(y,3)\}$

5) **Reflexive Relation**: A Relation R on set A is said to be reflexive if

$\forall x \in A \Rightarrow (x,x) \in R$

Ex: $A = \{1,2,3\}$

$R = \{(1,1),(1,3),(2,2),(2,3),(3,3)\}$

$1 \in A$ $(1,1) \in R$, $(2,2) \in R$, $(3,3) \in R$
$2 \in A$ $3 \in A$.

$\therefore$ R is Reflexive.

6). Irreflexive Relation: A Relation R on set A is said to be Irreflexive if

$\forall x \in A \Rightarrow (x,x) \notin R$

$A = \{1,2,3\}$

Ex:- $R = \{(1,2),(1,1),(2,3)\}$

$1 \in A \Rightarrow (1,1) \in R$ ✓

$(2 \in A) \Rightarrow (2,2) \notin R$ ✗, $(3 \in A \Rightarrow (3,3) \notin R$ ✗

$\therefore$ R is Irreflexive.

7) Symmetric Relation: for any $\overbrace{(x,y)}^{2\ elements} \in A$ if $(x,y) \in R \Rightarrow (y,x) \in R$ then R is said to be Symmetric relation on set A

Ex: $A = \{1,2,3\}$

$R = \{(1,1),(2,1),(1,2),(3,2),(2,3)\}$

$(1,2) \in A \Rightarrow (2,1) \in A$ ✓

$(3,2) \in A \Rightarrow (2,3) \in A$

8) **Asymmetric Relation**: for any $x, y \in A$ (2 elements)

if $(x, y) \in R \Rightarrow (y, x) \notin R$ then R is said to be Asymmetric Relation on set A

Ex: $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (1,3), (2,3)\}$

$(1,2) \in R$, $(2,1) \in R$ ✓

$(1,3) \in R$, $(3,1) \notin R$, $(2,3) \in R$ $(3,2) \notin R$

$\therefore$ R is Asymmetric.

---

9) **Anti-symmetric Relation**: for any $x, y \in A$

if $(x, y) \in R$ & $(y, x) \in R \Rightarrow x = y$

then R is said to be Anti-symmetric Relation on set A.

$R = \{(x, y) \mid x \leq y\}$ on set of Integers

$(x, y) \in R \Rightarrow x \leq y$.

$(y, x) \in R \Rightarrow y \leq x$ $\}$ $\Rightarrow x = y$.

$\therefore$ R is anti-symmetric

10. **Transitive Relation**: for any $x, y, z \in A$

if $(x, y) \in R$ & $(y, z) \in R$

$\Rightarrow (x, z) \in R$.

Ex:- $R = \{(x, y) \mid x \leq y\}$ on Set of Integers

$(x, y) \in R \Rightarrow x \leq y$

$(y, z) \in R \Rightarrow y \leq z$ } $x \leq z$

$\Downarrow$

$x \leq z \in R$

$\therefore (x, z) \in R$. $\therefore$ R is a Transitive Relation

11. **Equivalence Relation**: A Relation R on set A is said to be equivalence Relation if it's satisfying the following 3 conditions:

1) R is Reflexive
2) R is Symmetric
3) R is Transitive

P1: if R is a Relation "is greater than" from set A to set B where $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 6\}$ then

i) Find the elements in R

ii) Domain iii) Range iv) Find R^{-1}

Sol: Given $R = \{(x, y) \mid x$ is greater than $y\}$

i) $R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (5,1) (5,2)\}$

ii) Domain of R is $\{2, 3, 4, 5\}$

iii) Range of R is $\{1, 2\}$

iv) $R^{-1} = \{(1,2), (1,3), (2,3), (1,4), (2,4), (1,5), (2,5)\}$

P2:- if $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$ and $R = \{(a, b) \mid a$ is divisible by $b\}$ then

$a \mid b$

find i) Elements in R ii) Domain iii) Range iv) R^{-1}

$\frac{a}{b}$ $\frac{b}{a}$

Sol.

i) $R = \{(2,2), (4,2), (6,2), (6,3)\}$

ii) Domain $= \{2, 4, 6\}$

iii) Range $= \{2, 3\}$

iv) $R^{-1} = \{(2,2), (2,4), (2,6), (3,6)\}$

If $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ Let

R is the following relation

$R = \{(1, x), (1, z), (2, y), (2, z),$
$(3, z), (4, y), (4, z)\}$ then

i) Determine the matrix of the relation

ii) Draw the Arrow-diagram

iii) Find R^{-1}.

Sol. Matrix:

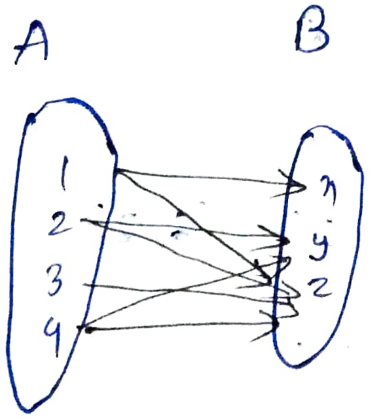
i)

$$\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$\mapsto$ set B as column wise

set A row wise

ii)



iii) $\bar{R}^1 = \{ (x,1), (z,1), (y,2), (z,2), (z,3),$
$(y,4), (z,4) \}$

* $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

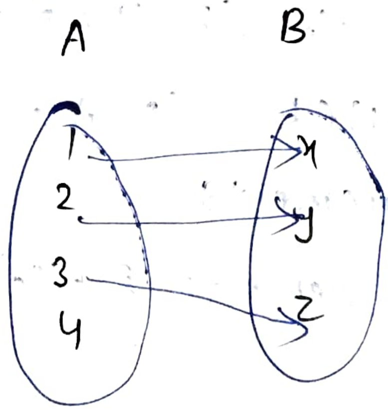
$R = \{ (1, x), (2, y), (3, z) \}$

i) matrix:

$$\begin{array}{c|ccc} & x & y & z \\ \hline 1 & \multicolumn{3}{c}{} \\ \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

ii)



iv) $\bar{R}^1 = \{ (x,1), (y,2), (z,3) \}$

01/03/2023:-

P1: If $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R is a Rel^n defined as $(x, y) \in R$ if $x - y$ is divisible by 3 then

assign

i) find the elements in R

ii) Show that R is an equivalence Reln

Sol

$A = \{1, 2, 3, 4, 5, 6, 7\}$

$R = \{(1,1), (2,2), (3,3), (4,1), (1,4), (1,7), (2,5), (3,6), (4,4)$

$(5,5), (6,6), (7,7), (4,7), (5,2), (6, (7,1), (7,4)\}$

$n(R) = 17$

ii) a) Reflexive Relation:-

$\forall x \in A \Rightarrow (x, x) \in R$

$1 \in A \Rightarrow (1,1) \in R$ $\quad (2 \in A \Rightarrow (2,2) \in R$

lly for all elements

All the elements satisfy the Reflexive Relation.

$\therefore$ R is Reflexive Relation

b) Symmetric Relation

for any $(x,y) \in A$ if $(x,y) \in R$ then

$\Rightarrow (y,x) \in R$.

i) $(1,1) \Rightarrow (1,1) \in R$

ii) $(1,4) \Rightarrow (4,1) \in R$

{ No need of Verification for same elements }

Detailed explanation:

$1, 4 \in A$ and if $(1,4) \in R \Rightarrow (4,1) \in R$ ✓

Illy for all elements.

$(1,7) \Rightarrow (7,1) \in R$ $(4,7) \Rightarrow (7,4) \in R$

$(2,5) \Rightarrow (5,2) \in R$

$(3,6) \Rightarrow (6,3) \in R$

$\therefore$ R is Symmetric Relation.

c) Transitive Relation:

for any $x, y, z \in A$ if $(x, y) \in R$ and $(y, z) \in R$

$\Longrightarrow (x, z) \in R$

$1, 4, 7 \in A$ and $(1,4) \in R$ & $(4,7) \in R$

$\hookrightarrow (1,7) \in R$ ✓

$\therefore$ R is Transitive Relation

$\therefore$ R is an Equivalence Relation

R satisfies Reflexive, Symmetric, Transitive

So R is an Equivalence Reln.

Q3* Let R is a Rel^n on set of integers and it is defined as

H.W

$(a,b) \in R$ if $a-b$ is divisible by 2

then S-T R is an Equivalence Relation.

Sol. $R = \{(a,b) \mid a-b$ is divisible by $2\}$ & $A = \mathbb{Z}$

i) Reflexive:

for every ($\forall$) $a \in A \Rightarrow (a,a) \in R$

W.K.T,

0 is divisible by 2

$\Rightarrow \underset{(=0)}{a-a}$ is divisible by 2

$\forall a \in \mathbb{Z}$

$\Rightarrow (a,a) \in R \ \forall a \in A$.

$\therefore$ R is Reflexive

ii) Symmetric Relation:

for any, if $a,b \in A$ and if $(a,b) \in R \Rightarrow (b,a) \in R$

if $(a,b) \in R \Rightarrow a-b$ is divisible b 2

$\Rightarrow b-a$ is also divisible by 2

$\Rightarrow (b,a) \in R$

$\therefore$ R is Symmetric Rel^n

$a-b = 2$

$b-a = -2$

iii) Transitive Relation:-

for any $(a, b, c) \in A$

if $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a, c) \in R.$

$(a, b) \in R \Rightarrow \quad a - b = 2K_1 \quad - (1)$

$(b, c) \in R \Rightarrow \quad b - c = 2K_2 \quad - (2)$

$(1)+(2) \Rightarrow \quad a - b + b - c \quad 2K_1 + 2K_2$

$a - c = 2(K_1 + K_2)$

$\therefore (a, c) \in R$

$\therefore$ R is a Transitive Relⁿ

$\therefore$ So R is Reflexive, Symmetric, Transitive, So it is an Equivalence Relⁿ

P3* If R is a Relⁿ on the set of integers defined by a R b if $a * b \geq 0$. Is R an Equivalence Relation.

Sol: $R = \{(a, b) \mid a \cdot b \geq 0\}$

i) Reflexive:-

$\forall a \in A \Rightarrow (a, a) \in R.$

/* If $a \in Z^+$ and $b \in Z^-$ then */

if $x > 0 \quad x \cdot x \geq 0$

$x = 0 \quad 0 \cdot 0 \geq 0$

$x < 0 \quad -x \cdot -x \geq 0$

ii) <u>Symmetric</u> :- for any $x, y \in A$ if

$(x, y) \in R \Rightarrow (y, x) \in R$

$(a, b) \in R$

$a \times b \geq 0 \Rightarrow b \times a \geq 0$ ✓

$\therefore$ R is symmetric Relⁿ.

iii) <u>Transitive</u> :

for any $(x, y, z) \in A$ if $(x, y) \in R$ &
$(y, z) \in R$

$\Rightarrow (x, z) \in R$

$(x, y) \in R \Rightarrow x \times y \geq 0$

$(y, z) \in R \Rightarrow y \times z \geq 0$

we may not conclude that $x \times z \geq 0$

<u>for ex:</u> $x\ y$

$-2, 0, 2$

$x \quad y \quad z$

$(-2, 0) \in R$ ✓

$(0, 2) \in R$ ✓

$(-2, 2) \notin R$

$\therefore$ R is not Transitive

$\therefore$ R is Reflexive, Symmetric but not Transitive So R is not an Equivalence Relation.

P4: Let R is a Rel^n on set of ~~Integers~~ (Real Numbers) and it is defined as $(x, y) \in R$ if $x - y$ is an Integer. Is R an Equivalence Relation.

$R = \{(x, y) \mid x - y \in \mathbb{Z}\}$

Sol: i) Reflexive:- $\forall x \in A \rightarrow x - x (= 0)$ is

$\forall x \in A \Rightarrow (x, x) \in R \rightarrow (x, x) \in R$

$(x, x) \Rightarrow x - x = 0 \rightarrow$ Integer

$\therefore R$ is Reflexive

ii) Symmetric:

for any $x, y \in A$ if $(x, y) \in R \Rightarrow (y, x) \in R$

$(x, y) \in R \Rightarrow x - y \in \mathbb{Z}$

$\Rightarrow y - x \in \mathbb{Z} \Rightarrow (y, x) \in R$

$\therefore R$ is Symmetric

iii) Transitive:

for any $x, y, z \in A$ if $(x, y) \in R$, $(y, z) \in R$

$\Rightarrow (x, z) \in R$

$(x, y) \in R \Rightarrow x - y \in \mathbb{Z} = z_1$

$(y, z) \in R \Rightarrow + \; y - z \in \mathbb{Z} = z_2$

$x - z = z_1 + z_2 = \mathbb{Z}$

$\therefore$ R is Transitive

Q4. Let R be a Relation, on Set of Real Number (R) and it is defined as $(x, y) \in R$ if $x - y$ is an integer. Is R an Equivalence Relation?

Sol: Given that,

$R = \{(x, y) \mid x - y \text{ is an Integer}\}$

i) Reflexive Relation:-

A Relation R is said to reflexive, if $\forall x \in A \Rightarrow (x, x) \in R$.

So, Let x be any real number i.e,

$x \in A, \Rightarrow x - x \,(=0)$

then, we know that,

0 is an Integer, we can write 0 as $\underset{(=0)}{x - x}$. So

$\underset{(=0)}{x - x}$ is also an integer

$\therefore (x, x) \in R \; (\because R = \{(x, y) \mid x - y \in$

$\therefore$ R is a Reflexive Relation

ii) Symmetric Relation :-

A Relation R is said to be Symmetric, if for any $x, y \in A$ if $(x, y) \in R \Rightarrow (y, x) \in R$

So, here

$(x, y) \in R \implies x - y$ is an Integer

$x - y = z_1$

if $x - y = z_1$

multiply '−' on B.S

$-(x - y) = -z_1$

$y - x = -z_1$

↳ ($-z_1$ is also an Integer)

So, $y - x$ is also an Integer

$\therefore (y, x)$ also belongs to R

$((y, x) \in R)$

$\therefore$ R is a Symmetric Relation.

---

iii) Transitive Relation :-

A Relation R is said to be transitive, if for any $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$

$\implies (x, z) \in R$

So,

$(x, y) \in R$

$\Rightarrow x - y = z_1$ (z_1 be an integer) — ①

and

$(y, z) \in R$

$\Rightarrow y - z = z_2$ (z_2 be an Integer) — ②

① + ②

$$x - y + y - z = z_1 + z_2$$

$$x - z = z_1 + z_2 = z$$

$x - z = z$ (Sum of any 2 Integers is also an Integer)

$\therefore (x, z) \in R$

$\therefore$ R is Transitive Relation.

So, R is Reflexive, Symmetric and Transitive. we can conclude that

R is an Equivalence Relation.

2/3/23.

Congruence Modulo m :- let a and b are 2 integers and $m \in Z^+$ then 2 integers a and b are congruence to modulo m

$\Leftrightarrow$ They have same remainder when divided by m and it is denoted by ($a \equiv b \pmod{m}$) read as "a is congruence to b mod m".

Note :- $a \equiv b \pmod{m} \Leftrightarrow a-b$ is divisible by m $\Rightarrow m / a-b$

$a-b = mK$ (K is any Integer)

P1. P.T congruence modulo m is an Equivalence Rel^n on the set of Integers

Sol: $R = \{ (a,b) | \ a \equiv b \pmod{m} \}$

Reflexive: $\forall a \in A \Rightarrow (a,a) \in R$

W.K.T

0 is divisible by m

$\underset{(=0)}{a-a}$ is divisible by m $\forall a \in A$

$\Rightarrow a \equiv a \pmod m$

$\Rightarrow (a,a) \in R \quad \forall a \in A$

$\therefore$ R is Reflexive.

ii) Symmetric:

for any $a,b \in A$ if $(a,b) \in R$
$\Rightarrow (b,a) \in R$

$(a,b) \in R \Rightarrow a \equiv b \pmod m$.
$\Leftrightarrow a-b$ is divisible by m
$\Rightarrow b-a$ is also divisiblye by m
$\Rightarrow b \equiv a \pmod m$
$\Rightarrow (b,a) \in R$

$\therefore$ R is Symmetric

iii) Transitive: Relation.

for $a,b,c \in R$ if $(a,b) \in R$ and $(b,c) \in R$
$\Rightarrow (a,c) \in R$.

$(a,b) \in R \Rightarrow a \equiv b \pmod m$
$\Leftrightarrow a-b = mk_1$

$(b,c) \in R \Rightarrow b \equiv c \pmod m$
$b-c = mk_2$

① + ② $a - c = m(K_1 + K_2)$

$a - c = mK$

$\Rightarrow a \equiv c \pmod{m}$

$\Rightarrow (a, c) \in R$

$\therefore$ R is Transitive Relation.

$\therefore$ So R is Reflexive, Symmetric, Transitive

Hence R is an Equivalence Relation.

Ordering Relations:-

i) partial order Relation or Partially Ordered Set :- (POSET) A Relation R on set A is called partial order Reln if it is satisfying following 3 properties

i) R is Reflexive

ii) R is Antisymmetric

iii) R is Transitive.

A set A together with the Partial Order R is called Partially ordered set and it is denoted by (A, R)

(or)

$(A, \leq)$

P1:- S.T "less than or equal" relation is a partially ordering on the set of Integers where $R = \{(a,b) \mid a \leq b\}$

Sol:-

$R = \{(a,b) \mid a \leq b\}$ on $A = \mathbb{Z}$

i) Reflexive Relation:

$\forall a \in A \Rightarrow (a,a) \in R$

let $a \in A$,

we know that, $a \leq a \quad \forall a \in A$.

$\therefore (a,a) \in R$

$\therefore R$ is Reflexive

ii) Antisymmetric Relation:

for any $x, y \in A$ if $(x,y) \in R$ and $(y,x) \in R$ $\Rightarrow y = x$

$$\left.\begin{array}{l}(x,y) \in R \Rightarrow x \leq y \\ (y,x) \in R \Rightarrow y \leq x\end{array}\right\} \Rightarrow y = x$$

$\therefore$ R is Anti-symmetric

iii) Transitive Relation:

for any $x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$

$(x,y) \in R \Rightarrow x \le y$ — ①

$(y,z) \in R \Rightarrow y \le z$ — ②

from ① and ② $x \le z$

$\therefore (x,z) \in R$

$\therefore$ R is Transitive

So R is Reflexive, Anti-Symmetric and Transitive

Hence R is an Partial order Relation and [A is a Partially ordered Set].

P2 ③ $R = \{(a,b) \mid a \text{ divides } b\}$

S.T "a divides b" relation is a partial ordering on the set of Integers

Sol $R = \{(a,b) \mid a \text{ divides } b\}$

i) Reflexive Relation:-

$\forall a \in A \Rightarrow (a,a) \in R$

(a divides b $=$ $\frac{b}{a}$ gives remainder 0)

$\frac{a}{a}$ always gives remainder 0

i.e, a always divides a $\forall a \in A$

$\therefore (a,a) \in R$

$\therefore$ R is Reflexive

ii) Anti- Symmetric Relation)

for any $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$

$\Rightarrow a = b$

$(a,b) \in R \Rightarrow \frac{b}{a}$ $b = a\cancel{k}$ — ①

$(b,a) \in R \Rightarrow$ $a = bK$ — ②

$$\frac{①}{②} \Rightarrow \frac{b}{a} = \frac{aK}{bK}$$

$$a^2 = b^2 \Rightarrow \boxed{a = b}$$

$\therefore$ R is Anti-symmetric

iii) <u>Transitive Reln.</u>

for any $a, b, c \in \mathbb{Z}$ if $(a,b) \in R$

$(b,c) \in R$

$\Rightarrow (a,c) \in R$

$(a,b) \in R \Rightarrow b = aK_1$ — ①

$(b,c) \in R \Rightarrow c = bK_2$ — ②

$$c = a(K_1 \times K_2)$$

$$c = aK$$

$\therefore$ a divides c

$(a,c) \in R$

$\therefore$ R is Transitive

So R is Reflexive, Anti symmetric and Transitive Hence R is a Partiall Ordereing Reln.

P3 Let R be the Rel^n on the Set of People Such that $(x, y) \in R$ if x and y are people x is older than y then check R is partially ordered or not.

P4: S.T The inclusion relation $\subseteq$ is a partially ordering on the power set of a set A..

Sol. $R = \{(x, y) \mid x \text{ is older than } y\}$

i) Reflexive. Relation:-

$\forall x \in A \Rightarrow (x, x) \in R$.

$(x, x) \notin R$ Since a person cannot be older than themselves

$\Rightarrow$ R is Anti-Symmetric and Transitive but not Reflexive

$x > y$ $\}$ $y \neq x$
$y > x$

ST the inclusion relation $\subseteq$ is a partial ordering on the power set of a set A

P4 Sol If A is any set, then the set of all subsets of A is called the power set of A.

i.e., $P(A)$

$R = \{ (X, Y) \mid X \subseteq Y \}$

i) <u>Reflexive Relation</u>: $\forall x \in A \Rightarrow (x, x) \in R$

For any subset X of A, we know that

$X \subseteq X$. (since every set is subset of itself)

$(X, X) \in R$

$\therefore$ R is Reflexive

ii) <u>Anti-Symmetric</u>: for any $x, y \in A$ if $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$

$(X, Y) \in R \Rightarrow X \subseteq Y$

$(Y, X) \in R \Rightarrow Y \subseteq X$

we know that if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$ (equal sets)

$\therefore$ R is Anti symmetric

iii) Transitive Relation:

for any x, y, z if $(x, y) \in R$,

$(y, z) \in R$

$\Rightarrow (x, z) \in R$

$(x, y) \in R \Rightarrow x \leq y$

$(y, z) \in R \Rightarrow y \leq z$

then we can conclude that

$x \leq z$

$\therefore (x, z) \in R$

$\therefore$ R is Transitive

So, R is Reflexive, Anti-Symmetric and Transitive

$\therefore$ R is an ~~Equivalence~~ Partial-order Relation.

03/03/2023

* **Comparable** :- The elements a and b of POSET (A, R) are called comparable if either a R b or b R a

* **Incomparable** :- The elements a and b of POSET (A, R) are called Incomparable if neither a R b nor b R a.

* Ex :- $R = \{(x, y) \mid x \leq y\}$ on set of $\mathbb{Z}$.

$\mathbb{Z} = \{ \ldots -2, 0, 1, 2 \ldots \}$

$x = 2, y = -3$

$2 \nleq -3 \mid 2 \geq -3$ ✓

$\therefore (-3, 2) \in R$

$\therefore (-3, 2)$ are comparable elements

$R = \{(x, y) \mid x \mid y\}$ or $R = \{(X, Y) \mid X \subseteq Y\}$ on $P(A)$

$(P(A), \subseteq)$ is a Poset

For example.

$A = \{1, 2, 3\}$

$P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\},$
$\{1,3\}, \{1,2,3\} \}$

$R = \{ (X, Y) \mid X \subseteq Y \}$

$\Rightarrow X = \{1\} \quad Y = \{3\}$

$X \not\subseteq Y, \; Y \not\subseteq X$

$\therefore X$ and Y are Incomparable
$\{1\}$ $\{3\}$

$\Rightarrow X = \{1\}, \; Y = \{1, 3\}$

$X \subseteq Y \checkmark, \; Y \not\subseteq X$

$\checkmark$
$(X, Y) \in R$

$\therefore \{1\}, \{1, 3\}$ are comparable

Total ordered relation :- if (A, R) is a POSET and every 2 elements of A are comparable then R is called Total ordered relation and A is called TOSET

Ex: Examples for TOSET are

(A, R) $\quad (\mathbb{Z}, \leq)$ is a Totally
and $(\mathbb{Z}, \geq)$

ordered relation because $a \leq b$ or $b \leq a$ whenever a and b are Integers

* The POSET $(\mathbb{Z}, |)$ is not a TOSET, it's a POSET. because it contains elements that are incomparable such as 3, 5 etc.

* The POSET $(P(A), \subseteq)$ is not a TOSET because it contains elements that are incomparable such as $\{1\}, \{3\}$

08/03/2023

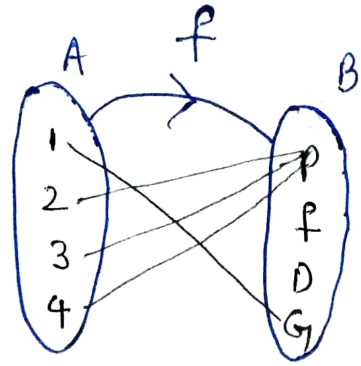
Functions:-

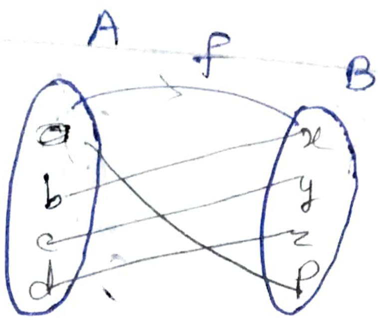
let A and B are non-empty sets then a function f from $A \to B$ is a relation from $A \to B$. such that for each element $x \in A$ $\exists$ a unique $y \in B$ such that $(x, y) \in f$ then we write it as $y = f(x)$,

here y is image of x and x is pre-image of y. It is denoted by $f: A \to B$ where A is domain of f and co-domain of f is B

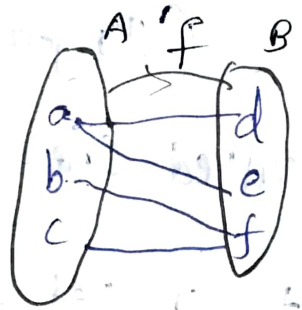
Ex:-

1)

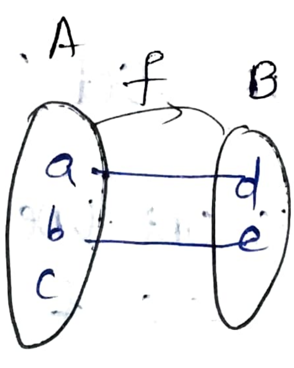


2) 

The belowing are the examples of non-functions

Case 1 :- 

$\therefore$ One element in A should not have 2 images ($a \to d, a \to e$)

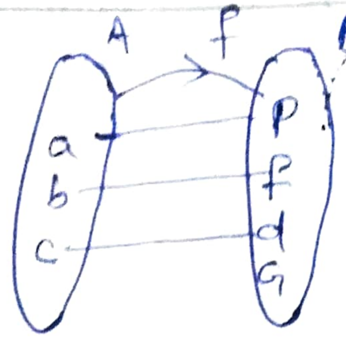
Case 2 :- 

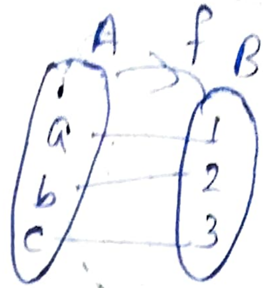
$\therefore$ Every element must have a image.

Range :- If $f : A \to B$ is a function, then the set of all f- images of elements in A is called the range of f.

It is denoted by $f(A)$

$$f(A) = \{ f(x) \mid \text{for some } x \in A \} \subseteq B$$





Range = $\{p, f, d\}$ Range = $\{1, 2, 3\}$

One - one function :- (Injective) (1 - 1)

A $f : A \to B$ is called one - one / injective if all elements in set A have distinct images in set B

Note: $f : A \to B$ is one - one

$\iff x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$

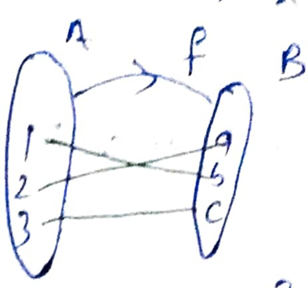
then $f(x_1) = f(x_2)$

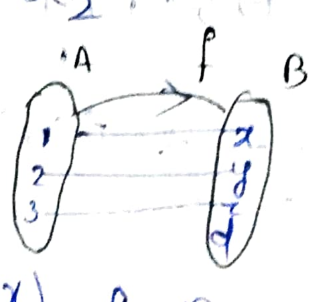
$\Rightarrow x_1 = x_2$

$\iff x_1, x_2 \in A$ & $f(x_1) \neq f(x_2)$

$\Rightarrow x_1 \neq x_2$

Ex:





Ex: $f(x) = x^3$ $(\forall x)$ $f : R \to R$

Onto function (Surjective) :- A $f: A \to B$ is onto function if each element of set B is mapped to atleast one element of set A.

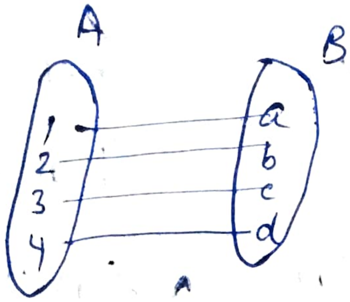
(or)

A function $f: A \to B$ is onto if and only if $\Longleftrightarrow$ Range of f = Co-domain of f.

Note: $f: A \to B$ is onto

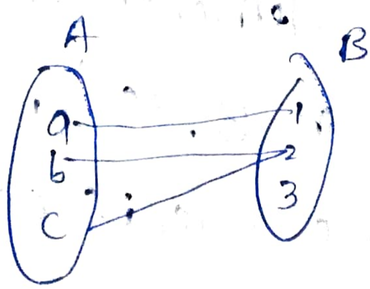
$\Longleftrightarrow \forall y \in B \ \exists$ atleast one $x \in A$ such that $y = f(x)$.

Ex:-



Range = {a, b, c, d} = co-domain

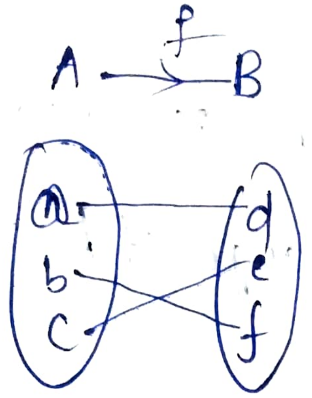
Example for not a Onto-function:-



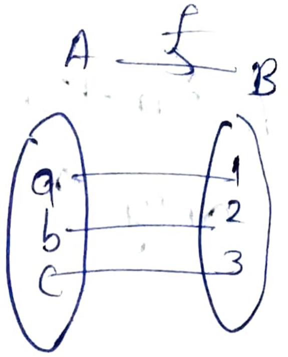
Bijective function:- A function $f: A \to B$ is said to be bijective function

$\Longleftrightarrow$ it is both injective and surjective
(one-one) (onto)

Ex 1.

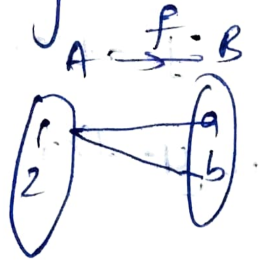


2.

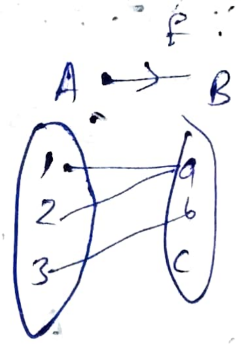


Not bijective:-

Ex:



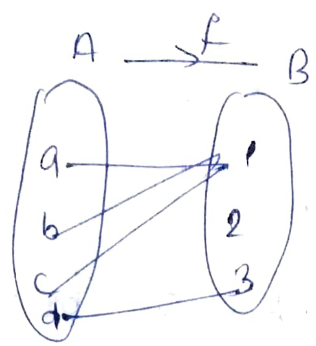
(not a function).



not one-one & not onto

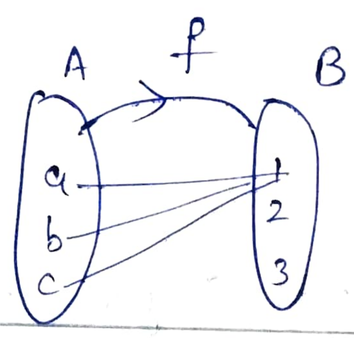
Many-one function: A function $f: A \to B$ is said to be many-one function if atleast 2 elements in A have (common) non-unique image in B. (Or) a function which is not one-one is called many-one function.

Ex:



Constant function: A function $f: A \to B$ is said to be constant function if the range of f contains one element. i.e; $f(x) = c$ $\forall x \in A$ for some constant c.

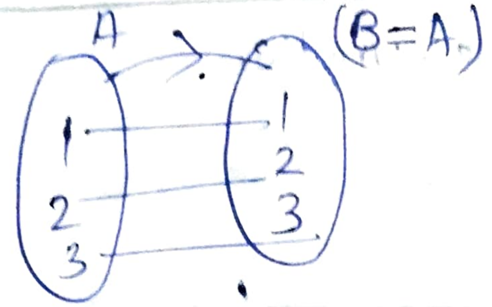
Ex:



Identity function:- Let A be a non-empty set then the function $f: A \to A$ is defined by $f(x) = x$ $(\forall x \in A)$ is called Identity function on 'A'.

and it is denoted by I_A.

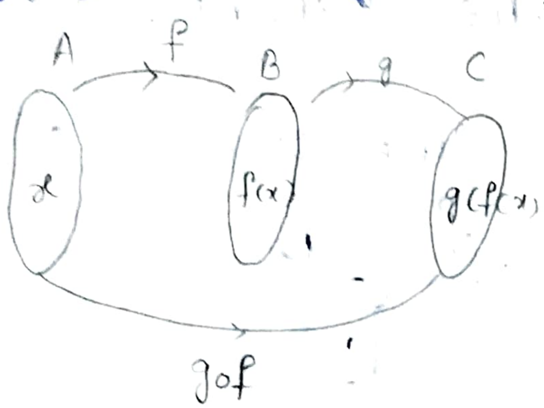
Ex:-



Composition function :-

let $f: A \to B$ and $g: B \to C$ be 2 functions then the composition of f & g denoted by gof is defined as $gof: A \to C$ given by

$$(g \circ f)(x) = g(f(x))$$



$(g of)(x)$

$= g(f(x))$

Q: If $f(x) = 5x+1$ & $g(x) = 2x^2+1$, then find i) fog ii) gof iii) fof iv) gog

Sol: Given, $f(x) = 5x+1$; $g(x) = 2x^2+1$

i) $(fog)(x) = f(g(x))$

$= f(2x^2+1)$

$= 5(2x^2+1)+1$

$= 10x^2+6$

ii) $gof(x) = g(f(x)) = g(5x+1)$

$= 2(5x+1)^2+1$

$= 2(25x^2+1+10x)+1$

$= 50x^2+20x+3$

iii) $fof(x) = f(f(x))$

$= f(5x+1)$

$= 5(5x+1)+1 = 25x+6$

iv) $gog(x) = g(g(x)) = g(2x^2+1)$

$= 2(2x^2+1)^2+1$

$= 2(4x^4+1+8x^2)+1$

$= 8x^4+16x^2+3$

$= 8x^4+8x^2+3$

Q. Let $I = \{0,1,2\}$ and define functions f & g from $I \to I$ as follows $\forall$ x in I, $f(x) = (x^2+x+1)$ mod 3, $g(x) = (x+2)^2$ mod 3 then PT, $f=g$ $\forall x \in I$

Sol:- Given $I = \{0,1,2\}$

$f(x) = (x^2+x+1)$ mod 3

$g(x) = (x+2)^2$ mod 3

when $x=0 \in I$

$f(0) = (0+0+1)$ mod 3

$= 1$ mod 3 $\Rightarrow 1 \% 3$

3) 1 (0

1

$= 1$

$g(0) = (0+2)^2 \bmod 3$ $\qquad 3 \overline{)4(} 1$, $\; 3$, $\; 1$

$= 4 \bmod 3$

$= 1.$

$\Rightarrow f(0) = g(0) \quad — (1)$

when $x = 1 \in I$

$f(1) = (1+1+1) \bmod 3$

$= 3 \bmod 3 = 0$

$\Rightarrow f(1) = g(1) \quad — (2)$

$g(1) = (1+2)^2 \bmod 3$

$= 9 \bmod 3 = 0$

when $x = 2 \in I$

$f(2) = (2^2+2+1) \bmod 3$

$= 7 \bmod 3$

$= 1.$

$\Rightarrow f(2) = g(2) \quad — (3)$

$g(2) = (2+2)^2 \bmod 3$

$= 16 \bmod 3$

$= 1$

from eq (1), (2), (3) $f = g \;\forall\, x \in I$

$\therefore$ Hence proved.

INVERSE Function:- if $f: A \rightarrow B$ is a bijective function then the relation $f^{-1} = \{(b, a) \mid (a, b) \in f\}$ then the f^{-1} is a function from $B \rightarrow A$ and it is called as The Inverse of f.

Since 'f' is bijective that is for each $y \in B \; \exists$ atleast $x \in A \Rightarrow f(x) = y$

$\Rightarrow x = f^{-1}(y)$

$\therefore f^{-1}(x)$ is Inverse function of $f(x)$

Ex:- If $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $f = \{(1, c), (2, b), (3, a)\}$ then find

"$f^{-1} = \{(c, 1), (b, 2), (a, 3)\}$".

* $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

$f = \{(1, a), (2, b), (3, a)\}$ find f^{-1}

f^{-1} doesn't exist, Since f is not one-one.

* $f = \{(1, a), (1, b), (1, c), (3, b)\}$ find f^{-1}

f^{-1} doesn't exist. Since f is not a function

4) If $f(x) = 2x+5$ find $f^{-1}(x)$.

$y = 2x+5 \Rightarrow x = f^{-1}(y)$

$\frac{y-5}{2} = x = f^{-1}(y)$

$f^{-1}(x) = \frac{x-5}{2}$

---

* Prove that $f(x) = 5x^3 - 1$ is a one-one function from $R \to R$, where R is the set of Real Numbers also prove that $f^{-1} o g^{-1} = (g o f)^{-1}$ for $f, g : Q \to Q$ such that $f(x) = 2x$, $g(x) = x+2$.

Sol:-

$f(x) = 5x^3 - 1$

$5x_1^3 - 1 = 5x_2^3 - 1$

$5x_1^3 = 5x_2^3$

$x_1^3 = x_2^3$

$x_1^3 - x_2^3 = 0$

$a^3 - b^3 = (a-b)(a^2+ab+b^2) = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) =$

$x_1 - x_2 = 0$ or $\underline{x_1^2 + x_1 x_2 + x_2^2 = 0}$

$x_1 = x_2$ $\notin R$

$\therefore$ f is one-one function

---

* $f^{-1} o g^{-1} = (gof)^{-1}$

$f(x) = 2x$ — (1) $g(x) = x+2$ — (2)

$2x = y$ $y = x+2$

LHS: $x = \frac{y}{2} = f^{-1}(y)$ $x = y-2 = f^{-1}(y)$

$f^{-1}(x) = \frac{x}{2}$ — (3) $g^{-1}(x) = x-2$ — (4)

$fog = f(g(x))$

$f^{-1} o g^{-1} = f^{-1}(g^{-1}(x))$ from (4)

$= f^{-1}(x-2)$

$= \frac{x-2}{2}$ — (5) from (3)

RHS: $(gof)^{-1} \Rightarrow gof = g(f(x))$ from eq 1

$= g(2x)$ eq (2)

$gof = 2x+2 = y$ $(gof)^{-1}(x) = \left(\frac{x-2}{2}\right)$

$x = \frac{y-2}{2}$ — (6)

eq (5) = eq (6)

$$f^{-1} o g^{-1}(x) = (gof)^{-1}(x) = \frac{x-2}{2}$$

$\therefore$ Hence proved

10/3/23 :-

S. T the function $f : R \to (1, \infty)$ and $g : (1, \infty) \to R$ defined by $f(x) = 3^{2x}+1$ and $g(x) = \frac{1}{2}\log_3(x-1)$ are Inverse

Sol. $f(x) = 3^{2x}+1 = y$

$$3^{2x} = y-1$$

$$2x \log 3 = \log(y-1)$$

$$x = \frac{1}{2\log_3}\log(y-1)$$

$$x = \frac{1}{2\log 3}\log(y-1)$$

$$= \frac{1}{2}\log_3(y-1)$$

$\left(\log \frac{1}{m} \log a\right) = \log_m a$ $(f^{-1} = g)$

$*\ f = g^{-1}$

$g(x) = \frac{1}{2}\log_3(x-1) = y \Rightarrow x = g^{-1}(y).$

$\log_3(x-1) = 2y$

$3^{\log_3(x-1)} = 3^{2y}$ $\left(\because a^{\log_a m} = m\right)$

$x - 1 = 3^{2y}$

$x = 3^{2y} + 1 = g^{-1}(y)$

$\Rightarrow g^{-1}(x) = 3^{2x} + 1 = f(x)$

$\therefore$ f and g are Inverse.

* Operations :- Unary operation:

A Unary operation is only one operation with only one element i.e, A single Input

Ex:- A^{-1}, A^T, Adj (A), $\sin\theta$, $\cos\theta$.

Binary operation: A binary operation is a calculation that combines 2 elements to produce another element.

Ex: $a+b$, $a\times b$, $a-b$, $a\div b$

$A\cup B$, $A\cap B$

n-ary operation: An operation that takes 'n' arguments for it's input is called an n-ary operation

Ex: $S = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots x_n$.

31/8/23

Total Order Relation:-

- Let R be a partial order on the set A, then R is called Total order on A if

$\forall x, y \in A$ iff either xRy or yRx

The poset (A, R) is called Total ordered set (or) chain (Toset).

Ex:- Let R is a relation "greater than or equal" is defined on the set of integers. Verify that $(Z, \geq)$ is a totally ordered set (TOSET).

Sol:- Given set is Z and $R = \{(a,b) / a \geq b, a, b \in Z\}$

reflexive :- let $a \in Z$

always $a \geq a$

$(a,a) \in R$.

Antisymmetric:- let $a, b \in Z$ and $(a,b) \in R$,

$(b,a) \in R$.

$a \geq b, b \geq a \Rightarrow a = b$

Transitive :-

$a, b, c \in Z$

$(a,b) \in R, (b,c) \in R$

$a \geq b ; b \geq c$

$a \geq c \rightarrow (a,c) \in R$.

Comparable :-

$-2, 4 \Rightarrow -2 \ngeq 4 ; 4 \geq -2$

$4R-2$

$\forall a, b \in Z$

we have aRb or bRa

$\therefore (Z, \geq)$ is Toset.

Ex:- let $S = \{a, b\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

either $\{a\} \not\subseteq \{b\}$ or $\{b\} \subseteq \{a\}$

→ its incomparable

$\therefore (P(S); \subseteq)$ is not a toset.

→ **Poset (Hasse diagram):-**

Since partial order is a relation on a set, we can think of diagraph of a partial order if the set is finite.

Since partial order is a reflexive, at every vertex in the digraph there would be a loop. But we need not exibit such loops explicitly.

In the digraph there is an edge from vertex a to vertex b and vertex b to vertex c. then there would be an edge from vertex a to vertex c. But we wou need not show vertex a to vertex c seperately.

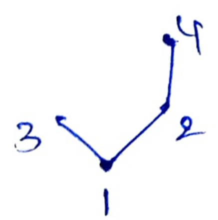
(exclude loop
exclude transitive)

— POSET → 3 conditions
i) reflexive (self mapping)
ii) relatn from bottom to top
iii)

Ex:- 'R' is a relation on the set $\{1, 2, 3, 4\}$ defined by xRy iff $x|y$. Prove that (A, R) is a poset. Draw its Hase diagram

$R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (2,4), (1,4)\}$

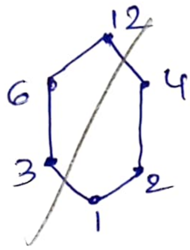
Sol:

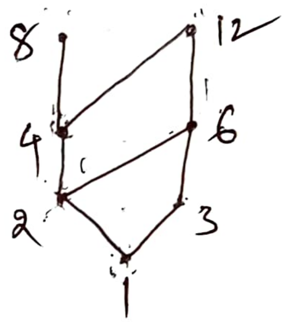


→ loop: self mapping
starting ending at same point
No other vertex in b/w

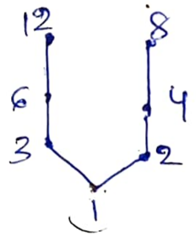
→ Cycle:
mapping,
starting ending is same
but other elements also comes in b/w.

Q:- $A = \{1, 2, 3, 4, 6, 12\}$, "/" → condition
The relation 'R' is defined by aRb iff a|b. Prove that R is a ~~particular~~ rel partial order relation. Draw its Hasse Diagram.





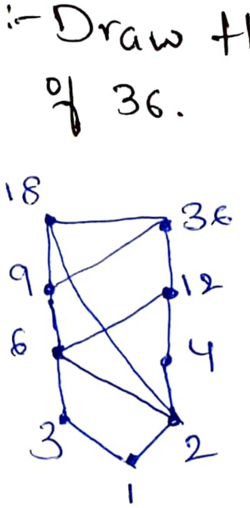
Q:- $A = \{1, 2, 3, 4, 6, 8, 12\}$. |

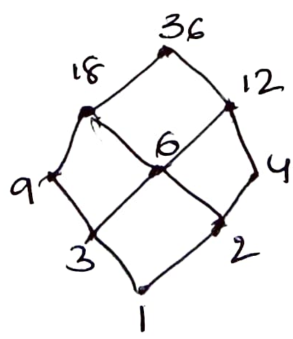


$R = \{(1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12), (1,2), (1,4), (1,8), (1,3), (1,6), (1,12), (2,4), (4,8), (3,6), (6,12)\}$

Q:- Draw the hasse diagram for positive divisors of 36.

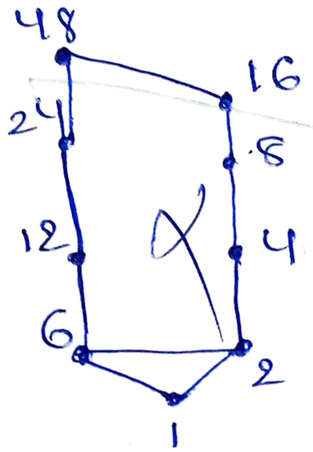
+ve divisors of 36:- $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

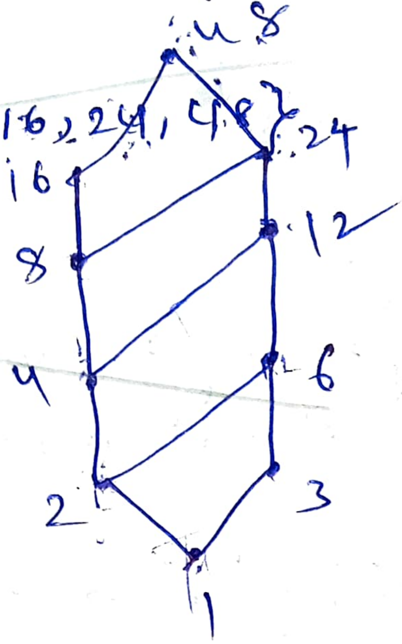




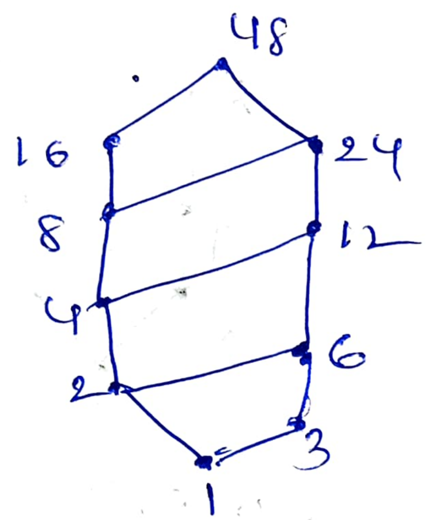
Q:- Draw the hasse diagram for positive divisors of 48.

{1, 2, 4, 6, 8, 12, 16, 24, 48}





{1, 2, 3, 4, 6, 8, 12, 16, 24, 48}.



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Lattice :-

Poset → Hasse diagram → lattice

(∧) Greatest lower bound — ~~Join~~ meet — GLB

Least upper bound (∨) — ~~Meet~~ join — LUB

[only single values considered if 2 elements has 2 LUB/GLB then it won't exist]

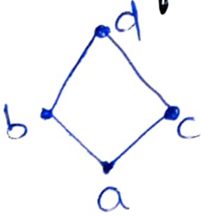
Lattices :-

→ Let (A,R) be a poset. This poset is called Lattice. If every two element subset of 'A' has a least upper bound and greatest lower bound in "A".

→ If (A,R) is a lattice, the least upper bound (LUB) of the two element subset $\{a,b\} \subseteq A$ is denoted by $a \vee b$, (read as a join b)

→ And the greatest lower bound (GUB) of the subset is denoted by $a \wedge b$, (read as a meet b).

Ex :- The following hasse diagram is a lattice or not.



LUB → $a \vee b$ (a join b)

GLB → $a \wedge b$ (a meet b)

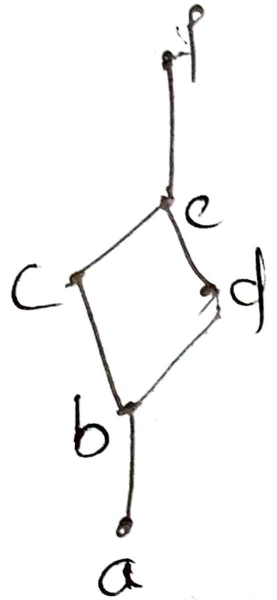
LUB

∨	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	d	c	d
d	d	d	d	d

GLB

∧	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

φ:-



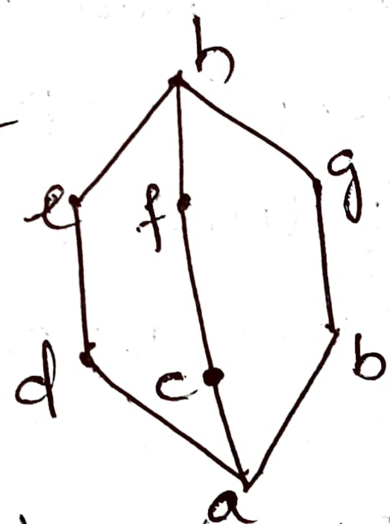
LUB

v	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	c	d	e	f
c	c	c	c	e	e	f
d	d	d	e	d	e	f
e	e	e	e	e	e	f
f	f	f	f	f	f	f

GLB

∧	a	b	c	d	e	f
a	a	a	b	b	d	c
b	a	b	b	b	b	b
c	a	b	c	b	c	c
d	a	b	b	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

φ:-



LUB of d, c = h

GLB of e, g = a

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Properties of Lattice :-

1) Idempotent :- $a \vee a = a$; $a \wedge a = a$.

2) Commutative :- $a \vee b = b \vee a$; $a \wedge b = b \wedge a$

3) Associative :- $a \vee (b \vee c) = (a \vee b) \vee c$.

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

4) Absorption :- $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

Bounded lattice :-

A lattice (L, R) is said to be bounded if it has a greatest element & a least element.

In a bounded lattice, the greatest element is denoted by 'I' and the least element is denoted by '∅' '0'

$a \vee 0 = a$, $a \wedge 0 = 0$

$a \vee I = I$, $a \wedge I = a$

Ex:- The lattice $\{Z^+, \mid\}$ has a least element i.e, 1 but greatest element doesn't exist therefore it is not a bounded lattice.

H/w → $\{\{1, 2, 3, 4, 5\}, \mid\}$
$\{\{1, 2, 4, 8, 16\}, \mid\}$
draw the hasse diagram and check whether distributive lattice or not.

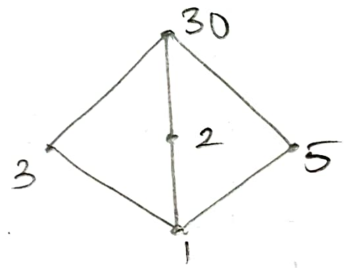
Distributive Lattice :-

A lattice (L, R) is said to be distributive for any (a, b, c) ∈ L the following distributive laws hold.

① $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

② $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Ex: If A = {1, 2, 3, 5, 30} and R is the divisibility relation. Prove that (A, R) is lattice and verify for distributive lattice.



A = {1, 2, 3, 5, 30}

∨	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

∧	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

$2 \vee (3 \wedge 5) = 2 \vee 1 = 2$

$(2 \vee 3) \wedge (2 \vee 5) = 30 \wedge 30 = 30$

$2 \vee (3 \wedge 5) \neq (2 \vee 3) \wedge (2 \vee 5)$

∴ As it doesn't satisfy distributive ~~lattice~~ laws

It is not a distributive lattice.