Discrete Mathematics

Unit 1: Sets and operations

Sets, operations on sets, venn diagrams, multi sets, binary relations, equivalency relations, ordering relations, operations on relations, partial orders.

functions:

Defination & introduction, composition of functions, inverse functions, binary and n array operators.

Set:

A set is a well defined collection of objects. these objects are called elements or members of the set.

Ex: $E = \{2, 4, 6, 8, 10\}$ finite set

$D = \{1, 3, 5, 7, 9\}$ infinite set

Names = {laxmi, radha, chitti}

age = {18, 19, 17}

Here $2 \in E$, $5 \notin E$

→ The set v of all vowles can be written as

$V = \{a, e, i, o, u\}$

→ sets can have other sets as members

ex: $A = \{N, Z, Q, R\}$

→ In a set containing four elements each of which is a set.

The four elements of this set are

N: which is a set of natural numbers

Z: which is a set of integers

Q: which is a set of rational numbers

R: which is a set of real numbers

Importance of set: The concept of a data type or type in computer science is built upon the concept set.

In particular data type or type is the name of the set together with the set of operations that can be perform on objects on that set.

Ex: The boolean is the name of the set $\{0,1\}$

→ Together with operations on one or more elements of the set such as AND, OR, and NOT

→ There are five ways used to describe a set

1> Describe a set by describing the properties of the numbers of the set.

2> Describe a set by listing its elements.

3> Describe a set 'A' by its characteristic eq^n's.

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \end{cases}$$

4> Describe a set by a recursive formula.

5> Describe a set by a operation (union, intersection, compliment, etc).

Ex: describe the set containing all the non negative integers ≥ or equal to 5.

1) lets suppose the set is A

$A = \{x / x$ are non- negative integers less than or equal to $5\}$

2) $A = \{0, 1, 2, 3, 4, 5\}$ listing

3) $\mu_A(x) = \begin{cases} 1 & \text{if } x \in \{0, 1, 2, 3, 4, 5\} \,/\, 0 \le x \le 5 \text{ (or)} \\ 0 & \text{o.w} \end{cases}$

4) $A = \{x_{i+1} = x_i + 1;\ i = 0, 1, 2, 3, 4;\ x_0 = 0\}$

<u>Sub set</u>: let the sets A and B are not empty then the set A is said to be subset of B if each and every element of A must present in B.

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$C = \{1, 2, 3\}$

$A \subseteq B \rightarrow$ subset

$A \subset B \rightarrow$ proper subset

$C \subset B$, proper subset

$A \subseteq B$ $\cancel{A} \subseteq B$

Proper set   improper set

<u>Ex</u>:

$A = \{a, e, i, o, u\}$

$B = \{a, e, i\}$

$B \subseteq A$

Proper sub set: let the set A is said to be proper sub set of B if first it must be subset of B and there exists atleast one element in B which is not there in A

$A=\{1,2,3\}$ $\quad B=\{1,2,3,4\}$ $\quad C=\{1,2,3\}$

$A \subseteq B$ $\qquad$ if $A \not\subset B$

Proper subset $\qquad$ improper subset

Ex: $A=\{a,e,i\}$ $\quad B=\{a,e,i,o,u\}$ $\quad C=\{a,e,i\}$

$A \subset B$ $\qquad$ if $A \not\subset B$

Properties of sets

let A, B, C are sets then

i) $A \subseteq A$ $\quad A=\{1,2,3\}$ $\quad A=\{1,2,3\}$

A set it self is a subset of same set.

ii) If A is the subset of B and B is the subset of C then C is subset of A.

$A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

$A=\{1,2,3\}$ $\quad B=\{1,2,3,4\}$ $\quad C=\{1,2,3,4,5\}$

$A \subseteq B$ $\qquad B \subseteq C$

$A \subseteq C$

iii) If A is subset of B, B is proper subset of C then A is proper subset of C [$A \subseteq B$, $B \subset C \Rightarrow A \subset C$

Ex: let $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3, 4\}$

$\underbrace{\underbrace{A = \{1,2,3\}, B = \{1,2,3\}}_{A \subseteq B}, \underbrace{C = \{1,2,3,4\}}_{B \subset C}}_{A \subset C}$

Equal sets

The sets A and B are set to be equal if and only if $A \subseteq B$ and $B \subseteq A$.

Ex:

case i) $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4\}$

Here,

$A \subseteq B$, but $B \not\subseteq A$

case ii) $A = B$ if

when $A \subseteq B$ & $B \subseteq A$

Ex:

$A = \{1, 2, 3\}$ $B = \{1, 2, 3\}$,

$A \subseteq B$ $B \subseteq A$

4th Property

iv) $A \subseteq B$ & $A \not\subset C$ then $B \not\subset C$

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4\}$ $C = \{1, 2, 3\}$

$A \subseteq B$ here B is not a subset and not a

$\cancel{A \not\subseteq B}$ $B \not\subseteq A$ proper set.

$B \not\subset C$

Null set or Empty set:

The set containing no elements is called null set or empty set.

→ It is denoted by ϕ or $\{\}$

Single-ton set

The set containing single elements is called single-ton set.

Ex: $\{1\}$, $\{2\}$, $\{a\}$, $\{b\}$

operations on sets:

We have many operators in set theory know we discuss the basic operators of set theory

i) union

ii) intersection

iii) complementary

i) union: If A and B are sets then union of A & B is denoted by $A \cup B$

$A \cup B = \{x : x \in A \text{ or } x \in B \text{ are both}\}$

ex: $A = \{1, 2, 3\}$; $B = \{3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

ii) Intersection: If A and B are 2 sets then

$A \cap B$ is denoted by $A \cap B$ and is defined as

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

ex: $A = \{a, b, c, d\}$; $B = \{a, c, d\}$

$A \cap B = \{a, c, d\}$

iii) complementary: In complements we have 2 types

a) Absolute complement: let A be the set and U be the universal set then complement of A

i.e $\bar{A} = \{x : x \notin A\} = \{x : x \in (U - A)\}$

b) Relative complement: If A & B are 2 sets then the relative complement of A w.r.t B is B−A

$B - A = \{x : x \in \{B - A\}$ (or) $x \in B$ but not A (or)

$x \in B, x \notin A\}$

ex: $A = \{1, 2, 3, 6\}$, $B = \{1, 2, 3, 4\}$

$B - A = \{4\}$

$A = \{a, e, i, w\}$, $B = \{a, e, i, o, u\}$

$B - A = \{o, u\}$

Basic properties of set operators

1) union : $A \cup A = A$ (Idempodent Property)

ex: $A = \{1, 2, 3\}$ $A = \{1, 2, 3\}$

$A \cup A = \{1, 2, 3\}$

2) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property)

ex: $A = \{1, 2, 3\}$; $B = \{3, 4, 5\}$ $C = \{5, 6, 7\}$

L.H.S	R.H.S
$B \cup C = \{3, 4, 5, 6, 7\}$	$A \cup B = \{1, 2, 3, 4, 5\}$
then $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 7\}$	$(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 7\}$

3) $A \cup B = B \cup A$ (commutative)

$A = \{a, b, c\}$, $B = \{c, d, e\}$

$A \cup B = \{a, b, c, d, e\}$

$B \cup A = \{a, b, c, d, e\}$

Intersection (Idempodent property)

$A \cap A = A$

$A = \{1, 2, 3\}$ $A = \{1, 2, 3\}$

$A \cap A = \{1, 2, 3\}$

2) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative)

ex: $A = \{1, 2, 3\}$; $B = \{3, 4, 5\}$ $C = \{5, 6, 7\}$

L.H.S

$B \cap C = \{5\}$

$A \cap (B \cap C) = \{\}$

R.H.S

$A \cap B = \{3\}$

$(A \cap B) \cap C = \{\}$

3.) $A \cap B = B \cap A$ (commutative)

$A \cap B = \{c\}$

$B \cap A = \{c\}$

Note: If A, B & C are sets then $(A \cup B) \cap C \neq (A \cup (B \cap C))$

Symmetrical difference (or) boolean sum:

let A & B are 2 sets then the symmetrical difference is $A \Delta B$ which is defined as

$A \Delta B = \{x : x \in A$ (or) $x \in B$ but not both$\}$

ex: $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$A \Delta B = \{1, 2, 5, 6\}$

Disjoint sets: let A and B are 2 sets are said to be disjoint if $A \cap B = \phi$ (null set)

$A = \{1, 2, 3\}$

$B = \{4, 5, 6\}$

$A \cap B = \phi$

Distributive laws: let A, B and C are 3 sets then

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Power set: let A be the set then the power set is denoted by P(A) which contains all subsets of A which is defined as

$P(A) = \{x : x \subseteq A\}$ $\qquad 2^n = 2^3 = 8$

ex: $A = \{1, 2, 3\}$

$P(A) = \phi$ $\{1\}, \{2\}, \{3\}$ $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$

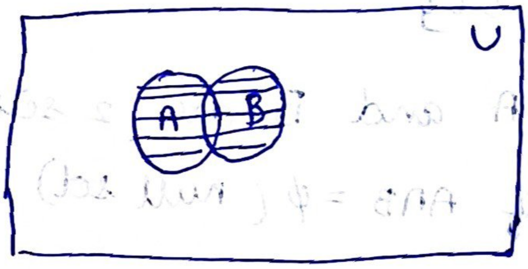
$A = \{1, 2, 3, 4\}$

$P(A) = \phi, \{1\} \{2\} \{3\} \{4\} \{1,2\} \{1,3\} \{1,4\} \{2,3\}$
$\{2,4\} \{3,4\} \{1,2,3\} \{2,3,4\} \{1,2,4\}$
$\{1,2,3,4\}$ $\{3,4,1\}$

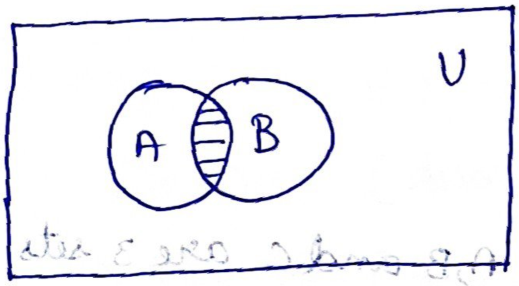
Venn diagram:

→ let A, B, C are sets and U be the universal set

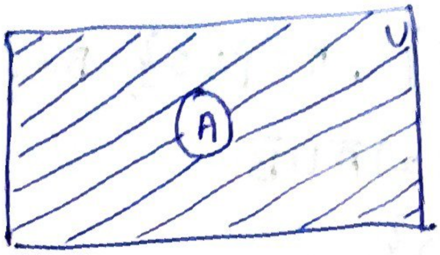
1) $A \cup B =$



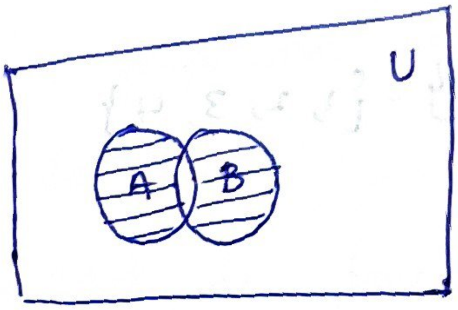
2) $A \cap B =$



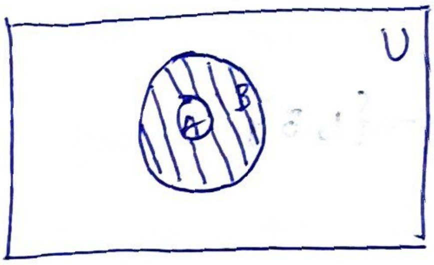
3.) $\overline{A}$ =



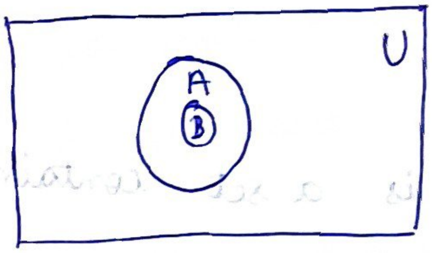
4.) $A \Delta B$ =



5.) $B - A$ =



6.) $B \subset A$ =



Demorgan's law

→ let A & B are 2 sets then

1.) $(\overline{A \cup B}) = \overline{A} \cap \overline{B}$

2.) $(\overline{A \cap B}) = \overline{A} \cup \overline{B}$

→ Prove that ϕ (empty set) is uniquie

Proof: let suppose ϕ_1, ϕ_2 are empty sets we know that empty set is a sub set of every set.

$\phi_1 \subseteq \phi_2$: $\phi_2 \subseteq \phi_1$ this is possible only when

$\phi_1 = \phi_2$ => emty set is unique

hence proved.

→ let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 5\}$, $B = \{1, 2, 3, 4\}$

then find 1) $A \cap \overline{B}$ 2) $\overline{A} \cup \overline{B}$

Sol i) $\overline{B} = \{x : x \in (U - B)\}$

$\overline{B} = \{1, 2, 3, 4, 5\} - \{1, 2, 3, 4\}$

$= \{5\}$

$A \cap \overline{B} = \{5\}$

ii) $\overline{A} = \{1, 2, 3, 4, 5\} - \{1, 5\}$

$= \{2, 3, 4\}$

$\overline{A} \cup \overline{B} = \{2, 3, 4, 5\}$

Multiset: A multiset is a set containing repeated objects

ex: $A = \{1, 2, 1, 1, 2, 3, 3, 4, 5, 5\}$

$1 \overset{3 \rightarrow \text{multiplicity}}{\in} A$, $2 \overset{2}{\in} A$, $3 \overset{2}{\in} A$, $4 \in A$, $5 \in$

The multiplicity of 1 is 3

The multiplicity of 2 is 2

The multiplicity 3 is 2

The multiplicity 4 is 1

The multiplicity 5 is 2

Importance of multiset:

→ multiset is used in fundamental in combinatories

→ It is an important tool in theory of rational data bases.

→ The multiset is implemented in data base system relations & SQL operators on multisets and return identical records.

Operations of multi set:

<u>Union</u>: Here we will take maximum multicity of elements in both sets. for example

$A = \{a, a, a, a, b, b, b, c, c, c, d, e, e\}$

$B = \{a, a, a, b, b, b, b, b, c, c, c, d, d, e, e, e\}$

1) $A \cup B = \{a, a, a, a, b, b, b, b, b, c, c, c, d, d, e, e, e\}$ max

2) $A \cap B = \{a, a, a, b, b, b, c, c, c, d, e, e\}$ min

$A - B = \{x : x \in A \notin B\}$

3.) $A - B = \{a\}$

4.) $B - A = \{2b, d, e\}$

5.) $A + B = \{a, a, a, a, a, a, a, b, b, b, b, b, b, b, b, c, c, c, c, c, d, d, d, e, e, e, e, e\}$

(2) Intersection : Here we have to take minimum multiplicity of elements in both set for example

$A \cap B = \{a, a, a, b, b, b, c, c, c, d, e, e\}$

(3.) The difference of multiset: the difference of multiset ~~of elements~~ is a multiset of difference of multiplicity of elements of

$A - B = \{a\}$

(4.) Addition of multiset is a multiset of the addition of all multiplicity of each element in each set

$A + B = \{a, a, a, a, a, a, a, b, b, b, b, b, b, b, b, c, c, c, c, c, c, d, d, e, e, e, e, e\}$

Properties of multiset operation

1) $A \cup B = B \cup A$

2) $A \cap B = B \cap A$

3) $A - B \neq B - A$

4) $A + B = B + A$

Relations

ordered pair: In ordered pair each set is specified by 2 objects in prescribed orderd. The ordered pair of a, b is represented as (a, b)

→ If $(a, b) = (c, d)$ iff $a = c$, $b = d$

Cartesian product

→ let A and B are sets then the cartesian product is denoted by $A \times B$ and which is defined as $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

for example: $A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$ then

$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)$
$(3,3), (3,4), (3,5)\}$

→ In generall the cartesian product of n sets is

$A_1 \times A_2 \times A_3 \cdots A_n = \{(a_1, a_2, \cdots a_n) : a_1 \in A_1, a_2 \in A_2$
$\cdots a_n \in A_n\}$

Here the expression $(a_1, a_2, \cdots a_n)$ is called

orderd ~~and~~ n-tuple.

Definitions and Properties

→ A binary relation R from set A to B (written as xRy where $x \in A$, $y \in B$) is a subset of the cartesian product $A \times B$. If the orderd pair is reversed the relation also changes

Domain of R: let R be a relation from A to B

the domain of R is denoted by

$\text{dom } R = \{x : x \in A \text{ and } y \in B \text{ and } (x, y) \in R\}$

Range: let R be a relation from A to B and the range of R is denoted by

$\text{ran} R = \{y : y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$

For ex: $A = \{1, 2, 9\}$, $B = \{1, 3, 7\}$

Case I: let take the relation R is "equal to" $(=R)$

$A \times B = \{(1,1), (1,3), (1,7), (2,1), (2,3), (2,7), (9,1), (9,3), (9,7)\}$

Then $R = \{(1,1)\}$ $(=R)$

$\text{Dom} R = \{1\}$, $\text{Range}\{1\}$

Case II: If relation R is "less than" $(=R)$

$R = \{(1,3), (1,7), (2,3), (2,7)\}$

$\text{Dom} R = \{1, 2\}$ $\text{Range } R = \{3, 7\}$

Case III: If relation R is "greater than" $(=R)$

$R = \{(2,1)\ (9,1)\ (9,3), (9,7)\}$

$\text{Dom} R = \{2, 9\}$ $\text{Range } R = \{1, 3, 7\}$

Properties of binary relation:

i) Reflexive: $\forall x \in A$, and the relation xRx holds then the set is reflexive

Ex: $R = \{(1,1), (2,2), (3,3)\}$

$R = \{(a,a), (b,b), (c,c)\}$

ii) Symmetric: $\forall x, y \in A$ then if $xRy \Rightarrow yRx$

(or) if xRy then yRx

ex: $R = \{(1,2), (2,1)\}$

iii) Transitive: $\forall x, y, z \in A$ if xRy and yRz then xRz

ex: $R = \{(\overset{x}{1}, \overset{y}{2}), (\overset{y}{2}, \overset{z}{3}), (\overset{x}{1}, \overset{z}{3})\}$

iv) irreflexive: $\forall x \in A$, $x \not{R} x$

$R = \{(a,b), (b,c), (a,c)\}$.

v) Antisymmetric: $\forall x, y \in A$ if xRy & yRx then $x = y$

ex: $R = \{(1,1), (2,2), (1,3), (2,3)\}$

vi) equivalence relation: let R be a relation on A. R will be an equivalence relation on A if R satifies

1) reflexive
2) symmetric
3) transitive

ex: $A = \{1, 2, 3\}$

$R = \{(1,1)\ (2,2)\ (3,3)\}$ Ref

$(1,2),\ (2,1)$ Sym

$(2,3)\ (1,3)$ Tran

equivalence class let A be given set & R is a equivalence relation on A the equivalence class of x is denoted by $[x] = \{y \mid y \in A \text{ and } (x,y) \in R\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)$
$(1,2)\ (2,1)\ (4,5)\ (5,4)$ of

the set $A = \{1, 2, 3, 4, 5\}$

$[1] = \{1, 2\}$, $[2] = \{2, 1\}$ (or) $\{1, 2\}$

$[3] = \{3\}$, $[4] = \{4, 5\}$, $[5] = \{5, 4\}$ or $\{4, 5\}$

$A = \{1, 2, 3\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)\ (1,3)\ (3,1)\}$

$[1] = \{1, 2, 3\}$

$[1] = \{1, 2, 3\}$

$[2] = \{2, 1\}$

$[3] = \{3, 1\}$

NOTE: We have equivalence class of x = equivalence class of y $[x] = [y]$ even if $x \neq y$ provided by the relation xRy

ex: $[1] = [2]$ (Previous example 1)

Partesian: let A be a given set a partesian of A is a collection of P of disjoint subsets from equivalence classess whose union is A.

$A = \{1, 2, 3, 4, 5\}$ $R = \begin{cases} (1,1) & (2,2) & (3,3) & (4,4) \\ (5,5) & (1,2) & (2,1) & (4,5) & (5,4) \end{cases}$

$[1] = \{1, 2\} = P_1$

$[2] = \{2, 1\}$ (or) $\{1, 2\}$

$[3] = \{3\}$ — P_2

$[4] = \{4, 5\}$ — P_3

$[5] = \{5, 4\}$

$P_1 \cup P_2 \cup P_3 = \{1, 2\} \cup \{3\} \cup \{4, 5\}$

$= \{1, 2, 3, 4, 5\}$

$= A$.

$P_1 \cap P_2 \cap P_3 = \phi$

~~Congruhense~~ congrucies: If a, b are congruent modulo m if and only if they have the same reminder when divided by m

1) $a \equiv b \ (\text{modulo } m) \Rightarrow a \bmod m = b \bmod m$

2) $a \equiv b \pmod{m}$ means $a-b$ is divisible by 'm'

(or) m divides $a-b$

ex: $a=3$, $b=6$, $m=3$

gives reminder '0'

ex: $a=4$ $b=7$ $m=3$

gives reminder '1'

Imp

Fact: congruence modulo m is an equivalence relation. i.e., $R = \{(a,b) / a \equiv b \pmod{m}\}$ is an

equivalence relation

1) Reflexive: $\forall a \in A$

we P.T $(a,a) \in R$

by relation $a \equiv a \pmod{m}$.

$\Rightarrow a-a$ is divisible by m

$\Rightarrow 0$ is divisible by m so that

$(a,a) \in R$

$\therefore R$ is reflexive

2) Symmetric: $\forall a,b \in A$

W.h.P.T $(a,b) \in R \Rightarrow (b,a) \in R$

so by definition of R, we can write

$a \equiv b \pmod{m}$

$\Rightarrow (a-b)$ is divisible by m

$\Rightarrow (a-b) = k \cdot m$

$\Rightarrow -(b-a) = km$

$\Rightarrow b-a = (-k)m$

$\Rightarrow b-a$ is divisible by m

$\Rightarrow b \equiv a \pmod{m}$

$\therefore$ R is symmetric

(3) Transitive : $\forall a, b, c \in A$

$(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$

By def^n of R

$a \equiv b \pmod{m} \Rightarrow (a-b) = k_1 m$ — (1)

$b \equiv c \pmod{m} \Rightarrow (b-c) = k_2 m$ — (2)

add (1) & (2)

$a-b+b-c = k_1 m + k_2 m$

$a-c = (k_1+k_2)m$

$a-c = k_3 m$ where $k_3 = (k_1+k_2)$

$\Rightarrow a \equiv c \pmod{m}$

$\therefore$ R is transitive

$\because$ R is satisfying reflexive, symmetric & Transitive $\therefore$ R is equivalence relation

$\Rightarrow$ The congruent is equivalent relation

Ordering relations

Partial ordering relations: A relation R is said to be a partial ordering relation, if R is Reflexive, antisymmetric, transitive.

Partial ordering set (Poset): a set A with partial ordering relation R, defined on A is called Poset which is denoted by.

$$[\underset{\text{set}}{A} : \underset{\text{relation}}{R}]$$

ex: $A = \{1, 2, 3\}$ $R_1 = \{\underbrace{(1,1), (2,2), (3,3)}_{\text{Reflexive}}, (1,2), (2,3), (1,3)\}$

$[A : R_1]$ is poset

→ (1, 2) is there but not (2, 1) so antisymmetric

ex: xRy & $yRz \rightarrow xRz$ (transitive)

$R_2 = \{(1,1)\ (2,2)\ (3,3)\}$

Characteristic properties of partial ordering

1.) Reflexive: i.e $\forall a \in A$, $a \leq a$.

2.) Antisymmetric: $\forall a, b \in A$, if $a \leq b$ and $b \leq a$ then $a = b$

3.) Transitive: $\forall a, b, c \in A$ if $a \leq b$ & $b \leq c$ then $a \leq c$

ex: Why $[Z : \leq]$ is Poset

i) consider reflexive condition here if we take

$(2,2)$ $2 \in Z$, $2 \leq 2$ so $(2,2) \in R(\leq)$

ii) consider antisymmetric condition if we take

$2 \leq 3$ it is valid but not $3 \leq 2$

so $(2,3) \in R(\leq)$

iii) consider transitive condition if we take

$\{1,2,3\} \in R$ then $1 \leq 2$, $2 \leq 3$ then there

exist $1 \leq 3$ so it is Poset.

Comparable : Two elements a & b in a set A

are said to be comparable under $\leq$ if

either $a \leq b$ (or) $b \leq a$ other wise they are

incomparable.

Totally orderd set: If every pair of elements of

a set A are comparable then we say

that the Poset $[A : \leq]$ is totally orderd

or A is a totally orderd set (or) a chain.

ex: If Z is set of integers and $\leq$ is a

usually ordering on Z then (Poset) $[Z : \leq]$

by division algorithm

$x = mq + r$ | r is remainder

where q and r are integers $0 \leq r < m$

Thus, $x \equiv r \pmod{m}$ and $[x] = [r]$

So, Each equivalence class for this relation is on of the classes $[0], [1], \ldots [m-1]$,

Ordering Relations

Partial ordering Relation: A relation R is said to be a partial ordering Relation, if R is Reflexive, Antisymmetric and Transitive

Partial ordering Set (POSET): A set 'A' with partial ordering Relation 'R' defined on 'A' is called POSET. Denoted by $[A; R]$ ⇒ (1)

Ex: 1) $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$.

$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$

Ex: 2) If "A" is any set of Real Numbers then

[A; ≤] is a poset.

$\because$ R_1 have (1,1), (2,2), (3,3)

Ex 1 R_1 is Reflexive ✓ xRx ✓

Antisymmetric $\because$ (1,2) will not be repeat like (2,1) but we can write (1,1), (2,2) (3,3) [$\because$ $x=y$ same element can repeat] but not like (1,2) (2,1) in the same set

Transitive

In R_1 there is no (1,2) (2,3) to check (1,3) exisits or not

→ (1,1) (1,2) are not there in R_1 to check (1,2) exisits or not so that it is Transitive.

R_2 = {(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)}

R_2 is Reflexive $\because$ (1,1), (2,2), (3,3) are there

R_2 is Antisymmetric since (1,2) is there but not (2,1)

(2,3) is there but not (3,2)

(1,3) is there but not (3,1)

R_3 is reflexive $\because$ (1,1), (2,2) are there

(1,2) present

(1,2) (2,3) are there

(2,3) is present in set so it is reflexive.

Ex:2: $[A ; \leq]$ is poset

1) **Reflexive**: take any real number

$1.5 \leq 1.5$ yes $[\because 1.5 = 1.5$

(Comparision b/w same elements with relation $\leq$ holds because of "=")

2) **Antisymmetric** take any two real numbers

$(1 \leq 2)$ ✓ but $(2 \leq 1)$ ✗

if $2 \leq 1$ then it becomes symmetric

So it is Antisymmetric

3) **Transitive** Ex: $1 \leq 2$, $2 \leq 3$ then

$1 \leq 3$ possible

So it is Transitive

$\therefore$ A is any set of real numbers with relation $\leq$ is a poset or

$[A ; \leq]$ is a poset

(2,2) (4,4), (6,6, (8,8)

$A = \{(2,4)\ (4,6),\ (6,8)\}$ partial order

'divide 4/6 6/8

① Note: The relation $\leq$ on the set of real numbers is the prototype of partial ordering, so it is common to write $\leq$ to represent an arbitrary partial order on A.

Characteristic properties of partial ordering:

1) Reflexivity i.e $\forall a \in A$, $a \leq a$

2) Antisymmetry i.e $\forall a, b \in A$ if $a \leq b$ and $b \leq a$ then $a = b$.

3) Transitivity i.e $\forall a, b, c \in A$ if $a \leq b$, $b \leq c$ then $a \leq c$.

Comparable: Two elements a and b in a set A are said to be comparable under $\leq$ if either $a \leq b$ or $b \leq a$; otherwise they are incomparble.

Totally ordered set: If every pair of elements of a set A are comparable, then we say that $[A, \leq]$ is totally ordered or that A is a totally ordered set or a chain.

Ex: $A = \{2, 4, 6, 8\}$ divid.

not totally ordered $2/4$ $4 \not| 6$ $6 \not| 8$.

In this case, the relation $\leq$ is called a total order. Ex: 2

Ex: If Z is the set of integers and $\leq$ is the usual ordering on Z, then $[Z; \leq]$

is partially ordered and totally ordered.

2) The relation $<$ on $\mathbb{Z}$ is not a partial order because it is not reflexive [$\because$ $2<2$ ✗ reflexive xRx. $2 \in \mathbb{Z}$.]

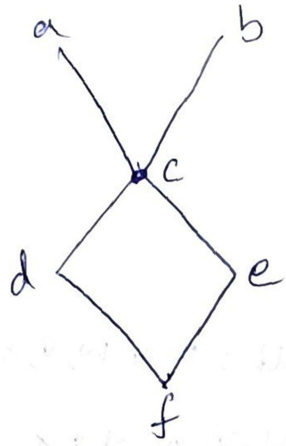
Functions:

S, Cycle:

Lattices, Hasse diagrams:

A Lattice is a partially ordered set $(L, \leq)$ in which Every Subset $\{a, b\}$ consisting of two elements has a least upper bound and a greatest lower bound.

Ex: 1)

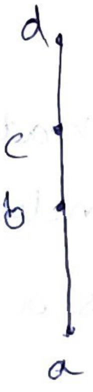


$\{a, b\}$ don't have least upper bound

$\therefore$ It is not lattice

X

2)



It is a lattice

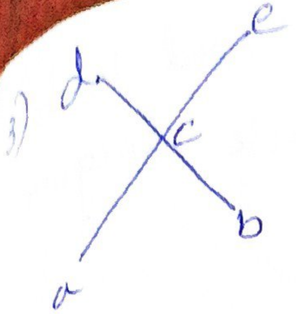
$\{c, d\}$; d is least U.B

a is G.L.B

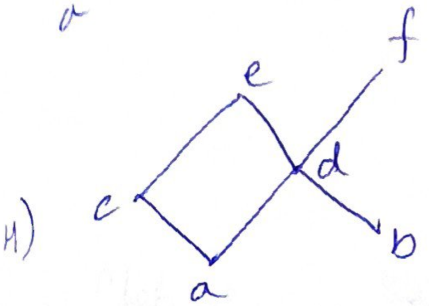
✓

If you take any pair here you'll get L.U.B and G.L.B.

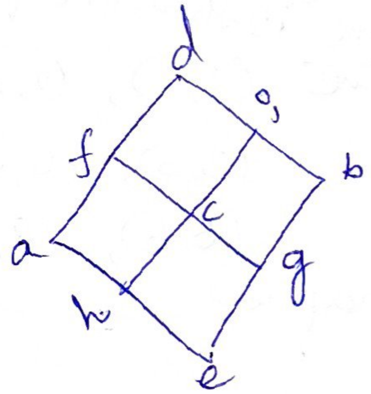
$\therefore$ It is lattice.

3) 

not a lattice
$\{d, e\}$ there $\nexists$ L.U.b

4) 

not a lattice
$\{b, f\}$ $\nexists$ L.U.b

5) 

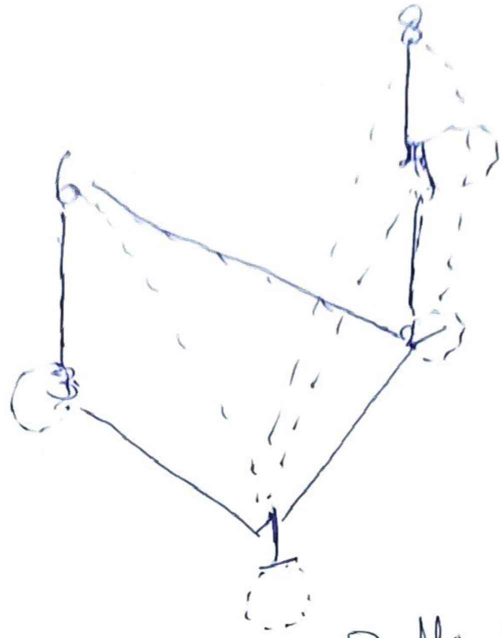
It is a lattice.

HASSE DIAGRAM:

A graphical representation of a partial order relation in which all arrowhead are understood to be pointing upward is known as the Hasse diagram.

Ex: Let $A = \{1, 2, 3, 4, 6, 8\}$ be ordered by the relation "a divides b"

$\therefore R = \{ (1,1), (1,2), (1,3), (1,4), (1,6), (1,8)$
$(2,2), (2,4), (2,6), (2,8)$
$(3,3), (3,6), (4,4), (4,8)$
$(6,6), (8,8) \}$



we need to draw basic diagram

Poset means: ① ~~Transitive~~ Reflexive is there in defⁿ so (1,1), (2,2) (3,3), (4,4) (6,6), (8,8)

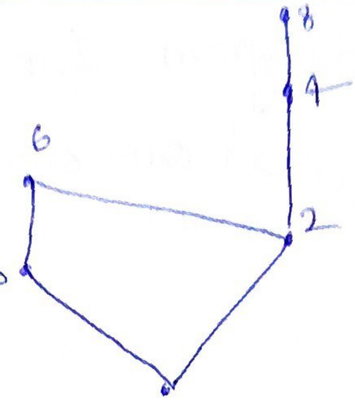
So no-need to map.

② Antisymmetric: means 1/2, 2/4 4/8 which is upwards only. satisfies defⁿ.

③ Transitive here 1/3, 3/6 and 1/6 so no-need to draw 1/6

Similarly no-need to draw 1/4 & 1/8 and 2/8.

So The Hasse diagram is

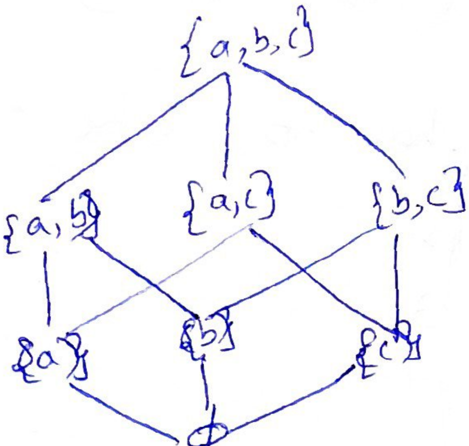


(1) The Hasse diagram of the subsets of $S = \{a, b, c\}$ with the inclusion relation $\subseteq$

Soln: we know that subset of S is
$\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}$
$\{c, a\}, \{a,b,c\}\}$

Now we write a relation $(R = \subseteq)$

$R = \{(\phi,\phi), (\phi, \{a\}), (\phi, \{b\}), \ldots$
$(a, \{a\}), (a, \{a,b\}), (a, \{a,c\})$
$(a, \{a,b,c\}), (b, \{b\}), (b, \{a,b\}), (b, \{b,c\})$
$(b, \{a,b,c\}), \ldots \}$



(*) we have to select top most element to draw hasse diagram not bottom

(3) Draw the Hasse diagram for the greater than or equal to relation on set $S = \{0, 1, 2, 3, 4, 5\}$

Sol^n Given $S = \{0, 1, 2, 3, 4, 5\}$

$$R = \{(a, b) \mid a \geq b\}$$

R is a partial order and hence (S, R) is a poset.

let take ~~R = {(0,5), (1,5), (2,5), (3,5) (4,5), (0,~~

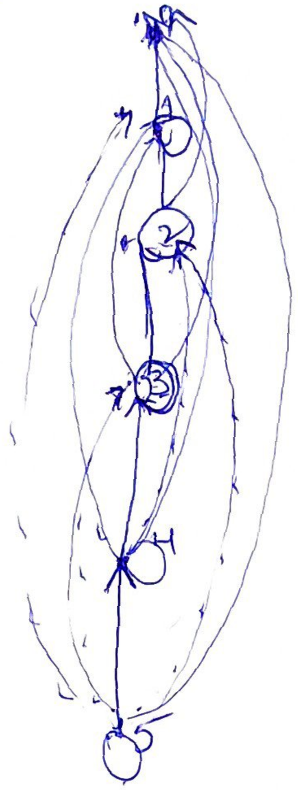
$$R = \{(5,5), (5,4), (5,3), (5,2), (5,1), (5,0)$$
$$(4,4)\ (4,3)\ (4,2), (4,1), (4,0)$$
$$(3,3), (3,2), (3,1), (3,0)$$
$$(2,2)\ (2,1), (2,0)$$
$$(1,1), (1,0), (0,0)\}$$

to draw we have to take top most element

or

usual procedure

1. Start with a directed graph
2. Remove self loops
3. Remove all transitive edges
4. Remove all ~~[illegible]~~ arrows.



is partially ordered and totally ordered

2) The relation $<$ on $\mathbb{Z}$ is not a partial order because it is not reflexive [$\because$ $2<2$ ✗ reflexive $\underline{xRx}$ $2\in\mathbb{Z}$.]

Functions:

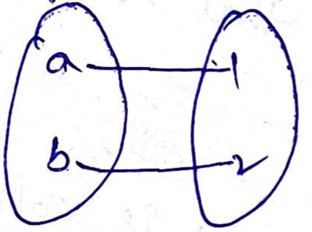
Def^n: 1. A function is a rule which maps a number or Entity to another unique number or Entity.

Def^n: A Relation 'f' from a set A to a set B is called function $f: A\to B$ if to Each element $a\in A$, we can assign unique element of B.

Ex: Let $A=\{a,b\}$, $B=\{1,2\}$

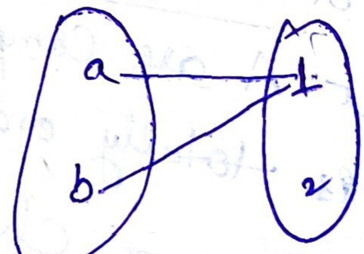
$A\times B=\{(a,1),(a,2),(b,1),(b,2)\}$

① codomain A B(codomain)



is function

② A B

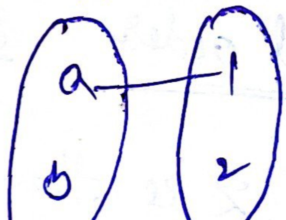


Yes function

③ A B

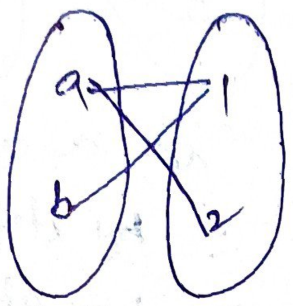
Yes function

(4) A B



no (not a function

Because in ~~dom~~ $b \in A$ is not assigned to any element, we can leave elements in B but not in A without assigning.

c)  is not a function, because $a \in A$ is assigned to two elements it is not possible in function.

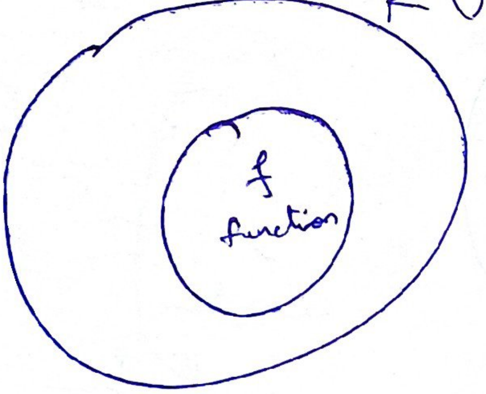
$a \in A$ ~~must~~ can assign unique element in B.

<u>Range</u>: the codomain (B) elements participating in relation in B (codomain) ~~from A to B~~ are called Range

<u>Ex</u>: ② Range $f = \{1\}$, codomain $= \{1, 2\}$

③ Range $f = \{1, 2\}$ codomain $\{1, 2\}$

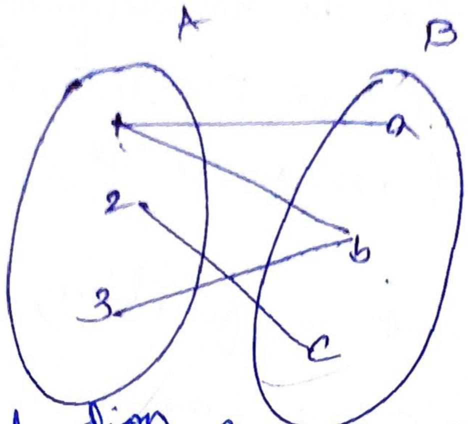
R (Relation)



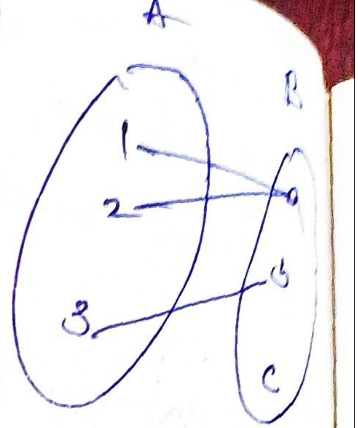
(function is subset of Relation.

<u>one to one</u> <u>function</u> (<u>Injection</u>): A function 'f' from set 'A' to set 'B' is one to one if <u>no two</u> <u>elements</u> <u>in A</u> <u>are</u> <u>mapped</u> to <u>same element</u> in B.

Ex: ①



②

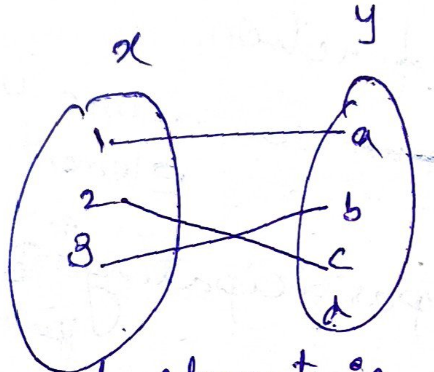


not a function ✗ for not a one-one ~~one to one~~

∵ The element (1) in A is mapped to a & b in B.

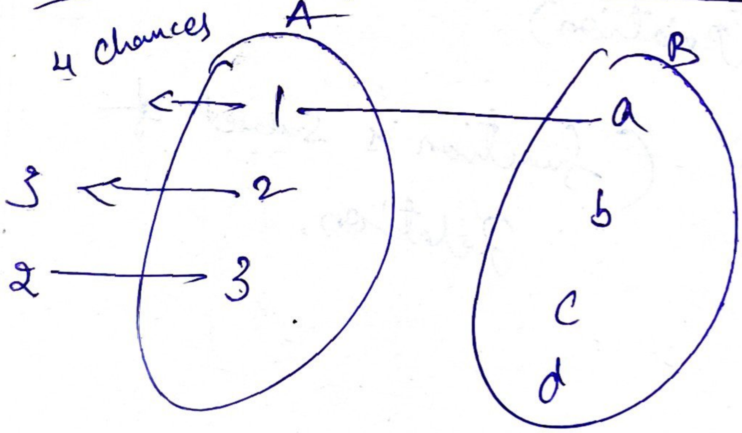
not a one-to-one ∵ by defn no two elements are mapped to B but here 1 and 2 are mapped to a ∴ The function is not one-to-one

Ex:



The no. of elements in $x \leq$ no. of elements in y then only the function is one-to-one function

Total no. of One-to-one functions



$4 \times 3 \times 2$

If A contains x no. of element, B contains y no. of ~~x no. of~~ element

$${}^{y}P_{x} = \{y(y-1)(y-2)(y-3)\ldots y-(x-1)\}$$

P → permutation

$^4P_x = {}^4P_3$ [∵ A contains 3 elements]

One-one function (Injective function): A function $f: A \to B$ is said to be one-to-one if

$$f(x_1) = f(x_2)$$
$$\Rightarrow x_1 = x_2.$$

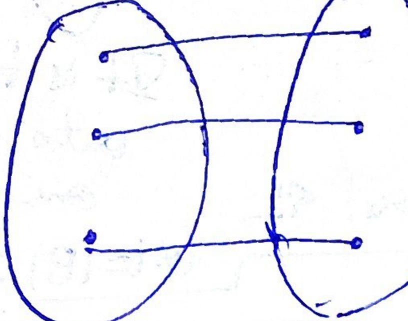
Thus, f is one-one iff for each $b \in \text{ran} f$, b has precisely one-preimage.

Onto function (Surjection): A function "f" from set 'A' to set 'B' is Onto if each element of "B" is mapped to atleast one element of A. [here we are talking about set B]

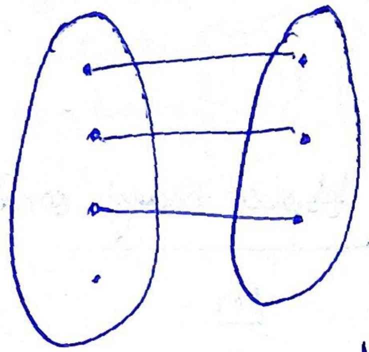
. Range of f = B

Ex:

A B

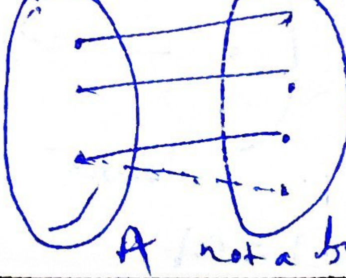


A B



is onto but → not on-to because it is not a function

(B)



B

A

not a onto function so a not a function.

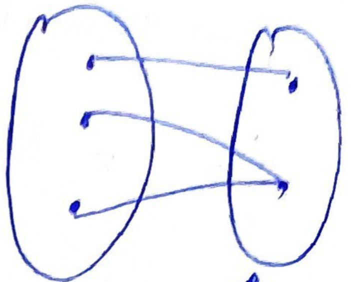
to become onto function first it must be function ∴ one element in A is not mapped to B so it is not function

(S) Example then onto

If $|A| = |B|$ ~ elements

$|A| \leq |B|$

$|A| \geq |B|$



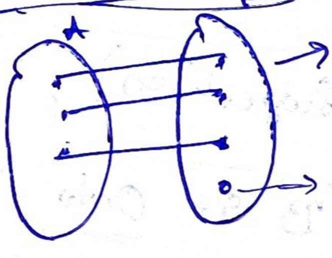
onto but not ...

Range of $f = B$ [∵ Since all elements B are participating

∴ Range of f = B in on-to function

* If ~~one-onto~~ one-to-one then onto? → (It is some times, not always)

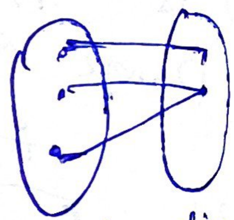
no → Ex:



→ It is one-one but not on-to because in B one element don't have preimage in A.

* If onto then one-one?

In always case 'no' Ex:



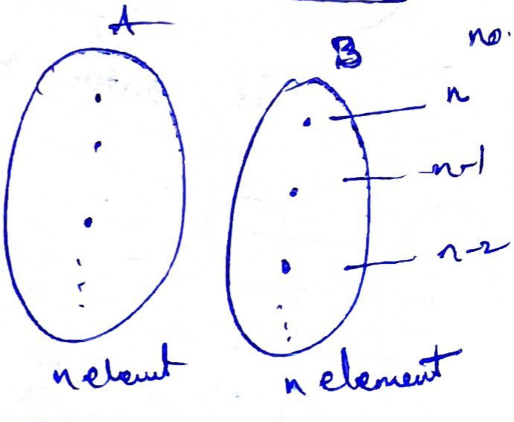
It is function and onto but not one-one function

* How many onto functions of $|A| = |B|$:

Ln

the no. of choices of 1st element have is n

The no. of choices of 2nd element have mapping from B → A is n-1



A

B

no. of ch...

n

n-1

n-2

n element

n element

∴ $n.(n-1)(n-2).\ldots 1 = n!$

Notes

On-to function or surjective function:

A function $f : A \to B$ is said to be onto iff $\text{ran} f = B$.

In otherwords, f is onto if each $b \in B$ has some pre-image in A.

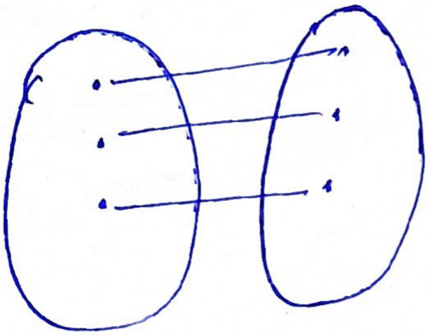
Bijective function: If a function is both one-to-one and onto then it is called a bijective function.

One - to one

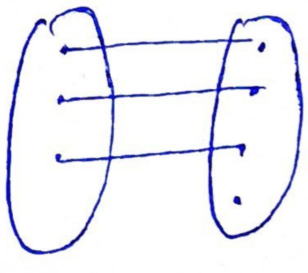
no two elements of A have same image in B

i.e no of elements of $A \leq$ no. of elements in B

i.e



$|A| = |B|$ case - (i)

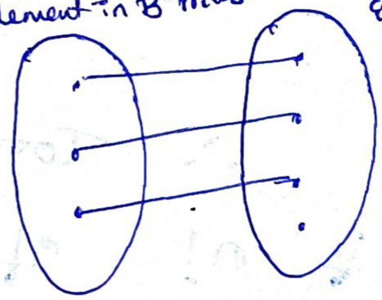


$|A| < |B|$ case - (ii)

from (i) & (ii) $\Rightarrow |A| \leq |B|$ — (I)

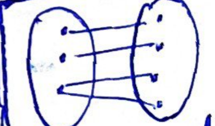
On to

Every element in B must have a preimage in A.



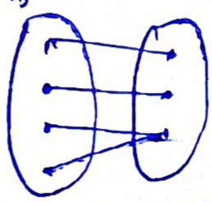
not on-to ∵ last element don't have preimage

it may be a function but not onto



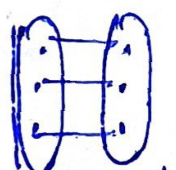
its not a function because here last element in A have two images in B. So not onto.

So there is a possibility of on to function is:



$|A| \geq |B|$

or



$|A| = |B|$

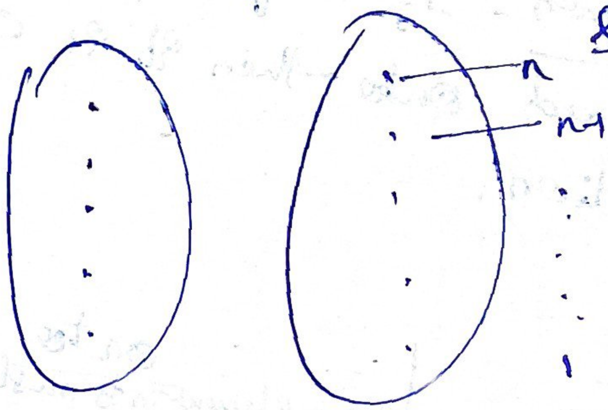
$\Rightarrow |A| \geq |B|$ — (II)

∴ from (I) & (II)

A function is Bijective if $|A| = |B|$

i.e no. of elements in A = no. of elements in B.

~~no. of~~ <u>number of bijections</u> from A to B having ~~the~~ Equal number of elements are



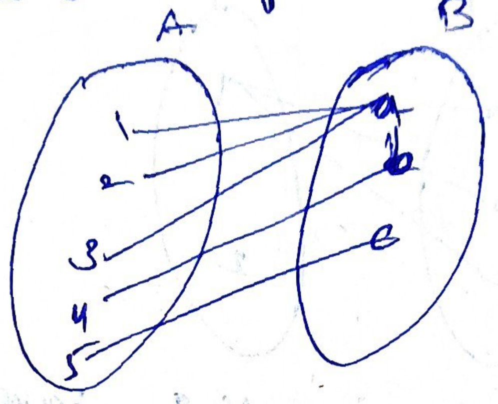
Suppose n elements are there

$\because |A| = |B|$

∴ Total no. of bijections present is $\underline{\underline{n!}}$ if $|A| = |B| = n$.

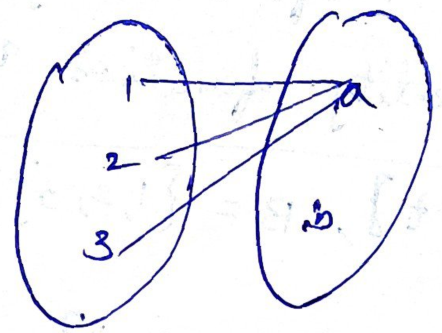
⇒ many-to-one function: A function will be known as many to one function when minimum of two elements of the domain has a connection with a single element of the codomain.

Ex:

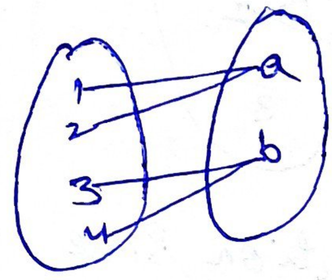


[A function which is not a one-one is many-to one function]

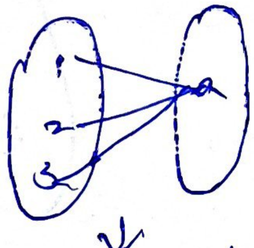
→ The many one functions can also be called a constant function if all the elements of the domain are connected to only one element.



→ The many one function is called an onto function if each element in the range has been utilized.



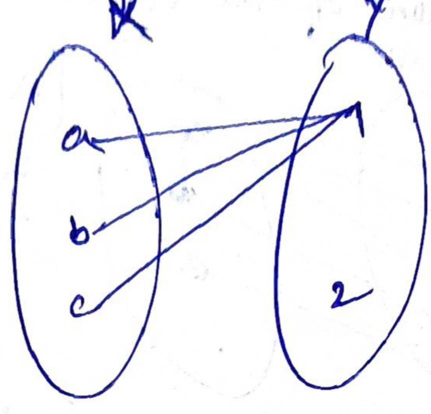
or



↓

Constant function and also onto funⁿ and also many to one.

Into function:- A function in which there must be an element of co-domain Y does not have a pre-image in domain X.



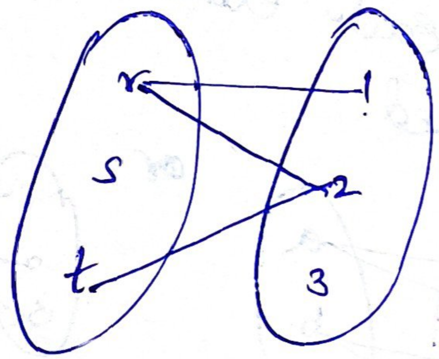
Here In codomain Y, 2 have no preimage in X. ∴ It is Into

Identy function: A function $f: X \to X$ is said to be an identity function if $f(x) = x \; \forall x \in X$

Ex: $A = \{r, s, t\}$, $B = \{1, 2, 3\}$ & $C = \{r, s, t, u\}$

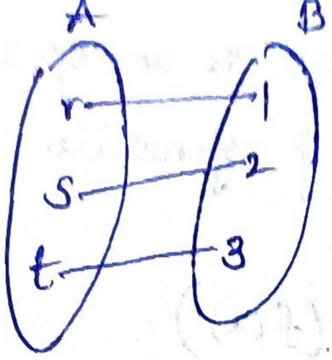
(1) $R = \{(r,1), (r,2), (t,2)\}$

$R: A \to B$



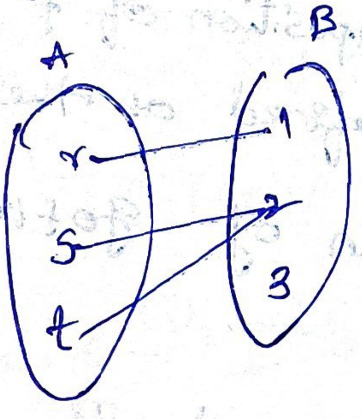
It is not a function since r have two 2 images and S is not mapped.

(2) $g = \{(r,1), (s,2), (t,3)\}$



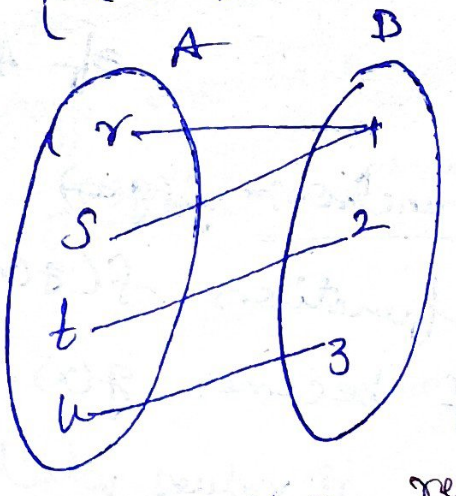
This is both one-one & onto since every element of A have unique image & every element in B have its preimage.

(3) $f = \{(r,1), (s,2), (t,2)\}$



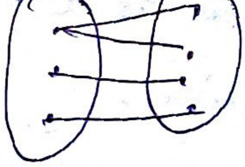
It is not one-one $\because$ s and t have same image.
It is not on-to $\because$ 3 in B don't have preimage.

(4) $h = \{(r,1), (s,1), (t,2), (u,3)\}$ ($\because$ two elements in A have same image)



It is not on-one function but it is onto function & ($\because$ every element in B have its preimage.

* Every function is a relation, but every relation is not a function Ex:



This is a relation but not a function.

W-3 D-2 Composition of Functions

The Composition of a function is an operation where two functions f and g generate a new function say h, i.e

$$h(x) = g(f(x))$$
$$= gof(x)$$

Defⁿ: $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function

$gof : A \to C$ given by $gof(x) = g(f(x))$ $\forall x \in A$

Note:- 1) $gof(x) \neq fog(x)$ always.

2) $gof(x) \neq g \cdot f(x)$. [∵ '•' shows ~~multipli~~ cation product of two functions

Domain in Composition Function:- ~~f(g(x))~~

In the Composition function $f(g(x)) = fog(x)$

The domain function f becomes g(x).

→ The domain is a set of all values which go into the function.

Ex: If $f(x) = 3x+1$ and $g(x) = x^2$, then f of g of x

$$f(g(x)) = f(x^2) = 3(x^2)+1.$$

If we reverse the function operation, such as g of f of x,

$$g(f(x)) = g(3x+1) = (3x+1)^2.$$

Properties of function Compositions:

1. Associative Property:- If there are three functions f, g and h, then they are said to be associative iff

$$f\circ(g\circ h) = (f\circ g)\circ h.$$

2. Commutative Property: Two functions f and g are said to be commute with each other, iff

$$f\circ g = g\circ f.$$

*

3. The function composition of one-to one function is always one-to-one.

4. The function composition of two onto function is always onto.

5. The inverse of the composition of two functions f and g is equal to the composition of the inverse of both the functions, such as

$$(f\circ g)^{-1} = g^{-1}\circ f^{-1}$$

Inverse function:

Definition: An inverse function or an anti-function is defined as a fimction, which can reverse into another function. In Simple words, if any function f takes x to y, then, the inverse of f will take y to x.

If the function is denoted by f or F, then the inverse function is denoted by f^{-1} or F^{-1}

Composition: Let $A = \{a, b, c\}$, $B = \{x, y, z\}$, $C = \{r, s, t\}$. Let $f: A \to B$ and $g: B \to C$ be defined by

$$f = \{(a, y), (b, x), (c, y)\} \text{ and } g = \{(x, s), (y, t), (z, r)\}$$

Find a) Composition function $g \circ f : A \to C$

b) $Im(f)$, $Img(g)$, $Im(g \circ f)$.

Sol $A = \{a, b, c\}$

$(g \circ f)(a) = g(f(a)) = g(y) = t$

$(g \circ f)(b) = g(f(b)) = g(x) = s$

$(g \circ f)(c) = g(f(c)) = g(y) = t.$

$\therefore g \circ f = \{(a, t), (b, s), (c, t)\}$

$Im(f) = \{y, x\}$

$Im(g \circ f) = \{s, t\}$

$Im(g) = \{s, t, r\}$

Note: If f and g are inverse functions, then $f(x) = y$ iff $g(y) = x$.

→ If any mapping is invertable then it must be both one-one and ~~on~~ on-to.

(or)

Any function is invertable then it must be both one-one and on-to.

→ Ex: If $f(x) = 2x+5$ then what will be the inverse of f?

Sol let $f(x) = 2x+5 = y \Rightarrow x = \frac{y-5}{2} = g(y)$.

then $f^{-1}(x) = \frac{x-5}{2}$.

Ex: ① Let $X = \{1,2,3,4\}$. Determine whether each relation on X is a function from X on to X.

a) $f = \{(2,3), (1,4), (2,1), (3,2), (4,4)\}$

b) $g = \{(3,1), (4,2), (1,1)\}$

c) $h = \{(2,1), (3,4), (1,4), (2,1), (4,4)\}$

a/b , a/b
$= \frac{b}{a}$ $= \frac{a}{b}$

Inverse function: An inverse function / An Anti function is defined as a function which can reverse into another function.

In simple words, If any function $f: X \to Y$ [f takes X to Y], then the inverse of f will take Y to X, i.e $f^{-1}: Y \to X$.

Let us consider

$$f(x) = y$$
$$x = f^{-1}(y)$$
$$\Rightarrow f^{-1}(y) = g$$
$$\boxed{g = f^{-1}}$$

here f, g are functions $\Rightarrow$ for then f^{-1}, g^{-1} are inverse function

~~here~~ **prob:** If $f(x) = 2x+5$ then what will be inverse of (f^{-1})?

Sol Let $f(x) = 2x+5 = y$

$$2x + y = y$$
$$2x = y - 5$$
$$x = \frac{y-5}{2} \Rightarrow f^{-1}(y) = g$$

for $f^{-1}(x) = \frac{x-5}{2}$.

Prb) P.T $f(x) = 5x^3 - 1$ is a one-one function from $R \to R$, where R is a set of real numbers also P.T $f^{-1} o g^{-1} = (g o f)^{-1}$ for $f, g : Q \to Q$, Such that $f(x) = 2x$ and $g(x) = x+2$.

Sol: Let $f(x) = 5x^3 - 1 \quad \forall \; x \in R$

→ To show that x is one-one

we need to prove that $\boxed{f(x_1) = f(x_2)} \Rightarrow \boxed{x_1 = x_2}$

i.e $f(x_1) = f(x_2) \Rightarrow 5x_1^3 - \cancel{1} = 5x_2^3 - \cancel{1}$

$\Rightarrow \cancel{5}x_1^3 = \cancel{5}x_2^3$

$\Rightarrow x_1^3 - x_2^3 = 0$

$[\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)]$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2)$

$\Rightarrow x_1 - x_2 = 0$ & | $x_1^2 + x_1 x_2 + x_2^2 = 0$

$\Rightarrow x_1 = x_2$ | Here roots are Complex. doesnot belongs to real number // So neglected

∴ The function $f(x)$ is one-one

part 2: Let $f(x) = 2x$, $g(x) = x+2$

and prove that $f^{-1} o g^{-1} = (g o f)^{-1}$

let take $f(x) = 2x = y$

$\Rightarrow x = y/2 \Rightarrow f^{-1}(y) = x$

$\Rightarrow f^{-1}(y) = y/2$

$\Rightarrow \boxed{f^{-1}(x) = x/2}$

By change of variable

and $g(x) = x+2 = y$

$x = y-2 \Rightarrow g^{-1}(y) = x$

$\Rightarrow g^{-1}(y) = y-2$

$\Rightarrow \boxed{g^{-1}(x) = x-2}$

L.H.S $f^{-1} o g^{-1} = f^{-1} o g^{-1}(x)$

$= f^{-1} \cdot (g^{-1}(x))$

$= f^{-1}(x-2)$

$= \frac{x-2}{2}$

R.H.S $(g o f)^{-1} = [g(f(x))]^{-1}$

So first find $g(f(x))$ then take inverse

$g(f(x)) = g(2x) =$

$= 2x+2$

let take $gof(x) = h(x) = y$

$\Rightarrow h(x) = 2x+2 = y$ — ①

$\Rightarrow 2x+2 = y$

$\Rightarrow 2x = y-2$

$\Rightarrow x = \frac{y-2}{2}$ — ②

from ① $x = h^{-1}(y)$ and ②

$\Rightarrow h^{-1}(y) = \frac{y-2}{2}$

$\Rightarrow h^{-1}(x) = \frac{x-2}{2}$

$\because (gof)(x) = h(x)$

$\Rightarrow (gof)^{-1} = h^{-1}$

$\Rightarrow (gof)^{-1}(x) = \frac{x-2}{2}$.

$\therefore$ L.H.S = R.H.S.

prb: S.T the function $f: R \to (1,\infty)$ and $g: (1,\infty) \to R$ defined by $f(x) = 3^{2x}+1$

$g(x) = \frac{1}{2}\log_3(x-1)$ are inverse.

$f(x) = 3^{2x}+1 = y \Rightarrow x = f^{-1}(y)$ — ①

$3^{2x} = y-1 \Rightarrow$ applying log o

$\log_3^{2x} = \log_3(y-1)$

$\Rightarrow 2x\log_3^3 = \log_3(y-1)$

$\Rightarrow 2x(1) = \log_3(y-1)$

$\left[\because \log_e^e = 1 \quad \log_a^x = x\log a\right.$

$x = \frac{\log_3(y-1)}{2} = \frac{1}{2}\log_3(y-1)$

from (1) $\quad f^{-1}(y) = \frac{1}{2}\log_3(y-1)$ — (3)

but given that $g(x) = \frac{1}{2}\log_3(x-1)$

So, by changing variable in (3)

we get $\quad f^{-1}(x) = \frac{1}{2}\log_3(x-1) = g(x)$

$\therefore$ ~~f(x)~~ $g(x)$ is inverse of $f(x)$

prb: If $X = \{1, 2, 3, 4, 5, 6, 7\}$ and R is a relation defined as $(x, y) \in R$ if $x-y$ is divisible by 3. find the elements of R & S.T R is an Equivalence relation.

$R = \{(x,y) / x-y \text{ is divisible by } 3\}$

Sol $\{(1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6)$, $(4,4), (4,1), (4,7), (5,5), (5,2), (6,6)$, $(6,3), (7,7), (7,1), (7,4)\}$

(1) Reflexive : $(a,a) \in R$

$\because$ $(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$

$(7,7) \in R$ $1-1=0$ divisibly 3.

$\vdots$

$7-7=0$ divisible by 3.

2) Transitive: $(a,b) \in R$, $(b,c) \in R$

$\Rightarrow (a,c) \in R$.

Consider $(\overset{a}{1},\overset{b}{4}).(\overset{b}{4},\overset{c}{7})$

$(a,c) \Rightarrow (1,7) \in R$

3) Symmetric : $(a,b) \in R$ Then $(b,a) \in R$

Consider $(1,4) \in R$

$\because (4,1) \in R$ . ...

$\therefore$ It is symmetry.

Operations: There are 3 kind of operations:

1) Unary (Transpose, inverse)

~~2)~~ Binary

~~3)~~ n-array

1) Unary: A unary operation is an operation with only one operand that is single in input.

Example: A^T, A^{-1}, $\sin\theta$, $\cos\theta$.

2) Binary: A Binary operation is calculation that combines two elements (operands) to produce another element.

Ex: $2+3$, $2 \% 3$, $2-3$, 2×3 → operation.

3) n-array: An operation that takes n-arguments for its input is an n-array operation.

Ex: $\sum_{i=1}^{10} x_j = x_1 + x_2 + \cdots + x_{10}$.