UNIT-II: Mathematical Logic

Syllabus

Statements and Notation, Connectives, Quantified Propositions, Logical Inferences, Methods of Proof of an Implication, First Order Logic and other Methods of Proof, Rules of Inference for Quantified Propositions

Self-study: Nested quantifiers, Strong Mathematical Induction

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1 Mathematical Logic : Statements and Notation

The rules of logic specify the meaning of mathematical statements. For instance, these rules help us understand and reason with statements such as "There exists an integer that is not the sum of two squares" and "For every positive integer n, the sum of the positive integers not exceeding n is $n(n+1)/2$." Logic is the basis of all mathematical reasoning, and of all automated reasoning. It has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

Usually sentences are classified in to four types.

- Declarative
- Interrogative
- Exclamatory
- Imperative

In logics, we pay our attention to those declarative sentences to which it is meaningful to assign one and only one of the truth values TRUE or FALSE. Such declarative sentences are called *Propositions* or *Statements*. For definiteness the following assumptions are made about propositions.

Assumption 1 : *The Law of Excluded Middle*
For every proposition p, either p is true or p is false
Assumption 2 : *The Law of Contradiction*
For every proposition p, it is not the case that p is both true and false

2 Connectives

Propostions are combined by means of connectives to form new propositions or statements. They are

(i) not (ii) and (iii) or (iv) if .. then (v) If and only if

2.1 Negation (*not*)

If p is a Proposition, then *"p is not true"* is also a proposition. This we represent as "$\sim p$" and it is referred as "*not p*" or "negation of p" or "denial of p". The proposition " not p" is true when p is false and false when p is true.

Example : p : Ramanujan is genius

$\sim p$: It is not the case that Ramanujan is genius

or $\sim p$: It is false that Ramanujan is genius

or $\sim p$: Ramanujan is not genius

The truth table for negation is as given below.

p	$\sim p$
T	F
F	T

2.2 Conjunction (*and*)

If p and q are propositions, then *"p and q"* is also a proposition, which we represent as "$p \wedge q$" and is called "Conjunction". The conjunction "$p \wedge q$" is true only when p is true and q is true. In all other cases it is false.

Example : p : $\sqrt{2}$ is an irrational number
q : 7 is a prime number
$p \wedge q$: $\sqrt{2}$ is an irrational number and 7 is a prime number

The truth table for conjunction is as given below.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2.3 Disjunction (*or*)

It is divided into two parts:

2.3.1 Inclusive Disjunction

If p and q are propositions, then *"p or q"* is also a proposition, represented by "$p \vee q$" and is called "Inclusive Disjunction". This disjunction "$p \vee q$" is true whenever at least one of the two propositions is true. In other words, this disjunction is false only when p is false and q is false. In all other cases, it is true.

Example : "Students who have taken calculus or computer science can take this class."

Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.

The truth table for inclusive disjunction is as given below.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

2.3.2 Exclusive Disjunction

If p and q are propositions, then *"p or q"* is also a proposition, represented by "$p \underline{\vee} q$" and is called "Exclusive Disjunction". This disjunction "$p \underline{\vee} q$" is true whenever one of the two propositions is true. In other words, this disjunction is false when p and q are both false or both true.

Example : "Students who have taken calculus or computer science, but not both, can enroll in this class." Here, we mean that students who have taken both calculus and computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, “Soup or salad comes with an entrée,” the restaurant almost always means that customers can have either soup or salad, but not both.

Hence, this is an exclusive, rather than an inclusive, or. The truth table for exclusive disjunction is given below.

p	q	$p\veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

2.4 Conditional (*if* p *then* q)

If p and q are propositions, then the proposition *“p implies q”* or *“if p then q”* is represented as “$p \to q$” and is called an “Implication” or a “Conditional”. In this case, p is called premise, hypothesis or antecedent of implication and q is called the conclusion or consequent of the implication. The conditional “$p \to q$” is false only when p is true and q is false, in all other cases it is true.

The statement “$p \to q$” is called a conditional statement because “$p \to q$” asserts that q is true on the condition that p holds. Note that the statement “$p \to q$” is true when both (i) p and q are true and (ii) p is false (no matter what truth value q has). Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express “$p \to q$”. In general, the following ways are used to express this conditional statement:

“if p, then q”	“p implies q”
“if p, q”	“q follows from p”
“p is sufficient for q”	“a sufficient condition for q is p”
“q if p”	“q whenever p”
“q when p”	

Example: A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

“If I am elected, then I will lower taxes.”

If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in “$p \to q$”.

Similarly, consider a statement that a professor might make:

“If you get 100% on the final, then you will get an A.”

If you manage to get a 100% on the final, then you would expect to receive an A. If you do not get 100% you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

The truth table for conditional is as given below.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, Contrapositive and Inverse: We can form some new conditional statements starting with a conditional statement "$p \to q$". In particular, there are three related conditional statements that occur so often that they have special names.

(i) The proposition "$q \to p$" is called the converse of "$p \to q$".

(ii) The contrapositive of "$p \to q$" is the proposition "$\neg q \to \neg p$".

(iii) The proposition "$\neg p \to \neg q$" is called the inverse of "$p \to q$".

Example: What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining"?

Solution: Since "q whenever p" is one of the ways to express the conditional statement "$p \to q$", the original statement can be rewritten as

"If it is raining, then the home team wins."

So, we can write as following:

Contrapositive: "If the home team does not win, then it is not raining."

Converse: "If the home team wins, then it is raining."

Inverse: "If it is not raining, then the home team does not win."

Note: Out of these three conditional statements formed from "$p \to q$", only the contrapositive always has the same truth value as "$p \to q$".

2.5 Biconditional

If p and q are propositions, then the conjunction of conditionals $p \to q$ and $q \to p$ is called "Biconditional" of p and q and is denoted by $p \leftrightarrow q$. Thus $p \leftrightarrow q$ is same as $(p \to q) \wedge (q \to p)$ and is read as *"p if and only if q"*.

Example: "You can take the flight if and only if you buy a ticket."
This statement is true if p and q are either both true or both false, that is, if you buy a

ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

The truth table for biconditional is as given below.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Combined Truth Table: The combined truth table is as follows:

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \underline{\vee} q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

3 Definitions

Compound Propositions: The new propositions obtained by using connectives are called molecular or compound propositions.

Simple Propositions: Propositions which don't contain any logical connectives are called simple propositions.

Propositional Function: A propositional function is a function whose variables are propositions.

Remark: While representing a statement (proposition) involving connectives in symbolic form, care has to be taken to ensure that the symbolic representation conveys the intended meaning of the statement without any ambiguity. Appropriate parenthesis are to be used at appropriate places to achieve this objective.

Well-formed Formulae: Statements represented in symbolic forms which cannot be interpreted in more than one way are called well-formed formulae.

Logically Equivalent or Equivalent: Two propositional functions p and q are logically equivalent or equivalent, if they have same truth values or truth tables. Then we write $p \equiv q$.

Problem 1 Construct the truth table for the following:

(i) $p \wedge \sim q$ (ii) $\sim p \vee q$ (iii) $\sim p \veebar \sim q$ (iv) $p \rightarrow \sim q$

Solution :

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$\sim p \veebar \sim q$	$p \rightarrow \sim q$
T	T	F	F	F	T	F	F
T	F	F	T	T	F	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	F	T

Problem 2 Construct the truth tables of the following compound propositions:

(i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$

Solution :

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Problem 3 Construct the truth tables of the following compound propositions:

(i) $(p \wedge q) \rightarrow r$ (ii) $q \wedge (\sim r \rightarrow p)$

Solution : Do it yourself.

Tautology: A compound proposition function which is true for all possible truth values of its components is called *"tautology"*.

Contradiction: A compound proposition function which is false for all possible truth values of its components is called *"contradiction"* or an *"absurdity"*

Contingency: A compund proposition which is neither a tautology nor a contradiction is called a *"contingency"*.

Example :

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Here, $p \vee \sim p$ is a tautology, and $p \wedge \sim p$ is a contradiction.

Problem 4 Show that

(i) $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology,

(ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction,

(iii) $(p \veebar q) \wedge (p \rightarrow q)$ is a contingency.

Solution :

p	q	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$	$(p \veebar q) \vee (p \leftrightarrow q)$	$(p \veebar q) \wedge (p \leftrightarrow q)$	$(p \veebar q) \wedge (p \rightarrow q)$
T	T	F	T	T	T	F	F
T	F	T	F	F	T	F	F
F	T	T	T	F	T	F	T
F	F	F	T	T	T	F	F

Problem 5 Prove that $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

Solution :

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence, $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a Tautology.

Problem 6 For any two Propositions p, q prove that $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$.

Solution :

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that $p \rightarrow q$ and $\sim p \vee q$ have identical truth values for all possible truth values of p and q.

$\therefore$ $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$ i.e., $p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.

Logical Implication: A proposition p logically implies a proposition q, and q is a logical consequence of p, if the implication $(p \rightarrow q)$ is true for all possible assignments of the truth values of p and q, that is, if $(p \rightarrow q)$ is a tautology.

Note: Much care must be taken not to confuse implication (or conditional) with logical implication. The conditional is only a way of connecting the two propositions p and q, whereasif p logically implies q then p and q are related to the extent that whenever p

has the truth value T then so does q. We do note that **every logical implication is an implication (conditional), but not all implications are logical implications.**

Valid and Faulty Inferences: We say that an inference is **valid** if the implication is a tautology, that is, if the implication is a logical implication otherwise, we say that the inference is **faulty** or **invalid**.

4 The Laws of Logic

The following results are known as Laws of Logic obtained based on the definition of logical equivalence.

1. ***Law of double negation*** : For any Proposition p, $\sim(\sim p) \Leftrightarrow p$
2. ***Idempotent Laws*** : For any Proposition p, $(a)\ p \vee p \Leftrightarrow p \quad (b)\ p \wedge p \Leftrightarrow p$
3. ***Identity Laws*** : For any Proposition p, $(a)\ p \vee F_0 \Leftrightarrow p \quad (b)\ p \wedge T_0 \Leftrightarrow p$
   Where F_0 is a Contradiction and T_0 is a Tautology.
4. ***Inverse Laws*** : For any Proposition p, $(a)\ p \vee \sim p \Leftrightarrow T_0 \quad (b)\ p \wedge \sim p \Leftrightarrow F_0$
5. ***Domination Laws*** : For any Proposition p, $(a)\ p \vee T_0 \Leftrightarrow T_0 \quad (b)\ p \wedge F_0 \Leftrightarrow F_0$
6. ***Commutative Laws*** : For any two Proposition p and q,
   $(a)\ (p \vee q) \Leftrightarrow (q \vee p) \quad (b)\ (p \wedge q) \Leftrightarrow (q \wedge p)$
7. ***Absorption Laws*** : For any two Proposition p and q,
   $(a)\ [p \vee (p \wedge q)] \Leftrightarrow p \quad (b)\ [p \wedge (p \vee q)] \Leftrightarrow p$
8. ***De Morgan Laws*** : For any two Proposition p and q,
   $(a) \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q \quad (b) \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
9. ***Associative Laws*** : For any three Proposition p, q and r
   $(a)\ p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \quad (b)\ p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
10. ***Distributive Laws*** : For any three Proposition p, q and r
    $(a)\ p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \quad (b)\ p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Problem 7 Obtain the truth table for $p \vee (q \wedge r)$ and $(p \wedge q) \vee (p \wedge r)$ for all possible truth values of p, q, r.
Solution : Do it yourself.

5 Fallacies

These are the statements which appear to be true but they are not. There are three forms of faulty inferences.

1. ***Fallacy of affirming the Consequent*** :

$$\begin{array}{l} p \to q \\ \underline{q \qquad} \\ \therefore \quad p \end{array} \qquad \text{is a Fallacy}$$

   since $[(p \to q) \wedge q] \to p$ is not a tautology.

2. ***Fallacy of denying the antecedent*** :

$$\begin{array}{l} p \to q \\ \underline{\sim p \qquad} \\ \therefore \quad \sim q \end{array} \qquad \text{is a Fallacy}$$

   since $[(p \to q) \wedge (\sim p)] \to (\sim q)$ is not a tautology.

3. ***Non sequitur fallacy*** (***Non sequitur means it doesn't follow***) :

$$\begin{array}{l} \underline{p \qquad} \\ \therefore \quad q \end{array} \qquad \text{is a Fallacy}$$

   since $p \to q$ is not a tautology.

Problem 8 Simplify the following Compound propositions using laws of logic.

(i) $(p \vee q) \wedge \sim(\sim p \vee q)$

(ii) $\sim [\sim [(p \vee q) \wedge r] \vee \sim q]$

Solution : The solution can be written as following:

(i)
$$\begin{aligned} (p \vee q) \wedge \sim(\sim p \vee q) &\equiv (p \vee q) \wedge (p \wedge \sim q) && \text{DeMorgan and Double Negation Laws} \\ &\equiv [(p \vee q) \wedge p\,] \wedge \sim q && \text{Associative Law} \\ &\equiv [\,p \wedge (p \vee q)\,] \wedge \sim q && \text{Commutative Law} \\ &\equiv p \wedge \sim q && \text{Absorption Law} \end{aligned}$$

(ii) $\sim [\sim [(p \vee q) \wedge r] \vee \sim q] \equiv \sim [\sim [(p \vee q) \wedge r]] \wedge \sim [\sim q]$ De Morgan Law

$$\begin{aligned} &\equiv [(p \vee q) \wedge r] \wedge q && \text{Law of double negation} \\ &\equiv (p \vee q) \wedge (r \wedge q) && \text{Associative Law} \\ &\equiv (p \vee q) \wedge (q \wedge r) && \text{Commutative Law} \\ &\equiv ((p \vee q) \wedge q) \wedge r && \text{Associative Law} \\ &\equiv q \wedge r && \text{Absorption Law} \end{aligned}$$

Remark : Logically equivalent propositions are treated as equivalent propositions. Some of the useful logically equivalent propositions are here under.

(i) $\sim (p \vee q) \equiv \sim p \wedge \sim q$ (ii) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(iii) $\sim (p \to q) \equiv p \wedge \sim q$

(iv) $(p \to q) \equiv \sim (\sim(p \to q)) \equiv \sim (p \wedge \sim q) \equiv \sim p \vee \sim\sim q \equiv \sim p \vee q$

Proposition	Negation
$\sim p$	p
$p \wedge q$	$\sim p \vee \sim q$
$p \vee q$	$\sim p \wedge \sim q$
$p \to q$	$p \wedge \sim q$
$\sim (p \to q)$	$\sim p \vee q$

Problem 9 Show that $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim(q \vee p)$.

Solution : This can be done by using truth tables of both LHS and RHS or obtain RHS from LHS by simplifying LHS.

$$
\begin{aligned}
(p \to q) \wedge [\sim q \wedge (r \vee \sim q)] &\equiv (p \to q) \wedge [\sim q \wedge (\sim q \vee r)] \quad \text{Commutative Law} \\
&\equiv (p \to q) \wedge \sim q \quad \text{Absorption Law} \\
&\equiv \sim [(p \to q) \to q] \quad \{ \because \sim (p \to q) \equiv p \wedge \sim q \} \\
&\equiv \sim [\sim (p \to q) \vee q] \quad \{ \because (p \to q) \equiv \sim p \vee q \} \\
&\equiv \sim [(p \wedge \sim q) \vee q] \quad \{ \because \sim (p \to q) \equiv p \wedge \sim q \} \\
&\equiv \sim [q \vee (p \wedge \sim q)] \quad \text{Commutative Law} \\
&\equiv \sim [(q \vee p) \wedge (q \vee \sim q)] \quad \text{Distributive Law} \\
&\equiv \sim [(q \vee p) \wedge T_0] \quad \{ \because q \vee \sim q \text{ is a Tautology } \} \\
&\equiv \sim (q \vee p) \quad \text{Identity Law}
\end{aligned}
$$

Problem 10 Show that $[(p \vee q) \wedge \sim \{ \sim p \wedge (\sim q \vee \sim r)\}] \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a Tautology.

Solution : Let $w = u \vee v$

where $u = (p \vee q) \wedge \sim \{ \sim p \wedge (\sim q \vee \sim r)\}$ and
$v = (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$

Now

$$
\begin{aligned}
u &\equiv (p \vee q) \wedge \sim \{ \sim p \wedge (\sim q \vee \sim r)\} \\
&\equiv (p \vee q) \wedge \sim \{ \sim p \wedge \sim (q \wedge r)\} \\
&\equiv (p \vee q) \wedge \sim \{ \sim (p \vee (q \wedge r))\} \\
&\equiv (p \vee q) \wedge \{ p \vee (q \wedge r)\} \\
&\equiv p \vee \{q \wedge (q \wedge r)\} \\
&\equiv p \vee (q \wedge r)
\end{aligned}
$$

Now

$$
\begin{aligned}
v &\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r) \\
&\equiv \sim (p \vee q) \vee \sim (p \vee r) \\
&\equiv \sim \{(p \vee q) \wedge (p \vee r)\} \\
&\equiv \sim \{p \vee (q \wedge r)\} \quad \equiv \sim \text{u}
\end{aligned}
$$

$$\therefore \quad w \equiv u \vee v \equiv u \vee \sim u \text{ , which is always true.}$$

$\therefore$ The given compound proposition is a Tautology.

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6 Rules of Inference

To construct proofs, we need a means of drawing conclusions or deriving new assertions from old ones; this is done using rules of inference. Rules of inference specify which conclusions may be inferred legitimately from assertions known, assumed, or previously established.

1. ***Modus Ponens or Rule of detachment*** : If the statement p is assumed as true, and also the statement $p \to q$ is accepted as true, then in these circumstances, we must accept q as true. Symbolically we write it as

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore \quad q \end{array}$$

or $p \wedge (p \to q) \to q$ is a tautology.

Example : It is 11.00 o'clock at Hyderabad.
If it is 11.00 o'clock at Hyderabad, then it is 9.30 o'clock at Newyork.
Then by rule of detachment, we can conclude that
It is 9.30 o'clock at Newyork.

2. ***Law of hypothetical syllogism or Transitive rule*** : Whenever the two implications $p \to q$ and $q \to r$ are accepted as true, we must accept the implication $p \to r$ as true. Symbolically we write it as

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore \quad p \to r \end{array}$$

This is because of the implication $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.
This can be extended to a larger number of variables, for example

$$\begin{array}{l} p \to q \\ q \to r \\ \underline{r \to s} \\ \therefore \quad p \to s \end{array}$$

3. ***Law of contrapositive*** : Symbolically it can be written as

$$(p \to q) \Longleftrightarrow \sim q \to \sim p$$

4. ***Demorgan Laws*** : Symbolically these laws can be written as

$$\sim (p \vee q) \Longleftrightarrow \sim p \wedge \sim q$$
$$\sim (p \wedge q) \Longleftrightarrow \sim p \vee \sim q$$

7 Methods of Proof of an Implication

The propositions that commonly appear in Mathematical discussions are conditionals of the form $p \to q$, where p and q are simple or compound propositions which may involve quantifiers. Given such a conditional, the process of establishing that the conditional is true by using the laws of logic (rules) and other known facts is called a *proof of conditional.* The process of establishing that a proposition is a false is called *disproof.*

1. ***Trivial Proof*** : If it is possible to establish that q is true, then regardless of the truth values of p, $p \to q$ is true.

2. ***Vacuous Proof*** : If p is shown to be false, then $p \to q$ is true for any proposition q.

3. ***Direct Proof*** : The direct proof begins by assuming p is true and then, from available information, from the frame of reference, the conclusion q is shown to be true by valid inference.

4. ***Indirect Proof*** : The implication $p \to q$ is equivalent to the implication $\sim q \to \sim p$. Hence we can establish the truth of $p \to q$ by prooving $\sim q \to \sim p$, which is called indirect proof. Hence, in the indirect proof
   (a) assume q is false
   (b) prove on the basis of available information, p is false.

5. ***Proof by Contradiction*** : We know the fact that $p \to q$ is true if and only if $p \wedge \sim q$ is false. This is derived from DeMorgan laws and the fact $p \to q \equiv \sim p \vee q$. Hence, in the proof by contradiction
   (a) assume $p \wedge \sim q$ is true
   (b) Discover some conclusion that is false or violates the existing / established rules.
   (c) Then based on the contradiction obtained in (b), we can say $p \wedge \sim q$ is false and hence $p \to q$ is true.

6. ***Proof by Cases*** : If Proposition p is in the form $p_1 \vee p_2 \vee p_3 \vee \ldots\ldots \vee p_n$ then $(p_1 \vee p_2 \vee p_3 \vee \ldots\ldots \vee p_n) \to q$ can be established by proving separately $p_1 \to q$, $p_2 \to q$, $p_3 \to q$, ... $p_n \to q$.

7. ***Proof by Elimination of Cases*** : This method of proof is nothing more than the law of disjunctive syllogism, given by $[(p \vee q) \wedge \sim p] \to q$. This can be extended to any finite number of cases.
   $\{ [(p_1 \vee p_2 \vee p_3 \vee p_4 \ldots.\vee p_n) \vee q] \wedge \sim p_1 \wedge \sim p_2 \wedge \sim p_3 \wedge \sim p_4 \ldots \wedge \sim p_n \} \to q$

8. ***Conditional Proof*** : We know that $p \to (q \to r) \equiv (p \wedge q) \to r$. Hence the proof of conditional $p \to (q \to r)$ is as follows.
   (a) combine the two antecedents p and q
   (b) Then prove r on the basis of available information.

9. ***Proof by Equivalence*** : To prove $p \leftrightarrow q$, we break it into two parts i.e., $p \to q$ and $q \to p$ and prove them separately. To prove these parts we can choose any one of the previous methods.

8 First Order Logic and Other Methods of Proof

Everyone enrolled in MRUH has lived in a hostel.
Sahithya has never lived in a hostel.

$\therefore$ Sahithya has never enrolled in MRUH.

Our target is to determine whether the aforementioned argument is correct or not. All three are statements and are independent as no implication is there. Propositional logic is not enough to handle these kinds of arguments therefore we'll move to first-order logic. We'll identify the validity of such arguments using first-order logic or predicate logic. So we'll understand **Predicates** and **Quantifiers**.

Predicates: Predicates are statements involving variables which are neither True nor False untill or unless the values of variables are specified.
Example:

1. x is an animal.
2. x is greater than 4.
3. x is less than 5
4. $x + y = 7$.

All the above statements are neither True nor False. They are not propositions because we can't identify thruth values for the argument.
In Predicate logic, a statement is devided into two parts: subject and predicate. Usually we denote such statements with shorthand notation; $G(x)$.

$$G(x) : x \text{ is greater than } 3,$$

where $G()$ denotes the predicate "is greater than 3" and x denotes the subject or variable. After assigning the values of a variable x, the statement $G(x)$ becomes proposition and has a truth value (either True or False)

$$G(5) : 5 \text{ is greater than } 3, \quad \text{(True)}$$

$$G(3) : 3 \text{ is greater than } 3, \quad \text{(False)}$$

Quantifiers: Quantifiers are words that refer to quantities such as "some" or "all". It tells for howmany elements a given Predicate is True. **In English**, Quantifiers are used to express the quantities without giving an exact number.
Ex. all. some, many, none, few etc.
Sentence like: —

1. "Jack has many friends"
2. "Can I have some water?"

Types of quantifiers are:

1. Universal quantifier: The Universal Quantifiers of $P(x)$ is the statement $P(x)$ *is true for all values of* x *in the domain.*
Example: Let $P(x)$ be a statement $x+1>x$

$$P(1) = 1+1>1 \quad = 2>1 \quad \text{True}$$
$$P(2) = 2+1>1 \quad = 3>1 \quad \text{True}$$

$P(x)$ is "True" for all natural numbers. Mathematically, we write: $(\forall x \in \mathbb{N}), P(x)$
Symbolically, we can write this as $\forall x,\ P(x)$, here "$\forall$" is a universal quantifier.

2. Existential quantifier The Existential Quantifiers of $Q(x)$ is the statement $Q(x)P$ *is True for some values of* x *in the domain.*
Example: Let $Q(x)$ be a statement $x<2$

$$Q(1) = 1<2 \quad \text{True}$$
$$Q(2) = 2<2 \quad \text{False}$$
$$Q(3) = 3<2 \quad \text{False}$$

There exists some$(x=1)$ in natural numbers for which $Q(x)$ is "True".
Symbolically, we can write this as $\exists x,\ Q(x)$, here "$\exists$" is an existential quantifier.

Remark: If we use a quantifier that appears within the scope of another quantifier, then it is called *nested quantifier.*
Example :

1. $O(x)$: x is an odd number; $P(x)$: x is a Prime number.
2. $B(x)$: x is a Bird.
3. $F(x)$: x can Fly.

Then "*all birds can fly*" can be written as
$$\forall x,\ B(x) \to F(x)$$
and "*not all birds can fly*" can be written as
$$\sim [\ \forall x,\ B(x) \to F(x)\ ] \text{ or } \exists x,\ B(x) \wedge \sim F(x)$$

4. There is a student who likes Mathematics but not Physics
$M(x)$: x likes Mathematics
$P(x)$: x likes Physics
$S(x)$: x is a student,
then the above statement can be written as $\exists x,\ [\ S(x) \wedge M(x) \wedge \sim P(x)\ ]$

The following table gives us the abbreviated meaning of the quantified statements.

No.	Sentence	Abbreviated Meaning
1	$\forall x,\ F(x)$	all True
2	$\exists x,\ F(x)$	at least one true
3	$\sim [\ \exists x,\ F(x)\ ]$	none true
4	$\forall x,\ [\sim F(x)\ ]$	all false
5	$\exists x,\ [\sim F(x)\ ]$	at least one false
6	$\sim [\ \exists x,\ [\sim F(x)\ ]\ ]$	none false
7	$\sim [\ \forall x,\ F(x)\ ]$	not all true
8	$\sim [\ \forall x,\ [\sim F(x)\ ]\ ]$	not all false

These can be further reduced to four, by listing the following equivalences.

No.	Sentence	Reference Numbers
1	$\forall x,\ F(x) \quad \equiv \quad \sim [\ \exists x,\ [\sim F(x)\ ]\ ]$	1 & 6
2	$\exists x,\ F(x) \quad \equiv \quad \sim [\ \forall x,\ [\sim F(x)\ ]\ ]$	2 & 8
3	$\sim [\ \exists x,\ F(x)\ ] \quad \equiv \quad \forall x,\ [\sim F(x)\ ]$	3 & 4
4	$\exists x,\ [\sim F(x)\ ] \quad \equiv \quad \sim [\ \forall x,\ F(x)\ ]$	5 & 7

Remark: The variable present in a quantified statement is called *bound variable.*

Other Methods of Proof: We can write a few more proof techniques as:

1. **Proof by example:** To show $\exists x,\ F(x)$ is true, it is sufficient to show $F(c)$ is true for some c in the universe. This type is the only situation where an example proves anything.

2. **Proof by exhaustion:** A statement of the form $\forall x,\ [\sim F(x)]$ that $F(x)$ is false for all x (all false) or, equivalently, that there are no values of x for which $F(x)$ is true (none true) will have been proven after all the objects in the universe have been examined and none found with property $F(x)$.

3. **Proof by counterexample:** To show that $\forall x,\ F(x)$ is false, it is sufficient to exhibit a specific example c in the universe such that $F(c)$ is false. This one value c is called a counterexample to the assertion $\forall x,\ F(x)$. In actual fact, a counterexample disproves the statement $\forall x,\ F(x)$.

9 Rules of Inference (Contd..)

5 ***Universal Specification*** : If a statement of the form $\forall x,\ P(x)$ is assumed to be true, then the Universal quantifier can be dropped to obtain $P(c)$ is true for an arbitrary object c in the Universe. This can be represented as

$$\frac{\forall x,\ P(x)}{\therefore \quad P(c) \text{ for all } c}$$

6 ***Universal Generalization*** : If a statement $P(c)$ is true for each element c of the Universe, then the Universal quantifier may be prefixed to obtain $\forall x,\ P(x)$. This can be represented as

$$\frac{P(c) \text{ for all } c}{\therefore \quad \forall x,\ P(x)}$$

7 ***Existential Specification*** : If $\exists x,\ P(x)$ is assumed to be true, then there is an element c in the universe such that $P(c)$ is true. This can be represented as

$$\frac{\exists x,\ P(x)}{\therefore \quad P(c) \text{ for some } c}$$

8 ***Existential Generalization*** : If $P(c)$ is true for some element c in the Universe, then $\exists X,\ p(X)$ is true. This can be represented as

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x,\ P(x)}$$

Example : 1. All Men are fallible
All Kings are Men
Therefore, all Kings are fallible

Let, $M(x)$: x is a Men
$F(x)$: x is fallible
$K(x)$: x is a King

The above argument is symbolized as

$$\begin{array}{c} \forall x,\ M(x) \rightarrow F(x) \\ \underline{\forall x,\ K(x) \rightarrow M(x)} \\ \therefore \forall x,\ K(x) \rightarrow F(x) \end{array}$$

The proof of this can be written as follows.

	Assertion	**Reason**
1.	$\forall x,\ M(x) \rightarrow F(x)$	Premise 1
2.	$M(c) \rightarrow F(c)$	Step 1 and Rule 5
3.	$\forall x,\ K(x) \rightarrow M(x)$	Premise 2
4.	$K(c) \rightarrow M(c)$	Step 3 and Rule 5
5.	$K(c) \rightarrow F(c)$	Step 2, 4 and Rule 2
6.	$\forall x,\ K(x) \rightarrow F(x)$	Step 5 and Rule 6

Example : 2. Lions are dangerous animals
There are Lions
Therefore, there are dangerous animals

Let, $L(x)$: x is a Lion
$D(x)$: x is dangerous

$\therefore$ The above argument is symbolized as

$$\begin{array}{l} \forall x,\ L(x) \to D(x) \\ \underline{\exists x,\ L(x) \qquad\qquad} \\ \therefore \exists x,\ D(x) \end{array}$$

The proof of this can be written as follows.

	Assertion	Reason
1.	$\exists x,\ L(x)$	Premise 2
2.	$L(c)$	Step 1 and Rule 7
3.	$\forall x,\ L(x) \to D(x)$	Premise 1
4.	$L(c) \to D(c)$	Step 3 and Rule 5
5.	$D(c)$	Step 2, 4 and Rule 1
6.	$\exists x,\ D(x)$	Step 5 and Rule 8

10 Proof by Mathematical Induction

Let $P(n)$ be a statemment which may be either true or false for each integer n. To prove $P(n)$ is true for all integers $n \geq 1$, the following conditions are required:

(i) $P(1)$ is true.

(ii) For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

If we generalise this process, then there are 3 steps to a proof using the principle of mathematical induction:

1. **Basis of Induction :** Show $P(n_0)$ is true where n_0 is starting point.
2. **Inductive hypothesis :** Assume $P(k)$ is true for $k \geq n_0$.
3. **Inductive step :** Show that $P(k+1)$ is true on the basis of the inductive hypothesis.

Example: Prove the formula $\frac{n(n+1)}{2}$ for the sum of the first n positive integer.

Proof: Let $S(n) = 1 + 2 + 3 + ... + n$. The proof consists of following steps:

1. **Basis of Induction :** Since $S(1) = 1 = 1(1+1)/2$, the formula is true for $n = 1$.
2. **Inductive hypothesis :** Assume the statement $S(n)$ is true for $n = k$, that is, $S(k) = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$.
3. **Inductive step :** Now, show that the formula is true for $n = k+1$, that is, $S(k+1) = \frac{(k+1)(k+2)}{2}$ follows from the inductive hypothesis.
   To do this, we have $S(k+1) = 1 + 2 + 3 + ... + k + (k+1) = S(k) + (k+1)$
   Now, using inductive hypothesis $S(k) = \frac{k(k+1)}{2}$ in above, we can write
   $S(k+1) = S(k) + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)(\frac{k}{2} + 1)$

$S(k+1) = \dfrac{(k+1)(k+2)}{2}$

Hence the formula holds for $(k+1)$ and the proof is complete by the principle of mathematical induction.

Strong Mathematical Induction: Let $P(n)$ be a statemment which may be either true or false for each integer n. Then $P(n)$ is true for all positive integers if there is an integer $q \geq 1$ such that

1. $P(1)$, $P(2)$,...,$P(q)$ are all true.
2. When $k \geq q$, the assumption that $P(i)$ is true for all integers $1 \leq i \leq k$ implies that $P(k+1)$ is true.

As in the case of the principle of mathematical induction, this form can be modified to apply to statements in which the starting value is an integer different from 1.

Thus, just as before, there are 3 steps to proof by strong mathematical induction:

1. **Basis of Induction :** Show $P(1)$, $P(2)$,...,$P(q)$ are all true.
2. **Strong Inductive hypothesis :** Assume $P(i)$ is true for all integers i such that $1 \leq i \leq k$ where $k \geq q$.
3. **Inductive step :** Show that $P(k+1)$ is true on the basis of the strong inductive hypothesis.

Remark: In a proof using the principle of mathematical induction we are allowed to assume $P(k)$ in order to establish $P(k+1)$. But in using strong mathematical induction we assume not only $P(k)$ but also $P(k-1)$, $P(k-2)$,...,$P(1)$ as well, to establish $P(k+1)$.

Example: Prove that the function $b(n) = 2(3)^n - 5$ is the unique function defined by

(1) $b(0) = -3$**,** $b(1) = 1$**, and**

(2) $b(n) = 4b(n-1) - 3b(n-2)$ **for** $n \geq 2$**.**

Solution: First it is easy to check that $b(n) = 2(3)^n - 5$ satisfies the given relations.

Next, we claim that if $a(n)$ is any other function satisfying relations (1) and (2), then $a(n) = b(n)$ for all n.

Let $P(n)$ be the statement: $a(n) = b(n)$.

We prove $P(n)$ is a true statement for all non-negative integers n by strong induction.

1. **Basis of Induction :** Since we know $b(0) = -3$ and $b(1) = 1$ and we are assuming $a(0) = -3$ and $a(1) = 1$. Thus, $a(0) = b(0)$ and $a(1) = b(1)$. Therefore, $P(0)$ and $P(1)$ are true.

2. **Strong Inductive hypothesis :** Assume $P(i)$ is true for all integers i such that $0 \leq i \leq k$ where $k \geq 1$. In other words, assume $a(i) = b(i)$ for all integers $0 \leq i \leq k$ where $k \geq 1$.

3. **Inductive step :** Now, show that $P(k+1)$ is true or $a(k+1) = b(k+1)$.

From (2), we can write $a(k) = 4a(k-1) - 3a(k-2)$
or, $a(k+1) = 4a(k) - 3a(k-1)$

But, by strong inductive hypothesis,
$a(k) = b(k)$ and $a(k-1) = b(k-1)$ since $k \geq 1$ and $k - 1 \geq 0$.

Thus, $a(k+1) = 4(2(3^k) - 5) - 3(2(3^{k-1}) - 5)$

$$= (8)(3^k) - (6)(3^{k-1}) - 5$$

$$= (8)(3^k) - 2(3^k) - 5$$

$$= (6)(3^k) - 5$$

$$= (2.3)(3^k) - 5$$

$$= (2)(3^{k+1}) - 5$$

$$= b(k+1)$$

and the result is proved by induction.