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Queueing Theory

"Delay is the enemy of efficiency" and "waiting is the enemy of utilisation"

21:1. INTRODUCTION

A flow of customers from infinite/finite population towards the service facility forms a *queue* (*waiting line*) on account of lack of capability to serve them all at a time. The queues may be of persons waiting at a doctor's clinic or at railway booking office, these may be of machines waiting to be repaired or of ships in the harbour waiting to be unloaded or of letters arriving at a typist's desk. In the absence of a perfect balance between the service facilities and the customers, waiting is required either of the service facilities or for the customer's arrival.

By the term '*customer*' we mean the arriving unit that requires some service to be performed. The customer may be of persons, machines, vehicles, parts, etc. ***Queues*** (***waiting line***) stands for a number of customers waiting to be serviced. The queue does not include the customer being serviced. The process or system that performs the services to the customer is termed by *service channel* or *service facility.*

The subject of queueing is not directly concerned with optimization (maximisation or minimization). Rather, it attempts to explore, understand, and compare various queueing situations and thus indirectly achieves optimization approximately.

21:2. QUEUEING SYSTEM

The mechanism of a queueing process is very simple. Customers arrive at a service counter and are attended to by one or more of the servers. As soon as a customer is served, it departs from the system. Thus a queueing system can be described as consisting of customers arriving for service, waiting for service if it is not immediate, and leaving the system after being served.

The general framework of a queueing system is shown in *Fig. 21.1* :

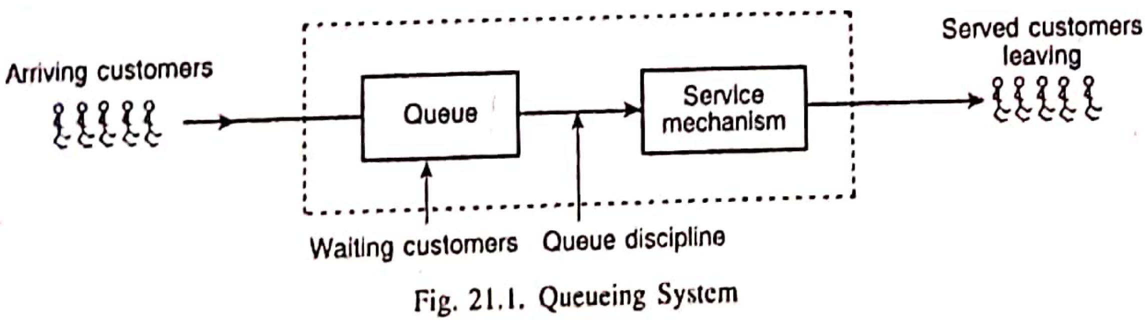

Fig. 21.1. Queueing System

21:3. ELEMENTS OF A QUEUEING SYSTEM

The basic elements of a queueing system are as follows :

1. **Input (or Arrival) Process.** This element of queueing system is concerned with the pattern in which the customers arrive for service. Input source can be described by following three factors :

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(*a*) ***Size of the queue.*** If the total number of potential customers requiring service are only few, then size of the input source is said to be *finite*. On the other hand, if potential customers requiring service are sufficiently large in number, then the input source is considered to be *infinite*.

Also, the customers may arrive at the service facility in batches of fixed size or of variable size or one by one. In the case when more than one arrival is allowed to enter the system simultaneously (entering the system does not necessarily mean entering into service), the input is said to occur in *bulk* or in *batches*. Ships discharging cargo at a dock, families visiting restaurants, etc. are the examples of bulk arrivals.

(*b*) ***Pattern of arrivals.*** Customers may arrive in the system at known (regular or otherwise) times, or they may arrive in a random way. In case the arrival times are known with certainty, the queueing problems are categorized as deterministic models. On the other hand, if the time between successive arrivals (inter-arrival times) is uncertain, the arrival pattern is measured by either mean arrival rate or inter-arrival time. These are characterised by the probability distribution associated with this random process. The most common stochastic queueing models assume that arrival rate follow a Poisson distribution and/or the inter-arrival times follow an exponential distribution.

(*c*) ***Customers' behaviour.*** It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes (patient customer), or if the queue is too long to suit him, may decide not to enter it (impatient customer). Machines arriving at the maintenance shop in a plant are examples of patient customers. For impatient customers,

(*i*) if a customer decides not to enter the queue because of its length, he is said to have *balked*.

(*ii*) if a customer enters the queue, but after some time loses patience and decides to leave, then he is said to have *reneged*.

(*iii*) if a customer moves from one queue to another (providing similar/different services) for his personal economic gains, then he is said to have *jockeyed* for position.

The final factor to be considered regarding the input process is the manner in which the arrival pattern changes with time. The input process which does not change with time is called a *stationary* input process. If it is time dependent then the process is termed as *transient*.

2. **Queue Discipline.** It is a rule according to which customers are selected for service when a queue has been formed. The most common queue discipline is the "first come, first served" (*FCFS*), or the "first in, first out" (*FIFO*) rule under which the customers are serviced in the strict order of their arrivals. Other queue discipline include : "last in, first out" (*LIFO*) rule according to which the last arrival in the system is serviced first.

This discipline is practised in most cargo handling situations where the last item loaded is removed first. Another example may be from the production process, where items arrive at a workplace and are stacked one on top of the other. Item on the top of the stack is taken first for processing which is the last one to have arrived for service. Besides these, other disciplines are : "selection for service in random order" (*SIRO*) rule according to which the arrivals are serviced randomly irrespective of their arrivals in the system; and a variety of *priority* schemes — according to which a customer's service is done in preference over some other customer.

Under *priority* discipline, the service is of two types : (*i*) *Pre-emptive priority*. Under this rule, the customers of high priority are given service over the low priority customers. That is, lower priority customer's service is interrupted (pre-empted) to start service for a priority customer. The interrupted service is resumed again as soon as the highest priority customer has been served.

(*ii*) *Non pre-emptive priority*. In this case the highest priority customer goes ahead in the queue, but his service is started only after the completion of the service of the currently being served customer.

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3. **Service Mechanism.** The service mechanism is concerned with service time and service facilities. Service time is the time interval from the commencement of service to the completion of service. If there are infinite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite, then the customers are served according to a specific order. Further, the customers may be served in batches of fixed size or of variable size rather than individually by the same server, such as a computer with parallel processing or people boarding a bus. The service system in this case is termed as *bulk service system.*

In the case of parallel channels "fastest server rule" (*FSR*) is adopted. For its discussion we suppose that the customers arrive before parallel service channels. If only one service channel is free, then incoming customer is assigned to free service channel. But it will be more efficient to assume that an incoming customer is to be assigned a server of largest service rate among the free ones.

Service facilities can be of the following types :

(*a*) ***Single queue-one server****, i.e.*, one queue-one service channel, wherein the customer waits till the service point is ready to take him in for servicing.

(*b*) ***Single queue-several servers*** wherein the customers wait in a single queue until one of the service channels is ready to take them in for servicing.

(*c*) ***Several queues-one server*** wherein there are several queues and the customer may join any one of these but there is only one service channel.

(*d*) ***Several servers.*** When there are several service channels available to provide service, much depends upon their arrangements. They may be arranged in *parallel* or in *series* or a more complex combination of both, depending on the design of the system's service mechanism.

By *parallel channels*, we mean a number of channels providing identical service facilities. Further, customers may wait in a single queue until one of the service channels is ready to serve, as in a barber shop where many chairs are considered as different service channels; or customers may form separate queues in front of each service channel as in the case of super markets.

For *series channels*, a customer must pass through all the service channels in sequence before service is completed. The situations may be seen in public offices where parts of the service are done at different service counters.

4. **Capacity of the System.** The source from which customers are generated may be finite or infinite. A *finite source* limits the customers arriving for service, *i.e.*, there is a finite limit to the maximum queue size. The queue can also be viewed as one with forced balking where a customer is forced to balk if he arrives at a time when queue size is at its limit. Alternatively, an *infinite source* is forever "abundant" as in the case of telephone calls arriving at a telephone exchange.

21:4. OPERATING CHARACTERISTICS OF A QUEUEING SYSTEM

Some of the operational characteristics of a queueing system, that are of general interest for the evaluation of the performance of an existing queueing system and to design a new system are as follows :

1. *Expected number of customers in the system* denoted by $E(n)$ or L is the average number of customers in the system, both waiting and in service. Here, n stands for the number of customers in the queueing system.

2. *Expected number of customers in the queue* denoted by $E(m)$ or L_q is the average number of customers waiting in the queue. Here $m = n - 1$, *i.e.*, excluding the customer being served.

3. *Expected waiting time in the system* denoted by $E(v)$ or W is the average total time spent by a customer in the system. It is generally taken to be the waiting time plus servicing time.

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easily show that, the time t to complete the service on a customer follows the exponential distribution :

$$s(t) = \begin{cases} \mu e^{-\mu t} \; ; & t > 0 \\ 0 \; ; & t < 0 \end{cases}$$

where μ is the mean service rate for a particular service channel. This shows that service times follows exponential distribution with mean $1/\mu$. The number, n, of potential services in time T will follow the *Poisson distribution* given by

$$\phi(n) = P[n \text{ service in time } T, \text{ if servicing is going on throughout } T] = \frac{(\mu T)^N}{n!} e^{-\mu T}.$$

Consequently, we can also show that

$$P[\text{no service in } \Delta t] = 1 - \mu\Delta t + o(\Delta t) \text{ and } P[\text{one service in } \Delta t] = \mu\Delta t + o(\Delta t).$$

21:7. CLASSIFICATION OF QUEUEING MODELS

Generally queueing model may be completely specified in the following symbolic form :

$$(a/b/c) : (d/e).$$

The first and second symbols denote the type of distributions of inter-arrival times and of inter-service times, respectively. Third symbol specifies the number of servers, whereas fourth symbol stands for the capacity of the system and the last symbol denotes the queue discipline.

If we specify the following letters as :

$M \equiv$ Poisson arrival or departure distribution,

$E_k \equiv$ Erlangian or Gamma inter-arrival for service time distribution,

$GI \equiv$ General input distribution,

$G \equiv$ General service time distribution,

then $(M/E_k/1) : (\infty/FIFO)$ defines a queueing system in which arrivals follow Poisson distribution, service times are Erlangian, single server, infinite capacity and "first in, first out" queue discipline.

21:8. DEFINITION OF TRANSIENT AND STEADY STATES

A queueing system is said to be in *transient state* when its operating characteristic (like input, output, mean queue length, etc.) are dependent upon time.

If the characteristic of the queueing system becomes independent of time, then the *steady-state* condition is said to prevail.

If $P_n(t)$ denotes the probability that there are n customers in the system at time t, then in the steady-state case, we have

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of } t).$$

Due to practical viewpoint of the steady-state behaviour of the systems, the present chapter is amply focused on studying queueing systems under the existence of steady-state conditions. However, the differential-difference equations which can be used for deriving transient solutions will be presented.

21:9. POISSON QUEUEING SYSTEMS

Queues that follow the Poisson arrivals (exponential inter-arrival time) and Poisson services (exponential service time) are called *Poisson queues*. In this section, we shall study a number of Poisson queues with different characteristics.

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Model I {$(M/M/1) : (\infty/FIFO)$}. This model deals with a queueing system having single service channel. Poisson input, Exponential service and there is no limit on the system capacity while the customers are served on a "first in, first out" basis.

The solution procedure of this queueing model can be summarized in the following three steps :

Step 1. Construction of Differential-Difference Equations. Let $P_n(t)$ be the probability that there are n customers in the system at time t. The probability that the system has n customers at time $(t+\Delta t)$ can be expressed as the sum of the joint probabilities of the four mutually exclusive and collectively exhaustive events as follows :

$$\begin{aligned} P_n(t+\Delta t) = {} & P_n(t) \cdot P[\text{no arrival in } \Delta t] \cdot P[\text{no service completion in } \Delta t] \\ & + P_n(t) \cdot P[\text{one arrival in } \Delta t] \cdot P[\text{one service completed in } \Delta t] \\ & + P_{n+1}(t) \cdot P[\text{no arrival in } \Delta t] \cdot P[\text{one service completed in } \Delta t] \\ & + P_{n-1}(t) \cdot P[\text{one arrival in } \Delta t] \cdot P[\text{no service completion in } \Delta t] \end{aligned}$$

This is re-written as :

$$\begin{aligned} P_n(t+\Delta t) = {} & P_n(t)[1 - \lambda \Delta t + o(\Delta t)][1 - \mu \Delta t + o(\Delta t)] + P_n(t)[\lambda \Delta t][\mu \Delta t] \\ & + P_{n+1}(t)[1 - \lambda \Delta t + o(\Delta t)][\mu \Delta t + o(\Delta t)] + P_{n-1}(t)[\lambda \Delta t + o(\Delta t)][1 - \mu \Delta t + o(\Delta t)] \end{aligned}$$

or $$P_n(t+\Delta t) - P_n(t) = -(\lambda + \mu)\Delta t\, P_n(t) + \mu \Delta t\, P_{n+1}(t) + \lambda \Delta t\, P_{n-1}(t) + o(\Delta t)$$

Since Δt is very small, terms involving $(\Delta t)^2$ can be neglected. Dividing the above equation by Δt on both sides and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_n(t) = -(\lambda + \mu) P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t); \quad n \geq 1.$$

Similarly, if there is no customer in the system at time $(t + \Delta t)$, there will be no service completion during Δt. Thus for $n = 0$ and $t \geq 0$, we have only two probabilities instead of four. The resulting equation is

$$P_0(t+\Delta t) = P_0(t)\{1 - \lambda \Delta t + o(\Delta t)\} + P_1(t)\{\mu \Delta t + o(\Delta t)\}\{1 - \lambda \Delta t + o(\Delta t)\}$$

or $$P_0(t+\Delta t) - P_0(t) = -\lambda \Delta t\, P_0(t) + \mu \Delta t\, P_1(t) + o(\Delta t).$$

Dividing both sides of this equation by Δt and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t); \quad n = 0.$$

Step 2. Deriving the Steady-State Difference Equations. In the steady-state, $P_n(t)$ is independent of time t and $\lambda < \mu$ when $t \to \infty$. Thus $P_n(t) \to P_n$ and

$$\frac{d}{dt} P_n(t) \to 0 \quad \text{as } t \to \infty.$$

Consequently the differential-difference equations obtained in *Step 1* reduce to

$$0 = -(\lambda + \mu) P_n + \mu P_{n+1} + \lambda P_{n-1}; \quad n \geq 1$$

and $$0 = -\lambda P_n + \mu P_1; \quad n = 0.$$

These constitute the steady-state difference equations.

Step 3. Solution of the Steady-State Difference Equations. For the solution of the above difference equations there exist three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straightforward and therefore the solution of the above equations will be obtained here by using the iterative method.

Using iteratively, the difference-equations yield

$$P_1 = \frac{\lambda}{\mu} P_0, \quad P_2 = \frac{\lambda + \mu}{\mu} P_1 - \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

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$$P_3 = \frac{\lambda + \mu}{\mu} P_2 - \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_0, \text{ and in general } P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.$$

Now,
$$P_{n+1} = \frac{\lambda + \mu}{\mu} P_n - \frac{\lambda}{\mu} P_{n-1}, \quad n \geq 1.$$

Substituting the values of P_n and P_{n-1}, the equation yields

$$P_{n+1} = \frac{\lambda + \mu}{\mu}\left(\frac{\lambda}{\mu}\right)^n P_0 - \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{n-1} P_0 = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0.$$

Thus, by the principle of mathematical induction, the general formulae for P_n, is valid for $n \geq 0$.

To obtain the value of P_0, we make use of the boundary condition $\sum_{n=0}^{\infty} P_n = 1$.

$$\therefore \quad 1 = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n; \quad \text{since } P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= P_0 \frac{1}{1 - \lambda/\mu}, \quad \text{since } \left(\frac{\lambda}{\mu}\right) < 1.$$

This gives
$$P_0 = 1 - \left(\frac{\lambda}{\mu}\right).$$

Hence, the steady-state solution is

$$P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1 - \frac{\lambda}{\mu}\right) = \rho^n (1 - \rho); \quad \rho = \left(\frac{\lambda}{\mu}\right) < 1, \text{ and } n \geq 0.$$

This expression gives us the probability distribution of queue length.

Characteristics of Model I

(*i*) Probability of queue size being greater than or equal to k, the number of customers is given by

$$P(n \geq k) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty} (1-\rho)\rho^k = (1-\rho)\rho^n \sum_{k=n}^{\infty} \rho^{k-n} = (1-\rho)\rho^n \sum_{k-n=0}^{\infty} \rho^{k-n}$$

$$= (1-\rho)\rho^n \sum_{x=0}^{\infty} \rho^x = \frac{(1-\rho)\rho^n}{1-\rho} = \rho^n.$$

(*ii*) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty}\frac{d}{d\rho}\rho^n = \rho(1-\rho)\frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n, \quad \text{since } \rho < 1$$

$$= \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}.$$

(*iii*) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n, \quad \text{where } m = n - 1$$

being the number of customers in the queue, excluding the customer which is in service.

$$\therefore \quad E(m) = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right]$$

$$= \frac{\rho}{1-\rho} - [1-(1-\rho)] = \frac{\rho}{1-\rho} - \rho$$

$$= \rho^2/(1-\rho) = \lambda^2/\mu(\mu-\lambda).$$

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(iv) Average length of non-empty queue is given by

$$E(m \mid m > 0) = \frac{E(m)}{P(m > 0)} = \frac{\lambda^2}{\mu(\mu - \lambda)} \times \frac{1}{(\lambda/\mu)^2} = \frac{\mu}{\mu - \lambda},$$

since

$$P(m > 0) = P(n > 1) = \sum_{n=0}^{\infty} P_n - P_0 - P_1 = \left(\frac{\lambda}{\mu}\right)^2$$

(v) The fluctuation (variance) of queue length is given by

$$V(n) = \sum_{n=0}^{\infty} [n - E(n)]^2 P_n = \sum_{n=0}^{\infty} n^2 P_n - [E(n)]^2.$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) = (1-\rho)\frac{\rho + \rho^2}{(1-\rho)^3} - \left[\frac{\rho}{1-\rho}\right]^2 = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}.$$

Waiting Time Distribution for Model I. The waiting time of a customer in the system is, for the most part, a continuous random variable except that there is a non-zero probability that the delay will be zero, that is a customer entering service immediately upon arrival. Therefore, if we denote the time spent in the queue by w and if $\Psi_w(t)$ denotes its cumulative probability distribution then from the complete randomness of the Poisson distribution, we have

$$\Psi_w(0) = P(w = 0) \quad \text{(No customer in the system upon arrival)}$$

$$= P_0 = (1 - \rho).$$

It is now required to find $\Psi_w(t)$ for $t > 0$.

Let there be n customers in the system upon arrival, then in order for a customer to go into service at a time between 0 and t, all the n customers must have been served by time t. Let $s_1, s_2, \ldots, s_n$ denote service times of n customers respectively. Then

$$w = \sum_{i=1}^{n} s_i, \; (n \geq 1) \quad \text{and} \quad w = 0 \quad (n = 0).$$

The distribution function of waiting time, w, for a customer who has to wait is given by

$$P(w \leq t) = P\left[\sum_{i=1}^{n} s_i \leq t\right]; \quad n \geq 1 \text{ and } t > 0.$$

Since, the service time for each customer is independent and identically distributed, therefore its probability density function is given by $\mu e^{-\mu t}$ $(t > 0)$, where μ is the mean service rate. Thus

$$\Psi_n(t) = \sum_{n=1}^{\infty} P_n \times P\,(n-1 \text{ customers are served at time } t) \times P\,(1 \text{ customer is served in time } \Delta t)$$

$$= \sum_{n=1}^{\infty}\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \frac{(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \cdot \mu\,\Delta t.$$

The expression for $\Psi_w(t)$, therefore, can be written as

$$\Psi_w(t) = P(w \leq t) = \sum_{n=1}^{\infty} P_n \int_0^t \Psi_n(t)\,dt$$

$$= \sum_{n=1}^{\infty} (1-\rho)\rho^n \int_0^t \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t} \cdot \mu\,dt = (1-\rho)\rho \int_0^t \mu e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\mu t \rho)^{n-1}}{(n-1)!} \cdot dt$$

$$= (1-\rho)\rho \int_0^t \mu e^{-\mu t(1-\rho)}\,dt.$$

Hence, the waiting time of a customer who has to wait is given by

$$\Psi(w) = \frac{d}{dt}[\Psi_w(t)] = \rho(1-\rho)\cdot\mu e^{-\mu t(1-\rho)} = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}.$$

Characteristic of Waiting Time Distribution

(*i*) Average waiting time of a customer (in the queue) is given by

$$E(w) = \int_0^\infty t\cdot\Psi(w)\,dt = \int_0^\infty t\,.\,\rho\mu(1-\rho)\,e^{-\mu t(1-\rho)}\,dt$$

$$= \rho\int_0^\infty \frac{xe^{-x}}{\mu(1-\rho)}\,dx,\quad \text{for } \mu(1-\rho)t = x$$

$$= \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

(*ii*) Average waiting time of an arrival who has to wait is given by

$$E(w\,|\,w>0) = \frac{E(w)}{P(w>0)} = \left\{\frac{\lambda}{\mu(\mu-\lambda)}\right\}\Big/\left(\frac{\lambda}{\mu}\right) = \frac{1}{\mu-\lambda}.$$

[Here $P(w > 0) = 1 - P(w = 0) = 1 - (1-\rho) = \rho$.]

(*iii*) For the busy period distribution, let the random variable v denote the total time that a customer has to spend in the system including service. Then the probability density of its cumulative density function is given by

$$\Psi(w\,|\,w>0) = \frac{\Psi(w)}{P(w>0)} = \left[\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}\right]\Big/\left(\frac{\lambda}{\mu}\right)$$

$$= (\mu-\lambda)\,e^{-(\mu-\lambda)t},\quad t>0.$$

(*iv*) Average waiting time that a customer spends in the system including service is given by

$$E(v) = \int_0^\infty t\,.\,\Psi(w\,|\,w>0)\,dt = \int_0^\infty t\,.\,(\mu-\lambda)\,e^{-(\mu-\lambda)t}\,dt$$

$$= \frac{1}{\mu-\lambda}\int_0^\infty xe^{-x}\,dx,\quad \text{for } (\mu-\lambda)t = x$$

$$= \frac{1}{\mu-\lambda}.$$

Relationships among Operating Characteristics

We have derived the following important characteristics of an $M/M/1$ queueing system :

$$E(n) = \frac{\lambda}{\mu-\lambda},\quad E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)},\quad E(w) = \frac{\lambda}{\mu(\mu-\lambda)} \text{ and } E(v) = \frac{1}{\mu-\lambda}.$$

Using these expressions, we observe some general relationships between the average system characteristics as follows :

(*i*) Expected number of customers in the system is equal to the expected number of customers in the queue plus a customer currently in service, *i.e.*,

$$E(n) = E(m) + \frac{\lambda}{\mu}$$

(*ii*) Expected waiting time of a customer in the system is equal to the expected waiting time in the queue plus the expected service time of a customer in service, *i.e.*,

$$E(v) = E(w) + \frac{1}{\mu}.$$

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(*iii*) Expected number of customers in the system is equal to the average number of arrivals per unit of time multiplied by the average time spent by the customer in the system, *i.e.*,

$$E(n) = \lambda E(v)$$

(*iv*) Expected number of customers in the queue is equal to the average number of arrivals per unit of time multiplied by the average time spent by a customer in the queue, *i.e.*,

$$E(m) = \lambda E(w).$$

Note. Relations between Average Queue Length and Average Waiting Time are known as *Little's Formulae*.

SAMPLE PROBLEMS

2101. *A TV repairman finds that the time spent on his jobs has an Exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought-in?*

[Kerala M.Sc. (Math.) 2001; Delhi M.B.A. (PT) 2008; Madurai M.B.A. 2009]

Solution. We are given,

$$\lambda = 10 \text{ sets per day, and } \mu = 16 \text{ sets per day.}$$

$$\therefore \quad \rho = \lambda/\mu = 10/16 = 0.625$$

The probability for the repairman to be idle is

$$P_0 = 1 - \rho = 1 - 0.625 = 0.375$$

(*i*) Expected idle time per day $= 8 \times 0.375 = 3$ hours.

(*ii*) Expected (or average) number of T.V. sets in the system

$$E(n) = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = \frac{5}{3} = 2 \text{ (approx.) T.V. sets.}$$

2102. *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following :*

(*i*) *the mean queue size (line length), and*

(*ii*) *the probability that the queue size exceeds 10.*

If the input of trains increases to an average 33 per day, what will be the change in (i) and (ii)?

[Meerut M.Sc. (Math.) 2000; Madras M.B.A. 2006; IGNOU M.B.A. (Dec.) 2006; Lucknow B.M.S. 2008]

Solution. Here, we have

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \quad \rho = \lambda/\mu = 36/48 = 0.75$$

(*i*) $$E(m) = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains.}$$

(*ii*) $$P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06.$$

When the input increases to 33 trains per day, we have

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{11}{480} \times 36 = 0.83$$

Then, we get

(*i*) $$E(n) = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.9 \text{ or } 5 \text{ trains (approx.)}$$

(*ii*) $$P(\geq 10) = \rho^{10} = (0.83)^{10} = 0.2 \text{ (approx.)}$$

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2103. *The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.*

(*i*) *What is the probability that a person arriving at the booth will have to wait?*

(*ii*) *What is the average length of the non-empty queues that form from time to time?*

(*iii*) *The Mahanagar Telephone Nigam Ltd. will install a second booth, when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth?*

[Delhi B.Sc. (Stat.) 1996; Visvesvaraya M.B.A. (June) 2011]

(*iv*) *Estimate the fraction of a day that the phone will be in use.*

[Delhi PG Dip. in Glob. Bus. Oper. 2010; Kerala M.B.A. 2010]

(*v*) *What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call?*

[Madras M.B.A. (Nov.) 2006]

Solution. Here, we are given :

$$\lambda = \frac{1}{10} \times 60 \text{ or } 6 \text{ per hour and } \mu = \frac{1}{3} \times 60 \text{ or } 20 \text{ per hour.}$$

(*i*) Probability that a person arriving at the booth will have to wait

$$P(w > 0) = 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0.3.$$

(*ii*) Average length of non-empty queues

$$E(m \mid m > 0) = \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 6} = 1.43.$$

(*iii*) The installation of a second booth will be justified, if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time is :

$$E(w) = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{3}{60} = \frac{\lambda'}{20(20 - \lambda')} \text{ or } \lambda' = 10.$$

Hence, the arrival rate should become 10 customers per hour to justify the second booth.

(*iv*) The fraction of a day that the phone will be busy = traffic intensity $\rho = \lambda/\mu = 0.3$.

(*v*) $$P(w \geq 0) = \int_{10}^{\infty} \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda)t}\, dt = \int_{10}^{\infty} (0.30)(0.23)\, e^{-0.23t}\, dt,$$

where $\lambda = 0.10$ per minute, and $\mu = 0.33$ per minute.

$$\therefore \quad P(w \geq 10) = (0.069) \left.\frac{e^{-0.23t}}{(-0.23)}\right|_{10}^{\infty} = 0.03.$$

This shows that 3 per cent of the arrivals on an average will have to wait for 10 minutes or more before they can use the phone.

2104. *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ a patient.*

[Delhi M.B.A. 2008]

Solution. Here,

$$\lambda = \frac{96}{24 \times 60} = \frac{1}{15} \text{ and } \mu = \frac{1}{10} \text{ patients per minute}$$

$$\therefore \qquad \rho = \lambda/\mu = 2/3.$$

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Average number of patients in the queue are given by,

$$E(m) = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3} = \frac{4}{3}$$

Fraction of the time for which there are no patients is given by,

$$P_0 = 1 - \rho = 1 - 2/3 = 1/3.$$

Now, when the average queue size is decreased from 4/3 patients to 1/2 patient, we are to determine the value of μ. So, we have

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} \Rightarrow \frac{1}{2} = \frac{(1/15)^2}{\mu(\mu - 1/15)^2}$$

i.e., $\mu = 2/15$ patients per minute.

$\therefore$ Average rate of treatment required $= 1/\mu = 15/2 = 7.5$ minutes.

i.e., a decrease in the average rate of treatment is (10 − 7.5) minutes or 2.5 minutes.

Budget per patient = Rs. (100 + 2.5 × 10) = Rs. 125.

Hence, in order to get the required size of the queue, the budget should be increased from Rs. 100 per patient to Rs. 125 per patient.

2105. *A road transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following :*

(*i*) *What is the average number of customers waiting for the service of the clerk?*

(*ii*) *What is the average time a customer has to wait before getting service?*

(*iii*) *The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to Rs. 50 per day. If the cost of goodwill of having to wait is estimated to be 12 paise per minute spent waiting before being served. Should the company install the computer system? Assume 8 hours working day.* ***[Madras M.B.A. 1997; Madurai M.B.A. 2010]***

Solution. We are given

$\lambda = 8$ customers per hour and $\mu = 12$ customers per hour.

(*i*) Average number of customers waiting for the service of the clerk (in the system) :

$$E(n) = \frac{\lambda}{\mu-\lambda} = \frac{8}{12-8} = 2 \text{ customers.}$$

The average number of customers waiting for the service of the clerk (in the queue) :

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{8\times 8}{12(12-8)} \quad \text{or} \quad 1.33 \text{ customers.}$$

(*ii*) The average waiting time of a customer (in the system) before getting service :

$$E(v) = \frac{1}{\mu-\lambda} = \frac{1}{12-8} \text{ hour or 15 minutes.}$$

The average waiting time of a customer (in the queue) before getting service :

$$E(w) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{ hour or 10 minutes.}$$

(*iii*) We now calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

An arrival waits for $E(w)$ hours before being served and there are λ arrivals per hour. Thus, expected waiting time for all customers in an 8-hour day with one system

$$= 8\lambda \times E(w) = 8 \times 8 \times \tfrac{1}{6} \text{ hrs. or } \tfrac{64}{6} \times 60 \text{ minutes, } i.e., \text{ 640 minutes.}$$

The goodwill cost per day with one system = $640 \times$ Re. 0.12 = Rs. 76.80.

The expected waiting time of a customer before getting service when there is an additional computer system is :

$$E(w^*) = \frac{8}{20(20-8)} = \frac{8}{20 \times 12} \text{ or } \frac{1}{30} \text{ hr.}$$

Thus, expected waiting time of customers in an 8-hour day with an additional computer system is :

$$8\lambda \times E(w^*) = 8 \times 8 \times \frac{1}{30} \text{ hr.} = 128 \text{ minutes.}$$

The total goodwill cost with an additional computer system = $128 \times$ Re. 0.12 = Rs. 15.36.

Hence, reduction in goodwill cost with the installation of a computer system

= Rs. 76.80 − Rs. 15.36 = Rs. 61.44.

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61.44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11.44 per day. It is, therefore, worthwhile to instal a computer.

2106. *In the production shop of a company the breakdown of the machines is found to be Poisson with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap, the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per hour. Which repairman should be hired?* *[Delhi M.Com. 2006]*

Solution. In this problem, we compare the total expected daily cost for both the repairmen. This would equal the total wages paid plus the downtime cost.

Case 1. Slow-cheap repairman

$\lambda = 3$ machines per hour and $\mu = 4$ machines per hour.

$\therefore$ Average downtime of a machine $= \frac{1}{\mu - \lambda} = \frac{1}{4-3} = 1$ hour.

$\therefore$ The downtime of 3 machines that arrive in an hour $= 1 \times 3 = 3$ hours.

Downtime cost = Rs. 40×3 = Rs. 120, charges paid to the repairman = Rs. 20×3 = Rs. 60

Total cost = Rs. 120 + Rs. 60 = Rs. 180.

Case 2. Fast-expensive repairman

$\lambda = 3$ machines per hour and $\mu = 6$ machines per hour.

$\therefore$ Average downtime of machine $= \frac{1}{\mu - \lambda} = \frac{1}{3}$ hours

$\therefore$ The downtime of 3 machines that arrive in an hour $= \frac{1}{3} \times 3 = 1$ hour.

Downtime cost = Rs. 40×1 = Rs. 40, charges paid to the repairman = Rs. 30×1 = Rs. 30

Total cost = Rs. 40 + Rs. 30 = Rs. 70.

From the above two cases, the decision of the company should be to engage the fast-expensive repairman.

PROBLEMS

2107. A supermarket has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Calculate :

(*i*) Probability that the cashier is idle.

(*ii*) Average number of customers in the queuing system. (*iii*) Queue size.

(*iv*) Average time a customer spends in the system.

(*v*) Average time a customer spends in the queue waiting for service. *[Gujarat M.B.A. 2008]*

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Model II $\{(M/M/1):(\infty/SIRO)\}$. This model is essentially the same as *Model I*, except that the service discipline follows the *SIRO*-rule (service in random order) instead of *FIFO*-rule. As the derivation of P_n for *Model I* does not depend on any specific queue discipline, it may be concluded that for the *SIRO*-rule case, we must have

$$P_n = (1-\rho)\,\rho^n,\ n \geq 0.$$

Consequently, whether the queue discipline follows the *SIRO*-rule or *FIFO*-rule the average number of customers in the system, $E(n)$, will remain the same. In fact $E(n)$ will remain the same for any queue discipline provided, of course, P_n remains unchanged. Thus, $E(v) = \frac{1}{\lambda}E(n)$ under the *SIRO*-rule is the same as under the *FIFO*-rule.

This result can be extended to any queue discipline as long as P_n remains unchanged. Specifically, the result applies to the three most common disciplines, namely, *FIFO, LIFO* and *SIRO*. The three queue disciplines differ only in the distribution of waiting time where the probabilities of long and short waiting times change depending upon the discipline used. Thus we can use the symbol *GD* (general discipline) to represent the disciplines *FIFO, LIFO* and *SIRO*, when the waiting time distribution is not required.

Model III $\{(M/M/1):(N/FIFO)\}$. This model differs from that of *Model I* in the sense that the maximum number of customers in the system is limited to N. Therefore, the difference equations of *Model I* are valid for this model as long as $n < N$.

The additional difference equation for $n = N$, is

$$P_N(t+\Delta t) = P_N(t)\,[1-\mu\,\Delta t] + P_{N-1}(t)\,.\,[\lambda\,\Delta t]\,[1-\mu\,\Delta t] + o(\Delta t).$$

This gives, after simplification, the differential-difference equation

$$\frac{d}{dt}P_N(t) = -\mu\,P_N(t) + \lambda\,P_{N-1}(t)$$

from which the resultant steady-state difference equation is

$$0 = -\mu\,P_N + \lambda\,P_{N-1}.$$

The complete set of steady-state difference equations for this model, therefore, can be written as

$$\mu P_1 = \lambda P_0$$

$$\mu P_{n+1} = (\lambda+\mu)\,P_n - \lambda\,P_{n-1}, \qquad 1 \leq n \leq N-1.$$

and

$$\mu P_N = \lambda P_{N-1}.$$

Using the iterative procedure (as in *Model I*), the first two difference equations give

$$P_n = (\lambda/\mu)^n\,P_0, \qquad n \leq N-1.$$

Also, we see that for this value of P_n, the third (last) difference equation holds for $n = N$. Therefore, we have

$$P_n = (\lambda/\mu)^n\,P_0 = \rho^n\,P_0, \qquad n \leq N \text{ and } (\lambda/\mu) = \rho.$$

For obtaining the value of P_0, we make use of the boundary conditions, $\sum_{n=0}^{N} P_n = 1$. Therefore

$$1 = P_0 \sum_{n=0}^{N}\rho^n = \begin{cases} P_0\,\dfrac{1-\rho^{N+1}}{1-\rho}, & (\rho \neq 1) \\ P_0\,(N+1), & (\rho = 1) \end{cases}$$

Thus

$$P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}}, & (\rho \neq 1) \\ \dfrac{1}{N+1}, & (\rho = 1) \end{cases}$$

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Hence,

$$P_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}}, & \rho \neq 1; \quad 0 \le n \le N \\ \dfrac{1}{N+1}, & (\rho = 1) \end{cases}$$

Remark. The steady-state solution exists even for $\rho \geq 1$. Intuitively this makes sense since the maximum limit prevents the process from "blowing up". If $N \to \infty$, then the steady-state solution is

$$P_n = (1-\rho)\rho^n; \qquad n < \infty.$$

This result is in complete agreement with that of *Model I.*

Characteristics of Model III

(*i*) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} n P_n = P_0 \sum_{n=0}^{N} n\rho^n = P_0 \rho \sum_{n=0}^{N} \frac{d}{d\rho}\rho^n$$

or

$$E(n) = P_0 \rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n P_0 \rho \frac{d}{d\rho}\left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

$$= P_0 \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2} = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})},$$

$$\text{since } P_0 = \frac{1-\rho}{1-\rho^{N+1}}; \ \rho \neq 1$$

(*ii*) Average queue length is given by

$$E(m) = \sum_{n=1}^{N} (n-1) P_n = E(n) - \sum_{n=1}^{N} P_n = E(n) - (1-P_0)$$

$$= E(n) - \frac{\rho(1-\rho^N)}{1-\rho^{N+1}}, \qquad \text{since } P_0 = \frac{1-\rho}{1-\rho^{N+1}}, \ (\rho \neq 1)$$

$$= \frac{\rho^2[1-N\rho^{N-1} + (N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})}.$$

(*iii*) The average waiting time in the system can be obtained by using Little's formula, that is, $E(v) = \{E(n)\}/\lambda'$, where λ' is the mean rate of customers entering the system and is equal to $\lambda(1-P_N)$. The average waiting time in the queue can be obtained by using the relations

$$E(w) = E(v) - 1/\mu \quad \text{or} \quad E(w) = \{E(m)\}/\lambda'.$$

SAMPLE PROBLEMS

2129. *At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady-state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.* ***[Delhi M.B.A. (Nov.) 2003]***

Solution. Here, $\lambda = 6$ and $\mu = 12$ so that $\rho = 6/12 = 1/2 = 0.5$.

The maximum queue length is 2, *i.e.*, the maximum number of trains in the system is 3 (= N).

The probability that there is no train in the system (both waiting and in service) is given by

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.53.$$

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Now, since $\qquad P_n = P_0 \rho^n$, therefore

$$P_1 = (0.53)(0.5) = 0.27, \; P_2 = (0.53)(0.5)^2 = 0.13, \text{ and } P_3 = (0.53)(0.5)^3 = 0.07.$$

Hence, we get

$$E(n) = 1(0.27) + 2(0.13) + 3(0.07) = 0.74.$$

Thus the average number of trains in the system is 0.74 and each train takes on an average $\frac{1}{12}$ (= .08) hours for getting service. As the arrival of new train expects to find an average of 0.74 trains in the system before it.

$$E(w) = (0.74)(0.08) \text{ hours} = 0.0592 \text{ hours or } 3.5 \text{ minutes}.$$

2130. *Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time (there being 10 lines, one of which is reserved for shunting purposes), calculate the probability that the yard is empty and find the average queue length.* *[Andhra M.E. (Mech. & Ind.) 1996]*

Solution. We have

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ and } \mu = \frac{1}{16} \text{ trains per minute.}$$

$$\therefore \qquad \rho = \lambda/\mu = 36/48 = 0.75.$$

The probability that the yard is empty is given by

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}}, \text{ since } N = 9$$

$$= \frac{0.25}{0.90} = 0.28.$$

Average queue length is given by

$$E(m) = \frac{\rho^2 [1 - N\rho^{N-1} + (N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})} = \frac{(0.75)^2 [1 - 9(0.75)^8 + 8(0.75)^9]}{0.25[(0.75)^{10}]}$$

$$= (2.22)\frac{(1-0.303)}{(1-0.005)} = (2.22)(0.70) = 1.55.$$

PROBLEMS

2131. Consider a single server queueing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible number of calling units in the system is two. Derive the steady-state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.

2132. If for a period of 2 hours in the day (8 to 10 a.m.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period : (*a*) the probability that the yard is empty, (*b*) average number of trains in the system; on the assumption that the line capacity of the yard is limited to 4 trains only. *[Meerut M.Sc. (Math.) 2001; Kerala M.Sc. (Math.) 2001]*

2133. Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.

(*i*) Find the effective arrival rate at the clinic.

(*ii*) What is the probability that an arriving patient will not wait?

(*iii*) What is the expected waiting time until a patient is discharged form the clinic? *[Pune M.C.A. (Oct.) 2007; Madras M.B.A. (Nov.) 2009]*

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2134. In a car-wash service facility, cars arrive for service according to a Poisson distribution with mean 5 per hour. The time for washing and cleaning each car varies but is found to follow an exponential distribution with mean 10 minutes per car. The facility cannot handle more than one car at a time and has a total of 5 parking spaces.

(i) Find the effective arrival rate.

(ii) What is the probability that an arriving car will get service immediately upon arrival?

(iii) Find the expected number of parking spaces occupied. *[Delhi B.Sc. (Stat.) 1997]*

2135. A petrol station has a single pump and space for not more than 3 cars (2 waiting, 1 being served). A car arriving when the space is filled to capacity goes elsewhere for petrol. Cars arrive according to a Poisson distribution at a mean rate of one every 8 minutes. Their service time has an exponential distribution with a mean of 4 minutes.

The proprietor has the opportunity of renting an adjacent piece of land, which would provide space for an additional car to wait (He cannot build another pump.) The rent would be Rs. 10 per week. The expected net profit from each customer is Re. 0.50 and the station is open 10 hours every day. Would it be profitable to rent the additional space?

Model IV ***(Generalized Model : Birth-Death Process).*** This model deals with a queueing system having single service channel, Poisson input with no limit on the system capacity. Arrivals can be considered as *births* to the system, whereas a departure can be looked upon as a *death.* Let

n = number of customers in the system,

λ_n = arrival rate of customers given n customers in the system,

μ_n = departure rate of customers given n customers in the system, and

P_n = steady-state probability of n customers in the system.

The model determines the values of P_n in terms of λ_n and μ_n. Now, from the axioms of Poisson process (*section 21:6*), we observe that an arrival during the small time interval Δt is negligible. This implies that for $n > 0$, state n can change only to two possible states : state $n - 1$ when a departure occurs at the rate μ_n and state $n + 1$ when an arrival occurs at rate λ_n. State 0 can only change to state *1* when an arrival occurs at the rate λ_0. Since no departure is possible when the system is empty, μ_0 is undefined.

Under steady-state conditions, for $n > 0$, the rates of flow into and out of state n must be equal. This is illustrated in the *transition-rate diagram* given below :

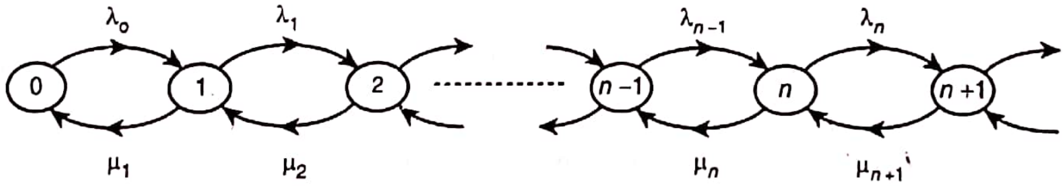

Fig. 21.2

The balance equation is :

Expected rate of flow into state n = Expected rate of flow out of state n

i.e., $$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = \lambda_n P_n + \mu_n P_n, \qquad n \geq 1$$

and $$\mu_1 P_1 = \lambda_0 P_0, \qquad n = 0$$

Using the iterative procedure (as in *Model I*), we have

$$P_1 = \frac{\lambda_0}{\mu_1} P_0, \qquad P_2 = \frac{\lambda_1 + \mu_1}{\mu_2} P_1 - \frac{\lambda_0}{\mu_2} P_0 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$$

$$P_3 = \frac{\lambda_2 + \mu_2}{\mu_3} P_2 - \frac{\lambda_1}{\mu_3} P_1 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0.$$

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In general, we can write the following formula

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_0}{\mu_n\mu_{n-1}\dots\mu_1}P_0,\ n \ge 1 \quad \text{or} \quad P_n = P_0 \prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}},\ n \ge 1.$$

Now
$$P_{n+1} = \frac{\lambda_n + \mu_n}{\mu_{n+1}}P_n - \frac{\lambda_{n-1}}{\mu_{n+1}}P_{n-1} = P_0\prod_{i=0}^{n}\frac{\lambda_i}{\mu_{i+1}}.$$

Thus, by mathematical induction the general value of P_n holds for all n.

To obtain the value of P_0, we use the boundary condition $\sum_{n=0}^{\infty} P_n = 1$ or $P_0 + \sum_{n=1}^{\infty} P_n = 1$, to get

$$P_0 = \left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)^{-1}.$$

Remark. $P_0 = 0$ if R.H.S. is a divergent series. In case R.H.S. is convergent, the value of P_0 will depend on λ_i's and μ_i's.

Special Cases

Case I. When $\lambda_n = \lambda$ *for* $n \ge 0$; *and* $\mu_n = \mu$ *for* $n > 1$

$$P_0 = \left[1 + \sum_{n=1}^{\infty}(\lambda/\mu)^n\right]^{-1} = 1 - \rho.$$

In this case, therefore

$$P_n = \rho^n(1-\rho),\ \text{for } n \ge 0.$$

This result is exactly the same as that of *Model I.*

Case II. When $\lambda_n = \dfrac{\lambda}{n+1}$ *for* $n \ge 0$ *and* $\mu_n = \mu$ *for* $n > 1$.

$$P_0 = \left[1 + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!\mu^n}\right]^{-1} = \left[1 + \rho + \frac{1}{2!}\rho^2 + \frac{1}{3!}\rho^3 + \dots\right]^{-1} = e^{-\rho}$$

$$\therefore \qquad P_n = \frac{1}{n!}\rho^n e^{-\rho} \text{ for } n \ge 0 \text{ and } \rho = \frac{\lambda}{\mu}.$$

which is a Poisson distribution with mean $E(n) = \rho$.

Case III. When $\lambda_n = \lambda$ *for* $n \ge 0$ *and* $\mu_n = n\mu$ *for* $n > 1$,

$$P_0 = \left[1 + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!\mu^n}\right]^{-1} = e^{-\rho}$$

$$\therefore \qquad P_n = \frac{1}{n!}\rho^n e^{-\rho},\ \text{for } n \ge 0 \text{ and } \rho = \frac{\lambda}{\mu}.$$

which is again Poisson with mean $E(n) = \rho$; and $E(m) = 0$, $E(w) = 0$.

In this case the service rate increases with increase in queue length and hence is known as a queueing problem with infinite number of channels, *i.e.*, $(M/M/\infty) : (\infty/FIFO)$. This model is known as a *Self-service Model.*

SAMPLE PROBLEM

2136. *Problems arrive at a computing centre in Poisson fashion at an average rate of five per day. The rules of the computing centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of* $\frac{1}{3}$ *day, and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line.*

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Solution. Here $\lambda = 5$ problems per day, and $\mu = 3$ problems per day.

It is given that the service rate increases with increase in the number of persons.

$$\therefore \quad \mu_n = n\mu \text{ when there are } n \text{ problems and } P_n = \frac{1}{n!}\rho^n e^{-\rho}$$

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\frac{1}{n!}\rho^n e^{-\rho} = e^{-\rho}.\rho.e^{\rho} = \rho = \frac{5}{3} \text{ or } 1.67.$$

Now, the average solving time, which is inversely proportional to the number of people working on the problem, is given by 1/5 day per problem.

$\therefore$ Expected time for a person entering the line is given by

$$\frac{1}{5}E(n) = \frac{1}{5}\times\frac{5}{3} \text{ days} = \frac{1}{3} \text{ day or 8 hours.}$$

PROBLEMS

2137. A shipping company has a single unloading berth with ships arriving in a Poisson fashion at an average rate of three per day. The unloading time distribution for a ship with n unloading crews is found to be exponential with average unloading time $n/2$ days. The company has a large labour supply without regular working hours, and to avoid long waiting line the company has a policy of using as many unloading crews on a ship as there are ships waiting in line or being unloaded. (*a*) Under these conditions what will be the average number of unloading crews working at any time? (*b*) What is the probability that more than 4 crews will be needed?

[Madurai M.Com. 2003]

2138. For a general queue system (discussed in model IV), find the steady-state probability when

(*a*) $$\lambda_n = \begin{cases} \lambda & \text{for } n = 0, 1, 2, \ldots, C-1 \\ 0 & \text{for } n = C, C+1, \ldots \end{cases} \quad \text{and} \quad \mu_n = n\mu \text{ for } n = 1, 2, \ldots, C$$

(*b*) $$\lambda_n = \begin{cases} C - n\lambda & \text{for } n = 0, 1, 2, \ldots, k \\ 0 & \text{for } n = k+1, k+2, \ldots \end{cases} \quad \text{and} \quad \mu_n = n\mu \text{ for } n > 1.$$

2139. An international airport operates with three check-out counters. The sign by the check-out area advises the customers that an additional counter will be opened any time the number of customers in the queue exceeds three. This means that for fewer than four customers, only one counter will be in operation. For four to six customers, two counters will be open. For more than six customers, all three counters will be open. The customers arrive at the counters area according to a Poisson distribution with a mean of 10 customers per hour. The average check-out time per customer is exponential with mean 12 minutes. Determine the following :

(*a*) Steady-state probability P_n of n customers in the check-out area.

(*b*) Average number of busy counters.

(*c*) Average number of idle counters.

[*Hint* : $\lambda_n = \lambda = 10$ customers per hour, $n = 0, 1, 2, \ldots$

$$\mu_n = \begin{cases} 60/12, \text{ i.e., } 5 \text{ customers per hour } (n = 0, 1, 2, 3) \\ 2\times 5 = 10 \text{ customers per hour } (n = 4, 5, 6) \\ 3\times 5 = 15 \text{ customers per hour } (n \geq 7)] \end{cases}$$

Model V $\{(M/M/C):(\infty/FIFO)\}$. This model is a special case of *Model IV* in the sense that here we consider C parallel service channels. The arrival rate is λ and the service rate per service channel is μ.

The effect of using C parallel service channels is a proportionate increase in the service rate of the facility to $n\mu$ if $n \leq C$ and $C\mu$ if $n > C$. Thus, in terms of the *generalized model (Model IV)*, λ_n and μ_n are defined as

$$\lambda_n = \lambda, \quad n \geq 0$$

and $$\mu_n = n\mu \text{ if } 1 \leq n \leq C \text{ and } C\mu, \text{ if } n \geq C.$$

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Utilizing the above values of λ_n and μ_n, the steady-state probabilities of *Model IV* becomes

$$P_n = \begin{cases} \dfrac{\lambda^n P_0}{n\mu\,(n-1)\,\mu \ldots (1)\,\mu}\,; & 1 \le n \le C, \\[2ex] \dfrac{\lambda^n P_0}{\underbrace{(C\mu)\,(C\mu) \ldots (C\mu)}_{n\,-\,C\text{ terms}}\,(C\mu)\,(C-1)\,\mu\,(C-2)\,\mu \ldots (1)\,\mu}\,; & n > C \end{cases}$$

$$= \frac{\lambda^n P_0}{n\,!\,\mu^n} \quad \text{if } 1 \le n \le C \quad \text{and} \quad \frac{\lambda^n P_0}{C^{n-C}\,C\,!\,\mu^n} \quad \text{if } n > C.$$

$$= \frac{1}{n\,!}\,\rho^n P_0 \ \text{ if } 1 \le n \le C \quad \text{and} \quad \frac{1}{C^{n-C}\,C\,!}\,\rho^n P_0 \ \text{ if } n > C.$$

To find the value of P_0, we use the boundary condition $\sum_{n=0}^{\infty} P_n = 1$. Thus, we have

$$\sum_{n=0}^{C-1} P_n + \sum_{n=C}^{\infty} P_n = 1$$

or

$$\left[\sum_{n=0}^{C-1} \frac{1}{n\,!}\,\rho^n + \sum_{n=C}^{\infty} \frac{1}{C^{n-C}\,C\,!}\,\rho^n\right] P_0 = 1$$

or

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n\,!}\,\rho^n + \rho^C \sum_{n=C}^{\infty} \frac{1}{C\,!}\left(\frac{\rho}{C}\right)^{n-C}\right]^{-1}$$

$$= \left[\sum_{n=0}^{C-1} \frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C\,!}\left(\frac{\lambda}{\mu}\right)^C \cdot \frac{C\mu}{C\mu - \lambda}\right]^{-1}$$

Remark. The result obtained above is valid only if $\dfrac{\lambda}{C\mu} < 1$; that is, the mean arrival rate must be less than the mean maximum potential service rate of the system. If $C = 1$, then the value of P_0 is in complete agreement with the value of P_0 for *Model I.*

Characteristics of Model V

(*i*) $P\,(n \ge C)$ = Probability that an arrival has to wait

$$= \sum_{n=C}^{\infty} P_n = \sum_{n=C}^{\infty} \frac{1}{C\,!\,C^{n-C}}\,(\lambda/\mu)^n P_0 = \frac{(\lambda/\mu)^C\,C\mu}{C\,!\,(C\mu - \lambda)}\,P_0$$

(*ii*) Probability that an arrival enters the service without wait

$$= 1 - P\,(n \ge C) \quad \text{or} \quad 1 - \frac{C\,(\lambda/\mu)^C}{C\,!\,(C - \lambda/\mu)}\,P_0$$

(*iii*) Average queue length is given by

$$E\,(m) = \sum_{n=C}^{\infty} (n - C)\,P_n = \sum_{x=0}^{\infty} x\,P_{x+C}, \quad \text{for } x = n - C$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{1}{C\,!\,C^x}\,(\lambda/\mu)^{C+x}\,P_0$$

or

$$E\,(m) = \frac{1}{C\,!}\,(\lambda/\mu)^C \sum_{x=0}^{\infty} x \cdot (\lambda/C\mu)^x\,P_0$$

$$= \frac{1}{C\,!}\,(\lambda/\mu)^C\,P_0 \sum_{x=0}^{\infty}\left(\frac{d}{dy}\,y^x\right)\cdot y, \quad \text{where } y = \frac{\lambda}{C\mu}$$

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$$= \frac{1}{C!} (\lambda/\mu)^C P_o\, y \frac{d}{dy}\left(\frac{1}{1-y}\right)$$

$$= \frac{\lambda\mu (\lambda/\mu)^C P_o}{(C-1)!\,(C\mu-\lambda)^2}$$

(iv) Average number of customers in the system is given by

$$E(n) = E(m) + \frac{\lambda}{\mu} = \frac{\lambda\mu (\lambda/\mu)^C P_o}{(C-1)!\,(C\mu-\lambda)^2} + \frac{\lambda}{\mu}.$$

(v) Average waiting time of an arrival is given by

$$E(w) = \frac{1}{\lambda} E(m) = \frac{\mu (\lambda/\mu)^C.\ P_o}{(C-1)!\,(C\mu-\lambda)^2}.$$

(vi) Average waiting time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu} = \frac{\mu (\lambda/\mu)^C P_o}{(C-1)!\,(C\mu-\lambda)^2} + \frac{1}{\mu} \quad \text{or} \quad E(v) = E(n)/\lambda.$$

(vii) Average number of idle servers is equal to

C – Average number of customers served.

SAMPLE PROBLEMS

2140. *A supermarket has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. Find (i) the probability that an arriving customer has to wait for service, (ii) the average number of customers in the system, and (iii) the average time spent by a customer in the super-market.* *[Delhi M.B.A. 2009; Kerala M.Sc. (Math.) 2001; Madras M.B.A. 2010]*

Solution. We are given

λ = 12 customers per hour, μ = 10 per hour, and C = 2 girls.

$$\therefore \quad P_o = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{12}{10}\right)^n + \frac{1}{2!}\left(\frac{12}{10}\right)^2 \frac{2\times 10}{20-12}\right]^{-1} = \frac{1}{4} \text{ (or 0.25).}$$

(i) Probability of having to wait for service

$$P(w > 0) = \frac{1}{C!} (\lambda/\mu)^C \frac{C\mu}{(C\mu-\lambda)} P_o$$

$$= \frac{1}{2} (12/10)^2 \frac{20}{20-12} \times \frac{1}{4} = 0.45$$

(ii) Average queue length is

$$E(m) = \frac{\lambda\mu (\lambda/\mu)^C P_o}{(C-1)!\,(C\mu-\lambda)^2} = \frac{12\times 10\times (1.2)^2\times 0.25}{(2-1)!\,(20-12)^2} = \frac{27}{40}$$

Average number of customers in the system

$$E(n) = E(m) + \frac{\lambda}{\mu} = \frac{27}{40} + \frac{12}{10} = 1.87 \text{ (or 2 customers) approx.}$$

(iii) Average time spent by customer in supermarket

$$E(v) = E(n)/\lambda = 1.87/12 = 0.156 \text{ hours or } 9.3 \text{ minutes.}$$

2141. *A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for both deposits and withdrawals is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in Poisson fashion throughout the day with mean arrival rate of 16 per hour. Withdrawers also arrive in Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle*

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both withdrawals and deposits? What could be the effect if this could be accomplished by increasing the mean service time to 3.5 minutes? **[A.I.M.A. (Dip. in Management) (Dec.) 1998]**

Solution. Initially we have two independent queueing systems for withdrawers and depositors with input as Poisson distribution and service as exponential distribution.

For withdrawers : $\lambda = 14/\text{hour}$; $\mu = 3/\text{minute}$ or $20/\text{hour}$

Average waiting time in the queue

$$E(w) = \frac{14}{20(20-14)} = \frac{14}{20 \times 6} = \frac{7}{60} \text{ hour or 7 minutes.}$$

For depositors : $\lambda = 16/\text{hour}$; $\mu = 3/\text{minute}$ or $20/\text{hour}$.

Average waiting time in the queue

$$E(w) = \frac{16}{20(20-16)} = \frac{16}{20 \times 4} = \frac{1}{5} \text{ hour or 12 minutes.}$$

If each teller could handle both withdrawals and deposits, we have a common queue with two servers. The queueing system is thus with 2 service channels with $\lambda = 14 + 16 = 30/\text{hour}$ and $\mu = 20/\text{hour}$.

$$\therefore \qquad P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{30}{20}\right)^n + \frac{1}{2!}\left(\frac{30}{20}\right)^2 \frac{2 \times 20}{(2 \times 20 - 30)} \right]^{-1} = 1/7.$$

Average waiting time of arrivals in the queue

$$E(w) = \left(\frac{30}{20}\right)^2 \times \frac{20}{(40-30)^2} \times \frac{1}{7} = \frac{9}{140} \text{ hours or 3.86 minutes.}$$

When the service time is increased to 3.5 minutes,

$$\lambda = 30/\text{hour} \quad \text{and} \quad \mu = 120/7 \text{ or } 17.14/\text{hour}.$$

$$\therefore \qquad P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{21}{12}\right)^n + \frac{1}{2!}\left(\frac{21}{12}\right)^2 \frac{2 \times 17.14}{(2 \times 17.14 - 30)} \right]^{-1} = 1/15.$$

Average waiting time of arrivals in the queue

$$E(w) = \left(\frac{21}{12}\right)^2 \times \frac{17.14}{(34.28-30)^2} \times \frac{1}{15} = \frac{343}{30} \text{ hours or 11.43 minutes.}$$

2142. *A company currently has two tool cribs, each having a single clerk, in its manufacturing area. One tool crib handles only the tools for the heavy machinery, while the second one handles all other tools. It is observed that for each tool crib the arrivals follow a Poisson distribution with a mean of 20 per hour and the service time distribution is negative exponential with a mean of 2 minutes.*

The tool manager feels that, if tool cribs are combined in such a way that either clerk can handle any kind of tool as demand arises, would be more efficient and the waiting problem could be reduced to some extent. It is believed that the mean arrival rate at the two tool cribs will be 40 per hour; while the service time will remain unchanged.

Compare in status of queue and the proposal with respect to the total expected number of mechanics at the tool crib(s), the expected waiting time including service time for each mechanic and probability that he has to wait for more than five minutes. *[Delhi M.B.A. 2012]*

Solution. For each tool crib, we have $\lambda = 20$ mechanics per hour, and $\mu = 30$ mechanics per hour.

$$\therefore \qquad E(n) = \frac{\lambda}{\mu - \lambda} = \frac{20}{30 - 20} = 2 \text{ mechanics,}$$

and

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(20)^2}{30 \times (30 - 20)} = \frac{4}{3} \text{ or } 1.33$$

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Expected waiting time in the system is given by

$$E(v) = \frac{1}{\mu - \lambda} = \frac{1}{30 - 20} = \frac{1}{10} \text{ hour or 6 minutes.}$$

For the proposed system (when both the tool cribs are combined), we have $\lambda = 40$ mechanics per hour, and $\mu = 30$ mechanics per hour, $C = 2$.

$$\therefore \quad P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 \frac{2\mu}{2\mu - \lambda} \right]^{-1}$$

$$= \left[1 + \frac{40}{30} + \frac{1}{2!}\left(\frac{40}{30}\right)^2 \times \frac{2 \times 30}{2 \times 30 - 40} \right]^{-1} = \frac{1}{5} \text{ or } 0.2$$

$$E(n) = \frac{\lambda\mu(\lambda/\mu)^2 P_0}{(2-1)!(2\mu - \lambda)^2} + \frac{\lambda}{\mu} = 2.4$$

$$E(v) = E(n)/\lambda = (2.4)/40 = 0.06 \text{ hour or 3.6 minutes.}$$

From above, it is clear that combining the two tool cribs and asking both servers to attend the customers leads to a more efficient handling of the demand. This is not a surprising result. By separating the functions, certain advantages occur but by combining the two, the net waiting time is reduced because in the former system it is possible that one tool crib is idle, whereas the second tool crib may have a long queue.

2143. *A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrivals follow a Poisson distribution and on average 80 persons in an 8-hour service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate the average number of customers in the system, average number of customers waiting to be serviced, average time a customer spends in the system, and average waiting time for a customer. Calculate how many hours each week a tax adviser spends performing his job. What is the probability that a customer has to wait before he gets service? What is the expected number of idle tax advisers at any specified time?* ***[Panjab Tech. Univ. M.B.A. (Dec.) 2010]***

Solution. $\lambda = 80/8 = 10$ arrivals per hour, $\mu = 1/20 = 3$ services per hour, and $C = 4$.

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C \frac{C\mu}{C\mu - \lambda} \right]^{-1} = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{10}{3}\right)^3 + \frac{1}{4!}\left(\frac{10}{3}\right)^4 \frac{12}{12 - 10} \right]^{-1} = \frac{27}{1267} = 0.02.$$

$$P_1 = \frac{1}{1!}\left(\frac{10}{3}\right)P_0 = \frac{90}{1267}, \quad P_2 = \frac{1}{2!}\left(\frac{10}{3}\right)^2 P_0 = \frac{150}{1267}, \quad P_3 = \frac{1}{3!}\left(\frac{10}{3}\right)^3 P_0 = \frac{500}{1267 \times 3}$$

$\therefore$ (*i*) Average number of customers waiting to be served

$$E(m) = \frac{\lambda\mu(\lambda/\mu)^C P_0}{(C-1)!(C\mu - \lambda)^2} = \frac{30\,(10/3)^4}{3!\,(12 - 10)^2} \times \frac{27}{1267} = 3.3.$$

(*ii*) Average number of customers in the system $E(n) = 3.3 + (10/3) = 6.6$.

(*iii*) Average waiting time of customers $E(w) = 3.30 \times (1/10) = 0.33$ hour ≈ 20 minutes.

(*iv*) Average time a customer spends in the system

$$E(v) = 0.33 + 1/3 = 0.66 \text{ hours} \approx 40 \text{ minutes.}$$

(*v*) Probability that a customer has to wait before getting service

$$P(n \geq C) = \frac{3(10/3)^4}{(4-1)!\,(12-10)} \times \frac{27}{1267} = 0.66.$$

(*vi*) Expected number of idle tax advisers at any specified time

$$= 4P_0 + 3P_1 + 2P_2 + 1P_3$$

$$= 4 \times \frac{12}{1267} + 3 \times \frac{90}{1267} + 2 \times \frac{150}{1267} + \frac{500}{3 \times 1267} = \frac{2}{3} \text{ or } 0.67 \text{ advisers.}$$

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$\therefore$ Probability that any tax adviser is idle $= \dfrac{\text{Expected number of advisers}}{\text{Total number of advisers}} = \dfrac{0.67}{4} = 0.17$

Probability that no tax adviser is idle $= 1 - 0.17 = 0.83$.

The expected weekly time a tax adviser spends in performing job $= 0.83 \times 48 = 39.84$ or 40 hours by treating 6 working days in a week and a day is assumed to be of 8 hours.

PROBLEMS

2144. A petrol pump station has two pumps. The service times follows the exponential distribution with a mean of 4 minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pumps remain idle?

[Delhi B.Sc. (Stat.) 1999]

2145. Given an average arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rate of 22 customers or at one of two channels in parallel, with mean service rate of 11 customers for each of the two channels? Assume that both queues are of *M/M/S* type.

2146. A telephone exchange has two long-distance operators. The telephone company finds that during the peak load, long-distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length of 5 minutes.

(*a*) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(*b*) If the subscribers wait and are serviced in turn, what is the expected waiting time?

Establish the formulae used.

[Delhi M.B.A. 1998]

2147. A telephone company is planning to install telephone booths in a new airport. It has established the policy that a person should not have to wait more than 10 per cent of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 minutes. How many phone booths should be installed?

[Delhi B.Sc. (Stat.) 2000]

2148. A bank has two counters for withdrawals. Counter I handles withdrawals of value less than Rs. 3,000 and counter II Rs. 3,000 and above. Analysis of service time shows a negative exponential distribution with mean service time of 6 minutes per customer for both the counters. Arrival of customers follows Poisson distribution with mean 8 per hour for Counter I and 5 per hour for Counter II.

(*i*) What are the average waiting times per customer of each counter?

(*ii*) If each counter could handle all withdrawals irrespective of their value, how would the average waiting time change?

[Delhi M.B.A. (Nov.) 2003]

2149. A library wants to improve its service facilities in terms of the waiting time of its borrowers. The library has two counters at present and borrowers arrive according to Poisson distribution with arrival rate 1 every 6 minutes and service time follows exponential distribution with a mean of 10 minutes. The library has relaxed its membership rules and a substantial increase in the number of borrowers is expected. Find the number of additional counters to be provided if the arrival rate is expected to be twice the present value and the average waiting time of the borrower must be limited to half the present value.

2150. Assume that customers arrive according to Poisson fashion and service is exponentially distributed at a small post office. Determine the minimum number of parallel service channels needed in each of the following situations to guarantee that the operation of the queueing situation will be stable (*i.e.*, the queue length will not grow indefinitely) :

(*i*) Customers arrive every 5 minutes and are served at the rate of 10 customers per hour.

(*ii*) The average inter-arrival time is 2 minutes, and the average service time is 6 minutes.

(*iii*) The arrival rate is 30 customers per hour, and the service rate per server is 40 customers per hour.

2151. An oil company is constructing a service station on a highway. Traffic analysis indicates that customer's arrivals over most of the day would approximate a Poisson distribution with a mean of 30 automobiles per hour. Previous studies show that one pump could service a mean of 10 automobiles per hour, with the service time distribution approximating the negative exponential. If 4 pumps are installed :

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