Problems (4)

1) A. T.V mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately poisson with an average rate of 10 ~~time~~ ~~each~~ ~~day?~~ per eight-hour day, What is the Mechanics expected idle time each day? ~~How~~ many jobs are ahead of the average sets just brought in?

soln:

$$\mu = \frac{1}{30} \text{ per min}, \quad \lambda = \frac{10}{8} = \frac{10}{8 \times 60} = \frac{1}{48} \text{ per min}$$

Expected no. of jobs are,

$$L_s = \frac{\lambda/\mu}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} \qquad \left(\because \frac{\lambda}{\mu} = \frac{30}{48}\right.$$

$$L_s = \frac{1/48}{\frac{1}{30}-\frac{1}{48}} = \frac{\frac{1}{48}}{\frac{48-30}{48\times 30}} = \frac{\cancel{30}^{5}}{\cancel{18}_{3}}$$

$$L_s = \frac{5}{3}$$

$$L_s = 1\frac{2}{3} \text{ jobs}$$

Since the fraction of the time the machanics is busy $= \frac{\lambda}{\mu} = \frac{30}{48}$

$\therefore$ The no. of hours for which the repairman remains busy in an eight-hour day

$$= 8\left(\frac{\lambda}{\mu}\right) = \cancel{8} \times \frac{\cancel{30}^{5}}{\cancel{48}_{6}} = 5 \text{ hours}$$

$\therefore$ The time for which the machanic remain idle in eight-hour day $= (8-5)$ hours

$= 3$ hours

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(2) Arrivals at a telephone booth are considered to Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean three minutes.

(i) What is the average length of the queue that forms from time to time?

(ii) The telephone department will install a second booth when convinced that an arrival would have to wait at least three minutes for the phone. By how much time must the flow of arrivals be increased in order to justify a second booth?

(iii) Estimate the fraction of a day that the phone will be in use

(iv) Find the average number of units in the system.

soln: $\lambda = \frac{1}{10}$ ~~min~~ , $\mu = \frac{1}{3}$ min

(i) Average length of non-empty queue

$$(L/L>0) = \frac{\mu}{\mu-\lambda} = \frac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{10}} = \frac{\frac{1}{\cancel{3}}}{\frac{10-3}{10\times\cancel{3}}}$$

$$= \frac{10}{7} = 1.43 \text{ person.}$$

(ii) $W_q = 3$, $\mu = \frac{1}{3}$, $\lambda = \lambda'$.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3}-\lambda'\right)} \Rightarrow \cancel{3}\left(\frac{1}{\cancel{3}}\left(\frac{1}{3}-\lambda'\right)\right) = \lambda'$$

$$\Rightarrow \frac{1}{3} = 2\lambda' \Rightarrow \lambda' = \frac{1}{6}$$

$$\lambda' = 0.16$$

Hence, increase in the arrival rate $= \lambda' - \lambda$

$$\Rightarrow 0.16 - 0.10$$

$$= 0.06 \text{ arrival per min}$$

(iii) The fraction of a day that the phone will be in busy = Traffic Intensity

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10} = 0.3$$

(iv) Average no. of unit in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/10}{\frac{1}{3} - \frac{1}{10}} = \frac{\frac{1}{10}}{\frac{10-3}{10 \times 3}} = \frac{3}{7}$$

$$= 0.43 \text{ person.}$$

3Q. Customers arrive at a one-window drive-~~bank~~ according to poission distribution with mean 10 per hour. Service time per customer is exponential with mean five minutes. The space in front of the window including that for the serviced car can accomodate a maximum of three cars. Others can wait outside this space

(i) What is the probability that an arriving customer can drive directly to the space in front of the window?

(ii) What is the probability that an arriving customer will have to wait outside the indicated space?

iii) How long is an arriving customer expected to wait before starting service?

soln/ $\lambda = 10 /\text{hour}, \quad \mu = \frac{1}{5}, \quad \mu = \frac{1}{5} \times 60$

$\mu = 12$ per hour

(i) The probability that an arriving customer can drive directly to the space in front of the window.

$$P_0 + P_1 + P_2 = P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 . P_0$$

$$= P_0\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right] \quad \left[P_0 = 1 - \frac{\lambda}{\mu}\right]$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right]$$

$$= \left(1 - \frac{10}{12}\right)\left(1 + \frac{10}{12} + \frac{100}{144}\right)$$

$$= 0.42$$

(ii) Probability that an arriving customer will have to wait outside the space

$$= 1 - 0.42 = 0.58$$

(iii) Average time of a customer in a queue

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{12(12-10)} = \frac{\cancel{10}^{\,5}}{12 \times \cancel{2}}$$

$$= \frac{5}{12} = 0.417 \text{ hour}$$

Q(4): In a supermarket, the average arrival rate of customers is 10 every 30 minutes following poission process. The average time taken by a cashier to list and calculate the customer's purchase is two and a half minutes following exponential distribution. What is the probability that the queue length exceeds six? What is the expected time spent by a customer in the system?

Soln: $\lambda = \frac{10}{30} = \frac{1}{3}$ per min

$\mu = \frac{1}{2.5} =$ per min

Traffic Intensity $\rho = \frac{\lambda}{\mu} = \frac{1/3}{1/2.5} = 0.833$

(i) Probability of queue size $> 6 = \rho^6$

$= (0.8333)^6 = 0.3348$

(ii) Expected waiting time $W_s = \frac{1}{\mu - \lambda}$

$= \frac{1}{\frac{1}{2.5} - \frac{1}{3}} = \frac{1}{\frac{0.5}{3 \times 2.5}} = 14.99$ mi

Q15. In a public telephone booth, the arrivals ~~on~~ rate on an average are 15 per hour. A call on average takes three minutes. If there is just one phone, find

(i) The expected no. of callers in the booth at any time.

(ii) The proportion of the time, the booth is expected to be idle?

Soln: $\lambda = 15$ per hour,

$\mu = \frac{1}{3}$ per min, $= \frac{1}{3} \times 60 = 20$ per hour

(i) Expected no. of non-empty queue $= \frac{\mu}{\mu - \lambda}$

$= \frac{20}{20-15} = \frac{20}{5} = 4$ person

(ii) The service is busy $= \frac{\lambda}{\mu} = \rho = \frac{15}{20} = \frac{3}{4}$

The booth is expected to be idle for $1 - \frac{3}{4} = \frac{1}{4}$ hours or $\frac{1}{4} \times 60 = 15$ min.

Model II : (M/M/1) : (N : FCFS)

This model differs from model I in the sense that the maximum number of customers in the system is limited to N. Arrivals will not exceed N in any case. The various measures of this model are,

1. $$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$ where $\rho = \frac{\lambda}{\mu}$ ($\frac{\lambda}{\mu} > 1$ is allowed)

2. $$P_N = \frac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n$$, for $n = 0, 1, 2 \ldots N$

3. $$L_q = L_s - \frac{\lambda}{\mu}$$

4. $$L_s = P_0 \sum_{n=0}^{N} n\rho^n$$

5. $$W_q = W_s - \frac{1}{\mu}$$

6. $$W_s = \frac{L_s}{\lambda}$$

Q1. In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the inter-arrival time follows an exponential distribution and the service time is also to be assumed as exponential with mean of 36 minutes. Calculate,

i) the probability that the yard is empty

ii) the average queue length, assuming that the line capacity of the yard is nine trains (consider N = 9)

Soln:

$$\lambda = \frac{30}{24 \times 60} = \frac{1}{48} \text{ train per min}$$

$$\mu = \frac{1}{36} \text{ per min} \quad \& \quad \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

i) The probability that the queue is empty is given by

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \quad (\text{where } N = 9)$$

$$= \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.943} = 0.265$$

ii) Average no. of trains in the system

$$L_s = P_0 \sum_{n=0}^{N} n\rho^n \quad (\because N = 9)$$

$$= (0.265)\left[0 + \rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5 + 6\rho^6 + 7\rho^7 + 8\rho^8 + 9\rho^9\right]$$

$= 0.265\,[0 + (0.75) + 2(0.75)^2 + 3(0.75)^3 + 4(0.75)^4 + 5(0.75)^5 + 6(0.75)^6 + 7(0.75)^7 + 8(0.75)^8 + 9(0.75)^9]$

$= 2.5529$

2Q: A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customer randomly arrive at an average rate $\lambda = 10$ per hour and the barber's service time is negative exponential with an average of $1/\mu = 5$ min per customer. Find P_0 and P_n

Soln $N = 10, \quad \lambda = 10$ per hour

$\lambda = \frac{10}{60} = \frac{1}{6}$ per min

$\mu = \frac{1}{5}$ per min.

Traffic Intensity $\rho = \frac{\lambda}{\mu} = \frac{1/6}{1/5} = \frac{5}{6}$

$\rho = 0.8334$

(i) $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}} = \dfrac{1-0.8334}{1-(0.8334)^{11}} = 0.1925$

(ii) $P_n = \dfrac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n = P_0 \rho^n$

$P_n = (0.1925)(0.8334)^n \quad [\because n = 0, 1, 2, \dots 10]$

3Q: If for a period of 2 hours in the day (8-10) am trains arrive at the yard every 20 min but the service time continues to remain 36 min, the calculate for this period

ⓐ the probability that the yard is empty, and

ⓑ the average no. of trains in the system, on the assumption that line capacity of the yard is limited to 4 trains only.

Soln

$\lambda = \dfrac{1}{20}$ per min

$\mu = \dfrac{1}{36}$ per min

$\rho = \dfrac{\lambda}{\mu} = \dfrac{1/20}{1/36} = \dfrac{36}{20} = 1.8$

$N = 4$ trains

(a) $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}} = \dfrac{1-(1.8)}{1-(1.8)^5} = 0.04$

(b) ~~L_s~~ = Average no. of train in the system

$$L_s = P_0 \sum_{n=0}^{N} n\rho^n$$

$$= (0.04) \sum_{n=0}^{4} n\rho^n$$

$$= 0.04\left[0 + \rho + 2\rho^2 + 3\rho^3 + 4\rho^4\right]$$

$$= 0.04\left[1.8 + 2(1.8)^2 + 3(1.8)^3 + 4(1.8)^4\right]$$

$$= 2.71 \simeq 3 \text{ (approx)}$$

$L_s = 3$ trains