

Discrete Structures (Mathematics)
Final Term Examination (Duration: 2Hours)

Total Marks: 60

Course Code: CMP-2111

Note: All questions carry equal marks except Q11, which of 10 marks.

Q1: Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.” (Hint: Using rules of inference to build arguments)

Solution: Let p be the proposition “You send me an e-mail message,” q the proposition “I will finish writing the program,” r the proposition “I will go to sleep early,” and s the proposition “I will wake up feeling refreshed.” Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$. We need to give a valid argument with premises $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$ and conclusion $\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

Q2: Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution: Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement “If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ ” is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false. Using the meaning of disjunction together with De Morgan’s law, we see that this implies that both $a > \sqrt{n}$ and $b > \sqrt{n}$ are false. This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. We can multiply these inequalities together (using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$) to obtain $ab > \sqrt{n} \cdot \sqrt{n} = n$. This shows that $ab \neq n$, which contradicts the statement $n = ab$.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Q3: Prove that the sum of two rational numbers is rational.

Solution: We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q , with $q \neq 0$, such that $r = p/q$, and integers t and u , with $u \neq 0$, such that $s = t/u$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = p/q$ and $s = t/u$, to obtain

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}.$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu , where $qu \neq 0$. This means that $r + s$ is rational. We have proved that the sum of two rational numbers is rational; our attempt to find a direct proof succeeded.

Q4: Give a proof by contradiction of the theorem “If $3n + 2$ is odd, then n is odd.”

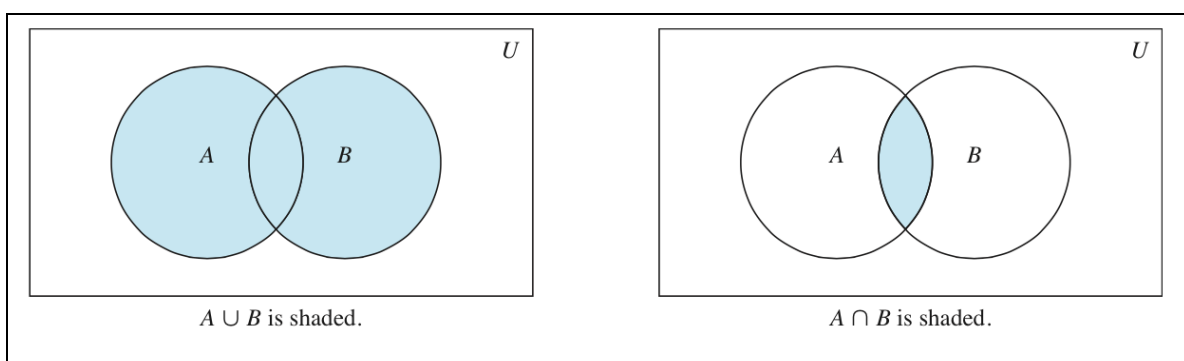
Solution: Let p be “ $3n + 2$ is odd” and q be “ n is odd.” To construct a proof by

contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n + 2$ is odd and that n is not odd. Because n is not odd, we know that it is even. Because n is even, there is an integer k such that $n = 2k$. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Because $3n + 2$ is $2t$, where $t = 3k + 1$, $3n + 2$ is even. Note that the statement “ $3n + 2$ is even” is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd.

Q5: Can you use straight triominoes to tile a standard checkerboard?

Solution: The standard checkerboard contains 64 squares and each triomino covers three squares. Consequently, if triominoes tile a board, the number of squares of the board must be a multiple of 3. Because 64 is not a multiple of 3, triominoes cannot be used to cover an 8×8 checkerboard.

Q6: Draw a Venn diagram that represents the union and intersection of A and B sets.



Q7: What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, the ordered pairs in R are $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$, and $(3,3)$.

Q8: Let A , B , and C be sets. Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Solution: We have

$$\begin{aligned}
 \overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law} \\
 &= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
 &= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
 &= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
 \end{aligned}$$

Q9: Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer. This graph is displayed in Figure 9.



Q10: Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n - 1$ years plus interest for the n th year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$.

We can use an iterative approach to find a formula for P_n . Note that

$$P_1 = (1.11)P_0, P_2 = (1.11)P_1 = (1.11)^2P_0, P_3 = (1.11)P_2 = (1.11)^3P_0$$

$$\vdots P_n = (1.11)P_{n-1} = (1.11)^nP_0.$$

When we insert the initial condition $P_0 = 10,000$, the formula $P_n = (1.11)^n 10,000$ is obtained. Inserting $n = 30$ into the formula $P_n = (1.11)^n 10,000$ shows that after 30 years the account

contains $P_{30} = (1.11)^{30} 10,000 = \$228,922.97$.

Q11: If a and r are real numbers and $r \neq 0$, then $\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$

Solution: See the lecture slides.