UNIT-V A.Meghamsh

Graph Theory

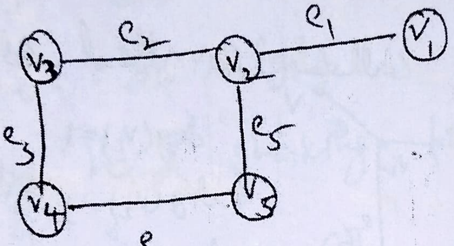
Definition: Graph is a collection of vertices 'V' and edges E. It is denoted by $G = (V, E)$

where $V \to$ set of vertices (or nodes)
$E \to$ set of edges

Line | branch $\to$ edge (e)
Point | joint $\to$ vertex (v)

Each edge has either one or two vertices associated with it, is called its end points. An edge is said to connect its end points.

Ex:-



$G = (V, E)$

where $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

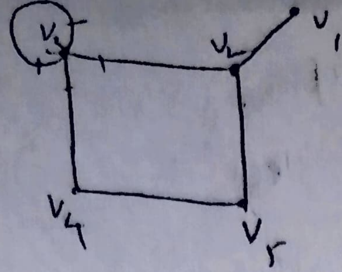
$= \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_2)\}$

$\to$ Each edge must be associated with unorder ~~an order~~ pair of vertices.

Ex: $e_1 - (v_1 v_2) / (v_2, v_1)$

Degree | valency: No. of edges associated with vertex is called degree of vertex.

Ex:-



$deg(v_1) = 1$ $\quad deg(v_4) = 2$

$deg(v_2) = 3$ $\quad deg(v_5) = 2$

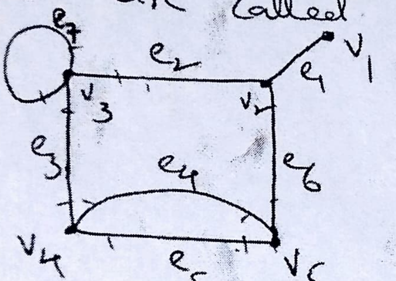
$deg(v_3) = 4$

→ Each self loop given 2 degree.

self loop | loop :- If starting & ending vertices are same then the edge is called as loop.

Parallel edges :- Both edges are having same end ~~vertex~~ vertices (or) If two or more edges associated with same end vertices are called parallel edges.

Ex:



$deg(v_1) = 1$ $\quad deg(v_4) = 3$

$deg(v_2) = 3$ $\quad deg(v_5) = 3$

$deg(v_3) = 4$

→ e_4, e_5 having same end vertices

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 1+3+4+3+3$$
$$= 14$$
$$= 2 \times 7$$
$$= 2 \times \text{no. of edges}$$

Hand Shaking Theorem :-

The hand shaking theory states that the sum of degree of all the vertices for a graph will be double the number of edges contained by that graph. The Symbolic representation of handshaking theory is described as follows.

$$\sum_{i=1}^{n} d(v_i) = 2 \times |E|$$
$$= 2 \times \text{no. of edges.}$$

Prob:- In a graph that has the degree of each vertex as 4 & 24 edges. Then find no. of vertices in this graph.

Sol^n let 'n' be the total no. of vertices

and given each vertex have degree 4

and given 24 edges

Hence By hand shaking theorem

$$\sum_{i=1}^{n} d(v_i) = 2 \times |E|$$
$$n \times \cancel{4} = 2 \times \cancel{24}^{6} \Rightarrow \boxed{n = 12}$$

∴ Thus, in graph, the no. of vertices = 12

Prob:- In a graph that has 21 edges, 3 vertices of degree 4, and all other vertices of degree 2 Now find total no. of vertices in this graph?

Given Total edges = 21

let 'n' be a total no. of vertices

3 vertices having degree 3

remaining (n-3) vertices having degree 2

Hence by Hand shaking theorem,

$3\times4 + (n-3)\times2 = 2\times21$

$12 + 2n-6 = 24 \Rightarrow 2n = 36$

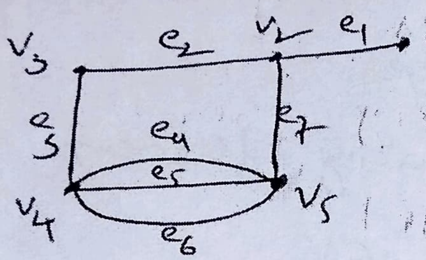
$\Rightarrow \boxed{n = 18}$

$\therefore$ Total no. of vertices = 18

Order of the Graph :- No. of vertices in the graph is called order of the graph.

Size of the graph :- No. of edges in the graph is called size of the graph.

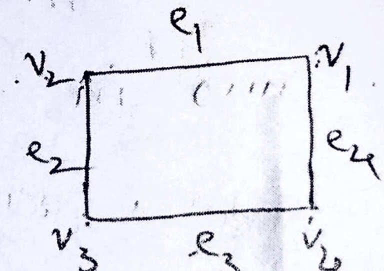
Ex :-



order = 5

size = 7

Adjacent vertex/nodes :- If there is an edge b/w two vertices,



$\left.\begin{array}{l} v_1, v_2 \\ v_2, v_3 \\ v_3, v_4 \end{array}\right\} \rightarrow$ Adjacent nodes

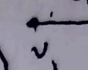
v_4, v_1

Incident edges :- An edge which joins two vertices is called Incident edge.

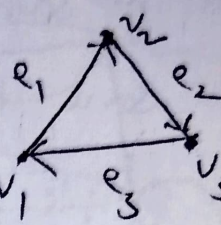
e_1 is incident edge on v_1 & v_2

e_2 " " " " v_2 & v_3

Directed Edge:- In a graph $G = (V, E)$ if an edge is associated with an ordered pair of vertices (or) if the edge contain some direction is called Directed edge.



Directed Graph:- In a graph $G = (V, E)$ if every edge is directed edge is called Directed graph.



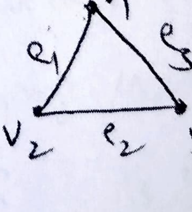
$e_1 \rightarrow (v_1, v_2)$ $e_3 \rightarrow (v_3, v_1)$

$e_2 \rightarrow (v_2, v_3)$

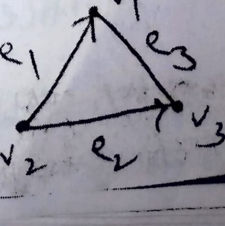
~~Un direc~~

Undirected Edge :- In a graph $G = (V, E)$ if an edge is associated with an unordered pair of vertices (or) if the edge donot do not contain any direction is called undirected edge.

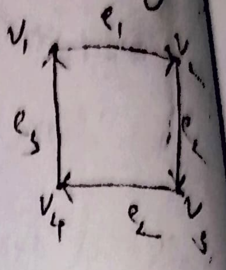
Undirected graph:- In a graph $G = (V, E)$ if every edge is undirected edge is called undirected graph.



Mixed graph:- If some edges are ~~edges are~~ directed edges and some edges are undirected edges is called mixed graph.



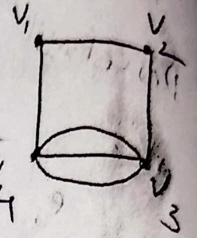
Cycle:- cycle is a collection of edges where the starting vertex and ending vertices are same.



Ex. $v_1 \to v_2 \to v_3 \to v_4 \to v_1$

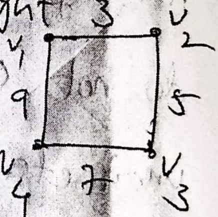
Multigraph:- A Graph $G = (V, E)$ is said to be Multigraph if contains some ||el edges

Ex



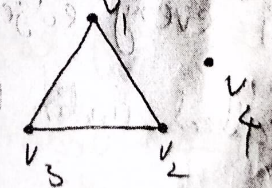
Simple graph:- If a graph doesn't contain any ||el edges is called simple graph.

Weighted graph:- In a graph $G = (V, E)$ if every edge contain some weight.



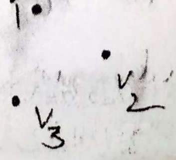
Isolated vertex:- Ex

In a graph $G = (V, E)$ if ~~an~~ ~~is~~ a vertex that has no edges, a vertex with degree zero.



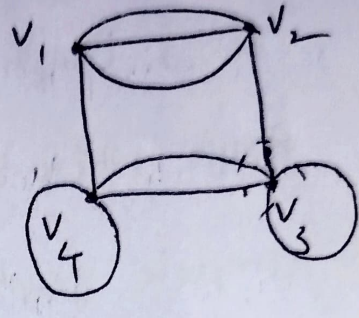
Null graph (isolated graph)

If all the vertices are isolated vertices then the graph $G = (V, E)$ is called Null graph



Degree sequence of a graph:-

Degree sequence of a graph is the list of degree of all the vertices of the graph. Usually we list the degrees in decreasing order ie from largest degree to smallest degree.



$d(v_1) = 4$, $d(v_2) = 4$

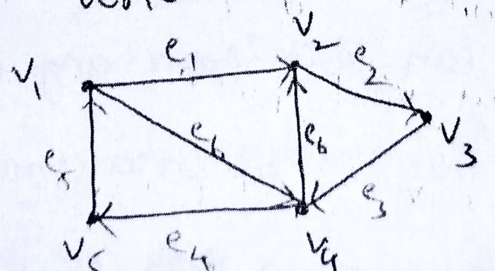
$d(v_3) = 5$, $d(v_4) = 5$

degree sequence = 5, 5, 4, 4

In-degree and out degree

In-degree of a vertex is the number of edges coming to the vertex

out-degree of a vertex is the number of edges coming which are coming out from the vertex



out degre $v_1^- = 2$

$v_2^- = 1$

$v_3^- = 1$

$v_4^- = 1$

$v_5^- = 1$

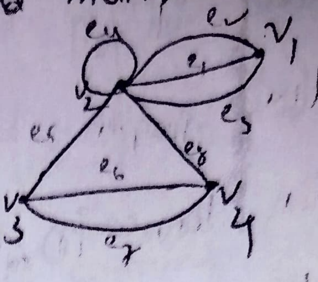
Indegree of vertex $v_1^+ = 1$ $v_3^+ = 1$

$v_2^+ = 2$ $v_4^+ = 2$

$v_5^+ = 1$

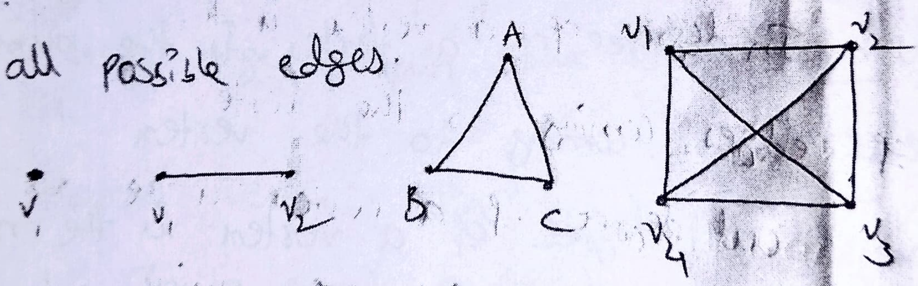
P.S.C.E

Pseudograph:- A pseudograph is a non-simple graph in which both loops and multiple edges are permitted



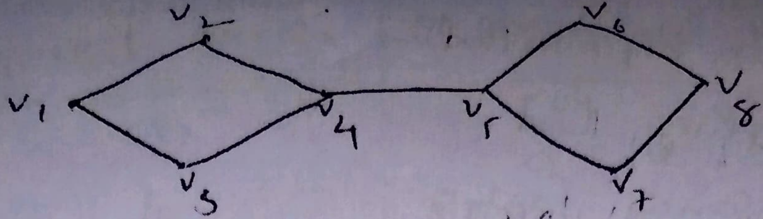
Trivial Graph:- A trivial graph is the graph which has only one vertex $\cdot v_1$

Complete graph:- A graph in which every pair of vertices is joined by exactly one edge is called complete graph. It contains all possible edges.



Connected graph:-

A connected graph is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, atleast one edge or path exists b/w every pair of vertices.



Disconnected graph:- A disconnected graph is a graph in which any path does not exist between every pair of vertices.

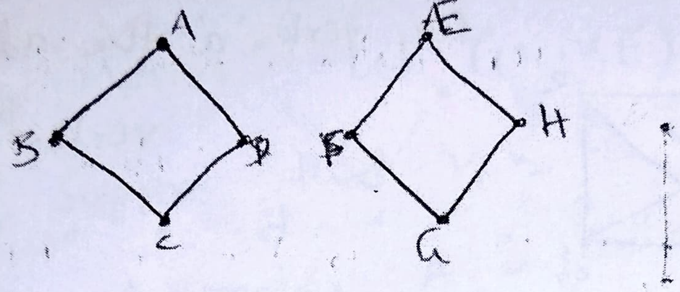


Table:-

Type	Edge	Multiple edges Allowed?	Loops Allowed?
1. Simple graph	undirected	NO	NO
2. Multi graph	undirected	yes	NO
3. Pseudograph	undirected	yes	yes
4. Directed graph	directed	NO	yes
5. Directed Multigraph	directed	yes	yes

Graph Representation

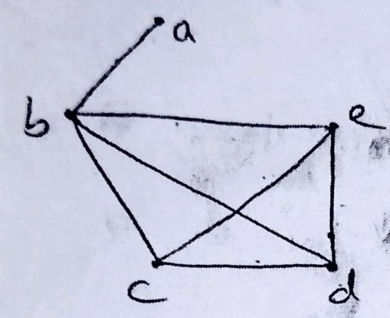
① Adjacency List

② Adjancy Matrix

③ Incidence Matrix

① Adjancy List

→ For undirected graph



For vertex 'a' the adjacency vertex is b

" " b " " a, c, d, e

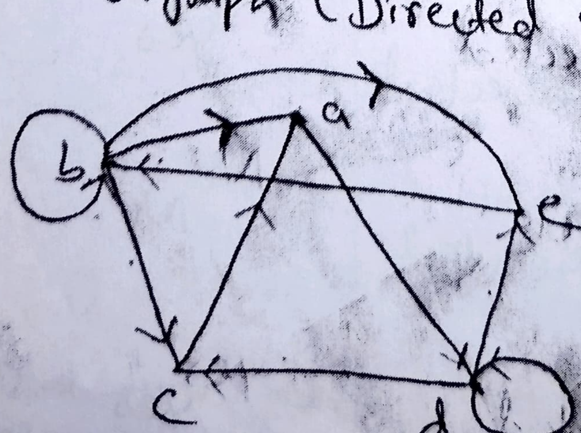
" " c " " b, d, e

" " d " " b, c, e

" " e " " b, c, d

For the Adjacency vertices means vertices connected by edge to the particular vertice.

→ For digraph (Directed graph) → (graph with direction)



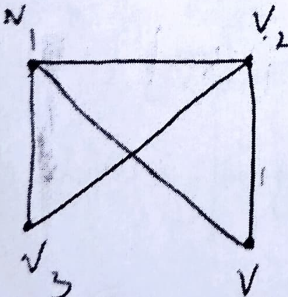
Initial vertex	terminal vertices
a	d
b	a, c, e, b
c	a
d	d, c, e
e	b

Adjancy Matrix :- Let $G=(V,E)$ be a simple graph. Suppose that $V_1, V_2, \ldots V_n$ are the vertices. A matrix A is called the adjacency matrix of G if $A = [a_{ij}]_{n \times n}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (V_i, V_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

→ if V_i, V_j have a edge b/w them

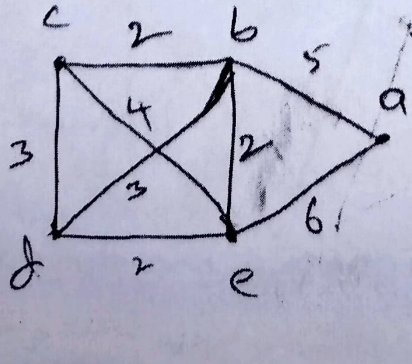
Ex :-

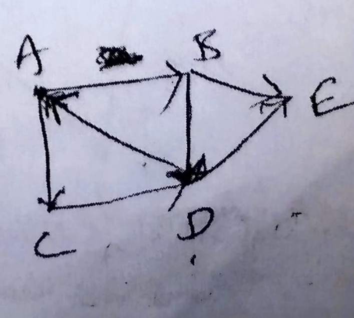


no edge b/w a_{ii}

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

H.W (1)





NOTE:-

① There are n! different adjacency matrices for a graph with n vertices

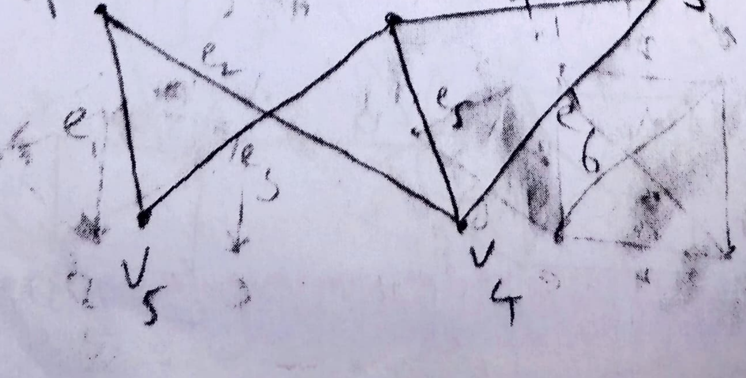
② The adjacency matrix of an undirected graph is symmetric

③ $a_{ii} = 0$ (Simple graph has no loop)

Incidence Matrix:-

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \dots v_n$ are the vertices and $e_1, e_2, \dots e_n$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1; & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$



$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 1 & 0 \\
v_3 & 0 & 0 & 0 & 1 & 0 & 1 \\
v_4 & 0 & 0 & 1 & 0 & 1 & 1 \\
v_5 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

- In this matrix, rows represents vertices and columns represents edges.

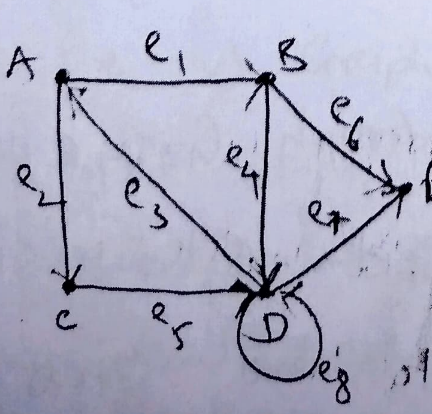
For directed graph, this matrix is filled with either 0 or 1 or -1

Here '0' represents edge is not connected to vertex

'1' represents edge is connected to outgoing edge

'-1' represents edge is connected to incoming edge.

e_1 e_2 e_3 e_4 e_5



A

$A \neq B$

C

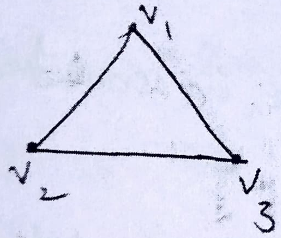
D

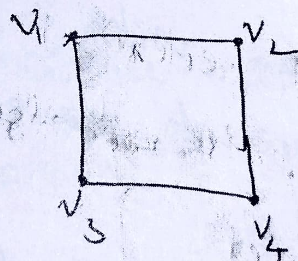
B

$$
\begin{array}{c|cccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
\hline
A & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
B & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
C & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
D & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\
E & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0
\end{array}
$$

Regular graph:- A Regular graph is a graph in which degree of all the vertices is same. If the degree of all the vertices is K, then it is called K-regular graph.

Ex:





In the above examples, all the vertices have degree 2. Therefore they are called 2-regular graph.

Bipartite graph:- A Graph $G=(V,E)$ is a Bipartite graph. If the vertex set V can be partitioned into two subset

V_1 and V_2 such that every edge in E connects to a vertex in V_1 and a vertex in V_2 (no edge in G connects either two vertices in V_1 or two vertices in V_2) is called a Bipartite graph.

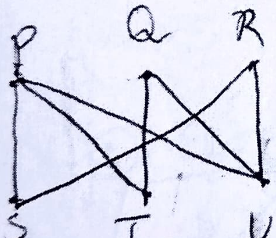
→ A Bipartite graph Graph does not contain any self loops.

→ A Bipartite graph can be denoted by $K_{m,n}$ where m contains the vertices in subset V_1

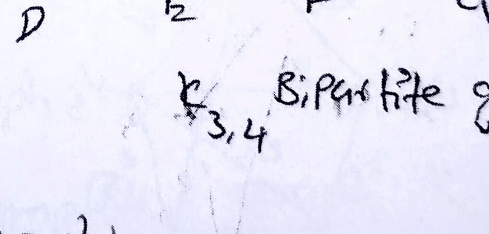
" " " " " V_2

n " "

Ex Draw $K_{3,3}$ and $K_{3,4}$ Bipartite graph.



$K_{3,3}$ Bipartite graph

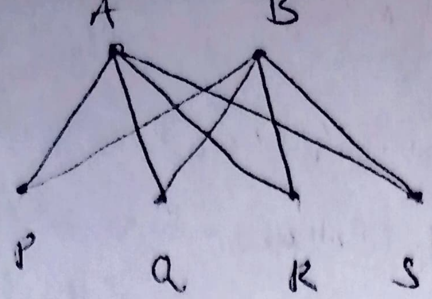


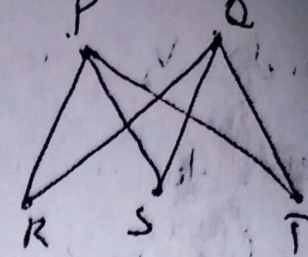
$K_{3,4}$ Bipartite graph

Complete Bipartite graph:-

A Graph $G = (V, E)$ is a complete Bipartite graph, if the vertex set V can be partitioned into two subsets V_1 and V_2 such that every vertex in V_1 subset is

Ex:





$K_{2,4}$ complete Bipartite graph

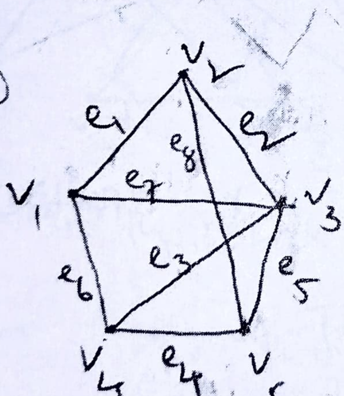
$K_{2,3}$ C.B.G

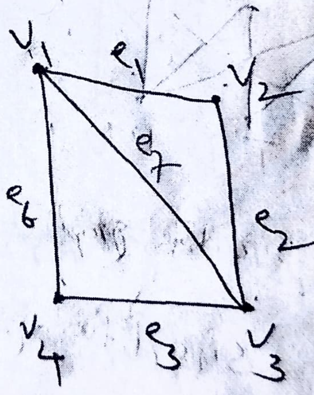
Sub Graph :- Given two Graphs $G=(V,E)$ and $G_1=(V_1,E_1)$, we say that G_1 is a sub graph of G if the following conditions hold.

→ ① All the vertices and all the edges of G_1 are in G i.e $V_1 \subseteq V$ and $E_1 \subseteq E$

→ ② Each edge of G_1 has the same end vertices in G as in G_1

Ex: ①





Graph- $G=(V,E)$

Graph $G_1=(V_1,E_1)$

$V = \{V_1, V_2, V_3, V_4, V_5\}$

$E = \{e_1, e_2, \ldots e_8\}$

$V_1 = \{V_1, V_2, V_3, V_4\}$

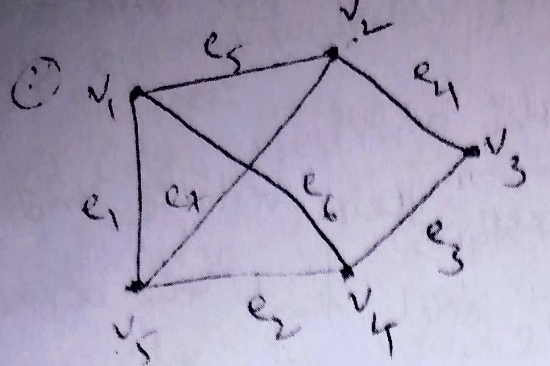
$E_1 = \{e_1, e_2, e_3, e_6, e_7\}$

$V_1 \subseteq V$
$E_1 \subseteq E$
$\}$ 1st condition is satisfied

→ Each edge of G_1 has same end vertices in G

$\therefore G_1=(V_1,E_1)$ is a sub graph of $G(V,E)$

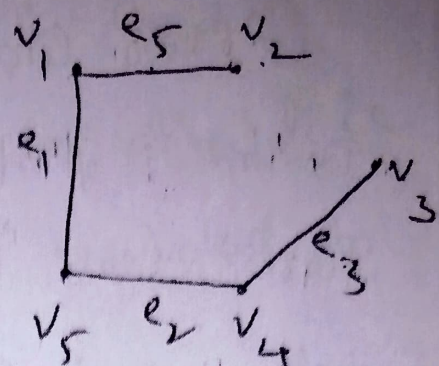
(ii)



Graph $G=(V,E)$

$V=\{v_1,v_2,v_3,v_4,v_5\}$

$E=\{e_1,e_2,\dots e_7\}$



Graph $G_1=(V_1,E_1)$

$V_1=\{v_1,v_2,v_3,v_4,v_5\}$

$E_1=\{e_1,e_2,e_3,e_5\}$

$\therefore V_1\subseteq V,\ E_1\subseteq E \to 1^{st}$ condition is satisfied

(2) Each egde of G_1 has the same end vertices in G

$\therefore G_1=(V_1,E_1)$ is a sub graph of $G=(V,E)$

NOTE:- (1) Every Graph is a subgraph of itself.

(2) Every simple graph of n vertices is a subgraph of the complete graph K_n.

(3) If G_1 is a sub graph of a graph G_2 and G_2 " " " " G

then G_1 is a sub graph of G

Graph Isomorphism :-

Two Graphs G_1 and G_2 are said to be Isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that the adjancency of vertices is preserved. Such Graphs will have the same structure, differing only in the way their vertices and edges are labelled or only in the way they are represented geometrically.

When G_1 and G_2 are isomorphic we can write it as $G_1 \cong G_2$

Two Graphs G_1 & G_2 are said to be isomorphic if they satisfies following condition.

→ ① Both graphs must contain same no. of vertices

→ ② " " " " " " Edges

→ ③ " " " " " " degree sequence

→ ④ One-to-one correspondence between the vertices of 2 graphs must be same. i.e

for. e

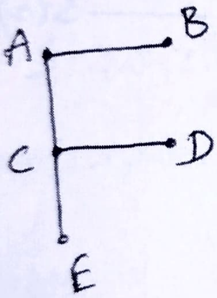
b

for every vertex in Graph 1 there should be an equivalent vertex in Graph 2.

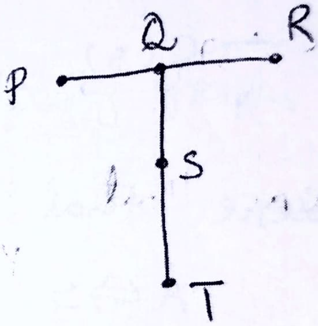
→ ⑤ Edge preserving should be satisfied. i.e each edge in Graph 1 is equivalent to an edge in Graph 2.

→ ⑥ Adjacency matrix of both graphs must be same.

Prob: Check whether the following Graphs are Isomorphic or not.



Graph $G_1 = (V_1, E_1)$.



Graph $G_2 = (V_2, E_2)$

① Number of vertices = 5

② No. of edges = 4

③ Degree sequence

$(A, B, C, D, E) = (2, 1, 3, 1, 1)$

D.S $= 3, 2, 1, 1, 1$

① No. of vertices = 5

② No. of edges = 4

③ Degree sequence

$(P, Q, R, S, T) = (1, 3, 1, 2, 1)$

D.S $= 3, 2, 1, 1, 1$

(ii) One-to-One mapping (ii) One-to-One mapping

A (2) — B (1)
A (2) — C (3)

P (1) — Q (3)

Q (3) — P (1)
Q (3) — R (1)
Q (3) — S (2)

B (1) — A (2)

C (3) — A (2)
C (3) — D (1)
C (3) — E (1)

R (1) — Q (3)

S (2) — T (1)
S (2) — Q (3)

D (1) — C (3)

(1) E — C (3)

T (1) — S (2)

We observe that

A ⇔ S

B ⇔ T

C ⇔ Q

D ⇔ P

E ⇔ R

This is called one-to-one mapping b/w G_1 & G_2

(c) Edge ~~mapping~~ Preserving property (c) Edge Preserving Property.

A - B ⇔ S - T

A - C ⇔ S - Q

C - D ⇔ Q - P

C - E ⇔ Q - R

P - Q ⇔ D - C

Q - R ⇔ C - E

Q - S ⇔ C - A

S - T ⇔ B - A

⑥ Adjancy matrix (6)

6 Adjany matrix

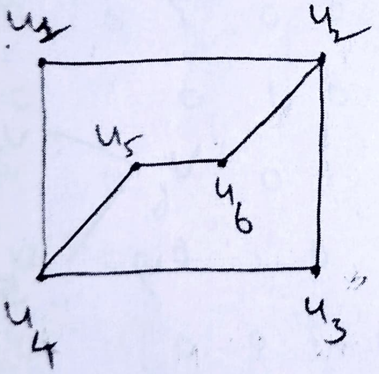
$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \hline \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(6) Adjany matrix

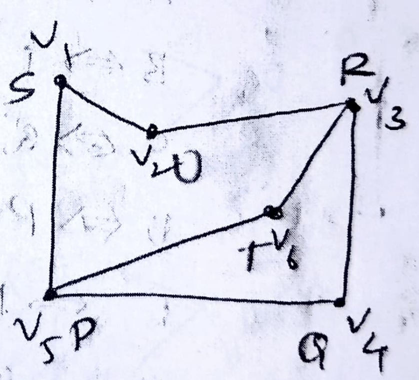
$$\begin{array}{c|c} & \begin{array}{ccccc} S & T & Q & P & R \end{array} \\ \hline \begin{array}{c} S \\ T \\ Q \\ P \\ R \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore$ G_1 & G_2 are isomorphic Graphs i.e $G_1 \cong G_2$

Prob:- Show that the following graphs are isomorphic.



Graph $G_1 = (V_1, E_1)$



Graph $G_2 = (V_2, E_2)$

① No. of vertices = 6

② No. of edges = 7

③ Degree sequence

$(u_1, u_2, u_3, u_4, u_5, u_6) = (2, 3, 2, 3, 2, 2)$

① No. of vertices = 6

② No. of edges = 7

③ Degree sequence

$(v_1, v_2, v_3, v_4, v_5, v_6) = (2, 2, 3, 2, 3, 2)$

D.S = 3,3,2,2,2,2

(4) One-to-one mapping

u_1 (2): u_2 (3), u_4 (3)

u_2 (3): u_1 (2), u_6 (2), u_3 (2)

u_3 (2): u_2 (3), u_4 (3)

u_4 (3): u_1 (2), u_3 (2), u_5 (2)

u_5 (2): u_4 (3), u_6 (2)

u_6 (2): u_2 (3), u_5 (2)

D.S = 3,3,2,2,2,2

(4) One-to-one mapping

v_1 (2): v_2 (2), v_5 (3)

v_2 (2): v_1 (2), v_3 (3)

v_3 (3): v_2 (2), v_4 (2), v_6 (2)

v_4 (2): v_3 (3), v_5 (3)

v_5 (3): v_1 (2), v_4 (2), v_6 (2)

v_6 (2): v_3 (3), v_5 (3

We obsere that $u_1 \Leftrightarrow v_4$

$u_2 \Leftrightarrow v_3$

$u_3 \Leftrightarrow v_6$

$u_4 \Leftrightarrow v_5$

$u_5 \Leftrightarrow v_1$

$u_6 \Leftrightarrow v_2$

(5) Edge preserving property

$u_1 - u_2 \Leftrightarrow v_4 \; v_3$

$u_2 - u_3 \Leftrightarrow v_3 - v_6$

$u_3 - u_4 \Leftrightarrow v_6 - v_5$

$u_4 - u_1 \Leftrightarrow v_5 - v_4$

$u_4 - u_5 \Leftrightarrow v_5 - v_1$

$u_5 - u_6 \Leftrightarrow v_1 - v_2$

$u_6 - u_2 \Leftrightarrow v_2 - v_3$

(5) Edge preserving Property.

$v_1 - v_2 \Leftrightarrow u_5 - u_6$

$v_2 - v_3 \Leftrightarrow u_6 - u_2$

$v_5 \; v_4 \Leftrightarrow u_2 - u_1$

$v_4 - v_5 \Leftrightarrow u_1 - u_4$

$v_5 - v_1 \Leftrightarrow u_4 - u_5$

$v_5 - v_6 \Leftrightarrow u_4 - u_3$

$v_6 - v_3 \Leftrightarrow u_3 - u_2$

(6) Adjacency matrix

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(6) Adjacency matrix

$$\begin{array}{c} \\ v_4 \\ v_3 \\ v_6 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_4 & v_3 & v_6 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

∴ G_1 & G_2 satisfies all the conditions,

hence, G_1 & G_2 are isomorphic graphs

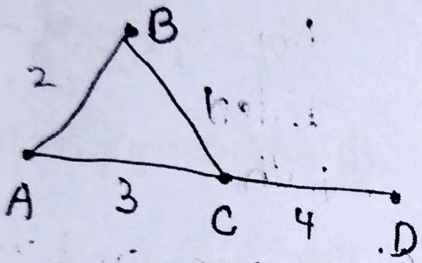
Eulerpath: The Euler path is a path by which i) we can visit every edge of the graph exactly once

ii) Starting and ending vertices are different

iii) Vertices are repeated but an edge can't be repeated

iv) At most or max 2 vertices have odd

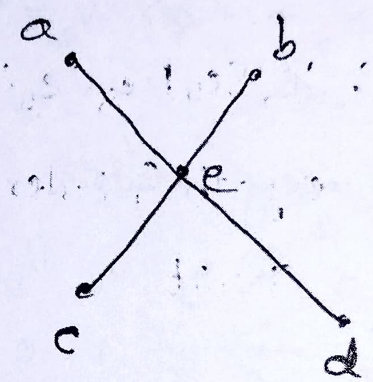
Ex:



C1B2A3C4D

or

D4C3A2B1C



No Euler Path

Euler Circuit:

→ Visit each edge of graph exactly once

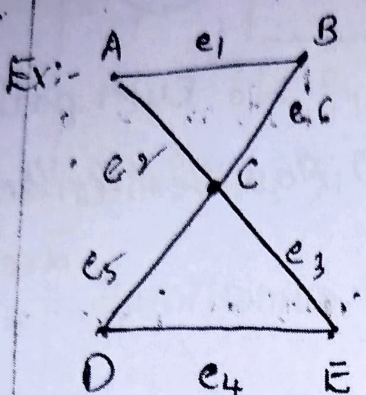
→ Vertex can be repeated; but an edge can't be repeated

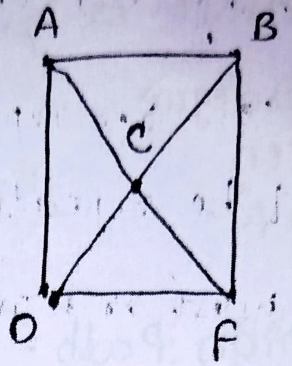
→ Starting & ending vertices are same

→ Each Vertex edge is even

Euler graph: A graph is said to be Euler graph if it contains Euler circuit.

Ex:-



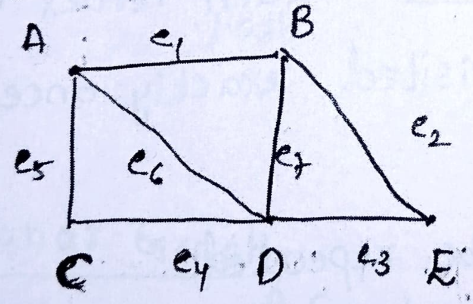


$A e_2 C e_3 E e_4 D e_5 C e_6 B e_1 A$

Euler path & Euler circuit.

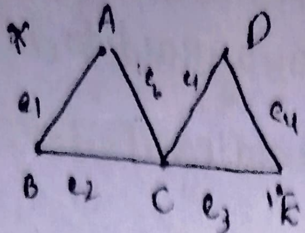
No Euler path.

(because we can't visit each edge exactly once)



$A e_1 B e_2 E e_3 D e_4 C e_5 A e_6 D e_7 B$

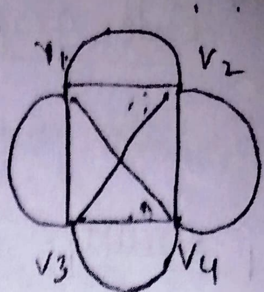
Euler path exists but not Euler circuit



Euler circuit

All vertices degree is even

5.



No Euler path

(all vertices degree are odd)

Hamiltonian Path:

→ Each vertex of the graph will visited exactly once

→ An edge can't be repeated

→ Start and end vertex must be different

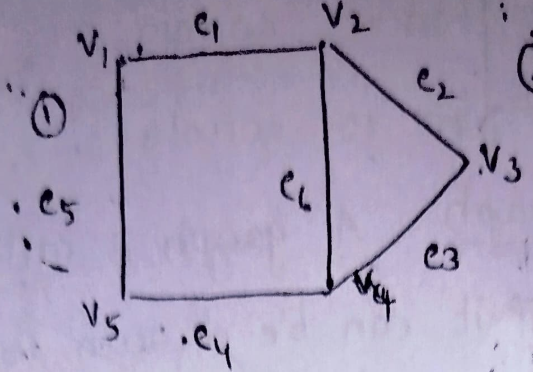
Hamiltonian Circuit: Each vertex of the graph will be visited exactly once except start vertex.

→ An edge can be repeated

→ Start and end vertex must be same

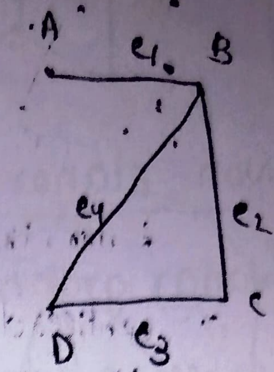
Hamiltonian graph: A graph is said to be hamiltonian graph if it contains hamiltonian circuit

Ex:

① 

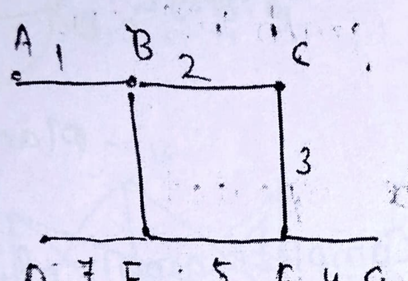
$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5 e_5 V_1$

Hamiltonian Circuit

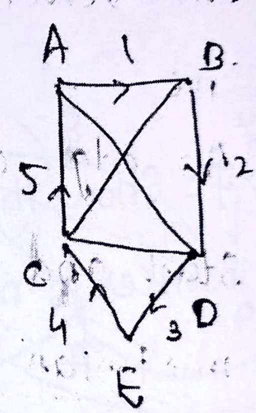
② 

$A e_1 B e_2 C e_3 D$

Hamiltonian path

③ 

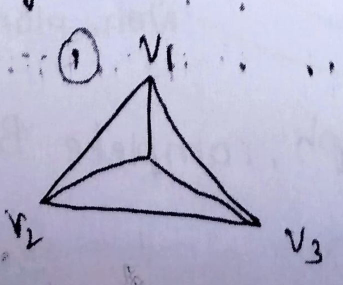
No Hamiltonian path

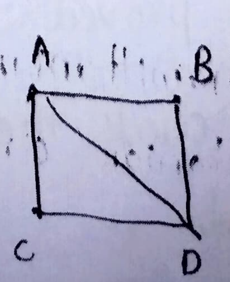
④ 

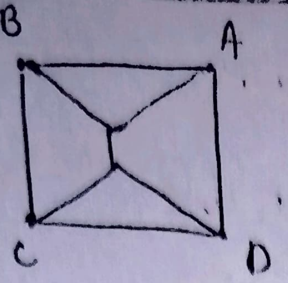
A 1 B 2 D 3 E 4 C 5 A

Hamiltonian circuit

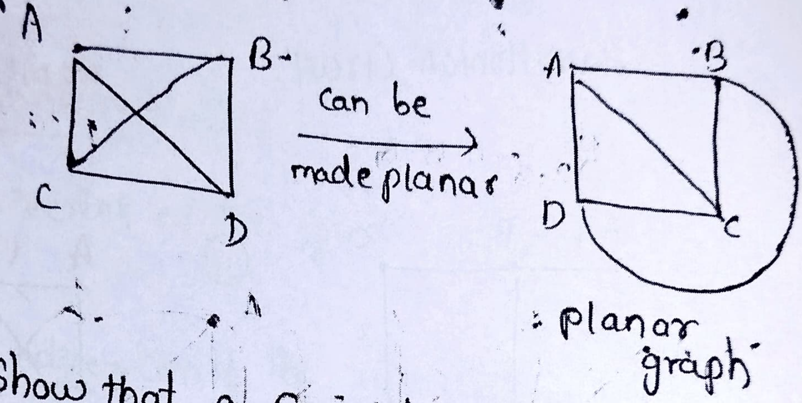
Planar Graph: A Graph is called planar graph if it can be drawn without any crossing edges

Ex. ① 

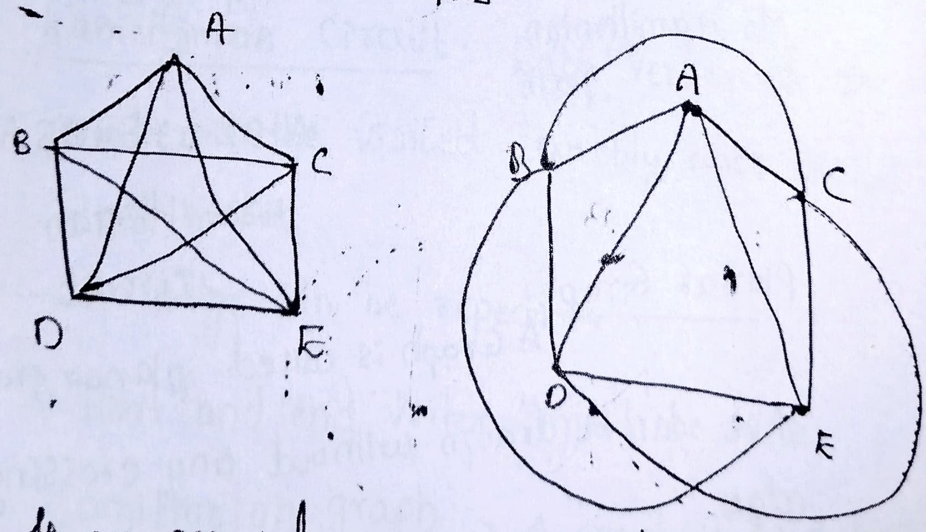
② 

3. 

Non-planar graph: A graph is called Non planar graph if it can be drawn with any crossing edges.

Ex: 

Proof: Show that a complete graph of 5 vertices is non-planar



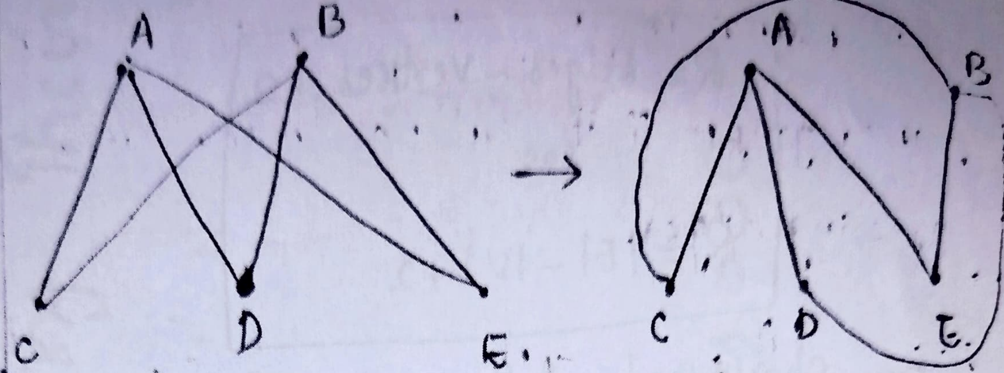
Hence proved. Non-planar

Prob: check the graph, complete Bipartite Graph.

① $K_{2,3}$ is planar or not.

② $K_{3,3}$ is planar or not.

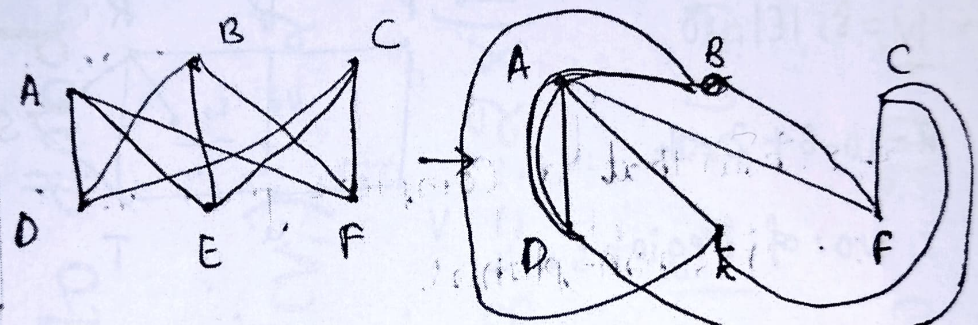
①



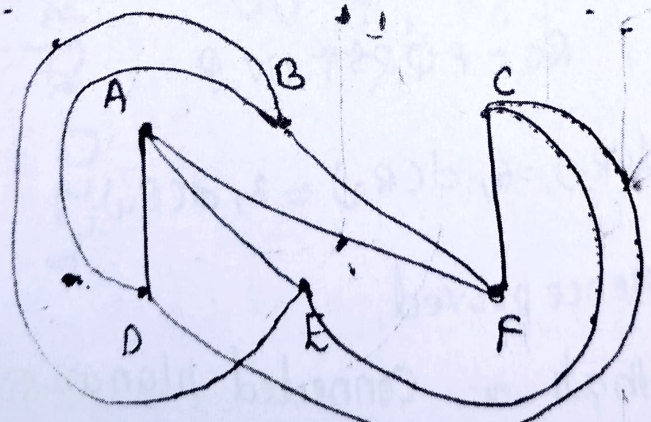
$K_{2,3}$ - C.B.G.

Planar Graph

2.



Non-planar

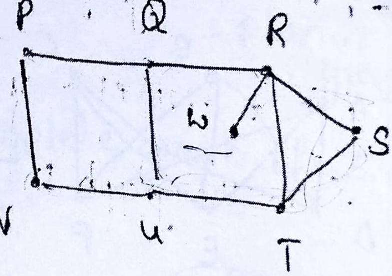


Euler's Formula: let G be a connected planar graph with edges and vertices. Let R be the no. of regions in the planar Graph of G

$$\boxed{\begin{array}{c} R = \text{Edges} - \text{vertices} + 2 \\ \text{(or)} \\ R = |E| - |V| + 2 \end{array}}$$

Ex:- Show that following graph is verified by Euler's Formula.

$|V| = 8$, $|E| = 10$



$R = 10 - 8 + 2 = 4$

$\therefore$ No. of Regions $= 4$

R_1 = PQVUP , R_2 = QRWRTUQ

R_3 = RSTR , R_4 = PQRSTUVP

$d(R_1) = 4$, $d(R_2) = 6$, $d(R_3) = 3$, $d(R_4) = 7$

$\therefore$ Hence proved.

Prob: Suppose that a connected planar simple graph has 20 vertices each of degree 3. How many regions does a representation of this

Sol: According to Hand Shaking Theorem,

$$\sum_{i=1}^{n} d(v_i) = 2 \times |E|$$

$$20 \times 3 = 2 \times |E|$$

$$|E| = 30 \Rightarrow \text{edges} = 30$$

$$R = 30 - 20 + 2$$

$$R = 12$$

Proof for Euler's Formula:-

Euler Formula: For a planar graph G

$$V - E + R = 2$$

V = No. of vertices

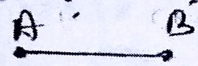
E = No. of Edges

R = No. of Regions

We have to prove Euler's formula using mathematical Induction.

Step 1: Basis of Induction:

Prove that it is True for $n = 1$ i.e, $e = 1$



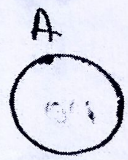
$V = 2$

$r = 1$

$V - e + r = 2$

$2 - e + 1 = 2$

$e = 1$



$V = 1$

$r = 2$

$V - e + r = 2$

$1 - e + 2 = 2$

$e = 1$

$\therefore$ Hence $e = 1$ is true.

Step 2: Induction Hypothesis:

Assume true for $e = k$

$V - k + r = 2$

$V + r = k + 2$ ——— (1)

Step 3: Induction Step

Prove that it is true for $e = K+1$.

Case 1: If a graph G has vertex of degree '1' then remove the vertex of degree '1' with the associated edge denoted in the result by G'.

No. of edges in $G' = e-1$

No. of vertices in $G' = v-1$

No. of regions in G = No. of regions in $G' = r$

There is no change in r

We have $e = K$

using Induction Hypothesis,

$$v + r = K+2$$

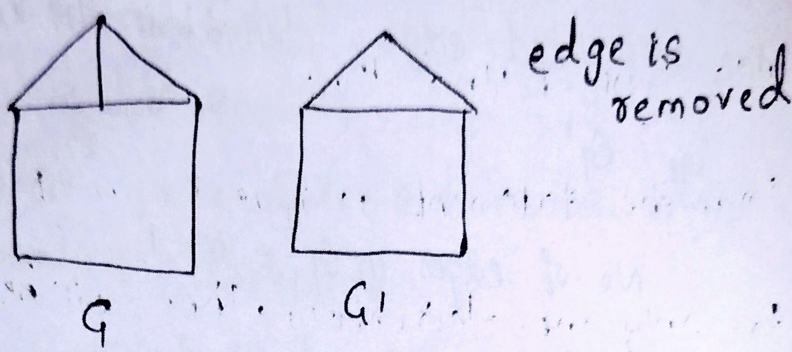
$$(v-1) + r = (e-1) + 2$$

$$v - e + r = 2$$

$e = K+1$ is true.

Using mathematical Induction we proved Euler's formula $v - e + r = 2$

Case 2: If a Graph G has no vertex of degree 1 that means the Graph G is finite, then remove one edge common to adjacent finite regions and denote in result by G'



No. of edges $= e - 1$; No. of regions $= r - 1$

No. of vertices $= v$

we have $e = K$

Using Induction Hypothesis;

$v + r = K + 2$

$v + (r - 1) = (e - 1) + 2$

$v + r - 1 = e + 2 - 1$

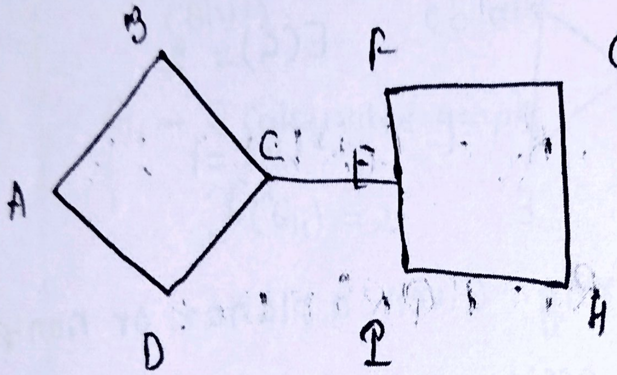
$v - e + r = 2.$

$e = K + 1$ is true

Using Mathematical Induction we proved Euler's formula $V - e + r = 2$

Hence Euler's formula is proved

Connectivity: To measure the connectedness of a graph 'G', we consider the minimum no. of vertices and edges to be removed from the graph in order to disconnect it.

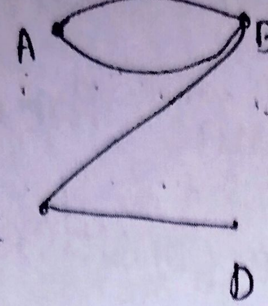


$E(G) = 1$

$V(G) = 1$

Edge Connectivity: It is the minimum no. of edges whose removal results in a disconnected graph. It is denoted by $E(G)$.

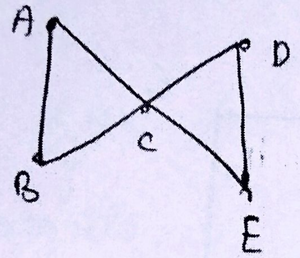
Ex:



E(G) = 1, V(G) = 1

Vertex Connectivity : It is the minimum no. of vertices whose removal result is a disconnected graph. It is denoted by V(G)

Ex:



E(G) = 2

V(G) = 1

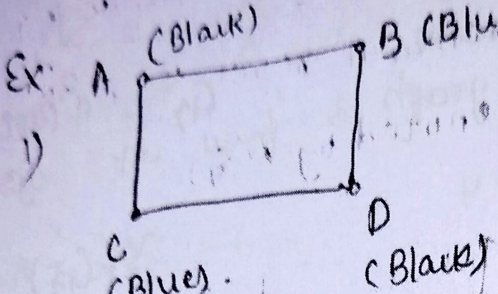
Graph-colouring: Given a planar or non-planar graph. If we assign colours to it's vertices in such a way that no 2 adjacent vertices have the same colour, then we can say that the graph 'G' is properly coloured.

Proper colouring of a graph means assigning colours to its vertices such that adjacent vertices have different colours.

Chromatic Number: The minimum no. of colour required to colour all the vertices of a given graph is called a chromatic number of a given graph.

Ex:

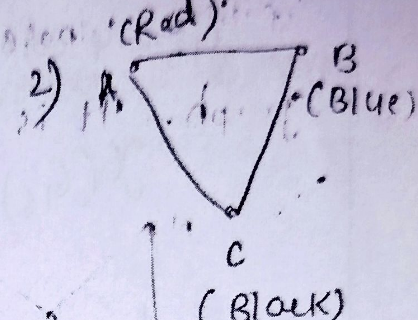
1)



G_1 - 2 colourable graph

$\chi(G_1) = 2$

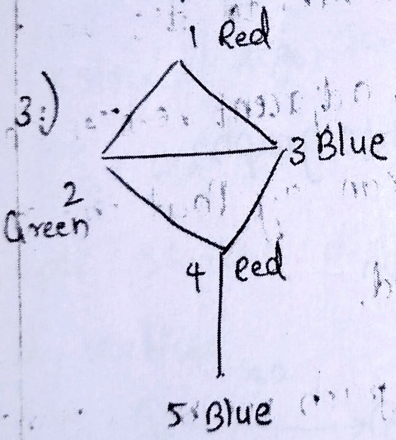
2)



G_2 - 3 colourable Graph

$\chi(G_2) = 3$

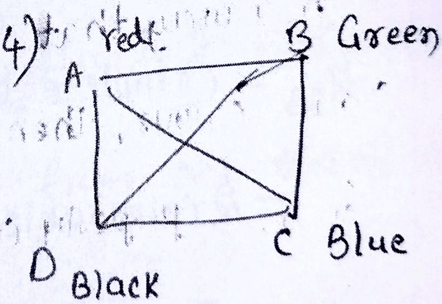
3)



G_3 = 4 colourable graph

$\chi(G_3) = 4$

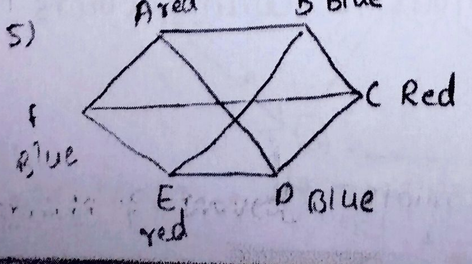
4)



G_4 = 4 colourable graph

$\chi(G_4) = 4$

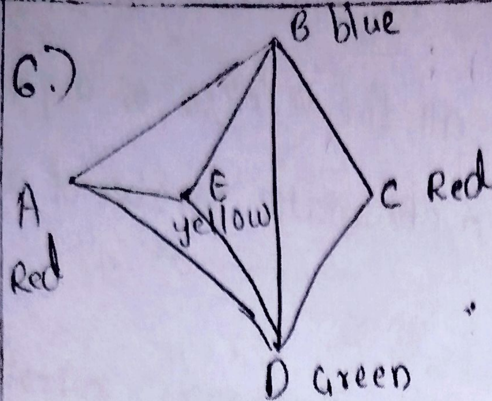
5)



G_5 - 2 colourable graph

$\chi(G_5) = 2$

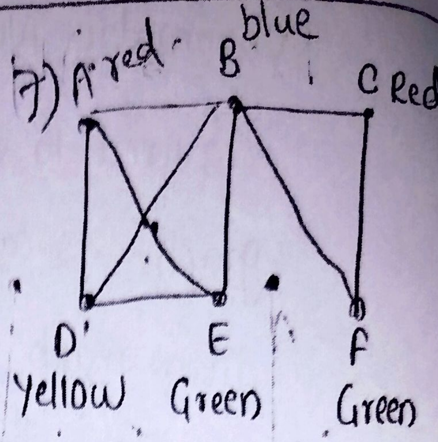
6.)



G_6 – 4 colourable graph

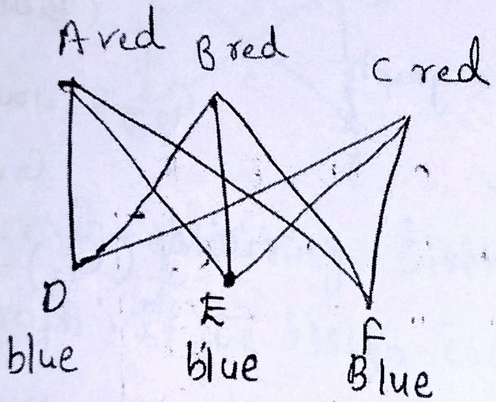
$\chi(G_6) = 4$

7)



G_7 – 4 colourable graph

$\chi(G_7) = 4$



K_{33} – complete bipartite graph

– 2 colourable graph

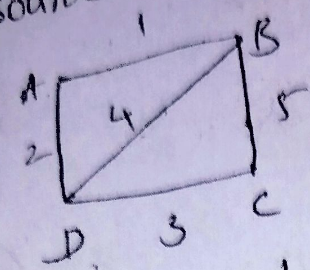
$\chi(K_{3,3}) = 2$

Note: A Graph G is said to be K-colourable, if we can properly colour it with K colors.

Shortest path problems:

Let G = (V, E) be a weighted graph. Now we can find shortest path from a source vertex to destination vertex.

Ex:



Now you want to go vertex A to vertex C

A - B - C = 6

A - D - C = 5

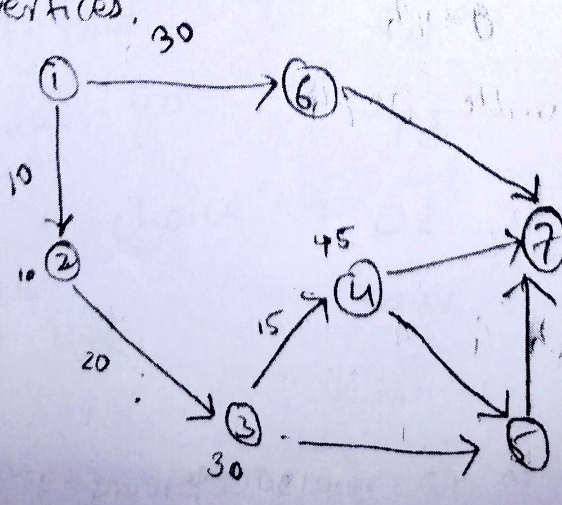
A - B - D - C = 8

A - D - B - C = 11

∴ shortest path is A - D - C

Dijkstra's Algorithm

Let G = (V, E) be directed weighted graph. Shortest distance from one vertex to all vertices.



selected vertex	visited set	d(2)	d(3)	d(4)	d(5)	d(6)	d(
1	{1}	10 min	∞	∞	∞	30	∞
2	{1, 2}	10	30 min	∞	∞	30	∞
3	{1, 2, 3}	10	30	45	35	30 min	∞
6	{1, 2, 3, 6}	10	30	45	35 min	30	65
5	{1, 2, 3, 6, 5}	10	30	45	35	30	42 min
7	{1, 2, 3, 6, 5, 7}	10	30	45 min	35	30	42
4	{1, 2, 3, 6, 5, 7, 4}	10	30	45	35	30	42

vertex	S.D
2	10
3	30
4	45
5	35
6	30
7	42