DISCRETE MATHEMATICS

[MR22-1BS0107]

DIGITAL NOTES

B.TECH II YEAR – I SEM (R22)

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DEPARTMENT OF

COMPUTER SCIENCE AND ENGINEERING

MALLA REDDY UNIVERSITY

(Telangana state private universities Act No.13 of 2020 and G.O.Ms.No.14, Higher Education (UE) Department)

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UNIT-I: Binary Relations on Sets

Syllabus

Binary Relations, Equivalence Relations, Partial Order Sets, Total Order Sets, Lattices, Hasse Diagrams, Functions, Binary and n-ary Operations.

Self-study: Transitive Closure, Fundamental theorem of Arithmetic.

Faculty -- **Dr. Sridhar V**

Sets

Definition: "A set is a well-defined collection of objects." These objects are called elements or members of the set.

So, we can say that it is the collection of all such type of elements or objects which satisfy some rule and it is possible to say whether a particular object belongs to the collection or not.

We write $a \in A$ to denote that "a is an element of the set A". Similarly, $a \notin A$ is used to denote that "a is not an element of the set A". It is common for sets to be denoted by using uppercase letters. Lowercase letters are usually used to denote elements of sets.

Examples: 1. The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

2. The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

3. Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a, 2, \text{Fred}, \text{New Jersey}\}$ is the set containing the four elements a, 2, Fred, and New Jersey.

4. The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

Remark: Sets can have other sets as members. For example, the set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers.

Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0,1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.

Notations: There are five ways used to describe a set.

1. Describe a set by describing the properties of the members of the set **(Set builder notation)**.
2. Describe a set by listing its elements **(Roster method).**
3. Describe a set A by its characteristic function, defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all x in U where U is the universal set, sometimes called the "universe of discourse" or just the "universe" which is a fixed specified set describing the context for the duration of the discussion.

4. Describe a set by a recursive formula. This is to give one or more elements of the set and a rule by which the rest of the elements of the set may be generated.
5. Describe a set by an operation (such as union, intersection, complement, etc.) on some other sets.

Problem: Describe the set containing all the nonnegative integers less than or equal to 5.
Solution: Let A denote the set. Then the set A can be described in the following ways:

1. $A = \{x \mid x$ is a nonnegative integer less than or equal to $5\}$.
2. $A = \{0,1,2,3,4,5\}$.
3. $\mu_A(x) = \begin{cases} 1 & \text{for } x = 0,1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$
4. $A = \{x_{i+1} = x_i + 1,\ i = 0,1,2,3,4, \text{ where } x_0 = 0\}$.

Remark: Sometimes in roster method, a set is described without listing all its members. Some members of the set are listed, and then ***ellipses*** **(. . .)** are used when the general pattern of the elements is obvious.

Subset: Let *A* and *B* be two sets. Then *A* is said to be a subset of *B* if every element of *A* is an element of *B*.

If *A* is a subset of *B*, we say *A* is contained in *B*. Symbolically, we write $A \subseteq B$.

Proper Subset: *A* is said to be a proper subset of *B* if *A* is a subset of *B* and there is at least one element of *B* which is not in *A*.

If *A* is a proper subset of *B*, then we say A is strictly contained in *B*. Symbolically, we write $A \subset B$.

Properties: Let *A*, *B*, and *C* are sets. Then we have
1. $A \subseteq A$.
2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
3. If $A \subseteq B$ and $B \subset C$ then $A \subset C$.
4. If $A \subseteq B$ and $A \not\subset C$ then $B \not\subset C$.

Equal Sets: Two sets *A* and *B* are equal iff $A \subseteq B$ and $B \subseteq A$. We write $A = B$.

Result: To show that two sets *A* and *B* are equal, we must show that each element of *A* is also an element of *B*, and conversely.

Null Set or Empty Set: A set containing no elements is called the empty set or null set, denoted by ϕ.

For example, given the universal set *U* of all positive numbers, the set of all positive numbers *x* in *U* satisfying the equation $x + 1 = 0$ is an empty set since there are no positive numbers which can satisfy this equation.

Note: The empty set is a subset of every set.

Note: It is important to note that the sets ϕ and $\{\phi\}$ are very different sets. The former has no elements, whereas the latter has the unique element ϕ.

Singleton: A set containing a single element is called a singleton.

Operations on Sets

There are several operations on set. A few important ones are discussed in this section. Consider U as the universal set and now we define the following operations.

Absolute Complement: Let A be any subset of U. The absolute complement of A is defined as $\overline{A} = \{x : x \notin A\}$.

Relative Complement: If A and B are sets, the relative complement of A with respect to B is given as $B - A = \{x : x \in B \text{ and } x \notin A\}$.

Note: $\overline{\phi} = U$, $\overline{U} = \phi$ and the complement of the complement of A is equal to A i.e. $\overline{\overline{A}} = A$.

Union: Let A and B be two sets. The union of A and B is given as

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}.$$

More generally, if $A_1, A_2, ..., A_n$ are sets, then their union is the set of all objects which belong to at least one of them, and is denoted by

$$A_1 \cup A_2 \cup ... \cup A_n \text{ or } \bigcup_{j=1}^{n} A_j.$$

Intersection: Let A and B be two sets. The intersection of A and B is given as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

More generally, if $A_1, A_2, ..., A_n$ are sets, then their intersection is the set of all objects which belong to every one of them, and is denoted by

$$A_1 \cap A_2 \cap ... \cap A_n \text{ or } \bigcap_{j=1}^{n} A_j.$$

Basic Properties	**Union**	**Intersection**
Idempotent	$A \cup A = A$	$A \cap A = A$
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$

Remark: It should be noted that, *in general*,

$$(A \cup B) \cap C \neq A \cup (B \cap C).$$

Symmetrical Difference or Boolean Sum: The symmetrical difference of two sets A and B is given as

$$A \Delta B = \{x : x \in A, \text{or } x \in B, \text{but not both}\}.$$

Disjoint Sets: Two sets A and B are said to be disjoint if they do not have a member in common.

Mathematically, $A \cap B = \phi$.

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Distributive Laws: Let A, B and C are three sets. Then,

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B),$$
$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B).$$

Power Set: Let A be a given set. The power set of A, denoted by $P(A)$, is the family of sets such that $X \subseteq A$ iff $X \in P(A)$.

Symbolically,

$$P(A) = \{X : X \subseteq A\}.$$

Example: Let $A = \{a, b, c\}$. Then the power set of A is

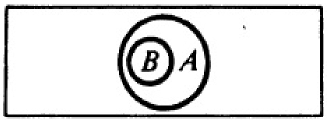
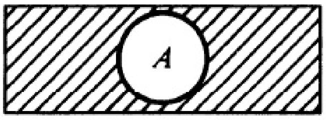
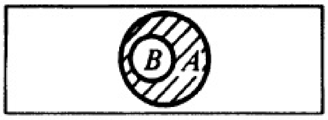
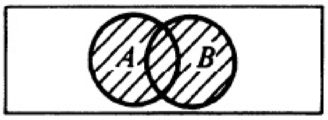
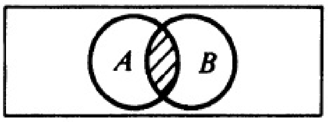
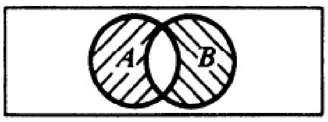
$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}.$$

Venn Diagram

It is often helpful to use Venn diagram [after JohnVenn (1834-1883)], to visualize the various properties of the set operations. The main points of Venn Diagram are:

- The universal set is represented by a large rectangular area.
- Subsets within this universe are represented by circular areas.

A summary of set operations and their Venn diagrams is given below:

Set Operation	Symbol	Venn Diagram
Set B is contained in set A	$B \subset A$	
The absolute complement of set A	$\bar{A}$	
The relative complement of set B with respect to set A	$A - B$	
The union of sets A and B	$A \cup B$	
The intersection of sets A and B	$A \cap B$	
The symmetrical difference of sets A and B	$A \Delta B$	

De Morgan's Laws: Let *A* and *B* be two sets. Then,

$$\overline{(A\cup B)}=\overline{A}\cap\overline{B}$$

$$\overline{(A\cap B)}=\overline{A}\cup\overline{B}$$

Note: The proof of the above laws can be given with the help of Venn diagram.

Problem: Prove that empty set is unique.

Proof: Let there are two empty sets, ϕ_1 and ϕ_2.

Since ϕ_1 and ϕ_2 are included in every set.

Therefore, $\phi_1 \subseteq \phi_2$ and $\phi_2 \subseteq \phi_1$

$\Rightarrow$ $\phi_1 = \phi_2$. Hence Proved.

Problem: Let $U=\{1,2,3,4,5\}$, $A=\{1,5\}$, $B=\{1,2,3,4\}$. Find the following sets.

(a) $A\cap\overline{B}$

(b) $\overline{A}\cup\overline{B}$

Solution: (a) $A\cap\overline{B}=\{1,5\}\cap(U-B)$

$=\{1,5\}\cap\{5\}$

$=\{5\}$

(b) $\overline{A}\cup\overline{B}=(U-A)\cup(U-B)$

$=\{2,3,4\}\cup\{5\}$

$=\{2,3,4,5\}$

Multiset

In mathematics, a multiset (or bag, or mset) is a modification of the concept of a set that, unlike a set, allows for multiple instances for each of its elements.

The number of instances given for each element is called the **multiplicity** of that element in the multiset.

As a consequence, an infinite number of multisets exist which contain only elements *a* and *b*, but vary in the multiplicities of their elements:

- The set {*a*, *b*} contains only elements *a* and *b*, each having multiplicity 1 when {*a*, *b*} is seen as a multiset.
- In the multiset {*a*, *a*, *b*}, the element *a* has multiplicity 2, and *b* has multiplicity 1.
- In the multiset {*a*, *a*, *a*, *b*, *b*, *b*}, *a* and *b* both have multiplicity 3.

These objects are all different, when viewed as multisets, although they are the same set, since they all consist of the same elements. As with sets, and in contrast to tuples, order does not matter in discriminating multisets, so {*a*, *a*, *b*} and {*a*, *b*, *a*} denote the same multiset.

Note: To distinguish between sets and multisets, a notation that incorporates square brackets is sometimes used: the multiset {*a*, *a*, *b*} can be denoted as [*a*, *a*, *b*].

Remark: The cardinality of a multiset is constructed by summing up the multiplicities of all its elements. For example, in the multiset {*a*, *a*, *b*, *b*, *b*, *c*} the multiplicities of the members *a*, *b*, and *c* are respectively 2, 3, and 1, and therefore the cardinality of this multiset is 6.

Example: One of the simplest and most natural examples is the multiset of prime factors of a natural number *n*. Here the underlying set of elements is the set of prime factors of *n*. We know that $120 = 2^3\, 3^1\, 5^1$ which gives the multiset {2, 2, 2, 3, 5}.

Notation: We can write $a \in^n A$ if element *a* occurs in the multiset *A* at least *n* times. For example: $1 \in^3 \{1,2,3,1,1\}$, $4 \in^0 \{1,2,3,1,1\}$.

Applications: Multiset Concept is useful in following areas:

- fundamental in combinatorics
- an important tool in the theory of relational databases
- used to implement relations in database systems
- SQL operates on multisets and return identical records

Relations

Ordered Pair: In ordered pair, each set is specified by two objects in a prescribed order. The ordered pair of *a* and *b*, with first coordinate *a* and second coordinate *b*, is the set (a,b).

Note: $(a,b)=(c,d)$ iff $a=c$ and $b=d$.

Cartesian Product: Let *A* and *B* are two sets. The Cartesian product of *A* and *B* is defined as

$$A\times B=\{(a,b): a\in A \text{ and } b\in B\}.$$

More generally, the Cartesian product of *n* sets $A_1, A_2, \ldots, A_n$ is defined as

$$A_1\times A_2\times\ldots\times A_n=\{(a_1,a_2,\ldots,a_n): a_i\in A_i, i=1,2,\ldots,n\}.$$

The expression $(a_1,a_2,\ldots,a_n)$ is called an ordered n-tuple.

Example: Let $A=\{0,1,2\}$ and $B=\{a,b\}$ Then,

$$A\times B=\{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b)\}.$$

Note: From the definition of the Cartesian product we have seen that any element (a,b) in a Cartesian product $A\times B$ is just an ordered pair.

Binary Relation: A binary relation R from A to B is a subset of Cartesian product $A \times B$.

n-ary Relation: An n-ary relation is a subset of a Cartesian product of n sets $A_1, A_2, \ldots, A_n$.
In case $n = 1$, a subset R of A is called a **unary** relation on A.

Example: Let all the points inside a unit circle whose centre is at the origin. Then the set

$$R = \{(x, y) : x \text{ and } y \text{ are real numbers and } x^2 + y^2 < 1\}$$

is a relation on the set of real numbers.

Domain of R: Let R be a relation from A to B. The **domain** of R denoted by **dom *R***, is defined as

$$\text{dom}\, R = \{x : x \in A \text{ and } (x, y) \in R \text{ for some } \; y \in B\}.$$

Range of R: Let R be a relation from A to B. The **range** of R denoted by **ran *R***, is defined as

$$\text{ran}\, R = \{y : y \in B \text{ and } (x, y) \in R \text{ for some } \; x \in A\}.$$

Note: 1- Here, $\text{dom}\, R \subseteq A$ and $\text{ran}\, R \subseteq B$. Moreover, the domain of R is the set of first coordinates in R and the range of R is the set of second coordinates in R.

Note: 2- We sometimes write $(x, y) \in R$ as $x\,R\,y$ which reads "x relates to y".

Properties of Binary Relations

1	**Transitivity**	$\forall x, y, z$ if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$;
2	**Reflexivity**	$\forall x \quad x\,R\,x$;
3	**Irreflexivity**	$\forall x \quad x \not R\, x$;
4	**Symmetry**	$\forall x, y$ if $x\,R\,y$, then $y\,R\,x$;
5	**Antisymmetry**	$\forall x, y$ if $x\,R\,y$ and $y\,R\,x$, then $x = y$;
6	**Asymmetry**	$\forall x, y$ if $x\,R\,y$, then $y \not R\, x$;

Equivalence Relation

Definition: Let R be a relation on A. R will be an equivalence relation on A if the following conditions are satisfied:

1. If $xRx \quad \forall\; x \in A$ (R is reflexive).
2. If xRy, then $yRx \quad \forall\; x, y \in A$ (R is symmetric).
3. If xRy and yRz, then $xRz \quad \forall\; x, y, z \in A$ (R is transitive).

Example: Let $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Define a relation R in **N** as follows:

$$R = \{(x, y);\; x, y \in N \text{ and } x + y \text{ is even}\}.$$

R is an equivalence relation in N because the first two conditions are clearly satisfied and to prove the third condition, if $x+y$ and $y+z$ are divisible by 2, then $x+(y+y)+z$ is divisible by 2. Therefore, $x+z$ is divisible by 2. Hence Proved.

Note: In this equivalence relation all the odd numbers are equivalent and so are all the even numbers.

Partition: Let A be a given set. A partition of A is a collection P of disjoint subsets whose union is A.
Mathematically,

1. For any $B \in P,\ B \subseteq A$;
2. For any $B, C \in P,\ B \cap C = \phi$, or $B = C$; and
3. For any $x \in A,\ \exists\ B \in P$ such that $x \in B$.

Equivalence Class: Let A be a given set and R is an equivalence relation on A, the equivalence class $[x]$ of each element x of A is defined as

$$[x] = \{y \in A:\ xRy\}.$$

Note: We can have $[x]=[y]$ even if $x \neq y$ provided xRy.

Remark: Let A be a given set and R is an equivalence relation on A, then $S = \{[x] : x \in A\}$ is a partition of A into disjoint nonempty subsets. Conversely, if P is a partition of A into nonempty disjoint subsets, then P is the set of equivalence classes for the equivalence relation E defined on A by aEb iff a and b belong to the same subset of P.

Congruence modulo 'm': Let *'m'* be any positive integer then the relation congruence modulo *'m'* $[\equiv (\text{mod}\ m)]$ is defined by $x \equiv y(\text{mod}\ m)$ iff $x = y + a.m$ for some integer a.

Theorem: For any positive integer m, the relation $\equiv(\text{mod}\ m)$ is an equivalence relation on the integers, and partitions the integers into 'm' distinct equivalence classes: [0], [1],…,[m-1].

Proof: We have $a \equiv b(\text{mod}\ m)$ which implies $a-b=m.k$.

First, we prove R is an equivalence relation. For this,

1. Reflexivity: $a-a=m.0$

$\Rightarrow \quad a \equiv a(\text{mod}\ m)$

$\Rightarrow \quad aRa$.

2. Symmetry: If $a \equiv b(\text{mod}\ m)$ then $a-b=m.k$

$\Rightarrow \quad -(a-b) = -m.k$

$\Rightarrow \quad b-a = m.(-k) = m.k'$ where $k' = k$

$\Rightarrow \quad b \equiv a(\text{mod}\ m)$

$\Rightarrow \quad bRa$.

3. Transitivity: If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

   then $a - b = m.k_1$ and $b - c = m.k_2$

   Adding these, we get

$\Rightarrow \quad a - b + b - c = m.k_1 + m.k_2$

$\Rightarrow \quad a - c = m.k_3$ where $k_3 = k_1 + k_2$

$\Rightarrow \quad a \equiv c \pmod{m}$.

Therefore, it is an equivalence relation.

Now, if x is any integer, then by division algorithm

$$x = mq + r$$

where q and r are integers and $0 \le r < m$.

Thus, $\quad x \equiv r \pmod{m}$ and $[x] = [r]$.

So, each equivalence class for this relation is one of the classes [0],[1],…,[m-1].

Moreover, if $[x] = [y]$ where $0 \le x \le y \le m\text{-}1$,

Then $y = ma + x$ for some integer a.

Therefore, $\quad 0 \le y - x = m.a < m$ implies $a = 0$ and $x = y$.

Ordering Relations

In this topic, we need to understand the following concepts.

Order Theory: It studies various kinds of binary relations that capture the intuitive notion of ordering expressed usually using phrases "less than" or "precedes".

Partial Order: A relation R on a set A is called a partial order on A when R is reflexive, antisymmetric, and transitive, and then the set A is called a **partially ordered set** or a **poset**.

Notation: $[A; R]$ is used to denote that A is partially ordered by the relation R.

Note: The relation $\le$ on the set of real numbers is the prototype of a partial order, so it is common to write $\le$ to represent an arbitrary partial order on A.

Characteristic Properties of a Partial Order

1. Reflexivity i.e. $\forall\ a \in A, \quad a \le a$

2. Antisymmetry i.e. ∀ a,b∈ A, if a ≤ b and b ≤ a then a=b
3. Transitivity i.e. ∀ a,b,c∈ A, if a ≤ b and b ≤ c then a≤ c

Comparable: Two elements a and b in a set A are said to be **comparable** under ≤ if either A ≤ b or b≤ a; otherwise, they are **incomparable**.

Totally Ordered Set: If every pair of elements of a set A are comparable, then we say that [A; ≤] is **totally ordered** or that A is a **totally ordered set** or a **chain.**

In this case, ≤ the relation is called a **total order**.

Examples: 1. If **Z** is the set of integers and ≤ is the usual ordering on **Z**, then [**Z**; ≤ **]**is partially ordered and totally ordered.

2. The relation < on **Z** is not a partial order because it is not reflexive.

Lattices

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A lattice is a partially ordered set (L, ≤) in which every pair of elements a, b ∈ L has a greatest lower bound and a least upper bound.

Example: Let Z^+ denote the set of all positive integers and let R denote the relation 'division' in Z^+, such that for any two elements a, b ∈ Z^+, aRb, if a divides b. Then (Z^+, R) is a lattice in which the join of a and b is the least common multiple of a and b, i.e.

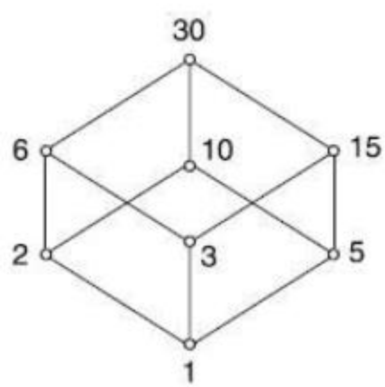
$$a \vee b = a \oplus b = \text{LCM of a and b},$$

and the meet of a and b, i.e. a ∗ b is the greatest common divisor (GCD) of a and b i.e.,

$$a \wedge b = a * b = \text{GCD of a and b}.$$

We can also write a+b = a∨b = a⊕b=LCM of a and b and a.b = a∧b = a∗b=GCD of a and b.

Example: Let n be a positive integer and Sn be the set of all divisors of n If n = 30, S30 = {1, 2, 3, 5, 6, 10, 15, 30}. Let R denote the relation division as defined in Example 1. Then (S30, R) is a Lattice see Fig:



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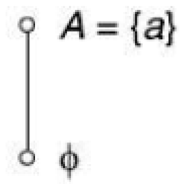
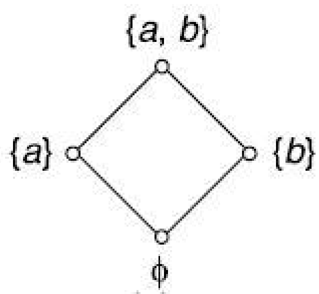
Example: Let A be any set and $P(A)$ be its power set. The poset $(P(A), \subseteq)$ is a lattice in which the meet and join are the same as the operations ∩ and ∪ on sets respectively.

$S = \{a\}$, $P(A) = \{\phi, \{a\}\}$



$S = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{a\}, S\}$.



Some Properties of Lattice

Let $(L, \preceq)$ be a lattice and $*$ and $\oplus$ denote the two binary operation meet and join on $(L, \preceq)$. Then for any $a, b, c \in L$, we have

(L1): $a*a = a$,	(L1)′ : $a\oplus a = a$ (Idempotent laws)
(L2): $b*a = b*a$,	(L2)′: $a \oplus b = b + a$ (Commutative laws)
(L3) : $(a*b)*c = a*(b*c)$,	(L3)′: $(a\oplus b)\oplus c = a\oplus(b + c)$ (Associative laws)
(L4) : $a*(a + b) = a$,	(L4)′: $a\oplus(a*b) = a$ (Absorption laws).

The above properties (L1) to (L4) can be proved easily by using definitions of meet and join. We can apply the principle of duality and obtain (L1)′ to (L4)′.

Complemented lattice:

A complemented lattice is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement, i.e. an element b satisfying a ∨ b = 1 and a ∧ b = 0. Complements need not be unique.

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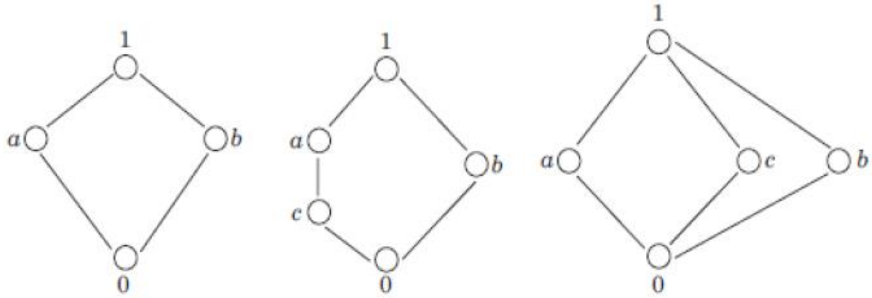
Example: Lattices shown in Fig (a), (b) and (c) are complemented lattices.



(a) (b) (c)

Sol.
For the lattice (a) GLB(a, b) = 0 and LUB(x, y) = 1. So, the complement a is b and vise versa. Hence, a complement lattice.

For the lattice (b) GLB(a, b) = 0 and GLB(c, b) = 0 and LUB(a, b) = 1 and LUB(c, b) = 1; so both a and c are complement of b.
Hence, a complement lattice.

In the lattice (c) GLB(a, c) = 0 and LUB(a, c) = 1; GLB(a, b) = 0 and LUB(a, b) = 1. So, complement of a are b and c.
Similarly complement of c are a and b also a and c are complement of b.
Hence lattice is a complement lattice.

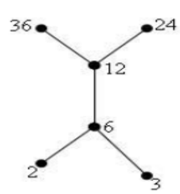
Hasse Diagrams

A partial order $\preceq$ on a set P can be represented by means of a diagram known as Hasse diagram of (P, $\preceq$). In such a diagram,

(i). Each element is represented by a small circle or dot.
(ii). The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, and a line is drawn between x and y if y covers x.
(iii). If $x < y$ but y does not cover x, then x and y are not connected directly by a single line.

Note: For totally ordered set (P, $\preceq$), the Hasse diagram consists of circles one below the other. The poset is called a chain.

Example: Let $P = \{1, 2, 3, 4, 5\}$ and $\leq$ be the relation "less than or equal to" then the Hasse diagram is:

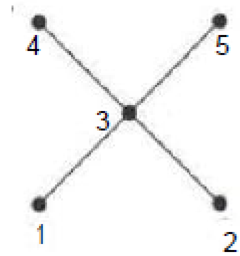


Is not a total order set

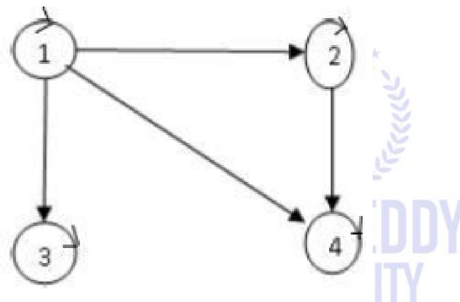
Example: Draw the Hasse diagram for the relation R on $A = \{1, 2, 3, 4, 5\}$ whose relation matrix given as: $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Sol: R= $\{(1, 1), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (5.5)\}$.

Hasse diagram for M_R is

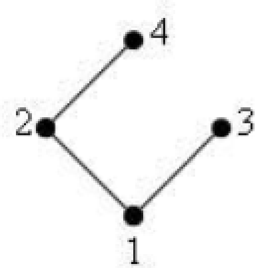


Example: A partial order R on the set $A = \{1, 2, 3, 4\}$ is represented by the following digraph. Draw the Hasse diagram for R.



Sol: By examining the given digraph , we find that
R= $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.
We check that R is reflexive, transitive and antisymmetric. Therefore, R is partial order relation on A.

The hasse diagram of R is shown below:



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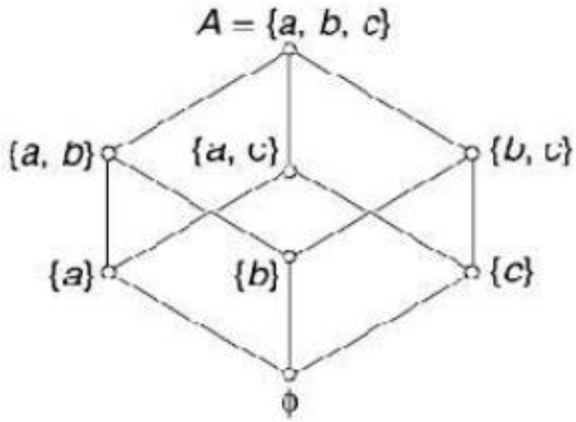
Example: Draw the Hasse diagram for the partial ordering ⊆ on the power set *P* (*S*) where $S = \{a,b, c\}$.

Sol: $S = \{a, b, c\}$.

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Hasse diagram for the partial ordered set is shown in fig:



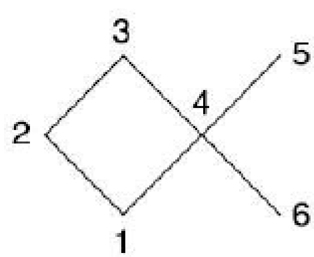
Minimal and Maximal elements(members):

Let $(P, \leq)$ denote a partially ordered set. An element $y \in P$ is called a *minimal member* of *P* relative to $\leq$ if for no $x \in P$, is $x < y$. Similarly an element $y \in P$ is called a maximal member of *P* relative to the partial ordering $\leq$ if for no $x \in P$, is $y < x$.

Note:

(i). The minimal and maximal members of a partially ordered set need not unique.
(ii). Maximal and minimal elements are easily calculated from the Hasse diagram.
They are the *'top'* and *'bottom'* elements in the diagram.

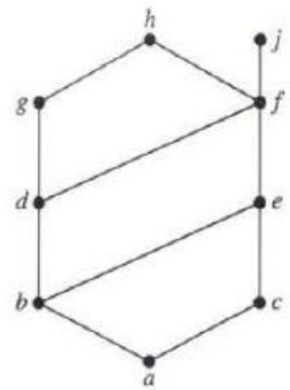
Example:



In the Hasse diagram, there are two maximal elements and two minimal elements.
The elements 3, 5 are maximal and the elements 1 and 6 are minimal.

Upper and Lower Bounds: Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. Any element $x \in P$ is called an *upper bound* for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is called a *lower bound* for A if for all $a \in A$, $x \leq a$.

Example: Find the great lower bound and the least upper bound of $\{b, d, g\}$, if they exist in the poset shown in fig:



Solution: The upper bounds of $\{b, d, g\}$ are g and h. Since $g < h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Since $a < b$, b is the greatest lower bound.

Functions

A function is a rule which maps a number or entity to another unique number or entity.
Let A and B be two non-empty sets. A function f from A to B is a relation from A to B such that:

(i) dom f = A i.e. f is defined at each a $\in$ A.

(ii) if ($x, y \in f$) and (x ,z $\in$f) then y= z i.e. no element of A is related to two elements of B.

Note: 1. The words mapping, transformation, correspondence, and operator are used as synonyms of function.

2. If f is a function from A to B, we write f: A$\to$B.

One-One Function or Injective Function: A function f A B : $\to$ is said to be one-to-one if $f(x_1)= f(x_2) \Rightarrow x_1= x_2$.
Thus, f is one-one iff for each b $\in$ran f , b has precisely one pre-image.

Onto Function or Surjective Function: A function f: A$\to$ B is said to be onto iff ran f = B.
In other words, f is onto if each b$\in$B has some pre-image in A.

Bijective Function: If a function is both one-to-one and onto then it is called a bijective function.

Remark: A one-to-one, onto function $f: A \to B$ is usually called a one-to-one correspondence between *A* and *B*.

Note: 1. If a function $f: A \to B$ is not one-to-one, we call it a **many-to-one** function.

2. If $f: A \to B$ is not necessarily onto B, then it is said to be **into** B.

3. Since a function is a set, two functions f and g from A to B are equal if they are equal as sets. In other words, $f = g$ iff $f(a) = g(a)$ for each $a \in A$.

Constant Function: A function $f: X \to X$ is said to be a constant function if

$$f(x) = k \quad \forall\ x \in X \quad \text{where } k \text{ is fixed.}$$

Identity Function: A function $f: X \to X$ is said to be an identity function if

$$f(x) = x \quad \forall\ x \in X .$$

Example: Let $A = \{r, s, t\}$, $B = \{1, 2, 3\}$ and $C = \{r, s, t, u\}$.

Now, $R = \{(r,1), (r,2), (t,2)\}$ is a relation from *A* to *B* but *R* is not a function since image of *r* is not unique.

The set $f = \{(r,1), (s,2), (t,2)\}$ is a function from *A* to *B* but f is not one-to-one since 2 has two pre-images.. Likewise, f is not onto *B* since 3 has no pre-image.

The function $g = \{(r,1), (s,2), (t,3)\}$ is both a one-to-one and an onto function from *A* to *B*.

On the other hand, the function $h: C \to B$ defined as $h = \{(r,1), (s,1), (t,2), (u,3)\}$ is onto, but not one-to-one since 1 has two pre-images.

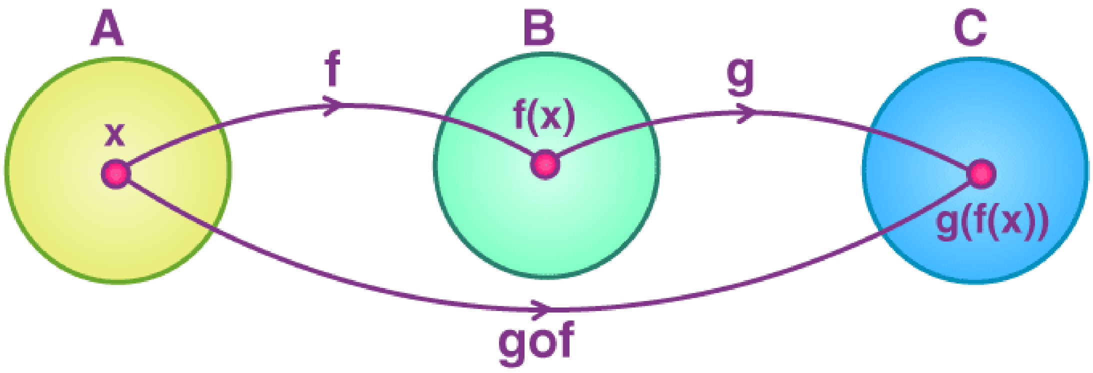
Composition of Functions

The composition of a function is an operation where two functions say f and g generate a new function say h in such a way that $h(x) = g(f(x))$.

It means here function g is applied to the function of x. So, basically, a function is applied to the result of another function.

Definition: Let $f: A \to B$ and $g: B \to C$ be two functions. Then the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x))$, $\forall\ x \in A$.

The below figure shows the representation of **composite functions**.



Note: The order of function is an important thing while dealing with the composition of functions since $f \circ g(x)$ is not always equal to $g \circ f(x)$.

Symbol: It is also denoted as $g \circ f(x)$, where ∘ is a small circle symbol. We cannot replace ∘ with a dot (.), because it will show as the product of two functions, such as $(g.f)(x)$.

Domain in Composition Function: $f(g(x))$ is read as f of g of x. In the composition of $f \circ g(x)$ the domain of function f becomes $g(x)$. The domain is a set of all values which go into the function.

Example: If $f(x) = 3x + 1$ and $g(x) = x^2$, then f of g of x, $f(g(x)) = f(x^2) = 3x^2 + 1$.

If we reverse the function operation, such as g of f of x, $g(f(x)) = g(3x+1) = (3x+1)^2$.

Properties of Function Compositions

1. **Associative Property:** If there are three functions f, g and h, then they are said to be associative if and only if
$$f \circ (g \circ h) = (f \circ g) \circ h$$
2. **Commutative Property:** Two functions f and g are said to be commute with each other, if and only if
$$f \circ g = g \circ f$$
3. The function composition of one-to-one function is always one to one.
4. The function composition of two onto function is always onto.
5. The inverse of the composition of two functions f and g is equal to the composition of the inverse of both the functions, such as $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Example: If $f(x)=3x^2$, then $f\circ f(x)=f(f(x))=f(3x^2)=3.9x^4=27x^4$.

Inverse Function

Definition: An inverse function or an anti-function is defined as a function, which can reverse into another function. In simple words, if any function f takes x to y then, the inverse of f will take y to x.

If the function is denoted by f or F, then the inverse function is denoted by f^{-1} or F^{-1}.

One should **not confuse** (-1) with exponent or reciprocal here.

Note: If f and g are inverse functions, then $f(x)=y$ iff $g(y)=x$.

Example: If $f(x)=2x+5$ then what will be the inverse of f ?

Let $f(x)=2x+5=y\Rightarrow x=\dfrac{y-5}{2}=g(y)$

Then, $f^{-1}(x)=\dfrac{x-5}{2}$.

Remark: In trigonometry, the inverse sine function is used to find the measure of angle for which sine function generated the value.

For example, $\sin^{-1}(1) = \sin^{-1}(\sin 90) = 90$ degrees. Hence, sin 90 degrees is equal to 1.

Note: A function that consists of its inverse fetches the original value.

Types of Inverse Function: There are various types of inverse functions like the inverse of trigonometric functions, rational functions, hyperbolic functions and log functions. The inverses of some of the most common functions are given below.

Function	**Inverse of the Function**	**Comment**
+	–	
×	/	Don’t divide by 0
1/x	1/y	x and y not equal to 0
x^2	$\sqrt{y}$	x and y ≥ 0
x^n	$y^{1/n}$	n is not equal to 0
e^x	ln(y)	y > 0
a^x	log a(y)	y and a > 0
Sin (x)	Sin^{-1} (y)	– π/2 to + π/2

Function	Inverse of the Function	Comment
Cos (x)	Cos^{-1} (y)	0 to π
Tan (x)	Tan^{-1} (y)	– π/2 to + π/2

Operations

We can classify the operations into following main categories:

Unary Operation: A unary operation is an operation with only one operand, i.e. a single input.

Examples: The function $f : A \to A$, (where A is a set) is a unary operation on A. The trigonometric functions are unary operations.

Notations: Common notations are prefix notation (e.g. +, −, ¬), postfix notation (e.g. factorial $n!$), functional notation (e.g. $\sin x$ or $\sin(x)$), and superscripts (e.g. transpose A^T).

Binary Operation: A binary operation or dyadic operation is a calculation that combines two elements (called operands) to produce another element.

Examples: The familiar arithmetic operations of addition, subtraction, and multiplication. Other examples are readily found in different areas of mathematics, such as vector addition and matrix multiplication.

Remark: More precisely, a binary operation on a set S is a mapping of the elements of the Cartesian product $S \times S$ to S i.e. $f : S \times S \to S$.

Note: Because the result of performing the operation on a pair of elements of S is again an element of S, the operation is called a closed (or internal) binary operation on S (or sometimes expressed as having the property of closure).

n-ary Operation: An operation that takes n arguments for its input is an n-ary operation.

Example: Adding a bunch of numbers together would be a simple n-ary operation, i.e.

$$\text{Sum}\left(x_1, x_2, \ldots, x_n\right) = \sum_{i=1}^{n} x_i$$

Also, picking out a maximum, minimum, mean, medium, the first object in a list and so on are also its examples.

Remark: In general, a function takes an object from space X and maps it to space Y. If the objects in X are tuples (an ordered collection of sub-objects) then the function is an n-ary operation.

$$f : X \to Y$$

$$X = X_1 \times X_2 \times \ldots \times X_n$$

then f is an n-ary operation.

SOLVED PROBLEMS

Example 1. Which of these sets are equal: $\{x, y, z\}$, $\{z, y, z, x\}$, $\{y, x, y, z\}$, $\{y, z, x, y\}$?

Solution: They are all equal. Order and repetition do not change a set.

Example 2. List the elements of each set where $\mathbf{N} = \{1, 2, 3, \ldots\}$.

(*a*) $A = \{x \in \mathbf{N} | \; 3 < x < 9\}$

(*b*) $B = \{x \in \mathbf{N} | \; x \text{ is even}, x < 11\}$

(*c*) $C = \{x \in \mathbf{N} | \; 4 + x = 3\}$

Solution: (*a*) A consists of the positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.

(*b*) B consists of the even positive integers less than 11; hence $B = \{2, 4, 6, 8, 10\}$.

(*c*) No positive integer satisfies $4 + x = 3$; hence $C = \emptyset$, the empty set.

Example 3. In a survey of 120 people, it was found that:

65 read *Newsweek* magazine, 20 read both *Newsweek* and *Time*,

45 read *Time*, 25 read both *Newsweek* and *Fortune*,

42 read *Fortune*, 15 read both *Time* and *Fortune*,

8 read all three magazines.

(*a*) Find the number of people who read at least one of the three magazines.

(*b*) Draw its Venn diagram.

(*c*) Find the number of people who read exactly one magazine.

Solution: (*a*) We want to find $n(N \cup T \cup F)$.

Using Inclusion–Exclusion Principle, we can write

$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$

$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$

(*b*) To draw the required Venn diagram, we need to obtain the following:

8 read all three magazines,

$20 - 8 = 12$ read *Newsweek* and *Time* but not all three magazines,

$25 - 8 = 17$ read *Newsweek* and *Fortune* but not all three magazines,

$15 - 8 = 7$ read *Time* and *Fortune* but not all three magazines,

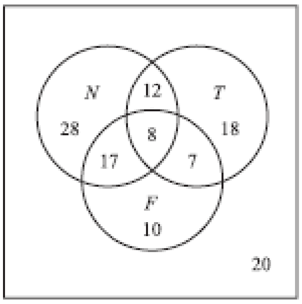
$65 - 12 - 8 - 17 = 28$ read only *Newsweek*,

$45 - 12 - 8 - 7 = 18$ read only *Time*,

$42 - 17 - 8 - 7 = 10$ read only *Fortune*,

$120 - 100 = 20$ read no magazine at all.

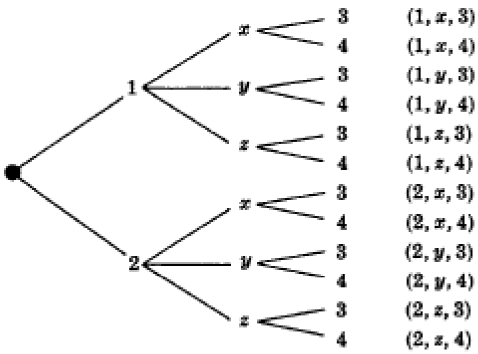
Hence, it can be drawn as following



(c) From Venn diagram, we can clearly see that 28+ 18 + 10 = 56 read exactly one of the magazines.

Example 4. Given: $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find: $A \times B \times C$.

Solution: $A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram as shown below. The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.



Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C).$$

Example 5. Find x and y given $(2x, x + y) = (6, 2)$.

Solution: Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x = 6 \text{ and } x + y = 2$$

from which we derive the answers $x = 3$ and $y = -1$.

Example 6. If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

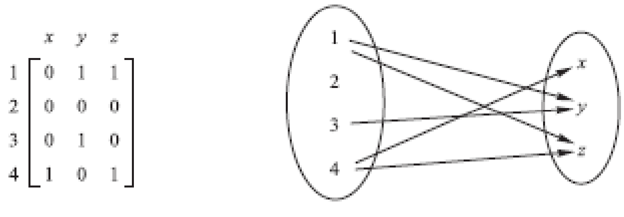
(a) Determine the matrix of the relation.

(b) Draw the arrow diagram of R.

(c) Find the inverse relation R^{-1} of R.

(*d*) Determine the domain and range of *R*.

Solution:



(*a*) (*b*)

(*c*) Reverse the ordered pairs of *R* to obtain R^{-1}:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

Observe that by reversing the arrows in above figure, we obtain the arrow diagram of R^{-1}.

(*d*) The domain of *R*, Dom*(R)*, consists of the first elements of the ordered pairs of *R*, and the range of *R*, Ran*(R)*, consists of the second elements. Thus,

$$\text{Dom}(R) = \{1, 3, 4\} \text{ and } \text{Ran}(R) = \{x, y, z\}$$

Example 7. Let $X = \{1, 2, 3, 4\}$. Determine whether each relation on *X* is a function from *X* into *X*.

(*a*) $f = \{(2, 3), (1, 4), (2, 1), (3.2), (4, 4)\}$

(*b*) $g = \{(3, 1), (4, 2), (1, 1)\}$

(*c*) $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$

Solution: Recall that a subset *f* of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in *f*.

(*a*) No. Two different ordered pairs (2, 3) and (2, 1) in *f* have the same number 2 as their first coordinate.

(*b*) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in *g*.

(*c*) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in *h*, these two ordered pairs are equal.

Example 8. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$,$C = \{r, s, t\}$. Let $f: A \to B$ and $g: B \to C$ be defined by:

$$f = \{(a, y), (b, x), (c, y)\} \text{ and } g = \{(x, s), (y, t), (z, r)\}.$$

Find: (*a*) composition function $g \circ f: A \to C$; (*b*) Im*(f)*, Im*(g)*, Im*(g∘f)*.

Solution: (*a*) Use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$

$$(g \circ f)(b) = g(f(b)) = g(x) = s$$

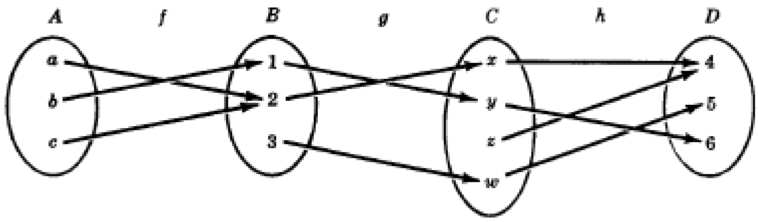
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

That is $g \circ f = \{(a, t), (b, s), (c, t)\}$.

(b) Find the image points (or second coordinates):

$$\text{Im}(f) = \{x, y\}, \text{Im}(g) = \{r, s, t\}, \text{Im}(g \circ f) = \{s, t\}.$$

Example 9. Let the functions $f : A \to B$, $g : B \to C$, $h : C \to D$ be defined as in the following figure: Determine if each function is: (a) onto, (b) one-to-one, (c) invertible.



Solution: (a) The function $f : A \to B$ is not onto since $3 \in B$ is not the image of any element in A.

The function $g : B \to C$ is not onto since $z \in C$ is not the image of any element in B.

The function $h : C \to D$ is onto since each element in D is the image of some element of C.

(b) The function $f : A \to B$ is not one-to-one since a and c have the same image 2.

The function $g : B \to C$ is one-to-one since 1, 2 and 3 have distinct images.

The function $h : C \to D$ is not one-to-one since x and z have the same image 4.

(c) No function is one-to-one and onto; hence no function is invertible.