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8/10/23

Assignment - II

1) Let $S = \{1,2,3\}$ and $P(S)$, the power set of S. On $P(S)$, define the relation R by xRy iff $x \subseteq y$. Show that $(P(S), \subseteq)$ is Poset, check whether it is Toset or not and also draw its Hasse diagram.

Sol: Given set $S = \{1,2,3\}$

$$R = \{(A,B) \mid A \subseteq B ; A,B \in P(S)\}$$

Reflexive: Let $A \in P(S)$

Always $A \subseteq A$.

$\Rightarrow (A,A) \in R$ $\quad \therefore$ Reflexive

Antisymmetric: Let $A,B \in P(S)$ & $(A,B) \in R$, $(B,A) \in R$

$\Rightarrow A \subseteq B ;\ B \subseteq A$

$\Rightarrow A = B$ (only possible case)

$\therefore$ Antisymmetric

Transitive: Let $A,B,C \in P(S)$ & $(A,B) \in R$, $(B,C) \in R$

$\Rightarrow A \subseteq B ;\ B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow (A,C) \in R$

$\therefore$ Transitive

$\therefore$ R is a partially ordering relation

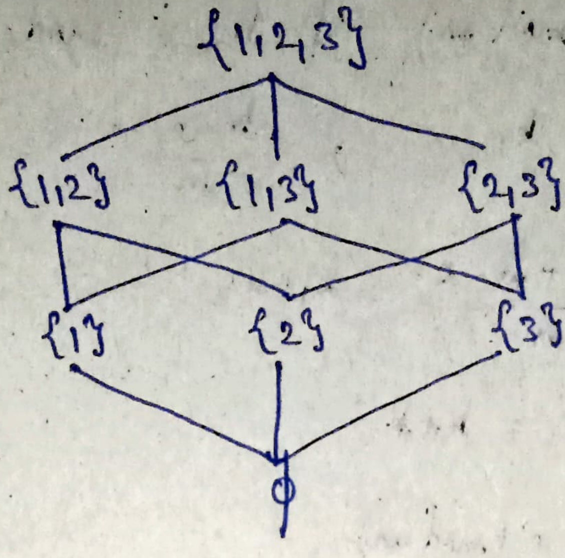
$\therefore P(S)$ is a POSet

$\therefore \forall A,B \in R$ we have either A related to B or B related to A.

$\therefore P(S)$ is a TOSet

$S = \{1,2,3\}$

$R = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$



2) Prove that for any positive integer m, the relation congruence modulo m is an equivalence relation on integers.

Sol: Given $R = \{(a,b) \mid a \equiv b \pmod{m};\ a,b,m \in Z\}$

Reflexive: $\forall x \in Z$, $(x,x) \in R \Rightarrow a \equiv a \pmod{m}$

$$\Rightarrow \frac{a-a}{m} = \frac{0}{m} = 0$$

$\therefore (a,a) \in R$

Symmetric: $\forall a,b \in Z$

$a \equiv b \pmod{m}$

$$\Rightarrow \frac{a-b}{m} = k$$

$$\Rightarrow -\frac{(b-a)}{m} = k$$

$\Rightarrow \frac{b-a}{m} = -k$

$\Rightarrow (b, a) \in R$

<u>Transitive</u>:- $\forall\, a, b, c \in Z$

$a \equiv b \pmod{m}$ & $b \equiv c \pmod{m}$

$\Rightarrow \frac{a-b}{m} = k_1$, $\frac{b-c}{m} = k_2$

$\Rightarrow \frac{a-b+b-c}{m} = k_1 + k_2$

$\Rightarrow \frac{a-c}{m} = k_1 + k_2$

$\Rightarrow a \equiv c \pmod{m}$

$\therefore (a, c) \in R$

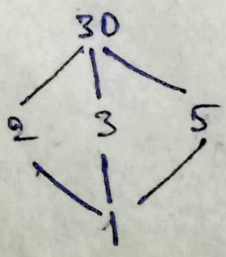
$\therefore R$ is an equivalence relation.

3) If $A = \{1, 2, 3, 5, 30\}$ and R is the divisibility relation, draw its Hasse diagram and verify that (A, R) is lattice or not?

<u>Sol</u>: Given $A = \{1, 2, 3, 5, 30\}$

$R = \{(x, y) \mid x \text{ is divisible by } y\}$

Hasse diagram



$\vee$	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	2	30	3	30	30
5	5	30	30	5	30
50	30	30	30	30	30

$\wedge$	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

$\therefore$ It is a lattice

Let $a = 2$, $b = 3$, $c = 5$

LHS

$a \vee (b \wedge c)$

$\Rightarrow 2 \vee (3 \wedge 5)$

$\Rightarrow 2 \vee 1$

$\Rightarrow 2$

RHS

$(a \vee b) \wedge (a \vee c)$

$\Rightarrow (2 \vee 3) \wedge (2 \vee 5)$

$\Rightarrow 30 \wedge 30$

$\Rightarrow 30$

$\because$ LHS $\neq$ RHS

$\therefore$ It is not a distributive lattice.

4) If the functions $f, g : Q \to Q$ are defined by $f(x) = 2x$ and $g(x) = x - 2$ then prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Solr $f : A \to B \Rightarrow f^{-1} : B \to A$ i.e $f^{-1} = \{(b, a) \mid (a, b) \in f\}$

$f(x) = 2x$

$g(x) = x - 2$

$\Rightarrow f \circ g = f(g(x))$

$= f(x-2)$

$= 2(x-2) = 2x-4 = F(x)$

$F(x) = 2x-4 = y$

$\Rightarrow 2x = y+4$

$\Rightarrow x = \frac{y+4}{2}$

$\Rightarrow F^{-1}(y) = \frac{y+4}{2}$

$\Rightarrow F^{-1}(x) = \frac{x+4}{2} \rightarrow (1)$

$f(x) = 2x = y$

$\Rightarrow x = \frac{y}{2}$

$\Rightarrow f^{-1}(y) = \frac{y}{2}$

$\Rightarrow f^{-1}(x) = \frac{x}{2}$

$g(x) = x-2 = y$

$\Rightarrow x = y+2$

$\Rightarrow g^{-1}(y) = y+2$

$\Rightarrow g^{-1}(x) = x+2$

$\Rightarrow g^{-1} \circ f^{-1} = g^{-1}(f^{-1}(x))$

$= g^{-1}\left(\frac{x}{2}\right)$

$= \frac{x}{2} + 2$

$= \frac{x+4}{2} \rightarrow (2)$

$\because (1) = (2)$

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

5) Prove using laws of logic $(P \vee Q) \wedge \sim(\sim P \vee R)$

$\Leftrightarrow P \wedge \sim Q$

Sol: LHS

$\Rightarrow (P \vee Q) \wedge [\sim(\sim P) \wedge \sim Q]$ [$\because$ Demorgan]

$\Rightarrow (P \vee Q) \wedge (P \wedge \sim Q)$ [$\because$ Double negation]

$\Rightarrow [(P \vee Q) \wedge P] \wedge \sim Q$ [$\because$ Associative]

$\Rightarrow P \wedge \sim Q$ [$\because$ Absorption]

6) Obtain PDNF and PCNF of $P \rightarrow (Q \rightarrow R)$.

Sol:

P	Q	R	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
F	T	T	T	T
T	F	F	T	T
F	F	T	T	T
F	T	F	F	T
F	F	F	T	T

PDNF :- True

$(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee$
$(\sim P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge \sim R)$

PCNF :- False

$(\sim P \vee \sim Q \vee R)$

$\therefore$ PCNF does not exist.

7) Verify the validity of following arguments.

① It is not sunny this afternoon and it is colder than yesterday.

② We will go swimming only if it is sunny.

③ If we do not go swimming then we will take a Hyderabad trip.

④ If we take the Hyderabad trip then we will be home by sunset

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∴ We will be home by sunset.

Sol: P: It is sunny

Q: It is colder than yesterday.

R: We will go swimming

S: We will take a Hyderabad trip

T: We will be home by sunset.

$\sim P \wedge Q$

$R \rightarrow P$

$\sim R \rightarrow S$

$S \rightarrow T$

$\therefore T$

Steps	Reason
1) $\sim P \wedge Q$	P-I
2) $R \rightarrow P$	P-II
3) $\sim R \rightarrow S$	P-III
4) $S \rightarrow T$	P-IV

Steps | **Reason**

Steps	Reason
5) $\sim R \rightarrow T$	③ & ④ with Transitive
6) $\sim P \rightarrow \sim R$	② with contrapositive
7) $\sim P \rightarrow T$	⑥ & ⑤ with Transitive
8) $\sim P$	① with Simplification
9) T	⑦ & ⑧ with Modus Ponens

$\therefore$ The given argument is valid

8) Verify the validity of the following argument.

① If you send me an email, then I will finish ~~writing~~ writing the program.

② If you do not send me an email, then I will go to sleep early.

③ If I go to sleep early then I will wake up feeling refreshed.

$\therefore$ If I do not finish writing the program, then I will wake up feeling refreshed.

Sol: P: You send me an email

Q: I will finish writing the program.

R: I will go to sleep early

S: I will wake up feeling refreshed.

$$\begin{array}{l} P \rightarrow Q \\ \sim P \rightarrow R \\ \underline{R \rightarrow S} \\ \therefore \sim Q \rightarrow S \end{array}$$

Steps	Reason
1) $P \to Q$	P-I
2) $\sim P \to R$	P-II
3) $R \to S$	P-III
4) $\sim Q \to \sim P$	① with Contrapositive
5) $\sim Q \to R$	④ & ② with Transitive
6) $\sim Q \to S$	⑤ & ③ with Transitive

∴ The given argument is valid.

9) ① All integers are rational numbers
② Some integers are powers of 2

∴ Some rational numbers are powers of 2

Sol: $I(x)$: x is an integer.
$R(x)$: x is a rational numbers
$P(x)$: x is a power of 2.

$\forall x, I(x) \to R(x)$
$\exists x, I(x) \wedge P(x)$

~~∴ ∃x~~

$\therefore \exists x, R(x) \wedge P(x)$

Steps	Reason
1) $\forall x, I(x) \to R(x)$	P-I
2) $\exists x, I(x) \wedge P(x)$	P-II
3) $I(c) \to R(c)$	(1) with Universal Specification
4) $I(c) \wedge P(c)$	(2) with Existential Specification
5) $I(c)$	(4) with Simplification
6) $R(c)$	(3) & (5) with Modus Pones
7) $P(c)$	(4) with Simplification
8) $R(c) \wedge P(c)$	(6) & (7) with Conjunction
9) $\exists x, R(x) \wedge P(x)$	(8) with Existential Generalization.

10) (1) All men are mortal
(2) Sachin is a man

$\therefore$ Sachin is mortal.

Sol: $M(x)$: x is mortal
$N(x)$: x is a man
c: Sachin

$\forall x, N(x) \to M(x)$
$N(c)$

$\therefore M(c)$

Steps	Reason
1) $\forall x, N(x) \rightarrow M(x)$	P-I
2) $N(c)$	P-II
3) $N(c) \rightarrow M(c)$	① with Universal Specification
4) $M(c)$	③ & ② with Modus Pones

11) Solve the recurrence relation $a_n = a_{n-1} + \frac{1}{n(n+1)}$, $a_0 = 1$

Sol: ~~n = 0 ⇒ a_0 =~~

$n=1 \Rightarrow a_1 = a_0 + \frac{1}{1(2)} = 1 + \frac{1}{2}$

$n=2 \Rightarrow a_2 = a_1 + \frac{1}{2(3)} = 1 + \frac{1}{2} + \frac{1}{2\times 3}$

$= 1 + \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{3}\right]$

$n=3 \Rightarrow a_3 = a_2 + \frac{1}{3(4)} = 1 + \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right]$

$n=4 \Rightarrow a_4 = a_3 + \frac{1}{4(5)}$

$= 1 + \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right]$

$\vdots$

$\Rightarrow a_n = 1 + \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right] + \dots + \left[\frac{1}{n} - \frac{1}{n+1}\right]$

$\Rightarrow a_n = 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{n+1}$ $\Rightarrow \boxed{a_n = 2 - \frac{1}{n+1}}$

12) Solve the recurrence relation $a_n = 2a_{n-1} + 1$ for $n \geq 2$ and $a_1 = 2$.

Sol^r $n=2 \Rightarrow a_2 = 2a_1 + 1 = 2(2) + 1 = 5 \cancel{= 2^4 - 1}$

$= 2^2 + 2 - 1$

$n=3 \Rightarrow a_3 = 2a_2 + 1 = 2(5) + 1 = 11 = 2^3 + 2^2 - 1$

$n=4 \Rightarrow a_4 = 2a_3 + 1 = 2(11) + 1 = 23 = 2^4 + 2^3 - 1$

$n=5 \Rightarrow a_5 = 2a_4 + 1 = 2(23) + 1 = 47 = 2^5 + 2^4 - 1$

$\vdots$

$\therefore \boxed{a_n = a^n + a^{n-1} - 1}$

$\boxed{\therefore a_n = 2^n + 2^{n-1} - 1}$

13) What is the solution of the recurrence relation,

$a_n = a_{n-1} + 2a_{n-2}$ and $a_0 = 2$; $a_1 = 7$.

Solⁿ $\Rightarrow a_n - a_{n-1} - 2a_{n-2} = 0 \rightarrow (1)$

$\Rightarrow \cancel{\quad} \; r^n - r^{n-1} - 2r^{n-2} = 0$

$\Rightarrow r^{n-2}[r^2 - r - 2] = 0$

$\Rightarrow r^2 - r - 2 = 0 \rightarrow (2)$

$\Rightarrow \cancel{r^2 + 2r} \; r^2 - 2r + r - 2 = 0$

$\Rightarrow r(r-2) + 1(r-2) = 0$

$\Rightarrow (r+1)(r-2) = 0$

$\Rightarrow r = -1, 2$

$\Rightarrow a_n = c_1(-1)^n + c_2(2)^n \rightarrow (3)$

Sub $n = 0$ in (3).

$\Rightarrow a_0 = c_1(-1)^0 + c_2(2)^0$

$\Rightarrow 2 = c_1 + c_2 \rightarrow (4)$

Sub $n = 1$ in (3)

$\Rightarrow a_1 = c_1(-1)^1 + c_2(2)^1$

$\Rightarrow 7 = -c_1 + 2c_2 \rightarrow (5)$

Solve (4) & (5).

$$x_1 + x_2 = 2$$

$$(+)\quad -x_1 + 2x_2 = 7$$

$$3x_2 = 9$$

$$\boxed{x_2 = 3}$$

$$\Rightarrow \quad \cancel{x_1 = 2} \qquad \boxed{x_1 = -1}$$

$$\Rightarrow \boxed{a_n = (-1)(-1)^n + 3(2)^n}$$

14) Find the solution of the recurrence relation

$a_n + 4a_{n-1} + 4a_{n-2} = 0$ and $a_0 = 2$; $a_1 = 1$.

Sol: $\Rightarrow a_n + 4a_{n-1} + 4a_{n-2} = 0 \rightarrow (1)$

$\Rightarrow r^n + 4r^{n-1} + 4r^{n-2} = 0$

$\Rightarrow r^{n-2}\left[r^2 + 4r + 4\right] = 0$

$\Rightarrow r^2 + 4r + 4 = 0 \rightarrow (2)$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 4}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{0}}{2}$$

$$= \frac{-4}{2}$$

$\Rightarrow r = -2, -2$

$\Rightarrow a_n = (c_1 + c_2 n) r^n \rightarrow (3)$

Sub $n=0$ in (3)

$\Rightarrow a_0 = (c_1 + c_2(0))(-2)^0$

$\Rightarrow \boxed{2 = c_1}$

Sub $n=1$ in (3)

$\Rightarrow a_1 = (c_1 + c_2(1))(-2)^1$

$\Rightarrow 1 = (2 + c_2)(-2)$

$\Rightarrow 1 = -4 - 2c_2$

$\Rightarrow 5 = -2c_2$

$\Rightarrow \boxed{c_2 = -\frac{5}{2}}$

$\Rightarrow \boxed{a_n = \left(2 + \left(-\frac{5}{2}\right)n\right)(-2)^n}$