Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Assignment 3 [Posted Sept 25, 2023]
Faculty Instructor: Dr. Phani Motamarri
TAs: Kartick Ramakrishnan, G Sundaresan, Sayan Dutta

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

Problem 1

This exercise will walk you through the steps in proving the existence of SVD of any rectangular matrix $\mathbf{A}$ of size $m \times n$ with rank r.

(a) Matrices of the form $\mathbf{G} = \mathbf{A}^T\mathbf{A}$ are called Gram matrices where $\mathbf{A} \in \mathbb{R}^{m\times n}$.
Show that $\mathbf{x}^T\mathbf{G}\mathbf{x} \geqslant 0 \ \forall\ \mathbf{x} \in \mathbb{R}^n$ and hence show that all eigen-values of $\mathbf{G}$ are non-negative.

(b) Show that $\mathbf{A}$ and $\mathbf{A}^T\mathbf{A}$ have the same rank.

(c) Show that a vector $\mathbf{u}$ of the form $\mathbf{A}\mathbf{v}/\sigma$ $(\sigma > 0)$ is a unit eigen-vector of $\mathbf{A}\mathbf{A}^T$ where $\mathbf{v}$ and σ^2 form the eigen-vector, eigen-value pair of $\mathbf{A}^T\mathbf{A}$.

(d) Note that i^{th} eigen-vector, eigen-value pair of $\mathbf{A}^T\mathbf{A}$ can be written as $\mathbf{A}^T\mathbf{A}\mathbf{v}_i = (\sigma_i^2)\mathbf{v}_i$.

Consider the case of a full rank matrix $\mathbf{A}$ ie. $(\sigma_i > 0 \ \ \forall i)$, if we define a new vector $\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$, show that $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{m\times m}$ is an orthogonal matrix, $\mathbf{\Sigma} \in \mathbb{R}^{m\times n}$ is diagonal matrix and $\mathbf{V} \in \mathbb{R}^{n\times n}$ is again an orthogonal matrix, $\mathbf{u}_i$ is i^{th} column of $\mathbf{U}$, and $\mathbf{v}_i$ is i^{th} column of $\mathbf{V}$

[Note: In low rank scenario some $\sigma_i = 0$ if other non-zero σ_i are sorted, we can compute $\mathbf{U}$ by adding additional column vectors that span $\mathbb{R}^m$ and add rows of 0-vector to $\mathbf{\Sigma}$.]

Problem 2 [15 marks]

You are one of the scientists working at NASA's Goddard Space Flight Center in Greenbelt, Maryland and have been researching Wide-Field Slitless Spectroscopy to capture galaxy spectra of the distant universe. With the help of NASA's James Webb Space Telescope, you have successfully captured the deepest and sharpest infrared image of the distant universe to date. It is an image of the galaxy cluster SMACS 0723 and has been named Webb's First Deep Field.

Unfortunately, due to some technical difficulties, the space telescope has not been able to transmit full-resolution images to Earth. However, an onboard computer can be programmed remotely from Earth to transmit the image in a compressed format until the difficulties are resolved. The control station on Earth has decided to use SVD to compress the image. As a scientist tasked with programming the onboard computer, think about the following:

(a) How many singular values are required to approximate the image i.e., make it look indistinguishable from the original image? (Hint: Load the image in Python or Matlab or Octave or Julia and the matrix representation of the image will be accessible to you.

For $r \times r$ pixel image, the image will have $r \times r \times 3$ matrix entries with the number 3 corresponding to color depth of the image representing Red, Blue, Green.)

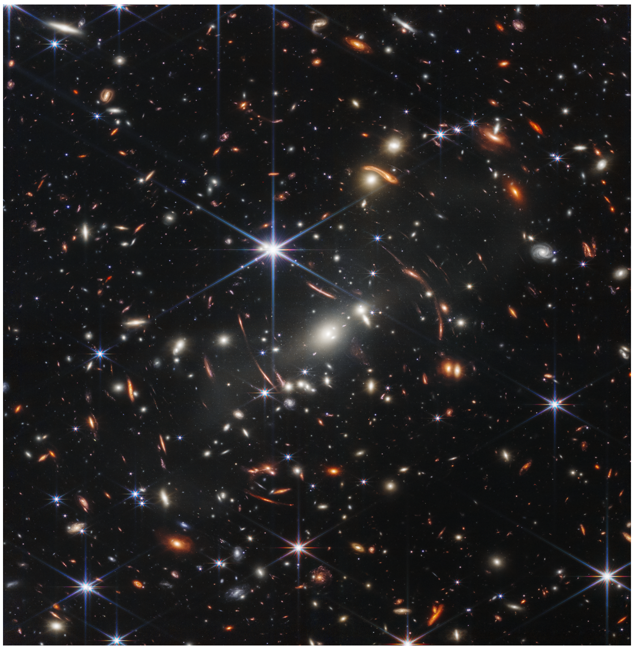
(b) Based on your observation in (a), how many entries need to be transmitted to earth to reconstruct the approximate image as opposed to sending the original image?

[Perform the tasks in a programming environment comfortable to you like Matlab/Octave/Python/Julia. You can use inbuilt functions for computing SVD.]

(c) What is the 2-Norm and Frobenius-Norm error between the matrix representation of the original image and the approximate image obtained for different number of singular values. Check if the following theorems hold for these errors:

For the matrix $\mathbf{A}$ of rank r, with singular values $\sigma_1,\ \sigma_2,\ \sigma_3,\ \dots\ \sigma_r$, $\mathbf{A}_v$ is the v-rank approximation of $\mathbf{A}$. ($\mathbf{A}_v = \sum_{i=1}^{v} \sigma_i \mathbf{u}_i \mathbf{v}_i$) such that $1 < v < r$, then: $\|\mathbf{A} - \mathbf{A}_v\|_2 = \sigma_{v+1}$, $\|\mathbf{A} - \mathbf{A}_v\|_F = \sqrt{\sigma_{v+1}^2 + \sigma_{v+2}^2 + \dots + \sigma_r^2}$

The image Webb's First Deep Field is as below and also downloadable from Teams assignment page as a PNG file.



Problem 3

(a) Geometrically, the orthogonal matrix is a matrix transformation that preserves 2-Norm of a matrix and causes rotation / reflection.

Can you justify $\mathbf{I} - 2\mathbf{P}$ is orthogonal matrix if $\mathbf{P}$ is orthogonal projector?

Prove the same algebraically as well.

(b) Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and its projector $\mathbf{P}$ which projects all vectors orthogonally on to column space of $\mathbf{A}$, then answer the following questions:

- If $\mathbf{A}$ is full rank, what is $\mathbf{P}$?
- Given $\mathbf{P}$ is there any way to find out the null space of $\mathbf{A}$?
- What can you say about the eigen-values of $\mathbf{P}$?

(c) If $\mathbf{P} \in \mathbb{R}^{m \times m}$ be a non-zero projection. Show that $\|\mathbf{P}\|_2 \geqslant 1$ with equality, if and only if $\mathbf{P}$ is orthogonal projector.

Problem 4

Given below is a magic matrix of size 3.

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

Find **Q** and **R** from QR factorization of given matrix by hand. Now do the following in Matlab/ Octave/ Python. For Matlab the command is:

$[\mathbf{Q}, \mathbf{R}] = \text{qr}(\text{magic}(3))$

Do these **Q** and **R** match your **Q** and **R**? Is the QR factorization unique? If not unique, can you impose a condition on **R** to make the factorization unique?

Problem 5

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 \mid & x \mid & x^2 \mid & \ldots \mid & x^{n-1} \end{bmatrix}$$

Each column is a function in $L^2[-1, 1]$ i.e., a vector space of real-valued function on $[-1, 1]$ which has inner-product of two functions f and g defined as:

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx \tag{1}$$

If the QR factorization of **A** using the above definition of inner product can be written as

$$\mathbf{A} = \mathbf{QR} = \begin{bmatrix} q_0(x) \mid & q_1(x) \mid & q_2(x) \mid & \ldots \mid & q_{n-1}(x) \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ 0 & r_{22} & \ldots & r_{2n} \\ 0 & 0 & \ldots & r_{3n} \\ . & & & \\ . & & & \\ 0 & 0 & \ldots & r_{mn} \end{bmatrix}$$

where columns of **Q** are functions of x, and are orthonormal with respect to inner product defined in equation (1). Now answer the following:

(a) Consider $n = 4$, derive expressions of $q_0(x), q_1(x), q_2(x), q_3(x)$ by using Gram Schmidt orthogonalization procedure.

(b) Show that $\int_{-1}^{1} q_{n-1}(x)dx = 0$ for $n \geqslant 2$.

Note that these $q_{n-1}(x)$ are called Legendre polynomials and the roots of these polynomials are called Gauss Legendre points and play a very important role in numerical integration as quadrature rules.

Problem 6

Let matrix:

$$\mathbf{A} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

(a) Consider a computer which rounds all computed results to five digits of relative accuracy. Using CGS or MGS, what will be the matrix $\mathbf{Q}$ associated with QR decomposition of $\mathbf{A}$ assuming that you are working on such a computer.

(b) Apply Householder's method to compute QR factorization of $\mathbf{A}$ using the same 5 digit arithmetic.

Compare the $\mathbf{Q}'s$ obtained in (a) and (b) and comment on the orthogonal nature of the $\mathbf{Q}$ matrix.