Numerical Methods (DS288): Assignment 5

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Question 1:

The exact solution is given by: $y(x) = x + 3e^{-x/2} - 2$

Table for Comparison:

$h = 1$

x	y exact	Euler	ModifiedEuler	Taylor	RK	Euler Error	Modified Error	Taylor Error	RK Error
0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
1	0.8195919791	0.5	0.875	0.8203125	0.8203125	0.3195919791	0.0554080209	0.0007205209	0.0007205209
2	1.1036383235	0.75	1.171875	1.1045125326	1.1045125326	0.3536383235	0.0682366765	0.000874209	0.000874209
3	1.6693904804	1.375	1.732421875	1.6701859898	1.6701859898	0.2943904804	0.0630313946	0.0007955094	0.0007955094

$h = 0.5$

x	y exact	Euler	ModifiedEuler	Taylor	RK	Euler Error	Modified Error	Taylor Error	RK Error
0.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
0.5	0.8364023492	0.75	0.84375	0.8364257812	0.8364257812	0.0864023492	0.0073476508	2.3432e-05	2.3432e-05
1.0	0.8195919791	0.6875	0.8310546875	0.8196284771	0.8196284771	0.1320919791	0.0114627084	3.6498e-05	3.6498e-05
1.5	0.9170996582	0.765625	0.9305114746	0.9171422954	0.9171422954	0.1514746582	0.0134118164	4.26372e-05	4.26372e-05
2.0	1.1036383235	0.94921875	1.1175870895	1.1036825982	1.1036825982	0.1544195735	0.013948766	4.42747e-05	4.42747e-05
2.5	1.3595143906	1.2119140625	1.3731149137	1.3595574923	1.3595574923	0.1476003281	0.0136005231	4.31017e-05	4.31017e-05
3.0	1.6693904804	1.5339355469	1.6821210263	1.6694307618	1.6694307618	0.1354549336	0.0127305459	4.02814e-05	4.02814e-05

$h = 0.25$

x	y exact	Euler	ModifiedEuler	Taylor	RK	Euler Error	Modified Error	Taylor Error	RK Error
0.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
0.25	0.8974907078	0.875	0.8984375	0.8974914551	0.8974914551	0.0224907078	0.0009467922	7.473e-07	7.473e-07
0.5	0.8364023492	0.796875	0.8380737305	0.8364036682	0.8364036682	0.0395273492	0.0016713813	1.319e-06	1.319e-06
0.75	0.8118678364	0.759765625	0.8140807152	0.8118695824	0.8118695824	0.0521022114	0.0022128788	1.7461e-06	1.7461e-06
1.0	0.8195919791	0.7585449219	0.8221962564	0.8195940337	0.8195940337	0.0610470573	0.0026042772	2.0545e-06	2.0545e-06
1.25	0.8557842856	0.7887268066	0.8586576326	0.8557865519	0.8557865519	0.0670574789	0.002873347	2.2664e-06	2.2664e-06
1.5	0.9170996582	0.8463859558	0.9201430663	0.9171020583	0.9171020583	0.0707137024	0.003043408	2.4001e-06	2.4001e-06
1.75	1.000586059	0.9280877113	1.0037200507	1.0005885301	1.0005885301	0.0724983477	0.0031339916	2.4711e-06	2.4711e-06
2.0	1.1036383235	1.0308267474	1.1067997322	1.1036408158	1.1036408158	0.0728115761	0.0031614087	2.4923e-06	2.4923e-06
2.25	1.2239574021	1.151973404	1.2270966386	1.2239598764	1.2239598764	0.0719839981	0.0031392365	2.4743e-06	2.4743e-06
2.5	1.3595143906	1.2892267285	1.3625931263	1.3595168168	1.3595168168	0.0702876621	0.0030787357	2.4262e-06	2.4262e-06
2.75	1.5085187874	1.4405733874	1.5115079943	1.5085211426	1.5085211426	0.0679454	0.0029892069	2.3552e-06	2.3552e-06
3.0	1.6693904804	1.604251714	1.6722687762	1.6693927479	1.6693927479	0.0651387664	0.0028782958	2.2674e-06	2.2674e-06

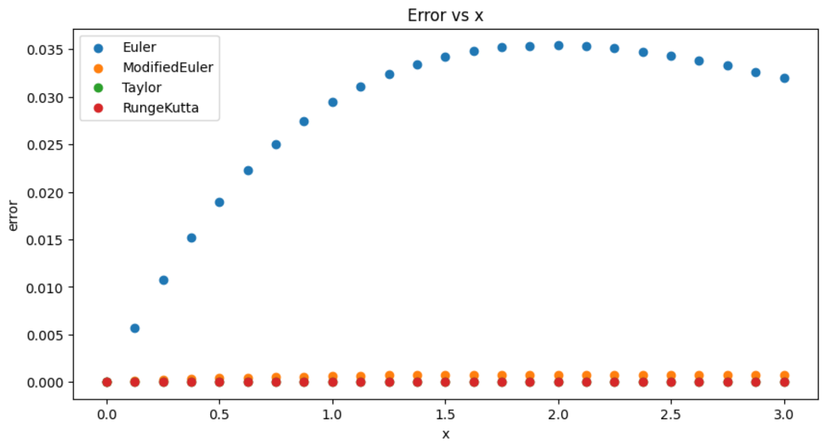
$h = 0.125$

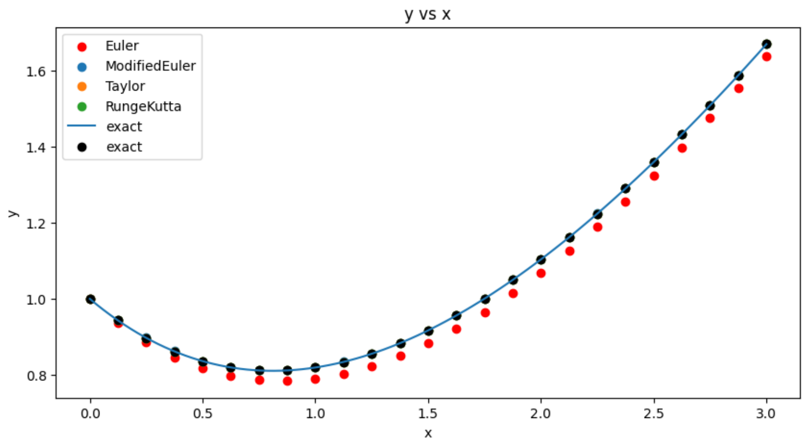
x	y exact	Euler	ModifiedEuler	Taylor	RK	Euler Error	ModEuler Error	Taylor Error	RK Error
0.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
0.125	0.9432391884	0.9375	0.943359375	0.943239212	0.943239212	0.0057391884	0.0001201866	2.36e-08	2.36e-08
0.25	0.8974907078	0.88671875	0.8977165222	0.8974907521	0.8974907521	0.0107719578	0.0002258145	4.43e-08	4.43e-08
0.375	0.8620873545	0.8469238281	0.8624055609	0.862087417	0.862087417	0.0151635264	0.0003182064	6.25e-08	6.25e-08
0.5	0.8364023492	0.8174285889	0.8368009273	0.8364024275	0.8364024275	0.0189737603	0.0003985781	7.82e-08	7.82e-08
0.625	0.8198468868	0.7975893021	0.8203149337	0.8198469787	0.8198469787	0.0222575848	0.0004680469	9.19e-08	9.19e-08
0.75	0.8118678364	0.7868024707	0.8123954748	0.81186794	0.81186794	0.0250653657	0.0005276384	1.036e-07	1.036e-07
0.875	0.8119455793	0.7845023163	0.8125238738	0.8119456928	0.8119456928	0.027443263	0.0005782945	1.135e-07	1.135e-07
1.0	0.8195919791	0.7901584215	0.820212858	0.819592101	0.819592101	0.0294335576	0.0006208789	1.219e-07	1.219e-07
1.125	0.8343484742	0.8032735202	0.8350046576	0.834348603	0.834348603	0.031074954	0.0006561834	1.288e-07	1.288e-07
1.25	0.8557842856	0.8233814251	0.8564692194	0.85578442	0.85578442	0.0324028604	0.0006849338	1.344e-07	1.344e-07
1.375	0.8834947339	0.8500450861	0.8842025283	0.8834948728	0.8834948728	0.0334496478	0.0007077944	1.389e-07	1.389e-07
1.5	0.9170996582	0.8828547682	0.9178250315	0.9170998006	0.9170998006	0.03424489	0.0007253733	1.424e-07	1.424e-07
1.625	0.9562419302	0.9214263452	0.9569801566	0.9562420751	0.9562420751	0.0348155851	0.0007382263	1.449e-07	1.449e-07
1.75	1.000586059	0.9653996986	1.0013329205	1.0005862056	1.0005862056	0.0351863604	0.0007468615	1.466e-07	1.466e-07
1.875	1.04981688	1.0144372174	1.0505686226	1.0498170276	1.0498170276	0.0353796626	0.0007517426	1.475e-07	1.475e-07
2.0	1.1036383235	1.0682223914	1.1043916161	1.1036384714	1.1036384714	0.0354159322	0.0007532926	1.478e-07	1.478e-07
2.125	1.1617722577	1.1264584919	1.162524155	1.1617724053	1.1617724053	0.0353137658	0.0007518973	1.476e-07	1.476e-07
2.25	1.2239574021	1.1888673362	1.2247053097	1.2239575489	1.2239575489	0.0350900659	0.0007479076	1.468e-07	1.468e-07
2.375	1.2899483061	1.2551881276	1.2906899491	1.2899484517	1.2899484517	0.0347601785	0.000741643	1.455e-07	1.455e-07
2.5	1.3595143906	1.3251763697	1.3602477843	1.3595145345	1.3595145345	0.0343380209	0.0007333937	1.439e-07	1.439e-07
2.625	1.4324390462	1.3986028466	1.4331624692	1.4324391882	1.4324391882	0.0338361996	0.000723423	1.42e-07	1.42e-07
2.75	1.5085187874	1.4752526687	1.5092307572	1.5085189271	1.5085189271	0.0332661188	0.0007119698	1.397e-07	1.397e-07
2.875	1.5875624573	1.5549243769	1.5882617074	1.5875625945	1.5875625945	0.0326380804	0.0006992502	1.372e-07	1.372e-07
3.0	1.6693904804	1.6374291033	1.67007594	1.669390615	1.669390615	0.0319613771	0.0006854596	1.345e-07	1.345e-07

Adams-Bashforth Four Step Explicit vs Adams-Moulton Three Step Implicit:

x	y exact	Adam Bashforth	Absolute Error (Bash)	Adam Moulton	Absolute Error (Moulton)
0.375	0.8620873545	-	-	0.8123553871	0.0497319674
0.5	0.8364023492	0.8364032909485142	9.41734299653163e-07	0.8620873297	2.48e-08
0.625	0.8198468868	0.8198485297052864	1.6428653613065336e-06	0.8364022643	8.49e-08
0.75	0.8118678364	0.8118701634853724	2.3271124558466028e-06	0.8198467491	1.378e-07
0.875	0.8119455793	0.8119484688210935	2.88953741722775e-06	0.8118676524	1.84e-07
1.0	0.8195919791	0.819595364341713	3.3852038124893014e-06	0.8119453553	2.24e-07
1.125	0.8343484742	0.8343522792897314	3.805096962516963e-06	0.8195917206	2.586e-07
1.25	0.8557842856	0.8557884492382046	4.163681233793071e-06	0.8343481861	2.881e-07
1.375	0.8834947339	0.8834991979588801	4.464046057361948e-06	0.8557839724	3.131e-07
1.5	0.9170996582	0.9171043712991692	4.713076125262283e-06	0.8834943999	3.34e-07
1.625	0.9562419302	0.9562468456479026	4.915404662964029e-06	0.9170993069	3.513e-07
1.75	1.000586059	1.0005911350090526	5.075973527501887e-06	0.956241565	3.652e-07
1.875	1.04981688	1.0498220790488677	5.19901847084725e-06	1.0005856829	3.762e-07
2.0	1.1036383235	1.1036436120451347	5.288530807723291e-06	1.0498164956	3.844e-07
2.125	1.1617722577	1.1617776058391371	5.348108213398817e-06	1.1036379332	3.903e-07
2.25	1.2239574021	1.2239627831302407	5.381055191433504e-06	1.1617718636	3.941e-07
2.375	1.2899483061	1.2899536965107503	5.390377572522809e-06	1.2239570061	3.96e-07
2.5	1.3595143906	1.3595197693986438	5.378818073387137e-06	1.28994791	3.962e-07
2.625	1.4324390462	1.4324443950599517	5.348872400112725e-06	1.3595139957	3.949e-07
2.75	1.5085187874	1.5085240902253085	5.302811069363145e-06	1.4324386539	3.923e-07
2.875	1.5875624573	1.5875676999831005	5.242696726304175e-06	1.5085183988	3.886e-07
3.0	1.6693904804	1.6693956508466221	5.170401332410535e-06	1.5875620734	3.839e-07

Note: $xe-y$ within the table stands for $x \times 10^y$.

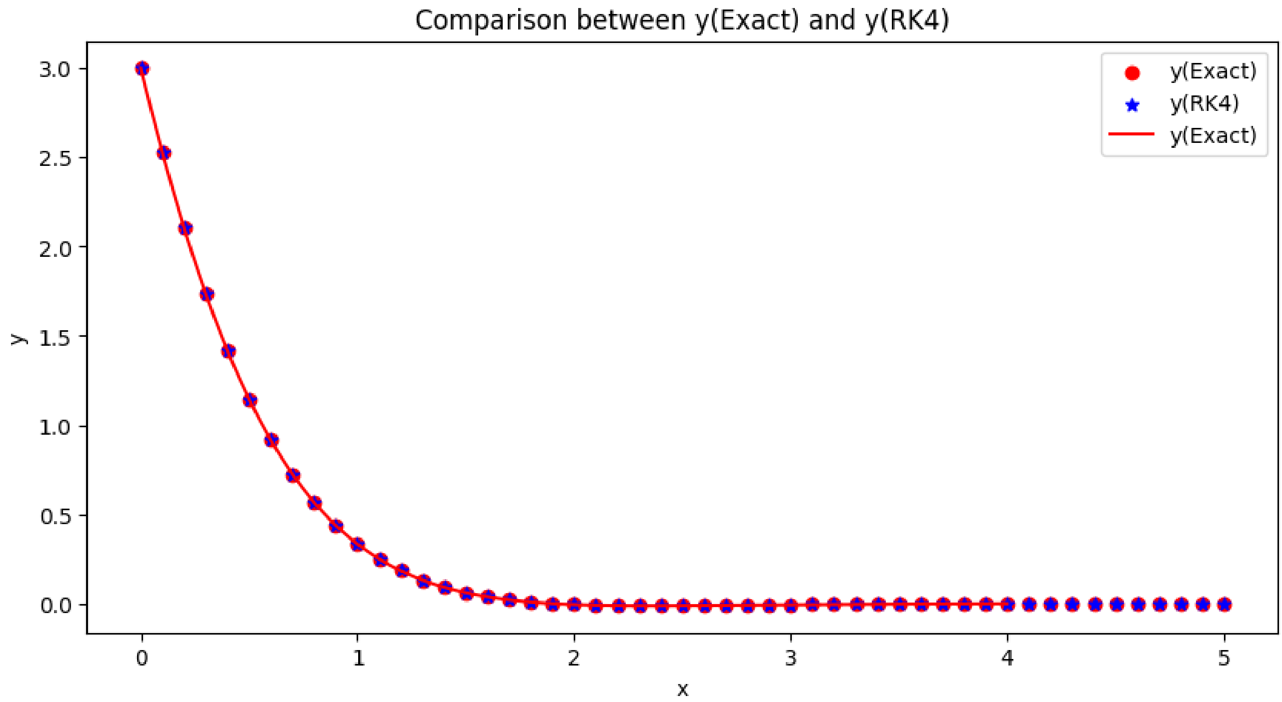


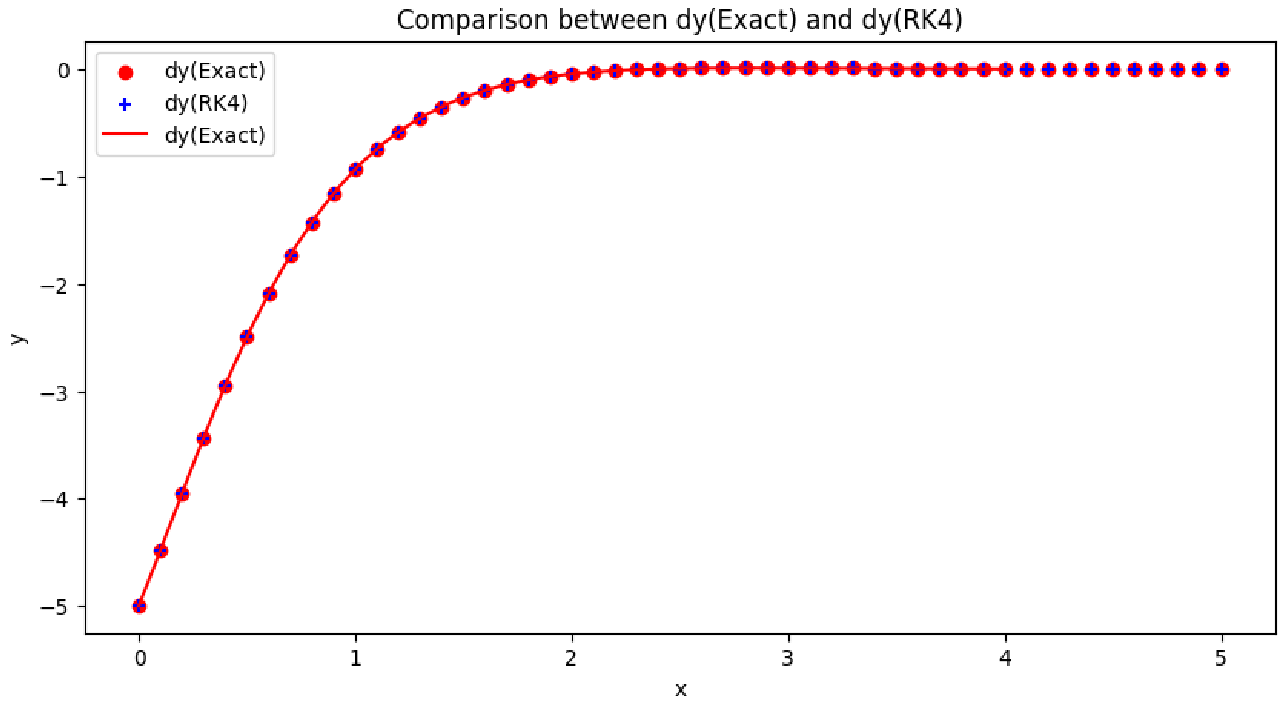


Question 2:
Put $u_1 = y$ (eq 1) and $u_2 = y'$ (eq 2).
Differentiating both the equations and combining, we obtain:
$u_1' = y' = u_2$
$u_2' = y'' = -(4y' + 5y) = -(4u_2 + 5u_1)$
It's been given that $u_1(0) = 3$ and $u_2(0) = -5$.

So, the given second order differential equation can be written as:
$u_1' = u_2$ with the initial condition $u_1(0) = 3$
$u_2' = -(4u_2 + 5u_1)$ with the initial condition $u_2(0) = -5$.
Table of outcome:

x	y(Exact)	y(RK4)	dy(Exact)	dy(RK4)	$\lvert y(RK4) - y(Exact)\rvert$	$\lvert dy(RK4) - dy(Exact)\rvert$
0.0	3.0	3.0	-5.0	-5.0	0.0	0.0
0.1	2.525658217055026	2.525645833333334	-4.481885989737062	-4.481854166666666	1.2383721693e-05	3.1823070395e-05
0.2	2.104046855694013	2.104027832552083	-3.950651542704738	-3.950599976475694	1.902314193e-05	5.1566229044e-05
0.3	1.735084272968695	1.735062685057882	-3.432423548670007	-3.432361519751276	2.1587910814e-05	6.2028918731e-05
0.4	1.416555087330076	1.41653369346863	-2.944181613409432	-2.944115995078026	2.1393861445e-05	6.5618331406e-05
0.5	1.144904546575131	1.14488509380768	-2.496076908375324	-2.496012568675801	1.9452767452e-05	6.4339699524e-05
0.6	0.915825975073754	0.91580945332808	-2.093266774410837	-2.093206959357834	1.6521745674e-05	5.9815053004e-05
0.7	0.724685409626735	0.72467226002807	-1.737349435345552	-1.737296116657582	1.3149598665e-05	5.331868797e-05
0.8	0.566819672956992	0.566809954399726	-1.427471777977892	-1.427425956962153	9.718557266e-06	4.5821015738e-05
0.9	0.437737377263264	0.437730896557476	-1.161172519690354	-1.161134484614697	6.480705788e-06	3.8035075656e-05
1.0	0.333246610858547	0.333243021925761	-0.935013398312139	-0.934982936351778	3.588932786e-06	3.0461960361e-05
1.1	0.249528238875141	0.24952711626036	-0.745042365627117	-0.745018932776962	1.122614781e-06	2.3432850155e-05
1.2	0.183169739798695	0.183170631338272	-0.587125160546764	-0.587108014150552	8.91539577e-07	1.7146396212e-05
1.3	0.131171199758918	0.131173662699689	-0.457175047093682	-0.457163346171334	2.462940771e-06	1.1700922349e-05
1.4	0.090932397606581	0.09093602188309	-0.351304861992977	-0.351297740585782	3.624276509e-06	7.121407195e-06
1.5	0.060227744781402	0.060232166573397	-0.265920744951194	-0.26591736342825	4.421791996e-06	3.381522944e-06
1.6	0.03717411244622	0.037179020588598	-0.197772931072261	-0.197772509881038	4.908142378e-06	4.21191223e-07
1.7	0.020195211156307	0.020200348476017	-0.143975676691798	-0.143977516483534	5.13731971e-06	1.839791737e-06
1.8	0.007985121447557	0.007990282665256	-0.102005676986159	-0.102009168467046	5.161217699e-06	3.491480887e-06
1.9	-0.000527248062221	-0.000522220601641	-0.069686131126713	-0.069690757420861	5.02746058e-06	4.626294148e-06
2.0	-0.006211622236465	-0.006206844052916	-0.045161840646539	-0.045167174432836	4.778183549e-06	5.333786297e-06
2.1	-0.009767053262399	-0.009762603753652	-0.026869319128465	-0.026875016345117	4.449508748e-06	5.697216652e-06
2.2	-0.011749499589537	-0.011745428078236	-0.013504784551111	-0.013510576055304	4.0715113e-06	5.791504193e-06
2.3	-0.012596185072761	-0.01259251655886	-0.003992045901124	-0.00399772812744	3.668513901e-06	5.682226316e-06
2.4	-0.012646800146462	-0.012643540560656	0.002548363187895	0.002542937810869	3.259585806e-06	5.425377026e-06
2.5	-0.012161716073231	-0.012158856920055	0.006827948164047	0.006822880509207	2.859153176e-06	5.067654841e-06
2.6	-0.011337449501512	-0.011334971848457	0.009416427413708	0.00941178031491	2.477653055e-06	4.647098798e-06
2.7	-0.010319649199824	-0.010317527016287	0.010765095925305	0.01076090199288	2.122183537e-06	4.193932426e-06
2.8	-0.009213888893341	-0.009212091774775	0.011227346402525	0.011223614892111	1.797118567e-06	3.731510414e-06
2.9	-0.00809454655939	-0.008093041891441	0.011076442792351	0.011073165500933	1.504667949e-06	3.277291418e-06
3.0	-0.007012036640346	-0.007010791268083	0.010520722643411	0.010517878859849	1.245372262e-06	2.843783561e-06
3.1	-0.005998641381074	-0.005997622852131	0.009716452067725	0.009714012640291	1.018528943e-06	2.439427434e-06
3.2	-0.005073163819057	-0.005072341268631	0.008778579640354	0.008776510244893	8.22550426e-07	2.069395461e-06
3.3	-0.004244599950501	-0.004243944692254	0.007789640585412	0.007787904288373	6.55258246e-07	1.736297039e-06
3.4	-0.003515002728319	-0.003514488609427	0.006807055642833	0.006805614856281	5.14118892e-07	1.440786552e-06
3.5	-0.002881686793139	-0.002881290364997	0.005869054319063	0.005867872242363	3.96428143e-07	1.1820767e-06
3.6	-0.002338900804413	-0.002338601353437	0.00499943295029	0.004998474587115	2.99450975e-07	9.58363175e-07
3.7	-0.001879074262368	-0.001878853738382	0.004211336450868	0.004210569282059	2.20523985e-07	7.6716881e-07
3.8	-0.001493727934283	-0.001493570807416	0.003510230419085	0.003509624802625	1.57126867e-07	6.0561646e-07
3.9	-0.001174121424983	-0.001174014496077	0.002896208550892	0.00289573791071	1.06928906e-07	4.70640183e-07
4.0	-0.000911697974179	-0.000911630158438	0.002365759803229	0.002365400659162	6.7815741e-08	3.59144067e-07
4.1	-0.000698375076843	-0.000698337175885	0.001913100894006	0.001912832776475	3.7900958e-08	2.68117531e-07
4.2	-0.000526719828896	-0.000526704302541	0.001531162754032	0.001530968038963	1.5526355e-08	1.94715069e-07
4.3	-0.000390039805948	-0.000390040551865	0.001212304532968	0.001212168225377	7.45917e-10	1.36307591e-07
4.4	-0.000282413592574	-0.000282425739327	0.000948815677172	0.000948725165654	1.2146754e-08	9.0511518e-08
4.5	-0.00019867960524	-0.000198699322204	0.0007332553431	0.000733200142195	1.9716964e-08	5.5200904e-08
4.6	-0.000134397417326	-0.000134421743339	0.00055866884427	0.000558640337267	2.4326013e-08	2.8507003e-08
4.7	-8.5792238936e-05	-8.5818930352e-05	0.000418712788594	0.00041870397973	2.6691416e-08	8.808864e-09
4.8	-4.9690383516e-05	-4.9717781413e-05	0.000307713873504	0.000307719155605	2.7397897e-08	5.282101e-09
4.9	-2.3451341141e-05	-2.3478256839e-05	0.00022068079769	0.000220695738992	2.6915698e-08	1.4941302e-08
5.0	-4.900364828e-06	-4.92598244e-06	0.000153284257105	0.000153305409234	2.5617612e-08	2.1152129e-08


Comparison between y(Exact) and y(RK4)
y(Exact)
y(RK4)
y(Exact)
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0
1
2
3
4
5
x


Comparison between dy(Exact) and dy(RK4)
dy(Exact)
dy(RK4)
dy(Exact)
0
−1
−2
−3
−4
−5
y
0
1
2
3
4
5
x

Question 3:

Fixed Point Iteration:

The general iteration formula is given by: $\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)})$ for $k = 1, 2, 3...$
A screenshot from the textbook (page 634), which describes the iteration scheme in this particular problem.

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Solution Solving the ith equation for x_i gives the fixed-point problem

$$\begin{aligned} x_1 &= \frac{1}{3}\cos(x_2x_3) + \frac{1}{6}, \\ x_2 &= \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ x_3 &= -\frac{1}{20}e^{-x_1x_2} - \frac{10\pi - 3}{60}. \end{aligned} \tag{10.4}$$

Let $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}))^t$, where

$$\begin{aligned} g_1(x_1, x_2, x_3) &= \frac{1}{3}\cos(x_2x_3) + \frac{1}{6}, \\ g_2(x_1, x_2, x_3) &= \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ g_3(x_1, x_2, x_3) &= -\frac{1}{20}e^{-x_1x_2} - \frac{10\pi - 3}{60}. \end{aligned}$$

Result:
$\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$

Iteration	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lVert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \rVert_\infty$
0	0.1	0.1	-0.1	-
1	0.4999833335	0.0222297936	-0.5230461262	0.4230461261913656
2	0.4999774683	2.81537e-05	-0.5235980718	0.022201639896641867
3	0.5	3.76e-08	-0.5235987747	2.8116039917230884e-05
4	0.5	1e-10	-0.5235987756	3.757173792917623e-08

Newton's Method:
$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - \mathbf{J}(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)})$ for $k = 1, 2, 3...$, where $\mathbf{J}$ is the Jacobian of the set of functions. In this case, the computation from the example is attached below.

Example 1 The nonlinear system

$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,$$
$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$
$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

was shown in Example 2 of Section 10.1 to have the approximate solution $(0.5, 0, -0.52359877)^t$. Apply Newton's method to this problem with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

Solution Define

$$\mathbf{F}(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))^t,$$

where

$$f_1(x_1, x_2, x_3) = 3x_1 - \cos(x_2x_3) - \frac{1}{2},$$
$$f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06,$$

and

$$f_3(x_1, x_2, x_3) = e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3}.$$

The Jacobian matrix $J(\mathbf{x})$ for this system is

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_3 \sin x_2x_3 & x_2 \sin x_2x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20 \end{bmatrix}.$$

Let $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$. Then $\mathbf{F}(\mathbf{x}^{(0)}) = (-0.199995, -2.269833417, 8.462025346)^t$ and

$$J(\mathbf{x}^{(0)}) = \begin{bmatrix} 3 & 9.999833334 \times 10^{-4} & 9.999833334 \times 10^{-4} \\ 0.2 & -32.4 & 0.9950041653 \\ -0.09900498337 & -0.09900498337 & 20 \end{bmatrix}.$$

Solving the linear system, $J(\mathbf{x}^{(0)})\mathbf{y}^{(0)} = -\mathbf{F}(\mathbf{x}^{(0)})$ gives

$$\mathbf{y}^{(0)} = \begin{bmatrix} 0.3998696728 \\ -0.08053315147 \\ -0.4215204718 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{y}^{(0)} = \begin{bmatrix} 0.4998696782 \\ 0.01946684853 \\ -0.5215204718 \end{bmatrix}.$$

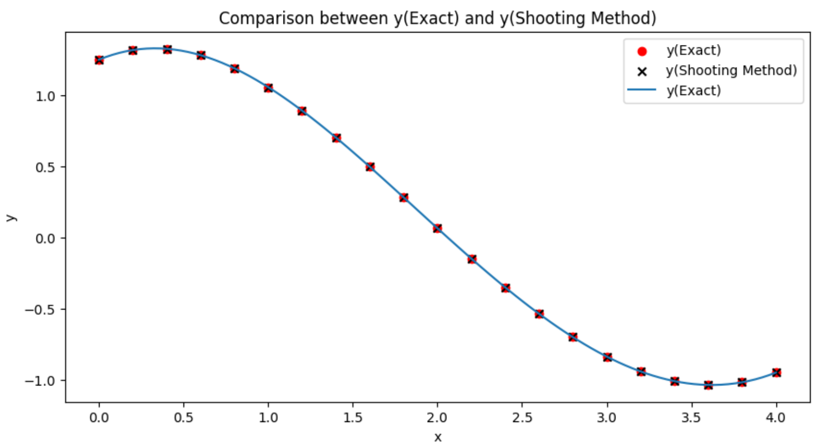
Result:
$\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$

Iteration	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\| \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \|\|_\infty$
0	0.1	0.1	-0.1	-
1	0.4998696729	0.0194668485	-0.5215204719	0.42152047193583064
2	0.5000142402	0.0015885914	-0.5235569643	0.017878257167124205
3	0.5000001135	1.24448e-05	-0.5235984501	0.0015761465869723395
4	0.5	8e-10	-0.5235987756	1.2444007535830790e-05
5	0.5	0.0	-0.5235987756	7.757857127143585e-10

Question 4:

a) Table for comparison

	x	y(Exact)	y(Shooting)	Absolute Error
0	0.0	1.25	1.25	0.0
1	0.2	1.317350212	1.3173080971	4.21149e-05
2	0.4	1.3265045603	1.3264260933	7.84671e-05
3	0.6	1.2817620876	1.281652347	0.0001097406
4	0.8	1.1894119056	1.1892756055	0.0001363002
5	1.0	1.0568859788	1.0567277081	0.0001582707
6	1.2	0.8920864725	0.8919107277	0.0001757448
7	1.4	0.7029475159	0.7027586739	0.000188842
8	1.6	0.4971871048	0.4969894112	0.0001976936
9	1.8	0.2821843958	0.2819819742	0.0002024216
10	2.0	0.0649310438	0.064727916	0.0002031278
11	2.2	-0.1479767675	-0.1481766606	0.0001998931
12	2.4	-0.3503253628	-0.3505181421	0.0001927793
13	2.6	-0.5362608188	-0.5364426521	0.0001818332
14	2.8	-0.7002624926	-0.7004295826	0.0001670901
15	3.0	-0.8371160376	-0.8372646138	0.0001485762
16	3.2	-0.9418875183	-0.9420138298	0.0001263115
17	3.4	-1.0098993544	-1.0099996655	0.0001003111
18	3.6	-1.0367083603	-1.0367789468	7.05865e-05
19	3.8	-1.0180858928	-1.0181230393	3.71465e-05
20	4.0	-0.9500000022	-0.95	2.2e-09



The algorithm has been given below:

Linear Shooting

To approximate the solution of the boundary-value problem

$$-y'' + p(x)y' + q(x)y + r(x) = 0, \quad \text{for } a \le x \le b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta,$$

(*Note: Equations* (11.3) *and* (11.4) *are written as first-order systems and solved.*)

INPUT endpoints a, b; boundary conditions α, β; number of subintervals N.

OUTPUT approximations $w_{1,i}$ to $y(x_i)$; $w_{2,i}$ to $y'(x_i)$ for each $i = 0, 1, \ldots, N$.

Step 1 Set $h = (b - a)/N$;
$u_{1,0} = \alpha$;
$u_{2,0} = 0$;
$v_{1,0} = 0$;
$v_{2,0} = 1$.

Step 2 For $i = 0, \ldots, N - 1$ do Steps 3 and 4.
(*The Runge-Kutta method for systems is used in Steps* 3 *and* 4.)

Step 3 Set $x = a + ih$.

Step 4 Set $k_{1,1} = hu_{2,i}$;

$$
\begin{aligned}
k_{1,2} &= h\left[p(x)u_{2,i} + q(x)u_{1,i} + r(x)\right];\\
k_{2,1} &= h\left[u_{2,i} + \tfrac{1}{2}k_{1,2}\right];\\
k_{2,2} &= h\left[p(x + h/2)\left(u_{2,i} + \tfrac{1}{2}k_{1,2}\right)\right.\\
&\qquad \left. + q(x + h/2)\left(u_{1,i} + \tfrac{1}{2}k_{1,1}\right) + r(x + h/2)\right];\\
k_{3,1} &= h\left[u_{2,i} + \tfrac{1}{2}k_{2,2}\right];\\
k_{3,2} &= h\left[p(x + h/2)\left(u_{2,i} + \tfrac{1}{2}k_{2,2}\right)\right.\\
&\qquad \left. + q(x + h/2)(u_{1,i} + \tfrac{1}{2}k_{2,1}) + r(x + h/2)\right];\\
k_{4,1} &= h\left[u_{2,i} + k_{3,2}\right];\\
k_{4,2} &= h\left[p(x + h)(u_{2,i} + k_{3,2}) + q(x + h)(u_{1,i} + k_{3,1}) + r(x + h)\right];\\
u_{1,i+1} &= u_{1,i} + \tfrac{1}{6}\left[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right];\\
u_{2,i+1} &= u_{2,i} + \tfrac{1}{6}\left[k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right];\\
k'_{1,1} &= hv_{2,i};\\
k'_{1,2} &= h\left[p(x)v_{2,i} + q(x)v_{1,i}\right];\\
k'_{2,1} &= h\left[v_{2,i} + \tfrac{1}{2}k'_{1,2}\right];\\
k'_{2,2} &= h\left[p(x + h/2)\left(v_{2,i} + \tfrac{1}{2}k'_{1,2}\right) + q(x + h/2)\left(v_{1,i} + \tfrac{1}{2}k'_{1,1}\right)\right];\\
k'_{3,1} &= h\left[v_{2,i} + \tfrac{1}{2}k'_{2,2}\right];\\
k'_{3,2} &= h\left[p(x + h/2)\left(v_{2,i} + \tfrac{1}{2}k'_{2,2}\right) + q(x + h/2)\left(v_{1,i} + \tfrac{1}{2}k'_{2,1}\right)\right];\\
k'_{4,1} &= h\left[v_{2,i} + k'_{3,2}\right];\\
k'_{4,2} &= h\left[p(x + h)(v_{2,i} + k'_{3,2}) + q(x + h)(v_{1,i} + k'_{3,1})\right];\\
v_{1,i+1} &= v_{1,i} + \tfrac{1}{6}\left[k'_{1,1} + 2k'_{2,1} + 2k'_{3,1} + k'_{4,1}\right];\\
v_{2,i+1} &= v_{2,i} + \tfrac{1}{6}\left[k'_{1,2} + 2k'_{2,2} + 2k'_{3,2} + k'_{4,2}\right].
\end{aligned}
$$

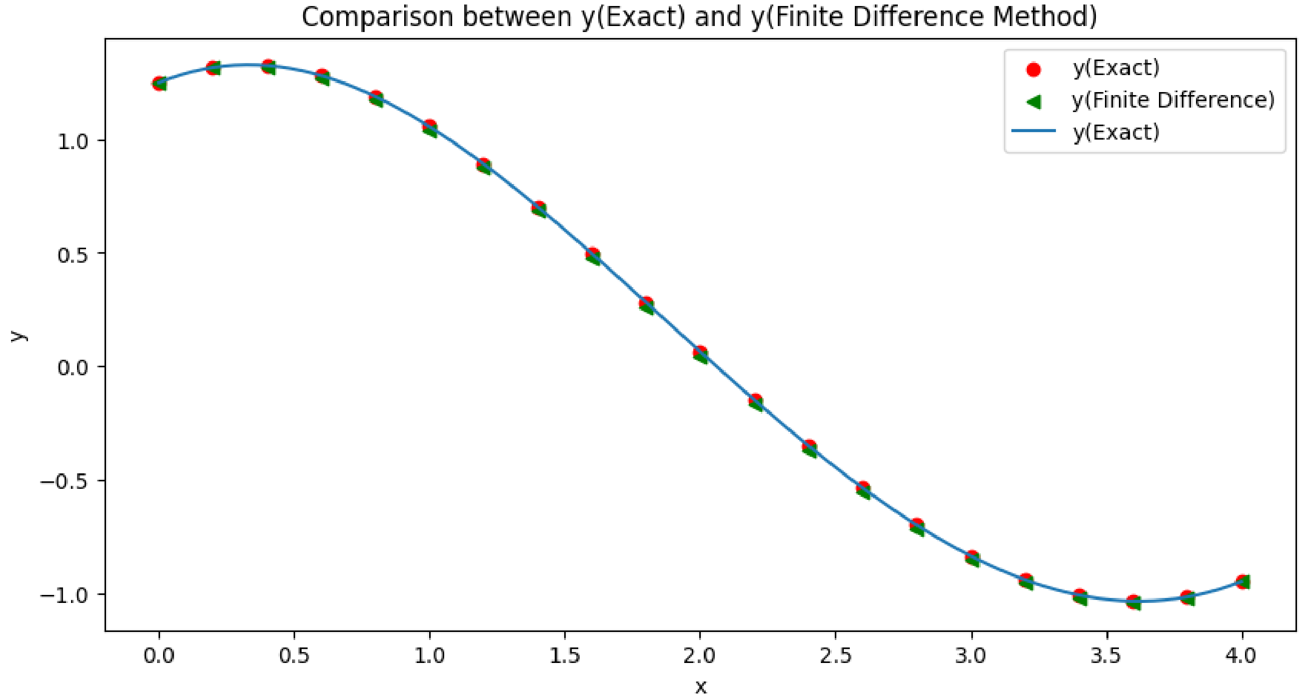
Step 5 Set $w_{1,0} = \alpha$;
$$w_{2,0} = \frac{\beta - u_{1,N}}{v_{1,N}};$$
OUTPUT $(a, w_{1,0}, w_{2,0})$.

Step 6 For $i = 1, \ldots, N$
set $W1 = u_{1,i} + w_{2,0}v_{1,i}$;
$W2 = u_{2,i} + w_{2,0}v_{2,i}$;
$x = a + ih$;
OUTPUT $(x, W1, W2)$. (*Output is* $x_i, w_{1,i}, w_{2,i}$.)

Step 7 STOP. (*The process is complete.*) ■

b) Table for comparison

	x	y(Exact)	y(Finite Difference)	Absolute Error
0	0.0	1.25	1.25	0.0
1	0.2	1.317350212	1.3145034486	0.0028467634
2	0.4	1.3265045603	1.3206068973	0.0058976631
3	0.6	1.2817620876	1.2727547904	0.0090072972
4	0.8	1.1894119056	1.1773987408	0.0120131648
5	1.0	1.0568859788	1.0421063162	0.0147796626
6	1.2	0.8920864725	0.8748775164	0.017208956
7	1.4	0.7029475159	0.6837123416	0.0192351743
8	1.6	0.4971871048	0.4763719857	0.0208151191
9	1.8	0.2821843958	0.2602638562	0.0219205396
10	2.0	0.0649310438	0.0423982623	0.0225327815
11	2.2	-0.1479767675	-0.1706161003	0.0226393327
12	2.4	-0.3503253628	-0.3725570094	0.0222316466
13	2.6	-0.5362608188	-0.5575645924	0.0213037735
14	2.8	-0.7002624926	-0.7201139873	0.0198514948
15	3.0	-0.8371160376	-0.854987803	0.0178717654
16	3.2	-0.9418875183	-0.9572498692	0.0153623509
17	3.4	-1.0098993544	-1.0222209406	0.0123215862
18	3.6	-1.0367083603	-1.0454565727	0.0087482125
19	3.8	-1.0180858928	-1.0227271584	0.0046412655
20	4.0	-0.9500000022	-0.95	2.2e-09



The algorithm is as follows:

ALGORITHM 11.3

Linear Finite-Difference

To approximate the solution of the boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x), \quad \text{for } a \le x \le b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta:$$

INPUT endpoints a, b; boundary conditions α, β; integer $N \ge 2$.

OUTPUT approximations w_i to $y(x_i)$ for each $i = 0, 1, \ldots, N+1$.

Step 1 Set $h = (b-a)/(N+1)$;
$x = a + h$;
$a_1 = 2 + h^2 q(x)$;
$b_1 = -1 + (h/2)p(x)$;
$d_1 = -h^2 r(x) + (1 + (h/2)p(x))\alpha$.

Step 2 For $i = 2, \ldots, N-1$
set $x = a + ih$;
$a_i = 2 + h^2 q(x)$;
$b_i = -1 + (h/2)p(x)$;
$c_i = -1 - (h/2)p(x)$;
$d_i = -h^2 r(x)$.

Step 3 Set $x = b - h$;
$a_N = 2 + h^2 q(x)$;
$c_N = -1 - (h/2)p(x)$;
$d_N = -h^2 r(x) + (1 - (h/2)p(x))\beta$.

Step 4 Set $l_1 = a_1$; (*Steps 4–8 solve a tridiagonal linear system using Algorithm* 6.7.)
$u_1 = b_1/a_1$;
$z_1 = d_1/l_1$.

Step 5 For $i = 2, \ldots, N-1$ set $l_i = a_i - c_i u_{i-1}$;
$u_i = b_i/l_i$;
$z_i = (d_i - c_i z_{i-1})/l_i$.

Step 6 Set $l_N = a_N - c_N u_{N-1}$;
$z_N = (d_N - c_N z_{N-1})/l_N$.

Step 7 Set $w_0 = \alpha$;
$w_{N+1} = \beta$.
$w_N = z_N$.

Step 8 For $i = N-1, \ldots, 1$ set $w_i = z_i - u_i w_{i+1}$.

Step 9 For $i = 0, \ldots, N+1$ set $x = a + ih$;
OUTPUT (x, w_i).

Step 10 STOP. (*The procedure is complete.*) ■