Numerical Methods (DS288): Assignment 1

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Question 1: There are multiple definitions of "Algebraic Functions", however we wish to proceed with the definition as provided in the Wikipedia article: `https://en.wikipedia.org/wiki/Algebraic_function` as well as the Wolfram Mathematica article: `https://mathworld.wolfram.com/AlgebraicFunction.html`

- $f_1(x) = x + \pi$, continous on entire $\mathbb{R}$. Suppose that there exists $p_1(x, y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_0(x) = 0$ when $y = f_1(x)$, with each $a_i(x)$ being polynomials in x with integer coefficients, $a_n(x) \neq 0$.[i.e we are assuming that f_1 is an algebraic function of degree n, as per the Wikipedia Definition]

Clearly, $a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_0(x) = \sum_{i=0}^{n}(x+\pi)^i a_i(x) = 0 \implies$

$\sum_{i=0}^{n} x^i a_i(x) = -\sum_{i=0}^{n}({}^iC_1 x^{i-1}\pi +^i C_2 x^{i-2}\pi^2 + ... + \pi^i)a_i(x)$

Evidently, LHS is a polynomial over the integers, while the Right Hand Side cannot be expressed in any way over the set of integers, since π is transcendental over $\mathbb{Z}$, i.e for any set, say $\mathcal{M} = \{\pi^{i_k} : i_k \in \{1, 2, ...\}, k \in \{1, 2, ...p\}\} = \{\pi^{i_1}, \pi^{i_2}, ...\pi^{i_p}\}$ is linearly independent over $\mathbb{Z}$. Even $\mathcal{M} \cup \{\gamma\}$ is L.I over $\mathbb{Z}$ as well, where $\gamma \in \mathbb{Q} \setminus \{0\}$.
Since LHS and RHS are identically equal, it forces $a_i(x) = 0$ for all i, which contradicts our assumption that $a_n(x) \neq 0$. This can be seen by plugging in any integer x such that the LHS doesn't vanish, say $x = \theta$ (without loss of generality) on both sides. LHS becomes an integer, RHS is a linear combination of distinct powers of π over $\mathbb{Z}$, an equality which cannot hold.
Hence, $f_1(x)$ cannot be algebraic.

[Note: As per the Wiki article, "If transcendental numbers occur in the coefficients the function is, in general, not algebraic..."]

- $f_2(x) = x + \sin(\pi) = x$, which is continuous on its domain $\mathbb{R}$. Thus, we are to construct a non-zero polynomial p_2 in two variables x, y over the integers such that $p_2(x, f_2(x)) = 0$. Consider $p_2(x, y) = x - y$. Evidently $p_2(x, f_2(x)) = p_2(x, x) = 0$; which implies that f_2 is indeed an algebraic function.

- $f_3(x) = x!$, $x \in \mathbb{Z}^+$. Consider a polynomial $p_3(x, y) = a_n(x)y^n + ... + a_0(x)$ with $a_n(x) \neq 0$ over $\mathbb{Z}$ such that $p_3(x, x!) = 0 \ \forall \ x \in \mathbb{Z}$.
Hence, $a_n(x)(x!)^n + ... + a_1(x)x! = -a_0(x)$ ... (*)

Now, let us compute which one between the terms $c(x!)^m$ and $dx^\alpha(x!)^{m-1}$ with $m \in \mathbb{Z}$, will grow faster (in magnitude) and by how much, with both c and $d \neq 0$, α being a constant. Dividing the first term by the latter, we obtain $K = \frac{|c|x!}{|d|x^\alpha}$. Now, despite of x^α being present in the denominator, and regardless of how big α is, K still grows faster than the exponential function as $x \to \infty$.

Having shown how much $c(x!)^m$ dominates over the other term, it's safe to assert that even the term with the least degree in $a_n(x)$, say $c_p x^p$ multiplied with $(x!)^n$ will dominate over all the entire LHS of $(*)$. Now, note that the RHS of $(*)$ is just a polynomial which is supposed to be identical to the left hand side. This is an impossibility given the growth rate of the LHS (faster than exponential). So, the only possible scenario is that the LHS is indentically $0 \implies a_0(x) = 0$. Afterwards, consider $a_n(x)(x!)^n + ... + a_1(x)(x!) = 0 \implies a_n(x!)^{n-1} + ... + a_1(x) = 0$ on $\mathbb{Z}$. Repeating the same argument, we arrive at $a_1(x) = 0$. Iterating the scheme, we finally get $a_n(x)x! = 0$ everywhere on $\mathbb{Z}$, which in turn implies $a_n(x) = 0$ on $\mathbb{Z}$. A polynomial cannot have infinitely many roots, so $a_n(x)$ must be 0, contradicting our initial assumption that $p_3(x, y)$ has $a_n(x) \neq 0$. Thus, no such polynomial p_3 exists, i.e $x!$ is transcendental over $\mathbb{Z}^+$.

- $f_4(x) = \dfrac{1}{x + x^2}$ is continous on its domain $\mathbb{R} \setminus \{0, -1\}$. For this function, consider the polynomial $p_4(x, y) \in \mathbb{Z}[x, y]$ (i.e a polynomial over integers in two varibles.) defined as $p_4(x, y) = xy + x^2y - 1$. Clearly, $p_4(x, f(x)) = 0$ identically on the domain of definition of f_4.

- $f_5(x) = x + \sin(x)[1 + 2\cos(2x)] - \sin(3x) = x + \sin(x) + 2\sin(x)\cos(2x) - \sin(3x) = x + \sin(x) + \sin(3x) - \sin(x) - \sin(3x) = x$ is continous on $\mathbb{R}$. Since $f_5(x)$ is "Identically Equal" to x, i.e holds regardless of the value of x on the entire real line, we can take $f_5(x) = x$. Since we've already established the algebraic nature for $f_2(x) = x$, the same argument holds here as well.

Question 2:

1. Choose interval $[a_n, b_n]$ such that $f(a_n)f(b_n) < 0$.
2. Select the midpoint $c_n = \dfrac{(a_n + b_n)}{2}$
3. If $f(a_n)f(c_n) < 0$, set $b_{n+1} = c_n$ (i.e, the new right endpoint becomes c_n)
4. Otherwise, check if $f(c_n) = 0$.
5. If not, set $a_{n+1} = c_n$
6. Repeat until c_n is within tolerance, $\quad n = 0, 1, 2...$

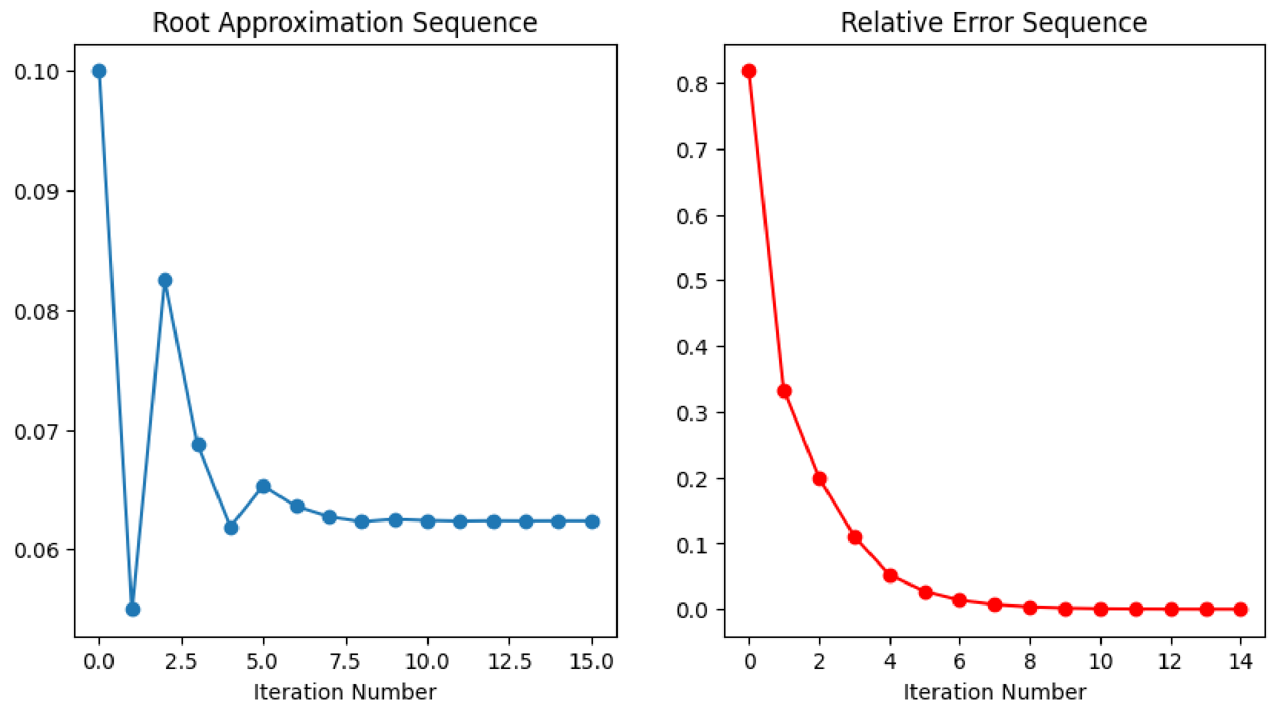
Error Term: $\epsilon_n = \dfrac{|x_n - x_{n-1}|}{|x_n|}, \quad n = 1, 2, ...$

Prescribed Error: 10^{-4}, i.e. 4 significant figures.

The function here is: $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$

The specific gravity of the ball has been given as 0.6, while water's is 1. Therefore, the ball must float on water, and the maximum possible depth is its diameter, i.e $0.055 \times 2 = 0.11$ metres.

Depth is a $+ve$ quantity, so the root must lie within $[0, 0.11]$.
Taking this interval as the starting interval $[a_0, b_0]$, we get:
The root rounded off to 4 significant figures is 0.06238, obtained with 15 iterations



Question 3:

The iteration scheme for the Newton Raphson Method is:
1. Take an initial guess x_0.
2. Use the iteration scheme: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,
3. If $f(x_{n+1}) = 0$ or error term is less than tolerance: stop iteration, else continue.

The iteration scheme for the Modified Newton's Method is:
1. Take an initial guess x_0.
2. Use the iteration scheme: $x_{n+1} = x_n - \dfrac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$
3. If $f(x_{n+1}) = 0$ or error term is less than tolerance: stop iteration, else continue.

$n = 0, 1, 2, ...$

Error Term: $\epsilon_n = \dfrac{|x_n - x_{n-1}|}{|x_n|}, \quad n = 1, 2, ...$

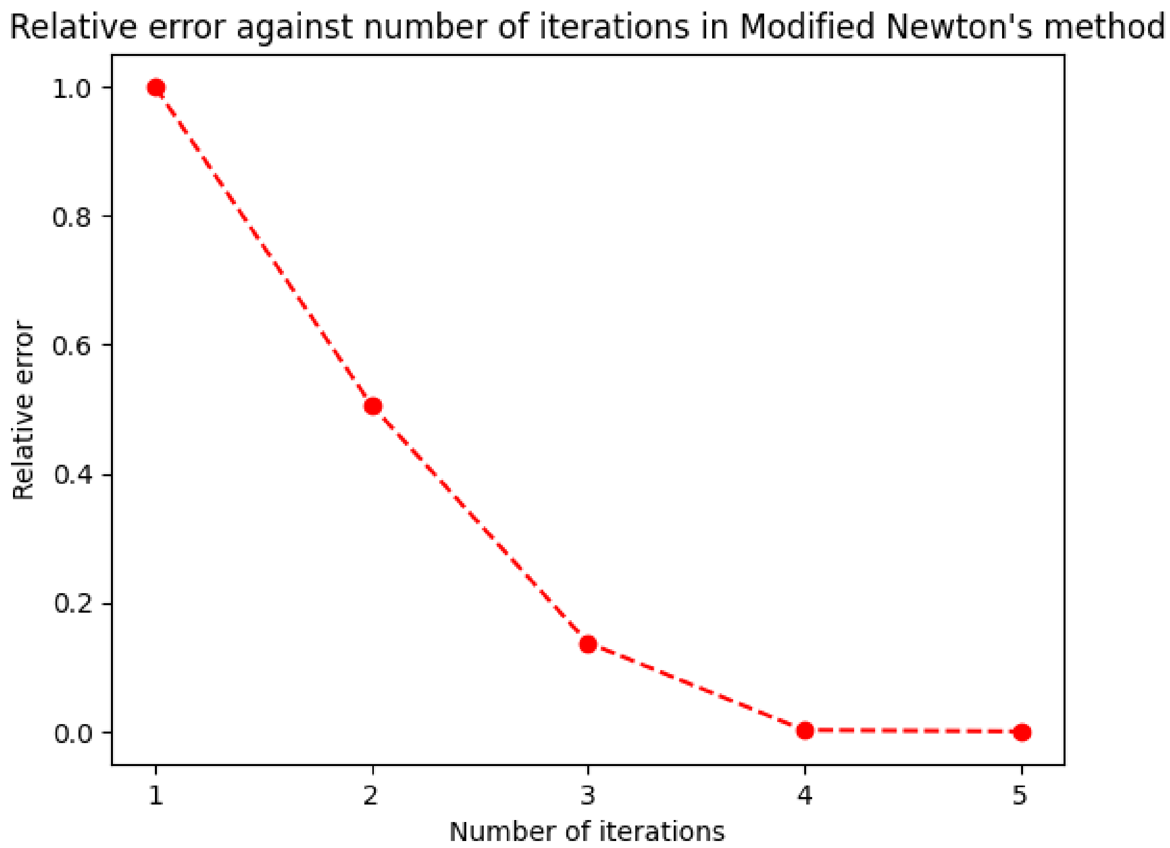
Computing the roots with initial approximation $x_0 = 0$ and relative tolerance 10^{-6}:

- ### $f(x) = (x + e^{-x^2}\cos x)^2$

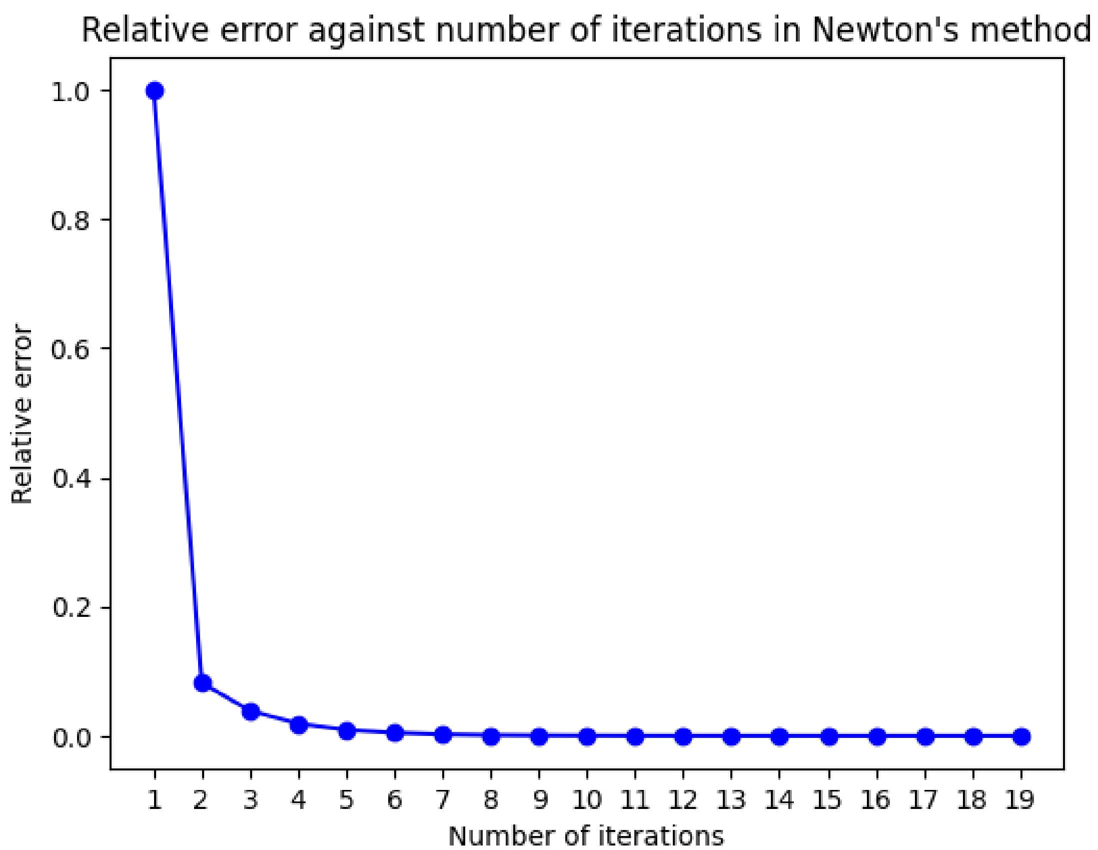
$f'(x) = 2e^{-2x^2}(e^{x^2}x + \cos x)(e^{x^2} - \sin x - 2x\cos x)$
$f''(x) = 2(-e^{-x^2}\sin x - 2e^{-x^2}x\cos x + 1)^2 + 2(x + e^{-x^2}\cos x)(4e^{-x^2}x\sin x + 4e^{-x^2}x^2\cos x - 3e^{-x^2}\cos x)$

By the Newton's Method:
Root rounded off to 6 significant figures: -0.588401
Number of iterations required: 19

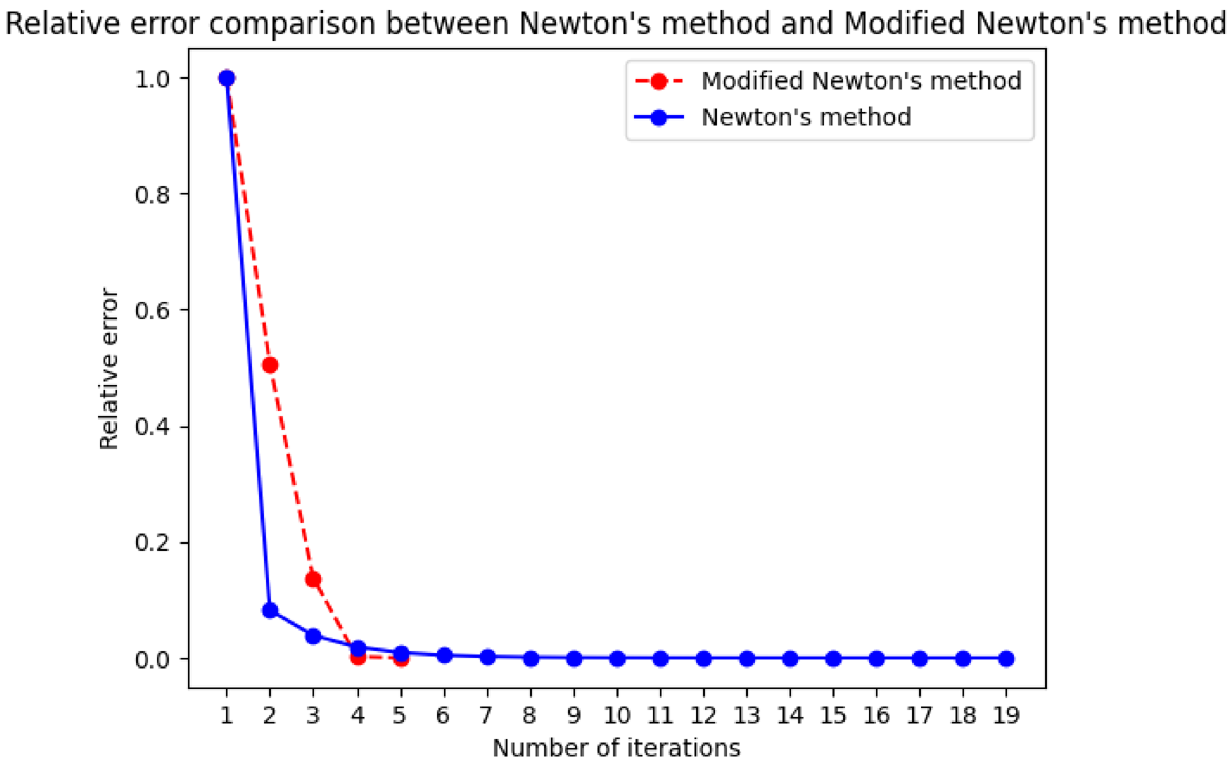
By the Modified Newton's Method:
Root rounded off to 6 significant figures: -0.588402
Number of iterations required: 5



Relative error against number of iterations in Modified Newton's method
Relative error
1.0
0.8
0.6
0.4
0.2
0.0
1
2
3
4
5
Number of iterations



Relative error against number of iterations in Newton's method
Relative error
1.0
0.8
0.6
0.4
0.2
0.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
Number of iterations



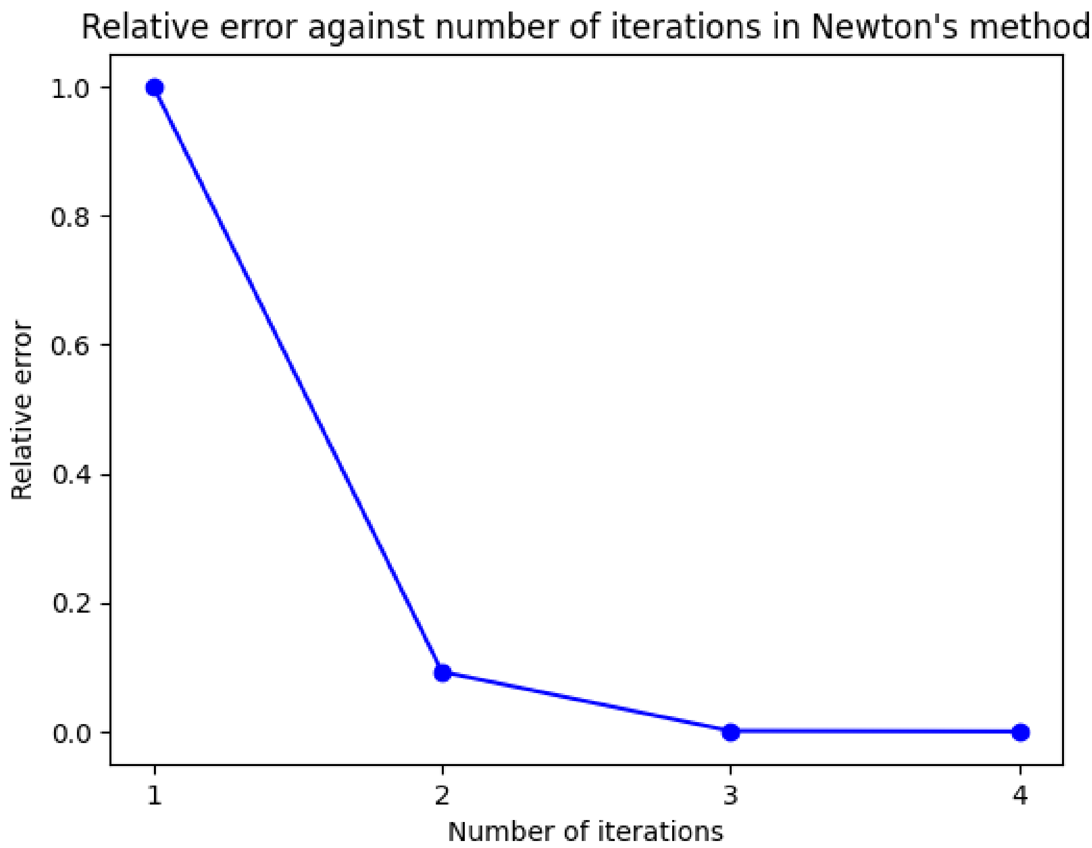
Explanation:
This huge amount of discrepancy between the number of iterations taken by the two different methods is due to the fact that the iteration converges to a root of multiplicity more than one $f(x)$. This can be verified directly by checking whether $f(x_n), f'(x_n) \approx 0$. The Newton's method has a quadratic rate of convergence when converging to a simple root. However, the rate drops to a linear rate when converging to a root with more multiplicity than 1.
We have $f(-0.5884014433804820) = 4.824271571144612 \times 10^{-13} \approx 0$, $f'(-0.588401443380482) = 2.896412171037962 \times 10^{-06} \approx 0$. Thus, we've verified that -0.588401 is indeed a root of greater multiplicity than 1.

- $f(x) = x - 2\log(1 + e^{-x})$

$$f'(x) = \frac{1 + 3e^{-x}}{1 + e^{-x}}$$
$$f''(x) = \frac{-2e^{x}}{(1 + e^{x})^2}$$

By the Newton's Method:
Root rounded off to 6 significant figures: 0.76449
Number of iterations required: 4

By the Modified Newton's Method:
Root rounded off to 6 significant figures: 0.76449
Number of iterations required: 4


Relative error against number of iterations in Newton's method
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0.6
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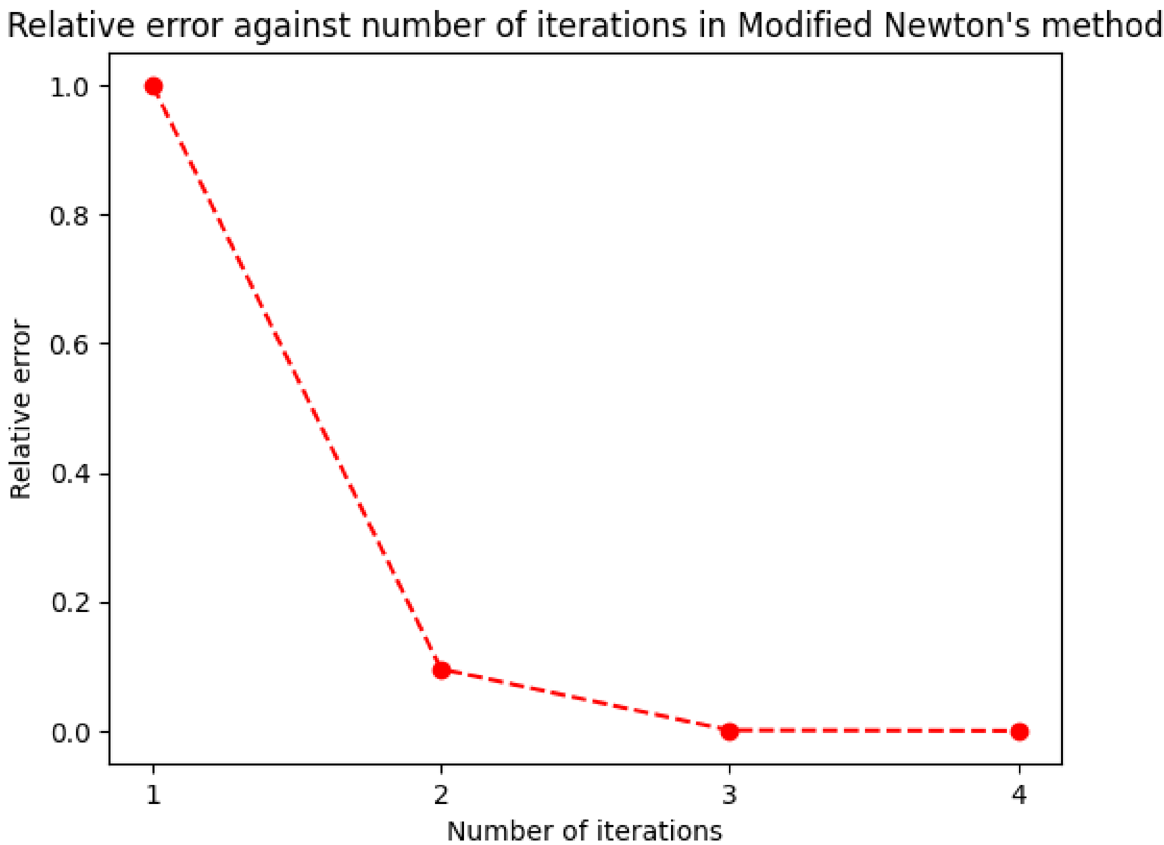
Relative error against number of iterations in Modified Newton's method
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0.8
0.6
0.4
0.2
0.0
Relative error
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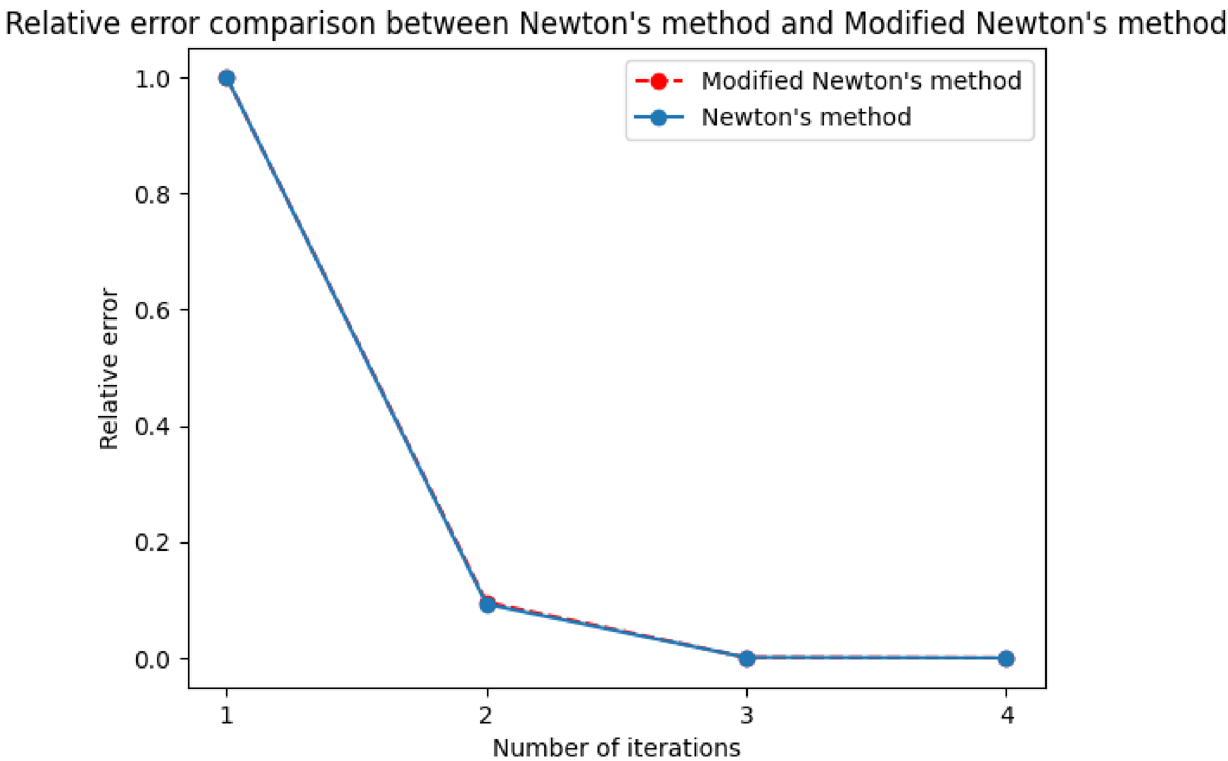
3
4
Number of iterations



As it can be seen from the plot, the error sequence is almost identical for both the methods. Thus, the root must have been simple.
The simplicity of the root can also be verified by evaluating:
$f(0.7644901716800719) = 9.992007221626409 \times 10^{-16} \approx 0$
$f'(0.7644901716800719) = 1.635344392343961 \neq 0$

Question 4:

Given $P(t) = \dfrac{\alpha/\beta}{1+Ae^{-\alpha t}}$, with $P(0) = 1000$ and $P(10) = 10,000$. $\alpha = k$ and $\beta = 5\times 10^{-5}k$. Therefore, $\alpha/\beta = 2\times 10^4$

We can compute $A = \dfrac{2\times 10^4 - P(0)}{P(0)} = 19$

Thus, $P(t) = \dfrac{2\times 10^4}{1+19e^{-kt}}$. By the given information, $P(10) = 10,000 = \dfrac{2\times 10^4}{1+19e^{-10k}}$

Now, consider the function $f(x) = \dfrac{2\times 10^4}{1+19e^{-10x}} - 10000$, for which our desired k is supposed to be a root, since $f(k) = 0$

It has been given that $\dfrac{dp}{dt}$ is "positively proportional" to the population P, with the proportionality constant being α. Hence, this mandates $\alpha > 0$, thus $\alpha = k > 0$.

Now that we've established a lower bound for k, we need to establish a suitable upper bound.

We know the fact that $\dfrac{x}{x+a} = \dfrac{1}{1+\frac{x}{a}} \to 1$, since $\dfrac{x}{a} \to 0$.

Our initial choice was to select the right endpoint as 1, since, $\dfrac{19}{e^{10}}$ is almost 0.

However the Secant Method doesn't converge in this interval given that the the value of f at the successive iteration points

$x_4 = 16.077921625358748$

$x_5 = 7.809090406962769$ are equal to $f(x_4) = f(x_5) = 10000$. Thus, the denominator becomes zero in the iteration scheme.

Now, trying with a slightly smaller right endpoint, say $x = 0.9$, we can still say that: $\dfrac{2\times 10^4}{1+19e^{-9}} = 2\times 10^4 \dfrac{e^9}{e^9+19} \approx 2\times 10^4$. It can be seen by $512 = 2^9 < e^9$ hence $19/e^9 < 19/512 \approx 0$.

Therefore, $f(0.9) \approx 10,000$ and $f(0) = -9000$. So, we need to search the interval $[0, 0.9]$ for the root.

Algorithm:

1. Pick a starting interval $[x_0, x_1]$.
2. Check if $f(x_n) = f(x_{n-1})$. If yes, then break the program, else continue.
3. Iterate through $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$ for $n = 1, 2, ...$
4. If $f(x_{n+1}) = 0$, break the loop, else continue until it reaches the desired precision.

Error Term: $\epsilon_{n+1} = \dfrac{|x_{n+1} - x_n|}{|x_{n+1}|}$ for $n = 0, 1, ...$

Here the prescribed error is: 10^{-4}

Result:

The root rounded to 4 decimal places is $k = 0.2944$, obtained through 8 iterations of the Secant Method on the interval $[0, 0.9]$