Unit-1 19.05.2022

Floating point representation

A real number x can be represented as floating point form as $x = \pm (0.f_1 f_2 f_3 \cdots f_n) \times \beta^e$

where $f_1 f_2 \ldots f_n$ are the digits of the system with base 'β' and 'e' is an integer which is the power of β and $0 \le f_i \le \beta - 1$ (for $i = 1, 2, 3 \ldots n$)

Ex-) (1) Binary floating point variable representation:-

$(0.10111) \times 2^5$ $\quad 0 \le \quad < 2-1$

(2) Octal floating point representation:-

$(0.51204) \times 8^7$

(3) Decimal floating point representation:-

$(0.07598) \times 10^5$

(4) Hexa decimal floating point representation:-

$0.A2B5C \times 16^4$

Q) Convert to the binary number $(1101.101)_2$ into decimal no. system.

$(1101.101)_2 = (1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 + 1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3})_{10}$

$= (13 + 0.5 + 0.1 + 125)_{10}$

$= (13.625)_{10}$

Convert the decimal no. into binary:- $(83.625)_{10}$

$$\begin{array}{r|l|l} 2 & 83 & \\ 2 & 41 & 1 \\ 2 & 20 & 1 \\ 2 & 10 & 0 \\ 2 & 5 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \\ & 0 & 1 \end{array}$$

$$\begin{array}{c|l} & .625 \\ \times & 2 \\ \hline 1 & .250 \\ & \times\ 2 \\ \hline 0 & .500 \\ \times & 2 \\ \hline 1 & .000 \end{array}$$

$\Rightarrow (83)_{10} = (1010011)_2$ $\qquad (.625)_{10} = (.101)_2$

$\Rightarrow (1010011.101)_2$

Normalised floating point representation:-

A real number 'x' $x = \pm (0.f_1 f_2 f_3 \ldots f_n) \times \beta^e$ is said to be normalised floating point form if $f_1 \neq 0$ where $f_1 f_2 f_3 \ldots f_n$ are the digits of the system with Base 'β' and 'e' an integer which is the power of β.

Q) write the decimal no. (0.00051267) into normalized floating point form.

ANS- ~~(0.51267×10^{3})~~

0.51267×10^{3}

Significant figure

The first non zero digits and all the digits to its right of a number are defind to be the significant digit of the number for decimal number.

Q) write the significant digits of the number (i) 7.000
(ii) 0.02507
(iii) 0.00123

A -> (i) 7,0,0,0

(ii) 2,5,0,7

(iii) 1,2,3

-> for non decimal number significant figure starts from 1st non-zero to last non-zero number.

Errors 23.05.2022

(1) Absolute error:-

If x^* be the absolute value of a real number 'x' then absolute error is represented as E_A and defined as

$$E_A = |x - x^*|$$

(2) Relative error:-

If x^* be the approximate value of real no 'x' then a relative error is represented as $E_R = \frac{|x-x^*|}{x}$

$$\Rightarrow E_R = \frac{E_A}{x}$$

(3) Percentage error:-

If x^* be the approximate value of a real no 'x' then a percentage error is represented as E_p and defined as

$$E_p = \left(\frac{|x-x^*|}{x} \times 100\right)\% \quad \text{or} \quad E_p = \left(\frac{E_A}{x} \times 100\right)\%$$

$$E_p = (E_R \times 100)\%$$

Q) If 0.0025 be the approximate value of the number 0.002 then find absolute error, relative error, percentage error.

Soln Let $x = 0.002$ & $x^* = 0.0025$

Absolute error $= E_A = |x - x^*| = |0.002 - 0.0025| = |-0.0005|$
$= 0.0005$

Relative error $= E_R = \frac{E_A}{x} = \frac{0.0005}{0.002} = \frac{\frac{5}{10000}}{\frac{2}{1000}} = \frac{5}{20} = \frac{25}{100}$
$= 0.25$

Percentage error $= E_p = (E_R \times 100)\%$
$= (0.25 \times 100)\%$
$= 25\%$

Truncation error

It is the quantity which must be added to the representation of the quantity in order that the result be exactly equal to the quantity we are seeking to generate.

ex-) $1000 = 999 + T = 1000$

K-digit arithmatic

If a real number in floating point representation is represented as $x = \pm(0.f_1 f_2 f_3 \cdots f_k f_{k+1} \cdots f_n) \times \beta^e$

while doing arithmatic such as addition, substraction, multiplication or division then first of all the no. change into there corresponding K-digit floating point form. by terminating the mantissa at the k^{th} decimal digit.

Note:- This termination is done by 2 methods.

(i) Chopping

(ii) Rounding or round up.

Chopping:- In this method we simply chop off the digits after the k^{th} decimal digit.

ex-) chop off the no. 0.213527×10^5 after 3 decimal place.

A-) 0.213×10^5

Rounding:- If a real no. in normalised floating point form is written as $x = \pm(0.f_1 f_2 f_3 \cdots f_k f_{k+1} \cdots f_n)10^e$ then the following rules followed to round up the number after k^{th} decimal place.

(i) If $f_{k+1} > 5$

then f_k is increse by 1

(iii) If $f_{k+1} < 5$

then f_k is not changed.

(iv) If $f_{k+1} = 5$

then f_k is increse by 1 when f_k is odd.

f_k is not change when f_k is even.

Q) Round off the following no after 3 decimal place.

① 0.235789 = 0.236

② 0.285482 = 0.235

③ 0.234527 = 0.234

(4) 0.231523 = 0.232

DT | 26/05/22

Q) multiply the numbers $\frac{1}{3}$ & $\frac{5}{7}$ using 4 digit arithmatic.

Let $x = \frac{1}{3}$ & $y = \frac{5}{7}$

Remark

$\Rightarrow x = 0.333333$ & $y = 0.714285$

$\Rightarrow fl(x) = 0.3333$ & $fl(y) = 0.7143$

$\Rightarrow fl(x) \times fl(y) = 0.3333 \times 0.7143$

$= 0.23807619$

$= 0.2381$

$\Rightarrow fl(x) \times fl(y) = 0.2381$ (ANS)

Roots of quardatic equation:-

If $ax^2+bx+c=0$ is a quadatic eqn then largest root

$$= \frac{-b+\sqrt{b^2-4ac}}{2a}$$

& smalest root: $\frac{-b-\sqrt{b^2-4ac}}{2a}$

largest root $= \frac{-b+\sqrt{b^2-4ac}}{2a}$

$$= \frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{2a(-b-\sqrt{b^2-4ac})}$$

$$= \frac{(-b)^2-(b^2-4ac)^2}{-2a(b+\sqrt{b^2-4ac})}$$

~~largest root~~ $\cancel{= \frac{-2c}{b+\sqrt{b^2-4ac}}}$ $= \frac{b^2-(b^2-4ac)}{-2a(b+\sqrt{b^2-4ac})} = \frac{4ac}{-2a(b+\sqrt{b^2-4ac})}$

$$\boxed{\text{largest root} = \frac{-2c}{b+\sqrt{b^2-4ac}}}$$

smallest root $= \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$= \frac{(-b-\sqrt{b^2-4ac})-(-b+\sqrt{b^2-4ac})}{2a(-b+\sqrt{b^2-4ac})}$$

$$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{2a(-b+\sqrt{b^2-4ac})}$$

$$= \frac{b^2-(b^2-4ac)}{2a(-b+\sqrt{b^2-4ac})} = \frac{4ac}{2a(-b+\sqrt{b^2-4ac})}$$

$$\boxed{\text{smallest root} = \frac{2c}{-b+\sqrt{b^2-4ac}}}$$

21.7.2

Q) find the largest root of the eq^n $x^2-400x+1=0$ using 4 digit arithmatic

Sol^n Given that the eq^n $x^2-400x+1=0$

the largest root of the eq^n $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ------(1)

Here $a = 1 = 0.1000\times10^1$

$b = -400 = -0.4000\times10^3$

$c = 1 = 0.1\times10^1 = 0.1000\times10^1$

Now $b^2 = (-0.4000\times10^3)^2 = 0.16000000\times10^6 = 0.1600\times10^6$

$4ac = 4\times0.1000\times10^1\times0.1000\times10^1$

$= 4\times0.01000000\times10^2$

$= 4\times0.1000\times10^1$

$= 0.4000\times10^1$

$\therefore\ b^2-4ac = 0.1600\times10^6 - 0.4000\times10^1$

$= 0.1600\times10^6 - 0.4000\times\dfrac{10^6}{10^5}$

$= 0.1600\times10^6 - \dfrac{0.4000\times10^6}{10^5} = \left(0.1600 - \dfrac{0.4000}{100000}\right)10^6$ {common}

$= (0.1600 - 0.000004000)10^6$

$= (0.1600 - 0.0000)10^6$

largest root $= \dfrac{-b+\sqrt{b^2-4ac}}{2a} = \dfrac{-(-0.4000\times10^3)+\sqrt{0.1600\times10^6}}{2\times0.1000\times10^1}$

$= \dfrac{0.4000\times10^3+0.04\times10^3}{0.2000\times10^1}$

$= \dfrac{(0.4000+0.04)10^3}{0.2000\times10^1} = \dfrac{0.4400\times10^2}{0.2000} = 2.2\times10^2$

$= 0.2200\times10^3$

Order of method

If a real number x is expressed in terms of normalised floating point form and each express as $x = \pm (0.f_1 f_2 f_3 \ldots f_n) \times 10^{\alpha}$ then the order of the number is 10^{α}

(Q) What is the order of the number 64000 k.m in terms of meter.

$64000\ km = 64000000\ m$

$= 0.64 \times 10^8$

the order of the no. $= 10^8$

(Q) The order of the no. 24000 k.m in terms of meter.

$24000 = 24000\ 000\ m$

$= 0.24 \times 10^8$ order of method 10^8

$\boxed{= 0.1 \times 10^8 = 1 \times 10^7}$

Convergents :-

If $\{x_n\}$ is a sequence of real numbers and $\lim_{n \to \infty} \{x_n\} = \alpha$, then the sequence called as convergent sequence

Rate of convergence

Let the numerical method produces the iteration $x_1, x_2, \ldots, x_n$ and the method is said to be convergent with rate of convergence 'p', where 'p' is the largest (+ve) integer s.t

$$\boxed{\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} \le \lambda}$$

Where 'λ' is a constant.

Unit-2

Introduction →

A frequently occurring problem is to find the roots of eqⁿs of the form $f(x)=0$.

→ If $f(x)$ is quadratic, cubic or Bi-quadratic expression, then algebraic formulae are available for expressing the roots in terms of the co-efficients.

→ Algebraic funⁿs of the form :-

$$f_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$ are

called Polynomials & we discuss some special methods for determining their roots.

→ A non-algebraic funⁿ is called as a Transcendental funⁿ. For example -: $f(x) = \ln x^3 - 0.7$,

$e^x - 5x$,

$\sin^2 x - e^x - x^2$, etc.

→ The roots of eqⁿ $f(x)=0$ may be real or complex.

→ We discuss methods of finding a real root of algebraic or transcendental eqⁿs & also methods of determining all real or complex roots of Polynomials.

Some Special Methods :-

① Bisection Method →

If a funⁿ $f(x)$ is cont. betⁿ a & b and $f(a)$ & $f(b)$ are of opposite signs, then ∃ at least one root betⁿ a & b ∋ $f(a) \cdot f(b) < 0$.

Computational Steps :-

① Choose two real no.s a & b $\ni$ $f(a).f(b) < 0$.

② Set $x_r = \frac{a+b}{2}$, $r \geq 0$.

③ ⓐ If $f(a).f(x_r) < 0$, then the roots lies in the interval (a, x_r) & set $b = x_r$ & go to step-2.

ⓑ If $f(a).f(x_r) > 0$, then the roots lies in the interval (x_r, b) & set $a = x_r$ & go to step-2.

ⓒ If $f(a).f(x_r) = 0$, then it means that x_r is a root of the eq^n $f(x) = 0$ & the computation may be terminated. □

① Ex:- Find a real root of the eq^n $x^2 - 3 = 0$ & $[a,b] = [1,2]$ by applying Bisection method ?

Sol^n:- Given an eq^n $f(x) = x^2 - 3 = 0$, $[a,b] = [1,2]$.

Step-1 Since, $f(1) = 1 - 3 = -2 < 0$

$f(2) = 2^2 - 3 = 1 > 0$

$\Rightarrow f(1).f(2) < 0$

Two real no.s 1 & 2 $\ni$ $f(1).f(2) < 0$.

Step-2 Set $x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

Step-3 Then, $f(x_0) = f(1.5) = -0.75 < 0$

& $f(a) = f(1) = -2 < 0$

$\Rightarrow f(1).f(1.5) > 0$

$\Rightarrow$ the root lies in bet^n 1.5 & 2.

Again, $x_1 = \frac{1.5+2}{2} = \frac{3.5}{2} = 1.75$

$\Rightarrow f(1.75) = 0.0625 > 0$

$\Rightarrow f(1.5) \cdot f(1.75) < 0$

$\Rightarrow$ the root lies bet^n 1.5 & 1.75.

As $x_2 = \frac{1.5+1.75}{2} = \frac{3.25}{2} = 1.625$

$\Rightarrow f(1.625) = -0.359375 < 0$

$\Rightarrow f(1.5) \cdot f(1.625) > 0$

$\Rightarrow$ the root lies in bet^n 1.625 & 1.75.

Again, $x_3 = \frac{1.625+1.75}{2} = 1.6875$

$\Rightarrow f(1.6875) = -0.15234375 < 0$

$\Rightarrow f(1.625) \cdot f(1.6875) > 0$

$\Rightarrow$ the root lies in bet^n 1.6875 & 1.75.

As $x_4 = \frac{1.6875+1.75}{2} = 1.71875$

$\Rightarrow f(1.71875) = -0.0458984375 < 0$

$\Rightarrow f(1.6875) \cdot f(1.71875) > 0$

$\Rightarrow$ the root lies in bet^n 1.71875 & 1.75.

Similarly, $x_5 = \frac{1.71875+1.75}{2} = 1.734375$

$\Rightarrow f(1.734375) = 0.00806 > 0$

We know, $x^2 - 3 = 0 \Rightarrow x = \sqrt{3} = 1.7320508 \simeq 1.73$.

So, the correct 2-decimal place is 1.73.

$\square$

(2) Ex:- Find an approximation to $\sqrt{7}$ correct to 2-decimal place using the Bisection method?

Sol^n:- Let $x = \sqrt{7} \Rightarrow x^2 - 7 = 0 \Rightarrow f(x) = x^2 - 7$.

Since, $f(0) = -7 < 0$, $f(1) = -6 < 0$, $f(2) = -3 < 0$

$f(3) = 2 > 0$

$\Rightarrow f(2) \cdot f(3) < 0$.

Step-1 The interval $[a, b] = [2, 3] \ni f(2) \cdot f(3) < 0$

Step-2 Set $x_0 = \frac{2+3}{2} = 2.5$

Step-3 As $f(2.5) = -0.75 < 0$

$\Rightarrow$ the root lies in bet^n 2.5 & 3.

Again, $x_1 = \frac{2.5+3}{2} = 2.75$

$\Rightarrow f(2.75) = 0.5625 > 0$

$\Rightarrow$ the root lies in bet^n 2.5 & 2.75.

As $x_2 = \frac{2.5+2.75}{2} = 2.625$

$\Rightarrow f(2.625) = -0.10938 < 0$

$\Rightarrow$ the root lies in bet^n 2.625 & 2.75.

Again, $x_3 = \frac{2.625+2.75}{2} = 2.6875$

$\Rightarrow f(2.6875) = 0.2227 > 0$

$\Rightarrow$ the root lies in bet^n 2.625 & 2.6875.

As $x_4 = 2.65625 \Rightarrow f(2.65625) = 0.05567 > 0$

$\Rightarrow$ the root lies in bet^n 2.625 & 2.65625.

Again, $x_5 = 2.640625 \Rightarrow f(2.640625) = -0.02709 < 0$

$\Rightarrow$ the root lies in bet^n 2.640625 & 2.65625

We know, $\sqrt{7} \approx 2.6458$.

So, the Correct 2-decimal place is 2.64. □

(3) Ex:- Find a real root of the eqⁿ $f(x) = x^3 - x - 1 = 0$ & $[a,b] = [1,2]$ by applying Bisection method?

Solⁿ:- Step-1 Two real no.s 1 & 2 ∋ $f(1).f(2) < 0$.

where, $f(1) = -1$, $f(2) = 5$.

Step-2 Set $x_0 = 1.5$

Step-3 Then, $f(1.5) = 0.875 > 0$

$\Rightarrow$ the root lies in betⁿ 1 & 1.5.

Again, $x_1 = 1.25 \Rightarrow f(1.25) = -0.2968 < 0$

$\Rightarrow$ the root lies in betⁿ 1.25 & 1.5.

As, $x_2 = 1.375 \Rightarrow f(1.375) = 0.2246 > 0$

$\Rightarrow$ the root lies in betⁿ 1.25 & 1.375.

Again, $x_3 = 1.3125 \Rightarrow f(1.3125) = -0.0515 < 0$

$\Rightarrow$ the root lies in betⁿ 1.3125 & 1.375.

As, $x_4 = 1.34375 \Rightarrow f(1.34375) = 0.0826 > 0$

$\Rightarrow$ the root lies in betⁿ 1.3125 & 1.34375.

Again, $x_5 = 1.328125 \Rightarrow f(x_5) = 0.01457 > 0$

$\Rightarrow$ the root lies in betⁿ 1.3125 & 1.328125.

Similarly, Proceeding in this manner & find more Successive approximations. □

(4) Ex:- Find a real root of the eqⁿ $x^3 - 2x - 5 = 0$

by applying Bisection Method ?

Sol^n:- Given that $f(x) = x^3 - 2x - 5 = 0$

Since, $f(0) = -5 < 0$, $f(1) = -6 < 0$

$f(2) = -1 < 0$, $f(3) = 16 > 0$

So, the interval $[a, b] = [2, 3]$.

Step-1 Let $f(2) \cdot f(3) < 0$

Step-2 Set $x_0 = 2.5$

Step-3 Then, $f(2.5) = 5.6250 > 0$

$\Rightarrow$ the root lies in bet^n 2 & 2.5.

As, $x_1 = 2.25 \Rightarrow f(2.25) = 1.890625 > 0$

$\Rightarrow$ the root lies in bet^n 2 & 2.25.

Again, $x_2 = 2.125 \Rightarrow f(2.125) = 0.3457 > 0$.

$\Rightarrow$ the root lies in bet^n 2 & 2.125.

As, $x_3 = 2.0625 \Rightarrow f(2.0625) = -0.3513 < 0$

$\Rightarrow$ the root lies in bet^n 2.0625 & 2.125.

Again, $x_4 = 2.09375 \Rightarrow f(x_4) = -0.0089 < 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.125.

$\Rightarrow x_5 = 2.10938 \Rightarrow f(x_5) = 0.16689 > 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.10938.

As $x_6 = 2.10156 \Rightarrow f(x_6) = 0.0785 > 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.10156.

$\Rightarrow x_7 = 2.09766 \Rightarrow f(x_7) = 0.03475 > 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.09766.

Again, $x_8 = 2.09570 \Rightarrow f(x_8) = 0.01282 > 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.09570.

As $x_9 = 2.09473 \Rightarrow f(x_9) = 0.00199 > 0$

$\Rightarrow$ the root lies in bet^n 2.09375 & 2.09473.

$\Rightarrow x_{10} = 2.09424 \Rightarrow f(x_{10}) = -0.0034 < 0$

$\Rightarrow$ the root lies in bet^n 2.09424 & 2.09473.

Hence, the correct to 3-decimal places is 2.094. □

② Secant Method → (Chord-Method)

Geometrical Interpretation :-

In Secant method, the approximation x_{n+1} is the point where the secant or chord joining the points (x_{n-1}, f_{n-1}) & (x_n, f_n) on the curve $y = f(x)$ meets the x-axis $(\because y=0)$ (this is why the method is so called) & this can be shown in the following way :-

The eq^n of the chord joining the points (x_{n-1}, f_{n-1}) & (x_n, f_n) is $y - f_n = \left(\frac{f_n - f_{n-1}}{x_n - x_{n-1}}\right)(x - x_n)$

Putting $y=0$, $x = x_{n+1}$ in this eq^n $\left[\because y - y_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2)\right]$

$$-f_n = \left(\frac{f_n - f_{n-1}}{x_n - x_{n-1}}\right)(x_{n+1} - x_n)$$

$$\Rightarrow f_n = \left(\frac{f_n - f_{n-1}}{x_n - x_{n-1}}\right)(x_n - x_{n+1})$$

$$\Rightarrow x_n - x_{n+1} = f_n \left(\frac{x_n - x_{n-1}}{f_n - f_{n-1}}\right)$$

$$\Rightarrow \boxed{x_{n+1} = x_n - f_n\left(\frac{x_n - x_{n-1}}{f_n - f_{n-1}}\right)} \quad \text{--- (1)}$$

$$\Rightarrow x_{n+1} = \frac{x_n(f_n - f_{n-1}) - f_n(x_n - x_{n-1})}{f_n - f_{n-1}}$$

$$\Rightarrow x_{n+1} = \frac{\cancel{x_n f_n} - x_n f_{n-1} - \cancel{x_n f_n} + x_{n-1} f_n}{f_n - f_{n-1}}$$

$$\Rightarrow \boxed{x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}} \quad \text{--- (2)}$$

OR

Derivation $\rightarrow$ Let x_{-1} & x_0 be two initial approximation to the real root 'α' of the eq^n $f(x) = 0$.

Then, we approximate the fun^n $f(x)$ bet^n x_{-1}, x_0 by a linear interpolating polynomial $P_1(x) = a_0 x + a_1$ S.t.

$$f(x_{-1}) = P_1(x_{-1}), \quad f(x_0) = P_1(x_0).$$

$\rightarrow$ Also we can write, $f(x_0) = f_0$ & $f(x_{-1}) = f_{-1}$.

These Cond^{n}s determine a_0 & a_1 as $a_0 = \dfrac{f_0 - f_{-1}}{x_0 - x_{-1}}$

& $a_1 = \dfrac{x_0 f_{-1} - x_{-1} f_0}{x_0 - x_{-1}}$.

As the polynomial $P_1(x)$ is an approximation to the fun^n $f(x)$, the root of the polynomial eq^n $P_1(x) = 0$, would be an approximation to that $f(x) = 0$.

Hence, $x_1 = \dfrac{-a_1}{a_0} = -\left(\dfrac{x_0 f_{-1} - x_{-1} f_0}{f_0 - f_{-1}}\right)$

$$\Rightarrow x_1 = \frac{x_{-1} f_0 - x_0 f_{-1}}{f_0 - f_{-1}}$$ is an approximate

sol^n of the eq^n $f(x) = 0$.

Next we obtained, 2nd approximation :-

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$; where $f(x_1) = f_1$ & $f(x_0) = f_0$.

Continuing in this manner, we obtain the $(n+1)$th approximation as $x_{n+1} = \dfrac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}$, where $f(x_n) = f_n$ & $f(x_{n-1}) = f_{n-1}$ for $n = 0, 1, 2, \ldots$. □

(5) **Ex:-** Let $f(x) = x^2 - 3$ & $x_{-1} = 1$, $x_0 = 2$ by using Secant method to find the approximate solⁿ ?

Solⁿ:- We know that, the formulae of Secant method

$$x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}} \quad \text{--- (1)}$$

Since, $f(x) = x^2 - 3$.

As $x_{-1} = 1 \Rightarrow f(x_{-1}) = x_{-1}^2 - 3$

$\Rightarrow f(1) = 1^2 - 3 = -2 \Rightarrow f_{-1} = -2$

& $x_0 = 2 \Rightarrow f(x_0) = x_0^2 - 3$

$\Rightarrow f(2) = 2^2 - 3 = 1 \Rightarrow f_0 = 1$

From eqⁿ(1), for $n = 0 \Rightarrow x_1 = \dfrac{x_{-1} f_0 - x_0 f_{-1}}{f_0 - f_{-1}}$

$$\Rightarrow x_1 = \frac{1.1 + 2.2}{1 + 2} = \frac{5}{3} = 1.667$$

$\Rightarrow f(x_1) = (1.667)^2 - 3 \Rightarrow f_1 = -0.2212$

For $n = 1$, $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \dfrac{2 \times (-0.2212) - 1.667 \times 1}{-0.2212 - 1}$

$$\Rightarrow x_2 = \frac{2.1094}{1.2212} = 1.72732 \Rightarrow f_2 = -0.01637$$

Similarly, find $x_3 = 1.7321429 \simeq 1.73215$, $f_3 = 0.00035$ and $x_4 = 1.7320508 \simeq 1.73205$. □

Note →

(1) The method is said to be cgt if the seq. $x_0, x_1, ---, x_n,$ --- cgs to the true solⁿ i.e. for a given $\epsilon > 0$ $\exists$ a +ve int. n_0 s.t. $|x_n - \alpha| < \epsilon$; $n \geq n_0$.

Meaning	Cgt → Convergent , Cgs → Converges

(2) Let $e_n = x_n - \alpha$, e_n is called the error in the nth iterate. The method which iterates $x_1, x_2, ---, x_n, ---$ is said to have rate of convergences p if p is the largest +ve real no. s.t. $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \lambda$, is a constant. The constant 'λ' is defined as asymptotic error constant.

→ If $p=1$, the convergence is described as linear & if $p=2$, the convergence is described as quadratic & so on. □

Thm 3.1 (Intermediate value Theorem) → If a funⁿ 'f' is cont. on a closed interval $[a,b]$ & $f(a).f(b)$ are of opposite signs, then $\exists$ at least one point $\alpha \in (a,b)$ s.t. $f(\alpha) = 0$. □

Meaning — Cont. → Continuous

(6) **Ex:** Secant method is used to find 1st 7th – approximations to the largest root of the equation,

$$f(x) = x^6 - x - 1 = 0, \quad x_{-1} = 1, \; x_0 = 2 \; ?$$

Soln:– We know that the formulae of Secant method

is $$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n)$$

Or $$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

For $n = 0$, $$x_1 = \frac{x_{-1} f_0 - x_0 f_{-1}}{f_0 - f_{-1}}$$

$$\Rightarrow x_1 = \frac{1 \times 61 - 2 \times (-1)}{61 - (-1)} = \frac{63}{62} = 1.0161290$$

$$\Rightarrow f(x_1) = -0.91537 = f_1$$

For $n = 1$, $x_2 = 1.1905778$

$$\Rightarrow f(x_2) = 0.65747 = f_2$$

For $n = 2$, $x_3 = 1.1176558$

$$\Rightarrow f(x_3) = -0.16849 = f_3$$

For $n = 3$, $x_4 = 1.1325316$

$$\Rightarrow f(x_4) = -0.02244 = f_4$$

For $n = 4$, $x_5 = 1.1348168$

$$\Rightarrow f(x_5) = 0.0009\,5 = f_5$$

For $n = 5$, $x_6 = 1.1347236$

$$\Rightarrow f(x_6) = -5.53887 = f_6$$

For $n = 6$, $x_7 = 1.1347241$, which is equal to the 7th – approximations by the Secant Method. □

(3) Newton – Raphson Method → (NR-Method)

Let x_0 be an initial approximation to a root of the eq^n $f(x)=0$, then the eq^n of the tangent line at the point (x_0, f_0), where $f(x_0)=f_0$ to the curve $y=f(x)$ is given by

$$y - f_0 = f'(x_0)(x - x_0)$$

$\Rightarrow -f_0 = f'(x_0)(x_1 - x_0)$, this tangent line meets the x-axis at the point $P(x_1, 0)$.

$$\Rightarrow x_1 - x_0 = \frac{-f_0}{f_0'}$$

$$\Rightarrow x_1 = x_0 - \frac{f_0}{f_0'}.$$

Again, starting from x_1 & following the same procedure we get the approximation x_2 as

$$x_2 = x_1 - \frac{f_1}{f_1'}.$$

Proceeding in this way we obtain a seq. of approximations $x_0, x_1, x_2, ---, x_n, ---$.

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \text{ for } n = 0, 1, 2, ----$$

[Rough → In Secant & NR – Method, $\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = \frac{1}{f'(x_n)}$

$\Rightarrow f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$.

Tangent foremulae, $y - f_n = \frac{f_n - f_{n-1}}{x_n - x_{n-1}} (x - x_n)$

$$\Rightarrow y - f_n = f'(x_n) \cdot (x - x_n).$$ □

* **Algorithm →**

Given $f(x)$ is continuously differentiable fun^n & a Point x_0.

Step-1 Set $n = 0$.

Step-2 Set $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Step-3 If the approximation is satisfactory, go to Step-5. Otherwise, go to Step-4.

Step-4 Add 1 to n & go to Step-2.

Step-5 The procedure is complete. □

Note → Secant method can be obtained from NR-Method if $f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$; $n \geq 1$. □

(7) **Ex:-** Finding the square root of 3 by using NR-Method?

Sol^n:- Setting $f(x) = x^2 - 3$ $\left[\because x = \sqrt{3} \Rightarrow x^2 - 3 = 0\right]$

$$\Rightarrow f'(x) = 2x$$

We know that the formulae of NR-Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{x_n^2 + 3}{2x_n} \; ; \; n \geq 0 \quad \text{---} \; (1)$$

For $x_0 = 0.5 \Rightarrow f(x_0) = -2.75 = f_0$.

Since, for $n = 0 \Rightarrow x_1 = \dfrac{x_0^2 + 3}{2x_0} = 3.25$

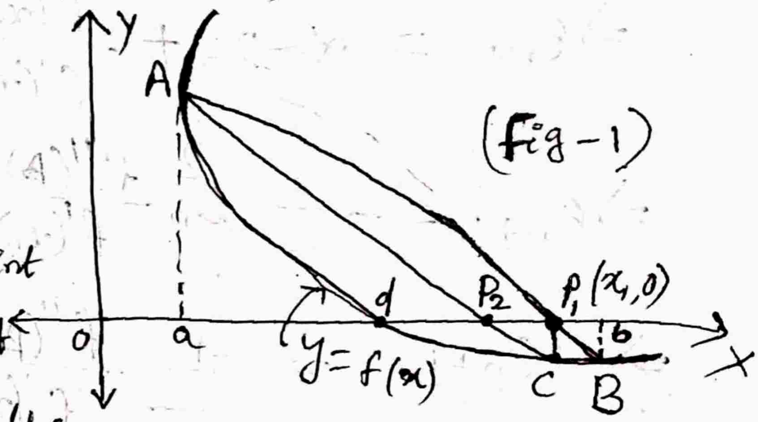
For $n = 1$, $x_2 = \dfrac{x_1^2 + 3}{2x_1} = 2.086$

For $n = 2$, $x_3 = 1.762$

For $n = 3$, $x_4 = 1.732$.

④ Regula-Falsi Method → (RF-Method)

As in the Previous Bisection method, in this method we choose 2-real no.s a & b s.t. $f(a)$ & $f(b)$ are in opposite signs. Then, the curve $y=f(x)$; $a \le x \le b$ in approximate by the straight line joining the Pt.s $A(a,f(a))$, $B(b,f(b))$, $P_1(x_1,0)$,



The no. x_1, the x co-ordinate of the point P_1 at which the straight line AB intersects the x-axis is taken as the 1st iteration/approximation to the root α of the eq^n $f(x)=0$ lying in the interval (a,b).

The no. x_1 is determined by equating the slopes of AB & AP_1. This gives

$$\frac{f(b)-f(a)}{b-a} = \frac{f(a)-0}{a-x_1} \qquad \left[\because \frac{y_2-y_1}{x_2-x_1} = m\right]$$

$$\Rightarrow a-x_1 = \frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow x_1 = a - f(a)\frac{b-a}{f(b)-f(a)} = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

$$\Rightarrow \boxed{x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}} \longrightarrow (1)$$

Next the interval in which the root lies is determined according as $f(a).f(x_1) < 0$ or $f(x_1).f(b) < 0$.

→ For the fun^n shown in the fig-1, we have $f(a).f(x_1) < 0$. In this case that root lies in the interval (a, x_1). Now assuming as before the line AC to be an approximation x_2, the x-coordinate of the pt. P_2 at which the line AC intersects the X-axis.

The no. x_2 is given by, $x_2 = a - f(a) . \dfrac{x_1 - a}{f(x_1) - f(a)}$ which is obtained by equating the slopes of the lines AC & AP_2.

Proceeding in this way we get x_{n+1}, the $(n+1)$th iterate as : $x_{n+1} = a - f(a)\left[\dfrac{x_n - a}{f(x_n) - f(a)}\right]$, where $x_0 = b$, $n = 0, 1, 2, \ldots$ — (1)

$$\Rightarrow \boxed{x_{n+1} = \frac{a f(x_n) - x_n f(a)}{f(x_n) - f(a)}} \quad ; \; x_0 = b, \; n = 0, 1, 2, \ldots \quad \text{— (2)}$$

→ Similarly, if the 1st approximation lies to the right of the actual root, then all the successive approximations also lies to the right of it. In this case the iterations are given by the eq^n (2) (See fig-1).

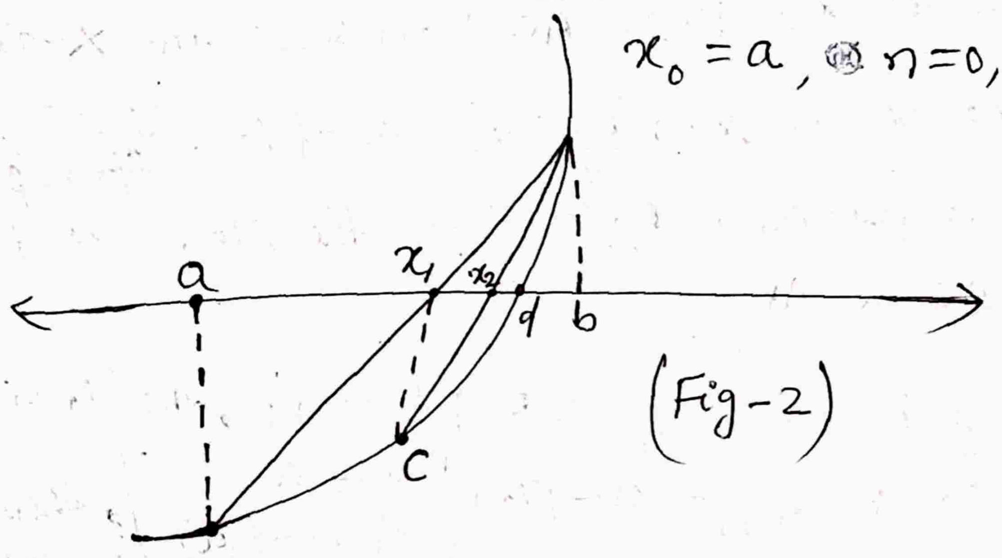
→ However, if the 1st approximation lies to the left of the exact root, then the successive approximations also lie on the same side of it (See fig-2). And in this case the approximations are generated by the following formula :

$$x_{n+1} = b - f(b)\left[\frac{b - x_n}{f(b) - f(x_n)}\right] \quad \text{(ii)}$$

$$\text{OR} \quad b - f(b)\left[\frac{x_n - b}{f(x_n) - f(b)}\right]$$

$$\Rightarrow \boxed{x_{n+1} = \frac{b f(x_n) - x_n f(b)}{f(x_n) - f(b)}} \quad (3)$$

; $x_0 = a$, $n = 0, 1, 2, \ldots$



(Fig-2)

$\square$

Algorithm $\longrightarrow$

Given the cont. funⁿ f on the interval $[a, b]$ with $f(a) \cdot f(b) < 0$. To find a solⁿ to $f(x) = 0$.

Step-1 Set $a_1 = a$ & $b_1 = b$.

Step-2 Set $n = 1$

Step-3 Set $x_n = \dfrac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$

Step-4 If x_n is a satisfactory approximation, go to step-10.

If x_i is not a satisfactory approximation go to Step-5.

Step-5 If $f(x_n) \cdot f(a_n) > 0$, then go to step-6.

If $f(x_n) \cdot f(a_n) < 0$, then go to step-8.

Step-6 Set $a_{n+1} = x_n$ & $b_{n+1} = b_n$.

Step-7 Add 1 to n & go to step-3.

Step-8 Set $a_{n+1} = a_n$ & $b_{n+1} = x_n$.

Step-9 Add 1 to n & go to step-3.

Step-10 Stop & write x_n. □

* Divided difference formulas $\rightarrow$

$\rightarrow f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ or $\dfrac{f(x_0) - f(x_1)}{x_0 - x_1}$

$\rightarrow f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

$\rightarrow f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$

$\rightarrow f[x_0, x_1, x_2, x_3, x_4] = \dfrac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0}$

$\rightarrow f[x_0, x_1, x_2, ---, x_n] = \dfrac{f[x_1, x_2, ---, x_n] - f[x_0, x_1, ---, x_{n-1}]}{x_n - x_0}$

$\rightarrow f[x_0, x_1, ---, x_{n+1}] = \dfrac{f[x_1, x_2, ---, x_{n+1}] - f[x_0, x_1, ---, x_n]}{x_{n+1} - x_0}$ $\square$

* Properties of divided difference →

① $f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\psi_n'(x_i)}$, where

$\psi_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$.

For $n=2$, $f[x_0, x_1, x_2] = \frac{f(x_0)}{\psi_n'(x_0)} + \frac{f(x_1)}{\psi_n'(x_1)} + \frac{f(x_2)}{\psi_n'(x_2)}$

$\Rightarrow f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$

$= \frac{f(x_0)}{(x_1-x_0)(x_2-x_0)} - \frac{f(x_1)}{(x_1-x_0)(x_2-x_1)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$

$= \frac{f(x_0)(x_2-x_1) - f(x_1)(x_2-x_0) + f(x_2)(x_1-x_0)}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}$

$= \frac{\{f(x_2)-f(x_1)\}(x_1-x_0) - \{f(x_1)-f(x_0)\}(x_2-x_1)}{(x_1-x_0)(x_2-x_0)(x_2-x_1)} = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$

② If $(i_0, i_1, \ldots, i_n)$ is a permutation of $(0,1,2,\ldots,n)$, then $f[x_0, x_1, \ldots, x_n] = f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]$.

③ $f[\underbrace{x_0, x_1, \ldots, x_n}_{(n+1)\text{ nodes}}] = \frac{1}{n!} f^{(n)}(x^*)$, where

$\text{Min}\{x_0, x_1, \ldots, x_n\} < x^* < \text{Max}\{x_0, x_1, \ldots, x_n\}$

④ $\frac{d}{dx}\{f[x_0, x_1, \ldots, x_n, x]\} = f[x_0, x_1, x_2, \ldots, x_n, x, x]$

⑤ Irrespective of position of nodes the value of divided difference is always equal $f[x_0, x_1] = f[x_1, x_0]$. ▯

⑧ Q:- Using the polynomial $f(x) = x^3$, find x^*, where $f[0,1,2] = \frac{1}{2!} f''(x^*)$?

Solⁿ:- Given that $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f''(x) = 6x \Rightarrow f''(x^*) = 6x^*$.

Since, $f[0,1,2] = \frac{1}{2!} f''(x^*)$

$\Rightarrow \frac{f[1,2]-f[0,1]}{2-0} = \frac{1}{2} \times 6x^* \Rightarrow f[1,2]-f[0,1] = 6x^*$

$\Rightarrow \frac{f(2)-f(1)}{2-1} - \frac{f(1)-f(0)}{1-0} = 6x^* \Rightarrow \frac{2^3-1^3}{1} - \frac{1^3-0^3}{1} = 6x^*$

$\Rightarrow 7-1 = 6x^* \Rightarrow 6 = 6x^* \Rightarrow x^* = 1$.

▯

9 Q:- Find the Sq. root of 3 using RF-Method ?

Ans.:- Given that $f(x) = x^2 - 3$

$\Rightarrow f(1) = -2 < 0,\ f(2) = 1 > 0$

$\Rightarrow f(1) \cdot f(2) < 0 \Rightarrow [a,b] = [1,2]$.

$\Rightarrow$ the root lies in bet^n 1 & 2.

Since, we know that the formula of RF-Method

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{1+4}{2+1} = \frac{5}{3} = 1.667$$

$\Rightarrow f(x_1) = -0.221$

$\Rightarrow f(a) \cdot f(x_1) > 0 \Rightarrow$ the root lies in bet^n x_1 & b.

$\Rightarrow [a, b] = [x_1, b] = [1.667, 2]$ at $x_0 = a = 1$.

Again, $$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)} = \frac{1.667 \times 1 + 2 \times 0.221}{1 + 0.221}$$

$$= \frac{2.109}{1.221} = 1.7273$$

$\Rightarrow f(x_2) = -0.015$

$\Rightarrow f(a) \cdot f(x_2) > 0 \Rightarrow [a, b] = [x_2, b] = [1.7273, 2]$

Since, $$x_3 = \frac{1.7273 \times 1 + 2 \times 0.015}{1 + 0.015}$$

$$= 1.7312$$

$\Rightarrow f(x_3) = -0.0028$

$\Rightarrow f(a) \cdot f(x_3) > 0 \Rightarrow [a,b] = [x_3, b] = [1.7312, 2]$

$$\therefore x_4 = \frac{1.7312 \times 1 + 2 \times 0.0028}{1 + 0.0028} = 1.732$$

$\Rightarrow f(x_4) = -0.00018$

$\Rightarrow f(a) \cdot f(x_4) > 0 \Rightarrow [a,b] = [x_4, b] = [1.732, 2]$

$$\therefore x_5 = \frac{1.732 + 2 \times 0.00018}{1 + 0.00018} = 1.732048231$$

$$= 1.73205$$

So, the value of $\sqrt{3} = 1.73205080 8 \approx 1.73205 = x_5$. ◻

(10)

Q:- Solve $x^3-9x+1=0$ for the root lying betn 2 & 4 by the RF-Method ?

Soln:- Given $a=2$ & $b=4$ & $f(x)=x^3-9x+1$.

$\Rightarrow f(a)=-9,\ f(b)=29$

$\Rightarrow f(a)\cdot f(b)<0$

Since, $x_1=\dfrac{af(b)-bf(a)}{f(b)-f(a)}=2.47$

$\Rightarrow f(x_1)=-6.1607$

$\Rightarrow f(a)\cdot f(x_1)>0 \Rightarrow [a,b]=[2.47,4]=[x_1,b]$

$\therefore x_2=\dfrac{2.47\times 29+4\times 6.1607}{29+6.1607}=2.73.$

$\Rightarrow f(x_2)=-3.2 \Rightarrow f(a)\cdot f(x_2)>0 \Rightarrow [a,b]=[x_2,b]$
$=[2.73,4]$

$\therefore x_3=\dfrac{2.73\times 29+4\times 3.2}{29+3.2}=2.85$

$\Rightarrow f(x_3)=-1.5 \Rightarrow f(a)\cdot f(x_3)>0 \Rightarrow [a,b]=[x_3,b]$
$=[2.85,4]$

$\Rightarrow x_4=2.91,\ f(x_4)=-0.55$

$\Rightarrow f(a)\cdot f(x_4)>0 \Rightarrow [a,b]=[2.91,4]$

$\Rightarrow x_5=2.93,\ f(x_5)=-0.22$

$\Rightarrow f(a)\cdot f(x_5)>0 \Rightarrow [a,b]=[2.93,4]$

$\Rightarrow x_6=\dfrac{2.93\times 29+4\times 0.22}{29+0.22}=2.937$
$=2.94$

Hence, the required root of the eqn $f(x)$ is 2.94 at $x_0=a=2$. □

(11) Q:- Using NR-Method find a root of the eq^n $f(x) = x^6 - x - 1$ in the interval $(1,2)$, which is correct to 3-decimal places ?

Sol^n:- Given that $f(x) = x^6 - x - 1$

$$\Rightarrow f'(x) = 6x^5 - 1$$

Now, we know that the formula of NR-Method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1} = \frac{6x_n^6 - x_n - x_n^6 + x_n + 1}{6x^5 - 1}$$

$$\Rightarrow \boxed{x_{n+1} = \frac{5x_n^6 + 1}{6x_n^5 - 1}} \quad \text{——— (1)}$$

AS $[a,b] = [1,2]$. Let us take $x_0 = 1$.

Then $x_1 = \frac{5x_0^6 + 1}{6x_0^5 - 1} = \frac{5 \cdot 1^6 + 1}{6 \cdot 1^5 - 1} = \frac{5+1}{6-1} = \frac{6}{5} = 1.2$

for $n = 0$.

For $n = 1$, $x_2 = \frac{5x_1^6 + 1}{6x_1^5 - 1} = \frac{5(1.2)^6 + 1}{6(1.2)^5 - 1}$

$$\Rightarrow x_2 = 1.143576$$

For $n = 1$, $x_3 = \frac{5x_2^6 + 1}{6x_2^5 - 1} = 1.134909$

For $n = 2$, $x_4 = \frac{5x_3^6 + 1}{6x_3^5 - 1} = 1.1347242$

For $n = 3$, $x_5 = \frac{5x_4^6 + 1}{6x_4^5 - 1} = 1.1347241$

Thus, the correct to 3-decimal place is 1.134. □

(12) Q:- Find the root of the polynomial eq^n $f(x) = x^3 - 8x - 4 = 0$ which lies in $(3,4)$ by using NR-Method ?

Sol^n:- Given that $f(x) = x^3 - 8x - 4$

$$\Rightarrow f'(x) = 3x^2 - 8$$

Now $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - 8x_n - 4}{3x_n^2 - 8}$

$$= \frac{3x_n^3 - 8x_n - x_n^3 + 8x_n + 4}{3x_n^2 - 8}$$

$$\Rightarrow \boxed{x_{n+1} = \frac{2x_n^3 + 4}{3x_n^2 - 8}} \quad \text{--- (1)}$$

As $[a,b] = [3,4]$, Choosing the initial guess $x_0 = 3$.

For $n=0$, $x_1 = \dfrac{2x_0^3 + 4}{3x_0^2 - 8} = \dfrac{2\cdot 3^3 + 4}{3\cdot 3^2 - 8}$

$$\Rightarrow x_1 = \frac{58}{19} = 3.05263$$

For $n=1$, $x_2 = \dfrac{2x_1^3 + 4}{3x_1^2 - 8} = 3.05137$

For $n=2$, $x_3 = \dfrac{2x_2^3 + 4}{3x_2^2 - 8} = 3.05137$

There-fore, the root is approximately equal to 3.051, which is correct to 3-decimal place.

* Th^m-3.3 $\rightarrow$

If $f \in C^2[a,b]$, $f(\alpha) = 0$, $f'(\alpha) \neq 0$ & $\alpha \in (a,b)$ then $\exists$ an $\epsilon > 0$ such that for every $x_0 \in I = (\alpha - \epsilon, \alpha + \epsilon)$, the sequence generated by NR-Method converges to α i.e. $\lim_{n\to\infty} x_n = \alpha$. □

Pf:- We know that, for large n

$$\frac{|e_{n+1}|}{|e_n|^2} \simeq \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

Taking $M = \dfrac{1}{2} \dfrac{\max_{x\in I} f''(x)}{\min_{x\in I} f'(x)}$

$$\Rightarrow \frac{|e_{n+1}|}{|e_n|^2} \leq M \Rightarrow |e_{n+1}| \leq M|e_n|^2 \quad \text{--- (i)}$$

$\therefore |e_n| \le M|e_{n-1}|^2$ [Replace n by $n-1$]

$\Rightarrow |e_n|^2 \le M^2 |e_{n-1}|^4$

$\Rightarrow M|e_n|^2 \le M^3 |e_{n-1}|^{2^2}$

$\Rightarrow |e_{n+1}| \le M^3 |e_{n-1}|^{2^2}$ [By eq^n(i)]

$\Rightarrow |e_{n+1}| \le \frac{1}{M}\left(M^4 |e_{n-1}|^{2^2}\right) = \frac{1}{M}\left(M^{2^2} |e_{n-1}|^{2^2}\right)$

$\Rightarrow |e_{n+1}| \le \frac{1}{M}\left(M|e_{n-1}|\right)^{2^2} = \frac{1}{M}\left(M|e_{n-1}|\right)^{2^{(n+1)-(n-1)}}$

Proceeding inductively we can write

$$|e_{n+1}| \le \frac{1}{M}\left(M|e_0|\right)^{2^{(n+1)-0}}$$

$$\Rightarrow |e_{n+1}| \le \frac{1}{M}\left(M|e_0|\right)^{2^{n+1}}$$

Replace n by $n-1$, we get

$$|e_n| \le \frac{1}{M}\left(M|e_0|\right)^{2^n}$$

Let us choose 'x_0' in such a way that $M|e_0| < K < 1$.
As $n \to \infty$, $(M|e_0|)^{2^n} \to 0$ because a quantity which is < 1, if we take the power of this quantity very large that tends to ∞, then the quantity is < 1 (very small) with its power tends to 0.

For example – $M|e_0| = 0.5 = \frac{1}{2}$

$$\Rightarrow \lim_{n\to\infty} (M|e_0|)^{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{2^n}$$

$$= \lim_{n\to\infty} \frac{1}{2^{2^n}}$$

$$\Rightarrow (M|e_0|)^{2^n} \to 0 \text{ as } n\to\infty. \quad = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$$

So, $\lim_{n\to\infty} |e_n| \le \frac{1}{M} \lim_{n\to\infty} (M|e_0|)^{2^n} \to 0$

$$\Rightarrow \lim_{n\to\infty} |e_n| = 0 \Rightarrow \lim_{n\to\infty} e_n = 0$$

$$\Rightarrow \lim_{n\to\infty} (\alpha - x_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \alpha = \lim_{n\to\infty} x_n$$

$$\Rightarrow \alpha = \lim_{n\to\infty} x_n \Rightarrow \lim_{n\to\infty} x_n = \alpha \qquad \square$$

(13) **Q:-** Let the eq^n $x^{2.2} = 69$ has a root bet^n 5 & 8. Use method of RF to determine it?

Sol^n:- Let $f(x) = x^{2.2} - 69$ & $(a,b) = (5,8)$.

As $f(a) = f(5) = 5^{2.2} - 69 = -34.50676 < 0$

& $f(b) = f(8) = 8^{2.2} - 69 = 28.00586 > 0$

$\Rightarrow f(5).f(8) < 0$

$\Rightarrow$ the root lies in bet^n 5 & 8.

So, $$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = 6.655990062$$

$\Rightarrow f(x_1) = -4.2756 < 0$

$\because f(b).f(x_1) < 0 \Rightarrow f(x_1).f(b) < 0$

$\Rightarrow$ the lies in bet^n x_1 & b i.e.

$[a,b] = [x_1, b] = [6.65599, 8]$.

So, $$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)} = 6.83400179$$

$\Rightarrow f(x_2) = (6.83400179)^{2.2} - 69$

$\Rightarrow f(x_2) = -0.406148 < 0$

As $f(x_2) \cdot f(b) < 0$

$\Rightarrow$ the root lies in betn x_2 & b

$\Rightarrow [x_1, b] = [x_2, b] = [6.834002, 8]$

So, $x_3 = \dfrac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = 6.85067$

$\Rightarrow f(x_3) = -0.0375\cancel{4}6$ & so on. □

(14) Q:- The eqn $2x = \log_{10} x + 7$ has a root betn 3 & 4. Find its root, correct to 3-decimal places by RF-Method ?

Soln:- Let $f(x) = 2x - \log_{10} x - 7$, $a = 3, b = 4$.

$\Rightarrow f(a) = -1.4771 < 0$

$f(b) = 0.3979 > 0$

$\Rightarrow f(a) \cdot f(b) < 0 \Rightarrow$ the root lies in $[a, b] = [3, 4]$.

$\because x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)} = 3.7878$

$\Rightarrow f(x_1) = -0.002787 < 0$ & $f(b) > 0$.

$\Rightarrow f(x_1) \cdot f(b) < 0$

$\Rightarrow$ the root lies in $[a, b] = [x_1, b] = [3.7878, 4]$.

So, $x_2 = \dfrac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = 3.7893$

$\Rightarrow f(x_2) = 0.000041 > 0$ & So on.

Thus, correct to 3-decimal place is 3.789. □

(15) Q:- Find a root of the eqn $4e^{-x} \sin x - 1 = 0$ by regula-falsi method given that the root lies

in betn 0 & 0.5 ?

Soln:- Given that $f(x) = 4e^{-x}\sin x - 1$

& $a = 0,\ b = 0.5$

$\Rightarrow f(a) = -1$ & $f(b) = 4e^{-0.5}\sin(0.5) - 1$
$= 0.163145$

$\Rightarrow f(a).f(b) < 0$

There-force, $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ $\{\because [a,b] = [0, 0.5]\}$

$= \dfrac{0 + 0.5}{0.163145 + 1} = \dfrac{0.5}{1.163145} = 0.4298690$

$\Rightarrow f(x_1) = 0.084545 > 0$

$\Rightarrow f(a).f(x_1) < 0$

$\Rightarrow$ the root lies in bet $[a,b] = [a, x_1]$
$= [0, 0.4299]$

So, the 1st approximation lies to the right of the root i.e. $[a, x_n]$

$\Rightarrow x_{n+1} = \dfrac{af(x_n) - x_n f(a)}{f(x_n) - f(a)}$; $n \geq 1$ —— (1)

For $n=1$, $x_2 = \dfrac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$ [By (1)]

$= \dfrac{0 + 0.4298690}{0.084545 + 1} = 0.39636$

$\Rightarrow f(x_2) = 0.038919 > 0$ ..

For $n=2$, $x_3 = \dfrac{af(x_2) - x_2 f(a)}{f(x_2) - f(a)} = 0.381512$ [By (1)]

For $n=3$, $x_4 = \dfrac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)} = 0.375159$

By (1),

For $n=4$, $x_5 = 0.37248$, $x_6 = 0.37136 \simeq 0.371$

$x_7 = 0.37089$, $x_8 = 0.370697$

$\simeq 0.371$ $\simeq 0.371$

It follows that the required root is 0.371. □

* Rate of Convergence in Bisection method →

Let $f(x)$ be a fun^n & 'α' is the root of eq^n $f(x) = 0$, then $f(\alpha) = 0$.

Let us considere the errore $e_{n+1} = x_{n+1} - \alpha$

$$e_n = x_n - \alpha$$

$$e_{n-1} = x_{n-1} - \alpha$$

In general, $x_{n+1} = \dfrac{x_{n-1} + x_n}{2}$

$$\Rightarrow e_{n+1} + \alpha = \frac{e_{n-1} + \alpha + e_n + \alpha}{2}$$

Th^m - 3.2 → Bisection method is linearly Convergent.

$$\Rightarrow e_{n+1} + \alpha = \frac{e_{n-1} + e_n}{2} + \alpha$$

$$\Rightarrow e_{n+1} = \left(\frac{e_{n-1} + e_n}{2}\right)\frac{e_n}{e_n}$$

$$= \left(\frac{e_{n-1} + e_n}{e_n}\right)\frac{e_n}{2}$$

$$= \left(\frac{e_{n-1}}{e_n} + 1\right)\frac{e_n}{2}$$

$\Rightarrow e_{n+1} \approx (0+1)\dfrac{e_n}{2}$, $\dfrac{e_{n-1}}{e_n}$ is verey small & tends to 0

$$\Rightarrow e_{n+1} \approx \frac{e_n}{2}$$

$$\Rightarrow \frac{e_{n+1}}{e_n} \approx \frac{1}{2}$$

$$\Rightarrow \frac{|e_{n+1}|}{|e_n|^1} = \frac{1}{2} \Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^1} = \frac{1}{2}.$$

Where, the rate of cgt $p = 1$ & the constant $\lambda = \frac{1}{2}$.

So, the Bisection Method is linearly cgt i.e. 1. □

* Rate of Convergence in Secant Method $\longrightarrow$

Let e_{n+1} be the error associated with $(n+1)$th approximation to the root.

Then, $e_{n+1} = \alpha - x_{n+1}$

$$= \alpha - \left\{ x_n - f(x_n) \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] \right\}$$

$$= \alpha - x_n + \frac{f(x_n)}{\{f(x_n) - f(x_{n-1})\} / (x_n - x_{n-1})}$$

$$\Rightarrow e_{n+1} = e_n + \frac{f(x_n)}{f[x_{n-1}, x_n]}$$ [$\because e_n = \alpha - x_n$ Using divided difference notation]

$$= e_n \left\{ 1 + \frac{f(x_n)}{e_n f[x_{n-1}, x_n]} \right\}$$

$$= \frac{e_n e_{n-1}}{e_{n-1}} \left\{ \frac{e_n f[x_{n-1}, x_n] + f(x_n)}{e_n f[x_{n-1}, x_n]} \right\}$$

$$= \frac{e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{e_n f[x_{n-1}, x_n] + f(x_n)}{e_n e_{n-1}} \right\}$$

$$= \frac{e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{f[x_{n-1}, x_n] + \frac{f(x_n)}{e_n}}{e_{n-1}} \right\}$$

$$= \frac{e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{f[x_{n-1}, x_n] + \frac{f(x_n) - f(\alpha)}{e_n}}{e_{n-1}} \right\}$$ [$\because f(\alpha) = 0 \Rightarrow f(\alpha) = 0$]

$$= \frac{e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{f[x_{n-1}, x_n] - \frac{f(\alpha) - f(x_n)}{\alpha - x_n}}{\alpha - x_{n-1}} \right\}$$ [$\because e_n = \alpha - x_n$, $e_{n-1} = \alpha - x_{n-1}$]

$$= \frac{e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{f[x_{n-1}, x_n] - f[x_n, \alpha]}{\alpha - x_{n-1}} \right\}$$

$$= \frac{-e_n e_{n-1}}{f[x_{n-1}, x_n]} \left\{ \frac{f[x_n, \alpha] - f[x_{n-1}, x_n]}{\alpha - x_{n-1}} \right\}$$

$$\Rightarrow e_{n+1} = \frac{-e_n e_{n-1}}{f[x_{n-1}, x_n]} \cdot f[x_{n-1}, x_n, \alpha]$$ [Using divided difference notation]

By using Property-3 of divided difference,

$$f[x_{n-1}, x_n] = f'(\lambda_n)$$

$$f[x_{n-1}, x_n, \alpha] = \frac{1}{2!} f''(\mu_n) = \frac{1}{2} f''(\mu_n).$$

where, $\text{Min}(x_{n-1}, x_n) < \lambda_n < \text{Max}(x_{n-1}, x_n)$

& $\text{Min}(x_{n-1}, x_n, \alpha) < \mu_n < \text{Max}(x_{n-1}, x_n, \alpha)$.

So we obtained, $e_{n+1} = -e_n e_{n-1} \cdot \dfrac{\frac{1}{2} f''(\mu_n)}{f'(\lambda_n)}$

$$\Rightarrow e_{n+1} = -\frac{e_n e_{n-1}}{2} \frac{f''(\mu_n)}{f'(\lambda_n)}$$

$$\Rightarrow \frac{e_{n+1}}{e_n e_{n-1}} = -\frac{1}{2} \frac{f''(\mu_n)}{f'(\lambda_n)}$$

$$\Rightarrow \left|\frac{e_{n+1}}{e_n e_{n-1}}\right| = \frac{1}{2}\left|\frac{f''(\mu_n)}{f'(\lambda_n)}\right| = \frac{1}{2}\frac{f''(\mu_n)}{f'(\lambda_n)}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{e_{n+1}}{e_n e_{n-1}}\right| = \frac{1}{2}\lim_{n\to\infty}\left(\frac{f''(\mu_n)}{f'(\lambda_n)}\right)$$

As $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = \alpha \Rightarrow \lim_{n\to\infty}\left[\dfrac{f''(\mu_n)}{f'(\lambda_n)}\right] = \dfrac{f''(\alpha)}{f'(\alpha)}$.

$$\Rightarrow \lim_{n\to\infty}\left|\frac{e_{n+1}}{e_n e_{n-1}}\right| = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}.$$

Let 'p' be the order of convergence. Then

$$\frac{|e_n|}{|e_{n-1}|^p} \simeq \lambda \Rightarrow \frac{|e_n|}{\lambda} \simeq |e_{n-1}|^p$$

$\Rightarrow \left(\dfrac{|e_n|}{\lambda}\right)^{1/p} \simeq |e_{n-1}|$ & Similarly $\dfrac{|e_{n+1}|}{|e_n|^p} \simeq \lambda$

$\Rightarrow |e_{n+1}| \simeq \lambda |e_n|^p$.

Then, we obtained

$$\Rightarrow \lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n||e_{n-1}|} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\lambda|e_n|^p}{|e_n|\left|\frac{e_n}{\lambda}\right|^{1/p}} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\lambda^1|e_n|^p}{|e_n|^1 . |e_n|^{1/p} . \lambda^{-1/p}} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$\Rightarrow \lim_{n\to\infty}\left[(\lambda)^{1+\frac{1}{p}} . |e_n|^{p-1-\frac{1}{p}}\right] = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \cdot 1$$

$$\Rightarrow (\lambda)^{1+\frac{1}{p}} . \lim_{n\to\infty}|e_n|^{\frac{p^2-p-1}{p}} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \lim_{n\to\infty}|e_n|^0$$

$$\Rightarrow (\lambda)^{1+\frac{1}{p}} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \quad \& \quad \frac{p^2-p-1}{p} = 0$$

$$\Rightarrow (\lambda)^{\frac{p+1}{p}} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \quad \& \quad p^2-p-1=0$$

$$\Rightarrow \lambda = \left[\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]^{\frac{p}{p+1}}$$ is called as asymptotic error constant.

& $p = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow p = \frac{1+\sqrt{5}}{2} = 1.618 \simeq 1.62$.

But $p \neq \frac{1-\sqrt{5}}{2}$ because p can not be negative.

Hence, the rate of convergence of Secant method is $p = 1.62$.

$\square$

$\longrightarrow$ From the definition of rate of convergence,

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \lambda$$

$$\Rightarrow \frac{|e_{n+1}|}{|e_n|^p} \simeq \lambda \Rightarrow |e_{n+1}| \simeq \lambda |e_n|^p.$$

$\square$

* Rate of Convergence in NR-Method →

Assuming $f \in C^2[a,b]$ & expanding $f(x)$ about the nth iterate x_n in Taylor Series with remainder term we have

Rough

$$\boxed{f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{1}{2}(x - x_n)^2 f''(x_n)}$$

$$\Rightarrow f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{1}{2}(x - x_n)^2 f''(\beta) \quad \text{—(1)}$$

, where the no. β lies betn x & x_n and $[a,b]$ is the interval containing the root α & the iterates x_n ; $n = 0, 1, 2, \ldots$

For $x = \alpha$, the eqn (1) reduces to :

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(\beta)$$

$$\Rightarrow 0 = f(x_n) + e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\beta)$$

$$\Rightarrow \frac{f(x_n) + e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\beta)}{f'(x_n)} = \frac{0}{f'(x_n)} = 0$$

$$\Rightarrow \frac{f(x_n)}{f'(x_n)} + e_n + \frac{1}{2} e_n^2 \frac{f''(\beta)}{f'(x_n)} = 0$$

$$\Rightarrow e_n + \frac{f(x_n)}{f'(x_n)} = -\frac{1}{2} e_n^2 \frac{f''(\beta)}{f'(x_n)} \quad \text{--- (2)}$$

But we know that $e_{n+1} = \alpha - x_{n+1}$

$$\Rightarrow e_{n+1} = \alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$$

$$\Rightarrow e_{n+1} = \alpha - x_n + \frac{f(x_n)}{f'(x_n)} = e_n + \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow e_{n+1} = -\frac{1}{2} e_n^2 \frac{f''(\beta)}{f'(x_n)} \quad [\text{By eq}^n (2)]$$

$$\Rightarrow \frac{e_{n+1}}{e_n^2} = -\frac{1}{2} \frac{f''(\beta)}{f'(x_n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2} \lim_{n\to\infty} \left|\frac{f''(\beta)}{f'(x_n)}\right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$$, Provided $x_n \to \alpha$ as $n \to \infty$. $[\because \beta \to \alpha]$

Therefore, NR-Method cgs quadratically with asymptotic error constant equals to $\frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} = \lambda$ and the rate of cgs $p = 2$. ▯

* Rate of Convergence of RF-Method ⟶

Let α be a root of the eq^n $f(x)=0$ in the interval (a,b) & $e_{n+1} = \alpha - x_{n+1}$ —— (4), be the error at the $(n+1)$th step i.e. the error committed in approximating the root α by x_{n+1}.

Then eq^{n}s (ii) & (4), we have

$$e_{n+1} = \alpha - \left\{ b - f(b)\left[\frac{b-x_n}{f(b)-f(x_n)}\right] \right\}$$

$$\boxed{\text{Let the 1st approximation } x_1 \text{ lie to the left of the root, then it is obvious that all the approximations lie to the left of the root.}}$$

$$\Rightarrow e_{n+1} = (\alpha - b) + f(b)\left[\frac{b-x_n}{f(b)-f(x_n)}\right]$$

$$= (\alpha - b) + \frac{f(b)}{\frac{f(b)-f(x_n)}{b-x_n}} = (\alpha - b) + \frac{f(b)}{f[b,x_n]}$$

{Using divided difference notation $\boxed{\because f[b,x_n] = f[x_n,b]}$}

$$= (\alpha - b)\left[1 + \frac{f(b)}{(\alpha-b)f[b,x_n]}\right]$$

$$= \frac{(\alpha-b)e_n}{e_n}\left\{\frac{(\alpha-b)f[b,x_n] + f(b)}{(\alpha-b)f[b,x_n]}\right\}$$

$$\Rightarrow e_{n+1} = (\alpha-b)e_n\left\{\frac{(\alpha-b)f[b,x_n]+f(b)}{e_n(\alpha-b)f[b,x_n]}\right\} = \frac{(\alpha-b)e_n}{f[b,x_n]}\left\{\frac{(\alpha-b)f[b,x_n]+f(b)}{e_n(\alpha-b)}\right\}$$

$$= \frac{(\alpha-b)e_n}{f[b,x_n]}\left[\frac{\dfrac{(\alpha-b)f[b,x_n]+f(b)}{\alpha-b}}{\dfrac{e_n(\alpha-b)}{\alpha-b}}\right]$$

$$= \frac{(\alpha-b)e_n}{f[b,x_n]}\left\{\frac{f[b,x_n]+\dfrac{f(b)}{\alpha-b}}{e_n}\right\}$$

$$= \frac{(\alpha-b)e_n}{f[b,x_n]}\left\{\frac{f[b,x_n]-\dfrac{f(b)}{b-\alpha}}{(\alpha-x_n)}\right\} \quad [\because e_n=\alpha-x_n]$$

$$= \frac{(\alpha-b)e_n}{f[b,x_n]}\left\{\frac{f[b,x_n]-\dfrac{f(b)-f(\alpha)}{b-\alpha}}{\alpha-x_n}\right\} \quad \left[\begin{array}{l}\because f(x)=0\\ \Rightarrow f(\alpha)=0\end{array}\right]$$

$$\cancel{= \frac{(\alpha-b)e_n}{f[b,x_n]}\left(\frac{f(b)+f(\alpha)}{-b+\alpha}+f[b,x_n]\right)}$$

$$= \frac{(\alpha-b)e_n}{f[b,x_n]}\left[\frac{f[b,x_n]-f[\alpha,b]}{-(x_n-\alpha)}\right]$$

$$= -\frac{(\alpha-b)e_n}{f[b,x_n]}\left[\frac{f[b,x_n]-f[\alpha,b]}{x_n-\alpha}\right]$$

$$= \frac{(b-\alpha)e_n}{f[b,x_n]}\,f[\alpha,b,x_n]$$

$$\Rightarrow e_{n+1} = (b-\alpha)e_n\frac{f[x_n,b,\alpha]}{f[b,x_n]} \quad \{\because f[\alpha,b,x_n]=f[x_n,b,\alpha]\}$$

$$\Rightarrow \frac{e_{n+1}}{e_n} = (b-\alpha)\frac{f[x_n,b,\alpha]}{f[b,x_n]}$$

$$\Rightarrow \left|\frac{e_{n+1}}{e_n}\right| = |b-\alpha|\left|\frac{f[x_n,b,\alpha]}{f[b,x_n]}\right|$$

$$\Rightarrow \frac{|e_{n+1}|}{|e_n|} = (b-\alpha)\frac{|f[x_n,b,\alpha]|}{|f[x_n,b]|} \quad \{\because f[x_n,b]=f[b,x_n]\}$$

By using $f[x_n, b] = f'(\lambda_n)$ & $f[x_n, b, \alpha] = \frac{1}{2} f''(\mu_n)$ [Property-3 of divided difference], where λ_n & μ_n lies between x_n & b $\left[\begin{array}{l} \text{Min}(x_n, b) < \lambda_n < \text{Max}(x_n, b) \\ \text{Min}(x_n, b, \alpha) < \mu_n < \text{Max}(x_n, b, \alpha) \end{array}\right]$.

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = (b-\alpha) \cdot \lim_{n\to\infty} \frac{|f[x_n, b, \alpha]|}{|f[x_n, b]|}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = (b-\alpha) \cdot \lim_{n\to\infty} \frac{|\frac{1}{2} f''(\mu_n)|}{|f'(\lambda_n)|}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \frac{(b-\alpha)}{2} \cdot \lim_{n\to\infty} \frac{f''(\mu_n)}{f'(\lambda_n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \frac{b-\alpha}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}$$

$\boxed{\because \mu_n \text{ \& } \lambda_n \text{ are always tend towards the root } \alpha.}$

- The method convergence linearly with asymptotic error constant $\lambda = \frac{b-\alpha}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}$ & rate of convergent $p=1$ by using the def^n of rate of converegence

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \lambda$$

$\square$

Newton's method for Non-linear Systems →

Here we will study the numerical solⁿ of systems of non-linear eqⁿ's in several variables.

Consider, two eqⁿ's with two unknowns :-

$$f_1(x,y) = 0 \quad \text{——— (1)}$$

$$f_2(x,y) = 0 \quad \text{——— (2)}$$

Let $\alpha = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ be the exact solⁿ and $w_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be the initial guess.

Taylor's series for fcnⁿ of one-variable :-

$$f(x) = f(x_0 + (x-x_0)) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

Taylor's series for fcnⁿ of several variables :-
(For two variables)

$$f(x+h, y+k) = f(x,y) + \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2\right] f(x,y)$$

or $$f(x+h, y+k) = f(x,y) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(x,y) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f\begin{pmatrix} x+\theta h, \\ y+\theta k \end{pmatrix}$$

Now expanding $f_1(\xi,\eta) = 0$ & $f_2(\xi,\eta) = 0$ about (x_0, y_0)

$$\Rightarrow f_1(\xi,\eta) = f_1(\underbrace{x_0}_{x} + \underbrace{(\xi - x_0)}_{h}, \underbrace{y_0}_{y} + \underbrace{(\eta - y_0)}_{k})$$

$$= f_1(x_0,y_0) + \left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right] f_1(x_0,y_0)$$
$$+ \frac{1}{2!}\left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right]^2 f_1\begin{pmatrix} x_0 + \theta(\xi - x_0), \\ y_0 + \theta(\eta - y_0) \end{pmatrix}$$
$$= 0$$

& $f_2(\xi,\eta) = f_2(x_0,y_0) + \left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right] f_2(x_0, y_0)$

$+ \frac{1}{2!}\left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right]^2 f_2\begin{pmatrix} x_0 + \theta(\xi - x_0) \\ y_0 + \theta(\eta - y_0)\end{pmatrix}$

$= 0$

for $0 < \theta < 1$.

Now dropping the 2nd order terms, we get

$$f_1(\xi,\eta) = f_1(x_0,y_0) + \left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right] f_1(x_0,y_0) = 0$$

& $f_2(\xi,\eta) = f_2(x_0,y_0) + \left[(\xi - x_0)\frac{\partial}{\partial x_0} + (\eta - y_0)\frac{\partial}{\partial y_0}\right] f_2(x_0,y_0) = 0$

i.e. $f_1(\xi,\eta) = f_1(x_0,y_0) + (\xi - x_0)\frac{\partial}{\partial x_0}(f_1(x_0,y_0)) + (\eta - y_0)\frac{\partial}{\partial y_0}(f_1(x_0,y_0))$

& $f_2(\xi,\eta) = f_2(x_0,y_0) + (\xi - x_0)\frac{\partial}{\partial x_0}(f_2(x_0,y_0)) + (\eta - y_0)\frac{\partial}{\partial y_0}(f_2(x_0,y_0))$ $= 0$

$= 0$

i.e. $f(\alpha) = f(\omega_0) + F(\omega_0)(\alpha - \omega_0) = 0$;

where, $F = \begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial y_0} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial y_0} \end{pmatrix}$ is the Jacobian Matrix.

So, $f(\omega_0) + F(\omega_0)(\alpha - \omega_0) = 0$

$\Rightarrow F(\omega_0)^{-1}\left[f(\omega_0) + F(\omega_0)(\alpha - \omega_0)\right] = 0$

$\Rightarrow F(\omega_0)^{-1} f(\omega_0) + (\alpha - \omega_0) = 0$

$\Rightarrow \alpha = \omega_0 - f(\omega_0).\,F(\omega_0)^{-1}$

i.e. $\omega_{n+1} = \omega_n - f(\omega_n).\,F(\omega_n)^{-1}$.

$\Rightarrow \omega_{n+1} - \omega_n = -f(\omega_n).\,F(\omega_n)^{-1}$

$\Rightarrow \boxed{\Delta\omega_n = -f(\omega_n).\,F(\omega_n)^{-1}}$

or $\Rightarrow$ $\boxed{\begin{array}{l} -f(w_n) = \Delta w_n \cdot F(w_n) \\ \text{or} \\ f(w_n) = -F(w_n) \cdot \Delta w_n \end{array}}$ is called NR-method for system.

(16) Q:- Let $f_1(x,y) = x^2 - 2x - y + 0.5 = 0$
$f_2(x,y) = x^2 + 4y^2 - 4 = 0$.

Use Newton's method with starting value $(x_0, y_0) = (2, 0.25)$?

Sol^n:- Now $\frac{\partial f_1}{\partial x} = 2x - 2$, $\frac{\partial f_2}{\partial x} = 2x$

$\frac{\partial f_1}{\partial y} = -1$, $\frac{\partial f_2}{\partial y} = 8y$

So, $F(x,y) = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}$ — (1) & given $(x_0, y_0) = (2, 0.25)$ — (i)

$\Rightarrow F(x,y)\Big]_{(x_0,y_0)} = \begin{pmatrix} 2x_0 - 2 & -1 \\ 2x_0 & 8y_0 \end{pmatrix}$

$= \begin{bmatrix} 2 \cdot 2 - 2 & -1 \\ 2 \cdot 2 & 8(0.25) \end{bmatrix}$ [By (i)]

$\Rightarrow F(w_0) = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$

Since, $f_1\Big]_{(x_0,y_0)} = 0.25$ & $f_2\Big]_{(x_0,y_0)} = 0.25$

$\because \Delta x_0 = x_1 - x_0$ and $\Delta y_0 = y_1 - y_0$

$\Rightarrow \Delta w_0 = \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix}$.

We have to find x_1 & y_1.

As we know $f(w_n) = -F(w_n) \cdot \Delta w_n$

$\Rightarrow \Delta w_n = -\frac{f(w_n)}{F(w_n)} = -F(w_n)^{-1} \cdot f(w_n)$ — (2)

$$\Rightarrow \Delta w_0 = -\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = -\begin{bmatrix} 1/4 & 1/8 \\ -1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} -1/4 & -1/8 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4} \times 0.25 - \frac{1}{8} \times 0.25 \\ \frac{1}{2} \times 0.25 - \frac{1}{4} \times 0.25 \end{bmatrix}$$

Rough $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow |A| \neq 0$

$\Rightarrow A^{-1}$ exist.

$\therefore A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for 2×2 matrix.

$\rightarrow$ For any matrix, $A^{-1} = \frac{\text{Adj}A}{|A|}$ for $|A| \neq 0$.

$$\Rightarrow \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} -3/32 \\ 1/16 \end{bmatrix} = \begin{bmatrix} -0.09375 \\ 0.0625 \end{bmatrix}$$

$\therefore x_1 - x_0 = -0.09375$ & $y_1 - y_0 = 0.0625$

$\Rightarrow x_1 = x_0 - 0.09375$ & $y_1 = y_0 + 0.0625$

$\Rightarrow x_1 = 2 - 0.09375$ & $y_1 = 0.25 + 0.0625$

$\Rightarrow x_1 = 1.90625$ & $y_1 = 0.3125$

$\Rightarrow (x_1, y_1) = (1.90625, 0.3125)$ —— (ii)

Again, From eq^n(1) Jacobian matrix

$$F(x,y) = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}$$

$$\Rightarrow F(x,y)\Big]_{(x_1,y_1)} = \begin{pmatrix} 2x_1-2 & -1 \\ 2x_1 & 8y_1 \end{pmatrix}$$

$$\Rightarrow F(w_1) = \begin{bmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{bmatrix} \quad [\text{By (ii)}]$$

From eq^n(2), we get

$$\Delta w_1 = -F(w_1)^{-1} \cdot f(w_1) \text{ —— (iii)}$$

Since, $f_1\Big]_{(x_1,y_1)} = x_1^2 - 2x_1 - y_1 + 0.5 = 0.0087$

$f_2\Big]_{(x_1,y_1)} = x_1^2 + 4y_1^2 - 4 = 0.0244$,

$\because \Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$

$\Rightarrow \Delta w_1 = \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

$\Rightarrow$ we have to find x_2, y_2.

AS $F(w_1) = \begin{bmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{bmatrix} \Rightarrow |F(w_1)| = 8.34375 \neq 0$

$\Rightarrow F(w_1)^{-1} = \frac{1}{8.34375}\begin{bmatrix} 2.5 & 1 \\ -3.8125 & 1.8125 \end{bmatrix}$

$\Rightarrow F(w_1)^{-1} = \begin{bmatrix} 0.29962 & 0.11985 \\ -0.45693 & 0.21723 \end{bmatrix}$

From (iii), $\Delta w_1 = -\begin{bmatrix} 0.29962 & 0.11985 \\ -0.45693 & 0.21723 \end{bmatrix}\begin{bmatrix} 0.0087 \\ 0.0244 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} -0.29962 & -0.11985 \\ 0.45693 & -0.21723 \end{bmatrix}\begin{bmatrix} 0.0087 \\ 0.0244 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} -0.00531 \\ -0.001325 \end{bmatrix}$

$\therefore x_2 - x_1 = -0.00531$ & $y_2 - y_1 = -0.001325$

$\Rightarrow x_2 = x_1 - 0.00531$ & $y_2 = y_1 - 0.001325$

$\Rightarrow x_2 = 1.90094$ & $y_2 = 0.31118$

$\therefore (x_2, y_2) = (1.90094, 0.31118)$

and So on. $\square$

(17) Q:- Find $F(w_0)$? Let $f(x,y) = 4x^2 - 15x\sin y + y^2$. Using Newton's method with starting value $\left(\frac{3}{4}, \frac{3}{4}\right) = (x_0, y_0)$?

Sol^n:- Given that $f(x,y) = 4x^2 - 15x\sin y + y^2$

$$f_1(x,y) = \frac{\partial f}{\partial x} = 8x - 15\sin y$$

$$f_2(x,y) = \frac{\partial f}{\partial y} = -15x\cos y + 2y.$$

$$\& \quad w_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}.$$

Now $\frac{\partial f_1}{\partial x} = 8$, $\frac{\partial f_1}{\partial y} = -15\cos y$

$\frac{\partial f_2}{\partial x} = -15\cos y$, $\frac{\partial f_2}{\partial y} = 15x\sin y + 2$.

$$\Rightarrow F(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 8 & -15\cos y \\ -15\cos y & 15x\sin y + 2 \end{bmatrix}$$

$$\Rightarrow F(x,y)\Big]_{(x_0,y_0)} = \begin{bmatrix} 8 & -15\cos y_0 \\ -15\cos y_0 & 15x_0\sin y_0 + 2 \end{bmatrix}$$

$$\Rightarrow \mathbf{F}(w_0) = \begin{bmatrix} 8 & -15\cos\frac{3}{4} \\ -15\cos\frac{3}{4} & 15.\frac{3}{4}\sin\frac{3}{4} + 2 \end{bmatrix}$$

$$\Rightarrow F(w_0) = \begin{bmatrix} 8 & -10.98 \\ -10.98 & 9.67 \end{bmatrix}.$$

□

(18) Q:- Let $f_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$

$f_2(x_1, x_2) = x_1^2 - x_2^2 = 0$. Using Newton's method with starting value $(x_0, y_0) = (3/4, 3/4)$. Find two iterations ?

Sol^n:- Given that, $w_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}$

$$\& \quad \frac{\partial f_1}{\partial x_1} = 2x_1, \quad \frac{\partial f_1}{\partial x_2} = 2x_2$$

$$\frac{\partial f_2}{\partial x_1} = 2x_1 \,, \quad \frac{\partial f_2}{\partial x_2} = -2x_2 \,.$$

$$\Rightarrow F(x_1,x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix} \quad \text{—(1)}$$

$$\Rightarrow F(x_1,x_2)\Big]_{(x_0,y_0)} = \begin{bmatrix} 2x_0 & 2y_0 \\ 2x_0 & -2y_0 \end{bmatrix}$$

$$\Rightarrow F(w_0) = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & -3/2 \end{bmatrix}$$

$$\Rightarrow |F(w_0)| = \begin{vmatrix} 3/2 & 3/2 \\ 3/2 & -3/2 \end{vmatrix} = -\frac{9}{4} - \frac{9}{4} = -\frac{9}{2} \neq 0 \,.$$

$$\Rightarrow (F(w_0))^{-1} = \frac{-1}{9/2} \begin{bmatrix} -3/2 & -3/2 \\ -3/2 & 3/2 \end{bmatrix} = \frac{2}{9} \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & -3/2 \end{bmatrix}$$

$$\Rightarrow (F(w_0))^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \quad \text{—(i)}$$

And $f_1(x_1,x_2)\Big]_{(x_0,y_0)} = x_0^2 + y_0^2 - 1 = \frac{9}{16} + \frac{9}{16} - 1 = \frac{1}{8} = f_1(w_0)$

$f_2(x_1,x_2)\Big]_{(x_0,y_0)} = x_0^2 - y_0^2 = 0 = f_2(w_0)$ —(ii)

We have to find x_1, y_1 i.e. $\Delta w_0 = \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$.

As we know, $\Delta w_n = -F(w_n)^{-1} \cdot f(w_n)$.

For $n=0$, $\Delta w_0 = -F(w_0)^{-1} \cdot f(w_0)$

$$\Rightarrow \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = -\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} f_1(w_0) \\ f_2(w_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} -1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/8 \\ 0 \end{bmatrix} \quad [\text{By (i) \& (ii)}]$$

$$\Rightarrow \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} -1/24 \\ -1/24 \end{bmatrix}$$

$$\Rightarrow x_1 - x_0 = -1/24 \; , \; y_1 - y_0 = -1/24$$

$$\Rightarrow x_1 = x_0 - \frac{1}{24} \; , \; y_1 = y_0 - \frac{1}{24}$$

$$\Rightarrow x_1 = \frac{3}{4} - \frac{1}{24} \; , \; y_1 = \frac{3}{4} - \frac{1}{24}$$

$$\Rightarrow x_1 = \frac{17}{24} \; , \; y_1 = \frac{17}{24} \Rightarrow w_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.7083 \\ 0.7083 \end{bmatrix}.$$

Again, we have to find x_2, y_2 i.e. $\Delta w_1 = \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix}$.

As $\Delta w_1 = -(F(w_1))^{-1} \cdot f(w_1)$ — (2)

By eq^n (1), $F(x_1, x_2) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}$ [By (1)]

$$\Rightarrow F(x_1, x_2)\Big]_{(x_1, y_1)} = \begin{bmatrix} 2x_1 & 2y_1 \\ 2x_1 & -2y_1 \end{bmatrix}$$

$$\Rightarrow F(w_1) = \begin{bmatrix} 2 \cdot \frac{17}{24} & 2 \cdot \frac{17}{24} \\ 2 \cdot \frac{17}{24} & -2 \cdot \frac{17}{24} \end{bmatrix} = \begin{bmatrix} 17/12 & 17/12 \\ 17/12 & -17/12 \end{bmatrix}$$

$$\Rightarrow |F(w_1)| = -\left(\frac{17}{12}\right)^2 - \left(\frac{17}{12}\right)^2 = -2 \cdot \left(\frac{17}{12}\right)^2 = = -2 \cdot \frac{289}{144}$$

$$= \frac{-289}{72} \neq 0$$

$$\therefore (F(w_1))^{-1} = \frac{1}{-289/72} \begin{bmatrix} -17/12 & -17/12 \\ -17/12 & 17/12 \end{bmatrix}$$

$$= \frac{72}{289} \begin{bmatrix} 17/12 & 17/12 \\ 17/12 & -17/12 \end{bmatrix}$$

$$\Rightarrow F(w_1)^{-1} = \begin{bmatrix} 6/17 & 6/17 \\ 6/17 & -6/17 \end{bmatrix}$$

and $f_1(x_1, x_2)\Big]_{(x_1, y_1)} = x_1^2 + y_1^2 - 1 = \frac{1}{288} = f_1(w_1)$

$$f_2(x_1, x_2)\Big]_{(x_1, y_1)} = x_1^2 - y_1^2 = 0 = f_2(\omega_1).$$

So, By (2) $\Rightarrow \Delta\omega_1 = -\left(F(\omega_1)\right)^{-1} \cdot f(\omega_1)$

$$\Rightarrow \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = -\begin{bmatrix} 6/17 & 6/17 \\ 6/17 & -6/17 \end{bmatrix} \begin{bmatrix} f_1(\omega_1) \\ f_2(\omega_1) \end{bmatrix}$$

$$= \begin{bmatrix} -6/17 & -6/17 \\ -6/17 & 6/17 \end{bmatrix} \begin{bmatrix} 1/288 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} -\frac{6}{17} \cdot \frac{1}{288} \\ -\frac{6}{17} \cdot \frac{1}{288} \end{bmatrix} = \begin{bmatrix} -1/816 \\ -1/816 \end{bmatrix}$$

$$\Rightarrow x_2 - x_1 = -\frac{1}{816}, \quad y_2 - y_1 = \frac{-1}{816}$$

$$\Rightarrow x_2 = x_1 - \frac{1}{816} = \frac{17}{24} - \frac{1}{816} = \frac{577}{816}$$

$$y_2 = y_1 - \frac{1}{816} = \frac{17}{24} - \frac{1}{816} = \frac{577}{816}$$

$$\Rightarrow \omega_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

$\square$

UNIT-II (POLYNOMIAL INTERPOLATION)

Interpolating Polynomial:-

Def^n:- The polynomial $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$ is called an interpolating polynomial to the function $f(x)$ at $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$ if $P_n(x_i) = f(x_i)$, for $i = 0, 1, 2, \ldots, n$.

* The points $x_0, x_1, \ldots, x_n$ are called nodal points or tabular points or nodes.

Note:-

If the no. of nodal points are $(n+1)$ then the degree of the interpolating polynomial is $\leq n$

Q:- Prove that the interpolating polynomial exist and is unique.

Pf:-

Suppose $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$ ——(1) is the polynomial interpolating to the function $f(x)$ at a set of $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$

Then we have, $P_n(x_i) = f_i = f(x_i)$, $i = 0, 1, 2, \ldots, n$

$$P_n(x_0) = f_0$$
$$P_n(x_1) = f_1$$
$$\vdots$$
$$P_n(x_n) = f_n$$

$$\Rightarrow a_0 + a_1x_0 + \cdots + a_nx_0^n = f_0$$
$$a_0 + a_1x_1 + \cdots + a_nx_1^n = f_1$$
$$\vdots$$
$$a_0 + a_1x_n + \cdots + a_nx_n^n = f_n$$

This is a system of $(n+1)$ eq^{n}s in $(n+1)$ unknowns $a_0, a_1, \ldots, a_n$

The above system will have a unique sol^n if

Determinant $\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} \neq 0$

Indeed the value of the determinant $\Delta \neq 0$.
Since, $\Delta = \pi(x_i - x_j)$, $0 \leq j < i \leq n$
where, $x_i \neq x_j$ for $i \neq j$
And the points $x_0, x_1, \ldots, x_n$ are distinct.
Therefore an unique interpolating polynomial exists whose co-efficients are the solⁿ of the system of above eqⁿs for a given function $f(x)$ & a given set of nodal points.

Find the interpolating polynomial for the following data,

$f(-1) = 0, f(0) = 1, f(1) = 2$.

Solⁿ:- Given that the data is,

$f(-1) = 0, f(0) = 1, f(1) = 2$.

The nodal points are, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$
Since there are 3 nodal points, so the degree of the interpolating polynomial $P(x)$ is ≤ 2. &

$P(x) = a_0 + a_1 x + a_2 x^2$ ——— ①

Now, $P(-1) = f(-1) \Rightarrow a_0 + a_1(-1) + a_2(-1)^2 = 0$
$P(0) = f(0) \Rightarrow a_0 + a_1(0) + a_2(0) = 1$
$P(1) = f(1) \Rightarrow a_0 + a_1(1) + a_2(1) = 2$

$\Rightarrow a_0 + a_1(-1) + a_2(-1)^2 = 0$
$a_0 + a_1(0) + a_2(0) = 1$
$a_0 + a_1(1) + a_2(1) = 2$

In matrix form the above system of eqⁿs can be written as,

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{———} (2)$$

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$\det. A = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -1(-1-1)$$

$$= -1 \times (-2)$$

$$= 2 \neq 0$$

So, A^{-1} exist.

So from eq^n(2) we have,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{———} (3)$$

But $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$$= \frac{\text{adj } A}{2}$$

Now we have to find out the ~~coefficient~~ co-factor of the elements of the matrix A

Let C_{ij}, $i = 1, 2, 3$

$j = 1, 2, 3$ denote the cofactors of the elets of A.

$$\therefore \quad C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = (-1)^2 (0-0) = 0$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1)^3 (1-0) = -1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1)^4 (1-0) = 1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^3 (-1-1) = (-1)(-2) = 2$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^4 (1-1) = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^5 (1+1) = (-1) \times (2) = -2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = (-1)^4 (0-0) = 0$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)^5 (0-1) = 1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = (-1)^6 (0+1) = 1$$

Now $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Hence $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

$$= \frac{\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

Substituting this value in eqn(3) we have.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0 \\ 0+0+1 \\ 0+(-1)+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow a_0 = 1,\ a_1 = 1,\ a_2 = 0$

Therefore the required interpolating polynomial is given by,

$$P(x) = a_0 + a_1 x + a_2 x^2$$

$$= 1 + x + 0$$

$$\Rightarrow P(x) = x + 1$$

Lagrange Interpolating Polynomial.

Let $P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ be an nth degree interpolating polynomial for a function $f(x)$ at $(n+1)$ distinct nodal points $x_0, x_1, x_2, \dots, x_n$

$$\left.\begin{aligned}\therefore f(x_0) &= P(x_0) = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n \\ f(x_1) &= P(x_1) = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n \\ &\vdots \\ f(x_n) &= P(x_n) = a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n\end{aligned}\right\} \quad (2)$$

Eliminating $a_0, a_1, a_2, \ldots, a_n$ from the eq^{n}s (1) & (2), we get,

$$\begin{vmatrix} P(x) & 1 & x & x^2 & \cdots & x^n \\ f(x_0) & 1 & x_0 & x_0^2 & \cdots & x_0^n \\ f(x_1) & 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & & \\ f(x_n) & 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = 0.$$

Expanding the above determinant we obtain.

$$\boxed{P(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)} \quad (3)$$

where, $l_i(x) = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$

$i = 0, 1, 2, \ldots, n$

are called Lagrange Fundamental polynomials. The polynomial given by rel^n (3) is called the Lagrange Interpolating polynomial.

Note :-

1. For $n=1$, $\boxed{P(x) = l_0(x)f(x_0) + l_1(x)f(x_1)}$, is called linear Lagrange Interpolating polynomial.

where, $l_0(x) = \dfrac{x-x_1}{x_0-x_1}$, $l_1(x) = \dfrac{x-x_0}{x_1-x_0}$

①

$l_0(x)$, $l_1(x)$ are called Lagrange Fundamental polynomial.

Q:- Prove that,

(i) $l_0(x) + l_1(x) = 1$.

(ii) $l_0(x_0) = 1$ & $l_1(x_1) = 1$

(iii) $l_0(x_1) = 0$ & $l_1(x_0) = 0$

Pf:-

(i) We know that $l_0(x) = \frac{x - x_1}{x_0 - x_1}$, $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

$$\therefore l_0(x) + l_1(x) = \frac{x - x_1}{x_0 - x_1} + \frac{x - x_0}{x_1 - x_0}$$

$$= \frac{x - x_1}{x_0 - x_1} - \frac{x - x_0}{x_0 - x_1}$$

$$= \frac{x - x_1 - x + x_0}{x_0 - x_1}$$

$$= \frac{x_0 - x_1}{x_0 - x_1}$$

$$= 1$$

(ii) We know that $l_0(x) = \frac{x - x_1}{x_0 - x_1}$

$$\Rightarrow l_0(x_0) = \frac{x_0 - x_1}{x_0 - x_1} = 1$$

Again we know that, $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

$$\Rightarrow l_1(x_1) = \frac{x_1 - x_0}{x_1 - x_0} = 1$$

(iii) $l_0(x_1) = \frac{x_1 - x_1}{x_0 - x_1} = \frac{0}{x_0 - x_1} = 0$

Similarly, $l_1(x_0) = \frac{x_0 - x_0}{x_1 - x_0} = \frac{0}{x_1 - x_0}$ 0

Q:- Construct Lagrange interpolation polynomial using the following data,

$$f(0)=1,\ f(-1)=2,\ f(1)=3.$$

Sol^n:- Given that the data is,

$$f(0)=1,\ f(-1)=2,\ f(1)=3.$$

Here $x_0=0,\ x_1=-1,\ x_2=1.$

$f(x_0)=1,\ f(x_1)=2,\ f(x_2)=3$

Now the Lagrange interpolating polynomial is given by, $P(x)=l_0(x)f(x_0)+l_1(x)f(x_1)+l_2(x)f(x_2)$

$$\Rightarrow P(x)=l_0(x)+2l_1(x)+3l_2(x) \quad \text{——(1)}$$

Now $l_0(x)=\dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$

$$=\frac{(x+1)(x-1)}{(0+1)\cdot(0-1)}$$

$$=\frac{x^2-1}{-1}$$

$$=1-x^2$$

$l_1(x)=\dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$

$$=\frac{(x-0)(x-1)}{(-1-0)(-1-1)}$$

$$=\frac{x(x-1)}{-1\times(-2)}$$

$$=\frac{x^2-x}{2}$$

$l_2(x)=\dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

$$=\frac{(x-0)(x+1)}{(1-0)(1+1)}$$

$$=\frac{x(x+1)}{2}$$

$$=\frac{x^2+x}{2}$$

(3)

Substituting these values in eq^n(1) we get

$$P(x) = (1-x^2) + 2\left(\frac{x^2-x}{2}\right) + 3\left(\frac{x^2+x}{2}\right)$$

$$= 1 - \cancel{x^2} + \cancel{x^2} - x + \frac{3}{2}x^2 + \frac{3}{2}x$$

$$= 1 + \frac{3}{2}x - x + \frac{3}{2}x^2$$

$$= 1 + \frac{x}{2} + \frac{3}{2}x^2$$

Lagrange Interpolating polynomial with equally spaced nodal points :-

If the nodal points are equally spaced with constant spacing $h > 0$ i.e, $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots$ then

$$l_i(x) = \frac{(-1)^{n-i} u(u-1)(u-2)\ldots(u-i+1)(u-i-1)\ldots(u-n)}{i!\,(n-i)!}$$

where, $x = x_0 + uh$ for $i = 1, 2, \ldots, n$ are called Lagrange fundamental polynomials. In this case the Lagrange interpolating polynomial is given by,

$$\boxed{P_n(x) = (n+1)\,{}^{u}C_{n+1} \sum_{i=0}^{n} \frac{(-1)^{n-i}}{u-i} \left({}^{n}C_i\right) f(x_i)}$$

Ex:- Find the approximate value of $e^{0.7}$ following data

x	0.4	0.6	0.8	1.0
e^x	1.492	1.822	2.226	2.718

Sol^n:- Given that the data is,

x	0.4	0.6	0.8	1.0
e^x	1.492	1.822	2.226	2.718

Here, $x_0 = 0.4$, $x_1 = 0.6$, $x_2 = 0.8$, $x_3 = 1.0$

Here $f(x) = e^x$

We have to find out the approximate of $e^{0.7}$

(4) Also given that,

$f(x_0) = 1.492$

$f(x_1) = 1.822$

$f(x_2) = 2.226$

$f(x_3) = 2.718$

Since there are 4 nodal points, so the degree of the interpolating polynomial is atmost 3.

The Lagrange interpolating polynomial is given by, $P_3(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) + l_3(x) f(x_3)$ ——— (1)

Here $l_0(x) = \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$

$= \dfrac{(x-0.6)(x-0.8)(x-1)}{(0.4-0.6)(0.4-0.8)(0.4-1.0)}$

for $x = 0.7$

$l_0(x) = \dfrac{(0.7-0.6)(0.7-0.8)(0.7-1)}{(-0.2)\times(-0.4)\times(-0.6)}$

$= \dfrac{0.1 \times (-0.1) \times (-0.3)}{-0.048}$

$= \dfrac{0.003}{-0.048}$

$= -0.0625$

$l_1(x) = \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$

for $x = 0.7$;

$l_1(x) = \dfrac{(0.7-0.4)(0.7-0.8)(0.7-1)}{0.2 \times (-0.2) \times (-0.4)}$

$= \dfrac{0.3 \times (-0.1) \times (-0.3)}{0.016}$

$\dfrac{0.009}{0.016}$

0.5625

(5)

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(x-0.4)(x-0.6)(x-1)}{(0.8-0.4)(0.8-0.6)(0.8-1)}$$

for $x = 0.7$,

$$l_2(x) = \frac{(0.7-0.4)(0.7-0.6)(0.7-1)}{0.4 \times 0.2 \times (-0.2)}$$

$$= \frac{0.3 \times 0.1 \times (-0.3)}{-0.016}$$

$$= \frac{-0.009}{-0.016}$$

$$= 0.5625$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-0.4)(x-0.6)(x-0.8)}{(1.0-0.4)(1-0.6)(1-0.8)}$$

for $x = 0.7$,

$$l_3(x) = \frac{(0.7-0.4)(0.7-0.6)(0.7-0.8)}{0.6 \times 0.4 \times 0.2}$$

$$= \frac{0.3 \times 0.1 \times (-0.1)}{0.048}$$

$$= \frac{-0.003}{0.048}$$

$$= -0.0625$$

for $x = 0.7$, the rel^n becomes

$$P_3(0.7) = (-0.0625) \times (1.492) + 0.5625 \times 1.222 + 0.5625 \times 2.226 + (-0.0625) \times 2.718$$

$$= -0.09325 + 1.024875 + 1.252125 - 0.169875$$

$$= 2.013875$$

$$\Rightarrow e^{0.7} = 2.013875$$

40(2)

Error in interpolation:-

If $P_n(x)$ is the interpolating polynomial to the function $f(x)$ at $(n+1)$ distinct nodal points $x_0, x_1, ---, x_n$ then

$$f(x) - P_n(x) = 0 \quad , \text{ for } x = x_i, \; i = 0, 1, 2, ---, n$$

$$f(x) - P_n(x) \neq 0 \; , \text{ for } x \neq x_i, \; i = 0, 1, 2, ---, n.$$

In this case the error exist and it is given by,

$$E(x) = f(x) - P_n(x)$$

$$= \frac{(x-x_0)(x-x_1) --- (x-x_n)}{(n+1)!} f^{n+1}(x^*)$$

where x^* lies in the interval (x_0, x_n) which is known as the truncation error.

* The bound for the truncation error is given by,

$$|E(x)| = \left| \frac{(x-x_0)(x-x_1) --- (x-x_n)}{(n+1)!} f^{n+1}(x^*) \right|$$

$$= \frac{1}{(n+1)!} \left| (x-x_0)(x-x_1) --- (x-x_n) \right| \left| f^{n+1}(x^*) \right|$$

If $\quad M = \max\limits_{x_0 \le x \le x_n} \left| f^{n+1}(x) \right|$, then

$$\left| f^{n+1}(x) \right| \le M$$

$$|E(x)| \le \frac{\left| (x-x_0)(x-x_1) --- (x-x_0) \right|}{(n+1)!} M$$

— x —

Q:- Find the bound for the error in linear interpolation

Sol:- Let the nodal points be x_0, x_1. The linear Lagrange interpolating polynomial is,

$$P(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$\therefore P(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1)$$

The error in this interpolation is,

$$E(x) = \frac{(x-x_0)(x-x_1)}{2!} f''(x^*), \text{ where } x_0 < x^* < x_1$$

$$\Rightarrow E(x) = \frac{(x-x_0)(x-x_1)}{2} f''(x^*) \quad \text{———— (I)}$$

Let $w(x) = (x-x_0)(x-x_1)$

$$= x^2 - xx_1 - xx_0 + x_0x_1$$

$$w'(x) = 2x - x_1 - x_0$$

for max^m/ min^m values of $w(x)$ we must have,

$$w'(x) = 0$$

$$\Rightarrow 2x - x_1 - x_0 = 0$$

$$\Rightarrow 2x = x_1 + x_0$$

$$\Rightarrow x = \frac{x_1 + x_0}{2}$$

$$\therefore w_{max} = \left(\frac{x_1+x_0}{2} - x_0\right)\left(\frac{x_1+x_0}{2} - x_1\right)$$

$$= \left(\frac{x_1}{2} + \frac{x_0}{2} - x_0\right)\left(\frac{x_1}{2} + \frac{x_0}{2} - x_1\right)$$

$$= \left(\frac{x_1}{2} - \frac{x_0}{2}\right)\left(\frac{x_0}{2} - \frac{x_1}{2}\right)$$

$$= \left(\frac{x_1 - x_0}{2}\right)\left(\frac{x_0 - x_1}{2}\right)$$

$$= -\left(\frac{x_1 - x_0}{2}\right)\left(\frac{x_1 - x_0}{2}\right)$$

$$= -\frac{(x_1 - x_0)^2}{4}$$

Now the bound for the error in linear interpolation is given by,

$$|E(x)| = \left| \frac{(x-x_0)(x-x_1)}{2} f''(x^*) \right|$$

$$= \frac{1}{2} \left| (x-x_0)(x-x_1) \right| \left| f''(x^*) \right|$$

$$\leq \frac{1}{2} \left| -\frac{(x_1-x_0)^2}{4} \right| \left| f''(x^*) \right|$$

$$= \frac{1}{8}(x_1 - x_0)^2 |f''(x^*)|$$

If $M = \max\limits_{x_0 \le x \le x_1} |f''(x)|$, then we have,

$$|f''(x)| \le M$$

$\Rightarrow |f''(x^*)| \le M$, where $x_0 < x^* < x_1$

$$|E(x)| \le \frac{1}{8}(x_1 - x_0)^2 M$$

i.e, $E(x) \le \frac{M}{8}(x_1 - x_0)^2$

Newton's divided differences :-

We shall define divided differences for an arbitrary set of nodal points $x_0, x_1, \ldots, x_n$ of a function $f(x)$ whose values at these points are $f(x_0), f(x_1), \ldots, f(x_n)$. The divided difference of zeroth order for the argument x_0 is denoted by $f[x_0]$ and is defined by $f[x_0] = f(x_0)$. The divided difference of order 1 for the arguments x_0, x_1 is denoted by $f[x_0, x_1]$ & is defined as,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The divided difference of order 2 for the arguments x_0, x_1 & x_2 is denoted by, $f[x_0, x_1, x_2]$ & is defined as

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

In general the nth order divided difference of the function $f(x)$ for $(n+1)$ nodal pts, denoted by, $f[x_0, x_1, \ldots, x_n]$ & is defined as,

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

- **Linear Newton's Interpolating Polynomial:**

Let $p(x) = a_0 + a_1 x$ —— (1)

be an interpolating polynomial to a function $f(x)$ at 2 distinct nodal points x_0 & x_1, where a_0 & a_1 are arbitrary constants.

Now
$$\left.\begin{aligned} f(x_0) &= p(x_0) = a_0 + a_1 x_0 \\ f(x_1) &= p(x_1) = a_0 + a_1 x_1 \end{aligned}\right\} \text{—— (2)}$$

Eliminating a_0 & a_1 from (1) & (2), the required linear interpolation is,

$$\begin{vmatrix} p(x) & 1 & x \\ f(x_0) & 1 & x_0 \\ f(x_1) & 1 & x_1 \end{vmatrix} = 0$$

Expanding this determinant interms of 1st row we get,

$$p(x)\begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix} - 1\begin{vmatrix} f(x_0) & x_0 \\ f(x_1) & x_1 \end{vmatrix} + x\begin{vmatrix} f(x_0) & 1 \\ f(x_1) & 1 \end{vmatrix} = 0$$

$$\Rightarrow p(x)(x_1 - x_0) - \{x_1 f(x_0) - x_0 f(x_1)\} + x\{f(x_0) - f(x_1)\} = 0$$

$$\Rightarrow p(x)(x_1 - x_0) = x_1 f(x_0) - x_0 f(x_1) - x\{f(x_0) - f(x_1)\}$$

$$= x_1 f(x_0) - x_0 f(x_1) - x f(x_0) + x f(x_1)$$

$$= x_1 f(x_0) - x_0 f(x_0) + x_0 f(x_0) - x_0 f(x_1) - x f(x_0) + x f(x_1)$$

$$= (x_1 - x_0) f(x_0) + \{x_0 f(x_0) - x f(x_0)\} + \{x f(x_1) - x_0 f(x_1)\}$$

$$= (x_1 - x_0) f(x_0) + (x_0 - x) f(x_0) + (x - x_0) f(x_1)$$

$$= (x_1 - x_0) f(x_0) + (x - x_0) f(x_1) - (x - x_0) f(x_0)$$

$$= (x_1 - x_0) f(x_0) + (x - x_0) \{f(x_1) - f(x_0)\}$$

$$\Rightarrow P(x) = \frac{(x_1 - x_0) f(x_0)}{x_1 - x_0} + (x - x_0) \left\{\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right\}$$

$$\Rightarrow \boxed{P(x) = f(x_0) + (x - x_0) f[x_0, x_1]} \quad \text{--- (3)}$$

Where, $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ is called the 1st divided difference of $f(x)$ relating to x_0 & x_1.

The eqⁿ (3) is called the linear Newton interpolating polynomial with divided difference

Higher order (Newton's divided difference interpolation):-

The interpolating polynomial $P_n(x)$ that interpolates a funcⁿ $f(x)$ at $(n+1)$ distinct nodal points can be written as,

$$P_n(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_0, x_1, \cdots, x_n] \quad \text{--- (7)}$$

This polynomial is called nth order Newton interpolating polynomial with divided differences

Ex:- Find the unique polynomial of deg. 2 or less such that $f(0)=1$, $f(1)=3$, $f(3)=55$, using the Newton divided difference interpolation.

Sol:- Given that the data is,

$$f(0)=1,\ f(1)=3,\ f(3)=55$$

Here $x_0=0$, $x_1=1$, $x_2=3$.

& $f(x_0)=1$, $f(x_1)=3$, $f(x_2)=55$

Since there are 3 nodal points, so the deg. of the interpolating polynomial is ≤ 2.

~~By Newton's~~ The Newton's divided difference interpolating polynomial is given by,

$$P(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2]$$

$$= 1 + (x-0)f[x_0,x_1] + (x-0)(x-1)f[x_0,x_1,x_2]$$

$$= 1 + xf[x_0,x_1] + x(x-1)f[x_0,x_1,x_2] \quad \text{(1)}$$

Now, $f[x_0,x_1] = \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

$$= \frac{3-1}{1-0}$$

$$= 2$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$= \frac{f[x_1,x_2]-f[x_0,x_1]}{3-0}$$

$$= \frac{1}{3}\left[\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}\right]$$

$$= \frac{1}{3}\left\{\left(\frac{55-3}{3-1}\right) - \left(\frac{3-1}{1-0}\right)\right\}$$

$$= \frac{1}{3}\left(\frac{52}{2} - \frac{2}{1}\right)$$

$$= \frac{1}{3}(26-2)$$

$$= \frac{1}{3} \times 24$$

$$= 8$$

Substituting these values in eqⁿ, we get

$$P(x) = 1 + 2x + 8x(x-1)$$

$$= 1 + 2x + 8x^2 - 8x$$

$$= 1 - 6x + 8x^2$$, which is the required interpolating polynomial.

Divided difference table:-

x	$f(x)$	1st order	2nd order	3rd order
x_0	$f(x_0)$			
		$f[x_0, x_1]$		
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f(x_2)$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f(x_3)$			

Ex:-

Construct the divided difference table for the following data. $f(10) = 355$, $f(0) = -5$, $f(8) = -21$, $f(1) = -14$, $f(4) = -125$

Sol:

Given that the data is,

$f(10) = 355$, $f(0) = -5$, $f(8) = -21$, $f(1) = -14$, $f(4) = -125$

x	$f(x)$	1st order	2nd order	3rd order	4th order
$x_0 = 10$	355				
		36			
$x_1 = 0$	-5		19		
		-2		~~17~~ 2	
$x_2 = 8$	-21		1		0
		-1		2	
$x_3 = 1$	-14		9		
		-37			
$x_4 = 4$	-125				

Forward difference operator:-

For a real valued func^n f of a real variable x, the forward difference operator denoted by Δ (read as del) is defined as

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

Note :-

→ $\Delta f_i = f_{i+1} - f_i$, where $f_i = f(x_i)$

→ $\Delta f(x_i) = f(x_i + h) - f(x_i)$

* $\Delta^2 f(x) = \Delta\{\Delta f(x)\}$

$= \Delta\{f(x+h) - f(x)\}$

$= \Delta f(x+h) - \Delta f(x)$

$= \{f(x+h+h) - f(x+h)\} - \{f(x+h) - f(x)\}$

$= f(x+2h) - f(x+h) - f(x+h) + f(x)$

$= f(x+2h) - 2f(x+h) + f(x)$

Forward difference table :-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

Ex:- Construct the forward difference table for the function $f(x) = 3x^3 + 2x^2 + x + 1$ for a set of arguments (nodal points) : 0, 2, 4, 6, 8.

Sol?-

Given that the function is

$$f(x) = 3x^3 + 2x^2 + x + 1$$

The set of arguements are 0, 2, 4, 6, 8

$f(0) = 1$

$f(2) = 3 \times 2^3 + 2 \times 2^2 + 2 + 1 = 3 \times 8 + 2 \times 4 + 2 + 1 = 24 + 8 + 3 = 35$

$f(4) = 3 \times 64 + 2 \times 16 + 4 + 1 = 192 + 32 + 5 = 229$

$f(6) = 3 \times 216 + 2 \times 36 + 6 + 1 = 648 + 72 + 7 = 727$

$f(8) = 3 \times 512 + 2 \times 64 + 8 + 1 = 1536 + 128 + 9 = 1673$

Forward difference table :-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		34			
2	35		160		
		194		144	
4	229		304		0
		498		144	
6	727		448		
		946			
8	1673				

Ex:- ~~Find~~ The following data hold good a polynomial. Find its degree.

x	-2	-1	0	1	2	3
$P(x)$	-5	1	1	1	7	25

Sol?:- Given that the data is,

x	-2	-1	0	1	2	3
$P(x)$	-5	1	1	1	7	25

Since the nodal pts are equispaced, we construct the following forward difference table.

Forward difference table

x	$P(x)$	ΔP	$\Delta^2 P$	$\Delta^3 P$	$\Delta^4 P$	$\Delta^5 P$
-2	-5					
		6				
-1	1		-6			
		0		6		
0	1		0		0	
		0		6		0
1	1		6		0	
		6		6		
2	7		12			
		18				
3	25					

As the 3rd differences of the polynomial are constant, the degree of the polynomial is 3.

Ex:- Construct the forward difference for the function $f(x) = (0,0,0,\epsilon,0,0,0)$

Sol^n:- Forward difference table

$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
0						
	0					
0		0				
	0		ϵ			
0		ϵ		-4ϵ		
	ϵ					
ϵ		-2ϵ	-3ϵ		10ϵ	
	$-\epsilon$			6ϵ		-20ϵ
0		ϵ	3ϵ			
	0				-10ϵ	
0			$-\epsilon$	-4ϵ		
		0				
	0					
0						

Th^m:-

If $x_i = x_0 + ih$, $i = 0, 1, 2, ---, n$ are equally spaced nodes with constant spacing h then.

$$f[x_0, x_1, ----, x_n] = \frac{\Delta^n f(x_0)}{n!\, h^n}$$

Pf:-

Let $x_i = x_0 + ih$, $i = 0, 1, 2, ---, n$ are equally spaced nodes with constant spacing h.

We shall prove that

$$f[x_0, x_1, ----, x_n] = \frac{\Delta^n f(x_0)}{n!\, h^n} \quad \text{--- (1)}$$

We prove it by method of induction.

For $n=1$

L.H.S. of (1) $= f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$= \frac{f_1 - f_0}{x_1 - x_0}$$

$$= \frac{f_1 - f_0}{h}$$

R.H.S. of (1) $= \dfrac{\Delta^1 f(x_0)}{1!\, h^1} = \dfrac{\Delta f_0}{h} = \dfrac{f_1 - f_0}{h}$

$\therefore$ L.H.S. of (1) = R.H.S. of (1)

So the th^m is true for $n=1$.

Let us assume that the th^m is true for $n=k$

$k \le n-1$.

i.e, $f[x_0, x_1, \ldots, x_k] = \dfrac{\Delta^k f(x_0)}{k!\, h^k}$

In particular for $k = n-1$, we have

$$f[x_0, x_1, \ldots, x_{n-1}] = \frac{\Delta^{n-1} f(x_0)}{(n-1)!\, h^{n-1}}$$

Now
$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

$$= \frac{1}{nh}\left\{ \frac{\Delta^{n-1} f(x_1)}{(n-1)!\, h^{n-1}} - \frac{\Delta^{n-1} f(x_0)}{(n-1)!\, h^{n-1}} \right\}$$

(By assumption).

$$= \frac{\Delta^{n-1}}{nh(n-1)!\, h^{n-1}} \left\{ f(x_1) - f(x_0) \right\}$$

$$= \frac{\Delta^{n-1}}{(n-1)!\, n\, h\, h^{n-1}} \Delta f(x_0)$$

$$= \frac{\Delta^{n-1}\, \Delta f(x_0)}{n!\, h^n}$$

$$= \frac{\Delta^n f(x_0)}{n!\, h^n}$$

This completes the proof of the th^m by induction method

Newton's forward difference interpolation:

Derive Newton's divided difference interpolating polynomial in terms of forward difference operator.

or

Derive Gregory-Newton interpolating polynomial

Pf:-

We know that the Newton's divided difference interpolating polynomial that interpolates a function $f(x)$ at $(n+1)$ distinct nodal points is given by,

$$P_n(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + \cdots + (x-x_0)(x-x_1) \cdots (x-x_{n-1}) f[x_0, x_1, \cdots, x_n] \quad \text{--- (1)}$$

Where, $f[x_0, x_1, \cdots, x_i] =$

$$= \frac{f[x_1, \cdots, x_i] - f[x_0, \cdots, x_{i-1}]}{x_i - x_0}$$

$i = 1, 2, \cdots, n$ are called Newton's divided differences.

If the nodal points $x_0, x_1, \cdots, x_n$ are equally spaced nodal points with constant spacing h,

then $x_i = x_0 + ih$; for $i = 0, 1, 2, \cdots, n$

Now $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$= \frac{\Delta f(x_0)}{h}$$

$$\frac{\Delta f(x_0)}{1! \, h^1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{1}{2h}\left[\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}\right]$$

$$= \frac{\Delta}{2h^2}\left[f(x_1) - f(x_0)\right]$$

$$= \frac{\Delta}{2h^2}\,\Delta f(x_0)$$

$$= \frac{\Delta^2 f(x_0)}{2!\,h^2}$$

Proceeding inductively we can show that

$$f[x_0, x_1, \ldots, x_n] = \frac{\Delta^n f(x_0)}{n!\,h^n}$$

Substituting these values in eq^n (1) we get,

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!\,h^1} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2} + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n f(x_0)}{n!\,h^n} \quad (2)$$

which is known as Newton's Forward difference interpolating polynomial.

Ex:-

Compute $f(x) = e^x$ for $x = 0.92$ from the table of values

x	0	0.1	0.2	0.3	0.4
e^x	1	1.1052	1.2214	1.3499	1.4918

Sol^n:-

Given that, $f(x) = e^x$

The data is.

x	0	0.1	0.2	0.3	0.4
e^x	1	1.1052	1.2214	1.3499	1.4918

The forward difference table for the given data is as follows:-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		0.1052			
0.1	1.1052		0.0110		
		0.1162		0.0013	
0.2	1.2214		0.0123		−0.0002
		0.1285		0.0011	
0.3	1.3499		0.0134		
		0.1419			
0.4	1.4918				

The Newton's forward difference interpolating polynomial is given by,

$$P(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{h} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2} +$$

$$(x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!\,h^3} + (x-x_0)(x-x_1).$$

$$(x-x_2)(x-x_3)\frac{\Delta^4 f(x_0)}{4!\,h^4}$$

$$= 1 + (x-0)\frac{0.1052}{0.1} + (x-0)(x-0.1)\frac{0.0110}{2(0.1)^2} + (x-0)(x-0.1).$$

$$(x-0.2)\frac{0.0013}{6(0.1)^3} + (x-0)(x-0.1)(x-0.2)(x-0.3)$$

$$\left\{\frac{-0.0002}{24(0.1)^4}\right\}$$

At $x = 0.92$

$$P(x) = 1 + (0.92-0)\frac{0.1052}{0.1} + (0.92-0)(0.92-0.1)\times\frac{0.0110}{2(0.01)}$$

$$+ (0.92-0)(0.92-0.1)(0.92-0.2)\frac{0.0013}{6(0.001)}$$

$$(0.92-0)(0.92-0.1)(0.92-0.2)(0.92-0.3)\left\{\frac{0.0002}{24(0.0001)}\right\}$$

$$= 1 + \left(0.92 \times \frac{0.1052}{0.1}\right) + \left\{(0.92)(0.82)\left(\frac{0.0119}{0.02}\right)\right\} + \left(0.92 \times 0.82 \times 0.72 \times \frac{0.0013}{0.006}\right) - \left(0.92 \times 0.82 \times 0.72 \times 0.62\right)\left(\frac{0.0002}{0.0024}\right)$$

$$= 1 + 0.96784 + 0.41492 + 0.1176864 - 0.02806368$$

$$= 2.47238272$$

Backward difference formula:

For a real valued function f of a real variable x, the backward difference operator denoted by ∇ (nabla) & is defined as,

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

* $\nabla^2 f(x) = \nabla\{\nabla f(x)\} = \nabla\{f(x) - f(x-h)\}$

$$= \nabla f(x) - \nabla f(x-h)$$

$$= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-h-h)\}$$

$$= f(x) - f(x-h) - f(x-h) + f(x-2h)$$

$$= f(x) - 2f(x-h) + f(x-2h)$$

* If $x_0, x_1, \ldots, x_n$ are equally spaced nodal pts with constant spacing h, then $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots$

Then $\nabla f(x_i) = f(x_i) - f(x_i - h)$

i.e, $\nabla f_i = f_i - f_{i-1}$

Backward difference table:-

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

Ex:- Construct the backward difference table for the data

x	1	2	3	4
f	63	52	43	38

Sol:- Given that the data is,

x	1	2	3	4
f	63	52	43	38

For this data the backward difference table is as follows :-

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$
1	63			
		-11		
2	52		2	
		-9		2
3	43		4	
		-5		
4	38			

For the following data calculate the difference & obtain the forward & backward difference polynomial at $x = 0.25$ & $x = 0.35$

x : 0.1 0.2 0.3 0.4 0.5

f : 1.40 1.56 1.76 2 2.28

Sol:-

Given that the data is,

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2	2.28

For this data the forward difference table is as follows:-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.2		−1.8	
0.3	1.76		−1.76		3.6
		0.24		1.8	
0.4	2		0.04		
		0.28			
0.5	2.28				

Now the Newton's forward difference interpolating polynomial is,

$$P(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{h} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2}$$

$$+ (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!\,h^3} + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\frac{\Delta^4 f}{4!\,\ldots}$$

$$= 1.40 + (x-0.1)\frac{0.16}{0.1} + (x-0.1)(x-0.2)\frac{0.04}{2(0.1)^2} + (x-0.1)(x-0.2)(x-0.3)\left\{\frac{-1.8}{3(0.1)^3}\right\} + (x-0.1)(x-\ldots)(x-0.3)(x-\ldots)(3\ldots$$

For $x = 0.25$,

$$P(x) = \cancel{\text{forward}}\ 1.40 + (0.25-0.1)\frac{0.16}{0.1} + (0.25-\ldots)(0.25-0.2)\frac{0.04}{2\times(0.1)^2} + (0.25-0.1)(0.25-\ldots)\times\left\{\frac{-1.8}{3\times(0.1)^3}\right\} + (0.25-0.1)(0.25-\ldots)(0.25-\ldots)\ldots\times\left\{\frac{3.6}{4(0.\ldots}\right.$$

$= 1.40 + 0.24 + 0.15 + 0.225 + 0.50625$

$= 2.52125$

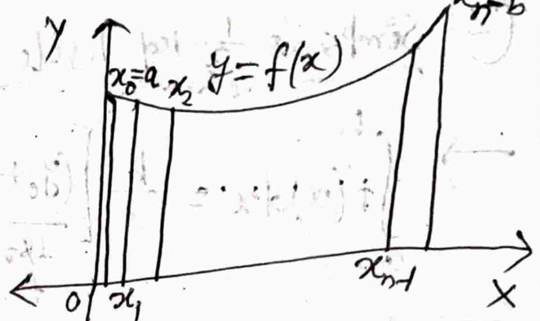
Unit-4

Numerical Integration

→ The Area bounded by the Curve $f(x)$ & X-axis betⁿ the limits a & b denoted by

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx \quad \text{--- (1)} \quad [\because y = f(x)]$$

→ Divide the interval (a,b) into n-equal interval with length h (step size).



i.e. $[a = x_0, x_1, \ldots, x_{n-1}, x_n = b] = [a,b]$.

$\Rightarrow a = x_0$

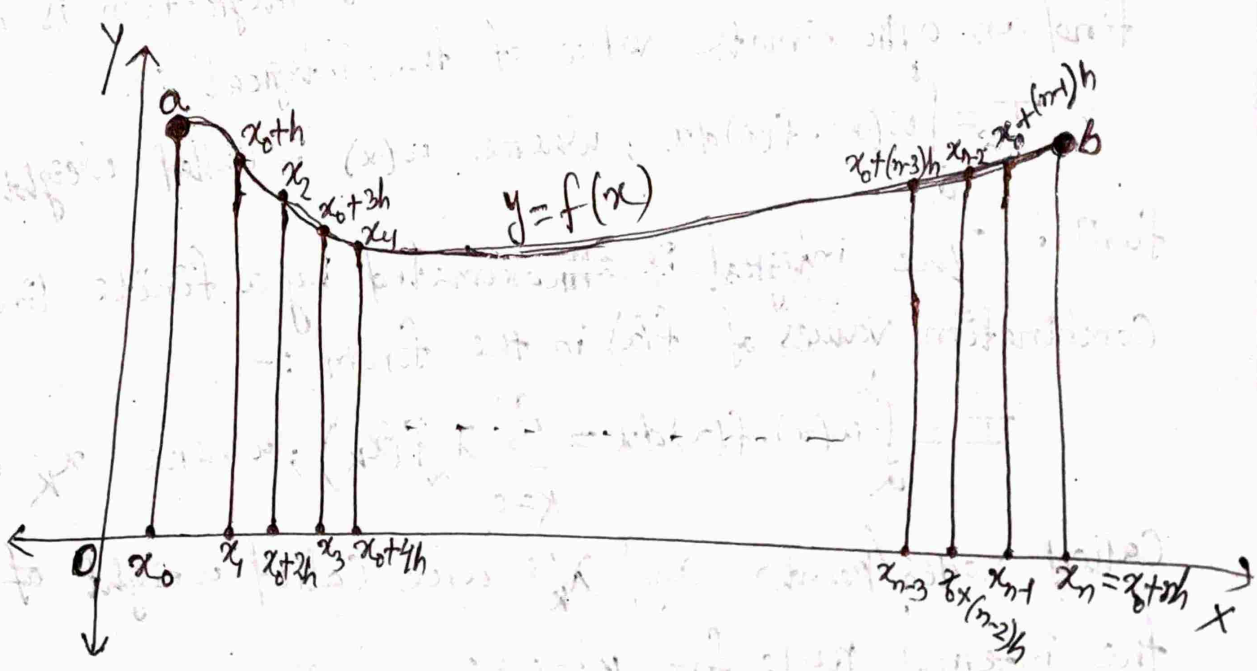
$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_1 + 2h = x_0 + 3h$

$\vdots$

$x_n = x_{n-1} + h = x_0 + nh$; $h = \frac{b-a}{n}$ or $n = \frac{b-a}{h}$.



→ Eq^n(1) Can be evaluated by using :-

① Trapezoidal Rule (Any no. interval) →

→ $\int_a^b f(x)dx = \frac{h}{2}\left[\underbrace{(y_0+y_n)}_{\text{1st + Last term}} + 2\underbrace{(y_1+y_2+\cdots+y_{n-1})}_{\text{Remaining terms}}\right]$ for n-intervals with $h=\frac{b-a}{n}$ & $y_n = f(x_n)$ for $n \geq 0$, $f(x)=y$.

→ It is applicable on any no. of interval.

② Simpson's $\frac{1}{3}$rd Rule (Even interval) →

→ $\int_a^b f(x)dx = \frac{h}{3}\left[\underbrace{(y_0+y_n)}_{\text{1st + last term}} + 2\underbrace{(y_2+y_4+\cdots)}_{\text{Even terms}} + 4\underbrace{(y_1+y_3+\cdots)}_{\text{odd / Remaining terms}}\right]$

→ It is applicable if the total no. of interval is even.

③ Simpson's $\frac{3}{8}$th Rule (3-Multiple) →

$\int_a^b f(x)dx = \frac{3h}{8}\left[\underbrace{(y_0+y_n)}_{\text{1st + last term}} + 2\underbrace{(y_3+y_6+y_9+\cdots)}_{\text{Multiple of 3}} + 3\underbrace{(y_1+y_2+y_4+\cdots)}_{\text{Remaining terms}}\right]$

*

→ The general problem of numerical integration is it find an approximate value of the integral :-

$$I = \int_a^b \omega(x).f(x)dx$$, where $\omega(x)$ is called weight funⁿ. The integral is approximated by a finite linear combination values of $f(x)$ in the form :-

$$I = \int_a^b \omega(x).f(x)dx = \sum_{K=0}^{n} \lambda_K f(x_K)$$; where x_K is called nodes/points and λ_K's are called weight of the integral rule for $K = 0, 1, 2, ---, n$.

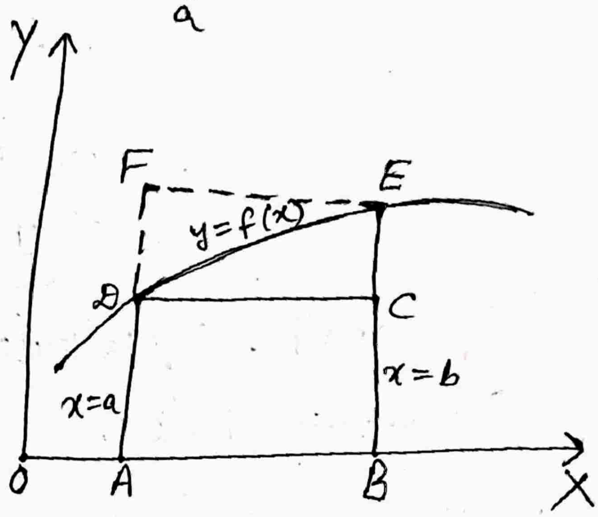
The error of approximation is given by,

$$R_n = \int_a^b \omega(x)f(x)dx - \sum_{K=0}^{n} \lambda_K f(x_K).$$

* Some Simple Quadrature rule :-

(i) Left Rectangular Rule →

Let us Consider the integral $I(f) = \int_a^b f(x)dx$.



Clearly it represents the area under the Curve $y=f(x)$ bounded by $x=a$, $x=b$ & x-axis $(y=0)$.

If the Point B Close to the point A, then ABCD or ABEF are nearly equal area which is equivalent to $\int_a^b f(x)dx$. So, we have either $\int_a^b f(x)dx = (b-a)f(a)$

i.e. $I(f) = (b-a)f(a)$

We write $R'(f) = (b-a)f(a)$.

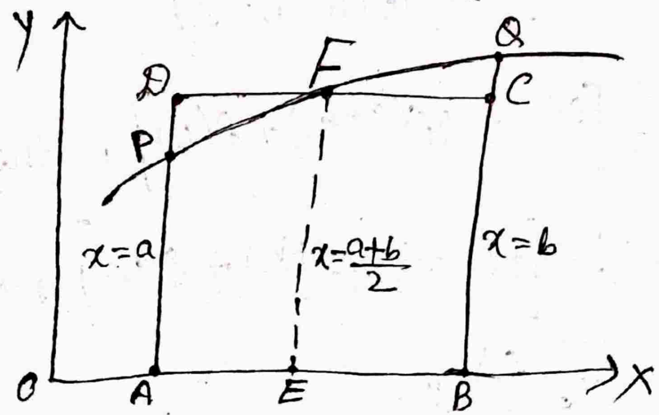
$\Rightarrow I(f) \simeq R'(f)$, $R'(f)$ is Called as left rectangular rule.

(ii) Right Rectangular Rule →

Writting $R''(f) = (b-a)f(b)$ & also $I(f) = (b-a)f(b)$

$\Rightarrow I(f) \simeq R''(f)$, $R''(f)$ is Called as right rectangular rule.

(iii) Mid-Point Quadrature Rule →



Now the area of the rectangle ABCD whose sides are AB & ordinate (E, F).

Then we obtained, $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

$$\Rightarrow I(f) \approx (b-a)f\left(\frac{a+b}{2}\right).$$

$$\& \quad M(f) = (b-a)f\left(\frac{a+b}{2}\right).$$

$\Rightarrow \boxed{I(f) \approx M(f)}$ when $(b-a)$ is very very small. This rule is known as the Mid-Point quadrature rule.

(iv) Trapezoidal rule → (For $n=1$)

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[y_0+y_1\right] \quad \{\because f(x)=y\}$$

→ In the interval $[a,b] \Rightarrow \int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right] = T(f)$

for $h = \frac{b-a}{n} \Rightarrow h = b-a$ (step size / Length) $\Rightarrow \boxed{I(f) \approx T(f)}$.

Rough: Area of trapezium

$= \frac{1}{2}$ (Distance betn them) × (Sum of two parallel Sides)

$$\Rightarrow T(f) = \frac{1}{2}(b-a)\left[f(a)+f(b)\right]$$

is called as Trapezoidal rule for $n=1$.

(v) Simpson's $\frac{1}{3}$rd rule → (For $n=2$)

$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$ in the interval $[a,b]$.

$\Rightarrow S_{1/3}(f) \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$ for $h = \frac{b-a}{2}$.

$$\Rightarrow \boxed{I(f) \approx S_{1/3}(f)}.$$

(vi) Simpson's $\frac{3}{8}$th rule $\rightarrow$ (For $n=3$)

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right] \text{ for } h=\frac{b-a}{3}, y_n=f(x_n)$$

$$\Rightarrow S_{3/8}(f)=\frac{b-a}{8}\left[(y_0+y_3)+3(y_1+y_2)\right]; \ 0\le n\le 3 \Rightarrow \boxed{I(f)\approx S_{3/8}(f)}.$$

(1) Q:- Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using

(i) Trapezoid rule

(ii) Simpson's $\frac{1}{3}$rd rule

(iii) Simpson's $\frac{3}{8}$th rule ?

Solⁿ:-

Let us take $n=6$ because 'n' is even & also multiple of 3. Then $h=\frac{b-a}{n}=\frac{1-0}{6}=1/6$; $a=0, b=1$.

& $f(x)=\frac{1}{1+x^2}$.

Since, $x_0=0=a$

$x_1=x_0+h=0+\frac{1}{6}=1/6$

$x_2=x_0+2h=2/6=1/3$

$x_3=x_0+3h=3/6=1/2$

$x_4=x_0+4h=4/6=2/3$

$x_5=x_0+5h=5/6$

$x_6=x_0+6h=6/6=1=b$.

$y_0=f(x_0)=\frac{1}{1+x_0^2}=\frac{1}{1+0}=1$

$y_1=\frac{1}{1+\left(\frac{1}{6}\right)^2}=\frac{36}{37}$

$y_2=\frac{1}{1+\left(\frac{1}{3}\right)^2}=\frac{9}{10}=0.9$

$y_3=\frac{1}{1+\left(\frac{1}{2}\right)^2}=\frac{4}{5}=0.8$

$y_4=\frac{1}{1+\left(\frac{2}{3}\right)^2}=\frac{9}{13}$

$y_5=\frac{1}{1+\left(\frac{5}{6}\right)^2}=\frac{36}{61}$

$y_6=\frac{1}{1+1^2}=\frac{1}{2}=0.5$

(i) $T(f)=\frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$

$=\frac{1}{12}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$

$=0.78423$

(ii) $S_{1/3}(f)=\frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$

$=\frac{1}{18}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{9}{10}+\frac{9}{13}\right)+4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right)\right]$

$=0.785396\simeq 0.7854$

(iii) $S_{3/8}(f) = \frac{3h}{8}\left[(y_0+y_6)+2y_3+3(y_1+y_2+y_4+y_5)\right]$

$= \frac{1}{16}\left[1+\frac{1}{2}+2\cdot\frac{4}{5}+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)\right]$

$= 0.785394 \approx 0.7854$

(iv) Direct integration,

$\int_a^b f(x)dx = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \approx 0.785398 \approx 0.7854$ $\square$

(2) Q:- Evaluate, $\int_1^4 \sqrt{x}\,dx$. Using left, right rectangular rules.

State which of the rules give the better approximation ?

Solⁿ:- Given $f(x)=\sqrt{x}$ & $a=1, b=4$.

$\therefore f(a)=1,\ f(b)=2$

(i) Left rectangular rule →

$R'(f) = (b-a)f(a)$

$= (4-1)f(1)$

$= 3.1 = 3$

$\Rightarrow I(f) = 3$

(ii) Right rectangular rule →

$R''(f) = (b-a)f(b)$

$= 3.2 = 6$

$\Rightarrow I(f) = 6$

(iii) Direct integration →

$\int_1^4 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4$

$= \frac{4^{3/2}}{3/2} - \frac{1}{3/2}$

$= \frac{2}{3}\left[2^3-1\right]$

$= \frac{14}{3}$

≈ 4.667

(iv) Error →

$(L.R.R) = 4.667 - 3 = 1.667$ & $(R.R.R) = 6 - 4.667 = 1.333$

Errore in approximation →

The quantity $\boxed{E(f) = I(f) - R_n(f)}$ is called the error associated with the rule $R_n(f)$. □

*** Th^m →** Derive the trapezoidal rule by integrating the polynomial $P_1(x) = f(a) + f[a,b](x-a)$ from $x = a$ to $x = b$.

Pf :– We know that $P_1(x) = \frac{h}{2}[f(a) + f(b)]$, where

$$h = \frac{b-a}{n} = \frac{b-a}{1} \Rightarrow h = b-a \quad \& \quad [a,b] = [\underbrace{x_0, x_1}_{1=n}]$$

(interval)

Then we have to show that

$$P_1(x) = \int_a^b [f(a) + f[a,b](x-a)]\,dx$$

R.H.S. $= \int_a^b [f(a) + f[a,b](x-a)]\,dx$

$$= \int_a^b f(a)\,dx + f[a,b]\int_a^b (x-a)\,dx$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)\,dx$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\left[\frac{x^2}{2} - ax\right]_a^b$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right)$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\left(\frac{b^2 - 2ab + 2a^2 - a^2}{2}\right)$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cdot \frac{(b-a)^{\cancel{2}}}{2}$$

$$= \frac{(b-a)}{2}\left[2f(a) + \{f(b) - f(a)\}\right]$$

$$= \frac{1}{2}(b-a)[f(b) + f(a)] = \frac{h}{2}[f(a) + f(b)] = \text{R.H.S.}$$

* Newton's-Cotes Rule →

This rule depends upon the interpolatory formulas. Let us consider the integral,

$$I(f) = \int_a^b f(x)\,dx \quad \text{— (1)}$$ to be approximated.

Let the points at which $f(x)$ is interpolated be the equispaced nodes $x_0 = a \le x_1 < x_2 < \dots < x_n = b$.

Where, $x_i = x_0 + ih$, $0 \le i \le n$ or $0(1)n = i$.

Then, the polynomial in Lagrange's form of interpolation to $f(x)$ is

$$P_n(x) = l_0(x) f(x_0) + \dots + l_n(x) f(x_n) \quad \text{— (2)}$$

where, $$l_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \quad \text{— (3)}$$

Then $f(x) \simeq P_n(x)$, $a \le x \le b$.

So, $I(f) = \int_a^b f(x)\,dx \simeq \int_a^b P_n(x)\,dx = I(P_n)$.

Let us we write, $w_i = \int_a^b l_i(x)\,dx$, — (4) $0 \le i \le n$.

Since, $I(P_n) = \int_a^b P_n(x)\,dx$

$$= \int_a^b \sum_{i=0}^{n} l_i(x) f(x_i)\,dx$$

$$= \sum_{i=0}^{n} \left[\int_a^b l_i(x)\,dx\right] f(x_i)$$

$$= \sum_{i=0}^{n} w_i f(x_i)$$

$$= w_0 f(x_0) + \cdots + w_n f(x_n)$$

$$= R_{n+1}(f)$$

The rule $R_{n+1}(f)$ is called $(n+1)$ points of Newton-Cotes quadrature formula. □

<u>Degree</u> <u>of</u> <u>Precision</u> →

Let $R_n(f)$ be the 'n' point rule to evaluate $I(f)$. Then the degree of precision of the rule is defined to be natural no. d such that $R_n(f)$ integrates exactly every monomial of deg. $\le d$. □

Note

Fore the more simplification, let us take

$$x = x_0 + \alpha h \quad \& \quad a \le x \le b$$

$$\therefore a \le x \le b \Rightarrow a \le x_0 + \alpha h \le b$$

$$\Rightarrow a \le a + \alpha h \le b$$

$$\Rightarrow 0 \le \alpha h \le b - a \qquad \left[\because h = \frac{b-a}{n} \Rightarrow b - a = nh \right]$$

$$\Rightarrow 0 \le \alpha h \le nh$$

$$\Rightarrow 0 \le \alpha \le n.$$

So, $W_i = \int_a^b l_i(x)\,dx$

$$= \int_a^b \prod_{\substack{j=0 \\ j \ne i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) dx \qquad \left[\begin{array}{l} \text{Put } x = x_0 + \alpha h \\ \Rightarrow dx = h\,d\alpha \\ a \le x \le b \\ \Rightarrow 0 \le \alpha \le n \end{array} \right]$$

$$= \int_0^n \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{(x_0 + \alpha h) - (x_0 + jh)}{(x_0 + ih) - (x_0 + jh)}\, h\,d\alpha$$

$$= h \int_0^n \prod_{\substack{j=0 \\ j \ne i}}^{n} \left(\frac{\alpha - j}{i - j} \right) d\alpha$$

$$= h \int_0^n \frac{(\alpha - 0)(\alpha - 1) \cdots (\alpha - (i-1))(\alpha - (i+1)) \cdots (\alpha - n)}{(i-0)(i-1) \cdots (i - (i-1))(i - (i+1)) \cdots (i - n)}\, d\alpha$$

$$= h \int_0^n \frac{\alpha(\alpha-1) \cdots (\alpha - i + 1)(\alpha - i - 1) \cdots (\alpha - n)}{\{i(i-1) \cdots 1\}\{(-1)(-2) \cdots (-(n-i))\}}\, d\alpha$$

$$= h \int_0^n \frac{\alpha(\alpha-1) \cdots (\alpha - i + 1)(\alpha - i - 1) \cdots (\alpha - n)}{i!\,(-1)^{n-i}\,(n-i)!}\, d\alpha$$

$$= \frac{h}{i!(-1)^{n-i}(n-i)!} \int_0^n \alpha(\alpha-1)\dots(\alpha-n)\,d\alpha$$

$$\Rightarrow \boxed{w_i = \frac{h(-1)^{i-n}}{i!(n-i)!} \int_0^n \alpha(\alpha-1)\dots(\alpha-n)\,d\alpha \;;\; 0 \le i \le n.} \quad \square$$

* Th^m – 6.1 (Error in Newton's-Cotes quadrature rule)

Let $f \in C^{n+1}[a,b]$. Then the error in approximating $I(f)$ by Newton's-Cotes method is given by,

$$\boxed{E(f) = I(f) - R_{n+1}(f) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi)\, \phi_n(x)\,dx.}$$

where, $\phi_n(x) = \prod_{i=0}^{n} (x - x_i)$ & $\xi \in (a,b)$.

Pf :- Since $f \in C^{n+1}[a,b]$.

So, $f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\left[(x-x_0)(x-x_1)\dots(x-x_n)\right]$

$$\Rightarrow f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\phi_n(x) \;;\; \xi, \mu \in (a,b).$$

Now integrating both sides, we get

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx$$

$$\Rightarrow I(f) = I(P_n) + \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx$$

$$\Rightarrow I(f) - I(P_n) = \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx$$

$$\Rightarrow \boxed{E(f) = \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx = \frac{f^{(n+1)}(\mu)}{(n+1)!}\int_a^b \phi_n(x)dx.} \quad \blacksquare$$

* Th^m – 6.2

Let $f \in C^{n+1}[a,b]$. If $\phi_n(x)$ does not change its sign in $[a,b]$, then $\boxed{E(f) = \frac{f^{(n+1)}(\mu)}{(n+1)!}\int_a^b \phi_n(x)\,dx},$

where $\mu \in (a,b)$.

Proof :- From the mean value thm for integrals, we know that if f is a cont. funn on $[a,b]$ & $g:[a,b]\to R$ is of one sign for all $x\in[a,b]$, then

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx \text{ for some } a<c<b.$$

Here we take $f(x)=f^{(n+1)}(\xi)$ & $g(x)=\phi_n(x)$.

Then $$\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx = f^{(n+1)}(\mu)\int_a^b \phi_n(x)\,dx \;;\; a<\mu<b. \quad (1)$$

Now $$E(f)=\frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\,\phi_n(x)\,dx$$

$$\Rightarrow E(f)=\frac{f^{(n+1)}(\mu)}{(n+1)!}\int_a^b \phi_n(x)\,dx, \text{ [By eqn (1)]}$$

for $\mu\in(a,b)$. $\square$

$Th^m - 6.3$

a) Let the index of the rule be even & $f \in C^{n+2}[a,b]$. Then $\boxed{E(f) = \dfrac{h^{n+3} f^{(n+2)}(\eta_1)}{(n+2)!} \varphi_n}$;

where, $\varphi_n = \int_0^n x^2(x-1)(x-2)\cdots(x-n)\,dx$.

b) Let the index of the rule be odd & $f \in C^{n+1}[a,b]$. Then $\boxed{E(f) = \dfrac{h^{n+2} f^{(n+1)}(\eta_2)}{(n+1)!} \psi_n}$;

where, $\psi_n = \int_0^n x(x-1)\cdots(x-n)\,dx$.

Where, $\eta_1, \eta_2 \in [a,b]$ & $h = \dfrac{b-a}{n}$. □

$Th^m - 6.4$

Let $f \in C^{n+2}[a,b]$ & $\int_a^b \phi_n(x)\,dx = 0$.

Then $\boxed{E(f) = \dfrac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b \phi_{n+1}(x)\,dx}$.

Where, $\phi_{n+1}(x) = (x - x_{n+1})\phi_n(x)$ is of one sign on (a,b) for a suitable node x_{n+1} & $\eta \in (a,b)$. □

* Th^m 6.5 (Trapezoidal rule, $n=1$)

If $f \in C^2[a,b]$, then $I(f) \simeq \frac{b-a}{2}[f(a)+f(b)]$

i.e. $\boxed{I(f) = \frac{b-a}{2}[f(a)+f(b)] - \frac{1}{12}(b-a)^3 f''(\eta)}$;

where, $a \le \eta \le b$ & $T(f) = \frac{b-a}{2}[f(a)+f(b)]$

i.e. $I(f) \simeq T(f)$.

& $E(f) = -\frac{(b-a)^3}{12} f''(\eta) = -\frac{h^3}{12} f''(\eta)$.

Pf:-

From Newton's Cotes rule, we know that the integral $I(f) = \int_a^b f(x)dx$ —— (1) is approximated by $I(f) \approx w_0 f(x_0) + w_1 f(x_1) + \cdots + w_n f(x_n)$

i.e. $I(f) \approx w_0 f_0 + w_1 f_1 + \cdots + w_n f_n$ —— (2)

$$w_i = \frac{(-1)^{i-n} \cdot h}{(n-i)!\, i!} \int_0^n \alpha(\alpha-1)\ldots(\alpha-i-1)(\alpha-i+1)\ldots(\alpha-n)\,d\alpha \quad \text{--- (3)}$$

If $x_0, x_1, \ldots, x_n$ are equispaced nodes with spacing h & $x = x_0 + \alpha h$.

Taking $n=1$, we get

$$I(f) \approx w_0 f_0 + w_1 f_1 = w_0 f(x_0) + w_1 f(x_1)$$

Since, $x_0 = a$ & $x_1 = b \Rightarrow I(f) \approx w_0 f(a) + w_1 f(b)$ --- (4)

AS $$w_0 = \frac{(-1)^{0-1} \cdot h}{(1-0)! \times 0!} \int_0^1 (\alpha-1)\,d\alpha \quad [\text{By } (3)]$$

$$= -h \int_0^1 (\alpha-1)\,d\alpha = -h\left[\frac{\alpha^2}{2} - \alpha\right]_0^1$$

$$= -h\left(\frac{1}{2} - 1\right) = \frac{h}{2}$$

$$w_1 = \frac{(-1)^{1-1} \cdot h}{(1-1)! \times 1!} \int_0^1 \alpha\,d\alpha = h\left[\frac{\alpha^2}{2}\right]_0^1 \quad [\text{By } (3)]$$

$$= \frac{h}{2}$$

Substituting these values in eq^n (4), we obtained

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] \qquad \left\{\because h = \frac{b-a}{n} = b-a\right\}$$

$$\Rightarrow \boxed{I(f) \approx \frac{b-a}{2}[f(a) + f(b)] = T(f)} \quad \text{--- (i)}$$

* Error in Trapezoidal rule →

We know that, the error in '$n+1$' point rule is $E(f) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi)\, \phi_n(x)\,dx$. Where

$$\phi_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n).$$

Here, $n=1 \Rightarrow E(f) = \frac{1}{2!}\int_a^b f''(\xi)\,\phi_1(x)\,dx$

$$\Rightarrow E(f) = \frac{1}{2}\int_a^b (x-x_0)(x-x_1)\; f''(\xi)\,dx$$

$$= \frac{1}{2}\int_a^b (x-a)(x-b)\, f''(\xi)\,dx$$

$$= \frac{f''(\mu)}{2}\int_a^b (x-a)(x-b)\,dx \;;\; \mu \in (a,b) = (x_0, x_1).$$

[By th^m-6.2]

$$= \frac{f''(\mu)}{2}\int_a^b \{x^2-(a+b)x+ab\}\,dx$$

$$= \frac{f''(\mu)}{2}\left[\frac{x^3}{3} - \frac{(a+b)}{2}x^2 + abx\right]_a^b$$

$$= \frac{f''(\mu)}{2}\left[\frac{b^3}{3} - \frac{(a+b)b^2}{2} + ab^2 - \frac{a^3}{3} + \frac{(a+b)a^2}{2} - a^2b\right]$$

$$= \frac{f''(\mu)}{12}\left[2b^3 - 3(a+b)b^2 + 6ab^2 - 2a^3 + 3(a+b)a^2 - 6a^2b\right]$$

$$= \frac{f''(\mu)}{12}\left[2b^3 - 3ab^2 - 3b^3 + 6ab^2 - 2a^3 + 3a^3 + 3a^2b - 6a^2b\right]$$

$$= \frac{f''(\mu)}{12}\left[a^3 - b^3 + 3ab^2 - 3a^2b\right]$$

$$= \frac{(a-b)^3}{12} f''(\mu)$$

Or $\quad -\frac{(b-a)^3}{12} f''(\mu)$

$$= -\frac{h^3}{12} f''(\mu) \;;\; h = b-a.$$

$$\Rightarrow \boxed{E(f) = \frac{-h^3}{12} f''(\mu)}\ ;\ \mu \in (a,b) = (x_0, x_1) \quad \text{(ii)}$$

[By (i) & (ii)]

Hence, $I(f) = T(f) + E(f)$

$$\Rightarrow \boxed{I(f) = \frac{h}{2}\left[f(a) + f(b)\right] - \frac{h^3}{12} f''(\mu)}. \quad \square$$

*** Th^m — 6.6** (Simpson's $\frac{1}{3}$rd rule, $n=2$)

If $f \in C^4[a,b]$, then $S_{1/3}(f) + E(f) \approx I(f)$.

i.e. $I(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^5}{90} f^{iv}(\eta)$,

$\eta \in [a,b]$ or $a \le \eta \le b$ ..

Pf :- We know that $I(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + E(f)$ (1)

Where, $w_i = \frac{(-1)^{i-n} \cdot h}{i!\,(n-i)!} \int_0^n \alpha(\alpha-1) \cdots (\alpha - i + 1)(\alpha - i - 1) \cdots (\alpha - n)\, d\alpha$ (2)

Here, $[a,b] = [x_0, x_2]$

$\Rightarrow x_0 = a$ & $x_2 = b$.

$$\Rightarrow x_0 = a$$

$$x_1 = x_0 + h = a + h = a + \frac{b-a}{2}$$

$$= \frac{a+b}{2}$$

$$\& \ x_2 = x_0 + 2h = a + (b-a) = b.$$

Since, $w_0 = \frac{(-1)^{0-2} h}{0!\,(2-0)!} \int_0^2 (\alpha - 1)(\alpha - 2)\, d\alpha$

$$= \frac{h}{2} \int_0^2 (\alpha^2 - 3\alpha + 2)\, d\alpha$$

$$\Rightarrow w_0 = \frac{h}{2}\left[\frac{\alpha^3}{3} - \frac{3\alpha^2}{2} + 2\alpha\right]_0^2$$

$$= \frac{h}{2}\left(\frac{8}{3} - 6 + 4\right) = \frac{h}{2}\cdot\frac{2}{3} = \frac{h}{3}$$

$$\Rightarrow w_0 = \frac{h}{3}$$

$$\because w_1 = \frac{(-1)^{1-2}}{1!\,(2-1)!}\, h \int_0^2 \alpha(\alpha-2)\, d\alpha$$

$$= \frac{-h}{\cancel{2}}\left[\frac{\alpha^3}{3} - \alpha^2\right]_0^2$$

$$= -h\left(\frac{8}{3} - 4\right) = \frac{4h}{3}$$

$$\& \; w_2 = \frac{(-1)^{2-2}.h}{2!\,(2-2)!}\int_0^2 \alpha(\alpha-1)\, d\alpha$$

$$= \frac{h}{2}\left[\frac{\alpha^3}{3} - \frac{\alpha^2}{2}\right]_0^2$$

$$= \frac{h}{2}\left(\frac{8}{3} - 2\right) = \frac{h}{2}\cdot\frac{2}{3} = \frac{h}{3}$$

$$\Rightarrow w_2 = \frac{h}{3}$$

Now eq^n (1) becomes

$$I(f) = \frac{h}{3} f(a) + \frac{4h}{3} f\left(\frac{a+b}{2}\right) + \frac{h}{3} f(b) + E(f).$$

$$\Rightarrow I(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + E(f) \quad \text{—— (3)}$$

* Error in Simpson's $\frac{1}{3}$rd rule →

Now index of a rule is $n=2$ = even.

Then by th^m – 6.3(a), for evaluation of error

$$E(f) = \frac{h^{n+3} f^{(n+2)}(\eta)}{(n+2)!}\int_0^n x^2(x-1)(x-2)\ldots(x-n)\, dx$$

$$\Rightarrow E(f) = \frac{h^5 f^{iv}(\eta)}{4!} \int_0^2 x^2(x-1)(x-2)\,dx \quad [\because n=2]$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \int_0^2 (x^4 - 3x^3 + 2x^2)\,dx$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left[\frac{x^5}{5} - \frac{3}{4}x^4 + \frac{2}{3}x^3\right]_0^2$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left[\frac{2^5}{5} - \frac{3}{4}\cdot 2^4 + \frac{2}{3}\cdot 2^3\right]$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left(\frac{32}{5} - 12 + \frac{16}{3}\right)$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left(\frac{96 - 180 + 80}{15}\right)$$

$$= \frac{h^5 f^{iv}(\eta)}{\cancel{24}\,6} \cdot \frac{-\cancel{4}}{15}$$

$$\Rightarrow E(f) = -\frac{h^5}{90} f^{iv}(\eta)$$

By $eq^n(3)$, $\boxed{I(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^5}{90} f^{iv}(\eta)}$;

for $a \leq \eta \leq b$.

□

Note

→ The integral $I(f) = \int_a^b f(x)\,dx$ is approximated by $S_{1/3}(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$.

This rule is also known as Parabolic rule.

→ Index of a rule :- If a quadrature rule is based on 'n+1' nodes, then the no. n is called the index of a rule.

□

$\ast$ $Th^m - 6.7$ (Simpson's $\frac{3}{8}$ th rule, $n=3$)

Let $f \in C^4[a,b]$, then $I(f) = S_{3/8}(f) + E(f)$.

Where, $S_{3/8}(f) = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$

$$E(f) = \frac{-3h^5}{80} f^{iv}(\eta), \quad a \le \eta \le b$$

$$x_i = x_0 + ih, \quad 1 \le i \le 3$$

$$\& \quad h = \frac{b-a}{3}.$$

Pf:- We know that $I(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + E(f)$ —— (1)

Where, $w_i = \frac{(-1)^{i-n} \cdot h}{i!(n-i)!} \int_0^n \alpha(\alpha-1)\ldots(\alpha-i+1)(\alpha-i-1)\ldots(\alpha-n)\,d\alpha$ —— (2)

Since, $x_0 = a$

$x_1 = x_0 + h = a + h$

$x_2 = x_0 + 2h = a + 2h$

$x_3 = x_0 + 3h = b$

Now $w_0 = \frac{(-1)^{0-3} \cdot h}{0!(3-0)!} \int_0^3 (\alpha-1)(\alpha-2)(\alpha-3)\,d\alpha$

$$= \frac{-h}{6}\int_0^3 (\alpha^3 - 6\alpha^2 + 11\alpha - 6)\,d\alpha$$

$$= \frac{-h}{6}\left[\frac{\alpha^4}{4} - 2\alpha^3 + \frac{11\alpha^2}{2} - 6\alpha\right]_0^3$$

$$= \frac{-h}{6}\left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right)$$

$$= \frac{-h}{\cancel{6}_{\,2}} \cdot \frac{-\cancel{9}^{\,3}}{4} = \frac{3h}{8}$$

$w_1 = \frac{(-1)^{1-3} \cdot h}{1!(3-1)!} \int_0^3 \alpha(\alpha-2)(\alpha-3)\,d\alpha$

$$= \frac{h}{2}\int_0^3 (\alpha^3 - 5\alpha^2 + 6\alpha)\,d\alpha$$

$$\Rightarrow w_1 = \frac{h}{2}\left[\frac{\alpha^4}{4} - \frac{5\alpha^3}{3} + 3\alpha^2\right]_0^3$$

$$= \frac{h}{2}\left(\frac{81}{4} - 45 + 27\right)$$

$$= \frac{h}{2} \cdot \frac{9}{4} = \frac{9h}{8}$$

$$w_2 = \frac{(-1)^{2-3} \cdot h}{(3-2)! \cdot 2!} \int_0^3 \alpha(\alpha-1)(\alpha-3)\, d\alpha$$

$$= \frac{-h}{2}\int_0^3 (\alpha^3 - 4\alpha^2 + 3\alpha)\, d\alpha$$

$$= \frac{-h}{2}\left[\frac{\alpha^4}{4} - \frac{4\alpha^3}{3} + \frac{3\alpha^2}{2}\right]_0^3$$

$$= \frac{-h}{2}\left(\frac{81}{4} - 36 + \frac{27}{2}\right)$$

$$= \frac{-h}{2} \cdot \frac{-9}{4} = \frac{9h}{8}.$$

$$\& \quad w_3 = \frac{(-1)^{3-3} \cdot h}{3!\,(3-3)!} \int_0^3 \alpha(\alpha-1)(\alpha-2)\, d\alpha$$

$$= \frac{h}{3!}\int_0^3 (\alpha^3 - 3\alpha^2 + 2\alpha)\, d\alpha$$

$$= \frac{h}{6}\left[\frac{\alpha^4}{4} - \alpha^3 + \alpha^2\right]_0^3$$

$$= \frac{h}{6}\left(\frac{81}{4} - 27 + 9\right)$$

$$= \frac{h}{\cancel{6}_2} \cdot \frac{\cancel{9}^3}{4} = \frac{3h}{8}$$

From eq^n(1), $I(f) = \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3) + E(f).$ —— (3)

* Error in Simpson's $\frac{3}{8}$th rule →

Now the index of the rule $n = 3$ is odd.

So, for the evaluation of error [By th^m - 6.3(b)]

$$E(f) = \frac{h^{n+2} f^{(n+1)}(\eta)}{(n+1)!} \int_0^n u(u-1)(u-2)\dots(u-n)\,du$$

$$= \frac{h^5 f^{iv}(\eta)}{4!} \int_0^3 u(u-1)(u-2)(u-3)\,du$$

$[\because n=3]$

$$= \frac{h^5 f^{iv}(\eta)}{24} \int_0^3 (u^4 - 6u^3 + 11u^2 - 6u)\,du$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left[\frac{u^5}{5} - \frac{3u^4}{2} + \frac{11u^3}{3} - 3u^2\right]_0^3$$

$$= \frac{h^5 f^{iv}(\eta)}{24} \left[\frac{243}{5} - \frac{243}{2} + 99 - 27\right]$$

$$\Rightarrow E(f) = \frac{h^5 f^{iv}(\eta)}{\cancel{24}\,8} \cdot \frac{-\cancel{9}\,3}{10} = \frac{-3h^5}{80} f^{iv}(\eta), \quad a \le \eta \le b.$$

From eq^n (3), we get

$$I(f) = \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3) - \frac{3h^5}{80} f^{iv}(\eta)$$

$$\Rightarrow \boxed{I(f) = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{iv}(\eta)};$$

$a \le \eta \le b$. ☐

3 **Ex:- Q:-** Show that Simpson's $\frac{1}{3}$rd rule integrates all polynomials of degree ≤ 3 exactly?

Pf:- Let $f(x)$ be a fun^n of x such that Simpson's $\frac{1}{3}$rd rule integrates it exactly.

Then $E(f) = 0 \Rightarrow \frac{-h^5}{90} f^{iv}(x) = 0$

$$\Rightarrow f^{iv}(x) = 0 \Rightarrow f^{iii}(x) = C_0$$

$$\Rightarrow f^{ii}(x) = C_0 x + C_1$$

$\Rightarrow f'(x) = c_0 \frac{x^2}{2} + c_1 x + c_2$

$\Rightarrow f(x) = c_0 \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3$; where $c_0, c_1, c_2,$ & c_3 are arbitrary constants.

$\Rightarrow f(x)$ is a polynomial of degree ≤ 3. $\square$

Note $\rightarrow$

$\rightarrow$ The rule $S_{1/3}(f)$ fails to be exact if $f(x) = x^4$. [$\because$ May be some of c_1, c_2, c_3, c_4 is zero]

$\rightarrow$ Trapezoidal rule integrates all polynomials of deg. ≤ 1 exactly & $S_{3/8}(f)$, deg ≤ 3.

Note →

Rule	Deg. of Precision
1. Trapezoidal rule	1
2. Simpson's $\frac{1}{3}$rd rule	3
3. Simpson's $\frac{3}{8}$th rule	3

(4)

Q:- Show that the deg. of precision of both Simpson's $\frac{1}{3}$rd rule & $\frac{3}{8}$th rule is 3 ?

Pf :- Let $f(x)$ be the funⁿ of x such that Simpson's $\frac{1}{3}$rd rule integrates it exactly.

Then $E(f) = 0$

$$\Rightarrow -\frac{h^5}{90} f^{iv}(x) = 0$$

$$\Rightarrow f(x) = c_0 \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3 \text{; where}$$

c_0, c_1, c_2 & c_3 are arbitrary constants.

$\Rightarrow f(x)$ is a poly. of deg. ≤ 3.

Again, let $g(x)$ be the funⁿ of x such that Simpson's $\frac{3}{8}$th rule integrates it exactly.

Then $E(g) = 0$

$$\Rightarrow -\frac{3h^5}{80} g^{iv}(x) = 0$$

$$\Rightarrow g(x) = c_0 \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3 \text{; where}$$

C_0, C_1, C_2, C_3 are arbitrary constants.

$\Rightarrow g(x)$ is a poly. of deg. ≤ 3

Hence, both $S_{1/3}(f)$ & $S_{3/8}(f)$ integrate the poly. exactly of deg. ≤ 3.

Thus, deg. of precision of both $S_{1/3}$ & $S_{3/8}$ rules is 3. $\square$

Note -

→ *Error in Newton-cotes rule* →

It is the difference of exact value to the approximation value $E(f) = I(f) - R_{n+1}(f)$

$$\Rightarrow E(f) = \frac{1}{(n+1)!} \int_a^b (x-x_0)(x-x_1)\dots(x-x_n)\,dx$$ $\square$

* Gaussian Quadrature Rule →

For $n=1$, the nodes are x_0 & x_1.

Hence, the integral is $\int_{-1}^{1} f(x)dx = \sum_{K=0}^{1} \lambda_K f_K$.

$$\Rightarrow \int_{-1}^{1} f(x)dx = \lambda_0 f_0 + \lambda_1 f_1 \quad \text{—(1)}$$

In this case, we have 4-unknowns λ_0, λ_1 & x_0, x_1.

The method will be exact for $f(x) = 1, x, x^2$ & x^3.

Taking $f(x) = 1 \Rightarrow f_0 = f_1 = 1$ $[\because f_0 = f(x_0), f_1 = f(x_1)]$

By eq^n(1), $\int_{-1}^{1} dx = \lambda_0 + \lambda_1 \Rightarrow \lambda_0 + \lambda_1 = 2$ —(2)

If the range of integration is not $[-1,1]$ in an integral, then we can always transform to $[-1,1]$ using the transformation :

$$\boxed{x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)}$$

Consider the integral of the form $\int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{n} \lambda_k f_k$.

In this case, all nodes x_k and weights λ_k are unknown which to be determined in the following way :-

Taking $f(x)=x$, $f_0=x_0$, $f_1=x_1$. $\left[\because \begin{array}{l} f_0=f(x_0) \\ f_1=f(x_1) \end{array}\right]$.

By (1), $\int_{-1}^{1} x\,dx = \lambda_0 x_0 + \lambda_1 x_1$

$\Rightarrow \lambda_0 x_0 + \lambda_1 x_1 = 0$ —— (3)

Similarly, taking $f(x)=x^2$, $f_0=x_0^2$, $f_1=x_1^2$.

By (1), $\int_{-1}^{1} x^2\,dx = \lambda_0 x_0^2 + \lambda_1 x_1^2$

$\Rightarrow \lambda_0 x_0^2 + \lambda_1 x_1^2 = 2/3$ —— (4)

And taking $f(x)=x^3$, $f_0=x_0^3$, $f_1=x_1^3$.

By (1), $\int_{-1}^{1} x^3\,dx = \lambda_0 x_0^3 + \lambda_1 x_1^3$

$\Rightarrow \lambda_0 x_0^3 + \lambda_1 x_1^3 = 0$ —— (5)

Eq^n (2) satisfied for $\lambda_0 = 1 = \lambda_1$

Then, taking $\lambda_0 = 1 = \lambda_1$ & solving we get, $x_0 = \frac{1}{\sqrt{3}}$, $x_1 = -\frac{1}{\sqrt{3}}$

By (1), $\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$ $[\because \lambda_0 = 1 = \lambda_1]$

$= f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \boxed{\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)}$

This is the 2-Point Gauss Legendre Rule.

* **Error in 2-Point Gauss Legendre Rule →**

$$E(f) = I(f) - R_2(f)$$

$$\Rightarrow \boxed{E_2(f) = \int_{-1}^{1} f(x)\,dx - \left\{ f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right) \right\}}$$

□

* **Gauss Legendre 3-Point Rule →**

The method is given by

$$\int_{-1}^{1} f(x)\,dx = \sum_{K=0}^{2} \lambda_K f_K \quad [\because n=2]$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \quad \text{—— (1)}$$

There are 6-unknowns in this case $(\lambda_0, \lambda_1, \lambda_2, x_0, x_1, x_2)$.

The method will be exact for

$$f(x) = x^i, \quad 0 \le i \le 5$$

Now taking $f(x) = 1$ i.e. $f_0 = 1 = f_1 = f_2$

By (1), $\int_{-1}^{1} dx = \lambda_0 + \lambda_1 + \lambda_2$

$$\Rightarrow \lambda_0 + \lambda_1 + \lambda_2 = 2 \quad \text{—— (2)}$$

Again, taking $f(x) = x$ i.e. $f_0 = x_0, f_1 = x_1, f_2 = x_2$.

By (1), $\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0$ —— (3)

Similarly, $f(x)=x^2 \Rightarrow \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 2/3$ — (4)

$f(x)=x^3 \Rightarrow \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0$ — (5)

$f(x)=x^4 \Rightarrow \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = 2/5$ — (6)

$f(x)=x^5 \Rightarrow \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 = 0$ — (7)

Solving the above eqⁿs, we get

$\lambda_0 = \frac{8}{9}, \quad \lambda_1 = \lambda_2 = \frac{5}{9}$

& $x_0 = 0, \quad x_1 = \sqrt{\frac{3}{5}}, \quad x_2 = -\sqrt{\frac{3}{5}}$

Hence, the 3-point Gauss Legendre rule is $\cancel{\int_{-1}^{1} f(x)\,dx = \frac{8}{9} f(0) + \sqrt{\frac{3}{5}} f}$

$$\int_{-1}^{1} f(x)\,dx = \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) + \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right)$$

$$\Rightarrow \boxed{\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right]}$$

* Error in 3-Point Gauss Legendre Rule →

$$E(f) = I(f) - R_3(f)$$

$$\Rightarrow \boxed{E_3(f) = \int_{-1}^{1} f(x)\,dx - \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right]}$$

(5) Q:- Evaluate approximately the integral $\int_0^1 xe^{-x}dx$

by using (i) 2-Point Gauss Legendre rule

(ii) 3-point Gauss Legendre rule ?

Solⁿ:- Given that, $f(x) = xe^{-x}$, $a=0$, $b=1$.

1st of all we transform the interval $[a,b] = [0,1]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$= \frac{t}{2} + \frac{1}{2} = \frac{t+1}{2}$

As $x = \frac{t+1}{2}$

$x \to 0,\ t \to -1$

$x \to 1,\ t \to 1$

$\Rightarrow \boxed{x = \frac{t+1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2} \Rightarrow \boxed{dx = \frac{dt}{2}}$

Now $\int_0^1 f(x)dx = \int_0^1 xe^{-x}dx = \int_{-1}^1 \left(\frac{t+1}{2}\right)e^{-\left(\frac{t+1}{2}\right)} \cdot \frac{1}{2}dt$

$\Rightarrow \int_{-1}^1 f(t)dt = \int_{-1}^1 \left(\frac{t+1}{4}\right) e^{-((t+1)/2)} dt$

Here, $f(t) = \frac{t+1}{4} e^{-\left(\frac{t+1}{2}\right)}$.

(i) 2-Point Gauss Legendre Rule →

By using 2-point Gauss Legendre rule, we get

$\int_{-1}^1 f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \int_{-1}^1 f(t)dt = \left(\frac{\frac{1}{\sqrt{3}}+1}{4}\right) e^{-\left(\left(\frac{1}{\sqrt{3}}+1\right)/2\right)} + \left(\frac{-\frac{1}{\sqrt{3}}+1}{4}\right) e^{-\left(\left(-\frac{1}{\sqrt{3}}+1\right)/2\right)}$

$$\Rightarrow \int_{-1}^{1} f(t)\,dt = \frac{\sqrt{3}+1}{4\sqrt{3}} e^{-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)} + \frac{\sqrt{3}-1}{4\sqrt{3}} e^{-\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)}$$

$$= 0.3943375673 \times e^{-0.78867513459}$$

$$+ 0.105662432702 \times e^{-0.211324865405}$$

$$= 0.1792053182 + 0.085534906$$

$$= 0.2647402242$$

$$\Rightarrow \int_{-1}^{1} f(t)\,dt \approx 0.26474$$

(ii) 3-Point Gauss Legendre Rule $\rightarrow$

By using 3-Point Gauss Legendre Rule, we get $\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right]$

$$\Rightarrow \int_{-1}^{1} f(t)\,dt = \frac{1}{9}\left[8 \times \left(\frac{0+1}{4}\right) e^{-\left(\frac{0+1}{2}\right)} + 5 \times \left(\frac{\sqrt{3/5}+1}{4}\right) e^{-\left(\frac{\sqrt{3/5}+1}{2}\right)} + 5 \times \left(\frac{-\sqrt{3/5}+1}{4}\right) e^{-\left(\frac{-\sqrt{3/5}+1}{2}\right)}\right]$$

$$= \frac{1}{9}\left[2e^{-1/2} + 5 \times 0.4436492 \times e^{-0.8872983} + 5 \times 0.0563508 \times e^{-0.11270166}\right]$$

$$= \frac{1}{9}\left[1.218061319 + 0.9133997839 + 0.2517240039\right]$$

$$= 0.264242 7896$$

$$\Rightarrow \int_{-1}^{1} f(t)\,dt \approx 0.26424$$

□

* Example - 2 : Compute the integral $\int_0^1 e^{-x^2} dx$ by (i) Trapezoidal rule
(ii) Simpson's 1/3rd rule
(iii) Simpson's 3/8th rule.

* Solution: Given that the integral is :-

$$I(f) = \int_0^1 e^{-x^2} dx.$$

Here: $a=0$, $b=1$ and $f(x) = e^{-x^2}$.

(i) Trapezoidal rule :-
By using trapezoidal rule;

$$T(f) = \left(\frac{b-a}{2}\right)[f(a)+f(b)]$$

$$= \left(\frac{1-0}{2}\right)[f(0)+f(1)] = \frac{1}{2}[e^{-0} + e^{-1}]$$

$$= \frac{1}{2}[e^{0}+e^{-1}] = \frac{1}{2}[1+0.367879441] = 0.683939972$$

$$= 0.68394 \text{ (Rounding being used)}$$

(ii) Simpson's 1/3rd rule:
By using Simpson's 1/3rd rule;

$$\int_a^b f(x)\, dx = \left(\frac{b-a}{6}\right)\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

$$= \left(\frac{1-0}{6}\right)\left\{f(0) + 4f\left(\frac{0+1}{2}\right) + f(1)\right\}$$

$$= \frac{1}{6}\left\{f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right\}$$

$$= \frac{1}{6}\left\{e^{-0} + 4\times e^{-\frac{1}{4}} + e^{-1}\right\}$$

$$= \frac{1}{6}\{1 + 3.115203132 + 0.367879441\}$$

$$= 0.747180429$$

$$= 0.74718 \text{ (Rounding being used)}$$

(iii) Simpson's 3/8th rule :-
By using Simpson's 3/8th rule;

$$\int_a^b f(x)\, dx = \frac{3h}{8}\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}$$

Here: $h = \frac{b-a}{3} = \frac{1-0}{3} = \frac{1}{3}$.

$x_0 = a = 0$
$x_1 = x_0 + h = 0 + \frac{1}{3} = \frac{1}{3}$
$x_2 = x_0 + 2h = 0 + 2/3 = 2/3$
$x_3 = b = 1.$

So: $\int_0^1 e^{-x^2}\,dx = \frac{3}{8}\left(\frac{1}{3}\right)\left\{f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right\}$

$= \frac{1}{8}\left\{e^{-0}+3\times e^{-\frac{1}{9}}+3\times e^{-\frac{4}{9}}+e^{-1}\right\}$

$= \frac{1}{8}\left\{1+2.684517951+1.923541164+0.367879441\right\}$

$= 0.74699232 = 0.746700$ (Rounding being used)

* Example :- Find the value of $\ln(1.1)$ by evaluating the integral $I_2(f) = \int_1^{1.1} \frac{dx}{x}$ approximately using the simpson's 1/3 rule.

* Solution :- Given that, the integral is :-

$$I(f) = \int_1^{1.1} \frac{dx}{x}.$$

Here: $a = 1$, $b = 1.1$, $f(x) = \frac{1}{x}$.

By using Simpson's 1/3rd rule :-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{1.1-1}{6}\right)\left[f(1) + 4f\left(\frac{1+1.1}{2}\right) + f(1.1)\right]$$

$$= \left(\frac{0.1}{6}\right)\left[f(1) + 4f\left(\frac{2.1}{2}\right) + f(1.1)\right] = \left(\frac{0.1}{6}\right)\left[f(1) + 4f(1.05) + f(1.1)\right]$$

$$= \left(\frac{0.1}{6}\right)\left[1 + \frac{.4}{1.05} + \frac{1}{1.1}\right] = \left(\frac{0.1}{6}\right)\left[1 + 3.80952381 + 0.9090909\right]$$

$$= 0.095310245. \text{ (Ans).}$$

* Example:- Evaluate $I(f) = \int_{-1}^{1} x^4\,dx$ (approximately) by Gauss-Legendre 2-point rule. Find also the error in this approximation.

* Solution: Given that the integral is:-

$$I(f) = \int_{-1}^{1} x^4\,dx \qquad \text{Here: } f(x) = x^4.$$

By using 2-point Gauss Legendre rule;

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{\sqrt{3}}\right)^4 + \left(-\frac{1}{\sqrt{3}}\right)^4 = 0.22222$$

But the real ans is:- $\int_{-1}^{1} x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^{1}$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = 0.4$$

So, the error of approximation is given by:-

$$0.4 - 0.22222 = 0.1778. \text{ (Ans).}$$

EXERCISE - VI

3. Evaluate $I(f) = \int_0^1 e^{-x^2}\,dx$, numerically by (i) midpoint rule
(ii) trapezoidal rule
(iii) Simpson's 1/3rd rule.

* Solution: Given that the integral is:-

$$I(f) = \int_0^1 e^{-x^2}\,dx.$$

Here; $a = 0$, $b = 1$, $f(x) = e^{-x^2}$

(i) Midpoint rule:- By using midpoint rule;

$$\int_a^b f(x)\,dx = (b-a)\, f\left(\frac{a+b}{2}\right)$$

$$= (1-0)\, f\left(\frac{0+1}{2}\right) = 1 \cdot f\left(\frac{1}{2}\right)$$

$$= e^{-\frac{1}{4}} = 0.7788007831.$$

(ii) Trapezoidal rule:- By using trapezoidal rule;

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left[f(a) + f(b)\right]$$

$$= \left(\frac{1-0}{2}\right)\left[f(0) + f(1)\right] = \frac{1}{2}\left[e^{-0} + e^{-1}\right]$$

$$= \frac{1}{2}\left[1 + 0.3678794412\right] = 0.6839397206.$$

(iii) Simpson's $1/3$rd rule:- By using Simpson's $1/3$rd rule:-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

$$= \left(\frac{1-0}{6}\right)\left\{f(0) + 4f\left(\frac{0+1}{2}\right) + f(1)\right\}$$

$$= \frac{1}{6}\left\{f(0) + 4f\left(\tfrac{1}{2}\right) + f(1)\right\}$$

$$= \frac{1}{6}\left\{e^{-0} + 4\times e^{-\frac{1}{4}} + e^{-1}\right\} = \frac{1}{6}\left(1 + 3.115203132 + 0.3678794412\right)$$

$$= 0.7471804289.$$

(4) Find the approx. value of $I(f) = \int_0^1 e^{-x}\,dx$, by Simpson's $\frac{1}{3}$ rule

Sol: Given that the integral is :-

$$I(f) = \int_0^1 e^{-x}\,dx. \qquad \text{Here: } a=0,\ b=1,\ f(x) = e^{-x}.$$

By using Simpson's $1/3$rd rule:-

$$\int_a^b f(x)\,dx = \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\tfrac{3}{5}}\right) + 5f\left(-\sqrt{\tfrac{3}{5}}\right)\right].$$

$$= \frac{1}{9}\left[8\times e^{-0} + 5\times e^{-\sqrt{\frac{3}{5}}} + 5\times e\right.$$

$$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{1-0}{6}\right)\left[f(0) + 4f\left(\frac{0+1}{2}\right) + f(1)\right] = \frac{1}{6}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + f(1)\right]$$

$$= \frac{1}{6}\left[e^{-0} + 4\times e^{-\frac{1}{2}} + e^{-1}\right] = \frac{1}{6}\left[1 + 2.426122639 + 0.3678794412\right]$$

$$= 0.63233368. \quad \text{(Ans).}$$

(7) Evaluate each of the following integrals by (a) trapezoidal (b) Simpson's $1/3$ rules.

(i) $\int_0^{1/2} x e^{x^2}\,dx$:- Here: $a=0,\ b=1/2,\ f(x) = xe^{x^2}$.

(a) Trapezoidal rule: By using trapezoidal rule:-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left[f(a) + f(b)\right]$$

$$= \left(\frac{\frac{1}{2}-0}{2}\right)\left[f(0) + f\left(\tfrac{1}{2}\right)\right] = \frac{1}{4}\left[0 + \frac{1}{2}e^{\frac{1}{4}}\right]$$

$$= \frac{1}{8}\times e^{1/4} = 0.1605031771 \quad \text{(Ans).}$$

(b) Simpson's $1/3$ rule:- By using Simpson's $1/3$ rule:-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$= \left(\frac{\frac{1}{2}-0}{6}\right)\left[f(0)+4f\left(\frac{0+\frac{1}{2}}{2}\right)+f\left(\frac{1}{2}\right)\right]$

$= \frac{1}{12}\left[f(0)+4f\left(\frac{1}{4}\right)+f\left(\frac{1}{2}\right)\right] = \frac{1}{12}\left[0+4\times\frac{1}{4}\times e^{\frac{1}{16}}+\frac{1}{2}\times e^{\frac{1}{4}}\right]$

$= \frac{1}{12}\left[e^{\frac{1}{16}}+\frac{1}{2}e^{1/4}\right] = \frac{1}{12}\left[1.064494459+0.6420127083\right]$

$= 0.1422089306.$

(ii) Given that the integral is: $\int_0^1 x\sin x\,dx$.

Here: $a=0$, $b=1$, $f(x)=x\sin x$.

(a) By using trapezoidal rule:-

$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left[f(a)+f(b)\right]$.

$= \left(\frac{1-0}{2}\right)\left[f(0)+f(1)\right] = \frac{1}{2}\left[0+\sin 1\right] = 0.4207354924$

(b) By using Simpson's 1/3rd rule:-

$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

$= \left(\frac{1-0}{6}\right)\left[f(0)+4f\left(\frac{0+1}{2}\right)+f(1)\right] = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$

$= \frac{1}{6}\left[0+4\times\frac{1}{2}\sin\frac{1}{2}+\sin 1\right] = \frac{1}{6}\left[0.9588510772+0.8414709848\right]$

$= 0.300053677.$

(iii) Given that the integral is: $\int_0^1 x\sqrt{1+x^2}\,dx$.

Here: $a=0$, $b=1$, $f(x)=x\sqrt{1+x^2}$.

(a) By using trapezoidal rule:-

$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left[f(a)+f(b)\right]$

$= \left(\frac{1-0}{2}\right)\left[f(0)+f(1)\right] = \frac{1}{2}\left[0+\sqrt{1+1}\right] = \frac{1}{2}\left[\sqrt{2}\right]$

$= 0.7071067812.$

(b) By using Simpson's 1/3rd rule:-

$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

$= \left(\frac{1-0}{6}\right)\left[f(0)+4f\left(\frac{0+1}{2}\right)+f(1)\right] = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$

$= \frac{1}{6}\left[0+4\times\frac{1}{2}\sqrt{1+\frac{1}{4}}+\sqrt{1+1}\right] = \frac{1}{6}\left[2\sqrt{\frac{5}{4}}+\sqrt{2}\right]$

$= \frac{1}{6}\left[3.65028154\right] = 0.6083802566.$

(iv) Given that the integral is :- $\int_0^{-1} \sqrt{1+x}\, dx$.

Here: $a=0$, $b=-1$, $f(x)=\sqrt{1+x}$.

(a) By using trapezoidal rule :-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)[f(a)+f(b)]$$

$$= \left(\frac{-1-0}{2}\right)[f(0)+f(-1)] = \left(-\frac{1}{2}\right)\left[\sqrt{1+0}+\sqrt{1+(-1)}\right]$$

$$= \left(-\frac{1}{2}\right)(1+0) = -\frac{1}{2} = -0.5.$$

(b) By using Simpson's 1/3rd rule:-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$= \left(\frac{-1-0}{6}\right)\left[f(0)+4f\left(\frac{0-1}{2}\right)+f(-1)\right]$$

$$= \left(-\frac{1}{6}\right)\left[\sqrt{1}+4\times\sqrt{1-\frac{1}{2}}+\sqrt{1-1}\right] = \left(-\frac{1}{6}\right)\left[1+4\times\sqrt{\frac{1}{2}}+0\right]$$

$$= -0.6380711875$$

(v) Given that the integral is :-

$$\int_{1.1}^{1.5} e^{x}\,dx.$$

Here: $a=1.1$, $b=1.5$, $f(x)=e^{x}$.

(a) By using trapezoidal rule :-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)[f(a)+f(b)]$$

$$= \left(\frac{1.5-1.1}{2}\right)[f(1.1)+f(1.5)] = \left(\frac{0.4}{2}\right)\left[e^{1.1}+e^{1.5}\right]$$

$$= 1.497171019.$$

(b) By using Simpson's 1/3rd rule :-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$= \left(\frac{1.5-1.1}{6}\right)\left[f(1.1)+4f\left(\frac{1.1+1.5}{2}\right)+f(1.5)\right]$$

$$= \left(\frac{0.4}{6}\right)\left[f(1.1)+4f(1.3)+f(1.5)\right]$$

$$= \left(\frac{0.4}{6}\right)\left[e^{1.1}+4\times e^{1.3}+e^{1.5}\right]$$

$$= 1.477536118.$$

(vi) Given that the integral is :-

$$\int_1^3 \frac{1}{3}\,dx \quad . \text{ Here: } a=1, b=3, f(x)=\frac{1}{3}.$$

(a) By Trapezoidal rule :- $\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)[f(a)+f(b)]$

$$= \left(\frac{3-1}{2}\right) [f(1)+f(3)] = \left(\frac{2}{2}\right)\left(\frac{1}{3}+\frac{1}{3}\right) = 1\times\frac{2}{3} = 0.6666667$$

(b) By Simpson's 1/3rd rule :-

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

$$= \left(\frac{3-1}{6}\right)\left[f(1)+4f\left(\frac{1+3}{2}\right)+f(3)\right] = \left(\frac{2}{6}\right)[f(1)+4f(2)+f(3)]$$

$$= \left(\frac{1}{3}\right)\left[\frac{1}{3}+\frac{4}{3}+\frac{1}{3}\right] = 0.66666667.$$

(19) Evaluate the integral, $I(f) = \int_0^1 \frac{dx}{1+x}$ by subdividing the interval $[0,1]$ into two equal parts and then applying the Gauss-Legendre three point rule to each part.

* Solution: Given that the integral is :-

$$I(f) = \int_0^1 \frac{dx}{1+x}.$$

First of all we transform the interval $[0,1]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$$= \left(\frac{1-0}{2}\right)t + \left(\frac{1+0}{2}\right) = \frac{1}{2}t+\frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{t+1}{2}} \qquad \Rightarrow \boxed{dx = \frac{1}{2}\,dt}.$$

$$\text{So; } \int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{\frac{1}{2}\,dt}{1+\frac{t+1}{2}} = \int_{-1}^1 \frac{\frac{1}{2}}{\frac{t+3}{2}}\,dt$$

$$= \int_{-1}^1 \frac{1}{t+3}\,dt$$

Here; $f(t) = \frac{1}{t+3}$.

By applying Gauss-Legendre 3 point rule :-

$$\int_{-1}^1 f(t)\,dt = \frac{1}{9}\left[8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)+5f\left(-\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[8\times\frac{1}{0+3} + 5\times\frac{1}{\sqrt{\frac{3}{5}}+3} + 5\times\frac{1}{-\sqrt{\frac{3}{5}}+3}\right]$$

$$= \frac{1}{9}\left[\frac{8}{3}+\frac{5}{3.774596669}+\frac{5}{2.225403331}\right] = 0.693121693$$

20. Integrate numerically, $I(f) = \int_1^3 \frac{\sin 2x}{x} dx$ using Gauss-Legendre 3-point rule.

* Solution: Given that the integral is:

$$I(f) = \int_1^3 \frac{\sin 2x}{x} \quad . \text{ Here: } a = 1, b = 3.$$

First of all we transform the interval $[1,3]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right) t + \left(\frac{b+a}{2}\right)$

$$= \left(\frac{3-1}{2}\right) t + \left(\frac{3+1}{2}\right) = \left(\frac{2}{2}\right) t + \left(\frac{4}{2}\right)$$

$$\Rightarrow \boxed{x = t + 2} \quad \Rightarrow \boxed{dx = dt}$$

So; $\int_1^3 \frac{\sin 2x}{x} dx = \int_{-1}^{1} \frac{\sin 2(t+2)}{(t+2)} \cdot dt$.

By using 3-point Gauss-Legendre for rule.

$$\int_{-1}^{1} f(x) dx = \frac{1}{9}\left[8 f(0) + 5 f\left(\sqrt{\tfrac{3}{5}}\right) + 5 f\left(-\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[8 \times \frac{\sin 2(0+2)}{(0+2)} + 5 \times \frac{\sin 2\left(\sqrt{\frac{3}{5}}+2\right)}{\left(\sqrt{\frac{3}{5}}+2\right)} + 5 \times \frac{\sin 2\left(-\sqrt{\frac{3}{5}}+2\right)}{\left(-\sqrt{\frac{3}{5}}+2\right)}\right]$$

$$= \frac{1}{9}\left[3.307287242 + 5 \times \frac{0.128746825}{2.774596669} + 5 \times \frac{0.885372732}{1.225403331}\right]$$

$$= \frac{1}{9}\left[3.307287242 + 0.232009982 + 3.612576813\right]$$

$$= 0.79465267$$

(21) Find the approx. value of $I(f) = \int_0^1 e^{-x^2} dx$, by (i) 2-point (ii) 3-pt. Gauss Legendre rule

Sol: Given that the integral is:-

$$I(f) = \int_0^1 e^{-x^2} dx.$$

Here: $a = 0$, $b = 1$, $f(x) = e^{-x^2}$.

First of all we transform the interval $[0,1]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right) t + \left(\frac{b+a}{2}\right)$

$$= \left(\frac{1-0}{2}\right) t + \left(\frac{1+0}{2}\right) = \frac{1}{2} t + \frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{t+1}{2}} \quad \Rightarrow \boxed{dx = \frac{1}{2} dt}.$$

Now; $\int_0^1 e^{-x^2} dx = \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)^2} \cdot \frac{1}{2} dt.$

Here: $f(t) = \frac{1}{2} e^{-\left(\frac{t+1}{2}\right)^2}$.

(i) By using Gauss-Legendre 2-point rule;

$$\int_{-1}^{1} f(t)\,dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{2} e^{-\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right)^2} + \frac{1}{2} e^{-\left(\frac{-\frac{1}{\sqrt{3}}+1}{2}\right)^2}$$

$$= \frac{1}{2} e^{-\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right)^2} + \frac{1}{2} e^{-\left(\frac{-1+\sqrt{3}}{2\sqrt{3}}\right)^2}$$

$$= \frac{1}{2} \times (0.536865078 + 0.956324299) = 0.746594688$$

(ii) By using Gauss-Legendre 3-point rule;

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[8 f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[8 \times \frac{1}{2} \times e^{-\left(\frac{0+1}{2}\right)^2} + 5 \times \frac{1}{2} \times e^{-\left(\frac{\sqrt{\frac{3}{5}}+1}{2}\right)^2} + 5 \times \frac{1}{2} \times e^{-\left(\frac{-\sqrt{\frac{3}{5}}+1}{2}\right)^2}\right].$$

$$= \frac{1}{9}\left[\cancel{4\times e}\ 3.115203132 + 1.137681475 + 2.468449652\right]$$

$$= 0.746814584 \text{ (Ans)}$$

(22) Integrate numerically, $I(f) = \int_0^1 \frac{\sin x}{x}\,dx$ by Gauss Legendre 2 pt. formula

Sol: Given that the integral is:-

$$I(f) = \int_0^1 \frac{\sin x}{x}\,dx.$$

Here: $a = 0$, $b = 1$.

First of all we transform the interval $[0,1]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$$= \left(\frac{1-0}{2}\right)t + \left(\frac{1+0}{2}\right) = \frac{1}{2}t + \frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{t+1}{2}} \Rightarrow \boxed{dx = \frac{1}{2}\,dt}$$

Now; $$\int_0^1 \frac{\sin x}{x}\,dx = \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{2}\right)}{\left(\frac{t+1}{2}\right)} \cdot \frac{1}{2}\,dt$$

$$= \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{2}\right)}{(t+1)}\,dt.$$

Here: $f(t) = \dfrac{\sin\left(\frac{t+1}{2}\right)}{(t+1)}$.

By using Gauss - Legendre 2-point formula:-

$$\int_{-1}^{1} f(t)\,dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\sin\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right)}{\left(\frac{1}{\sqrt{3}}+1\right)} + \frac{\sin\left(\frac{-\frac{1}{\sqrt{3}}+1}{2}\right)}{\left(-\frac{1}{\sqrt{3}}+1\right)}$$

$$= \frac{0.709420148}{1.577350269} + \frac{0.209755475}{0.422649773} = 0.946041135 \text{ (Ans)}$$

(23) Show that $I(f) = \int_{0.2}^{1.5} e^{-x^2}\,dx \simeq 0.6586$ when evaluated by Gauss - Legendre 3-point rule.

Sol: Given that the integral is:-

$$I(f) = \int_{0.2}^{1.5} e^{-x^2}\,dx.$$

Here: $a = 0.2$, $b = 1.5$.

First of all, we transform the interval $[0.2, 1.5]$ to $[-1, 1]$.

Let $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$$= \left(\frac{1.5-0.2}{2}\right)t + \left(\frac{1.5+0.2}{2}\right) = 0.65t + 0.85$$

$\Rightarrow dx = 0.65\,dt$.

So, $$\int_{0.2}^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} e^{-(0.65t+0.85)^2} \cdot (0.65)\,dt.$$

Here: $f(t) = (0.65) \times e^{-(0.65t+0.85)^2}$

By using Gauss Legendre 3-point rule:-

$$\int_{-1}^{1} f(t)\,dt = \frac{1}{9}\left[8 f(0) + 5 f\left(\sqrt{\tfrac{3}{5}}\right) + 5 f\left(-\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[8 \times (0.65)\, e^{-(0.85)^2} + 5 \times (0.65) \times e^{-\left(0.65\times\sqrt{\frac{3}{5}}+0.85\right)^2} + 5\times(0.65)\times e^{-\left(0.65\left(-\sqrt{\frac{3}{5}}\right)+0.85\right)^2}\right]$$

$$= \frac{1}{9}\left[2.524791855 + 0.520339225 + 2.882287692\right]$$

$= 0.658602085$

$\simeq 0.6586$. (Proved).

(21) Show that $I(f) = \int_0^1 \frac{dx}{1+x} \simeq 0.693122$ by Gauss-Legendre 3-point formula.

Ans:- Given that the integral is: $I(f) = \int_0^1 \frac{dx}{1+x}$.

Here: $a=0, b=1$.

First of all we transform the interval $[0,1]$ to $[-1,1]$.

Let $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$= \left(\frac{1-0}{2}\right)t + \left(\frac{1+0}{2}\right) = \frac{1}{2}t + \frac{1}{2}$

$\Rightarrow x = \frac{t+1}{2} \quad \Rightarrow \boxed{dx = \frac{1}{2}dt}$.

Now; $\int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}dt}{1+\frac{t+1}{2}} = \int_{-1}^{1} \frac{1}{t+3}\,dt$.

Here: $f(t) = \frac{1}{t+3}$.

By using Gauss-Legendre 3-point formula:-

$$\int_{-1}^{1} f(t)\,dt = \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\tfrac{3}{5}}\right) + 5f\left(-\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[8 \times \frac{1}{0+3} + 5 \times \frac{1}{\sqrt{\frac{3}{5}}+3} + 5 \times \frac{1}{-\sqrt{\frac{3}{5}}+3}\right]$$

$$= \frac{1}{9}\left[2.666666667 + 1.32464484 + 2.246783791\right]$$

$= 0.693121693 \simeq 0.693122$. (Ans).

CHAPTER-7 : NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

1st order initial value problem

An ordinary differential equation is a relation between function, it's derivatives and the variable upon which they depend.

The most general form of an ordinary differential equation is given by :-

$$f(x, y, y', y'', \ldots, y^n) = 0$$

But we want to findout the numerical solution of 1st order initial value problem

$$\boxed{\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0}$$

We discuss the following numerical method to find the approximate solution :

(1) Picard's method

(2) Euler's method

(3) Modified Euler's method

(4) Runge- Kutta method (RK-method).

(1) PICARD'S Method :-

In this method, we compute the integral in an iterative way.

Given, differential equation:

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0 \, , \quad x_0 \le x \le b.$$

We have: $\dfrac{dy}{dx} = f(x,y)$

$$\Rightarrow dy = f(x,y)\,dx$$

Integrating both sides and considering $y = y(x)$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow [y(x)]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx.$$

$$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx .$$

Now, writing y_1 as the 1st approximation ; we get :-

$$y_1 = y(x_0) + \int_{x_0}^{x} f(x, y_0)\,dx$$

The 2nd approximation is :-

$$y_2 = y(x_0) + \int_{x_0}^{x} f(x, y_1)\,dx$$

Similarly ;

$$y_3 = y(x_0) + \int_{x_0}^{x} f(x, y_2)\,dx$$

....

....

$$\boxed{y_{n+1} = y(x_0) + \int_{x_0}^{x} f(x, y_n)\,dx}$$ This is known as Picard's method.

* Example - 1 : Determine the approximate solution y_1, y_2, y_3 of the differential equation : $\frac{dy}{dx} = x+y$ with $y(0)=1$ by Picard's method at $x=0.1$, compare the approximate sol^n to the exact sol^n.

Sol^n: Given that ;

$$\frac{dy}{dx} = x+y , \quad y(0) = 1 .$$

Here: $f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$.

$y_{n+1} = y(x_0) + \int_{x_0}^{x} f(x, y_n)\,dx$

We know, by Picard's method ;

$$y_{n+1} = y(x_0) + \int_{x_0}^{x} f(x, y_n)\,dx \quad \text{——— (1)} .$$

Putting $n=0$ in eq^n(1), we have :-

$$y_1 = y(x_0) + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= y(0) + \int_{0}^{x} f(x, 1)\,dx$$

$$= y(0) + \int_{0}^{x} (x+1)\,dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$= 1 + x + \frac{x^2}{2} .$$

Putting $n=1$ in eq(1), we have:-

$$y_2 = y(x_0) + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= y(0) + \int_{0}^{x} (x+y_1)\, dx \quad = \quad y(0) + \int_{0}^{x} \left(x + 1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + \int_{0}^{x} \left(1 + 2x + \frac{x^2}{2}\right) dx \quad = 1 + \left[x + x^2 + \frac{x^3}{6}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{6}.$$

Putting $n=2$ in eq(1), we have:-

$$y_3 = y(x_0) + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$= y(0) + \int_{0}^{x} (x+y_2)\, dx \quad = 1 + \int_{0}^{x} \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + \left[x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Again; we have:-

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} - y = x.$$

It is a linear differential equation.

Here: $P(x) = -1$, $Q(x) = x$.

$$I.F. = e^{\int P(x)\, dx} = e^{-x}.$$

Solution of the given diff. is:-

$$ye^{-x} = \int x e^{-x}\, dx$$

$$= x(-e^{-x}) - \int -e^{-x}\, dx \quad = -xe^{-x} - e^{-x} + c$$

$$\Rightarrow y = -(x+1) + ce^{x} \quad \text{—— (2)}$$

Given that; $y(0) = 1$. Then from eq(2), we have:-

$$1 = -(0+1) + ce^{0} \quad \Rightarrow 1 = -1 + c$$

$$\Rightarrow c = 2.$$

$\therefore$ The exact soln is:-

$$y = -x - 1 + 2e^{x}.$$

Given that, $x = 0.1$:

The approximate solution is :-

$$y_1 = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$$

$$y_2 = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{6} = 1.110166667.$$

$$y_3 = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{24} = 1.1103375.$$

Exact solution is :-

$$y = -0.1 - 1 + 2e^{0.1}$$

$$= -1.1 + 2.210341836 = 1.110341836. \text{ (Ans).}$$

(2) EULER'S METHOD:-

Let us consider a linear differential equation of type :-

$$\frac{dy}{dx} = f(x,y), \quad x_0 \le x \le b \quad \text{with } y(x_0) = y_0.$$

By Taylor's theorem of expansion, we have :-

$$y(x_1) = y(x_0 + h) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!} f''(\xi_0),$$

where $x_0 \le \xi_0 \le x_1$.

Dropping the remainder term, we get :-

$$\Rightarrow y(x_1) \simeq y(x_0) + h\,y'(x_0)$$

$$= y_0 + h\,f(x_0, y_0) \quad \left(\text{since; } \frac{dy}{dx} = f(x,y)\right)$$

$$= y_1.$$

The approximate solution of the given differential equation at $x = x_1$ is 'y_1'

$$\Rightarrow \boxed{y(x_1) \simeq y_1 = y_0 + h\,f(x_0, y_0)}$$

Similarly, By Taylor's theorem of expansion, we have :-

$$y(x_2) \simeq y(x_1 + h) = y(x_1) + h\,y'(x_1) + \frac{h^2}{2!} f''(\xi_1),$$

where $x_1 \le \xi_1 \le x_2$.

Dropping the remainder term, we get :-

$$y(x_2) \simeq y(x_1) + h\,y'(x_1)$$

$$= y_1 + h\,f(x_1, y_1)$$

$$= y_2$$

'y_2' be the approximate solution of the given diff. equation at $x = x_0$.

$$\Rightarrow \boxed{y(x_2) \simeq y_2 = y_1 + h f(x_1, y_1)}$$

Proceeding in this manner, we get :-

$$\boxed{y(x_{n+1}) \simeq y_{n+1} = y_n + h f(x_n, y_n)}$$

where $n = 0, 1, 2, \ldots\ldots$

* **Question:** Determine the approximate solution of the differential equation : $\frac{dy}{dx} = x + y$, for $y(0) = 1$ with $0 \le x \le 0.3$ where $h = 0.1$ by using Euler's method.

* **Solution:** Given that, the differential equation is :-

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad 0 \le x \le 0.3, \quad h = 0.1.$$

Here: $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$

By Euler's method, we have :-

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)} \quad \text{—— (1)}$$

Putting $n = 0$ in eq(1); we have:-

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h (x_0 + y_0)$$
$$= 1 + (0.1)(0+1) = 1 + 0.1$$
$$= 1.1.$$

Putting $n = 1$ in eq(1), we have:-

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.1 + (0.1)(x_0 + h + y_1) = 1.1 + (0.1)(0 + 0.1 + 1.1)$$
$$= 1.1 + (0.1)(1.2) = 1.1 + 0.12$$
$$= 1.22.$$

Putting $n = 2$ in eq(1), we have :-

$$y_3 = y_2 + h f(x_2, y_2)$$
$$= 1.22 + (0.1)(0 + 0.2 + 1.22) \quad (\because x_2 = x_0 + 2h)$$
$$= 1.22 + (0.1)(1.42) = 1.22 + 0.142 = 1.362.$$

Hence, the approximate solutions are :-

$y(0.1) = 1.1$, $y(0.2) = 1.22$ and $y(0.3) = 1.362$. (Ans)

* Example-2 :- Solve $\frac{dy}{dx} = x+y$; $y(0)=1$ by Euler's method with $h=0.1$ and 0.01.

Solution: Given that the differential equation is:-

$$\frac{dy}{dx} = x+y \; ; \; y(0)=1 \, , \; h=0.1 .$$

~~By Euler's~~ Here:- $f(x,y)=x+y$, $x_0=0$, $y_0=1$, $h=0.1$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \text{——— (1)}.$$

Putting $n=0$ in eq^n(1), we have:-

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= y_0 + h\,(x_0+y_0)$$
$$= 1 + (0.1)(0+1) = 1+(0.1)$$
$$= 1.1 .$$

Putting $n=1$ in eq^n(1), we have:-

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= y_1 + h\,(x_1+y_1) = 1.1 + (0.1)(0.1+1.1)$$
$$= 1.22 .$$

* Exercise: Solve numerically $\frac{dy}{dx} = 2x$, $y(0)=0$ by Euler's method for $0 \le x \le 0.1$ taking $h=0.01$ using 4-digit arithmetic.

* Solution: Given that the differential equation is:-

$$\frac{dy}{dx} = 2x \, , \quad y(0)=0 \, , \quad h=0.01 .$$

Here: $f(x,y) = 2x$, $x_0=0$, $y_0=0$, $h=0.01$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \text{——— (1)}.$$

Putting $n=0$ in eq^n(1), we have:-

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= y_0 + h\,(2x_0) = 0 + (0.01)(2.0)$$
$$= 0.01 .$$

Putting $n=1$ in eq(1), we have:-

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(2x_1)$$
$$= y_1 + h \times 2(x_0 + h) = 0.01 + (0.01)\, 2(0 + 0.1)$$
$$= 0.01 + (0.02)(0.1) = 0.01 + 0.002 = 0.012.$$

Putting $n=2$ in eq(1), we have:-

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + h(2x_2)$$
$$= y_2 + h \times 2(x_0 + 2h) = (0.012) + (0.01) \times 2(0 + 2 \times 0.01)$$
$$= (0.012) + (0.02)(0.02) = 0.012 + 0.0004$$
$$= 0.0124.$$

3. Modified Euler's Method:-

In this method, we have to determine the approximate solution of linear differential equation of type:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad \text{for} \quad x_0 \le x \le b.$$

$$\Rightarrow dy = f(x, y)\, dx$$

$$\Rightarrow \int_{x_n}^{x_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

$$\Rightarrow [y]_{x_n}^{x_{n+1}} = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

$$\Rightarrow y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

$$\Rightarrow y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

$$\Rightarrow y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\, dx \quad \text{———— (1)}$$

Applying trapezoidal rule in R.H.S of eq(1), we have:-

$$y_{n+1} = y_n + \frac{x_{n+1} - x_n}{2} \left\{ f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right\}$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2} \left\{ f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right\}$$

Here the approximation Y_{n+1} be the modified solution when we modified y_n by Y_n; then we have :-

$$\boxed{Y_{n+1} = Y_n + \frac{h}{2}\left\{f(x_n, Y_n) + f(x_{n+1}, y_{n+1})\right\}}$$

This is known as modified Euler's method, where $n = 0, 1, 2, \ldots$ and $Y_0 = y_0$.

* Question: Determine the approximate solution of the 1st order linear differential equation :-

$$\frac{dy}{dx} = x + y, \quad y(0) = 1 \quad \text{for } 0 \le x \le 0.3 \text{ where } h = 0.1$$

by modified Euler's method.

Solution: Given that the differential equation is :-

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad 0 \le x \le 0.3, \quad h = 0.1.$$

Here: $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

By Euler's method; we have :-

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \text{---(*)}$$

$$\Rightarrow \boxed{y_{n+1} = y_n + h(x_n + y_n)} \quad \text{--- (1)}$$

Putting $n = 0$; we have :-

$$y_1 = y_0 + h(x_0 + y_0)$$
$$= 1 + (0.1)(0 + 1) = 1 + 0.1$$
$$= 1.1.$$

Putting $n = 1$, we have :- from eq (1), we have :-

$$y_2 = y_1 + h(x_1 + y_1)$$
$$= (1.1) + (0.1)(0 + 0.1 + 1.1)$$
$$= (1.1) + (0.1)(1.2) = (1.1) + (0.12)$$
$$= 1.22.$$

Putting $n = 2$ from eq (1) we have :-

$$y_3 = y_2 + h(x_2 + y_2)$$
$$= (1.22) + (0.1)(0.2 + 1.22) \quad (\because x_2 = x_0 + 2h).$$
$$= 1.362$$

By modified Euler's method, we have:-

$$Y_{n+1} = Y_n + \frac{h}{2}\{f(x_n, Y_n) + f(x_{n+1}, y_{n+1})\}$$

$$\Rightarrow Y_{n+1} = Y_n + \frac{h}{2}\{x_n + Y_n + x_{n+1} + y_{n+1}\} \quad —(2).$$

Putting $n=0$ in eq^n(2), we have:-

$$Y_1 = Y_0 + \frac{h}{2}\{x_0 + Y_0 + x_1 + y_1\}$$

Taking $Y_0 = y_0$;

$$Y_1 = 1 + \frac{(0.1)}{2}\{0 + 1 + 0.1 + 1.1\} \quad (\because x_1 = x_0 + h)$$

$$= 1.11.$$

Putting $n=1$ in eq^n(2), we have:-

$$Y_2 = Y_1 + \frac{h}{2}\{x_1 + Y_1 + x_2 + y_2\}$$

$$= 1.11 + \left(\frac{0.1}{2}\right)\{0.1 + 1.11 + 0.2 + 1.22\}$$

$$= 1.2415$$

Putting $n=2$ in eq^n(2), we have:-

$$Y_3 = Y_2 + \frac{h}{2}\{x_2 + Y_2 + x_3 + y_3\}$$

$$= 1.2415 + \left(\frac{0.1}{2}\right)\{0.2 + 1.2415 + 0.3 + 1.362\}$$

$$= 1.396675.$$

4. Runge-Kutta Methods:-

1. Runge-Kutta method for order '2':-

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

Otherwise;

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}(K_1 + K_2) \\ \text{where, } K_1 &= hf(x_n, y_n) \\ K_2 &= hf(x_n + h, y_n + K_1) \end{aligned}}$$

(2) Runge-Kutta method for order '4':-

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where, $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

Question:- Determine the approximate solution $y(0.1)$, $y(0.2)$ for the differential equation: $\frac{dy}{dx} = y$, $y(0) = 1$, $h = 0.1$ by RK-method for order '2'.

Solution: Given that, the differential equation is:-

$$\frac{dy}{dx} = y, \quad y(0) = 1, \; h = 0.1$$

Here: $f(x, y) = y$, $x_0 = 0$, $y_0 = 1$; $h = 0.1$.

By RK-method for order '2', we have:-

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2) \\ \text{where, } k_1 &= hf(x_n, y_n) \\ \& \; k_2 &= hf(x_n + h, y_n + k_1) \end{aligned}\right\} \text{———} (1).$$

Putting $n = 0$ in eq(1), we have:-

$k_1 = h f(x_0, y_0)$

$= h(y_0) = (0.1)(1)$

$= 0.1.$

$k_2 = h f(x_0 + h, y_0 + k_1)$

$= h(y_0 + k_1) = (0.1)(1 + 0.1)$

$= (0.1)(1.1)$

$= 0.11.$

Now; $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$\Rightarrow y(x_1) = 1 + \frac{1}{2}(0.1 + 0.11)$

$\Rightarrow y(x_0 + h) = 1 + \frac{1}{2}(0.21)$

$\Rightarrow y(0 + 0.1) = 1 + \frac{1}{2}(0.21)$

$\Rightarrow y(0.1) = 1.105.$

Putting $n=1$ in eq^n (1), we have :-

$k_1 = h f(x_1, y_1)$

$= h(y_1) = (0.1)(1.105)$

$= 0.1105$.

$k_2 = h f(x_1+h, y_1+k_1)$

$= h(y_1+k_1) = (0.1)(1.105 + 0.1105)$

$= 0.12155$.

Now; $y_2 = y_1 + \frac{1}{2}(k_1+k_2)$

$\Rightarrow y(x_2) = 1.105 + \frac{1}{2}(0.1105 + 0.12155)$

$\Rightarrow y(0.2) = \underline{1.221025}$.

Qus: Determine the approximate solution $y(0.1)$, $y(0.2)$ of diff. equation $\frac{dy}{dx} = y$, $y(0) = 1$, $h = 0.1$ by RK - method of order '4'

Solution: Given that the diff. eqn is :-

$$\frac{dy}{dx} = y, \quad y(0) = 1, \quad h = 0.1.$$

Here: $f(x,y) = y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

By RK. method of order '4'; we have :-

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \\ \text{where,} \quad k_1 &= h f(x_n, y_n) \\ k_2 &= h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= h f(x_n + h, y_n + k_3) \end{aligned}\right\} \quad \text{--- (1).}$$

Putting $n = 0$ in eqn (1), we have :-

$$k_1 = h f(x_0, y_0)$$
$$= h(y_0) = (0.1)(1)$$
$$= 0.1.$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= h\left(y_0 + \frac{k_1}{2}\right) = (0.1)\left(1 + \frac{0.1}{2}\right)$$
$$= 0.105.$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= h\left(y_0 + \frac{k_2}{2}\right) = (0.1)\left(1 + \frac{0.105}{2}\right)$$
$$= 0.10525.$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= h(y_0 + k_3) = (0.1)(1 + 0.10525)$$
$$= 0.110525.$$

Now; $y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$$\Rightarrow y(x_1) = 1 + \frac{1}{6}\left[0.1 + 2 \times 0.105 + 2 \times 0.10525 + 0.110525\right]$$

$$\Rightarrow y(x_0 + h) = 1.105170833$$

$$\Rightarrow y(0.1) = 1.105170833$$

Again; Putting $n=1$ in eq(1), we have :-

$$k_1 = hf(x_1, y_1) = h(y_1)$$
$$= (0.1)(1.105170833) = 0.110517083.$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left(y_1 + \frac{k_1}{2}\right)$$
$$= (0.1)\left(1.105170833 + \frac{0.110517083}{2}\right)$$
$$= 0.116042937.$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left(y_1 + \frac{k_2}{2}\right)$$
$$= (0.1)\left(1.105170833 + \frac{0.116042937}{2}\right) = 0.11631923.$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h(y_1 + k_3)$$
$$= (0.1)(1.105170833 + 0.11631923) = 0.122149006.$$

Now; $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$\Rightarrow y(x_2) = 1.105170833 + \frac{1}{6}[0.110517083 + 2 \times 0.116042937 + 2 \times 0.11631923 + 0.122149006]$$

$$\Rightarrow y(x_0 + 2h) = 1.22140257$$

$$\Rightarrow y(0.2) = 1.22140257. \text{ (Ans).}$$

SUMMERY :-

* For $\frac{dy}{dx} = f(x,y)$, $f(x_0) = y_0$: different methods are :-

(1) Picard's method : $y_{n+1} = y(x_0) + \int_{x_0}^{x} f(x, y_n)\, dx$

(2) Euler's method : $y_{n+1} = y_n + h f(x_n, y_n)$

(3) Modified Euler's method : $y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y_{n+1})\}$

(4) Runge-Kutta Method for order '2' :

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2);$$

where, $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + h, y_n + k_1)$

(5) RK- method for order '4' :

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where; $k_1 = h f(x_n, y_n)$

$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = h f(x_n + h, y_n + k_3)$

Q. 3. Find the exact sol^n of $y' + y = 0$; $y(0) = 1$. Using Euler's method findout $y(0.04)$ with $h = 0.04$. compare your answer with exact sol^n & find the error. (carry out all calculations to 4-dec places)

Solution:- Given that;

$$\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -y, \quad , y(0) = 1, \quad , h = 0.04.$$

Here: $f(x, y) = -y$, $x_0 = 0$, $y_0 = 1$, $h = 0.04$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h(-y_n) \quad \text{———} (1).$$

Putting $n=0$ in eq(1), we have:-

$$y_1 = y_0 + h(-y_0)$$

$\Rightarrow y(x_1) = 1 - (0.04)(1) = 1 - 0.04 = 0.96$.

$\Rightarrow y(x_0 + h) = 0.96$

$\Rightarrow y(0.04) = 0.96$.

The exact solution is given by:-

$\frac{dy}{dx} = -y \Rightarrow \int \frac{dy}{y} = -\int dx$

$\Rightarrow \ln y = -x + c$

Putting $x=0$ & $y=1$: $\Rightarrow \ln 1 = -0+c$

$\Rightarrow 0 = -0+c$

$\Rightarrow c = 0$.

$\therefore \ln y = -x$

$\Rightarrow y = e^{-x}$

$\Rightarrow y(0.04) = e^{-(0.04)} = 0.9607$. (Ans)

Q. Using Euler's method findout $y(0.02)$, Given that $\frac{dy}{dx} = x^2+y$, $y(0)=1$ Take $h=0.01$

Sol:- Given, $\frac{dy}{dx} = x^2+y$, $y(0)=1$, $h=0.01$.

Here: $f(x,y) = x^2+y$, $x_0=0$, $y_0=1$, $h=0.01$

By Euler's method, we have:-

$y_{n+1} = y_n + h f(x_n, y_n)$

$\Rightarrow y_{n+1} = y_n + h(x_n^2+y_n)$ ——— (1).

Putting $n=0$ in eq(1), we have:-

$y_1 = y_0 + h(x_0^2+y_0)$

$\Rightarrow y(x_1) = 1 + (0.01)(0+1)$

$\Rightarrow y(x_0+h) = 1+0.01$

$\Rightarrow y(0.01) = 1.01$.

Putting $n=1$ in eq(1), we have:-

$y_2 = y_1 + h(x_1^2+y_1)$ $\Rightarrow y(x_2) = y_1 + h\{(x_0+h)^2+y_1\}$

$\Rightarrow y(x_0+2h) = 1.01 + (0.01)\{(0.01)^2 + 1.01\}$.

$\Rightarrow y(0.02) = 1.020101$.

Q. Solve: $\frac{dy}{dx} = x+y$, $y(0)=0$ by Euler's method taking $h=0.2$ and find out $y(0.4)$.

Sol: Given, $\frac{dy}{dx} = x+y$, $y(0)=0$, $h=0.2$

Here: $f(x,y) = x+y$, $x_0=0$, $y_0=0$, $h=0.2$.

By Euler's method, we have:-

$y_{n+1} = y_n + h f(x_n, y_n)$

$\Rightarrow y_{n+1} = y_n + h(x_n+y_n)$ ——— (1).

Putting $n=0$ in eq(1), we have:-

$y_1 = y_0 + h(x_0+y_0)$ $\Rightarrow y(x_1) = 0 + (0.2)(0+0)$

$\Rightarrow y(0.2) = 0$

Putting $n=1$ in eq(2), we have:-

$$y_2 = y_1 + h(x_1+y_1) \Rightarrow y(x_2) = 0+(0.2)(0.2+0)$$

$\Rightarrow y(x_0+2h) = 0.04$

$\Rightarrow y(0.4) = 0.04$. (Ans).

Q. Solve by Euler's method: $\frac{dy}{dx} = xy^2$; $y(2)=1$. Determine $y(2.1)$ by choosing $h=0.1$ and then $h=0.05$. Compare results with the exact sol^n. Carry out all calculations to 4-s

Sol^n: Given; $\frac{dy}{dx} = xy^2$, $y(2)=1$, $h=0.1$.

Here: $f(x,y) = xy^2$, $x_0 = 2$, $y_0 = 1$, $h=0.1$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h(x_n y_n^2) \quad \text{——— (1)}.$$

Putting $n=0$ in (1), we have:-

$$y_1 = y_0 + h(x_0 \times y_0^2)$$

$\Rightarrow y(x_1) = 1+(0.1)(2\times 1) \Rightarrow y(x_0+h) = 1+0.2$

$\Rightarrow y(2+0.1) = 1.2$

$\Rightarrow y(2.1) = 1.2$. ——— (A)

Again, for $h=0.05$;

$$y_1 = y_0 + h(x_0 \times y_0^2)$$

$\Rightarrow y(x_1) = 1+(0.05)(2\times 1)$.

$\Rightarrow y(x_0+h) = 1+0.1$

$\Rightarrow y(2+0.05) = 1.1 \Rightarrow y(2.05) = 1.1$.

Putting $n=1$ in eq(1), we have:-

$$y_2 = y_1 + h(x_1 \times y_1^2)$$

$\Rightarrow y(x_2) = 1.1 + (0.05)(2.05 \times (1.1)^2)$

$\Rightarrow y(x_0+2h) = 1.224025$

$\Rightarrow y(2+0.1) = 1.224025$

$\Rightarrow y(2.1) = 1.224025$ ——— (B)

The exact solution is given by :-

$$\frac{dy}{dx} = xy^2 \Rightarrow \frac{dy}{y^2} = x\,dx$$

$$\Rightarrow \int y^{-2}dy = \int x\,dx \Rightarrow \frac{y^{-1}}{-1} = \frac{x^2}{2}+c$$

$$\Rightarrow -\frac{1}{y} = \frac{x^2}{2}+c.$$

Putting $x=2,\ y=1$

$$\Rightarrow -1 = 2+c \Rightarrow c = -3.$$

So; $-\frac{1}{y} = \frac{x^2}{2}-3. \Rightarrow \frac{1}{y} = 3-\frac{x^2}{2} \Rightarrow \frac{1}{y} = \frac{6-x^2}{2}$

$$\Rightarrow y = \frac{2}{6-x^2}$$

$$\Rightarrow y(2.1) = \frac{2}{6-(2.1)^2} = 1.257861635$$
$$= 1.258$$

Q. (1) Find the exact solution of $\frac{dy}{dx} = -2+\frac{y}{x}$, $y(1)=2$. Use Euler's method to determine $y(1.2)$ with $h=0.1$ and $h=0.05$. Compare the approximate results with the exact values.

Sol:- Given; $\frac{dy}{dx} = -2+\frac{y}{x}$, $y(1)=2$, $h=0.1$. Here: $f(x,y) = -2+\frac{y}{x}$

$x_0=1,\ y_0=2.$

By Euler's method, we have:-

$$y_{n+1} = y_n + h\,f(x_n,y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\left\{-2+\frac{y_n}{x_n}\right\} \quad \text{—— (1)}$$

Putting $n=0$ in eq(1), we have:-

$$y_1 = y_0 + h\left\{-2+\frac{y_0}{x_0}\right\}$$

$$\Rightarrow y(x_1) = 2+(0.1)\left\{-2+\frac{2}{1}\right\} \Rightarrow y(x_0+h) = 2+(0.1)(-2+2)$$

$$\Rightarrow y(1+0.1) = 2+0$$

$$\Rightarrow y(1.1) = 2.$$

Putting $n=1$ in eq(1), we have:-

$$y_2 = y_1 + h\left\{-2+\frac{y_1}{x_1}\right\}$$

$$\Rightarrow y(x_2) = 2+(0.1)\left\{-2+\frac{2}{1.1}\right\} \Rightarrow y(x_0+2h) = 1.981818182$$

$$\Rightarrow y(1+0.2) = 1.981818182$$

$$\Rightarrow y(1.2) = 1.9818182. \quad \text{—— (A)}$$

Again; Given $\frac{dy}{dx} = -2+\frac{y}{x}$, $y(1)=2$, $h=0.05$.

Here: $f(x,y) = -2+\frac{y}{x}$, $x_0=1,\ y_0=2$ & $h=0.05$.

By Euler's method, we have :-

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\left\{-2 + \frac{y_n}{x_n}\right\} \quad \text{—— (2)}$$

Putting $n=0$ in eq(2), we have :-

$$y_1 = y_0 + h\left\{-2 + \frac{y_0}{x_0}\right\} \Rightarrow y(x_1) = 2 + (0.05)\left\{-2 + \frac{2}{1}\right\}$$

$$\Rightarrow y(x_0+h) = 2 + (0.05)(-2+2) \Rightarrow y(1+0.05) = 2+0$$

$$\Rightarrow y(1.05) = 2.$$

Putting $n=1$ in eq(2), we have :-

$$y_2 = y_1 + h \left\{-2 + \frac{y_1}{x_1}\right\}$$

$$\Rightarrow y(x_2) = 2 + (0.05)\left\{-2 + \frac{2}{1.05}\right\} \Rightarrow y(x_0+2h) = 1.995238095$$

$$\Rightarrow y(1+0.1) = 1.995238095$$

$$\Rightarrow y(1.1) = 1.995238095$$

Putting $n=2$ in eq(2), we have :-

$$y_3 = y_2 + h\left\{-2 + \frac{y_2}{x_2}\right\}$$

$$\Rightarrow y(x_3) = 1.995238095 + (0.05)\left\{-2 + \frac{1.995238095}{1.1}\right\}$$

$$\Rightarrow y(x_0+3h) = 1.985930736 \Rightarrow y(1+0.15) = 1.985930736$$

$$\Rightarrow y(1.15) = 1.985930736.$$

Putting $n=3$ in eq(2), we have :-

$$y_4 = y_3 + h\left\{-2 + \frac{y_3}{x_3}\right\}$$

$$\Rightarrow y(x_4) = 1.985930736 + (0.05)\left\{-2 + \frac{1.985930736}{1.15}\right\}$$

$$\Rightarrow y(x_0+4h) = 1.972275551$$

$$\Rightarrow y(1+0.2) = 1.972275551$$

$$\Rightarrow y(1.2) = 1.972275551 \quad \text{—— (B)}.$$

The exact sol^n is given by :-

$$\frac{dy}{dx} = -2 + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} + \left(-\frac{1}{x}\right)y = -2$$

$$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}.$$

$$\Rightarrow \frac{1}{x}\frac{dy}{dx} + \left(-\frac{1}{x^2}\right)y = -\frac{2}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = -\frac{2}{x} \Rightarrow \frac{y}{x} = \int -\frac{2}{x}\,dx \Rightarrow \frac{y}{x} = -2\ln x + c$$

Putting $x=1$ & $y=2$;

$$\frac{2}{1} = -2\ln 1 + c \Rightarrow c = 2$$

$$\therefore \frac{y}{x} = -2\ln x + 2$$

$$\Rightarrow y = -2x\ln x + 2x$$

$$\Rightarrow y(1.2) = -2(1.2)\ln(1.2) + 2(1.2)$$

$$\Rightarrow y(1.2) = 1.962428264 \text{ (Ans)}.$$

Q.(8) Solve the i.v.p: $\frac{dy}{dx} = \sqrt{y}$; $y(0)=1$ using Euler's method with $h=0.2$ and $h=0.1$; determine $y(0.4)$ approximately. compare the approximate results with the exact solution.

Sol:- Given; $\frac{dy}{dx} = \sqrt{y}$, $y(0)=1$, $h=0.2$

Here: $f(x,y) = \sqrt{y}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\sqrt{y_n} \quad \text{——— (1)}$$

Putting $n=0$ in eq(1), we have:-

$y_1 = y_0 + h\sqrt{y_0}$ $\Rightarrow y(x_1) = 1 + (0.2)\sqrt{1}$

$\Rightarrow y(x_0+h) = 1 + 0.2$ $\Rightarrow y(0+0.2) = 1.2$

$\Rightarrow y(0.2) = 1.2$;

Putting $n=1$ in eq(1), we have:-

$y_2 = y_1 + h\sqrt{y_1}$ $\Rightarrow y(x_2) = 1.2 + (0.2)\sqrt{1.2}$

$\Rightarrow y(x_0+2h) = 1.419089023$ $\Rightarrow y(0+0.4) = 1.419089023$

$\Rightarrow y(0.4) = 1.419089023$.

Again, given; $\frac{dy}{dx} = \sqrt{y}$, $y(0)=1$, $h = 0.1$.

Putting $n=0$ in eq(1), we have:-

$y_1 = y_0 + h\sqrt{y_0}$ $\Rightarrow y(x_1) = 1 + (0.1)\sqrt{1}$

$\Rightarrow y(0+0.1) = 1 + 0.1$ $\Rightarrow y(0.1) = 1.1$.

Putting $n=1$ in eq(1), we have:-

$y_2 = y_1 + h\sqrt{y_1}$ $\Rightarrow y(x_2) = 1.1 + (0.1)\sqrt{1.1}$

$\Rightarrow y(x_0+2h) = \cancel{1.17}$ 1.204880885

$\Rightarrow y(0.2) = 1.204880885$.

Putting $n=2$; $y_3 = y_2 + h\sqrt{y_2}$ $\Rightarrow y(x_3) = 1.204880885 + (0.1)\sqrt{1.204880885}$

$\Rightarrow y(0.3) = 1.314647951$.

& $y(0.4) = 1.314647951 + (0.1)\sqrt{1.314647951} = 1.429306049$.

$y^{-\frac{1}{2}+1} / (-\frac{1}{2}+1)$

The exact solution is given by:-

$$\frac{dy}{dx} = \sqrt{y} \Rightarrow \int \frac{dy}{\sqrt{y}} = \int dx$$

$$\Rightarrow 2\sqrt{y} = x + c.$$

Putting $x=0$ & $y=1$ $\Rightarrow 2 = 0 + c$

$\Rightarrow c = 2.$

$$\therefore 2\sqrt{y} = x+2 \Rightarrow \sqrt{y} = \frac{x+2}{2}$$

$$\Rightarrow y = \left(\frac{x+2}{2}\right)^2$$

$$\Rightarrow y(0.4) = \left(\frac{0.4+2}{2}\right)^2 = 1.4400. \text{ (Ans)}$$

Q. Determine 'y' for $x = 0.1, 0.2, 0.3, 0.4, 0.5$ where 'y' is the solution of the differential equation: $\frac{dy}{dx} = 2(y+1)$, $y(0)=0$ by using Euler's method with $h=0.1$. Find the exact solution and compare the numerical results with exact values to see how good the results are.

Sol: Given; $\frac{dy}{dx} = 2(y+1)$, $y(0)=0$, $h=0.1$.

Here; $f(x,y) = 2(y+1)$, $x_0=0$, $y_0=0$, $h=0.1$.

By Euler's method, we have:-

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\Rightarrow y_{n+1} = y_n + h\{2(y_n+1)\} \quad \text{—— (1)}.$$

Putting $n=0$ in eq(1), we have:-

$$y_1 = y_0 + h\{2(y_0+1)\} \Rightarrow y(x_1) = 0 + (0.1)\{2(0+1)\}$$

$$\Rightarrow y(x_0+h) = 0.2$$

$$\Rightarrow y(0.1) = 0.2.$$

Putting $n=1$ in eq(1), we have:-

$$y_2 = y_1 + h\{2(y_1+1)\} \Rightarrow y(x_2) = 0.2 + (0.1)\{2(0.2+1)\}$$

$$\Rightarrow y(0.2) = 0.44.$$

Putting $n=2$ in eq(1), we have:-

$$y_3 = y_2 + h\{2(y_2+1)\} \Rightarrow y(x_3) = 0.44 + (0.1)\{2(0.44+1)\}$$

$$\Rightarrow y(0.3) = 0.728.$$

Putting $n=3$ in eq(1),

$$y_4 = 0.728 + (0.1)\{2(0.728+1)\}$$

$$\Rightarrow y(0.4) = 1.0736.$$

Putting $n=4$; $y(0.5) = 1.0736 + (0.1)\{2(1.0736+1)\}$

$$\Rightarrow y(0.5) = 1.48832.$$

The exact solution is given by:-

$\frac{dy}{dx} = 2(y+1) \Rightarrow \int \frac{dy}{y+1} = 2\int dx \Rightarrow \ln(y+1) = 2x + C.$

Putting $x=0, y=0$

$\Rightarrow \ln 1 = 0 + C \Rightarrow C = 0.$

$\therefore \ln(y+1) = 2x \Rightarrow y+1 = e^{2x}$

$\Rightarrow y = e^{2x} - 1.$

$\therefore y(0.1) = e^{2\times 0.1} - 1 = 0.221402758$

$y(0.2) = e^{2\times 0.2} - 1 = 0.491824697.$

$y(0.3) = e^{2\times 0.3} - 1 = 0.8221188.$

$y(0.4) = e^{2\times 0.4} - 1 = 1.225540928.$

$y(0.5) = e^{2\times 0.5} - 1 = 1.718281828.$

Q. Let $\frac{dy}{dx} = x^2 + y$; $y(0) = 1$. Using Euler's method with $h = 0.05$ findout the approximate value y_1 of $y(0.05)$. Then applying modified Euler's method determine $y(0.05)$ correct to five significant figures.

Sol: Given; $\frac{dy}{dx} = x^2 + y$; $y(0) = 1$, $h = 0.05$.

$x_0 = 0$, $y_0 = 1$, $h = 0.05$.

Here: $f(x,y) = x^2 + y$;

By Euler's method, we have:-

$y_{n+1} = y_n + h f(x_n, y_n)$ ——— (1)

$\Rightarrow y_{n+1} = y_n + h(x_n^2 + y_n)$

Putting $n=0$ in eq(1), we have:-

$y_1 = y_0 + h(x_0^2 + y_0) \Rightarrow y(x_1) = 1 + (0.05)(0+1)$

$\Rightarrow y(0.05) = 1.05$

By modified Euler's method we have:-

$Y_{n+1} = Y_n + \frac{h}{2}\{f(x_n, Y_n) + f(x_{n+1}, y_{n+1})\}$

$\Rightarrow Y_{n+1} = Y_n + \frac{h}{2}\{x_n^2 + Y_n + (x_{n+1})^2 + y_{n+1}\}$ ——— (2).

Putting $n=0$ in eq(2) & taking $Y_0 = y_0$;

$Y_1 = Y_0 + \frac{h}{2}\{x_0^2 + Y_0 + x_1^2 + y_1\}$

$\Rightarrow Y(x_1) = 1 + \frac{0.05}{2}\{0 + 1 + 0.0025 + 1.05\}$

$\Rightarrow Y(0.05) = 1.0513125.$ (Ans).

$= 1.0513.$

Q. Solve numerically: $\frac{dy}{dx} = y - x$; $x = 0$, $y = 2$ by RK-me... of order '2' choosing $h = 0.1$ for $y(0.1)$ and $y(0.2)$. Com... your answers with the exact sol of the given i.v.p..

Sol:- Given; $\frac{dy}{dx} = y - x$, $x_0 = 0$, $y_0 = 2$ $\Rightarrow y(0) = 2$, $h = 0.1$.

Here: $f(x,y) = y - x$, $x_0 = 0$, $y_0 = 2$, $h = 0.1$.

By RK-method of order '2'; we have:-

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}[k_1 + k_2]. \\ \text{where; } k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_n + h, y_n + k_1) \end{aligned}\right\} \text{——} (1).$$

Putting $n = 0$;

$$\begin{aligned} k_1 &= h f(x_0, y_0) \\ &= h(y_0 - x_0) = (0.1)(2 - 0) \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} \& \quad k_2 &= h f(x_0 + h, y_0 + k_1) \\ &= h\{(y_0 + k_1) - (x_0 + h)\} = (0.1)\{(2 + 0.2) - (0 + 0.1)\} \\ &= 0.21. \end{aligned}$$

Now; $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$\Rightarrow y(x_1) = 2 + \frac{1}{2}[0.2 + 0.21]$

$\Rightarrow y(0.1) = 2.205$.

Putting $n = 1$;

$$\begin{aligned} k_1 &= h f(x_1, y_1) \\ &= h(y_1 - x_1) = (0.1)(2.205 - 0.1) \\ &= 0.2105. \end{aligned}$$

$$\begin{aligned} \& \quad k_2 &= h f(x_1 + h, y_1 + k_1) \\ &= h\{(y_1 + k_1) - (x_1 + h)\} = (0.1)\{(2.205 + 0.2105) - (0.1+ \\ &= 0.22155 \end{aligned}$$

Now; $y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$,

$\Rightarrow y(x_2) = 2.205 + \frac{1}{2}[0.2105 + 0.22155]$

$\Rightarrow y(0.2) = 2.421025$

The exact solution is given by:

$$\frac{dy}{dx} = y - x \Rightarrow \frac{dy}{dx} + (-1)y = -x.$$

I.F: $e^{\int -1\,dx} = e^{-x}$:

$$\Rightarrow e^{-x}\frac{dy}{dx} - ye^{-x} = -xe^{-x}$$

$$\Rightarrow ye^{-x} = \int -xe^{-x}\,dx$$

$$= -\left[x\int e^{-x} - \int -e^{-x}\,dx\right] = -\left[x \times -e^{-x} + (-e^{-x})\right] + c$$

$$= xe^{-x} + e^{-x} + c$$

$$\Rightarrow y = x + 1 + ce^{x}.$$

Putting $x = 0, y = 2$

$$\Rightarrow 2 = 0 + 1 + ce^{0} \Rightarrow 2 = 1 + c \Rightarrow c = 1.$$

$$\therefore y = x + 1 + e^{x}.$$

$$\Rightarrow y(0.1) = 0.1 + 1 + e^{0.1} = 2.205170918.$$

$$y(0.2) = 0.2 + 1 + e^{0.2} = 2.421402758.$$

Q 2. Solve: $\frac{dy}{dx} = y - x$; $y(0) = 2$ by RK-method of order four choosing $h = 0.1$.

Sol:- Given; $\frac{dy}{dx} = y - x$, $y(0) = 2$, $h = 0.1$.

$$\frac{dy}{dx} = y - x, \quad x_0 = 0, \quad y_0 = 2, \quad h = 0.1.$$

Here; $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$, $h = 0.1$.

By RK-method of order '4'; we have:-

$$\left.\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\
\text{where; } k_1 &= hf(x_n, y_n) = h(y_n - x_n) \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = h\left\{\left(y_n + \frac{k_1}{2}\right) - \left(x_n + \frac{h}{2}\right)\right\} \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) = h\left\{\left(y_n + \frac{k_2}{2}\right) - \left(x_n + \frac{h}{2}\right)\right\} \\
k_4 &= hf(x_n + h, y_n + k_3) = h\{(y_n + k_3) - (x_n + h)\}
\end{aligned}\right\} \quad - (1)$$

Putting $n = 0$ in eq(1); we have:-

$$k_1 = h(y_0 - x_0) = (0.1)(2 - 0) = 0.2$$

$$k_2 = h\left\{\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right\} = (0.1)\left\{\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right\} = 0.205.$$

$$k_3 = (0.1)\left\{\left(2 + \frac{0.205}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right\} = 0.20525$$

$$k_4 = (0.1)\{(2 + 0.20525) - (0 + 0.1)\} = 0.210525.$$

Now; $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$\Rightarrow y(x_1) = 2 + \frac{1}{6}[0.2 + 2 \times 0.205 + 2 \times 0.20525 + 0.210525]$$

$$\Rightarrow y(0.1) = 2.205170833.$$

$x_0 = 0, y_0 = 2$

Similarly, for $h = 0.2$;

$$k_1 = (0.2)(2-0) = 0.4$$

$$k_2 = (0.2)\left\{\left(0+\frac{0.4}{2}\right) - \left(0+\frac{0.2}{2}\right)\right\} = 0.02$$

$$k_3 = (0.2)\{(2+$$

Putting $n=1$ in eq(2); for $h=0.1$;

$$k_1 = (0.1)(2.205170833 - 0) = 0.2205170833.$$

$$k_2 = (0.1)\left\{\left(2.205170833 + \frac{0.2205170833}{2}\right) - \left(0.1+\frac{0.1}{2}\right)\right\} = 0.2165429...$$

$$k_3 = (0.1)\left\{\left(2.205170833 + \frac{0.2165429937}{2}\right) - \left(0.1+\frac{0.1}{2}\right)\right\} = 0.2163442...$$

$$k_4 = (0.1)\{(2.205170833 + 0.21634423) - (0.1+0.1)\} = 0.222151506.$$

Now; $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$\Rightarrow y(x_2) = 2.205170833 + \frac{1}{6}[0.2205170833 + 2\times 0.2165429937 + 2\times 0.21634423 + 0.222151506]$$

$$\Rightarrow y(0.2) = 2.423244654.$$

Q. Find the constants k_1, k_2, k_3, k_4 in RK-method of order four for the numerical evaluation of $y(0.2)$ where 'y' is the exact solution of $\frac{dy}{dx} = 1+y^2$; $y(0)=0$. Take $h=0.2$. Also evaluate $y(0.2)$ approximately.

Sol: Given; $\frac{dy}{dx} = 1+y^2$; $y(0)=0$; $h=0.2$.

Here: $f(x,y) = 1+y^2$, $x_0=0$, $y_0=0$, $h=0.2$.

By RK-method of order '4'; we have:-

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\ \text{where; } k_1 &= h f(x_n, y_n) = h[1+y_n^2] \\ k_2 &= h f\left(x_n+\frac{h}{2}, y_n+\frac{k_1}{2}\right) = h\left[1+\left(y_n+\frac{k_1}{2}\right)^2\right] \\ k_3 &= h f\left(x_n+\frac{h}{2}, y_n+\frac{k_2}{2}\right) = h\left[1+\left(y_n+\frac{k_2}{2}\right)^2\right] \\ k_4 &= h f(x_n+h, y_n+k_3) = h[1+(y_n+k_3)^2] \end{aligned}\right\} \text{--- (1)}$$

Putting $n=0$ in eq(1), we have:-

$$k_1 = h[1+y_0^2] = (0.2)[1+0] = 0.2$$

$$k_2 = h\left[1+\left(y_0+\frac{k_1}{2}\right)^2\right] = (0.2)[1+(0+0.1)^2] = 0.202$$

$$k_3 = h\left[1+\left(y_0+\frac{k_2}{2}\right)^2\right] = (0.2)\left[1+\left(0+\frac{0.202}{2}\right)^2\right] = 0.2020402$$

$$k_4 = h[1+(y_0+k_3)^2] = (0.2)[1+(0+0.2020402)^2] = 0.208164048$$

Putting these values:-

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\Rightarrow y(x_1) = 0 + \frac{1}{6}[0.2 + 2\times 0.202 + 2\times 0.2020402 + 0.208164048]$$

$$\Rightarrow y(0.2) = 0.202707408.$$

Q. Use RK-method of order '4' to solve the following eq for $0 \le x \le 0.5$ choosing $h=0.1$; $\frac{dy}{dx} = \frac{1}{10}(x^2+y^2)$; $y(0)=1$.

Sol:- Given; $\frac{dy}{dx} = \frac{1}{10}(x^2+y^2)$; $y(0)=1$, $h=0.1$.

Here: $f(x,y) = \frac{x^2+y^2}{10}$, $x_0=0$, $y_0=1$, $h=0.1$.

By RK-method of order '4'; we have:-

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}[k_1+2k_2+2k_3+k_4]. \\ \text{where; } k_1 &= hf(x_n, y_n) = h\left(\frac{x_n^2+y_n^2}{10}\right) \\ k_2 &= hf\left(x_n+\frac{h}{2}, y_n+\frac{k_1}{2}\right) = h\left\{\frac{(x_n+\frac{h}{2})^2+(y_n+\frac{k_1}{2})^2}{10}\right\} \\ k_3 &= hf\left(x_n+\frac{h}{2}; y_n+\frac{k_2}{2}\right) = h\left\{\frac{(x_n+\frac{h}{2})^2+(y_n+\frac{k_2}{2})^2}{10}\right\} \\ k_4 &= hf(x_n+h, y_n+k_3) = h\left\{\frac{(x_n+h)^2+(y_n+k_3)^2}{10}\right\} \end{aligned}\right\} \quad -(1)$$

Putting $n=0$;

$$k_1 = (0.1)\left\{\frac{0+1}{10}\right\} = 0.01.$$

$$k_2 = (0.1)\left\{\frac{(0+\frac{0.1}{2})^2+(1+\frac{0.01}{2})^2}{10}\right\} = 0.01012525.$$

$$k_3 = (0.1)\left\{\frac{(0+\frac{0.1}{2})^2+(1+\frac{0.01012525}{2})^2}{10}\right\} = 0.010126508.$$

$$k_4 = (0.1)\left\{\frac{(0+0.1)^2+(1+0.010126508)^2}{10}\right\} = 0.010303555.$$

Now; $y(x_1) = 1 + \frac{1}{6}[0.01 + 2\times 0.01012525 + 2\times 0.010126508 + 0.010303555]$

$$\Rightarrow y(0.1) = 1.010134512$$

$$y(x_2) = 1.010134512 + 0.010134512 = 1.020269024 = y(0.2)$$

Ex-3:- (Page-288):- For the eq^n $\frac{dy}{dx} = cy$; $y(0) = 1$, show that the formula (modified Euler's) leads to $y_n = \left\{\frac{(1+\frac{ch}{2})}{(1-\frac{ch}{2})}\right\}^n$; provided $\left|\frac{ch}{2}\right| < 1$. Further, show that $\lim_{h\to 0} y_n = e^{cx_n}$, for a fixed value of $x = x_0 + nh$.

Imp

Solution:- Given that, $\frac{dy}{dx} = cy$; $y(0) = 1$.

Here: $f(x,y) = cy$ $\Rightarrow f(x_n, y_n) = cy_n$

& $f(x_{n+1}, y_{n+1}) = cy_{n+1}$.

By Modified Euler's method, we have:-

$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y_{n+1})\}$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\{cy_n + cy_{n+1}\} \quad \Rightarrow y_{n+1} = y_n + \frac{ch}{2}y_n + \frac{ch}{2}y_{n+1}$$

$$\Rightarrow y_{n+1}\left(1-\frac{ch}{2}\right) = y_n\left(1+\frac{ch}{2}\right)$$

$$\Rightarrow y_{n+1} = \left[\frac{(1+\frac{ch}{2})}{(1-\frac{ch}{2})}\right]y_n.$$

Applying induction on 'n', we obtain:-

$$y_n = \left[\frac{(1+\frac{ch}{2})}{(1-\frac{ch}{2})}\right]^n \quad \because y_0 = y(0) = 1 \qquad (P_{rv}).$$

Substituting $h = \frac{x_n}{n}$ in the above eq^n, we get:-

Rough

$h = \frac{x_n}{n}$

$\Rightarrow$ when $h\to 0$, $n\to\infty$

$$y_n = \left[\frac{(1+\frac{cx_n}{2n})}{(1-\frac{cx_n}{2n})}\right]^n$$

$$= \left[\frac{(1+\frac{cx_n}{2}\cdot\frac{1}{n})}{(1-\frac{cx_n}{2}\cdot\frac{1}{n})}\right]^n$$

$$\therefore \lim_{h\to 0} y_n = \lim_{n\to\infty} y_n$$

$$= \lim_{n\to\infty}\left[\frac{(1+\frac{\alpha}{n})}{(1-\frac{\alpha}{n})}\right]^n, \text{ where, } \alpha = \frac{cx_n}{2} = \text{a const.}$$

$$= \frac{\lim_{n\to\infty}(1+\frac{\alpha}{n})^n}{\lim_{n\to\infty}(1-\frac{\alpha}{n})^n} = \frac{e^{\alpha}}{e^{-\alpha}} = e^{2\alpha}$$

$$= e^{2\cdot\frac{cx_n}{2}}$$

$$= e^{cx_n}. \quad (P_{rv}).$$