Logic

1.1 Propositional Logic :-

Proposition → A Proposition is a declarative sentence (i.e. a sentence that declares a fact) i.e. either true or false, but not both.

Ex:- $1+1=2$ (True Proposition/statement)
$2+2=3$ (False Proposition/statement)

* **Propositional Logic** →

Def^n → The truth value of a Proposition is true denoted by T, if it is a true Proposition & false denoted by F, if it is false Proposition. The area of logic that deals with Propositions is called the Propositional Calculus or Propositional Logic.

→ Note :- Proposition is also called as Statement.

* Def^n → **Not Statement / Not Proposition** →
Interrogative, exclamatory, order, wishes etc. sentences are not example of statements.

Ex:- Where are you going ?
$x+y=z$, $x+y=2$.
Read this carefully.
Oh ! What a Scenary ?

→ The Sentences which involves variables like x, y, z etc. are not examples of Statement.

→ The sentences which contain adjectives like good, bad, young old, fat, thin, ~~etc~~ small, big etc. are not examples of Statement. → This statement is called as **Fuzzy Statement**.

Logic → Logic is the branch of Mathematics as well as logical science which deals with the study of Proposition or statements & its truth values.

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OR Logic is the branch of Mathematics that studies the principles of correct reasoning, describing relationships among propositions in terms of implication, contradiction, conversion etc.

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Note → All statements are Sentences.

But all Sentences are not statements.

* Truth Values → True & false Propositions are known as truth values. 'T' is denoted as true statement & 'F' is denoted as false statement.

* Connectives → The four words or Phrases like "or", "and", "iff" & "only if" are known as Connectives.

* Prime statements → If a statement does not contain any connectives, then the statement known as Prime statements.

Ex:- Today is Monday.

* Composite statements → If a statement contains one or more ~~statements~~ Connectives, then the statement known as Composite statements.

Ex:- * Today is Monday and it is raining today : $p \wedge q$.

* $(p \vee q) \rightarrow (p \wedge \sim q)$.

* Negation → (Not (~))

When a statement modified by word "not" then a new statement is found & the new statement known as negation of the given statement.

In other word, let 'p' be a proposition. The negation of p denoted by $\sim p$ or $\bar{p}$ is the statement ~~"not p" or~~ "It is not the case that p" & read as "not p".

Ex:- Today is Monday, Find the negation of this Proposition ?

Solⁿ:- The negation is

"It is not the case that today is Moday".

or "Today is not Monday".

or "It is not Monday today".

OR p: Today is Monday.

$\sim p$: Today is not Monday.

*** Axiom** → If 'p' is true, then $\sim p$ is false & if 'p' is false, then $\sim p$ is true.

→

p	$\sim p$
T	F
F	T

The truth table for the negation of Proposition 'p'.

*** Defⁿ** → Conjunction ($(\wedge)$, and)

Let p & q be Propositions. The Conjunction of p and q, denoted by $p \wedge q$, is the Proposition "p and q".

The Conjunction $p \wedge q$ is true when both p & q are true & is false otherwise.

→

p	T	T	F	F
q	T	F	T	F
$p \wedge q$	T	F	F	F

The truth table for the Conjunction of two Propositions.

Ex:- Find the Conjunction of the Propositions p & q, where p: Today is Friday & q: It is raining today ?

Ans:- Here, $p \wedge q$: Today is Friday and it is raining today is the Conjunction of these Propositions p & q.

This Proposition is true on rainy Fridays & is false on any day that is not a Friday & on Fridays when it does not rain.

*** Defⁿ** → Disjunction ($(\vee)$, or)

Let p & q be Propositions. The disjunction of p & q, denoted by $p \vee q$, is the Proposition "p or q".

The disjunction $p \vee q$ is false when both p & q are

false & is true otherwise.

→

p	T	T	F	F
q	T	F	T	F
p∨q	T	T	T	F

The truth table for the disjunction of two propositions.

Ex:- Find the disjunction of the propositions p & q, where
p: Today is Friday & q: It is raining today ?

Ans:- The disjunction of these propositions, p∨q is written as
p∨q : Today is Friday or it is raining today.

This proposition is true on any day i.e. either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it also does not rain.

* **Exclusive or (XOR)** Let p & q be two propositions. The exclusive or of p & q, denoted by $p \oplus q$, is the proposition i.e. true when exactly one of p & q is true & is false otherwise.

That means this proposition is true when p is true & q is false, and when p is false & q is true. It is false when both p & q are false and when both p & q are true.

Ex:- Consider the following statement:

1) "In order to get a job in this multinational company, experience with C++ or Java is mandatory".

Ans.- If p: experience with C++
q: experience with Java.

i.e. p∨q : experience with C++ or Java.

Here,

p	T	T	F	F
q	T	F	T	F
p∨q	T	T	T	F

So, here 'or' is simply means "inclusive or" / "disjunction".

'Inclusive or' it simply means that the whole compound proposition will get satisfied whethere p is satisfied or q is satisfied or both of them get satisfied.

If both of them are satisfied then also the whole compound proposition will get satisfied.

2) "When you buy a car from XYZ Company, you get \$2500 cashback or accessories worth \$2500".

Ans.- You may get cashback or you may get accessories. But it is not possible to get both cashback & accessories. Here, this particular operator 'or' is simply means "exclusive or" / "XOR".

i.e. XOR simply means exactly one of p & q will be true but both cannot be true.

p	T	T	F	F
q	T	F	T	F
$p \oplus q$	F	T	T	F

→ The truth table for the XOR of two propositions.

* Def^n → Conditional statements ($\rightarrow$ / $\Rightarrow$), only if)

Let p & q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q".

The cond^{n}al statement $p \rightarrow q$ is false when p is true & q is false and true otherwise.

Where, 'p' is called the hypothesis (or antecedent or premise) & 'q' is called the conclusion (or consequence).

→ The statement $p \rightarrow q$ is called a cond^{n}al statement because $p \rightarrow q$ asserts that 'q' is true on the cond^n that 'p' holds. A cond^{n}al statement is also called as an "implication".

→ $p \rightarrow q$ is true when both p & q are true (or when both p & q are false) or when p is false (no matter what the truth value q has), otherwise $p \rightarrow q$ is false.

→

p	T	T	F	F
q	T	F	T	F
$p \to q$	T	F	T	T

The truth table for the Cond^{n}al Statement $(p \to q)$ of two Propositions.

→ The Proposition $p \to q$ may be expressed as :

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary cond^n for p is q"
"a unless $\sim p$"
"q Provides p"
"p implies q"
"p only if q"
"a Sufficient cond^n for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"
"q is Consequence of p".

Ex:- The Pledge many Politicians make when running for office is "If I am elected, then I will lower taxes".

Ans.- If the politician is elected, then voters would expect this politician to lower taxes. If the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes.

It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the Campaign Pledge. It corresponds to the case that when p is true but q is false, then $p \to q$ is false.

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Note → In logic word "but" sometimes is used instead of "and" in a conjunction.

Ex:- "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining".

→ The use of the connective 'or' in a disjunction corresponds to one of the two ways the word 'or' is used in English, namely in an 'inclusive way'. Thus a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use 'or' in an 'exclusive sense'. When the "exclusive or" is used to connect the propositions p & q, the proposition "p or q (but not both)" is obtained.

→ If no. of prime statement in a composite statement 'n', then there are 2^n possible assignment rows of truth values.

→ "p only if q" expresses the same thing as "if p then q". It says that p cannot be true when q is not true. i.e. the statement is false if p is true, but q is false. When p is false, q may be either true or false because the statement says nothing about the truth value of q.

→ "q unless ~p" means that if ~p is false, then q must be true. i.e. the statement "q unless ~p" is false when p is true & q is false, but it is true otherwise. Consequently "q unless ~p" & "p → q" always have the same truth value.

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Def^n: Biconditional Statements $((\leftrightarrow / \Leftrightarrow), (\text{iff}))$

Let p & q be propositions. The bicond^{n}al statement $p \leftrightarrow q$ is the proposition "p if & only if q". The bicond^{n}al statement $p \leftrightarrow q$ is true when p & q have the same truth values, & is false otherwise.

→ Bicond^{n}al statements are also called bi-implications.

→

p	T	T	F	F
q	T	F	T	F
$p \leftrightarrow q$	T	F	F	T

The truth table for the bicond^{n}al $p \leftrightarrow q$.

→ The statement $p \leftrightarrow q$ is true when both the cond^{n}al statements $p \to q$ & $q \to p$ are true & is false otherwise.

→ There are some other common ways to express $p \leftrightarrow q$:

"p is necessary & sufficient for q"

"if p then q & conversely"

"p iff q".

→ $p \leftrightarrow q$ has exactly the same truth value as $(p \to q) \wedge (q \to p)$ i.e. $(p \to q)$ and $(q \to p)$.

→

p	T	T	F	F
q	T	F	T	F
$p \to q$	T	F	**T**	**T**
$q \to p$	T	T	**F**	**T**
$p \leftrightarrow q = (p \to q) \wedge (q \to p)$	T	F	F	T

We have to concluded this $p \leftrightarrow q = (p \to q) \wedge (q \to p)$.

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Ex:- Let p : You can take the flight & q : You buy a ticket. Then $p \leftrightarrow q$ is the statement :

Ans.- $p \leftrightarrow q$: You can take the flight iff you buy a ticket. This statement is true if p & q are either both true or both false, i.e. if you buy a ticket & can take the flight or if you do not buy a ticket & you cannot take the flight.

It is false when p & q have opposite truth values, i.e. when you do not buy a ticket, but you can take the flight (such as when you get a free trip) & when you buy a ticket & cannot take the flight (such as when the airline bumps you).

*** Def^n →** Converse Statement :-

Converse of a cond^{n}al statement "$p \to q$" is "$q \to p$".

*** Def^n →** Inverse Statement :-

Inverse of a cond^{n}al statement "$p \to q$" is "$\sim p \to \sim q$".

*** Def^n →** Contrapositive Statement :-

Contrapositive of a cond^{n}al statement "$p \to q$" is "$\sim q \to \sim p$".

Ex:- Construct the truth table of the compound proposition $(p \vee \sim q) \to (p \wedge q)$?

Ans.-

p	q	$\sim q$	$p \vee \sim q$	$p \wedge q$	$(p \vee \sim q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

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* 1.2 Propositional Equivalences →

* Tautology →

A Compound Proposition that is always true, no matter what the truth values of the Propositions that occur in it, is called as a tautology.

→

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$\therefore$ $p \vee \sim p$ is a tautology.

[$\sim(p \wedge \sim p)$ is also a tautology.]

* Contradiction → (Absurdity)

A Compound Proposition i.e. always false, no matter what the truth values of the Propositions that occur in it, is called as a Contradiction.

→

p	$\sim p$	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	T
F	T	F	T

$\therefore$ $p \wedge \sim p$ is a Contradiction, where, $\sim(p \wedge \sim p)$ is a tautology.

So, the truth tables of $p \vee \sim p$ & $p \wedge \sim p$ shown in above.

Here, $p \vee \sim p$ is always true it is a tautology & $p \wedge \sim p$ is always false it is a Contradiction.

* Contingency → A Compound Proposition i.e. neither a tautology nor a Contradiction is called a Contingency.

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* Def^n → Logical Equivalences → (≡)

Compound Propositions that have the same truth values in all possible cases are called logically equivalent.

OR The compound propositions p & q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p & q are logically equivalent.

* De Morgan's law →

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Ex:- Show that $\sim(p \vee q)$ & $\sim p \wedge \sim q$ are logically equivalent ?

Ans.

p	q	~p	~q	p∨q	~(p∨q)	~p∧~q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Ex:- Show that $\sim(p \wedge q)$ & $\sim p \vee \sim q$ are logically equivalent ?

Ans.

p	q	~p	~q	p∧q	~(p∧q)	~p∨~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Ex:- Show that $p \rightarrow q$ & $\sim p \vee q$ are logically equivalent.

Ans.

p	q	~p	p→q	~p∨q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

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Ex:- Show that $p \vee (q \wedge r)$ & $(p \vee q) \wedge (p \vee r)$ are logically equivalent

This is the distributive law of disjunction over conjunction ?

Ans.

$2^3 = 8$
$p \to \frac{8}{2} = 4$
$q \to \frac{4}{2} = 2$
$r \to \frac{2}{2} = 1$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

* Logical Equivalence →

	Equivalence	Name
1.	$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
2.	$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
3.	$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
4.	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
5.	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
6.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
7.	$\sim(\sim p) \equiv p$	Double negation law

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8.	$\sim(p \wedge q) \equiv \sim p \vee \sim q$ $\sim(p \vee q) \equiv \sim p \wedge \sim q$	De Morgan's laws
9.	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
10.	$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$	Negation laws

* Logical equivalences involving cond^{n}al statements :-

1. $p \to q \equiv \sim p \vee q$
2. $p \to q \equiv \sim q \to \sim p$
3. $p \vee q \equiv \sim p \to q$
4. $p \wedge q \equiv \sim(q \to \sim p)$
5. $\sim(p \to q) \equiv p \wedge \sim q$
6. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.

* Logical equivalences involving bicond^{n}al statements :-

10. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
11. $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$
12. $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
13. $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$.

Ex:- Show that $\sim(p \to q)$ & $p \wedge \sim q$ are logically equivalent ?

Ans.- As $\sim(p \to q) \equiv \sim(\sim p \vee q)$ [Cond^{n}al statement no. (1)]

$\equiv \sim(\sim p) \wedge (\sim q)$ (2nd De Morgan's law)

$\equiv p \wedge \sim q$ (Double negation law)

$\Rightarrow \sim(p \to q) \equiv p \wedge \sim q$.

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Ex:- Show that $\sim(p \vee (\sim p \wedge q))$ & $\sim p \wedge \sim q$ are logically equivalent by developing a series of logical equivalences?

Ans.-

$$\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim(\sim p \wedge q) \quad [\text{2nd De Morgan's law}]$$

$$\equiv \sim p \wedge (\sim(\sim p) \vee (\sim q)) \quad [\text{1st De Morgan's law}]$$

$$\equiv \sim p \wedge (p \vee \sim q) \quad [\text{Double negation}]$$

$$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) \quad [\text{2nd Distributive law}]$$

$$\equiv F \vee (\sim p \wedge \sim q) \quad [\because \sim p \wedge p \equiv F \text{ (Negation law)}]$$

$$\equiv (\sim p \wedge \sim q) \vee F \quad [\text{Commutative law}]$$

$$\equiv \sim p \wedge \sim q \quad [\text{Identity law for } F]$$

Ex:-

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology & $(p \wedge q) \rightarrow (p \vee q) \equiv T$?

Ans. Here $(p \wedge q) \rightarrow (p \vee q)$

$$\equiv \sim(p \wedge q) \vee (p \vee q) \quad [\because p \rightarrow q \equiv \sim p \vee q]$$

$$\equiv (\sim p \vee \sim q) \vee (p \vee q) \quad [\text{1st De Morgan's law}]$$

$$\equiv (\sim p \vee p) \vee (\sim q \vee q) \quad [\text{Associative \& Commutative Laws}]$$

$$\equiv T \vee T \quad (\text{Domination law})$$

$$\equiv T$$

1.3 * Predicates & Quantifiers →

*** Def^n → Predicate Logic →**

Predicate logic is the most powerful type of logic that used to express the meaning of a wide range of statements in Mathematics & in Computer science.

This logic permit us reason & explore rel^{n}ships bet^n objects.

*** Def^n:- Predicate →**

Statements involving variables such as $x > 3$, $x + y = z$, $x = y + 3$ are often found in mathematical assertions in computer programs and in system specifications.

→ These statements are neither true nor false when the values of the variable are not specified.

→ Consider statement, "x is greater than 3". Above statement contains 2 parts: the 1st part contain the subject of the statement i.e. the variable x. The 2nd part contain the predicate i.e. "is greater than 3" which refers to a property that subject of the statement can have.

We can denote the above statement by $P(x)$, where P denote the predicate & x is variable.

→ The statement $P(x)$ is also said to be the value of propositional fun^n P at x.

→ Once a value has been assigned to the ~~vector~~ variable x, the statement $P(x)$ becomes a proposition & has a truth value.

Ex:- Let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(4)$ & $P(2)$?

Solⁿ:- We obtain the statement $P(4)$ by setting $x=4$ in the statement "$x>3$", which is the statement "$4>3$" is true. However, $P(2)$ by setting $x=2$ is the statement "$2>3$" is false.

Ex: Let $Q(x,y)$ denote the statement "$x=y+3$". What are the truth values of the propositions $Q(1,2)$ & $Q(3,0)$?

Solⁿ:- To obtain $Q(1,2)$, set $x=1$ & $y=2$ in the statement $Q(x,y)$. Hence, $Q(1,2)$ is the statement "$1=2+3$" which is false [because $1=5$ is false].

The statement $Q(3,0)$ is the statement "$3=0+3$" which is true [because $3=3$ is true].

Note → In general, a statement involving the n-variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$. A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional funⁿ P at the n-tuple $(x_1, x_2, \ldots, x_n)$ & P is also called an n-place predicate or a n-ary predicate.

* Defⁿ → Quantifiers → Certain declarative sentences involves words that indicate quantity such as all, some, many, none & few are used in quantifications. Since, such words indicates quantity they are called quantifiers.

→ There are 2 - types of quantifications :

① Universal quantification

② Existential quantification

* ① Universal →

Universal quantification which tells us that a Predicate is true for every element under consideration.

Def^n → * The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain".

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifiere.

We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$".

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An element for which $P(x)$ is false is called a counter example of $\forall x P(x)$. [~~$x \in P(x)$~~].

* ② Existential →

Existential quantification which tells us that there is one or more element under consideration for which the predicate is true.

* Def^n → The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$".

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Ex:- ① Let $x, y \in R$

Then $x+y = y+x$, $\forall x, y \in R$.

② If $a \in F$ $\exists$ $0 \in F$ such that $a+0 = 0+a = a$.

Statement	When true	When false
$\forall x P(x)$	$P(x)$ is true $\forall x$.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false $\forall x$.

Ex:- Let $P(x)$ be the statement "$x+1 > x$". What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real no.s ?

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Sol^n:- Here $x \in R \Rightarrow x+1 > x, \forall x$.

So, $P(x)$ is true for all real no.s x, the quantification $\forall x P(x)$ is true.

* Def^n → The area of logic that deals with Predicates & quantifiers is called as the Predicate Calculus.

* Def^n → Many Mathematical Statements assert that a Property is true for all values of a variable in a particular domain is called as the domain of discourse (or the universe of discourse) often just referred ~~as~~ to as the domain.

Ex:- Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real no.s ?

Sol^n:- Given $x \in R \Rightarrow x > 3$ is sometimes true.

If $x = 4 \Rightarrow 4 > 3$ is the existential quantification of $P(x)$, which is $\exists x P(x)$ is true.

→ The statement $\exists x P(x)$ is false iff there is no element x in the domain for which $P(x)$ is true.

i.e. $\exists x P(x)$ is false iff $P(x)$ is false for every element of the domain.

Ex:- Let $Q(x)$ denote the statement "$x = x+1$". What is the truth values of the quantification $\exists x Q(x)$, where the domain consists of all real no.s ?

Sol^n:- Here, $Q(x)$ is false for every real no., then the truth value of quantification $\exists x Q(x)$ is false because $x \in R$, if $x = 1 \Rightarrow 1 = 1+1 = 2 \Rightarrow 1 = 2$ is false. Hence proved.

Ex:- Let $Q(x)$ denote the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real no.s ?

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Sol^n:- Here $Q(x)$ is not true for every real no. x.
Because $x \in R$ if $x = 3 \nless 2$ i.e. $3 \nless 2$.
Thus $x = 3$ is a counter example for the statement $\forall x Q(x)$. So, $\forall x Q(x)$ is false.

Note $\rightarrow$ A statement $\forall x P(x)$ is false where $P(x)$ is a propositional fun^n iff $P(x)$ is not always true when x is in domain.

* Binding variables $\rightarrow$

Def^n $\rightarrow$ When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.

$\rightarrow$ An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

$\rightarrow$ The part of a logical expression to which a quantifier is applied is called the scope of this quantifier.

Ex:- In the statement $\exists x\ (x+y=1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free bcz it is not bound by a quantifier & no value is assigned to this variable. So, the statement $\exists x\ (x+y=1)$, x is bound, but y is free. This variable x is refers to the binding variables.

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→ Using modifiers $\sim$ & the quantifiers $\forall$ & $\exists$, we can form 8 different expressions involving the open proposition $P(x)$.

Sentence	Abbreviated Meaning
$\forall x, P(x)$	all true
$\exists x\, P(x)$	at least one true
$\sim[\forall x\, P(x)]$	not all true
$\exists x\, [\sim P(x)]$	at least one false
$\sim[\exists x\, P(x)]$	none true
$\forall x\, [\sim P(x)]$	all false
$\sim[\forall x\, [\sim P(x)]]$	not all false
$\sim[\exists x\, [\sim P(x)]]$	none false

→ From the above we conclude that,

"all true" ~~means~~ is same as "none false".
"all false" " " "none true".
"not all true" " " "at least one false".
"not all false" " " "at least one true".

Thus the 8 expressions can be grouped into 4 equivalences :-

(1) $\forall x P(x) \equiv \sim[\exists x\, [\sim P(x)]]$

(2) $\sim[\forall x P(x)] \equiv$ ~~$\sim[\exists x\, P(x)]$~~ $\exists x\, [\sim P(x)]$

(3) $\forall x\, [\sim P(x)] \equiv \sim[\exists x\, P(x)]$

(4) $\sim[\forall x\, [\sim P(x)]] \equiv \exists x\, P(x)$.

Equivalent Statement	Negation
$\sim[\forall x P(x)]$	$\exists x\, [\sim P(x)]$
($P(x)$ is false for every x (not all true))	(There is an x for which $P(x)$ is false (At least one false))
$\sim[\exists x P(x)]$	$\forall x\, [\sim P(x)]$
(There is an x for which $P(x)$ is false)	($P(x)$ is false for every x (all false))

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* Negating quantified expressions →

Ex:- Consider the negation of the statement :

"Every student in your class has taken a course in Calculus" ?

Ans.- This statement is a universal quantification, namely $\forall x P(x)$; where $P(x)$ is the statement "x has taken a course in Calculus" & the domain consists of the students in your class. The negation of this statement ~~that~~ is

"It is not case that every student in your class has taken a course in Calculus" OR "There is a student in your class who has not taken a course in Calculus" which simply means the existential qualification of the negation of the original propositional fun^n, namely $\exists x[\sim P(x)]$ that Means at least one student in your class who has not taken a course in Calculus.

Note → $\sim[\forall x P(x)] \equiv \exists x[\sim P(x)]$.

$\sim[\exists x P(x)] \equiv \forall x[\sim P(x)]$.

The rules for negations for quantifiers are called De Morgan's laws for quantifiers.

Statement	Negation
1. $\forall x P(x)$ (All true)	1. $\exists x[\sim P(x)]$ (At least one false)
2. $\exists x P(x)$ (At least one true)	2. $\forall x[\sim P(x)]$ (All false)
3. $\forall x[\sim P(x)]$ (All false)	3. $\exists x P(x)$ (At least one true)
4. $\exists x[\sim P(x)]$ (At least one false)	4. $\forall x P(x)$ (All true)

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* Truth Value of $\forall x P(x)$ ⟶

① $\forall x P(x)$ is true if $P(x)$ is true for every x. ② $\forall x P(x)$ is false if $P(x)$ is false at least one x. Note ⟶ When all the elements in the domain can be listed as $x_1, ---, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction of $P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$ because this conjunction is true iff $P(x_1), ---, P(x_n)$ are all true.

→ Ex:- What is the truth value of $\forall x P(x)$, where $P(x): x^2 < 10$ & the domain consist of the +ve integers not exceeding 4 ?

Sol^n:- Given that, $P(x): x^2 < 10$.

The statement $\forall x P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ because the domain consists of the +ve integers 1, 2, 3, 4.

So, $\forall x P(x)$ is false because $P(4)$: $16 < 10$, which is false.

* Truth ~~table~~ value of $\exists x P(x)$ ⟶

① $\exists x P(x)$ is true if $P(x)$ is true for at least one x in the domain.

② $\exists x P(x)$ is false if $P(x)$ is false for every x in the domain.

Ex:- Let $P(x): x^2 = x$, where $x \in Z$. What is the truth value of $\exists x P(x)$?

Sol^n:- Given that, $P(x): x^2 = x$.

So, $\exists x P(x)$ because at least one statement $P(x)$ is true. In fact exactly two of them are true namely $P(0)$ & $P(1)$.

where, $P(0): 0^2 = 0$ & $P(1): 1^2 = 1$.

Note → When all the elements in the domain can be listed as $x_1, x_2, ---, x_n$; the existential quantification $\exists x P(x)$ is same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$ because this disjunction is true iff at least one of $P(x_1), \ldots, P(x_n)$ is true.

Ex:- What is the truth value of $\exists x P(x)$, where $P(x): x^2 > 10$ & the domain consists of the +ve integers not exceeding 4 ?

Sol^n:- Given that, $P(x): x^2 > 10$. The statement $\exists x P(x)$ is same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$. Because the domain $= \{1, 2, 3, 4\}$. So, $\exists x P(x)$ is true as $P(4): 16 > 10$, which is true.

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1.6 * Mathematical induction :→

→ Principle of Mathematical induction :-

To Prove that $P(n)$ is true $\forall$ +ve int. n, where $P(n)$ is a Propositional fun^n.

* → Basis Step :- we verify that $P(1)$ is true for $n=1$.

* → Inductive Step :- We show that the Cond^{n}al statement $P(K) \longrightarrow P(K+1)$ is true $\forall$ +ve int. K i.e. $\forall K \in \mathbb{Z}_+$

To Complete the inductive step of a Proof using the Principle of mathematical induction. We assume that $P(K)$ is true for an arbitrary +ve int. K & show that under this ~~Condition~~ assumption, $P(K+1)$ must also be true for an arb. +ve int. K. ~~show that this assumption~~ The The assumption that $P(K)$ is true is called the inductive hypothesis. Once, we complete both steps in a Proof by mathematical induction, we have shown that $P(n)$ is true $\forall$ +ve int. i.e. we have to shown that $\forall n P(n)$ is true where the quantification is over the set of +ve int.

Q:- Show that if n is a +ve int., then $1+2+\cdots+n = \frac{n(n+1)}{2}$?

Sol^n:- Let $P(n)$ be the Proposition that the sum of 1st n +ve int. $= \frac{n(n+1)}{2}$ i.e. $P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Basis Step :- Let $n=1$, then $P(1): \frac{1(1+1)}{2}$ is true.

Inductive step :- For the inductive hypothesis we assume that $P(K)$ holds for an arbitrary +ve int. K i.e. we assume that ~~$P(2)$~~ $P(K): 1+2+\cdots+K = \frac{K(K+1)}{2}$ is true. Under this assumption, it must be shown that $P(K+1)$ is true i.e. $P(K+1): 1+2+\cdots+K+(K+1) = \frac{(K+1)(K+2)}{2}$ is true.

$L.H.S. = 1+2+\cdots+K+(K+1) = \frac{K(K+1)}{2} + (K+1)$

$= \frac{K(K+1)+2(K+1)}{2}$

$= \frac{(K+1)(K+2)}{2}$

$= R.H.S.$

OR

$P(K): 1+2+3+\cdots+K = K(K+1)/2$

$\Rightarrow 1+2+3+\cdots+K+(K+1) = \frac{K(K+1)}{2} + (K+1)$

$\Rightarrow P(K+1): 1+2+\cdots+K+(K+1) = \frac{(K+1)(K+2)}{2}$ $\Rightarrow P(K+1)$ is true.

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This completes the inductive step. We have completed the basis & inductive steps. So, by mathematical induction we know that $P(n)$ is true $\forall$ +ve int. n i.e. we have proven that

$P(n): 1+2+\dots+n = \frac{n(n+1)}{2}$ $\forall$ +ve int. n.

Q:- Show that (i) $1+3+5+\dots+(2n-1) = n^2$ [Conjecture formula]

(ii) $1+2+2^2+\dots+2^n = 2^{n+1}-1$; $\forall n \geq 0$

(iii) Geometric progression series:

$$\sum_{j=0}^{n} ar^j = a+ar+\dots+ar^n = \frac{ar^{n+1}-a}{r-1}.$$

(iv) inequality $n < 2^n$; $\forall n \in \mathbb{Z}_+$.

(v) $2^n < n!$; $\forall n \geq 4$.

(vi) n^3-n is divisible by 3.

(vii) n^2+n is divisible by 2 or 2 divides n^2+n; $\forall n \in \mathbb{Z}_+$

by using mathematical induction?

Sol^n:- (i) Let $P(1): 1^2 = 1$ is true.

Assume that $P(k): 1+3+5+\dots+(2k-1) = k^2$ is true.

We have to show $P(k+1): 1+3+5+\dots+(2k-1)+(2k-1+2) = (k+1)^2$.

$L.H.S = 1+3+\dots+(2k-1)+(2k-1+2)$

$= [1+3+\dots+(2k-1)] + (2k+1) = k^2+(2k+1) = (k+1)^2 = R.H.S.$

i.e. $P(k+1): 1+3+\dots+(2k-1)+(2k+1) = (k+1)^2$ is true.

Hence, $P(n): 1+3+\dots+(2n-1) = n^2$ is true $\forall n$.

(ii) Let $P(n): 1+2+2^2+\dots+2^n = 2^{n+1}-1$; $\forall n \geq 0$.

Here, $P(0): 2^{0+1}-1 = 2-1 = 1$ is true.

$P(1): 2^{1+1}-1 = 2^2-1 = 3 = 1+2$ is true.

Assume that $P(k): 1+2+2^2+\dots+2^k = 2^{k+1}-1$ is true.

We have to show that $P(k+1)$ is true $\forall k \geq 0$.

i.e. we have to show that

$P(k+1): 1+2+\cdots+2^k+2^{k+1} = 2^{k+2}-1$ is true.

$L.H.S. = 1+2+2^2+\cdots+2^k+2^{k+1}$

$= (2^{k+1}-1)+2^{k+1} = 2^{k+1}+2^{k+1}-1$

$= 2\cdot 2^{k+1}-1 = 2^{k+2}-1 = R.H.S.$

Hence, $P(n)$ is true $\forall n \geq 0$.

(iii) Let $P(n): \sum_{j=0}^{n} ar^j = a+ar+\cdots+ar^n = \frac{ar^{n+1}-a}{r-1}, \forall n \geq 0$.

Since, $P(0): \frac{ar^1-a}{r-1} = \frac{ar-a}{r-1} = \frac{a(r-1)}{r-1} = a$ is true for $n=0$.

Assume $P(k)$ is true i.e. $P(k): a+ar+\cdots+ar^k = \frac{ar^{k+1}-a}{r-1}$

To show $P(k+1): a+\cdots+ar^k+ar^{k+1} = \frac{ar^{k+2}-a}{r-1}$ is true.

$L.H.S. = a+\cdots+ar^k+ar^{k+1}$

$= \frac{ar^{k+1}-a}{r-1}+ar^{k+1} = \frac{ar^{k+1}-a+ar^{k+2}-ar^{k+1}}{r-1}$

$= \frac{ar^{k+2}-a}{r-1} = R.H.S.$

Hence, $P(n)$ is true $\forall n \geq 0$.

(iv) By similar Process, $P(1): 1<2^1$ i.e. $1<2$ is always true.

Assume $P(k): k<2^k$ is true.

To show $P(k+1): (k+1)<2^{k+1}$ is true.

$L.H.S. = k+1 < 2^k+2^k = 2\cdot 2^k = 2^{k+1} = R.H.S.$

i.e. $(k+1) < 2^{k+1}$.

Where, $1<2^1 \Rightarrow 1<2^k \ \forall k \geq 1$

& $k<2^k \Rightarrow k+1 < 2^k+2^k = 2^{k+1}$.

Hence, $P(n)$ is true $\forall n \in \mathbb{Z}_+$.

(v) Let $P(4): 2^4 < 4! \Rightarrow 16<24$ is true.

Assume, $P(k): 2^k < k!$ is true.

To show $P(k+1): 2^{k+1} < (k+1)!$ is true.

$L.H.S. = 2^{k+1} = 2^k\cdot 2$

$< k!\times 2 \quad [\because 2^k < k!]$

$< k!\times(k+1) \quad [\because 2<(k+1), \forall k \geq 4]$

$< (k+1)! = R.H.S.$

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So, $P(n)$ is true $\forall n \geq 4$.

(vi) Let $P(0)$ & $P(1)$: 0 is divisible by 3.

Assume $P(k)$ is divisible by 3 & say $k^3 - k = 3m$; $m \in \mathbb{Z}$.

To show $P(k+1)$ is divisible by 3; where $P(k+1)$: $(k+1)^3 - (k+1) = 3n$; $n \in \mathbb{Z}$.

$$L.H.S. = (k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

$$= k^3 + 3k^2 + 3k - k$$

$$= (k^3 - k) + 3(k^2 + k)$$

$$= 3m + 3(k^2 + k)$$

$$= 3(m + k^2 + k),$$ which is divisible by 3

$$= R.H.S.$$

$\Rightarrow$ $P(k+1)$ is divisible by 3. So, $P(n)$ is true $\forall n$.

(vii) Let $P(0)$: 0 is divisible by 2 & $P(1)$: 2 is divisible by 2.

Assume, $P(k)$: $n^2 + n$ is divisible by 2 & say $n^2 + n = 2m$; $m \in \mathbb{Z}$

To show $P(k+1)$: $(n+1)^2 + (n+1)$ is divisible by 2.

$$L.H.S. = (k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1$$

$$= (k^2 + k) + (2k + 2)$$

$$= 2m + 2(k+1)$$

$$= 2(m + k + 1),$$ which is divisible by 2

$$= R.H.S.$$

So, $P(n)$ is divisible by 2. (Proved)

2.1 * **Set :-** A collection of well defined & distinct objects is called as set.

* **Object :-** The objects in a set are called the elements or members of the set. A set is said to contain its elements.

* **Equality of two sets :-** Two sets are equal iff they have the same elements. i.e. if A & B are sets then A & B are equal iff $\forall x\ (x \in A \leftrightarrow x \in B)$. We write $A = B$ if A & B are equal sets.

Ex:- The sets $\{1,3,5\}$ & $\{3,5,1\}$ are equal because they have the same elements.

→ The sets $\{1,3,5\}$ & $\{3,5,1,1,5,3\}$ [is same as the $\{3,5,1\}$ because they have the same elements] are equal i.e. $\{1,3,5\}$ & $\{3,5,1,1,5,3\} = \{3,5,1\}$ are equal sets.

Subset :- The set A is said to be a subset of B iff every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. $A \subseteq B$ iff the quantification $\forall x\ (x \in A \rightarrow x \in B)$ is true.

Ex:- Set of all computer science majors at your school is a subset of the set of all students at your school.

Th^m-1 For every set S, (i) $\phi \subseteq S$ & (ii) $S \subseteq S$.

(i) Let S be a set. To show that $\phi \subseteq S$, we must show that $\forall x\ (x \in \phi \rightarrow x \in S)$ is true because the empty set contains no elements. It follows that $x \in \phi \rightarrow x \in S$ is always true, because its hypothesis is always false & a cond^{n}al statement with a false hypothesis is true i.e. $\forall x\ (x \in \phi \rightarrow x \in S)$ is true $\Rightarrow \phi \subseteq S$.

(ii) Let 'S' be the set of all people in China is a subset of the set of all people in China i.e. 'S' $\Rightarrow S \subseteq S$.

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→ $A \subset B$ we say that A is a proper subset of B.

→ If $A \subseteq B$ & $B \subseteq A$, then $A = B$.

* **Cardinality of S :-** Let S be a set. If there are exactly n-distinct elements in S, where 'n' is a non-negative integer, we say that S is a finite set & that 'n' is the cardinality of S. The cardinality of S is denoted by $|S|$.

Ex:- Let A be the set of odd +ve int. < 10 i.e.

i.e. $A = \{1, 3, 5, 7, 9\} \Rightarrow |A| = 5$.

→ A set is said to be infinite if it is not finite.

* **The Power Set :-**

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

Ex:- What is the power set of the set $\{0, 1, 2\}$?

Sol^n:- The power set $P(\{0,1,2\})$ is the set of all subsets of $\{0, 1, 2\}$.

Hence, $P(\{0,1,2\}) = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

→ The empty set & the set itself are members of this set of subsets.

→ If a set has n elements, then its power set has 2^n elements.

* **Cartesian Products :-** The ordered ~~set~~ n-tuple $(a_1, \ldots, a_n)$ is the ordered collection that has a_1 as its 1st element, a_2 as its 2nd element, ... and a_n as its nth element.

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→ We say that two ordered n-tuples are equal iff each corresponding pair of their elements is equal.
In particular 2-tuples are called ordered pair. The ordered pairs (a,b) & (c,d) are equal iff $a=c$ & $b=d$.

→ Let A & B be sets. The Cartesian Product of A & B, denoted by $A \times B$ is the set of all ordered pairs (a,b) where $a \in A$ & $b \in B$. Hence, $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

Ex:- What is the Cartesian Product of $A=\{1,2\}$ & $B=\{1,2,3\}$?

Sol^n:- The Cartesian product of $A \times B$ is

$$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}.$$

→ Cartesian Product of $A \times B \neq$ Cartesian Product of $B \times A$.

Ex:- What is the Cartesian Product $A \times B \times C$. ~~where~~ where $A=\{0,1\}$, $B=\{1,2\}$, $C=\{0,1,2\}$?

Sol^n:- The Cartesian Product $A \times B \times C$ consists of all ordered ~~set~~ triples (a,b,c), where $a \in A$, $b \in B$ & $c \in C$.
Hence, $A \times B \times C = \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),$
$(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\}$

* Proper Subset :-

A Proper Subset of a set A is ~~called~~ a subset of A i.e. not equal to A.

Ex:- Let $A=\{1,3,5\}$ & $B=\{1,5\}$.
$\Rightarrow$ B is a ~~Sub~~ Proper subset of A.

→ Let $C=\{1,3,5\}$ $\Rightarrow$ C is a subset of A.
But it is not a Proper subset of A because $C=A$.

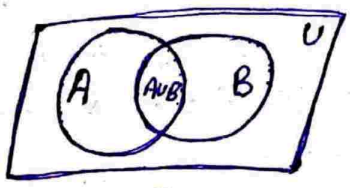
→ The set $D=\{1,4\}$ is not a subset of A because 4 is not an element of A.

2.2 Set Operations :→

* **Union of two Sets** :- Let A & B are two sets. The union of the sets A & B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B or in both.

→ An element $x \in A \cup B$ iff $x \in A$ or $x \in B$ i.e.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$



Venn Diagram of $A \cup B$.

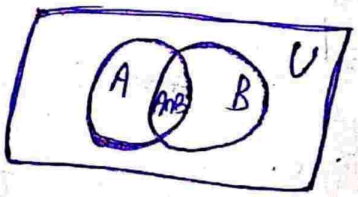
Ex:- The union of the sets $\{1,3,5\}$ & $\{1,2,3\}$ is the sets

$$\{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}.$$

* **Intersection of two sets** :- Let A & B be two sets. The intersection of the sets A & B, denoted by $A \cap B$, is the set containing those elements in both A & B.

→ An element $x \in A \cap B$ iff $x \in A$ & $x \in B$ i.e.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



Venn Diagram of $A \cap B$.

Ex:- The intersection of the sets $\{1,3,5\}$ & $\{1,2,3\}$ is the sets

$$\{1,3,5\} \cap \{1,2,3\} = \{1,3\}.$$

* **Disjoint set** :- Two sets are called disjoint if their intersection is the empty set. We can write $A \cap B = \phi$.

Ex:- Let $A = \{1,3,5,7,9\}$ & $B = \{2,4,6,8,10\}$.

Here, $A \cap B = \phi$ because A & B are disjoint.

Note :- ① $|A \cup B| = |A| + |B| - |A \cap B|$

Where, this symbol '| |' is read as cardinality of the set.

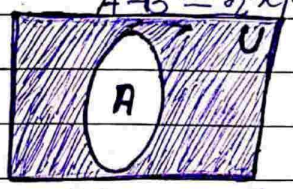
② The generalization of this result to unions of an arbitrary no. of sets is called the <u>Principle of inclusion – exclusion</u>.

* <u>Complement Set B w.r.t. A OR Difference Sets</u> :-

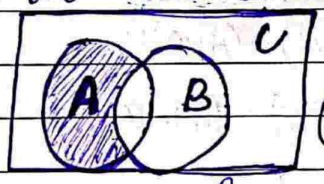
Let A & B be two sets. The difference of A & B, denoted by A − B, is the set containing those elements that are in A but not in B. The difference of A & B is also called the complement of B with respect to (w.r.t.) A.

→ An element $x \in A-B$ iff $x \in A$ & $x \notin B$ i.e.

$A-B = \{x / x \in A \wedge x \notin B\}$.



($\bar{A}$ is shaded)



(A−B is Shaded)

① Venn Diagram of A−B means eliminate B.

② Venn Diagram of Complement set A.

<u>Ex:-</u> Let $A = \{1,3,5\}$ & $B = \{1,2,3\}$.

Since, $A-B = \{1,3,5\} - \{1,2,3\} = \{5\}$.

& $B-A = \{1,2,3\} - \{1,3,5\} = \{2\}$

* <u>Complement Set</u> :- Let U be the universal set. The Complement of the set A, denoted by $\bar{A}$, is the complement of A w.r.t. U. In other words, the Complement of the set A is U−A.

→ An element $x \in \bar{A}$ iff $x \notin A$ i.e. $\bar{A} = \{x / x \notin A\} = \{x / \sim(x \in A)\}$

To see the figure ② / venn diagram of $\bar{A}$.

<u>Ex:-</u> 1) Let $A = \{a,e,i,o,u\}$, then $\bar{A} = \{x / x \notin A\}$.

i.e. $\bar{A} = \{b,c,d,f,g,h,j,k,l,m,n,p,q,r,s,t,v,w,x,y,z\}$.

2) Let A is the set of all positive integers greater than 10 i.e. $A = \{x / x > 10, x \in Z_+\}$.

Then, Complement of A, $\bar{A} = \{x / x \leq 10, x \in Z_+\}$.

i.e. $\bar{A} = \{1,2,3,4,5,6,7,8,9,10\}$.

* Set identities - table

Identity	Name
$A \cup \phi = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement laws

Ex:- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$?

Sol^n:- Let $x \in \overline{A \cap B}$, by def^n of Complement $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$.

By def^n of intersection, $x \notin A \cap B \Rightarrow \sim(x \in A \wedge x \in B)$ is true.

By De Morgan's law, $\sim(x \in A) \vee \sim(x \in B)$.

By def^n of negation, $x \notin A \vee x \notin B$

By def^n of Complement, $x \in \bar{A} \vee x \in \bar{B}$ i.e. $x \in \bar{A}$ or $x \in \bar{B}$

It follows by def^n of union, $x \in \bar{A} \cup \bar{B}$.

This shows that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ —— ①.

Again, let $x \in \bar{A} \cup \bar{B}$. By def^n of union, $x \in \bar{A}$ or $x \in \bar{B}$.

Using def^n of Complement, $x \notin A$ or $x \notin B$.

i.e. $\sim(x \in A) \vee \sim(x \in B)$ is true.

By De Morgan's law, $\sim(x \in A \wedge x \in B)$ is true.

By def^n of intersection, $\sim(x \in A \cap B)$

By def^n of Complement, $x \in \overline{A \cap B}$.

This shows that $\bar{A} \cup \bar{B} \subset \overline{A \cap B}$ —— ②

Hence, from ① & ② we get $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

→ We can prove this by identity with following steps :-

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ [By Complement]

$= \{x \mid \sim(x \in A \cap B)\}$ [By negation $x \notin A \Rightarrow \sim(x \in A)$]

$= \{x \mid \sim(x \in A \wedge x \in B)\}$ [By intersection]

$= \{x \mid \sim(x \in A) \vee \sim(x \in B)\}$ [By De Morgan's law]

$= \{x \mid x \notin A \vee x \notin B\}$ [By negation]

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ [By Complement]

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$ [By union]

$\Rightarrow \overline{A \cap B} = \bar{A} \cup \bar{B}$.

Ex:- Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

Sol^n:- $A \cap (B \cup C) = \{x \mid x \in A \cap (B \cup C)\}$

$= \{x \mid x \in A \wedge x \in (B \cup C)\}$ [By intersection].

$= \{x \mid x \in A \wedge (x \in B \vee x \in C)\}$ [By union]

$= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}$ [We know by logic]

$= \{x \mid x \in A \cap B \vee x \in A \cap C\}$ [By intersection]

$= \{x \mid x \in (A \cap B) \cup (A \cap C)\}$ [By union]

$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

OR Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

By def^n of union, $x \in A$ and $x \in B$ or $x \in C$ (or both).

We know that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$.

By def^n of intersection, $x \in A \cap B$ or $x \in A \cap C$

By def^n of union, $x \in A \cap B \vee x \in A \cap C$ i.e. $x \in (A \cap B) \cup (A \cap C)$.

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ —— (1)

Again, let $x \in (A \cap B) \cup (A \cap C)$, then by def^n of union $x \in A \cap B$ or $x \in A \cap C$. By def^n of intersection, $x \in A$ ~~or~~ and $x \in B$ or $x \in A$ and $x \in C$ i.e. $x \in A$ and $x \in B$ or $x \in C$.

By def^n of union, $x \in A$ and $x \in B \cup C$.

By def^n of intersection, $x \in A \cap (B \cup C)$.

So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ —— (2)

Hence, from (1) & (2) we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

* A membership table for the distributive Property :-

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Ex :- Let A, B & C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$?

Sol^n :- We have, $\overline{A \cup (B \cap C)} = \bar{A} \cap (\overline{B \cap C})$ [By De Morgan's law]

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ [By De Morgan's law]

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ [By Commutative law]

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ [By Commutative law]

Hence, $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$.

* **Def^n** → The union of collection of sets is the set that contains those elements that are members at least one set in the collection. We use the notation,

$A_1 \cup A_2 \cup \ldots \cup A_m = \bigcup_{n=1}^{m} A_n$ to denote the union of the sets $A_1, A_2, ---, A_n$.

* **Def^n** → The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection. We use the notation,

$A_1 \cap A_2 \cap --- \cap A_m = \bigcap_{n=1}^{m} A_n$ to denote the intersection of the sets $A_1, A_2, ---, A_n$.

Ex:- Let $A_n = \{n, n+1, n+2, ---\}$. Then,

$\bigcup_{n=1}^{m} A_n = \{1, 2, 3, ---\}$ & $\bigcap_{n=1}^{m} A_n = \{m, m+1, m+2, ---\}$.

Ex:- The bit strings for the sets $\{1,2,3,4,5\}$ & $\{1,3,5,7,9\}$ are 11111 00000 & 1010101010 respectively. Use bit strings to find the union & intersection of these sets?

Sol^n:- The bit string for the union of these sets is

11111 00000 $\vee$ 10101 01010 = 1111101010, which corresponds to the set $\{1,2,3,4,5,7,9\}$.

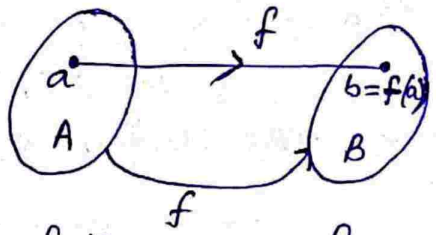
The bit string for the intersection of these sets is 11111 00000 $\wedge$ 10101 01010 = 10101 00000, which corresponds to the set $\{1,3,5\}$.

2.3 * Functions →

Def^n → Let A & B be non-empty sets. A fun^n 'f' from A to B is an assignment of exactly one element of B to each element of A. We write $f(a)=b$ if b is the unique element of B assigned by the fun^n f to the element 'a' of A. If f is a fun^n from A to B, we write $f: A \to B$.

→ Fun^{n}s are sometimes also called mappings or transformations.

→ If 'f' is a fun^n from A to B, we say that A is the domain of 'f' & B is the co-domain of 'f'. If $f(a)=b$, we say that 'b' is the image of 'a' & 'a' is a pre-image of 'b'. The range of 'f' is the set of all images of elements of A. Also, if 'f' is a fun^n from A to B, we say that 'f' maps A to B.



The fun^n f maps from A to B.

→ Two fun^{n}s are equal, when they have the same domain, have the same co-domain & map elements of their common domain to the same elements in their common co-domain.

⇒ If we change either the domain or the co-domain of a fun^n, then we obtain a different fun^n.

⇒ If we change the mapping of elements, then we also obtain a different fun^n.

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*** Def^n** → Let f_1 & f_2 be fun^{n}s from A to R. Then, f_1+f_2 & f_1f_2 are also fun^n from A to R defined by,

$$(f_1f_2)x = f_1(x).f_2(x) \quad \& \quad (f_1+f_2)x = f_1(x)+f_2(x).$$

Ex:- Let f_1 & f_2 be two fun^{n}s from R to R s.t. $f_1(x)=x^2$ & $f_2(x)=x-x^2$. What are the fun^{n}s f_1+f_2 & f_1f_2?

Sol^n:- We know, $(f_1f_2)(x) = f_1(x).f_2(x)$

$$= x^2(x-x^2) = x^3-x^4.$$

$$\& \quad (f_1+f_2)(x) = f_1(x)+f_2(x)$$

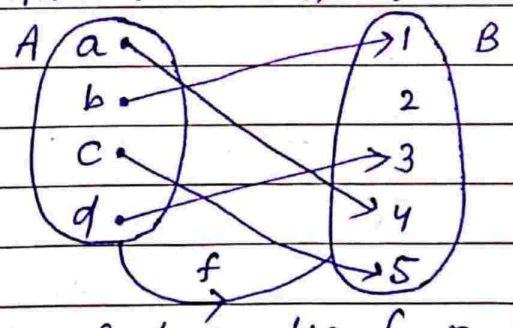
$$= x^2+x-x^2 = x.$$

*** Def^n** → A fun^n f is said to be one-to-one or injective iff $f(a)=f(b)$ implies $a=b$ $\forall a,b$ in the domain of f. A fun^n is said to be an injection if it is one-to-one.

→ A fun^n 'f' is one-to-one iff $f(a)\neq f(b)$ whenever $a\neq b$.

Ex:- Determine whether the fun^n 'f' from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ with $f(a)=4$, $f(b)=1$, $f(c)=5$, $f(d)=3$ is one-to-one?

Sol^n:- The fun^n f is one-to-one because f takes on different values at the 4-elements of its domain.



Ex:- Determine whether the fun^n $f(x)=x^2$ from the set of integers to the set of integers is one-to-one?

Sol^n:- The fun^n $f(x)=x^2$ is not one-to-one because $f(1)=1=f(-1)$ but $1\neq -1$.

i.e. if $f(x_1)=x_1^2$ & $f(x_2)=x_2^2$, then

$$\Rightarrow f(x_1)=f(x_2) \Rightarrow x_1^2=x_2^2$$

$$\Rightarrow x_1=\pm x_2.$$

$\Rightarrow x_1=x_2,\ x_1=-x_2$, which is ~~on~~ not injective.

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The funⁿ $f(x)=x^2$ with its restricted to Z^+ or Z^+ is one-to-one.

* **Defⁿ** → A funⁿ $f: A \to B$ is called onto or surjective iff for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$. A funⁿ f is called a surjection if it is onto.

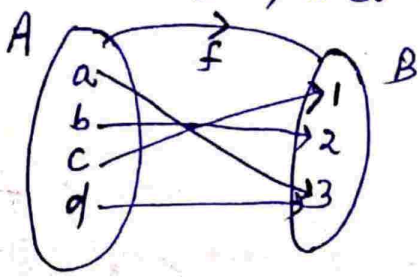
Note →

→ A funⁿ f is one-to-one if $\forall a \forall b (f(a)=f(b) \to a=b)$, or equivalently $\forall a \forall b (a \neq b \to f(a) \neq f(b))$. where, the domain is the domain of the funⁿ.

→ A funⁿ f is onto if $\forall b \exists a (f(a)=b)$, where the domain for 'a' is the domain of the funⁿ & the domain for 'b' is the co-domain of the funⁿ.

Ex:- Let f be the funⁿ from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a)=3$, $f(b)=2$, $f(c)=1$, $f(d)=3$. Is 'f' an onto funⁿ?

Solⁿ:- Because all three elements of the co-domain are images of elements in the domain, so f is onto.



If the co-domain were $\{1,2,3,4\}$ then would not be onto.

Ex:- Is the funⁿ $f(x)=x^2$ & $f(x)=x+1$ from the set of integers to the set of int. onto?

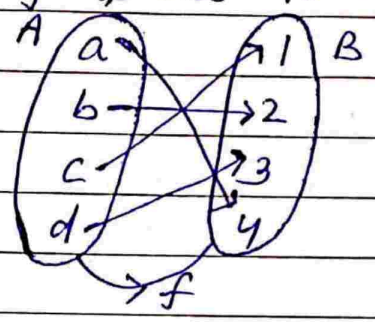
Solⁿ:- The funⁿ f is not onto because, there is no integer x with $x^2=-1$, for instance.

And, this funⁿ $f(x)=x+1$ is onto because for every int. y there is an int. x s.t. $f(x)=y$ iff $x+1=y$ which holds iff $x=y-1$.

*** Def^n** → "The fun^n f is one-to-one correspondence or a bijection if it is both one-to-one & onto.

Ex :- Let f be the fun^n from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ & $f(d) = 3$. Is f a bijection?

Sol^n:- "The fun^n f is one-to-one & onto.



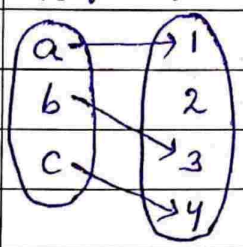
It is one-to-one because no two values in the domain are assigned the same fun^n value.

It is onto because all 4-elements of the co-domain are images of elements in the domain.

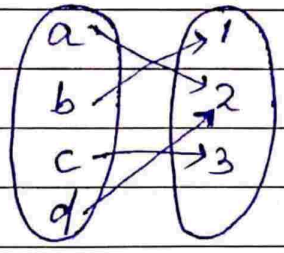
Hence, f is a bijection.

→ Examples of different types of correspondences :-

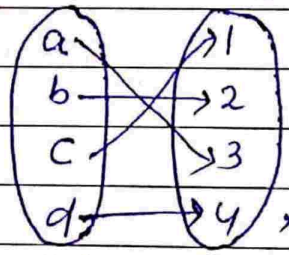
(a) one-to-one, not onto

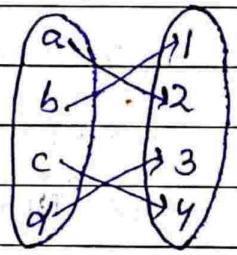


(b) onto, not one-to-one

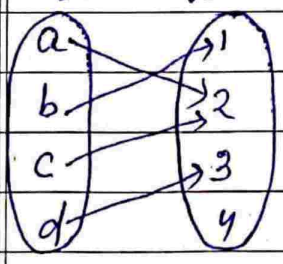


(c) Onto, one-to-one

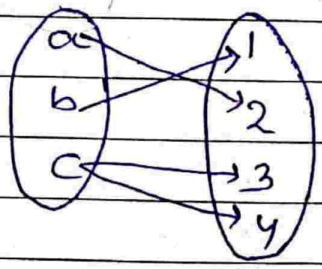




(d) Neither one-to-one, nor onto



(e) Not a fun^n



* **Inverse funⁿ :→** Let f be one-to-one correspondence from A to B. The inverse funⁿ of f is the funⁿ that assigns to an element b belonging to B the unique element a in A s.t. $f(a) = b$. The inverse funⁿ of f is denoted by f^{-1}.

Hence, $f^{-1}(b) = a$ when $f(a) = b$.

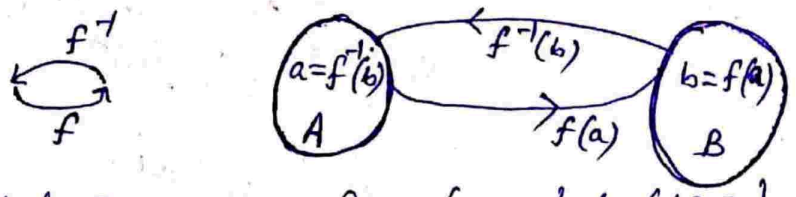
Note →

→ Let A be a set. The identity funⁿ on A is the funⁿ $I_A : A \to A$ where $I_A(x) = x$; $\forall x \in A$. In other words, the identity funⁿ I_A is the funⁿ that assigns each element to itself. The funⁿ I_A is one-to-one & onto, so it is a bijection.

[Note that $\iota = I$ is the Greek letter iota.]

→ If a funⁿ f is not a one-to-one correspondence, we cannot define an inverse funⁿ of f. When f is not one-to-one correspondence either it is not one-to-one or it is not onto.

→ A one-to-one correspondence is called invertible because we can define an inverse of this funⁿ. A funⁿ is not invertible if it is not a one-to-one correspondence because the inverse of such a funⁿ does not exist.



Ex :- Let f be a funⁿ from $\{a, b, c\}$ to $\{1, 2, 3\}$ s.t. $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible & if it is, what is its inverse?

Solⁿ :- The funⁿ f is invertible because it is one-to-one correspondence. The inverse funⁿ f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, $f^{-1}(3) = b$.

Ex :- Let f be the funⁿ from R to R with $f(x) = x^2$. Is invertible

Sol^n:- Here, f is not invertible because $f(1) = 1 = f(-1) \Rightarrow 1 = -1$ is false; f is not one-to-one. If an inverse fun^n were defined, it would have to assign two elements to 1. Hence, f is not invertible. But, the fun^n $f(x) = x^2$ from the set of non-negative real no.s to the set of non-negative real no.s is one-to-one. If $f(x) = f(y)$, then $x^2 = y^2 \Rightarrow x = \pm y$ i.e. $x = y, x = -y$. Since, both x & y are non-negative, we must have $x = y$. So, this fun^n is one-to-one. Furthermore, $f(x) = x^2$ is onto when the co-domain is the set of all non-negative real no.s because each non-negative real no. has a square root. i.e. if y is a non-negative real no., $\exists$ a non-negative (≥ 0) real no. x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the fun^n $f(x) = x^2$ from the set of non-negative real no.s to the set of non-negative real no.s is one-to-one & onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$.

$\boxed{2.4}$ * **Relations** $\longrightarrow$ Rel^n from a set A to itself are of special interest. In other words, A rel^n on the set A is a rel^n from A to A means a rel^n on a set A is a subset of $A \times A$.

Ex:- Let $A = \{1, 2, 3, 4\}$ be the set. Which ordered pairs are in the rel^n $R = \{(a,b) \mid a \text{ divides } b\}$?

Sol^n:- Here, (a,b) is in R iff a & b are +ve int. not exceeding 4 s.t. a divides b.

Since, $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.
$= \{(a,b) \mid a \text{ divides } b\}$.

* **Properties of rel^{n}s** :-

There are several properties that are used to classify rel^{n}s on a set. We will introduce the most important properties here.

(1) **Reflexive rel^n** :- A rel^n on R on a set A is called reflexive

if $(a,a) \in R$ for every element $a \in A$.

Ex:- Let the rel^n on $\{1,2,3,4\}$ i.e. $R = \{1,2,3,4\}$.

Then $R_1 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

are reflexive rel^n because both contain all pairs of the form (a,a), namely $(1,1), (2,2), (3,3), (4,4)$.

Ex:- Is the "divides" rel^n on the set of +ve integers reflexive?

Sol^n:- Yes, divides rel^n on the set of +ve int. is reflexive rel^n.

Because, $a \mid a$ whenever 'a' is a +ve int., the "divides" rel^n is reflexive [if we replace the set of +ve int. with the set of all int.s then the rel^n is not reflexive because 0 does not divide 0].

(2) Symmetric rel^n :- A rel^n R on a set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ $\forall a,b \in A$. A rel^n R on a set A s.t. $\forall a,b \in A$ if $(a,b) \in R$ & $(b,a) \in R$, then $a = b$ is called antisymmetric.

→ Using quantifiers, we see that the rel^n R on the set A is symmetric if $\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)$.

→ Similarly, the rel^n R on the set A is antisymmetric if $\forall a \forall b (((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b))$.

→ i.e. a rel^n is symmetric iff a is related to b implies that b is related to a.

→ A rel^n is antisymmetric iff there are no pairs of distinct elements 'a' & 'b' with 'a' related to 'b' & 'b' related to 'a'.

→ The terms symmetric & antisymmetric are not opposites, because a rel^n can have both. ... of these properties or may lack both of them. A rel^n cannot be both symmetric & antisymmetric if it contains some pair of the form (a,b), where $a \neq b$.

Ex:- Let $R = \{1,2,3,4\}$.

$R_1 = \{(1,1),(1,2),(2,1)\}$

$R_2 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

$R_3 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

$R_4 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

Here, the rel^{n}s R_1 & R_2 are symmetric.

$R_5 = \{(3,4)\}$.

Here, the rel^{n}s R_3, R_4 & R_5 are antisymmetric. For each of these rel^{n}s there is no pair of elements a & b with $a \neq b$ s.t. both (a,b) & (b,a) belong to the rel^n. The reader should verify that none of the other rel^{n}s is antisymmetric.

Ex:- Is the "divides" rel^n on the set of +ve int.s symmetric & is it antisymmetric?

Sol^n:- This rel^n is not symmetric because $1|2$, but $2 \nmid 1$. It is antisymmetric, because for if a & b are +ve int.s with $a|b$ & $b|a$, then $a=b$.

(3) Transitive rel^n:- A rel^n R on a set A is called transitive if whenever $(a,b) \in R$ & $(b,c) \in R$, then $(a,c) \in R$ $\forall a,b,c \in A$.

→ Using quantifiers, ~~we~~ the rel^n R on a set A is transitive if we have $\forall a \forall b \forall c \left(((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R \right)$.

Ex:- Let $R = \{1,2,3,4\}$. Then, $R_1 = \{(3,4)\}$

$$R_2 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$

$$R_3 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$

are transitive.

Ex:- Is "divides" rel^n on the set of +ve int.s transitive?

Sol^n:- Let $a|b$ & $b|c$, then there are +ve int.s k & l s.t. $b = ak$ & $c = bl = (ak)l$.

$$\Rightarrow c = akl \Rightarrow a|c.$$

It follows that, this rel^n is transitive.

Def^n → Let R be a rel^n on the set A. The powers R^n, $n = 1,2,3,\ldots$ are defined recursively by $R^1 = R$ & $R^{n+1} = R^n \circ R$.

The def^n shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$ & so on.

Th^m-1 The rel^n R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3,\ldots$.

Sol^n:- Let $R^n \subseteq R$ for $n = 1,2,3,\ldots$.

We have to show that R is a transitive.

If $(a,b) \in R$ & $(b,c) \in R$, then by the def^n of composition $(a,c) \in R^2$ $[\because (a,b) \circ (b,c) \in R \circ R]$. Here, $R^2 \subseteq R$ this means that $(a,c) \in R$. Hence, R is transitive.

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Conversely, let R is transitive.
We have to show that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.
We have to prove it by mathematical induction.
For $n = 1$, $R^1 \subseteq R$ which is trivially true.
Assume that for $n = K$, $R^K \subseteq R$ is true.
We must show that for $n = K+1$, $R^{K+1} \subseteq R$ is true.
Assume that $(a,b) \in R^{K+1}$ because $R^{K+1} = R^K \circ R$.
There is an element x with $x \in A$ s.t. $(a,x) \in R$ & $(x,b) \in R^n$ for $n = K+1$.
Since, $R^K \subseteq R$ (inductive hypothesis) & R is transitive, then we obtained $(a,x) \in R$ & $(x,b) \in R$, it follows that $(a,b) \in R$. This shows that $R^{K+1} \subseteq R$ is true for $n = K+1$.
Hence, $R^n \subseteq R$ is true for $n = 1, 2, 3, \ldots$

2.5 Closures of Equivalence Rel^{n}s :-

* Equivalence Rel^{n}s $\longrightarrow$ A rel^n on a set A is called an equivalence rel^n if it is reflexive, symmetric & transitive.

→ The notation $a \sim b$ is denote that a & b are equivalent elements w.r.t. a particular equivalence rel^n.

* We can obtain closures of rel^{n}s with respect to (w.r.t) ~~to~~ property in the following way :-

① Reflexive Closure →
The reflexive closure of rel^n R on set A is $R \cup \Delta$. Where, $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal rel^n on set A.

② Symmetric Closure → Let R be a rel^n on set A & let R^{-1} be the inverse of R. The symmetric closure of rel^n R on set A is $R \cup R^{-1}$.

③ Transitive Closure → Let R be a rel^n on set A. The

Connectivity relⁿ is defined as $R^* = \bigcup_{n=1}^{\infty} R^n$. The transitive Closure of R is R^*.

Ex:- Let R be the relⁿ on set $\{1,2,3,4\}$ with

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$. Find the reflexive, symmetric & transitive closure of R ?

Solⁿ:- ① Reflexive Closure → Given that $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

Since, ~~$R^{-1} = \{(1,1), (4,1), (3,2), (1,3), (3,4)\}$~~ $\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

So, the reflexive closure of R is $R \cup \Delta$, where Δ is the diagonal relⁿ on $\{1,2,3,4\}$ $\Rightarrow R = R \cup \Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

$\cup \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

$\Rightarrow R = \{(1,1), (1,4), (2,2), (2,3), (3,1), (3,3), (3,4), (4,4)\}$ [$\because R \cup \Delta = \Delta \cup R$]

② Symmetric Closure → Given that $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

Since, $R^{-1} = \{(1,1), (4,1), (3,2), (1,3), (4,3)\}$.

So, the Symmetric Closure of R is $R \cup R^{-1}$

i.e. $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\} \cup \{(1,1), (4,1), (3,2), (1,3), (4,3)\}$

$= \{(1,1), (1,3), (1,4), (2,3), (3,1), (3,2), (3,4), (4,1), (4,3)\}$

③ Transitive Closure → Given that $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

Since, transitive closure of R is R^*

i.e. $R^* = R^{next} = \{(1,1),(1,4),(2,1),(2,3),(2,4),(3,1),(3,4)\}$.

Ex:- Congruence modulo m → Let m be a +ve int. with $m>1$. ~~Where~~ Show that the relⁿ $R = \{(a,b) \mid a \equiv b \pmod m\}$ is an equivalence relⁿ on the set of +ve int. ?

Solⁿ:- Given that $a \equiv b \pmod m \Rightarrow m \mid a-b$.

① Reflexive relⁿ → $a \equiv a \pmod m \Rightarrow m \mid a-a \Rightarrow m \mid 0$ is true because $0 = 0 \cdot m$. Hence $a \equiv a \pmod m$ is reflexive relⁿ.

② Symmetric relⁿ → Let $a \equiv b \pmod m \Rightarrow m \mid a-b \Rightarrow a-b = km$, where k is an integer. It follows that $b-a = -km$
$\Rightarrow b-a = (-k)m$ i.e. $m \mid b-a \Rightarrow b \equiv a \pmod m$.
Hence, the Congruence modulo m is Symmetric relⁿ.

③ Transitive relⁿ → Let $a \equiv b \pmod m$ & $b \equiv c \pmod m$. Then $m \mid a-b$ & $m \mid b-c \Rightarrow a-b = km$ & $b-c = lm$, where k & l are integers. Adding two eqⁿs, we obtained
$a-c = (a-b)+(b-c) = km + lm \Rightarrow (a-c) = (k+l)m$
$\Rightarrow m \mid a-c \Rightarrow a \equiv c \pmod m$.
Thus, Congruence modulo m is transitive relⁿ.
Hence, Congruence modulo m is an equivalence relⁿ.

2.6 Partial Ordering :-

* Defⁿ → A relⁿ R on a set S is called a partial ordering or partial order if it is reflexive, anti-symmetric & transitive.

→ A set S together with a partial ordering R is called a partially ordered set or poset & denoted by (S,R).

→ Members of S are called elements of the poset.

Ex:- Show that the "greater than or equal" relⁿ $(\geq)$ is a poset on the set of integers ?

Solⁿ:- ① Reflexive relⁿ → Here, $a \geq a$ for every int. a, $\geq$ is reflexive relⁿ.

② Anti-symmetric relⁿ → If $a \geq b$ & $b \geq a$, then $a = b$.

So, $\geq$ is antisymmetric relⁿ.

③ Transitive relⁿ → If $a \geq b$ & $b \geq c$, ~~then~~ imply that $a \geq c$.

It follows that $\geq$ is transitive relⁿ.

Hence, $\geq$ is poset on Z.

→ By similar process, the inclusion relⁿ $\subseteq$ is a poset on $P(S)$ (power set) i.e. $(P(S), \subseteq)$ is a poset.

* Defⁿ → The elements a & b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When a & b are elements of S s.t. neither $a \preceq b$ nor $b \preceq a$, a & b are called incomparable.

Note :- In different posets different symbols such as $\leq, \subseteq, |$ are used for a partial ordering.

→ The notation $a \preceq b$ is used to denote that $(a,b) \in R$ in an arbitrary poset (S, R). The symbol $\preceq$ is similar to $\leq$ symbol.

[The symbol $\preceq$ is used to denote the relⁿ in any poset, not just "$\leq$" relⁿ.]

Ex :- In the poset $(Z^+, |)$ are the int. 3 & 9 comparable? Are 5 & 7 comparable?

Solⁿ :- Here, the int. 3 & 9 are comparable because $3|9$.

And the int. 5 & 7 are incomparable because $5 \nmid 7$ & $7 \nmid 5$.

* Defⁿ → If $(S, \preceq)$ is a poset & every two elements of S are comparable, S is called a totally ordered or linearly ordered set & $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Ex:- The Poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a & b ~~int~~ are integers.

Ex:- The Poset $(Z^+, |)$ is not totally ordered, because it contains elements that are incomparable such as 5 & 7.

* Def^n → $(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering & every non-empty subset of S has a least element.

Ex:- The Poset $(Z^+, \leq)$ is well-ordered, where $\leq$ is the usual "less than or equal to" rel^n.

Th^m-1 (The Principle of well-ordered Induction)

Suppose that S is a well-ordered set. Then $P(x)$ is true $\forall x \in S$ if for every $y \in S$ & if $P(x)$ is true $\forall x \in S$ with $x \prec y$, then $P(y)$ is true (inductive step).

Pf:- We have to proof it by method of contradiction.

Let it is not case that $P(x)$ is true $\forall x \in S$. Then, there is an element $y \in S$ s.t. $P(y)$ is false. Consequently, the set $A = \{x \in S \mid P(x) \text{ is false}\}$ is non-empty.

Because S is well-ordered, A has a least element 'a'. By the choice of 'a' as a least element of A, we know that $P(x)$ is true $\forall x \in S$ with $x \prec a$. This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true $\forall x \in S$ with $x \prec y$.

* Hasse Diagram →

Hasse diagram is a type of Mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction. Such a diagram with labeled vertices, uniquely determines its partial order.

Ex:- Let $A = \{1, 2, 3, 4\}$.

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$ is a poset.

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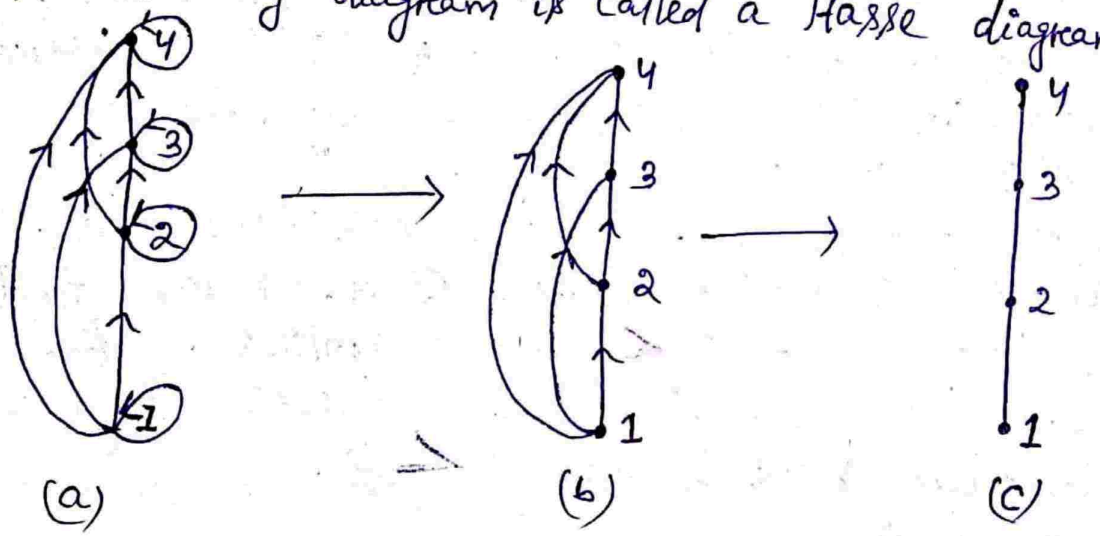
Procedure -

(a) Here, 1st start with the directed graph for this relⁿ. Because a partial ordering is reflexive, a loop is present at every vertex.

(b) Remove these loops.

Then, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. For instance, (1,3) & (3,4), (1,2) & (2,3) and (2,3) & (3,4) are in the partial ordering, then remove the edges (1,4), (1,3) & (2,4) respectively. (c) Finally, arrange each edge so that its initial vertex is below its terminal vertex. Remove all the arrows on the directed edges because all edges point "upward" toward their terminal vertex.

The resulting diagram is called a Hasse diagram.



Ex:- Draw the Hasse diagram representing the partial ordering:

(1) $\{(a,b) \mid a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$?

Sol:- Here, $R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12)\}$ is a Poset.

Rough- (1,2) & (2,4) → (1,4), (1,3) & (3,6) → (1,6), (1,2) & (2,8) → (1,8), 5-25

~~(1,3) & (3,12) → (1,12), (2,4) & (4,8) → (2,8), (2,6) & (6,12) → (2,12),~~

(3,6) & (6,12) → (3,12).

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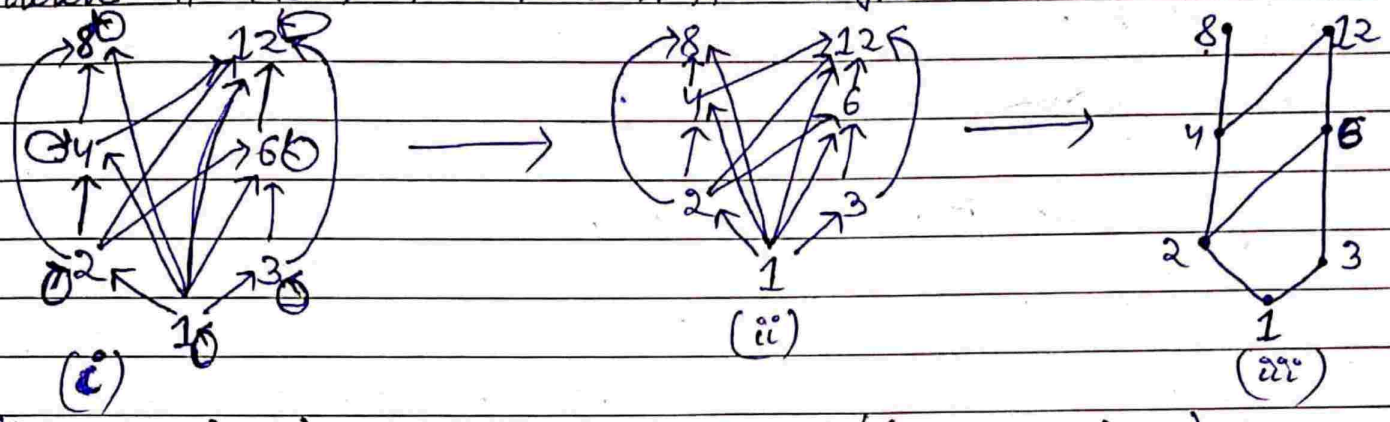
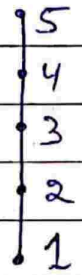
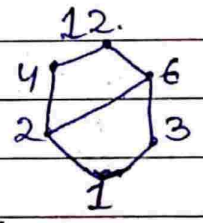
(i) Remove all loops. (ii) Then delete all the edges implied by the transitive property. These are (1,4), (1,6), (1,8), (1,12), (2,8), (2,12) & (3,12). (iii) Therefore, arrange all edges to point upward & delete all arrows to obtain Hasse diagram.



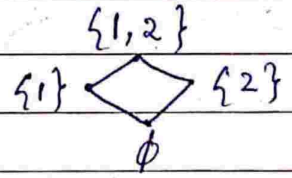
(2) ({1,2,3,4,5}, ≤) :



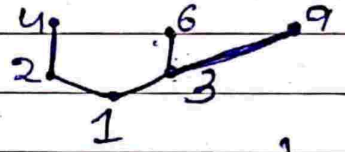
(3) ({1,2,3,4,6,12}, /) :



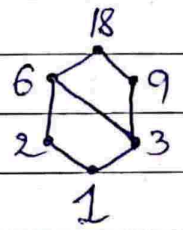
(4) ({ϕ, {1}, {2}, {1,2}}, ⊆) :



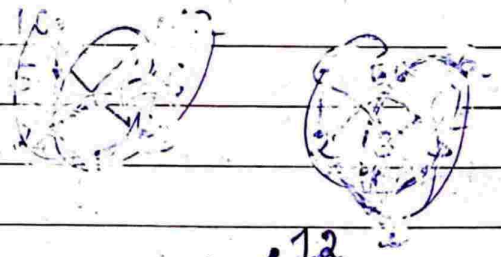
(5) ({1,2,3,4,6,9}, /) :



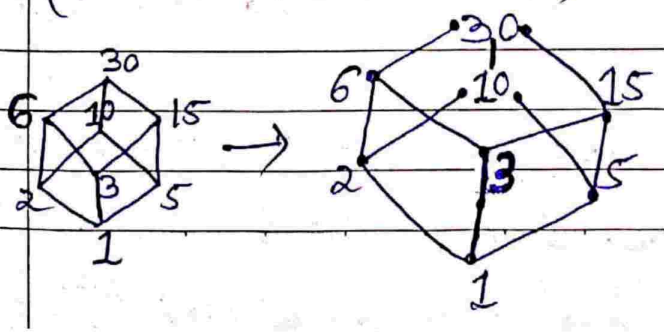
(6) ({1,2,3,6,9,18}, /) :



(7) ({1,2,3,5,6,10,12}, /) :



(8) ({1,2,3,5,6,10,15,30}, /) :



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* Def^n →

Maximal & minimal elements :-

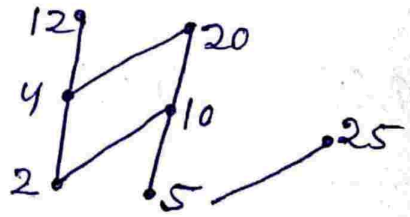
→ An element of a poset is called maximal if it is not less than any elements of the poset (it is not related any elements).
i.e. 'a' is maximal in the poset $(S, \preceq)$ if there is no $b \in S$ s.t. $a \prec b$.

→ An element of a poset is called minimal if it is not greater than any element of the poset. i.e. 'a' is minimal if there is no $b \in S$ s.t. $b \prec a$.

→ Maximal & minimal elements are the top & bottom diagram elements in the Hasse diagram.

Ex:- Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal & minimal ?

Sol^n:- Here, the maximal elements are 12, 20 & 25 and the minimal elements are 2 & 5.



Note :-

→ Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. i.e. 'a' is the greatest element of the poset $(S, \preceq)$ if $b \preceq a$, $\forall b \in S$. The greatest element is unique when it exists.

→ An element is called the least element if it is less than all the other elements in the poset. i.e. 'a' is the least element of $(S, \preceq)$ if $a \preceq b$, $\forall b \in S$. The least element is unique when it exists.

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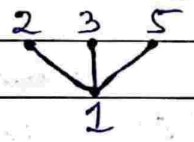
Ex:- Let S be a Set. Determine whether there is a greatest element & a least element in the Poset $(P(S), \subseteq)$?

Solⁿ:- The least element is empty set ϕ because $\phi \subseteq T$ for any subset T of S and the set S is the greatest element in this Poset because $T \subseteq S$ whenever T is a subset of S.

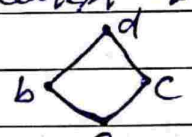
Ex:- Is there a greatest element & a least element in the Poset $(Z^+, |)$?

Solⁿ:- The integer 1 is the least element because $1 | n$ $(n = 1 \cdot n)$ whenever n is a +ve int.. Because there is no int. i.e. divisible by all +ve ints and there is no greatest element.



Here 1 is least element & 2,3,5 are greatest elements.

But d may be 2 or 3 or 4 or 5 or ---like this.



Here, 'a' is least element & d is greatest element.

* **Upper bound & lower bound :-**

→ Sometimes it is possible to find an element i.e. greater than or equal to all the elements in a subset A of a Poset $(S, \preceq)$. If u is an element of S s.t. $a \preceq u$; $\forall a \in A$, then 'u' is called an upper bound of A.

→ There may be an elements less that or equal to all the elements in A. If l is an element of S s.t. $l \preceq a$; $\forall a \in A$, then 'l' is called a lower bound of A.

* **Least upper bound (Lub) & Greatest lower bound (Glb) :-**

→ The element x is called the lub of the subset A if x is an upper bound i.e. less than every other upper bound of A. Because there is only one such element, if it exists it makes sense to call this element the lub.

i.e. x is the lub of A if $a \preceq x$ whenever $a \in A$ & $x \preceq z$ whenever z is upper bound of A.

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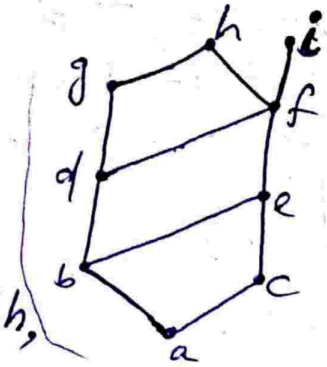
→ Similarly, the element y is called the glb of A if y is a lower bound of A is unique if it exists.

→ The glb & lub of a subset A are greatest lower bound & least upper bound respectively.

Ex:- Find the glb & lub of $\{b, d, g\}$ in the Poset ?

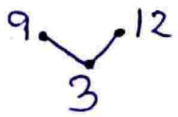


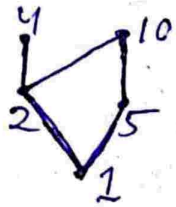
Sol^n:- The upper bounds are g & h and $g < h$, then g is the lub.

The lower bounds are a & b and $a < b$, then b is the glb.

Ex:- Find the glb & lub of the sets $\{3, 9, 12\}$ & $\{1, 2, 4, 5, 10\}$, if they exist in the Poset $(z^+, |)$?

Sol^n:- An integer is a lower bound of $\{3, 9, 12\}$ if $3, 9$ & 12 are divisible by this integer.





The glb of $\{3, 9, 12\}$ is 3 because $1|3 \Rightarrow g.c.d.(3, 9, 12) = 3$.

An int. is a lower bound of $\{1, 2, 4, 5, 10\}$ is 1.

Hence, 1 is the glb for $\{1, 2, 4, 5, 10\} \Rightarrow g.c.d(1, 2, 4, 5, 10) = 1$.

Again, an int. is an upper bound for $\{3, 9, 12\}$ iff it is divisible by $3, 9, 12$. The int.s with this property are those divisible by the $l.c.m(3, 9, 12) = 36$.

Hence, 36 is the lub of $\{3, 9, 12\}$.

An int. is an upper bound for $\{1, 2, 4, 5, 10\}$ iff it is divisible by $1, 2, 4, 5, 10$. The int.s with this property are those

divisible by the l.c.m $(1,2,4,5,10) = 20$. ~~& g.c.d(1,2,4,5,10) = 1~~.
Hence, 20 is the lub of $\{1,2,4,5,10\}$.

2.7 Lattices →

* **Def^n** → A partially ordered set in which every pair of elements has both a lub & a glb is called a lattice.

Ex:- Is the poset $(z^+, |)$, a lattice ?

Sol^n:- Let a & b be two +ve int.s. The lub & glb of these two int.s are the least common multiple (l.c.m) & greatest common divisor (g.c.d) of these int.s respectively. It follows that this poset is a lattice.

Ex:- Determine whether the posets $(\{1,2,3,4,5\}, |)$ & $(\{1,2,4,8,16\}, |)$ are lattice ?

Sol^n:- Here, 1st poset $(\{1,2,3,4,5\}, |)$ is not a lattice because 2 & 3 have no upper bounds [$\because$ lcm$(2,3) = 6$ is not present in this set].
Every two elements of the 2nd poset have both a lub & a glb. The lub of two elements in this poset $(\{1,2,4,8,16\}, |)$ is the largere of the elements [$\because$ lcm$(8,16) = 16$ is present in this set] & the glb of two elements in this poset is the smallere of the elements [$\because$ gcd$(1,2) = 1$ is also present in this set]. Hence, the 2nd poset is a lattice.

Ex:- Determine, whether $(P(S), \subseteq)$ is a lattice where S is a set?

Sol^n:- Let A & B be two subsets of S. The lub & the glb of A & B are $A \cup B$ & $A \cap B$ respectively.
Hence, $(P(S), \subseteq)$ is a lattice.

* **Topological Sorting** → A total ordering $\preceq$ is said to be compatible with the partial ordering R if $a \preceq b$ whenever aRb.

Constructing a compatible total ordering from a partial ordering is called topological sorting.

Lemma-1 Every finite non-empty poset $(S, \preceq)$ has at least one minimal element.

Pf.:- Choose the element a_0 of S. If a_0 is not minimal, then there is an element a_1 with $a_1 < a_0$.

If a_1 is not minimal, then there is an element a_2 with $a_2 < a_1$.

Continue in this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} < a_n$. Because, there are only a finite no. of elements in the poset. This process must end with a minimal element a_n.

Ex:- Find a compatible total ordering for the poset $(\{1,2,4,5,12,20\}, |)$?

Sol^n:- (a) The 1st step is to choose a minimal element. This must 1, because it is only minimal element.

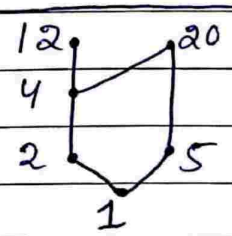
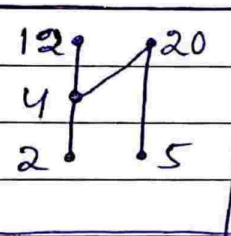
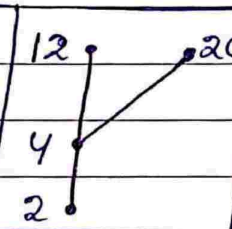
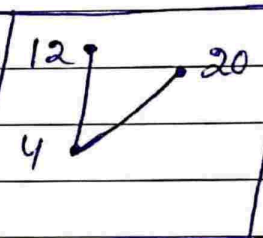
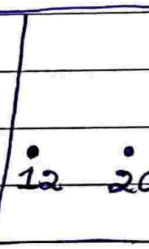
(b) Next, select a minimal element of $(\{2,4,5,12,20\}, |)$. There are two minimal elements in this poset namely 2 & 5. We select 5. (c) The remaining elements are $(\{2,4,12,20\}, |)$.

The only minimal element at this stage is 2.

(d) Next 4 is choosen because it is the only minimal element of $(\{4, 12, 20\}, |)$. Because both 12 & 20 are minimal elements of $(\{12, 20\}, |)$. Either can be chosen next. We select 20.

(e) Then which leaves 12 as the last element left. This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12 .$$

					
Minimal element chosen 1	5	2	4	20	12

[A topological sort of $(\{1, 2, 4, 5, 12, 20\}, |)$]

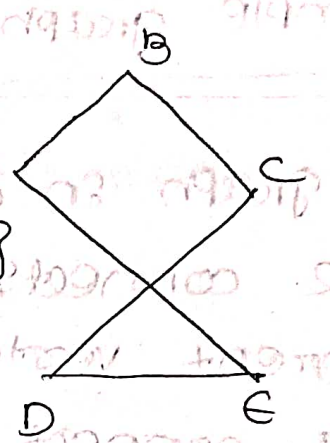
Unit-3

Graph Theory

Graph theory is the study of graphs which are mathematical structures used to model pair wise relation between objects.

defn of Graph

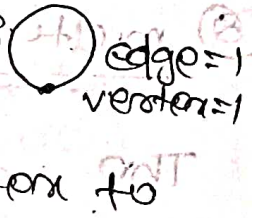
A Graph G(V, E) consist of 'V', the set of all vertices. or nodes and 'E' is the set of all ~~nodes~~ edges.



vertices = {A, B, C, D, E}

edges = {AB, BC, AE, CD, DE}

each edge has either one or two vertices associated with edge.



Loop

Loop is an edge that connects a vertex to itself.

based on no. of vertex, the graph is divided into 2 types.

1. finite graph
2. infinite graph.

1. finite graph

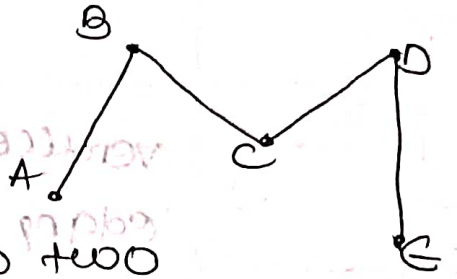
The graph with finite no. of vertex is called a finite graph.

② The graph with infinite no. of vertex is called infine graph.

Based on no. of edges the graph is divided into 3 types.

① simple graph
② multi "
③ pseudo "

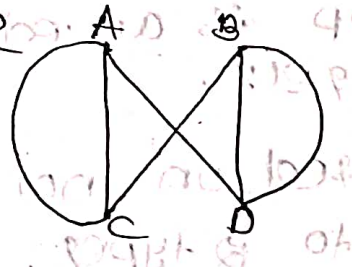
① simple graph



A graph in which each edge connects with different vertices and no two edges connect the same pair of vertices is called a simple graph.

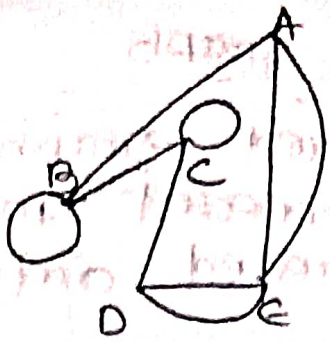
② multi graph

The graph that has multiple edges connecting the same pair of vertices is called multi graph.

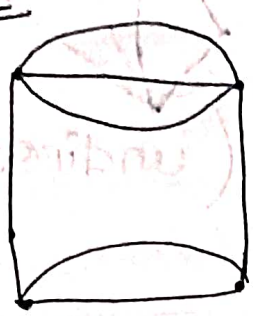


③ pseudo graph

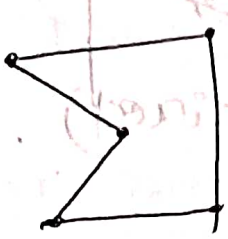
Graph includes loops and multiple edges connecting the same pair of vertices is called pseudo graph.



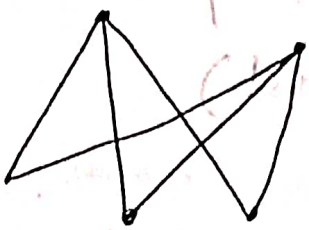
Ex:-



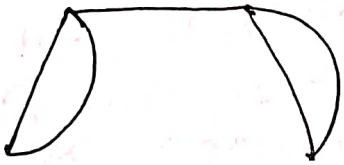
(multi)



(simple)

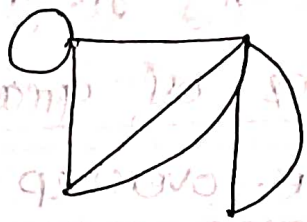


(simple)



(multi)

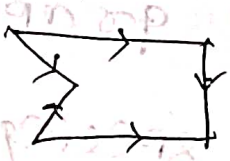
(pseudo)



(~~multi~~ pseudo)

Directed graph

A directed graph $G(V, E)$ consists of a non-empty set of vertices 'V' and a set of directed edges 'E'.

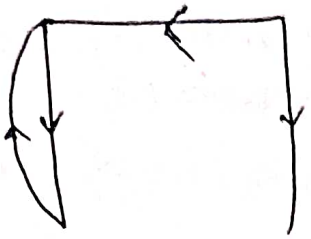


Un-directed graph

An un-directed graph is a set of 'V' & 'E' that are connected together where all edges are bidirectional.

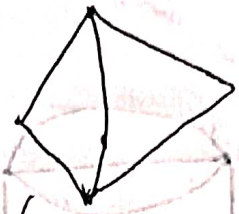
mixed graph

A mixed graph is a set of 'v', 'E' that are connected together where some edges are directed and some edges are undirected.



(directed)

(mixed)



(undirected)

Simple - directed

The directed graphs with no loops & multiple edges are called simple directed graph.

Examples of graph

Niche-overlap graph in ecology

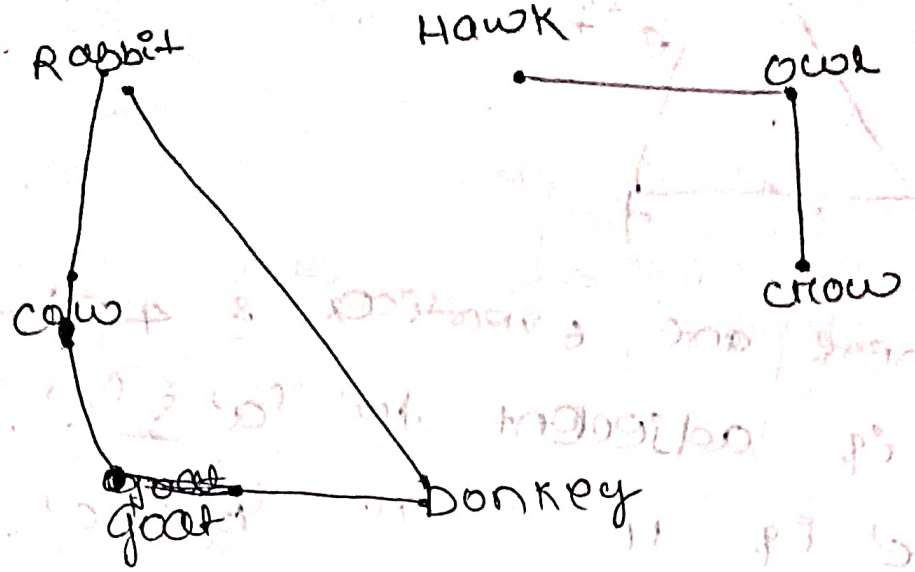
This graph ~~are~~ is used in many mene-models involving interaction of different species of animals.

Ex:- The competetion between species in an ecosystem can be modelling using a Niche-overlap graph.

→ ~~This~~ Each species represents vertex.

→ It is a un-directed and also a simple graph.

ex:-

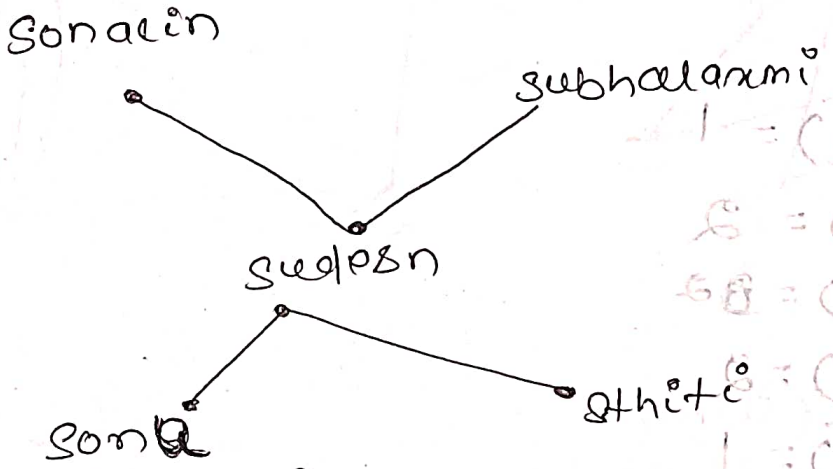


Acquaintanceship graph (simple undirected)

This graph used to represent relationship between people.

ex:- whether two people know each other or not.

→ The individuals are represented by vertices and an undirected edge means the people know each other.

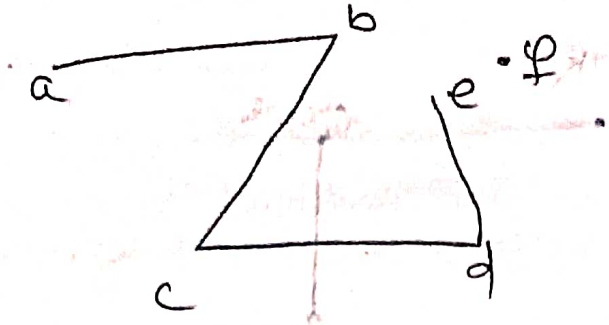


(Graph terminology)

Adjacent vertices

2 vertices 'A' & 'B' are said to be adjacent, if there is an edge between them.

Ex:-



Here, there are 6 vertices & 4 edges.

→ vertex 'b' is adjecent to 'a' & 'c'.

→ vertex 'c' is " " 'b' & 'c'.

→ " 'd' is " " 'c' & 'e'.

But, here, 'a' & 'c' are not adjecent.

Degree of a vertex

The degree of a vertex in a graph 'g', is written as deg( ). is equal to the no. of edges which incident on the vertex.

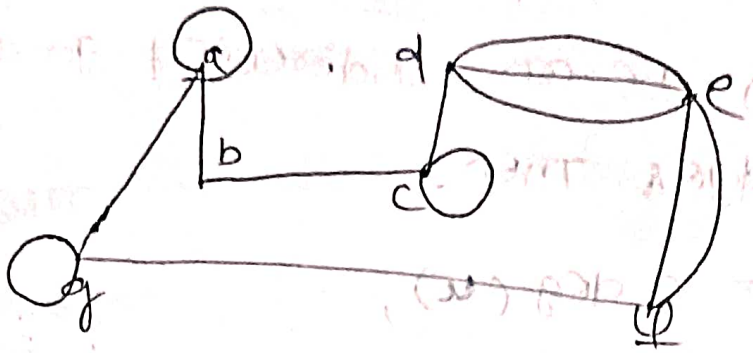
deg(a) = 1
deg(b) = 2
deg(c) = 2
deg(d) = 2
deg(e) = 1
deg(f) = 0.

→ A LOOP at a vertex contributes twice to the degree of vertex

→ A vertex of degree zero is called **isolated**.
and vertex of degree 1 is called **pendant**.

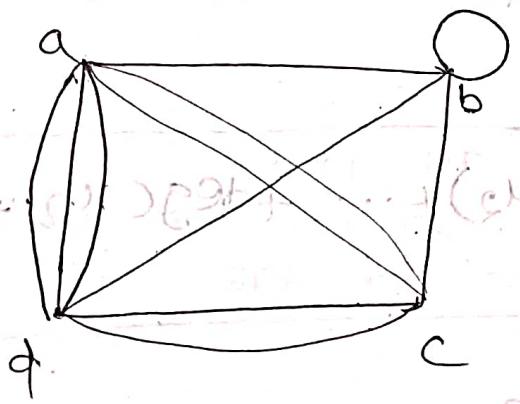
Ex:-

(1)

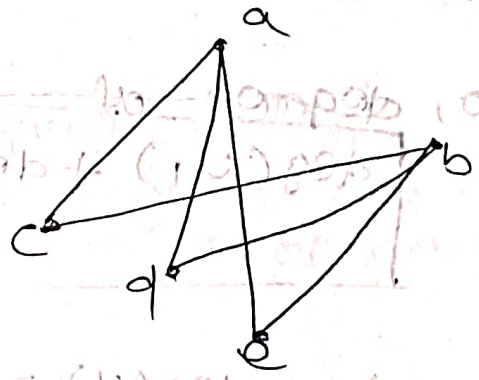


deg(a) = 4
deg(b) = 2
deg(c) = 4
deg(d) = 4
deg(e) = 5
deg(f) = 3
deg(g) = 4

(2)



deg(a) = 6
deg(b) = 5
deg(c) = 5
deg(d) = 6



deg(a) = 3
deg(b) = 3
deg(c) = 2
deg(d) = 2
deg(e) = 2

The Handshaking Theorem

Statement

Let, $G(V,E)$ be an undirected graph with 'e' edges. Then,

$$2e = \sum_{u \in V} \deg(u),$$

twice of edge = sum of degrees of each vertex.

Let 'G' be a graph with 'e' edges and 'N' vertices. Let $u_1, u_2, \ldots u_n$ are 'N' vertices.

Since, one edge contributes two degree as each edge has two end points.

So, ~~degree of u_1~~

$$\boxed{\deg(u_1) + \deg(u_2) + \cdots + \deg(u_n) = 2 \times \text{no. of edges}}$$

$$\Rightarrow \sum_{u \in V} \deg(u) = 2e.$$

Proved ($\square$).

Note :-

This is known as hand-shaking theorm, because, the no. of hands shaken in a path is always even.

Q:- How many edges are there in a graph with 10 vertices each of degree 6.

Ans:-

$$60 = 2 \times e$$
$$\Rightarrow e = 30.$$

The sum of degrees of vertices is

$$6 \times 10 = 60.$$

So, by hand-shaking theorm,

$$2e = 60.$$
$$\Rightarrow e = 30.$$

So, no. of edges = 30.

Theorm-2

An undirected graph has an even no. of vertices of odd degree.

Proof

Let $G = (V, E)$ is an undirected graph. Let, V_1 be the set of even vertices and V_2 be the set of odd vertices of graph G.

from Hand-shaking theorm,

$$2E = \sum_{u \in V} \deg(u).$$

$$2E = \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u). \quad (1)$$

Because, degree of 'u' is even in 'V_1'. So, the sum,

$\sum_{u \in V_1} \deg(u)$ is even as L.H.S of the eq^n (1)

is even. So, the sum $\sum_{u \in V_2} \deg(u)$ must be even.

Because, all terms in the sum $\sum_{u \in V_2} \deg(u)$ is odd. So, there must be an even no. of such terms. Thus, there are an even no. of vertices of odd degree.

Q:- Draw a graph with following data.

$\deg(v_1) = 4$
$\deg(v_2) = 3$
$\deg(v_3) = 5$
$\deg(v_4) = 3$
$\deg(v_5) = 2$.

graph cannot form.

Degree of vertices in a directed graph

In a directed graph, the edges are represented by, (u, v). where, 'u' is the initial vertex and 'v' is the terminal/end vertex.

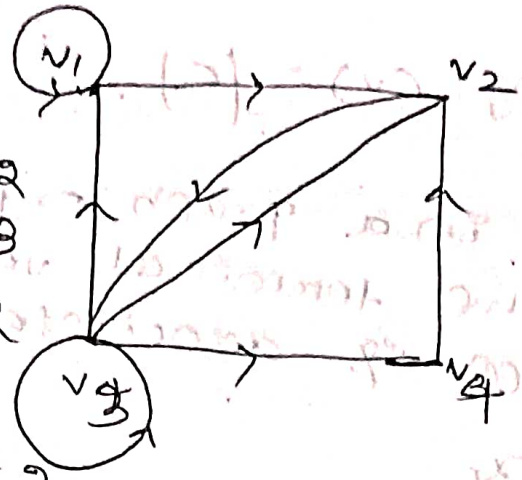
→ In a directed edge (u, v), we said 'u' is adjacent to 'v' and 'v' is adjacent from 'u'.

→ In a directed graph, indegree is denoted by $\deg^-(u)$ and defined as, the no. of edges with 'u' as terminal vertex

→ The outdegree is denoted by $\deg^+(u)$ and is defined as the no. of edge with 'u' as initial vertex.

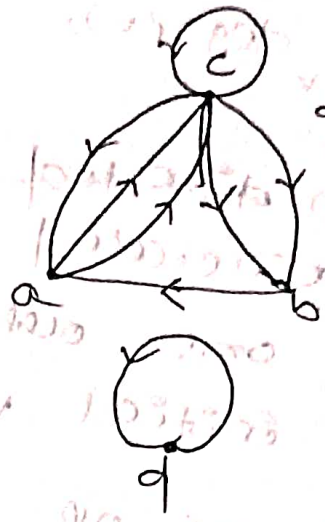
→ In Loop, the initial vertex and the terminal vertex are same. So, loop contributes both 1 to indegree and outdegree.

Ex:-

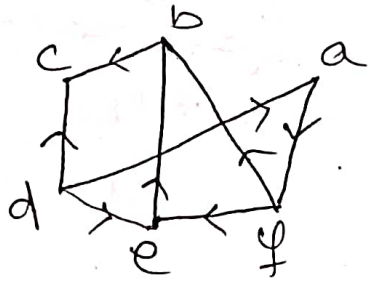


$deg^-(v_1) = 2$
$deg^-(v_2) = 3$
$deg^-(v_3) = 2$
$deg^-(v_4) = 1$

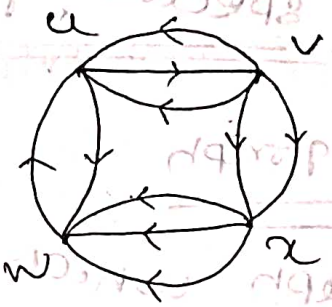
$deg^+(v_1) = 2$
$deg^+(v_2) = 1$
$deg^+(v_3) = 4$
$deg^+(v_4) = 1$



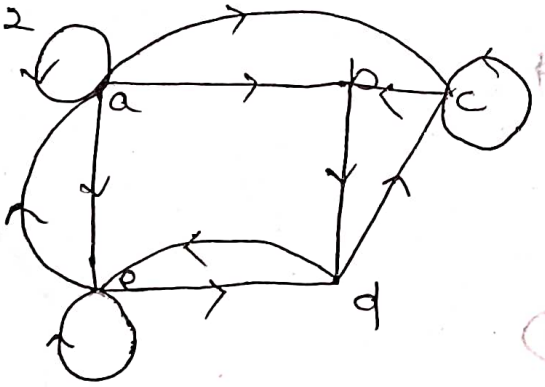
$deg^-(a) = 2$
$deg^-(c) = 3$
$deg^-(b) = 2$
$deg^-(d) = 1$
$deg^+(a) = 2$
$deg^+(b) = 1$
$deg^+(c) = 4$
$deg^+(d) = 1$



$deg^-(a) = 1$
$deg^-(b) = 2$
$deg^-(c) = 2$
$deg^-(d) = 0$
$deg^-(e) = 2$
$deg^-(f) = 1$
$deg^+(a) = 1$
$deg^+(b) = 1$
$deg^+(c) = 0$
$deg^+(d) = 3$
$deg^+(e) = 1$
$deg^+(f) = 2$



$deg^-(u) = 3$
$deg^-(v) = 1$
$deg^-(w) = 4$
$deg^-(x) = 2$
$deg^+(u) = 2$
$deg^+(v) = 4$
$deg^+(w) = 1$
$deg^+(x) = 3$.



$deg^-(a) = 2$
$deg^-(b) = 2$
$deg^-(c) = 3$
$deg^-(d) = 2$
$deg^-(e) = 3$
$deg^+(a) = 4$
$deg^+(b) = 1$
$deg^+(c) = 2$
$deg^+(d) = 2$
$deg^+(e) = 3$

Theorem - 3

Let $G = (V, E)$ is a directed graph. Then,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

For an ~~directed~~ edge in a graph, 1 indegree is associated at the terminal vertex and one outdegree is associated at the initial vertex

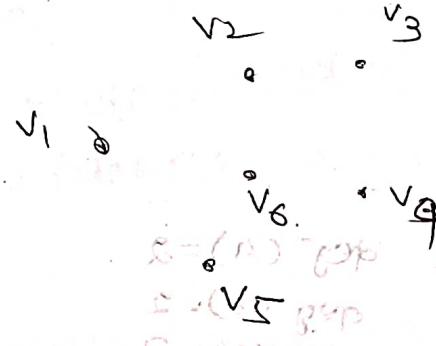
Total no. of indegree = Total no. of outdegree = no. of edges.

Some special simple graph

NULL graph

A graph which contains isolated vertices is called a NULL graph.

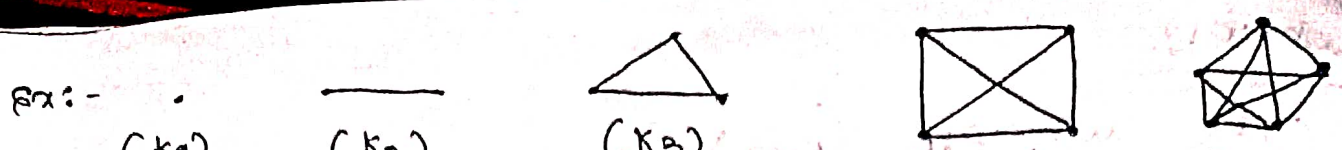
Ex:-



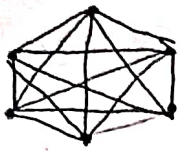
complete graph (K_n)

The complete graph (K_n) is a simple graph with 'n' vertices in which there is exactly one edge between each pair of distinct vertices.

Ex:-



(K_1) (K_2) (K_3) (K_4) (K_5)



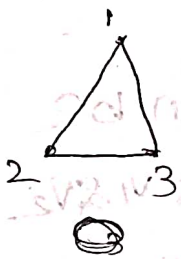
(K_6)

degree of each vertex in K_n is $n-1$.

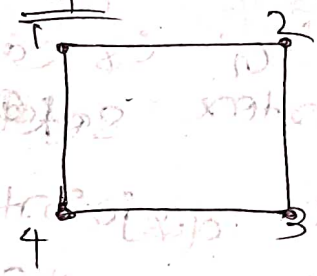
Cycle (C_N):-

The cycle ($C_N \geq 3$) consists of 'N' vertices $1, 2, \ldots, N$. and edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}$.

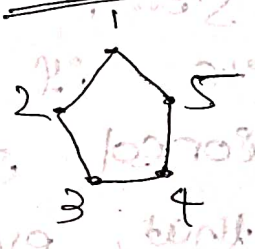
C_3



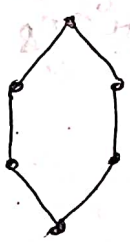
C_4



C_5



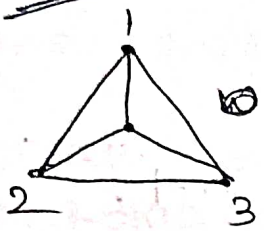
C_6



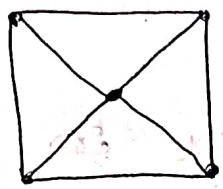
Wheels

The wheels (W_n) is obtained when we add an additional vertex to cycle (C_N, for $N \geq 3$) and connect this new vertex to each of 'N' vertices of C_N by new edges.

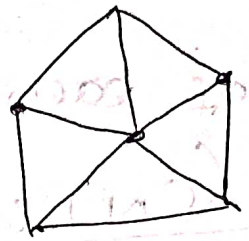
W_3



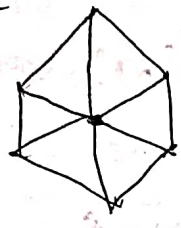
W_4



W_5



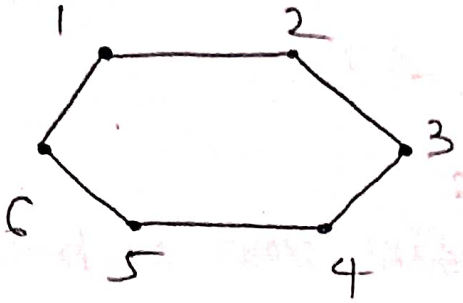
W_6



Bipartite graph

A simple graph 'G' is called bipartite if its vertex set 'V' can be partitioned into two disjoint sets V_1 & V_2 such that, every edge in the graph connects a vertex of 'V_1' to a vertex of 'V_2' so that, no two edge in 'G' connects either two vertices in 'V_1' or two vertices in 'V_2'.

Ex:- C_6 is a bipartite graph.



$V = \{1, 2, 3, 4, 5, 6\}$

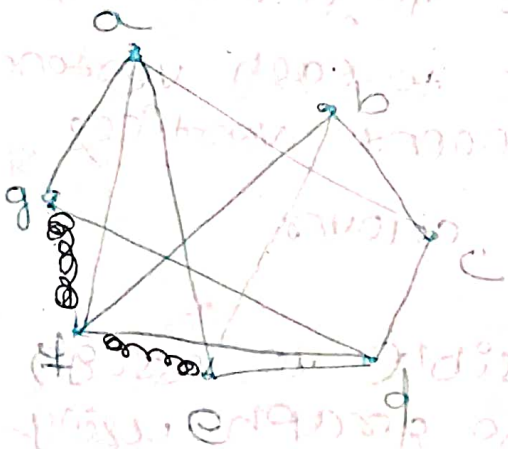
$V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$

Every edges of 'C_6' connects a vertex in 'V_1' and a vertex in 'V_2' and there is no edges between two vertices of 'V_1' and two vertices of 'V_2'

"K_3" is not bipartite graph. (K_3)

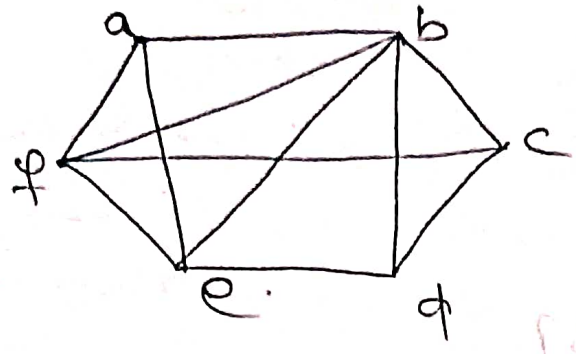
because, if we divide the vertex of "K_3" into two disjoint sets, then one of the set must contain two vertices which are connected by an edge.

which of the graph is bipartite



$V = \{a, b, c, d, e, f, g\}$

$V_1 = \{a, b, d\}$, $V_2 = \{c, e, f, g\}$



Theorm-4

A simple graph is bipartite iff it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices are assign the same colour.

Suppose $G=(V,E)$ is a bipartite graph. then, $V=V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in 'E' connects a vertex in 'V_1' and a vertex in 'V_2'.

If we assign one colour to each vertex in 'V_1' and second colour to each vertex in 'V_2', then no two adjacent vertices are assigned the same colour.

Conversely, Let, it is possible to assign colours to vertices of the graph using just two colours, so that, no two adjacent vertices are assign the same colour.

Let V_1 be the set of vertices assigned one colour and 'V_2' be the set of vertices assigned the 2nd colour. then $V_1 \cap V_2 = \phi$.

$\Rightarrow$ V_1 and V_2 are disjoint. and $V = V_1 \cup V_2$

Again, As, there is no two adjacent vertices, at either both in 'V_1' or both in 'V_2'

So, every edge connects a vertex in 'V_1' and a vertex in 'V_2'.

Hence, 'G' is bipartite.

Complete bipartite graph

It is denoted by $K_{m,n}$. The complete bipartite graph $K_{m,n}$ is the graph that has its vertex partitioned into two subsets with 'm' and 'n' vertices respectively.

There is an edge between two vertices iff one vertex is in 1st subset and other vertex in the 2nd subset.

$K_{2,3}$

$K_{3,3}$

$K_{2,6}$

Draw the following graphs.

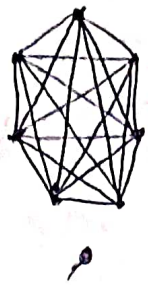
$K_{3,5}$, $K_{1,8}$, $K_{4,4}$, K_7, C_7, W_7.

$K_{3,5}$

$K_{1,8}$

$K_{4,4}$

K_7



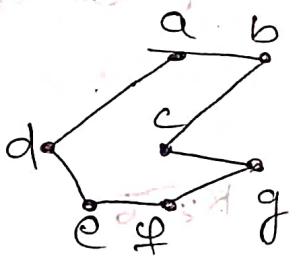
C_7

W_7

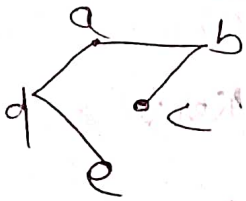
Sub-graph

A sub-graph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subset V$ and $F \subset E$.

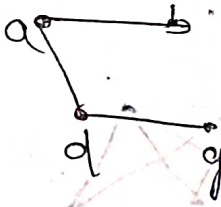
Ex:-



$G=(V,E)$



$H=(W,F)$



$H_1=(W_1,F_1)$

$V=\{a,b,c,d,e,f,g\}$

$E=\{ab, bc, cg, fg, ef, de, ad\}$

$W=\{a,b,d,e,g\}$

$F=\{ab, bc, de, ad\}$

$W_1=\{a,b,d,g\}$

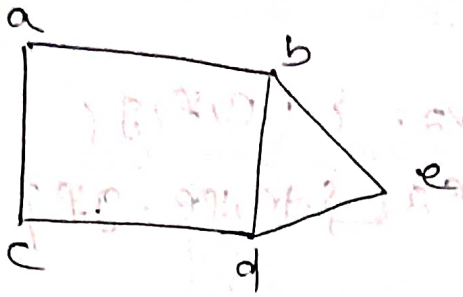
$F_1=\{ab, ad, dg\}$

Here, H is the subgraph of 'G' but H_1 is not.

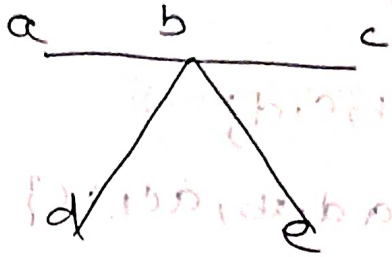
GRAPH UNION

Let, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the graph union is denoted by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Ex:-



G_1



G_2

$V_1 = \{a, b, c, d, e\}$

$E_1 = \{ab, ac, cd, bd, be, de\}$

$V_2 = \{a, b, c, d, e\}$

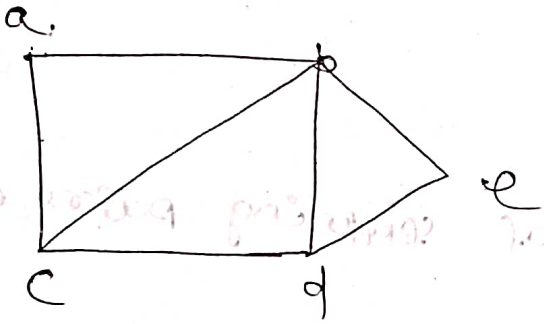
$E_2 = \{ab, bc, bd, be\}$.

on $G_1 \cup G_2$

$V_1 \cup V_2$

~~$G_1 \cup G_2$~~ $= \{a, b, c, d, e\}$.

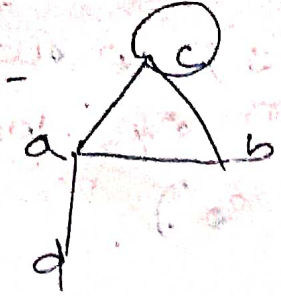
$E_1 \cup E_2 = \{ab, ac, cd, bd, be, de, bc\}$



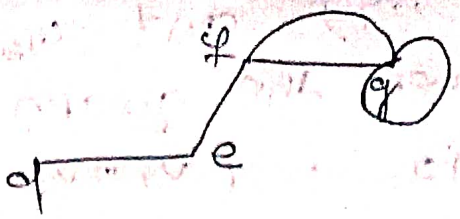
$G_1 \cup G_2$

Ex:-

Ex:-



G1



G2.

$V_1 = \{a, b, c, d\}$

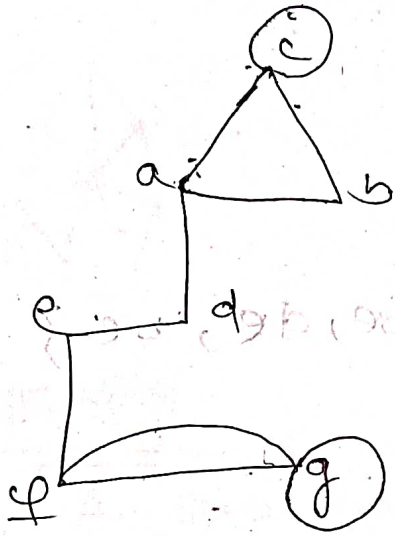
$E_1 = \{ad, ab, ac, bc\}$

$V_2 = \{d, e, f, g\}$

$E_2 = \{de, fe, gf\}$

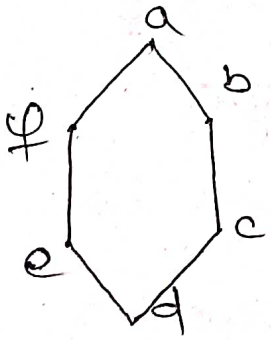
$V_1 \cup V_2 = \{a, b, c, d, e, f, g\}$

$E_1 \cup E_2 = \{ab, ad, ac, bc, de, fe, gf\}$

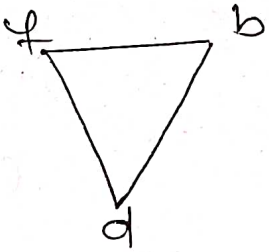


Q:- find the union of following pairs of simple graph.

(1)

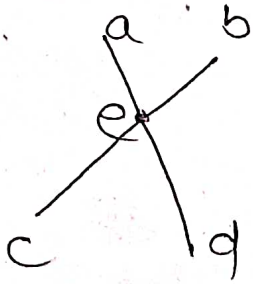


G1

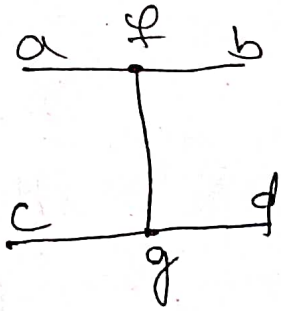


G2

(2)

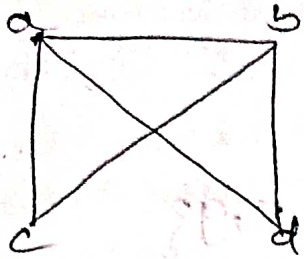


G1

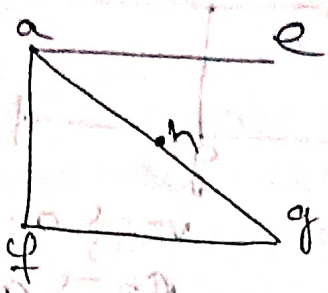


G2.

③

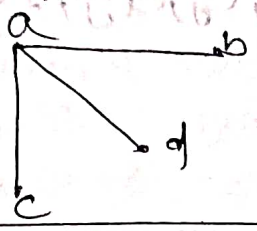


G_1



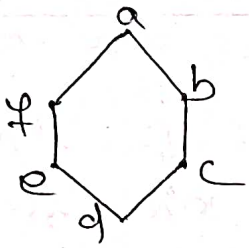
G_2

④ Draw the subgraph of following graph.



Ans:-

~~Rep~~ ①



G_1

G_2

$V_1 = \{a, b, c, d, e, f\}$

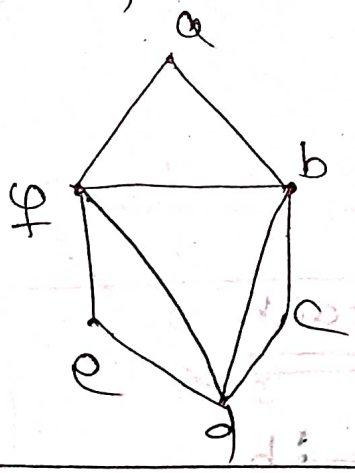
$E_1 = \{ab, bc, cd, de, ef, af\}$.

$V_2 = \{d, b, f\}$

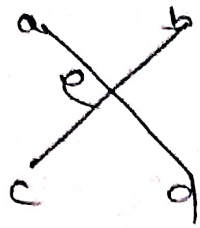
$E_2 = \{df, bf, bd\}$

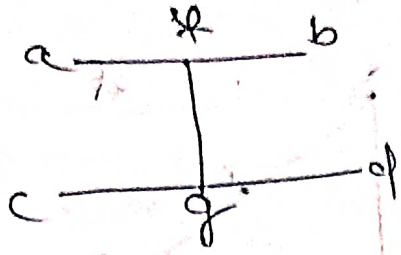
$V_1 \cup V_2 = \{a, b, c, d, e, f\}$

$E_1 \cup E_2 = \{ab, bc, cd, de, ef, af, df, bf, bd\}$.



(2)





$V_1 = \{a, b, c, d, e\}$

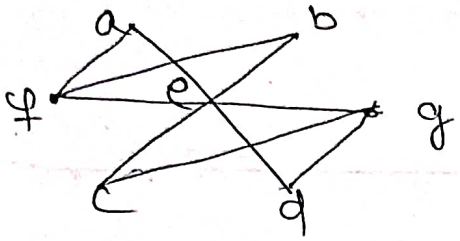
$E_1 = \{ae, be, ce, de\}$

$V_2 = \{a, b, c, d, f, g\}$

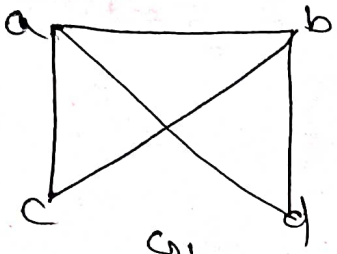
$E_2 = \{af, bf, cg, dg, fg\}$

$V_1 \cup V_2 = \{a, b, c, d, e, f, g\}$

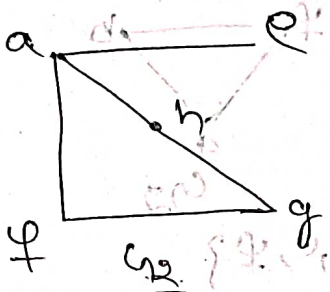
$E_1 \cup E_2 = \{ae, be, ce, de, af, bf, cg, dg, fg\}$



(3)



G_1



G_2

$V_1 = \{a, b, c, d\}$

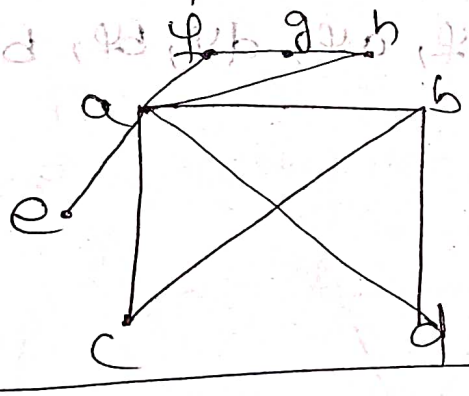
$E_1 = \{ab, bd, bc, ad, ac\}$

$V_2 = \{a, e, h, g, f\}$

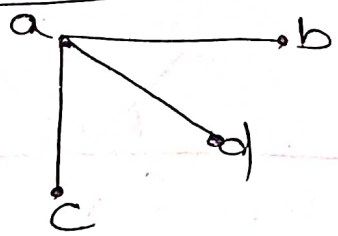
$E_2 = \{ae, ah, gh, fg, af$

$V_1 \cup V_2 = \{a, b, c, d, e, f, g, h\}$

$E_1 \cup E_2 = \{ab, bd, bc, ad, ac, ae, ah, gh, fg, af\}$



(4)



subgraph

Representation of graphs

Adjacency matrix

Let, $G = (V, E)$ is a simple graph and $v_1, v_2 \dots v_n$ be its n vertices. Then its adjacency matrix of this graph is an $n \times n$ matrix of the graph.

$$A = (a_{ij})_{n \times n}, \text{ where } a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

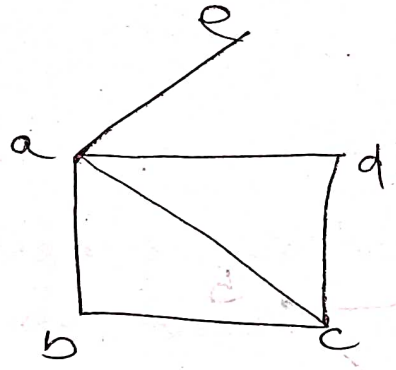
→ In a graph theory, an adjacency matrix is a square matrix used to represent a finite graph.

→ The elements of the matrix indicates whether the pairs of vertices are adjacent or not in the graph.

→ In simple graph, the adjacency matrix is a (0, 1) matrix, with zeros on its diagonal.

→ In a undirected graph, the adjacency matrix is a symmetric matrix.

ex:-



vertex	adjacent
a	b, c, d, e
b	a, c
c	a, b, d
d	a, c
e	a

matrix:

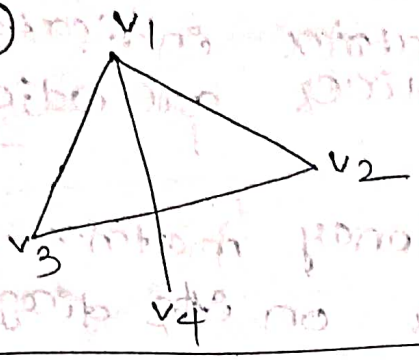
	a	b	c	d	e
a	0	1	1	1	1
b	1	0	1	0	0
c	1	1	0	1	0
d	1	0	1	0	0
e	1	0	0	0	0

Hence, the adjacency matrix is,

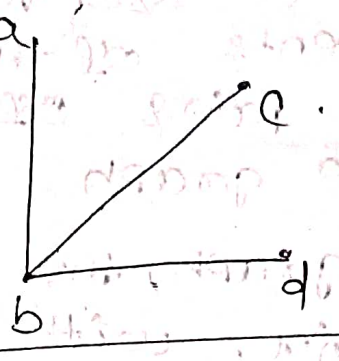
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q:-

(a)



(b)

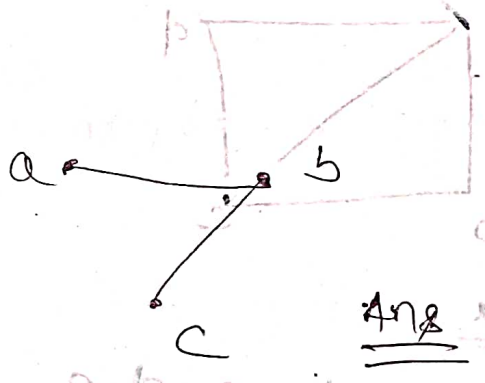


Q:- Draw the graph with the given adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Ans:-

	a	b	c
a	0	1	0
b	1	0	1
c	0	1	0



Ans

(a) Ans:-

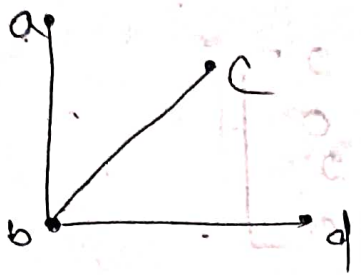
vertex	adjacent
v_1	v_2, v_3, v_4
v_2	v_3, v_1
v_3	v_1, v_2
v_4	v_1

	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	0
v_3	1	1	0	0
v_4	1	0	0	0

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ans

(b)



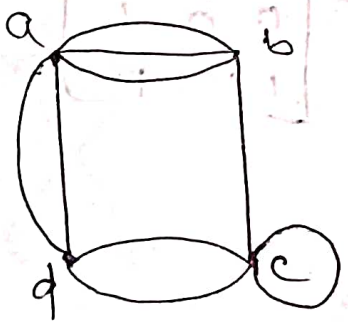
vertex	adjacent
a	b
b	a, c, d
c	b
d	b

	a	b	c	d
a	0	1	0	0
b	1	0	1	1
c	0	1	0	0
d	0	1	0	0

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Ans.

Q:- using adjacency matrix to represent the pseudo graph shown in the figure.

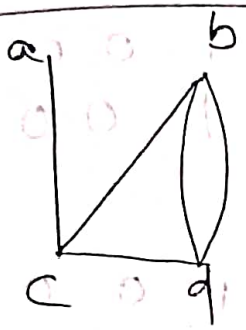


vertex	Adjacent
a	b, d
b	a, c
c	b, d, c
d	a, c

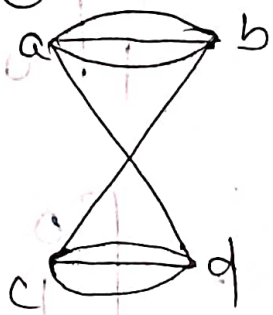
	a	b	c	d
a	0	3	0	2
b	3	0	1	0
c	0	1	1	2
d	2	0	2	0

So, Adjacency matrix is, $\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$.

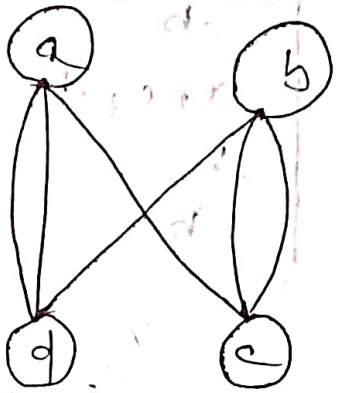
Q:- ①



②



③



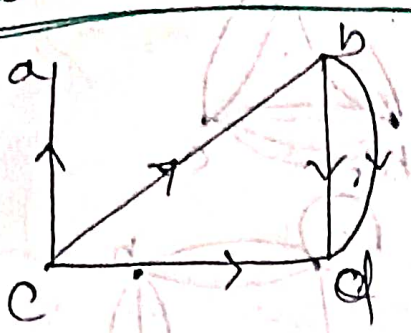
find the adjacency matrix of the given graphs.

Q:-2:- Construct the ~~adjacency matrix from the~~ graph following matrix

④ $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ ⑤ $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ ⑥ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

~~④ $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$~~

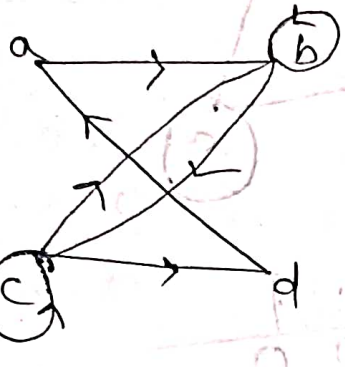
Adjacency of directed graph



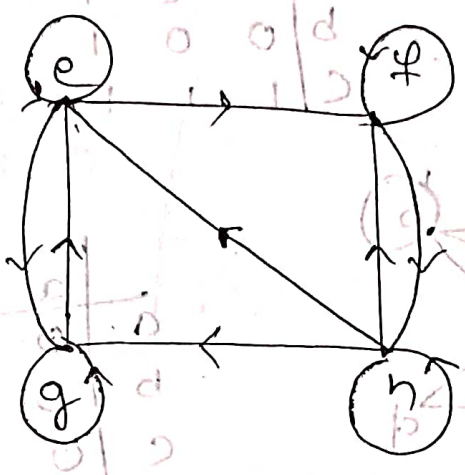
	a	b	c	d
a	0	0	0	0
b	0	0	0	2
c	1	1	0	1
d	0	0	0	0

matrix = $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q) (7)

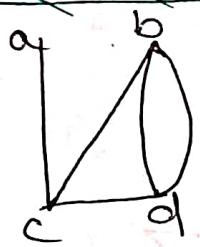


(8)



Incident matrix

(1)

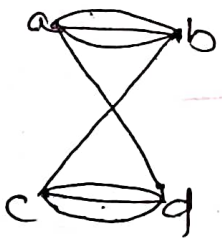


	a	b	c	d
a	0	0	1	0
b	0	0	1	2
c	1	1	0	1
d	0	2	1	0

matrix

$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

(2)

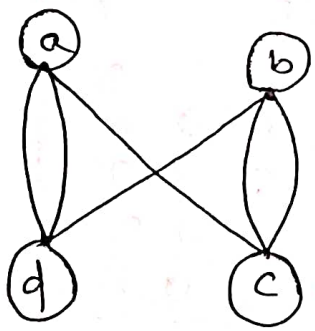


	a	b	c	d
a	0	3	0	1
b	3	0	1	0
c	0	1	0	3
d	1	0	3	0

matrix

$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$

(3)



	a	b	c	d
a	1	0	1	2
b	0	1	2	1
c	1	2	1	0
d	2	1	0	1

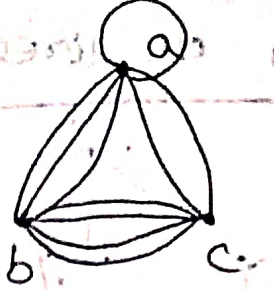
matrix

$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

(4)

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$$

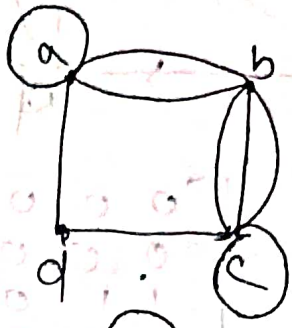
	a	b	c
a	1	3	2
b	3	0	4
c	2	4	0



(5)

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

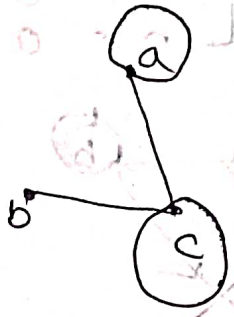
	a	b	c	d
a	1	2	0	1
b	2	0	3	0
c	0	3	1	1
d	1	0	1	0



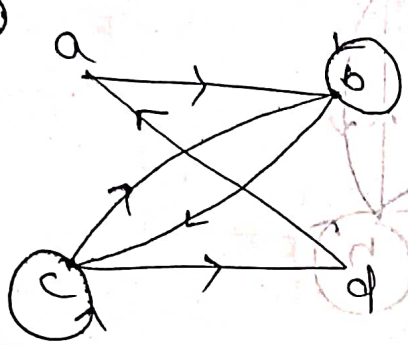
(6)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

	a	b	c
a	1	0	1
b	0	0	1
c	1	1	1



(7)

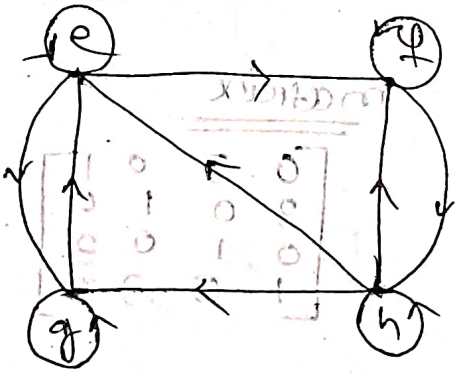


	a	b	c	d
a	0	1	0	0
b	0	1	1	0
c	0	1	1	1
d	1	0	0	0

matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(8)

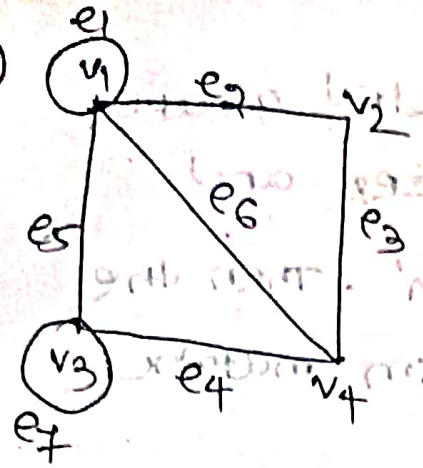


	e	f	g	h
e	1	1	1	0
f	0	1	0	1
g	1	0	1	0
h	1	1	1	1

matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(9)

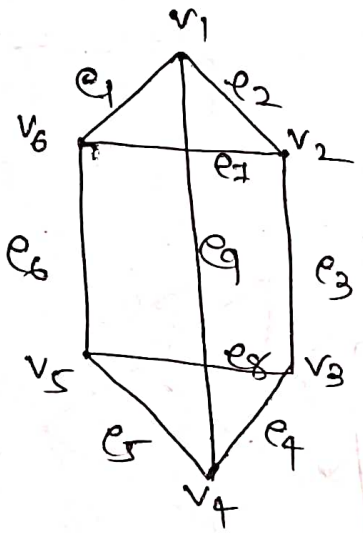


	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	0	0	1	1	0
v_2	0	1	1	0	0	0	0
v_3	0	0	0	1	1	0	1
v_4	0	0	0	1	0	1	0

matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

(10)



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
v_1	1	1	0	0	0	0	0	0	1
v_2	0	1	1	0	0	0	1	0	0
v_3	0	0	1	1	0	0	0	1	0
v_4	0	0	0	1	1	0	0	0	1
v_5	0	0	0	0	1	1	0	1	0
v_6	1	0	0	0	0	1	1	0	0

matrix

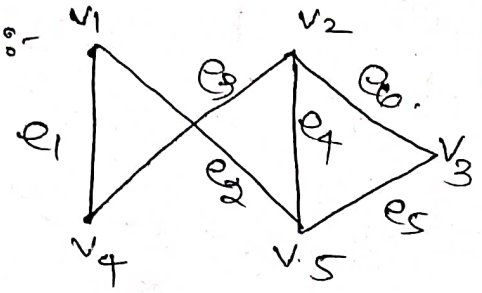
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Incident matrix

Let $G = (V, E)$ is an undirected graph. Let $v_1, v_2 \dots v_n$ are 'n' vertices and $e_1, e_2, \dots, e_m$ are edges of 'G'. Then the incident matrix is an ~~mxn~~ $n \times m$ matrix ~~with~~ $M = [m_{ij}]_{n \times m}$ with

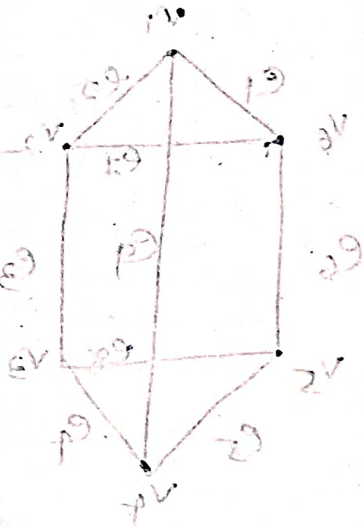
$$m_{ij} = \begin{cases} 1, & \text{when } e_j \text{ is incident on } v_i. \\ 0, & \text{otherwise.} \end{cases}$$

Ex:-

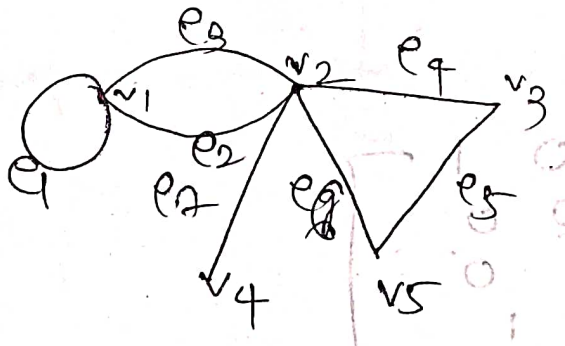


	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

matrix =
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$



Ex:-

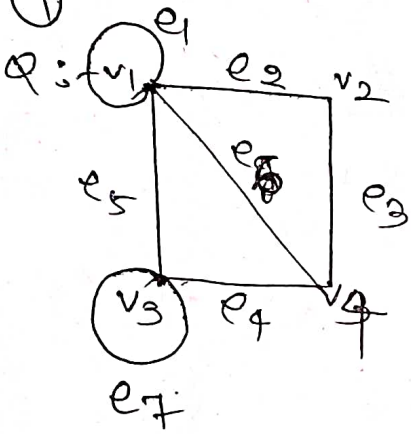


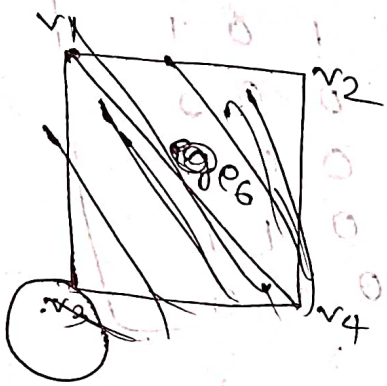
	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	1	0	0	0	0
v_2	0	1	1	1	0	1	1
v_3	0	0	0	1	1	0	0
v_4	0	0	0	0	0	0	1
v_5	0	0	0	0	1	1	0

$$\text{matrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{5\times 7}$$

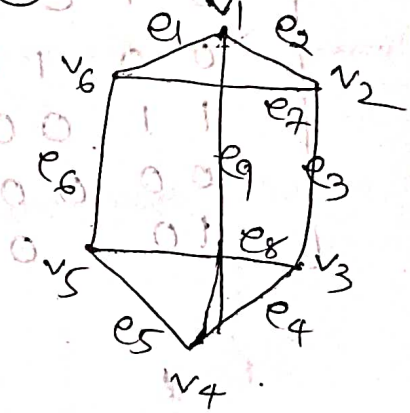
(9)

Q:

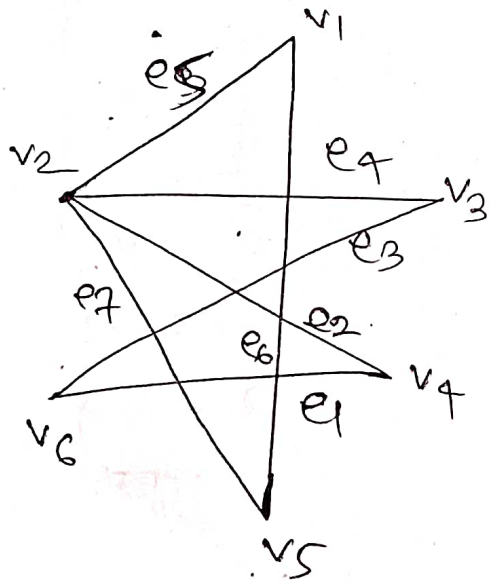




(10)

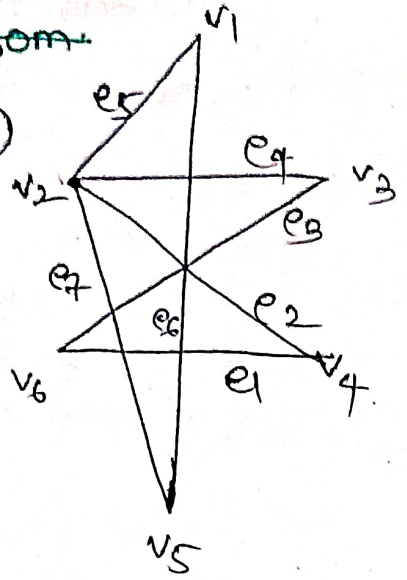


(11)



~~Isom.~~

(4)



	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	0	0	0	0	1	1	0
v_2	0	1	0	1	1	0	1
v_3	0	0	1	1	0	0	0
v_4	1	1	0	0	0	0	0
v_5	0	0	0	0	0	1	1
v_6	1	0	1	0	0	0	0

<u>matrix</u>

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Isomorphism

A simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, if there is one to one and onto function 'f' from V_1 to V_2 with the property that 'a' and 'b' are adjacent in 'G_1' iff f(a) and f(b) are adjacent in 'G_2', for all a, b in V_1.

Such a function 'f' is called an isomorphism

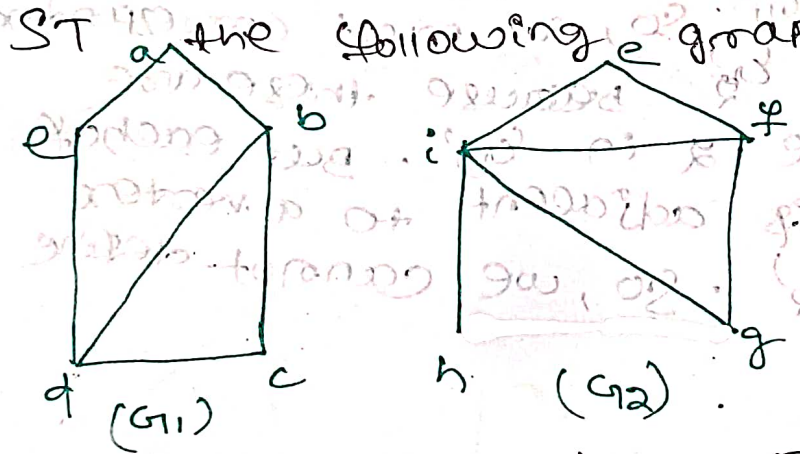
→ The property preserved by isomorphism of a graph is called graph invariant.

① $|V_1| = |V_2|$

② $|E_1| = |E_2|$.

③ degree of vertices must be same.

Q) ST the following graphs are not isomorphic



$|V_1| = 5$

$|E_1| = 6$

~~deg~~

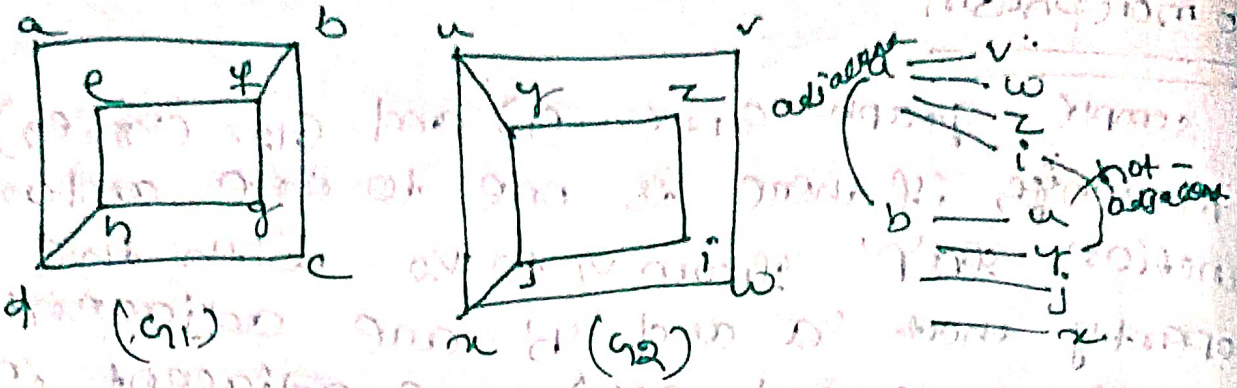
$|V_2| = 5$

$|E_2| = 6$.

$deg(h) = 1$

In graph 'G_2', there is a vertex of degree 1, but graph 'G_1' does not contain any vertex of degree 1. So, the graphs are not isomorphic.

a)



$|V_1| = 8$ $\quad$ $|V_2| = 8$

$|E_1| = 10$ $\quad$ $|E_2| = 10$

$\deg(e) = 2$ $\quad$ $\deg(a) = 2$

$\deg(f) = 3$ $\quad$ $\deg(b) = 3$

$\deg(g) = 2$ $\quad$ $\deg(c) = 2$

$\deg(h) = 3$ $\quad$ $\deg(d) = 3$

$\deg(i) = 2$

$\deg(j) = 3$

$\deg(y) = 3$

$\deg(z) = 2$

$\deg(u) = 3$

$\deg(v) = 2$

$\deg(w) = 2$

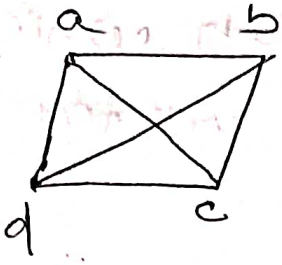
$\deg(x) = 3$

Here, $\deg(a) = 2$ in 'G_1'. So, 'a' must correspondence to v, w, z, i in G_2 because these are vertices of degree 2 in 'G_2'. But each of these 4 vertices is adjacent to a vertex of degree 2 in 'G_2'. So, we cannot define a mapping.

Connectivity

Path

A path is a finite or infinite sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.



ab – bc – cd

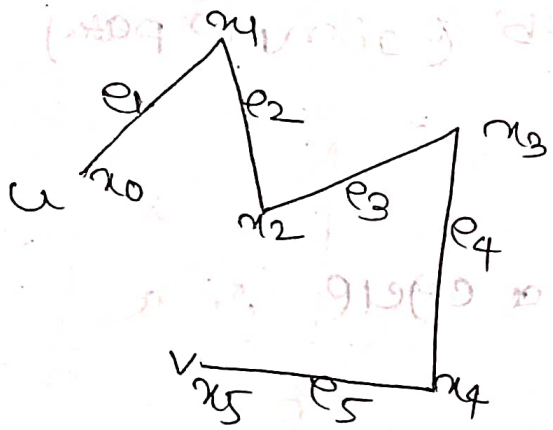
a – b – c.

Path of undirected graph

Let, 'n' be a non-ve integer and 'G' is an undirected graph. A path of length 'l' from 'u' to 'v' in 'G' is a sequence of 'n' edges $e_1, e_2, \ldots, e_n$ of 'G' such that, e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1 x_2\}$ ..... e_n is associated with (x_{n-1}, x_n).

where, $x_0 = u$.

$x_n = v$.



length = 5.

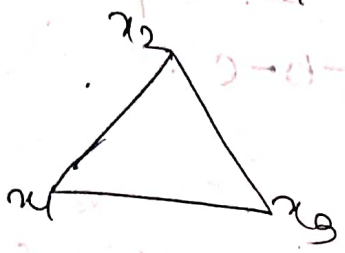
Path = $x_0, x_1, x_2, x_3, x_4, x_5$.

when the graph is simple, we denote ~~this~~ path by vertex sequence $x_0, x_1, \ldots x_n$.

circuit

A circuit is a path which ~~are start~~ has same starting & ending vertex.

Ex:-

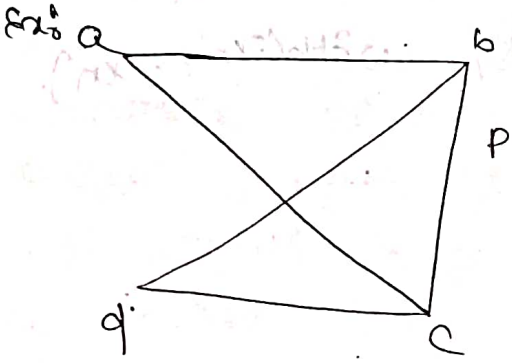


path = x_1, x_2, x_3, x_1.

simple path or simple circuit

A path or a circuit is simple if it does not contain the same edge more than once.

Ex:

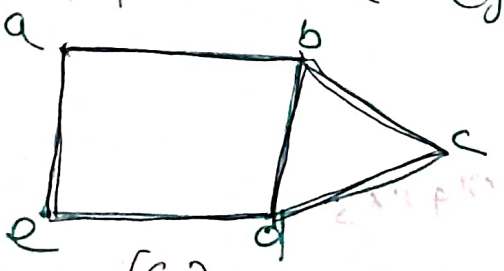


path - d-c-a-c-b (not a simple path.

d-c-a-b (simple path).

A circuit is also called a cycle in a graph.

Q:- find all the cycles of graph 'G'.



(G).

circuit = 3.

walk

A walk is defined as an alternating sequence of vertices and edges of a graph

$v_0, e_1, v_1, e_2 \ldots\ldots v_{n-1}, e_n, v_n$.

where, v_{i-1} and v_i are the end points of e_i for $i = 1$ to n

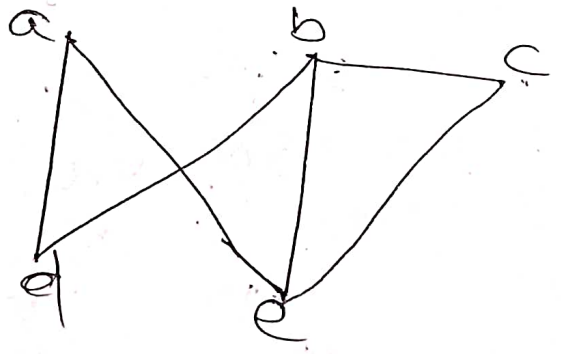
closed walk

It is defined as a walk that is begins and ends at same vertex.

Trail

It is defined as a walk that has not repeated edge

Q:- Does ~~each~~ each list of these list of vertices from a path in the following graph which paths are simple, which paths are circuit, what are the length of those paths.



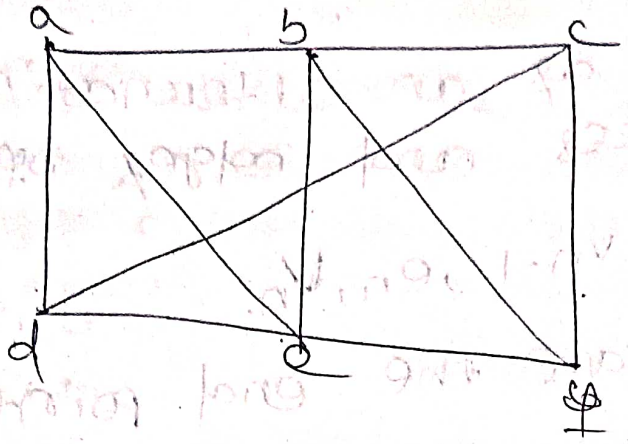
(a) a, e, b, c, b — path — length = 4

(b) a, e, a, d, b, c, a — ~~not a~~ path not defined

(c) e, b, a, d, b, e - path not defined

(d) c, b, d, a, e, c - circuit, length = 5

(2)



(a) a, d, c, f, e - path,
length = 4

(b) d, e, c, a - path not defined

(c) b, c, f, e, b - circuit
length = 4

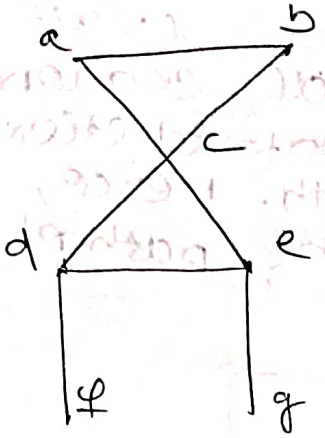
(d) a, b, e, d, a - circuit
length = 4

Connected graph

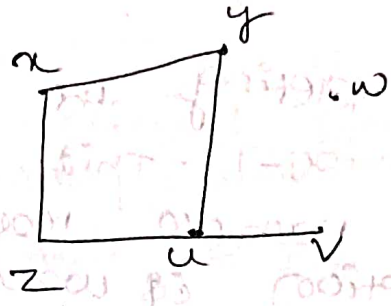
An undirected graph is connected if there is a path between every pair of vertices.

The graph which is not connected is called disconnected.

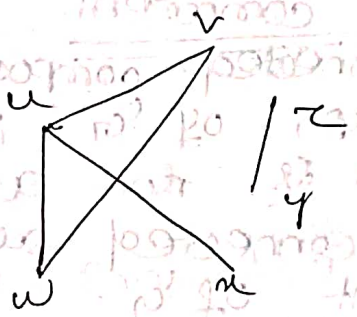
→ A graph with just one vertex is connected.



(connected)



(disconnected).



(disconnected)

Theorem-1

There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof

Let $G(V, E)$ be a connected undirected graph, and u, v are two distinct vertices of 'G'. Since, 'G' is connected, $\exists$ at least one path between u and v.

Let, $x_0, x_1, \ldots, x_n$ be the vertex sequence of path 'p' of least length. where, $x_0 = u$ and $x_n = v$.

Claim :- This path 'p' is a simple path of least length.

suppose, 'p' is not simple. so, two edges associated with vertices $\{x_k, x_{k+1}\}$ & $\{x_i, x_{i+1}\}$ are same. so, we have two vertices x_i and x_j such that $x_i = x_j$.

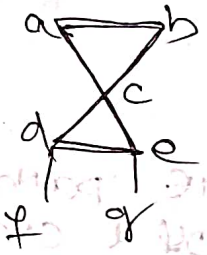
This means, there is a path 'u' to 'v' of shorter length with vertex sequence $x_0, x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n$.

$(\because x_i = x_j)$

obtained by deleting the edges of sequence, $x_i, x_{i+1}, \ldots, x_{j-1}$. This is a contradiction as 'p' is the path of least length. Hence, our assumption is wrong. So, the path 'p' is simple path.

Connected component

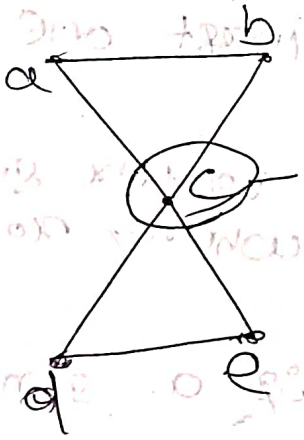
A connected component is a maximal connected sub-graph of 'G'. i.e. A connected subgraph of 'G' that is not a proper sub-graph of any other connected subgraph of 'G' is a connected component of 'G'.



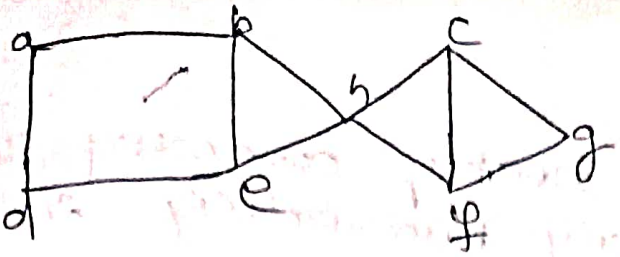
a, b, c, d, e is the connected component

Cut vertices

A cut vertex of a connected graph 'G', is a vertex whose removal makes the graph disconnected.



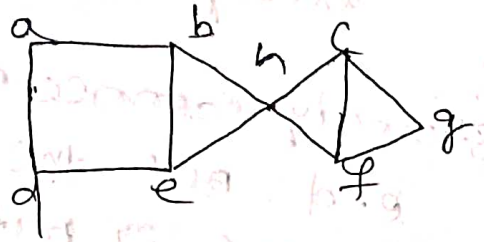
cut vertex.



Here, 'h' is the cut vertex.

Cut ~~shape~~ set

In a connected graph 'G', a cut ~~shape~~ set is a set of edges whose removal from graph 'G' leaves 'G' disconnected provided removal of no proper subset of this edges. This disconnects 'G'



{(b,h), (e,h)} - cut set.

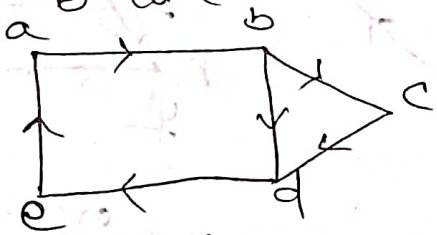
{(a,b), (b,e), (b,h), (e,h)} = not a cut set

Connectedness in a directed graph

Strongly connected

A directed graph is strongly connected if there is a path from A to B and from B to A where A and B are vertices in the graph.

Ex:-



(G)

a - b
b - a
b - c
c - b
c - d
d - c

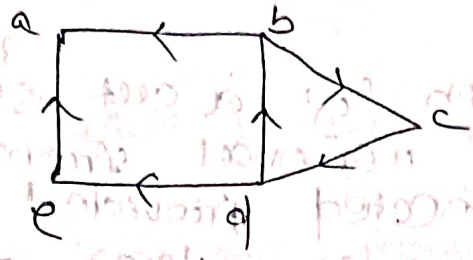
Graph 'G' is strongly connected as there is a path between any two vertices in this directed

graph.

Weakly connect

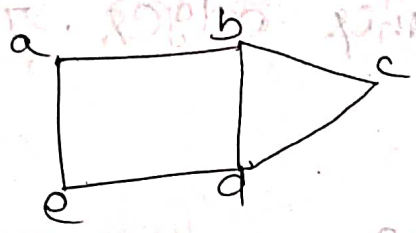
A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Ex:-



not strongly connected.

d-e.
~~d-e~~
e-d X.

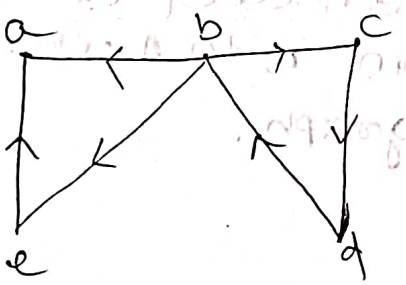


weakly connected graph.

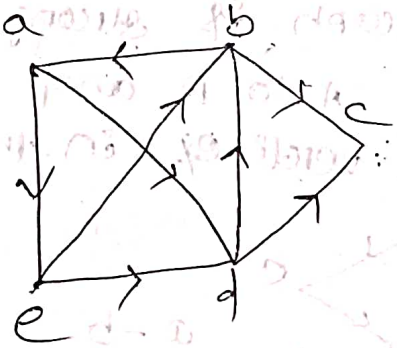
This graph is not strongly connected as there is no path between e-d. But the graph is weakly connected. because there is path between any two vertices in the underlying undirected graph.
↳ not bidirectional

Note

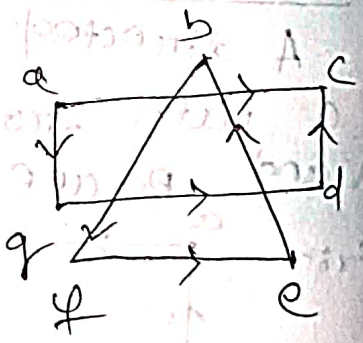
A strongly connected directed graph is also weakly connected.



(weakly connected)



(weakly connected)



(not connected) as, there is no path between g & f

Theorem-2

Let, 'G' be a graph with adjacency matrix A w.r.t ordering 1, 2, ..., n (with directed or undirected edges, with multiple edges and loops allowed). The no. of different path of length 'r' from i to j, where r is a positive integer, equals the (i, j)th entry of A^r.

Proof

This theorem will proved by using method of induction.

Let, 'G' be a graph with adjacency matrix 'A', Assume an order 1, 2, ..., n of vertices of 'G'.

Now, the no. of path of length 'r' from i to j is the (i, j)th entry of 'A'.

This is because of the defination of adjacency matrix.

Let us assume that (i, j)th entry of A^r is the no. of different paths of length r from i to j.

Now, for A^{r+1}

$$A^{r+1} = A^r \cdot A.$$

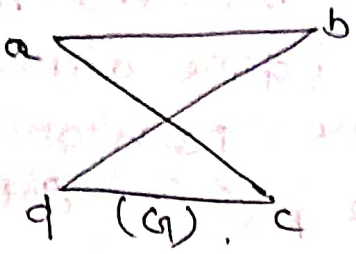
Now, (i, j)th entry of A^{r+1}

$$= b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}.$$

where, b_{ik} is nothing but the no. of paths of length r from i to k.

By product and sum rule of counting no. of paths of length 'r' from i to k namely b_{ik} and the no. of path of length 1, from k to j namely a_{kj} when there product added we get the no. of paths of length r+1.

Ex:- How many paths of length 4 are there from A to D in the simple graph 'G' in the given figure.



Adj(G)

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

no. of paths of length 4 = $(a,d)^{th}$ entry in A^4.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 8 & 0 & 0 & 8 \\ b & 0 & 8 & 8 & 0 \\ c & 0 & 8 & 8 & 0 \\ d & 8 & 0 & 0 & 8 \end{array}$$

path from 'a' to 'd' = 8.

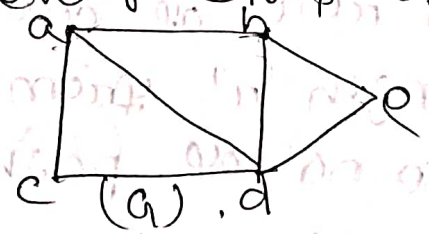
paths are:-

a-b-a-b-d, a-b-a-c-d, a-b-d-b-d, a-b-d-c-d, a-c-a-b-d, a-c-a-c-d, a-c-d-b-d, a-c-d-c-d.

Euler and Hamilton path

Euler path

An euler path in a graph 'G' is a simple path containing every edges of 'G'.

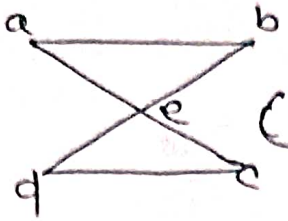


In graph 'G', acdebdab is an euler path.

Euler circuit

An Euler circuit in a graph 'G' is a simple circuit containing every edge of the graph.

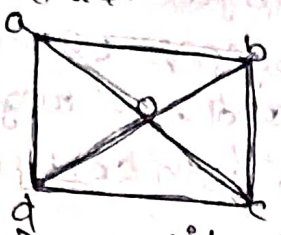
Ex:



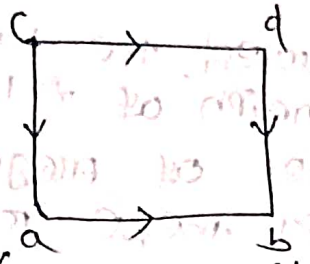
(G).

In graph(G), abedcea is an euler circuit.

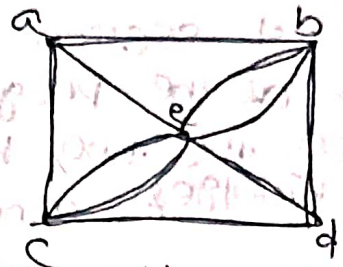
Q) Determine whether the given graph has an euler-circuit. construct such a circuit when there exists. If no euler circuit exist, determine whether the graph has an euler path and construct such a path if one exist.



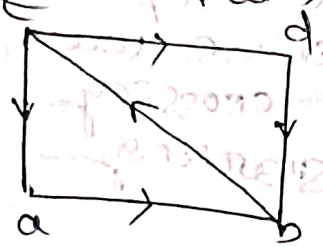
~~Euler circuit~~
~~= aabcbadcdaa.~~
(neither circuit nor path)



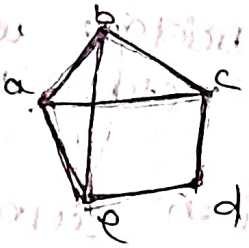
(neither circuit nor path)



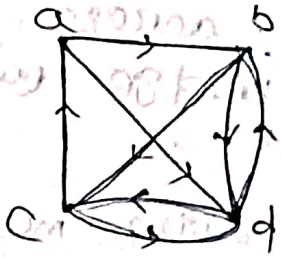
circuit
= aebecedcabde



Euler circuit
= cabcdbc.

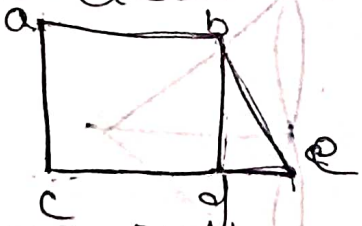


~~Euler path~~ (neither path nor circuit)
~~= aedcabe~~
~~abeadebc~~

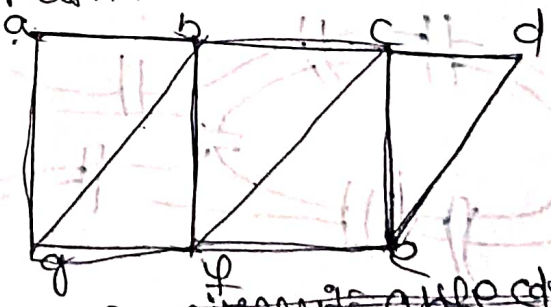


Euler circuit
= abcdcadbd.

Q) Which of the following undirected graph have euler circuit or euler path.

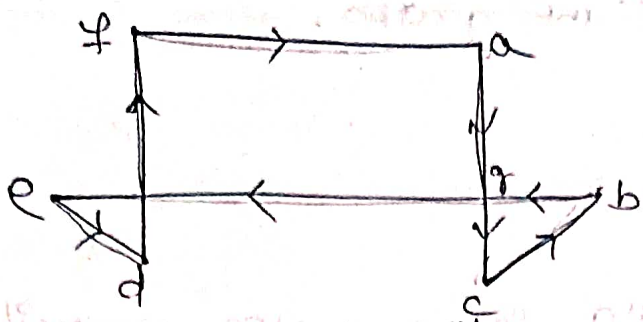


Euler path
= bacdbed.



~~Euler circuit abfedcbfga~~
(NOT)

a) check whether it has euler path or circuit



It is an euler circuit

a g c b g e d f a.

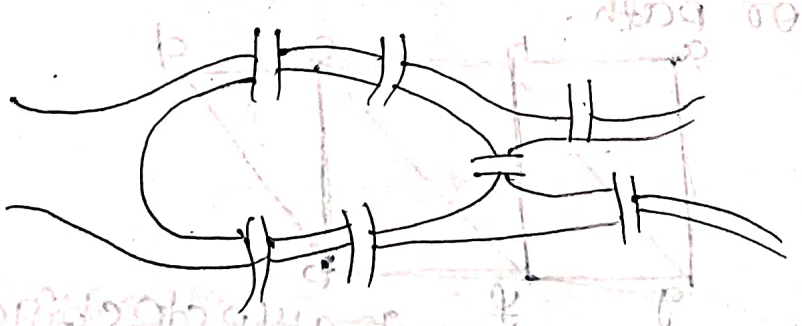
The seven bridges of konigsberg :-

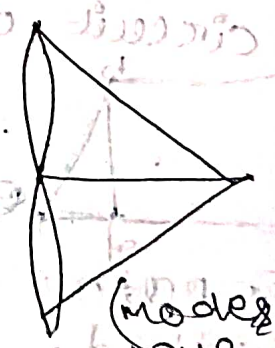
The town of Konigsberg, Prussia was divided into icetions by the branches of the Pregel River

→ These 4 section included the two regions on the banks of the Pregel, kneiph of island and the regi between the two branches of Pregel. on 15th centu seven bridges connected these region.

→ The town people took along walks through town on sundays. They wonder whether it is possible to start at some location in the town trave across all the bridges without crossing any bridge twice and return to the starting point.

The swiss mathematician Euler solving this problem using graph theory. euler studied the problem using the multigraph obtained when the four region are represented by vertices and bridges by edges.

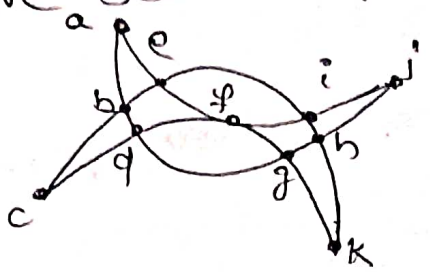




(modes of above proble

Theorem - 1

A connected multigraph with at least two vertices has an euler circuit iff each of its vertices have ~~been~~ even degree.



deg(a) = 2.
deg(b) = 4.
deg(c) = 2.
deg(d) = 4.
deg(e) = 4
deg(f) = 4
deg(g) = 4
deg(h) = 4
deg(i) = 4.
deg(j) = 2.
deg(k) = 2.

we can solve this problem and it has an euler circuit since each vertices of 'G' has even degrees, so we can construct euler circuit.

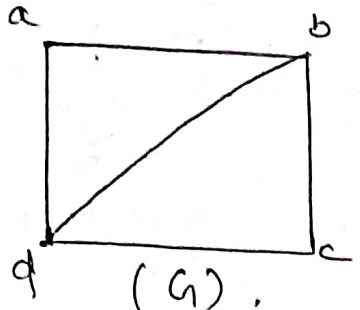
The euler circuit is -

a b d g h j i h k g f d c b e i f e a.

→ The circuit gives a way to draw the similar without using the pencil.

Theorem-11

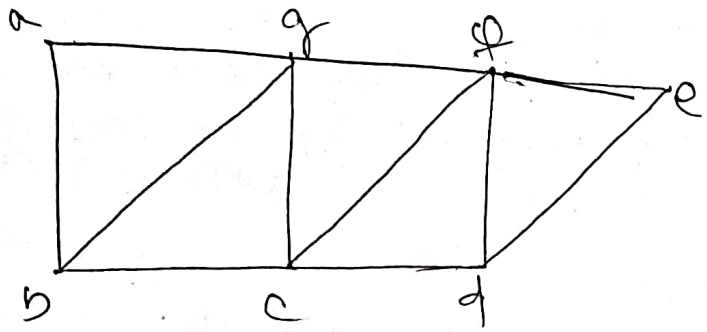
The connected multigraph has an euler path but not an euler circuit iff it has exactly two vertices of odd degree.

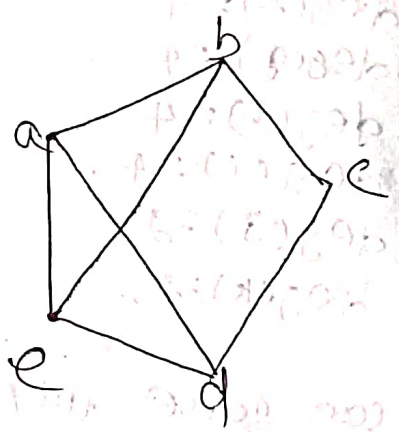


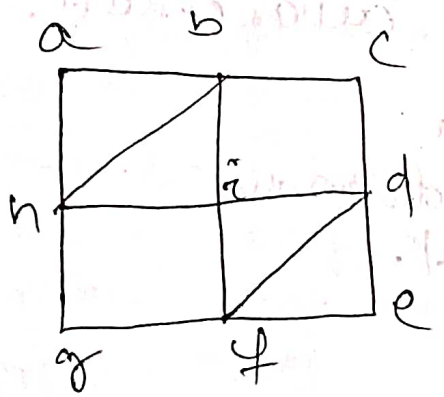
(G).

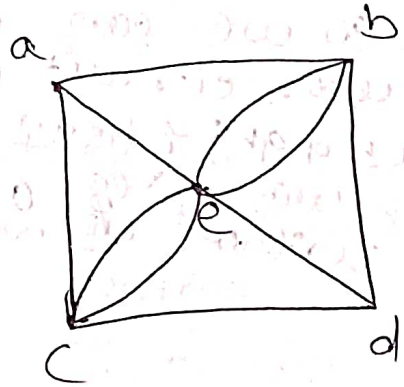
It has exactly two vertices of odd degree, so, it has euler path but not euler circuit.

→ Euler path is bad bcd.







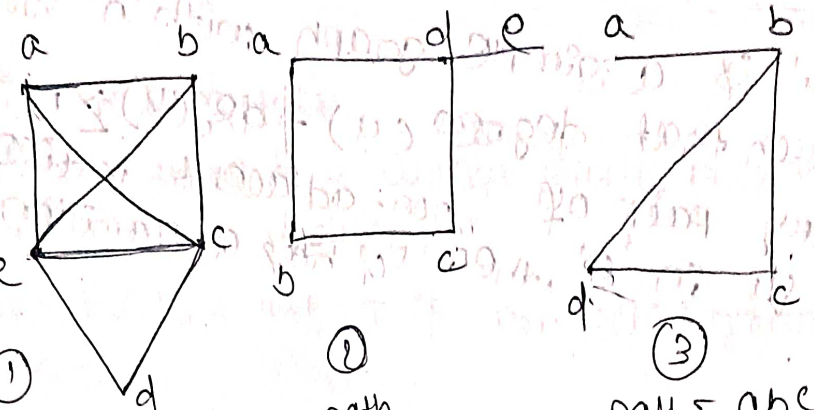


Hamilton path and circuit

A simple path in a graph 'G', that passes through every vertex exactly once.

→ A simple circuit in a graph 'G', that passes through every vertex exactly once.

Ex:-

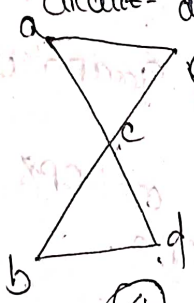


① ② ③

~~path = abcd~~
circuit = abcdea

path = edabc.

path = abcd



④

path = aecbd.

~~Pes~~
~~A ve~~

Test

A graph with a vertex of degree one cannot have a Hamilton circuit.

② Hamilton circuit cannot have a smaller circuit within it.

Theorem-III (Dirac's theorem)

If 'G' is a simple graph with 'n' vertices such that the degree of each vertex in 'G' is at least $n/2$, then 'G' has a Hamilton circuit.

$n \geq 3$

Theorem-IV (Ore's theorem)

If 'G' is a simple graph with 'n' vertices, $n \geq 3$, such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices 'u' and 'v' in 'G', then 'G' has a Hamilton circuit.

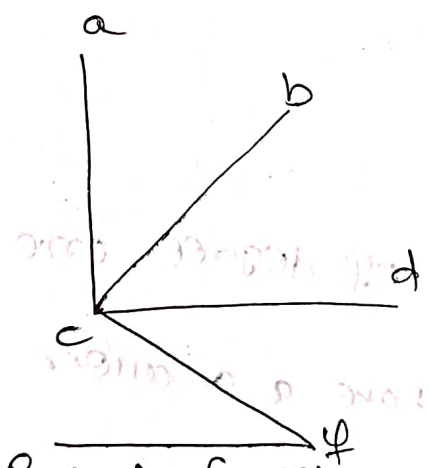
TREE

A Tree is a connected undirected graph with no simple circuit.

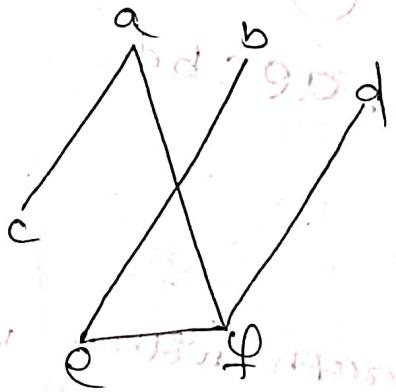
→ Tree cannot contain multiple edges or loops.

So, any tree must be a simple graph.

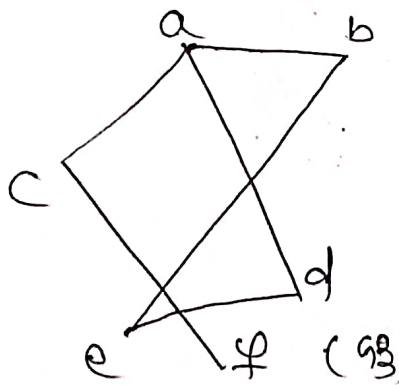
Ex:-



(G1) ~~(Tree)~~.



(G2) (Tree)



(G3) (not a tree).

'a' is a tree because it is connected graph and doesnot have a circuit.

Theorem-1

An undirected graph is a tree iff there is a unique simple path between any two of its vertices.

Proof:-

Let 'T' is a tree

To show there is a unique simple path between any two of its vertices,

By defn of tree, T is connected graph with no simple circuit.

Let x and y are two vertices of T.

Since, T is connected, there is a path between x and y. To check uniqueness of path,

let there is another simple path from x to y. Then the path formed by combining 1st path from x to y and by the path from y to x obtained by reversing the order of 2nd path form circuit.

→ T has a simple circuit which is a contradiction, Hence, there is a unique simple path between any two vertices of T.

Conversely, To show T is a tree,

let, there is a unique simple path between any two vertices.

T is connected. Next to show there is no simple circuit, let there is a simple circuit contained the vertices x and y.

Then there is two simple paths between x to y.

Hence, a graph with a unique simple path between any two vertices is a tree.

Terminology of Tree

Rooted tree

A rooted tree is a tree in which one vertex has been designed as the root and every edge is directed away from the root.

a - root.

Parent

Suppose, 'T' is a rooted tree. Parent of vertex 'v' in a rooted tree is the vertex 'u' such that (u,v) is the edge of root tree.

Child

Child of a vertex 'v' in a rooted tree is any vertex with 'v' as its parent.

Sibling

Sibling of a vertex 'v' in a rooted tree is a vertex with same parent as 'v'.

Ancestor

Ancestor of a vertex 'v' in a rooted tree is any vertex on the path from the root to 'v'.

Descendant

Descendant of a vertex 'v' in a rooted tree is any vertex that has 'v' as ancestor.

Internal vertex

A vertex that has children is called internal vertex.

Leaf

A vertex with no children is called leaf.

Subtree

If 'a' is a vertex in a tree, the subtree with a as its root is the subgraph of tree consisting of 'a' and its descendant and an edge incident to this descendant.

ex:- find the parent of 'e' = b

children of d = h, i, j.

sibling of c = b, d.

ancestor of g = e, b, a.

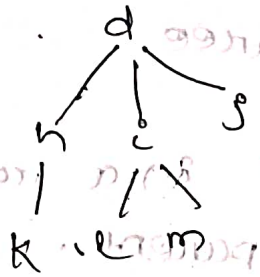
descendant of b = e, f, g.

all the internal vertices = a, b, d, e, h, i.

all the leaf = f, g, c, k, l, m, j.

subtree rooted at 'd'

=



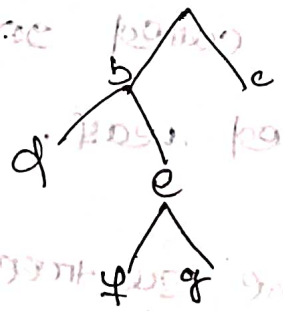
M-ary tree

A rooted tree is called M-ary tree if every internal vertex has no more than M children.

full M-ary tree

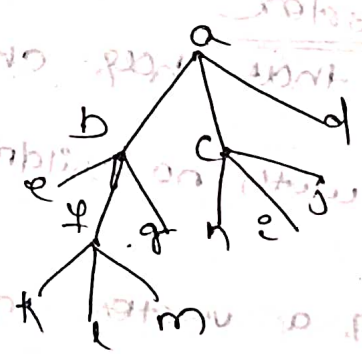
A tree is called a full M-ary tree if every internal vertex has exactly M children.

→ An M-ary tree with M = 2 is called a binary tree.



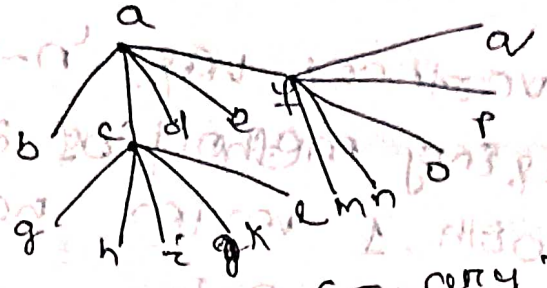
T1

(binary)

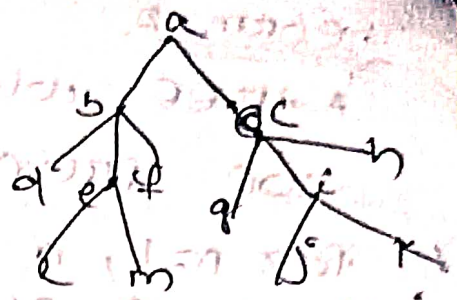


T2

(3-ary)



T3 (5-ary)



T4 (3-ary).

full m-ary tree for some integer m.

T_1 - full-m-ary with children = 2.

T_2, T_3 = full m-ary.

T_4 - is not a full m-ary tree for any m because some of its internal vertices have 3 children and some other have 3 children.

<u>ordered rooted tree</u>

An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered.

<u>Note:-</u>

In an ordered binary tree

- If an internal vertex has two children, then the 1st child is called left child and the 2nd child is called right child.
- The tree rooted at left child of a vertex is called left subtree.
- The tree rooted at right child of a vertex is called right subtree.

Theorem - 2

"A tree with 'n' vertices has 'n-1' edges"

We prove this using method of induction.

for $n=1$, a tree with 1 vertex has no edges. So it is a tree for $n=1$.

Let, it is tree for $n=k$.

i.e. tree with 'k' vertices has $k-1$ edges.

Now to show it is tree for $n=k+1$.

Suppose, a tree T has $(k+1)$ vertices.

So, there is atleast one vertex of ~~degree~~ deg 1.

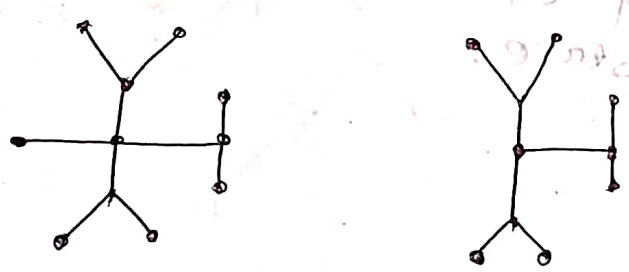
Let remove this vertex and its associated edge from 'T' and consider

$T' = T - v$.

Now, T' has k vertices because resulting graph is still connected and has no simple circuit so, it has $k-1$ edges.

So, T has $k-1+1$ edges because it has one more edge than T'.

$\Rightarrow$ T has k edges.



Theorem - 3

A full m-way tree with 'i' internal vertices contains $n = mi + 1$ vertices.

Proof :-

Let 'T' is an ~~m-ary~~ m-ary tree with 'i' internal vertices.

So, every vertex, except root is the child of an internal vertex.

Because each of the internal vertices has 'm' children, then there are mi vertices except root. So, total no. of vertices = mi+1.

Tree contain, n = mi+1 vertices.

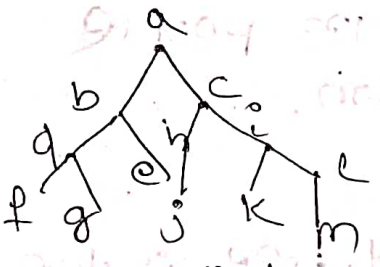
Theorem-4 :-

A full m-ary tree with

(i) n vertices has $i = \frac{(n-1)}{m}$ internal vertices and $l = \frac{[(m-1)n+1]}{m}$ leaves.

(ii) i internal vertices has n = mi+1 vertices and l = (m-1)i+1 leaves.

(iii) Leaves has $n = \frac{ml-1}{m-1}$ vertices and $i = \frac{l-1}{m-1}$ internal vertices.

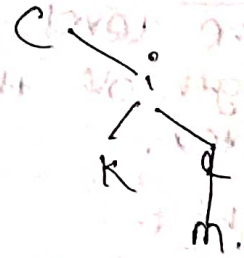


left-child of d = f

right " " d = g

left subtree of 'c'

right subtree of 'c'



Properties of tree

ex:-

Suppose that someone starts a chain letter, each person who receives the letters is asked to send it in to four other people. Some people do this but other donot send any letter. How many people have scan letter, including 1st person, if one one recieves more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out? How many people send out the letter.

Soln

The chain letter can be represented using a 4-ary tree. $m=4$

The internal vertices (i) are those people who send out the letter.

The leaves are those people who didn't send it out.

Given leaves $(l)=100$.

The vertices (n) are those people who scan letter.

$$n = \frac{ml-1}{m-1} = \frac{4\times 100-1}{4-1} = \frac{399}{3} = 133.$$

and $$i = \frac{l-1}{m-1} = \frac{99}{3} = 33.$$

33 people send out the letter and 133 people seen the letter including 1st person.

Level :-

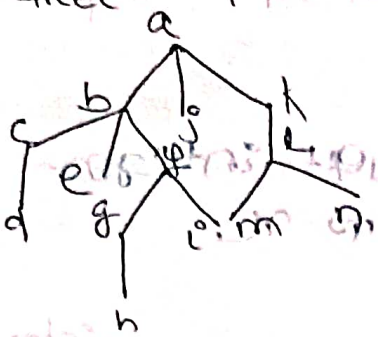
The level of a vertex in a rooted tree is the length of the unique path from the root to this vertex.

Hight :-

The hight of the rooted tree is the maximum of the leaves of vertices.

→ The height of a rooted tree is the lengthof the longest path from the root to any vertex.

Ex:- what is the level of each vertex in the rooted tree and what is the height of tree.



level of a = 0
level of b, j, k = 1
" " c, e, f, l = 2
" " d, g, i, m, n = 3
" " h = 4.
Height of tree = 4.

Note:- The level of a root is defined as 0.

Balanced tree

A rooted m-ary tree is balanced if all leaves are at level 'h' or 'h-1'.

Ex:-

Theorem-5 :-

There are atmost m^h leaves in an m-ary tree of height 'h'.

Proof:-

This is proved by using method of induction.

consider an m-ary tree of height 1.

This tree consists of a root with no more than 'm' children. each children are leaf.

so there are $m' = m$ leaves in an m-ary tree of height 1.

Let us assume it is true for an m-ary tree of height h-1.

$\Rightarrow$ there are atmost m^{h-1} leaves.

Now, Let us consider an 'm' ary tree of height 'h'.

The leaves of T are the leaves of the subtree T obtained by deleting the edges from the root to each of the vertices of level 1.

so, each of subtree has height less than or equal to h-1.

So, this subtree has atmost m^{h-1} leaves.

because there are atmost 'm' subtree, each with a maximum m^{h-1} leaves.

So, there are atmost $m \cdot m^{h-1} = m^h$ leaves of the rooted tree.

① If an m-ary tree of height 'h' has 'l' leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced then $h = \lceil \log_m l \rceil$

($\lceil\ \rceil$ - ceiling function. = smallest integer greater than or equal to x.

<u>Proof :-</u>

we know that in a m-ary tree of height 'h' has atmost m^h leaves

$\Rightarrow l \leq m^h$.

Taking log on both sides.

$\Rightarrow \log l \leq \log m^h$

$\Rightarrow \log l \leq h \log m$.

$\Rightarrow h \geq \dfrac{\log l}{\log m}$

$\Rightarrow h \geq \log_m l$.

because, 'h' is an integer.

$h \geq \lceil \log_m l \rceil$

Now, suppose the tree is balanced
Then each leaf is at level 'h' or 'h-1'.
and there is atleast one leaf at level 'h' as height is 'h'. So, ~~the~~ here must be more than m^{h-1} leaves.

$\Rightarrow l \leq m^h$.

$\Rightarrow m^{h-1} < l \leq m^h$.

$\Rightarrow h-1 \log m < \log l \leq h \log m$.

$\Rightarrow h-1 < \frac{\log l}{\log m} \leq h$.

$\Rightarrow h-1 < \log_m l \leq h$.

$\Rightarrow \boxed{\log_m l = h}$.

Depth-First Search :—

We can build a spanning tree for a connected simple graph using depth-first search.

We will form a rooted tree and the spanning tree ~~form~~ will be underlying undirected graph of this rooted tree.

Procedure :—

1. Arbitrarily choose a vertex of the graph as root.
2. Form a path starting at this vertex by successively adding vertices & edges where each new edge is incident with last vertex in the path and a vertex not already in path.
3. Continue adding vertices and edges to this path as long as possible.
4. If the path goes through all the vertices of the graph, then the tree consisting of this path is a spanning tree.

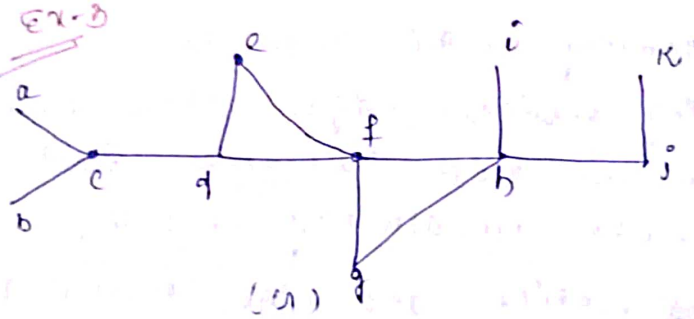
   If the path doesnot go through all vertices, then more vertices & edges must be added.
5. Move back to next to last vertex in path, & form a new path starting at this vertex passing through vertices that were not visited
6. Repeate this procedure
7. Because the graph has a finite no. of edges & is connected this process ends with the production of spanning tree.

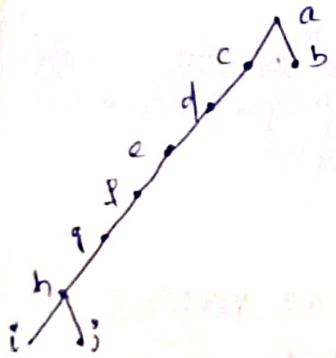
→ DFS is also called backtracking because the algorithm returns to vertices previously visited to add paths.

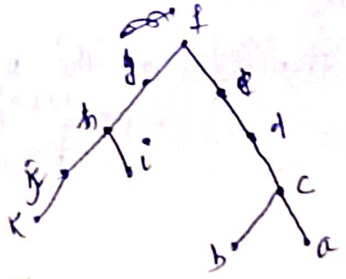
→ The edges selected by depth-first search of a graph are called tree edges.

→ All other edges of the graph must connect a vertex to an ancestor or descendant of this vertex in the tree. These edges are called backedges.

Ex-3







Breadth - First Search :—

We can also produce a spanning tree of a simple graph by use of BFS.

Procedure :—

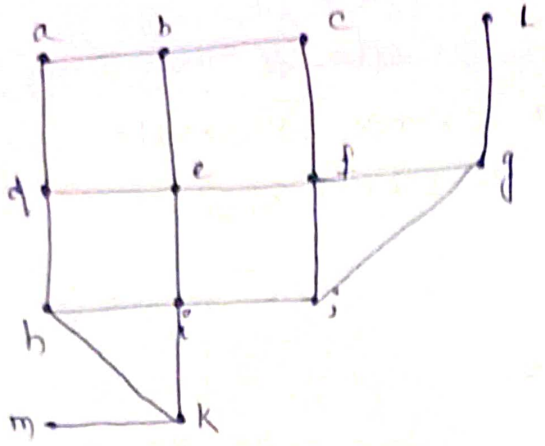
1) Arbitarily choose a root from the vertices.

2) add all the ~~vertices~~ edges incident to this vertex. This new vertices added at this stage become the vertices at level 1. Arbitarily ordered them.

3) For each vertex at level 1 visited in order add each edge incident to this vertex as long as it doesn't produce simple circuit.

4) Arbitarily order the children of each vertex at level 1. This procedures vertices at level 2.

5) Follow the procedure until all the vertices in the tree have been added.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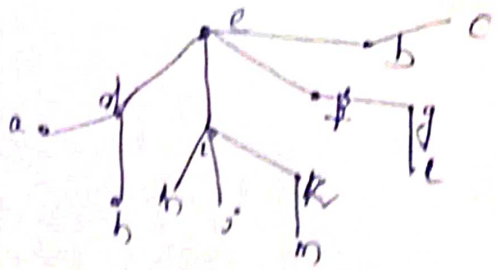
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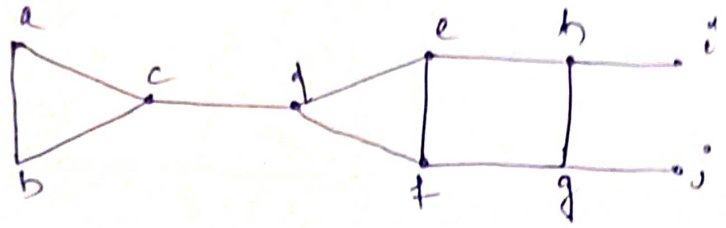
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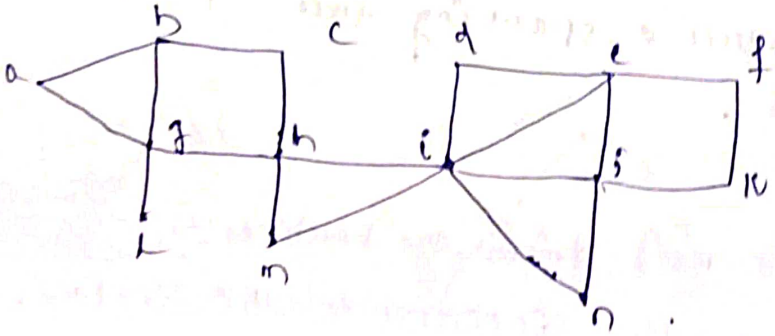
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Tree Traversal :— (a listing of all vertices of a tree)

Ordered rooted tree are often used to store information. We need procedures for visiting each vertex of an ordered rooted tree to access data.

Universal address system :—

It is a procedure for traversing all vertices of an ordered rooted tree on the ordering of children.

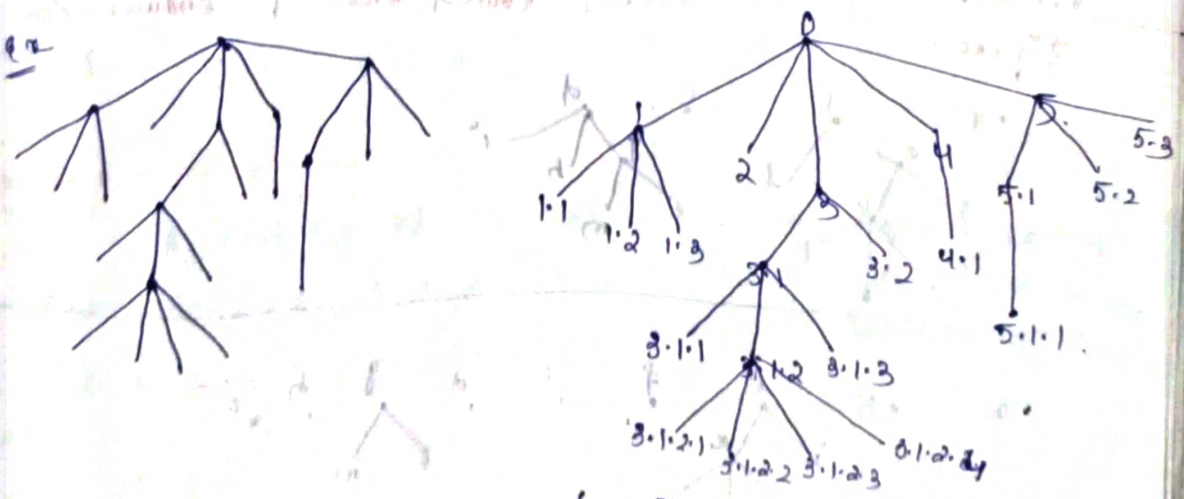
Procedure

1) label the root with integer 0.
2) Then label its k children (at level 1) from left to right with 1, 2, 3 ... k.
3) For each vertex at level n, with label A, label its k children, from left to right as A·1, A·2, ... A·k.

Following this procedure, a vertex at level n, for $n \geq 1$, is labeled $x_1 \cdot x_2 \cdots x_n$ where the unique path from root to goes through the x_1 st vertex of level 1, x_2 nd vertex at level 2, & so on.

This labeling is called the universal address system of the ordered rooted tree.

Ex



(universal Address system of an ordered rooted tree)

The lexicographic ordering of the labeling is :

$0 < 1 < 1.1 < 1.2 < 1.3 < 2 < 3 < 3.1 < 3.1.1 < 3.1.2 < 3.1.2.1 < 3.1.2.2 < 3.1.2.3 < 3.1.2.4 < 3.1.3 < 3.2 < 4 < 4.1 < 5 < 5.1 < 5.1.1 < 5.2 < 5.3$.

Traversal Algorithm:-

Procedures for systematically visiting every vertex of an ordered rooted tree are called traversal algorithms.

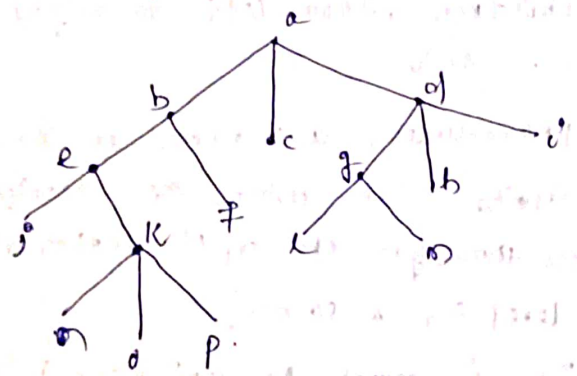
Three most commonly used algorithms are

1) Preorder traversal
2) inorder traversal
3) Postorder traversal.

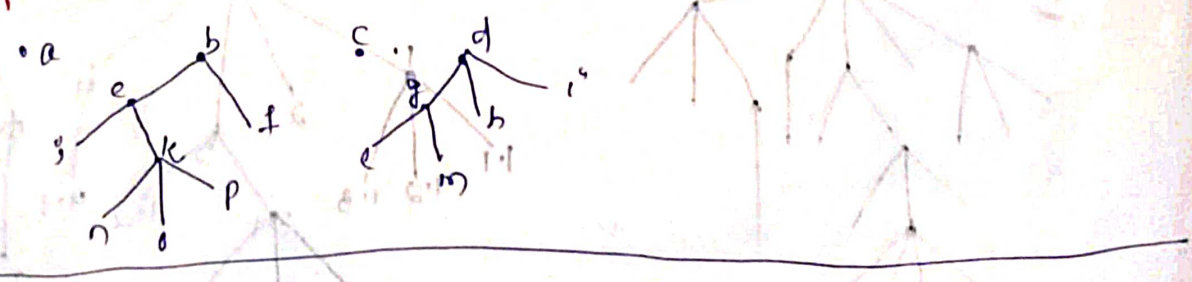
Preorder traversal:-

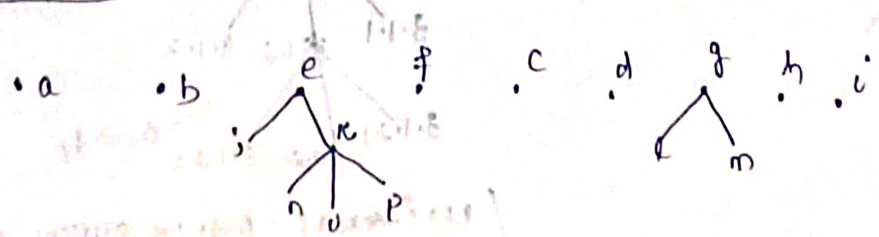
A listing of the vertices of an ordered rooted tree defined recursively - the root is listed, followed by the 1st subtree followed by the other subtree in the order they occurs from left to right.

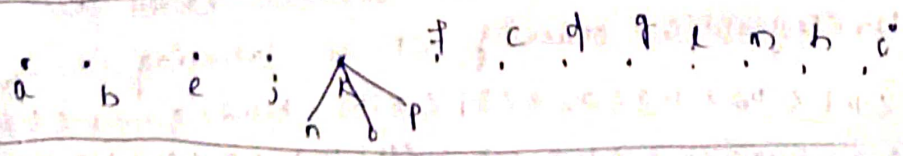
Ex-2



In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown in figure.







a b e j k n o p f c d g l m h i

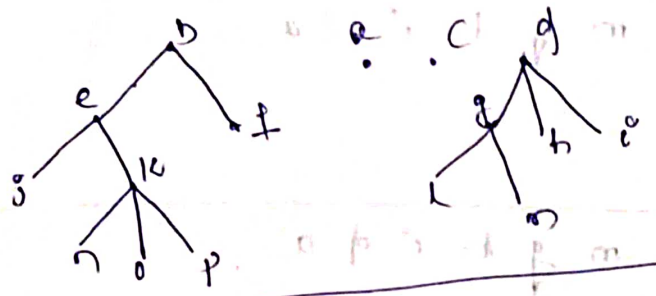
The preorder traversal T is

a b e j k n o p f c g l m h i .

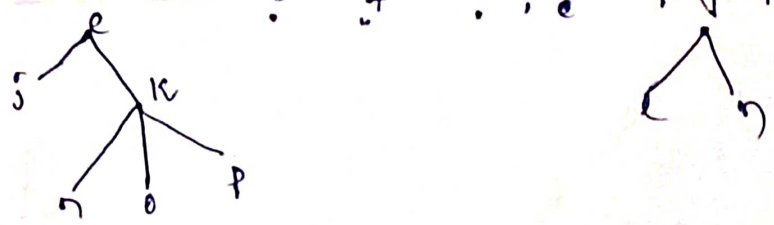
In order traversal:-

A listing of the vertices of an ordered rooted tree defined recursively - the 1st subtree is listed, followed by the root, followed by the other subtrees in the order they occur from left to right.

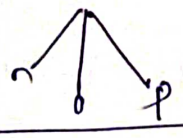
In which order does an inorder traversal visit the vertices of the ordered rooted tree T in above fig.



j e k b f a c g l m d h i



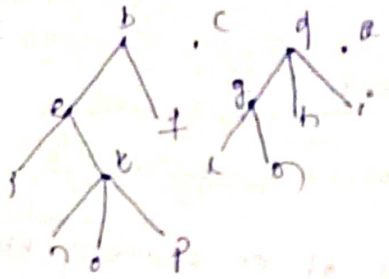
j e k b f a c g m d h i

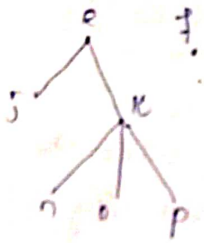


j e n k o p b f a c e g m d h i

Post order traversal :-

A listing of the vertices of an ordered rooted tree defined recursively, the subtree are listed in the order they occur from left to right followed by root-





f b c g h i d a

---

j

k e f b c l m g h i d a

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j n o p k e f b c l m g h i d a