Operation research :-

Defn :-

→ OR is the Scientific knowledge through ~~interdesciplinary~~ interdesciplinary for the purpose of determining the best utilization of limited ~~re~~ resources.

→ OR was 1st developed in 1940 by Mccloskey and Trefthen.

Optimization :- The process of maximization of profit & minimization of cost.

Linear Programming :-

→ L.P. deals with optimization (maximization or minimization) of a function of ~~a~~ variables known as objective function.

→ It is subject to a set of linear equalities or inequalities ($\geq, \leq$) known as contraints.

→ Linear Programming is a ~~mathm~~ mathematical techniques which involves the allocation of limited resource in an optimal maner on the basis of a given criteria of optimality.

→ The graphical method to solving a LPP is applicable where two variables are involved.

General formulation of L.P.P :-

→ The value of n decission variables $x_1, x_2, x_3, \ldots, x_n$ to maximize/minimize the objective function. & subject to constraints.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq = \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq = \geq)\ b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq = \geq)\ b_n$$

→ Finally satisfy the non-negative restriction:

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Ex → 1

A manufatun Produces 2 types of model. Each model of the type m_1 required 4 hrs of grinding & 2hrs of policing. Each model of the type m_2 required 2hrs of grinding & 5hrs of policing. The manufacturer has ~~two~~ 2 grinders & 3 policers. each grinder works 40hrs. for a week. Each policers work 60hrs for a week Profit on m_1 model is rupees 3/- & the model m_2 model rupees 4/-. What ever is produce in a week is sold in the market. How should be the manufacture allocate its production capacity to the two types of model. So that ~~he may~~ may ~~it~~ team A make the maximum Profit in a week.

Soln → Decision variable

Let, x_1 & x_2 be the no. of units of m_1 & m_2 model.

$$Max\ Z = 3x_1 + 4x_2$$

S.t.C,

their are two Constraints are is grindings & policing.

→ The no. of hrs available in each grinder for a ~~week~~ week = 40hrs.

→ There are 2 grinders.

∴ Hence the manufacture doesn't have more than $2 \times 40 = 80$ hrs of grinding.

→ m_1 requires 4 hrs of grinding & m_2 requires 2 hrs of grinding.

The grinding constraints is given by

$$4x_1 + 2x_2 \leq 80.$$

→ There are 3 Policers.

The no. of hrs available in each Policer for a week = 60 hrs.

Hence the manufacture doesn't have more than

$3 \times 60 = 180$ hrs. of policing.

→ m_1 requires 2 hrs. of policing; m_2 requires 5 hrs. of policing.

→ The policing constraints is given by

$$2x_1 + 5x_2 \leq 180$$

we have,

$$\text{Max } Z = 3x_1 + 4x_2$$

$$\text{s.t.c. } 4x_1 + 2x_2 \leq 80$$

$$2x_1 + 5x_2 \leq 180$$

$$x_1, x_2 \geq 0.$$

Ex-2 A company manufactures two Products A and B. These products are processed in the same machine. It takes 10 minute to process one unit of product A and 2 minutes of each unit of product B and the machine operates for a maximum of 35 hrs. in a week. Product A requires 1 kg. and B 0.5 kg of raw material per unit the supply of which is 600 kg. per week. Market constraint on product B is known to be 800 unit every week. Product A costs Rs. 5 per unit and sold at Rs. 10. Product B costs Rs. 6 per unit and can be sold in the market at a unit price of Rs. 8. Determine the number of units of A and B per week to maximise the profit.

Sol^n → ~~Let x_1 & x_2 be the number of products A and B.~~

Decision variables →

Let x_1 and x_2 be the number of products of A & B.

Objective function :→ Costs of product A per unit is Rs. 5 and sold at Rs. 10 per unit.

∴ Profit on one unit of product A.

$$= 10 - 5 = 5$$

∴ x_1 units of products A contributes a profit of Rs. $5x_1$

x_1 Profit contribution from one unit of product

$$B = 8 - 6 = 2$$

∴ x_2 units of products B contribute a profit of Rs. $2x_2$

∴ The objective function is given by

$$\text{Max } z = 5x_1 + 2x_2$$

Constraints Time requirement constraint is given by

$$10x_1 + 2x_2 \leq (35 \times 60)$$

$$10x_1 + 2x_2 \leq 2100$$

Raw material constraint is given by

$$x_1 + 0.5x_2 \leq 600$$

Market demand on product B is 800 units every week ∴ $x_2 \geq 800$.

The Complete LPP is

$$\max z = 5x_1 + 2x_1$$

Subject to $10x_1 + 2x_2 \leq 2100$

$x_1 + 0.5x_2 P \leq 600$

$x_2 \geq 800$

$x_1, x_2 \geq 0$

Procedure to solving LPP in Graphical method :-

Step-1 Consider each inequality Constraint as equation.

Step-2 Plot each equ^n on the graph. each will geo geometrically represent a straight line.

Step-3 mark the region

i) if the inequality Constraint Corresponding to that line is less than equal to ($\leq$) below the line lying in the 1st ~~quard~~ quadrant is shaded.

ii) if the inequality Constraint greater than equal to ($\geq$) sign the region above the line in the 1st quadrant is shaded.

iii) The Points laying in Common region will satisfy all the constraint simultaneously. the Common region is often is called the feasible region.

Step-4

→ Assign & arbitary value say 0 for the objective function.

Step-5

Draw the straight ~~the~~ line who represents the objective function with the arbitary value.

Step-6

→ Strecth objective function line till the extreme Point of feasible region.

Step-7

→ Find the Corresodinates of the extreme Points selected in step-6 & find the max^n or min^n value of Z.

Exo→

Solve the following LPP by G.M.

$$\text{Min } z = 20x_1 + 10x_2$$

S.t.C

$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Sol^n→ Replace all the inequalities by the Constraints by equ^n.

$$x_1 + 2x_2 = 40$$

Put, $x_1 = 0 \Rightarrow x_2 = 20$

Put, $x_2 = 0 \Rightarrow x_1 = 40$

$x_1 + 2x_2 = 40$ Passes through the Point $(0,20)$ & $(40,0)$.

$$3x_1 + x_2 = 30$$

Put $x_1 = 0 \Rightarrow x_2 = 30$

Put $x_2 = 0 \Rightarrow x_1 = 10$

$3x_1 + x_2 = 30$ Passes through $(0,30)$ & $(10,0)$.

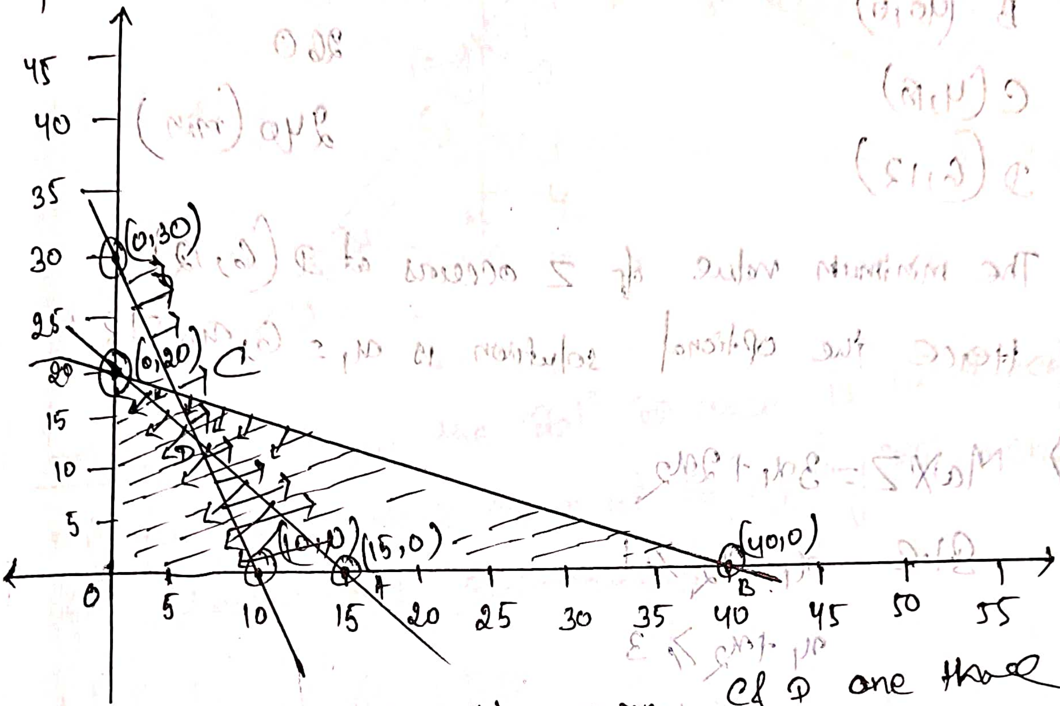
$4x_1 + 3x_2 = 60$ ~~Passes through (0,20) & (15,0)~~

Put $x_1 = 0 \Rightarrow x_2 = 20$

Put $x_2 = 0 \Rightarrow x_1 = 15$

$4x_1 + 3x_2$ Passes through $(0,20)$ & $(15,0)$.

Plot each eqn on the graph.



→ Hence ABCD is a feasible region. Cf D one thae

Points of intersection of lines.

Eqn (1) → $x_1 + 2x_2 = 40$ — (1)

Eqn (2) × 2 → $6x_1 + 2x_2 = 60$ — (2)

$\quad\quad (-) \quad (-) \quad (-)$

$-5x_1 = -20$

$\Rightarrow \boxed{x_1 = \frac{20}{5}} \Rightarrow \boxed{x_1 = 4}$

~~Put $2x_2 = 40 \Rightarrow 2x_2 = 36 \Rightarrow x_2$~~

Put the value of x_1 in eqn (1)

$x_1 + 2x_2 = 40$

$\Rightarrow 4 + 2x_2 = 40$

$\Rightarrow 2x_2 = 36 \Rightarrow \boxed{x_2 = 18}$

we get $C(4,18)$

Similarly eqn ② & ③ we get $D(6,12)$.

Corner Points	value of $z = 20x_1 + 10x_2$
$A(15,0)$	300
$B(40,0)$	800
$C(4,18)$	260
$D(6,12)$	240 (min)

→ The minimum value of Z occurs at $D(6,12)$.

→ Hence the optional solution is $x_1 = 6, x_2 = 12$.

Q) $\text{Max } Z = 3x_1 + 2x_2$

s.t.c $x_1 - x_2 \geq 1$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Solⁿ →

$x_1 - x_2 = 1$

$x_1 + x_2 = 3$

replace all the unequalities by the eqⁿ $x_1 - x_2 = 1$

~~passes through point~~ $\cancel{(0,-1)}$

$x_1 - x_2 = 1$

Put $x_1 = 0 \Rightarrow x_2 = -1$

Put $x_2 = 0 \Rightarrow x_1 = 1$

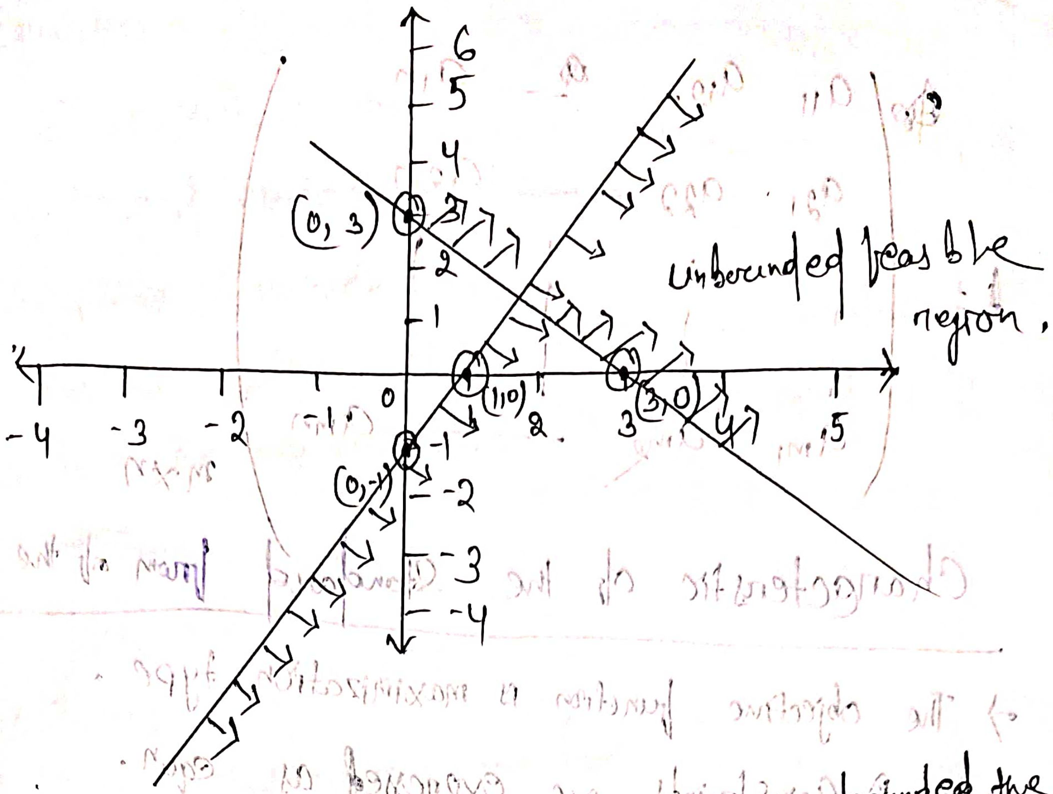
$x_1 - x_2 = 1$ passes through point $(0,-1)$ & $(1,0)$.

$x_1 + x_2 = 3$

Put $x_1 = 0 \Rightarrow x_2 = 3$

Put $x_2 = 0 \Rightarrow x_1 = 3$

$x_1 + x_2 = 3$ passes through the point $(0,3)$ & $(30,0)$.



Conclusion :- Hence the Sol^n space is unbounded the maximum value of Z occur at infinity (∞) hence the problem has unbounded Sol^n.

Matrix form of LPP

→ The LPP can be expressed in the matrix form.

Maximized / minimize.

$Z = Cx$

Subject to $Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$C = (C_1, C_2, \text{---} C_n)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

Characterstic of the Standard from of the L.P.P

-> The objective function is maximization type.

-> All Constraints are expressed as eqn.

-> Write hand side of each constraint is non-negative all Constraints are of less than equal to type.

-> Write hand side of each Constraint is non-negative all Constraints are of less than equal to type.

-> All the variable are non-negative.

Slack variable :-

-> Slack variable means the non-negative variables which are added in the L.H.S of the Constraints to Convert the inequit ($\leq$) in to an equation.

$$3 < 5$$

$$3+2 \leq 5 \Rightarrow 2 \text{ is the Slack variable.}$$

$$\left(\sum a_{ij} x_i \leq b_i\right)$$

-> Then the non-negative variable S_i which are ~~Intrd~~ interoduced to Convert the inequalite ($\leq$) to the equalities.

$\Sigma a_{ij} x_i + S_i = b_i$

we called 'slack variable'.

② Surplus Variables

→ The non-negative variable which are introduced to content the inequalities ($\geq$) into ~~equality~~ equality in equation.

$\Sigma a_{ij} x_i \geq b_i$

$\Sigma a_{ij} x_i - S_i \geq b_i$

→ S_i is called surplus variable

→ Surplus variable is defined as the non-negative variable, which are substracted from the LHS of the constraints to convert the equality ($\geq$) in to an eqn.

Simplex method

→ for the solution of any LPP by Simplex algorithm the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follow:

Step-1 → Check whether the objective func^n of the given LPP is to be maximized or minimized.

→ If it is to be minimized then we convert it into a problem of ~~maxim~~ minimization by $\min Z = -\text{Max}(-Z)$

Step-2 → Check whether all $b(i = 1, 2, \ldots, m)$ are positive. if any one of b_i is negative then multiply the ~~equation~~ inequation of the constraint by -1

So as to get all b_i to be positive.

Step-3 → Express the problem in standard form by introducing slack / surplus variables to convert the inequality constraints in to equations.

Step-4 → obtain an initial basic feasible solⁿ to the problem in the form $x_B = B^{-1} b$ and put it in the first column of the simplex table. Form the ~~initial~~ initial simplex table as given below.

C_B	S_B	C_j / X_B	c_1 / x_1	c_2 / x_2	c_3 / x_3	--- / x_4	-- / --x_n	0 0 -- 0 / S_1 -- S_2 -- S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	--a_{1n}	1 0 -- 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	--a_{2n}	1 0 -- 0

Step-5 Compute the net evaluations $z_j - c_j$ by using the relation $z_j - c_j = C_B(a_j - c_j)$

Examine the sign of $z_j - c_j$

i) If all $z_j - c_j \geq 0$, then the initial basic solution X_B is an optimum basic feasible solution.

ii) If at least one $z_j - c_j > 0$, then proceed to next step as the solution is not optimal.

Step-6 (To find the entering variable i.e. key column)

If there are more than one negative $z_j - c_p$ choose the most negative of them. Let it be ~~$z_p - c_p$~~ $z_r - c_r$ for some $j = r$. This gives the entering variable x_r and is indicated by an arrow at the bottom of the r^{th} column. If there are more than one variable having the same most negative

$Z_j - C_j$ then any, one of the variable can be selected arbitarily as the entering variable.

c) If all $X_{ir} \leq 0$ $(i = 1, 2, \ldots m)$ then there is an unbounded solution to the given Problem.

ii) If at least one $X_{ir} > 0$ $(i = 1, 2, \ldots m)$ then the Corresponding vector X_r enters the basis.

Step-7 (To find the leaving variable or key row)

Compute the ratio (X_{Bi} / X_{kr} $X_{ir} > 0$)

If the minimum of these ratios be X_{Bi}/X_{kr} then choose the variable X_k to leave the basis called the key row and the element at the intersection of key row and key column is called the key element.

Step-8 Form a new basis by droping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula.

$$\text{New element} = \text{old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

Step-9 Go to step(5) and repeat the produce until either an optimum solution is obtained or there is an indication of unbounded solution.

Simplex method

Solve the LPP by using S.M.

$Max.z = 3x_1 + 2x_2 + 5x_3$

s.t.c

$x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0.$

Soln: By introducing slack variable S_1, S_2 & S_3 convert the problem in standard form

$Max\, z = 3x_1 + 2x_2 + 5x_3 + 0.S_1 + 0.S_2 + 0.S_3$

s.t.c

$x_1 + 2x_2 + x_3 + S_1 = 430$

$3x_1 + 0x_2 + 2x_3 + S_2 = 460$

$x_1 + 4x_2 + 0x_3 + S_3 = 420$

$x_1, x_2, x_3, S_1, S_2, \& S_3 \geq 0$

An initial basic feasible solⁿ is given by.

$x_1 = x_2 = x_3 = 0$

$S_1 = 430$

$S_2 = 460$

$S_3 = 420$

Righting in matrix form

$AX = b$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & S_1 & S_2 & S_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Construct initial simplex table :->

C_j			3	2	5	0	0	0	$\min X_B/x_3,\ x_3>0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	430	1	2	1	1	0	0	430/1 = 430
0	S_2	460	3	0	2	0	1	0	460/2 = 230 (min)
0	S_3	420	1	4	0	0	0	1	
$Z_j - C_j$			-3	-2	-5 ↑	0	0	0	

key element

(most negative value)

$$Z_j - C_j = C_B X_j - C_j$$

$$Z_1 C_1 = C_B X_1 - C_1 = -3$$

$$Z_2 C_2 = C_B X_2 - C_2 = -2$$

$$Z_3 C_3 = C_B X_3 - C_3 = -5$$

1st iteration simplex table :->

C_j			3	2	5	0	0	0	$\min X_B/x_2,\ x_2>0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	200	-1/2	2	0	1	-1/2	0	200/2 = 100
5	x_3	230	3/2	0	1	0	-1/2	0	—
0	S_3	420	1	4	0	0	0	1	$\frac{420}{4}$ = 105
$Z_j - C_j$			9/2	-2 ↑	0	0	5/2	0	—

2nd Iteration Simplex table.

		C_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_2	100	$-1/4$	1	0	$1/2$	$-1/4$	0
5	x_3	230	$3/2$	0	1	0	$1/2$	0
0	S_3	20	2	0	0	-2	1	1
$Z_j - C_j$			4	0	0	1	2	0

Since all $Z_j - C_j \geq 0$. The sol^n is optimum.

It is given by $x_2 = 100$, $x_3 = 230$, $x_1 = 0$

$\text{Max } z = C_B X_B$

$= 2 \times 100 + 5 \times 230$

$= 200 + 1150$

$= 1350$

$\text{Max } z = 3x_1 + 2x_2 + 5x_3$

$= 0 + 2 \times 100 + 5 \times 230$

$= 1350$

Q) $\text{Min } Z = x_2 - 3x_3 + 2x_5$

S.t.C $3x_2 - 8x_3 + 2x_5 \leq 7$

$-2x_2 + 4x_3 \leq 12$

$-4x_2 + 3x_3 + 8x_5 \leq 10$

$x_2, x_3, x_5 \geq 0.$

~~$\text{Max } z = -x_2 + 3x_3 - 2x_5$~~

Solⁿ $\text{Max } z = -x_2 + 3x_3 - 2x_5$

By introducing slack variable s_1, s_2, s_3

Convert the problem in standard form.

$\text{Max } z = -x_1 + 3x_3 - 2x_5 + 0.s_1 + 0.s_2 + 0.s_3$

S.t.C $3x_2 - x_3 + 2x_5 + s_1 = 7$

$-2x_2 + 4x_3 + 0.x_5 + s_2 = 12$

$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

An initial feasible solⁿ is given by

$x_2 = x_3 = x_5 = 0.$

$s_1 = 7$

$s_2 = 12$

$s_3 = 10$

Righting in maxstrix form

$AX = b$

$$\begin{pmatrix} x_2 & x_3 & x_5 & S_1 & S_2 & S_3 \\ 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \\ S_2 \\ S_3 \\ S_5 \end{pmatrix}$$

~~Construct initial Complex table :-~~

			-1	3	-2	0	0	0
C_B	B	X_B	3	-1	2	1	0	0
0	S_1	7	-2	4	0	0	1	0
0	S_2	12	-4	3	8	0	0	1
0	S_3	10						

Construct initial Complex table :-

									min X_B/X_3, $x_3>0$
C_j			-1	3	-2	0	0	0	
C_B	B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	
0	S_2	7	3	-1	2	1	0	0	—
0	S_3	12	-2	$\boxed{4}$	0	0	1	0	$12/4 = 3$ (min)
0	S_5	10	-4	3	8	0	0	1	3.333
$Z_j - C_j$			1	-3 ↑	2	0	0	0	

$Z_j - C_j = C_B X_J - C_j$

Key element

IV

1st iteration simplex table

		C_j	-1	3	-2	0	0	0	$x_B/x_2,\ x_2>0$
C_B	B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	
0	S_1	10	5/2	0	2	1	1/4	0	64
3	x_3	3	-1/2	1	0	0	1/4	0	—
0	S_3	1	-5/2	0	8	0	~~2~~ -3/4	1	—
$Z_j - C_j$			-1/2 ↑	0	2	0	3/4	0	

key element

$$Z_j - C_j = C_B X_j - C_j$$

2nd iteration simplex table

		C_j	-1	3	-2	0	0	0
C_B	B	X_B	x_2	x_3	x_5	S_1	S_2	S_3
-1	x_2	4	1	0	4/5	2/5	1/10	0
3	x_3	5	0	1	-2/5	1/5	3/10	0
0	S_3	14/5	0	0	-8/5	0	2/5	1
$Z_j - C_j$			0	0	0	1/5	1/10	0

Since all $Z_j - C_j \geq 0$ ∴ The sol^n is optimum,

It is given by $x_2 = 4,\ x_3 = 5,\ x_5 = 0$.

$$\text{Max } z = C_B X_B = -1\times 4 + 3\times 5 = -4 + 15 = 11$$

$$\text{Max } z = -x_2 + 3x_3 - 2x_5 = -4 + 3\times 5 - 2\times 0 = -4 + 15 = 11$$

$$\text{Min } z = -11$$

Degeneracy of LPP

The Phenomenon of obtaining a degenerate basic feasible ~~soln~~ region of L.P.P

Degeneracy L.P.P may arise

(1) at the initial stage

(2) at any subsequent iteration stage.

In case (i), at least one of the basic variable should be zero in the initial basic feasible solution.

(ii) at any iteration of the simplex method more than one variable is eligible to leave the basis, and hence the next simplex iteration produces a degenerate solution in which at least one basic variables zero i.e there subsequent iteration may not produce improvements in the value of the objective function. As a result, it is possible to repeat the same sequence of simplex iteration endlessly without improving the solution. This concept is known as cycling ~~tie~~ (tie).

Artificial variable technique / Big-M method

LPP in which constraints may also have ≥ and = sign, after ensuring that all $b_i \geq 0$ are considered. In such cases basis matrix cannot be obtained as an identity matrix in the starting table, therefore we introduce a new type of variable called artificial variable.

The artificial variable technique is a device to get the starting basic feasible solution, so that simplex ~~produce~~ procedure may be adopted as ~~usual~~ usual until the optimal solution is obtained. Cost of artificial variable is $-M$.

There are two methods to solve such LPP.

→ Big M method or method of penalties.

→ The two-phase simplex method.

Simply method	Big-M method
to convert general to standard form ≤ add a slack variable. ≥ substract a surplus variable.	≤ add a slack variable. ≥ substract a surplus variable and add a artificial variable.
= Nothing	= add a artifical variable.

Q) Use Big-M method to solve

$$\text{Max } z = x_1 - x_2 + 3x_3$$

Subject to $x_1 + x_2 \leq 20$

$x_1 + x_3 = 5$

$x_2 + x_3 \geq 10$

$x_1, x_2, x_3 \geq 0$.

Soln) By introducing slack variable S_1 & surpluse variable S_2 and Artifical variable A_1, A_2. The given L.P.P can reformulated as

$$\text{Max } z = x_1 - x_2 + 3x_3 + 0 \cdot S_1 + 0 \cdot S_2 - M \cdot A_1 - M A_2$$

Subject to. C.

$x_1 + x_2 + S_1 = 20$

$x_1 + x_3 + A_1 = 5$

$x_2 + x_3 - S_2 + A_2 = 10$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$.

An initital basic feasible solution is given by,

$x_1 = x_2 = x_3 = 0$.

$S_1 = 20, \ A_1 = 5, \ A_2 = 10$.

writing in matrix form, $AX = B$

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & S_1 & S_2 & A_1 & A_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 10 \end{pmatrix}$$

The Big M method extends the simplex algorithm to problems that contain greater than constraints.

Initial table →

		C_j	1	-1	3	0	0	-M	-M	$x_3 > 0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	min $\frac{x_B}{x_{3j}}$
0	S_1	20	1	1	0	1	0	0	0	—
-M	A_1	5	1	0	1	0	0	1	0	$5/1 = 5$ min
-M	A_2	10	0	1	1	0	-1	0	1	$10/1 = 10$
$Z_j - C_j$			-M-1	-M+1	-2M-3	0	M	0	0	

$Z_j - C_j = C_B X_j - C_j$

$Z_1 - C_1 = 0 \times 1 + (-M) \times 1 + ((-M) \times 0) - 1$

$= -M - 1$

$Z_2 - C_2 = C_B X_2 - C_2$

$= -M + 1$

$Z_3 - C_3 = C_B X_3 - C_3 = -2M - 3$

Since $Z_j - C_j < 0$, the feasible solution is not optimize. Choose the most negative $Z_j - C_j = -2M - 3$, x_3 variable enter the basis and artifical variable A_1 leaves the basis.

		C_j	1	-1	3	0	0	-M	-M	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Min x_B/x_2, $x_2>0$
0	S_1	20	1	1	0	1	0	0	0	$20/1=20$
3	x_3	5	1	0	1	0	0	1	0	-
-M	A_2	5	-1	$\boxed{1}$	0	0	-1	-1	1	$5/1=5$
	Z_j-C_j		M+2	-M+1	0	0	M	3+2M	0	

key element

O.V

$$Z_j - C_j = C_B X_j - C_j$$

		C_j	1	-1	3	0	0	-M	-M
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2
0	S_1	15	2	0	0	1	1	1	-1
3	x_3	5	1	0	1	0	0	1	0
-1	x_2	5	-1	1	0	0	1	-1	1
	Z_j-C_j		3	0	0	0	1	4+M	M-1

<u>Note :-</u> ① If no artifical variable appear in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

② If at least one artificial variable is there in the basis at zero level and the optimality conditions are satisfied, then the current solution is a optimal basic feasible solution. (through degenerated solution)

③ If atleast one artifical variable is appear in the basis at positive level and optimality conditions are satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimise the objective function. It contains a very large penalty M & is called pseudo ~~optimal~~ optimal solution. since all $z_j - c_j \geq 0$, the solution is optimum and is given by,

$x_1 = 0,\ x_2 = 5,\ x_3 = 5.$

$Max\ z = x_1 - x_2 + 3x_3 = 0 - 5 + 3\times5 = 10$

Q) Using S.M to solve the L.P.P

$Max\ z = x_1 + 2x_2 + x_3$

St.C

$2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6 \Rightarrow -2x_1 + x_2 - 5x_3 - s_2 = -6$

$\Rightarrow 2x_1 - x_2 + 5x_3 + s_2 = 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Solⁿ → $Max\ z = x_1 + 2x_2 + x_3$

By introducing slack variable s_1, s_2, s_3

Conver the problem in standard form,

$Max\ z = x_1 + 2x_2 + x_3 + 0.s_1 + 0.s_2 + 0.s_3$

St.C = $2x_1 + x_2 - x_3 + s_1 = 2$

$2x_1 - x_2 + 5x_3 + s_2 = 6$

$4x_1 + x_2 + x_3 + s_3 = 6 \quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

An initial feasible sol^n is given by

$$x_1 = x_2 = x_3 = 0$$
$$s_1 = 2$$
$$s_2 = 6$$
$$s_3 = 6$$

Righting in matrix form

$$Ax = b$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

Construct initial table

		c_j	1	2	1	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min X_B/x_2
0	s_1	2	2	1	-1	1	0	0	$2/1 = 1$ min
0	s_2	6	2	-1	5	0	1	0	—
0	s_3	6	4	1	1	0	0	1	$6/1 = 6$
$z_j - c_j$			-1	$-2\uparrow$	-1	0	0	0	

IN

ON

key element.

$$z_j - c_j = C_B X_j - c_j$$

1st iteration simplex table

	C_j		1	2	1	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	$x_B/x_3>0$
2	x_2	2	2	1	-1	1	0	0	-
0	s_2	8	4	0	4	1	1	0	2
0	s_3	4	2	0	2	-1	0	1	2 min
	$Z_j - C_j$		3	0	-3 ↑	2	0	0	

x_2	s_2	s_3
1	0	0
0	1	0
0	0	1

$x_2/x_3 = 0$

$= 0$

$s_2/x_3 = 1/4 = 0.25$

$s = 0$

2nd iteration table

	C_j		1	2	1	0	0	0
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	4	3	1	0	1	0	1
0	s_2	0	0	0	0	1	1	-4
1	x_3	2	1	0	1	-1/2	0	1/2
	$Z_j - C_j$		12	0	0	3/2	0	5/2

$$\max z = C_B X_B$$
$$= 2 \times 8 + 1 \times 2$$
$$= 16 + 2$$
$$= 18$$

$$\max z = x_1 + 2x_2 + x_3$$
$$= 8 + 2 \times 0 + 2$$
$$= 10$$

x_2	s_2	s_3
1	0	0
0	1	0
0	0	1

2nd iteration table

	C_j		1	2	1	0	0	0
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	4	3	1	0	1/2	0	0
0	s_2	0	0	0	0	3	1	0
1	x_3	2	1	0	1	-1/2	0	1/2
	$Z_j - C_j$		6	0	0	1/2	0	1/2

$$Z_j - C_j = C_B X_j - C_j$$

$$\max z = C_B X_B$$
$$= 4 \times 2 + 0 \times 0 + 1 \times 2 = 10$$

$$\max z = x_1 + 2x_2 + x_3$$
$$= 0 + 2 \times 4 + 2 = 10$$

Q) Using two-Phase method to solve the given L.P.P.

$$Max\ z = 5x_1 - 4x_2 + 3x_3$$

S.t.C

$$2x_1 + x_2 - 6x_3 = 20$$

$$6x_1 + 5x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0.$$

Solⁿ → we convert the L.P.P in to standard form by using ~~Stang~~ slack variable 'S_1' & 'S_2' and artificial variable 'A_1'

$$Max\ z = 5x_1 - 4x_2 + 3x_3 + 0S_1 + 0S_2$$

S.t.C

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$

$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$

$$8x_1 - 3x_2 + 6x_3 + S_2 = 50.$$

$$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0.$$

$x_1 = x_2 = x_3 = 0$,

$A_1 = 20$

$S_1 = 76$

$S_2 = 50$

Phase-1 Assuming a cost -1 to the artificial variable A_1 and cost 0 to the other variables.

The objective function of the auxiliary L.P.P is

$$\text{Max } z^* = 0x_1 - 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1$$

s.t.c

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$

$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$

$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

$$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$$

Writing in matrix form

$Ax = b$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & A_1 & S_1 & S_2 \end{array}$$

$$\begin{pmatrix} 2 & 1 & -6 & 1 & 0 & 0 \\ 6 & 5 & 10 & 0 & 1 & 0 \\ 8 & -3 & 6 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ A_1 \\ S_1 \\ S_2 \end{pmatrix}$$

Initial simplex table

		Cj	0	0	0	0	0	-1
C_B	B	X_B	-2	~~0~~1	-6	0	0	1
-1	A_1	-20	6	5	10	1	0	0
0	S_1	76						
0	S_2	50						

Initial simplex table

I.V

		C_j	0	0	0	0	0	-1	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	min Ratio $\frac{X_B}{x_1} > 0$
-1	A_1	20	2	1	-6	0	0	1	10
0	S_1	76	6	5	10	1	0	0	12.66
0	S_2	50	8	-3	6	0	1	0	6.25 min
O.V		$Z_j - C_j$	$-2\uparrow$	-1	6	0	0	0	

$$Z_j - C_j = C_B X_j - C_j$$

1st iteration table

I.V

		C_j	0	0	0	0	0	-1	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	min $\frac{X_B}{x_2} > 0$
-1	A_1	$15/2$	0	$7/4$	$-15/2$	0	$-1/4$	1	4.28
0	S_1	$77/2$	0	$29/4$	$11/2$	1	$-3/4$	0	5.31
0	x_1	$\frac{25}{4}$	1	$\frac{-3}{8}$	$\frac{3}{4}$	0	$1/8$	0	—
O.V		$Z_j - C_j$	0	$-7/4\uparrow$	$15/2$	0	$1/4$	0	

$$Z_j - C_j = C_B X_j - C_j$$

A_1	S_1	x_1
1	0	0
0	1	0
0	0	1

2nd iteration table

		C_j	0	0	0	0	0	-1
C_B	B	X_B	x_1	x_2	x_3	s_1	s_2	A_1
0	x_2	$30/7$	0	1	$-30/7$	0	$-7/7$	$4/7$
0	s_1	$52/7$	0	0	$250/7$	1	$2/7$	$-29/7$
0	x_1	$55/7$	1	0	$-6/7$	0	$1/14$	$3/14$
$z_j - C_j$			0	0	0	0	0	1

$$z_j - C_j = C_B X_B - C_j$$

$$x_2 = 0, x_1 = 0, x_3 = 0$$

$$\text{Max } z^* = 3x_1 - 4x_2 + 3x_3 = 0.$$

Since all $z_j - C_j \geq 0$ an optimum sol^n to the auxilary L.P.P has been obtained and $\max z^* = 0$ and no artificial variable appear in the basis. So we go the phase 2.

Phase-2

Consider the final simplex table of phase-1. Consider the actual cost associated with the original variables deleted the artificial variable A_1 Coloumn from the table as it is elemnated in phase-1.

	C_j		5	-4	3	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
-4	x_2	30/7	0	1	-30/7	0	-1/7
0	S_1	52/7	0	0	250/7	1	2/7
5	x_1	55/7	1	0	-6/7	0	1/14
	$Z_j - C_j$		0	0	69/7	0	13/14

$x_1 = 5, \quad x_2 = -4, \quad x_3 = 0$

$Max\,z = 5x_1 - 4x_2 + 3x_3$

$= \cancel{5 \times 5} \quad \cancel{= 5 \times 5 - 4 \times (-4) + 0}$

$\cancel{= 41}$

$= 5 \times \frac{55}{7} - 4 \times \frac{30}{7}$

$= \frac{275}{7} - \frac{120}{7} = \frac{155}{7}$

$Max\,z = C_B X_B$

$= -4 \times \frac{30}{7} + 5 \times \frac{55}{7}$

$= \frac{-120}{7} + \frac{275}{7}$

$= \frac{155}{7}$

Integer Programming Problem :->

→ In L.P.P in which all the decision variable are to assume non-negative integer values is constrain called an integer programming problem.

Ex: A ~~for~~ furniture ~~dealer~~ daler deals in 2 items tables & chair. He has spent 10,000 and store almost 60 pices a table cost is 500 per piece and chair cost is 200 per piece. He can ~~sell~~ sell a table at profit of 50 rupees and ~~sale~~ sell a chair at profit of 20 rupees assume that he can sell all the items that he byes formulate the problem as an L.P.P. so that he can maximize the profit.

→ There are two methods to solve IPP.
a) Gomory's cutting plane method.
b) Branch & Bound method
~~Dual Programming~~.

Sol^n :->

$$Max\ z = 50x_1 + 20x_2$$

$$= 10000.$$

s.t.c. $x_1 + x_2 \leq 60.$

$500x_1 + 200x_2 \leq 100,00.$

$x_1, x_2 \geq 0.$

In the optimal of a L.P.P are restricted to assume non-negative integer value while the remaining variable are free to take any non-negative values then it is called mixed integer programming problem.

→ If all the variables ~~of~~ in the optimal soln are allow to take values 0 or 1 the problem is called 0-1 Programming problem.

Duality :-

→ Every LPP (called the Primal) is associated with another LPP (called the dual).

The importance of the duality concept is due to main reasons.

i) If the primal contains a large number constraints and a smaller number of variables, the labour of computation can be reduced by converting it in to the dual problem.

ii) The interpretation of the dual variables from the cost or economic point of view, the dual problem is more useful in making further decisions.

Formation of Dual Problem

For formulating dual problem, first we bring the problem in the cannonical form. The following changes are used in formulating the dual problem.

Steps

1) First we convert the problem in cannonical form.

Min → Min $\geq$

Max → Max $\leq$

2) Change the objective function of maximization in the primal in to minimization one in the dual and vice-versa

Max Z → Min Z $\geq$

Min Z → Max Z $\leq$

(2) Change the objective function of maximization in the Primal into minimization one in the dual and vice-versa.

$$Max\ z \rightarrow Min\ z \geq$$
$$Min\ z \rightarrow Max\ z \leq$$

(3) The number of variable in the Primal will be the no of Constraints in the dual and vice-versa.

~~(4) In forming the constraints for the dual, we consider the~~

(4) The cost coefficien $c_1, c_2 ---- c_n$ in the objective function of the Primal will be the RHS Constant of the Constraints in the dual & vice-versa.

(5) In forming the constraints for the dual, we consider the transpose if body matrix of the Primal Problem.

(6) If variables in both Problems are non-negative

(7) If the variable in the Primal is unrestricted in sign, then the corresponding constraints in the dual will be an equation & vice-versa.

~~Defined~~ Definition on the Dual Problem

Primal	Dual
$Max\ z = c_1x_1 + c_2x_2 --- + c_nx_n$	$Min\ z = b_1w_1 + b_2w_2 + --- + b_nw_n$
Subject to	Subject to
$a_{11}x_1 + a_{12}x_2 + --- + a_{1n}x_n \leq b_1$	$a_{11}w_1 + a_{21}w_2 + a_{m1}w_m \geq c_1$
$a_{21}x_1 + a_{22}x_2 + --- + a_{2n}x_n \leq b_2$	$a_{12}w_1 + a_{22}w_2 + --- + a_{m2}w_m \geq c_2$
$\vdots$	$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + --- + a_{mn}x_n \leq b_m$	$a_{1n}w_1 + a_{2n}w_2 + --- + a_{mn}w_m \geq c_n$
$x_1, x_2, --- x_n \geq 0$	$w_1, w_2 ---- w_m \geq 0$

Max Z	Min Z
$\leq$	$\geq$
$\leq$	$\geq$
$x_1, x_2 \geq 0$	$w_1, w_2 \geq 0$

Column into Row

Example 1

Write the dual of the following Primal LP Problem

$\text{Max } z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Soln: Since the Problem is not in the Canonical form, we interchange the inequality of second constraint.

$\text{Max } z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \leq 2$

$(-1) \quad 2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

Dual Let w_1, w_2, w_3 be the dual variables.

$\text{Min } Z' = 2w_1 + 6w_2 + 6w_3$

Subject to

$2w_1 + 2w_2 + 4w_3 \geq 1$

$w_1 - w_2 + w_3 \geq 2$

$-w_1 + 5w_3 + w_3 \geq 1$

$w_1, w_2, w_3 \geq 0.$

Example-2

Write the dual of the following primal LP problem

~~Max $z = x_1 + 2x_2 + x_3$~~

~~Subject to $2x_1 + x_2 - x_3 \leq 2$~~

~~$-2x_1 + x_2 - 5x_3 \geq -6$~~

~~$4x_1 + x_2 + x_3 \leq 6$~~

~~$x_1, x_2, x_3 \geq 0$~~

Min $Z = 2x_2 + 5x_3$

subject to

$x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

~~Soln~~

Soln

Since the given primal problem is not in the canonical form, we interchange the inequality of the constraints. Also the third constraints in an equation. This equation can be converted into two inequations.

$\text{Min } z = 0x_1 + 2x_2 + 5x_3$

Subject to

$x_1 + x_2 + 0x_3 \geq 2$

multiply (-1). $-2x_1 - x_2 - 6x_3 \geq -6$

$x_1 - x_2 + 3x_3 \leq 4$

$x_1 - x_2 + 3x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Again on rearranging the constraint, we are

$\text{Min } z = 0x_1 + 2x_2 + 5x_3$

Subject to

$x_1 + x_2 + 0x_3 \geq 2$

$-2x_1 - x_2 - 6x_3 \geq -6$

$-x_1 + x_2 - 3x_3 \geq -4$

$x_1 - x_2 + 3x_3 \geq 4$ $\quad x_1, x_2, x_3 \geq 0$

<u>Dual</u> : Since there are 4 Constraints in the Primal, we 4 dual variable namely w_1, w_2, w_3 & w_4 :

$$Max\ z = 2w_1 - 6w_2 - 4w_3 + 4w_4$$

$$St.C = w_1 - 2w_2 - w_3 + w_4 \leq$$

Two-PHASE SIMPLEX METHOD steps 1-7

The two-phase simplex method is another method to solve a given LPP involving some artificial variable.

The solution is obtained in two phases.

Phase-1

In this phase, we construct an auxiliary LPP leading to a final simplex table containing a basic feasible to solution to the original problem.

Step-1 Assign a cost -1 to each artificial variable and a cost 0 to all other variables and get a new objective function $z^* = -A_1 - A_2 - A_3$ ---- where A_i are artificial variable.

Step-2 Write down the auxiliary LPP in which the new objective function is to be maximized subject to the given set of constraints.

Step-3 Solve the auxiliary LPP by simplex method until either of the following three cases arise:-

i) max $z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level.

ii) max $z^* = 0$ and at least one artificial variable appears in the optimum basis at zero level.

iii) max $z^* = 0$ and no artificial variable appears in the optimum basis.

In case (I), given LPP doesnot possess any feasible solution, whereas in case (ii) & (iii) we go to phase (II)

Phase-2

Use the optimum basic feasible solution of phase 1 as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and zero cost to every ~~arti~~ artificial variable in the basis at zero level. Delete the artificial variable column ~~fro~~ from the table which is ~~elo~~ eliminated from the basis in Phase 1. Apply simplex method to the modified simplex table obtained at the end of phase 1 till an ~~opti~~ optimum basic feasible is obtained or till there is an indication of unbounded solution.

Steps for a Standard Primal form :-

1. Change the objective function to maximization form.

2. If the constraints $\geq$ sign then convert into $\leq$ sign.

3. If the constraints has an $=$ sign then replace it two constraints involving the inequalities going in opposite direction.

Ex:- $x_1 + x_2 = 4$

$\Rightarrow x_1 + x_2 \leq 4$
$x_1 + x_2 \geq 4$

4. Every unrestricted is replaced by the differences two non-negative variables.

Ex:- $x_1, x_2 \geq 0$, x_3 is unrestricted

$x_3 = x_3' - x_3''$ when x_3' & x_3'' both non-negative variables.

5. The standard primal from of the given L.P.P in which.

i) all constraints have $\leq$ sign, where the objective function is maximization form

ii) all constraints have $\geq$ sign, where the objective function is ~~minimization~~ minimization form.

Q) ~~Min Z~~ Using duality to solve the L.P.P

$\text{Min } Z = 2x_2 + 5x_3$

St: C

$x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

$\text{Max } z = -2x_2 - 5x_3$

St.C $= x_1 + x_2 \geq 2 \Rightarrow (-1)* \; -x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$\Rightarrow x_1 - x_2 + 3x_3 \leq 4$

$x_1 - x_2 + 3x_3 \geq 4 \Rightarrow (-1)* \; -x_1 + x_2 - 3x_3 \leq -4$

$\text{Max } z = -2x_2 - 5x_3$

St C $= -x_1 - x_2 \leq -2$

~~$x_1 - x_2 + 3x_3$~~

$2x_1 + x_2 + 6x_3 \leq 4$

$x_1 - x_2 + 3x_3 \leq 4$

$-x_1 + x_2 - 3x_3 \leq -4$

<u>Duality</u> → let w_1, w_2, w_3, w_4 the dual variables

$\text{Min } Z' = -2w_1 + 4w_2 + 4w_3 - 4w_4$

St.C $= -w_1 + 2w_2 + w_3 - w_4 \geq 0$

$-w_1 + w_2 - w_3 + w_4 \geq -2$

$6w_2 + 3w_3 - 3w_4 \geq -5$

$w_1, w_2, w_3, w_4 \geq 0$

Q) $\text{Min } Z = x_1 + x_2 + x_3$

s.t.c

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted.

Sol^n Standard Primal

$\text{Max } z = -x_1 - x_2 - x_3$

s.t.c

$x_1 - 3x_2 + 4x_3 \leq 5$ — (1)

$x_1 - 3x_2 + 4x_3 \geq 5$

$\Rightarrow -x_1 + 3x_2 - 4x_3 \leq -5$ — (2)

$x_1 - 2x_2 \leq 3$ — (3)

$-2x_2 + x_3 \leq -4$ — (4)

Again ~~rearrange~~ rearranging the constraints

$\text{Max } z = -x_1 - x_2 - (x_3' - x_3'')$

s.t.c. $x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$

$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$

$x_1 - 2x_2 \leq 3$

$-2x_2 + (x_3' - x_3'') \leq -4$

$x_1, x_2, x_3', x_3'' \geq 0$.

Duality → w_1, w_2, w_3, w_4 are the dual variable.

$$\text{Min } Z' = 5w_1 - 5w_2 + 3w_3 - 4w_4$$

$$\text{s.t.c} = w_1 - w_2 + w_3 \geq -1$$

$$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$$

$$+4w_1 - 4w_2 + w_4 \geq -1$$

$$\cancel{w_1, w_2, w_3, w_4 \geq 0}$$

$$-4w_1 + 4w_2 - w_4 \geq 1$$

$$w_1, w_2, w_3, w_4 \geq 0$$

Important Results in Duality :→

1. The dual of the dual is Primal.
2. If one is maximization Problem then the other is a minimization one.
3. The neccessary and sufficient Condition for any LPP and its dual to hava an optimal Solution is that both must have feasible solution.
4. Fundamental duality theorem states if either the Primal or dual Problem has a finite optimal solution, then the other Problem also has a finite optimal solution and also the optimal values of the objective function in both the Problems are the same is $\max z = \min z'$. The solution of the other Problem can be read from the $z_j - c_j$ row below the Columns of slack, surplus variables.
5. Existence Theorem states that, if either problem has an unbounded solution then the other problem has no feasible solution.

6) Complementary slackness theorem: According to which

i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa.

ii) If a primal constraint is a strict inequality then the corresponding dual variable is zero at optimum and vice versa.

Goal Programming :-

→ Goal Programming is a extension of linear programming to handle multiple ~~ob~~ objective measures.

UNIT-2

Transportation problem and program :-

The transportation problem is one of the subclasses of LPP in which the objective is to transport various quantities of a single homogeneous comodity that are initially stored at a various origins to difference destinactions.

In such a way that the transportation cost is in.

Mathematical formulation :-

- Consider a transportation problem if M objects (rows) and n destinations (columns).
- Let cj be the cost of transporting one unit of the Product from the ith origin to jth destination. ai is the quality of commodity available at origin i. bj is the quality of commodity needed at destination j. xij is the quality transported from ith origin to jth destination.
- The Linear Programming model representing the transportation problem is given by,

$$\text{minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i = 1, 2 ---- n)$$

(Row sum)

$$\sum_{i=1}^{m} x_{ij} = b_j \ (i = 1, 2, --- n)$$

(column Sum)

$$x_{ij} \geq 0 \ \forall i \& j$$

- The transportation problem is said to be balanced if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ i.e if the total supply is equal to the total demand.

Destinations

Origin	1	2	3	------ n	Supply
1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	---- x_{1n} c_{1n}	a_1
2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	---- x_{2n} c_{2n}	a_2
3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	---- x_{3n} c_{3n}	a_3
⋮ m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}	---- x_{mn} c_{mn}	⋮ a_m
Demand	b_1	b_2	b_3	---- b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Feasible Solution :-

A set of non-negative allocation ($x_{ij} > 0$) which satisfies the row and column sum is called a feasible solution

$$\sum \text{Demand} = \sum \text{Supply}$$

Basic feasible Solution :-

- A feasible solution is called a basic feasible solution is the no of non-negative allocations is equal to $m+n-1$.
- where m is the no of rows and n is the no of columns in transportation table.

Not-degenerate basic feasible solution :-

Any feasible solution to a transportation problem containing m origin of n destination is said to be non-degenerate. If it contains $m+n-1$ occupied cells and each allocation is in independent position.

Transportation problem:-

- The transportation problem is one of subclasses of LPP in which the objective is to transport various quantities of a single homogeneous commodity that are initially store at a various origin to different destination is such a way that the transportation cost is minimum.

	*	*
	*	*

*		*
*		*

	*	*	
	*		
	*	*	

The allocation in the following table are in independent position.

	*	
*	*	*
*		

*	*		
	*		*
		*	*

Degenerate basic feasible solution→

→ If a basic feasible solution contains less than m+n-1 non negative allocation, it is said to be degenerate basic feasible solution.

Optimal solution→

→ optimal solution is a feasible solution (not necessary basic) which minimizes the total cost.

→ The solution of transportation problem can be obtained in two stages namely, initial soln & optimum solution initial solution can be obtained by using one of the three Methods.

(i) North - west corner Rule (NWCR)

(ii) Least cost Method or Matrix minimal method.

(iii) Vogle's approximation Method (VAM).

- VAM is performed over the other 2 Method, since the initial basic feasible solution obtained by this method is either optimal or very close to optimal solution.

- The cells in the transportation table can be classified as occupied cells & unoccupied cells. The allocated cell is unoccupied cells.

- The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution, that can be obtained by MODI (Modified distribution Method)

North - west Corner Rule →

Step -

1. Starting with the cell at the upper left corner (north west) of the transportation matrix. we allocate as much as possible so that either the capacity of the first row is exhausted or destination requirement of the first column is satisfied i.e, $x_{11} = \min(a_1, b_1)$

2. If $b_1 \geq a$, we move down vertically to the second row & make the second allocation of magnitude $x_{22} = \min(a_2 b_1 - x_{11}$ in the cell (2, 1)

- If $b_1 < a_1$ move right horizontally to the second column & make the second allocation of magnitude $x_{12} = \min(a_1, x_{11} - b_1)$ in the cell (1,2)

$x_{12} = \min(a_1 - a_1, b_1) = 0$ in cell (1,2)

are $x_{21} = \min(a_2 - b_1 - b_1) = 0$ in the cell (2,1)

3. Repeat step 1 & 2 moving down towards the lower right corner of the transportation table until all the requirement are satisfied.

Ex- obtain the initial basic feasible solution of a transportation problem whose cost & requirements table is given below.

Origin/destination	A1	A2	A3	Supply
O1				
O2				
O3				
O4				
Demand	7	4	18	34

Soln-

Since $\Sigma a_i = 34 = \Sigma b_j$ there exists a feasible solution to the transportation problem. We obtain initial solution as follows.

The first allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(5, 7) = 5$. The second allocation is made in the cell (2,1) & the magnitude of the allocation is given by $x_{21} = \min(8, 2) = 2$

North-west Corner Rule →

N

O \ D	D1	D2	D3	Supply
O1	2 $\boxed{5}$	7	4	~~5~~ 0
O2	3 $\boxed{2}$	3 $\boxed{6}$	1	~~8~~ ~~6~~ 0
O3	5	4 $\boxed{3}$	7 $\boxed{4}$	~~7~~ ~~4~~ 0
O4	1	6	2 $\boxed{14}$	~~14~~ 0
Demand	~~7~~ ~~2~~0 5	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	

Σ supply = Σ demand = 34

The 1st allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(5, 7) = 5$

Total cost =

→ The third allocation is made in the cell (2,2) the magnitude $x_{22} = \min(8-2, 9)$
$= (6, 9)$

→ The magnitude of 4th allocation is made in the cell (3,2) ~~the magnitude $x_{22} = \min$~~ ~~is~~ given by $\min(7, 9-6) = (7, 3)$

→ The 5th allocation is made in the cell (3,3) with magnitude $x_{33} = \min(7-3, 14) = (4, 14)$

→ The final allocation is made in the cell (4,3) with magnitude $x_{43} = \min(14, 18-4)$
$= (14, 14)$

Hence, we get the initial basic feasible sol^n to the given T.P & is given by $x_1 = 5$,
$x_{21} = 2$, $x_{22} = 6$, $x_{32} = 3$, $x_{33} = 4$, $x_{13} = 14$

Que) find the solution using NWCR method.

S \ D	D_1	D_2	D_3	Dummy column	Supply
S_1	4	8	8	0	76
S_2	16	24	16	0	82
S_3	8	16	24	0	77
Demand	72	102	41	26	235

Total supply = 235

Total Demand = 215

Σ Supply = Σ demand = 34

Σ Supply $\neq$ Σ demand

The problem in unbalanced TP as the Total Supply is greater than total demand.

TP

- unbalanced TP
  - Demand > Supply: add a dummy row with cost zero & Supply equal to (demand - supply)
  - Demand < Supply: add a dummy Column with cost zero & demand equal to (Supply - demand)
- Balanced TP

S \ D	D_1	D_2	D_3	column	Supply
S_1	4	8	8	0	~~76~~ 40
S_2	10	24	16	0	~~82~~ 0
S_3	8	16	24	0	~~74~~ 36
demand	72	102	41	20	235

$$\text{Total cost} = (4\times12) + (8\times4) + (24\times82) + (16\times16) + (24\times4) + 0$$

$$= 288 + 32 + 1968 + 256 + 184$$

$$= 3{,}528$$

Vogel's Approximation Method (VAM) :-

Step. 1

1. find the penalty cost namely the difference between the smallest & the next smallest cost in each Row and column.
2. Among the penalties as found in step. 1, chose the maximum penalty. If this penalty is more than 1, chose any one arbitarly.
3. In the selected row or column as by step-2 find out the cell having the least cost allocate to this cell as much as possible depending upon the capacity & requirements
4. Delete the row or column which is fully exhausted again compute the column & row Pendalties called the reduce transportation table & then go to step 2 repeat the Procedure until all the requirements & are satisfied.

Ex:-

Step:- 1

	D_1	D_2	D_3	D_4	Supply	P_1	P_2	P_3	P_4	P_5
O_1	11 $\boxed{200}$	13 $\boxed{150}$	17	14	~~350~~ ~~150~~ 0	2	1	-	-	-
O_2	16	18 $\boxed{175}$	14	10 $\boxed{125}$	300	4	4	4	4	-
O_3	21	24	13 $\boxed{275}$	10 $\boxed{125}$	400 250	3	3	3	3	3
Demand	~~200~~ 0	~~225~~ ~~175~~ 0	275	250						
P_1	5↑	5	1	0						
P_2	-	5↑	1	0						
P_3	-	6↑	1	0						
P_4	-	-	1	0						
P_5	-	-	13↑	10						

No of allocated cell $= m+n-1$

$= 3+4-1$

$= 6$

- There are 6 positive independent allocation so the solution is non-degenerate basic feasible solution.

The total transportation cost is $11 \times 200 + 13 \times 150 + 18 \times 175 + 13 \times 275 + 10 \times 125$

Optimality test :-

- once the initial basic feasible solution has been computed the next step in the problem is to determine whether the solution obtained is optimum or not.
- optimality test can be conducted any initial basic feasible solution TP provided such allocation has exactly $(m+n-1)$ allocation where M is the no of origin of n is no of destination also this allocation

must be independent position.

- To perform the optimality test we shall discuss modi (modified distribution) method.

Step-1

1. Find the initial basic feasible solution of a TP by using any one of the 3 method.

2. Find out a set of numbers u_i & v_j for each row & columns satisfying $u_i+v_j = c_{ij}$ for each optimal cell who start we assign a number 0 to any row or column having maximum no of allocations is more than one choose any one arbitary.

3. For each empty an occupied cell we find the sum u_i & v_j written at the bottom right corner of that cell. this steps gives the optimality conclusion.

4. Findout for each empty cell the net evaluation value which is written at the bottom right corner of that cell. this steps gives the optimality conclusion.

   (i) If all $\Delta_{ij} > 0$, the solution is optimum & a unique solution exists.

   (ii) If $\Delta_{ij} \geq 0$, then solution is optimum but an alternate solution exists.

   (iii) If atleast one $\Delta_{ij} < 0$, the solution is not optimum. in this case we go to the next step, to improve the total transporcation cost.

5. Select the empty cell having the most negative value of A_{ij}.

From this cell we draw a closed path by drawing horizontal and vertical lines with the corenerc cell occupied assigned sign (+) & (-) alternatively and find the minimum allocation from the cell having -ve sign. this allocation should be added to the allocation having +ve sign & subtracted from the allocation having -ve sign.

6. The above step in a better solution by making one or more occupied for this new setup basic Feasible allocation. repeat from step 2 it is an optimum basic feasible solution is occupied.

Q) Find the optimal solution of the transportation Problem (by using MODI method)

O \ D	D_1	D_2	D_3	D_4	Supply
O_1	3 [200]	1 [50]	7	4	250
O_2	2	6 [250]	5 [100]	9	350
O_3	8	3	3 [250]	2 [150]	400
Demand	200	300	350	150	

Total supply = 250 + 350 + 400 = 1000

Total demand = 200 + 300 + 350 + 150 = 1000

Total supply = total balanced

So, the Tp is balanced

- We find the initial basic feasible solution by a north west corner rule.

O \ D	D_1	D_2	D_3	D_4	Supply	
O_1	3 $\boxed{200}$	1 $\boxed{50}$	7	4	~~250~~ ~~50~~ 0	$u_1 = 0$
O_2	2	6 $\boxed{250}$	5 $\boxed{100}$	9	~~350~~ ~~100~~ 0	$u_1 = 5$
O_3	8	3	3 $\boxed{250}$	2 $\boxed{150}$	~~400~~ ~~150~~	$u_3 = 3$
	200	~~300~~ ~~250~~	~~350~~ ~~250~~ 0	~~150~~	100	

$V_1 = 3 \quad V_2 = 1 \quad V_3 = 0 \quad V_4 = 1$

$(3 \times 200) + (1 \times 50) + (6 \times 250) + (5 \times 100) + (3 \times 250) + (2 \times 150)$

$= 600 + 50 + 1500 + 1500 + 750 + 300$

$= 3700$

We apply modi method to find out a set of number u_i & v_i for which $u_i + v_j = c_{ij}$ only for occupied cell.

<u>Step-4</u>

$\Delta_{ij} = c_{ij} - (u_i - v_j)$

$\Delta_{13} = c_{13} - (u_i + V_3)$

$= 7 - (0+0) = 7$

$\Delta_{14} = c_{13} - (u_1 + V_4)$

$= 4 - (0+0-1)$

$= 4+1 = 5$

$\Delta_{21} = C_{21} - (u_2 + v_1)$

$= 2 - (5 + 3)$

$= 2 - 8 = 6$

$\Delta_{24} = C_{24} - (u_2 + v_4)$

$= 9 - (5 + (-1))$

$= 9 - 4 = 5$

$\Delta_{31} = C_{31} - (u_3 + v_1)$

$= 8 - (3 + 3)$

$= 2$

$\Delta_{32} = C_{32} - (u_3 + v_2)$

$= 3 - (3 - 1)$

$= -1$

If all $\Delta_{ij} \geq 0$, the solution

Assignment problem :-

- Suppose there n jobs to be Perform of N persons are availables for doing job. Assume that each person can do each job at a time through with varying degree of efficiency.
- The objective of assignment problem is to assign a number of jobs to the equal number of persons at a minimum cost or maximum profit.

Ex using the following cost matrix determining the optimal job assignment & the cost of assignment.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{bmatrix}
10 & 3 & 3 & 2 & 8 \\
9 & 7 & 8 & 2 & 7 \\
7 & 5 & 6 & 2 & 4 \\
3 & 5 & 8 & 2 & 4 \\
9 & 10 & 9 & 6 & 10
\end{bmatrix}
\end{array}
$$

Mechanic

Soln:-

→ No. of row = 5

→ No. of column = 5

No. of rows & No. of columns are same.
So, the cost matrix is a square matrix.

1. → Select the smallest element in each rows & substract this smallest element from all the elements in its row.

$$
\begin{bmatrix}
8 & 1 & 1 & 0 & 6 \\
7 & 5 & 6 & 0 & 5 \\
5 & 3 & 4 & 0 & 2 \\
1 & 3 & 6 & 0 & 2 \\
3 & 4 & 3 & 0 & 4
\end{bmatrix}
$$

3 → Select the smallest element in each column Substracted this smallest element from all the elements in this column.

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

3 → Draw minimum no of horizontal & vertical lines (N) to cover all zero.

(a) If N=n, n = order of matrix, then an optimal solution can be made.

(b) If N<n, then go to next step.

$$\begin{bmatrix} 7 & \boxed{0} & 0 & 0 & 4 \\ 6 & 4 & 5 & \boxed{0} & 3 \\ 4 & 2 & 3 & 0 & \boxed{0} \\ \boxed{0} & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

No of lines drawn assignment problem to cover all zeros is N = 4.

The order of the matrix is n = 5. N<n, the solution is not optimum.

4 → Determine the smallest uncovered value (x)

(a) write uncovered value = uv − x

(b) Intercast value = intersection value + x

(c) line value (other values) at same.

The modified is given by

where $x = 2$ (smallest no)

(second)

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \end{array}$$

$$\begin{bmatrix} 9 & \boxed{0} & 0 & 2 & 6 \\ 6 & 2 & 3 & \boxed{0} & 3 \\ 4 & 0 & 1 & 0 & \boxed{0} \\ \boxed{0} & 0 & 3 & 0 & 0 \\ 2 & 1 & \boxed{0} & 0 & 2 \end{bmatrix}$$

(first)

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 9 & 0 & 0 & 2 & 6 \\ B & 6 & 2 & 3 & 0 & 3 \\ C & 4 & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 3 & 0 & 0 \\ E & 2 & 1 & 0 & 0 & 2 \end{array}$$

No of lines drawn to cover all zeros, $N = 5$

The order of matrix $n = 5$

$N = n$, the solution is optimum, we determine the optimum Assignment.

Job	Mechanic	cost
1	D	3
2	A	3
3	E	4
4	B	2
5	C	4

minimum total cost $= 3+3+9+2+4$

$=$ RS $21/-$

Network Analysis

Construction of the network diagram :-

1. Introduction:-

It is a technique used for planning & scheduling large projects in the field of purchasing construction, maintanance of fabrication of computer systems etc.

2. Projects:-

A Project is defined as combination of interrelated activities, all of which must be executed in a certain order its completion.

3. Phases of Project Management :-

(a) Planning :-

- Divided /decided the project into distinct activities.
- Estimate time requirement for activities.
- Men, Machines & materials required for the projects in addition to the estimates costs.
- construct arrow diagram.

(b) Scheduling :-

- Determine the start & end time for activity.
- Determine the Critical path on which the activity require attention
- Determine the slack & float for non-critical path.

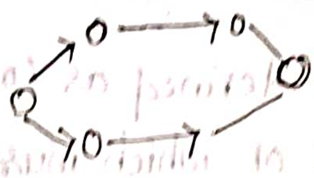
(c) Controlling :-

- making the Periodical progress report.
- Reviewing the progress.
- Analyzing the status of project.

4. Basic terms :-

(a) Network diagram :-

It is the graphical representation of logically & Sequencially connected arrows (→) & nodes (o), representing activities & events in a project.



(b) Types of Activity :-

(i) Prerceeding Activity :-

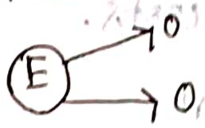
Activity that must be accombleished.

(ii) Succeeding Activity :-

Activity that can't be accombleished until an event has occurred/happened

(iii) Concurrent Activity :-

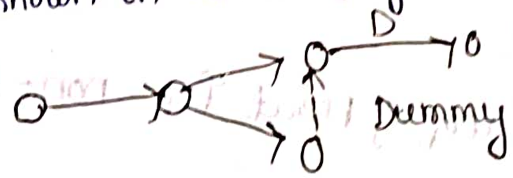
Activity taking place at the same time or in the same direction.



(iv) Dummy Activity :-

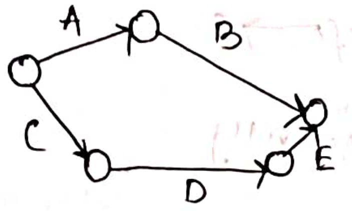
Activity which neither consume time or resources but are used simply to represent a connection or a link between the events are known as dummies.

It is shown in network by a dotted lines.



(b) Activity :-

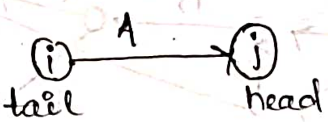
It represents some actions and in a time consuming effort, necessary to complete a particular part of overall project.



A, B, C, D & E are activity.

(c) Event :-

Beginning and end points of an activity are called events or nodes.



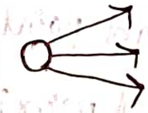
(i) Merge event :-

It is not necessary for an event to be the ending of only one activity but can be the ending event of two or more activities. such event is called as a Merge event.


ending point

(ii) Burst event :-

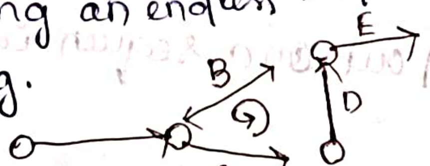
If the event happen to be the beginning event of two or more activities. it is defined as a burst event.

Starting point


(d) Common errors :-

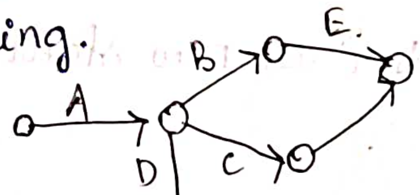
(i) Looping error :-

Drawing an endless Loop in a network is known as error of looping.



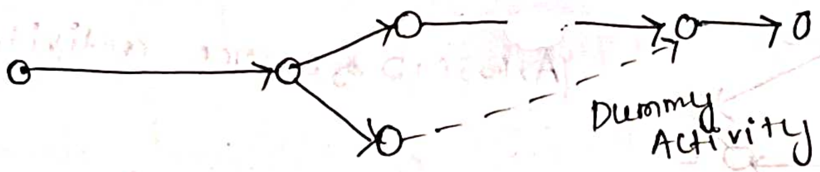
(ii) Dangling :-

To disconnect an activity before the completion of all activities called error of dangling.



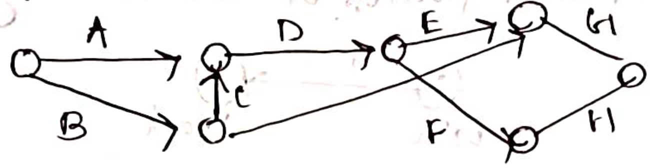
(iii) Redudancy :-

If a dummy actifity is the only activity emanating from an event & can be eliminated called redundancy.



Rules of Network construction :-

→ Try to avoid the arrow that cross each other.



→ Use always straight arrow.

→ No event can occure until every activity preceding it has been completed.

→ An event can't occure twice.

→ Dummies should be introduced only, if it is extremely neccessary.

→ Network has only one entry point called start event & one end point is called end point.

→ Use arrow left to right. Avoid mixing two directions.

Numbering the events :-

→ Number must be unique

→ Number should be carried out on a sequencial basis from left to right.

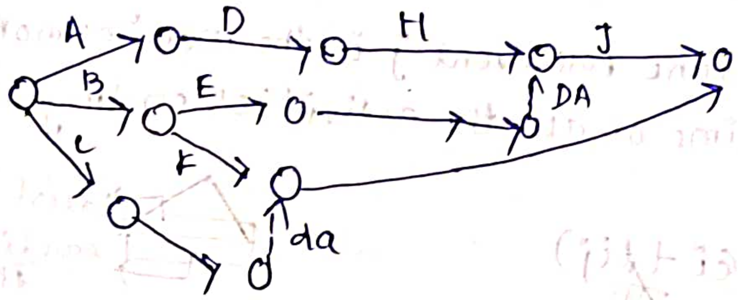
→ Initial event must be numbered as 1.

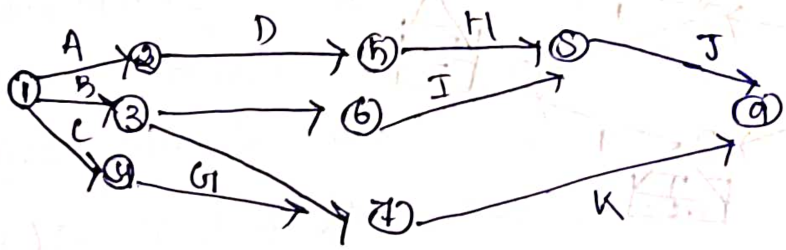
→ Number all new start event 2, 3 & so on.

Construction of Network :-

Q. Construct a Network for the Project whose activities and precedence relationships are as given below.

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	-	A	B	B	C	D	E	H, I	F, G





Time analysis :-

Once the network of a Project is constructed the time analysis of the network becomes essential for planning various activities of the Project. An activity time is a forecast of the time an activity is expected to take from its starting point to its completion. For basic scheduling we shall use the following notation for basic scheduling computations.

( ) = Activity (i, j) with tail event i & head event j

= Activity (i,j) with estimated completion time of activity (i, j)

- Estimated completion time of activity (i, j)

= Earliest starting time of activity (i, j)

= Earliest finishing time of activity (i, j)

= Latest starting time of activity (i, j)

= Latest finishing time of activity (i, j)

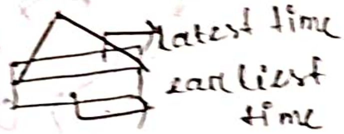
(a) **Forward pass computation :-**

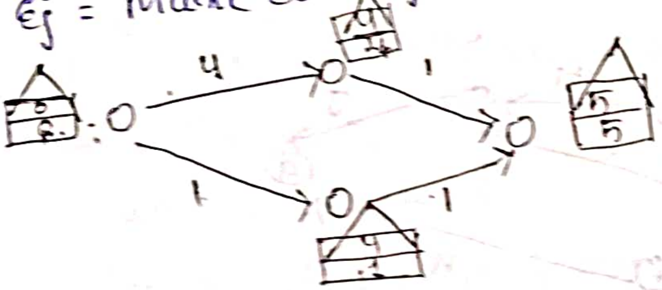
Step 1 :-
The computations begin from the start node & move towards the end node.
Let zero be the starting time for the project.

Step 2 :- $(EF)_{ij} = (ES)_{ij} + t_{ij}$

Step 3 :-
Earliest event time for event j is the maximum of the earliest finish time of all the activities ending of that event.

$E_j = Max(E_i + t_{ij})$





(b) **Backward pass Computations :-**

Step 1 :- For ending event assume $E = L$

Step 2 :- Latest finish time for activity (i, j) is the target time for completing the project.

Step 3 :- Latest starting time of the activity (i, j)
= Latest completion time of (i, j) − the activity time

$LS_{ij} = LF_{ij} - t_{ij}$
$= L_j - t_{ij}$

Step 4 :- Latest event time for event i is the minimum of the latest start time of all activities originating from the event

$$\boxed{L_i = \min_j (L_j - t_{ij})}$$

Determination of Floats & slack times:-

(i) Float :-

It is defined as the difference between the latest and earliest activity time.

(ii) Slack :-

It is defined as difference between the latest and the earliest event time.

Types of Float :-

1. Total Float :-

It refers to the amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

$(TF)_{ij}$ = (Latest start - earliest start) for activity (i, j)

i.e Total Float $(TF)_{ij}$ = Latest start $(LS)_{ij}$ - Earliest Start $(ES)_{ij}$

or

$$(TF)_{ij} = (L_j - E_i) - t_{ij}$$

2. Free Float :-

The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent succeeding activity.

Earliest Finish − Earliest Start

$FF_{ij} = (E_j - E_i) - t_{ij}$

FF_{ij} = Total Float − Head event slack

(Free float activity)

Head event slack $= L_j - E_i$

3. Independent float :-

- The amount of time by which the start of any activity can be delayed without affecting the earliest time of any.

- Immediatly following activity assuming that the preceeding activity has finished at its latest finish time.

Mathematically we can write,

$$IF_{ij} = (E_j - L_i) - t_{ij}$$

or

IF_{ij} = Free float − Total event slack

The negative independent float is always taken as zero

$$IF_{ij} \leq FF_{ij} \leq TF_{ij}$$

Activity	Normal time	Earliest		Latest		Total Float (TF)	Free Float (FF)	Independent Float	e_j	L_i
	t_{ij}	Start (ES) ES (e_i)	Finishing $EF = e_i + t_{ij}$	Start $LS = L_j - t_{ij}$	Finish LF (L_j)	LS-ES	$e_j - L_i - t_{ij}$	$e_j - L_i - t_{ij}$		
1-2	4	0	4	0	4	0	0	0	4	0
1-3	1	0	1	3	4	3	0	0	1	0
2-4	1	4	5	4	5	0	0	0	5	4
3-4	1	1	2	4	5	3	3	0	5	4

The critical activities are 1-2, 2-4, the tx given by 1-2-4.

Critical Activity :-

(i) An activity is said to be critical, If the total Float $(TF)_{ij}$ for any activity (i,j) is zero.

Activity 1-2 & 2-4 are critical Activity.

(ii) The Float can be used to reduced project duration.

(iii) The activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire Project.

Critical path :-

- The sequence of critical activities in a network is called the critical path. It is the longest path in the network from the starting event to the ending event & defines the minimum time required to complete the project.
- In the network, it is denoted by double line (=).
- This path identifies all the critical activities of the project.
- Hence, the activity (i,j) to lies in the critical path. following conditions must be satisfied.
  - $E_{si} = L_{fi}$
  - $E_{sj} = L_{fj}$
  - $E_{sj} - E_{si} = L_{fj} - L_{fi} = t_{ij}$

$E_{si}, E_{sj} \rightarrow$ earliest start time of the event i & j

$L_{fi}, L_{fj} \rightarrow$ latest finish time of the event i & j.

Critical Path Method (CPM) →

Step-1 → list all the jobs & the draw arrow (Network) diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs.

The length of the arrows has no significance. The arrows are placed based on the predecessor, successor & concurrent relation within the job.

Step-2 →

Indicate the normal time (t_{ij}) for each activity (i, j) above the arrow which is deterministic.

Step-3 →

Calculate the earliest stat time & the earliest finish time for each event & write the earliest time (E_i) for each event (i) in the □.

Also calculate the latest finish & latest start time from this we calculate the latest time L_j for each event j & put it in the Δ.

Step-4 →

Tabulate the various times namely normal time, earliest time & latest time on the arrow diagram.

Step-5 →

Determine the total float for each activity by taking the difference bet^n the earliest start & the latest start time.

Step-6 -

- Identify the critical activites & connect them with the beginning event & the ending event in the network diagram by double line arrows. which gives the critical path.

Step. 7 -

- calculate the total project duration.

PERT (Program Evaluation & Review Technique) :-

- PERT is a probabilistic method, where the activity time are represented by a probability distribution.
- This distribution of activity times is based on three different times estimates made for each activity which are as follows.

(i) Optimistic time estimates :-

- It is the smallest time taken to complete the activity is everything goes on well.
- There is very little chance, that activity can be done in less time.
- It is denoted by to or a.

(ii) Most Likely time estimates:-

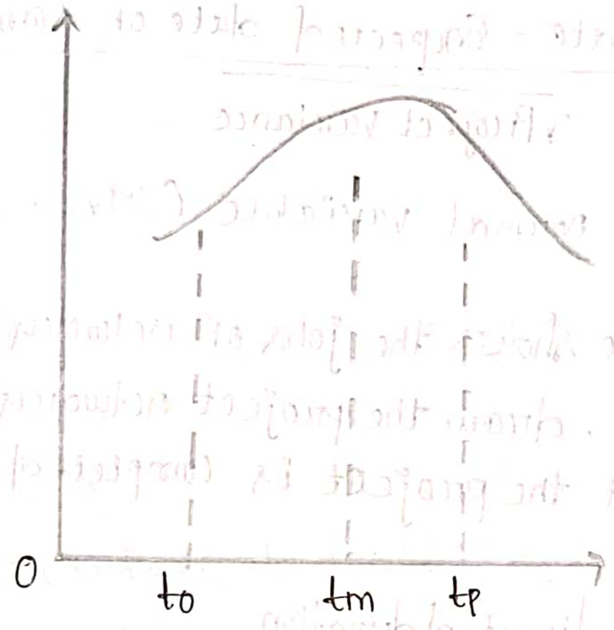
- It refers to the estimate of the normal time, the activity would take.
- It is denoted by tm or m.

(iii) Pessimistic time estimates:-

- It is the longest time that an activity would take if everything goes wrong.

- It is denoted by t_p or $\frac{b}{6}$.

* This ~~figure~~ three 3 time values, are shown in the following figure.



Time distribution curve

- By using the three time values, we have to calculate the expected time of an activity.

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

(expected time)

- The variance of an activity is given by,

$$\sigma^2 = \left[\frac{t_p - t_o}{6}\right]^2$$

- The expected length or durations is denoted by Te of the entire project to the length of the critical path. i.e, the sum of the te's of all the activity along the critical path.
- The main objective in the analysis to PERT.
- To find the completion for the particular event within specified Ts is given by

$P(Z \leq D)$

where $D = \dfrac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$

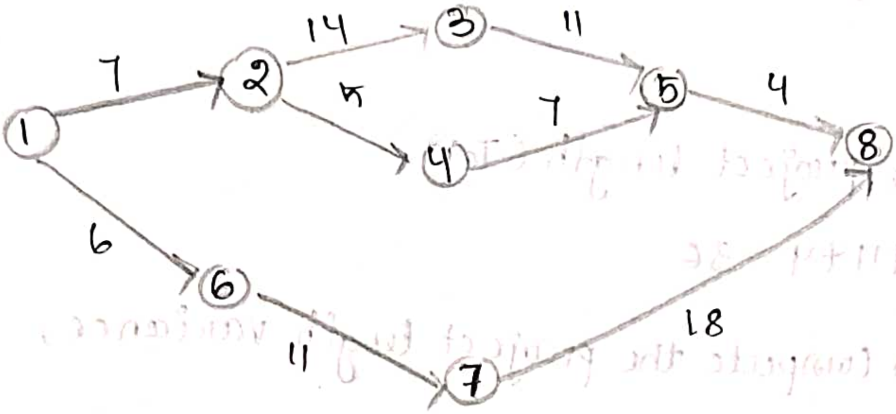
- Z is the standard normal variable (SNV).

Ex:- The following table shows the jobs of network along their time estimates. draw the project network find the probability that the project is completed in 40 days.

Activity	Estimated duration to (a)	tm (m)	tp (b)	te	σ^2
1-2	1	7	13	7	4
1-2	2	5	14	6	4
2-3	2	14	26	14	16
2-4	2	5	8	5	1
3-5	7	10	19	11	4
4-5	5	5	17	7	4
6-7	5	8	29	11	16
5-8	3	3	9	4	1
7-8	8	17	32	18	16

Solution :-

Step 1 :- Draw the project Network



Step 2 :- compute the expected duration for each activity using the formulas.

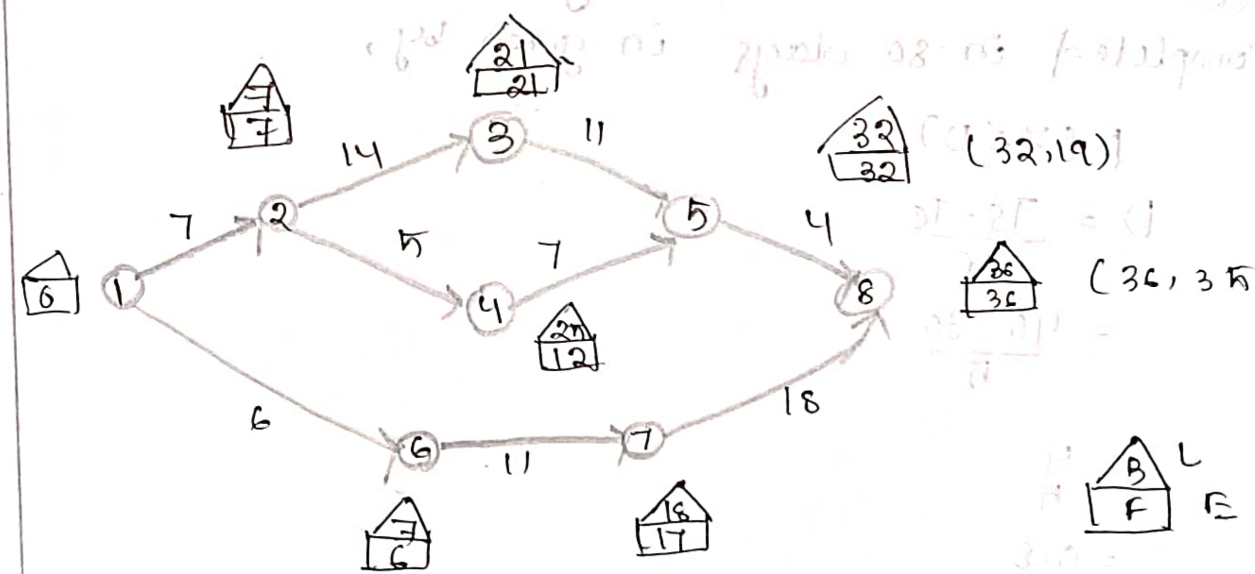
$$te = \frac{to + 4tm + tp}{6}$$

Also, calculate expected variance σ^2 for each activity

$$\sigma^2 = \left(\frac{tp - to}{6}\right)^2$$

Step 3 :- calculate the earliest & latest occurance for each events.

$\boxed{te = \text{expected time}}$



Step 4:- Find the critical path of identifing the critical activities.

1 — 2 — 3 — 5 — 6

Step 5:-

compute the project length (Te)

7+14+11+4 = 36

Step-6:- Then compute the project length variance,

$\sigma^2 = 4+16+4+1$

$= 25$

Standard deviation $= \sqrt{\sigma^2} = \sqrt{25} = 5$

Step 7:- Then calculate the standard normal variable

$D = \frac{TS - Te}{\sigma}$ [TS = Scheduling time]

TS = Scheduling time to compute the project

Te = Normal expected project length duration.

σ = Project standard deviation.

Step 8:- Then the probability of project will be completed in 80 days in given by,

$P(Z \leq D)$

$D = \frac{TS - Te}{\sigma}$

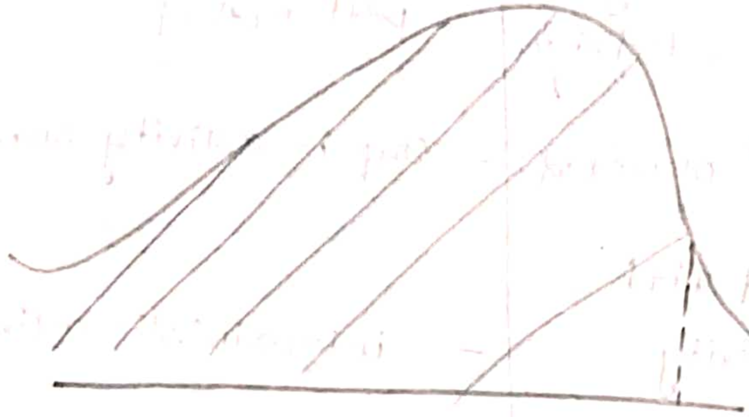
$= \frac{40 - 36}{5}$

$= \frac{4}{5}$

$= 0.8$

$P(Z \leq 0.8) = 0.7881$

$= 78.81\%$



conclusion:-

If the project is performed 100 times under the same condition there will be 78.81 % chance for this job to be completed in 40 days.

Difference between PET & CPM :-

- It stands for program evalution & review technique.	- It stands for critical path method
- PERT is event oriented.	- CMP is activity oriented
- It is associated with Probabilistic activity	- interminisitic activity.
- It is based on 3 times estimates mainly optimastics.	- It is based on single time complete.
- Labour equipments materials are limited	- No limitation of resources.
- mainly used for research & development project	- mainly used for construction Project.
crashing concept is no appriciable/ to PERT (apply)	- crashing concept as a completion technique/ apply to cpm to shorten the project duration along with least additional cost.

Cost consideration in pert or CPM :-

- The total cost of any projet comprises direct and indirect cost. (consist of)

Direct cost :- This cost is directly dependend upon the amount of resources in the execution of individual activity, man power, material consume etc.

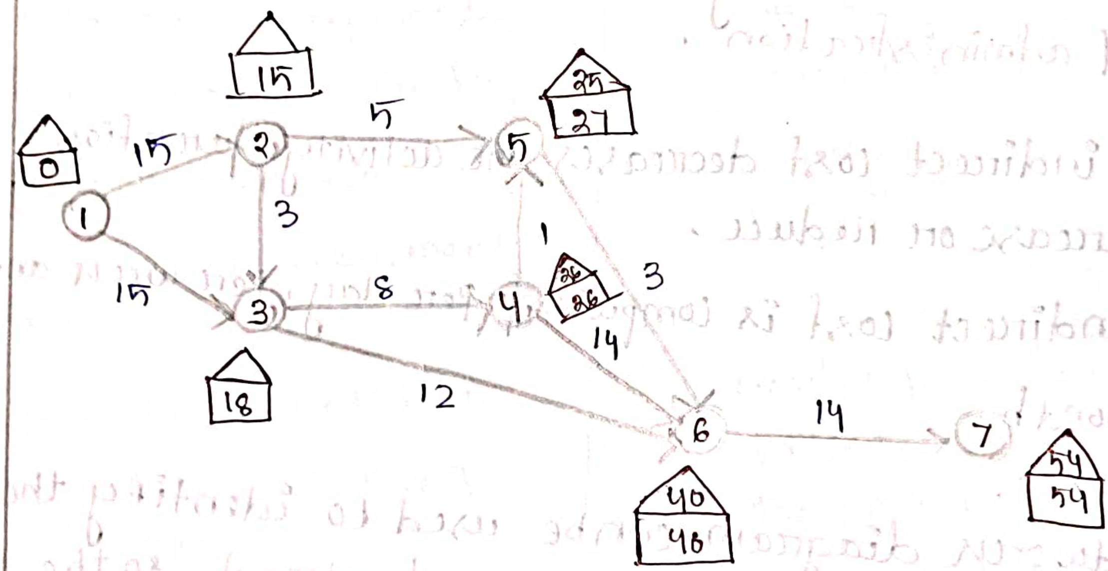
- The direct cost increases the activity duration is decrease or reduced.

Indirect cost :- This cost is associated with overhead expenses. such as managerial services, indirect supply, general administration.

- The indirect cost decreases the activity duration is also decrease or reduce.
- The indirect cost is computed per day, per week and per month.

* The network diagram can be used to identify the activity ~~whose duration~~ should be shortened. so the the compeletation time of the project should be shortened.
* The process of reducing the activity duration by putting on extra effort is called crashing the activity

Job	duration
1-2	15
1-3	15
2-3	3
2-5	5
3-4	8
3-6	12
4-5	1
4-6	14
5-6	3
6-7	14



(28, 26, 37)

Crash Time (Tc) :-

- It is denoted by Tc.
- It is the shortest possible activity time.
- Then the activity cost corresponding to the crash time is called the crash cost (Cc).

Normal cost :- (CN)

It is equal to the absolute minimum of the direct cost required to perform an activity.

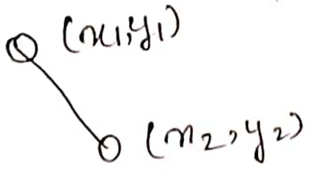
Normal time :- (TN)

The corresponding time duration taken by an activity is known as normal time

cost slope :-

It is increase in the cost of per unit of ~~time~~ time saved by crashing. It is defined as cost slop.

$$\boxed{\text{cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}} = \frac{C_c - C_N}{T_N - T_c}$$

Ex:- 

$\Rightarrow \left(\frac{y_2 - y_1}{x_2 - x_1} \right)$

Q/ An activity takes 4 days to complete at a normal cost of RS 500 if it is possible to complete the activity in 2 days with an additional cost of RS 700 what is the incremental cost of ~~Rs 700~~ 'an activities.

A:- $T_N = 4$ days, $T_c = 2$ days, $C_N = 500$, $C_c = 700$

$$\text{cost slope} = \frac{C_c - C_N}{T_N - T_c} = \frac{700 - 500}{4 - 2} = 100$$

UNIT-4

Decision Theory :-

what is it ?

It is a statistical tool or technique which is used to select the best way of doing any work.

Basic Terms:-

* **The decision maker:-**

The decision maker is charged with responsibility of making the decision. that is he has to select one form a set of possible courses of actions.

* **The acts:-**

The acts are the alternatives courses of action of startagies that are available to the among two or more alternative courses of action. the problem is to choose the best of these alternative to achieve the objective.

* **Pay off table:-**

- The pay of table represents the economics of a problem i.e revenue costs associated with any action with a particular outcome. it is an ordered statement of Profit or ~~coats~~ costs resulting under the given situation.

- The payoff can be interpreted as the outcome in quantitative from, If the decision maker adopts a particular strategy under a particular state of nature.

* **Opportunity Loss table:-**

- An opportunity loss is the loss incurred because of failure to ~~take~~ take the best possible action.

- opportunity cost losses are calculated separately for each states of nature.

- Given the possible state of nature. we can determine the best possible cost.

- For a given state of nature, the opportunity loss of an ~~act~~ act is the difference between the payoff of that ~~act~~ act & the payoff for best ~~act~~ act that could have been selected.

Decision Environment :-

Decision Environment is divided into 3 types:-

* **DE under Certainty :-**

The process of choosing an act or startegy ~~when~~ when the state of nature is completely known, is called DE under Certainly.

Ex:-

Let us suppose that a person has to travel from one place say (A) to another place say (B). He can follow any one of the routes R_1, R_2, R_3 (Say)

Acts / Routs	States of Nature: Fuel Saving (S_1)	Time Saving (S_2)	Enjoyment (S_3)
R_1	5	0	6
R_2	0	4	3
R_3	8	2	1

The result of each of the acts (routes) are unown with certainty.

- If a person wants to economics on petrol, he will use route R_3.
- It the values of time, he would prefer to take route R_2.
- If he gives more weight to enjoyment and function he would like to take route R_1.

DE under uncertainty :-

The process of choosing an ~~act~~ act / strategy out of the various courses of action at hand, when the outcome i.e the state of nature of any ~~act~~ act is unown is termed as DE under uncertainty.

Ex:- will the new plant or industrial unit be successful ?

will the new products be able to complete with others in the market ?

Decision making under risk :-

- In decision making under risk, there are several possible outcomes for each alternatives, the decision maker knows the probability of occurrence of each outcome.

Ex:-
we know that, the probability of rolling a 5 in a dice is 1/6. In decision making under risk, the decision maker usually attempts to maximize his/her expected well being.

- There are 3 methods are used in decision making under risk.

Expected monetary value (EMV) :-

- when the probabilities can be assigned to the various state of nature, it is possible to calculate the statistical expection of gain for each course of action.
- The conditional value of each event in the payoff table is multiplied by its probability and the product is summed up. the resulting number is the EMV for the act.
- The decision maker selects from the available alternative activity, the action that leads to the maximum expected gain.

Ex :-

Let the states of nature be S_1 & S_2 and the alternative strategies A_1 & A_2 let the probabilities for the states of nature S_1 & S_2 be respectivity 0.6 & 0.4 let the payoff table be as shown below.

	A_1	A_2
S_1	30	20
S_2	35	30

then,

EMV for A_1 = $(30 \times .6) + (35 \times .4) = 18 + 14 = 32$

EMV for A_2 = $(20 \times .6) + (30 \times .4) = 12 + 12 = 24$

$\therefore$ EMV for A1 is greater.

$\therefore$ The decision maker will choose the strategy A1.

Expected opportunity Loss (EOL) :-

- The difference between the greater payoff & actual Payoff is known as opportunity loss under this Criterion the strategy which has minimum expected opportunity loss (EOL) is choosen.
- The calculation of EOL is similar to that EMV.

Ex:-

consider an opportunity loss table A1 & A2 are the strategies & S1 & S2 are the state of nature.

	A1	A2
S1	0	10
S2	2	-5

Let the probability for 2 states be 0.6 & 0.4

EOL for $A_1 = (0 \times 0.6) + (2 \times 0.4) = 0.8$

EOL for $A_2 = (10 \times 0.6) + (-5 \times 0.4) = 4$

EOL for A1 is least. Therefore, the strategy A1 may be choosen.

Expected value of perfect Information (EVPI) :-

- The expected value with perfect information is the average return in the long run, if we have perfect information before a decision is to be made.

- In order to calculate EVPI, we choose the best alternative with the probability of their state of nature. the expected value of perfect information (EVPF) is the expected outcome with perfect information minimum the outcome with max EMV.

$$\boxed{EVPI = EOPI - Max.\ EMV}$$

Ex :-

A_1, A_2, A_3 are the ~~costs~~ acts and S_1, S_2, S_3 are the states of nature.

Also known that $P(S_1) = 0.5$

$P(S_2) = .4$

$P(S_3) = .1$

States of nature payoff table

	A_1	A_2	A_3
S_1	30	25	22
S_2	20	35	20
S_3	40	30	35

calculate EVPI.

Soln

EMV for $A_1 = (30 \times 0.5) + (20 \times .4) + (40 \times .1) = 27$

EMV for $A_2 = (25 \times 0.5) + (35 \times 0.4) + (30 \times .1)$
$= 12.5 + 14 + 3 = 29.5$

EMV for $A_3 = (22 \times 0.5) + (20 \times .4) + (35 \times .1)$
$= 11 + 8 + 3.5 = 22.5$

The highest EMV is for the strategy A_2 & it is 29.5

Now, to find EVPI, work out expected value for maximum payoff under all states of nature

	Max. profit of each state	Probability	Expect value (profit × prob.)
S_1	30	.5	$30 \times .5 = 15$
S_2	35	.4	$35 \times .4 = 14$
S_3	40	.1	$40 \times .1 = 4$
			33

∴ Expected payoff with profit information = 33

Thus, the expected value of perfect information

(EVPI) = EVPI - Max . EMV

= 33 - 29.5

= 3.5

Hungarian Method

Q/

	machine				
Jobs	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Sol/ Substract the minimum element from all the element in respective row, we get row reduction matrix as;

Jobs	A	B	C	D	E
1	5	0	8	10	11
2	0	6	15	0	3
3	8	5	0	0	0
4	0	6	4	2	7
5	3	5	6	0	8

Substract the minimum element from all the element in ~~req~~ respective column, we get column redution matrix as;

Jobs	A	B	C	D	E
			machine		
1	5	0	8	10	11
2	0	6	15	0	3
3	8	5	0	0	0
4	0	6	4	2	7
5	3	5	6	0	8

3. Draw minimum no of horizontal and vertical linear (N) to cover all zero.

   (a) If N=n, n= order of matrix, then an optimal sol^n can be made.

   (b) If N<n, then go to the next step.

Jobs	A	B	C	D	E
1	5	$\boxed{0}$	8	10	11
2	$\boxed{0}$	6	15	0	3
3	8	5	$\boxed{0}$	0	0
4	$\boxed{0}$	6	4	2	7
5	3	5	6	$\boxed{0}$	8

row → 1 zero → marked

row → max zero → skip

4. determine the smallest uncovered element

jobs	A	B	C	D	E
1	5	$\boxed{0}$	85	10	8
2	0	6	12	0	$\boxed{0}$
3	8	5	$\boxed{0}$	0	0
4	$\boxed{0}$	6	1	2	4
5	3	5	3	$\boxed{0}$	5

Jobs	Machinary
1	B
2	E
3	C
4	A
5	D

minimum total cost = 8 + 12 + 4 + 6 + 12

= 42