06/01

Complex Variable

- $z = a + ib$

  $\bar{z} = a - ib$

  $Re(z) = \dfrac{z+\bar{z}}{2}$ $\quad Im(z) = \dfrac{z-\bar{z}}{2i}$

- z is purely ~~im~~ real if $z = \bar{z}$

  & z is purely img if $z = -\bar{z}$

  $|z|^2 = z\bar{z}$

- Limit of a f^n $f(z)$

  $$\underset{z \to z_0}{Lt}\ f(z) = l$$

  for a given $\varepsilon > 0$, $\exists\ \delta > 0$

  $|f(z) - l| < \varepsilon \quad \forall\ z$ such that $|z - z_0| < \delta$

- Continuity

  $f(z)$ is said to be cont. at $z = z_0$ if

  $$\underset{z \to z_0}{Lt}\ f(z) = f(z_0)$$

* Derivative of a f^n

let $f(z)$ be a single-valued f^n of the ~~variable~~ from $z = x+iy$

then $f(z)$ is said to be differentiable if

$$\underset{\delta z \to 0}{Lt}\ \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists as $\delta z \to 0$ along any path & is denoted by $f'(z)$.

* Theorem:

- The necessary and sufficient cond^n for a f^n $f(z) = u(x,y) + iv(x,y)$ to be differentiable for all points in a region D are

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ } Cauchy - Reimann eq^{ns}

(ii) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ & $\dfrac{\partial v}{\partial y}$ ~~exist~~ continuous in D.

* Analytic f^n :-

→ A single-valued f^n $f(z)$ which has a unique derivative w.r.t z at all points of a region D then $f(z)$ is said to be analytic.

Q Check $f(z) = \sqrt{|xy|}$ is analytic at origin or not.

Sol^n: Here, $u = \sqrt{|xy|}$,

$v = 0$

$\therefore \dfrac{\partial u}{\partial x} = \cancel{\dfrac{y}{\sqrt{xy}}}$ $\lim\limits_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{0-0}{x} = 0$

$\dfrac{\partial u}{\partial y} = \lim\limits_{y \to 0} \dfrac{u(0,y) - u(0,0)}{y} = \lim\limits_{y \to 0} \dfrac{0-0}{y} = 0$

$\dfrac{\partial v}{\partial x} = \lim\limits_{x \to 0} \dfrac{v(x,0) - v(0,0)}{x} = 0$

$\dfrac{\partial v}{\partial y} = 0$.

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\because f(z) = \sqrt{|xy|}$ is analytic at origin.

Now,

$f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$

$= \lim\limits_{z \to 0} \dfrac{\sqrt{|xy|} - 0}{z}$

Let $x \to 0$ along the path $y = mx$ | $z = x + iy$

$= x + imx$

$z = x(1+im)$

$z \to 0$ means $x \to 0$

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|x \cdot mx|}}{x(1+im)}$$

$$= \lim_{x \to 0} \frac{\sqrt{|m|}}{1+im}$$

$$= \frac{\sqrt{|m|}}{1+im},$$

which depends on m

$\therefore$ $f'(0)$ is not unique

$\therefore$ $f(z)$ is not diff^r at origin.

Q2. $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$

$$= 0, \quad z = 0$$

Soln:
$$f(z) = \left(\frac{x^3 - y^3}{x^2+y^2}\right) + i \cdot \left(\frac{x^3+y^3}{x^2+y^2}\right)$$

Here,

$$u = \frac{x^3 - y^3}{x^2+y^2}$$

$$v = \frac{x^3+y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{y - 0}{y} = 1$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \& \quad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

Now,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

Let $z \to 0$ along $y = mx$ path

Here, $z \to 0 \Rightarrow x \to 0$.

$$z = x + iy = x + imx$$
$$= x(1+im)$$

$$f'(0) = \lim_{x \to 0} \frac{\frac{x^3 - y^3}{x^2+y^2} + i \cdot \frac{x^3+y^3}{x^2+y^2}}{x(1+im)}.$$

$$= \lim_{x \to 0} \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+im)} \qquad (y = mx)$$

It depends on m.

$\therefore$ $f'(0)$ is not unique, ie $f(z)$ is not diff at origin

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* How to derive CR eqⁿ

* Derive C-R eqⁿ in cartesian form :-

let $f(z) = u(x,y) + iv(x,y)$ be analytic.

$\therefore$ $f(z)$ possesses a unique derivative.

Now, $$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - (u(x,y) + iv(x,y))}{\delta z}$$

exists along any path as $\delta z \to 0$

Case 1: let $\delta z \to 0$ along real axis.

then, $\delta z = \delta x + i\delta y$

$= \delta x$ $(\because \delta y = 0)$

$\therefore$ $\delta z \to 0 \Rightarrow \delta x \to 0$

$$\therefore f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - (u(x,y) + iv(x,y))}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Case 2. let $\delta z \to 0$ along imaginary axis

then, $\delta z = \delta x + i\delta y$

$= i\delta y$ $(\because \delta x = 0)$

$\therefore$ $\delta z \to 0 \Rightarrow \delta y \to 0$

Now, $f'(z) = \lim_{\delta y \to 0} \frac{u(x,y+\delta y)+iv(x,y+\delta y) - (u(x,y)+iv(x,y))}{i\delta y}$

$$= \lim_{\delta y \to 0} \frac{u(x,y+\delta y) - u(x,y)}{i\delta y} + \lim_{\delta y \to 0} \frac{v(x,y+\delta y) - v(x,y)}{\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{--- ②}$$

Since, $f'(z)$ has a unique derivative

$\therefore$ from ① & ②, we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real & img. parts we get;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

which are C-R eq^n in cartesian form.

*<u>Theorem</u>:-

- If $f(z)$ is an analytic f^n with constant modulus then $f(z)$ is constant.

<u>Proof</u>: let $f(z) = u+iv$ be an analytic f^n

$\therefore \quad |f(z)| = \sqrt{u^2+v^2}$

$= c$ (say)

$\therefore \quad u^2+v^2 = c^2$ --- ①

Differentiating ① partially wrt x we get,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{--- ②}$$

Similarly, diff ① partially w.r.t y,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \quad - ③$$

Since $f(z)$ is analytic f^n, so it satisfies C-R eq^n

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Using CR eq^n in eq^n ③, we get,

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad - ④$$

Squaring & adding ② & ④ we get,

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + \cancel{2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}} + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$- \cancel{2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}} = 0.$$

$$\Rightarrow \quad (u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad (\because u^2+v^2 \neq 0) \quad - ⑤$$

Now, $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\Rightarrow \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad (\text{by } ⑤)$$

$$\therefore \quad |f'(z)| = 0$$

$\Rightarrow \quad f(z)$ is constant.

* Harmonic f^n :

→ A f^n $u(x,y)$ is said to be harmonic if u satisfies Laplace Eq^n.

i.e. $\boxed{\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0}$

Theorem :

If $f(z) = u + iv$ is analytic then show that both u and v are harmonic f^n.

Proof.

Let $f(z) = u + iv$ be analytic

∴ By Cauchy-Riemann Eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -① \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -②$$

Diff ① partially wrt x & ② partially wrt y we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad -③$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad ④$$

But $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$

③ + ④ ⇒ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

∴ u is harmonic

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Again, Diff partially ① wrt y & ② wrt x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad - ⑤$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad - ⑥$$

But $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore$ ⑤ + ⑥ $\Rightarrow$ $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$

$\therefore$ v is harmonic.

— x —

Q. Derive C-R eq^{n}s in Polar form i.e.

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$$

Hence deduce $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = 0$

Proof: let (r, θ) be the polar coordinates of a point (x, y).

Then $z = x + iy = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\Rightarrow \boxed{z = re^{i\theta}}$

Now,

$$u + iv = f(z) = f(re^{i\theta})$$

Diff^r partially wrt r and θ we get,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \quad - ①$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta} \quad - ②$$

① ÷ ② $\Rightarrow$ $\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir} \cdot \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$

$$= -\frac{i}{r} \cdot \left(\frac{\partial u}{\partial \theta}\right) + \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

Comparing,

$$\frac{\partial u}{\partial r} = +\frac{1}{r}\cdot\frac{\partial v}{\partial \theta}. \quad - (3)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad - (4)$$

Diff partially (3) wrt r and (4) wrt θ.

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\cdot\frac{\partial v}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 v}{\partial r\partial \theta} \quad - (5)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta\partial r} \quad - (6)$$

(5) $\Rightarrow$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\cdot\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} = 0.}$$

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Q1. Find the value of p if $f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$ is analytic

Sol^n

Q2. Show that:

$u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Sol^n (1)

Polar form of CR eq^n are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \quad (1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad (2)$$

Here, $u = r^2\cos 2\theta$

$v = r^2 \sin p\theta$

Now,

$$\frac{\partial u}{\partial r} = (2r)\cos 2\theta \qquad \frac{\partial v}{\partial r} = (2r\sin p\theta)$$

$$\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta \qquad \frac{\partial v}{\partial \theta} = r^2 p\cos p\theta.$$

Now $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$.

$\Rightarrow (2r \cdot \cos 2\theta) = \frac{1}{r} \cdot (r^2 p \cos p\theta)$

$\Rightarrow \quad 2\cos 2\theta = p\cos p\theta \quad - (1)$.

Again, $\frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$.

$\Rightarrow \quad -2r^2 \sin 2\theta = -r \cdot 2r\sin p\theta$.

$\Rightarrow \quad \sin 2\theta = \sin p\theta. \quad - (2)$.

$(1)^2 + (2)^2$

$\Rightarrow \quad \cos^2 2\theta + \sin^2 2\theta = \left(\frac{p\cos p\theta}{2}\right)^2 + (\sin^2 p\theta)^2$

$\Rightarrow \quad 1 = \frac{p^2}{4}\cos^2 p\theta + \sin^2 p\theta$.

$\therefore \quad \frac{p^2}{4} = 1$.

$\Rightarrow \quad p^2 = 4$

$\Rightarrow \quad \boxed{p = \pm 2}$

But $p = -2$ does not satisfy the ~~given~~ eq^{n}s (1) & (2).

$\therefore \quad \boxed{p = 2}$.

(2) $u = e^{-x}(x\sin y - y\cos y)$

$\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x\sin y - y\cos y)$

$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - [e^{-x}\sin y \cancel{*} - e^{-x}(x\sin y - y\cos y)]$

$= -2e^{-x}\sin y + e^{-x}(x\sin y - y\cos y)$

$\frac{\partial u}{\partial y} = e^{-x}[(x\cos y - (\cos y \cancel{+} - y\sin y)]$

$= e^{-x}[x\cos y - \cos y + y\sin y]$

$\frac{\partial^2 u}{\partial y^2} = e^{-x}[-x\sin y + \sin y + y\cos y + \sin y]$

$\therefore \quad \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \Rightarrow u$ is a harmonic f^n

* Milne-Thomson

→ Method for finding analytic f^n $f(z)$ when one part (either real or img part is given).

$\boxed{\text{Case 1}}$ Real part of f. u is given.

$$\frac{\partial u}{\partial x} = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = \phi_2(x,y)$$

Now we find,

$\phi_1(z,0)$ & $\phi_2(z,0)$

∴ According to this method,

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$

Ex find the analytic f^n whose real part is $u = e^{-x}(x\sin y - y\cos y)$. Also, find complex conjugate of u. f^n

Soln:

$$\frac{\partial u}{\partial x} = \phi_1(x,y) = e^{-x}(\sin y) - e^{-x}(x\sin y - y\cos y)$$

$$\frac{\partial u}{\partial y} = \phi_2(x,y) = e^{-x}(x\cos y - \cos y + y\sin y)$$

$$\phi_1(z,0) = 0$$

$$\phi_2(z,0) = e^{-z}(z-1)$$

∴ By Milne-Thomson method,

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$

$$= \int -ie^{-z}(z-1)\,dz$$

$$= -i\left[\int e^{-z} z\,dz - \int e^{-z}\,dz\right]$$

$$= -i[-e^{-z} \cdot z] + c$$

∴ $f(z) = ize^{-z} + c$

Putting, $z = x+iy$.

$$f(z) = i(x+iy)e^{-(x+iy)} + c$$

$$= i(x+iy)\,e^{-x} \cdot e^{-iy} + c$$

$$= e^{-x}(ix - y)(\cos y - i\sin y) + c$$

$$= e^{-x}(-y\cos y + x\sin y) + ie^{-x}(x\cos y + y\sin y) + c$$

∴ $v = e^{-x}(x\cos y + y\sin y) + c'$, where $c' = ic$.

~~Part 2~~

$\boxed{\text{Case 2}}$ v is given (Img part of $f(z)$ is given)

$$\frac{\partial v}{\partial x} = \psi_1(x,y)$$

$$\frac{\partial v}{\partial y} = \psi_2(x,y)$$

find $\psi_1(z,0)$ & $\psi_2(z,0)$

By Milne-Thomson method, * Note

$$f(z) = \int [-\psi_2(z,0) + i\psi_1(z,0)]\,dz$$

Q2. find the analytic f^n $f(z)$ whose img. part is $v = e^{-x}(x\cos y + y\sin y)$

Soln:

$$\frac{\partial v}{\partial x} = \psi_1(x,y) = -e^{-x}(x\cos y + y\sin y) + e^{-x}(\cos y)$$

$$\frac{\partial v}{\partial y} = \psi_2(x,y) = e^{-x}[-x\sin y + \sin y + y\cos y]$$

$$\psi_1(z,0) = -e^{-z} \cdot z + e^{-z} = e^{-z}(1-z)$$

$$\psi_2(z,0) = 0$$

NOTEBOOK
SMO

$\therefore$ By Milne-Thomson method,

$$f(z) = \int [-\psi_2(z,0) + i\psi_1(z,0)]\,dz$$
$$= \int +ie^{-z}(1-z)\,dz$$
$$= \int -ie^{-z}(z-1)\,dz$$
$$= i\,[+e^{-z}z] + c$$

$e^{-z}z - \int e^{-z}$
$-e^{-z}z - \int e^{-z}$

Put $z = x+iy$

$$f(z) = +i(x+iy)e^{-(x+iy)} + c$$
$$= e^{-x}(-ix+y)(\cos y - i\sin y) + c$$
$$= e^{-x}(y\cos y - x\sin y) - ie^{-x}(x\cos y + y\sin y)\ \cancel{A/c}$$

18/01

* In an electric field,
if $\phi = u+iv$ denotes an analytic f^n, then u is called velocity potential f^n & v is called

Imp

Ex1. If $w = \phi + i\psi$ denotes complex potential for an electric field & $\psi = x^2-y^2+\frac{x}{x^2+y^2}$, determine ϕ.

Ex2. If $u-v = (x-y)(x^2+4xy+y^2)$ & $f(z) = u+iv$ is analytic find $f(z)$ in terms of z.

Ex3. If $f(z)$ is analytic, P.T. $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Sol^n

(1) Here,
$$v = x^2-y^2+\frac{x}{x^2+y^2}$$
$$\frac{\partial v}{\partial x} = \psi_1(x,y) = 2x + \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}$$
$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \psi_2(x,y) = -2y + \frac{-x(2y)}{(x^2+y^2)^2}$$
$$= -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\psi_1(z,0) = 2x - \frac{1}{x^2} = 2z - \frac{1}{z^2}$$
$$\psi_2(z,0) = 0.$$

$\therefore$ By Milne-Thomson method,
$$f(z) = \int (\psi_2(z,0) + i\psi_1(z,0))\,dz$$
$$= \int +i\left(2z-\frac{1}{z^2}\right)dz$$
$$= +i\frac{2z^2}{2} + i\left(-\frac{1}{z}\right) + c$$
$$= +i\left(z^2+\frac{1}{z}\right) + c$$

Put $z = x+iy$.
$$f(z) = i(x^2+2ixy-y^2) + i\frac{x-iy}{x^2+y^2} + c$$
$$\therefore \phi = Re(f(z)) = \left(-2xy + \frac{y}{x^2+y^2} + c\right)$$

* Q2. Sol^n

$$u-v = (x-y)(x^2+4xy+y^2)$$
$$f(z) = u+iv$$
$$if(z) = ui - v$$
(+)
$$(1+i)f(z) = (u-v) + i(u+v)$$
$$= U + iV$$
$$= F(z) \text{ say}$$

Now, $U = u - v$ is given
which is $Re(F(z))$

$$\therefore \frac{\partial U}{\partial x} = \phi_1(x,y) = x^2 + 4xy + y^2 + (x-y)(2x+4y)$$

$$\frac{\partial U}{\partial y} = \phi_2(x,y) = -(x^2+4xy+y^2) + (x-y)(4x+2y)$$

& $\phi_1(z,0) = z^2 + 2z^2 = 3z^2$

$\phi_2(z,0) = -(z^2) + 4z^2 = 3z^2$

$$\therefore F(z) = \int \left(\phi_1(z,0) - i\phi_2(z,0)\right) dz$$

$$= \int (3z^2 - i\,3z^2)\,dz$$

$$= z^3 - iz^3 + c$$

$$= z^3(1-i) + c.$$

Here,

$F(z) = (1+i) f(z)$

$$\Rightarrow f(z) = \frac{F(z)}{1+i}$$

$$= \frac{z^3(1-i)}{1+i} + c'$$

$$= -iz^3 + c'$$

Q3. sol^n

$f(z) = u + iv$ is analytic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$\phi = |f(z)|^2 = u^2 + v^2$ — (i)

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}.$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2 \quad \text{— (ii)}$$

$$\frac{\partial \phi}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2 \quad \text{— (iv)}$$

$$= 2u\frac{\partial^2 u}{\partial y^2} + 2\left(-\frac{\partial v}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 \quad \text{— (iii)}$$

(using C-R eq^{ns})

$\cancel{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}} = 2.$

Since u & v are harmonic

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(ii) + (iii)

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2|f'(z)|^2 + 2|f'(z)|^2$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2}$$

Hence proved

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Complex Integration :-

let $f(z) = u(x,y) + iv(x,y)$

$z = x + iy$

$dz = dx + idy$

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C \left[(udx - vdy) + i\int_C (udy + vdx)\right]$$

which shows that the evaluation of integrals of a complex f^n can be reduced to the evaluation of two line integrals of real f^n.

18/ Note The value of the integral is independent of the path of the integration when the integrant $f(z)$ is analytic.

Q. $\int_C \frac{dz}{z-a}$ where C is $|z-a| = r$

$\Rightarrow z - a = re^{i\theta}$

$\Rightarrow dz = ire^{i\theta}d\theta$, $0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}}$$

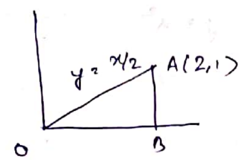
$= i(2\pi)$ (Remember the result)

Q. Evaluate

$$\int_0^{2+i} (\bar{z})^2 dz$$

along (i) $y = x/2$
(ii) real axis to 2 & then vertically to $2+i$

Sol:



Along OA, $y = x/2$
$\Rightarrow x = 2y$

$z = x + iy = 2y + iy = y(2+i)$

$\Rightarrow dz = dy(2+i)$

$\bar{z} = x - iy$
$= (2-i)y$

$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2-i)^2 y^2 (2+i)dy$$

$$= \int_0^1 (2-i)5y^2 dy$$

$$= \frac{5}{3}(2-i)$$

$2+iy$

(ii). along BA, $z = 2 + iy$, $\bar{z} = 2 - iy$
$x = 2$
$dz = idy$

along OB, $z = x$ $\Rightarrow dz = dx$
$\bar{z} = x$

$(4-y^2-i4y$

$$\therefore \text{Given integral} = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 idy$$

$$= \frac{x^3}{3}\Big|_0^2 + \int_0^1 4i\cancel{v} (4 - y^2 - 4yi)\, idy$$

$$= \frac{8}{3} + 4i - \frac{1}{3}i + 2$$

$$= \frac{14}{3} + \frac{11i}{3}$$

Q $\int_0^{1+i} (x^2 - iy)\,dz$ along the path $y = x^2$.

Sol^n

$z = x + iy$

$\Rightarrow z = \sqrt{y} + iy$

$\Rightarrow dz = \left(\frac{1}{2\sqrt{y}} + i\right) dy$

Given $= \int_0^1 (y - iy)\left(\frac{1}{2\sqrt{y}} + i\right) dy$

$= \int_0^1 \left[\frac{1}{2}(\sqrt{y} - i\sqrt{y}) + (yi + y)\right] dy$

$= \frac{1}{2} \times \frac{2}{3} - i.\frac{2}{3}\times\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}$

$= 5/6 + i\frac{1}{6} = \frac{1}{6}(5 + i)$ *

$\neq$ ~~8(~~

Q. $\int_c (z - z^2)\,dz$, c is the upper half of the circle $|z| = 1$

Q. $\int_{1+i}^{2+3i}$

* Cauchy's Th^m

→ If $f(z)$ is an analytic f^n and $f'(z)$ is continuous at each point within and on a closed curve c then

$$\boxed{\int_c f(z)\,dz = 0}$$

* Extension of Cauchy's Th^m :-

→ If $f(z)$ is analytic in the region D bet^n two closed curve c and c_1 then,

$$\boxed{\int_c f(z)\,dz = \int_{c_1} f(z)\,dz}$$

* Cauchy's Integral formula :-

Statement :- If $f(z)$ is analytic within and on a closed curve c and a is any point within c then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z - a}.$$

Generalisation :-

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^2}$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^3}$$ *

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}}$$ *

Q1. Evaluate -

$\int_c \frac{z^2 - z + 1}{z - 1}\,dz$ where c is (i) $|z| = 1$ (ii) $|z| = \frac{1}{2}$

Q2. $\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where c is $|z| = 3$

Sol^n (i) Here, $f(z) = z^2 - z + 1$, which is analytic within and on c, $|z| = 1$ & $a = 1$ lies within c

$a = 1$

$f(a) = f(1) = 1$

$\therefore$ By C.I.F,

$f(a) = f(1) = \frac{1}{2\pi i}\int_c \frac{(z^2 - z + 1)\,dz}{z - 1} = 1$

$\Rightarrow \int_c \frac{(z^2 - z + 1)\,dz}{z - 1} = 2\pi i$ #

(ii) $c = |z| = \frac{1}{2}$

$a = 1$ lies outside c

NOTEBOOKS
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$\therefore f(z) = \dfrac{z^2-z+1}{z-1}$ is analytic within & on C.

$\therefore f(a) = \dfrac{1}{2\pi i}\int_C \left(\dfrac{z^2-z+1}{z-1}\right)dz = 0.$

(2) $-\int \dfrac{(\sin\pi z^2+\cos\pi z^2)}{(z-1)}dz + \int \dfrac{(\sin\pi z^2+\cos\pi z^2)}{z-2}dz$

$\Downarrow$

Here $f(z) = +(\sin\pi z^2+\cos\pi z^2)$

$a = 1, 2$ lies inside the circle.

$f(1) = -1$

$f(2) = 1$

$\therefore \int_C \dfrac{f(z)dz}{(z-1)(z-2)} = \int_C \dfrac{\sin\pi z^2+\cos\pi z^2}{z-2}dz - \int_C \dfrac{\sin\pi z^2+\cos\pi z^2}{z-1}dz$

$= 2\pi i(1) - 2\pi i(-1)$

$= 4\pi i$

#

Q. $\int_C \dfrac{\sin^2 z\, dz}{(z-\frac{\pi}{6})^3}dz$ where c is a circle $|z|=1$

Soln. $f(z) = \sin^2 z$ is analytic inside $C: |z|=1$

Here, $a = \pi/6$ lies within c.

By C.I.F, $f''(a) = \dfrac{2!}{2\pi i}\int_C \dfrac{f(z)dz}{(z-a)^3}$

$\Rightarrow \int_C \dfrac{\sin^2 z\, dz}{(z-\pi/6)^3} = \pi i\left(\dfrac{d^2}{dz^2}(\sin^2 z)\right)\Big|_{z=\pi/6}$

$= \pi i\,(2\cos 2z)\big|_{z=\pi/6}$

$= \pi i$

#

22/01

Q Evaluate

$\int_C \dfrac{e^{2z}dz}{(z+1)^4}$ c is $|z| = 3$

$f(z) = e^{2z}$

$a = -1$

$f'''(a) = \dfrac{3!}{2\pi i}\int_C \dfrac{f(z)dz}{(z-a)^4}$

$\Rightarrow \cancel{\int_C \dfrac{e^{2z}dz}{(z+1)^4}} = \cancel{\dfrac{2}{\pi i}}$

$\Rightarrow \int_C \dfrac{e^{2z}dz}{(z+1)^4} = \dfrac{\pi i}{3}\dfrac{d^3}{dz^3}(e^{2z})\Big|_{z=-1}$

$= \dfrac{\pi i}{3}(8e^{2z})\Big|_{z=-1}$

$= \dfrac{8\pi i}{3}e^{-2}$

#

(2) $\int_C \dfrac{e^z dz}{(z^2+\pi^2)^2}$, c is $|z|=4$

$= \int_C \dfrac{e^z dz}{(z+\pi i)^2(z-\pi i)^2}$ —(1)

$\dfrac{1}{(z+\pi i)^2(z-\pi i)^2} = \dfrac{A}{z+\pi i} + \dfrac{B}{(z+\pi i)^2} + \dfrac{C}{z-\pi i} + \dfrac{D}{(z-\pi i)^2}$

$\Rightarrow 1 = A(z+\pi i)(z-\pi i)^2 + B(z-\pi i)^2 + C(z-\pi i)(z+\pi i)^2 + D(z+\pi i)^2$

$z = -\pi i$,

$B(-4\pi^2) = 1$

$\Rightarrow B = -\dfrac{1}{4\pi^2} = D$

$2\pi i$

$4\pi^2 i$

$-4\pi^2$

$A = \dfrac{7}{2\pi^3 i}$, $C = -\dfrac{7}{2\pi^3 i}$

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Given integral $= \int_c \frac{A e^z dz}{z+\pi i} + \int_c \frac{B e^z dz}{(z+\pi i)^2} + \int_c \frac{C e^z dz}{(z-\pi i)}$

$+ \int_c \frac{D e^z dz}{(z-\pi i)^2}$

$= \frac{A}{1} \times 2\pi i f(-\pi i) + \frac{B}{1}(2\pi i) f'(-\pi i)$

$+ \frac{C}{1}(2\pi i) f(\pi i) + \frac{D}{1}(2\pi i) f'(\pi i)$

$\frac{1}{2\pi i}$

* Morera's Thm :-

→ If $f(z)$ is continuous in a region D and $\int_c f(z)dz = 0$ around every closed curve c in D then $f(z)$ is analytic.

* Liouville's Thm :-

→ If $f(z)$ is analytic and bounded for all z in the entire complex plane then $f(z)$ is const.

Q. Evaluate.

① $\int_c \frac{5z-2}{z^2-z} dz$ where c is $|z| = 2$ Ans. $10\pi i$

②. $\int_c \frac{3z^2+z}{z^2-1} dz$ where c is (i) $|z-1| = 1$

(ii) $|z| = 2$

③ $\int_c \frac{e^{2z} dz}{z^2-3z+2}$ where c is $|z| = 3$

① Soln: $\int \frac{5z-2}{z-1} dz - \int \frac{5z-2}{z} dz$

$= 2\pi i \times 3 + 4\pi i$

$= 10\pi i$

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* Gauss - Jordan

05/02

* Zeros of an analytic f^n

→ A zero of an analytic f^n $f(z)$ is that value of z for which $f(z) = 0$

Singularity

→ A singular pt of a f^n is that point at which the f^n ceases to be analytic.

Diff^t types of singularity

Lorenz series for expansion of $f(z)$ in powers of $(z-a)$

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots \infty$

1. Isolated singularity

→ If $z=a$ is a singular pt of $f(z)$ such that $f(z)$ is analytic at each pt in its neighbourhood then $z=a$ is called isolated singularity.

2. Removable singularity

→ If all the negative powers of $z-a$ in Lorenz series are zero then $z=a$ is called removable singular and then $\lim_{z\to a} f(z)$ exists finitely.

3. Essential singularity

→ If the negative powers of $z-a$ in Lorenz series is infinite then $z=a$ is called an essential singularity

$\lim_{z\to a} f(z)$ is infinite

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4. If all the negative powers of $f(z)$ in Lorenz series after the n^{th} are missing then $z=a$ is called a pole of order n.

* A pole of 1st order is called simple pole.

Q. find the nature of singularity of the following f^n

i) $\frac{z-\sin z}{z^2}$

Sol^n Here, $z=0$ is a singularity

$$\frac{z-\sin z}{z^2} = \frac{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\infty\right)}{z^2}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \ldots\infty$$

So, there are no negative powers of z in the expansion

$\therefore$ $z=0$ is a removable singularity.

(2) $(z+1)\sin\left(\frac{1}{z-2}\right)$

Put $z-2=t$

$$\therefore f(z) = (t+3)\sin\left(\frac{1}{t}\right)$$

$$= (t+3)\left[\left(\frac{1}{t}-\frac{1}{t^3 3!}+\frac{1}{t^5 5!}-\ldots\infty\right)\right]$$

$$= 1+3/t-\frac{1}{6t^2}-\frac{1}{2t^3}\ldots\infty$$

$$= 1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}\ldots\infty$$

So, there are infinite no. of terms in the negative power of $(z-2)$.

So, $z=2$ is called essential singularity.

3) $\frac{1}{1-e^z}$

$\rightarrow$ Poles are given by $1-e^z=0 \Rightarrow e^z = e^{i2n\pi}$

$\Rightarrow \boxed{z = i2n\pi}$

$z = 2\pi i$ is a simple pole.

4) $$\frac{e^{1/z}}{z^2} = \frac{1+\frac{1}{z}+\frac{1}{z^2.2!}+\frac{1}{z^3.3!}+\ldots\infty}{z^2}$$

$$= \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{2z^4}+\ldots\infty$$

There are infinite no. of negative power terms of z

$\Rightarrow$ $z=0$ is called essential singularity.

Q. find poles of the following f^n

i) $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Here, singular pts are $z=1$ and $z=-2$

$$\lim_{z\to1}(z-1)^2 f(z) = \frac{1}{3}$$

$\Rightarrow z=1$ is a pole of order 2

$$\lim_{z\to-2}(z+2)f(z) = \frac{4}{9}$$

$\Rightarrow z=-2$ is a simple pole

* Residue of $f(z)$

The coeff of $(z-a)^{-1}$ in Lorenz series expansion of $f(z)$ is called residue of $f(z)$ at $z=a$.

$\therefore$ a_{-1} is called residue at $z=a$

Cauchy's Residue Theorem :

→ If $f(z)$ is analytic in a closed curve C except a finite no of singular pt within C then

$$\int_C f(z)\,dz = 2\pi i \sum R_i$$

where R_i denote residues at the poles within C.

① If $f(z)$ has a simple pole at $z=a$, then

$$\text{Residue} = \lim_{z\to a} (z-a) f(z)$$

② If $f(z)$ has a pole of order n at $z=a$, then

$$\text{Residue} = \frac{1}{(n-1)!}\left\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] \right\}_{z=a}$$

Q. find residues of the f^n at its poles and hence evaluate $\int_C f(z)$ where c is the circle

i) $|z| = 2.5$ $\qquad f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

ii) $|z| = 1.5$

Soln:

Residue at $z=-2$, $R_1 = \lim_{z\to -2} (z+2) f(z)$

$$= \frac{4}{9}$$

Residue at $z=1$, $R_2 = \frac{1}{1!}\left\{ \frac{d}{dz} \left((z-1)^2 f(z) \right) \right\}_{z=1}$

$$= \frac{d}{dz}\left(\frac{z^2}{z+2} \right)\Bigg|_{z=1}$$

$$= \frac{(z+2)(2z) - z^2}{(z+2)^2}\Bigg|_{z=1}$$

$$= \frac{3\times 2 - 1}{9}$$

$$= \frac{5}{9}$$

i) $c: |z| = 2.5$

Here, both the poles 1 and -2 lie within c.

∴ By CRT,

$$\int_C f(z)\,dz = 2\pi i \sum R_i$$

$$= 2\pi i \left(\frac{4}{9} + \frac{5}{9} \right)$$

$$= 2\pi i.$$

ii) $c: |z| = 1.5$

Here, only 1 lie within c.

∴ By CRT,

$$\int_C f(z)\,dz = 2\pi i \sum R_i$$

$$= 2\pi i\,(5/9)$$

$$= \frac{10\pi i}{9}$$

Q. find residues of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ and hence evaluate $\int_C f(z)\,dz$ where c is $|z| = 2.5$