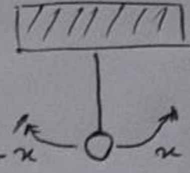
Simple Harmonic Motion: [N.K. Bajaj - The physics of waves and oscillations]



⊳ Factors:

- Elasticity of the material, $F = ma$
- Inertia., f (restoring force) $= -kx$.

So,

$$f = ma = -kx$$

$$\text{or, } m\frac{d^2x}{dt^2} = -kx$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{kx}{m} = 0$$

$$\therefore \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

SHM can be defined as (i) a force acting on an oscillating body is always in direction opposite to the displacement of the body, from the mean position,

(ii) the displacement vs. time graph occur on an oscillating body must be sinusoidal in nature.

(iii) The elastic potential energy is proportional to the square of its displacement.

$$PE = kx^2$$

*
$$\frac{d^2x}{dt^2} = \frac{d}{dt}\cdot\frac{dx}{dt}$$

multiplying and dividing by dx.

$$= \frac{d}{dt}\cdot\frac{dx}{dt}\cdot\frac{dx}{dx}$$

$$= \frac{dx}{dt}\,\frac{d}{dx}\,\frac{dx}{dt}$$

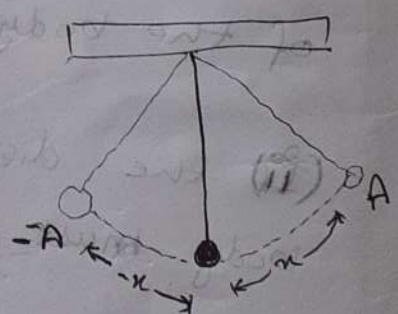
$$\therefore \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

Also,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Or, $\frac{d}{dx}\left(\frac{v^2}{2}\right) + \omega^2 x = 0$

Or, $\frac{d}{dx}\left(\frac{v^2}{2} + \frac{\omega^2 x^2}{2}\right) = 0$



Or $\frac{1}{2dx}\int d\left(v^2 + \omega^2 x^2\right) = 0$

$$\therefore v^2 + \omega^2 x^2 = C$$

At extremes, say 'A' and '-A'
$v = 0$,

So, $0^2 + \omega^2(\pm A)^2 = C$

$$\therefore C = \omega^2 A^2$$

$$v^2 = \omega^2(A^2 - x^2)$$

$$v = \pm\,\omega\sqrt{A^2 - x^2}$$

$$\frac{dx}{dt} = \pm\,\omega\sqrt{A^2 - x^2}$$

$$\pm\,\omega\, dt = \frac{dx}{\sqrt{A^2 - x^2}}$$

$$\boxed{\begin{aligned} \therefore\ \sin^{-1}\frac{x}{A} &= \omega t + \delta_1 \\ \therefore\ \cos^{-1}\frac{x}{A} &= \omega t + \delta_2 \end{aligned}}$$

$\therefore$ SHM, $\boxed{x(t) = A\sin(\omega t + \delta)}$

Q. A 50 gm mass vibrates in SHM at the end of a spring. The amplitude of the motion is 12cm and the period is 0.1 minutes. find the max. speed of the mass what will be the speed at $x = A/2$.

Soln:

$$\cancel{|V_{max}| = \omega A = 2\pi\frac{1}{t}A = 2\pi\frac{1}{6}\times 12\times10^{-2} = 0.125\ m/s.}$$

~~At $x = A/2$,~~

$$\cancel{V_{max} = \omega\frac{A}{2} = \frac{\omega A}{2} = \frac{0.125}{2} = 0.0625}$$

$$\cancel{x = 0.12\sin\left(\frac{2\pi}{6} + \delta\right)}$$

$$V_{max} = \omega A$$

$$= 2\pi\frac{1}{t}A$$

$$= 2\pi\times\frac{1}{6}\times12\times10^{-2}$$

$$= 0.125\ m/s$$

At $x = A/2$.

$$V = \omega\sqrt{A^2 - x^2}$$

$$= 2\pi\frac{1}{t}\sqrt{A^2 - \frac{A^2}{4}}$$

$$= \frac{\pi}{t}\cdot A\sqrt{3}$$

$$= 0.108\ m/s.$$

In order to describe SHM 180°, a particular instance is assigned.

$$x(t) = A \sin(\omega t + \delta)$$

$$t = 0$$

$$\phi = \boxed{\omega t + \delta}$$

$$x(t) = A \sin \omega t$$

$$A \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$= A \cos \omega t.$$

Q. A particle starts at $t=0$ from the mean position with a velocity, $v = 3\pi$ m/s in the positive direction.

If the time period of oscillation is 2 seconds (a) write the expression for displacement of the particle.

(b) what minimum time does the particle take from the mean position to a point P which lies midway between mean position and the right extreme pos.

(c) what minimum time does the particle take to reach the right extreme ~~end~~ position from mean position.

* $x(t) = A\sin(\omega t + \delta)$

$$\frac{dx(t)}{dt} = A\omega\cos(\omega t + \delta)$$

$$\therefore v = \pm\,\omega\sqrt{A^2(1-\sin^2(\omega t+\delta))}$$

* P.E. $du = -F(x)\,dx$

$$F(x) = -m\omega^2$$

$$u = m\omega^2\int_0^x x\,dx$$

$$= \frac{1}{2}m\omega^2x^2$$

$$= \frac{1}{2}KA^2\sin^2(\omega t+\delta) \qquad \left[\because K = m\omega^2,\ \because x = A\sin(\omega t+\delta)\right]$$

average $= \frac{1}{2}\int KA^2\sin^2(\omega t+\delta)\,dt$

$$= \frac{1}{2}KA^2\,\frac{1}{2}$$

$\therefore$ $P.E._{avg} = \frac{1}{4}KA^2$

K.E.

$$KE = \frac{1}{2}mv^2$$

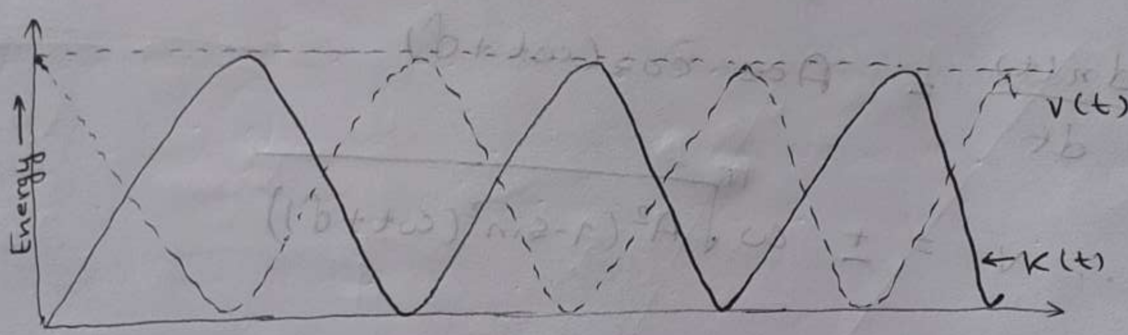
$$= \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\delta)$$

$$= \frac{1}{2}\int KA^2\cos^2(\omega t+\delta)\,dt$$

$$\therefore K.E._{avg} = \frac{1}{4}KA^2$$

* $\because$ T.E. = P.E. + K.E.

$$\boxed{\therefore T.E. = \frac{1}{2}KA^2}$$



Q. A block whose mass is 680 gm is fastened to a spring whose spring constant is $K = 65$ N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless horizontal surface and released from rest at $t = 0$. (a) What is the phase angle for the motion.

(b) What is the total M.E. of the oscillatory motion

(c) Find out P.E. and K.E. of this oscillation when the block is half-way to its end-point.

Soln:-

$m = 680\ gm = 0.68\ kg$

$F = 0.680 \times 9.8\ m/s =$

$K = 65\ N/m$

$x_f = 11\ cm = 0.11\ m$

$x_i = 0$

we have,

(a) → $x(t) = A \sin(\omega t + \delta)$

Or, $x(0) = A \sin \delta$

(b) → $M.E._{total} = \frac{1}{4} KA^2 = \frac{1}{4} \times 65 \times (0.11)^2 =$

* Angular Simple Harmonic Motion:

> Angular oscillation:

$$\tau = -k\theta$$

$$\alpha = -\omega^2\theta$$

$$\alpha = \frac{\tau}{I}$$

$$\omega = \sqrt{\frac{k}{I}}$$

$$\theta = \theta_0 \sin(\omega t + \delta)$$

Q. A uniform disk of radius 5 cm and mass 200 gm is fixed at its centre by a metal wire. The hanging disk is rotated about the wire through an angle and released. If the disk makes torquitional oscillation with the time period 0.25 seconds. Find spring constant.

601 N/m

Damped Oscillation.

$$F = -\gamma v$$

$$F = ma = m\frac{d^2x}{dt^2} = \frac{m\,dv}{dt}$$

$$\frac{m\,dv}{dt} = -\gamma v$$

$$\therefore \frac{dv}{dt} - \frac{\gamma}{m} v = 0$$

$$\frac{dv}{dt} + \frac{1}{\tau} v = 0$$ (τ is relaxation time)

$$\therefore \frac{dv}{dt} + 2bv = 0$$

Rearranging and integrating,

$$\int \frac{dv}{v} = \int -\frac{1}{\tau}\,dt$$

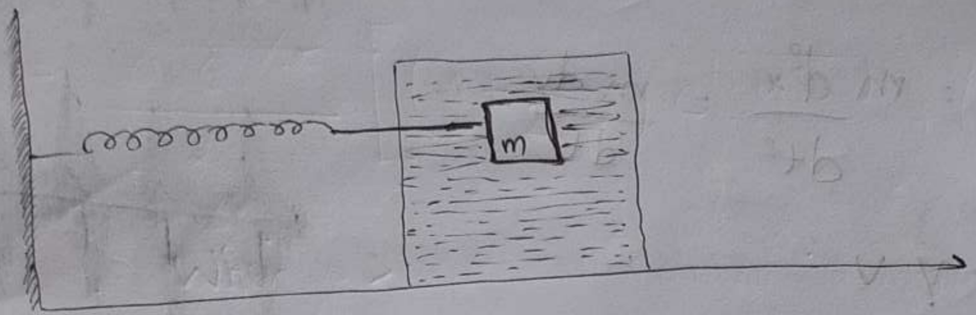
$$\therefore \ln v = -\frac{t}{\tau} + c$$

at, $t = 0$, $v = v_0$

$$\ln v_0 = c$$

$$\boxed{v = v_0 e^{-t/\tau}}$$

Differential eqⁿ of Damped oscillation:



$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

$x = Ae^{\alpha t}$
'α' and 't' are arbitrary constants.

$$\frac{dx}{dt} = \alpha Ae^{\alpha t}$$

$$\frac{d^2x}{dt^2} = \alpha^2 Ae^{\alpha t}$$

$$\alpha^2 Ae^{\alpha t} + 2b\alpha Ae^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0$$

Quadratic eqⁿ $\longrightarrow$ $\alpha^2 + 2b\alpha + \omega_0^2 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2}$$

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2}$$

So, $\alpha t = (-b + \sqrt{b^2-\omega_0^2})\, t$

$$x = Ae^{\alpha t}$$

$$\therefore x = Ae^{(-b \pm \sqrt{b^2-\omega_0^2})t}$$

$$x_1 = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t}$$

$$x = A_1 e^{-\frac{1}{2\tau}+\beta t} + A_2 e^{-1/2\tau - \beta t}$$

$\beta = \sqrt{b^2-\omega_0^2}$

$b = \frac{1}{2\tau}$

for findinding A_1 and A_2,

Differentiating (iii) w.r.t. t

$$\frac{dx}{dt} = \left(-\frac{1}{2\tau}+\beta\right) A_1 e^{-\frac{1}{2\tau}+\beta t} + \left(\frac{1}{2\tau}-\beta\right) A_2 e^{\frac{1}{2\tau}-\beta t}$$

Now, $t = 0$,

$$x_{max} = A_1 + A_2 \quad \frac{dx}{dt} = 0$$

$$\left(-\frac{1}{2\tau}+\beta\right) A_1 + \left(\frac{-1}{2\tau}-\beta\right) A_2 = 0.$$

$$\frac{-1}{2\tau}(A_1+A_2) + \beta(A_1 - A_2) = 0.$$

$$-\frac{1}{2\tau} a_0 + \beta(A_1 - A_2) = 0$$

$$(A_1 - A_2) = \frac{a_0}{2\tau\beta}$$

$$A_1 + A_2 = a_0$$

$$A_1 + A_2 + A_1 - A_2 = \frac{a_0}{2\tau\beta} + a_0$$

$$A_1 = \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

$$A_2 = \frac{a_0}{2}\left(1 - \frac{1}{2\beta\tau}\right)$$

$$x = \frac{a_0 e^{+\frac{t}{2\tau}}}{2}\left[\left(1+\frac{1}{2\tau\beta}\right)e^{\beta t} + \left(1-\frac{1}{2\tau\beta}\right)e^{-\beta\tau}\right]$$

$$\left[\therefore x = \frac{a_0 e^{t/2\tau}}{2}\left[\left(1+\frac{1}{2\tau\beta}\right)e^{(\sqrt{b^2-\omega_0^2})t} + \left(1-\frac{1}{2\tau\beta}\right)e^{-(\sqrt{b^2-\omega_0^2})t}\right]\right.$$

Case (i), When $b > \omega_0$ (overdamping)

↓

(aperiodic motion)

→ same formula.

∴ The displacement of the oscillator attacks maximum and dies off exponentially w.r.t. time.

Case (ii), When $b = \omega_0$ (critical damping)

Assume that, $\sqrt{b^2-\omega_0^2} = h$

$$x = \frac{a_0 e^{t/2\tau}}{2}\left[\left(1+\frac{1}{2\tau\beta}\right)e^{ht} + \left(1-\frac{1}{2\tau\beta}\right)e^{-ht}\right]$$

$$x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$$

$$= e^{-bt}\left(A_1 e^{ht} + A_2 e^{-ht}\right)$$

$$= e^{-bt}\left(A_1\left(1+ht+\frac{h^2t^2}{2!}+\frac{h^3t^3}{3!}\right) + A_2\left(\qquad\right)\right)$$

$$= e^{-bt}\left(A_1(1+ht) + A_2(1-ht)\right)$$

$$= e^{-bt}(M+Nt) \rightarrow M = A_1 + A_2$$

$$N = (A_1 - A_2)h$$

at, $t = 0$, $x = x_{max} = a_0$

$$\frac{dx}{dt} = 0 \quad ; \quad a_0 = M$$

$$\frac{dx}{dt} = \frac{d}{dt}(Me^{-bt}) + \frac{d}{dt}(Nte^{-bt})$$

$$0 = -bMe^{-bt} + Ne^{-bt} - Nte^{-bt}$$

$$-bM + N$$

$$bM = N \quad ; \quad N = ba_0$$

$$\therefore \quad x = e^{-bt}(a_0 + ba_0 t)$$

$$= a_0 e^{-bt}(1 + bt)$$

$$= a_0 e^{-t/2\tau}\left(1 + \frac{1}{2\tau}\right)$$

This is the required eq^n for critical damping.

From the above equation, we see that the second term decreases less rapidly than first term. In such cases, ~~&~~ the displacement of the oscillator first increases and quickly ~~rep~~ returns back to its equilibrium position. This is also known as just-aperiodic damping or critical damping.

case (iii) :— when $b < \omega_0$, (weak damping)

$\sqrt{b^2-\omega_0^2}$ will be imaginary.

Let, $\sqrt{b^2-\omega_0^2} = i\omega$; $\omega = \sqrt{\omega_0^2 - b^2}$

$$x = A_1 e^{(-b+i\omega)t} + A_2 e^{(-b-i\omega)t}$$

$$= e^{-bt}\left(\cos\omega t\,(A_1+A_2) + \sin\omega t\,[i(A_1-A_2)]\right)$$

$$= e^{-bt}\left(A\cos\omega t + B\sin\omega t\right)$$

$$= e^{-bt}\left(a_0\cos\omega t \cdot \frac{A}{a_0} + a_0\sin\omega t\,\frac{B}{a_0}\right)$$

$$= e^{-bt}\left(a_0\left[\cos\omega t \cdot \sin\phi + \sin\omega t\cos\phi\right]\right)$$

$$= a_0 e^{-bt}\sin(\omega t + \phi)$$

$A = A_1 + A_2$

$B = i(A_1 - A_2)$

$\therefore$ we can conclude that, the frequency of damped oscillator is smaller than its natural frequency,

i.e., $\dfrac{\omega}{2\pi} < \dfrac{\omega_0}{2\pi}$

The amplitude of the oscillator does not remain constant but decays exponentially with the time according to the value e^{-bt}

There are 3 parameters characterising weak damping:

1) Relaxation time
2) logarithmic Decrement
3) Quality Factor.

- Relaxation Time: It is defined as the time in which the amplitude of the weakly damped system reduces to $\frac{1}{e}$ of its initial value.

- Energy, $E = E_0 e^{-t/\tau}$

E ← Energy at time τ

E_0 ← initial

$\omega t = \tau \rightarrow E = \frac{E_0}{e}$

- Logarithmic Decrement: Due to damping, the amplitude of the oscillator decreases exponentially w.r.t. time. Suppose,

a_n and a_{n+1} be two successive amplitudes of oscillation of the particles on two sides of the equilibrium position respectively. The time interval between two successive amplitudes will be $\frac{T}{2}$.

So, $a_n = a_0 e^{-bkt}$

and, $a_{n+1} = a_0 e^{-b(t+\frac{T}{2})}$

$\therefore \frac{a_n}{a_{n+1}} = e^{\frac{bt}{2}} = d$

'd' is a constant.

- Quality Factor: It is referred to as figure of merit of a harmonic oscillator and defined as 2π times the ratio between energy stored and energy lost per period. $Q = 2\pi = \frac{\text{Energy stored}}{\text{Energy lost}} = \frac{2\pi E}{PT}$

where, P is average loss of energy per cycle $= \frac{E}{\tau}$

$$Q = \frac{E\omega}{P} = \omega\tau$$

FORCED DAMPED OSCILLATION:

If on the application of external periodic force, ~~and~~ the oscillator oscillates with the driving frequency ~~under~~ and constant amplitude and phase ~~as~~ long as the applied force is operative, then the system is said to be under forced damped oscillation.

Diff eq^n :

$$m \cdot \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = F(t)$$

If $F(t) = f\cos(nt)$, f and n are constants.

Let, $\frac{\gamma}{m} = 2b$; $\frac{k}{m} = \omega_0^2$

or, $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f\cos(nt)$$

unhomogenous second order

General sol^n:- $$x(t) = \underbrace{X_i(t)}_{\text{Particular Solution}} + \underbrace{X_m(t)}_{\text{Complementary Solution.}}$$

$$x(t) = \underbrace{x_H(t)}_{\text{Homogenous part}} + \underbrace{x_p(t)}_{\text{Solution for damped oscillation.}}$$

for Particular Solution:-

Let us assume,

$$X_p(t) = A\cos(nt - \phi)$$, where, ϕ is phase difference bet^n applied force and displacement of oscillation.

$$\frac{dx}{dt} = -An\sin(nt - \phi)$$

$$\frac{d^2x}{dt^2} = -An^2\cos(nt - \phi)$$

$$-An^2\cos(nt-\phi) - 2bAn\sin(nt-\phi) + A\omega_0^2\cos(nt-\phi)$$
$$= a\cos[(nt-\phi)+\phi]$$

a, $-An^2\cos(nt-\phi) - 2bAn\sin(nt-\phi) + A\omega_0^2\cos(nt-\phi) = a[\cos(nt-\phi)\cos\phi - \sin(nt-\phi)\sin\phi]$

$$A(\omega_0^2-n^2)\cos(nt-\phi) - 2bAn\sin(nt-\phi) = a[\cos(nt-\phi)\cos\phi - \sin(nt-\phi)\sin\phi]$$

$$\therefore \quad A(\omega_0^2-n^2) = a\cos\phi$$
$$\therefore \quad 2Abn = a\sin\phi$$

Squaring and Adding,

$$A^2[(\omega_0^2-n^2)^2+4b^2n^2] = a^2$$
$$\therefore A = \frac{a}{\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}}$$

Phase, $\tan\phi = \frac{\cancel{2bn}}{\cancel{\omega_0^2-n^2}} = \frac{2bn}{\omega_0^2-n^2}$.

Now, a particular solution is given by,

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}} \times \cos(nt-\phi)$$

And, General Solution,

$x(t) =$

Steady State Solution:

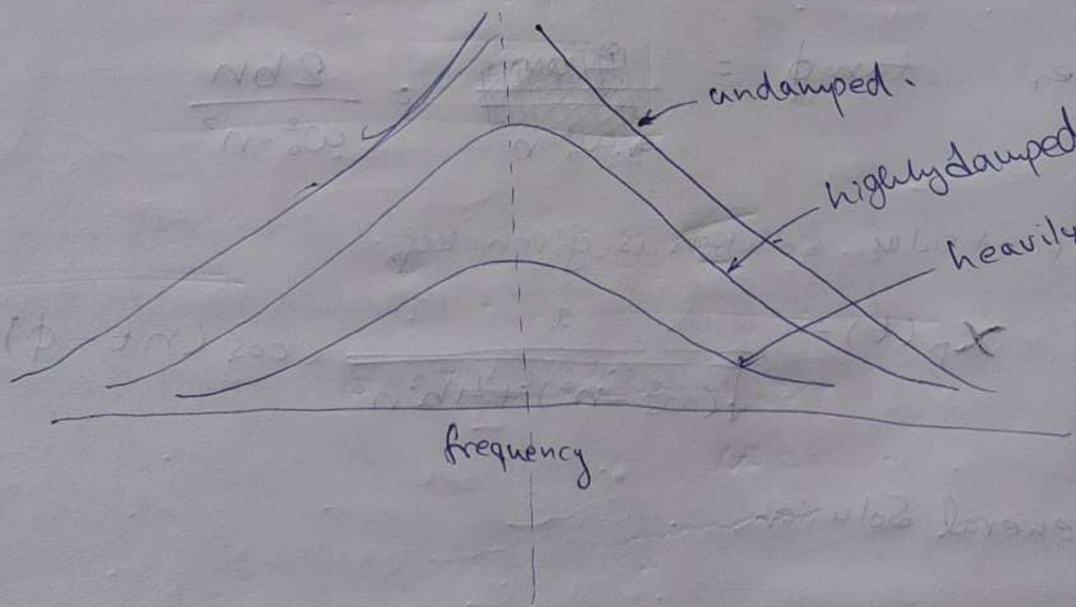
When, the tussle between damping and externally applied force tends an oscillator ~~oscillate~~ oscillates & drives with the driving frequency, then it is said to be in steady state.

In steady state, the homogenous ~~solution~~ term vanishes. as $t \rightarrow \infty$.

$$x(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2 n^2}} \cos(nt - \phi)$$

$$= A \cos(nt - \phi)$$

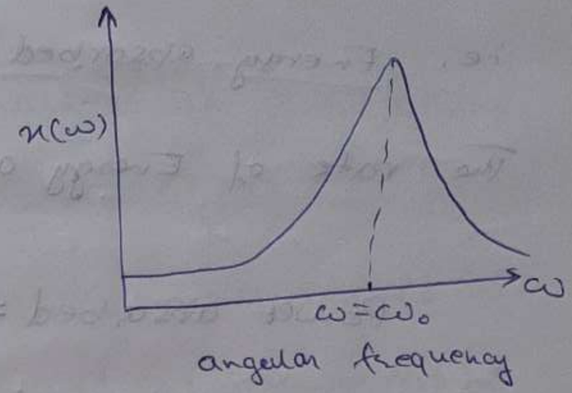
$$= \frac{t}{m} \cos(nt - \phi)$$



Resonance: for a weakly damped forced oscillation after a transient period, the object will oscillate with the same frequency as that of the driving force.

The plot of amplitude and angular frequency shows that...

$x(\omega)$ reaches at maximum when the angular frequency of the driving force is equal to the natural frequency of the undamped oscillation.



This phenomenon is known as resonance.

$$A = \frac{b}{m\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} \qquad -(i)$$

From this equation, it is clear that the amplitude of the oscillator not only depends upon amplitude of driving force but also depends upon b, ω_0, n and m.

When, $n \to 0$, $A =$

When, $n \to n$, $A \to 0$

Diff. (i) w.r.t. n,

$$\frac{d}{dn}(\omega_0^2 - n^2 + 4b^2n^2) = -4n(\omega^2 - n^2) + 8b^2n = 0$$

$$\therefore n = \sqrt{\omega_0^2 - 2b^2}$$

Power absorbed by a forced oscillator :

Whenever an oscillator is driven by an external force, energy is absorbed by the oscillator. This energy absorbed is equal to the energy dissipated during damping.

i.e, Energy absorbed = Energy dissipated during damping.

The rate of Energy absorbed is maximum at ~~reson~~. resonant frequency.

Power absorbed = Damping force × Velocity

$$= \gamma \frac{dx}{dt}$$

or,
$$\boxed{P_t = \frac{n^2 \gamma A^2}{2}}$$

Quality factor: $\quad Q = \omega_d \tau$

Damped oscillation, $\omega_d = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}$

COUPLED OSCILLATION:

An undamped harmonic oscillator has only one natural frequency of oscillation, $\omega_0 = \sqrt{\frac{k}{m}}$, but when two or more oscillations interact, several natural frequencies are possible.

Let us consider a system of masses as given below,

All possible motions in which every mass moves in SHM with same frequency is called as normal mode motion.

when, $k' = 0$ (i.e, there is no spring in between masses)
we have two uncoupled harmonic motion.
(i.e., both bobs will have their own natural frequency)

when, k' is included, there are two cases where the masses can oscillate (2 normal modes are possible).

Case I: $x_1 = x_2$ (i.e, both bobs are oscillating in same direction in same phase)
[k' is never stretched or compressed.]

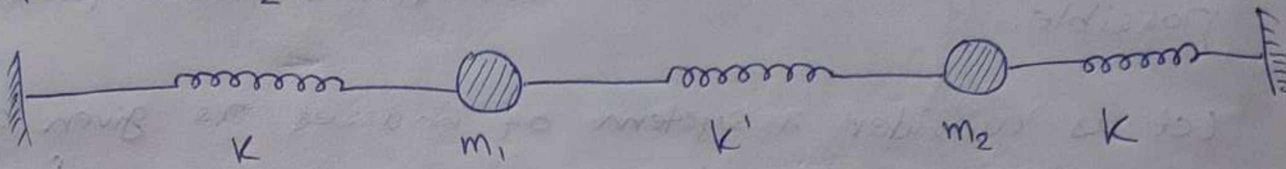
$$\omega = \sqrt{\frac{k}{m}}$$, with equal amplitudes.

Case II: $x_1 = -x_2$ (i.e, both bobs are oscillating in opposite direction)

the mid-point of spring k' is in stationary and the force ~~exerted on~~ it exerts on same mass is like that of a spring with a force constant $2k'$. The total force on each mass is same for a spring with force constant $k+2k'$.

i.e., $$\omega = \sqrt{\frac{k+2k'}{m}}$$

x_1 and x_2 are the displacements from the masses m_1 and m_2. $(m_1 = m_2 = m)$



$F = ma$,

I: $$-Kx_1 + K'(x_2 - x_1) = m\frac{d^2x_1}{dt^2}$$

II: $$-Kx_2 - K'(x_2 - x_1) = m\frac{d^2x_2}{dt^2}$$

$$-Kx_2 + K'(x_1 - x_2) = m\frac{d^2x_2}{dt^2}$$

We assume,
These equations have the solutions of the form

$$x_1 = a_1 \cos(\omega t + \phi)$$
$$x_2 = a_2 \cos(\omega t + \phi)$$

where, ω is unknown.

$$-Ka_1 + K'(a_2 - a_1) = -m\omega^2 a_1$$
$$-Ka_2 - K'(a_2 - a_1) = -m\omega^2 a_2$$

$$\Rightarrow (K + K' - m\omega^2)a_1 - K'a_2 = 0$$

$$\Rightarrow -K'a_1 + (K + K' - m\omega^2)a_2 = 0$$

This is simultaneous homogenous equation.

They always have a trivial solution, $a_1 = a_2 = 0$. For this equation, if we do not want a trivial solution, the only condition is that the determinant of the eqn^{s} must be zero.

$$\begin{vmatrix} k + k' - m\omega^2 & -k' \\ -k' & k + k' - m\omega^2 \end{vmatrix} = 0$$

$$\text{or,} \quad (k + k' - m\omega^2)^2 = k'^2$$

$$\therefore \quad k + k' - m\omega^2 = \pm k'$$

when, $\omega > 0$,

$$\omega = \sqrt{\frac{k}{m}} \checkmark \quad , \quad \omega = \sqrt{\frac{k+2k'}{m}}$$

Hence, our assumption is correct.

For, $\omega^2 = \frac{k}{m}$, $k'a_1 - k'a_2 = 0$

$-k'a_1 + k'a_2 = 0$

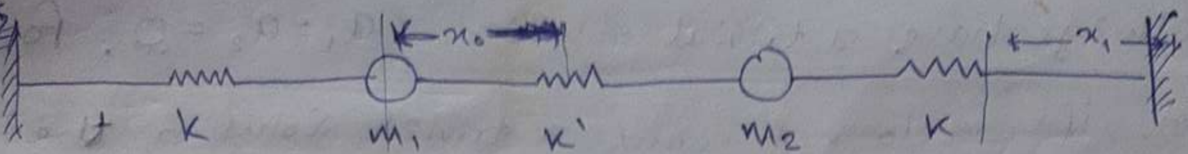
$\therefore a_1 = a_2 \rightarrow$ Mode I

for. $\omega^2 = \frac{k+2k'}{m}$, $k'a_1 + k'a_2 = 0$

$k'a_2 + k'a_1 = 0$

$\therefore a_1 = -a_2 \rightarrow$ Mode II

Q.8



$K = 10\,Nm^{-1}$, $m_1, m_2 = m$

$k' = 30\,Nm^{-1}$ x_0 by 40 cm

$m = 1\,kg$ $x_1 = 0$

Both particles are initially at rest. The system is set into oscillation by displacing x_0 by 40 cm while, $x_1 = 0$. what is the angular frequency of faster normal mode?

(i) Calculate the avg Kinetic Energy of x_1.

~~Calculate avg.~~

Q. (ii) Calculate the avg. Kinetic Energy of x_1.

(iii) A coupled oscillator has $K = 9\,Nm^{-1}$, $K' = 0.1\,Nm^{-1}$, $m = 1\,kg$. Initially, both particles have 0 velocity with $x_0 = 5\,cm$ and $x_1 = 0$. After how many oscillations in x_0, does it completely die down?

VECTOR ALGEBRA

$\frac{d}{dx}$ → derivation

$\frac{\delta}{\delta x}$ → Partial derivation

* Dot Product:

$$\Rightarrow \text{Let } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \quad (\because \theta = 0^\circ)$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0 \quad (\because \theta = \tfrac{\pi}{2})$$

$$\theta = \cos^{-1} \frac{a.b}{|a|.|b|}$$

* ∇ - Del Operator:

$$\nabla = \hat{i}.\frac{\delta}{\delta x} + \hat{j}.\frac{\delta}{\delta y} + \hat{k}.\frac{\delta}{\delta z}$$

* Gradient:

If $\phi(x,y,z)$ is a scalar function, then grad. ϕ is given by

$$\nabla\phi = \hat{i}\frac{\delta\phi}{\delta x} + \hat{j}\frac{\delta\phi}{\delta y} + \hat{k}\frac{\delta\phi}{\delta z}$$

The operator gradient operates on only scalar function.
The operator gradient is always applied on a scalar field.
So the resultant will be a vector field.
Physically, gradient gives rate of change of ϕ w.r.t x, y, z respectively. Geometrically, it gives normal to the level surface.

DIVERGENCE OF A VECTOR

If $\vec{F}$ is a vector point function then, $\nabla \cdot f$ is called as divergence of a vector. If,

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Divergence of a vector is a scalar.

Physically, divergence measures outflow or inflow.

CURL OF A VECTOR:

$\nabla \times \vec{F}$ is called curl of a vector.

If, $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Then,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & f_2 & f_3 \end{vmatrix}$$

If $\nabla \times \vec{F} = 0$, then it is said to be irrotational.

If $\nabla \times F \neq 0$, then it is rotational.

Physically, curl operation gives the angular velocity

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})$$

~~Note~~,

* $\nabla\cdot(\nabla\phi) = \nabla^2\phi$ (Divergent of gradient)

* $\nabla\cdot(\nabla\times\vec{F}) = 0$ (Divergent of curl)

* $\nabla\times(\nabla\times F) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$ (Curl of Curl)

Q. Find the value of a, b, c such that the vector

$\vec{v} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

is irrotational.

Soln: As it is irrotational, its curl is zero.

i.e.,
$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right) + \hat{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right)$$

$$= \hat{i}(c-1) - \hat{j}(4-a) + \hat{k}(b-2)$$

So,

$c-1 = 0 \Rightarrow$ ~~b=2~~ $c = 1$

$-4+a = 0 \Rightarrow a = 4$

$b-2 = 0 \Rightarrow b = 2$

$\therefore a = 4;\ b = 2;\ c = 1$

Q. If θ is the angle between two surfaces denoted by ϕ_1 and ϕ_2, then, $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|}$

Angle bet^n two surfaces

* $div(\vec{A} \times \vec{B}) = \vec{B} \cdot curl\vec{A} - \vec{A} \cdot curl\vec{B}$

* $curl(\vec{a} \times \vec{b}) = \vec{a}\, div.\vec{b} - \vec{b}.\, div.\vec{a} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

Vector Integration:

Line Integral $\int_c \vec{F} \cdot dr$

If $\vec{r}$ is (x, y, z)

If $F = b_1 i + b_2 \hat{j} + b_3 \hat{k}$

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\int F\, dr = \int (b_1 dx + b_2 dy + b_3 dz)$$

If the integral is evaluated over a surface, then it is called as surface integral.

If 'S' is any surface and 'N' is the unit normal vector to the surface 'S', then

$\int_S F.N\, dS$ is called as surface integral.

Gauss's divergence theorem: $\iint - \iiint$

Green's ~~dives~~ theorem: $\int - \iint$ given surfaces, x-y plane

Stoke's theorem : $\int - \iint$ any plane xy-yz-zx

Green's Theorem:

Let S be a closed region in x-y plane bounded by a curve C.
If $P(x,y)$ and $Q(x,y)$ be two continuous and continuous diff. scalar point functions in (x,y), then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$

NOTE: This theorem is used if the surface is in x-y plane only
This theorem converts single integration problem to double integration problem.

Gauss divergence Theorem

Let 'S' is a closed surface enclosing a volume 'V',
if $\vec{F}$ is continuous and differentiable vector point function then,

$$\int_S \vec{F}.\underbrace{\vec{N}}_{\hat{n}}\,ds = \int_V \text{div.}\,\vec{F}\,dV$$

where $\vec{N}$ is the outward drawn Unit Normal vector.

Note : This theorem is used to convert double integration problem to triple integration problem.

Stoke's Theorem

Let 'S' is a surface enclosed by a closed C and $\vec{F}$ is a continuous and differentiable vector point function then,

$$\oint_C \vec{F}.\,d\vec{r} = \int_S \text{curl}\,\vec{F}.\underbrace{\vec{N}}_{\hat{n}}\,ds$$

Where, $\vec{N}$ is outward drawn Unit Normal vector over S.

Note: This theorem is used for any Surface (or) plane.
This theorem is used to convert single integration to triple integration problem

Griffith's

Maxwell's Equation:

1) Gauss law in electrostatics:

$$E = \lim_{q_0 \to 0} \frac{F}{q_0}$$

$$= \lim_{q_0 \to 0} \frac{KQq_0}{r^2} \times \frac{1}{q_0}$$

$$\therefore E = \frac{KQ}{r^2} \qquad \left(K = \frac{1}{4\pi\varepsilon_0}\right)$$

Electric flux, $\boxed{\Phi_e = \int_s E \cdot ds}$

The net electric flux through any closed surface is $\frac{1}{\varepsilon_0}$ times of charge enclosed in it.

i.e., $\boxed{\Phi_e = \frac{q_{net}}{\varepsilon_0}}$

Gauss law

The differential form of Gauss law

$$q = \int_v \rho \, dv \qquad (\rho \text{ is charge density})$$

So, $$\int_s E \cdot ds = \int_v \frac{\rho}{\varepsilon_0} dv$$

Applying Gauss divergence theorem,

$$\int_v \nabla \cdot E \, dv = \int_v \frac{\rho}{\varepsilon_0} dv$$

Diff. w.r.t. 'v' on both sides, we get

$$\text{①} \quad \boxed{\nabla \cdot E = \frac{\rho}{\varepsilon_0}}$$

Or, $\nabla \cdot E \varepsilon_0 = \rho$

$$\boxed{\therefore \nabla \cdot D = \rho}$$

(where $D = E\varepsilon_0$ is displacement current)

2) Gauss law of magnetism:

$$\phi_m = \int_s B \cdot ds$$

The net magnetic flux through any closed surface is zero.

i.e., $\phi_m = 0$.

So, $\int_s B \cdot ds = 0$

Applying Gauss Divergence Theorem,

$$\int_v \nabla \cdot B \, dv = 0$$

$$\text{②} \quad \boxed{\therefore \nabla \cdot B = 0}$$

Continuity Eq^n:

$$I = -\frac{dq}{dt}$$

$q = \int_v \rho \, dv$

$I = \int_s J \cdot dv$

J is current density

Or, $\int_s J \cdot ds = -\frac{d}{dt} \int_v \rho \, dv$

Or, $\int_s J \cdot ds = -\int_v \frac{\partial \rho}{\partial t} \cdot dv$

Applying Gauss divergence theorem,

$$\int_v \nabla\cdot J\, dv = -\int_v \frac{\delta \rho}{\delta t}\, dv$$

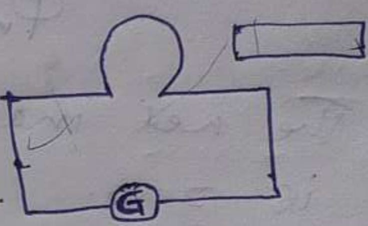
diff. w.r.t. v,

$$\boxed{\nabla\cdot J = -\frac{\delta \rho}{\delta t}}$$

$\nabla B = 0$
$\nabla \varepsilon_0 E = \rho$
$\nabla D = \rho$

(iii) Faraday's Law of Electromagnetic Induction:

The reduction in EMF is equal to rate of change of magnetic flux.



EMf, $e = -\frac{d}{dt}\phi_B$

$$\int_c E\cdot dl = -\frac{d}{dt}\int_s B\cdot ds$$

Applying Stoke's theorem,

$$\int_s \nabla\times E\, ds = -\frac{d}{dt}\int_s B\cdot dt$$

diff. w.r.t. s,

(III) $$\boxed{\nabla\times E = -\frac{\delta B}{\delta t}}$$

iv) Ampere's Circuit Law

The line integral of magnetic field over a closed loop is μ_0 times of the current passing through the loop.

$$\oint_l B.dl = \mu_0 I$$

or,

$$\oint_l \frac{B}{\mu_0} dl = I$$

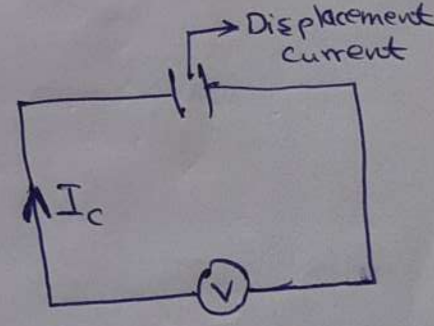
$$\therefore \int_l H\, dl = I \qquad \left(\because H = \frac{B}{\mu_0}\right)$$

$$\int_l H\, dl = \int_s J.ds$$

Applying Stoke's theorem,

$$\int_s \nabla . H ds = \int_s J.ds$$

v) Modified Ampere's Circuit Law:



Consider a capacity circuit, the current flow in this circuit may occur by two methods

i) Conduction Current, I_c

ii)

Electric field intensity due to a thin sheet is given by

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{\varepsilon_0 A} \qquad \left(\sigma = \frac{q}{A}\right)$$

$$\frac{\delta E}{\delta t} = \frac{1}{A\varepsilon_0} . I_D \qquad (I_D \text{ is displacement current})$$

or, $$\frac{\delta E.\varepsilon_0}{\delta t} = \frac{I_D}{A}$$

or, $$\frac{\delta D}{\delta t} = \frac{I_D}{A}$$

or, $$\boxed{\frac{\delta D}{\delta t} = J}$$

According to maxwell's modified ampere's circuit law,

$$I_{net} = I_c + I_D$$

or, $$\int_l B.dl = \mu_0 (I_c + I_D)$$

$$\int_l H.dl = \int_s I_c + I_D$$

Applying stoke's theorem,

$$\int_s \nabla \times H \, ds = \int_s (J_c + J_D) \, ds$$

diff. w.r.t. s,

$$\nabla . H = J_c + J_D$$

(iv) $$\boxed{\therefore \nabla H = J_c + \frac{\delta D}{\delta t}}$$

Vector Calculus:

Q. Find the gradient of $\phi = x^2y - y^2z - xyz$ at $(+1, -1, 0)$ in the direction of vector $(\hat{i} - \hat{j} + 2\hat{k})$

Sol^n Gradient, $\nabla\phi = \frac{\delta}{\delta x}\phi\,\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$

$= -2\hat{i} + \hat{j}$

$(2xy - zy)\hat{i} + (x^2 - 2yz - xz)\hat{j} + (-y^2 - xy)\hat{k}$

$x = 1$
$y = -1$
$z = 0$

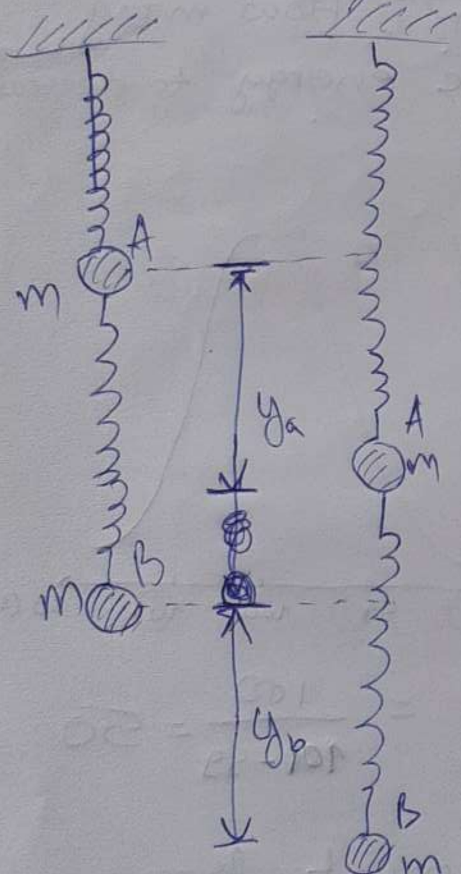
Now, $\hat{n} = \frac{\hat{i} - \hat{j} + 2\hat{k}}{|\sqrt{x^2}|}$

$\nabla\phi . \hat{n} =$

Q. Two equal masses of mass m are connected with two identical massless springs of spring const. K as shown in the fig. Show that the angular frequencies of the two normal modes of the vertical oscillations are given by $\omega^2 = (3 \pm \sqrt{5})\frac{K}{2m}$. Also, show that in the slower mode, the ratio of the amplitude of mass A to the mass B is $\frac{1}{2}(\sqrt{5}-1)$ while in the faster mode, their ratio is $\frac{1}{2}(\sqrt{5}+1)$.

Solⁿ: Rest force acting on A $= -ky_a + k(y_b - y_a)$

on B $= -k(y_b - y_a)$



$$\ddot{y}_a = -\frac{2}{m}k\,y_a + \frac{k}{m}y_b \quad \text{— (i)}$$

$$\ddot{y}_b = -\frac{k}{m}y_b + \frac{k}{m}y_a \quad \text{— (ii)}$$

we know in normal mode,

$$y_a = A\cos(\omega t + \phi)$$
$$\ddot{y}_a = -A\omega^2 y_a$$
$$y_b = B\cos(\omega t + \phi)$$
$$\ddot{y}_b = -B\omega^2 y_b$$

Substitute in (i) & (ii)

$$-A\omega^2 = \frac{-2k}{m}A + \frac{k}{m}B \quad \text{— (iii)} \Rightarrow \frac{A}{B} = \frac{\frac{k}{m}}{-\omega^2 + \frac{2k}{m}} = \frac{+k}{-\omega^2 m + 2k}$$

$$-B\omega^2 = \frac{-k}{m}B + \frac{k}{m}A \quad \text{— (iv)} \Rightarrow \frac{A}{B} = \frac{-\omega^2 + \frac{k}{m}}{\frac{k}{m}} = \frac{k - \omega^2 m}{k}$$

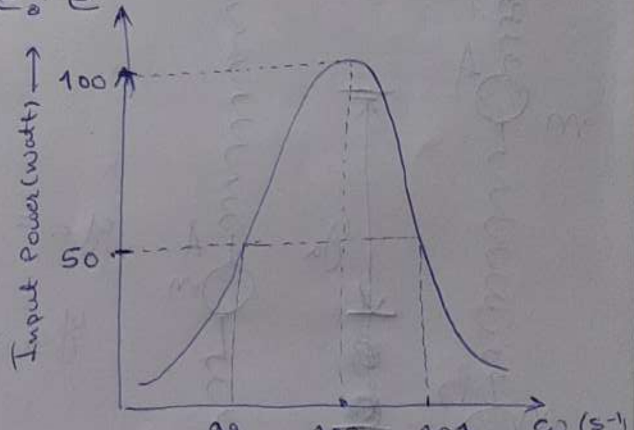
Q. The figure shows the power resonance curve of a certain mechanical system which is driven by a force of constant magnitude but variable angular frequency 'ω'. Answer the following on the basis of the information given in graph.

⟨1⟩ What is the resonant angular frequency ω_0?

⟨2⟩ What is the Q?

⟨3⟩ At exact resonance, what is the total energy, E_0 of the oscillator?

⟨4⟩ If the driving force is removed, the energy decreases according to the eq^n $E = E_0 e^{-\gamma t}$. How many seconds does it take for the energy to decrease to a value $E \longrightarrow E_0 e^{-1}$



Sol^n:

1) → At Max. Power

Ang. frequency, ω = Resonant freq. ω_0.

$\therefore \omega_0 = 100$ rad/s

2) → $Q = \dfrac{\text{Resonant freq. } \omega_0}{\text{Angular freq. width } \Delta\omega \text{ at half max power}} = \dfrac{\omega_0}{\Delta\omega} = \dfrac{100}{101 - 99} = 50$

3) → Energy of the oscillator at resonant frequency

$$E_0 = \frac{1}{2} m\omega_0^2 \frac{f_0^2}{\omega_0^2 \gamma^2} \quad ; \quad \gamma = 2b, \quad f_0 = \frac{F_0}{m}$$

$$= \frac{1}{2} \cdot \pi \frac{f_0^2}{m \cdot \gamma^2}$$

$$= \frac{F_0^2}{2m\gamma^2}$$

Q. A massless spring suspended from a rigid support carries a flat disc of mass 100 g at its lower end. It is observed that the system oscillates with a frequency of 10 Hz and the amplitude of the damped oscillation reduces to half its ~~uniform~~ undamped value in 1 min.

1. Calculate Resistive force constant.
2. Relaxation time.
3. Quality factor.

Soln:-

(1) $A = A_0 e^{-\gamma t}$ $\qquad \gamma = \frac{b}{2m}$

$$\frac{A_0}{2} = A_0 e^{-\gamma t}$$

$$\frac{1}{2} = e^{-\frac{b}{2m}t}$$

$$\therefore b =$$

(2) $A = \frac{1}{e} = e^{-1} = e^{-\frac{bt}{2m}} = e^{-\frac{b\tau}{2m}} = e^{-\frac{b \cdot 2m}{2m \cdot b}}$

$$\left(\because \tau = \frac{2m}{b}\right)$$

(3) $Q = \omega_0 \tau$

Dumped Oscillation:

Q. A Spring compressed by 10 cm develops a restoring force of 10 N. A body of mass 4 kg is attached to it. Calculate the compression of the spring due to the weight and calculate the period of oscillation.

Solⁿ:-

$$K = \frac{F}{y} = \frac{10\,N}{10\times10^{-2}}$$

$$\therefore K = 100\,N/m$$

$$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$\boxed{K \rightarrow \text{Spring const.}}$

$$mg = Kx$$

$$\text{or, } 100x = 40 \Rightarrow x = 0.4 = 40\,cm$$

$$w = \frac{2\pi}{T}$$

$$\text{So, } T = \frac{2\pi}{\omega} = \frac{2\pi}{5} = 1.255$$

Q. A mass of 1 kg is suspended from a spring at stiffness constant 25 N/m. If the undamped frequency is $2\sqrt{3}$ times the damped frequency, calculate the damping factor or damping const.

$\boxed{U.f. = 2\sqrt{3}\ D.F.}$

Solⁿ:-

$$\omega_0 = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\omega^2 = \omega_0^2 - b^2$$

damped

undamped

$$\omega^2 = \frac{K}{m} - \frac{b^2}{4m^2}$$

Q. A point moves along the X-axis according to the law $x = a\sin^2(\omega t - \frac{\pi}{2})$. Find:

(i) amplitude, period of oscillation.

(ii) Velocity v_x as in function of x; also draw V_x.

Sol^n:- (i) → we know,

$$x = a\sin^2(\omega t - \frac{\pi}{2})$$

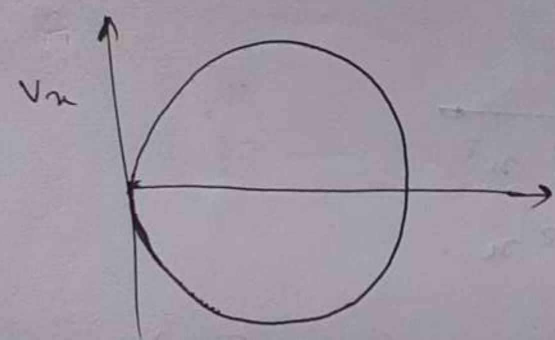
$$= a\left[\frac{1-\cos\{2(\omega t - \frac{\pi}{2})\}}{2}\right]$$

$$\therefore x = \frac{a}{2} - \frac{a}{2}\cos(2\omega t - \pi)$$

implies, amplitude $= \frac{a}{2}$ and period of oscillation $= \pi$.

(ii) → $\therefore V_x^2 = \frac{\omega^2 a^2}{4} - \omega^2 x^2$

$$v_x^2 = 4\omega^2 x(a-x)$$



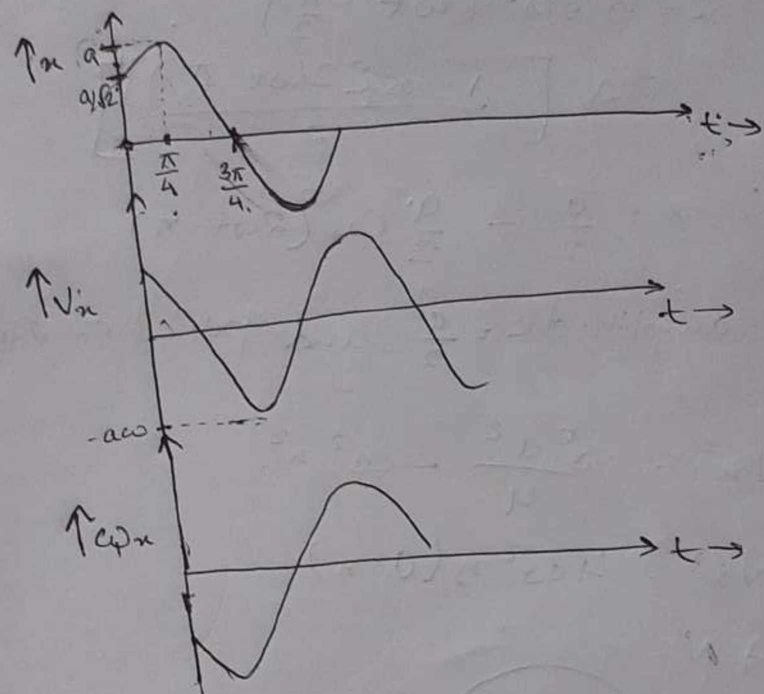
Q. A particle performs harmonic oscillations along x-axis about

* **TUTORIAL** ⊲ 30th Jan, 2020

Q. WAVES AND OSCILLATIONS ALONG THE X-AXIS according to the law of $x = a\cos(\omega t - \frac{\pi}{4})$. (i) Draw the appropriate plots of displacement 'x', velocity projection V_x, acceleration projection 'ω_x' as a function of 't'..

(ii) Express V_x and ω_x as a function of coordinate 'x'.

Solⁿ:- (i) →



$x = a\cos(\omega t - \frac{\pi}{4})$

$\frac{dx}{dt} = V = -a\omega\sin(\omega t - \frac{\pi}{4})$

$\frac{dV}{dt} = \omega_x = -a\omega^2\cos(\omega t - \frac{\pi}{4})$

(ii) → $V_x = \omega\sqrt{a^2 - x^2}$

$\omega_x = -\omega^2 x.$

$V_x = \dot{x} = -a\omega\sin(\omega t - \pi/4)$

$V_x^2 = a^2\omega^2 \sin^2(\omega t - \pi/4)$

$x = a\cos(\omega t - \pi/4)$

$x^2 = a^2\cos^2(\omega t - \pi/4)$

$\cos^2(\omega t - \pi/4) = \frac{x^2}{a^2}$

$\sin^2(\omega t - \pi/4) = 1 - \frac{x^2}{a^2}$

$V_x^2 = a^2\omega^2\left(1 - \frac{x^2}{a^2}\right)$

$V_x = \omega\sqrt{a^2 - x^2}$