INDEX

NAME: Mohit Kumar Singh ROLLNO.: 1815052

STD.: Ist SEM DIV./SEC.: SUBJECT: MATHS

S. No.	Date	Title	Page No.	Teacher's Sign/Remarks

Sequence and Series

1. Def.ⁿ of Sequence and series
2. Infinite Series
3. Conversions of sequences and infinite series.
4. Comparision Test
5. Ratio "
6. Raabee's "
7. Cauchy's root "
8. Integral "
9. Alternating series & Leibnitz's Test.

---

* **Sequence** → A sequence in $\mathbb{R}$ is a function from $\mathbb{N}$ into $\mathbb{R}$

Notation:- $x = \{x_n\}$; where x_n denotes n^{th} term of an sequence.

Ex:- $\left\{\frac{1}{n}\right\}$

→ **Limit of a Sequence** :-

$\{x_n\}$ is a sequence

and $\underset{n\to\infty}{Lt}\ x_n = l$

for a given $\varepsilon > 0$, $\Rightarrow M > 0$ such that;

$|x_n - l| < \varepsilon, \forall\ n \geq M.$

→ **Convergent Sequence** :- A sequence $\{x_n\}$ is said to be convergent if it has limit (a finite number)

* Limit of a sequence is unique

→ <u>Divergent Sequence</u> :- A sequence is divergent if it is not convergent i.e. it has no limits.

→ <u>Bounded Sequence</u> :- A sequence $\{x_n\}$ is bounded if there exists $(\exists)$ $K>0$ such that $|x_n|<K \ \forall \ n\in\mathbb{N}$.

Ex:- (1) $\left\{\frac{1}{n}\right\}$

(2) $\left\{\frac{3n}{n+\sqrt{n}}\right\}$

(1) Let $\varepsilon>0$ be given

we have to find $M>0$ such that $|x_n-0|<\varepsilon \ \forall \ n\geq M$

$$\Rightarrow \left|\frac{1}{n}-0\right|<\varepsilon \ \forall \ n\geq M$$

$$\Rightarrow \frac{1}{n}<\varepsilon$$

$$\Rightarrow n>\frac{1}{\varepsilon} \ \forall \ n\geq M$$

So; Choosing $M>\frac{1}{\varepsilon}$;

we have $\left|\frac{1}{n}-0\right|<\varepsilon \ \forall \ n\geq M$

$\lim_{n\to\infty}\frac{1}{n}=0$; i.e. 0 is the limit

* **Theorem:-** Every convergence sequence is bounded. Does the converse true? Justify.

Proof:- Let the seq. $\{x_n\}$ be convergent and converges to a finite no. l

$\lim_{n\to\infty} x_n = l$

i.e, for $\varepsilon = 1$, $\exists\, M > 0$ such that

$|x_n - l| < 1 \;\forall\, n \geq M$.

$$|x_n| = |x_n - l + l|$$
$$\leq |x_n - l| + |l|$$
$$\leq 1 + |l|$$

Let $k = \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}$

$\therefore |x_n| < k + 1 + |l|$

Let $K = k + 1 + |l|$

$\therefore |x_n| < K \;\forall\, n \in \mathbb{N}$

$\therefore \{x_n\}$ is bounded.

But converse is not necessarily true.

Ex:- Let $\{x_n\}$ be defined by

$x_n = 0$; if n is even
$\;\;\;\; = 1$; if n is odd.

It stays inside the interval $[0,1]$

i.e, the sequence is bounded. The sequence has no limit i.e, it is not convergent.

* Infinite Series :-

Let $\sum_{n=1}^{\infty} U_n = U_1 + U_2 + \dots + \infty$ be an infinite series.

Let $S_n = U_1 + U_2 + \dots + U_n$ denotes the sum of first n terms of the series.

* If $\underset{n\to\infty}{Lt}\ S_n = l$ (a finite number)

then $\sum U_n$ is convergent and l is sum of the series.

* Divergent series :- If $\lim_{n\to\infty} S_n = \pm\infty$; then the series is divergent.

* Oscillatory series :- If a series is neither convergence nor divergent; then the series is oscillatory.

* Standard result :-

(i) The harmonic series $\left(\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \infty\right)$ is convergent if $p > 1$; and divergent if $p \leq 1$.

Ex :- (i) $\sum \frac{1}{n^{3/2}} \rightarrow$ Convergent

(ii) $\sum \frac{1}{n} \rightarrow$ divergent

(iii) $\sum \frac{1}{\sqrt{n}} \rightarrow$ divergent

Theorem:-

The necessary condition for ~~convergence~~ convergent of infinite series $\sum_{n=1}^{\infty} U_n$ is $\lim_{n\to\infty} U_n = 0$.
Does the converse true ? Justify.

Proof:- Let the infinite series $\sum U_n$ be convergent and converge to a finite number l.

By def^n; $\lim_{n\to\infty} S_n = l$ ——(1)

$$\Rightarrow U_n = S_n - S_{n-1}$$

$$\Rightarrow \lim_{n\to\infty} U_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \quad \left\{ \text{as } \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} \right.$$

$$= l - \lim_{n\to\infty} S_{n-1}$$

$$= l - l$$

$$= 0$$

$$\boxed{\lim_{n\to\infty} U_n = 0}$$

The converse of the result is not necessarily true.

Ex:- Let $\sum_{n=1}^{\infty} \frac{1}{n}$

Here $U_n = \frac{1}{n}$

$\therefore \lim_{n\to\infty} U_n = 0$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as $P \leq 1$.

So, the converse is not true.

Test 1:-

* Comparison test :- (limit form)

Statement :- for two +ve term series, ΣU_n and ΣV_n ; If

$$\lim_{n\to\infty} \frac{U_n}{V_n} = l \ (\neq 0)$$

Then both the series converge or diverge together.

* Test the convergence of :-

1) $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots + \infty$

2) $\sum \frac{\sqrt{n}}{n^3+1}$

3) $\sum \frac{1}{n} \sin\frac{1}{n}$

1) $U_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{n^{\cancel{3}2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

Let us choose ΣV_n where $V_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = 2 \neq 0$$

$\therefore$ By comparision test, both the series ΣU_n and ΣV_n converges or diverges together.

But ΣV_n is convergent .. so; U_n will be also convergent.

2) $\sum \frac{\sqrt{n}}{n^3+1}$

$$U_n = \frac{n^{1/2}}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{1}{n^{5/2}\left(1+\frac{1}{n^3}\right)}$$

Let $V_n = \frac{1}{n^8}$.

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = 1.$$

$\therefore$ By comparison test, both the series converges or diverges together.

But V_n is convergent so, U_n is also convergent.

3) $\sum \frac{1}{n} \sin \frac{1}{n}$

Soln: $U_n = \frac{1}{n} \sin \frac{1}{n}$

$V_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

So, both the series converges or diverges together.

But, V_n is convergent.

So, U_n is convergent.

* Test 2 :-

D'Alembert's Ratio Test :- (Ratio test)

Statement :- In a +ve term series $\sum U_n$;

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = l \text{ (a finite no.)}$$

If $l > 1 \longrightarrow$ series converges.
If $l < 1 \longrightarrow$ series diverges.
If $l = 1 \longrightarrow$ Ratio test fails.

Ex :-

(1) $\sum \frac{2^n}{n^3+1}$

(2) $\sum \frac{n!}{n^n}$

(3) $1 + 3x + 5x^2 + \dots + \infty$

(4) $1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \dots + \infty \; ; \; \alpha > 0, \beta > 0.$

(1) $U_n = \frac{2^n}{n^3+1} \qquad U_{n+1} = \frac{2^{n+1}}{(n+1)^3+1}$

$$\frac{U_n}{U_{n+1}} = \frac{1}{2} \times \frac{(n+1)^3+1}{n^3+1}$$

$$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt} \frac{1}{2} \times \frac{\left(1+\frac{1}{n}\right) + \frac{1}{n^3}}{1+\frac{1}{n^3}}$$

$$= \frac{1}{2} < 1$$

$\therefore$ Divergent series

② $\sum \frac{n!}{n^n}$

$u_n = \frac{n!}{n^n}$ $\quad U_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{U_n}{U_{n+1}} = \frac{1}{(n+1)} \times \frac{(n+1)^{n+1}}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e^{\lim_{n\to\infty} \cancel{n}\left(\cancel{1}+\frac{1}{\cancel{n}}-\cancel{1}\right)}$$

$$= e > 1$$

$\therefore$ convergent series.

③ ~~$1+3x+5x^2+ \dots +\infty$~~

④ $1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\dots+\infty$

$$U_n = \frac{(\alpha+1)(2\alpha+1)\dots[(n-1)\alpha+1]}{(\beta+1)(2\beta+1)\dots[(n-1)\beta+1]}$$

$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{\beta}{\alpha}$; Therefore; by ratio term;

→ Series converges if $\frac{\beta}{\alpha} > 1$.

→ Series diverges if $\frac{\beta}{\alpha} < 1$.

→ Ratio test fails if $\frac{\beta}{\alpha} = 1$.

When $\beta = \alpha$; then the given series becomes:-

$1+1+ \dots +\infty$

$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt}\, n = \infty$ $\quad \therefore$ Series will diverges when $\beta = \alpha$

Series will convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

③ $1+3x+5x^2+\dots+\infty$

$U_n = (2n-1)x^{n-1}$

$U_{n+1} = (2n+1)x^n$

$\frac{U_n}{U_{n+1}} = \frac{(2n-1)}{(2n+1)} \cdot \frac{1}{x}$

$\Rightarrow \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)} \cdot \frac{1}{x}$

$= \frac{1}{x}$.

By ratio test; Series is convergent if $\frac{1}{x}>1$ or $x<1$

Series is divergent if $\frac{1}{x}<1$ or $x>1$.

Ratio test fails when $x=1$.

When $x=1$;

the series $1+3+5+\dots+\infty$

$S_n = n^2$

$\lim_{n\to\infty} S_n = \infty$. So, series is divergent.

So; when $x<1$; Series is convergent.

when $x\geq 1$; Series divergent.

Q. What is the limit of the sequence $\left\{\frac{1+2+3+\dots+n}{n^2}\right\}$

Soln: Let $U_n = \left(\frac{1+2+3+\dots+n}{n^2}\right)$

$$= \frac{n(n+1)}{2n^2}$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

Q.

* **Test 3: Raabee's Test :-**

In a +ve term series ΣU_n

$$\underset{n\to\infty}{Lt} n\left(\frac{U_n}{U_{n+1}} - 1\right) = l \text{ (say)}$$

If $l > 1 \Rightarrow$ Series is convergent.

If $l < 1 \Rightarrow$ Divergent series

If $l = 1 \Rightarrow$ test fails

* **Test 4: Cauchy's Root test :-**

In a +ve term series ΣU_n

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = l \text{ (say)}$$

If $l < 1 \Rightarrow$ Series is convergent

If $l > 1 \Rightarrow$ Series is divergent.

If $l = 1 \Rightarrow$ Test fails.

* (i) $\sum \frac{4.7 \ldots (3n+1)}{n!} x^n$

(ii) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$

(iii) $\sum \frac{n^3}{3^n}$

(iv) $\sum \frac{1}{(\log n)^{n}}$

(v) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \ldots \infty$; $x > 0$

(i) $U_n = \frac{4.7 \ldots (3n+1)\, x^n}{n!}$

$U_{n+1} = \frac{4.7 \ldots (3n+4) \, x^{n+1}}{(n+1)!}$

$\frac{U_n}{U_{n+1}} = \frac{(3n+1)(n+1)}{(3n+4)} \times \frac{1}{x}$

$\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = \frac{1}{3x}$

By ratio test; Series is convergent when $\frac{1}{3x} > 1 \Rightarrow x < \frac{1}{3}$

Series is divergent when $\frac{1}{3x} < 1 \Rightarrow x > \frac{1}{3}$

Ratio test fails when $\frac{1}{3x} = 1 \Rightarrow x = \frac{1}{3}$

When $x = \frac{1}{3}$,

$\frac{U_n}{U_{n+1}} = \frac{(3n+1)(n+1)}{(3n+4)} \times \frac{1}{3}$

(11) $U_n = \frac{(n!)^2}{(2n)!} \times x^{2n}$

$$U_{n+1} = \frac{\{(n+1)!\}^2 \times x^{2n+2}}{(2n+2)!}$$

$$\frac{U_n}{U_{n+1}} = \frac{1}{(n+1)^2} \times (2n+1)(2n+2) \times \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{4}{x^2}$$

By ratio test; Series is convergent if $\frac{4}{x^2} > 1 \Rightarrow x^2 < 4 \Rightarrow x \in ($

Series is divergent if $\frac{4}{x^2} < 1 \Rightarrow x^2 > 4$

When $x^2 = 4$, ratio test fails.

When $x^2 = 4$;

$$\frac{U_n}{\cancel{U_{n+1}}} = \frac{\cancel{(2n+1)}\cancel{(2n+2)}(n!)^2}{(2n)!} \times 4^n$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+1)(2n+2)}{(n+1)^2} \times \frac{1}{4}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = n\left(\frac{\cancel{4n^2} + 6n + 2 - \cancel{4n^2} - 4 - 8n}{4(n+1)^2}\right)$$

$$= \cancel{n\left(\frac{3n^2+4n+1}{(n+1)^2}\right)}\ n\left(\frac{-2-2n}{4(n+1)^2}\right) = \frac{-n}{2}\left(\frac{\cancel{1+n}}{(n+1)^{\cancel{2}}}\right) = \frac{-1}{2}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{-1}{2} < 1$$

So, It is a divergent series when $x = 2$.

When $\begin{cases} x^2 < 4 \longrightarrow \text{convergent} \\ x^2 > 4 \longrightarrow \text{divergent} \\ x^2 = 4 \longrightarrow \text{divergent.} \end{cases}$

(iii) $U_n = \dfrac{n^3}{3^n}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \frac{n^{3/n}}{3}$$

$$= \frac{1}{3} \underset{n\to\infty}{Lt} \left(n^{1/n}\right)^3$$

$$= \frac{1}{3} < 1$$

So, the series is convergent.

(iv) $U_n = \dfrac{1}{(\log n)^n}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\log n} = 0$$

$\because l < 1$. So, the series is convergent

(v) $U_n = \left(\frac{n+1}{n+2}\right)^n x^n$

$U_{n+1} = \left(\frac{n+2}{n+3}\right)^{n+1} x^{n+1}$

$\frac{U_n}{U_{n+1}} = \cancel{\left\{\frac{(n+1)(n+3)}{(n+2)^2}\right\}^n \left(\frac{1}{x}\right)}\ \frac{(n+1)^n (n+3)^{n+1}}{(n+2)^{2n+1}} \times \frac{1}{x}$

$= \frac{1}{x}$

By root test ; Series is convergent if $x < 1$
Series is divergent if $x > 1$
If $x = 1$, then Cauchy's root test fails;

When $x = 1$;

$U_n = \left(\frac{n+1}{n+2}\right)^n$; $V_n = 1$

$\lim_{n\to\infty} U_n = e^{\lim_{n\to\infty} n\left(\frac{n+1-n-2}{n+2}\right)}$
$= e^{\lim_{n\to\infty} n\left(\frac{-1}{n+2}\right)} = e^{-1} = \frac{1}{e}$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{e} \neq 0$; $\therefore$ series is not convergent by necessary condition of convergence. but, series being positive term; the must be divergence when $x = 1$.

or; $\lim_{n\to\infty} U_n =$

(Q) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\cdots+\infty$

Soln: $U_n=\left\{\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right\}^{-n}$

$\left(\frac{n+1}{n}\right)^{n+1}-\frac{(n+1)}{n}$

$$U_n=\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}$$

$$\lim_{n\to\infty}(U_n)^{1/n}=\frac{1}{e-1}<1$$

Series is convergent.

$$\underset{n\to\infty}{Lt}(U_n)^{1/n}=\underset{n\to\infty}{Lt}\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-1}$$

$$=\underset{n\to\infty}{Lt}\left\{\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right\}^{-1}$$

$$=\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)}$$

$$=\frac{1}{e^{\underset{n\to\infty}{Lt}(n+1)\times\frac{1}{n}}-\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)}$$

$$=\frac{1}{e-1}<1$$

* **Alternating series:-** A series whose series are alternatively negative and Positive.

Leibnitz's Test: In an alternating series

$$U_1 - U_2 + U_3 - \cdots \infty$$

if (i) $|U_{n+1}| < |U_n| \; \forall n$

(ii) $\underset{n\to\infty}{Lt} U_n = 0$

then the series is convergent.

(1) Test the convergence of

(i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots \infty$

(ii) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \cdots \infty$

(iii) $1 - 2 + 3 - 4 + \cdots \infty$

(i) soln- $U_n = (-1)^{n-1} \frac{1}{\sqrt{n}}$

$U_{n+1} = (-1)^n \cdot \frac{1}{\sqrt{n+1}}$

$|U_{n+1}| < |U_n| \; \forall n$

also; $\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} (-1)^n \frac{1}{\sqrt{n}} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

(ii) soln- $|U_{n+1}| < |U_n| \; \forall n$ $\quad U_n = (-1)^{n-1} \cdot \frac{1}{n\sqrt{n}}$

also; $\underset{n\to\infty}{Lt} U_n = 0$

By Leibnitz's test, the series is convergent.

(iii) $1-2+3-4+\cdots\infty$

$S_n = 1-2+3-4+\cdots$ to n term

$= (1-2)+(3-4)+\cdots$

$= -1 - 1 \cdots\cdots \frac{n}{2}$ terms

$= \frac{-n}{2}$ if n is even

$\underset{n\to\infty}{Lt}\ S_n = -\infty$, if n is even.

~~So, the series is convergent~~

if n is odd;

$1+(-2+3)+(-4+5)+\cdots$ n terms

$= 1+\{(+1)+(+1)+\cdots\cdots \frac{(n-1)}{2}$ term$\}$

$= 1+\frac{n-1}{2}$

$\underset{n\to\infty}{Lt}\ S_n = \infty$, if n is odd.

~~So, the series~~

So, the series oscillate.

Integral test :

A series $f(1) + f(2) + \ldots f(n) + \ldots \infty$ is convergent or divergent according as $\int_1^\infty f(x)\,dx$ is finite or infinite.

Proof of that Harmonic series result :-

Let $f(x) = \frac{1}{x^p}$

Its convergence or divergent depends on $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite.

$$\int_1^\infty \frac{dx}{x^p} = \lim_{m\to\infty} \int_1^m \frac{dx}{x^p}$$

$$= \lim_{m\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^m = \lim_{m\to\infty} \left(\frac{m^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

If $p>1$; $\lim_{m\to\infty} \left(\frac{m^{-p+1}}{-p+1} + \frac{1}{p-1}\right) = \frac{1}{p-1}$ (finite)

If $p<1$; $\lim_{m\to\infty} \left(\frac{m^{-p+1}}{1-p} + \frac{1}{p-1}\right) = \infty$

If $p=1$; $\int_1^\infty \frac{dx}{x} = [\log x]_1^\infty = \infty$

* Asymptotes :- A straight line at finite distance from ~~infinity~~ origin is said to be asymptote to an infinite branch of curve if the perpendicular distance of a point P on the curve approaches '0'. as the point P moves to infinity along the curve.

General eq. of ~~curve~~

$$y = mx + C$$

NOTE → The no. of asymptotes to a curve (cartesian curve) does not exceeds the degree of the equation of the curve

There are three types of asymptotes :-

(i) Asymptotes parallel to x axis

(ii) Asymptotes parallel to y axis

(iii) Inclined or obliqued asymptotes.

Case-I. Asymtotes parallel to x axis can be found by equating the coefficient of highest power of x in the curve to zero provided the coefficient is not a constant.

Case-II. Asymptotes parallel to y axis can be found by equating the coefficient of highest power of y in the curve to zero provided the coefficient is not a constant.

(8) $2x^2y^2 - x^2y + xy^2 - x + y + 5 = 0$

Asymptotes parallel to x axis:-

$x^2 y(2y-1) + xy^2 - x + y + 5 = 0$

$\Rightarrow y(2y-1) = 0$

$\therefore y = 0,\ y = \frac{1}{2}$

→ Asymptotes parallel to y axis:-

$2x^2 + x = 0$

$x(2x+1) = 0$

$\therefore x = -\frac{1}{2},\ x = 0$

Case (III) Inclined or obliqued asymptotes:-

Let the eq^n of curve

$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + \dots = 0$ where

$\phi_i\left(\frac{y}{x}\right)$ denotes an expression of degree i in $\left(\frac{y}{x}\right)$.

Let $y = mx + c$ be the obliqued asymptote :

Step 1:- Putting $x = 1$ and $y = m$ in the n^{th} degree term (highest degree) we get $\phi_{(n)}(m)$

Step 2:- Putting $x = 1$ and $y = m$ in the $(n-1)^{th}$ degree terms we get $\phi_{n-1}(m)$

Step 3:- $\phi_3(m) = 0$

$m = m_1, m_2, m_3$

Step 4:- c can be calculated from the formula:- $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

If c takes the form $\frac{0}{0}$ for some value of m then c can be calculated from $\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

where (ϕ_{n-2}) can be obtained by putting $x = 1$, $y = m$ in $(n-2)^{th}$ degree term.

Q. $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2(y+x) + 1 = 0$

Soln. There is no asymptotes parallel to x-axis and y axis.

Let $y = mx + c$ be the oblique asymptote.

Putting $x = 1$ and $y = m$ in 3rd degree terms we get

~~$\phi_3(m) = m^3 - 2m^2 - m + 2 + 3m^2 - 7m + 2 + 2m + 2 + 1$~~

$\phi_3(m) = m^3 - 2m^2 - m + 2$

Putting $x = 1$ and $y = m$ in 2nd degree term;

$\phi_2(m) = 3m^2 - 7m + 2$

Now, $\phi_3(m) = 0$

$\Rightarrow m^3 - 2m^2 - m + 2 = 0$

$\Rightarrow m = \pm 1, 2$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$$c = \frac{-(3m^2 - 7m + 2)}{(3m^2 - 4m - 1)}$$

$c = -1, -2, 0$

$\therefore$ Three asymptotes are (i) $y = -x - 1$

(ii) $y = -x - 2$

(iii) $y = 2x$

(Q) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

Soln. There is no asymptotes parallel to x-axis and y-axis.

Let $y = mx + c$ be the oblique asymptote.

Putting $x = 1$ and $y = m$ in 3rd degree terms;

$\phi_3(m) = 1+3m-4m^3 = 0$

$\Rightarrow 4m^3 - 3m - 1 = 0$

$\Rightarrow 4m^3 - 4m^2 + 4m^2 - 4m + m - 1 = 0$

$\Rightarrow 4m^2(m-1) + 4m(m-1) + 1(m-1) = 0$

$\Rightarrow (m-1)(4m^2+4m+1) = 0$

$\Rightarrow (m-1)(2m+1)^2 = 0$

$m = 1, -\frac{1}{2}, -\frac{1}{2}$

~~Putting $x=1$ and $y=m$ in 1st degree term;~~

$\phi_1(m) = -1+m$.

$\phi_3'(m) = 3 - 12m^2$

$\phi_3''(m) = -24m$

Then, $\frac{c^2}{2!} \times (-24m) + (-1+m) = 0$

$\Rightarrow -12c^2m + m - 1 = 0$

$\Rightarrow c^2 = \frac{m-1}{12m}$

$m = 1 \Rightarrow c^2 = 0 \Rightarrow c = 0$

$m = -\frac{1}{2} \Rightarrow c^2 = \frac{+\frac{3}{2}}{+6} = \frac{1}{4} \Rightarrow c = \pm\frac{1}{2}$

$y = x$

$y = -\frac{x}{2} + \frac{1}{2}$

$y = -\frac{x}{2} - \frac{1}{2}$

* Partial Differentiation :-

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

$$\frac{\partial z}{\partial y} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

$$\frac{\partial^2 z}{\partial x^2} \qquad \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Ex:(1) $z = x^3 + y^3 + xy$

$$\frac{\partial z}{\partial x} = 3x^2 + y$$

$$\frac{\partial z}{\partial y} = 3y^2 + x$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 1$$

Ex:-2 If $u = \frac{y}{z} + \frac{z}{x}$

find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Soln. $\frac{\partial u}{\partial x} = \frac{-z}{x^2}$, $\frac{\partial u}{\partial y} = \frac{1}{z}$, $\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x}$

So, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$

$= 0$

Ex:-(3) If $x^x y^y z^z = c$

show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1}$

Soln. $x^x y^y z^z = c$

$x\log x + y\log y + z\log z = \log c$

$z \log z = \log c - y\log y - x\log x$

Diff. p. w.r.t y;

$\frac{\partial z}{\partial y} \cdot \log z + \cancel{z} \cdot \frac{1}{\cancel{z}}\frac{\partial z}{\partial y} = -\cancel{y} \cdot \frac{1}{\cancel{y}} - \log y$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)}$ —(i)

From symmetry; $\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)}$

$\frac{\partial^2 z}{\partial x \partial y} = +(1+\log y) \cdot \frac{1}{(1+\log z)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial x}$

$= -\frac{1}{z}\frac{(1+\log y)(1+\log x)}{(1+\log z)^3}$

when $x = y = z$;

$\frac{\partial^2 y}{\partial x \partial y} = -\frac{1}{x} \cdot \frac{(1+\log x)(1+\log x)}{(1+\log x)^3} = -(x \log_e x)^{-1}$

(4) $u = f(r)$

$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Soln. $u = f(r)$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\frac{\partial r}{\partial x}\right)^2 + f'(r) \frac{\partial^2 r}{\partial x^2} = f'(r)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = f''(r) \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} f'(r)$$

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r) \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1 \times \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \qquad \frac{\partial r}{\partial y} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{x^2+y^2} - \frac{1 \times \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{(x^2+y^2)} , \quad \frac{\partial^2 r}{\partial y^2} = \frac{\sqrt{x^2+y^2} - \frac{y}{\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$= \frac{x^2+y^2-x}{(x^2+y^2)^{3/2}} \qquad = \frac{x^2+y^2-y}{(x^2+y^2)^{3/2}}$$

* A function $U = f(x, y)$ is said to be homogenous of degree n in x and y if u can be expressed as

$$U = x^n \phi\left(\frac{y}{x}\right)$$

1) $U = \dfrac{x^3+y^3}{x^2+y^2}$

2) $U = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

3) $U = \dfrac{xy^2}{x^3+y^3}$

4) $\sin^{-1} \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$

1) $U = \dfrac{x^3\left(1+\frac{y^3}{x^3}\right)}{x^2\left(1+\frac{y^2}{x^2}\right)} = x\dfrac{\left\{1+\left(\frac{y}{x}\right)^3\right\}}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}$ → Homogeneous

degree = 1

2) $U = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$ → Homogeneous

→ degree = 1/2

3) $U = \dfrac{xy^2}{x^3+y^3} = x^0\dfrac{\left(\frac{y^2}{x^2}\right)}{1+\left(\frac{y}{x}\right)^3}$ → Homogeneous

→ degree = 0

4) $U = \sin^{-1} \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$ → not homogeneous

$\sin U = \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$ → homogene

* Euler's theorem :-

If u is a homogeneous function of degree n in (x & y) then;

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Proof:- $u=x^n\phi\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial y}=x^n\phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)+\cancel{x^n\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x}\right)}$$
$$+\cancel{\frac{y}{x}x^n\phi'\left(\frac{y}{x}\right)}$$

$$=nx^n\phi\left(\frac{y}{x}\right)=nu$$

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Ex: 1 → If $u=\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$ find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{3}{2}u$

Ex: 2 :- If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, find " " "

Ex: 3: If $u=\sin^{-1}\left(\frac{xy^2}{x^3+y^3}\right)$, " " " "

Ex: 4 :- If $u=\frac{y}{x}+\frac{z}{x}$, " " $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

$\textcircled{1} \underline{n v} + n' w$

~~Soln 2:- $x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin$~~

soln:2:- $\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$\Rightarrow x\dfrac{\partial}{\partial x}(\sin u) + y\dfrac{\partial}{\partial y}(\sin u) = \dfrac{1}{2}\sin u$

$\Rightarrow x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin u$

$\Rightarrow \boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u}$

Q.1. If $u = \sin^{-1}\left(\dfrac{x+2y+3z}{x^8+y^8+z^8}\right)$

Find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

Q.2. If $u = \dfrac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$; find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

Soln. Q.1. $\sin u = \dfrac{x+2y+3z}{x^8+y^8+z^8}$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = -7\tan u$

Q.2. soln. $u = v + w$

$v = \dfrac{x^3y^3z^3}{x^3+y^3+z^3}$

Degree $= 6$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 6v$

$w = \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

$e^w = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$

degree $= 0$.

Q. If u is a homogeneous equation of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n(n-1)u$$

Soln.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \text{—— (i)}$$

Diff. (i) w.r.t. x;

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial y\partial x}=n\frac{\partial u}{\partial x} \quad \text{—— (ii)}$$

Diff. (i) w.r.t. y;

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+x\frac{\partial^2 u}{\partial y\partial x}=n\frac{\partial u}{\partial y} \quad \text{—— (iii)}$$

Multiplying x to (ii) & y to (iii) and adding; we get;

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2xy\frac{\partial^2 u}{\partial x\partial y}=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n(n-1)u$$

Q If $u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$

P.T. $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

$\sin u$ is homogeneous with degree $1/2$, By euler's th^m;

$\therefore x \frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Diff w.r.t x;

$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{2}\sec^2 u \frac{\partial u}{\partial x}$ —(i)

$\Rightarrow$ diff w.r.t. y;

$\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \frac{\partial u}{\partial y}$ — (ii)

Multiplying x to (i) & y to (ii) and adding; we get;

$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= \frac{1}{2}\frac{\partial u}{\partial x}\left(x\sec^2 u + y \sec^2 u\right)$

$= \frac{1}{2}\sec^2 u\,(x+y)\frac{\partial u}{\partial x}$

$= \frac{1}{2}\sec^2 u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$

$= \frac{1}{2}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

$= \frac{1}{2}\frac{\sin u}{\cos^3 u}$

$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y}$

$= \frac{1}{2}\frac{\sin u}{\cos^3 u}$

Total derivative :-

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$, then total derivative is defined as

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

when $t = x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

Suppose $f(x, y) = c$

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\partial f / \partial y}$$

(Q).1. If $u = \cos\left(\frac{x}{y}\right)$ where $x = e^t$, $y = t^2$, find $\frac{du}{dt}$ as a function of t.

Q.2. If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$. Find $\frac{du}{dx}$

Q.3. $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \cancel{-\sin y \cdot \frac{dy}{dx} \cdot e^t + (-\sin y) \cdot 2t} \; - \sin\left(\frac{x}{y}\right) \cdot \frac{1}{y} \cdot e^t + \left(+\sin\left(\frac{x}{y}\right)\right) \cdot \frac{+x}{y^2} \cdot 2t$$

$$= \cancel{-\sin y \left(e^t \frac{dy}{dx} + 2t\right)} = -\sin\left(\frac{x}{y}\right) \cdot \frac{1}{y} \cdot e^t + \sin\left(\frac{x}{y}\right) \cdot \frac{x}{y^2} \cdot 2t$$

$$=\frac{1}{y}\sin\left(\frac{x}{y}\right)\times\left\{\cancel{e}\,e^{t}+\frac{x}{y}\times 2t\right\}$$

Q9) Let $f = x^3+y^3+3xy-1=0$

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\times\frac{dy}{dx}$$

$$= \log xy + \cancel{x}\times\frac{1}{\cancel{xy}}\times\cancel{y} + x\times\frac{1}{\cancel{x}y}\times\cancel{x}\times-\left(\frac{x^2+y}{y^2+x}\right)$$

Change of variable:-

If $u = f(x, y)$ where $x = \phi(r, s)$, $y = \psi(r, s)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Q.1. If $z = f(x, y)$ where $x = e^{u} + e^{-v}$, $y = e^{-u} + e^{v}$

prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Q.2. If $u = f(x-y, y-z, z-x)$ find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Solⁿ. $x - y = r$, $y - z = s$, $z - x = t$

$$u = f(r, s, t)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \overset{1}{\frac{\partial r}{\partial x}} + \frac{\partial u}{\partial s} \cdot \overset{0}{\frac{\partial s}{\partial x}} + \frac{\partial u}{\partial t} \cdot \overset{(-1)}{\frac{\partial t}{\partial x}}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{—(i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{—(ii)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \quad \text{—(iii)}$$

Adding (i), (ii) & (iii);
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

(1) $z = f(x, y)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^{u}) + \frac{\partial z}{\partial y} \cdot (e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= -\frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} \cdot e^{v}$$

$$\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

- Jacobian

If u and v are functions of x and y then Jacobian of u and v w.r.t. x & y is defined and denoted as:-

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)}$$

$$J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

* Properties :-

(1) $J \cdot J' = 1$

(2) $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$. If (x,y) is a function of (r,s)

Q1. If $x = r\cos\theta$, $y = r\sin\theta$. Find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$

(ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

Q.2 If $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, find $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$

Q.3 If $x = r\cos\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

2. soln.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & 1 \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho(\cos^2\phi + \sin^2\phi)$$

$$= \rho$$

3. soln.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= r^2\cos\theta$$

* **Result:** If $\frac{\partial(u,v)}{\partial(x,y)} = 0$, then u and v are dependent i.e, there exist a functional relation between u and v.

Q. If $u = \tan^{-1}x - \tan^{-1}y$

& $V = \frac{x-y}{1+xy}$

examine whether u and v are dependent and if so, find the relation betn u and v.

Soln.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{-1}{1+y^2} \\ \frac{1(1+xy)-(y)(x-y)}{(1+xy)^2} & \frac{-1(1+xy)-x(x-y)}{(1+xy)^2} \end{vmatrix}$$

$1-xy-xy-y^2$

$-1-xy-x^2+xy$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{-1}{1+y^2} \\ \frac{1+y^2}{(1+xy)^2} & \frac{-(1+x^2)}{(1+xy)^2} \end{vmatrix} = 0$$

Hence, there exist a functional relation between u and v i.e, they are dependent and relation is

$U = \tan^{-1} V$

* Laplace Transformations

Let $f(t)$ be a f^n of t, where $t>0$

Then Lap. transformation of $f(t)$ is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

provided the integral exists and s is a parameter real or complex number.

$f(t) = L^{-1}$.

* $f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}$, where $s>0$
t^n	$\frac{n!}{s^{n+1}}$, $n=0,1,2,3\ldots$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{a^2+s^2}$
$\cos at$	$\frac{s}{a^2+s^2}$
$\sinh at$	$\frac{a}{a^2-s^2}$
$\cosh at$	$\frac{s}{a^2-s^2}$

(i) $$L(1) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$$

(ii) $$L(t^n) = \int_0^\infty e^{-st} t^n\,dt$$

Put $st = z$

$s\,dt = dz$

$dt = \frac{dz}{s}$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-z} z^n\,dz$$

$$= \frac{1}{s^{n+1}}\Gamma(n+1) = \frac{n!}{s^{n+1}}, \quad n=0,1,2,3\ldots$$

(iii) $L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\, dt$

$$= \int_0^\infty e^{t(a-s)}\, dt$$

$$= \frac{+1}{s-a}$$

(iv) $L(\sin at) = \int_0^\infty e^{-st} \cdot \sin at\, dt$

* Properties of Laplace transformations

1. Linear Property — If ~~~~ f, g, h are f^n of t and a, b, c are constants.

Then, $L(af(t) + bg(t) - ch(t)) = aL(f(t)) + bL(g(t)) - cL(h(t))$

2. First shifting property.

If $L(f(t)) = L(s)$

$$L(e^{at}f(t)) = f(s-a)$$

Proof:- $$L(e^{at}f(t)) = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= f(s-a)$$

using first shifting property, we have the following results,

(I) $L(e^{at}) = \dfrac{1}{s-a}$

(II) $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}$

(III) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

(IV) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

(V) $L(e^{at}\sinh bt) = \dfrac{b}{b^2-(s-a)^2}$

* Inverse Laplace ~~first~~ Transformation formulae

(I) $L^{-1}\left(\frac{1}{s}\right) = 1$

(II) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(III) $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

(IV) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

(V) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

(VI) $L^{-1}\left(\frac{1}{a^2-s^2}\right) = \frac{\sinh at}{a}$

(VII) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{e^{at} \sin bt}{b}$

(VIII) $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at} \cos bt$

$\cos^2 x$

(Q) Find Laplace transform of following:-

(i) $\sin 5t \cos 2t$

(ii) $\sin^2 t$

(iii) $\cos^2 t$

(iv) $\sin^3 t$

(v) $e^{-t} \sin 2t$

(i) $L\left(\frac{1}{2} \times 2 \sin 5t \cos 2t\right)$

$= L\left(\frac{1}{2} \cdot (\sin 7t + \sin 3t)\right)$

$= \frac{1}{2}\left(\frac{7}{5^2+7^2} + \frac{3}{s^2+3^2}\right)$

(ii) $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$

$= \frac{1}{2} L(1-\cos 2t)$

$= \frac{1}{2} L(1) - \frac{1}{2} L(\cos 2t)$

$= \frac{s}{2} - \frac{1}{2} \times \frac{s}{s^2+4}$

(III) $L(\cos^2 t) = L\left(\frac{\cos 2t + 1}{2}\right)$

$= \frac{1}{2} L(\cos 2t + 1)$

$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+4}\right\}$

(IV) $L(\sin^3 t) = L\left(\frac{3\sin t - \sin 3t}{4}\right)$

$= \frac{3}{4}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{4} \cdot \frac{3}{s^2+9}$

8 $f(t) = \pi$ if $0 < t < \pi$
$= 1$ if $t > \pi$

$$L(f(t)) = \int_0^{\pi} e^{-st}\pi\, dt + \int_{\pi}^{\infty} e^{-st} dt$$

$$= \frac{-\pi}{s}\left[e^{-st}\right]_0^{\pi} + \cdot e^{-s\pi}$$

$=$

9 $e^{2t} \times \cos^2 t$

$$= \frac{e^{2t}}{2}\left(\cos 2t + 1\right)$$

$$= \frac{1}{2}\left(e^{2t}\cos 2t + e^{2t}\right)$$

$$L(e^{2t}\cos^2 t) = \frac{1}{2} \times \frac{1}{(s-2)} + \frac{(s-2)}{2\{(s-2)^2+4\}}$$

3. Change of scale property :- If $L(f(t)) = L(s)$

then $L(f(at)) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Put $at = z$ $\qquad \int_0^\infty e^{-st}\, at$

$$L(f(at)) = \int_0^\infty e^{-st} f(at)\, dt$$

$at = z$

$dt = \frac{dz}{a}$

$$= \int_0^\infty e^{-sz/a} f(z) \frac{dz}{a}$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

* Laplace transform of periodic function :-

If $f(t)$ is a periodic function with period T, i.e., $f(t+T) = f(t)$

then $L(f(t)) = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

* Laplace transform of a derivative of a function :

If $f'(t)$ is continuous function and $L(f(t)) = \bar{f}(s)$ then

$L(f'(t))$ becomes $s\bar{f}(s) - f(0)$

$L(f''(t))$ becomes $s^2\bar{f}(s) - sf(0) - f'(0)$

$$L(f^n(t)) = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

Proof. $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\bar{f}(s) - f(0)$$

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

* If $L(f(t)) = \bar{f}(s)$

then (i) $L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(\bar{f}(s))$

(ii) $L\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)\,ds$

Eg:-1: 1. $t\cos at$

$$L(\cos at) = \frac{s}{a^2+s^2}$$

$$L(t\cos at) = (-1)\frac{d}{ds}\left(\frac{s}{a^2+s^2}\right)$$

$$= (-1)\cdot \frac{(s^2+a^2)-2s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

2. $t^2 \sin at$

$$L(\sin at) = \frac{a}{a^2+s^2}$$

$$L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{a^2+s^2}\right)$$

3. $\dfrac{(1-e^t)}{t}$

$$L(1-e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \log\left[\frac{s}{s-1}\right]_s^\infty = \log\left(\frac{s-1}{s}\right)$$

Find inverse laplace transformation :-

(1). $L^{-1}\left(\dfrac{s^2-3s+5}{s^3}\right)$ (2). $L^{-1}\left(\dfrac{s+2}{s^2-4s+13}\right)$ (3) $L^{-1}\left(\dfrac{4s+5}{(s-1)^2(s+2)}\right)$

(4). $L^{-1}\left(\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right)$

(1) $L^{-1}\left(\dfrac{1}{s} - \dfrac{3}{s^2} + \dfrac{5}{s^3}\right) = 1 - 3t + \dfrac{5t^2}{2!}$

(2) $L^{-1}\left(\dfrac{s+2}{(s-2)^2+3^2}\right) = L^{-1}\left(\dfrac{s-2+4}{(s-2)^2+3^2}\right)$

$$= L^{-1}\left(\frac{s-2}{(s-2)^2+(3)^2} + \frac{4}{(s-2)^2+(3)^2}\right)$$

$$= e^{2t} \times \cos 3t + \frac{4}{3} \times e^{2t} \sin 3t$$

(3) $L^{-1}\left(\dfrac{4s+5}{(s-1)^2(s+2)}\right)$ $\dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+2}$

$$A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$$A = \frac{1}{3},\ B = 3,\ C = \frac{-1}{3}$$

$L^{-1} = \frac{1}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$

4) $L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)$

$\frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$

$(A+B)s^2 + (2A-B+C)s + 5A-C$

$A+B=0, \quad 2A-B+C = 5, \quad 5A-C=3$

$A=1, \quad B=-1, \quad C=2$

$L^{-1}\left(\frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}\right)$

$= e^{t} - \frac{s-2}{(s+1)^2+2^2}$

$= e^{t} - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2} - \frac{3}{(s-1)^2+2^2}\right)$

$= e^{t} - e$

* $L\left(\int_0^t f(t)\,dt\right) = \frac{1}{s}\bar{f}(s)$; if $L(f(t)) = \bar{f}(s)$

* $L(t\,f(t)) = -\frac{d}{ds}(\bar{f}(s))$

* $L\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)\,ds$

* Solution of differential equation by Laplace transform

1) working rule-

a) Take Laplace transfoom in both side of given eqn
b) Keep $\bar{y}$ terms in LHS and all other terms in RHS
c) Express $\bar{y}$ in terms of x in RHS
d) Take inverse Laplace transform to get value of y.

Q. Solve the d. eqn.

$$y''' + 2y'' - y' - 2y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 6$$

$$y' = \frac{dy}{dt}$$

Soln Taking L.T on both sides;

$$s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0) + 2\left(s^2\bar{y} - s y(0) - y'(0)\right)$$
$$- \left(s\bar{y} - y(0)\right) - 2\bar{y} = 0$$

$$\Rightarrow s^3\bar{y} - 6 + 2s^2\bar{y} - s\bar{y} - 2\bar{y} = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\Rightarrow \bar{y} = \frac{6}{s^3 + 2s^2 - s - 2} = \frac{6}{(s^2-1)(s+2)} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \left(\frac{A}{(s-1)} + \frac{B}{s+1} + \frac{C}{s+2}\right)$$

$$A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

$$A = 1, \quad B = -3, \quad C = 2$$

$$L^{-1}(\bar{y}) = y = e^{t} - 3e^{-t} + 2e^{-2t}$$

⇒ Convolution theorem of Laplace theorem

If $L^{-1}(\bar{f}(s)) = f(t)$

& $L^{-1}(\bar{g}(s)) = g(t)$

then $L^{-1}(\bar{f}(s) \cdot \bar{g}(s))$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$= F * G$

where $F * G$ is called convolution of f & g.

Q. using convolution theorem, find (i) $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

(ii) $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$

(i) we know $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = f(t)$

$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a} = g(t)$

using convolution theorem

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{\cos au \cdot \sin a(t-u)}{a}\, du$$

$$= \frac{1}{2a}\int_0^t \{\sin at - \sin(2au-at)\}\, du$$

$$= \frac{1}{2a} \sin at [u]_0^t + \left[\frac{\cos(2au - at)}{2a}\right]_0^t$$

$$= \frac{1}{2a} t \sin at$$

* Proof of convolution theorem :-

Let $\phi(t) = \int_0^t f(u) g(t-u)\, du$

$$L(\phi(t)) = \int_0^\infty e^{-st} \left(\int_0^t f(u)\, g(t-u)\, du \right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

Changing the order of integration $\qquad t - u = v$

$$L(\phi(t)) = \int_0^\infty \int_0^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-su} f(u) \cdot \left(\int_0^\infty e^{-sv}\, g(v)\, dv \right) du$$

$$= \int_0^\infty e^{-su} f(u)\, \bar{g}(s)\, du = \bar{f}(s)\, \bar{g}(s)$$