Vector: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} = (a_1, a_2, a_3)$$

Vectors in $\mathbb{R}^n$:

Let $\mathbb{R}$ be a set of all real numbers and 'n' is a fixed positive integer (we assume $n \geq 2$) then, $\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, 2, 3, \ldots n\}$ is the set of all ordered n-tuples of real numbers.

If, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then,

x is called a point or a row vector or simply a vector in $\mathbb{R}^n$ - space, which is also denoted by, $x = (x_1, x_2, x_3 \ldots x_n)$ or $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$

and in latter case, it is called a column vector.

* Each component in a vector $x = (x_1, x_2, \ldots x_n)$ is a real number and is called a scalar.
* Two vectors $x = (x_1 x_2 \ldots x_n)$ and $y = (y_1 y_2 \ldots y_n)$ in $\mathbb{R}^n$ are said to be equal and is written as $x = y$, if $x_i = y_i \ \forall\ i = 1, 2, \ldots, n$.
* The vector $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$ is called the zero-vector and is denoted by 0 (zero).
* The sum of the two vectors $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is denoted and defined by $x + y = (x_1 + y_1, x_2 + y_2 + \ldots + x_n + y_n)$ which is also a unique vector in $\mathbb{R}^n$ i.e., $\mathbb{R}^n$ is closed under vector addition.

* The product of a scalar $K \in \mathbb{R}$ an a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is denoted and defined as $Kx = (Kx_1, Kx_2, \ldots, Kx_n)$ which is also a unique vector in $\mathbb{R}^n$ i.e, $\mathbb{R}^n$ is closed under scalar multiplication.

* for any vetors $x, y, \in \mathbb{R}^n$ and for any scalars $\alpha, \beta \in \mathbb{R}$, it is easy to see that:

(i) $(x+y)+z = x+(y+z)$

(ii) $x+0 = x = 0+x$

(iii) $-x+x = 0 = x+(-x)$

where, $-x = (-x_1, -x_2, \ldots, -x_n)$

(iv) $x+y = y+x$

(v) $\alpha(x+y) = \alpha x + \alpha y$

(vi) $(\alpha+\beta)x = \alpha x + \beta x$

(vii) $(\alpha\beta)x = \alpha(\beta x)$

(viii) $1x = x$

The set $\mathbb{R}^n$ which is closed under vector addition an scalar multiplication and also all the above conditions from (i) to (viii) hold $\forall\, x, y, z \in \mathbb{R}^n$ and $\forall\, \alpha, \beta \in \mathbb{R}$, is called a vector-space or linear-space of dimension 'n' and $\mathbb{R}^n$ is also called Euclidean n-space.

In particular, $R^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ is two-dimensional Euclidean space or plane.

$R^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ is three-dimensional Euclidean space or plane.

NOTE:

$-y = (-1)y = (-y_1, -y_2, \ldots, -y_n)$

$x - y = x + (-y) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$

$0x = (0x_1, 0x_2, \ldots, 0x_n) = (0, 0, \ldots, 0) = 0 \rightarrow$ zero-vector in $\mathbb{R}^n$

* Vector (Linear) Subspace or Simply Subspace:

A non-empty subset W of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if W itself is a vector space under the same vector addition and scalar multiplication as in $\mathbb{R}^n$.

Theorem: A necessary and sufficient condition for a non-empty subset W of $\mathbb{R}^n$ to be a subspace of $\mathbb{R}^n$ is that $\alpha x + \beta y \in W$ $\forall x, y \in W$ and $\forall \alpha, \beta \in \mathbb{R}$

For e.g.: (i) $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

(ii) $W_2 = \{(x, 0, z) \mid x, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

In particular, $\{0\}$ and $\mathbb{R}^n$ are always subspace of $\mathbb{R}^n$

Linear combination, Linear dependence and Linear independence of vectors in $\mathbb{R}^n$:

* If $v_1, v_2, \dots, v_m$ are finite numbers of vectors in $\mathbb{R}^n$ and $\alpha_1, \alpha_2, \dots, \alpha_m$ are scalars in $\mathbb{R}$ then a vector of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$ is called a linear combination of $v_1, v_2, \dots, v_m$.

* A finite no. of vectors $v_1, v_2, \dots, v_m$ in $\mathbb{R}^n$ are said to be <u>linearly dependent</u> (L.D.) if $\exists$ scalars $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$, not all zeros i.e., at least one $\alpha_i \neq 0$, such that,

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0, \text{ zero vector in } \mathbb{R}^n$$

* $v_1, v_2, \dots, v_m$ are said to be linearly independent (L.I.) if they are not L.D., i.e., if, $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$, where $\alpha_i \in \mathbb{R}$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_m = 0.$$

For e.g.:

(i) In $\mathbb{R}^3$, $e_1 = (1,0,0)$

$e_2 = (0,1,0)$

$e_3 = (0,0,1)$

are L.I., because,

$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$, where $\alpha_i \in \mathbb{R}$

$\Rightarrow \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (0,0,0)$

$\Rightarrow (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0,0,\alpha_3) = (0,0,0)$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$

$\therefore \alpha_1 = 0;\ \alpha_2 = 0;\ \alpha_3 = 0.$

(ii) In $\mathbb{R}^3$, $v_1 = (1,0,-2)$

$v_2 = (2,1,0)$

$v_3 = (-1,-1,-2)$

are L.D., because,

$-v_1 + v_2 + v_3 = (-1,0,2) + (2,1,0) + (-1,-1,-2)$

Or, $(-1)v_1 + 1(v_2) + 1(v_3) = (-1+2-1,\ 0+1-1,\ 2+0-2)$

$= (0,0,0)$

$= 0$

Here, $\alpha_1 = -1$

$\alpha_2 = 1$

$\alpha_3 = 1$

i.e, $\alpha_i \in \mathbb{R}^*$

* L.D. and L.I. Sol:

If $S = \{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}^n$, then the set S is called L.D. or L.I. ~~respectively~~ accordingly as $v_1, v_2, \ldots, v_m$ are L.D. or L.I. respectively

E.g.: (i) $S_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a L.I. set in $\mathbb{R}^3$

(ii) $S_2 = \{(1,0,-2), (2,1,0), (-1,-1,-2)\}$ is a L.D. set in $\mathbb{R}^3$.

* Linear Span: If $S = \{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}^n$, then the set of all linear combination of vectors in S is denoted by $L(S)$ and is called linear span of S.

i.e., $L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m \mid \alpha_i \in \mathbb{R}\}$

If $L(S) = \mathbb{R}^n$, then S is called a generating system of $\mathbb{R}^n$ i.e, if every vector v in $\mathbb{R}^n$ can be ~~expanded~~ expressed as a linear combination, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$ then, S will be called a generating system of $\mathbb{R}^n$.

In this case, we also say that $\mathbb{R}^n$ is generated by S.

E.g.: (i) $S = \{e_1 = (1,0,0,\ldots,0)$

$e_2 = (0,1,0,\ldots,0), \ldots, e_n = (0,0,\ldots,0,1)\}$

is a generating system of $\mathbb{R}^n$.

(ii) $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a g.s. of $\mathbb{R}^3$

(iii) $S = \{e_1 = (1,0), e_2 = (0,1)$ is a g.s. of $\mathbb{R}^2\}$.

Sgu

Justification:

Let $v = (a,b,c) \in \mathbb{R}^3$

$$= (a,0,0) + (0,b,0) + (0,0,c)$$

$$= a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$= ae_1 + be_2 + ce_3$$

= all linear combination of vectors in S.
(A.L.C)

* Basis of $\mathbb{R}^n$:

A subset S of $\mathbb{R}^n$ is called basis of $\mathbb{R}^n$ if S is L.I. and $L(S) = \mathbb{R}^n$

i.e., a linearly independent g.s. of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Alternatively, a basis of $\mathbb{R}^n$ can be defined as a maximal L.I. subset of $\mathbb{R}^n$ i.e., a subset of $\mathbb{R}^n$ having maximum no. of L.I. vectors and this no. is called dimension of $\mathbb{R}^n$

This set $S = \{e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_n = (0,0,\dots,0,1)\}$ is a L.I. subset of $\mathbb{R}^n$ which is also a g.s. of $\mathbb{R}^n$. So that S is a basis of $\mathbb{R}^n$, which is called standard or canonical basis of $\mathbb{R}^n$. Also, max. no. of L.I. vectors in $\mathbb{R}^n$ is n so that dimension of $\mathbb{R}^n$ is n.

* Row Space, Column Space and Rank of a matrix:

Let A be a $m \times n$ matrix ~~over~~ every real numbers and let $R_1, R_2, \ldots, R_m$ be the m rows of A and $C_1, C_2, \ldots, C_n$ be the n columns of A. Each row of A consists of n and can be viewed as a vector in $\mathbb{R}^n$, each colum of A consists of m elements and can be viewed as vector in $\mathbb{R}^m$.

The subspace of $\mathbb{R}^n$ is spaned (generated) by the rows $R_1, R_2, \ldots, R_m$ ~~of~~ A is called Row space, and the subspace of $\mathbb{R}^m$ generated by the columns C_1, C_2, C_n of A is called column space of A.

The dimension of the row space and the column space of the matrix A are same, i.e, max. no. of L.I. rows and L.I. columns of A are same and this number is called the rank of the matrix A, and we denote it by rank (A) or ρ (A).

Sometimes, max. no. of L.I. rows (column) of A is called row-rank (column-rank) of A, because.

$$\text{row-rank}(A) = \text{column-rank}(A).$$

* Null Space and nullity of a matrix

Let A be $m\times n$ matrix. The null space of A is the set of all column vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that $Ax = 0$, and is denoted by $N(A)$, which is the set of all solutions of Linear Homogenous equations $Ax = 0$. $N(A)$ is subspace of $\mathbb{R}^n$

If $\text{rank}(A) = r$, then the system of equations $Ax = 0$ has $n-r$ ~~aligned~~ L.I. solutions and this no. $n-r$ is called the nullity of A and is denoted by nullity (A) or $\nu(A)$.

* Alternative defn. of rank of a matrix:

Rank of a non-zero matrix is defined as the order of the largest non-singular square sub-matrix of A.

The rank of a zero matrix is defined to be zero.

E.g: (i) if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then, $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

is a sq. submatrix of A and

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = 3 \neq 0$$

$\therefore \text{Rank}(A) = 2$.

(ii) if $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$

Then, there are ${}^4C_2 = 6$ nos of 2×2 submatrices of A and all are singular, so rank(A)=1

NOTE: Rank of a matrix in a non-negative integer. Rank of a non-zero matrix is a +ve integer

Rank of a non-singular sq. matrix of order n is n

Rank of a unit matrix of order n is n.

Eg: Find the rank of $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{pmatrix}$

Sol^n: There are ${}^4C_3 = 4$ nos. of 3×3 submatrices of A

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ -1 & 0 & -2 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \\ -1 & 0 & 7 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & -1 & 3 \\ 3 & 0 & -1 \\ -1 & -2 & 7 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 0 & -2 & 7 \end{vmatrix} =$$

$\therefore$ all 3×3 submatrices are singular

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$$

$\therefore$ rank$(A) = 2$

Elementary Operations or Elementary Transformations:

The following operations when applied to a matrix are called elementary row-operations.

<i> Interchange of any two rows: $R_i \longleftrightarrow R_j$

<ii> Multiplication of each element of a row by a non-zero scalar k : $R_i \longleftrightarrow kR_j$

<iii> Addition to the elements of a row, a constant multiple of corresponding elements of another row: $R_i \longrightarrow R_i + kR_j$

Similarly, elementary column operations are defined, which are denoted by C_i, C_j.

(i) $C_i \longleftrightarrow C_j$ (ii) $C_i \longrightarrow kC_j$ (iii) $C_i \rightarrow C_i + kC_j$

× ——— × ——— × ——— × ——— ×

Echelon form of a matrix or Echelon matrix:

A matrix is said to be in Echelon form if the number of zeros preceeding the first non-zero entry of any row increases row by row until only zero rows remain (if zero-row exists).

The first non-zero entries of the rows of an Echelon matrix are called distinguished elements And if each of them is unity and also it is the only non-zero element in the corresponding column, then the matrix is said to be in Row reduced Echelon form.

For Eg.: (i) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is an Echelon matrix.

(ii) $\begin{pmatrix} 0 & 2 & -3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is an Echelon matrix

(iii) $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a row-reduced Echelon matrix

(iv) $\begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ is not an Echelon Matrix.

THEOREM: Any set of vectors containing the zero vector is L.D.

PROOF: Let, $S = \{0, v_1, v_2, \dots, v_m\}$ is a set of vectors in $\mathbb{R}^n$ containing the zero vector 0.

Then, $1.0 + 0.v_1 + 0.v_2 + \dots + 0.v_m = 0$

where $1 \neq 0$

S is a L.D.

THEOREM: Non-zero rows of an Echelon matrix are L.I..

PROOF: Let $R_1, R_2, \dots, R_p$ are non-zero rows of a ~~matrix~~ $m \times n$ echelon matrix (starting from 1st to pth row).

If possible, let $R_1, R_2, \dots, R_p$ are L.D.

where, $1 \le p \le m$

Then,

$\exists$ scalar $\alpha_1, \alpha_2, \dots, \alpha_p$ not all zero such that

$$\alpha_1 R_1 + \alpha_2 R_2 + \dots + \alpha_p R_p = 0 \qquad \text{-(i)}$$

Let, k be the least suffix such that $\alpha_k \neq 0$.

Then (i) becomes,

$$\alpha_k R_k + \alpha_{k+1} R_{k+1} + \dots + \alpha_p R_p = 0.$$

$$\Rightarrow -\alpha_k R_k = \alpha_{k+1} R_{k+1} + \dots + \alpha_p R_p$$

$$\Rightarrow R_k = (-\alpha_k^{-1} \alpha_{k+1}) R_{k+1} + (-\alpha_k^{-1} \alpha_{k+2}) R_{k+2} + \dots + (-\alpha_k^{-1} \alpha_p) R_p \qquad \text{-(ii)}$$

Let, jth component of R_k is the 1st non-zero entry, so that jth component $R_{k+1}, R_{k+2}, \dots, R_p$ are all zeros and hence jth component of the vector in RHS of (ii) is zero whereas jth component of the vector in LHS of (ii) is non-zero and thus we meet a contradiction. Therefore, our initial assumption is wrong and hence,

$R_1, R_2, \dots, R_p$ must be L.I.

NOTE: ~~We~~ Since the non-zero rows of a matrix A in Echelon are L.I., So, they constitute a maximal L.I. set of vectors and hence, they form a basis of the row-space of the matrix A. Therefore, the row-rank or rank of A = no. of non-zero rows of A when it is in Echelon form.

THEOREM: Elementary Operations do not alter the rank of a matrix.

NOTE: To find the rank of a matrix, we reduce it to Echelon form using Elementary operations, and then the rank of the matrix = no. of non-zero rows in its Echelon form.

Q. Find the rank of the matrix:-

(i) $\begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

Sol^n:- $\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$ $\quad R_1 \longleftrightarrow R_2$

$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$ $\quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix} \quad R_2 \to R_2 - R_3$$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 9R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is in Echelon form with three non-zero row, so that rank of the matrix is 3.

NOTE: If A be the above given matrix then

$r = \text{rank}(A) = 3$

Also, n = no. of columns = 4.

$\therefore$ Nullity $(A) = n - r$
$= 4 - 3$
$= 1$

Q. Find the real values of λ for which the following matrix has rank 2.

(i) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{pmatrix}$

Soln:- (i) $\rightarrow$ $\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 2 & -4 & \lambda^2-5 \end{pmatrix}$ $R \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 5R_1$

$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-2\lambda+2 \end{pmatrix}$ $R_3 \rightarrow R_3 - 2R_2$

which is in Echelon form and rank(A) will be 2 if $\lambda^2 - 2\lambda - 3 = 0$

or, $(\lambda+1)(\lambda-3) = 0$

$\therefore \lambda = -1$ or 3

SYSTEM OF LINEAR EQUATIONS:

Consider the following m linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

where, a_{ij} and b_i are constants.

If at least one $b_i \neq 0$, then the system is said to be a non-homogeneous system of linear equations and if $b_i = 0$ $\forall i = 1, 2, 3, \ldots, m$, then the system is said to be homogeneous.

Object Science

In matrix form, the system can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or, $AX = B$

where, $A = (a_{ij})_{m \times n}$ is called coefficient matrix or associated matrix of the system,

and, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Also, the matrix,

$$[A \vdots B] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

is called the augmented matrix of the system and it is also denoted by $[AB]$ or $[A|B]$.

The system of eq^ns. $AX = B$ is said to be consistent or solvable if it has a solution i.e., if $\exists$ values of the unknown $x_1, x_2, \ldots, x_n$ which satisfy all the equations of the system simultaneously, otherwise the system is said to be inconsistent. The homogenous system $Ax = 0$ is always consistent because $X = O$ i.e., $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is clearly a solution and it is called the trivial solution or zero solution and any other solution is called a non-trivial solution.

e.g., (i) $2x + 3y = 12$
$x + y = 5$

is a consistant System because $x = 3, y = 2$ is the solution.

(ii) $2x + 3y = 12$
$4x + 6y = 5$

is an inconsistant system (no solution)

THEOREM: The necessary and sufficient condition for a non-homogenous system of linear eqns $Ax = B$ (in matrix form) to be consistent (i.e., to have a soln) is that $\text{rank}([A:B]) = \text{rank}(A)$.

Nature of Solution:

Consider a system of m linear equations in n unknowns given by $Ax = B$ in matrix form.

(i) If $\text{rank}([A:B]) \neq \text{rank}(A)$, then, the system is inconsistent, i.e., has no soln.

(ii) If $\text{rank}([A:B]) = \text{rank}(A) = n =$ no. of unknowns then, the system ~~has~~ is consistent and has a unique solution.

(iii) If $\text{rank}([A:B]) = \text{rank}(A) = r < n$ then, the system is consistent and has infinite no. of solutions, but only $(n-r+1)$ of them are alike (L.I.) provided, $V \neq 0$,

and, if $V = 0$, i.e., if the system is homogenous & consistent, then in this case, it has $n-r$ alike solutions

If $\text{rank}(A) =$ no. of vectors, L.I.
If $\text{rank}(A) <$ no. of vectors, L.D.

NOTE: If there are n linear eqns in n unknowns, given by $AX = B$ and $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$ exists, and in this case the system has the unique solⁿ.

$$X = A^{-1}B$$

Gaussian Elimination Method:

In this method, ~~the~~ a given system of linear equations is first written in matrix form $AX = B$ and then the augmented matrix $[A:B]$ is reduced to Echleon form using elementary raw operations only, and then rank of A and rank of $[A:B]$ are compared, and if they are equal, the system is consistent and we write the system again in individual eqⁿ form and then the unknowns are obtained by back substitution. This method is general and can be applied when the number of equations and the number of unknown are equal or unequal, and can also ~~A/ ... can be obta~~ be obtained by matrix inversion ~~method~~ fails.

Q. Solve by Gaussian elimination method.

$$3x + 2y + z = 3$$
$$2x + y + z = 0$$
$$6x + 2y + 4z = 6$$

Soln:- In matrix form, the system is

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

or, $AX = B$ (say)

Now, $[A:B] = \begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 6 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 3R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -1 & 1 & 6 \end{pmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{pmatrix} \quad R_3 \to R_3 - R_1$$

which is in Echelon form.

$\therefore$ rank$(A) = 2$ and rank$([A:B]) = 3$

$\therefore$ rank$(A) \neq$ rank$([A:B])$

Q. Solve:

$$x + 2y + 3z = 9$$
$$2x - y + z = 8$$
$$2x - z = 3$$

$\rightarrow$ $\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 2 & 0 & -1 & 3 \end{bmatrix}$ $\begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -4 & -7 & -15 \end{bmatrix}$ $R_3 \to 5R_3 - 4R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & 0 & -15 & -35 \end{bmatrix}$

$\text{Rank}(A) = 3$

$\text{Rank}([A:B]) = 3$

$\text{Rank}(A) = \text{Rank}([A:B]) = 3 =$ no. of unk

$x = \frac{8}{3}$

$y = \frac{-1}{3}$

$z = \frac{7}{3}$

Q. Test Consistency and Solve:

$$2x + 5y + 6z = 13$$
$$3x + y - 4z = 0$$
$$x - 2y + 8z = -10$$

Sol^n:- In matrix form, the system is:

$$\begin{pmatrix} 2 & 5 & 6 \\ 3 & 1 & -4 \\ 1 & -3 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 0 \\ -10 \end{pmatrix}$$

or, $AX = B$ (say)

$$[A:B] = \left(\begin{array}{ccc|c} 2 & 5 & 6 & 13 \\ 3 & 1 & -4 & 0 \\ 1 & -3 & -8 & -10 \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{pmatrix} \quad R_1 \longleftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 - \frac{11}{10} R_2$$

which is in echelon form.

$\therefore$ rank $[A:B]$ = rank $(A) = 2$ < no. of unknowns

$\therefore$ The system is consistent and has infinite no. of solutions.

$\therefore$ The reduced equivalent system is:

$$x - 3y - 8z = -10 \quad \text{-(i)}$$
$$10y + 30z = 30$$
$$y + 2z = 3 \quad \text{-(ii)}$$

$\therefore y = 3 - 2z$; $\therefore x = 2z - 1$

$\therefore$ Sol^n is given by,

$\therefore x = 2k - 1$

$\therefore y = 3 - 2k$

$\therefore z = k$ (where k is any real number)

NOTE: $n - r + 1 = 3 - 2 + 1 = 2$

The system has only two L.I. $sol^n s$.

Taking $k = 0$, we get two sol^n vectors as

$$x_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ which are L.I.

Let,

$$\alpha_1 x_1 + \alpha_2 x_2 = 0$$

$$\Rightarrow \alpha_1 \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\alpha_1 + \alpha_2 \\ 3\alpha_1 + \alpha_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\alpha_1 + \alpha_2 = 0, \quad 3\alpha_1 + \alpha_2 = 0, \quad \alpha_2 = 0$$

$$\therefore \alpha_1 = 0, \ \alpha_2 = 0$$

Q. Examine consistency and solve.

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

Soln:- $[A:B] = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}$ $R_2 \to R_2 - R_3$

$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{pmatrix}$ $R_4 \to R_4 - \frac{3}{5}R_3$

$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

which is in echelon form.

$\therefore$ rank([A:B]) = rank(A) = 3 = no. of unknowns,

$\therefore$ The system is consistent and has a unique soln.

$\therefore$ The reduced system is:

$$x + 2y - z = 3 \quad \text{-(i)}$$

$$-y = -4$$

$$\text{or } y = 4 \quad \text{-(ii)}$$

$$5z = 20$$

$$\text{or } z = 4 \quad \text{-(iii)}$$

$$\therefore x = 3 - 2y + z$$
$$= 3 - 8 + 4$$
$$= -1$$

$\therefore$ Sol^n is $x = -1$, $y = 4$, $z = 4$.

Q. Investigate for what values of λ and μ, the foll^n system has, (i) no sol^n. (ii) a unique sol^n. (iii) many sol^n.

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

Sol^n:- $[A:B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \end{matrix}$$

Which is in echelon form.

(i) if $\lambda = 3$ and $\mu \neq 10$, then,

$\text{rank}(A) = 2$ and $\text{rank}([A:B]) = 3$

$\therefore$ $\text{rank}(A) \neq \text{rank}([A:B])$

in this case, the system is inconsistant i.e., has no solutions.

(ii) If, $\lambda \neq 3$, then for any value of μ,

$\text{rank}(A) = \text{rank}[A:B] = 3$ = no. of unknowns

$\therefore$ in this case, the system is consistand has a unique solⁿ:

(iii) If $\lambda = 3$ and $\mu = 10$, then,

$\text{rank}(A) = \text{rank}([A:B]) = 2 <$ no. of unknowns

$\therefore$ in this case the system is consistant and has many solⁿs.

Q. Consider the system:

$$x + y + z = 1$$

$$x + 2y + z = b$$

$$5x + 7y + az = b^2$$

Determine the condition under which the system has (i) no solⁿ. (ii) only one solⁿ (iii) many solⁿs.

Solⁿ:

$$[A:B] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 5 & 7 & a & b^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & b-1 \\ 0 & 0 & a-5 & b^2-2b-3 \end{pmatrix}$$

Which is in echelon form.

(i) If $a=5$ and $b^2-2b-3 \neq 0$

$\Rightarrow (b+1)(b-3) \neq 0$

$\Rightarrow b \neq -1,\ b \neq 3.$

then, rank $(A) = 2$ & rank$([A:B]) = 3$

$\therefore$ In this case, the system is inconsistant and has no solⁿ.

(ii) If $a \neq 5$, then for any value of b,

rank (A) = rank$([A:B]) = 3$ = no. of unknowns

$\therefore$ in this case, the system is consistant and has only one soln.

(iii) If $a=5$ and $b=-1$ or $b=3$, then

rank (A) = rank$([A:B]) = 2 <$ no. of unknowns

$\therefore$ it is consistant and has many solⁿ.

Q. Consider,

$$x + 2y + z = 3$$

$$ay + 5z = 10$$

$$2x + 7y + az = b$$

(i) For what value of a, the system has unique solⁿ?

(ii) for what values of a and b, the system has more than one solⁿ?

Gauss - Jordan Method of Solving the System of Linear Equations:

In this method of solving a system of n-linear eq^{n}s in n-unknowns. The system is first written in matrix form $AX = B$ and then using elementary row operations only the augmented matrix $[A:B]$ is converted to Echelon form and if consistent, then A is converted to a diagonal matrix / unit matrix and then the unknowns are readily obtained.

Q. Solve by Gauss Jordan Method:

$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

Sol^{n}: In matrix form, the system is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

or, $AX = B$ (say)

Now, $$[A:B] = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{pmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to \frac{1}{12} R_3 \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \end{array}$$

$\therefore$ A is converted to unit matrix and hence soln is $x = 1,\ y = 3,\ z = 5$.

Q. Solve by Gauss-Jordan method.

$$3x + 2y + z = 3$$
$$2x + y + z = 0$$
$$6x + 2y + 4z = 6$$

Soln.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

or, $AX = B$ (say)

$$[A:B] = \begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 3 \\ 6 & 2 & 4 & 6 \end{pmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 6 \\ 0 & -1 & 1 & 6 \end{pmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

It is inconsistent.

Q. Solve by Gauss - Jordan Method.

$$x - 2y + z = 3, \quad -x + 3y = -4, \quad 2x - 5y + 5z = 17$$

Solⁿ:

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 2 & -5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 17 \end{pmatrix}$$

$$A X = B$$

$$[A:B] = \begin{pmatrix} 1 & -2 & 1 & 3 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & 11 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & 10 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5/2 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to \frac{1}{4} R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -13/2 \\ 0 & 1 & 0 & -7/2 \\ 0 & 0 & 1 & 5/2 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$\therefore$ A is converted into unit matrix and hence solⁿ is $x = -13/2$, $y = -7/2$, $z = 5/2$

Matrix Inversion by Gauss-Jordan Method:

Let A be an invertible matrix i.e., a non-singular matrix, then A^{-1} exists, we can write

Now applying elementary row operations only, we convert A of LHS to I and accordingly I of RHS will be converted to P (say) so that,

$$\therefore I = PA$$

$$\therefore A^{-1} = P$$

Elementary Matrix: A matrix obtained from a unit matrix by an application of a single elementary operation is called an Elementary method.

Theorem: Every elementary row (column) operation of a matrix can be obtained by pre (post) multiplication of the matrix by the corresponding Elementary-matrix.

E.g. (i) Applying $R_2 \to R_2 - 3R_1$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

we get an E-matrix, $E_R = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now,

consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix}$ then applying

$R_2 \to R_2 - 3R_1$ to A, we get,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -9 \\ 1 & 4 & 5 \end{pmatrix}$$

Also, $E_R A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 \\ -3+3 & -6+6 & -9 \\ 1 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -9 \\ 1 & 4 & 5 \end{pmatrix}$$

~~Theorem~~

Eg.: (ii) Applying $C_2 \to C_2 - 2C_1$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

we get an E-matrix $E_c = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now, consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix}$ then applying,

$C_2 \to C_2 - 2C_1$ to A, we get $\begin{pmatrix} 1 & 0 & 3 \\ 3 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix}$

Also, $A E_c = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 3 \\ 3 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix}$$

Q. Find inverse of A by Gauss - Jordan Method:

where, ① $A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix}$ ② $A = \begin{pmatrix} 8 & 7 & 9 \\ 4 & 8 & 8 \\ 7 & 5 & 6 \end{pmatrix}$

Soln:- We have,

$$A = IA$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_3 \to R_3 - 2R_1$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} A$$

$R_1 \to R_1 + 2R_2$ & then, $R_2 \to R_2 - 2R_3$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 9 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{pmatrix} A$$

$R_1 \to R_1 - 9R_3$, $R_2 \to -R_2$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix} A$$

as $I = PA$

$$\therefore A^{-1} = P = \begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$$

Q. Examine if the following vectors are L.D. or L.I. :

(i) $(3,-2,0,4),(5,0,0,1),(-6,1,0,1),(2,0,0,3)$

Soln:- Let, $A = \begin{pmatrix} 3 & -2 & 0 & 4 \\ 5 & 0 & 0 & 1 \\ -6 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \end{pmatrix}$ be the matrix

formed by given vectors as rows.

$$\therefore A' = \begin{pmatrix} 3 & 5 & -6 & 2 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -5 & 2 \\ -2 & 0 & 1 & 0 \\ 4 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \leftrightarrow R_4 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -5 & 2 \\ 0 & 10 & -9 & 4 \\ 0 & -19 & 21 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -5 & 2 \\ 0 & 10 & -9 & 4 \\ 0 & 0 & 39/10 & \cancel{-\frac{27}{5}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 + \frac{19}{10} R_2$$

which is in echelon form.

$\therefore$ rank$(A') = 3$

$\therefore$ rank$(A) = 3$

$\therefore$ ~~each~~ given matrix is L.D.

NOTE: If rank(A) = No. of ~~eqns~~ vectors, L.I.

If rank$(A) <$ No. of ~~eqns~~ vectors, L.D.

(ii) $(1,1,0),\ (1,0,0),\ (1,1,1)$

Sol^n: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

which is in echelon form with three non-zero rows.

$\therefore$ rank$(A) = 3$ and we are given three vectors, so given vectors are L.I.

Alternatively,

$$v_1 = (1,1,0),\ v_2 = (1,0,0),\ v_3 = (1,1,1)$$

which are vectors in $\mathbb{R}^3$.

Let, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_2 v_3 = 0$, where $\alpha_i \in \mathbb{R}$

$\Rightarrow \alpha_1(1,1,0) + \alpha_2(1,0,0) + \alpha_3(1,1,1) = (0,0,0)$

$\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_3,\ \alpha_3) = (0,0,0)$

$\therefore\ \alpha_1 + \alpha_2 + \alpha_3 = 0\ ;\ \alpha_1 + \alpha_3 = 0\ ;\ \alpha_3 = 0$

$\therefore\ \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0$

$\therefore$ All scalar coefficient are zero.

$\therefore$ Given vectors are L.I.

(iii) $(3,4,7), (2,0,3), (8,2,3), (5,5,6)$

Solⁿ:- Using rank concept method,

Let, $A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 0 & 3 \\ 8 & 2 & 3 \\ 5 & 5 & 6 \end{pmatrix}$ be the matrix formed by given vectors as rows.

$$\sim \begin{pmatrix} 2 & 0 & 3 \\ 3 & 4 & 7 \\ 5 & 5 & 6 \\ 8 & 2 & 3 \end{pmatrix} \quad \begin{matrix} R_1 \longleftrightarrow R_2 \\ R_3 \longleftrightarrow R_4 \end{matrix}$$

$$\sim \begin{pmatrix} 2 & 0 & 3 \\ 0 & 8 & 5 \\ 0 & 10 & 3 \\ 0 & 2 & -9 \end{pmatrix} \quad \begin{matrix} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - 5R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 2 & 0 & 3 \\ 0 & 12 & 5 \\ 0 & 0 & 48 \\ 0 & 0 & -59 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 - 5R_4 \\ R_4 \to 6R_4 - R_2 \end{matrix}$$

$\therefore$ rank$(A) = 3 <$ no. of vectors.

$\therefore$ The vectors are L.D.

Q. Express the vector $v=(4,9,19)$ as a linear combination of $v_1=(1,-2,3)$, $v_2=(3,-7,10)$, $v_3=(2,1,9)$.

Sol^n:- Let, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where, $\alpha_i \in \mathbb{R}$

$\Rightarrow (4,9,19) = \alpha_1(1,-2,3) + \alpha_2(3,-7,10) + \alpha_3(2,1,9)$

$= (\alpha_1 + 3\alpha_2 + 2\alpha_3, -2\alpha_1 - 7\alpha_2 + \alpha_3, 3\alpha_1 + 10\alpha_2, 9\alpha_3)$

$\therefore \alpha_1 + 3\alpha_2 + 2\alpha_3 = 4$

$-2\alpha_1 - 7\alpha_2 + \alpha_3 = 9$

$3\alpha_1 + 10\alpha_2 + 9\alpha_3 = 19$

In matrix form, this system is,

$$\begin{pmatrix} 1 & 3 & 2 \\ -2 & -7 & 1 \\ 3 & 10 & 9 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 19 \end{pmatrix}$$

$\Rightarrow \quad R_2 \to R_2 + 2R_1, \; R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \\ 7 \end{pmatrix}$$

$\Rightarrow \quad R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 24 \end{pmatrix}$$

$\therefore$ The system is in echelon form.

$\therefore$ rank (A) = rank $[A:B]$ = 3 = no. of unknowns,

$\therefore$ The system is consistant and has a unique sol^n.

The equivalent system is:

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 4 \quad$ -(i)

$-\alpha_2 + 5\alpha_3 = 17 \quad$ -(ii)

$8\alpha_3 = 24 \quad$ -(iii)

$\therefore \alpha_3 = 3$

$\alpha_2 = 5\times 3 - 17 = -2$

$\alpha_1 = 4 - 3(-2) - 2\times 3 = 4$

$\therefore v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$= 4v_1 - 2v_2 + 3v_3$

which is the required eq^n.

Eigen Value and Eigen Vector of a square matrix:

Let A be a square matrix of order n, then a scalar λ is called an eigen value of A if there exists a non-zero colum vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that

$AX = \lambda X$ — (i)

and in this case, X is called an eigen-vake of A corresponding to the eigen value λ.

<u>THEOREM</u>: λ is an eigen value of a square matrix A iff $|A - \lambda I| = 0$

<u>PROOF</u>: λ is an eigen value of A, sq^n matrix of order n

$\Leftrightarrow AX = \lambda X$, for some non-zero column vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\Leftrightarrow AX - \lambda X = 0$

$\Leftrightarrow AX - \lambda X = 0$, when I is unit matrix of order 'n'

$\Leftrightarrow (A - \lambda I)X = 0$, where $X \neq 0$.

$\Leftrightarrow A - \lambda I$ is not invertible, otherwise $X = 0$ which is a contradiction.

$\Leftrightarrow A - \lambda I$ is singular

$\Leftrightarrow |A - \lambda I| = 0$

<u>NOTE</u>: These eigen values of A are roots of the eq^n $|A - \lambda I| = 0$ in λ and the eigen vectors are non-zero solutions of $(A - \lambda I)X = 0$

Definition – Let A be a square-matrix of order 'n', then–

(i) $A - \lambda I$ is called characteristic matrix of A where I is unit matrix of order n and λ is a scalar indeterminate.

(ii) $|A - \lambda I|$ is called the characteristic function of A and when expanded, it becomes a polynomial of degree 'n' in λ and is called characteristic polynomial of A.

(iii) $|A - \lambda I| = 0$ is called the characteristic eq^n of A and the roots of this eq^n are called characteristic roots or latent roots of A and a non-zero column vector X satisfying $AX = \lambda X$ or $(A - \lambda I)X = 0$ is called characteristic vector of A corresponding to λ.

NOTE: Characteristic roots (vectors) are same as eigen values (roots)

Q. Find the eigen values and eigen vectors of the following matrices:

(i) $A = \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

(iv) $A = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix}$ (v) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sol^n: (i) Eigen Values of A are roots of

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda-5)(\lambda+6)+18 = 0$

$\Rightarrow \lambda^2+\lambda-12 = 0$

$\Rightarrow (\lambda-3)(\lambda+4) = 0$

$\Rightarrow \lambda = 3, -4$ which are eigen values

Eigen vectors are non-zero sol^n vectors of,

$$(A-\lambda I)x = 0$$

or $\begin{pmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ — (i)

when, $\lambda = 3$, (i) becomes –

$$\begin{pmatrix} 2 & -2 \\ 9 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $R_1 \to \frac{1}{2}R_1$, $R_2 \to \frac{1}{9}R_2$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $R_2 \to R_2 - R_1$

The system is in echelon form. The ~~returned~~ reduced equivalent ~~system~~ is,

$$x - y = 0$$

$$\therefore x = y.$$

Clearly, $x=1, y=1$ is a non-zero sol^n $\therefore$ eigen-vector corresponding to $\lambda = 3$ is $\underline{X_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

In fact eigen vector corresponding to $\lambda = 3$ may be taken as $\lambda_1 = \begin{pmatrix} k \\ k \end{pmatrix}$ where, $k \neq 0$.

When, $\lambda = -4$, (i) becomes,

$$\begin{pmatrix} 9 & -2 \\ 9 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 - R_1$$

The system is in echelon form. The reduced equivalent system is,

$$9x - 2y = 0$$

Clearly, $x = 2,\ y = 9$ is a non-zero solution.

$\therefore$ Eigen vector corresponding to $\lambda = -4$ is $X_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

In fact eigen vector corresponding to $\lambda = -4$ may be taken as, $X_2 = \begin{pmatrix} 2k \\ 9k \end{pmatrix}$, where $k \neq 0$.

<u>NOTE</u>: $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ are L.I.

i.e., eigen vectors corresponding to distinct eigen values are L.I.

<u>Solnº</u>:- (iv) Eigen values of $A = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix}$

are roots of the eqn:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 & 3 \\ 5-\lambda & 1-\lambda & 6 \\ 5-\lambda & 2 & 2-\lambda \end{vmatrix} = 0 \qquad C_1 \to C_1+C_2+C_3$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 3-\lambda & 3 \\ 0 & 4 & -1-\lambda \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\Rightarrow (5-\lambda)\{(\lambda-3)(\lambda+1) - 12\} = 0$$

or, $(5-\lambda)(\lambda^2 - 2\lambda - 15) = 0$

or, $(5-\lambda)(\lambda - 5)(\lambda+3) = 0$

$\therefore \lambda = 5, 5, -3$ which are eigen values.

Eigen vectors of A are non-zero solution vectors of the system $(A - \lambda I)X = 0$

or, $$\begin{pmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -(i)$$

when $\lambda = 5$, (i) becomes,

$$\begin{pmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

This system is in echelon form.

The rank of coefficient matrix is $r = 1$ and no. of unknowns is $n = 3$, so the system has $n - r = 3 - 1 = 2$ no. of L.I. solutions. The reduced equivalent system is

$$-x - 2y + 3z = 0$$

Clearly, $x = 1, y = 1, z = 1$ and $x = 2, y = -1, z = 0$ are two non-zero L.I. solutions.

i.e., Eigen vectors corresponding to $\lambda = 5$ are,

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

When $\lambda = -3$, (i) becomes,

$$\begin{pmatrix} 7 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 4 & 6 \\ 7 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_1 \longleftrightarrow R_3$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 8 & 16 \\ 0 & -16 & -32 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 \to R_3 + 2R_2$$

The system is in echelon form.

Rank of coefficient matrix, $r = 2$

and, no. of unknowns is $n = 3$.

$\therefore$ The system has $n - r = 3 - 2 = 1$ L.I.

The reduced equivalent system is

$$x + 2y + 5z = 0 \qquad \text{-(ii)}$$

$$8y + 16z = 0$$

$$\Rightarrow y + 2z = 0$$

$$\Rightarrow y = -2z \qquad \text{-(iii)}$$

$\therefore$ from (ii) and (iii), $x = -2y - 5z$

$= 4z - 5z$

$\Rightarrow x = -z$

Clearly, $x = -1,\ y = -2,\ z = 1$.

$\therefore$ Eigen vector corresponding to $\lambda = -3$ is,

$$X_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Cayley-Hamilton Theorem:

Every Square matrix satisfies its own characteristic eq^n.

Explanation: If A is a square matrix of order n, then CH-eq^n of A is $|A - \lambda I| = 0$

or, $a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \dots + a_n\lambda^n = 0$ (say) —(i)

Then, CH-theorem says that,

$a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + \dots + a_n A^n = O$. (null matrix of size $n \times n$) —(ii)

If $|A| \neq 0$, then A^{-1} exists, and in this case,

$|A| = a_0 \neq 0$

$\therefore$ Multiplying (ii) by A^{-1}, we get,

$a_0 I A^{-1} + a_1 A A^{-1} + a_2 A^2 A^{-1}, a_3 A^3 A^{-1} + \dots + a_n A^n A^{-1} = O A^{-1}$

$\Rightarrow a_0 A^{-1} = -(a_1 I + a_2 A + a_3 A^2 + \dots + a_n A^{n-1})$

$\Rightarrow A^{-1} = -\frac{1}{a_0}(a_1 I + a_2 A + a_3 A^2 + \dots + a_n A^{n-1})$

Q. Verify Cayley-Hamilton theorem for the following matrices and find A^{-1} if it exists.

(i) $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$

Soln: (i) CH-eqn of A ie,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda+1) - 4 = 0$$

$$\Rightarrow \lambda^2 - 5 = 0$$

To verify: $A^2 - 5I = 0$,

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 5I$$

$\therefore A^2 - 5I = 0$

$\therefore$ Cayley-Hamilton theorem is verified.

Now, $A^2 - 5I = 0$

or, $A^2 A^{-1} - 5 I A^{-1} = 0 A^{-1}$

or, $A - 5A^{-1} = 0$

or, $A^{-1} = \frac{1}{5} A$

$\therefore A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Soln- (ii) CH-eqn of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)(2-\lambda) - 1\} + 1(\lambda-2+1) + 1\{1-(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4\lambda+3) + \lambda - 1 + \lambda - 1 = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 + 2\lambda - 2 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify, $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = AA = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ +5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$= 0$ $\therefore$ CH theorem is verified.

Now. $A^3 - 6A^2 + 9A - 4I = 0$

$\Rightarrow A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0A^{-1}$

$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

Q. Find A^{-1} using C-H theorem where,

(i) $A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Solⁿ: (i) CH eqⁿ of A is,

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow$

$\Rightarrow \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

By C-H theorem,

$$A^3 - 7A^2 + 15A - 9I = 0$$

$$\Rightarrow A^3A^{-1} - 7A^2A^{-1} + 15AA^{-1} - 9IA^{-1} = 0A^{-1}$$

$$\Rightarrow A^2 - 7A + 15I - 9A^{-1} = 0$$

$$\Rightarrow 9A^{-1} = A^2 - 7A + 15I$$

$\vdots$

$$\therefore A^{-1} =$$

Soln:- (iii) CH eqn of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1-\lambda & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda-1 & 1 & 0 \\ 0 & -1 & -\lambda & 1 \\ 0 & -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} \lambda-1 & 1 & 0 \\ -1 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[-(\lambda+1)(\lambda^2-0) - 1(\lambda+1)\right] = 0$$

$$\Rightarrow (\lambda-1)(\lambda+1)\lambda^2 + (\lambda-1)(\lambda+1) = 0$$

$$\Rightarrow (\lambda^2-1)\lambda^2 + (\lambda^2-1) = 0$$

$$\therefore \lambda^4 - 1 = 0$$

By CH-theorem $A^4 - I = O$

$\Rightarrow A^4 = I$

$\Rightarrow A^{-1}A^4 = A^{-1}I$

$\Rightarrow A^3 = A^{-1}$

Similar Matrices: A square matrix B of order 'n' is said to be similar to a square matrix A of order n if there exists a non-singular matrix P such that $B = P^{-1}AP$

The transformation of a matrix A by a non-singular matrix P to $B = P^{-1}AP$ is called a similarity transformation.

Theorem: Similar matrices have the same eigen value.

Proof: Let A and B are two similar matrices then $\exists$ a non-singular matrix P such that $B = P^{-1}AP$.

Now, $|B - \lambda I| = |P^{-1}AP - \lambda I|$

$= |P^{-1}AP - \lambda P^{-1}P|$

$= |P^{-1}AP - \lambda P^{-1}IP|$

$= |P^{-1}(A - \lambda I)P|$

$= |P^{-1}||A - \lambda I||P|$

$= \frac{1}{|P|}|A - \lambda I||P|$

$= |A - \lambda I|$

$\because |A - \lambda I| = 0$

$\Rightarrow |B - \lambda I| = 0$

$\therefore$ A and B have the same CH-eqn and hence the same CH-roots i.e, same eigen values.

NOTE: If X is an eigen vector of A corresponding to eigen value λ, then,

$$AX = \lambda X \quad , \text{ where } X \neq 0$$

$$\Rightarrow P^{-1}AX = P^{-1}\lambda X$$

$$\Rightarrow P^{-1}AIX = \lambda P^{-1}X$$

$$\Rightarrow P^{-1}APP^{-1}X = \lambda P^{-1}X$$

$$\Rightarrow BP^{-1}X = \lambda P^{-1}X$$

$$\Rightarrow BY = \lambda Y, \text{ where } Y = P^{-1}X \neq 0$$

$\therefore Y = P^{-1}X$ is eigen values of B corresponding to λ

Diagonalisation by similarity transformation:

If A is a square matrix of order n having n linearly independent eigen ~~values~~ vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix, whose diagonal elements are eigen values of A and the matrix P consist of column vectors, which are eigen vectors of A.

Justification:

Let A is a sq. matrix of order 3 and also $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$$X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}; X_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

be the corresponding eigen vectors.

Let. $$P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

$$= (X_1 \; X_2 \; X_3)$$

$$\therefore AP = (AX_1 \; AX_2 \; AX_3)$$

$$= (\lambda_1 X_1 \; \lambda_2 X_2 \; \lambda_3 X_3)$$

$\because AX_1 = \lambda_1 X_1 \,;\; AX_2 = \lambda_2 X_2 ;\; AX_3 = \lambda_3 X_3$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\Rightarrow AP = PD$, where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is diagonal matrix.

$$\Rightarrow P^{-1}AP = D$$

* The resulting diagonal matrix D is called the spectral matrix of A.

NOTE: $D = P^{-1}AP$

$$\therefore D^2 = DD = P^{-1}APP^{-1}AP$$

$$= P^{-1}AIAP$$

$$= P^{-1}AAP$$

$$= P^{-1}A^2P$$

$$D^3 = P^{-1}APP^{-1}A^2P$$

$$= P^{-1}AIA^2P$$

$$= P^{-1}A^3P$$

and so on.

In general, $D^n = P^{-1} A^n P$

$\therefore A^n = P D^n P^{-1}$

Q. Diagonalise the following matrices by similarity transformation and hence find A^5 where,

(i) $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

Solⁿ:- (i) CH-eqⁿ of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)(\lambda-6) - 4 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 14 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-7) = 0$$

$\Rightarrow \lambda = 2, 7$ which are eigen values of A

Eigen values of A are non-zero solution vectors of

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{pmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{— (i)}$$

when $\lambda = 2$ (i) because,

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 - 2R_1$$

$\therefore$ The reduced equivalent system is

$$x + 2y = 0$$

Clearly, $x = 2, y = -1$ is a non-zero solution.

$\therefore$ Eigen vector corresponding to $\lambda = 2$ is

$$X_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

When $\lambda = 7$, (i) becomes,

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_2 \to R_2 + \frac{1}{2} R_1$$

$$\Rightarrow \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reduced eqv. system is

$$-4x + 2y = 0.$$

Clearly, $x = 1, y = 2$, is a non-zero solⁿ.

$\therefore$ Eigen vector corresponding to $\lambda = 7$ is

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let, $P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

$$\therefore |P| = 4 + 1 = 5$$

$$\therefore P^{-1} = \frac{1}{|P|} (\text{adj. } P) = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\therefore P^{-1} A P = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\therefore A = P D P^{-1}$$

$$\therefore A^5 = P D^5 P^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2^5 & 0 \\ 0 & 7^5 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Q. Diagonalise the matrix :

$$A = \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$ by similarity transformation and hence find A^6.

Sol^n: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -3 & -3 \\ -3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & -3 \\ 1-\lambda & -2-\lambda & -3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 1, 2$$

which are eigen values of A.

Eigen vectors of A are non-zero sol^n vectors of the system $(A - \lambda I)X = 0$.

$$\begin{pmatrix} 4-\lambda & -3 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(i)}$$

when $\lambda = 1$, (i) becomes –

$$\begin{pmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_2 \to R_2 - R_1 \; ; \; R_3 \to R_3 + \frac{1}{3} R_1$$

$$\Rightarrow \begin{pmatrix} 3 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system is now in echelon for

The reduced equivalent System is,

$$3x - 3y - 3z = 0$$

or, $$x - y - z = 0.$$

The system has $n - r = 3 - 1 = 2$ nos. of L.I. Sol^{n}s.

Clearly, $x = 1, y = 1, z = 0$

and, $x = 1, y = 0, z = 1$

are two L.I. Sol^{n}s.

$\therefore$ Eigen Vectors corresponding to $\lambda = 1$ are:

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 2$, (i) becomes,

$$\begin{pmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_1 \longleftrightarrow R_3$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 3 & -4 & -3 \\ 2 & -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_2 \rightarrow R_2 + 3R_1 ; \; R_3 \rightarrow R_3 + 2R_1$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system is in echelon form and has $n-r = 3-2 = 1$ L.I. sol^n. The reduced equ. system,

$$x+y=0$$
$$-y-3z=0$$
$$x=y=3z$$

Clearly, $x=3,\ y=3,\ z=-1$, is a non-zero sol^n. e.vector corresponding to $\lambda=2$ is $X_3 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$

Let $P = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$\therefore \underline{P^{-1}AP} = D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

which is required diagonal matrix, similar to A.

Also, $A = PDP^{-1}$

$$\therefore A^6 = PD^6P^{-1}$$
$$= P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}P^{-1}$$

~~Note: $P^{-1}AP = \begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -3 & 3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}P$~~

~~$= (-3\ 4$~~

Now, $P^{-1} = \frac{1}{|P|} \text{adj of } P$

$$|P| = -3 - 1(-1) + 3(1)$$
$$\therefore |P| = 1.$$

Matrix of Cofactors,

of $P = \begin{pmatrix} (-1)^2(-3) & (-1)^3 \cdot 1(-1) & (-1)^4 \cdot 3 \cdot 1 \\ -1 \cdot 1(-4) & 1 \cdot 0 \cdot (-1) & -1 \cdot 3 \cdot (1) \\ 1 \cdot 0 \cdot 3 & -1 \cdot 1 \cdot 0 & 1(-1)(-1) \end{pmatrix}$

$= \begin{pmatrix} -3 & 1 & 1 \\ 4 & -1 & -1 \\ 3 & 0 & 1 \end{pmatrix}$

adjoint of $P = (\text{M. of C.})^T$

$= \begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{|P|}\begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$A^6 = PD^6P^{-1}$

$= \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix} P^{-1}$

$= \begin{pmatrix} 1 & 1 & 192 \\ 1 & 0 & 192 \\ 0 & 1 & -64 \end{pmatrix}\begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$A^6 = \begin{pmatrix} 190 & -195 & 195 \\ 189 & 188 & 195 \\ -63 & 63 & -64 \end{pmatrix}$

NOTE: $P^{-1}AP = \begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix} P$

$$= \begin{pmatrix} -3 & 4 & 3 \\ 1 & -1 & 0 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

New Chapter

Scalar Point function, Scalar field, Vector point function and vector field:

If to each point $P(x,y,z)$ is a region $E \subseteq \mathbb{R}^3$, there corresponds a unique scalar $\phi(x,y,z)$ [vector $\vec{f}(x,y,z)$] by some rule ϕ (rule f) then ϕ is called ($\vec{f}$ is called) a scalar point function (vector point function) and the region E together with ϕ (with $\vec{f}$) is called a scalar field (vector field).

Eg: (i) If $\phi(x,y,z)$ denotes temperature at any Point in a region then ϕ is a scalar point function.

Also, $\phi(x,y,z) = x^2+y^2+z^2$ defines a scalar function.

(ii) If $\vec{f}(x,y,z)$ denotes velocity of a fluid particle at any point (x,y,z) then $\vec{f}$ is a vector point function.

Also, $\vec{f}(x,y,z) = x^2\hat{i} + yz\hat{j} - zx\hat{k}$ defines a vector point function.

Level Surface: If $\phi(x,y,z)$ is a scalar point function defined at each point in a region $E \subseteq \mathbb{R}^3$ and C is an arbitrary constant then the eqn $\phi(x,y,z) = C$, represents a family of surfaces and any surface of this family is called a level surface or an iso-ϕ surface.

Eg.: if $\phi(x,y,z) = x^2+y^2+z^2$

then, $\phi(x,y,z) = C = a^2$ (say)

i.e. $x^2+y^2+z^2 = a^2$ represents a family of spheres with centre origin and radius $a>0$.

In particular, $\phi(x,y,z) = 2^2$

i.e. $x^2+y^2+z^2 = 4$ is a level surface.

Del or nabla operator: The vector differential operator del or nabla is denoted and defined as $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \sum \hat{i}\frac{\partial}{\partial x}$

Also, for any vector $\vec{a}$, we define

$$\vec{a}\cdot\vec{\nabla} = \vec{a}\cdot\hat{i}\frac{\partial}{\partial x} + \vec{a}\cdot\hat{j}\frac{\partial}{\partial y} + \vec{a}\cdot\frac{\partial}{\partial z}$$

$$= \sum \vec{a}\cdot\hat{i}\frac{\partial}{\partial y}$$

Note $\vec{a}\cdot\vec{\nabla} \neq \vec{\nabla}\cdot\vec{a}$

Laplacian Operator: It is denoted and defined by, $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

The eqn $\nabla^2\phi = 0$

$$\text{or, } \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

is known as Laplace's eqn and any scalar point function satisfying Laplacian eqn is called a Harmonic function.

E.g.: if $\phi = x^2 - y^2 + 2xz$

then, $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2 - 2 + 0 = 0$

$\therefore \phi$ is a harmonic function.

Gradient of a Scalar Point function:

The gradient of a differentiable scalar point function $\phi(x, y, z)$ is a vector denoted and defined as grad $\phi = \vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \sum \hat{i}\frac{\partial\phi}{\partial x}$$

Geometrical interpretation:

The gradient of a scalar point function $\phi(x, y, z)$ at a point $P(x_1, y_1, z_1)$ on the level surface $\phi(x, y, z) = c$ is a vector normal to the surface at P.

NOTE: ① The unit normal to the surface $\phi(x, y, z) = C$ at $P(x_1, y_1, z_1)$ is $\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

② $\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ at $P(x_1, y_1, z_1)$ is normal to the surface $\phi(x, y, z) = c$ at P and therefore $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are ~~ratio~~ different ratios of the normal and hence the eqn of tangent plane to the surface at (x_1, y_1, z_1) is given by:

$$(x - x_1)\frac{\partial\phi}{\partial x} + (y - y_1)\frac{\partial\phi}{\partial y} + (z - z_1)\frac{\partial\phi}{\partial z} = 0$$

Also, eq^n of the normal to the surface at (x_1, y_1, z_1) is

$$\frac{x - x_1}{\frac{\delta \phi}{\delta x}} = \frac{y - y_1}{\frac{\delta \phi}{\delta y}} = \frac{z - z_1}{\frac{\delta \phi}{\delta z}}$$

The values of partial derivatives at (x_1, y_1, z_1) are to be used in the above eq^ns.

Q. 1 Find the unit vector normal to the surface $x^2 y + 2xz = 4$ at the point $(2, -2, 3)$. Also, find the eq^ns of the tangent

Sol^n Given surface is,

$$x^2 y + 2xz = 4$$

Here, $\phi(x, y, z) = x^2 y + 2xz$

At the point $(2, -2, 3)$:

$$\frac{\delta \phi}{\delta x} = 2xy + 2z = -8 + 6 = -2$$

$$\frac{\delta \phi}{\delta y} = x^2 = 4$$

$$\frac{\delta \phi}{\delta z} = 2x = 4$$

$$\therefore \vec{\nabla} \phi = \hat{i} \frac{\delta \phi}{\delta x} + \hat{j} \frac{\delta \phi}{\delta y} + \hat{k} \frac{\delta \phi}{\delta z} = -2\hat{i} + 4\hat{j} + 4\hat{j}$$

$$= 2(-\hat{i} + 2\hat{j} + 2\hat{k})$$

$\therefore$ Required unit normal vector is $\hat{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \dfrac{2(-\hat{i}+2\hat{j}+2\hat{k})}{2\sqrt{(-1)^2+(2)^2+(2)^2}}$

$$= \frac{-1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Eq^n of tangent place at $(2,-2,3)$ is given by,

$$(x-x_1)\frac{\delta\phi}{\delta x} + (y-y_1)\frac{\delta\phi}{\delta y} + (z-z_1)\frac{\delta\phi}{\delta z} = 0$$

or, $(x-2)(-2) + (y+2)4 + (z-3)4 = 0$

or, $x-2 \quad -2(y+2) - 2(z-3) = 0$

$\therefore \; x - 2y - 2z = 0$

Also, eq^n of normal at $(2,-2,3)$ is given by

$$\frac{x-x_1}{\frac{\delta\phi}{\delta x}} = \frac{y-y_1}{\frac{\delta\phi}{\delta y}} = \frac{z-z_1}{\frac{\delta\phi}{\delta z}}$$

$$\Rightarrow \frac{x-2}{-2} = \frac{y+2}{4} = \frac{z-3}{4}$$

Q.2. If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i)

THEOREM: If $f(x,y,z)$ and $g(x,y,z)$ are two scalar point functions and C is a scalar constant, then –

(i) $\vec{\nabla} c = \vec{0}$ (ii) $\vec{\nabla}(Cf) = C\vec{\nabla} f$

(iii) $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$ (iv) $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$

(v) $\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}$

Proof: (i) $\vec{\nabla} c = \hat{i}\frac{\partial c}{\partial x} + \hat{j}\frac{\partial c}{\partial y} + \hat{k}\frac{\partial c}{\partial z} = \hat{i}0 + \hat{j}0 + \hat{k}0$
$= \vec{0}$

(v) $\vec{\nabla}\left(\frac{f}{g}\right) = \Sigma \hat{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \Sigma \hat{i}\left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right)$

$= \frac{1}{g^2}\Sigma\left(\hat{i} g\frac{\partial f}{\partial x} - \hat{i} f\frac{\partial g}{\partial x}\right)$

$= \frac{1}{g^2}\left(g\Sigma\hat{i}\frac{\partial f}{\partial x} - f\Sigma\hat{i}\frac{\partial g}{\partial x}\right)$

$= \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}$

Q. If, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then show that
$\vec{\nabla} r^n = n r^{n-2}\vec{r}$, where $r = |\vec{r}|$

Soln: $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

Diff practically w.r.t. x

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \vec{\nabla} r^n = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\, n.r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= nr^{n-2}\vec{r}$$

Divergence and Curl of a vector point function:

Let, $\vec{f}(x,y,z)$ a differentiable vector point function.

The divergence of $\vec{f}$ is a scalar denoted and defined as,

$$\text{div.}\vec{f} = \vec{\nabla}.\vec{f} = \hat{i}.\frac{\partial \vec{f}}{\partial x} + \hat{j}.\frac{\partial \vec{f}}{\partial y} + \hat{k}.\frac{\partial \vec{f}}{\partial z}$$

$$= \sum \hat{i}.\frac{\partial \vec{f}}{\partial x}$$

If, $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, then,

$$\text{div.}\vec{f} = \vec{\nabla}.\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

The curl of $\vec{f}$ is a vector denoted and defined as

$$\text{curl } \vec{f} = \vec{\nabla} \times \vec{f} = \hat{i} \times \frac{\partial \vec{f}}{\partial x} + \hat{j} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z}$$

$$= \sum \hat{i} \times \frac{\partial \vec{f}}{\partial x}$$

If, $\vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$, then,

$$\text{curl } \vec{f} = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

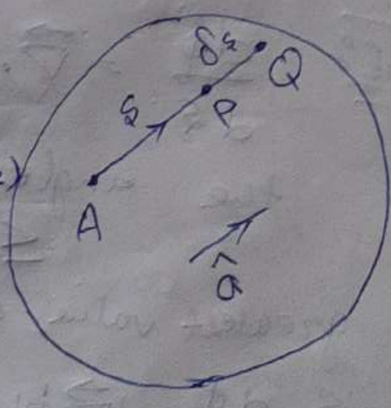
Def^n: A vector function $\vec{f}(x, y, z)$ is called

(i) Solenoidal if $\text{div } \vec{f} = 0$

(ii) irrotational if $\text{curl } \vec{f} = \vec{0}$

Directional derivative:

Let $\phi(x, y, z)$ is a scalar point function defined at each point in a region $\mathbb{E} \subseteq \mathbb{R}^3$ and A be a fixed point in $\mathbb{E}$ and also $\hat{a}$ is a given unit vector.



Let $P(x, y, z)$ and $Q(x+\delta x, y+\delta y, z+\delta z)$ are two neighbouring points in $\mathbb{E}$ on the line through A in the direction of $\hat{a}$ such that

$AP = s$, $AQ = s + \delta s$, so that $PQ = \delta s$

$$\underset{Q \to P}{Lt} \frac{\phi(Q) - \phi(P)}{PQ}$$

$$= \underset{\delta s \to 0}{Lt} \frac{\phi(x+\delta x, y+\delta y + z+\delta z) - \phi(x,y,z)}{\delta s}$$

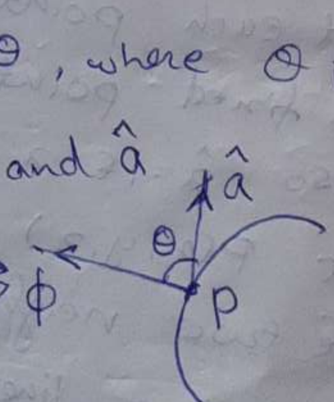
$= \underset{\delta s \to 0}{Lt} \dfrac{\delta \phi}{\delta s}$, if it exists, is called the directional derivative of ϕ at the point P in the direction of $\hat{a}$ and is denoted by $\dfrac{\delta \phi}{\delta s}$

Eg: $\dfrac{\delta \phi}{\delta x}$ is the directional derivative of ϕ in the direction of $\hat{i}$.

Theorem: The directional derivative of a scalar point function $\phi(x,y,z)$ at any point $P(x,y,z)$ in the direction of a unit vector $\hat{a}$ is given by, $\dfrac{d\phi}{ds} = \vec{\nabla}\phi \cdot \hat{a}$

NOTE: $\dfrac{d\phi}{ds} = \vec{\nabla}\phi \cdot \vec{a} = |\vec{\nabla}\phi| \cos\theta$, where θ is the angle between $\vec{\nabla}\phi$ and $\hat{a}$

$\leq |\vec{\nabla}\phi|$, $\because \cos\theta \leq 1$



$\therefore$ greatest value of directional derivative is $\dfrac{d\phi}{ds} = |\vec{\nabla}\phi|$ and it occurs in the direction of $\vec{\nabla}\phi$ i.e., in the direction of normal to the surface at the point considered.

Q. Find the directional derivative of $\phi = x^2+y^2+4xz$ at $(1,-2,2)$ in the direction of the vector $2\hat{i}-2\hat{j}-\hat{k}$

Soln:- $\phi = x^2+y^2+4xz$

At the point $(1,-2,2)$:

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(2x+4z)+\hat{j}(2y)+\hat{k}(4x)$$

$$= 10\hat{i}-4\hat{j}+4\hat{k}$$

unit vector in the direction of $2\hat{i}-2\hat{j}-\hat{k}$ is

$$\hat{a} = \frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{4+4+1}} = \frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$$

$\therefore$ required directional derivative is

$$\frac{d\phi}{ds} = \vec{\nabla}\phi\cdot\hat{a}$$

$$= (10\hat{i}-4\hat{j}+4\hat{k})\cdot\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$$

$$= \frac{1}{3}(20+8-4)$$

$$= 8$$

Q. Find the rate of change of the function $x^2y+2xy+z^2$ at the point $P(-1,1,2)$ in the direction of $\vec{PQ}$, where Q is the point $(1,2,0)$. Is it increasing or decreasing in this direction?

Soln:- Let, $\phi = x^2y+2xy+z^2$

At the point $P(-1,1,2)$:

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$= 2+\lambda$

$= \hat{i}(2xy+2y) + \hat{j}(x^2+2x) + \hat{k}(2z)$

$= 0\hat{i} - \hat{j} + 4\hat{k}$

Now $\vec{PQ} = (1+1)\hat{i} + (2-1)\hat{j} + (0-2)\hat{k}$

$= 2\hat{i} + \hat{j} - 2\hat{k}$

$\therefore$ unit vector in the direction of $\vec{PQ}$ is

$\hat{a} = \frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$

$\therefore$ Required directional derivative is,

$\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \hat{a}$

$= (0\hat{i} - \hat{j} + 4\hat{k}) \cdot \frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$

$= \frac{1}{3}(0 - 1 - 8)$

$= -3$

Since $\frac{d\phi}{ds} < 0$, so, ϕ is decreasing at P in the direction of $\vec{PQ}$

Q. Find the value of λ for which $\vec{V} = (x+3y)\hat{i} + (y-2x)\hat{j} + (x+\lambda z)\hat{k}$ is a solenoidal vector.

Soln div. $\vec{V} = \vec{\nabla} \cdot \vec{V}$

$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(x+\lambda y)$

$= 1 + 1 + \lambda$

∴ $\vec{v}$ will be solenoidal,

if, div. $\vec{v} = 0$

or, $2 + \lambda = 0$

$\therefore \lambda = -2.$

Q. for what value of λ and μ

$$\vec{v} = (y^2 + 2\mu xz)\hat{i} + (\lambda xy + \mu yz)\hat{j} + (y^2 + \mu x^2)\hat{k}$$

is irrotational.

Solⁿ:-

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2\mu xz & \lambda xy + \mu yz & y^2 + \mu x^2 \end{vmatrix}$$

$$= \hat{i}(2y - \mu y) + \hat{j}(2\mu x - 2\mu x) + \hat{k}(\lambda y - 2y)$$

$$= y(2 - \mu)\hat{i} + 0\hat{j} + y(\lambda - 2)\hat{k}$$

∴ $\vec{v}$ will be irrotational if,

$$\text{curl } \vec{v} = \vec{0}$$

i.e, if, $\mu = 2$

$\lambda = 2$

Ex If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is a constant vector then show that curl $\vec{v} = 2\vec{\omega}$.

Soln

$$\text{curl } \vec{v} = \sum \hat{i} \times \frac{\partial \vec{v}}{\partial x} = \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{\omega} \times \vec{r})$$

$$= \sum \hat{i} \times \left(\vec{\omega} \times \frac{\partial \vec{r}}{\partial x}\right)$$

($\vec{\omega}$ is constant vector)

$$= \sum \hat{i} \times (\vec{\omega} \times \hat{i})$$

$$= \hat{i} \times (\vec{\omega} \times \hat{i}) + \hat{j} \times (\vec{\omega} \times \hat{j}) + \hat{k} \times (\vec{\omega} \times \hat{k})$$

$$= (\hat{i}.\hat{i})\vec{\omega} - (\hat{i}.\vec{\omega})\hat{i} + (\hat{j}.\hat{j})\vec{\omega} - (\hat{j}.\vec{\omega})\hat{j} + (\hat{k}.\hat{k})\vec{\omega} - (\hat{k}.\vec{\omega})\hat{k}$$

$$= \vec{\omega} + \vec{\omega} + \vec{\omega} - \{(\hat{i}.\vec{\omega})\hat{i} + (\hat{j}.\vec{\omega})\hat{j} + (\hat{k}.\vec{\omega})\hat{k}\}$$

$$= 3\vec{\omega} - \vec{\omega}$$

$$= 2\vec{\omega}$$

($\because$ if $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$ then $\hat{i}.\vec{\omega} = \omega_1$, $\hat{j}.\vec{\omega} = \omega_2$, $\hat{k}.\vec{\omega} = \omega_3$)

Ex. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $f(r)$ is a scalar function of r then, examine if $f(r).\vec{r}$ is irrotational.

Soln

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

Diff. partially w.r.t. x.

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\delta r}{\delta y} = \frac{y}{r}$; $\frac{\delta r}{\delta z} = \frac{z}{r}$

$$f(r)\vec{r} = f(r)(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= x f(r)\hat{i} + y f(r)\hat{j} + z f(r)\hat{k}$$

$$\text{curl}\{f(r)\vec{r}\} = \vec{\nabla} \times \{f(r)\vec{r}\}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \left[\frac{\delta}{\delta y}\{z f(r)\} - \frac{\delta}{\delta z}\{y f(r)\}\right]\hat{i} + \left[\frac{\delta}{\delta z}\{x f(r)\} - \frac{\delta}{\delta r}\{z f(r)\}\right]$$
$$+ \hat{k}\left[\frac{\delta}{\delta x}\{y f(r)\} - \frac{\delta}{\delta y}\{x f(r)\}\right].$$

$$= \left[z f'(r)\frac{\delta r}{\delta y} - y f'(r)\frac{\delta r}{\delta z}\right]\hat{i} + \left[x f'(x)\frac{\delta r}{\delta z} - z f'(r)\frac{\delta r}{\delta x}\right]$$
$$\hat{j} + \left[y f'(r)\frac{\delta r}{\delta x} - x f'(r)\frac{\delta r}{\delta y}\right]\hat{k}$$

$$= f'(r)\left[\left(z\frac{y}{r} - y\frac{z}{r}\right)\hat{i} + \left(x\frac{z}{r} - z\frac{x}{r}\right)\hat{j} + \left(y\frac{x}{r} - x\frac{y}{r}\right)\right]$$

$$= f'(r)(0\hat{i} + 0\hat{j} + 0\hat{k})$$
$$= \vec{0}$$

$\therefore$ $f(r)\vec{r}$ is an irrotational vector.

Ex Show that $|\vec{r}|^n \vec{r}$ is irrotational for any n and it is solenoidal only when $n = -3$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Soln. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore |\vec{r}| = \sqrt{x^2+y^2+z^2}$

or, $r^2 = x^2+y^2+z^2$ where, $r = |\vec{r}|$

Diff. partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

and, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$|\vec{r}|^n \vec{r} = r^n \vec{r}$$

$$= r^n (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= x r^n \hat{i} + y r^n \hat{j} + z r^n \hat{k}$$

$$\therefore \text{curl}(|\vec{r}|^n \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$= \vec{0} \quad \forall n$$

$\therefore |\vec{r}|^n \vec{r}$ is irrotational $\forall n$.

$$\operatorname{div}(|\vec{r}|^n\vec{r}) = \frac{\partial}{\partial x}(x r^n) + \frac{\partial}{\partial y}(y r^n) + \frac{\partial}{\partial z}(z r^n)$$

$$= r^n + x \cdot n r^{n-1}\frac{\partial r}{\partial x} + r^n + y n r^{n-1}\frac{\partial r}{\partial y} + r^n + z \cdot n r^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + n r^{n-1}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2} r^2$$

$$= (n+3) r^n.$$

Ex. Verify div . curl $\vec{F} = 0$ where,

$$\vec{F} = x^2 y\,\hat{i} + zx\,\hat{j} + 2yz\,\hat{k}$$

Soln:- Curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & zx & 2yz \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(zx)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(x^2 y) - \frac{\partial}{\partial x}(2yz)\right\}\hat{j}$$

$$+ \left\{\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}\right\}\left\{\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(x^2 y)\right\}\hat{k}$$

$$= (2z - x)\hat{i} + (0-0)\hat{j} + (z - x^2)\hat{k}$$

$$\therefore \operatorname{div} \operatorname{curl} \vec{F} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(z - x^2)$$

$$= -1 + 0 + 1$$

$$= 0$$

NOTE: div curl $\vec{F} = 0$ for any vector f^n $\vec{F}$.

Ex: Verify curl. grad $\phi = \vec{0}$ for $\phi = xy^2z^3$

Soln: $\text{grad } \phi = \vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$= y^2z^2\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$

$\therefore$ curl grad $\phi = \vec{\nabla} \times (\vec{\nabla}\phi)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$= (6xyz^2 - 6xyz^2)\hat{i} + (3y^2z^2 - 3y^2z^2)\hat{j}$

$+ (2yz^3 - 2yz^3)\hat{k}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= \vec{0}$

Ex. Find div $(\phi\vec{r})$ where, $\phi = x^2 + y^2 + z^2$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Soln: $\phi\vec{r} = \phi(x\hat{i} + y\hat{j} + z\hat{k})$

$= x\phi\hat{i} + y\phi(\hat{j}) + z\phi\hat{i}$

$\therefore \text{div}(\phi\vec{r}) = \vec{\nabla}\cdot(\phi\vec{r})$

$= \frac{\partial}{\partial x}x\phi + \frac{\partial}{\partial y}y\phi + \frac{\partial}{\partial z}z\phi$

$= \phi + x\frac{\partial \phi}{\partial x} + \phi + y\frac{\partial \phi}{\partial y} + \phi + z\frac{\partial \phi}{\partial y}$

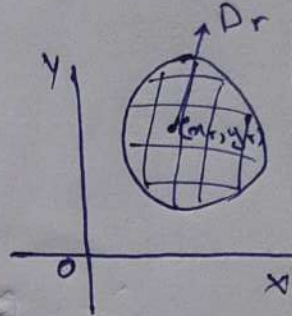
$= 3\phi + x(2x) + y(2y) + z(2z)$

$= 3\phi \cdot 2(x^2+y^2+z^2)$

$= 5(x^2+y^2+z^2)$

Double Integration :-

Let, $f(x,y)$ is a function of two variables which is defined in at each point in a finite plane region D in the xy- plane.



and let D is divided into n-subregion $D_1, D_2, \ldots, D_n$ with respective areas $\delta A_1, \delta A_2, \ldots, \delta A_n$ and let (x_r, y_r) be any point in D_r $\forall\, r = 1, 2, \ldots, n$

then, the limit of the sum $\sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$ as $n \to \infty$ and simultaneously tangent of $\delta A_r \to 0$, if it exists. is called double integral of $f(x,y)$ over the region D and is denoted by $\iint_D f(x,y)\,dA$ or $\iint_D f(x,y)\,dx\,dy$

Then,
$$\iint_D f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

In particular, if $f(x,y) = 1$ $\forall\,(x,y) \in D$

then, $$\iint_D \delta x\,\delta y = \lim_{n\to\infty} \sum_{r=1}^{n} \delta A_r$$

= area of the region D.

COMPLEX VARIABLES:

- **Limit of a function:** $\underset{z \to z_0}{Lt} f(z) = l$

For a given $\varepsilon > 0$, $\exists\ \delta > 0$

such that, $|f(z) - l| < \varepsilon \ \forall\ z$ satisfies, $|z - z_0| < \delta$

Continuity of $f(z)$ at $z = z_0$.

if, $\underset{z \to z_0}{Lt} f(z) = f(z_0)$

- Differentibility:

Let $f(z)$ be a single valued function of the variable,

$$z = x + iy$$

Then,

$f(z)$ is said to be differentiable if

$$f'(z) \to \underset{z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists as $\delta z \to 0$ along any path.

- **Analytic functions:**

If a single valued function $f(z)$ which is differentiable at all points of a region R then, $f(z)$ is said to be analytic in the region R.

Imp theorem: The necessary and sufficient conditions for a function $f(z) = u(x,y) + i\,v(x,y)$ to be analytic in a region R are:

(i) All partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in R.

(ii)
$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right\} \text{Cauchy - Reimann equation (C-R eq}^n\text{)}$$

Q. Show that $f(z) = \sqrt{|xy|}$, $z \neq 0$
$= 0$, $z = 0$

is not analytic ~~function~~ at origin although it is a C-R eq^n.

Sol^n:- $u = \sqrt{|xy|}$

$v = 0$

At (0,0)

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Hence, C-R-eqn satisfies at origin.

Now,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{x\to 0} \frac{\sqrt{|xy|} - 0}{z}$$

Let $z \to 0$ along the path $y = mx$.

$\therefore$ Now, $z = x + iy$

$z = x + imx = x(1+im)$

$\therefore$ $z \to 0$ implies $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{\sqrt{|x \cdot mx|}}{x(1+im)}$$

$$= \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

$$\therefore f'(0) = \frac{\sqrt{|m|}}{1+im}$$

which depends on m

$\therefore$ $f'(0)$ is not unique, i.e., it is not differentiable at origin.

Hence, it is not analytic.

Q. Examine C-R eqn at origin and analytic condition for function: $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$

$= 0, \quad z = 0$

Soln:- $f(z) = \dfrac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$

$= \dfrac{(x^3 - y^3) + i(x^3+y^3)}{x^2+y^2}$

Here, $u = \dfrac{x^3 - y^3}{x^2+y^2}, \quad v = \dfrac{x^3+y^3}{x^2+y^2}$

At $(0,0)$

$\dfrac{\partial u}{\partial x} = \lim_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x}$

$= \lim_{x \to 0} \dfrac{x - 0}{x} = 1$

$\dfrac{\partial v}{\partial y} = \lim_{y \to 0} \dfrac{v(0,y) - v(0,0)}{y}$

$= \lim_{x \to 0} \dfrac{y - 0}{y} = 1$

$\dfrac{\partial u}{\partial y} = \lim_{y \to 0} \dfrac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \dfrac{-y - 0}{y} = -1$

$\dfrac{\partial v}{\partial x} = \lim_{x \to 0} \dfrac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \dfrac{x - 0}{x} = 1$

Hence, it satisfies ~~obeys~~ C-R eqn.

Now,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} - 0}{z}$$

$$= \lim_{z\to 0} \frac{\dfrac{x^3(1+i) - (mx)^3(1-i)}{x^2+(mx)^2}}{x + imx} \qquad \left(\begin{array}{l} \because z = x+iy \\ y = mx \end{array}\right)$$

$$= \lim_{z\to 0} \frac{x^3 + ix^3 - (mx)^3 + i(mx)^3}{x^2(1+m^2) \times x(1+ix)}$$

Q. $f(z) = \bar{z}$

Soln:- $f(z) = x - iy$

$u = x, \quad v = -y.$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Since, $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$,

C-R eqn is not Satisfied.

Derive C-R eq^ns in cartesian form R.

$$z = u + iv$$

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + i v(x+\delta x, y+\delta y) - (u(x,y) + iv(x,y))}{\delta z}$$

exists, as $\delta z \to 0$ along any path.

→

Let $\delta z \to 0$ along real axis,

Then, $\delta y = 0$,

and $\delta z = \delta x + i\delta y = \delta x$

$\therefore \delta z \to 0 \Rightarrow \delta x \to 0$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - (u(x,y) + iv(x,y))}{\delta x}$$

* ~~Derive C-R in car~~

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x}$$

$$+ i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$= \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \quad -(i)$$

$$f'(z) = \lim_{i\delta y \to 0} \frac{u(x, y+i\delta y) + iv(x, y+\delta y) - (u(x,y) + iv(x,y))}{i\delta y}$$

Derive C-R eqⁿ in polar form:

Let (r, θ) is the polar coordinates of a point (x, y) where, $z = x + iy$

$$= r\cos\theta + i r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

Let, $u + iv = f(z)$

$$= f(re^{i\theta}) \quad \text{— (i)}$$

be analytic.

Diff. (i) partially w.r.t. r and θ, we get,

<u>w.r.t., r</u>

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \quad \text{— (ii)}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot rie^{i\theta} \quad \text{— (iii)}$$

Dividing (ii) by (iii), we get,

$$ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

Now, equating real and imaginary parts, we get,

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

and, $$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

Polar form is deduced.

Q. Hence, prove that, $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Soln:- we have,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad -(i)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad -(ii)$$

Diff. (i) and (ii) w.r.t. θ and r respectively

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

and $$\frac{\partial^2 u}{\partial \theta^2} = -r \frac{\partial v}{\partial \theta \partial r}$$

Now, putting the obtained eqns in main eqn,

$$-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{-r\partial v}{\partial \theta \partial r}$$

$$= 0$$

Harmonic function

A function $u(x,y)$ is said to be harmonic if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Q. Show that,

$$u = e^{-x}(x \sin y - y \cos y) \text{ is harmonic.}$$

Soln:-

$$\frac{\partial u}{\partial x} = -e^{-x}(x \sin y - y \cos y) + e^{-x} \sin y$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y)$$

Now,

$$\frac{\partial^2 u}{\partial x^2} =$$

• THEOREM:

If, $f(z) = u+iv$ is analytic then both u and v are harmonic function.

$\boxed{\text{Then, both } u \text{ and } v \text{ are Harmonic functions.}}$

Sol^n Since, $u+iv$ is analytic function,

By, C-R eq^n,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{—(i)}$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{—(ii)}$$

Diff. (i) and (ii) partially w.r.t. x and y respectively,

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 v}{\delta x \delta y}$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{-\delta^2 v}{\delta y \delta x}$$

$$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Milne-Thomson Method:

when only one part real or imaginary part is given.

CASE I: Suppose real part u is given, then from u, we can find,

$$\frac{\partial u}{\partial x} = \phi_1(x,y)$$

and, $$\frac{\partial u}{\partial y} = \phi_2(x,y)$$

Then, find $\phi_1(z,0)$ and $\phi_2(z,0)$ by M.T. method.

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$$

$$= \ldots\ldots + c$$

Putting $z = x+iy$,

CASE II: Suppose imaginary part v is given, then from v, we can find,

$$\frac{\partial v}{\partial x} = \psi_1(x,y)$$

and $$\frac{\partial v}{\partial y} = \psi_2(x,y)$$

Then, find, $\psi_1(z,0)$ and $\psi_2(z,0)$ by M.T. method.

$$f(z) = \int \left[-\psi_2(z,0) + i\psi_1(z,0) \right] dz$$

$$= \ldots\ldots + c$$

Putting, $z = x + iy$.

Complex Integration:

If $f(z) = u(x,y) + iv(x,y)$ is a f^n of $z = x+iy$.

Then, $\int_c f(z)\,dz = \int_c (u+iv)(dx+idy)$

$= \int_c ($

where c is any path complex integral of a function is reduced to sum of integrals of real function along C.

E.g.: ① Evaluate.

$\int_c^{2\pi} \frac{dz}{z-a}$, where, C is $|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$, $0 \le \theta \le 2$

$dz = rie^{i\theta} d\theta$

E.g.: ② $\int_0^{2+i} (\bar{z})^2 dz$ along (i) $y = x/2$

(ii) horizontally to 2 and vertically to $2+i$

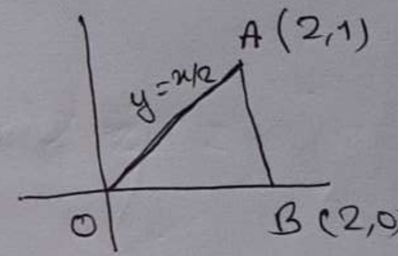
① Soln:- $\int_0^{2\pi} \frac{r i e^{i\theta} d\theta}{r e^{i\theta}} = 2\pi i$

② Soln:- $y = x/2$ or $x = 2y$.

$$z = x + iy = 2y + iy = (2+i)y.$$

$$dz = (2+i)\,dy. \quad 0 \le y \le 1.$$

$$\int_0^{2+i} (\bar{z})^2 \, dz$$



$$\Rightarrow \int_0^1 (2+i)^2 y^2 (2+i)\, dy.$$

$$= \frac{5}{3}(2-i)$$

Along OB, $x = 0$,

$\bar{z} = x$, x varies from 0 - 2.

Along BA, $x = 2$, $z = 2 + iy$; $\bar{z} = 2 - iy$

$$= \int_{OB} x^2 \, dx + \int_{BA} (2 - iy)^2 \, dy$$

$$= \frac{x^3}{3} + \int_{BA} (4 - 4iy - y^2)\, dy$$

$$= \frac{x^3}{3} + 4y - 2yi - \frac{y^3}{3}$$

Cauchy-Goursat Theorem

(1) If $f(z)$ is analytic and $f'(z)$ is continuous at each point within and on a closed cure C,

Then,
$$\int_c f(z)\,dz = 0$$

(2) If $f(z)$ is analytic ~~within~~ in a region D bounded by two closed curves c and C_1, then

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz$$

Cauchy's Integral formula

If $f(z)$ is analytic within and on a closed curve C, and 'a' is any point within C,

Then,
$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-a}$$

OR
$$\int \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a)$$

Extension of Cauchy's Integral formula

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-a)^3}\,dz$$

$$\vdots$$

$$f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Q. Evaluate :

① $\int_c \frac{dz}{z-a}$; c is $|z-a| = r$

② $\int_c \frac{z^2-z+1}{z-1}\, dz$

where, c is (i) $|z| = 1$

(ii) $|z| = 1/2$

③ $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$

where, c is $|z| = 3$

Soln ①

$$f(1) = \frac{1}{2\pi i}\int_c \frac{dz}{z-a}$$

$$a,\ 1 = \frac{1}{2\pi i}\int_c \frac{dz}{z-a}$$

$$2\pi i = \int_c \frac{dz}{z-a}$$

3Soln 2

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

Q. Show that analytic function with constant modulus is constant.

Soln:- $f(z) = u+iv$ is analytic.

given, $|f(z)| = \sqrt{u^2+v^2} = c$ (say)

i.e., $u^2+v^2 = c^2$ —(i)

Diff. (i) partially w.r.t. x and y, we get,

$$2u\frac{\delta u}{\delta x} + 2v\frac{\delta v}{\delta x} = 0 \quad \text{—(ii)}$$

$$2u\frac{\delta u}{\delta y} + 2v\frac{\delta v}{\delta y} = 0 \quad \text{—(iii)}$$

Since, $f(z)$ is analytic,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{—(iv)}$$

and, $$\frac{\delta u}{\delta y} = \frac{-\delta v}{\delta x} \quad \text{—(v)}$$

(iii) and (v) gives,

$$-u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta y} = 0 \quad \text{—(vi)}$$

Squaring and adding (ii) and (vi), we get,

$$u^2\left(\frac{\delta u}{\delta x}\right)^2 + v^2\left(\frac{\delta v}{\delta x}\right)^2 + u^2\left(\frac{\delta v}{\delta x}\right)^2 + v^2\left(\frac{\delta v}{\delta y}\right)^2 = 0$$

or, $$\left[\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2\right](u^2+v^2) = 0$$

Since, $u^2+v^2 \neq 0$, $u^2+v^2 = c \Rightarrow u^2+v^2 = c^2$

NOTE:- If $f(z) = u + iv$ is analytic function,

Then, u is called velocity potential function and v is called stream f^n.

Q. If $z = \phi + i\psi$ represents complex partial for an electric field and, $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, find ϕ.

Soln:- $f(z) = \int [-\psi_2(z,0) + i\psi_1(z,0)]\, dz$ —(i)

$$\frac{\partial \psi}{\partial x} = \psi_1(x,y) = 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\psi_1 = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = \psi(x,y) =$$

$$\psi_1(z,0) =$$

$$\psi_2(z,0) =$$

So, from (i),

$$f(z) = i\left(z^2 + \frac{1}{z}\right) + c$$

$$= \left(-2xy + \frac{y}{x^2+y^2} + c\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right)$$

Q. If $u-v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is analytic, find $f(z)$ in terms of z.

Soln:- Let, $f(z) = u+iv \quad -(i)$

$\Rightarrow \quad if(z) = iu - v \quad -(ii)$

(i) + (ii) gives,

$$u-v+i(u+v) = f(z)(1+i)$$
$$= F(z)$$
$$= U+iV \quad (\text{say})$$

$\therefore F(z) = U+iV$ is analytic

$$U = u-v = (x-y)(x^2+4xy+y^2)$$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} =$$

$$\phi_2(x,y) = \frac{\partial V}{\partial x} =$$

$$\phi_1(z,0) =$$

$$\phi_2(z,0) =$$

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz$$

$\Rightarrow (1+i)f(z) = (1+i)z^3 + c$

or, $f(z) = \dfrac{\qquad}{1+i} + c$

$\therefore \quad f(z) = -iz^3 + c$

Q. If $f(z)$ is analytic f^n of z, prove that,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4 \, |f'(z)|^2$$

Sol^n-

$$f(z) = u + iv$$

$$|f(z)|^2 = u^2 + v^2$$

Q. $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, find $f(z)$

Sol^n:- $f(z) = \int [-\psi_2(z,0) + i\psi_1(z,0)] dz$

$$\psi_1 = \frac{\delta v}{\delta x} = 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\psi_2 = \frac{\delta v}{\delta y} = -2y + \frac{-x}{(x^2+y^2)^2}$$

Morrera's Theorem:

It can be treated as converse of Cauchy's theorem.

If $f(z)$ is continuous in region B and $\int_C f(z)dz = 0$ about closed curve C, then $f(z)$ is analytic in B.

Liouville's Theorem:

If $f(z)$ is analytic and bounded for all z in an entire complex plane, the $f(z)$ is a constant.

Q. Evaluate $\int_C \frac{e^z dz}{(z^2+\pi^2)^2}$, C is $|z|=4$

Soln:-

$$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)(z-\pi i)^2}$$

$$= \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

$$\int \frac{e^z dz}{(z^2+\pi^2)^2} = A.\ 2\pi i f(-\pi i) + C\ 2\pi i f(\pi i) + B \frac{2\pi i}{2!} f'(-\pi i) + D \frac{2\pi i}{2!} f'(\pi i)$$

Morrera's Theorem:

It can be treated as converse of Cauchy's theorem.

If $f(z)$ is continuous in region D and $\int_C f(z)\,dz = 0$ about closed curve C, then $f(z)$ is analytic in B.

Liouville's Theorem:

If $f(z)$ is analytic and bounded for all z in an entire complex plane, the $f(z)$ is a constant.

Q. Evaluate $\int_C \frac{e^{z}\,dz}{(z^2+\pi^2)^2}$, C is $|z|=4$

Soln:- $\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)(z-\pi i)^2}$

$$= \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

$$\int \frac{e^z\,dz}{(z^2+\pi^2)^2} = A.\,2\pi i\, f(-\pi i) + C\; 2\pi i f(\pi i) + B\,\frac{2\pi i}{2!} f'(-\pi i) + D\,\frac{2\pi i}{2!} f'(\pi i)$$

Laurent Series for expansion of f(z) in powers of (z-a).

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$
$$\ldots + a_{-1}(z-a) + a_{-2}$$

* Zeros

* Singularities of an analytic fn:

A singular point of a function is that point at which the function ceases to be analytic, i.e., the discontinuous function can be taken as singular function.

Different types of Singularities:

(1) Isolated Singularity:

If, $z = a$, is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood then, $z = a$ is called an isolated singularity.

(2) Removable Singularity:

If, $\lim_{z \to a} f(z)$ is finite, then 'a' is called a removable singularity.

Also, if $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$

(3) Poles Singularity:

If all the negative powers of $(z-a)$ in Lauren's series after the n^{th} are missing then 'a' is called a pole of order 'n.'

A pole of first order is called a simple pole.

(4) Essential Singularity:

If the negative powers of $(z-a)$ in Lauren's series are infinite, then 'a' is called an essential singularity, and in such case,

$\lim_{z \to a} f(z)$ is infinite.

Q. Find the nature of the singularity of the following functions.

(i) $\frac{z-\sin z}{z^2}$

(ii) $(z+1)\sin\left(\frac{1}{z-2}\right)$

(iii) $\frac{1}{1-e^z}$

(i) → Here, $z=0$ is singularity.

$$\frac{z-\sin z}{z^2} = \frac{z-\left(z-\frac{z^3}{3!}+\dots\right)}{z^2}$$

Since there are no negative powers of z in the expansion, $\therefore z=0$ is a removable singularity.

(ii) → Here, $z-2=t \Rightarrow z=t+2$

then, $f(z) = (z+3)\sin\left(\frac{1}{z}\right)$

$$(z+3)\sin\left(\frac{1}{z}\right) = 1+\frac{3}{t}-\frac{1}{3!\,t^2}+\dots$$

$$= 1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}+\dots$$

Since there are infinite no. of terms in -ve powers of $(z-2)$, $\therefore z=2$ is an essential singularity

(iii) → Poles of $1-e^z=0$

$$\Rightarrow e^z = 1 = e^{2n\pi i}$$

$$\therefore z = 2n\pi i, \quad (n\in \mathbb{Z})$$

$f(z)$ has simple poles and $z=2\pi i$.

(iv) $\dfrac{e^{2z}}{(z-1)^4}$

Soln: Let, $z-1=t \Rightarrow z=t+1$

So, $f(z) = e^2 \cdot \dfrac{e^{t+1}}{t^4}$

There are finite no.s of -ve powers of z.

Evaluate: $\int_c \frac{e^z dz}{\cos \pi z}$, c is $|c| = 1$.

Solⁿ:

$$\lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{e^z}{\cos \pi z}$$

$$R_1 = Res\left(\frac{1}{2}\right)$$

$$R_2 = Res\left(-\frac{1}{2}\right) = \frac{e^{-1/2}}{\pi} = \frac{-e^{1/2}}{\pi}$$

$$\hookrightarrow = 2\pi i (R_1 + R_2)$$

$$= -4i \sinh \frac{1}{2}$$

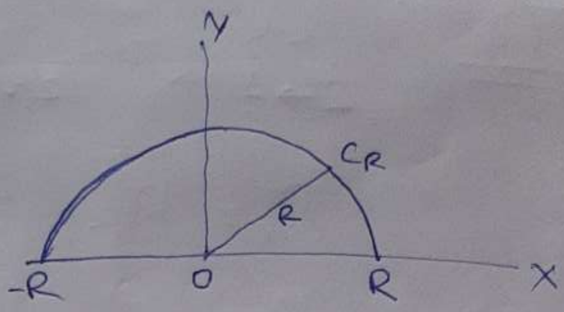
Q. $\int_c \tan z \, dz$, c is $|z| = 2$

Q. $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$, c is $|z| = 3$ $\boxed{= -4\pi i}$

Case: Integration around a semicircle.

Form : $\int_{-\infty}^{\infty} f(x)\,dx.$

$\int_c f(z)\,dz$



$$= \int_{C_R} f(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx.$$

Numerical Analysis:

Calculus of finite difference:

It deals with the changes that takes place in the value of the function due to the changes in the

If, $y = f(x)$ is a function of x,

then, ~~calculus of finite difference~~

Difference between two ~~consequtive~~ consecutive arguments is called and is denoted by 'h'.

Operators:

(i) Δ :- Delta is known as forward difference operator.

Let $y = f(x)$ be a function of x and 'h' be the interval of differencing,

then, $$\Delta f(x) = f(x+h) - f(x) \quad \text{-(i)}$$

(ii) ∇ : It is known as backward difference operator

Let, $y = f(x)$ be a function of x and 'h' be the interval of differencing,

then, $$\nabla f(x) = f(x) - f(x-h) \quad -(ii)$$

(iii) E : Displacement or shift operator

$$Ef(x) = f(x+h) \quad -(iii)$$

(iii) E^{-1} : Inverse operator defined as,

$$E^{-1} f(x) = f(x-h) \quad -(iv)$$

Relation betn different operators:

(i) Δ and E :

from (i) and (iii), we get

$$\Delta - E = -1$$

$$\therefore \Delta = E - 1$$

$$\therefore E = 1 + \Delta$$

(ii) ∇ and E^{-1}:

from (ii) and (iv), we get,

$$\nabla + E^{-1} = 1$$

$$\therefore E^{-1} = 1 - \nabla$$

$$\therefore \nabla = 1 - E^{-1}$$

Arguments are equally spaced:

Forward Difference Table:

x	y	Δy	$\Delta^2 y$
x_1	y_1		
		$y_2 - y_1 = \Delta y_1$	
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$
		$y_3 - y_2 = \Delta y_2$	
x_3	y_3		

Backward Difference Table

x	y	∇y	$\nabla^2 y$
x_1	y_1		
		$y_2 - y_1 = \nabla y_2$	
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$
		$y_3 - y_2 = \nabla y_3$	
x_3	y_1		

Theorem: The n^{th} order difference of a polynomial of degree n is constant and all its higher order differences are 0.

$$y = f(x) = a_0 x^n + a_1 x^{n-1} + \dots + kx + l$$

$$\Delta^{n+1} f(x) = 0$$

$$\Delta^n f(x) = \text{constant}$$
$$= a_0 \cdot n! \, h^n$$
$$= a_0 n!$$

Q. Find

Q. Find a function whose first order difference is

(i) e^x

Soln. Let $f(x)$ be the function of x.

$$\Delta f(x) = e^x \qquad -(i)$$

we know,

$$\frac{d}{dx}(e^x) = e^x$$

Q.1 Evaluate $\int (x+y+ix^3)\,dz$ along path joining $z=0$ to $z=1+i$.

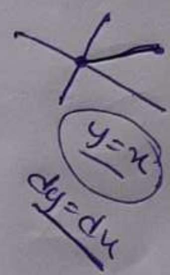
Q.2 $\int_C \frac{z^2+1}{z^2+2z+5}\,dz$ where C is $|z+1-i|=2$

Q.3 Find residue of $f(z) = \frac{\cot \pi z}{z}$ over $|c|=1$

Soln:- 1 $\int_0^1 (x+x+ix^3)\,dx.(1+i)$

$dz = dx+idy$

$\int_0^1 (x+x+ix^3)(dx+idx)$



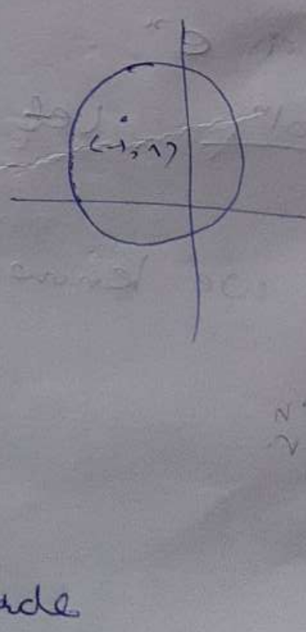
$= (1+i)\int_0^1 (2x+ix^3)\,dx$

$= (1+i)\left[x^2 + \frac{ix^4}{4}\right]_0^1$

$= (1+i)\left[1+\frac{i}{4}\right]$

$= 1+i+\frac{i}{4}-\frac{1}{4} = \frac{3}{4}+\frac{5}{4}i = \frac{1}{4}(3+5i)$

Soln:- 2 $\int \frac{z^2+1}{z^2+2z+1+4}\,dz$



$= \int \frac{z^2+1}{(z+1)^2+(2)^2}\,dz$

$= \int \frac{z^2+1}{(z+1+2i)(z+1-2i)}$

Here, $z = -1 \pm 2i$

$z = -1+2i$, lies inside the circle

$z = -1-2i$ lies outside the circle

$\operatorname{Ref}(-1-2i)=0$

$$\operatorname{Ref}(-1+2i)=\lim_{z\to(-1+2i)} (z+1-2i)\,\frac{z^2+1}{(z+1+2i)(z+1-2i)}$$

Missing Terms:

Case I: Suppose one term (entry) is missing

Say,

x;	x_1	x_2	x_3	x_4
y;	y_1	y_2	?	y_4

Here, three entries are given,

$\therefore$ 2nd order diff. is constant.

$$\Delta^3 y_1 = 0$$

$$\Rightarrow (E-1)^3 y_1 = 0$$

$$\Rightarrow (E^3 - 3E^2 + 3E - 1) y_1 = 0$$

$$\Rightarrow y_4 - 3y_3 + 3y_2 - y_1 = 0$$

Case II: Suppose two entries are missing

Say

x	x_1	x_2	x_3	x_4	x_5	x_6
y	y_1	y_2	?	y_4	?	y_6

So, $\Delta^4 y_1 = 0$ — (i)

$\Delta^4 y_2 = 0$ — (ii)

for (i), $(E-1)^4 y_1 = 0$

or, $(E^4 - 4E^3 + 6E^2 - 4E + 1) y_1 = 0$

$\Rightarrow y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0$ — (iii) for (iv) $y_6 - 4y_5 + 6y_4 - 4y_3 + y_2 = 0$

Q. x: 1 2 3 4

y: 1 8 ? 64

Sol^n: Since three entries are given

$\Delta^3 y_1 = 0$

or $(E-1)^3 y_1 = 0$

or $(E^3 - 3E^2 + 3E - 1) y_1 = 0$

or $y_4 - 3y_3 + 3y_2 - y_1 = 0$

or $3y_3 = y_4 - y_1 + 3y_2$

$y_3 = \frac{64 - 1 + 24}{3}$

$= \frac{63 + 24}{3}$

$= \frac{87}{3}$

$y_3 = 29$

> Formulas to find missing entry y at any point x.

(i) Newton's forward interpotention formula:

This formula will be used when arguments are equally spaced and the value of y to be calculated at some point x which lies near the beginning of the given data.

Let, $y = f(x)$ be a function of x

Taking values $y_0, y_1, y_2, \ldots, y_n$

while x takes $x_0, x_1, x_2, \ldots, x_n$

Let, 'x_i' is a equally spaced and 'h' is the interval of spacing

Then,

$$y_x = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-\overline{n-1})}{n!}\Delta^n y_0$$

where, $u = \frac{x - x_0}{h}$

(ii) Newton's backward inter potention formula:

$$y_x = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots +$$

$$\frac{u(u+1)(u+2)(u+(n-1)}{n!}\nabla^n y_n$$

(iii) Lagrangds formula:

Taking values $y_1, y_2, y_3, \dots, y_n$
while x takes, $x_1, x_2, x_3, \dots, x_n$

where, x_i are not necessarily equally spaced.

Then

$$y_x = \frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} \times y_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} y_2$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)\dots(x-x_n)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)\dots(x_3-x_n)} y_3 + \dots +$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_{n-1})}{(x_{(n)}-x_1)(x_n-x_2)(x_n-x_3)\dots(x_n-x_{(n-1)})} \times y_n$$