# Graph Theory CS305 Notes

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* $G = (V, E)$

where,

$V$ = Vertex Set $\{V_1, V_2, V_3, V_4 \dots V_n\}$

$E$ = Edge Set $\{E_1, E_2, E_3, E_4 \dots E_n\}$

$|V|$ = Order of Graph

$|E|$ = Size of Graph

- Undirected Graph:

  $l_x \in E$ and $l_x = \{V_i, V_j\}$


Undirected edge.

- Directed Graph:


directed edge.

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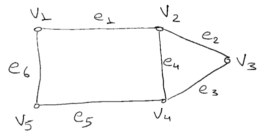



$G = (V, E)$

$V = \{V_1, V_2, V_3, V_4, V_5\}$

$E = \{e_1, e_2; e_3, e_4, e_5, e_6\}$

$e_1 = \{V_1, V_2\}$

$e_6 = \{V_1, V_5\}$

- Adjacent Vertex → Common Edge.

  ↳ Example: $V_5$ and $V_2$ are adjacent vertices of $V_1$

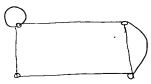
- Adjacent Edge → Two edges connected by a common vertex.

  ↳ Example: $e_1$ and $e_2$ are adjacent edges connected by $V_1$.

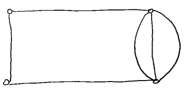
- Self Loop →  Edge starting and ending on same vertex.

- Multiedges (or Parallel Edges): More than one edge between two vertices.

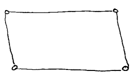
- Pseudograph: Graph containing self loop as well as multiedges.



- Multigraph: Must not contain self loop but must contain multiedges.



- Simple Graph: Must not contain neither Self loop nor multiedge.



* Handshaking Theorem (Handshaking Lemma)

$$\sum_{u \in V} d(v) = 2 * |E|$$

where,

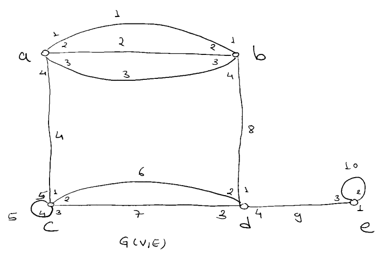
$d(v)$ is degree of vertex.

NOTE: The number of odd vertices in any graph is always even.

In the given graph in right side,

$$\sum_{u \in V} d(v) = 2 * |E|$$

DEGREE AND EDGE




Self loop counts as 2x degrees.

Degree of C = 5

A = 4

In the given graph in right side,

$$\sum_{u \in V} d(v) = 2 * |E| = 2 * 10 = 20$$

As counted, d(v) is indeed 20.
$\therefore$ 20 = 20.

Number of odd vertices = { C (5 degree), e (3 degree) }
= 2 {which is even}

Self loop counts as 2x degrees.

Degree of C = 5
A = 4
B = 4
D = 4
E = 3
$\therefore$ Total Degree = 20

Total Edges = 10 (Edge is not counted 2x for self loop)

## * Regular Graph vs Complete Graph

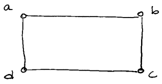
### Regular Graph

→ Graph in which every vertex has same degree.

→ No. of edges, $E = \frac{n \times d}{2}$

where, n is number of vertices
d is degree of each vertex.

E.g.:




d(a) = d(b) = d(c) = d(d) = 2
No. of vertices = {a, b, c, d} = 4
No. of edges, $E = \frac{4 \times 2}{2} = 4$

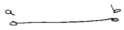
### Complete Graph.

→ A complete graph ($K_n$) is a simple graph in which every pair of vertices are adjacent

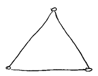
→ No. of egdes = ${}^{n}C_2 = \frac{n(n-1)}{2}$

→ Degree of Every vertex = (n-1)
where, n is total number of vertices

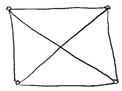
Eg: $K_1 \equiv$ o . $K_2 \equiv$




$K_3 \equiv$



$K_4 \equiv$



## * Havel-Hakimi Theorem:

→ For a given degree sequence, whether a simple graph exists or not.

Example:

- Does graph exist for the given degree sequence:

(i) (2, 2, 2, 2) (ii) (3, 2, 1, 1, 0) (iii) (6, 5, 4, 3, 3, 1) (iv) (7, 6, 5, 4, 3, 4, 2, 1)

Step I: Arrange in decreasing order of degree.
Step II: Remove first element (v1).
Step III: Decrease 1 degree in next |v1| elements., i.e., if v1 = 6, the reduce 1 in following 6 elements.
Step IV: For the newly obtained degree sequence, follow step I to III.
Step V: Repeat until all elements are zero.

If zero obtained, then yes, graph exists.
Else, then no, graph is not possible.

Example: (i) (2, 2, 2, 2) (Already in descending order)
→ ($\cancel{2}$, 2, 2, 2) (Remove 1st element)
⇒ ($\underset{-1}{2}$, $\underset{-1}{2}$, 2) (Decrease 1 in next n-elements corresponding value to 1st element, i.e, next 2 elements)

⇒ (1, 1, 2) (Newly obtained degree sequence)
⇒ (2, 1, 1) (in descending order)
⇒ ($\cancel{2}$, 1, 1) (Remove 1st element)
⇒ ($\underset{-1}{1}$, $\underset{-1}{1}$) (Decrease 1 in next 2-elements)

⇒ (0, 0) (Zero degree sequence obtained)

Hence, graph exists.

(ii) (3, 2, 1, 1, 0) (Already in desc. order)
→ ($\cancel{3}$, 2, 1, 1, 0) (Remove 1st element)
⇒ ($\underset{-1}{2}$, $\underset{-1}{1}$, $\underset{-1}{1}$, 0) (Decrease 1 in next 3-elements, because 3 is the value of the deleted element)

⇒ (1, 0, 0, 0) (Newly obtained degree sequence already in desc. order)
⇒ ($\cancel{1}$, 0, 0, 0) (Remove 1st element)
⇒ ($\underset{-1}{0}$, 0, 0) (Decrease 1 in next 1-elements)

= (-1, 0, 0) (Zero degree sequence not obtained)

Negative degree sequence obtained.
Hence, graph does not exist

Alt. method} Also, (3, 2, 1, 1, 0). Here, No. of vertices with odd degree = {3, 1, 1} = 3
As this does not satisfy handshake theorem, the graph cannot exist.

(iii) (6, 5, 4, 3, 3, 1) (Already in desc. order)
→ ($\cancel{6}$, 5, 4, 3, 3, 1)
⇒ (5, 4, 3, 3, 1)

(iii) $(6,5,4,3,3,1)$ (Already in desc. order)

→ $(\cancel{6},\underline{5,4,3,3,1})$

⇒ $(5,4,3,3,1)$

Not sufficient elements.
Hence, graph does not exist.

(iv) $(7,6,5,4,3,4,2,1)$

→ $(\cancel{7},\underbrace{6,5,4,4,3,2,1}_{-1})$

⇒ $(\cancel{5},\underbrace{4,3,3,2,1}_{-1},0)$

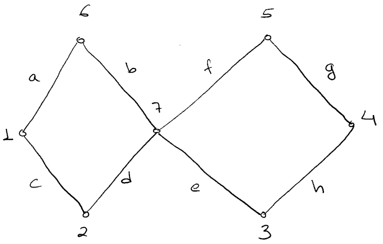
⇒ $(\cancel{3},\underbrace{2,2,1}_{-1},0,0)$

⇒ $(\cancel{1},\underbrace{1}_{-1},0,0,0)$

⇒ $(0,0,0,0)$

Hence, Graph Exists.

## * Walk, Trail, Path, Circuit and Cycle.




Walk = 1a2b6b7f5f7e3h4
↳ Can be repeated - both vertex and edge
↳ Open walk and Closed walk
Starts and Ends on different vertices — Starts and Ends on same vertex.

Trail = 7f5g4h3e7d2c1a6b7
↳ Vertex can be repeated but edge cannot be repeated.
↳ Open Trail and Closed Trail.
↳ Also a circuit.

* Circuit is same as Closed trail
* Cycle is same as Closed path

Path = 2d7b6a1c2
↳ Except for starting and ending vertex, other vertices cannot be repeated
↳ Open path and Closed Path
2d7b6a1 — ↳ Also a cycle.

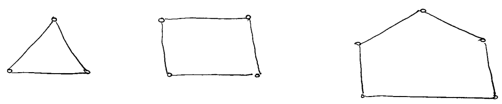
## * Types of Simple Graph:

→ 1. Null Graph ($N_n$) : Graph with 'n' vertices and zero edges.

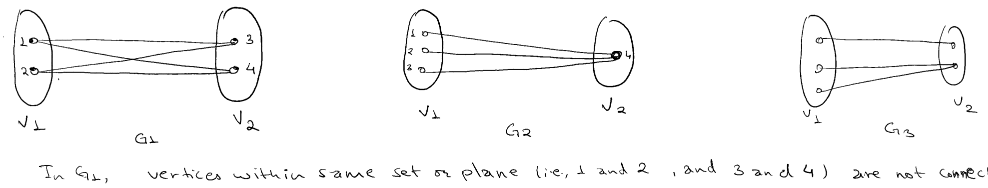
$v_{1\circ}$ $v_{1\circ}$ $v_{2\circ}$ $v_{1\circ}$ $v_{2\circ}$ $v_{3\circ}$

i.e., $v = n$
$e = 0$
$d = 0$

→ Cyclic Graph ($C_n$, $n \geq 3$): Graph with 'n' vertices $\{v_1, v_2, v_3, \dots v_n\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots \{v_{n-1}, v_n\}\{v_n, v_1\}$



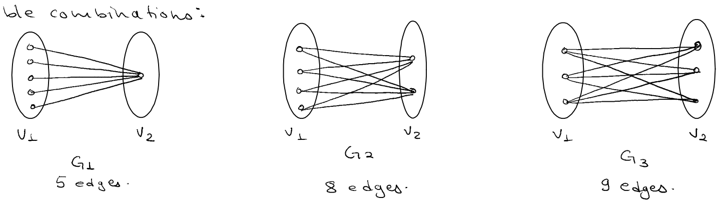
→ Bipartite Graph: Graph where vertex can be partitioned into two sets $V_1$ and $V_2$ such that every vertex is in between a vertex of $V_1$ to $V_2$




In $G_1$, vertices within same set or plane (i.e., 1 and 2, and 3 and 4) are not connected.
In $G_2$, vertices (1, 2 and 3) fall under same set, hence they are not connected to one another.

Here, maximum edges possible = $\left\lfloor \frac{n^2}{4} \right\rfloor$ , i.e., ⌊ ⌋ brackett denotes lower value.
where, n is no. of vertices.
in case of odd, answer comes in decimal,
so, lower $(4.7) = 4$.
not ceiling value. but, floor or lower value.

Example.,
Given, $n = 6$.
Possible combinations:




Max. edge possible = $\lfloor 6^2 \rfloor = \lfloor 36 \rfloor = 9$

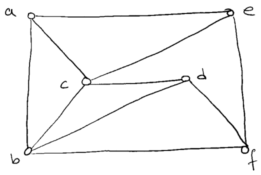
$G_1$ 5 edges. $G_2$ 8 edges. $G_3$ 9 edges.

Max. edge possible $= \left\lfloor \frac{6^2}{4} \right\rfloor = \left\lfloor \frac{36}{4} \right\rfloor = 9$

→ Complete Bipartite Graph ($K_{m,n}$) : Bipartite graph in which every vertex in $V_1$ is adjacent to every vertex in $V_2$ set.

Degree of graph = max $(m,n)$
where, $m$ and $n$ are no. of vertices in set $V_1$ and $V_2$ respectively.

* Chromatic : Minimum number of colours that can be used to fill up each node/vertex such that the neighbouring/adjacent vertex distinguish in colour.




- starting with highest degree
$\{ b, c, a, d, e, f \}$
$\;\; 4, 4, 3, 3, 3, 3$

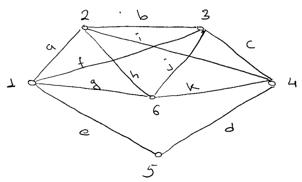
Give b colour red (1)
c colour yellow (2)
a colour purple (3)
d colour black (4)
e colour red (4) (not adjacant with b)
f colour yellow (3) (not adjacant with c)

Therefore, chromatic number = 4.

* Euler Graph :
↳ Euler path exists : Covering all edges.
↳ Euler Circuit exists : Cover all edges and finish in the same point (vertex) as starting point.
↳ No repetition of edges.

* Hamiltonian Graph:
↳ Hamiltonian path exists : Covering all vertices
↳ Hamiltonian cycle/circuit exists : Cover all vertices and finish in the same vertex as started.
↳ No repetition of vertices, except for starting & finishing vertex.

Example:




Q. What graph is it?
(a) Euler Graph
(b) Hamiltonion Graph
(c) Both
(d) None

Solution: Start from a point with highest degree.

for Euler : 1 a 2 b 3 c 4 d 5 e 1 f 3 j 6 h 2 i 4 k 6 g 1
↳ all edges must be covered but all vertices need not be covered.
↳ vertices can be repeated but edges cannot be repeated.

for Hamiltonian : 1 a 2 b 3 j 6 k 4 d 5 e 1
↳ all vertices must be covered but all edges need not be covered.
↳ edges can be repeated but vertices cannot be repeated, except for starting vertex.

∴ (c) Both.

* Planar Graph : → Graph that can be embedded in the plane.

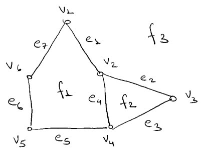
max. no. of faces, $f \le 2n - 4$
where, $n$ is number of vertices.

$\sum d(f) = 2e$
where, $d(f)$ is degree of all faces or regions $d(r)$
$e$ is number of edges.

Euler formula: $n - e + f = 2$

where, $d(f)$ is degree of all faces or regions $d(r)$
$e$ is number of edges.

Euler formula: $n - e + f = 2$
where, $n$ is number of vertices
$e$ is number of edges
$f$ is number of faces or regions, $r$



degree of faces = number of edges in that plane

degree of $f_1 = \{e_1, e_4, e_5, e_6, e_7\} = 5$
degree of $f_2 = \{e_2, e_3, e_4\} = 3$
degree of $f_3 = \{e_1, e_2, e_3, e_5, e_6, e_7\} = 6$

$\therefore$ Total degree, $\Sigma d(f) = 5 + 3 + 6 = 14$
Also, No. of edges, $e = 7$

So, $\Sigma d(f) = 2e = 2 \times 7 = 14$

**Example** The maximum number of faces that are possible for a simple connected planar graph with 12 vertices are ____ if degree $(f) \geq 3$

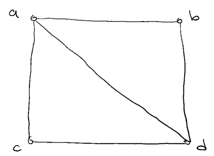
$\rightarrow$ we know, $f \leq 2n - 4$
or, $f \leq 2 \times 12 - 4$
$\therefore f \leq 20$

$\therefore$ Max. number of faces = 20.

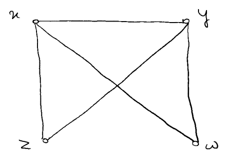
## * Isomorphism:

- Number of vertex equal
- Number of edges equal
- Degree Sequence equal
- Mapping (of vertex)
  - check degree of each vertex and degree of adjacent vertices.
  - check edges between mapped vertices

**Example:** Check isomorphism of these two Graphs.



$G_1$



$G_2$

$\rightarrow$

No. of vertices = 4
No. of edges = 5
Degree Sequence = {3, 3, 2, 2}

No. of vertices = 4
No. of edges = 5
Degree Sequence = {3, 3, 2, 2}

for mapping:

$a \rightarrow x$
$b \rightarrow z$
$c \rightarrow w$
$d \rightarrow y$

Checking if mapping is correct.

Degree of:
$a = 3 \rightarrow x = 3$
$b = 2 \rightarrow z = 2$
$c = 2 \rightarrow w = 2$
$d = 3 \rightarrow y = 3$

Degrees of adjacent vertices of:
$a = \{3,2,2\} \rightarrow x = \{3,2,2\}$
$b = \{3,3\} \rightarrow z = \{3,3\}$
$c = \{3,3\} \rightarrow w = \{3,3\}$
$d = \{3,2,2\} \rightarrow y = \{3,2,2\}$

Edge between:
a and b exists $\rightarrow$ x and z exists
b and c does not exist $\rightarrow$ z and w does not exist
c and d exists $\rightarrow$ w and y exists
d and a exists $\rightarrow$ y and x exists
a and c exists $\rightarrow$ x and w exists
b and d exists $\rightarrow$ z and y exists

Hence, mapping also verified.

Therefore, the given two graphs are isomorphic graphs