INDEX

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STD.: Ist SEM DIV./ SEC.: ______ SUBJECT: MATHS

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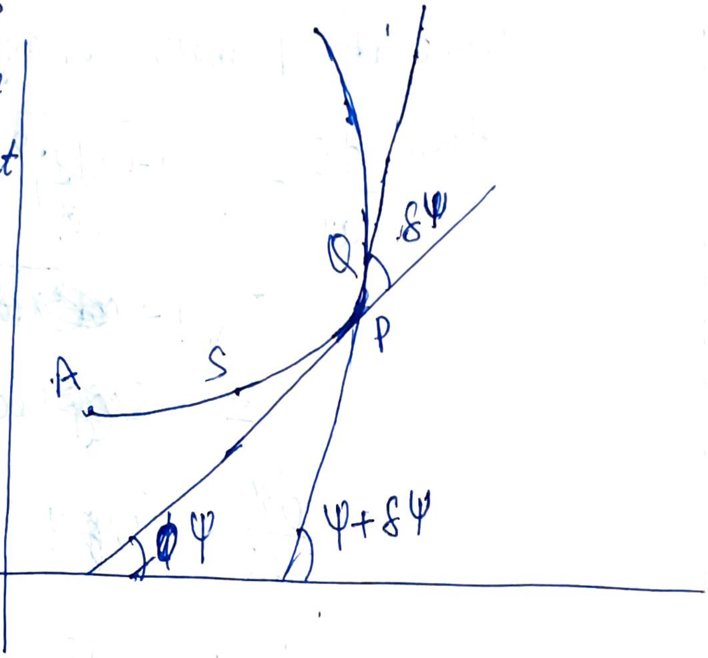
* Curvature and Radius of Curvature :-

Let A be a fixed point on a plane curve and let P and Q are two neighbouring points on it such that arc $AP = s$ & arc $AQ = (s + \delta s)$

so that arc $PQ = \delta s$

The $\delta\psi$ is the angle through which the tangent turns or rotate as a point moves along the curve from P to Q through an arcual distance 'δs' & this '$\delta\psi$' is called total bending or total curvature of the arc PQ.



The ratio $\frac{\delta\psi}{\delta s}$ is called the averge curvature of the arc PQ. The curvature of the curve at P is defined to be $\lim_{Q \to P} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$

Def^n: Let A be a fixed point on a plane curve and P be any point on it such that arc $AP = s$ & let the tangent to the curve at P makes an angle ψ to some fixed line OX. Then the curvature of the curve at point A is defined as $\frac{d\psi}{ds}$

and reciprocal of the curvature at P is called the radius of curvature at point P and radius is denoted as ρ which is usually positive.

Thus, $\rho = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}$

* Derivative of an Arc :-

If $P(x,y)$ is any point on a curve in the $x-y$ plane and A be a fixed point on it. arc $AP = s$ & the tangent to the curve at P makes an angle ψ with x-axis then

then $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\tan^2\psi}$

$= \sec\psi$

$\frac{dx}{ds} = \cos\psi$

Similarily, $\frac{ds}{dy} = \operatorname{cosec}\psi$

$\frac{dy}{ds} = \sin\psi$

The relation between s and ψ of a curve is called Intrinsic eqn operator

Ex:- Find radius of curvature at any pt (s, ψ) of the curve

(i) $s = a \sec\psi + a \log(\sec\psi + \tan\psi)$

(ii) $s = c \tan\psi$

(iii) $s = a e^{x/c}$ where (x, y) & (s, ψ) are Cartesian & intrinsic coordinates of the same pt.

Soln (i) $\frac{ds}{d\psi} = a \sec\psi \tan\psi + \frac{a}{(\sec\psi + \tan\psi)} \cdot (\sec\psi \tan\psi + \cancel{\tan} \sec^2\psi)$

$= a \sec\psi \tan\psi + a \sec\psi$

$= a \sec\psi (1 + \tan\psi)$

which is reqd. radius of curvature.

(iii) $s = a e^{x/c}$

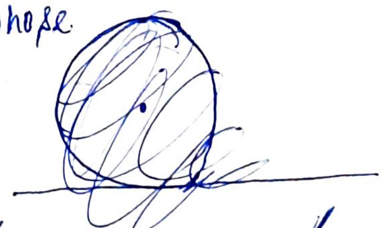
$\Rightarrow \frac{ds}{dx} = a \frac{1}{c} e^{x/c}$

$\Rightarrow \sec\psi = \frac{1}{c} \cdot s$

$\Rightarrow s = c \sec\psi$

$\frac{ds}{d\psi} = c \sec\psi \tan\psi$ which is reqd. radius of curvature ~~at~~ any ~~point of a circle is~~

Q.2. Prove that radius of curvature at any point of a circle is constant and it is equal to the radius of circle.



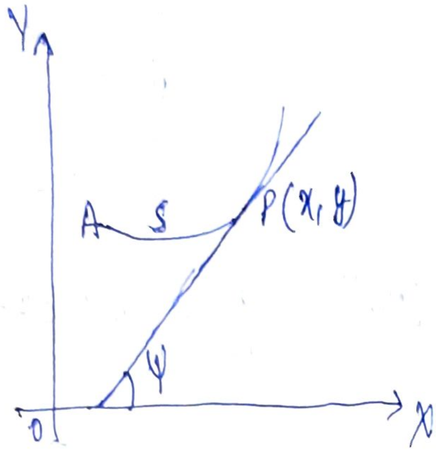
Soln:- Let A is a fixed point on the circle whose radius is a and centre is at C.
Let AX be the tangent to the circle at the point A which is a fixed line.
Let P be any point on the circle such that arc AP = s and Let tangent to the circle at P makes an angle ψ with AX

Then, $\angle ACP = \psi$

$s = a\psi$

$\therefore \frac{ds}{d\psi} = a$, which is constant & is the radius of the circle.

Formula for radius of curvature in cartesian co-ordinates :



Let A be a fixed ~~pt.~~ pt. on a curve $y = f(x)$ & $P(x, y)$ be any point on it such that arc $AP = s$ & the tangent to the curve at P makes an angle ψ with +ve x-axis then $\frac{dy}{dx} = \tan\psi$

$$\therefore \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{ds}(\tan\psi)$$

$$\cancel{\Rightarrow \frac{d^2y}{dx^2}\frac{dx}{ds} = \sec^2\psi\frac{d\psi}{ds}}$$

$$\Rightarrow \frac{d^2y}{dx^2}\cdot\frac{dx}{ds} = \sec^2\psi\,\frac{d\psi}{ds}$$

$$\Rightarrow \frac{d^2y}{dx^2}\cos\psi = \sec^2\psi \times \frac{1}{\frac{ds}{d\psi}}$$

$$\Rightarrow \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\Rightarrow \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$ which is the formula for radius of curvature

Since ρ is always taken +ve by $y_2 = \frac{d^2y}{dx^2}$ may be -ve

so in practice, $\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|}$

Note:- Since def^n of curvature is independent of the ~~curve of a~~ choice of axis.

So, we have the formula

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

~~where~~ which is suitable when $\frac{dx}{dy} = 0$ i.e, $\frac{dy}{dx}$ is infinite at the point considered.

* Formula for radius of curvature in parametric co-ordinates:-

The radius of curvature at any point of the curve $x = \phi(t), y = \psi(t)$ is given by $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'}$ where $x' = \frac{dx}{dt}, x'' = \frac{d^2x}{dt^2}$

$y' = \frac{dy}{dt}, y'' = \frac{d^2y}{dt^2}$

* Formula for radius of curvature in polar co-ordinates :-

The radius of curvature at any point (r, θ) on the polar curve $r = f(\theta)$ is given by $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$ where $r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$.

Ex-1 Find the radius of curvature at (2, 3) of the curve

$$9x^2+4y^2=36x$$

Soln:-

$$9x^2+4y^2=36x \quad \text{---(1)}$$

Diff. w.r.t. x;

$$18x+8y\frac{dy}{dx}=36$$

$$\Rightarrow 9x+4y\frac{dy}{dx}=18 \quad \text{---(2)}$$

Diff. w.r.t. x;

$$\Rightarrow 9+4\left(\frac{dy}{dx}\right)^2+4y\frac{d^2y}{dx^2}=0 \quad \text{---(3)}$$

At the point (2, 3) i.e, when $x=2$, $y=3$ from (2) & (3) we have;

$$18+12\frac{dy}{dx}=18$$

$$\Rightarrow \frac{dy}{dx}=0$$

$$9+0+12\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{-9}{12}=\frac{-3}{4}$$

$\therefore$ reqd. radius of curvature $=\rho=\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$

$$=\frac{1}{\left|-\frac{3}{4}\right|}=\frac{4}{3}$$

Q-② Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ at the point where it is cut by x-axis.

Sol^n:- The curve cuts x-axis where $y=0$.

$\therefore x = 2a$

$\therefore$ the curve cuts x axis at $(2a, 0)$

$$y^2 = \frac{4a^2(2a-x)}{x}$$

$$\Rightarrow y^2 = \frac{8a^3}{x} - 4a^2$$

Diff. w.r.t. x

$$2y\frac{dy}{dx} = -\frac{8a^3}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4a^3}{x^2 y}$$ which is undefined at $(2a, 0)$

$$\frac{dx}{dy} = \frac{-x^2 y}{4a^3}$$

$$\frac{d^2x}{dy^2} = -\frac{2xy\frac{dx}{dy}}{4a^3} - \frac{x^2}{4a^3}$$

$$= \frac{-1}{4a^3}\left(x^2 + 2xy\frac{dx}{dy}\right)$$

At the point $(2a,0)$ i.e, when $x = 2a$, $y=0$

$$\frac{dx}{dy} = 0, \quad \frac{d^2x}{dy^2} = \frac{-1}{4a^3}\cdot(4a^2+0)$$

$\therefore$ reqd. radius of curvature is $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$

$$= \frac{1}{\left|-\frac{1}{a}\right|} = a$$

Ex-(3) Find the radius of curvature at any point on the curve
$$x = a(t+\sin t),\ y = a(1-\cos t)$$

Sol^n.

$$x' = \frac{dx}{dt} = a + a\cos t,\quad x'' = \frac{d^2x}{dt^2} = -a\sin t$$

$$y' = \frac{dy}{dt} = a\sin t,\quad y'' = \frac{d^2y}{dt^2} = a\cos t$$

$\therefore$ reqd. radius of curvature is given by

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'} = \frac{\left[a^2(1+\cos t)^2 + a^2\sin^2 t\right]^{3/2}}{a(1+\cos t)(a\cos t) + a\sin t\cdot a\sin t}$$

$$= \frac{a^3\left[2+2\cos t\right]^{3/2}}{a^2(\cancel{\cos t}\cos t+1)}$$

$$= \cancel{a} \; a\times 2\sqrt{2}\,(1+\cos t)^{1/2}$$

$$= a\times 2\sqrt{2}\times\sqrt{2\cos^2 t/2}$$

$$\boxed{\rho = 4a\cos t/2}$$

Ex-(4) Find radius of curvature at any point of the curve

(i) $r = a(1-\cos\theta)$, (ii) $r = a(1+\cos\theta)$

~~Soln~~: (iii) $\frac{2a}{r} = (1+\cos\theta)$, (iv) ~~$r^n = a^n$~~ $r^2 = a^2\cos 2\theta$ at $\theta = 0$

(i) $r = a(1-\cos\theta)$

$$\therefore r_1 = \frac{dr}{d\theta} = a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = a\cos\theta$$

~~r_{12}~~

reqd. radius of curvature is:-

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{(r^2+2r_1^2-rr_2)} = \frac{[a^2(1-\cos\theta)^2+a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2+2a^2\sin^2\theta-a(1-\cos\theta)a\cos\theta}$$

$$= \frac{a^3[2-2\cos\theta]^{3/2}}{a^2(3-3\cos\theta)}$$

$$= \frac{a\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}\,a}{3}\sqrt{1-\cos\theta} \quad \text{— (1)}$$

$$= \frac{2\sqrt{2}\,a}{3} \cdot \sqrt{2\sin^2\theta/2}$$

$$\rho = \frac{4a}{3} \cdot \sin\left(\frac{\theta}{2}\right)$$

Note:-

Also $\rho = \frac{2\sqrt{2}\,a}{3} \cdot \sqrt{1-\cos\theta}$

$$= \frac{2\sqrt{2a}}{3}\sqrt{a(1-\cos\theta)} \qquad \therefore \boxed{r = a(1-\cos\theta)}$$

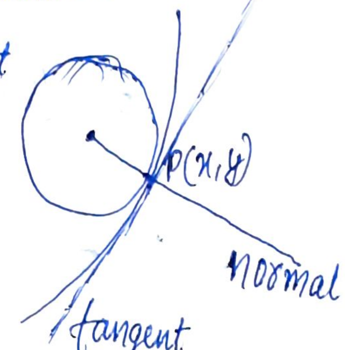
$$= \frac{2\sqrt{2a}}{3}\sqrt{r}$$

$$\therefore \rho \propto \sqrt{r}$$

* Centre of curvature & Circle of curvature :-

If ρ is the radius of curvature at a point P of a curve and C is the point on the normal to the curvature at P and C is the point on the normal to the curve at P at a distance $PC = \rho$ on the concave side of the curve, then C is called the <u>centre of curvature</u> at P and the circle with centre C & radius ρ is called the circle of curvature of the curve at P.

If $(\bar{x}, \bar{y})$ are the coordinates of the centre of curvature at the point $P(x, y)$ then the eqn of circle of curvature at P is



$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

where, $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$, where $y_1 = \frac{dy}{dx}$

$y_2 = \frac{d^2y}{dx^2}$

Q. Find the eqn of circle of curvature at the point $(3/2, 3/2)$ of the curve $x^3+y^3 = 3xy$.

Soln.

$x^3 + y^3 = 3xy$ —(i)

Diff. w.r.t. x;

$3x^2 + 3y^2 y_1 = 3y + 3x y_1$

$\Rightarrow x^2 + y^2 y_1 = y + x y_1$ —(2)

Diff. w.r.t. x;

$2x + 2y y_1^2 + y^2 y_2 = x y_2 + y_1 + y_1$

$\Rightarrow 2x + 2y y_1^2 + y^2 y_2 = x y_2 + 2y_1$ —(3)

At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ i.e, when $x = \frac{3}{2}$, $y = \frac{3}{2}$ from (2) and (3);

$\Rightarrow \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 y_1 = \frac{3}{2} y_1 + \frac{3}{2}$

$\Rightarrow \frac{1}{2} y_1 = -\frac{1}{2}$

$\Rightarrow y_1 = -1$

Also; $\cancel{2}\cdot\frac{3}{\cancel{2}} + \cancel{2}\cdot\frac{3}{\cancel{2}}(-1)^2 + \left(\frac{3}{2}\right)^2 \cdot y_2 = \left(\frac{3}{2}\right) y_2 + 2(-1)$

$\Rightarrow 3 + 3 + \frac{9}{4} y_2 = \frac{3}{2} y_2 - 2$

$\Rightarrow \frac{3}{4} y_2 = -8$

$\Rightarrow y_2 = -\frac{32}{3}$

$\therefore$ radius of curvature at $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} = \frac{(1+1)^{3/2}}{\frac{32}{3}} = \frac{3\sqrt{2}}{16}$$

If $(\bar{x}, \bar{y})$ is center of curvature at $\left(\frac{3}{2}, \frac{3}{2}\right)$

then

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= \frac{3}{2} - \frac{3}{32}\cdot\sqrt{2}$$

$$= \frac{48 - 3\sqrt{2}}{32}$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= \frac{3}{2} - \frac{3}{32}\sqrt{2} = \frac{48 - 3\sqrt{2}}{32}$$

Required eqn of circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

* Beta and Gramma function or Eulerian Integrals of first and 2nd kind:-

Beta function is denoted and defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{where } m,n>0$$

Gramma function is denoted and defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \quad \text{where } n>0$$

Properties:-

(I) $\beta(m,n) = \beta(n,m)$

i.e, Beta function is symmetrical in its arguments

Proof:- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \beta(n,m)$$

(II) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

Proof:- we know;

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta\,d\theta$

$x=0,\ \theta=0$

$x=1,\ \theta=\pi/2$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\,2\cos\theta\sin\theta\,d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

Proof - We know;

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Let $x = \frac{t}{1+t}$

$\Rightarrow x = 1 - \frac{1}{1+t}$

$dx = \frac{+1}{(1+t)^2}dt$

$x = 0 \Rightarrow t = 0$

$x \to 1 \Rightarrow t \to \infty$

$= \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(\frac{1}{1+t}\right)^{n-1}\frac{1}{(1+t)^2}dt$

$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$

$\therefore \beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}}dx$

But $\beta(m,n) = \beta(n,m)$. Hence proved.

(iv) $\Gamma 1 = 1$

Proof:- $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$\therefore \Gamma 1 = \int_0^\infty e^{-x} dx$

$= \left[-e^{-x}\right]_0^\infty$

$= 1$

$\therefore \Gamma 1 = 1$

(v) $\Gamma(n+1) = n\Gamma n = \lfloor n$ if $n \in N$

Proof:- $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} (-e^{-x}) dx$

$= 0 - 0 + n \int e^{-x} x^{n-1} dx$

$= n \Gamma n$

If n is a positive integer then,

$\Gamma(n+1) = n\Gamma n$

$= n \Gamma(n-1)+1$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\,\Gamma 1$

$= \lfloor n \quad , \because \Gamma 1 = 1$

Ex-① Prove that

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

Solⁿ: We know;

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{—} ①$$

Now, $\displaystyle\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Let $x = \frac{1}{t}$

$dx = -\frac{1}{t^2}$

$x = 1 \to t = 1$

$x \to \infty;\ t \to 0$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \cdot \frac{-1}{t^2}\,dt$$

$$= \int_0^1 \frac{t^{m+n}}{(1+t)^{m+n}\, t^{m-1+2}}\,dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \text{—} ②$$

From ① & ②;

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Ex-2 Prove that

(i) $\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$

(ii) $\dfrac{\beta(m+1, n)}{m} = \dfrac{\beta(m, n+1)}{n} = \dfrac{\beta(m, n)}{(m+n)}$

Soln. (i) $\beta(m+1, n) + \beta(m, n+1)$

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}\,dx$$

$$= \int_0^1 x^{m}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n}\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(\cancel{x} + 1 - \cancel{x})\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m, n).$$

(ii) $$\beta(m+1, n) = \int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 x^{m}(1-x)^{n-1}\,dx$$

$$= \left[-x^m \frac{(1-x)^n}{n}\right]_0^1 - \int_0^1 m x^{m-1}\left\{-\frac{(1-x)^n}{n}\right\}dx$$

$$= 0 - 0 + \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n+1-1}\,dx$$

$$\Rightarrow \beta(m+1, n) = \frac{m}{n}\beta(m, n+1)$$

$$\Rightarrow \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n}$$

$$\therefore \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)+\beta(m,n+1)}{m+n} = \frac{\beta(m,n)}{(m+n)}$$

(3) Prove that:

$$\Gamma n = \int_0^1 \log\left(\frac{1}{x}\right)^{n-1} dx$$

Let $t = \log\left(\frac{1}{x}\right)$

$\Rightarrow x = e^{-t}$

$\therefore dx = -e^{-t}dt$

$x \to 1,\ t \to 0$

$x \to 0 \Rightarrow t \to \infty$

Soln:

$$\int_0^1 \log\left(\frac{1}{x}\right)^{n-1} dx$$

$$= \int_\infty^0 -(e^{-t})\,dt\ t^{n-1}\,dt$$

$$= \int_0^\infty e^{-t}\,t^{n-1}\,dt$$

$$= \Gamma n.$$

* Relation between Beta and Gamma function —

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Q. Find the value of $\Gamma\left(\frac{1}{2}\right)$

Soln. we know

$$\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \beta(m,n)$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

when $m = n = \frac{1}{2}$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2 \times \frac{\pi}{2} = \pi$$

$$\Rightarrow \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$$

$$\Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Ex. Prove that :-

$$2^n \Gamma\left(n+\frac{1}{2}\right) = 1.3.5.\dots(2n-1)\sqrt{\pi}$$

Sol^n. $\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(n-\frac{1}{2}\right)+1$ $\quad \because \Gamma(n+1) = n\Gamma n$

$$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{3}{2}\right)+1$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\cdots\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{1.3.5\dots(2n-1)}{2^n}\sqrt{\pi}$$

$$\boxed{2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\dots(2n-1)\sqrt{\pi}}$$

Ex: Prove that

$$\int_0^a x^4\sqrt{a^2-x^2}\,dx = \frac{\pi}{3}\times\frac{a^6}{2}$$

Let $x = a\sin\theta$

$\therefore dx = a\cos\theta\,d\theta$

$x=0 \Rightarrow \theta = 0$

$x=a \Rightarrow \theta = \pi/2$

Soln: $\int_0^{\pi/2} a^4\sin^4\theta\sqrt{a^2(1-\sin^2\theta)}\,.\,a\cos\theta\,d\theta$

$$= a^6\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= \frac{a^6}{2}\times 2\int_0^{\pi/2}\sin^{2\times\frac{5}{2}-1}\theta\cos^{2\times\frac{3}{2}-1}\theta\,d\theta$$

$$= \frac{a^6}{2}\times\beta\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= \frac{a^6}{2}\times\frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{\Gamma 4} = \frac{a^6}{2}\times\frac{\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 4} = \frac{a^6}{2}\times\frac{3}{8}\times\pi\times\frac{1}{3!}$$

Q: Evaluate $\int_0^{\infty} 5^{-x^2}\, dx$

Soln: $\int_0^{\infty} 5^{-x^2}\, dx = \int_0^{\infty} e^{-x^2 \log 5}\, dx$

Note:- $e^{\log N} = N$

Let $t = x^2 \log 5$

$dt = 2x \log 5\, dx$

$= 2\sqrt{\frac{t}{\log 5}} \cdot \log 5\, dx$

$dx = \frac{dt}{2\sqrt{\log 5}} \cdot t^{-1/2}\, dt$

$= \int_0^{\infty} e^{-t} \cdot \frac{1}{2\sqrt{\log 5}}\, t^{-1/2}\, dt$

$= \frac{1}{2\sqrt{\log 5}} \int_0^{\infty} e^{-t}\, t^{-1/2}\, dt$

$= \frac{1}{2\sqrt{\log 5}} \cdot \Gamma\frac{1}{2} = \frac{1}{2\sqrt{\log 5}} \cdot \sqrt{\pi} = \frac{1}{2}\sqrt{\frac{\pi}{\log 5}}$

Q: Evaluate :- $\int_0^{\infty} \frac{x^c}{c^x}\, dx$, wher $c > 1$

Soln. $\int_0^{\infty} \frac{x^c}{c^x}\, dx$

$= \int_0^{\infty} c^{-x}\, x^c\, dx$

Let $t = x \log c$

$dt = \log c \cdot dx$

$= \int_0^{\infty} e^{-x \log c}\, x^c\, dx$

$= \int_0^{\infty} e^{-t} \left(\frac{t}{\log c}\right)^c \cdot \frac{dt}{(\log c)}$

$= \frac{1}{(\log c)^{c+1}} \int_0^{\infty} e^{-t}\, t^c\, dt = \frac{1}{(\log c)^{c+1}}\, \Gamma(c+1)$

7) Evaluate $\int_0^1 \sqrt{1-x^4}\, dx$

Let $x^2 = \sin\theta$

$2x\,dx = \cos\theta\, d\theta$

$2\sqrt{\sin\theta}\, dx = \cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

Soln. $\frac{1}{2}\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\ \frac{\cos\theta\, d\theta}{\sqrt{\sin\theta}}$

$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^2\theta\, d\theta$

$= \frac{1}{4}\times 2\int_0^{\pi/2} \sin^{-1/2}\theta \cos^2\theta\, d\theta$

$= \frac{1}{4}\times \beta\left(\frac{1}{4}, \frac{3}{2}\right)$

$= \frac{1}{4}\times \frac{\Gamma\frac{1}{4}\times\frac{1}{2}\times\sqrt{\pi}}{\Gamma\frac{7}{4}} = \frac{1}{\cancel{8}2}\,\frac{\Gamma\frac{1}{4}\sqrt{\pi}}{\frac{3}{\cancel{4}}\times\Gamma\frac{3}{4}}$

$= \frac{1}{6}\sqrt{\pi}\times\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

8) Evaluate $\int_0^1 x^m (\log x)^n\, dx$ where $m, n > -1$

Let $\log x = -t$

$\Rightarrow x = e^{-t}$, $dx = -e^{-t}dt$

$x \to 0 \Rightarrow t \to \infty$

$x = 1 \Rightarrow t = 0$

Soln. $-\int_\infty^0 e^{-mt}\times(-t)^n \times e^{-t}\, dt$

$= (-1)^n \int_0^\infty e^{-t(m+1)}\times(+t)^n\, dt$

Let $u = (m+1)t$

$du = (m+1)\,dt$

$u = 0 \Rightarrow t = 0$

$u \to \infty \Rightarrow t \to \infty$

$= (-1)^n \int_0^\infty e^{-u}\left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)}$

$= \frac{(-1)^n}{(m+1)^{n+1}}\times\int_0^\infty e^{-u} u^n\, du = \frac{(-1)^n}{(m+1)^{n+1}}\,\Gamma(n+1)$

Q. Prove that —

$$\Gamma n \, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2n}{2^{2n-1}}$$

Solⁿ we know;

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{—} (1)$$

When $m = \frac{1}{2}$;

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma n}{\Gamma n+\frac{1}{2}} = 2\int_0^{\pi/2} \cos^{2n-1}\theta \, d\theta \quad \text{—} (2)$$

When $m = n$ from (1) we get

$$\frac{\Gamma n \, \Gamma n}{\Gamma 2n} = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1} \, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} 2\theta \, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi} \sin^{2n-1} \phi \, \frac{d\phi}{2}$$

Let $\phi = 2\theta$

$\therefore d\phi = 2d\theta$

$\theta = 0 \Rightarrow \phi = 0$

$\theta \to \pi/2 \Rightarrow \phi = \pi$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi} \sin^{2n-1} \phi \, d\phi$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} \phi \, d\phi \qquad f(2a-x) = f(x)$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} \theta \, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1}\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma n \, \cancel{\Gamma n}}{\Gamma 2n} = \frac{1}{2^{2n-1}} \frac{\Gamma\frac{1}{2} \, \cancel{\Gamma n}}{\Gamma\left(\frac{1}{2}+n\right)} \quad , \text{ using } (2)$$

$$\Rightarrow \Gamma n \, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi} \times \Gamma 2n}{2^{2n-1}}$$

(Q) $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n(a+b)^m} \beta(m,n)$

Soln put $\dfrac{x}{a+bx} = \dfrac{y}{a+b} \Rightarrow x = \dfrac{y}{a+b(1-y)}$

$$dx = \frac{\{a+b(1-y)\}dy + y\{+b\,dy\}}{\{a+b(1-y)\}^2} = \frac{(a+b)\,dy}{\{a+b(1-y)\}^2}$$

$$\therefore \int_0^1 \frac{\left\{\frac{y}{a+b(1-y)}\right\}^{m-1}\left\{1-\frac{y}{a+b(1-y)}\right\}^{n-1}}{\left(\frac{y}{a+b}\right)^{m+n}} \times \frac{a(a+b)\,dy}{\{a+b(1-y)\}^2} = \int_0^1 \frac{y^{m-1}}{\{a+b(1-y)\}^{m-1}} \cdot \left\{\frac{a+b-by-y}{a+b(1-y)}\right\}$$

$$= \int_0^1 \frac{(ay)^{m-1}\left[(1-y)(a+b)\right]^{n-1}}{a(a+b)^{m+n-1}} dy$$

$$= \frac{1}{a^n(a+b)^m} \int_0^1 y^{m-1}(1-y)^{n-1} dy$$

$$= \frac{1}{a^n(a+b)^m} \beta(m,n)$$

$$* \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n(a+b)^m}\beta(m,n)$$

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b} \Rightarrow x = \dfrac{ay+bxy}{a+b}$

$$x = \frac{ay}{a+b(1-y)}$$

$$dx = \frac{(a+b(1-y))\,dy + yb\,dy}{(a+b(1-y))^2} \times a$$

$$= \frac{a(a+b)\,dy}{(a+b(1-y))^2}$$

$$= \int_0^1 \frac{\left(\frac{y}{(a+b(1-y))}\right)^{m-1}\left(1-\frac{y}{(a+b(1-y))}\right)^{n-1}}{\cancel{\left(\frac{y}{a+b}\right)^{m+n}}\left(\frac{a(a+b)}{a+b(1-y)}\right)^{m+n}} \times \frac{a(a+b)\,dy}{(a+b(1-y))^2}$$

$$= \int_0^1 \frac{\frac{y^{m-1}}{\cancel{(a+b(1-y))^{m-1}}}\frac{(a+b-by-y)^{n-1}}{\cancel{(a+b(1-y))^{n-1}}}}{\cancel{\left(\frac{y}{a+b}\right)^{m+n}}\left(\frac{a(a+b)}{\cancel{a+b(1-y)}}\right)^{m+n}} \times \frac{a(a+b)\,dy}{\cancel{(a+b(1-y))^2}}$$

$$= \int_0^1 \frac{y^{m-1}\,[(1-y)(a+b)]^{n-1}}{\{a(a+b)\}^{m+n-1}}\,dy$$

$$= \frac{1}{a^n(a+b)^m}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$= \frac{1}{a^n(a+b)^m}\beta(m,n)$$