I N D E X

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STD.: Ist SEM DIV./SEC.: SUBJECT: MATHS

S. No.	Date	Title	Page No.	Teacher's Sign/Remarks

* Hyperbolic functions :-

→ Hyperbolic cosine and Hyperbolic sine functions are real valued functions of a real variable x denoted & defined as :-

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \forall x \in R$$

where; $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots + \frac{x^n}{\lfloor n} + \dots \infty$

$$e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \dots \frac{(-1)^n x^n}{\lfloor n} + \dots \infty$$

→ Also hyperbolic tangent & hyperbolic secant functions are denoted & defined as ;

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad \forall x \in R$$

→ Lastly hyperbolic cotangent & hyperbolic cosecant functions are denoted & defined as;

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}; \quad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\forall x \in R - \{0\} \text{ i.e, } 0 \neq x \in R$$

* Series for cosh x & sinh x :-

1) $\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} + \dots \infty \; \{\cosh x \geq 1\}$

2) $\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots + \infty$

NOTE:- $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \dots \infty$

$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \dots \infty$

* Some Important results :-

1) $\cosh^2 x - \sinh^2 x = 1$ 2) $\cosh^2 x + \sinh^2 x = \cosh 2x$

3) $\operatorname{sech}^2 x + \tanh^2 x = 1$ 4) $\coth^2 x - \operatorname{cosech}^2 x = 1$

Proof:- ① $\cosh^2 x - \sinh^2 x$

$$= \left\{\frac{1}{2}(e^x + e^{-x})\right\}^2 - \left\{\frac{1}{2}(e^x - e^{-x})\right\}^2$$

$$= \frac{1}{\cancel{4}} \times \cancel{4} \times e^x \times e^{-x}$$

$$= 1$$

② $\cosh^2 x + \sinh^2 x = \cosh 2x$

$$= \left\{\frac{1}{2}(e^x + e^{-x})\right\}^2 + \left\{\frac{1}{2}(e^x - e^{-x})\right\}^2$$

$$= \frac{1}{4} \times 2 \times (e^{2x} + e^{-2x})$$

$$= \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x$$

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5) $(\cosh x + \sinh x) = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x$

6) $(\cosh x - \sinh x) = e^{-x}$

7) $(\cosh x + \sinh x)^n = (e^x)^n$

$$= e^{nx}$$

$$= \cosh nx + \sinh nx$$

8) $(\cosh x - \sinh x)^n = (\cosh nx - \sinh nx)$

* Inverse hyperbolic functions:-

If $\sinh u = x$ then; $u = \sinh^{-1}x$ which is called inverse hyperbolic sine of x.

Similarly $\cosh^{-1}x$, $\tanh^{-1}x$ etc. are defined.

* De Moivre's theorem :-

If n is an integer & θ is a real number then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where $i = \sqrt{-1}$ and if n is a rational number but not an integer, then $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$, the total number of values being equal to the denominator of n.

This is De Moivre's th^m for rational index.

If n is an irrational number then $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$, the total number of values being infinite.

* Derivative of hyperbolic function:-

1) $\frac{d}{dx}\cosh x = \sinh x$

2) $\frac{d}{dx}\sinh x = \cosh x$

3) $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$

4) $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

5) $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

6) $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$

Proof:- 1) $\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

2) $\frac{d}{dx}\operatorname{cosech} x = \frac{d}{dx}\frac{1}{\sinh x} = -\frac{1}{\sinh^2 x}\cosh x$

$= -\operatorname{cosech} x \cdot \coth x$

* Inverse hyperbolic functions & their deriatives :-

If $\sinh u = x$ then $u = \sinh^{-1} x$

1) $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$; 2) $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$

3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1-x^2}$; (4) $\frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2}$

5) $\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$; (6) $\frac{d}{dx}\operatorname{cosech}^{-1}x = \frac{-1}{x\sqrt{1+x^2}}$

Proof:- (1) Let $y = \sinh^{-1}x$

$$x = \sinh y$$

$$\therefore \frac{dx}{dy} = \cosh y = \sqrt{1+\sinh^2 y}$$

$$= \sqrt{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$$

(6) $y = \operatorname{cosech}^{-1}x$

$$x = \operatorname{cosech} y = \frac{1}{\sinh y}$$

$$\frac{dx}{dy} = \frac{-1}{\sinh^2 y} \times \cosh y$$

$$= -\operatorname{cosech} y \coth y$$

$$= -x\sqrt{1+x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{1+x^2}}}$$

* <u>Successive derivatives :-</u>

If $y=f(x)$ is a differentiable function of x then,

$y_1 = Dy = \frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, is called first derivative of y w.r.t. x and if it is also a differentiable function of x then,

$y_2 = D^2y = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is 2^{nd} derivative of y w.r.t. x.

Similarly,

$y_3 = D^3y = f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called 3^{rd} derivative of y w.r.t. x.

In general,

$y_n = D^n y = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$ is called n^{th} derivative of y w.r.t. x

where $D = \frac{d}{dx}$ is called <u>the differential operator.</u>

* n^{th} derivatives of some particular or standard n^{th} derivatives :-

① If $y = (ax+b)^m$ then;

$y_n = a^n m(m-1)(m-2)\dots(m-n+1)(ax+b)^{m-n}$

$= \begin{cases} a^n \dfrac{\lfloor m}{\lfloor m-n} (ax+b)^{m-n} & \text{if } m \text{ is a +ve integer} > n \\ a^n \lfloor n & \text{if } m = n \\ 0 & \text{if } m \text{ is a +ve integer} < n. \end{cases}$

In particular if $y = x^n$ then $y_n = \lfloor n$.

Proof:- ① $y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1} \cdot a$

$y_2 = am\cancel{(ax+b)}(m-1)(ax+b)^{m-2} a$

$= a^2 m(m-1)(ax+b)^{m-2}$

$y_3 = a^2 m(m-1)(m-2)(ax+b)^{m-3} a$

$= a^3 m(m-1)(m-2)(ax+b)^{m-3}$

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$y_n = a^n m(m-1)(m-2)(m-3)\ldots(m-n+1)(ax+b)^{m-n}$

If m is a +ve integer $>n$ then;

$$m(m-1)(m-2)\ldots\ldots(m-n+1) = {}^mP_n = \frac{\lfloor m}{\lfloor m-n}$$

$$\therefore \; y_n = a^n \frac{\lfloor m}{\lfloor m-n}(ax+b)^{m-n}$$

② $y = \dfrac{1}{ax+b}$ then

$$y_n = \frac{(-1)^n a^n \lfloor n}{(ax+b)^{n+1}}$$

In particular if $y = \dfrac{1}{x+1}$, then $y_n = \dfrac{(-1)^n \lfloor n}{(x+1)^{n+1}}$

Proof:- $y = \dfrac{1}{ax+b}$

$y_1 = \dfrac{-1 \cdot a}{(ax+b)^2} = (-1)a(ax+b)^{-2}$

$y_2 = (-1)a(-2)(ax-b)^{-3} a$

$= (-1)^2 a^2 \lfloor 2 (ax-b)^{-3}$

$y_3 = (-1)^2 a^2 \lfloor 2 (-3)(ax-b)^{-4} \cdot a$

$= (-1)^3 a^3 \lfloor 3 (ax+b)^{-4}$

In general, $y_n = (-1)^n a^n \lfloor n \, (ax+b)^{-(n+1)}$

$$\boxed{y_n = \frac{(-1)^n a^n \lfloor n}{(ax+b)^{n+1}}}$$

(3) If $y = \log(ax+b)$ then

$$y_n = \frac{(-1)^{n-1} a^n \lfloor n-1}{(ax+b)^n}$$

In particular if if $y = \log(1+x)$ then;

$$y_n = \frac{(-1)^{n-1} \lfloor n-1}{(1+x)^n}$$

Proof:-

$$y = \log(ax+b)$$

$$\therefore y_1 = \frac{1}{ax+b} \cdot a$$

$$y_2 = a\left\{\frac{-1}{(ax+b)^2} \cdot a\right\} = (-1)a^2 (ax+b)^{-2}$$

$$y_3 = (-1)a^2 (-2) \cdot (ax+b)^{-3} \cdot a$$

$$= (-1)^2 a^3 \lfloor 2 \, (ax+b)^{-3}$$

$$y_n = (-1)^{n-1} \lfloor n-1 \, (ax+b)^{-n}$$

$$\boxed{y_n = \frac{(-1)^{n-1} \lfloor n-1}{(ax+b)^n}}$$

④ If $y = e^{ax+b}$ then $y_n = a^n e^{ax+b}$

In particular if $y = e^{-x}$ then $y_n = (-1)^n e^{-x}$

⑤ (i) If $y = \sin(ax+b)$ then

$$y_n = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

(ii) If $y = \cos(ax+b)$ then;

$$y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

In particular if $y = \cos x$;

then $y_n = \cos\left(\frac{n\pi}{2} + x\right)$

Proof:- (ii) $y = \cos(ax+b)$

$$\Rightarrow y_1 = -\sin(ax+b)\cdot a$$

$$= a\cos\left(\frac{\pi}{2} + ax + b\right)$$

$$\Rightarrow y_2 = -a\cdot\sin\left(\frac{\pi}{2} + ax + b\right)\cdot a$$

$$= -a^2 \cos\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right)$$

$$\Rightarrow y_2 = -a^2\cos\left(\frac{2\pi}{2} + ax + b\right)$$

In general; $y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(6) (i) If $y = e^{ax+K} \sin(bx+c)$ then;

$$y_n = (a^2+b^2)^{n/2} e^{ax+K} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

(ii) If $y = e^{ax+K} \cos(bx+c)$ then;

$$y_n = (a^2+b^2)^{n/2} e^{ax+K} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Proof:- (ii) $y = e^{ax+\cancel{b}K} \cos(bx+c)$

$$y_1 = ae^{ax+b} \cos(bx+c) - e^{ax+K} b\sin(bx+c)$$

$$= e^{ax+b} \{a\cos(bx+c) - b\sin(bx+c)\}$$

Let $a = r\cos\theta$, $b = r\sin\theta$;

So, that $r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$; $\theta = \tan^{-1}\frac{b}{a}$

$$\therefore y_1 = e^{ax+K}[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)]$$

$$= re^{ax+K}[\cos(bx+c+\theta)]$$

$$\therefore y_2 = r[ae^{ax+K}\cos(bx+c+\theta) - e^{ax+K}.b\sin(bx+c+\theta)]$$

$$= re^{ax+K}[r\cos\theta\cos(bx+c+\theta) - r\sin\theta\sin(bx+c+\theta)]$$

$$= r^2e^{ax+K}\cos\{(bx+c+\theta)+\theta\}$$

$$\Rightarrow y_3 = r^2e^{ax+K}\cos(bx+c+2\theta)$$

In general,

$$y_n = r^n e^{ax+\cancel{b}K}\cos(bx+c+n\theta)$$

$$\boxed{y_n = (a^2+b^2)^{n/2} e^{ax+K}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)}$$

Ex. 1. Find the n^{th} derivative w.r.t. x of:-

(i) $\sin^2 x$, (ii) $\cos^3 2x$, (iii) $\cos x \sin 2x \cos 3x$

(iv) $\cos^3 x \sin^2 x$, (v) $e^{4x}\sin 3x$ (vi) $e^{-3x}\cos(4x-1)$

(vii) $e^x \sin x \sin 2x$, (viii) $\log(2-x-3x^2)$, (ix) $\dfrac{2x-1}{2x^2-3x-2}$

(x) $\dfrac{1}{x^2-a^2}$ (xi) $\dfrac{1}{(x^2+a^2)}$

(i) Let $y = \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\therefore y_n = \frac{1}{2}\left[0 - 2^n \cos\left(\frac{n\pi}{2}+2x\right)\right]$$

$$= -2^{n-1}\cos\left(\frac{n\pi}{2}+2x\right)$$

(ii) Let $y = \cos^3 2x = \frac{1}{4}\left[\cos 6x + 3\cos 2x\right]$

$$y_n = \frac{1}{4}\left[6^n \cos\left(\frac{n\pi}{2}+6x\right) + 3\cdot 2^n \cos\left(\frac{n\pi}{2}+2x\right)\right]$$

$$= 2^{n-2}\left[3^n \cos\left(\frac{n\pi}{2}+6x\right) + 3\cos\left(\frac{n\pi}{2}+2x\right)\right]$$

(iii) Let $y = \cos x \sin 2x \cos 3x = \frac{1}{2}\cdot(2\cos 3x \cos x)\sin 2x$

$$= \frac{1}{2}\cdot\left[\cos(3x+x)+\cos(3x-x)\right]\sin 2x$$

$$= \frac{1}{2}\cdot\left[2\cos 4x \sin 2x + 2\sin 2x \cos 2x\right]$$

$$= \frac{1}{2}\left[\sin 6x + \sin 2x + \sin 4x\right]$$

$$y_n = \frac{1}{4}\left[6^n \sin\left(\frac{n\pi}{2}+6x\right) + 2^n \sin\left(\frac{n\pi}{2}+2x\right) + 4^n \sin\left(\frac{n\pi}{2}+4x\right)\right]$$

$$= 2^{n-2}\left[3^n \sin\left(\frac{n\pi}{2}+6x\right) + \sin\left(\frac{n\pi}{2}+2x\right) + 2^n \sin\left(\frac{n\pi}{2}+4x\right)\right]$$

(iv) Let $y = \cos^3 x \sin^2 x = \frac{1}{4}(2\sin x \cos x)^2 \cdot \cos x$

$= \frac{1}{4}\sin^2 2x \cos x$

$\Rightarrow y = \frac{1}{8}(1 - \cos 4x)\cos x$

$= \frac{1}{8}\cos x - \frac{1}{16}\cdot 2\cos 4x \cos x$

$= \frac{1}{8}\cos x - \frac{1}{16}(\cos 5x + \cos 3x)$

$y_n = \frac{1}{8}\cdot \cos\left(\frac{n\pi}{2} + x\right) - \frac{1}{16}\cdot 5^n \cos\left(\frac{n\pi}{2} + 5x\right) - \frac{1}{16}\cdot 3^n \cos\left(\frac{n\pi}{2} + 3x\right)$

(v) Let $y = e^{4x}\sin 3x$

or $y = e^{ax+K}\sin(bx+c)$

Here; $a=4$, $K=0$, $b=3$, $c=0$;

$y_n = (a^2+b^2)^{n/2} e^{ax+K}\sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

$= (3^2+4^2)^{n/2} e^{4x}\sin\left(3x + n\tan^{-1}\frac{3}{4}\right)$

$= 5^n e^{4x}\sin\left(3x + n\tan^{-1}\frac{3}{4}\right)$

(vi) $e^{-3x}\cos(4x-1)$

or $y = e^{ax+K}\cos(bx+c)$

(vii) $y = e^x \sin x \sin 2x$

$= \frac{1}{2} \times e^x \times 2 \sin 2x \sin x$

$= \frac{1}{2} e^x \left[\cos(2x - x) - \cos(2x + x)\right]$

$= \frac{1}{2} e^x \left(\cos x - \cos 3x\right)$

$= \frac{1}{2}\left(e^x \cos x - e^x \cos 3x\right)$

$\therefore y_n = \frac{1}{2}\left[(1^2+1^2)^{n/2} e^x \cos(x + n\tan^{-1} 1) - (1^2+3^2)^{n/2} e^x \cos(3x + n\tan^{-1} 3)\right]$

$= \frac{1}{2} e^x \left[2^{n/2} \cos\left(x + \frac{n\pi}{4}\right) - 10^{n/2} \cos(3x + n\tan^{-1} 3)\right]$

(viii) Let $y = \log(2 - x - 3x^2)$

$y = \log(2 - 3x + 2x - 3x^2)$

$= \log\{1(2-3x) + x(2-3x)\}$

$= \log(2-3x) + \log(1+x)$

$y_n = \frac{(-1)^{n-1} \underline{|n-1}}{(1+x)^n} + \frac{(-1)^{n-1} (-3)^n \underline{|n-1}}{(2-3x)^n}$

$= (-1)^{n-1} \underline{|n-1} \left(\frac{1}{(1+x)^n} + \frac{(-1)^n 3^n}{(2-3x)^n}\right)$

(ix) $y = \frac{2x-1}{2x^2 - 3x - 2}$

$= \frac{(2x-1)}{(x-2)(2x+1)}$

Let $\dfrac{2x-1}{(x-2)(2x+1)} = \dfrac{A}{x-2} + \dfrac{B}{2x+1}$ —— (1)

$\Rightarrow 2x-1 = A(2x+1) + B(x-2)$ —— (2)

which is an identity.

Putting $x=2$ & $x=-\frac{1}{2}$ in (2) we get;

$\Rightarrow 3 = 5A \Rightarrow A = \dfrac{3}{5}$

& $-2 = B\left(-\dfrac{1}{2}-2\right) = -\dfrac{5}{2}B$

$\Rightarrow B = \dfrac{4}{5}$

$\therefore y = \dfrac{3/5}{x-2} + \dfrac{4/5}{2x+1}$

$y_n = \dfrac{3}{5} \cdot \dfrac{(-1)^n |\underline{n}}{(x-2)^{n+1}} + \dfrac{4}{5} \cdot \dfrac{(-1)^n 2^n |\underline{n}}{(2x+1)^{n+1}}$

$= \dfrac{(-1)^n |\underline{n}}{5}\left[\dfrac{3}{(x-2)^{n+1}} + \dfrac{2^{n+2}}{(2x+1)^{n+1}}\right]$

(x) Let $y = \dfrac{1}{x^2-a^2}$

$= \dfrac{1}{(x+a)(x-a)}$

$= \dfrac{(x+a)-(x-a)}{2a(x+a)(x-a)}$

$= \dfrac{1}{2a}\left(\dfrac{1}{x-a} - \dfrac{1}{x+a}\right)$

$\therefore y_n = \dfrac{1}{2a}\left[\dfrac{(-1)^n |\underline{n}}{(x-a)^{n+1}} - \dfrac{(-1)^n |\underline{n}}{(x+a)^{n+1}}\right]$

$= \dfrac{(-1)^n |\underline{n}}{2a}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}}\right]$

(XI) Let $y = \dfrac{1}{x^2+a^2} = \dfrac{1}{(x+ia)(x-ia)}$

$$= \frac{(x+ia)-(x-ia)}{2ia(x+ia)(x-ia)}$$

$$= \frac{1}{2ia}\left(\frac{1}{x+ia} - \frac{1}{x-ia}\right)$$

$$\therefore y_n = \frac{1}{2ia}\left[\frac{(-1)^n \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ia)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right] \quad \text{---(1)}$$

Let $x = r\cos\theta$, $a = r\sin\theta$

$r = \sqrt{a^2+x^2}$, $\theta = \tan^{-1}\dfrac{a}{x}$

$$\therefore (x-ia)^{-(n+1)} = \{r(\cos\theta - i\sin\theta)\}^{-(n+1)}$$

$$\Rightarrow (x-ia)^{-(n+1)} = r^{-(n+1)}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right] \quad \text{---(2)}$$

Similarly; $(x+ia)^{-(n+1)} = r^{-(n+1)}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$ ---(3)

From (1), (2) & (3);

$$y_n = \frac{(-1)^n \lfloor n}{2ia} \cdot r^{-(n+1)}\, 2i \sin(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{a} \cdot \frac{1}{r^{(n+1)}} \sin(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{a(a^2+x^2)^{\frac{n+1}{2}}} \sin\left\{(n+1)\tan^{-1}\frac{a}{x}\right\}$$

Ex-(2). If $y = \sin ax + \cos ax$; show that
$$y_n = a^n \{1 + (-1)^n \sin 2ax\}^{1/2}$$

Soln. $y = \sin ax + \cos ax$

$$y_n = a^n \sin\left(\frac{n\pi}{2} + ax\right) + a^n \cos\left(\frac{n\pi}{2} + ax\right)$$
$$= a^n \left[\sin\left(\frac{n\pi}{2} + ax\right) + \cos\left(\frac{n\pi}{2} + ax\right)\right]$$

$$= a^n [\sin\theta + \cos\theta] \quad ; \text{ where } \theta = \frac{n\pi}{2} + ax$$
$$= a^n \left[(\sin\theta + \cos\theta)^2\right]^{1/2}$$
$$= a^n \left[(1 + \sin 2\theta)\right]^{1/2}$$
$$= a^n \left[1 + \sin(n\pi + 2ax)\right]^{1/2}$$
$$= a^n \left[1 + (-1)^n \sin 2ax\right]$$

Ex-(3). If $y = x^{2n}$; show that
$$y_n = 2^n \{1.3.5 \ldots (2n-1)\} x^n$$

Sol.
$$y = x^{2n}$$
$$y_1 = 2n\, x^{2n-1}$$
$$y_2 = 2n(2n-1)\, x^{2n-2}$$
$$y_3 = 2n(2n-1)(2n-2)\, x^{2n-3}$$
& so on

$$y_n = 2n(2n-1)(2n-2) \ldots \{2n-(n-1)\}\, x^{2n-n}$$
$$= 2n(2n-1) \ldots (n+1) x^n$$
$$= \frac{2n(2n-1) \ldots (n+1)\, n(n-1) \ldots 3.2.1}{n(n-1) \ldots 3.2.1} x^n$$

$$= \frac{1\cdot 2\cdot 3\cdot 4 \cdots (2n-1)\,2n}{\lfloor n} \times x^n$$

$$= \frac{\{1\cdot 3\cdot 5 \cdots (2n-1)\}\; 2\cdot 4\cdot 6 \cdots 2n}{\lfloor n}\, x^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)\cdot 2^n \cancel{\lfloor n}}{\cancel{\lfloor n}} \times x^n$$

* Leibnitz's Theorem :-

If $y = UV$ where U and V are functions of x possessing derivatives w.r.t. x upto nth order term then $y_n = (UV)_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \dots + {}^nC_r U_{n-r} V_r + \dots + {}^nC_{n-1} U_1 V_{n-1} + U V_n$

By actual diff ; we get ;

$$y_1 = \frac{dy}{dx} = d(UV) = \frac{dU}{dx} \cdot V + U \cdot \frac{dV}{dx}$$

$$\Rightarrow y_1 = U_1 V + U V_1$$

Also, $y_2 = \frac{dy_1}{dx} = \frac{d}{dx}(U_1 V + U V_1)$

$$= \left(\frac{dU_1}{dx} V + U_1 \frac{dV}{dx}\right) + \left(\frac{dU}{dx} V_1 + U \frac{dV_1}{dx}\right)$$

$$= U_2 V + U_1 V_1 + U_1 V_1 + U V_2$$

$$\Rightarrow y_2 = U_2 V + {}^2C_1 U_{2-1} V_1 + U V_2$$

These shows that the theorem is true for $n=1$ & $n=2$,

Let the theorem is true for $n = m \in N$ then

$$y_m = U_m V + {}^mC_1 U_{m-1} V_1 + {}^mC_2 U_{m-2} V_2 + \dots + {}^mC_{m-1} U_1 V_{m-1} + U V_m$$

Differentiating both sides w.r.t. x ;

$$\Rightarrow y_{m+1} = (U_{m+1} V + U_m V_1) + {}^mC_1 (U_m V_1 + U_{m-1} V_2) + {}^mC_2 (U_{m-1} V_2 + U_{m-2} V_3) + \dots + {}^mC_{m-1}(U_2 V_{m-1} + U_1 V_m) + (U_1 V_m + U V_{m+1})$$

$$\Rightarrow y_{m+1} = U_{m+1} V + (1 + {}^mC_1) U_m V_1 + ({}^mC_1 + {}^mC_2) U_{m-1} V_2 + ({}^mC_2 + {}^mC_3) U_{m-2} V_3 + \dots + ({}^mC_{m-1} + 1) U_1 V_m + U V_{m+1}$$

$$\Rightarrow y_{m+1} = U_{m+1} V + {}^{m+1}C_1 U_{m+1-1} V_1 + {}^{m+1}C_2 U_{m+1-2} V_2 + \dots + {}^{m+1}C_{m+1-1} U_1 V_{m+1-1} + U V_{m+1}$$

This shows that the theorem is true for $n = m$.

$\left(\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r \;\; \& \;\; {}^mC_0 = {}^mC_m = 1\right)$

Hence, by the method of induction, it is true for any +ve integer n.

i.e, $y_n = (UV)_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \dots + {}^nC_{n-1} U_1 V_{n-1} + U V_n$

Ex:- Using Leibnitz's thm; find n^{th} derivative w.r.t. x of :-
(i) $x^3 \sin x$, (ii) $x^2 \cos 3x$, (iii) $x^3 e^{ax}$ (iv) $x^{n-1} \log x$ (v) $e^{a\sin^{-1}x}$ at $x=0$.

Soln. (i) Let $y = x^3 \sin x$

$= (\sin x) x^3$

$= UV$ (Say) where $U = \sin x$, $V = x^3$

$\therefore y_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \dots + U V_n$

$= \sin\left(\frac{n\pi}{2} + x\right) \cdot x^3 + {}^nC_1 \cdot \sin\left\{(n-1)\frac{\pi}{2} + x\right\} \cdot 3x^2 + {}^nC_2 \cdot \sin\left\{(n-2)\frac{\pi}{2} + x\right\}$

$+ {}^nC_3 \sin\left\{(n-3)\frac{\pi}{2} + x\right\} \cdot 6$

$= x^3 \sin\left(\frac{n\pi}{2} + x\right) + 3nx^2 \sin\left\{(n-1)\frac{\pi}{2} + x\right\} + 3n(n-1)x \sin\left\{(n-2)\frac{\pi}{2} + x\right\}$

$+ n(n-1)(n-2) \sin\left\{(n-3)\pi/2 + x\right\}$

(iv) Let $y = x^{n-1} \log x$ —(i)

$y_1 = x^{n-1} \cdot \frac{1}{x} + (n-1)x^{(n-2)} \cdot \log x$

$\Rightarrow x y_1 = x^{n-1} + (n-1) x^{n-1} \cdot \log x$

$\Rightarrow x y_1 = x^{n-1} + (n-1) y$ using (1)

Diff. $(n-1)$ times w.r.t. x & using Leibnitz's thm:-

$x y_n + {}^{n-1}C_1 \cdot 1 \cdot \cancel{y_{n-1}} = \lfloor n-1 + \cancel{(n-1) y_{n-1}}$

$\Rightarrow x y_n = \lfloor n-1$

$\Rightarrow \boxed{y_n = \frac{\lfloor n-1}{x}}$

(v) Let $y = e^{a\sin^{-1}x}$ — (1) ~~at x=0~~

Let y -

Sol: $y_1 = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = e^{a\sin^{-1}x} \times a$

$\Rightarrow \sqrt{1-x^2}\, y_1 = ay$ — (2) using (1)

Diff. w.r.t. x;

$\sqrt{1-x^2}\, y_2 + y_1 \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} \times (-\cancel{2}x) = ay_1$

$\Rightarrow (1-x^2) y_2 - x y_1 = a y_1 \sqrt{1-x^2}$

$\Rightarrow (1-x^2) y_2 - x y_1 - a \cdot ay$ using (2)

$\Rightarrow (1-x^2) y_2 - x y_1 - a^2 y = 0$ — (3)

Diff. n times w.r.t. x ; using Leibnitz theorem, we get ;

$\left\{(1-x^2) y_{n+2} + {}^nC_1 (-2x) y_{n+1} + {}^nC_2 (-2) y_n\right\} - \left(x y_{n+1} + {}^nC_1 \cdot 1 \cdot y_n\right) - a^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - n(n-1) y_n - x y_{n+1} - n y_n - a^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0$ — (4)

When $x=0$; from (1), (2), (3), (4) ; we get :-

$y = e^{a\sin^{-1}0} = e^0 = 1$

$y_1 = ay = a$

$y_2 = a^2 y = a^2$

$y_{n+2} - (n^2+a^2) y_n = 0$

$\Rightarrow y_{n+2} = (n^2+a^2) y_n$ — (5)

∴ Replacing n by 1, 2, 3 ... in (5) ; we get;

$y_3 = (1^2 + a^2) y_1 = a(a^2+1^2)$

$y_4 = (2^2 + a^2) y_2 = a^2(a^2+2^2)$

$y_5 = (3^2+a^2)\, y_3 = a(a^2+1)(a^2+3^2)$

$y_6 = (4^2+a^2)\, y_4 = a(a^2+2^2)(a^2+4^2)$

~~$y_7 = ((5^2+a$~~ & so on;

In general;

$$y_n(0) = \begin{cases} a(a^2+1^2)(a^2+3^2)(a^2+5^2)+\dots+\{a^2+(n-2)^2\} & \text{if } n \text{ is odd} \\ a^2(a^2+2^2)(a^2+4^2)(a^2+6^2)+\dots+\{a^2+(n-2)^2\} & \text{if } n \text{ is even} \end{cases}$$

Ex:- If $y = \tan^{-1}x$ show that

$$(1+x^2)\, y_{n+1} + 2nx\, y_n + n(n-1)\, y_{n-1} = 0$$

Soln. Let $y = \tan^{-1}x$ — (1)

$y_1 = \dfrac{1}{1+x^2} \Rightarrow (1+x^2)\, y_1 = 1$ — (2)

Now, diff. n times using leibnitz theorem;

$(1+x^2)\, y_{n+1} + {}^nC_1 (2x)\, y_n + {}^nC_2 (2)\, y_{n-1} = 0$

$\Rightarrow \boxed{(1+x^2)\, y_{n+1} + 2nx\, y_n + n(n-1)\, y_{n-1} = 0}$

Ex:- If $y = \sin(m\sin^{-1}x)$, show that $(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (m^2-n^2)\, y_n = 0$

Also find $(y_n)_0$ i.e, value of y_n when $x=0$

Soln Let $y = \sin(m\sin^{-1}x)$ — (1)

$y_1 = \cos(m\sin^{-1}x) \times \dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow y_1 \times \sqrt{1-x^2} = m\cos(m\sin^{-1}x)$ — (2)

$\Rightarrow y_2 \times \sqrt{1-x^2} + y_1 \times \dfrac{(-2x)}{2\sqrt{1-x^2}} = -m\sin(m\sin^{-1}x) \times \dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\, y_2 - x y_1 = -m^2 y$

$\Rightarrow (1-x^2)\, y_2 - x y_1 + m^2 y = 0$

Diff. n times w.r.t. x & using Leibnitz's thm:-

$$\left\{(1-x^2)\,y_{n+2} + {}^nC_1\cdot(-2x)\,y_{n+1} + {}^nC_2\,(-2)\,y_n\right\} - \left(x\,y_{n+1} + {}^nC_1\cdot 1\cdot y_n\right) + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - 2nx\,y_{n+1} - n(n-1)\,y_n - x\,y_{n+1} - n\,y_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)\,y_n = 0 \quad \text{——} (4)$$

When $x=0$ from (1), (2), (3) & (4) we get —

$$y=0,\quad y_1=m,\quad y_2+m^2y=0 \Rightarrow y_2=0$$

$$y_{n+2} + (m^2-n^2)\,y_n = 0$$

$$\Rightarrow y_{n+2} = (n^2-m^2)\,y_n \quad \text{——} (5)$$

Replacing n by 1, 2, 3, ... in (5) we get;

$$y_3 = (1^2-m^2)\,y_1 = m(1^2-m^2)$$

$$y_4 = (2^2-m^2)\,y_2 = 0$$

$$y_5 = (3^2-m^2)\,y_3 = m(1^2-m^2)(3^2-m^2)$$

$$y_6 = (4^2-m^2)\,y_4 = 0$$

$$y_7 = (5^2-m^2)\,y_5 = m(1^2-m^2)(3^2-m^2)(5^2-m^2)$$

In general,
$$(y_n)_0 = \begin{cases} 0 & \text{if } n \text{ is even} \\ m(1^2-m^2)(2^2-m^2)\cdots\{(n-2)^2-m^2\} & \text{if } n \text{ is odd.} \end{cases}$$

Ex:- If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ show that

$$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0$$

Sol^n:-

$$\frac{y}{b} = \cos\left\{\log\left(\frac{x}{n}\right)^n\right\}$$

$$y = b\cos\left\{\log\left(\frac{x}{n}\right)^n\right\}$$

$$y = b\cos\{n(\log x - \log n)\} \quad \text{--- (1)}$$

$$y_1 = -b\sin\{n(\log x - \log n)\} \times n \times \frac{1}{x}$$

$$y_1 = -\frac{nb}{x}\sin\{n(\log x - \log n)\}$$

$$\Rightarrow y_1 \times x = -nb\sin\{n(\log x - \log n)\}$$

$$\Rightarrow y_2 \times x + y_1 = -nb\cos\{n(\log x - \log n)\} \times \frac{n}{x}$$

$$\Rightarrow x^2 y_2 + x y_1 = -n^2 y \text{ - using (1)}.$$

$$\Rightarrow x^2 y_2 + x y_1 + n^2 y = 0 \quad \text{--- (2)}$$

diff. n times w.r.t. x & using leibnitz theorem;

$$\Rightarrow x^2 y_{n+2} + {}^nC_1 (2x)\, y_{n+1} + {}^nC_2 (2)\, y_n + x y_{n+1} + y_n + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + n(n-1)\, y_n + {}^nC_1 y_n + n^2 y_n$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0$$

Ex:- If $f(x) = \tan x$ show that

$$f^{(n)}(0) - {}^nC_2 f^{(n-2)}(0) + {}^nC_4 f^{(n-4)}(0) - \dots = \sin \frac{n\pi}{2}$$

Soln: Let $y = f(x) = \tan x$

$\Rightarrow y = \frac{\sin x}{\cos x}$

$\Rightarrow y \cos x = \sin x$

diff. n times w.r.t. x & using Leibnitz's theorem;

$$y_n \cos x + {}^nC_1 y_{n-1}(-\sin x) + {}^nC_2 y_{n-2}(-\cos x) + {}^nC_3 y_{n-3} \sin x + {}^nC_4 y_{n-4} \cos x - \dots - y \cos(\tfrac{n\pi}{2}+x) = \sin(\tfrac{n\pi}{2}+x)$$

When $x = 0$;

$$y_n - {}^nC_2 y_{n-2} + {}^nC_4 y_{n-4} - \dots = \sin\left(\frac{n\pi}{2}\right)$$

or; $f^{(n)}(0) - {}^nC_2 f^{(n-2)}(0) + {}^nC_4 f^{(n-4)}(0) - \dots = \sin\left(\frac{n\pi}{2}\right)$

Ex. If $y = e^x \log x$ show that

$$x y_{n+2} + (n-2x+1) y_{n+1} - (2n-x+1) y_n + n y_{n-1} = 0$$

Soln. $y = e^x \log x$ — (1)

$\therefore y_1 = e^x \log x + e^x \cdot \frac{1}{x}$

$\Rightarrow y_1 = y + \frac{e^x}{x}$ — using (1)

$\Rightarrow x y_1 = xy + e^x$ — (2)

Diff. w.r.t. x;

$x y_2 + y_1 = x y_1 + y + e^x$

$= x y_1 + y + (x y_1 - xy)$ using (2)

$\Rightarrow x y_2 + (1-2x) y_1 + (x-1) y = 0$ — (3)

Diff n times w.r.t. & using Leibnitz theorem;

$$(x y_{n+2} + {}^nC_1 y_{n+1}) + (1-2x) y_{n+1} - {}^nC_1 (2) y_n + (x-1) \cdot y_n + {}^nC_1 \cdot 1 \cdot y_{n-1} = 0$$

$$\Rightarrow x y_{n+2} + n y_{n+1} + (1-2x) y_{n+1} - 2n y_n + (x-1) y_n + n y_{n-1} = 0$$

* Rolle's theorem :-

Let $f:[a,b] \to R$ is a function such that

(i) f is continuous in the closed interval $[a,b]$

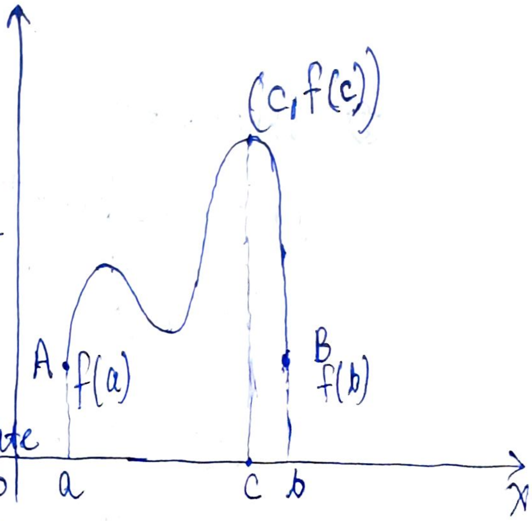
(ii) f is differentiable in the open interval (a,b)
i.e, $f'(x)$ exists $\forall\ x \in (a,b)$ i.e, $a<x<b$

(iii) $f(a) = f(b)$

then $\exists$ at least one $c \in (a,b)$ i.e; $a<c<b$ such that $f'(c)=0$

* Geometrical meaning :-

If $y=f(x)$ is a continuous curve & joining the points $A(a,f(a))$ & $B(b,f(b))$ with equal ordinates i.e, $f(a)=f(b)$, and the curve possesses tangent at each intermediate point betn A and B then Rolle's theorem geometrically says that at least one tangent to curve at some intermediate point $(c, f(c))$ is parallel to x axis.



Proof:- f is continuous in $[a,b]$ implies f is bounded and attains its bounds at least once in $[a,b]$ so $\exists$ (there exists) two points c & d in $[a,b]$ such that

$f(c) = M =$ exact upper bond
$=$ Supremum
$=$ least upper bound (lub)

& $f(d) = m =$ exact lower bound
$=$ infimum
$=$ greatest lower bound (glb)

of f in $[a,b]$

Case-I :- If $M = m = K$ (say)

then $f(x) = K \ \forall \ x \in [a,b]$

$\therefore f'(x) = 0$

$\therefore$ the theorem is trivially proved in this case.

Case-II :- If $M \neq m$

$\therefore$ either $f(a) = f(b) \neq M$ or $f(a) = f(b) \neq m$

Let $f(a) = f(b) \neq M$

$\Rightarrow f(a) = f(b) \neq f(c)$

$\therefore a \neq c$ & $b \neq c$

$\therefore a < c < b$

We claim $f'(c) = 0$

By def^n of Supremum;

$f(x) \leq M \quad \forall x \in [a,b]$

$\Rightarrow f(x) \leq f(c)$

$\Rightarrow f(c+h) \leq f(c)$ where $x = c+h \in [a,b]$

$\therefore f(c+h) - f(c) \leq 0$

$$\therefore \frac{f(c+h) - f(c)}{h} \begin{cases} \leq 0 & \text{if } h > 0 \\ \geq 0 & \text{if } h < 0 \end{cases}$$

$$\therefore \underset{h \to 0^+}{Lt} \frac{f(c+h) - f(c)}{h} \leq 0 \ \& \ \underset{h \to 0^-}{Lt} \frac{f(c+h) - f(c)}{h} \geq 0$$

$\therefore Rf'(c) \leq 0$ & $Lf'(c) \geq 0$

Since $f'(c)$ exists. So, we must have

$Rf'(c) = Lf'(c)$

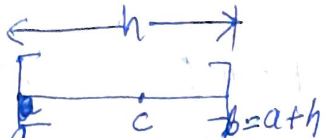
$\therefore f'(c) = 0$

Similarly if $f(a) = f(b) \neq m = f(d)$

Then we get $f'(d) = 0$ where $a < d < b$.

Hence the theorem.

Lec. 10



* **Note :-** If $b = a+h$ and $a < c < a+h$, then

$$c = a + \theta h \text{ for some } \theta \text{ with } 0<\theta<1$$

$$\therefore\ f'(c) = 0 \text{ becomes } f'(a+\theta h) = 0$$

* Lagrange's mean value theorem :-

Let $f : [a,b] \to \mathbb{R}$ is a function which is

(i) continuous in the closed interval $[a,b]$

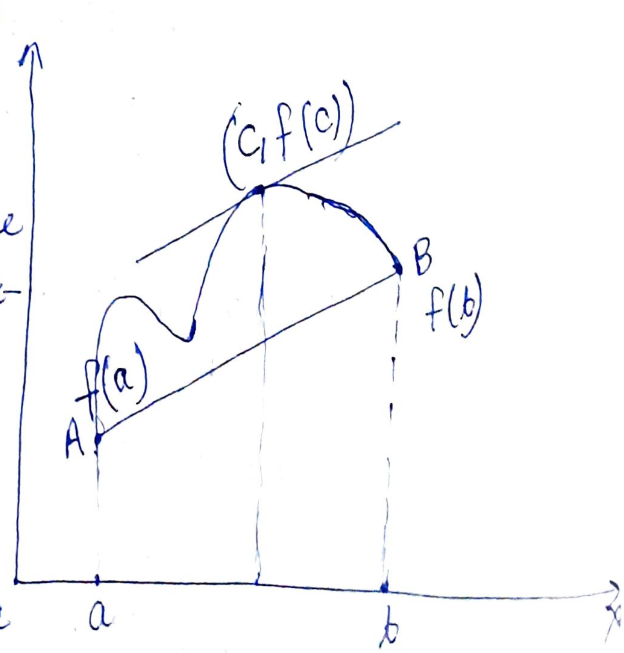
(ii) differentiable in the open interval (a,b)

i.e., $f'(x)$ exists $\forall\, x \in (a,b)$

then $\exists$ at least one point $c \in (a,b)$ i.e., $a<c<b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

* Geometrical interpretation :-



If $y = f(x)$ is a continuous curve joining the points $A(a, f(a))$ & $B(b, f(b))$ and the curve possesses a tangent at each intermediate point betn A & B, then LMV thm geometrically says that the tangent to the curve at some intermediate point $(c, f(c))$ is parallel to the ~~curve~~ ~~at some intermediate point~~ chord AB joining the end pts of the curve.

Proof:- We define a new function ϕ as follows

$$\phi(x) = f(x) + Kx \quad \forall\ x \in [a,b]$$

where K is a suitable constant chosen such that

$$\phi(a) = \phi(b)$$

or $f(a) + Ka = f(b) + Kb$

or $K(a-b) = f(b) - f(a)$

$$K = -\frac{f(b)-f(a)}{(b-a)}$$

With this choice of K we see that

(i) ϕ is continuous in $[a,b]$ being sum of two continuous function.

(ii) $\phi'(x) = f'(x) + K$ exists ~~for all~~ $\forall\ x \in (a,b)$
i.e., ϕ is differentiable in (a,b)

(iii) $\phi(a) = \phi(b)$

$\therefore$ ϕ satisfied all condition of Rolle's thm and therefore $\exists$ at least one point $c \in (a,b)$ i.e, $a<c<b$ such that

$$f'(c) = 0$$

$$\Rightarrow f'(c) + K = 0$$

$$\Rightarrow -K = f'(c)$$

$$\Rightarrow \frac{f(b)-f(a)}{(b-a)} = f'(c)$$

Note:- If $b = a+h$, then $c = a+\theta h$ where $0<\theta<1$

$$\therefore \frac{f(b)-f(a)}{(b-a)} = f'(c) \text{ becomes}$$

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$$

$$\Rightarrow \boxed{f(a+h) = f(a) + hf'(a+\theta h)}$$

* Cauchy's Mean Value theorem

Let f and g are two functions of x which are both continuous in closed interval $[a,b]$ and differentiable in open interval (a,b) and also

$g'(x) \neq 0 \;\forall\; x \in (a,b)$ then $\exists$ at least one point $c \in (a,b)$ i.e, $a<c<b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof:- If $g(a) = g(b)$ then g satisfies all the cond^{n}s of Rolle's theorem and so $\exists$ at least one point $\xi \in (a,b)$ such that $g'(\xi) = 0$ which is against the hypotheses $g'(x) \neq 0 \;\forall\; x \in (a,b)$ and hence in this case $g(a) \neq g(b)$

we define a new function ϕ as below;

$$\phi(x) = f(x) + Kg(x) \ \forall \ x \in [a,b]$$

where K is a suitable constant value chosen such that

$$\phi(a) = \phi(b)$$

ys

or $f(a) + Kg(a) = f(b) + Kg(b)$

or $K[g(a) - g(b)] = f(b) - f(a)$

or $K = -\dfrac{f(b) - f(a)}{g(b) - g(a)}$ —— (1)

with this choice of K we see that

(i) ϕ is continuous in $[a,b]$ being the sum of two continuous $f^n s$

(ii) $\phi'(x) = f'(x) + Kg'(x)$ exists $\forall \ x \in (a,b)$ i.e, ϕ is differentiable in (a,b)

(iii) $\phi(a) = \phi(b)$.

$\therefore$ ϕ satisfies all condition of Rolle's theorem and therefore $\exists$ at least one point $c \in (a,b)$ such that;

$$\phi'(c) = 0$$

$$f'(c) + Kg'(c) = 0$$

$$-K = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Ex-1. Verify Rolle's theorem for

(i) $f(x) = x(x-2)^2$ in $[0, 2]$

(ii) $f(x) = x^2(1-x^2)$ in $[0,1]$

(iii) $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

(i) $f(x) = x(x-2)^2, \forall x \in [0, 2]$

$\Rightarrow f(x) = x^3 - 4x^2 + 4x$

which is a polynomial of deg. 3

& so, it is continuous.

$f'(x) = 3x^2 - 8x + 4$ exists $\forall x \in (0, 2)$

$f(0) = f(2) = 0$;

$\therefore$ f satisfies all the condition of Rolle's theorem

To verify $\exists\ c \in (0, 2)$ such that $f'(c) = 0$

Now, $f'(x) = 0$

$\Rightarrow 3x^2 - 8x + 4 = 0$

$\Rightarrow 3x^2 - 6x - 2x + 4 = 0$

$\Rightarrow (x-2)(3x-2) = 0$

$\Rightarrow x = 2$ or $x = 2/3$

Here, $c = 2/3$ is such that $0 < c < 2$ and $f'(c) = 0$

$\therefore$ Rolle's th^m is verified.

(iii) $\cancel{f(x) = x^2 - x^4}$

$f(x) = x(x+3)\, e^{-x/2} \quad \forall\, x \in [-3, 0]$

$= (x^2 + 3x)\, e^{-x/2}$

which is continuous in $[-3, 0]$ being product of two continuous function.

$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}(x^2+3x)\cdot e^{-x/2}$

$= \frac{1}{2} e^{-x/2}(4x + 6 - x^2 - 3x)$

$= \frac{1}{2} e^{-x/2}(6 + x - x^2)$

exists $\forall\, x \in (-3, 0)$

$f(0) = 0, \quad f(-3) = 0$

$\therefore f(0) = f(-3)$

$\therefore$ f satisfies all the conditions of Rolle's thm.

NOW, $f'(x) = 0$

$\Rightarrow 6 + x - x^2 = 0$

$\Rightarrow x = -2$ or $x = 3$

Here $c = -2$

such that $-3 < c < 0$

for which $f'(c) = 0$

$\therefore$ Rolle's theorem satisfied.

Ex:- Is Rolle's th^m applicable to the function

(i) $f(x) = |x|$, in $[-1,1]$

(ii) $f(x) = \tan x$ in $[0,\pi]$

(iii) $f(x) = x^2 - 4x + 1$ in $[-2,2]$

(iv) $f(x) = 1-(x-1)^{2/3}$ in $[0,2]$

(v) $f(x) = (x-a)^m (b-x)^n$ in $[a,b]$ where m & n are constants

(i) $f(x) = |x|$ is not differentiable at $x=0 \in (-1,1)$
because $Lf'(0) = -1$, $Rf'(0) = 1$
so, Rolle's th^m can not be applied.

(ii) $f(x) = \tan x$ is not defined at $x = \pi/2$ so Rolle's th^m can not be applied

(iii) $f(-2) = 4+8+1 = 13$
$f(2) = 4-8+1 = -3$
$\therefore f(-2) \neq f(2)$.
$\therefore$ Rolle's theorem can not be applied.

(iv) $f'(x) = -\frac{2}{3}(x-1)^{-1/3}$ does not exist at $x=1$.
$\therefore$ Rolle's th^m can not be applied.

(v) If $m<0$ or $n<0$ then $f(x)$ is undefined at $x=a$ or $x=b$ respectively

If $m=0$ or $n=0$ then $f(a) \neq f(b)$.

If $m>0$ & $n>0$ then
$f(x) = (x-a)^m (b-x)^n$ is defined and continuous

$$f'(x) = m(x-a)^{m-1}(b-x)^n + (x-a)^m \cdot n(b-x)^{n-1}(-1)$$

$$= (x-a)^{m-1}(b-x)^{n-1}\{m(b-x) - n(x-a)\}$$

$$= (x-a)^{m-1}(b-x)^{n-1}\{mb + na - (n+m)x\}$$

which exists $\forall\, x \in (a,b)$

$f(a) = 0 = f(b)$

$\therefore$ all the cond^{n}s of Rolle's th^m are satisfied.

$\therefore$ Rolle's th^m can be applied for this function.

if $m > 0,\ n > 0$

Note:- Now $f'(x) = 0$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{mb + na - (n+m)x\} = 0$$

$$\Rightarrow x = a \text{ or } x = b \text{ or } x = \left(\frac{mb+na}{m+n}\right)$$

Here $c = \dfrac{mb+na}{m+n}$ is such that

$a < c < b$. for which $f'(c) = 0$

So, Rolle's th^m is verified.

Ex:- If f, g, ϕ are continuous functions in $[a,b]$ & they are derivable in (a,b) then show that $\exists$ a point $\xi \in (a,b)$ such that

$$\begin{vmatrix} f(a) & f(b) & f'(\xi) \\ g(a) & g(b) & g'(\xi) \\ \phi(a) & \phi(b) & \phi'(\xi) \end{vmatrix} = 0$$

Sol^n: We consider the function F defined by

$$F(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ \phi(a) & \phi(b) & \phi(x) \end{vmatrix} \quad \forall\, x \in [a,b]$$

which being of the form $A f(x) + B g(x) + C \phi(x)$ is continuous in $[a,b]$

$$F'(x) = \begin{vmatrix} f(a) & f(b) & f'(x) \\ g(a) & g(b) & g'(x) \\ \phi(a) & \phi(b) & \phi'(x) \end{vmatrix}$$ which exists $\forall\ x \in (a,b)$

$$F(a) = \begin{vmatrix} f(a) & f(b) & f(a) \\ g(a) & g(b) & g(a) \\ \phi(a) & \phi(b) & \phi(a) \end{vmatrix} = 0 \ ; \ \because\ C_1 = C_3$$

$F(b) = 0$ since $C_2 = C_3$

$\therefore\ F(a) = F(b)$

So, $F(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore\ \exists$ a point $\xi \in (a,b)$ such that $F'(\xi) = 0$

$$\Rightarrow \begin{vmatrix} f(a) & f(b) & f'(\xi) \\ g(a) & g(b) & g'(\xi) \\ \phi(a) & \phi(b) & \phi'(\xi) \end{vmatrix} = 0$$

Ex:- If $0<\alpha<\beta<\pi/2$ prove that $\exists\ \theta$ with $\alpha<\theta<\beta$ such that
$$\begin{vmatrix} \sin\alpha & \cos\alpha & \tan\alpha \\ \sin\beta & \cos\beta & \tan\beta \\ \cos\theta & -\sin\theta & \sec^2\theta \end{vmatrix} = 0$$

Sol^n:- Let $f(x) = \begin{vmatrix} \sin\alpha & \cos\alpha & \tan\alpha \\ \sin\beta & \cos\beta & \tan\beta \\ \sin x & \cos x & \tan x \end{vmatrix}$ $\forall\ x \in [\alpha, \beta]$

Since, $\sin x, \cos x, \tan x$ are continuous in $[\alpha, \beta]$

$f(x)$ is also continuous in $[\alpha, \beta]$

$f'(x) = \begin{vmatrix} \sin\alpha & \cos\alpha & \tan\alpha \\ \sin\beta & \cos\beta & \tan\beta \\ \cos x & -\sin x & \sec^2 x \end{vmatrix}$ which exists $\forall\ x \in (\alpha, \beta)$

$f(\alpha) = 0$; $\because R_1 = R_3$ $\begin{vmatrix} \sin\alpha & \cos\alpha & \tan\alpha \\ \sin\beta & \cos\beta & \tan\beta \\ \sin\alpha & \cos\alpha & \tan\alpha \end{vmatrix}$

$f(\beta) = 0$; $\because R_2 = R_3$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem

So, $\exists$ a point θ with $\alpha<\theta<\beta$ such that $f'(\theta) = 0$

$$\Rightarrow \begin{vmatrix} \sin\alpha & \cos\alpha & \tan\alpha \\ \sin\beta & \cos\beta & \tan\beta \\ \cos\theta & -\sin\theta & \sec^2\theta \end{vmatrix} = 0$$

Ex: Using Rolle's theorem prove that following function has not two different real roots.

$$2x^3+3x^2+6x-23=0$$

Sol^n:- given eq^n is

$$2x^3+3x^2+6x-23=0 \quad \text{---(i)}$$

Let $f(x) = 2x^3+3x^2+6x-23$

If possible let the eq^n (i) has two diff. roots α & β with $\alpha<\beta$.

Then $f(\alpha)=0=f(\beta)$

Clearly f is continuous in $[\alpha,\beta]$

$f'(x) = 6x^2+6x+6$

$= 6(x^2+x+1)$ exists $\forall\ x\in(\alpha,\beta)$

$\therefore$ all the conditions of Rolle's th^m are satisfied

$\therefore$ $\exists$ at least one real number c with $\alpha<c<\beta$ such that $f'(c)=0$

$\Rightarrow 6(c^2+c+1)=0$

$\Rightarrow c^2+c+1=0$

which is not possible for real numbers c.

So, our assumption is wrong.

The function has not two different real roots.

Ex: Verify LMV th^m for

(i) $f(x) = 2x^2 - 7x + 2$ in $[2, 5]$

(ii) $f(x) = \sqrt{x}$ in $[4, 9]$

(iii) $f(x) = e^{-x}$ in $[0, 1]$

Sol-(ii) $f(x) = \sqrt{x}$ is continuous in $[4, 9]$

$f'(x) = \frac{1}{2\sqrt{x}}$ exists $\forall x \in (4, 9)$

$\therefore$ f satisfies all condition of LMV th^m.

Here $a = 4$, $b = 9$.

So, we need to verify $\exists c$ with $4 < c < 9$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{f(9) - f(4)}{9 - 4}$$

$$\Rightarrow \sqrt{c} = \frac{5}{2}$$

$$\Rightarrow c = \frac{25}{4} \in (4, 9)$$

$\therefore c = \frac{25}{4}$ is such that $a < c < b$ for which $\frac{f(b) - f(a)}{b - a} = f'(c)$

$\therefore$ LMV th^m is verified.

(iii) $f(x) = e^{-x}$ is continuous in $[0, 1]$

$f'(x) = -e^{-x}$ exist in $(0, 1)$

$\therefore$ f satisfies all cond^ns of LMV th^m.

Here $a = 0$, $b = 1$.

So, we need to verify $\exists c$ with $0 < c < 1$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{f(1)-f(0)}{1} = -e^{-c}$$

$$\Rightarrow e^{-1}-1 = -\frac{1}{e^{c}}$$

$$\Rightarrow \frac{1}{e^{c}} = 1-\frac{1}{e}$$

$$\Rightarrow e^{c} = \frac{e}{e-1}$$

$\Rightarrow c = \log\left(\frac{e}{e-1}\right)$ which is such that $0<c<1$.

for which $\frac{f(b)-f(a)}{b-a} = f'(c)$ holds

$\therefore$ LMV th^m is verified.

Ex:- Prove that

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

if $b>a>0$. Hence prove that

$$\frac{3}{25}+\frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6}+\frac{\pi}{4}.$$

Soln- We consider $f(x) = \tan^{-1}x \ \forall x \in [a,b]$

which is continuous in $[a,b]$

$f'(x) = \frac{1}{1+x^2}$ exists $\forall x \in (a,b)$

$\therefore$ $f(x)$ satisfies both the conditions of LMV th^m.

So, $\exists\ c \in (a,b)$ i.e, $a<c<b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} = \frac{1}{1+c^2} \quad \text{--- (i)}$$

We have $a < c < b$

$\therefore \quad a^2 < c^2 < b^2 \quad \because \; b > a > 0$

$1+a^2 < 1+c^2 < 1+b^2$

$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \quad \text{--- (ii)}$$

from (i) & (ii);

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \quad \text{proved.}$$

Eg:- In the mean value theorem

$f(a+h) = f(a) + hf'(a+\theta h)$, $0 < \theta < 1$

if $f(x) = \sqrt{x}$, $a = 1$, $h = 3$ then find θ.

Solⁿ: $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$\therefore f(a+h) = f(a) + hf'(a+\theta h)$

$\Rightarrow f(1+3) = f(1) + 3f'(1+3\theta)$

$$\Rightarrow \sqrt{4} = \sqrt{1} + 3 \times \frac{1}{2\sqrt{1+3\theta}}$$

$$\Rightarrow \frac{1}{3} = \frac{1}{2\sqrt{1+3\theta}}$$

$$\Rightarrow \theta = \frac{5}{12}$$

Q.2) If f is continuous in $[a,b]$ & differentiable in (a,b) then prove that f is strictly monotonic

(i) increasing in $[a,b]$ if $f'(x)>0 \ \forall x\in(a,b)$

(ii) decreasing in $[a,b]$ if $f'(x)<0 \ \forall x\in(a,b)$

Solⁿ:- Let $x_1, x_2 \in [a,b]$ such that $x_2 > x_1$

then $[x_1, x_2] \subseteq [a,b]$

~~there~~ $\therefore$ by LMV thᵐ $\exists$ c with $x_1 < c < x_2$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$$

(i) If $f'(x)>0 \ \forall x\in(a,b)$

then $f'(c)>0$

$$\Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$$

$\Rightarrow f(x_2)-f(x_1)>0, \ \because x_2-x_1>0$

$\Rightarrow f(x_2)>f(x_1) \ \forall x_2>x_1$ in $[a,b]$

$\therefore f(x)$ is strictly monotic increasing in $[a,b]$

(ii) If $f'(x)<0 \ \forall x\in(a,b)$

then $f'(c)<0$

$$\Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} < 0$$

$\Rightarrow f(x_2)-f(x_1)<0, \ \because x_2-x_1>0$

$\Rightarrow f(x_2)<f(x_1) \ \forall x_2>x_1$ in $[a,b]$

$\therefore f(x)$ is strictly monotic decreasing in $[a,b]$

NOTE:-

If $f'(x)>0$ $(f'(x)<0)$ $\forall\, x\in(a,b)$ except at a finite number of points x where $f'(x)=0$ then also f is strictly increasing (decreasing) in $[a,b]$

Ex:- Prove that:-

(i) $\frac{\sin x}{x}$ decreases strictly in $0<x\leq \pi/2$

(ii) $\frac{\tan x}{x}$ increases strictly in $0<x<\pi/2$

Soln. (i) $f(x)=\frac{\sin x}{x}$, $0<x\leq\pi/2$

$$f'(x)=\frac{x\cos x-\sin x}{x^2}$$

Let $\phi(x)=x\cos x-\sin x$; $0\leq x\leq \pi/2$

$\therefore\ \phi'(x)=\cos x - x\sin x-\cos x$
$= -x\sin x<0\ \forall\ x\in\left(0,\frac{\pi}{2}\right)$

$\therefore\ \phi(x)$ is strictly decreasing in $\left[0,\frac{\pi}{2}\right]$

~~$\phi(x)>0\ \forall\ x\in[0,\pi/2]$~~

$\therefore\ \phi(x)<\phi(0)\ \forall\, x>0$ in $0<x\leq\frac{\pi}{2}$

$\Rightarrow x\cos x-\sin x<0$

$\Rightarrow \frac{x\cos x-\sin x}{x^2}<0$

$\Rightarrow f'(x)<0\ \forall\ x$ in $0<x\leq\pi/2$

$\therefore\ f(x)=\frac{\sin x}{x}$ decreases strictly in $0<x\leq\pi/2$

(ii) Let $f(x)=\frac{\tan x}{x}$; $0<x<\pi/2$

$$f'(x)=\frac{x\sec^2x-\tan x}{x^2}$$

Let $\phi(x)=x(\sec^2x)-\tan x$; $0\leq x<\pi/2$

$\phi'(x)=\sec^2x+x\cdot 2\sec x\cdot\sec x\tan x-\sec^2 x$
$=2x\sec^2x\tan x>0\ \forall\ x\in(0,\pi/2)$

$\phi(x)$ is strictly increasing in $0\leq x<\pi/2$

$\therefore \phi(x) > \phi(0) \; \forall x > 0$ in $0 < x < \pi/2$

$\Rightarrow x\sec^2 x - \tan x > 0$

$\Rightarrow \dfrac{x\sec^2 x - \tan x}{x^2} > 0$

$\Rightarrow f'(x) > 0 \; \forall \; x$ in $0 < x < \pi/2$

$\therefore f(x) = \dfrac{\tan x}{x}$ increases strictly in $0 < x < \pi/2$.

Ex:- Verify Cauchy's mean value thm for

(i) $f(x) = e^x$, $g(x) = e^{-x}$ in $x \in [a,b]$

(ii) $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x^2}$ in $[a,b]$ where $0 < a < b$

(iii) $f(x) = \sqrt{x}$, $g(x) = \dfrac{1}{\sqrt{x}}$ " " "

(i) $f(x) = e^x$, $g(x) = e^{-x} = \dfrac{1}{e^x}$

are continuous in $[a,b]$

$f'(x) = e^x$, $g'(x) = -e^{-x}$ exists $\forall x \in (a,b)$

Also, $g'(x) = -\dfrac{1}{e^x} \neq 0 \; \forall x \in (a,b)$

So, f and g satisfies all the condn of CMV thm.

To verify $\exists$ c with $a < c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\text{or, } \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

$$\text{or, } \frac{e^{(a+b)}(e^b - e^a)}{e^a - e^b} = -e^{2c}$$

$$\text{or, } -e^{(a+b)} = -e^{2c}$$

or, $a+b = 2c$

or, $c = \frac{a+b}{2}$ which is s.t. $a < c < b$

for which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \text{ holds}$$

(11) $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$

are continuous in $[a,b]$; $0 < a < b$

$f'(x) = -\frac{1}{x^2}$, $g'(x) = -\frac{2}{x^3}$ exists $\forall x \in (a,b)$

Also, $g'(x) = -\frac{2}{x^3} \neq 0 \ \forall x \in (a,b)$

$\therefore$ f and g satisfies all the condn of CMV thm

$$\text{Now, } \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\text{or, } \frac{\frac{1}{b} - \frac{1}{a}}{\frac{1}{b^2} - \frac{1}{a^2}} = \frac{-\frac{1}{c^2}}{-\frac{2}{c^3}}$$

$\Rightarrow c = \frac{2ab}{a+b}$ which is HM between a and b.

$\therefore a < c < b$ for which $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ holds.

(III)

Ex:- Deduce the result

$$f(b) - f(a) = \xi f'(\xi) \log \frac{b}{a}$$

under suitable condition to be stated by you.

Solⁿ. Let ~~f(x)~~ f is continuous in $[a,b]$ & diff. in (a,b)

Also, let $g(x) = \log x \ \forall x \in [a,b]$

which is defined if $0 < a < b$

under this condition, g is also continuous in $[a,b]$

$g'(x) = \frac{1}{x}$ exists $\forall x \in (a,b)$

Also, $g'(x) \neq 0 \ \forall x \in (a,b)$

$\therefore$ f and g satisfies all the condⁿ. of CMV thᵐ

$\therefore \exists \ \xi$ with $a < \xi < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

$$\Rightarrow \frac{f(b)-f(a)}{\log b - \log a} = \frac{f'(\xi)}{\frac{1}{\xi}}$$

$$\Rightarrow f(b)-f(a) = \xi f'(\xi) \log \frac{b}{a}.$$

Ex:- prove that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$

for some θ with $0 < \alpha < \theta < \beta < \pi/2$

Solⁿ: Let $f(x) = \sin x$ & $g(x) = \cos x$

$\forall x \in [\alpha, \beta]$ where $0 < \alpha < \beta < \pi/2$

Clearly f & g are continuous in $[\alpha, \beta]$

$f'(x) = \cos x$, $g'(x) = -\sin x$

exists $\forall x \in (\alpha, \beta)$

Also, $g'(x) = -\sin x \neq 0 \ \forall x \in (\alpha, \beta)$

$\therefore$ f and g satisfies all condⁿs of CMV thᵐ;

$\therefore \exists \theta$ with $\alpha < \theta < \beta$ such that

$$\Rightarrow \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

$$\Rightarrow \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$

Ex:- Show that

$\sin x < x < \tan x$ if $0 < x < \pi/2$

Solⁿ:- Let $f(x) = x - \sin x \ \forall x$ in $0 \leq x < \pi/2$

$f'(x) = 1 - \cos x > 0$ in $0 < x < \pi/2$

$\therefore f$ is strictly increasing in $0 \leq x < \pi/2$

$\therefore f(x) > f(0) \ \forall x > 0$ in $0 < x < \pi/2$

$\Rightarrow x - \sin x > 0$

$\Rightarrow \sin x < x \quad \text{—(i)}$

Let $f(x) = \tan x - x$ in $0 \leq x < \pi/2$

$f'(x) = \sec^2 x - 1 = \tan^2 x > 0$ in $0 < x < \pi/2$

$\therefore$ f is strictly increasing in $0 \leq x < \pi/2$

$\therefore$ $f(x) > f(0)$ $\forall$ $x > 0$ in $0 < x < \pi/2$

$\tan x - x > 0$

$\Rightarrow \tan x > x$ —— (2)

Combining (1) & (2),

$\sin x < x < \tan x$ $\forall$ $x \in (0, \pi/2)$

Taylor's Theorem (generalised mean value theorem) with Lagrange's form of remainder ~~theorem~~ term:-

If $f:[a,a+h]\to R$ is a function such that

(i) f and all its derivatives upto $(n-1)^{th}$ order are continuous in $[a,a+h]$

(ii) the n^{th} derivative $f^{(n)}(x)$ exists $\forall\, x\in(a,a+h)$

then $\exists$ a real number θ with $0<\theta<1$ such that

$$f(a+h)=f(a)+\frac{h}{\lfloor 1}f'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\dots+\frac{h^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\frac{h^n}{\lfloor n}f^{(n)}(a+\theta h)$$

The $(n+1)$th term in the RHS is denoted by R_n and it is

$R_n=\frac{h^n}{\lfloor n}f^{(n)}(a+\theta h)$, which is called Lagrange's form of remainder term in Taylor's th^m.

Note:- (1) If $n=1$, then Taylor's theorem is nothing but Lagrange's mean value theorem

If $n=2$, then $f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a+\theta h)$

which is called mean value th^m of 2^{nd} order.

(2) If f possesses continuous derivatives of every order n in $[a,a+h]$ & $R_n\to 0$ as $n\to\infty$ then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\dots\infty$$

which is known as <u>Taylor's infinite series.</u>

If also $a+h=x$ so that $h=x-a$ then

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\frac{(x-a)^3}{\lfloor 3}f'''(a)+\dots$$

which is <u>Taylor's expansion of $f(x)$ in an infinite series</u>

in power of $(x-a)$ in the neighbour hood of the point $x=a$.

In particular if $a=0$ then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots \infty$$

which is known as Maclaurin's expansion of $f(x)$ in power of x in an infinite series in the neighbourhood of $x=0$.

(3) If $a+h=x$ & $a=0$ so that $h=x$ in Taylor's theorem then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$$

which is known as Maclaurin's Theorem with Lagrange's form of remainder form $R_n = \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$; $0<\theta<1$

Maclaurin's Theorem :- If $f : [0,h] \longrightarrow R$ is a function such that —

(i) $f, f', f'', \ldots, f^{(n-1)}$ are all continuous in $[0,h]$

& (ii) $f^{(n)}(x)$ exists $\forall\, x \in (0,h)$

then for every $x \in (0,h]$ i.e, $0 < x \leq h$ $\exists$ a real number θ with $0<\theta<1$ such that

$$\underbrace{f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \frac{x^n}{\lfloor n} f^{(n)}(\theta x)}$$

where $R_n = \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$ is Lagrange's form of remainder term

Ex-1. Expand (i) $\sin x$, (ii) $\cos x$, (iii) $\cosh x$, (iv) $\sinh x$, (v) e^{ax} in a finite series as well as an infinite series in ascending power of x.

Soln. (i) Let $f(x) = \sin x$, $f(0) = 0$

$\therefore f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{iv}(x) = \sin x \quad f^{iv}(0) = 0$

$f^{v}(x) = \cos x \quad f^{v}(0) = 1$

& so on

In general $f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right)$

which exists $\forall x \in R$ & for every +ve integer

$\therefore$ by Maclaurin's theorem $\exists\, \theta$ with $0<\theta<1$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{iv}(0) + \frac{x^5}{\lfloor 5} f^{v}(0) + \dots + \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$$

$$\Rightarrow \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots + \frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right)$$

which is reqd. expansion of $\sin x$ in a finite series

Hence, $R_n = \frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right)$

$$\therefore |R_n| = \left|\frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right)\right| = \left|\frac{x^n}{\lfloor n}\right| \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$\Rightarrow |R_n| \leq \left|\frac{x^n}{\lfloor n}\right| \quad \because \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \leq 1$$

Also, $\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0 \quad \forall x \in R$

because $\sum_{n=0}^{\infty} \frac{x^n}{\lfloor n} = e^x$ is a convergent series

$\therefore R_n \to 0$ as $n \to \infty$

$\therefore$ all the conditions of Maclaurin's infinite expansions are satisfied.

$\therefore \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \dots \infty$

which is reqd expansion of $\sin x$ in an infinite series in power of x.

(iii) Let $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$, $f(0) = 1$

$f'(x) = \sinh x = \frac{e^x - e^{-x}}{2}$, $f'(0) = 0$

$f''(x) = \cosh x$, $f''(0) = 1$

$f'''(x) = \sinh x$, $f'''(0) = 0$

$f^{IV}(x) = \cosh x$, $f^{(IV)}(0) = 1$

& so on

In general, $f^{(n)}(x) = \begin{cases} \cosh x & \text{if } n \text{ is even} \\ \sinh x & \text{if } n \text{ is odd} \end{cases}$

which exists $\forall x \in R$ & for every +ve integer n.

$\therefore$ by Maclaurin's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \dots + \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$$

$$\Rightarrow \cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots + R_n \quad \text{— (1)}$$

when $R_n = \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$, $0 < \theta < 1$

$$= \begin{cases} \frac{x^n}{\lfloor n} \cosh \theta x & \text{if } n \text{ is even} \\ \frac{x^n}{\lfloor n} \sinh \theta x & \text{if } n \text{ is odd.} \end{cases}$$

which is reqd. expansion of $\cosh x$ in a finite series

Also $\underset{n\to\infty}{Lt} \frac{x^n}{\lfloor n} = 0 \ \forall\ x \in R$.

$\therefore R_n \to 0$ as $n \to \infty$

$\therefore$ all the cond^{n}s of Maclaurin's infinite expansion are satisfied.

$\therefore \cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots + \infty$

which is reqd. expansion of $\cosh x$ in an infinite series.

(v) Let $f(x) = e^{ax}$, $f(0) = e^0 = 1$

$f'(x) = ae^{ax}$, $f'(0) = a$

$f''(x) = a^2 e^{ax}$, $f''(0) = a^2$

$f'''(x) = a^3 e^{ax}$, $f'''(0) = a^3$

& so on

In general, $f^{(n)}(x) = a^n e^{ax}$ exists $\forall\ x \in R$ & +ve integer n.

$\therefore$ by Maclaurin's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \dots + \frac{x^n}{\lfloor n} f^{(n)}(\theta x)$$

; $0 < \theta < 1$

$$\Rightarrow e^{ax} = 1 + \frac{x}{\lfloor 1} a + \frac{x^2}{\lfloor 2} a^2 + \frac{x^3}{\lfloor 3} a^3 + \dots + \frac{x^n}{\lfloor n} a^n e^{a\theta x}$$

$$\Rightarrow e^{ax} = 1 + \frac{ax}{\lfloor 1} + \frac{a^2}{\lfloor 2} x^2 + \frac{a^3}{\lfloor 3} x^3 + \dots + \frac{a^n}{\lfloor n} x^n e^{a\theta x}$$

which is the expansion of e^{ax} in a finite series

Hence, $R_n = \frac{a^n}{\lfloor n} x^n e^{a\theta x}$

$= \frac{(ax)^n}{\lfloor n} e^{a\theta x}$

$\because \underset{n\to\infty}{Lt} \frac{(ax)^n}{\lfloor n} = 0$ so, $R_n \to 0$ as $x \to \infty$

$\therefore$ all cond^{n}s of maclaurin's theorem are satisfied.

$\therefore e^{ax} = 1 + \frac{a}{\lfloor 1} + \frac{a^2}{\lfloor 2} x^2 + \frac{a^3}{\lfloor 3} x^3 + \ldots \infty$

Ex-2. Expand (i) $\sin^{-1}x$, (ii) $\tan x$, (iii) $\tan^{-1}x$ in an infinite series giving at least three non-zero terms assuming Maclaurin's expansion valid.

Soln (i) Let $f(x) = \sin^{-1}x$; $f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$, $f'(0) = 1$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)$

$\Rightarrow f''(x) = x(1-x^2)^{-3/2}$, $f''(0) = 0$

$\Rightarrow f'''(x) = (1-x^2)^{-3/2} + x(1-x^2)^{-5/2}\left(-\frac{3}{2}\right)\cdot(-2x)$

$= (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$, $f'''(0) = 1$

$\Rightarrow f^{iv}(x) = -\frac{3}{2}(1-x^2)^{-5/2}(-2x) + 6x(1-x^2)^{-5/2} - 3x^2 \cdot \frac{5}{2}(1-x^2)^{-7/2}\cdot(-2x)$

$= 9x(1-x^2)^{-5/2} + 15x^3(1-x^2)^{-7/2}$

$f^{iv}(0) = 0$

$f^{v}(x) = 9(1-x^2)^{-5/2} + 9x\left(-\frac{5}{2}\right)(1-x^2)^{-7/2}(-2x) + 45x^2(1-x^2)^{-7/2} + 15x^3\left(\frac{-7}{2}\right)(1-x^2)^{-9/2}(-2x)$

$f^{v}(0) = 9$

& so on

$\therefore$ by Maclaurin's ~~theorem~~ infinite series —

$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots \infty$

$\Rightarrow \sin^{-1}x =$

(11) Let $f(x) = \tan x \quad , f(0) = 0$

$f'(x) = \sec^2 x \quad , f'(0) = 1$

$= 1 + \tan^2 x$

$f''(x) = 2\tan x \sec^2 x \quad , f''(0) = 0$

$= 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x , \quad f'''(0) = 2$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 8\tan^2 x + 6\tan^4 x$

$f^{iv}(x) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x \quad , f^{iv}(0) = 0$

$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$

$f^{v}(x) = 16\sec^2 x + 120\tan^2 x \sec^2 x + 120\tan^4 x \sec^2 x$

$f^{v}(0) = 16$

$\therefore$ by Maclaurin's ~~theorem~~ Infinite series :-

$\Rightarrow \tan x = x + \frac{x^3}{\lfloor 3} 2 + \frac{x^5}{\lfloor 5} \cdot 16 + \dots \infty$

$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \infty$

Ex. Expand $\log \cos x$ about $x = \pi/3$ in power of $(x - \pi/3)$ giving upto three non-zero terms.

Soln. Let $f(x) = \log \cos x$

Here $a = \pi/3$

$f(a) = f(\pi/3) = \log \cos \pi/3 = \log \frac{1}{2}$

$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x \quad , f'(\frac{\pi}{3}) = -\sqrt{3}$

$f''(x) = -\sec^2 x \quad , f''(\pi/3) = -\sec^2 \pi/3 = -4$

$f'''(x) = -2\sec x \sec x \tan x$

$= -2\sec^2 x \cdot \tan x$

$f'''\left(\frac{\pi}{3}\right) = -2 \times \sec^2\frac{\pi}{3} \times \tan\frac{\pi}{3} = -2\times 4\times\sqrt{3} = -8\sqrt{3}$

& so on

$\therefore$ by Taylor's ~~theorem~~ expansion:-

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \dots \infty$$

$$\log\cos x = \log\frac{1}{2} + \left(x-\frac{\pi}{3}\right) - \sqrt{3} - \frac{\left(x-\frac{\pi}{3}\right)^2}{\lfloor 2} \times 4 - \frac{\left(x-\frac{\pi}{3}\right)^3}{\lfloor 3} \times 8\sqrt{3} + \dots \infty$$

Q:- Prove that:-

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(1+\xi)^5} \text{ for some } \xi \text{ bet}^n \text{ 0 and } x.$$

Sol^n: Let $f(x) = \log(1+x)$, $f(0) = 0$.

$f'(x) = \frac{1}{1+x}$, $f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$, $f''(0) = -1$

$f'''(x) = \frac{+2}{(1+x)^3}$, $f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4}$, $f^{IV}(0) = -6$

$f^{V}(x) = \frac{+24}{(1+x)^5}$, $f^{V}(0) = 24$

which all exists $\forall\, x > 0$

$\therefore$ by maclaurin's theorem with $n = 5$, we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} \times f^{IV}(0) + \frac{x^5}{\lfloor 5} f^{V}(\theta x)$$

$$\log(1+x) = x + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3} \times 2 + \frac{x^4}{\lfloor 4}(-6) + \frac{x^5}{\lfloor 5} \times \frac{24}{(1+\theta x)^5}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(1+\xi)^5} \text{ where } \xi = \theta x \text{ bet}^n \text{ 0 and } x$$

Ex:- Prove that

$$\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{\lfloor 2}\sin a - \frac{(x-a)^3}{\lfloor 3}\cos \xi$$ for some ξ betⁿ a and x.

Solⁿ: Let $f(x) = \sin x$, $f(a) = \sin a$

$f'(x) = \cos x$, $f'(a) = \cos a$

$f''(x) = -\sin x$, $f''(a) = -\sin a$

$f'''(x) = -\cos x$

Which exists $\forall x$.

$\therefore$ by Taylor's ~~expansion~~ thᵐ with $n = 3$

We get ;

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a+\theta h) ; 0<\theta<1$$

Let $(a+h) = x$ so, $h = (x-a)$

$$\therefore f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \frac{(x-a)^3}{\lfloor 3}f'''(a+\theta h)$$

$$\therefore \sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{\lfloor 2}\sin a - \frac{(x-a)^3}{\lfloor 3}\cos \xi$$

where $\xi = (a+\theta h)$ is such that $a < \xi < a+h$

$a < \xi < x$