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STD.: Ist SEM DIV./ SEC.: ______ SUBJECT: MATHS

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* Exact Differential eqⁿ :-

Differential eqⁿ is said to be exact, If it can be obtained from its primitive just by differentiation without any further process of elimination or multiplication etc,

eg:- $x^2+y^2=c^2$

$\Rightarrow 2x+2y\frac{dy}{dx}=0$

Differential equation of the form $Mdx+Ndy=0$, if there exists a function $u(x,y)$ such that $Mdx+Ndy=du$ i.e, if $Mdx+Ndy$ is a perfect or exact differential.

eg: ~~xdx + ydy = 0~~

$xdy+ydx=0$ is an exact D. eqⁿ.

because $xdy+ydx=d(xy)$

*** Theorem:** Necessary & sufficient condition of the d eqⁿ $Mdx+Ndy=0$ to be exact is that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Proof:- Let $Mdx+Ndy$ is an exact D eqn.

so, ∃ a function $U(x,y)$

such that $Mdx+Ndy=dU=\frac{\partial u}{\partial x}dx+\frac{\partial U}{\partial y}\times dy$ —(2)

since x & y are independent variables for the function $U(x,y)$ so dx & dy are also independent and therefore equating the co-efficient of dx and dy from (2) we get

$M = \frac{\partial U}{\partial x}$, $N = \frac{\partial U}{\partial y}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial N}{\partial x}$$

$\therefore$ the condition is necessary.

Conversely let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To prove: $M dx + N dy = 0$ is exact.

Let $U = \int\limits_{y-\text{const}} M dx$

$$\therefore \frac{\partial U}{\partial x} = M$$

Now, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$$= \frac{\partial^2 u}{\partial y \partial x}$$

$$= \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

Integrating w.r.t x;

$N = \frac{\partial U}{\partial y} + \phi(y)$ where $\phi(y)$ is a function of y free from x.

$$\therefore M dx + N dy = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \phi(y)\, dy$$

$$= dU + d\int \phi(y)\, dy = d\left[U + \int \phi(y)\, dy\right]$$

$= dV$ where $V = U + \int \phi(y) dy$

$\therefore$ $M dx + N dy$ is a perfect or exact differential

$\therefore$ $M dx + N dy = 0$ is an exact d.eqn

$\therefore$ the condn is sufficient.

* Method of ~~solving~~ solution of an exact d.eqⁿ of first order:-

(I) When the eqⁿ is $Mdx + Ndy = 0$ —— (i)
is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

First we find $u = \int_{y\text{-const.}} M dx + \phi(y)$

Then diff. partially w.r.t. y and using $\frac{\partial u}{\partial y} = N$ and then integrating w.r.t. y ~~and~~ we can find $\phi(y)$ and the solution is given by $u(x, y) = c$.

(II) First find $\int_{y\text{-const.}} M dx$ —— (I)

then find $\int_{x\text{-const.}} N dy$ —— (II)

then adding (I) with (II) omitting those terms already present in (I) & then equating to an arbitray constant we get the solⁿ.

Ex-1: Solve $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$

Solⁿ:- $M = 2x^3 + 4y$, $N = 4x + y - 1$

$\frac{\partial M}{\partial y} = 4$, $\frac{\partial N}{\partial x} = 4$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ the given d.eqⁿ is exact.

Method I :-

$$\text{Let } U = \int_{y\text{-const.}} M\,dx + \phi(y)$$

$$= \int_{y\text{-const.}} (2x^3+4y)\,dx + \phi(y)$$

$$= \frac{x^4}{2} + 4xy + \phi(y)$$

$$\frac{\partial U}{\partial y} = 4x + \phi'(y)$$

$$\therefore N = 4x + \phi'(y)$$

$$\Rightarrow 4x+y-1 = 4x+\phi'(y)$$

$$\Rightarrow \phi'(y) = y-1$$

Integrating w.r.t. y,

$$\phi(y) = \frac{y^2}{2} - y$$

$$\therefore U = \frac{x^4}{2} + 4xy + \frac{y^2}{2} - y$$

$\therefore$ sol^n is given by $U = C$

$$\Rightarrow \frac{x^4}{2} + 4xy + \frac{y^2}{2} - y = C.$$

Method II :-

$$\int_{y\text{-const.}} M\,dx = \int_{y\text{-const}} (2x^3+4y)\,dx = \frac{x^4}{2} + 4xy \quad \text{—— (1)}$$

$$\int_{x\text{-const.}} N\,dy = \int_{x\text{-const}} (4x+y-1)\,dy = 4xy + \frac{y^2}{2} - y \quad \text{—— (11)}$$

Adding (i) & (ii) and omitting $4xy$,

$$\frac{x^4}{2} + 4xy + \frac{y^2}{2} - y = C$$

Ex-21:- Solve :-

$$y \sin 2x \, dx - (1 + y^{\circ} + \cos^2 x) \, dy = 0$$

Here $M = y \sin 2x$, $N = -(1 + y + \cos^2 x)$

$\therefore \frac{\partial M}{\partial y} = \sin 2x$, $\frac{\partial N}{\partial x} = -2\cos x(-\sin x)$

$= \sin 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$

$\therefore$ the d.eq^n is exact.

Let $u = \int_{y \text{ const}} M dx + \phi(y)$

$= \int_{y\text{-const.}} y \sin 2x \, dx + \phi(y)$

$u = -\frac{y}{2} \cos 2x \, dx + \phi(y)$

$\therefore \frac{\partial u}{\partial y} = -\frac{1}{2} \cos 2x + \phi'(y)$

$N = -\frac{1}{2} \cos 2x + \phi'(y)$

$\Rightarrow -(1 + y + \cos^2 x) = -\frac{1}{2} \cos 2x + \phi'(y)$

$\Rightarrow -1 - y = \frac{1}{2} + \phi'(y)$

$\Rightarrow \phi'(y) = -y - \frac{3}{2}$

Integrating w.r.t. y,

$\phi(y) = -\frac{y^2}{2} - \frac{3}{2} y$

$\therefore\ u = -\frac{y}{2} \cdot \cos 2x - \frac{y^2}{2} - \frac{3}{2} y.$

$\therefore$ sol^n is given by

$u = C$

$\Rightarrow\ -\frac{y}{2} \cos 2x - \frac{y^2}{2} - \frac{3}{2} y = C$

$\Rightarrow\ y \cos 2x + y^2 + 3y + C = 0$

(II)

$\int_{y\text{-const}} M\,dx = y \int \sin 2x\, dx = -\frac{y \cos 2x}{2}$

$= -y\cos^2 x + \frac{y}{2}$ —(i)

$\int_{x\text{-const}} N\,dy = -\int (1 + y + \cos^2 x)\, dy$

$= -\left(y + \frac{y^2}{2} + y\cos^2 x\right)$ —— (ii)

$\therefore\ -\frac{y}{2} - \frac{y^2}{2} - y\cos^2 x = c$

$\Rightarrow\ \frac{y^2}{2} + y\cos^2 x + \frac{y}{2} + c = 0$

(Q) (8) $(x^4-2xy^2+y^4)dx-(2x^2y-4xy^3+\sin y)\,dy=0$

Soln: Here $M=x^4-2xy^2+y^4$, $N=-(2x^2y-4xy^3+\sin y)$

$$\frac{\partial M}{\partial y}=-4xy+4y^3,\quad \frac{\partial N}{\partial x}=-4xy+4y^3$$

Method-I

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so, the given eq^n is exact.

Let $u=\int\limits_{y\text{-const}} M\,dx+\phi(y)$

$=\int\limits_{y\text{-const}}(x^4-2xy^2+y^4)\,dx+\phi(y)$

$\Rightarrow u=\frac{x^5}{5}-x^2y^2+xy^4+\phi(y)$ —— (i)

$\therefore \frac{\partial u}{\partial y}=-2x^2y+4xy^3+\phi'(y)$

$\therefore N=-2x^2y+4xy^3+\phi'(y)$

$-2x^2y+4xy^3+\phi'(y)=-2x^2y+4xy^3-\sin y$

$\therefore \phi'(y)=-\sin y$

$\Rightarrow \phi(y)=\cos y$ —— (ii)

$\therefore$ ~~The~~ eq^n (i) becomes;

$U(x,y)=c$

$\Rightarrow \frac{x^5}{5}-x^2y^2+xy^4+\cos y=c$

(4) $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol^n. Here $M = 1+e^{x/y}$, $N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$, $\frac{\partial N}{\partial x} = e^{x/y} \times \frac{-1}{y} + \frac{e^{x/y}}{y}\left(1-\frac{x}{y}\right)$

$= e^{x/y} \times \left(\frac{-x}{y^2}\right)$

$\therefore$ the d.eq^n is exact.

$$U = \int_{y\text{ const}} M\,dx + \phi(y)$$

$$= \left(x + \frac{e^{x/y}}{1/y}\right) + \phi(y)$$

$$= \left(x + ye^{x/y}\right) + \phi(y)$$

$$\frac{\partial U}{\partial y} = e^{x/y} + y \times e^{x/y} \times \frac{-x}{y^2} + \phi'(y)$$

$$N = \frac{\partial U}{\partial y} = e^{x/y} - e^{x/y}\frac{x}{y} + \phi'(y)$$

$$\Rightarrow e^{x/y} - e^{x/y} \times \frac{x}{y} = e^{x/y} - e^{x/y} \times \frac{x}{y} + \phi'(y)$$

$$\Rightarrow \phi'(y) = 0$$

$$\Rightarrow \phi(y) = c_1$$

$$\therefore U = x + ye^{x/y} + c_1$$

$\therefore$ sol^n is given by $U = c_2$

$\Rightarrow x + ye^{x/y} + c_1 = c_2 \Rightarrow x + ye^{x/y} = c$ where $c = c_2 - c_1$

Note:
$$\int_{y\text{-const}} M\,dx = \int_{y\text{-const}} \left(1+e^{x/y}\right)dx = x + y e^{x/y}$$

$$\int_{x\text{-const}} N\,dy = \int_{x\text{-const}} e^{x/y}\left(1-\frac{x}{y}\right)dy = \int \left\{1\cdot e^{x/y} + y\cdot e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)\right\}dy$$

$$= \int_{x\text{-const}} \frac{\partial}{\partial y}\left(y e^{x/y}\right)dy$$

$$= y e^{x/y}$$

Solⁿ is given by;

$$\boxed{x + y e^{x/y} = c}$$

(5) $(xy\cos xy + \sin xy)\,dx + x^2\cos xy\,dy$.

Sol. $M = xy\cos xy + \sin xy$, $N = x^2\cos xy$

$$\frac{\partial M}{\partial y} = x\cos xy - x^2y\sin xy + x\cos xy \quad , \quad \frac{\partial N}{\partial x} = 2x\cos xy - x^2y\sin xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, the deqⁿ is exact.

$$\cancel{U = \int_{y\text{-const}} M\,dx + \phi(y) = \int_{y\text{-const}} (xy\cos xy + \sin xy)\,dx + \phi(y)}$$

Let $u = \int_{x\text{-const}} N\,dy + \phi(x)$

$$= \int_{x\text{-const}} x^2\cos xy\,dy + \phi(x) = \frac{x^2\sin xy}{x} + \phi(x)$$

$$= x\sin xy + \phi(x)$$

$$\frac{\partial U}{\partial x} = \sin xy + xy\cos xy + \phi'(x) = M$$

$$\textcircled{1} \Rightarrow xy\cos xy + \sin xy = \sin xy + xy\cos xy + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \phi(x) = C_1$$

$$\therefore \quad U(x, y) = C_2$$

$$\Rightarrow \sin xy + xy\cos xy + C_1 = C_2$$

$$\Rightarrow \underline{\sin xy + xy\cos xy = C} \qquad \text{where } C = C_2 - C_1$$

* Integrating factor :-

If a d.eq^n is not exact but it becomes exact when multiplied by some factor then this factor is called I.F. of the d.eq^n.

eg. $y\,dx - x\,dy = 0$ —— (1)

is not an exact d-eq^n.

Multiplying the d.eq^n.(1) by $\frac{1}{y^2}$ we get

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0 \quad \text{—— (2)}$$

Here $M = \frac{1}{y}$, $N = -\frac{x}{y^2}$

$$\therefore \frac{\partial M}{\partial y} = \frac{-1}{y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ d.eq^n (2) is exact.

So, I.F. of the eqn (1) is $\frac{1}{y^2}$.

* Rules for finding I.F. :-

(I) If $Mdx + N dy = 0$ is not exact

i.e, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ but

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x), \text{ a function of } x \text{ above}$$

then $\text{I.F.} = e^{\int f(x)dx}$

Proof:- Let $Mdx + N dy = 0$ —— (1)

is not exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

$e^{\int f(x)dx}$ will be an I.F. of the d.eq^n (1) if

$$e^{\int f(x)dx} M\,dx + e^{\int f(x)dx} N\,dy = 0 \quad \text{—— (2)}$$

is exact.

i.e, if $\frac{\partial}{\partial y}\left(e^{\int f(x)dx} M\right) = \frac{\partial}{\partial x}\left(e^{\int f(x)dx} N\right)$

i.e, if $e^{\int f(x)dx}\frac{\partial M}{\partial y} = e^{\int f(x)dx}\frac{\partial N}{\partial x} + N\cdot e^{\int f(x)dx}\cdot f(x)$

i.e, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + Nf(x)$

i.e, if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

$\therefore e^{\int f(x)dx}$ is an I.F. of d.eq^n (1)

(II) If $Mdx + Ndy = 0$ is not exact but $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$, a f^n of y-alone

then $I.F. = e^{\int f(y)dy}$

(III) If $Mdx + Ndy = 0$ is not exact and M, N are homogeneous functions of same degree in x & y then

$$I.F. = \frac{1}{Mx + Ny}$$

(IV) If $Mdx + Ndy = 0$ is not exact where M is of the form $yf(xy)$ and N is of the form $xg(xy)$

then $I.F. = \dfrac{1}{Mx - Ny}$

NOTE: (i) $d\left(\frac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$

(ii) $d\left(\frac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$

(iii) $d(xy) = xdy + ydx$

(iv) $d(x^2+y^2) = 2x\,dx + 2y\,dy$.

Solve $\left(\frac{x}{x^2+y^2} + x^2y\right)dy + \left(xy^2 - \frac{y}{x^2+y^2}\right)dx = 0$

$$\Rightarrow \left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) + xy(x\,dy + y\,dx) = 0$$

$$\Rightarrow \frac{\frac{x\,dy - y\,dx}{x^2}}{1+\left(\frac{y}{x}\right)^2} + xy\,d(xy) = 0$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + xy\,d(xy) = 0$$

$$\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + \int xy\,d(xy) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{x^2y^2}{2} = c$$

(2) Solve

$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

(3) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

(4) $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$$

(2) $x\,d\cancel{x} + y\,dy + \dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{1+\dfrac{y^2}{x^2}} = 0$

$$x\,d\cancel{x} + y\,dy + \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = 0$$

$$\therefore \int x\,dx + \int y\,dy + \int \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = c$$

$$\Rightarrow \boxed{\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c}$$

(3) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x^2}\sqrt{x^2\left(1+\frac{y^2}{x^2}\right)}\,dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = \frac{1}{x}\sqrt{1+\left(\frac{y}{x}\right)^2}\,dx$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{1}{x}\,dx$$

$$\therefore \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x} + \log c$$

$$\Rightarrow \log\left\{\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right\} = \log x + c$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$$

(4) $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2+y^2}}$$

$$\Rightarrow \frac{\frac{1}{2}(2x\,dx + 2y\,dy)}{x^2 \cdot \frac{x\,dy - y\,dx}{x^2}} = \sqrt{\frac{a^2 - (x^2+y^2)}{x^2+y^2}}$$

$$\Rightarrow \frac{d(x^2+y^2)}{2x^2 \cdot d\left(\frac{y}{x}\right)} = \sqrt{\frac{a^2-(x^2+y^2)}{x^2+y^2}}$$

$$= \frac{\sqrt{a^2-(x^2+y^2)} \times \sqrt{x^2+y^2}}{x^2+y^2}$$

$$\Rightarrow \frac{d(x^2+y^2)}{2x^2 \cdot d\left(\frac{y}{x}\right)} = \frac{\sqrt{a^2-(x^2+y^2)} \times \sqrt{x^2+y^2}}{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$\Rightarrow \frac{du}{2dv} = \frac{\sqrt{a^2-u} \times \sqrt{u}}{(1+v^2)}$$

$u = x^2+y^2$

$v = \frac{y}{x}$

$$\Rightarrow \frac{2dv}{1+v^2} = \frac{du}{\sqrt{a^2-u}\,\sqrt{u}}$$

$$\Rightarrow 2\int \frac{dv}{1+v^2} = \int \frac{du}{\sqrt{a^2-u}\,\sqrt{u}} + c$$

$$\Rightarrow 2\tan^{-1} v = \int \frac{2\,dt}{\sqrt{a^2-t^2}} + c$$

Let $t = \sqrt{u}$

$dt = \frac{1}{2\sqrt{u}}\,du$

$$\Rightarrow 2\tan^{-1} v = 2\sin^{-1} t + c$$

$\Rightarrow 2\tan^{-1}\frac{y}{x} = 2\sin^{-1}\frac{\sqrt{u}}{a} + c$

$\Rightarrow 2\tan^{-1}\frac{y}{x} = 2\sin^{-1}\frac{\sqrt{x^2+y^2}}{a} + c$

(5) $(2x^2+y^2+x)\,dx + xy\,dy = 0$ ——(1)

Here $M = 2x^2+y^2+x$, $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the eqn (1) is not exact.

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x}$, a f^n of x-value

$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

∴ ~~the~~ Multiplying the eqn (1) by I.F;

$\Rightarrow (2x^3+y^2x+x^2)\,dx + x^2y\,dy = 0$ ——(2)

which is exact

$\Rightarrow 2x^3dx + x^2dx + xy(ydx + xdy) = 0$

$\Rightarrow 2x^3dx + x^2dx + xy\,d(xy) = 0$

$\Rightarrow 2\int x^3dx + \int x^2dx + \int (xy)\,d(xy) = c$

$\Rightarrow \frac{x^4}{2} + \frac{x^3}{3} + \frac{x^2y^2}{2} = c$ which is reqd. soln.

(6) Solve

$(y^2e^x + 2xy)dx - x^2dy = 0$ —— (1)

Here, $M = y^2e^x + 2xy$, $N = -x^2$

$\frac{\partial M}{\partial y} = 2ye^x + 2x$, $\frac{\partial N}{\partial x} = -2x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ & so, the eq^n (1) is not exact.

$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2x - 2ye^x - 2x}{y^2e^x + 2xy} = -\frac{2}{y}$

$I.F = e^{\int -\frac{2}{y}dy} = +\frac{1}{y^2}$

Multiplying the eq^n (1) by I.F;

$\Rightarrow \frac{1}{y^2}(y^2e^x + 2xy)dx - \frac{x^2}{y^2}dy = 0$

$\Rightarrow \left(e^x + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}dy = 0$ —— (2) which is exact.

$\Rightarrow e^x dx + \frac{2x}{y}dx - \frac{x^2}{y^2}dy = 0$

$\Rightarrow \int e^x dx + \int d\left(\frac{x^2}{y}\right) = c$

$\Rightarrow \boxed{e^x + \frac{x^2}{y} = c}$

(9) Solve:

$(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$ — (1)

This is a homogeneous d. equation

Here $M = (x^2y - 2xy^2)$, $N = (3x^2y - x^3)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$, $\frac{\partial N}{\partial x} = 6xy - 3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$; so, the eq^n (1) is not exact.

Now, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$

$= x^2y^2$

$I.F. = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

Multiplying the d. eq^n (1) by I.F.;

$\frac{1}{x^2y^2}(x^2y - 2xy^2)\,dx + \frac{1}{x^2y^2}(3x^2y - x^3)\,dy = 0$

$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$ — (2) which is exact.

$\Rightarrow \frac{1}{y}\,dx - \frac{x}{y^2}\,dy - \frac{2}{x}\,dx + \frac{3}{y}\,dy = 0$

$\Rightarrow d\left(\frac{x}{y}\right) - \frac{2}{x}\,dx + \frac{3}{y}\,dy = 0$

$\Rightarrow \int d\left(\frac{x}{y}\right) - 2\int \frac{dx}{x} + 3\int \frac{1}{y}\,dy = c$

$\Rightarrow \frac{x}{y} - 2\log x + 3\log y = c$

which is the sol^n.

(8) $(1+xy+x^2y^2)\,y\,dx + (1-xy+x^2y^2)\,x\,dy = 0$ —(1)

$M = (1+xy+x^2y^2)\,y$ is of the form $yf(xy)$

$N = (1-xy+x^2y^2)\,x$ which is of the form $xg(xy)$

clearly the d. eqn (i) is not exact

Now, $Mx - Ny = 2x^2y^2$

$\therefore$ I.F. $= \dfrac{1}{Mx-Ny} = \dfrac{1}{2x^2y^2}$

Multiplying the d. eqn (i) by I.F. we get;

$$\Rightarrow \frac{1}{2x^2y^2}(1+xy+x^2y^2)\,y\,dx + \frac{1}{2x^2y^2}(1-xy+x^2y^2)\,x\,dy = 0$$

$$\Rightarrow \left(\frac{1}{x^2y} + \frac{1}{x} + y\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y} + x\right)dy = 0 \quad \text{—(2)}$$

which is exact

$$\Rightarrow \frac{1}{x^2y^2}(y\,dx + x\,dx) + (y\,dx + x\,dy) + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$\Rightarrow \frac{d(xy)}{x^2y^2} + d(xy) + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$\Rightarrow \frac{-1}{xy} + xy + \log x - \log y = c$$

$$\Rightarrow \boxed{x^2y^2 + xy\log\left(\frac{x}{y}\right) - 1 = cxy}$$

* Linear d.eqⁿ of first order or Leibnitz's linear d.eqⁿ :-

A d.eqⁿ of the form $\frac{dy}{dx} + Py = Q$ is called linear d.eqⁿ of first order where P and Q are functions of x-alone (or constant)

eg :- $\frac{dy}{dx} + e^x y = x$ is a linear d.eqⁿ of 1st order.

Note :- A linear d.eqⁿ may be taken as

$$\frac{dx}{dy} + Px = Q$$

where P & Q are fⁿs of y-alone (or constant).

Method of solution :-

Linear d.eqⁿ is

$$\frac{dy}{dx} + Py = Q \quad \text{--- (1)}$$

where P, Q are fⁿs of x-alone.

Multiplying the d.eqⁿ (1) by $e^{\int P dx}$ we get ;

$$e^{\int P dx}\frac{dy}{dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow e^{\int P dx}\frac{dy}{dx} + y e^{\int P dx} \cdot \frac{d}{dx}\int P dx = Q e^{\int P dx}$$

$$\Rightarrow e^{\int P dx}\frac{dy}{dx} + y \frac{d}{dx} e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = \frac{d}{dx}\int\left(Q e^{\int P dx}\right) dx \quad \text{--- (2)}$$

$\therefore$ the d.eqⁿ become exact when multiplied by $e^{\int P dx}$

$\therefore e^{\int P dx}$ is an I.F. of (1) ;

Integrating (2);

$\Rightarrow y e^{\int P dx} = \int Q e^{\int P dx} + c$

$\Rightarrow y(I.F.) = \int Q(I.F.)\, dx + c$

which is the soln of linear eqn.

Note:- If the d. eqn is $\frac{dx}{dy} + Px = Q$

then $I.F. = e^{\int P dy}$

and solution is given by :-

$$x(I.F.) = \int Q(I.F.)\, dy + c$$

Note — (i) A d. eqn of the form

$f'(y)\frac{dy}{dx} + Pf(y) = Q$, where P and Q are functions of x-alone can be reduced to linear d. equation by the ~~form~~ substitution $z = f(y)$

$\therefore \frac{dz}{dx} = f'(y)\frac{dy}{dx}$

The above d. eqn becomes

$\frac{dz}{dx} + Pz = Q$ which is linear

$IF = e^{\int P dx}$

Soln is :- $z \times (I.F.) = \int Q(I.F.)\, dx + c$

* Bernoulli's d. eq^n :-

A d. eq^n of the form $\frac{dy}{dx} + Py = Qy^n$ where n is a constant is called Bernoulli's d. eq^n.

Method of sol^n :-

$$\frac{dy}{dx} + Py = Qy^n$$

$$\text{or } \frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q \quad \text{—— (1)}$$

$$\text{Let } z = \frac{1}{y^{n-1}} = y^{1-n}$$

$$\therefore \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx}$$

$\therefore$ (1) becomes ——

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

or $\frac{dz}{dx} + P_1 z = Q_1$ where $P_1 = (1-n)P$, $Q_1 = (1-n)Q$

It is a linear d. eq^n.

$I.F. = e^{\int P_1 dx}$ & sol^n is

$$z(I.F.) = \int Q_1 (I.F.)\, dx + c$$

* Solve :-

① $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solⁿ $\frac{dy}{dx} + \frac{2x}{(1+x^2)} y = \frac{4x^2}{(1+x^2)}$ — (i)

Here $P = \frac{2x}{(1+x^2)}$, $Q = \frac{4x^2}{(1+x^2)}$

$I.F. = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = (1+x^2)$

$\therefore$ solⁿ is given by

$y(I.F.) = \int Q(I.F) dx + c$

$\Rightarrow y(1+x^2) = \int 4x^2 dx + c$

$\Rightarrow y(1+x^2) = \frac{4x^3}{3} + c$ which is reqd. soln.

② $(1+y^2)dx - (\tan^{-1}y - x)dy = 0$

$\Rightarrow (\tan^{-1}y - x) dy = (1+y^2) dx$

$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$

$= \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{\tan^{-1}y}{1+y^2}$ — (i)

It is a linear d. eqn of 1st order.

Here, $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

$$I.F. = e^{\int P\,dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

$\therefore$ soln is given by

$$x(I.F.) = \int Q(I.F.)\,dy + c$$

$$\Rightarrow x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy + c$$

Let $t = \tan^{-1}y$

$\frac{dt}{dy} = \frac{1}{1+y^2}$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = \int t e^{t}\,dt + c$$

$$= (t-1)e^{t} + c$$

$$= (\tan^{-1}y - 1)e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$$

3) $\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^3}$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{2}{y^2 x} = \frac{1}{x^3}$$

$\frac{1}{y^2} = t$

$-\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow -\frac{1}{2}\frac{dt}{dx} + \frac{2t}{x} = \frac{1}{x^3}$$

$$\frac{dt}{dx} - \frac{4t}{x} = -\frac{2}{x^3}$$

It is a linear d. eqn of first order.

$$I.F. = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

$\therefore$ Sol^n is given by

$$\frac{1}{y^2}(I.F.) = \int Q(IF)\,dx + c$$

$$\Rightarrow \frac{1}{y^2}\times\frac{1}{x^4} = \int -\frac{2}{x^3}\times\frac{1}{x^4}\,dx + c$$

$$= -2\int x^{-7}\,dx + c$$

$$= \frac{-2}{-6}x^{-6} + c$$

$$\Rightarrow \frac{1}{x^4y^2} = \frac{1}{3x^6} + c$$

Q4) $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\Rightarrow \frac{1}{\cos^2 y}\frac{dy}{dx} + 2x\tan y = x^3 \quad \text{--- (1)}$$

(1) becomes —

Let $z = \tan y$

$$\frac{dz}{dx} = \sec^2 y\frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = x^3$$

It is a linear d. eq^n of 1st order

$P = 2x$, $Q = x^3$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

Sol^n is given by

$$\Rightarrow z\times IF = \int Q(IF)\,dx + c$$

$$\Rightarrow \tan y \times e^{x^2} = \int e^{x^2}\times x^3\,dx + c$$

$$\Rightarrow \tan y\times e^{x^2} = \frac{1}{2}\int e^{x^2}x^2\times 2x\,dx + c$$

$= \frac{1}{2}\int e^t t\,dt + c$ where $t = x^2$, $dt = 2x\,dx$

$= \frac{1}{2}(t-1)e^t + c$

$$\boxed{\tan y\, e^{x^2} = \frac{1}{2}(x^2-1)e^{x^2} + c}$$

5) $(x+2y^3)\frac{dy}{dx} = y$

$\Rightarrow y\frac{dx}{dy} - x = 2y^3$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$

It is a linear d. eqn of 1st order

$P = -\frac{1}{y}$, $Q = 2y^2$

$\Rightarrow I.F. = e^{\int -\frac{1}{y}dy} = \frac{1}{y}$

Soln is given by; $x \times (I.F.) = \int Q(IF)\,dy + c$

$\Rightarrow x \times \frac{1}{y} = \int 2y^2 \times \frac{1}{y} + c$

$\Rightarrow \frac{x}{y} = y^2 + c$

$$\Rightarrow \boxed{x = y^3 + cy}$$

(8) $x\frac{dy}{dx} + y = y^2 \log x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$ — (1)

$\Rightarrow$ Let $z = \frac{1}{y}$

$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$\therefore$ (1) becomes

$-\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x}\log x$ — (2)

It is a linear d. eqn of 1st order

$P = -\frac{1}{x}$, $Q = -\frac{1}{x}\log x$

$IF = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

solution is;

$z \times (I.F.) = \int Q(I.F.)\,dx + C$

$\Rightarrow \frac{1}{y} \times \frac{1}{x} = \int -\frac{1}{x}\log x \times \frac{1}{x}\,dx + C$ $\quad \log x = t$

$\Rightarrow \frac{1}{xy} = -\left[\log \int \frac{1}{x^2}dx - \int\left(\frac{d}{dx}\log x \int \frac{1}{x^2}dx\right)\right] + c$ $\quad x = e^t$ $\quad dt =$

$\Rightarrow \frac{1}{xy} = -(\log x)\left(-\frac{1}{x}\right) + \int \frac{1}{x}\left(-\frac{1}{x}\right)dx + C$

$\Rightarrow \frac{1}{xy} = \frac{1}{x}\log x + \frac{1}{x} + C$

(3) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

$\Rightarrow \frac{1}{y(\log y)^2}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}$ —(1)

Let $z = \frac{1}{\log y}$

$\frac{dz}{dy} = \frac{-1}{y(\log y)^2}\cdot\frac{dy}{dx}$

$\therefore$ (1) becomes —

$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$ —(2)

$I.F. = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

$\therefore$ Sol^n is :-

$z\cdot\frac{1}{x} = \int -\frac{1}{x^2}\cdot\frac{1}{x}dx + c$

$\Rightarrow \frac{1}{\log y}\cdot\frac{1}{x} = \frac{1}{2x^2} + c$

(8) $\frac{dy}{dx} + y\cos x = 2e^{\cos x}$ given $y = 4$ when $x = \pi/2$

It is a linear d. eqn.

$I.F. = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\therefore$ Soln is $y \times \sin x = \int 2e^{\cos x} \sin x\, dx + c$

$y \times \sin x = -2e^{\cos x} + c$

when $x = \pi/2$; $y = 4$

$\Rightarrow 4 = -2e^{0} + c$

$\Rightarrow c = 6$

$\boxed{y \times \sin x = -2e^{\cos x} + 6}$

$\cos x = t$
$-\sin x \frac{dx}{dt} = \frac{dt}{dx}$

(9) $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = (1-\sqrt{x})$

2^{nd} and higher order linear d.eq^n with const. coeff. :-

A linear d.eq^n of n^{th} order with const. coeff. in its standard form is given by

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-2}\frac{d^2y}{dx^2} + P_{n-1}\frac{dy}{dx} + P_n y = X$$

or $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-2}D^2 + P_{n-1}D + P_n)\,Y = X$

where $P_1, P_2, P_3 \ldots, P_n$ are constants & X is a function of x-alone.

& $D \equiv \frac{d}{dx}$

First we consider the 2^{nd} order linear d.eq^n :-

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \quad \text{—— (1)}$$

Let $y = e^{mx}$ is a trial solution of the d.eq^n (1)

$\therefore \frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2 e^{mx}$

Putting these in (1);

$(m^2 + P_1 m + P_2)\, e^{mx} = 0$

$\Rightarrow m^2 + P_1 m + P_2 = 0$ —— (2), $\because e^{mx} \neq 0$

which is called auxillary equation.

or; $(D^2 + P_1 D + P_2)\,Y = 0 \rightarrow$ auxillary eq^n.

& let m_1 & m_2 are its roots

Case I: Let m_1 and m_2 are real and different roots

Then the general soln of the d.eq^n (1) is given by

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ because it satisfies the d.eq^n (1) of 2^{nd} order and contains two arbitrary constants

* **Case II** :- Let $m_1 = m_2 = K$ (say) are real & equal roots then two gen. soln of ① is given by

$$y = (c_1 + c_2 x) e^{Kx}$$, because it satisfies the given d. eqn and contains two arbitary constants.

* **Case III** :- Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ are complex conjugate roots then the gen. soln is given by

$$y = e^{\alpha x}(A \cos\beta x + B \sin\beta x)$$ where A & B are arbitary const.

* **Note** :- If $0, m_1, m_2$, are real & diff. roots, K, K, K are real and equal roots, $\alpha \pm i\beta$ are two complex roots and $\gamma \pm i\delta, \gamma \pm i\delta$ are two pairs of equal complex roots of the auxiliary eqn of a linear d. eqn. of order 12 then its general soln is given by

$$y = c_1 + c_2 e^{m_1 x} + c_3 e^{m_2 x} + (c_4 + c_5 x + c_6 x^2) e^{Kx}$$
$$+ e^{\alpha x}[A \cos\beta x + B \sin\beta x] + e^{\gamma x}[(A_1 + A_2 x)\cos\delta x + (B_1 + B_2 x)\sin\delta x]$$

Solve

1. $\frac{d^4y}{dx^4} - y = 0$

Sol^n $(D^4-1)y = 0$

Auxiliary eq^n:- $m^4 - 1 = 0$

$m = 1, -1, i, -i$

$= 1, -1, 0 \pm i \cdot 1$

$\therefore$ gen sol^n is given by:-

$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

2. $\frac{d^3y}{dx^3} - 8y = 0$

Sol^n. $(D^3 - 8)y = 0$

Auxiliary eq^n:- $m^3 - 8 = 0 \Rightarrow m^3 = 2^3$

$m = 2, 2\omega, 2\omega^2$

$= 2, 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right), 2\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 2, (-1 \pm i\sqrt{3})$

$\therefore$ gen. sol^n is given by:-

$y = c_1 e^{2x} + e^{-x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]$

(3) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Soln $(D^3 + D^2 + D + 1)y = 0$

Auxiliary eqn :-

$(m^3 + m^2 + m + 1) = 0$

$m^2(m+1) + (m+1) = 0$

$\Rightarrow (m+1)(m^2+1) = 0$

$\Rightarrow m = -1, \pm i$

$\Rightarrow m = -1, 0 \pm i \cdot 1$

gen. soln is given by :-

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

(4) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

Soln. $(D^3 + 3D^2 + 3D + 1)y = 0$

Auxiliary eqn :-

~~$(D^3 + 3D^2 + 3D + 1) =$~~

$m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0$

$m = -1, -1, -1$

$\therefore$ gen. soln. is given by

$y = (c_1 + c_2 x + c_3 x^2)e^{-x}$

(5) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$

Sol- $(D^3 - D^2 - 4D + 4)y = 0$

A. eq^n is :- $m^3 - m^2 - 4m + 4 = 0$

$\Rightarrow m^2(m-1) - 4(m-1) = 0$

$\Rightarrow m = 1, 2, -2.$

gen. soln. is given by;

$y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x}$

(6) $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$

where a, b, g are +ve constants :-

given by $x = d$, $\frac{dx}{dt} = 0$, when $t = 0$

Soln. $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$ — (1)

Let $y = (x-a)$

$\frac{dy}{dt} = \frac{dx}{dt} \Rightarrow \frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}$

~~(1) is given by~~

$\therefore$ (1) becomes —

$\frac{d^2y}{dt^2} + \frac{g}{b}y = 0$ — (2)

or $\left(D^2 + \frac{g}{b}\right)y = 0$, where $D = \frac{d}{dt}$

A. eq^n is $m^2 + \frac{g}{b} = 0$

$\Rightarrow m = \pm\sqrt{-\frac{g}{b}} = 0 \pm i\sqrt{\frac{g}{b}}$

gen. sol^n is :- $y = A\cos\sqrt{\frac{g}{b}}\,t + B\sin\sqrt{\frac{g}{b}}\,t$

$$\Rightarrow (x-a) = A\cos\sqrt{\frac{g}{b}}\,t + B\sin\sqrt{\frac{g}{b}}\,t$$

$$\therefore \frac{dx}{dt} = -A\sqrt{\frac{g}{b}}\cdot\sin\sqrt{\frac{g}{b}}\,t + B\sqrt{\frac{g}{b}}\cos\sqrt{\frac{g}{b}}\,t$$

given $x=\alpha$, $\frac{dx}{dt}=0$ when $t=0$

$$\therefore \alpha - a = A$$

$$0 = B\sqrt{\frac{g}{b}} \Rightarrow B=0$$

$\therefore$ particular soln is

$$(x-a) = (\alpha-a)\cdot\cos\sqrt{\frac{g}{b}}\,t$$

$$\Rightarrow x = a + (\alpha-a)\cos\sqrt{\frac{g}{b}}\,t.$$

(7) $\frac{d^2x}{dt^2} + \mu x = 0$, $\mu > 0$ & given $x=a$, $\frac{dx}{dt}=0$ when $t=\frac{\pi}{\sqrt{\mu}}$.

Soln: $\frac{d^2x}{dt^2} + \mu x = 0$

or $(D^2+\mu)x = 0$

Auxillary eq^n is:

$$m^2+\mu = 0$$

$$\Rightarrow m^2 = -\mu$$

$$\Rightarrow m = 0 \pm i\sqrt{\mu}$$

$\therefore$ gen soln is

$$x = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t$$

$$\therefore \frac{dx}{dt} = -A\sqrt{\mu}\sin\sqrt{\mu}t + B\sqrt{\mu}\cos\sqrt{\mu}t$$

given $x = a$, $\frac{dx}{dt} = 0$ when $t = \frac{\pi}{\sqrt{\mu}}$

$\therefore a = A\cos\pi + B\sin\pi$

$\boxed{a = -A}$

$0 = -A\sqrt{\mu}\sin\pi + B\sqrt{\mu}\cos\pi$

$= -B\sqrt{\mu}$

$\Rightarrow \boxed{B = 0}$

$\therefore$ Particular solⁿ is:-

$x = -a\cos\sqrt{\mu}\,t$

(8) Show that solⁿ of $\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \mu x = 0$ is ~~$x = e^{-1/2Kt}(A\cos nt + B$~~

$x = e^{-1/2Kt}(A\cos nt + B\sin nt)$

if $K^2 < 4\mu$ & $n^2 = \mu - \frac{1}{4}K^2$

Ans→ $\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \mu x = 0$

$(D^2 + KD + \mu)x = 0$

Auxiliary eqⁿ:-

$m^2 + Km + \mu = 0$

$m = -\frac{K}{2} \pm \frac{\sqrt{K^2 - 4\mu}}{2}$

$= \frac{-K \pm \sqrt{-4n^2}}{2}$

$= -\frac{1}{2}K \pm in$

$\therefore$ gen. soln. is:-

$x = e^{-1/2Kt}(A\cos nt + B\sin nt)$

9 $\frac{d^2y}{dx^2} + m^2 y = 0$

10 $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$

Soln $(D^3 - D)y = 0$

Auxiliary eqn :- $m^3 - m = 0$

$m(m^2 - 1) = 0$

$m = 0, m = 1, m = -1$

$y = c_1 + c_2 e^{x} + c_3 e^{-x}$

Method of Solving $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X$ —— (1)

where P_1, P_2 are constants and $X \neq 0$ is a function of x-alone

Let $y = \phi(x, c_1, c_2)$ is the general sol^n of $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$ —— (2)

and $y = \psi(x)$ is a particular sol^n of (1)

then the gen. sol^n of (1) is given by

$y = \phi(x, c_1, c_2) + \psi(x)$ because it satisfies (1) & contains two arbitray constants.

In this case $\phi(x, c_1, c_2)$ is called the complementry function (C.F.) and $\psi(x)$ is called the particular integral (P.I.) and so the gen. sol^n of (1) is $y =$ C.F. + P.I.

Particular integral (P.I.) :-

Let the linear d. eq^n is

$$f(D)y = X.$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_{n-1}D + P_n$;

$P_1, P_2, \dots, P_n$ are constants and X is a function of x alone.

We define $\frac{1}{f(D)}X$ as that function of x not containing arbitray constant which when operated by $f(D)$ we get X again i.e,

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

$\therefore$ $\frac{1}{f(D)}X$ is a particular sol^n of (1)

i.e, $$\text{P.I.} = \frac{1}{f(D)}X$$

Note :- In particular if $f(D) = D$ then $P.I. = \frac{1}{D} X = \int X\,dx$

& if $f(D) = D^2$ then $P.I. = \frac{1}{D^2} X = \frac{1}{D} \cdot \frac{1}{D} \cdot X$

$$= \int \left(\int X\,dx \right) dx$$

* **Rules for finding P.I. in some particular cases :-**

(I) If $X = e^{ax+b}$ then

$$P.I. = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax+b}$$

$$= \begin{cases} \frac{1}{f(a)} e^{ax+b} \;; \text{ if } f(a) \neq 0 \\ x \cdot \frac{1}{f'(D)} e^{ax+b} = x \cdot \frac{1}{f'(a)} e^{ax+b} \;; \text{ if } f(a) = 0, f'(a) \neq 0 \\ x^2 \cdot \frac{1}{f''(D)} e^{ax+b} = x^2 \cdot \frac{1}{f''(a)} e^{ax+b} \;; \text{ if } f(a) = f'(a) = 0,\ f''(a) \neq 0 \\ \text{\& so on} \end{cases}$$

(II) If $X = e^{ax+b} \cdot V$ where V is a function of x

then $P.I. = \frac{1}{f(D)} \cdot X = \frac{1}{f(D)} e^{ax+b} V$

$$= e^{ax+b} \cdot \frac{1}{f(D+a)} \cdot V$$

(III) If $X = x^m$ where m is a +ve integer

$$P.I. = \frac{1}{f(D)} X = [f(D)]^{-1} x^m$$

$$= (a_0 + a_1 D + a_2 D^2 + \dots + a_m D^m + \dots) x^m$$

(IV) If $X = \sin(ax+b)$ or $\cos(ax+b)$

& $f(D) = \phi(D^2)$

then $P.I. = \frac{1}{\phi(D^2)} \sin(ax+b)$

$$= \begin{cases} \frac{1}{\phi(-a^2)} \sin(ax+b) & \text{— if } \phi(-a^2) \neq 0 \\ x \times \frac{1}{\phi'(D^2)} \sin(ax+b) = x \times \frac{1 \times \sin(ax+b)}{\phi'(-a^2)}; & \text{if } \phi(-a^2) = 0 \text{ \& } \phi'(-a^2) \neq 0 \\ x^2 \times \frac{1}{\phi''(D^2)} \sin(ax+b) = x^2 \times \frac{1}{\phi''(-a^2)} \sin(ax+b); & \text{if } \phi(-a^2) = \phi'(-a^2) = 0 \\ & \phi''(-a^2) \neq 0 \end{cases}$$

Solve :-

(1) $\frac{d^2y}{dx^2} + 4y = \sin 2x$

or $(D^2+4)y = \sin 2x$

Auxiliary eqn is

$m^2+4=0$

$m = 0 \pm i.2$

$C.F. = A\cos 2x + B\sin 2x$

$P.I. = \frac{1}{D^2+4}\sin 2x$

$= x \times \frac{1}{2D} \times \sin 2x$

$= \frac{x}{2}\int \sin 2x \, dx$

$= \frac{x}{2}\left(-\frac{1}{2}\cos 2x\right)$

$= -\frac{x}{4}\cos 2x$

$\therefore$ general soln is

$y = C.F. + P.I.$

$y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$

NOTE: $y = \psi(x) = -\frac{x}{4}\cos 2x$

$\frac{dy}{dx} = -\frac{1}{4}\cos 2x + \frac{x}{2}\sin 2x$

$\frac{d^2y}{dx^2} = \frac{1}{2}\sin 2x + \frac{1}{2}\sin 2x + \frac{x}{2}\cos 2x$

$= \sin 2x + x\cos 2x$ $\Rightarrow \frac{d^2y}{dx^2} + 4y = \sin 2x$

(2) $\frac{d^2y}{dx^2} + y = \cos^2 x$

or $(D^2+1)\,y = \cos^2 x$

A. eqn is $(m^2+1)=0$

$\Rightarrow m = 0 \pm i$

$\therefore$ C.F. $= A\cos x + B\sin x$

$$P.I. = \frac{1}{D^2+1}\cos^2 x$$

$$= \frac{1}{D^2+1}\cdot\frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cdot 1 + \frac{1}{D^2+1}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{-2^2+1}\cos 2x\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\cos 2x\right)$$

$\therefore$ gen. solution is $y =$ C.F. + P.I.

$$y = A\cos x + B\sin x + \frac{1}{2}\left(1 - \frac{1}{3}\cos 2x\right)$$

(3) $\frac{d^8y}{dx^8} - 2\frac{d^4y}{dx^4} + y = \sin(2x+3)$

or $(D^8 - 2D^4 + 1)\,y = \sin(2x+3)$

A. eqn is $m^8 - 2m^4 + 1 = 0$

$\Rightarrow (m^4-1)^2 = 0$

$\Rightarrow (m^2-1)^2(m^2+1)^2 = 0$

$\Rightarrow m = \pm 1, \pm 1, \pm i, \pm i$

C.F. $= (C_1 + C_2x)e^{x} + (C_3 + C_4x)e^{-x} + (A_1 + A_2x)\cos x + (B_1 + B_2x)\sin x$

$$P.I. = \frac{1}{D^8 - 2D^4 + 1} \sin(2x+3)$$

$$= \frac{1}{(D^4-1)^2} \sin(2x+3)$$

$$= \frac{1}{\{(-2^2)^2 - 1\}^2} \sin(2x+3)$$

$$= \frac{1}{225} \sin(2x+3)$$

$\therefore$ gen. solⁿ is

$$y = C.F. + P.I.$$

(4) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$$

Soln:- $(D^2 + 2D)y = x^2$

A eqⁿ is $(m^2 + 2m) = 0$

$\Rightarrow m(m+2) = 0$

$\Rightarrow m = 0, m = -2$

$\therefore C.F. = C_1 + C_2 e^{-2x}$

$$P.I. = \frac{1}{D^2 + 2D} x^2$$

$$= \frac{1}{2D\left(1 + \frac{D}{2}\right)} x^2 = \frac{1}{2D}\left(1 + \frac{D}{2}\right)^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \frac{D}{2} + \frac{D^2}{2^2} - \frac{D^3}{2^3} + \cdots\right] x^2$$

$$= \frac{1}{2}\left(\frac{1}{D} - \frac{1}{2} + \frac{D}{4} - \frac{D^2}{8} + \dots\right)x^2$$

$$= \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{2x}{4} - \frac{1}{8}\right) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{4} - \frac{1}{8}$$

$\therefore$ gen sol^n is $y = C.F. + P.I.$

(5) $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \cosh x$

$(D^3 + D)y = \cosh x$

A. eq^n is $m^3 + m = 0$

$m(m^2+1) = 0$

$m = 0, \pm i$

$\therefore$ $C.F. = C_1 + A\cos x + B\sin x$

$$P.I. = \frac{1}{D^3+D}\cosh x$$

$$= \frac{1}{D^3+D} \cdot \frac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{D^3+D}e^x + \frac{1}{D^3+D}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+1}e^x + \frac{1}{-1-1}e^{-x}\right]$$

$$= \frac{1}{4}\left[e^x - e^{-x}\right]$$

$$P.I. = \frac{1}{2}\sinh x$$

$\therefore$ gen. sol^n is

$y = C.F. + P.I.$

(6) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

or $(D^2 - 2D + 1)y = xe^x \sin x$

A. eqⁿ is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1.$

$\therefore$ C.F. $= y = (c_1 + c_2 x)e^x$

P.I. $= \frac{1}{(D^2 - 2D + 1)} xe^x \sin x$

$= \frac{1}{(D-1)^2} xe^x \sin x$

$= e^x \cdot \frac{1}{(D+1-1)^2} \cdot x \sin x$

$= e^x \cdot \frac{1}{D^2} \cdot x \sin x$

$= e^x \cdot \frac{1}{D} \int x \sin x \, dx$

$= e^x \cdot \frac{1}{D} \left[-x \cos x - \int 1 \, (-\cos x) \, dx \right]$

$= e^x \cdot \frac{1}{D} \left[-x \cos x + \sin x \right]$

$= e^x \int \left[-x \cos x + \sin x \right] dx$

$= -e^x \int x \cos x \, dx + e^x \int \sin x \, dx$

$= -e^x \left[x \sin x - \int 1 \cdot \sin x \, dx \right] + e^x(-\cos x).$

$= -e^x (x \sin x + 2 \cos x)$

$\therefore$ gen. solⁿ is $y = C.F. + P.I.$

(7) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$

(8) $\frac{d^2y}{dx^2} + y = \sin 2x \cdot \sin x$

(9) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^{2x}$

(10)* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x\sin x$

(10) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x\sin x$

or $(D^2 + 2D + 1)y = x\sin x$

A. eqⁿ is

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$\therefore C.F. = (C_1 + C_2 x)e^{-x}$

$P.I. = \frac{1}{D^2 + 2D + 1} x\sin x$

$= \frac{1}{(D+1)^2} \cdot x\sin x$

We know $e^{ix} = \cos x + i\sin x$

$\therefore xe^{ix} = x\cos x + xi\sin x$

We consider $\frac{1}{(D+1)^2} xe^{ix}$

$= e^{ix} \frac{1}{(D+i+1)^2} \cdot x = e^{ix} \cdot \frac{1}{(1+i)^2\left(1 + \frac{D}{1+i}\right)^2} x$

$$= \frac{e^{ix}}{2i}\left(1+\frac{D}{1+i}\right)^{-2} x$$

$$= -\frac{1}{2} i e^{ix}\left[1 - 2\frac{D}{1+1} + 3\frac{D^2}{(1+i)^2} - \cdots\right] x$$

$$= -\frac{1}{2} i e^{ix}\left(x - \frac{2}{i+1}\right)$$

$$= -\frac{1}{2} i(\cos x + i\sin x)\left(x - \frac{\cancel{2}(1-i)}{\cancel{2}}\right)$$

$$= -\frac{1}{2}(i\cos x - \sin x)(x-1+i)$$

$$= \frac{1}{2}(\sin x - i\cos x)(x-1+i)$$

$$= \frac{1}{2}\{(x-1)\sin x + \cos x\} + i\frac{1}{2}\{\sin x + (1-x)\cos x\}$$

$\therefore$ P.I. = imaginary part of $\frac{1}{(D+1)^2} x e^{ix}$

$$= \frac{1}{2}\{\sin x + (1-x)\cos x\}$$

$\therefore$ general solⁿ is

$$y = C.F. + P.I.$$

$$= (C_1 + C_2 x)e^{-x} + \frac{1}{2}\{\sin x + (1-x)\cos x\}$$

(11) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin x$

or $\cancel{D^2-2D}\,(D^2-2D+2)y = \sin x$

Aux. eq^n is:-

$m^2 - 2m + 2 = 0$

$\Rightarrow (m-1)^2 + 1 = 0$

$\Rightarrow (m-1) = \pm i$

$m = 1+i, 1-i$

C.F. =

$$P.I. = \frac{1}{D^2-2D+2}\sin x$$

$$= \frac{1}{-1^2-2D+2}\sin x$$

$$= \frac{1}{1-2D}\sin x$$

$$= (1+2D)\cdot\frac{1}{(1+2D)(1-2D)}\sin x$$

$$= (1+2D)\cdot\frac{1}{1-4D^2}\sin x$$

$$= (1+2D)\cdot\frac{1}{(1-4(-1^2))}$$

$$= (1+2D)\cdot\frac{1}{5}\sin x$$

$$= \frac{1}{5}(\sin x + 2\cos x)$$

$\therefore$ gen sol^n is

$y = C.F. + P.I.$

=

* Method of variation of parameter :-

Consider the 2^{nd} order linear d. eqn

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad \text{--- (1)}$$

Let $C.F. = C_1U + C_2V$ --- (2)

where C_1 and C_2 are arbitray constant and U, V are functions of x.

Let the complete soln of (1) is given by

$$Y = AU + BV \quad \text{--- (3)}$$

where A & B are functions of x.

$$\frac{dY}{dx} = A\frac{dU}{dx} + U\frac{dA}{dx} + B\frac{dV}{dx} + V\frac{dB}{dx}$$

we choose A & B such that

$$U\frac{dA}{dx} + V\frac{dB}{dx} = 0 \quad \text{--- (4)}$$

$$\therefore \frac{dY}{dx} = A\frac{dU}{dx} + B\frac{dV}{dx} \quad \text{--- (5)}$$

$$\therefore \frac{d^2Y}{dx^2} = A\frac{d^2U}{dx^2} + \frac{dA}{dx}\frac{dU}{dx} + B\frac{d^2V}{dx^2} + \frac{dB}{dx}\cdot\frac{dV}{dx} \quad \text{--- (6)}$$

Also, U & V satisfy

$$\frac{d^2U}{dx^2} + P\frac{dU}{dx} + QU = 0 \quad \text{--- (7)}$$

$$\frac{d^2V}{dx^2} + P\frac{dV}{dx} + QV = 0 \quad \text{--- (8)}$$

Substituting the values from ③, ⑤, ⑥ in ①; we get;

$$A\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+B\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv\right)+\frac{dA}{dx}\cdot\frac{du}{dx}+\frac{dB}{dx}\cdot\frac{dv}{dx}=X$$

$$\Rightarrow \frac{dA}{dx}\cdot\frac{du}{dx}+\frac{dB}{dx}\cdot\frac{dv}{dx}=X \quad \text{—— ⑨}$$

using ⑦ & ⑧

Solving ④ & ⑨ we can find $\frac{dA}{dx}$ & $\frac{dB}{dx}$ and then integrating we can find A & B and hence the complete solⁿ $y = Au + Bv$ is obtained.

Solve :-

① $\frac{d^2y}{dx^2}+a^2y=\tan ax$

$\Rightarrow (D^2+a^2)y=\tan ax$

A. eqⁿ is $(m^2+a^2)=0$

$\Rightarrow m=\pm ia$

$=0\pm ia$

$\therefore$ C.F. $= C_1\cos ax + C_2\sin ax$

$\therefore u=\cos ax,\ v=\sin ax$

Let the complete soln is given by $y = Au+Bv$ where A & B are functions of x chosen in such a way that (by the method of variation of parameters)

$$u\frac{dA}{dx}+v\frac{dB}{dx}=0$$

$$\Rightarrow \cos ax\frac{dA}{dx}+\sin ax\frac{dB}{dx}=0 \quad \text{—— ①}$$

Also $\frac{du}{dx}\cdot\frac{dA}{dx}+\frac{dv}{dx}\cdot\frac{dB}{dx}=X$

$\Rightarrow -a \sin ax \frac{dA}{dx} + a \cos ax \frac{dB}{dx} = \tan ax$ —— (2)

(1) $\times$ $a \cos ax$ — (2) $\times$ $\sin ax$ gives

$$a(\cos^2 ax + \sin^2 ax)\frac{dA}{dx} = 0 - \sin ax \times \tan ax$$

$$\Rightarrow \frac{dA}{dx} = -\frac{1}{a} \times \frac{\sin^2 ax}{\cos ax} = -\frac{1}{a} \times \frac{1-\cos^2 ax}{\cos ax}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{a}(\cos ax - \sec ax)$$

$$A = \frac{1}{a}\int(\cos ax - \sec ax)\,dx + c_1$$

$$= \frac{1}{a}\left[\frac{1}{a}\sin ax - \frac{1}{a}\log(\sec ax + \tan ax)\right] + c_1$$

$$= \frac{1}{a^2}\left[\sin ax - \log(\sec ax + \tan ax)\right] + c_1$$

(1) $\times$ $a \sin ax$ + (2) $\times$ $\cos ax$ gives:

$$a(\sin^2 ax + \cos^2 ax)\frac{dB}{dx} = 0 + \cos ax \times \tan ax$$

$$\Rightarrow a\frac{dB}{dx} = \sin ax$$

$$\Rightarrow \frac{dB}{dx} = \frac{1}{a}\sin ax$$

$$\Rightarrow B = \frac{1}{a}\int \sin ax + c_2$$

$$= -\frac{1}{a^2}\cos ax + c_2$$

$\therefore$ complete solⁿ is:

$$Y = AU + BV$$

$$= \left[\frac{1}{a^2}\{\sin ax - \log(\sec ax + \tan ax)\} + c_1\right]\cos ax$$

$$+ \left[\frac{-1}{a^2}\cos ax + c_2\right]\sin ax$$

$$\therefore\ y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}\left[\sin ax - \cos ax - \log(\sec ax + \tan ax)\right]$$

(2) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \log x$

(3) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

solⁿ. or $(D^2 - 1)y = \dfrac{2}{1+e^x}$

Aux. eqⁿ is $(m^2 - 1) = 0$

$$m = \pm 1$$

$$\therefore\ C.F. = c_1 e^x + c_2 e^{-x}$$

$$\therefore\ U = e^x,\ V = e^{-x}$$

$$U\frac{dA}{dx} + V\frac{dB}{dx} = 0$$

$$\Rightarrow e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \quad \text{--- (1)}$$

Also; $\dfrac{dU}{dx}\cdot\dfrac{dA}{dx} + \dfrac{dV}{dx}\cdot\dfrac{dB}{dx} = X$

$$\Rightarrow e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{1+e^x} \quad \text{--- (II)}$$

(1) + (II) $\Rightarrow$ $$2e^x\frac{dA}{dx} = \frac{2}{1+e^x}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{e^x(1+e^x)}$$

$$\therefore A = \int \frac{dx}{e^x(e^x+1)} + c_1$$

$$= \int \frac{e^x+1-e^x}{e^x(e^x+1)}\,dx + c_1$$

$$= \int\left(\frac{1}{e^x} - \frac{1}{e^x+1}\right)dx + c_1$$

$$A = \int e^{-x}dx - \int \frac{e^{-x}}{1+e^{-x}}\,dx + c_1$$

$$\underline{A = -e^{-x} + \log(1-e^{-x}) + c_1}$$

(I) − (II) $\Rightarrow \cancel{2}e^{-x}\frac{dB}{dx} = \frac{\cancel{[illegible]}}{1+e^x}$

$$\Rightarrow \frac{dB}{dx} = -\frac{e^x}{1+e^x}$$

$$\therefore B = -\int \frac{e^x\,dx}{1+e^x} + c_2$$

$$\underline{B = -\log(1+e^x) + c_2}$$

$\therefore$ Complete solⁿ $y = AU + BV$

* Method of undetermined coefficient for finding P.I. of a linear d-eq^n.

$f(D) = X$:-

→ This method is applicable only when X contains terms in particular forms. The form of P.I. depends on the form of X and the following table suggests the form trial P.I. to be used corresponding to special form of X.

SL.No.	Form of X	Form of trial P.I.
1.	Ke^{ax}	Ae^{ax}
2.	Kx^n or $a_0+a_1x+a_2x^2+\ldots+a_nx^n$	$A_0+A_1x+A_2x^2+\ldots+A_nx^n$
3.	Kx^ne^{ax} or $e^{ax}(a_0+a_1x+a_2x^2+\ldots+a_nx^n)$	$e^{ax}(A_0+A_1x+A_2x^2+\ldots+A_nx^n)$
4.	$(K\sin ax)$ or $(K\cos ax)$ or $(K_1\sin ax+K_2\cos ax)$	$A\sin ax+B\cos ax$
5.	$(Ke^{ax}\sin bx)$ or $(Ke^{ax}\cos bx)$ or $e^{ax}(K_1\sin bx+K_2\cos bx)$	$e^{ax}(A\sin bx+B\cos bx)$
6.	$Kx^n\sin ax$ or $Kx^n\cos ax$ or $(a_0+a_1x+a_2x^2+\ldots+a_nx^n)\sin ax$ or $(\ldots)\cos ax$	$(A_0+A_1x+\ldots+A_nx^n)\sin ax+(B_0+B_1x+\ldots+B_nx^n)\cos ax$

NOTE:- i) when X is a sum of functions in column II, then trial form of P.I. is sum of corresponding functions in column III.

ii) If a term of C.F., u(say) is also a term of X and u corresponds to a root of auxiliary eq^n which is repeated r times, then trial form of P.I. must contain the term ($x^r u$ terms arising from it by differentiation)

for eg:- consider the d.eq^n

$$(D-3)^2(D+2)y = e^{3x}+x^2$$

Auxiliary eq^n:-

$$(m-3)^2(m+2)=0 \Rightarrow m = 3, 3, -2$$

$C.F. = (C_1 + C_2 x) e^{3x} + C_3 e^{-2x}$

x^2 is not occuring in C.F. so, the contribution of x^2 in the trial P.I. is

$(A_0 + A_1 x + A_2 x^2 + \ldots) +$

$U = e^{3x}$ is present in both C.F. & X where $m = 3$ is twice repeated root. Hence, contribution of e^{3x} in trial P.I is

$A_3 e^{3x} + A_4 x e^{3x} + A_5 x^2 e^{3x}$

$\therefore$ Trial P.I. is

$y = A_0 + A_1 x + A_2 x^2 + A_3 e^{3x} + A_4 x e^{3x} + A_5 x^2 e^{3x}$

iii) If a term of X is $x^K U$ and u is a part of C.F corresponding to a root of auxiliary eqn which is repeated r times then the trial P.I. must contain the term $x^{K+r} u$ plus terms arising from it by differentiation.

eg:- consider the d. eqn

$(D-2)^3 (D+3) y = x^2 e^{2x}$

A. eqn is $(m-2)^3 (m+3) = 0$

$\Rightarrow m = 2, 2, -3, 2$

$\therefore$ C.F. is $(C_1 + C_2 x + C_3 x^2) e^{2x} + C_4 e^{-3x}$

The trial form of P.I. is

$y = A_0 e^{2x} + A_1 x e^{2x} + A_2 x^2 e^{2x} + A_3 x^3 e^{2x} + A_4 x^4 e^{2x} + A_5 x^5 e^{2x}$

Solve:-

(1) $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 3y = x^3 + \cos x$.

Soln or $(D^2 - 2D + 3)y = x^3 + \cos x$

$\therefore$ Aux. eqn is $m^2 - 2m + 3$

$\therefore m = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm i\sqrt{2}$

$\therefore$ C.F. $= e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x)$

Let the trial form of P.I. is

$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4\cos x + A_5\sin x$

where $A_0, A_1, A_2, A_3, \ldots A_5$ are undetermined co-efficient

$\therefore \frac{dy}{dx} = A_1 + 2A_2x + 3A_3x^2 - A_4\sin x + A_5\cos x$

$\frac{d^2y}{dx^2} = 2A_2 + 6A_3x - A_4\cos x - A_5\sin x$

Substituting terms in the given d.eqn we get;

$\Rightarrow (2A_2 + 6A_3x - A_4\cos x - A_5\sin x) - 2(A_1 + 2A_2x + 3A_3x^2 - A_4\sin x + A_5\cos x) + 3(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4\cos x + A_5\sin x) = x^3 + \cos x$

Which is an identity

$\Rightarrow (3A_3)x^3 + (3A_2 - 6A_3)x^2 + (6A_3 - 4A_2 + 3A_1)x + (3A_4 - 2A_5 - A_4)\cos x + (2A_4 + 3A_5 - A_5)\sin x + (2A_2 - 2A_1 + 3A_0) = x^3 + \cos x$

$\Rightarrow 3A_3 = 1$	$3A_2 - 6A_3 = 0$	$6A_3 - 4A_2 + 3A_1 = 0$
$\Rightarrow A_3 = \frac{1}{3}$	$\Rightarrow 3A_2 = 2$	$\Rightarrow 2 - \frac{8}{3} + 3A_1 = 0$
	$A_2 = \frac{2}{3}$	$\Rightarrow A_1 = \frac{2}{9}$

$2A_2 - 2A_1 + 3A_0 = 0 \Rightarrow 3A_0 = 2A_1 - 2A_2 = \frac{4}{9} - \frac{4}{3} = \frac{-8}{9}$

$\Rightarrow A_0 = \frac{-8}{27}$

$-A_4 - 2A_5 + 3A_4 = 1$

$\Rightarrow A_4 - A_5 = \frac{1}{2}$ —— (I)

$-A_4 + 2A_4 + 3A_5 = 0$

$\Rightarrow A_4 + A_5 = 0$ —— (II)

(I) + (II) $\Rightarrow 2A_4 = \frac{1}{2}$

$A_4 = \frac{1}{4}$

$A_5 = \frac{-1}{4}$

$\therefore$ complete solⁿ is

$y = C.F. + P.I.$

$y = e^{ax}\left(A\cos\sqrt{2}x + B\sin\sqrt{2}x\right) - \frac{8}{27} + \frac{2}{9}x + \frac{2}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{4}(\cos x - \sin x)$

(2) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x$

or $(D^2 - 5D + 6)y = e^{3x} + \sin x$

Aux. eq^n :-

$m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3.$

$\therefore$ C.F. :- $y = C_1 e^{2x} + C_2 e^{3x}$

$U = e^{3x}$ is a part of C.F. corresponding to $m = 3$. occuring once and U is also present in X.

$\therefore$ Trial P.I. is :-

$y = A_0 e^{3x} + A_1 x e^{3x} + A_2 \cos x + A_3 \sin x$

$\frac{dy}{dx} = 3A_0 e^{3x} + A_1 e^{3x} + 3A_1 x e^{3x} - A_2 \sin x + A_3 \cos x$

(3) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^3 e^{2x}$

or $(D^2 - 4D + 4)y = x^3 e^{2x}$

A. eqⁿ is :-

$m^2 - 4m + 4 = 0$

$\Rightarrow m = 2, 2$

$\therefore$ C.F. $= (C_1 + C_2 x) e^{2x}$

Here $u = e^{2x}$ is a part of C.F. corresponding to double root $m = 2$ & also

$x^3 u$ is in X.

$\therefore$ trial P.I. is

$y = A_2 x^2 e^{2x} + A_3 x^3 e^{2x} + A_4 x^4 e^{2x} + A_5 x^5 e^{2x}$

(because e^{2x} & $x e^{2x}$ terms are already present in C.F.)

$y = x^2 (A_2 e^{2x} + A_3 x e^{2x} + A_4 x^2 e^{2x} + A_5 x^3 e^{2x})$

$\dfrac{dy}{dx} = 2A_2 x e^{2x} + 2A_2 x^2 e^{2x} + 3A_3 x^2 e^{2x} + 2A_3 x^3 e^{2x} + 4A_4 x^3 e^{2x}$
$+ 2A_4 x^4 e^{2x} + 5A_5 x^5 e^{2x} + 2A_5 x^5 e^{2x}$