Comprehensive Problem Set: Chain Rule and Implicit Differentiation

From Basic to Advanced Applications

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1 Level 1: Basic Problems (Beginner)

1.1 Direct Chain Rule Applications

Problem 1.1 **(5 points)**

Given $z = x^2 + y^2$, with $x = 2t$ and $y = 3t$, find $\frac{dz}{dt}$.

Problem 1.2 **(5 points)**

If $z = \sin(xy)$, $x = t^2$, and $y = t^3$, find $\frac{dz}{dt}$.

Problem 1.3 **(5 points)**

For $w = e^{x+y}$, where $x = \cos t$ and $y = \sin t$, compute $\frac{dw}{dt}$.

1.2 Basic Implicit Differentiation

Problem 1.4 **(5 points)**

Find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$.

Problem 1.5 **(5 points)**

Find $\frac{dy}{dx}$ for $x^3 + y^3 = 3xy$ at the point $(1, 1)$.

Problem 1.6 **(5 points)**

Find $\frac{dy}{dx}$ for $e^{xy} = x + y$.

2 Level 2: Intermediate Problems

2.1 Multiple Variables Chain Rule

Problem 2.1 **(10 points)**

If $z = \ln(x^2 + y^2)$, $x = e^t$, and $y = e^{-t}$, find $\frac{dz}{dt}$.

Problem 2.2 **(10 points)**

Given $w = \sqrt{x^2 + y^2 + z^2}$, with $x = \sin t$, $y = \cos t$, $z = t$, find $\frac{dw}{dt}$.

Problem 2.3 **(10 points)**

If $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$.

2.2 Higher Order Implicit Differentiation

Problem 2.4 **(10 points)**

For the circle $x^2 + y^2 = r^2$, find $\frac{d^2y}{dx^2}$.

Problem 2.5 **(10 points)**

For $y^2 = x^3$, find $\frac{d^2y}{dx^2}$.

Problem 2.6 **(10 points)**

For $\sin(x + y) = y \cos x$, find $\frac{dy}{dx}$.

3 Level 3: Advanced Problems

3.1 Composite Functions with Three Variables

Problem 3.1 **(15 points)**

Let $w = f(x, y, z)$, where $x = u + v$, $y = u - v$, $z = uv$. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of f_x, f_y, f_z.

Problem 3.2 **(15 points)**

If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Problem 3.3 **(15 points)**

Given $u = f(x, y)$, $x = r \cos\theta$, $y = r \sin\theta$, show that:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

3.2 Implicit Systems

Problem 3.4 **(15 points)**

The system

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + 2y + 3z = 0 \end{cases}$$

defines y and z as functions of x. Find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Problem 3.5 **(15 points)**

If $F(x, y, z) = 0$ defines z as a function of x and y, show that:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Problem 3.6 **(15 points)**

For the surface $z^3 + xyz - 2 = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1, 1)$.

4 Level 4: Applied Problems

4.1 Physics Applications

Problem 4.1 **(20 points)**

A particle moves along the path $x = \cos t$, $y = \sin t$, $z = t$. The temperature at point (x, y, z) is $T = x^2 + y^2 + z^2$. Find the rate of change of temperature with respect to time.

Problem 4.2 **(20 points)**

In thermodynamics, the ideal gas law is $PV = nRT$. If pressure P and volume V are functions of time, find $\frac{dT}{dt}$ in terms of $\frac{dP}{dt}$ and $\frac{dV}{dt}$.

Problem 4.3 **(20 points)**

A ladder 10 ft long rests against a vertical wall. If the bottom slides away at 1 ft/s, how fast is the top sliding down when the bottom is 6 ft from the wall?

4.2 Geometry Applications

Problem 4.4 **(20 points)**

Find the equation of the tangent line to the curve $x^3 + y^3 = 9$ at the point $(1, 2)$.

Problem 4.5 **(20 points)**

For the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the points where the tangent line is horizontal.

Problem 4.6 **(20 points)**

Show that the curve $x^3 + y^3 = 3xy$ (Folium of Descartes) has a horizontal tangent at $(2^{2/3}, 2^{1/3})$ and a vertical tangent at $(2^{1/3}, 2^{2/3})$.

5 Level 5: Challenging Problems

5.1 Advanced Chain Rule

Problem 5.1 **(25 points)**

If $u = f(x, y)$, $x = e^s \cos t$, $y = e^s \sin t$, show that:

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^{2s}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Problem 5.2 **(25 points)**

Let $z = f(x, y)$, where $x = u^2 - v^2$ and $y = 2uv$. Find $\frac{\partial^2 z}{\partial u^2}$ in terms of partial derivatives of f.

Problem 5.3 **(25 points)**

If $w = f(u, v)$, $u = x^2 - y^2$, $v = 2xy$, show that:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$$

5.2 Advanced Implicit Differentiation

Problem 5.4 **(25 points)**

For the surface defined by $F(x, y, z) = 0$, show that:

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

(This is the cyclic rule in thermodynamics.)

Problem 5.5 **(25 points)**

The equation $z^3 - 3xyz = 1$ defines z as a function of x and y. Find $\frac{\partial^2 z}{\partial x^2}$.

Problem 5.6 **(25 points)**

Consider the system:

$$\begin{cases} u^2 + v^2 = x^2 + y^2 \\ uv = xy \end{cases}$$

Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

6 Level 6: Extremely Challenging Problems

Problem 6.1 **(30 points)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Define $g(t) = f(\cos t, \sin t)$. Show that:

$$[g'(t)]^2 \leq \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

with equality if and only if $(\cos t, \sin t)$ is parallel to ∇f.

Problem 6.2 **(30 points)**

The equations

$$\begin{cases} x = u^3 - 3uv^2 \\ y = 3u^2v - v^3 \end{cases}$$

define a transformation from (u, v) to (x, y). Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and its inverse.

Problem 6.3 **(30 points)**

A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$ for all $t > 0$.

(a) Prove Euler's theorem: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

(b) If f is homogeneous of degree n, show that $\frac{\partial f}{\partial x}$ is homogeneous of degree $n-1$.

(c) Show that if f is homogeneous of degree n, then:

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

7 Solutions

7.1 Solutions to Level 1 Problems

Solution 1.1

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x)(2) + (2y)(3) = 4x + 6y = 4(2t) + 6(3t) = 8t + 18t = 26t$

Solution 1.2

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y\cos(xy))(2t) + (x\cos(xy))(3t^2) = 2t^4\cos(t^5) + 3t^4\cos(t^5) = 5t^4\cos(t^5)$

Solution 1.3

$\frac{dw}{dt} = e^{x+y}(\frac{dx}{dt} + \frac{dy}{dt}) = e^{\cos t + \sin t}(-\sin t + \cos t)$

Solution 1.4

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Solution 1.5

$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$. At $(1,1)$: $3 + 3\frac{dy}{dx} = 3 + 3\frac{dy}{dx} \Rightarrow 0 = 0$, need second equation. Actually: $(3 - 3\frac{dy}{dx}) = (3 - 3\frac{dy}{dx}) \Rightarrow \frac{dy}{dx} = 1$.

Solution 1.6

$e^{xy}(y + x\frac{dy}{dx}) = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1-ye^{xy}}{xe^{xy}-1}$

7.2 Solutions to Level 2 Problems

Solution 2.1

$\frac{dz}{dt} = \frac{2x}{x^2+y^2}e^t + \frac{2y}{x^2+y^2}(-e^{-t}) = \frac{2e^{2t}-2e^{-2t}}{e^{2t}+e^{-2t}} = \frac{2\sinh(2t)}{\cosh(2t)} = 2\tanh(2t)$

Solution 2.2

$\frac{dw}{dt} = \frac{1}{2\sqrt{x^2+y^2+z^2}}(2x\cos t - 2y\sin t + 2z) = \frac{x\cos t - y\sin t + z}{\sqrt{x^2+y^2+z^2}}$

Solution 2.3

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,\ \frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta$

Solution 2.4

From $2x+2yy' = 0$, we have $y' = -x/y$. Differentiate: $y'' = -\frac{y-xy'}{y^2} = -\frac{y-x(-x/y)}{y^2} = -\frac{y^2+x^2}{y^3} = -\frac{r^2}{y^3}$

Solution 2.5

$2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y}$. Differentiate: $2(y')^2 + 2yy'' = 6x \Rightarrow y'' = \frac{6x-2(y')^2}{2y} = \frac{6x-\frac{9x^4}{2y^2}}{2y} = \frac{12xy^2-9x^4}{4y^3}$

Solution 2.6

$\cos(x+y)(1+y') = y'\cos x - y\sin x \Rightarrow y' = \frac{\cos(x+y)+y\sin x}{\cos x - \cos(x+y)}$

7.3 Solutions to Level 3 Problems

Solution 3.1

$\frac{\partial w}{\partial u} = f_x \frac{\partial x}{\partial u} + f_y \frac{\partial y}{\partial u} + f_z \frac{\partial z}{\partial u} = f_x(1) + f_y(1) + f_z(v) = f_x + f_y + v f_z$
$\frac{\partial w}{\partial v} = f_x \frac{\partial x}{\partial v} + f_y \frac{\partial y}{\partial v} + f_z \frac{\partial z}{\partial v} = f_x(1) + f_y(-1) + f_z(u) = f_x - f_y + u f_z$

Solution 3.2

$\frac{\partial z}{\partial s} = f_x g_s + f_y h_s$, $\frac{\partial z}{\partial t} = f_x g_t + f_y h_t$

Solution 3.3

From solution 2.3: $u_r = u_x \cos\theta + u_y \sin\theta$, $u_\theta = -u_x r \sin\theta + u_y r \cos\theta$
Then: $u_r^2 + \frac{1}{r^2} u_\theta^2 = (u_x \cos\theta + u_y \sin\theta)^2 + \frac{1}{r^2}(-u_x r \sin\theta + u_y r \cos\theta)^2$
$= u_x^2 \cos^2\theta + 2u_x u_y \cos\theta \sin\theta + u_y^2 \sin^2\theta + u_x^2 \sin^2\theta - 2u_x u_y \sin\theta \cos\theta + u_y^2 \cos^2\theta$
$= u_x^2(\cos^2\theta + \sin^2\theta) + u_y^2(\sin^2\theta + \cos^2\theta) = u_x^2 + u_y^2$

Solution 3.4

Differentiate both equations:

$$\begin{cases} 2x + 2yy' + 2zz' = 0 \\ 1 + 2y' + 3z' = 0 \end{cases}$$

From second: $z' = -\frac{1+2y'}{3}$. Substitute into first: $2x + 2yy' + 2z\left(-\frac{1+2y'}{3}\right) = 0$
Solve for y': $y' = \frac{3x-z}{2z-3y}$, then $z' = \frac{x-2y}{3y-2z}$

Solution 3.5

Differentiate $F(x, y, z(x, y)) = 0$ w.r.t. x: $F_x + F_z z_x = 0 \Rightarrow z_x = -\frac{F_x}{F_z}$
Similarly for y: $F_y + F_z z_y = 0 \Rightarrow z_y = -\frac{F_y}{F_z}$

Solution 3.6

Differentiate $z^3 + xyz - 2 = 0$ w.r.t. x: $3z^2 z_x + yz + xyz_x = 0 \Rightarrow z_x = -\frac{yz}{3z^2+xy}$
At $(1, 1, 1)$: $z_x = -\frac{1}{3+1} = -\frac{1}{4}$
Similarly: $z_y = -\frac{xz}{3z^2+xy} = -\frac{1}{4}$ at $(1, 1, 1)$

7.4 Solutions to Level 4 Problems

Solution 4.1

$\frac{dT}{dt} = 2x(-\sin t) + 2y(\cos t) + 2z(1) = -2\cos t \sin t + 2\sin t \cos t + 2t = 2t$

Solution 4.2

$P\frac{dV}{dt} + V\frac{dP}{dt} = nR\frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{1}{nR}\left(P\frac{dV}{dt} + V\frac{dP}{dt}\right)$

Solution 4.3

Let x = distance from wall, y = height. Then $x^2 + y^2 = 100$. Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 6$, $y = 8$, and $\frac{dx}{dt} = 1$: $2(6)(1) + 2(8)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{3}{4}$ ft/s.

Solution 4.4

$3x^2 + 3y^2y' = 0 \Rightarrow y' = -\frac{x^2}{y^2}$. At $(1, 2)$: $y' = -\frac{1}{4}$. Tangent line: $y - 2 = -\frac{1}{4}(x - 1)$

Solution 4.5

$\frac{x}{2} + \frac{2y}{9}y' = 0 \Rightarrow y' = -\frac{9x}{4y}$. Horizontal when $y' = 0 \Rightarrow x = 0$. Then $\frac{0}{4} + \frac{y^2}{9} = 1 \Rightarrow y = \pm 3$. Points: $(0, 3)$ and $(0, -3)$.

Solution 4.6

$3x^2 + 3y^2y' = 3y + 3xy' \Rightarrow y' = \frac{y-x^2}{y^2-x}$
Horizontal when $y' = 0 \Rightarrow y = x^2$. Substitute: $x^3 + x^6 = 3x^3 \Rightarrow x^6 = 2x^3 \Rightarrow x^3 = 2 \Rightarrow x = 2^{1/3}, y = 2^{2/3}$
Vertical when denominator $0 \Rightarrow y^2 = x \Rightarrow y = x^{1/2}$. Substitute: $x^3 + x^{3/2} = 3x^{3/2} \Rightarrow x^3 = 2x^{3/2} \Rightarrow x^{3/2} = 2 \Rightarrow x = 2^{2/3}, y = 2^{1/3}$

7.5 Solutions to Level 5 Problems

Solution 5.1

$x_s = e^s \cos t, x_t = -e^s \sin t, y_s = e^s \sin t, y_t = e^s \cos t$
$u_s = u_x x_s + u_y y_s = e^s(u_x \cos t + u_y \sin t)$
$u_{ss} = e^s(u_x \cos t + u_y \sin t) + e^{2s}(u_{xx} \cos^2 t + 2u_{xy} \cos t \sin t + u_{yy} \sin^2 t)$
$u_t = u_x x_t + u_y y_t = e^s(-u_x \sin t + u_y \cos t)$
$u_{tt} = e^{2s}(u_{xx} \sin^2 t - 2u_{xy} \sin t \cos t + u_{yy} \cos^2 t) - e^s(u_x \cos t + u_y \sin t)$
Adding: $u_{ss} + u_{tt} = e^{2s}(u_{xx} + u_{yy})$

Solution 5.2

$z_u = f_x(2u) + f_y(2v)$
$z_{uu} = 2f_x + 2u[f_{xx}(2u) + f_{xy}(2v)] + 2v[f_{yx}(2u) + f_{yy}(2v)]$
$= 2f_x + 4u^2 f_{xx} + 8uv f_{xy} + 4v^2 f_{yy}$

Solution 5.3

$w_x = f_u(2x) + f_v(2y)$, $w_y = f_u(-2y) + f_v(2x)$
$w_{xx} = 2f_u + 4x^2 f_{uu} + 8xy f_{uv} + 4y^2 f_{vv}$
$w_{yy} = -2f_u + 4y^2 f_{uu} - 8xy f_{uv} + 4x^2 f_{vv}$
Adding: $w_{xx} + w_{yy} = 4(x^2 + y^2)(f_{uu} + f_{vv})$

Solution 5.4

From $F(x, y, z) = 0$, we have: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$, $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$
Multiply: $\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right) = -1$

Solution 5.5

Differentiate $3z^2 z_x - 3yz - 3xyz_x = 0$: $z_x = \frac{yz}{z^2 - xy}$
Differentiate again: $z_{xx} = \frac{(yz_x + y_x z)(z^2 - xy) - yz(2zz_x - y - xy_x)}{(z^2 - xy)^2}$
Where $y_x = 0$ (treating y constant)

Solution 5.6

Differentiate both equations:

$$\begin{cases} 2uu_x + 2vv_x = 2x \\ vu_x + uv_x = y \end{cases}$$

Solve this linear system for u_x and v_x:

$$u_x = \frac{xu - yv}{u^2 - v^2}, \quad v_x = \frac{xv - yu}{u^2 - v^2}$$

7.6 Solutions to Level 6 Problems

Solution 6.1

$g'(t) = f_x(-\sin t) + f_y(\cos t) = \nabla f \cdot (-\sin t, \cos t)$
By Cauchy-Schwarz: $|g'(t)|^2 \leq \|\nabla f\|^2 \|(-\sin t, \cos t)\|^2 = f_x^2 + f_y^2$
Equality when $(-\sin t, \cos t)$ is parallel to ∇f

Solution 6.2

$J = \begin{pmatrix} 3u^2 - 3v^2 & -6uv \\ 6uv & 3u^2 - 3v^2 \end{pmatrix}$
$\det(J) = (3u^2 - 3v^2)^2 + 36u^2v^2 = 9(u^4 + 2u^2v^2 + v^4) = 9(u^2 + v^2)^2$
Inverse: $J^{-1} = \frac{1}{9(u^2+v^2)^2} \begin{pmatrix} 3u^2 - 3v^2 & 6uv \\ -6uv & 3u^2 - 3v^2 \end{pmatrix}$

Solution 6.3

(a) Differentiate $f(tx, ty) = t^n f(x, y)$ w.r.t. t: $xf_x(tx, ty) + yf_y(tx, ty) = nt^{n-1}f(x, y)$ Set $t = 1$: $xf_x + yf_y = nf$
(b) Differentiate $f(tx, ty) = t^n f(x, y)$ w.r.t. x: $tf_x(tx, ty) = t^n f_x(x, y) \Rightarrow f_x(tx, ty) = t^{n-1}f_x(x, y)$
(c) Differentiate Euler's theorem w.r.t. x: $f_x + xf_{xx} + yf_{xy} = nf_x \Rightarrow xf_{xx} + yf_{xy} = (n-1)f_x$
Differentiate w.r.t. y: $xf_{xy} + yf_{yy} = (n-1)f_y$
Multiply first by x, second by y, add: $x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = (n-1)(xf_x + yf_y) = n(n-1)f$

8 Summary

9 Additional Practice

Quick Practice Exercises

1. Find $\frac{dz}{dt}$ if $z = \frac{x}{y}$, $x = t^2$, $y = t^3$.
2. Find $\frac{dy}{dx}$ for $x^2y + xy^2 = 6$.
3. If $u = f(x - ct)$, show that $u_{tt} = c^2u_{xx}$.
4. For $w = \arctan(y/x)$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.
5. Find the tangent line to $y^2 = x^3(2 - x)$ at $(1, 1)$.

Learning Progression:

- **Levels 1-2**: Master basic chain rule and implicit differentiation
- **Level 3**: Handle multiple variables and systems
- **Level 4**: Apply concepts to real-world problems
- **Levels 5-6**: Tackle theoretical and advanced applications

Study Tips:

1. Start with Level 1 problems to build confidence
2. Move sequentially through levels For each problem, try solving without looking at solution first
3. Note patterns in solutions (e.g., cyclic rule appears in multiple contexts)
4. Practice derivative notation until it becomes natural