Efficient-UCBV: An Almost Optimal Algorithm using Variance Estimates

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Abstract

We propose a novel variant of the UCB algorithm (referred to as Efficient-UCB-Variance (EUCBV)) for minimizing cumulative regret in the stochastic multi-armed bandit (MAB) setting. EUCBV incorporates the arm elimination strategy proposed in UCB-Improved (Auer and Ortner 2010), while taking into account the variance estimates to compute the arms' confidence bounds, similar to UCBV (Audibert, Munos, and Szepesvári 2009). Through a theoretical analysis we establish that EUCBV incurs a *gap-dependent* regret bound of $O\left(\frac{K \log(T\Delta^2/K)}{\Delta}\right)$ after T trails, where Δ is the minimal gap between optimal and sub-optimal arms; the above bound is an improvement over that of existing state-of-the-art UCB algorithms (e.g., UCB1, UCB-Improved, UCBV, MOSS, and OCUCB). Further, EUCBV incurs a *gap-independent* regret bound of $O\left(\sqrt{KT}\right)$ which is an improvement over that of UCB1, UCBV and UCB-Improved, while being comparable with that of MOSS and OCUCB. Through an extensive numerical study we show that EUCBV significantly outperforms the popular UCB variants (like MOSS, OCUCB, etc.) as well as the Thompson sampling and Bayes-UCB algorithms.

1 Introduction

In this paper, we deal with the stochastic multi-armed bandit (MAB) setting. In its classical form, stochastic MABs represent a sequential learning problem where a learner is exposed to a finite set of actions (or arms) and needs to choose one of the actions at each timestep. After choosing (or pulling) an arm the learner receives a reward, which is conceptualized as an independent random draw from stationary distribution associated with the selected arm. Each of these rewards is random and drawn independently from the distribution associated with each arm. The mean of the reward distribution associated with an arm i is denoted by r_i whereas the mean of the reward distribution of the optimal arm $*$ is denoted by r^* such that $r_i < r^*, \forall i \in \mathcal{A}$. With this formulation the learner faces the task of balancing exploitation and exploration. In other words, should the learner pull the arm which currently has the best known estimates or explore arms more thoroughly to ensure that a correct decision

is being made. The objective in the stochastic bandit problem is to minimize the cumulative regret, which is defined as follows:

$$R_T = r^*T - \sum_{i\in\mathcal{A}} r_i z_i(T),$$

where T is the number of timesteps, and $z_i(T)$ is the number of times the algorithm has chosen arm i up to timestep T. The expected regret of an algorithm after T timesteps can be written as,

$$\mathbb{E}[R_T] = \sum_{i=1}^{K} \mathbb{E}[z_i(T)]\Delta_i,$$

where $\Delta_i = r^* - r_i$ is the gap between the means of the optimal arm and the i-th arm.

In recent years the MAB setting has garnered extensive popularity because of its simple learning model and its practical applications in a wide-range of industry defined problems, including, but not limited to, mobile channel allocations, active learning and computer simulation games.

1.1 Related Work

Bandit problems has been extensively studied in several earlier work such as (Thompson 1933), (Robbins 1952) and (Lai and Robbins 1985). Lai and Robbins in (Lai and Robbins 1985) established an asymptotic lower bound for the cumulative regret. Over the years stochastic MABs has seen several algorithms with strong regret guarantees. For further reference an interested reader can look into (Bubeck and Cesa-Bianchi 2012). The upper confidence bound algorithms balance the exploration-exploitation dilemma by linking the uncertainty in estimate of an arm with the number of times an arm is pulled, and therefore ensuring sufficient exploration. One of the earliest among these algorithms is UCB1 (Auer, Cesa-Bianchi, and Fischer 2002), which has a gap-dependent regret upper bound of $O\left(\frac{K \log T}{\Delta}\right)$, where $\Delta = \min_{i:\Delta_i>0} \Delta_i$. This result is asymptotically order-optimal for the class of distributions considered. But, the worst case gap-independent regret bound of UCB1 is found to be $O\left(\sqrt{KT \log T}\right)$. In the later work of (Audibert and Bubeck 2009), the authors propose the MOSS algorithm and showed that the worst case gap-independent regret bound of

MOSS is $O\left(\sqrt{KT}\right)$ which improves upon UCB1 by a factor of order $\sqrt{\log T}$. However, the gap-dependent regret of MOSS is $O\left(\frac{K^2 \log\left(T\Delta^2/K\right)}{\Delta}\right)$ and in certain regimes, this can be worse than even UCB1 (see (Audibert and Bubeck 2009; Lattimore 2015)).

The UCB-Improved algorithm, proposed in (Auer and Ortner 2010), is a round-based[1] variant of UCB1, that incurs a gap-dependent regret bound of $O\left(\frac{K \log(T\Delta^2)}{\Delta}\right)$, which is better than that of UCB1. On the other hand, the worst case gap-independent regret bound of UCB-Improved is $O\left(\sqrt{KT \log K}\right)$. Recently in (Lattimore 2015), the authors showed that the algorithm OCUCB achieves order-optimal gap-dependent regret bound of $O\left(\sum_{i=2}^{K} \frac{\log(T/H_i)}{\Delta_i}\right)$ where $H_i = \sum_{j=1}^{K} \min\left\{\frac{1}{\Delta_i^2}, \frac{1}{\Delta_j^2}\right\}$, and a gap-independent regret bound of $O\left(\sqrt{KT}\right)$. This is the best known gap-dependent and gap-independent regret bounds in the stochastic MAB framework. However, unlike our proposed EUCBV algorithm, OCUCB does not take into account the variance of the arms; as a result, empirically we find that our algorithm outperforms OCUCB in all the environments considered.

In contrast to the above work, the UCBV (Audibert, Munos, and Szepesvári 2009) algorithm utilizes variance estimates to compute the confidence intervals for each arm. UCBV has a gap-dependent regret bound of $O\left(\frac{K\sigma_{\max}^2 \log T}{\Delta}\right)$, where $\sigma_{\max}^2$ denotes the maximum variance among all the arms $i \in \mathcal{A}$. Its gap-independent regret bound can be inferred to be same as that of UCB1 i.e $O\left(\sqrt{KT \log T}\right)$. Empirically, (Audibert, Munos, and Szepesvári 2009) showed that UCBV outperforms UCB1 in several scenarios.

Another notable design principle which has recently gained a lot of popularity is the Thompson Sampling (TS) algorithm ((Thompson 1933), (Agrawal and Goyal 2011)) and Bayes-UCB (BU) algorithm (Kaufmann, Cappé, and Garivier 2012). The TS algorithm maintains a posterior reward distribution for each arm; at each round, the algorithm samples values from these distribution and the arm corresponding to the highest sample value is chosen. Although TS is found to perform extremely well when the reward distributions are Bernoulli, it is established that with Gaussian priors the worst case regret can be as bad as $\Omega\left(\sqrt{KT \log T}\right)$ (Lattimore 2015). The BU algorithm is an extension of the TS algorithm that takes confidence intervals into consideration while choosing arms.

The final design principle we will state is the information theoretic approach of DMED (Honda and Takemura 2010) and KL-UCB (Garivier and Cappé 2011) algorithms. The algorithm KL-UCB uses Kullback-Leibler divergence to compute the upper confidence bound for the arms. KL-UCB is stable for a short horizon and is known to reach the (Lai and

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[1]An algorithm is *round-based* if it pulls all the arms equal number of times in each round and then eliminates one or more arms that it deems to be sub-optimal.

Table 1: Regret upper bound of different algorithms

Algorithm	Gap-Dependent	Gap-Independent
EUCBV	$O\left(\dfrac{K\sigma_{\max}^2 \log\left(\frac{T\Delta^2}{K}\right)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$
UCB1	$O\left(\dfrac{K \log T}{\Delta}\right)$	$O\left(\sqrt{KT \log T}\right)$
UCBV	$O\left(\dfrac{K\sigma_{\max}^2 \log T}{\Delta}\right)$	$O\left(\sqrt{KT \log T}\right)$
UCB-Imp	$O\left(\dfrac{K \log(T\Delta^2)}{\Delta}\right)$	$O\left(\sqrt{KT \log K}\right)$
MOSS	$O\left(\dfrac{K^2 \log(T\Delta^2/K)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$
OCUCB	$O\left(\dfrac{K \log(T/H_i)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$

Robbins 1985) lower bound in the special case of Bernoulli distribution. However, (Garivier and Cappé 2011) showed that KL-UCB, MOSS and UCB1 algorithms are empirically outperformed by UCBV in the exponential distribution as they do not take the variance of the arms into consideration.

1.2 Our Contributions

In this paper we propose the Efficient-UCB-Variance (henceforth referred to as EUCBV) algorithm for the stochastic MAB setting. EUCBV combines the approach of UCB-Improved, CCB (Liu and Tsuruoka 2016) and UCBV algorithms. EUCBV by virtue of taking into account the empirical variance of the arms performs significantly better than the existing algorithms in the stochastic MAB setting. EUCBV outperforms UCBV (Audibert, Munos, and Szepesvári 2009) which also takes into account empirical variance but is less powerful than EUCBV because of the usage of exploration regulatory factor and arm elimination parameter by EUCBV. Also we carefully design the confidence interval term with the variance estimates along with the pulls allocated to each arm to balance the risk of eliminating the optimal arm against excessive optimism. Theoretically we refine the analysis of (Auer and Ortner 2010) and prove that for $T \geq K^{2.4}$ our algorithm is order optimal and enjoys a worst case gap-independent regret bound of $O\left(\sqrt{KT}\right)$ same as that of MOSS and OCUCB and better than UCBV, UCB1 and UCB-Improved. Also the gap-dependent regret bound of EUCBV is better than UCB1, UCB-Improved and MOSS but is poorer than OCUCB. However, EUCBV's gap-dependent bound matches OCUCB in the worst case scenario when all the gaps are equal. Through our theoretical analysis we establish the exact values of the exploration parameters for the best performance of EUCBV. Our proof technique is highly generic and can be easily extended to other MAB settings. An illustrative table containing the bounds is provided in Table 1.

Empirically we show that EUCBV owing to its estimating the variance of the arms performs significantly better than MOSS, OUCUB, UCB-Improved, UCB1, UCBV,

Thompson Sampling, Bayes-UCB, DMED, KL-UCB and Median Elimination algorithms. Note that except UCBV all the aforementioned algorithms do not take into account the empirical variance estimates of the arms. Also EUCBV is the first arm-elimination algorithm that takes into account the variance estimates of the arm for minimizing cumulative regret and thereby answers an open question raised by (Auer and Ortner 2010). In (Auer and Ortner 2010) the authors conjectured that an UCB-Improved like arm-elimination algorithm can greatly benefit by taking into consideration the variance of the arms. Also it is the first algorithm that follows the same proof technique of UCB-Improved and achieves a gap-independent regret bound of $O\left(\sqrt{KT}\right)$ thereby closing the gap of UCB-Improved (Auer and Ortner 2010) which achieved a gap-independent regret bound of $O\left(\sqrt{KT \log K}\right)$.

The rest of the paper is organized as follows. In Section 2 we present the EUCBV algorithm. Our main theoretical results are stated in Section 3, while the proofs are established in section 4. Section 5 contains results and discussions from our numerical experiments. Finally, we draw our conclusions in section 6 and discuss about future works.

2 Algorithm: Efficient UCB Variance

2.1 Notations: We denote the set of arms by $\mathcal{A}$, with the individual arms labeled i, where $i = 1, \ldots, K$. We denote an arbitrary round of EUCBV by m. For simplicity, we assume that the optimal arm is unique and denote it by $*$. We denote the sample mean of the rewards for an arm i at time instant t by $\hat{r}_i(t) = \frac{1}{z_i(t)} \sum_{\ell=1}^{z_i(t)} X_{i,\ell}$, where $X_{i,\ell}$ is the reward sample received when arm i is pulled for the ℓ-th time, and $z_i(t)$ is the number of times arm i has been pulled until timestep t. We denote the true variance of an arm by σ_i^2 while $\hat{v}_i(t)$ is the estimated variance, i.e., $\hat{v}_i(t) = \frac{1}{z_i(t)} \sum_{\ell=1}^{z_i(t)} (X_{i,\ell} - \hat{r}_i)^2$. Whenever there is no ambiguity about the underlaying time index t, for simplicity we neglect t from the notations and simply use $\hat{r}_i, \hat{v}_i$, and z_i to denote the respective quantities. We assume the rewards of all arms are bounded in $[0, 1]$.

2.2 The algorithm: Earlier round-based arm elimination algorithms like Median Elimination (Even-Dar, Mannor, and Mansour 2006) and UCB-Improved (Auer and Ortner 2010) mainly suffered from two basic problems:
(i) *Initial exploration:* Both of these algorithms pull each arm equal number of times in each round, and hence waste a significant number of pulls in initial explorations.
(ii) *Conservative arm-elimination:* In UCB-Improved, arms are eliminated conservatively, i.e, only after $\epsilon_m < \frac{\Delta_i}{2}$, where the quantity ϵ_m is initialized to 1 and halved after every round. In the worst case scenario when K is large, and the gaps are uniform ($r_1 = r_2 = \cdots = r_{K-1} < r^*$) and small this results in very high regret.

EUCBV algorithm which is mainly based on the arm elimination technique of the UCB-Improved algorithm remedies these by employing exploration regulatory factor ψ and arm elimination parameter ρ for aggressive elimination of sub-optimal arms. Along with these, similar to

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Algorithm 1 EUCBV

Input: Time horizon T, exploration parameters ρ and ψ.
Initialization: Set $m := 0$, $B_0 := A$, $\epsilon_0 := 1$, $M = \left\lfloor \frac{1}{2} \log_2 \frac{T}{e} \right\rfloor$, $n_0 = \left\lceil \frac{\log (\psi T \epsilon_0^2)}{2 \epsilon_0} \right\rceil$ and $N_0 = K n_0$.
Pull each arm once
for $t = K+1, .., T$ **do**

Pull arm $i \in \arg\max_{j \in B_m} \left\{ \hat{r}_j + \sqrt{\frac{\rho(\hat{v}_j + 2) \log (\psi T \epsilon_m)}{4 z_j}} \right\}$, where z_j is the number of times arm j has been pulled
   Arm Elimination
     For each arm $i \in B_m$, remove arm i from B_m if,

$$\hat{r}_i + \sqrt{\frac{\rho(\hat{v}_i + 2) \log (\psi T \epsilon_m)}{4 z_i}}$$

$$< \max_{j \in B_m} \left\{ \hat{r}_j - \sqrt{\frac{\rho(\hat{v}_j + 2) \log (\psi T \epsilon_m)}{4 z_j}} \right\}$$

   if $t \geq N_m$ and $m \leq M$ **then**
     Reset Parameters
      $\epsilon_{m+1} := \frac{\epsilon_m}{2}$
      $B_{m+1} := B_m$
      $n_{m+1} := \left\lceil \frac{\log (\psi T \epsilon_{m+1}^2)}{2 \epsilon_{m+1}} \right\rceil$
      $N_{m+1} := t + |B_{m+1}| n_{m+1}$
      $m := m + 1$

     Stop if $|B_m| = 1$ and pull $i \in B_m$ till T is reached.
   end if
end for

CCB (Liu and Tsuruoka 2016) algorithm, EUCBV uses optimistic greedy sampling whereby at every timestep it only pulls the arm with the highest upper confidence bound rather than pulling all the arms equal number of times in each round. Also, unlike the UCB-Improved, UCB1, MOSS and OCUCB algorithms (which are based on mean estimation) EUCBV employs mean and variance estimates (as in (Audibert, Munos, and Szepesvári 2009)) for arm elimination. Further, we allow for arm-elimination at every time-step, which is in contrast to the earlier work (e.g., (Auer and Ortner 2010); (Even-Dar, Mannor, and Mansour 2006)) where the arm elimination takes place only at the end of the respective exploration rounds.

3 Main Results

The main result of the paper is presented in the following theorem where we establish a regret upper bound for the proposed EUCBV algorithm.

Theorem 1 (*Gap-Dependent Bound*) *For $T \geq K^{2.4}$, $\rho =$*

$\frac{1}{2}$ and $\psi = \frac{T}{K^2}$, the regret R_T for EUCBV satisfies

$$\mathbb{E}[R_T] \leq \sum_{i \in \mathcal{A}: \Delta_i > b} \left\{ \frac{C_0 K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \left(\Delta_i + \frac{8\sigma_i^2 \log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i} \right) \right\}$$
$$+ \sum_{i \in \mathcal{A}: 0 < \Delta_i \leq b} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \max_{i \in \mathcal{A}: 0 < \Delta_i \leq b} \Delta_i T.$$

for all $b \geq \sqrt{\frac{e}{T}}$ and C_0, C_2' are integer constants.

Proof 1 (Outline) *The proof is along the lines of the technique in (Auer and Ortner 2010). It comprises of three modules. In the first module we prove the necessary conditions for arm elimination within a specified number of rounds. However, here we require some additional technical result (see Lemma 1, Lemma 2) to bound the length of the confidence intervals. Further, note that our algorithm combines the variance-estimate based approach of (Audibert, Munos, and Szepesvári 2009) with the arm-elimination technique of (Auer and Ortner 2010) (see Lemma 3). Also, while (Auer and Ortner 2010) uses Chernoff-Hoeffding bound to derive their regret bound whereas in our work we use Bernstein inequality (as in (Audibert, Munos, and Szepesvári 2009)) to obtain the bound. To bound the probability of non-uniform arm selection we use Lemma 4, Lemma 5 and Lemma 6. In the second module we bound the number of pulls required if an arm is eliminated on or before a particular number of rounds. Note that the number of pulls allocated in a round m for each arm is $n_m := \left\lceil \frac{\log(\psi T \epsilon_m^2)}{2\epsilon_m} \right\rceil$ which is much lower than the number of pulls of each arm required by UCB-Improved or Median-Elimination. Also we introduce the variance term in the bound by Lemma 7 and Lemma 8. Finally, the third module deals with case of bounding the regret, given that a sub-optimal arm eliminates the optimal arm (see Lemma 9). The detailed proof is available in Section 4.* ∎

Discussion: From the above result we see that the most significant term in the gap-dependent bound is of the order $O\left(\frac{K \log(T\Delta^2/K)}{\Delta} \right)$ which is better than the existing results for UCB1, UCBV, MOSS and UCB-Improved (see Table 1). Audibert et al. in (Audibert and Bubeck 2010) have defined the term $H_1 = \sum_{i=1}^K \frac{1}{\Delta_i^2}$, which is referred to as the hardness of a problem; Bubeck et al. in (Bubeck and Cesa-Bianchi 2012) have conjectured that the gap-dependent regret upper bound can match $O\left(\frac{K \log(T/H_1)}{\Delta} \right)$. However, in (Lattimore 2015) it is proved that the gap-dependent regret bound cannot be lower than $O\left(\sum_{i=2}^K \frac{\log(T/H_i)}{\Delta_i} \right)$, where $H_i = \sum_{j=1}^K \min\left\{ \frac{1}{\Delta_i^2}, \frac{1}{\Delta_j^2} \right\}$ (OCUCB proposed in (Lattimore 2015) achieves this bound). Further, in (Lattimore 2015) it is shown that only in the worst case scenario when all the gaps are equal (so that $H_1 = H_i = \sum_{i=1}^K \frac{1}{\Delta^2}$) the above two bounds match. In the latter scenario, we see that the gap-dependent bound of our proposed EUCBV simplifies to $O\left(\frac{K \log(T/H_1)}{\Delta} \right)$, thus matching the gap-dependent

bound of OCUCB which is order optimal.

Next, we specialize the result of Theorem 1 in Corollary 1 to obtain the gap-independent worst case regret bound.

Corollary 1 (*Gap-Independent Bound*) *When the gaps of all the sub-optimal arms are identical, i.e., $\Delta_i = \Delta = \sqrt{\frac{K \log K}{T}} > \sqrt{\frac{e}{T}}, \forall i \in \mathcal{A}$ and C_3 being an integer constant, the regret of EUCBV is upper bounded by the following gap-independent expression:*

$$\mathbb{E}[R_T] \leq \frac{C_3 K^3}{T^{\frac{1}{4}}} + 8\sigma_i^2 \sqrt{KT}.$$

The proof is given in Appendix 7.10.

Discussion: In the non-stochastic scenario, Auer et al. in (Auer et al. 2002) showed that the bound on the cumulative regret for EXP-4 is $O\left(\sqrt{KT \log K}\right)$. However, in the stochastic case, UCB1 proposed in (Auer, Cesa-Bianchi, and Fischer 2002) incurred a regret of order of $O\left(\sqrt{KT \log T}\right)$ which is clearly improvable. From the above result we see that in the gap-independent bound of EUCBV the most significant term is $O\left(\sqrt{KT}\right)$ which matches the upper bound of MOSS and OCUCB, and is better than UCB-Improved, UCB1 and UCBV (see Table 1).

4 Proofs

We first present a few technical lemmas that is required to prove the result in Theorem 1.

Lemma 1 *If $T \geq K^{2.4}$, $\psi = \frac{T}{K^2}$, $\rho = \frac{1}{2}$ and $m \leq \frac{1}{2}\log_2\left(\frac{T}{e}\right)$, then,*

$$\frac{\rho m \log(2)}{\log(\psi T) - 2m \log(2)} \leq \frac{3}{2}$$

Lemma 2 *If $T \geq K^{2.4}$, $\psi = \frac{T}{K^2}$, $\rho = \frac{1}{2}$, $m_i = \min\{m | \sqrt{4\epsilon_m} < \frac{\Delta_i}{4}\}$ and $c_i = \sqrt{\frac{\rho(\hat{v}_i + 2) \log(\psi T \epsilon_{m_i})}{4 z_i}}$, then, $c_i < \frac{\Delta_i}{4}$.*

Lemma 3 *If $m_i = \min\{m | \sqrt{4\epsilon_m} < \frac{\Delta_i}{4}\}$, $\bar{c}_i = \sqrt{\frac{\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2) \log(\psi T \epsilon_{m_i})}{4 z_i}}$ and $n_{m_i} = \frac{\log(\psi T \epsilon_{m_i})}{2\epsilon_{m_i}}$ then we can show that,*

$$\mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \leq \frac{2}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$$

Lemma 4 *If $m_i = \min\{m | \sqrt{4\epsilon_m} < \frac{\Delta_i}{4}\}$, $c_i = \sqrt{\frac{\rho(\hat{v}_i + 2) \log(\psi T \epsilon_{m_i})}{4 z_i}}$ and $n_{m_i} = \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}}$ then in the m_i-th round,*

$$\mathbb{P}\{c^* > c_i\} \leq \frac{32 K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}.$$

Lemma 5 *If $m_i = \min\{m | \sqrt{4\epsilon_m} < \frac{\Delta_i}{4}\}$, $c_i = \sqrt{\frac{\rho(\hat{v}_i + 2) \log(\psi T \epsilon_{m_i})}{4 z_i}}$ and $n_{m_i} = \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}}$ then in the m_i-th round,*

$$\mathbb{P}\{z_i < n_{m_i}\} \leq \frac{32 K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}.$$

Lemma 6 *For $T \geq K^{2.4}$, $\epsilon_{m_i} \geq \sqrt{\frac{e}{T}}$, $\psi = \frac{T}{K^2}$ and $\rho = \frac{1}{2}$,*

$$\frac{6K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} > \frac{K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}$$

Lemma 7 *For all bounded rewards in $[0,1]$, $\frac{\Delta_i}{4} \geq \frac{\Delta_i}{16\sigma_i^2+1}$.*

Lemma 8 *For two integer constants c_1 and c_2, if $c_1 < 20c_2$ then,*

$$c_1 \frac{4\sigma_i^2+1}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right) < c_2 \frac{\sigma_i^2}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right)$$

Lemma 9 *If m_* be the first round that the optimal arm $*$ gets eliminated, then we can show that the regret is upper bounded by,*

$$\sum_{m_*=0}^{max_{j \in \mathcal{A}'} m_j} \sum_{i \in \mathcal{A}'':m_i > m_*} \left(\frac{388K}{(\psi T \epsilon_{m_*})^{\frac{3\rho}{2}}}\right).T \max_{j \in \mathcal{A}'':m_j \geq m_*} \Delta_j$$

$$\leq \sum_{i \in \mathcal{A}'} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T\Delta_i}} + \sum_{i \in \mathcal{A}'' \backslash \mathcal{A}'} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{Tb}}$$

The proofs of lemmas 1 - 9 can be found in Appendix 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8 and 7.9 respectively.

Proof of Theorem 1

Proof 2 *For each sub-optimal arm $i \in \mathcal{A}$, let $m_i = \min\left\{m | \sqrt{4\epsilon_{m_i}} < \frac{\Delta_i}{4}\right\}$. Also, let $\mathcal{A}' = \{i \in \mathcal{A} : \Delta_i > b\}$ and $\mathcal{A}'' = \{i \in \mathcal{A} : \Delta_i > 0\}$. Note that as all rewards are bounded in $[0,1]$, it implies that $\sigma_i^2 \leq \frac{1}{4}$. Now, as in (Auer and Ortner 2010), we bound the regret under the following two cases:*

- *Case (a): some sub-optimal arm i is not eliminated in round m_i or before and the optimal arm $* \in B_{m_i}$*
- *Case (b): an arm $i \in B_{m_i}$ is eliminated in round m_i (or before), or there is no optimal arm $* \in B_{m_i}$*

The details of each case are contained in the following subsections.

Case (a): For simplicity, let $c_i := \sqrt{\frac{\rho(\hat{v}_i+2)\log(\psi T \epsilon_{m_i})}{4z_i}}$ denote the length of the confidence interval corresponding to arm i in round m_i. Thus, in round m_i (or before) whenever $z_i \geq n_{m_i} \geq \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}}$, by applying Lemma 2 we obtain $c_i < \frac{\Delta_i}{4}$. Now, the sufficient conditions for arm i to get eliminated by an optimal arm in round m_i is given by

$$\hat{r}_i \leq r_i + c_i, \hat{r}^* \geq r^* - c^*, c_i \geq c^* \text{ and } z_i \geq n_{m_i}. \quad (1)$$

Indeed, in round m_i suppose (1) holds, then we have

$$\hat{r}_i + c_i \leq r_i + 2c_i = r_i + 4c_i - 2c_i$$
$$< r_i + \Delta_i - 2c_i \leq r^* - 2c^* \leq \hat{r}^* - c^*$$

so that a sub-optimal arm $i \in \mathcal{A}'$ gets eliminated. Thus, the probability of the complementary event of these four conditions yields a bound on the probability that arm i is not

eliminated in round m_i. A bound on the complementary of the first condition is given by,

$$\mathbb{P}(\hat{r}_i > r_i + c_i) \leq \mathbb{P}(\hat{r}_i > r_i + \bar{c}_i) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \quad (2)$$

where

$$\bar{c}_i = \sqrt{\frac{\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T \epsilon_{m_i})}{4n_{m_i}}}.$$

From Lemma 3 we can show that $\mathbb{P}(\hat{r}_i > r_i + \bar{c}_i) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \leq \frac{2}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$. Similarly, $\mathbb{P}\{\hat{r}^* < r^* - c^*\} \leq \frac{2}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$. Summing the above two contributions, the probability that a sub-optimal arm i is not eliminated on or before m_i-th round is $\left(\frac{4}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}\right)$.

Again, from Lemma 4 and Lemma 5 we can bound the probability of the complementary of the event $c_i \geq c^*$ and $z_i \geq n_{m_i}$ is bounded by,

$$\frac{64K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}$$

Now, applying Lemma 6 we can show that $\frac{64K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}} \leq \frac{384K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$. Summing up over all arms in $\mathcal{A}'$ and bounding the regret for each arm $i \in \mathcal{A}'$ trivially by $T\Delta_i$, we obtain

$$\sum_{i \in \mathcal{A}'} \left(\frac{388KT\Delta_i}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}\right) \leq \sum_{i \in \mathcal{A}'} \left(\frac{388KT\Delta_i}{(\psi T \frac{\Delta_i^2}{4.16})^{\frac{3\rho}{2}}}\right)$$

$$\leq \sum_{i \in \mathcal{A}'} \left(\frac{388.2^{2+2\frac{3\rho}{2}}.16^{\frac{3\rho}{2}} KT^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}} \Delta_i^{2\frac{3\rho}{2}-1}}\right)$$

$$\overset{(a)}{\leq} \sum_{i \in \mathcal{A}'} \left(\frac{388.2^{2+\frac{3}{2}}.16^{\frac{3}{4}} KT^{1-\frac{3}{4}}}{(\frac{T}{K^2})^{\frac{3}{4}} \Delta_i^{2.\frac{3}{4}-1}}\right) \leq \sum_{i \in \mathcal{A}'} \frac{C_1 K^{\frac{5}{2}}}{\sqrt{T\Delta_i}}$$

Here in (a) we substitute the values of ρ and ψ and C_1 denotes a constant integer value.

*Case (b): Here, there are two sub-cases to be considered. Case $(b1)$ ($* \in B_{m_i}$ **and each** $i \in \mathcal{A}'$ **is eliminated on or before** m_i): Since we are eliminating a sub-optimal arm i on or before round m_i, it is pulled no longer than,*

$$z_i < \left\lceil \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}} \right\rceil$$

So, the total contribution of i till round m_i is given by,

$$\Delta_i \left\lceil \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}} \right\rceil \overset{(a)}{\leq} \Delta_i \left\lceil \frac{\log(\psi T (\frac{\Delta_i}{16 \times 256})^4)}{2(\frac{\Delta_i}{4\sqrt{4}})^2} \right\rceil$$

$$\leq \Delta_i \left(1 + \frac{32 \log \left(\psi T \left(\frac{\Delta_i^4}{16384}\right)\right)}{\Delta_i^2}\right) \leq \Delta_i \left(1 + \frac{512 \log \left(\psi T \Delta_i^4\right)}{\Delta_i^2}\right)$$

Here, (a) happens because $\sqrt{4\epsilon_{m_i}} < \frac{\Delta_i}{4}$. Summing over all arms in $\mathcal{A}'$ the total regret is given by,

$$\sum_{i \in \mathcal{A}'} \Delta_i \left(1 + \frac{512 \log \left(\psi T \Delta_i^4\right)}{\Delta_i^2}\right) = \sum_{i \in \mathcal{A}'} \left(\Delta_i + \frac{512 \log \left(\psi T \Delta_i^4\right)}{\Delta_i}\right)$$

$$\overset{(a)}{\leq} \sum_{i \in \mathcal{A}'} \Delta_i \left(1 + \frac{1024 \log \left(\frac{T \Delta_i^2}{K}\right)}{\Delta_i^2}\right)$$

$$\overset{(b)}{\leq} \sum_{i \in \mathcal{A}'} \left(\Delta_i + \frac{64(16\sigma_i^2 + 1) \log \left(\frac{T \Delta_i^2}{K}\right)}{\Delta_i}\right)$$

$$\overset{(c)}{\leq} \sum_{i \in \mathcal{A}'} \left(\Delta_i + \frac{8\sigma_i^2 \log \left(\frac{T \Delta_i^2}{K}\right)}{\Delta_i}\right)$$

We obtain (a) by substituting the value of ψ, (b) from Lemma 7 and (c) from Lemma 8.

Case (b2) (Optimal arm $*$ is eliminated by a sub-optimal arm): *Firstly, if conditions of Case a holds then the optimal arm $*$ will not be eliminated in round $m = m_*$ or it will lead to the contradiction that $r_i > r^*$. In any round m_*, if the optimal arm $*$ gets eliminated then for any round from 1 to m_j all arms j such that $m_j < m_*$ were eliminated according to assumption in Case a. Let the arms surviving till m_* round be denoted by $\mathcal{A}'$. This leaves any arm a_b such that $m_b \geq m_*$ to still survive and eliminate arm $*$ in round m_*. Let such arms that survive $*$ belong to $\mathcal{A}''$. Also maximal regret per step after eliminating $*$ is the maximal Δ_j among the remaining arms j with $m_j \geq m_*$. Let $m_b = \min \left\{m | \sqrt{4\epsilon_m} < \frac{\Delta_b}{4}\right\}$. Hence, the maximal regret after eliminating the arm $*$ is upper bounded by,*

$$\sum_{m_*=0}^{max_{j \in \mathcal{A}'} m_j} \sum_{i \in \mathcal{A}'' : m_i > m_*} \left(\frac{388 K}{(\psi T \epsilon_{m_*})^{\frac{3\rho}{2}}}\right) . T \max_{j \in \mathcal{A}'' : m_j \geq m_*} \Delta_j$$

Again applying Lemma 9 we can show that the above expression is upper bounded by

$$\sum_{i \in \mathcal{A}'} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T} \Delta_i} + \sum_{i \in \mathcal{A}'' \setminus \mathcal{A}'} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T} b}$$

*Finally, summing up the regrets in **Case a** and **Case b**, the total regret is given by*

$$\mathbb{E}[R_T] \leq \sum_{i \in \mathcal{A} : \Delta_i > b} \left\{\frac{C_0 K^{\frac{5}{2}}}{\sqrt{T} \Delta_i} + \left(\Delta_i + \frac{8\sigma_i^2 \log \left(\frac{T \Delta_i^2}{K}\right)}{\Delta_i}\right)\right\}$$

$$+ \sum_{i \in \mathcal{A} : 0 < \Delta_i \leq b} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T} \Delta_i} + \max_{i \in \mathcal{A} : 0 < \Delta_i \leq b} \Delta_i T.$$

where C_0, C_2' are integer constants and $C_0 = C_1 + C_2'$.

5 Experiments

In this section, we conduct extensive empirical evaluations of EUCBV against several other popular bandit algorithms. We use expected cumulative regret as the metric of comparison. We compare with the following algorithms: KL-UCB+ (Garivier and Cappé 2011), DMED (Honda and Takemura 2010), MOSS (Audibert and Bubeck 2009), UCB1 (Auer, Cesa-Bianchi, and Fischer 2002), UCB-Improved (Auer and Ortner 2010), Median Elimination (Even-Dar, Mannor, and Mansour 2006), Thompson Sampling (TS) (Agrawal and Goyal 2011), OCUCB (Lattimore 2015), Bayes-UCB (BU) (Kaufmann, Cappé, and Garivier 2012) and UCB-V (Audibert, Munos, and Szepesvári 2009)[2]. The parameters of EU-CBV algorithm for all the experiments are set as follows: $\psi = \frac{T}{K^2}$ and $\rho = 0.5$ (as in Corollary 1). Please note that KL-UCB+ empirically outperforms KL-UCB (as shown in (Garivier and Cappé 2011)).
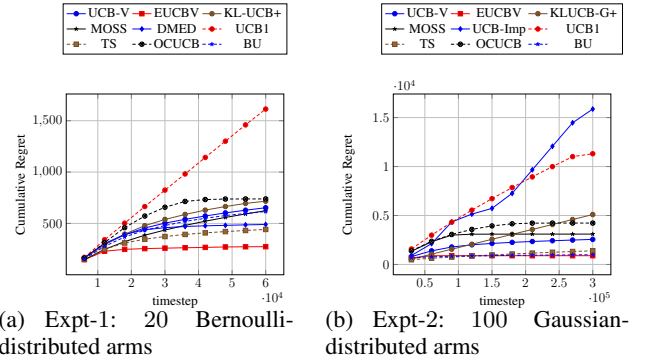

(a) Expt-1: 20 Bernoulli-distributed arms

(b) Expt-2: 100 Gaussian-distributed arms

Figure 1: A comparison of the cumulative regret incurred by the various bandit algorithms.

Experiment-1 (Bernoulli with uniform gaps): This experiment is conducted to observe the performance of EU-CBV over a short horizon. The horizon T is set to 60000. The testbed comprises of 20 Bernoulli distributed arms with expected rewards of the arms as $r_{1:19} = 0.07$ and $r_{20}^* = 0.1$ and these type of cases are frequently encountered in web-advertising domain (see (Garivier and Cappé 2011)). The regret is averaged over 100 independent runs and is shown in Figure 1(a). EUCBV, MOSS, OCUCB, UCB1, UCB-V, KL-UCB+, TS, BU and DMED are run in this experimental setup. Not only do we observe that EUCBV performs better than all the non-variance based algorithms such as MOSS, OCUCB, UCB-Improved and UCB1, but it also outperforms UCBV because of the choice of the exploration parameters. Because of the small gaps and short horizon T, we do not compare with UCB-Improved and Median Elimination for this test-case.

Experiment-2 (Gaussian 3 Group Setting): This experiment is conducted to observe the performance of EU-CBV over a large horizon in Gaussian distribution testbed. This setting comprises of a large horizon of $T = 3 \times 10^5$ timesteps and a large set of arms. This testbed comprises of

---

[2]The implementation for KL-UCB, Bayes-UCB and DMED were taken from (Cappe, Garivier, and Kaufmann 2012)

100 arms involving Gaussian reward distributions with expected rewards of the arms are in 3 groups, $r_{1:66} = 0.07$, $r_{67:99} = 0.01$ and $r^*_{100} = 0.09$ with variance set as $\sigma^2_{1:66} = 0.01, \sigma^2_{67:99} = 0.25$ and $\sigma^2_{100} = 0.25$. The regret is averaged over 100 independent runs and is shown in Figure 1(b). From the results in Figure 1(b), we observe that since the gaps are small and the variances of the optimal arm and the arms farthest from the optimal arm are the highest, EUCBV, which allocates pulls proportional to the variances of the arms, outperforms all the non-variance based algorithms MOSS, OCUCB, UCB1, TS, BU, UCB-Improved and Median-Elimination ($\epsilon = 0.1, \delta = 0.1$). The performance of Median-Elimination is extremely weak in comparison with the other algorithms and its plot is not shown in Figure 1(b). We omit its plot in order to more clearly show the difference between EUCBV, MOSS and OCUCB. Also note that the order of magnitude in the y-axis (cumulative regret) of Figure 1(b) is 10^4. KL-UCB-Gauss+ (denoted by KLUCB-G+) which is initialized with Gaussian priors perform much worse than TS. The performance of DMED is similar to KLUCB-G+ in this setup and its plot is omitted.
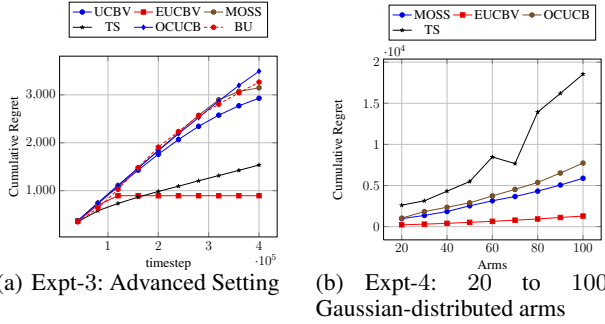

(a) Expt-3: Advanced Setting (b) Expt-4: 20 to 100 Gaussian-distributed arms

Figure 2: Further Experiments with EUCBV

Experiment-3 (Advanced Setting, failure of TS): This experiment is conducted to demonstrate that in certain environments when the horizon is large, gaps are small and the variance of the optimal arm is high, the Bayesian algorithms (like TS) do not perform well but EUCBV performs exceptionally well. This experiment is conducted on 100 Gaussian distributed arms such that expected rewards of the arms $r_{1:10} = 0.045$, $r_{34:99} = 0.04$, $r^*_{100} = 0.05$ and the variance is set as $\sigma^2_{1:10} = 0.01$, $\sigma^2_{100} = 0.25$ and $T = 4 \times 10^5$. The variance of the arms $i = 11 : 99$ are chosen uniform randomly between $[0.2, 0.249]$. We refer to this setup as Advanced Setting because here the chosen variance values are such that only variance-aware algorithms will perform very well because otherwise it will get bogged down in costly exploration. The algorithms that are not variance-aware will spend a significant amount of pulls trying to find the optimal arm. The result is shown in Figure 2(a). Predictably EUCBV, which allocates pulls proportional to the variance of the arms, outperforms its closest competitors TS, BU, UCBV, MOSS and OCUCB. The plots for KLUCB-G+, DMED, UCB1, UCB-Improved and Median Elimination are omitted from the figure as their performance is extremely weak in comparison with other algorithms. We omit their

plots to clearly show how EUCBV outperforms its nearest competitors. Note that EUCBV by virtue of its aggressive exploration parameters outperforms UCBV in all the experiments even though UCBV is a variance-based algorithm. The performance of TS is also weak and this is in line with the observation in (Lattimore 2015) that the worst case regret of TS when Gaussian prior is used is $\Omega\left(\sqrt{KT\log T}\right)$.

Experiment-4 (Gaussian with large horizon and uniform gaps): This experiment is conducted to show the stability and performance of EUCBV over a very large horizon and over a large number of arms. This testbed comprises of $20 - 100$ (interval of 10) arms with Gaussian reward distributions, where the expected rewards of the arms are $r_{i:i\neq*} = 0.05$ and $r^* = 0.1$ and variances are set as $\sigma^2_{i:i\neq*} = 0.25$ and $\sigma^2_* = 0.7$. For each of these testbeds of $20 - 100$ arms, we report the cumulative regret averaged over 100 independent runs. The horizon is set at $T = 10^5 + K^3_{20:100}$ timesteps. We report the performance of MOSS, TS and OCUCB who are the closest competitors of EUCBV over this uniform gap setup. From the results in Figure 2(b), it is evident that the growth of regret for EUCBV is much lower than that of TS, OCUCB and MOSS. Because of the poor performance of Bayes-UCB and KL-UCB-Gauss+ in the last two experiments we do not implement them in this setup. Bayes-UCB and KL-UCB-Gauss+ are not feasible to be run over such a large horizon as these algorithms are extremely slow in their execution. Also in all the experiments with Gaussian distributions EUCBV significantly outperforms all the Bayesian algorithms.

6 Conclusion and Future Works

In this paper, we studied the EUCBV algorithm which takes into account the empirical variance of the arms and employs aggressive exploration parameters to eliminate sub-optimal arms. Our theoretical analysis conclusively established that EUCBV exhibits an order-optimal gap-independent regret bound of $O\left(\sqrt{KT}\right)$. Empirically we show that EUCBV performs superbly across diverse experimental settings and outperforms most of the bandit algorithms in stochastic MAB setup. Our experiments show that EUCBV is extremely stable for larger horizons and performs consistently well across different types of distributions. One avenue for future work is to remove the constraint of $T \geq K^{2.4}$ required for EUCBV to reach the order optimal regret bound. Another future direction is to come up with an anytime version of EUCBV. An anytime algorithm does not need the horizon T as an input parameter.

References

[Agrawal and Goyal 2011] Agrawal, S., and Goyal, N. 2011. Analysis of thompson sampling for the multi-armed bandit problem. *arXiv preprint arXiv:1111.1797*.

[Audibert and Bubeck 2009] Audibert, J.-Y., and Bubeck, S. 2009. Minimax policies for adversarial and stochastic bandits. In *COLT*, 217–226.

[Audibert and Bubeck 2010] Audibert, J.-Y., and Bubeck, S. 2010. Best arm identification in multi-armed bandits. In *COLT-23th Conference on Learning Theory-2010*, 13–p.

[Audibert, Munos, and Szepesvári 2009] Audibert, J.-Y.; Munos, R.; and Szepesvári, C. 2009. Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. *Theoretical Computer Science* 410(19):1876–1902.

[Auer and Ortner 2010] Auer, P., and Ortner, R. 2010. Ucb revisited: Improved regret bounds for the stochastic multi-armed bandit problem. *Periodica Mathematica Hungarica* 61(1-2):55–65.

[Auer et al. 2002] Auer, P.; Cesa-Bianchi, N.; Freund, Y.; and Schapire, R. E. 2002. The nonstochastic multiarmed bandit problem. *SIAM Journal on Computing* 32(1):48–77.

[Auer, Cesa-Bianchi, and Fischer 2002] Auer, P.; Cesa-Bianchi, N.; and Fischer, P. 2002. Finite-time analysis of the multiarmed bandit problem. *Machine learning* 47(2-3):235–256.

[Bubeck and Cesa-Bianchi 2012] Bubeck, S., and Cesa-Bianchi, N. 2012. Regret analysis of stochastic and non-stochastic multi-armed bandit problems. *arXiv preprint arXiv:1204.5721*.

[Bubeck, Munos, and Stoltz 2011] Bubeck, S.; Munos, R.; and Stoltz, G. 2011. Pure exploration in finitely-armed and continuous-armed bandits. *Theoretical Computer Science* 412(19):1832–1852.

[Cappe, Garivier, and Kaufmann 2012] Cappe, O.; Garivier, A.; and Kaufmann, E. 2012. pymabandits. http://mloss.org/software/view/415/.

[Even-Dar, Mannor, and Mansour 2006] Even-Dar, E.; Mannor, S.; and Mansour, Y. 2006. Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *The Journal of Machine Learning Research* 7:1079–1105.

[Garivier and Cappé 2011] Garivier, A., and Cappé, O. 2011. The kl-ucb algorithm for bounded stochastic bandits and beyond. *arXiv preprint arXiv:1102.2490*.

[Honda and Takemura 2010] Honda, J., and Takemura, A. 2010. An asymptotically optimal bandit algorithm for bounded support models. In *COLT*, 67–79. Citeseer.

[Kaufmann, Cappé, and Garivier 2012] Kaufmann, E.; Cappé, O.; and Garivier, A. 2012. On bayesian upper confidence bounds for bandit problems. In *AISTATS*, 592–600.

[Lai and Robbins 1985] Lai, T. L., and Robbins, H. 1985. Asymptotically efficient adaptive allocation rules. *Advances in applied mathematics* 6(1):4–22.

[Lattimore 2015] Lattimore, T. 2015. Optimally confident ucb: Improved regret for finite-armed bandits. *arXiv preprint arXiv:1507.07880*.

[Liu and Tsuruoka 2016] Liu, Y.-C., and Tsuruoka, Y. 2016. Modification of improved upper confidence bounds for regulating exploration in monte-carlo tree search. *Theoretical Computer Science*.

[Robbins 1952] Robbins, H. 1952. Some aspects of the sequential design of experiments. In *Herbert Robbins Selected Papers*. Springer. 169–177.

[Thompson 1933] Thompson, W. R. 1933. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika* 285–294.

7   Appendix

7.1   Proof of Lemma 1

Lemma 10 *If* $T \geq K^{2.4}$, $\psi = \dfrac{T}{K^2}$, $\rho = \dfrac{1}{2}$ *and* $m \leq \dfrac{1}{2}\log_2\left(\dfrac{T}{e}\right)$, *then,*

$$\frac{\rho m \log(2)}{\log(\psi T) - 2m\log(2)} \leq \frac{3}{2}$$

Proof 3 *The proof is based on contradiction. Suppose*

$$\frac{\rho m \log(2)}{\log(\psi T) - 2m\log(2)} > \frac{3}{2}.$$

Then, with $\psi = \dfrac{T}{K^2}$ *and* $\rho = \dfrac{1}{2}$*, we obtain*

$$
\begin{aligned}
6\log(K) \quad &> \quad 6\log(T) - 7m\log(2) \\
&\overset{(a)}{\geq} \quad 6\log(T) - \frac{7}{2}\log_2\left(\frac{T}{e}\right)\log(2) \\
&= \quad 2.5\log(T) + 3.5\log_2(e)\log(2) \\
&\overset{(b)}{=} \quad 2.5\log(T) + 3.5
\end{aligned}
$$

where (a) *is obtained using* $m \leq \dfrac{1}{2}\log_2\left(\dfrac{T}{e}\right)$*, while* (b) *follows from the identity* $\log_2(e)\log(2) = 1$*. Finally, for* $T \geq K^{2.4}$ *we obtain,* $6\log(K) > 6\log(K) + 3.5$*, which is a contradiction.* ∎

7.2   Proof of Lemma 2

Lemma 11 *If* $T \geq K^{2.4}$, $\psi = \dfrac{T}{K^2}$, $\rho = \dfrac{1}{2}$, $m_i = min\{m | \sqrt{4\epsilon_m} < \dfrac{\Delta_i}{4}\}$ *and* $c_i = \sqrt{\dfrac{\rho(\hat{v}_i+2)\log(\psi T\epsilon_{m_i})}{4z_i}}$*, then,* $c_i < \dfrac{\Delta_i}{4}$*.*

Proof 4 *In the* m_i*-th round since* $z_i \geq n_{m_i}$*, by substituting* z_i *with* n_{m_i} *we can show that,*

$$
\begin{aligned}
c_i &\leq \sqrt{\frac{\rho(\hat{v}_i+2)\epsilon_{m_i}\log(\psi T\epsilon_{m_i})}{2\log(\psi T\epsilon_{m_i}^2)}} \overset{(a)}{\leq} \sqrt{\frac{2\rho\epsilon_{m_i}\log(\frac{\psi T\epsilon_{m_i}^2}{\epsilon_{m_i}})}{\log(\psi T\epsilon_{m_i}^2)}} \\
&= \sqrt{\frac{2\rho\epsilon_{m_i}\log(\psi T\epsilon_{m_i}^2) - 2\rho\epsilon_{m_i}\log(\epsilon_{m_i})}{\log(\psi T\epsilon_{m_i}^2)}} \\
&\leq \sqrt{2\rho\epsilon_{m_i} - \frac{2\rho\epsilon_{m_i}\log(\frac{1}{2^{m_i}})}{\log(\psi T\frac{1}{2^{2m_i}})}} \\
&\leq \sqrt{2\rho\epsilon_{m_i} + \frac{2\rho\epsilon_{m_i}\log(2^{m_i})}{\log(\psi T) - \log(2^{2m_i})}} \\
&\leq \sqrt{2\rho\epsilon_{m_i} + \frac{2\rho\epsilon_{m_i}m_i\log(2)}{\log(\psi T) - 2m_i\log(2)}} \\
&\overset{(b)}{\leq} \sqrt{2\rho\epsilon_{m_i} + 2.\frac{3}{2}\epsilon_{m_i}} < \sqrt{4\epsilon_{m_i}} < \frac{\Delta_i}{4}
\end{aligned}
$$

In the above simplification, (a) *is due to* $\hat{v}_i \in [0,1]$*, while* (b) *is obtained using Lemma 1.* ∎

7.3   Proof of Lemma 3

Lemma 12 *If* $m_i = min\{m | \sqrt{4\epsilon_m} < \dfrac{\Delta_i}{4}\}$, $\bar{c}_i = \sqrt{\dfrac{\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T\epsilon_{m_i})}{4z_i}}$ *and* $n_{m_i} = \dfrac{\log(\psi T\epsilon_{m_i})}{2\epsilon_{m_i}}$ *then we can show that,*

$$\mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \leq \frac{2}{(\psi T\epsilon_{m_i})^{\frac{3\rho}{2}}}$$

Proof 5 *We start by recalling that,*

$$\mathbb{P}(\hat{r}_i > r_i + c_i) \leq \mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \tag{3}$$

where

$$
\begin{aligned}
c_i &= \sqrt{\frac{\rho(\hat{v}_i+2)\log(\psi T\epsilon_{m_i})}{4z_i}} \\
\bar{c}_i &= \sqrt{\frac{\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T\epsilon_{m_i})}{4z_i}}.
\end{aligned}
$$

Note that, substituting $z_i \geq n_{m_i} \geq \dfrac{\log(\psi T\epsilon_{m_i})}{2\epsilon_{m_i}}$*,* $\bar{c}_i$ *can be simplified to obtain,*

$$\bar{c}_i \leq \sqrt{\frac{\rho\epsilon_{m_i}(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)}{2}} \leq \sqrt{\epsilon_{m_i}}. \tag{4}$$

The first term in the LHS of (3) can be bounded using the Bernstein inequality as below:

$$
\begin{aligned}
\mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) &\leq \exp\left(-\frac{(\bar{c}_i)^2 z_i}{2\sigma_i^2 + \frac{2}{3}\bar{c}_i}\right) \\
&\overset{(a)}{\leq} \exp\left(-\rho\left(\frac{3\sigma_i^2 + 3\sqrt{\epsilon_{m_i}} + 6}{6\sigma_i^2 + 2\sqrt{\epsilon_{m_i}}}\right)\log(\psi T\epsilon_{m_i})\right) \\
&\overset{(b)}{\leq} \exp\left(-\rho\log(\psi T\epsilon_{m_i})\right) \leq \frac{1}{(\psi T\epsilon_{m_i})^{\frac{3\rho}{2}}}
\end{aligned}
\tag{5}
$$

where, (a) *is obtained by substituting equation 4 and* (b) *occurs because for all* $\sigma_i^2 \in [0,1]$*,* $\left(\frac{3\sigma_i^2 + 3\sqrt{\epsilon_{m_i}} + 6}{6\sigma_i^2 + 2\sqrt{\epsilon_{m_i}}}\right) \geq \frac{3}{2}$*.*

The second term in the LHS of (2) can be simplified as follows:

$$
\begin{aligned}
&\mathbb{P}\left\{\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \\
&\leq \mathbb{P}\left\{\frac{1}{n_i}\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2 - (\hat{r}_i - r_i)^2 \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \\
&\leq \mathbb{P}\left\{\frac{\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \\
&\overset{(a)}{\leq} \mathbb{P}\left\{\frac{\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + \bar{c}_i\right\} \\
&\overset{(b)}{\leq} \exp\left(-\rho\left(\frac{3\sigma_i^2 + 3\sqrt{\epsilon_{m_i}} + 6}{6\sigma_i^2 + 2\sqrt{\epsilon_{m_i}}}\right)\log(\psi T\epsilon_{m_i})\right) \leq \frac{1}{(\psi T\epsilon_{m_i})^{\frac{3\rho}{2}}}
\end{aligned}
\tag{6}
$$

where inequality (a) is obtained using (4), while (b) follows from the Bernstein inequality.

Thus, using (5) and (6) in (3) we obtain $\mathbb{P}(\hat{r}_i > r_i + c_i) \leq \dfrac{2}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}.$ ∎

7.4   Proof of Lemma 4

Lemma 13 *If* $m_i = min\{m|\sqrt{4\epsilon_m} < \dfrac{\Delta_i}{4}\}$, $c_i = \sqrt{\dfrac{\rho(\hat{v}_i + 2)\log(\psi T \epsilon_{m_i})}{4z_i}}$ *and* $n_{m_i} = \dfrac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}}$ *then in the m_i-th round,*

$$\mathbb{P}\{c^* > c_i\} \leq \frac{32K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}.$$

Proof 6 *From the definition of c_i we know that $c_i \propto \frac{1}{z_i}$ as ψ and T are constants. Therefore in the m_i-th round,*

$$\mathbb{P}\{c^* > c_i\} \leq \mathbb{P}\{z^* < z_i\}$$
$$\leq \sum_{m=0}^{m_i} \sum_{z^*=1}^{n_{m_i}} \sum_{z_i=1}^{n_{m_i}} \left(\mathbb{P}\{\hat{r}^* < r^* - c^*\} + \mathbb{P}\{\hat{r}_i > r_i + c_i\} \right)$$

Again, we can show that

$$\sum_{m=0}^{m_i} \sum_{z^*=1}^{n_{m_i}} \sum_{z_i=1}^{n_{m_i}} \mathbb{I}_i \cup \mathbb{I}_* \leq \sum_{m=0}^{m_i} |B_{m_i}| n_{m_i}$$
$$\leq \sum_{m=0}^{m_i} \frac{4K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} \frac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}} \leq \sum_{m=0}^{m_i} \frac{4K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} \frac{2\log(\frac{T}{K})}{\epsilon_{m_i}}$$

Now, applying Bernstein inequality and following the same way as in Lemma 3 we can show that,

$$\mathbb{P}\{\hat{r}^* < r^* - c^*\} \leq \exp(-\frac{(c^*)^2}{2\sigma_*^2 + \frac{2c^*}{3}} z^*) \leq \frac{4}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$$

$$\mathbb{P}\{\hat{r}_i > r_i + c_i\} \leq \exp(-\frac{(c_i)^2}{2\sigma_i^2 + \frac{2c_i}{3}} z_i) \leq \frac{4}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}}$$

Hence, summing everything up,

$$\mathbb{P}\{c^* > c_i\}$$
$$\leq \sum_{m=0}^{m_i} \sum_{z^*=1}^{n_{m_i}} \sum_{z_i=1}^{n_{m_i}} \left(\mathbb{P}\{\hat{r}^* < r^* - c^*\} + \mathbb{P}\{\hat{r}_i > r_i + c_i\} \right)$$
$$\leq \sum_{m=0}^{m_i} \frac{32K \log T}{(\psi T \epsilon_{m_i})^{3\rho} \epsilon_{m_i}} \leq \frac{32K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}$$
∎

7.5   Proof of Lemma 5

Lemma 14 *If* $m_i = min\{m|\sqrt{4\epsilon_m} < \dfrac{\Delta_i}{4}\}$, $c_i = \sqrt{\dfrac{\rho(\hat{v}_i+2)\log(\psi T \epsilon_{m_i})}{4z_i}}$ *and* $n_{m_i} = \dfrac{\log(\psi T \epsilon_{m_i}^2)}{2\epsilon_{m_i}}$ *then in the m_i-th round,*

$$\mathbb{P}\{z_i < n_{m_i}\} \leq \frac{32K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}.$$

Proof 7 *Following a similar argument as in Lemma 4, we can show that in the m_i-th round,*

$$\mathbb{P}\{z_i < n_{m_i}\} \leq \sum_{m=0}^{m_i} \sum_{z_i=1}^{n_{m_i}} \sum_{z^*=1}^{n_{m_i}} \left(\mathbb{P}\{\hat{r}^* > r^* - c^*\} + \mathbb{P}\{\hat{r}_i < r_i + c_i\} \right)$$
$$\leq \frac{32K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}.$$
∎

7.6   Proof of Lemma 6

Lemma 15 *For $T \geq K^{2.4}$, $\epsilon_{m_i} \geq \sqrt{\frac{e}{T}}$, $\psi = \frac{T}{K^2}$ and $\rho = \frac{1}{2}$,*

$$\frac{6K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} > \frac{K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}$$

Proof 8 *We prove this lemma by contradiction. Suppose,*

$$\frac{6K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} < \frac{K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}} \qquad (7)$$

Again, we know that

$$\frac{6K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} \overset{(a)}{\leq} \frac{6K}{(\frac{T^2}{K^2}\epsilon_{m_i})^{\frac{3}{4}}} \overset{(b)}{\leq} \frac{6K}{(\frac{T^2}{K^2}\sqrt{\frac{e}{T}})^{\frac{3}{4}}} < \frac{6K^{\frac{5}{2}}}{T^{\frac{9}{8}}}$$

where, in (a) we substitute the values of ρ and ψ and (b) happens because $\epsilon_{m_i} \geq \sqrt{\frac{e}{T}}$. Similarly we can show that,

$$\frac{K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}} \overset{(a)}{\leq} \frac{K \log T}{(\frac{T^2}{K^2})^{\frac{3}{2}}} \left[1 + 2^{3\frac{1}{2}+1} \frac{2^{\frac{1}{2}\log_2 \frac{T}{e}} - 1}{2-1} \right]$$
$$\leq \frac{6K^4 \sqrt{T} \log T}{T^3} \leq \frac{6K^4 \log T}{T^{\frac{5}{2}}}$$

where, in (a) we substitute the values of ρ and ψ. Substituting the values in Equation 7 we get,

$$\frac{6K^{\frac{5}{2}}}{T^{\frac{9}{8}}} < \frac{6K^4 \log T}{T^{\frac{5}{2}}}$$
$$\frac{T^{\frac{11}{8}}}{\log T} < K^{\frac{3}{2}}$$
$$\frac{T^{\frac{11}{8}}}{\sqrt{T}} \overset{(a)}{<} K^{1.5}$$
$$K^{2.1} \overset{(b)}{<} K^{1.5}$$

where, (a) occurs because of the inequality $\log T < \sqrt{T}$ and (b) happens because of the condition that $T \geq K^{2.4}$. So clearly we can conclude that,

$$\frac{6K}{(\psi T \epsilon_{m_i})^{\frac{3\rho}{2}}} > \frac{32K \log T}{(\psi T)^{3\rho}} \sum_{m=0}^{m_i} \frac{1}{\epsilon_{m_i}^{3\rho+1}}$$
∎

7.7 Proof of Lemma 7

Lemma 16 *For all bounded rewards in* $[0,1]$, $\frac{\Delta_i}{4} \geq \frac{\Delta_i}{16\sigma_i^2 + 1}$.

Proof 9 *Since all rewards are bounded in* $[0,1]$, *we know that* $\sigma_i^2 \leq \frac{1}{4}$. *Hence we can show that,*

$$\frac{\Delta_i}{16\sigma_i^2 + 1} \leq \frac{\Delta_i}{16.\frac{1}{4} + 1}$$
$$\leq \frac{\Delta_i}{5}$$

But for all $\Delta_i \in [0,1]$ *we know that* $\frac{\Delta_i}{5} \leq \frac{\Delta_i}{4}$. *Hence,* $\frac{\Delta_i}{4} \geq \frac{\Delta_i}{16\sigma_i^2 + 1}$. ∎

7.8 Proof of Lemma 8

Lemma 17 *For two integer constants* c_1 *and* c_2, *if* $c_1 < 20c_2$ *then,*

$$c_1 \frac{16\sigma_i^2 + 1}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right) < c_2 \frac{\sigma_i^2}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right)$$

Proof 10 *We again prove this by contradiction. Suppose,*

$$c_1 \frac{16\sigma_i^2 + 1}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right) > c_2 \frac{\sigma_i^2}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right)$$

Further reducing the above two terms we can show that,

$$16c_1\sigma_i^2 + c_1 > c_2\sigma_i^2$$
$$16c_1.\frac{1}{4} + c_1 > \frac{c_2}{4}$$
$$20c_1 > c_2$$

But, we already know that $20c_1 < c_2$. *Hence,*

$$c_1 \frac{16\sigma_i^2 + 1}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right) < c_2 \frac{\sigma_i^2}{\Delta_i} \log\left(\frac{T\Delta_i^2}{K}\right)$$
∎

7.9 Proof of Lemma 9

Lemma 18 *If* m_* *be the round that the optimal arm* $*$ *gets eliminated, then we can show that the regret is upper bounded by,*

$$\sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} \left(\frac{388K}{(\psi T\epsilon_{m_*})^{\frac{3\rho}{2}}}\right).T \max_{j\in\mathcal{A}'':m_j\geq m_*} \Delta_j$$
$$\leq \sum_{i\in\mathcal{A}'} \frac{C_2'K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \frac{C_2'K^{\frac{5}{2}}}{\sqrt{T}b}$$

Proof 11

$$\sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} \left(\frac{388K}{(\psi T\epsilon_{m_*})^{\frac{3\rho}{2}}}\right).T \max_{j\in\mathcal{A}'':m_j\geq m_*} \Delta_j$$

$$\leq \sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} \left(\frac{388K\sqrt{4}}{(\psi T\epsilon_{m_*})^{\frac{3\rho}{2}}}\right).T.4\sqrt{\epsilon_{m_*}}$$

$$\leq \sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} C_2K\left(\frac{T^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}}\epsilon_{m_*}^{\frac{3\rho}{2}-\frac{1}{2}}}\right)$$

$$\leq \sum_{i\in\mathcal{A}'':m_i>m_*} \sum_{m_*=0}^{\min\{m_i,m_b\}} \left(\frac{C_2KT^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}}2^{-(\frac{3\rho}{2}-\frac{1}{2})m_*}}\right)$$

$$\leq \sum_{i\in\mathcal{A}'} \left(\frac{C_2KT^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}}2^{-(\frac{3\rho}{2}-\frac{1}{2})m_*}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{C_2KT^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}}2^{-(\frac{3\rho}{2}-\frac{1}{2})m_b}}\right)$$

$$\leq \sum_{i\in\mathcal{A}'} \left(\frac{C_2KT^{1-\frac{3\rho}{2}}.2^{\frac{3\rho}{2}-\frac{1}{4}}}{\psi^{\frac{3\rho}{2}}\Delta_i^{\frac{3\rho}{2}-\frac{1}{2}}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{C_2KT^{1-\frac{3\rho}{2}}.2^{\frac{3\rho}{2}-\frac{1}{4}}}{\psi^{\frac{3\rho}{2}}b^{\frac{3\rho}{2}-\frac{1}{2}}}\right)$$

$$\leq \sum_{i\in\mathcal{A}'} \left(\frac{C_2K2^{\frac{3\rho}{2}+\frac{19}{4}}.T^{1-\frac{3\rho}{2}}}{\psi^{\rho}\Delta_i^{2\frac{3\rho}{2}-1}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{C_2K2^{\frac{3\rho}{2}+\frac{19}{4}}.T^{1-\frac{3\rho}{2}}}{\psi^{\frac{3\rho}{2}}b^{2\frac{3\rho}{2}-1}}\right)$$

$$\overset{(a)}{\leq} \sum_{i\in\mathcal{A}'} \left(\frac{C_2'K.T^{1-\frac{3}{4}}}{(\frac{T}{K^2})^{\frac{3}{4}}\Delta_i^{2.\frac{3}{4}-1}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{C_2'KT^{1-\frac{3}{4}}}{(\frac{T}{K^2})^{\frac{3}{4}}b^{2.\frac{3}{4}-1}}\right)$$

$$\leq \sum_{i\in\mathcal{A}'} \frac{C_2'K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \frac{C_2'K^{\frac{5}{2}}}{\sqrt{T}b}$$

In the above simplification, (a) *is obtained by substituting the values of* ψ *and* ρ. ∎

7.10 Proof of Corollary 1

Proof 12 *From (Bubeck, Munos, and Stoltz 2011) we know that the function* $x \in [0,1] \mapsto x\exp(-Cx^2)$ *is decreasing on* $\left[\frac{1}{\sqrt{2C}}, 1\right]$ *for any* $C > 0$. *Thus, we take* $C = \lfloor\frac{T}{e}\rfloor$ *and choose* $\Delta_i = \Delta = \sqrt{\frac{K\log K}{T}} > \sqrt{\frac{e}{T}}$ *for all* i.

First, let us recall the result in Theorem 1 below:

$$\mathbb{E}[R_T] \leq \sum_{i\in\mathcal{A}:\Delta_i>b} \left\{\frac{C_0K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \left(\Delta_i + \frac{8\sigma_i^2\log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i}\right)\right\}$$
$$+ \sum_{i\in\mathcal{A}:0<\Delta_i\leq b} \frac{C_2'K^{\frac{5}{2}}}{\sqrt{T}\Delta_i} + \max_{i\in\mathcal{A}:0<\Delta_i\leq b} \Delta_i T.$$

Now, with $\Delta_i = \Delta = \sqrt{\frac{K\log K}{T}} > \sqrt{\frac{e}{T}}$ *we obtain,*

$$\sum_{i\in\mathcal{A}:\Delta_i>b} \frac{8\sigma_i^2\log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i} \leq \frac{8\sigma_i^2K\sqrt{T}\log\left(T\frac{K(\log K)}{TK}\right)}{\sqrt{K\log K}}$$

$$\leq \frac{8\sigma_i^2\sqrt{KT}\log\left(\log K\right)}{\sqrt{\log K}} \overset{(a)}{\leq} 2\sigma_i^2\sqrt{KT}$$

where (a) follows from the identity $\dfrac{\log\left(\log K\right)}{\sqrt{\log K}} \leq 1$ for $K \geq 2$.

For the term $\displaystyle\sum_{i\in\mathcal{A}:\Delta_i>b} \frac{C_0 K^{\frac{5}{2}}}{\sqrt{T\Delta_i}}$ by substituting the value of $\Delta_i = \Delta = \sqrt{\dfrac{K\log K}{T}}$ we get,

$$\frac{C_0 K^{\frac{5}{2}+1}}{\sqrt{T\Delta_i}} \leq \frac{C_0 K^{\frac{5}{2}}}{\sqrt{T\sqrt{\frac{K\log K}{T}}}}$$

$$\leq \frac{C_0 K^{\frac{5}{2}+1}}{(KT\log K)^{\frac{1}{4}}} \leq \frac{C_0 K^3}{T^{\frac{1}{4}}}$$

Similarly for the term $\displaystyle\sum_{i\in\mathcal{A}:0<\Delta_i\leq b} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T\Delta_i}}$ we can show that,

$$\sum_{i\in\mathcal{A}:0<\Delta_i\leq b} \frac{C_2' K^{\frac{5}{2}}}{\sqrt{T\Delta_i}} \leq \frac{C_2' K^{\frac{5}{2}+1}}{\sqrt{T\sqrt{\frac{e}{T}}}} \leq \frac{C_2' K^3}{T^{\frac{1}{4}}}$$

Thus, the total worst case gap-independent bound is given by

$$\mathbb{E}[R_T] \leq \frac{C_3 K^3}{T^{\frac{1}{4}}} + 8\sigma_i^2\sqrt{KT}.$$

where C_3 is an integer constant such that $C_3 = C_0 + C_2'$

∎