Efficient-UCBV: An Almost Optimal Algorithm using Variance Estimates

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Abstract. We propose a novel variant of the UCB algorithm (referred to as Efficient-UCB-Variance (EUCBV)) for minimizing cumulative regret in the stochastic multi-armed bandit (MAB) setting. EUCBV incorporates the arm elimination strategy proposed in UCB-Improved (Auer and Ortner, 2010), while taking into account the variance estimates to compute the arms' confidence bounds, similar to UCBV (Audibert et al., 2009). Through a theoretical analysis we establish that EUCBV incurs a *gap-dependent* regret bound of $O\left(\frac{K\log(T\Delta^2/K)}{\Delta}\right)$ after T trails, where Δ is the minimal gap between a sub-optimal and the optimal arm; the above bound is an improvement over that of existing state-of-the-art UCB algorithms (e.g., UCB1, UCBV, UCB-Improved, MOSS, and OCUCB). Further, EUCBV incurs a *gap-independent* regret bound of $O\left(\sqrt{KT}\right)$ which is an improvement over that of UCB1, UCBV and UCB-Improved, while being comparable with that of MOSS and OCUCB. Through an extensive numerical study we show that EUCBV significantly outperforms the popular UCB variants (like MOSS, OCUCB, Bayes-UCB, etc.) as well as the (non-UCB based) Thompson sampling algorithm.

Keywords: Stochastic multi-armed bandits, cumulative regret, UCB-Improved, UCBV.

1 Introduction

In this paper we deal with the stochastic multi-armed bandit (MAB) setting. In its classical form, stochastic MABs represent a sequential learning problem where a learner is exposed to a finite set of actions (or arms) and needs to choose one of the actions at each timestep. After choosing (or pulling) an arm the learner receives a reward, which is conceptualized as an independent random draw from stationary distribution associated with the selected arm. Each of these rewards is random and drawn independently from the distribution associated with each arm. The mean of the reward distribution associated with an arm i is denoted by r_i whereas the mean of the reward distribution of the optimal arm $*$ is denoted by r^* such that $r_i < r^*, \forall i \in \mathcal{A}$. With this formulation the learner faces the task of balancing exploitation and exploration. In other words, should the learner pull the arm which currently has the best known estimates or explore arms more thoroughly to ensure that a correct decision is being made. The objective in the stochastic bandit problem is to minimize the cumulative regret, which is defined as follows:

$$R_T = r^* T - \sum_{i \in \mathcal{A}} r_i z_i(T),$$

where T is the number of timesteps, $z_i(T)$ is the number of times the algorithm has chosen arm i up to timestep T. The expected regret of an algorithm after T timesteps can be written as,

$$\mathbb{E}[R_T] = \sum_{i=1}^{K} \mathbb{E}[z_i(T)] \Delta_i,$$

where $\Delta_i = r^* - r_i$ is the gap between the means of the optimal arm and the i-th arm.

In recent years the MAB setting has garnered extensive popularity because of its simple learning model and its practical applications in a wide-range of industry defined problems, including, but not limited to, mobile channel allocations, active learning and computer simulation games.

1.1 Related Work

Bandit problems has been extensively studied in several earlier work such as Thompson (1933), Robbins (1952) and Lai and Robbins (1985). The Lai and Robbins (1985) work established an asymptotic lower bound for the cumulative regret. Over the years stochastic MABs has seen several algorithms with strong regret guarantees. For further reference an interested reader can look into Bubeck et al. (2012). The upper confidence bound algorithms balance the exploration-exploitation dilemma by carefully tracking the uncertainty in estimates. One of the earliest among these algorithms is UCB1 (Auer et al., 2002a), which has a gap-dependent regret upper bound of $O\left(\frac{K \log T}{\Delta}\right)$, where $\Delta = \min_{i:\Delta_i>0} \Delta_i$. This result is asymptotically order-optimal for the class of distributions considered. But, the worst case gap-independent regret bound of UCB1 is found to be $O\left(\sqrt{KT \log T}\right)$. In the later work of Audibert and Bubeck (2009), the authors propose the MOSS algorithm and showed that the worst case gap-independent regret bound of MOSS is $O\left(\sqrt{KT}\right)$ which improves upon UCB1 by a factor of order $\sqrt{\log T}$. However, the gap-dependent regret of MOSS is $O\left(\frac{K^2 \log\left(T\Delta^2/K\right)}{\Delta}\right)$ and in certain regimes, this can be worse than even UCB1 (see Audibert and Bubeck (2009); Lattimore (2015)). The UCB-Improved algorithm, proposed in Auer and Ortner (2010), is a round-based algorithm[1] variant of UCB1 that has a gap-dependent regret bound of $O\left(\frac{K \log T\Delta^2}{\Delta}\right)$, which is better than that of UCB1. On the other hand, the worst case gap-independent regret bound of UCB-Improved is $O\left(\sqrt{KT \log K}\right)$. Recently in Lattimore (2015), the authors showed that the algorithm OCUCB achieves order-optimal

[1] An algorithm is *round-based* if it pulls all the arms equal number of times in each round and then eliminates one or more arms that it deems to be sub-optimal.

gap-dependent regret bound of $O\left(\sum_{i=2}^{K}\frac{\log(T/H_i)}{\Delta_i}\right)$ where $H_i=\sum_{j=1}^{K}\min\{\frac{1}{\Delta_i^2},\frac{1}{\Delta_j^2}\}$ and gap-independent regret bound of $O\left(\sqrt{KT}\right)$. This is the best known gap-dependent and gap-independent regret bounds in the stochastic MAB framework. However, OCUCB does not take into account the variance of the arms and we show that our algorithm outperforms OCUCB in all the environments considered.

Another interesting design principle is the UCBV (Audibert et al., 2009) algorithm which is quite different from the above algorithms owing to its utilization of variance estimates. UCBV has a gap-dependent regret bound of $O\left(\frac{K\sigma_{\max}^2\log T}{\Delta}\right)$, where $\sigma_{\max}^2$ denotes the maximum variance among all the arms $i\in\mathcal{A}$. Its gap-independent regret bound can be inferred to be same as that of UCB1 i.e $O\left(\sqrt{KT\log T}\right)$. Empirically, Audibert et al. (2009) showed that UCBV outperforms UCB1 in several scenarios.

Another notable design principle which has recently gained a lot of popularity is the Thompson Sampling (TS) (Thompson, 1933), (Agrawal and Goyal, 2011) algorithm and Bayes-UCB (BU) algorithm (Kaufmann et al., 2012) which employs the Bayesian approach in solving the MAB problem. The TS algorithm samples actions according to the posterior probability that they are optimal. Even though TS is found to perform extremely well in the Bernoulli distribution, it is established that when Gaussian priors are used the worst case regret can be as bad as $\Omega\left(\sqrt{KT\log T}\right)$ (Lattimore, 2015).

The final design principle we will state is the information theoretic approach of the DMED (Honda and Takemura, 2010) and KL-UCB (Garivier and Cappé, 2011) algorithms. The algorithm KL-UCB uses Kullbeck-Leibler divergence to compute the upper confidence bound for the arms. KL-UCB is stable for a short horizon and is known to reach the Lai and Robbins (1985) lower bound in the special case of Bernoulli distribution. But Garivier and Cappé (2011) showed that KL-UCB, MOSS and UCB1 algorithms are empirically outperformed by UCBV in the exponential distribution as they do not take the variance of the arms into consideration.

1.2 Contribution

In this paper we propose the Efficient UCB Variance (hence referred to as EUCBV) algorithm for the stochastic MAB setting. EUCBV combines the approach of UCB-Improved, CCB (Liu and Tsuruoka, 2016) and UCBV algorithms. EUCBV by virtue of taking into account the empirical variance of the arms performs significantly better than the existing algorithms in the stochastic MAB setting. EUCBV outperforms UCBV (Audibert et al., 2009) which also takes into account empirical variance but is less powerful than EUCBV because of the usage of exploration regulatory factor and arm elimination parameter by EUCBV. Also we carefully design the confidence interval term with the variance estimates along with the pulls allocated to each arm to balance the risk of eliminating the optimal arm against excessive optimism. Theoretically we refine the analysis of Auer and Ortner (2010) and prove that for $T\geq K^{2.4}$ our algorithm is order optimal and enjoys a worst case gap-independent regret bound of $O\left(\sqrt{KT}\right)$ same as that of MOSS and OCUCB and better than UCBV, UCB1 and UCB-Improved. Also the gap-dependent regret bound of EUCBV is better than UCB1, UCB-Improved and MOSS but is poorer than OCUCB. However, EUCBV's gap-dependent bound matches

OCUCB in the worst case scenario when all the gaps are equal. Through our theoretical analysis we establish the exact values of the exploration parameters for the best performance of EUCBV. Our proof technique is highly generic and can be easily extended to other MAB settings. An illustrative table containing the bounds is provided in Table 1.

Table 1: Regret upper bound of different algorithms

Algorithm	Gap-Dependent	Gap-Independent
EUCBV	$O\left(\dfrac{K\log(T\Delta^2/K)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$
UCB1	$O\left(\dfrac{K\log T}{\Delta}\right)$	$O\left(\sqrt{KT\log T}\right)$
UCBV	$O\left(\dfrac{K\sigma^2_{\max}\log T}{\Delta}\right)$	$O\left(\sqrt{KT\log T}\right)$
UCB-Imp	$O\left(\dfrac{K\log(T\Delta^2)}{\Delta}\right)$	$O\left(\sqrt{KT\log K}\right)$
MOSS	$O\left(\dfrac{K^2\log(T\Delta^2/K)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$
OCUCB	$O\left(\dfrac{K\log(T/H_i)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$

Empirically we show that EUCBV owing to its estimating the variance of the arms performs significantly better than MOSS, OUCUB, UCB-Improved, UCB1, UCBV, Thompson Sampling, Bayes-UCB, DMED, KL-UCB and Median Elimination algorithms. Please note that except UCBV all the aforementioned algorithms does not take into account the empirical variance estimates of the arms. Also EUCBV is the first arm-elimination algorithm that takes into account the variance estimates of the arm for minimizing cumulative regret and thereby answers an open question raised by Auer and Ortner (2010). In Auer and Ortner (2010) the authors conjectured that an UCB-Improved like arm-elimination algorithm can greatly benefit by taking into consideration the variance of the arms. Also it is the first algorithm that follows the same proof technique of UCB-Improved and achieves a gap-independent regret bound of $O\left(\sqrt{KT}\right)$ thereby closing the gap of UCB-Improved (Auer and Ortner, 2010) which achieved a gap-independent regret bound of $O\left(\sqrt{KT\log K}\right)$.

The rest of the paper is organized as follows. In section 2 we state the main algorithm EUCBV and in the next section 3 we state all the main results of the paper. In section 4 we establish the proofs of all the Lemma, Theorem and Corollaries and section 5 contains the numerical experiments. We conclude in section 6 and discuss about future works.

2 Algorithm: Efficient UCB Variance

2.1 Notations: We denote the set of arms by $\mathcal{A}$, with the individual arms labeled i, where $i = 1, \ldots, K$. We denote an arbitrary round of EUCBV by m. For simplicity, we

Algorithm 1 EUCBV

Input: Time horizon T, exploration parameters ρ and ψ.

Initialization: Set $m := 0$, $B_0 := A$, $\epsilon_0 := 1$, $M = \left\lfloor \frac{1}{2}\log_2 \frac{T}{e} \right\rfloor$, $n_0 = \left\lceil \frac{\log(\psi T \epsilon_0^2)}{2\epsilon_0} \right\rceil$ and $N_0 = K n_0$.

Pull each arm once

for $t = K+1, .., T$ **do**

Pull arm $i \in \arg\max_{j \in B_m} \left\{ \hat{r}_j + \sqrt{\frac{\rho(\hat{v}_j + 2)\log(\psi T \epsilon_m)}{4 z_j}} \right\}$, where z_j is the number of times arm j has been pulled

Arm Elimination

For each arm $i \in B_m$, remove arm i from B_m if,

$$\hat{r}_i + \sqrt{\frac{\rho(\hat{v}_i + 2)\log(\psi T \epsilon_m)}{4 n_m}} < \max_{j \in B_m} \left\{ \hat{r}_j - \sqrt{\frac{\rho(\hat{v}_j + 2)\log(\psi T \epsilon_m)}{4 n_m}} \right\}$$

if $t \geq N_m$ and $m \leq M$ **then**

Reset Parameters

$\epsilon_{m+1} := \frac{\epsilon_m}{2}$

$B_{m+1} := B_m$

$n_{m+1} := \left\lceil \frac{\log(\psi T \epsilon_{m+1}^2)}{2\epsilon_{m+1}} \right\rceil$

$N_{m+1} := t + |B_{m+1}| n_{m+1}$

$m := m + 1$

Stop if $|B_m| = 1$ and pull $i \in B_m$ till T is reached.

end if

end for

assume that the optimal arm is unique and denote it by $*$. We denote the sample mean of the rewards for an arm i at time instant t by $\hat{r}_i(t) = \frac{1}{z_i(t)} \sum_{\ell=1}^{z_i(t)} X_{i,\ell}$, where $X_{i,\ell}$ is the reward sample received when arm i is pulled for the z-th time. $z_i(t)$ is the number of times an arm i has been pulled till timestep t. We denote the true variance of an arm by σ_i^2 while $\hat{v}_i(t)$ is the estimated variance, i.e., $\hat{v}_i(t) = \frac{1}{z_i(t)} \sum_{\ell=1}^{z_i(t)} (X_{i,\ell} - \hat{r}_i)^2$. Whenever there is no ambiguity about the underlaying time index t, for simplicity we neglect t from the notations and simply use $\hat{r}_i$, $\hat{v}_i$, and z_i to denote the respective quantities. We assume the rewards of all arms are bounded in $[0, 1]$.

2.2 The algorithm: Earlier arm elimination algorithms like Median Elimination (Even-Dar et al., 2006) and UCB-Improved (Auer and Ortner, 2010) mainly suffered from two basic problems:
(i) *Initial exploration:* Both of these algorithms pull each arm equal number of times in each round, and hence waste a significant number of pulls in initial explorations.
(ii) *Conservative arm-elimination:* In UCB-Improved, arms are eliminated conservatively, i.e, only after $\epsilon_m < \frac{\Delta_i}{2}$, the sub-optimal arm i is discarded with high probability. The quantity ϵ_m is initialized to 1 and halved after every round. In the worst case scenario when K is large and the gaps are uniform ($r_1 = r_2 = \cdots = r_{K-1} < r^*$) and

small this results in very high regret.

EUCBV algorithm which is mainly based on the arm elimination technique of the UCB-Improved algorithm remedies these by employing exploration regulatory factor ψ and arm elimination parameter ρ for aggressive elimination of sub-optimal arms. Along with these, similar to CCB (Liu and Tsuruoka, 2016) algorithm, EUCBV uses optimistic greedy sampling whereby at every timestep it only pulls the arm with the highest upper confidence bound rather than pulling all the arms equal number of times in each round. Also, unlike the UCB-Improved, UCB1. MOSS and OCUCB algorithms (which are based on mean estimation) EUCBV employs mean and variance estimates (as in Audibert et al. (2009)) for arm elimination. Further, we allow for arm-elimination at every time-step, which is in contrast to the earlier work (e.g., Auer and Ortner (2010); Even-Dar et al. (2006)) where the arm elimination takes place only at the end of the respective exploration rounds.

3 Main Results

We present below the main theorem of the paper which establishes the regret upper bound for the EUCBV algorithm.

Main Theorem

Theorem 1 **(*Gap-dependent bound*).** *For $T \geq K^{2.4}$, $\rho = \frac{1}{2}$ and $\psi = \frac{T}{K^2}$, the regret R_T for EUCBV satisfies*

$$\mathbb{E}[R_T] \leq \sum_{i\in\mathcal{A}:\Delta_i > b} \left\{ 64K + \left(\Delta_i + \frac{64\log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i} \right) \right\} + \sum_{i\in\mathcal{A}:0<\Delta_i\leq b} 32K + \max_{i\in\mathcal{A}:0<\Delta_i\leq b} \Delta_i T$$

for all $b \geq \sqrt{\frac{e}{T}}$.

Proof. The proof comprises of three modules. In the first module we prove the necessary conditions for arm elimination within a specified number of rounds, which is motivated by the technique in Auer and Ortner (2010). We first refine the proof while breaking down the confidence interval by using Lemma 1. Then we combine the approach of Audibert et al. (2009) with that of Auer and Ortner (2010) for removing the estimated variance factor in the confidence interval term for an arm i. Note that even though Audibert et al. (2009) uses Bernstein inequality to obtain the bound, we use Chernoff-Hoeffding bound. This is because of our choice of ρ which has to be $\frac{1}{2}$ or it may lead to a regret polynomial in T and usage of Bernstein inequality will force the ρ to take a value higher than 1. The second module bounds the number of pulls required if an arm is eliminated on or before a particular number of rounds. Note that the number of pulls allocated in a round m for each arm is $n_m := \left\lceil \frac{\log(\psi T\epsilon_m^2)}{2\epsilon_m} \right\rceil$ which is much lower than the pulls of each arm required by UCB-Improved or Median-Elimination. The third module deals with bounding the regret given a sub-optimal arm eliminates the optimal arm. The detailed proof is given in Section 4.

From the above result we see that the most significant term in the gap-dependent bound is of the order $O\left(\dfrac{K\log(T\Delta^2/K)}{\Delta}\right)$ and it is better than UCB1, UCBV, MOSS and UCB-Improved. In Audibert and Bubeck (2010) the authors define the term $H_1 = \sum_{i=1}^{K}\frac{1}{\Delta_i^2}$ as the hardness of a problem and in Bubeck and Cesa-Bianchi (2012) the authors conjectured that the gap-dependent regret upper bound can match the quantity of $O\left(\dfrac{K\log(T/H_1)}{\Delta}\right)$. But Lattimore (2015) proved that the gap-dependent regret bound cannot be lower than $O\left(\sum_{i=2}^{K}\frac{\log(T/H_i)}{\Delta_i}\right)$, where $H_i = \sum_{j=1}^{K}\min\{\frac{1}{\Delta_i^2}, \frac{1}{\Delta_j^2}\}$ and only in the worst case scenario, when all the gaps are equal that $H_1 = H_i = \sum_{i=1}^{K}\frac{1}{\Delta^2}$. In such a scenario the EUCBV gap-dependent bound of $O\left(\dfrac{K\log(T\Delta^2/K)}{\Delta}\right)$ reduces to $O\left(\dfrac{K\log(T/H_1)}{\Delta}\right)$ and hence matches the gap-dependent bound of OCUCB.

Next, we specialize the result of Theorem 1 in Corollary 1 to obtain the gap-independent worst case regret.

Corollary 1 (***Gap-independent bound***). *With $\psi = \frac{T}{K^2}$, $\rho = \frac{1}{2}$ and substituting $\Delta_i = \Delta = \sqrt{\frac{K\log K}{T}}, \forall i \in \mathcal{A}$, we have the following gap-independent bound for the regret of EUCBV:*

$$\mathbb{E}[R_T] \le 96K^2 + 64\sqrt{KT}$$

Proof. In the non-stochastic scenario Auer et al. (2002b) showed that the bound on the cumulative regret can be $O\left(\sqrt{KT\log K}\right)$. But UCB1 suffered from a regret of order of $O\left(\sqrt{KT\log T}\right)$ which is clearly improvable. Also we know from Bubeck et al. (2011) that the function $x \in [0,1] \mapsto x\exp(-Cx^2)$ is decreasing on $\left[\frac{1}{\sqrt{2C}}, 1\right]$ for any $C > 0$. So, we take $C = \left\lfloor \frac{T}{e} \right\rfloor$ and choose $\Delta_i = \Delta = \sqrt{\frac{K\log K}{T}} > \sqrt{\frac{e}{T}}$ for all $i : i \neq * \in \mathcal{A}$.

We again recall the result of Theorem 1 below,

$$\begin{aligned}\mathbb{E}[R_T] \le &\sum_{i\in\mathcal{A}:\Delta_i>b}\left\{64K + \left(\Delta_i + \frac{64\log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i}\right)\right\} \\ &+ \sum_{i\in\mathcal{A}:0<\Delta_i\le b} 32K + \max_{i\in\mathcal{A}:0<\Delta_i\le b}\Delta_i T\end{aligned}$$

Substituting the parameter values $\psi = \frac{T}{K^2}$ and $\rho = \frac{1}{2}$ in the result of above Theorem 1 we get,

$$\sum_{i\in\mathcal{A}:\Delta_i>b}\frac{32\log(\psi T\Delta_i^4)}{\Delta_i}=\frac{32K\sqrt{T}\log(T^2\frac{K^2(\log K)^2}{T^2K^2})}{\sqrt{K\log K}}\leq\frac{64\sqrt{KT}\log(\log K)}{\sqrt{\log K}}$$
$$\overset{(a)}{\leq}64\sqrt{KT}$$

Where (a) happens because $\dfrac{\log(\log K)}{\sqrt{\log K}}\leq 1$ for $K\geq 2$. So, the total worst case gap-independent bound cannot be worse than,

$$\mathbb{E}[R_T]\leq 96K^2+64\sqrt{KT}$$

Here, in the gap-independent bound of EUCBV the most significant term is $O\left(\sqrt{KT}\right)$ which matches the upper bound of MOSS and OCUCB and is better than UCB-Improved, UCB1 and UCBV.

4 Proofs

A technical lemma used to prove Theorem 1 is presented below.

Lemma 1. *If* $T\geq K^{2.4}$, $\psi=\dfrac{T}{K^2}$, $\rho=\dfrac{1}{2}$ *and* $m\leq\dfrac{1}{2}\log_2(\dfrac{T}{e})$*, then,*

$$\frac{\rho m\log(2)}{\log(\psi T)-2m\log(2)}\leq\frac{3}{2}$$

Proof. We are going to prove this by contradiction. Let's say,

$$\begin{aligned}
&\frac{\rho m\log(2)}{\log(\psi T)-2m\log(2)}\geq\frac{3}{2}\\
&\Rightarrow 2\rho m\log(2)\geq 3\log(\psi T)-6m\log(2)\\
&\Rightarrow 2\rho m\log(2)\overset{(a)}{\geq}6\log(\frac{T}{K})-6m\log(2)\\
&\Rightarrow 7m\log(2)+6\log(K)\overset{(b)}{\geq}6\log(T)\\
&\Rightarrow 3.5\log(2)\log_2(\frac{T}{e})+6\log(K)\overset{(c)}{\geq}6\log(T)\\
&\Rightarrow\frac{3.5\log(2)\log(\frac{T}{e})}{\log(2)}+6\log(K)\geq 6\log(T)\\
&\Rightarrow 3.5\log(T)+6\log K-3.5\geq 6\log(T)\\
&\Rightarrow 6\log K\geq 2.5\log T+3.5
\end{aligned}$$

In the above inequalities, (a) happens because $\psi = \frac{T}{K^2}$, (b) occurs as $\rho = \frac{1}{2}$ and (c) happens because $m \leq \frac{1}{2}\log_2(\frac{T}{e})$. But, for $T \geq K^{2.4}$, we can see that $6\log K \geq 2.5\log T + 3.5$ is clearly not possible. Hence, $\dfrac{\rho m \log(2)}{\log(\psi T) - 2m\log(2)} \leq \dfrac{3}{2}$.

Proof of Theorem 1

Proof. Let, for each sub-optimal arm i, $m_i = \min\{m|\sqrt{4\epsilon_{m_i}} < \dfrac{\Delta_i}{4}\}$. Let $\mathcal{A}' = \{i \in \mathcal{A} : \Delta_i > b\}$ and $\mathcal{A}'' = \{i \in \mathcal{A} : \Delta_i > 0\}$. Also z_i denotes total number of times an arm i has been pulled. In the m-th round, n_m denotes the number of pulls allocated to the surviving arms in B_m.

Case a: *Some sub-optimal arm i is not eliminated in round m_i or before and the optimal arm $* \in B_{m_i}$*

An arbitrary sub-optimal arm $i \in \mathcal{A}'$ can get eliminated only when the event,

$$\hat{r}_i \leq r_i + c_i \text{ and } \hat{r}^* \geq r^* - c^* \tag{1}$$

takes place. So to bound the regret we need to bound the probability of the complementary event of these two conditions. We denote $c_i = \sqrt{\frac{\rho(\hat{v}_i+2)\log(\psi T\epsilon_{m_i})}{4n_{m_i}}}$ for the i-th arm. Note that as arm elimination condition is being checked in every timestep, in the m_i-th round whenever $z_i \geq n_{m_i} = \dfrac{\log(\psi T\epsilon_{m_i}^2)}{2\epsilon_{m_i}}$ we have,

$$
\begin{aligned}
c_i &\leq \sqrt{\frac{\rho(\hat{v}_i+2)\epsilon_{m_i}\log(\psi T\epsilon_{m_i})}{2\log(\psi T\epsilon_{m_i}^2)}} \overset{(a)}{\leq} \sqrt{\frac{2.\rho\epsilon_{m_i}\log(\frac{\psi T\epsilon_{m_i}^2}{\epsilon_{m_i}})}{\log(\psi T\epsilon_{m_i}^2)}} \\
&= \sqrt{\frac{2\rho\epsilon_{m_i}\log(\psi T\epsilon_{m_i}^2) - 2\rho\epsilon_{m_i}\log(\epsilon_{m_i})}{\log(\psi T\epsilon_{m_i}^2)}} \leq \sqrt{2\rho\epsilon_{m_i} - \frac{2\rho\epsilon_{m_i}\log(\frac{1}{2^{m_i}})}{\log(\psi T\frac{1}{2^{2m_i}})}} \\
&\leq \sqrt{2\rho\epsilon_{m_i} + \frac{2\rho\epsilon_{m_i}\log(2^{m_i})}{\log(\psi T) - \log(2^{2m_i})}} \leq \sqrt{2\rho\epsilon_{m_i} + \frac{2\rho\epsilon_{m_i} m_i\log(2)}{\log(\psi T) - 2m_i\log(2)}} \\
&\overset{(b)}{\leq} \sqrt{2\rho\epsilon_{m_i} + 2.\frac{3}{2}\epsilon_{m_i}} < \sqrt{4\epsilon_{m_i}} < \frac{\Delta_i}{4}
\end{aligned}
$$

In the above (a) happens because $\hat{v}_i \in [0,1]$, $\rho = \frac{1}{2}$ and (b) occurs by applying the result from Lemma 1. Similarly, we can also show that in the m_i-th round $c^* < \frac{\Delta_i}{4}$. Again, in the m_i-th round a sub-optimal arm $i \in \mathcal{A}'$ gets eliminated as,

$$
\begin{aligned}
\hat{r}_i + c_i &\leq r_i + 2c_i = r_i + 4c_i - 2c_i \\
&< r_i + \Delta_i - 2c_i \leq r^* - 2c^* \leq \hat{r}^* - c^*
\end{aligned}
$$

Thus, the probability that a bad arm is not eliminated correctly in the m_i-th round (or before) is given by ,

$$\mathbb{P}(\hat{r}_i > r_i + c_i) \leq \mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right) \tag{2}$$

where

$$\bar{c}_i = \sqrt{\frac{\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T \epsilon_{m_i})}{4 n_{m_i}}}$$

Note that, substituting $n_{m_i} \geq \frac{\log(\psi T \epsilon_{m_i})}{2\epsilon_{m_i}}$, $\bar{c}_i$ can be simplified to obtain,

$$\bar{c}_i \leq \sqrt{\frac{\rho\epsilon_{m_i}(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)}{2}} \leq \sqrt{\epsilon_{m_i}}. \tag{3}$$

The first term in the LHS of (2) can be bounded using the Chernoff-Hoeffding bound as below:

$$\begin{aligned}\mathbb{P}\left(\hat{r}_i > r_i + \bar{c}_i\right) &\leq \exp\left(-(\bar{c}_i)^2 z_i\right) \leq \exp\left(-\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T\epsilon_{m_i})\right) \\ &\overset{(a)}{\leq} \exp\left(-\rho\log(\psi T\epsilon_{m_i})\right) \leq \frac{1}{(\psi T\epsilon_{m_i})^\rho}\end{aligned} \tag{4}$$

where, (a) occurs because $(\sigma_i^2 + \sqrt{\rho\epsilon_{m_i}} + 2) \geq 1$.

The second term in the LHS of (2) can be simplified as follows:

$$\begin{aligned}&\mathbb{P}\left\{\hat{v}_i \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \leq \mathbb{P}\left\{\frac{1}{n_i}\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2 - (\hat{r}_i - r_i)^2 \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \\ &\leq \mathbb{P}\left\{\frac{\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + \sqrt{\epsilon_{m_i}}\right\} \overset{(a)}{\leq} \mathbb{P}\left\{\frac{\sum_{t=1}^{n_i}(X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + \bar{c}_i\right\} \\ &\overset{(b)}{\leq} \exp\left(-\rho(\sigma_i^2 + \sqrt{\epsilon_{m_i}} + 2)\log(\psi T\epsilon_{m_i})\right) \leq \frac{1}{(\psi T\epsilon_{m_i})^\rho}\end{aligned} \tag{5}$$

where inequality (a) is obtained using (3), while (b) follows from the Chernoff-Hoeffding bound.

Thus, using (4) and (5) in (2) we obtain $\mathbb{P}(\hat{r}_i > r_i + c_i) \leq \dfrac{2}{(\psi T\epsilon_{m_i})^\rho}$. Similarly, $\mathbb{P}\{\hat{r}^* \leq r^* - c^*\} \leq \dfrac{2}{(\psi T\epsilon_{m_i})^\rho}$. Summing the two up, the probability that a sub-optimal arm i is not eliminated on or before m_i-th round is $\left(\dfrac{4}{(\psi T\epsilon_{m_i})^\rho}\right)$.

Summing up over all arms in $\mathcal{A}'$ and bounding the regret for each arm $i \in \mathcal{A}'$ trivially by $T\Delta_i$, we obtain

$$\sum_{i\in\mathcal{A}'}\left(\frac{4T\Delta_i}{(\psi T\epsilon_{m_i})^\rho}\right) \leq \sum_{i\in\mathcal{A}'}\left(\frac{4T\Delta_i}{(\psi T\frac{\Delta_i^2}{4.16})^\rho}\right) \leq \sum_{i\in\mathcal{A}'}\left(\frac{2^{2+2\rho}.16^\rho T^{1-\rho}}{\psi^\rho\Delta_i^{2\rho-1}}\right)$$

$$\stackrel{(a)}{\leq} \sum_{i \in \mathcal{A}'} \left(\frac{2^{2+1}.16^{\frac{1}{2}} T^{1-\frac{1}{2}}}{(\frac{T}{K^2})^{\frac{1}{2}} \Delta_i^{2.\frac{1}{2}-1}} \right) = \sum_{i \in \mathcal{A}'} 32K$$

Here in (a) we substitute the values of ρ and ψ.

Case b: ***An arm $i \in B_{m_i}$ is eliminated in round m_i or before or there is no $* \in B_{m_i}$***

Case $b1$: ***$* \in B_{m_i}$ and each $i \in \mathcal{A}'$ is eliminated on or before m_i***

Since we are eliminating a sub-optimal arm i on or before round m_i, it is pulled no longer than,

$$z_i < \left\lceil \frac{\log{(\psi T \epsilon_{m_i}^2)}}{2\epsilon_{m_i}} \right\rceil$$

So, the total contribution of i till round m_i is given by,

$$\Delta_i \left\lceil \frac{\log{(\psi T \epsilon_{m_i}^2)}}{2\epsilon_{m_i}} \right\rceil \leq \Delta_i \left\lceil \frac{\log{(\psi T (\frac{\Delta_i}{16 \times 256})^4)}}{2(\frac{\Delta_i}{4\sqrt{4}})^2} \right\rceil, \text{since } \sqrt{4\epsilon_{m_i}} < \frac{\Delta_i}{4}$$

$$\leq \Delta_i \left(1 + \frac{32 \log{(\psi T (\frac{\Delta_i^4}{16384})})}{\Delta_i^2} \right) \leq \Delta_i \left(1 + \frac{32 \log{(\psi T \Delta_i^4)}}{\Delta_i^2} \right)$$

Summing over all arms in $\mathcal{A}'$ the total regret is given by,

$$\sum_{i \in \mathcal{A}'} \Delta_i \left(1 + \frac{32 \log{(\psi T \Delta_i^4)}}{\Delta_i^2} \right) = \sum_{i \in \mathcal{A}'} \left(\Delta_i + \frac{32 \log{(\psi T \Delta_i^4)}}{\Delta_i} \right)$$

$$\stackrel{(a)}{\leq} \sum_{i \in \mathcal{A}'} \Delta_i \left(1 + \frac{64 \log{(\frac{T\Delta_i^2}{K})}}{\Delta_i^2} \right)$$

We obtain (a) by substituting the value of ψ.

Case $b2$: ***Optimal arm $*$ is eliminated by a sub-optimal arm*** Firstly, if conditions of Case a holds then the optimal arm $*$ will not be eliminated in round $m = m_*$ or it will lead to the contradiction that $r_i > r^*$. In any round m_*, if the optimal arm $*$ gets eliminated then for any round from 1 to m_j all arms j such that $m_j < m_*$ were eliminated according to assumption in Case a. Let the arms surviving till m_* round be denoted by $\mathcal{A}'$. This leaves any arm a_b such that $m_b \geq m_*$ to still survive and eliminate arm $*$ in round m_*. Let such arms that survive $*$ belong to $\mathcal{A}''$. Also maximal regret per step after eliminating $*$ is the maximal Δ_j among the remaining arms j with $m_j \geq m_*$.

Let $m_b = \min\{m|\sqrt{4\epsilon_m} < \dfrac{\Delta_b}{4}\}$. Hence, the maximal regret after eliminating the arm $*$ is upper bounded by,

$$
\begin{aligned}
&\sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} \left(\frac{4}{(\psi T\epsilon_{m_*})^{\rho}}\right).T \max_{j\in\mathcal{A}'':m_j\geq m_*} \Delta_j \\
&\leq \sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} \left(\frac{4\sqrt{4}}{(\psi T\epsilon_{m_*})^{\rho}}\right).T.4\sqrt{\epsilon_{m_*}} \\
&\leq \sum_{m_*=0}^{max_{j\in\mathcal{A}'} m_j} \sum_{i\in\mathcal{A}'':m_i>m_*} 32\left(\frac{T^{1-\rho}}{\psi^{\rho}\epsilon_{m_*}^{\rho-\frac{1}{2}}}\right) \\
&\leq \sum_{i\in\mathcal{A}'':m_i>m_*} \sum_{m_*=0}^{\min\{m_i,m_b\}} \left(\frac{32T^{1-\rho}}{\psi^{\rho}2^{-(\rho-\frac{1}{2})m_*}}\right) \\
&\leq \sum_{i\in\mathcal{A}'} \left(\frac{32T^{1-\rho}}{\psi^{\rho}2^{-(\rho-\frac{1}{2})m_*}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{32T^{1-\rho}}{\psi^{\rho}2^{-(\rho-\frac{1}{2})m_b}}\right) \\
&\leq \sum_{i\in\mathcal{A}'} \left(\frac{32T^{1-\rho} * 2^{\frac{\rho}{2}-\frac{1}{4}}}{\psi^{\rho}\Delta_i^{\rho-\frac{1}{2}}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{32T^{1-\rho_a}}{\psi^{\rho}b^{\rho-\frac{1}{2}}}\right) \\
&\leq \sum_{i\in\mathcal{A}'} \left(\frac{2^{\frac{\rho}{2}+\frac{19}{4}}.T^{1-\rho}}{\psi^{\rho}\Delta_i^{2\rho-1}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{2^{\frac{\rho}{2}+\frac{19}{4}}.T^{1-\rho}}{\psi^{\rho}b^{2\rho_a-1}}\right) \\
&\overset{(a)}{\leq} \sum_{i\in\mathcal{A}'} \left(\frac{2^{\frac{1}{4}+\frac{19}{4}}.T^{1-\frac{1}{2}}}{(\frac{T}{K^2})^{\frac{1}{2}}\Delta_i^{2.\frac{1}{2}-1}}\right) + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} \left(\frac{2^{\frac{1}{4}+\frac{19}{4}}.T^{1-\frac{1}{2}}}{(\frac{T}{K^2})^{\frac{1}{2}}b^{2.\frac{1}{2}-1}}\right) \\
&\leq \sum_{i\in\mathcal{A}'} 32K + \sum_{i\in\mathcal{A}''\setminus\mathcal{A}'} 32K
\end{aligned}
$$

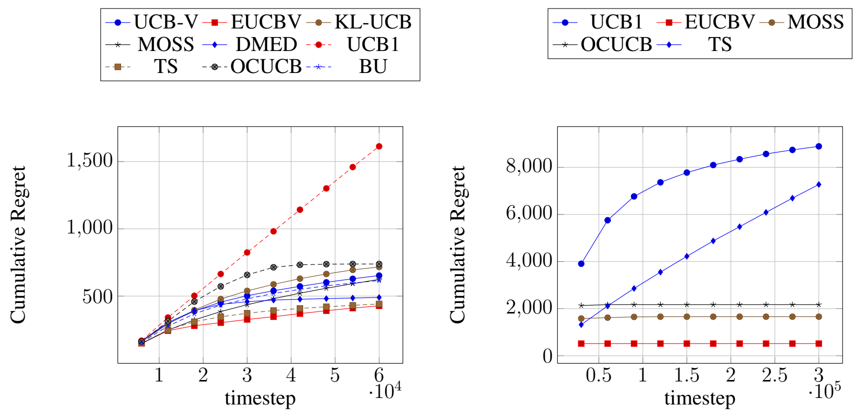
In the above section, (a) is obtained by substituting the values of ψ and ρ. Summing up **Case a** and **Case b**, the total regret is given by,

$$
\begin{aligned}
\mathbb{E}[R_T] \leq &\sum_{i\in\mathcal{A}:\Delta_i>b} \left\{64K + \left(\Delta_i + \frac{64\log\left(\frac{T\Delta_i^2}{K}\right)}{\Delta_i}\right)\right\} \\
&+ \sum_{i\in\mathcal{A}:0<\Delta_i\leq b} 32K + \max_{i\in\mathcal{A}:0<\Delta_i\leq b} \Delta_i T
\end{aligned}
$$

5 Experimental Section

In this section we conduct extensive empirical evaluations of EUCBV against several other popular bandit algorithms. We use cumulative regret as the metric of comparison. We implement the following algorithms: KL-UCB (Garivier and Cappé, 2011), DMED

(Honda and Takemura, 2010), MOSS (Audibert and Bubeck, 2009), UCB1 (Auer et al., 2002a), UCB-Improved (Auer and Ortner, 2010), Median Elimination (Even-Dar et al., 2006), Thompson Sampling (TS) (Agrawal and Goyal, 2011), OCUCB (Lattimore, 2015), Bayes-UCB (BU) (Kaufmann et al., 2012) and UCB-V (Audibert et al., 2009)[2]. The parameters of EUCBV algorithm for all the experiments are set as follows: $\psi = \frac{T}{K^2}$ and $\rho = 0.5$ (as in Corollary 1).



(a) Expt-1: 20 Bernoulli-distributed arms (b) Expt-2: 100 Gaussian-distributed arms

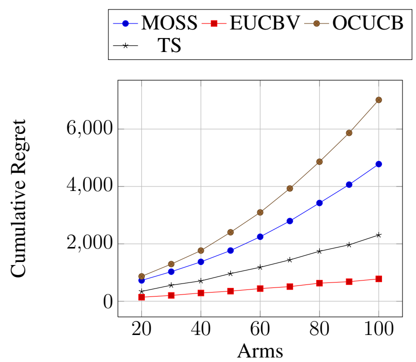
Fig. 1: Cumulative regret for various bandit algorithms on two stochastic K-armed bandit environments.

Experiment-1 (Bernoulli with uniform gaps): This experiment is conducted to observe the performance of EUCBV over a short horizon. The horizon T is set to 60000. These type of cases are frequently encountered in web-advertising domain. The testbed comprises of 20 Bernoulli distributed arms with expected rewards of the arms as $r_{i_{i \neq *}} = 0.07$ and $r^* = 0.1$. The regret is averaged over 100 independent runs and is shown in Figure 1(a). EUCBV, MOSS, UCB1, UCB-V, KL-UCB, TS, BU and DMED are run in this experimental setup. Here not only do we observe that EUCBV performs better than all the non-variance based algorithms like MOSS, OCUCB, UCB-Improved and UCB1, but it also outperforms UCBV because of the choice of the exploration parameters. Because of the small gaps and short horizon T, we do not implement UCB-Improved and Median Elimination on this test-case.

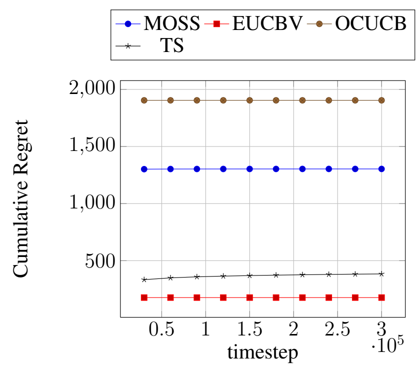
Experiment-2 (Failure of TS): This experiment is conducted on a large horizon and over a large set of arms. The horizon T is set for a large duration of 3×10^5. This testbed comprises of 100 arms involving Gaussian reward distributions with expected rewards of the arms $r_{1:33} = 0.7$, $r_{34:99} = 0.8$ and $r^*_{100} = 0.9$ with variance set as $\sigma^2_{1:33} = 0.7, \sigma^2_{34:99} = 0.1$ and $r^*_{100} = 0.7$. The regret is averaged over 100 independent runs and is shown in Figure 1(b). From the results in Figure 1(b), we observe that

[2] The implementation for KL-UCB, Bayes-UCB and DMED were taken from Cappe et al. (2012)

since the gaps are small and the variances of the optimal arm and the arms farthest from the optimal arm are the highest, EUCBV outperforms all the non-variance based algorithms MOSS, OCUCB, UCB1, TS, UCB-Improved and Median-Elimination($\epsilon = 0.1, \delta = 0.1$). The performance of UCBV, UCB-Improved and Median elimination is extremely weak in comparison with the other algorithms and their plots are not shown in Figure 1(b). Note that the performance of TS is also weak and this is in line with the observation in Lattimore (2015) that the worst case regret of TS when Gaussian prior is used is $\Omega\left(\sqrt{KT\log T}\right)$. Again both Bayes-UCB and KL-UCB-Gauss perform much worse than TS and their plots are omitted from the figure. We omit their plots in order to more clearly show the difference between EUCBV, MOSS and OCUCB.



(a) Expt-3: 20 to 100 Bernoulli-distributed arms



(b) Expt-4: Advanced Setting

Fig. 2: Further Experiments with EUCBV

Experiment-3 (Large horizon and uniform gaps): This experiment is conducted to show the stability and performance of EUCBV over a very large horizon and over a large number of arms. This testbed comprises of $20-100$ (interval of 10) arms with Bernoulli reward distributions, where the expected rewards of the arms are $r_{i_{i\neq *}} = 0.05$ and $r^* = 0.1$. For each of these testbeds of $20-100$ arms, we report the cumulative regret averaged over 100 independent runs. The horizon is set at $T = 10^5 + K^3_{20:100}$ timesteps. We report the performance of MOSS, TS, OCUCB and EUCBV only over this uniform gap setup. From the results in Figure 2(a), it is evident that the growth of regret for EUCBV is much lower than that of TS, OCUCB and MOSS.

Experiment-4 (Advance Setting): This experiment is conducted on 100 Gaussian distributed arms such that expected rewards of the arms $r_{1:33} = 0.4$, $r_{34:99} = 0.6$, $r^*_{100} = 0.9$ and the variance is set as $\sigma^2_{1:33} = 0.2$, $\sigma^2_{34:99} = 0.1$, $\sigma^2_{100} = 0.4$ and $T = 3 \times 10^5$. This experiment is conducted to show that in certain environments, when the variance of the optimal arm is higher than all the other sub-optimal arms, then EUCBV performs exceptionally well. We refer to this setup as Advanced Setting because here the chosen variance values are such that only variance-aware algorithms will perform well because the variance of the optimal arm is chosen to be higher than the other arms and so algorithms that are not variance-aware will spent a significant amount of pulls trying to find the optimal arm. The result is shown in Figure 2(b). Predictably EUCBV, which allocates pulls proportional to the variance of the arms, outperforms TS, MOSS and OCUCB. The plot for UCBV is omitted from the figure and it performs very poorly compared to other algorithms. Note that EUCBV by virtue of its aggressive exploration parameters outperforms UCBV in all the experiments even though UCBV is a variance based algorithm.

6 Conclusion and Future Works

In this paper, we studied the EUCBV algorithm which takes into account the empirical variance of the arms and employs aggressive exploration parameters to eliminate sub-optimal arms. Our theoretical analysis conclusively established that EUCBV exhibits an order-optimal gap-independent regret bound of $O\left(\sqrt{KT}\right)$. Empirically we show that EUCBV performs superbly across diverse experimental settings and outperforms most of the bandit algorithms in stochastic MAB setup. Our experiments show that EUCBV is extremely stable for larger horizons and performs consistently well across different types of distributions. One avenue for future work is to remove the constraint of $T \geq K^{2.4}$ required for EUCBV to reach the order optimal regret bound. Another future direction is to come up with an anytime version of EUCBV. An anytime algorithm does not need the horizon T as an input.

Bibliography

Agrawal, S. and Goyal, N. (2011). Analysis of thompson sampling for the multi-armed bandit problem. *arXiv preprint arXiv:1111.1797*.

Audibert, J.-Y. and Bubeck, S. (2009). Minimax policies for adversarial and stochastic bandits. In *COLT*, pages 217–226.

Audibert, J.-Y. and Bubeck, S. (2010). Best arm identification in multi-armed bandits. In *COLT-23th COLT-2010*, pages 13–p.

Audibert, J.-Y., Munos, R., and Szepesvári, C. (2009). Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. *TCS*, 410(19):1876–1902.

Auer, P., Cesa-Bianchi, N., and Fischer, P. (2002a). Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. (2002b). The nonstochastic multiarmed bandit problem. *SICOMP*, 32(1):48–77.

Auer, P. and Ortner, R. (2010). Ucb revisited: Improved regret bounds for the stochastic multi-armed bandit problem. *Periodica Mathematica Hungarica*, 61(1-2):55–65.

Bubeck, S. and Cesa-Bianchi, N. (2012). Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *arXiv preprint arXiv:1204.5721*.

Bubeck, S., Cesa-Bianchi, N., and Lugosi, G. (2012). Bandits with heavy tail. *arXiv preprint arXiv:1209.1727*.

Bubeck, S., Munos, R., and Stoltz, G. (2011). Pure exploration in finitely-armed and continuous-armed bandits. *TCS*, 412(19):1832–1852.

Cappe, O., Garivier, A., and Kaufmann, E. (2012). pymabandits. `http://mloss.org/software/view/415/`.

Even-Dar, E., Mannor, S., and Mansour, Y. (2006). Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *JMLR*, 7:1079–1105.

Garivier, A. and Cappé, O. (2011). The kl-ucb algorithm for bounded stochastic bandits and beyond. *arXiv preprint arXiv:1102.2490*.

Honda, J. and Takemura, A. (2010). An asymptotically optimal bandit algorithm for bounded support models. In *COLT*, pages 67–79. Citeseer.

Kaufmann, E., Cappé, O., and Garivier, A. (2012). On bayesian upper confidence bounds for bandit problems. In *AISTATS*, pages 592–600.

Lai, T. L. and Robbins, H. (1985). Asymptotically efficient adaptive allocation rules. *Advances in applied mathematics*, 6(1):4–22.

Lattimore, T. (2015). Optimally confident ucb: Improved regret for finite-armed bandits. *arXiv preprint arXiv:1507.07880*.

Liu, Y.-C. and Tsuruoka, Y. (2016). Modification of improved upper confidence bounds for regulating exploration in monte-carlo tree search. *TCS*.

Robbins, H. (1952). Some aspects of the sequential design of experiments. In *Herbert Robbins Selected Papers*, pages 169–177. Springer.

Thompson, W. R. (1933). On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, pages 285–294.