Improved Regret Bounds with Clustered UCB(Supplementary Material)

Student Paper

No Institute Given

In the following appendices we will prove the bounds based on the events ξ_1,ξ_2 and ξ_3. In ξ_1, we will assume two important assumptions $i)\hat{r}^* < \hat{r}_i, \forall i \in s_i$ and $ii)\exists a_i \in s_i$ such that $\sqrt{\dfrac{\epsilon_m}{w_{s_i}}} < \dfrac{\Delta_i}{5}$. For ξ_2, we will assume that $a^* \in s^*$ and $|s^*| = 1$, $a_i \in s_i \forall a_i \setminus a^* \in B_m$ and $\exists a_s$ such that $\sqrt{\epsilon_m} < \dfrac{2\Delta_s}{5}$, where $\hat{r}_s > \hat{r}_i, \forall i \in s_i$. ξ_3 be the event when the optimal arm a^* gets eliminated by a sub-optimal arm. At the start of any round m, we fix ϵ_m.

Appendix A

Theorem 1. *The probability that the optimal arm $a^* \in s_i$ will lie above $\hat{r}_b + \hat{\Delta}_{s_i}$ after $\left\lceil \dfrac{2\log(\psi(m)T\epsilon_m^2)}{\epsilon_m} \right\rceil$ pulls in the m-th round is given by $\left\{1 - \dfrac{2}{(\psi(m)T\epsilon_m^2)^{\ell_m^2\epsilon_m}}\right\}$ where $\hat{r}_b$ is the arm with the minimum payoff in s_i, $\hat{\Delta}_{s_i} = max_{i\in s_i}\hat{r}_i - min_{j\in s_i}\hat{r}_j, i \neq j$, $\epsilon_m = \max\left\{\dfrac{\hat{\Delta}_{s,m}}{\ell_m}, \dfrac{1}{\sqrt{\psi(m)T}}\right\}$, if $\hat{\Delta}_{s,m} \neq 0$ and T is the horizon.*

Proof. of Theorem 1:
We start by considering the worst case scenario that in the m-th round, in a cluster s_i, the optimal arm a^* has performed worst. This event is characterized by ξ_1 such that $\hat{r}^* < \hat{r}_i, \forall a_i \in s_i$. Let, $\hat{\Delta}_{s_i} = max_{i\in s_i}\hat{r}_i - min_{j\in s_i}\hat{r}_j$ and $i \neq j$. Also, let in ξ_1, $a_s, a_b \in s_i$, $|s_i| = k_{s_i}$ and $\hat{r}_b < \hat{r}_i, \forall i \in s_i$ also $\hat{r}_s > \hat{r}_i, \forall i \in s_i$.
Now, we have to bound the $\mathbb{P}\{\hat{r}^* \geq \hat{r}_s - \hat{\Delta}_{s_i}\}$
But, $\hat{\Delta}_{s_i} \leq (k_{s_i} - 1)\epsilon_m$, as $|\hat{r}_i - \hat{r}_j| \leq \epsilon_m, \forall i, j \in s_i$
$\leq (\ell_m)\epsilon_m$, as $k_{s_i} \leq \ell_m$

$$\mathbb{P}\{\hat{r}^* \geq \hat{r}_s - \hat{\Delta}_{s_i}\} \Rightarrow \mathbb{P}\{\hat{r}^* + \frac{\hat{\Delta}_{s_i}}{2} \geq \hat{r}_s - \frac{\hat{\Delta}_{s_i}}{2}\}$$

Now, applying Chernoff-Hoeffding bound and considering independence of events,

$$\mathbb{P}\{\hat{r}^* \leq r^* + \frac{\hat{\Delta}_{s_i}}{2}\} \Rightarrow \mathbb{P}\{\hat{r}^* \leq r^* + \frac{\ell_m\epsilon_m}{2}\} \leq exp(-2\frac{(\ell_m\epsilon_m)^2}{4}n^*)$$

Now, putting $n_{s_i} = n^* = \dfrac{2\log(\psi(m)T\epsilon_m^2)}{\epsilon_m}$

$$\mathbb{P}\{\hat{r}^* \leq r^* + \frac{\hat{\Delta}_{s_i}}{2}\} \leq exp(-\ell_m^2\epsilon_m \log(\psi(m)T\epsilon_m^2))$$

$$\mathbb{P}\{\hat{r}^* \leq r^* + \frac{\hat{\Delta}_{s_i}}{2}\} \leq \frac{1}{(\psi(m)T\epsilon_m^2)^{\ell_m^2\epsilon_m}}$$

Similarly, $\mathbb{P}\{\hat{r}_s \geq r_s - \frac{\hat{\Delta}_{s_i}}{2}\} \leq \frac{1}{(\psi(m)T\epsilon_m^2)^{\ell_m^2 \epsilon_m}}$

Hence, the probability that the optimal arm a^* after n_{s_i} pulls going above $\hat{\Delta}_{s_i}$ is $\left\{1 - \frac{2}{(\psi(m)T\epsilon_m^2)^{\ell_m^2 \epsilon_m}}\right\}$

Appendix B

Lemma 1. *The number of times an arm $a_i \in s_i$ is pulled in each round is $n_{s_i} = \left\lceil \frac{2\log(\psi m T \epsilon_m^2)}{\epsilon_m} \right\rceil$ and this eliminates the arm a_i such that $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$ by the condition $\left\{\hat{r}_i + \sqrt{\frac{\log(\psi(m)T\epsilon_m^2)}{2w_{s_i}n_{s_i}}} < \max_{j\in s_i} \hat{r}_j - \sqrt{\frac{\log(\psi(m)T\epsilon_m^2)}{2w_{s_i}n_{s_i}}}\right\}, \forall s_i \in S$ with probability $\left\{1 - \left(\frac{1}{2\psi(m)T\epsilon_m^2}\right)\right\}$.*

Proof. of Lemma 1:
In ξ_1 arm elimination condition, given the choice of confidence interval c_m, we want to bound the event $\hat{r}_i + c_m \leq \hat{r}^* - c_m$ with a more tighter event of $\hat{r}_i + \sqrt{w_{s_i}}c_m \leq \hat{r}^* - \sqrt{w_{s_i}}c_m$ which will result in faster elimination of arms within a cluster, given the choice of c_m and w_{s_i}.
Now, in ξ_1, $c_m = \sqrt{\frac{\log(\psi(m)T\epsilon_m^2)}{2w_{s_i}n_{s_i}}}$.
Putting the value of $n_{s_i} = \frac{2\log(\psi(m)T\epsilon_m^2)}{\epsilon_m}$ in c_m,
$c_m = \sqrt{\frac{\epsilon_m \log(\psi(m)T\epsilon_m^2)}{2w_{s_i} * 2\log(\psi(m)T\epsilon_m^2)}} = \frac{\sqrt{\epsilon_m}}{2\sqrt{w_{s_i}}}$
Again, $\exists a_i \in s_i$ such that,

$$\begin{aligned}
\hat{r}_i + c_m &\leq \hat{r}_i + 2c_m \\
&= \hat{r}_i - \sqrt{\epsilon_m} + 2c_m + \sqrt{\epsilon_m} \\
&= \hat{r}_i - 2\sqrt{w_{s_i}}c_m + 2c_m + 2\sqrt{w_{s_i}}c_m \\
&\leq \hat{r}_i - 2\sqrt{w_{s_i}}c_m + 8c_m, w_{s_i} \geq 4 \\
&\leq r_i - 2\sqrt{w_{s_i}}c_m + 10c_m \\
&\leq r_i - 2\sqrt{w_{s_i}}c_m + 10\frac{\sqrt{\epsilon_m}}{2\sqrt{w_{s_i}}} \\
&= r_i + 5\frac{\sqrt{\epsilon_m}}{\sqrt{w_{s_i}}} - 2\sqrt{w_{s_i}}c_m
\end{aligned}$$

Also in ξ_1, $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$, $\exists a_i \in s_i$
Hence, $\hat{r}_i + c_m \leq \hat{r}_i + 2c_m \leq r_i + \Delta_i - 2\sqrt{w_{s_i}}c_m \leq r^* - 2\sqrt{w_{s_i}}c_m$
$\Rightarrow \hat{r}_i \leq r^* - 2\sqrt{w_{s_i}}c_m - 2c_m \leq \hat{r}^* - 2\sqrt{w_{s_i}}c_m$
So, we can bound the event of $\hat{r}_i + c_m \leq \hat{r}^* - c_m$ by $\hat{r}_i + \sqrt{w_{s_i}}c_m \leq \hat{r}^* - \sqrt{w_{s_i}}c_m$

given that $\sqrt{\dfrac{\epsilon_m}{w_{s_i}}} < \dfrac{\Delta_i}{5}$ becomes true for some arm $a_i \in s_i$ after the m-th round and $c_m = \sqrt{\dfrac{\log(\psi(m)T\epsilon_m^2)}{2w_{s_i}n_{s_i}}}$.

Appendix C

Theorem 2. *With a probability of* $\left\{1 - \left(\dfrac{2}{\psi(m)T\epsilon_m^2)}\right)\right\}$ *a sub-optimal arm can be deleted within a cluster* s_i *in round* m *by the arm elimination condition, where* $\epsilon_m = \max\left\{\dfrac{\hat{\Delta}_{s,m}}{\ell_m}, \dfrac{1}{\sqrt{\psi(m)T}}\right\}$, *if* $\hat{\Delta}_{s,m} \neq 0$ *and* T *is the horizon.*

Proof. of Theorem 2:
From Lemma 1 we know that given the assumptions under ξ_1, we can bound $\hat{r}_i + c_m \leq \hat{r}^* - c_m$ by $\hat{r}_i + \sqrt{w_{s_i}}c_m \leq \hat{r}^* - \sqrt{w_{s_i}}c_m$ given that $\sqrt{\dfrac{\epsilon_m}{w_{s_i}}} < \dfrac{\Delta_i}{5}$ for some arm $a_i \in s_i$. So, in ξ_1, the arm deletion condition within a cluster s_i is satisfied when $\sqrt{\dfrac{\epsilon_m}{w_{s_i}}} < \dfrac{\Delta_i}{5}$ and so we need to bound the probability,

$\mathbb{P}\{\hat{r}^* \leq r^* - \sqrt{w_{s_i}}c_m\} \leq U_m$, where U_m is an arbitrary upper bound.

Applying Chernoff-Hoeffding bound and considering independence of events,

$$\begin{aligned}
\mathbb{P}\{\hat{r}^* \leq r^* - \sqrt{w_{s_i}}c_m\} &\leq exp(-2w_{s_i}c_m^2 n_{s_i}) \\
&\leq exp(-2w_{s_i} * \frac{\log(\psi(m)T\epsilon_m^2)}{2w_{s_i}n_{s_i}} * n_{s_i}) \\
&\leq \frac{1}{\psi(m)T\epsilon_m^2}
\end{aligned}$$

Similarly, $\mathbb{P}\{\hat{r}_i \geq r_i + \sqrt{w_{s_i}}c_m\} \leq \dfrac{1}{\psi(m)T\epsilon_m^2}$

Summing, the two up, the probability that a sub-optimal arm a_i is not eliminated in ξ_1 is $\left(\dfrac{2}{\psi(m)T\epsilon_m^2)}\right)$.

Appendix D

Theorem 3. *With a probability of* $\left(1 - \dfrac{4}{(\psi(m)T\epsilon_m^2)^{1+|B_m|^2\epsilon_m}}\right)$ *a sub-optimal arm can be deleted in round* m, *where* $\hat{\Delta}_{s,m} = \max_{i\in B_m}\{\hat{r}_i\} - \min_{j\in B_m}\{\hat{r}_j\}$ *for* $i \neq j$, $\epsilon_m = \max\left\{\dfrac{\hat{\Delta}_{s,m}}{\ell_m}, \dfrac{1}{\sqrt{\psi(m)T}}\right\}$ *for* $\hat{\Delta}_{s,m} \neq 0$, B_m *is the set of arms still not eliminated in the* m*-th round and* T *is the horizon.*

Proof. of theorem 3:
Here, we will consider the scenario which will make it possible to remove all the arms in

B_m at one go. Let in the event ξ_2, there be two clusters s_i and s^* such that $a^* \in s^*$ and $a_i \in s_i, \forall a_i \setminus a^* \in B_m$. So, $|s^*| = 1$ and $|s_i| = |B_m| - 1 \leq |B_m|$. Let $\hat{r}_b < \hat{r}_i, \forall i \in s_i$ and $\hat{r}_s > \hat{r}_i, \forall i \in s_i$ Thus, in ξ_2 we see that if

$$\hat{r}_s + c'_m < \hat{r}^* - c'_m, \text{ where } c'_m = \sqrt{\frac{|B_m|\epsilon_m \log(\psi(m)T\epsilon_m^2)}{2\ell_m n_{s_i}}}$$

then all the arms are eliminated and stopping condition reached. We divide the proof into two parts. The above said event is the product of two simultaneous events, in the first event $\hat{r}_s + c'_m < \hat{r}^* - c'_m$ given that $\hat{r}_s > \hat{r}_i \forall i \in s_i$. In the first part we establish the probability of $\hat{r}_s + c_m < \hat{r}^* - c'_m$ just as before in Lemma 1. In the second the part we prove the bounds on the probability that $\exists a_s \in s_i$ such that $\hat{r}_s > \hat{r}_i, \forall i \in s_i$ and $\sqrt{\epsilon_m} < \frac{2\Delta_s}{5}$ for the arm a_s.

Proceeding as like Lemma 1,

Now, in ξ_2, $c'_m = \sqrt{\frac{|B_m|\epsilon_m \log(\psi(m)T\epsilon_m^2)}{2\ell_m n_{s_i}}}$

Putting the value of $n_{s_i} = \frac{2\log(\psi(m)T\epsilon_m^2)}{\epsilon_m}$ in c'_m,

$$c'_m = \sqrt{\frac{\epsilon_m^2 |B_m| \log(\psi(m)T\epsilon_m^2)}{2\ell_m * 2\log(\psi(m)T\epsilon_m^2)}} = \frac{\epsilon_m\sqrt{|B_m|}}{2\sqrt{\ell_m}}$$

From the condition imposed in ξ_2, we know that $\ell_m \geq |B_m| - 1$ and $\frac{\ell_m}{|B_m|\epsilon_m} = \frac{\ell_m^2}{|B_m|\Delta_{s,m}}$ and so $\frac{\ell_m}{|B_m|\epsilon_m} \geq 1$.

Again, $\exists a_s \in s_i$ such that,

$$\begin{aligned}
\hat{r}_s + c'_m &\leq \hat{r}_s + 2c'_m \\
&\leq \hat{r}_s + 2\sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m \\
&\leq r_s + 3\sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m \\
&\leq r_s + 5\sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m - \sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m \\
&\leq r_s + 5\frac{\sqrt{\epsilon_m}}{2} - \sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m
\end{aligned}$$

Also in ξ_2, $\sqrt{\epsilon_m} < \frac{2\Delta_s}{5}$, $\exists a_s \in s_i$

We can immediately see that the condition in ξ_2 happens after ξ_1 because $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$ will occur before $\sqrt{\epsilon_m} < \frac{2\Delta_s}{5}$ with a higher probability given the value of $w_{s_i} \geq 4$. Hence, $w_{s_i} = \ell_m^2 k_{s_i}$ guarantees that ξ_2 happens after ξ_1 with high probability.

$$\begin{aligned}
\hat{r}_s + 2c'_m \leq \hat{r}_s + 2\sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m &\leq r_s + \Delta_s - \sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m \\
&\leq r^* - \sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m
\end{aligned}$$

$$\leq \hat{r}^*$$

So, from the condition imposed in ξ_2, given that $\exists a_i \in s_i$ such that $\sqrt{\epsilon_m} < \dfrac{2\Delta_s}{5}$ for the given c'_m, we will be upper bounding the probability of $\hat{r}_s + 2c'_m \leq \hat{r}^*$ with $\hat{r}_s + 2\sqrt{\dfrac{\ell_m}{|B_m|\epsilon_m}}c'_m \leq \hat{r}^* \Rightarrow \hat{r}_s + \sqrt{\dfrac{\ell_m}{|B_m|\epsilon_m}}c'_m \leq \hat{r}^* - \sqrt{\dfrac{\ell_m}{|B_m|\epsilon_m}}c'_m$.

Applying Chernoff-Hoeffding bound and considering independence of events,

$$\mathbb{P}\{\hat{r}^* \leq r^* - \sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m\} \leq exp(-2 * (\sqrt{\frac{\ell_m}{|B_m|\epsilon_m}}c'_m)^2 n_{s_i}) = \frac{1}{(\psi(m)T\epsilon_m^2)}$$

Similarly, $\mathbb{P}\{\hat{r}_s \geq r_s - \sqrt{\dfrac{\ell_m}{|B_m|\epsilon_m}}c'_m\} \leq \dfrac{1}{(\psi(m)T\epsilon_m^2)}$

Now, for bounding the event that for the arm $a_s \in s_i$, $\hat{r}_s \geq \hat{r}_i, \forall i \in s_i$ and also a_b be an arm such that $\hat{r}_b < \hat{r}_i, \forall i \in s_i$

$$\mathbb{P}\{\hat{r}_s \geq \hat{r}_i | \forall i \in s_i\} \leq \mathbb{P}\{\hat{r}_s \geq \hat{r}_b + (|B_m| - 2)\epsilon_m\} \leq \mathbb{P}\{\hat{r}_s \geq \hat{r}_b + |B_m|\epsilon_m\}$$

$$\Rightarrow \mathbb{P}\{\hat{r}_s - \frac{|B_m|\epsilon_m}{2} \geq \hat{r}_b + \frac{|B_m|\epsilon_m}{2}\}$$

Again applying Chernoff-Hoeffding bound and considering independence of events,

$$\mathbb{P}\{\hat{r}_s \leq r_s - \frac{|B_m|\epsilon_m}{2}\} \leq exp(-2*(\frac{|B_m|\epsilon_m}{2})^2 n_{s_i}) \leq exp\left(-2*(\frac{|B_m|\epsilon_m}{2})^2 \frac{2\log(\psi(m)T\epsilon_m^2)}{\epsilon_m}\right)$$

$$\leq \left(\frac{1}{(\psi(m)T\epsilon_m^2)^{|B_m|^2\epsilon_m}}\right)$$

Similarly, $\mathbb{P}\{\hat{r}_b \geq r_b + \dfrac{|B_m|\epsilon_m}{2}\} \leq \left(\dfrac{1}{(\psi(m)T\epsilon_m^2)^{|B_m|^2\epsilon_m}}\right)$

Considering two events and from the conditions imposed in ξ_2 and the stopping condition in the algorithm, we can see that the probability of the set s_i not getting deleted and the rounds not stopping is given by,

$$\left(\frac{2}{\psi(m)T\epsilon_m^2}\right) \times \left(\frac{2}{(\psi(m)T\epsilon_m^2)^{|B_m|^2\epsilon_m}}\right) = \left(\frac{4}{(\psi(m)T\epsilon_m^2)^{1+|B_m|^2\epsilon_m}}\right).$$

Appendix E

Theorem 4. *The upper bound on the total regret over horizon T after round m is given by* $R_T \leq \sum_{i\in B_m}\left(\max\left\{\left(\dfrac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right), \left(\dfrac{25\Delta_i}{(\psi(m)\Delta^2)(0.16\psi(m)T\Delta^2)^{2|B_m|^2\Delta/5}}\right)\right\} + \left(\Delta_i + \dfrac{27\log(\psi(m)T\dfrac{\Delta_i^{\frac{8}{5}}}{12})}{\Delta_i^{\frac{3}{5}}}\right)\right)$, *where B_m is the set of arms still not eliminated in the m-th round and Δ is the minimal gap.*

Proof. of theorem 4:

Case 1: Given that in round m, either ξ_1 or ξ_2 occurs and a sub-optimal arm $a_i \in s_i$ or the cluster s_i itself will get eliminated. So, for an arm a_i to survive, both these events should fail to delete the arm. In round m, the regret suffered for not eliminating an arm a_i is given by,

$$\max\left\{\left(\frac{2}{\psi(m)T\epsilon_m^2}\right),\left(\frac{4}{(\psi(m)T\epsilon_m^2)^{1+|B_m|^2\epsilon_m}}\right)\right\} \tag{1}$$

from **section 7.2, Theorem 2 and section 7.3, Theorem 3**.

Now, from ξ_1 we know that an arm gets eliminated within a cluster if $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$ and from ξ_2 we know that all arms get eliminated within a cluster if $\exists a_s \in s_i$ such that $\sqrt{\epsilon_m} < \frac{2\Delta_s}{5}$ and $\hat{r}_s > \hat{r}_i, \forall i \in s_i$. So, if $\sqrt{\epsilon_m} \geq \frac{\sqrt{w_{s_i}}\Delta_i}{5}$ in ξ_1 and $\sqrt{\epsilon_m} \geq \frac{2\Delta_s}{5}$ in ξ_2 then we can say that arm a_i will not be eliminated.

Again, in ξ_1 when arm a_i is eliminated, we can tightly bound ϵ_m such that,

$$\sqrt{\frac{\epsilon_m}{w_{s_i}}} \leq \frac{\Delta_i}{5} \Rightarrow \sqrt{\frac{\epsilon_m}{\ell_m^3}} \leq \frac{\Delta_i}{5}, \text{ as } w_{s_i} \leq \ell_m^3$$

$$\Rightarrow \sqrt{\frac{\epsilon_m}{D_m^3}} \leq \frac{\Delta_i}{5}, \text{ since } \ell_m \leq D_m$$

$$\Rightarrow \sqrt{\frac{\epsilon_m}{(1/\sqrt{\epsilon_m})^3}} \leq \frac{\Delta_i}{5}, \text{ since } D_m = \frac{1}{\sqrt{\epsilon_m}}$$

$$\Rightarrow \epsilon_m \leq (\frac{\Delta_i}{5})^{\frac{4}{5}}$$

Hence, in the m-th round if $\epsilon_m \geq (\frac{\Delta_i}{5})^{\frac{4}{5}}$, we can definitely say that arm a_i is not eliminated in ξ_1.

Proceeding similarly in ξ_2, we know that,

$$\sqrt{\epsilon_m} \leq \frac{2\Delta_s}{5} \Rightarrow \frac{\sqrt{\epsilon_m}}{\ell_m} \leq \frac{2\Delta_s}{5}$$

$$\Rightarrow \frac{\sqrt{\epsilon_m}}{D_m} \leq \frac{2\Delta_s}{5}, \text{ since } \ell_m \leq D_m$$

$$\Rightarrow \epsilon_m \leq \frac{2\Delta_s}{5}, \text{ since } D_m = \frac{1}{\sqrt{\epsilon_m}}$$

Hence, in the m-th round if $\epsilon_m \geq \frac{2\Delta_s}{5}$, a cluster s_i, such that $\exists a_s \in s_i$ and $\hat{r}_s \geq \hat{r}_i, \forall i \in s_i$, then s_i is not eliminated in ξ_2.

Bounding equation (1) trivially by $T\Delta_i$,

$$\Rightarrow \max\left\{\left(\frac{2T\Delta_i}{\psi(m)T\epsilon_m^2}\right),\left(\frac{4T\Delta_i}{(\psi(m)T\epsilon_m^2)^{1+|B_m|^2\epsilon_m}}\right)\right\}$$

$$\leq \max\left\{\left(\frac{2\Delta_i}{\psi(m)(\frac{\Delta_i}{5})^{\frac{8}{5}}}\right),\left(\frac{4T\Delta_i}{(\psi(m)T\frac{4\Delta_s^2}{25})\times(\psi(m)T\frac{4\Delta_s^2}{25})^{|B_m|^2\epsilon_m}}\right)\right\}$$

$$\leq \max\left\{\left(\frac{2\Delta_i}{\psi(m)(\frac{\Delta_i}{5})^{\frac{8}{5}}}\right),\left(\frac{25\Delta_i}{(\psi(m)\Delta_s^2)\times(0.16\psi(m)T\Delta_s^2)^{|B_m|^2 2\Delta_s/5}}\right)\right\}$$

$$\leq \max\left\{\left(\frac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right),\left(\frac{25\Delta_i}{(\psi(m)\Delta_s^2)\times(0.16\psi(m)T\Delta_s^2)^{2|B_m|^2\Delta_s/5}}\right)\right\}$$

$$\leq \max\left\{\left(\frac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right),\left(\frac{25\Delta_i}{(\psi(m)\Delta^2)(0.16\psi(m)T\Delta^2)^{2|B_m|^2\Delta/5}}\right)\right\}, \text{ as } \Delta_s \geq \Delta$$

Hence, total regret till round m is,

$$\leq \max\left\{\left(\frac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right), \left(\frac{25\Delta_i}{(\psi(m)\Delta^2)(0.16\psi(m)T\Delta^2)^{2|B_m|^2\Delta/5}}\right)\right\}$$

Summing over all arms left till round m,

$$\leq \sum_{i\in B_m} \max\left\{\left(\frac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right), \left(\frac{25\Delta_i}{(\psi(m)\Delta^2)(0.16\psi(m)T\Delta^2)^{2|B_m|^2\Delta/5}}\right)\right\},$$

Case 2: Again, given the arm $a_i \in s_i$ has survived till this round, a_i got pulled n_{s_i} number of times from $1, 2, .., m$ and so the regret suffered due to a_i is no less than,

$$n_{s_i} = \left\lceil \frac{2\log\left(\psi(m)T\epsilon_m^2\right)}{\epsilon_m} \right\rceil$$

Also, since we are exploring locally within a cluster and an arm is pulled no longer till it is eliminated by the arm elimination rule, so the conditions imposed in ξ_1 is valid here. So, the total contribution of a_i till round m is given by,

$$\begin{aligned}
\Delta_i\left\lceil \frac{2\log\left(\psi(m)T\epsilon_m^2\right)}{\epsilon_m} \right\rceil &\leq \Delta_i\left\lceil \frac{2\log\left(\psi(m)T(\frac{\Delta_i}{5})^{\frac{8}{5}}\right)}{(\frac{\Delta_i}{5})^{\frac{8}{5}}} \right\rceil, \text{ since } \epsilon_m \geq (\frac{\Delta_i}{5})^{\frac{4}{5}} \\
&\leq \Delta_i\left(1 + \frac{27\log\left(\psi(m)T(\frac{\Delta_i}{5})^{\frac{8}{5}}\right)}{\Delta_i^{\frac{8}{5}}}\right) \\
&\leq \Delta_i\left(1 + \frac{27\log\left(\psi(m)T\frac{\Delta_i^{\frac{8}{5}}}{12}\right)}{\Delta_i^{\frac{8}{5}}}\right)
\end{aligned}$$

Summing over all arms left till round m,

$$\begin{aligned}
&\leq \sum_{i\in B_m} \Delta_i\left(1 + \frac{27\log\left(\psi(m)T\frac{\Delta_i^{\frac{8}{5}}}{12}\right)}{\Delta_i^{\frac{8}{5}}}\right) \\
&\leq \sum_{i\in B_m} \left(\Delta_i + \frac{27\log\left(\psi(m)T\frac{\Delta_i^{\frac{8}{5}}}{12}\right)}{\Delta_i^{\frac{3}{5}}}\right)
\end{aligned}$$

Summing up **Case 1** and **Case 2**, the total regret till round m is given by,

$$R_T \leq \sum_{i\in B_m} \left(\max\left\{\left(\frac{27}{\psi(m)(\Delta_i)^{\frac{3}{5}}}\right), \left(\frac{25\Delta_i}{(\psi(m)\Delta^2)(0.16\psi(m)T\Delta^2)^{2|B_m|^2\Delta/5}}\right)\right\} + \left(\Delta_i + \frac{27\log\left(\psi(m)T\frac{\Delta_i^{\frac{8}{5}}}{12}\right)}{\Delta_i^{\frac{3}{5}}}\right)\right)$$

Appendix F

Theorem 5. *The error bound till round m is given by $e_t \leq \sum_{i \in A'} \left(\frac{51}{\psi(m)\Delta_i^{6/5}}\right) + \sum_{i \in A'' \setminus A'} \left(\frac{51}{\psi(m)\Delta_b^{6/5}}\right)$, where the arms surviving till m-th round belong to the set $A^{'}$, arms to still survive and eliminate arm a^* after round m belong to $A^{''}$.*

Proof. of theorem 5
Let, ξ_3 be the event that the optimal arm a^* was eliminated by a sub-optimal arm. In any round m_*, if the optimal arm a^* gets eliminated then for any round from 1 to m_j all arms a_j such that $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$ were eliminated. Let, the arms surviving till m_* round be denoted by $A^{'}$. This leaves any arm a_b such that $\sqrt{\frac{\epsilon_m}{w_{s_i}}} > \frac{\Delta_b}{5}$ to still survive and eliminate arm a^* in round m_*. Let, such arms that survive a^* belong to $A^{''}$. Also maximal regret per step after eliminating a^* is the maximal Δ_j among the remaining arms a_j with $m_j \geq m_*$. Let m_b be the round when $\sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_b}{5}$. Hence, the maximal regret after eliminating the arm a^* is upper bounded by,

$$\sum_{m_*=0}^{max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \left(\frac{2}{\psi(m) T \epsilon_m^2}\right) . T \max_{j \in A'' : m_j \geq m_*} \Delta_j$$

$$\leq \sum_{m_*=0}^{max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \left(\frac{2}{\psi(m) T \epsilon_m^2}\right) . T . 5\sqrt{\frac{\epsilon_m}{w_{s_i}}}, \text{ since } \sqrt{\frac{\epsilon_m}{w_{s_i}}} < \frac{\Delta_i}{5}$$

$$\leq \sum_{m_*=0}^{max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \frac{10}{2} \left(\frac{1}{\psi(m) \epsilon_m^{3/2}}\right), \text{ as } w_{s_i} \geq 4$$

$$\leq \sum_{m_*=0}^{max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \left(\frac{14}{\psi(m) \Delta_i^{6/5}}\right), \text{ as } \epsilon_m \geq \left(\frac{\Delta_i}{5}\right)^{\frac{4}{5}}$$

$$\leq \sum_{i \in A'' : m_i > m_*} \sum_{m_*=0}^{\min\{m_i, m_b\}} \left(\frac{14}{\psi(m) \Delta_i^{6/5}}\right)$$

$$\leq \sum_{i \in A'} \left(\frac{14}{\psi(m) \Delta_i^{6/5}}\right) + \sum_{i \in A'' \setminus A'} \left(\frac{14}{\psi(m) \Delta_b^{6/5}}\right)$$

$$\leq \sum_{i \in A'} \left(\frac{14 * 5^{4/5}}{\psi(m) \Delta_i^{6/5}}\right) + \sum_{i \in A'' \setminus A'} \left(\frac{14 * 5^{4/5}}{\psi(m) \Delta_b^{6/5}}\right)$$

$$\leq \sum_{i \in A'} \left(\frac{51}{\psi(m) \Delta_i^{6/5}}\right) + \sum_{i \in A'' \setminus A'} \left(\frac{51}{\psi(m) \Delta_b^{6/5}}\right)$$