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# UCB with clustering and improved exploration

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## Abstract

In this paper, we present a novel algorithm for the stochastic multi-armed bandit problem. Our proposed method, referred to as ClusUCB, partitions the arms into clusters and then follows the UCB-Improved strategy with aggressive exploration factors to eliminate sub-optimal arms as well as clusters. Through a theoretical analysis, we establish that ClusUCB achieves a better gap-dependent regret upper bound than UCB-Improved[4] and MOSS[2] algorithms. Further, numerical experiments on test-cases with small gaps between optimal and sub-optimal mean rewards show that ClusUCB results in lower cumulative regret than several popular UCB variants as well as MOSS and Thompson sampling.

## 1 Introduction

In this paper, we consider the stochastic multi-armed bandit problem, a classical problem in sequential decision making. In this setting, a learning algorithm is provided with a set of decisions (or arms) with reward distributions unknown to the algorithm. The learning proceeds in an iterative fashion, where in each round, the algorithm chooses an arm and receives a stochastic reward that is drawn from a stationary distribution specific to the arm selected. Given the goal of maximizing the cumulative reward, the learning algorithm faces the exploration-exploitation dilemma, i.e., in each round should the algorithm select the arm which has the highest observed mean reward so far (*exploitation*), or should the algorithm choose a new arm to gain more knowledge of the true mean reward of the arms and thereby avert a sub-optimal greedy decision (*exploration*).

Formally, let  $r_i, i = 1, \dots, K$  denote the mean rewards of the  $K$  arms and  $r^* = \max_i r_i$  the optimal mean reward. The objective in the stochastic bandit problem is to mini-

mize the cumulative regret, which is defined as follows:

$$R_T = r^*T - \sum_{i \in A} r_i N_i(T),$$

where  $T$  is the number of rounds,  $N_i(T) = \sum_{m=1}^T I(I_m = i)$  is the number of times the algorithm chose arm  $i$  up to round  $T$ . The expected regret of an algorithm after  $T$  rounds can be written as

$$\mathbb{E}[R_T] = \sum_{i=1}^K \mathbb{E}[N_i(T)] \Delta_i,$$

where  $\Delta_i = r^* - r_i$  denotes the gap between the means of the optimal arm and of the  $i$ -th arm.

An early work involving a bandit setup is [18], where the author deals the problem of choosing between two treatments to administer on patients who come in sequentially. Following the seminal work of Robbins [17], bandit algorithms have been extensively studied in a variety of applications. From a theoretical standpoint, an asymptotic lower bound for the regret was established in [13]. In particular, it was shown there that for any consistent allocation strategy, we have  $\liminf_{T \rightarrow \infty} \frac{\mathbb{E}[R_T]}{\log T} \geq \sum_{\{i: r_i < r^*\}} \frac{(r^* - r_i)}{D(p_i || p^*)}$ , where  $D(p_i || p^*)$  is the Kullback-Leibler divergence between the reward densities  $p_i$  and  $p^*$ , corresponding to arms with mean  $r_i$  and  $r^*$ , respectively.

There have been several algorithms with strong regret guarantees. The foremost among them is UCB1 [5], which has a regret upper bound of  $O\left(\frac{K \log T}{\Delta}\right)$ , where  $\Delta = \min_{i: \Delta_i > 0} \Delta_i$ . This result is asymptotically order-optimal for the class of distributions considered. However, the worst case gap independent regret bound of UCB1 can be as bad as  $O\left(\sqrt{TK \log T}\right)$ . In [2], the authors propose the MOSS algorithm and establish that the worst case regret of MOSS is  $O\left(\sqrt{TK}\right)$  which improves upon UCB1 by a factor of order  $\sqrt{\log T}$ . However, the gap-dependent regret of MOSS is  $O\left(\frac{K^2 \log(T\Delta^2/K)}{\Delta}\right)$  and in certain regimes, this can be worse than even UCB1 (see [2],[14]).

The UCB-Improved algorithm, proposed in [4], is a round-based algorithm<sup>1</sup> variant of UCB1 that has a gap-dependent regret bound of  $O\left(\frac{K \log T \Delta^2}{\Delta}\right)$ , which is better than that of UCB1. On the other hand, the worst case regret of UCB-Improved is  $O\left(\sqrt{TK \log K}\right)$ .

## Our Work

We propose a variant of UCB algorithm, henceforth referred to as ClusUCB, that incorporates clustering and an improved exploration scheme. ClusUCB is a round-based algorithm that starts with a partition of the arms into small clusters, each having same number of arms. The clustering is done at the start with a pre-specified number of clusters. Each round of ClusUCB involves both (individual) arm elimination as well as cluster elimination.

The clustering of arms provides two benefits. First, it creates a context where UCB-Improved like algorithm can be run in parallel on smaller sets of arms with limited exploration, which could lead to fewer pulls of sub-optimal arms with the help of more aggressive elimination of sub optimal arms. Second, the cluster elimination leads to whole sets of sub-optimal arms being simultaneously eliminated when they are found to yield poor results. These two simultaneous criteria for arm elimination can be seen as borrowing the strengths of UCB-Improved as well as other popular round based approaches.

The motivation for our work stems from the remark in Section 2.4.3 of [8], where the authors conjecture that one should be able to obtain a bandit algorithm with a gap-dependent regret bound that is better than MOSS [2] and UCB-Improved [4], in particular, with a regret bound of the order  $O\left(\frac{K \log(\frac{T}{H})}{\Delta}\right)$ , where  $H = \sum_{i: \Delta_i > 0} \frac{1}{\Delta_i^2}$ . While ClusUCB does not achieve the conjectured regret bound, the theoretical analysis establishes that the gap-dependent regret of ClusUCB is always better than that of UCB-Improved and better than that of MOSS when  $\sqrt{\frac{e}{T}} \leq \Delta \leq 1$  (see Table 1). Moreover, the gap-independent bound of ClusUCB is of the same order as UCB-Improved, i.e.,  $O(\sqrt{KT \log K})$ . However, ClusUCB is not able to match the gap-independent bound of  $O(\sqrt{KT})$  for MOSS.

On four synthetic setups with small gaps, we observe empirically that ClusUCB outperforms UCB-Improved[4] and MOSS[2] as well as other popular stochastic bandit algorithms such as DMED[12], UCB-V[3], Median Elimination[10], Thompson Sampling[1] and KL-UCB[11].

<sup>1</sup>An algorithm is *round-based* if it pulls all the arms equal number of times in each round and then proceeds to eliminate one or more arms that it identifies to be sub-optimal.

Table 1: Gap-dependent regret bounds for different bandit algorithms

Algorithm	Upper bound
UCB1	$O\left(\frac{K \log T}{\Delta}\right)$
UCB-Improved	$O\left(\frac{K \log(T \Delta^2)}{\Delta}\right)$
MOSS	$O\left(\frac{K^2 \log(T \Delta^2 / K)}{\Delta}\right)$
ClusUCB	$O\left(\frac{K \log\left(\frac{T \Delta^2}{\sqrt{\log(KT)}}\right)}{\Delta}\right)$

The rest of the paper is organized as follows: In Section 2, we present the ClusUCB algorithm. In Section 3, we present the associated regret bounds and prove the main theorem on the regret upper bound for ClusUCB in Section 4. In Section 5, we present the numerical experiments and provide concluding remarks in Section 6. Further proofs of corollaries and propositions presented in Section 4 are provided in the appendices.

## 2 Clustered UCB

**Notation.** We denote the set of arms by  $A$ , with the individual arms labeled  $i, i = 1, \dots, K$ . We denote an arbitrary round of ClusUCB by  $m$ . We denote an arbitrary cluster by  $s_k$ , the subset of arms within the cluster  $s_k$  by  $A_{s_k}$  and the set of clusters by  $S$  with  $|S| = p \leq K$ . Here  $p$  is a pre-specified limit for the number of clusters. For simplicity, we assume that the optimal arm is unique and denote it by  $*$ , with  $s^*$  denoting the corresponding cluster. The best arm in a cluster  $s_k$  is denoted by  $a_{max_{s_k}}$ . We denote the sample mean of the rewards seen so far for arm  $i$  by  $\hat{r}_i$  and for the best arm within a cluster  $s_k$  by  $\hat{r}_{a_{max_{s_k}}}$ . We assume that all arms' rewards are bounded in  $[0, 1]$ .

### The algorithm

As mentioned in a recent work [15], UCB-Improved has two shortcomings:

1. A significant number of pulls are spent in early exploration, since each round  $m$  of UCB-Improved involves pulling every arm an identical  $n_m = \left\lceil \frac{2 \log(T \epsilon_m^2)}{\epsilon_m^2} \right\rceil$  number of times. The quantity  $\epsilon_m$  is initialized to 1 and halved after every round.

**Algorithm 1** ClusUCB

**Input:** Number of clusters  $p$ , time horizon  $T$ , exploration parameters  $\rho_a, \rho_s$  and  $\psi$ .

**Initialization:** Set  $B_0 := A$ ,  $S_0 = S$  and  $\epsilon_0 := 1$ .

Create a partition  $S_0$  of the arms at random into  $p$  clusters of size up to  $\ell = \left\lceil \frac{K}{p} \right\rceil$  each.

**for**  $m = 0, 1, \dots, \left\lfloor \frac{1}{2} \log_2 \frac{T}{\epsilon} \right\rfloor$  **do**

Pull each arm in  $B_m$  so that the total number of times it has been pulled is  $n_m = \left\lceil \frac{2 \log(\psi T \epsilon_m^2)}{\epsilon_m} \right\rceil$ .

**Arm Elimination**

For each cluster  $s_k \in S_m$ , delete arm  $i \in s_k$  from  $B_m$  if

$$\hat{r}_i + \sqrt{\frac{\rho_a \log(\psi T \epsilon_m^2)}{2n_m}} < \max_{j \in s_k} \left\{ \hat{r}_j - \sqrt{\frac{\rho_a \log(\psi T \epsilon_m^2)}{2n_m}} \right\}$$

**Cluster Elimination**

Delete cluster  $s_k \in S_m$  and remove all arm  $i \in s_k$  from  $B_m$  if

$$\begin{aligned} & \max_{i \in s_k} \left\{ \hat{r}_i + \sqrt{\frac{\rho_s \log(\psi T \epsilon_m^2)}{2n_m}} \right\} \\ & < \max_{j \in B_m} \left\{ \hat{r}_j - \sqrt{\frac{\rho_s \log(\psi T \epsilon_m^2)}{2n_m}} \right\}. \end{aligned}$$

Set  $\epsilon_{m+1} := \frac{\epsilon_m}{2}$

Set  $B_{m+1} := B_m$

Stop if  $|B_m| = 1$  and pull  $\max_{i \in B_m} \hat{r}_i$  till  $T$  is reached.

**end for**

2. In UCB-Improved, arms are eliminated conservatively, i.e, only after  $\epsilon_m < \frac{\Delta_i}{2}$ , the sub-optimal arm  $i$  is discarded with high probability. This is disadvantageous when  $K$  is large and the gaps are identical ( $r_1 = r_2 = \dots = r_{K-1} < r^*$ ) and small.

To reduce early exploration, the number  $n_m$  of times each arm is pulled per round in ClusUCB is lower than that of UCB-Improved and also that of Median-Elimination, which used  $n_m = \frac{4}{\epsilon^2} \log\left(\frac{3}{\delta}\right)$ , where  $\epsilon, \delta$  are confidence parameters. To handle the second problem mentioned above, ClusUCB partitions the larger problem into several small sub-problems using clustering and then performs local exploration aggressively to eliminate sub-optimal arms within each clusters with high probability.

As described in the pseudocode in Algorithm 1, ClusUCB begins with a initial clustering of arms that is performed

by random uniform allocation. The set of clusters  $S$  thus obtained satisfies  $|S| = p$ , with individual clusters having a size that is bounded above by  $\ell = \left\lceil \frac{K}{p} \right\rceil$ . Each round of ClusUCB involves both individual arm as well as cluster elimination conditions. These elimination conditions are inspired by UCB-Improved. Notice that, unlike UCB-Improved, there is no longer a single point of reference based on which we are eliminating arms. Instead now we have as many reference points to eliminate arms as number of clusters formed.

The exploration regulatory factor  $\psi$  governing the arm and cluster elimination conditions in ClusUCB is more aggressive than that in UCB-Improved. With appropriate choice of  $\psi$  and  $\rho_a$  and  $\rho_s$  we can achieve aggressive elimination even when the gaps  $\Delta_i$  are small and  $K$  is large.

In [15], the authors recommend incorporating a factor of  $d_i$  inside the log-term of the UCB values, i.e.,  $\max\{\hat{r}_i + \sqrt{\frac{d_i \log T \epsilon_m^2}{2n_m}}\}$ . The authors there examine the following choices for  $d_i$ :  $\frac{T}{t_i}$ ,  $\frac{\sqrt{T}}{t_i}$  and  $\frac{\log T}{t_i}$ , where  $t_i$  is the number of times an arm  $i$  has been sampled. Unlike [15], we employ cluster as well as arm elimination and establish from a theoretical analysis that the choice  $\psi = \frac{T}{\log(KT)}$  helps in achieving a better gap-dependent regret upper bound for ClusUCB as compared to UCB-Improved and MOSS (see Corollary 1 in the next section).

### 3 Main results

We now state the main result that upper bounds the expected regret of ClusUCB.

**Theorem 1 (Regret bound).** *The regret  $R_T$  of ClusUCB satisfies*

$$\begin{aligned} \mathbb{E}[R_T] & \leq \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \left\{ \frac{C_1(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \Delta_i \right. \\ & \quad + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \left. \right\} + \sum_{\substack{i \in A, \\ \Delta_i > b}} \left\{ 2\Delta_i + \frac{C_1(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right. \\ & \quad + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} + \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \left. \right\} \\ & \quad + \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \sum_{\substack{i \in A_{s^*}, \\ 0 < \Delta_i \leq b}} \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}} \\ & \quad + \sum_{\substack{i \in A \setminus A_{s^*}, \\ \Delta_i > b}} \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \sum_{\substack{i \in A \setminus A_{s^*}, \\ 0 < \Delta_i \leq b}} \frac{2C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}} \\ & \quad + \max_{i: \Delta_i \leq b} \Delta_i T, \end{aligned}$$

where  $b \geq \sqrt{\frac{e}{T}}$ ,  $C_1(x) = \frac{2^{1+4x}x^{2x}}{\psi^x}$ ,  $C_2(x) = \frac{2^{2x+\frac{3}{2}}x^{2x}}{\psi^x}$  and  $A_{s^*}$  is the subset of arms in cluster  $s^*$  containing optimal arm  $a^*$ .

*Proof.* See Section 4.  $\square$

We now specialize the result in the theorem above by substituting specific values for the exploration constants  $\rho_s$ ,  $\rho_a$  and  $\psi$ .

**Corollary 1 (Gap-dependent bound).** With  $\psi = \frac{T}{\log(KT)}$ ,  $\rho_a = \frac{1}{2}$  and  $\rho_s = \frac{1}{2}$ , we have the following gap-dependent bound for the regret of ClusUCB:

$$\begin{aligned} \mathbb{E}[R_T] \leq & \sum_{\substack{i \in A_{s^*}: \\ \Delta_i > b}} \frac{4\sqrt{\log(KT)}}{\Delta_i} + \sum_{i \in A: \Delta_i > b} \left\{ 3\Delta_i \right. \\ & + \frac{6.8\sqrt{\log(KT)}}{\Delta_i} + \frac{96 \log\left(\frac{T\Delta_i^2}{\sqrt{\log(KT)}}\right)}{\Delta_i} \Big\} \\ & + \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \frac{2.8\sqrt{\log(KT)}}{\Delta_i} + \sum_{\substack{i \in A_{s^*}, \\ 0 \leq \Delta_i \leq b}} \frac{2.8\sqrt{\log(KT)}}{\Delta_i} \\ & + \sum_{\substack{i \in A, \\ 0 \leq \Delta_i \leq b}} \frac{2.8\sqrt{\log(KT)}}{\Delta_i} + \max_{i \in A: \Delta_i \leq b} \Delta_i T \end{aligned}$$

for all  $b \geq \sqrt{\frac{e}{T}}$ .

*Proof.* See Appendix C.  $\square$

The most significant term in the bound above is  $\sum_{i \in A: \Delta_i \geq b} \frac{96 \log\left(T \frac{\Delta_i^2}{\sqrt{\log(KT)}}\right)}{\Delta_i}$  and hence, the regret upper bound for ClusUCB is of the order  $O\left(\frac{K \log\left(\frac{T\Delta^2}{\sqrt{\log(KT)}}\right)}{\Delta}\right)$ . As shown in Table 1, the gap-dependent bound of ClusUCB is always better than UCB1 and UCB-Improved. In comparison to the gap-dependent bound of MOSS, we observe that ClusUCB will be better if  $\frac{T^{K-1}\Delta^{2K-2}}{K^K} > \frac{1}{\sqrt{\log(KT)}}$ , which is equivalent to  $\Delta > \frac{2}{\sqrt{T}(\log 2T)^{\frac{1}{4}}}$  for  $K \geq 2$ . Since Corollary 1 holds for all  $\Delta \geq \sqrt{\frac{e}{T}} > \frac{2}{\sqrt{T}(\log 2T)^{\frac{1}{4}}}$ , it can be clearly seen that for all  $\sqrt{\frac{e}{T}} \leq \Delta \leq 1$  and  $K \geq 2$ , the gap-dependent bound is better than MOSS.

**Corollary 2 (Gap-independent bound).** Considering the same gap of  $\Delta_i = \Delta = \sqrt{\frac{K \log K}{T}}$  for all  $i: i \neq *$  and with  $\psi = K^2T$ ,  $\rho_a = \frac{1}{4}$  and  $\rho_s = \frac{1}{2}$ , we have the following gap-independent bound for the regret of ClusUCB:

$$\mathbb{E}[R_T] \leq 6\frac{\sqrt{KT}}{p} + 128\sqrt{KT \log K}$$

$$+ \frac{64\sqrt{KT} \log(\log K)}{\sqrt{\log K}} + 6.8\sqrt{\frac{T}{K \log K}} + 2.8\sqrt{\frac{T}{e}}$$

*Proof.* See Appendix D.  $\square$

From the above result, we observe that the order of the regret upper bound of ClusUCB is  $O(\sqrt{KT \log K})$  and this matches the order of UCB-Improved. However, this is not as low as the order  $O(\sqrt{KT})$  of MOSS.

## Analysis of elimination error

Let  $\tilde{R}_T$  denote the contribution to the expected regret in the case when the optimal arm  $*$  gets eliminated during one of the rounds of ClusUCB. This can happen if a sub-optimal arm eliminates  $*$  or if a sub-optimal cluster eliminates the cluster  $s^*$  that contains  $*$  - these correspond to cases b2 and b3 in the proof of Theorem 1 (see Section 4). We shall denote variants of ClusUCB that include arm elimination condition only and cluster elimination condition only as ClusUCB-AE and ClusUCB-CE, respectively, while ClusUCB corresponds to Algorithm 1, which uses both arm and cluster elimination conditions.

For ClusUCB-AE, the quantity  $\tilde{R}_T$  can be extracted from the proofs (in particular, case b2 in Appendix A) and simplified using the values  $\rho_a = \frac{1}{4}$  and  $\psi = K^2T$ , to obtain  $\tilde{R}_T = 2\sqrt{KT}$ .

A similar exercise for ClusUCB-CE (see Case b2 in Appendix B) with  $\rho_s = \frac{1}{4}$  and  $\psi = K^2T$  yields  $\tilde{R}_T = 2\sqrt{KT}$ .

Finally, for ClusUCB, the relevant terms from Theorem 1 that corresponds to  $\tilde{R}_T$  can be simplified with  $\rho_a = \frac{1}{4}$ ,  $\rho_s = \frac{1}{2}$  and  $\psi = K^2T$  (as in Corollary 2 to obtain

$$\tilde{R}_T = 4\frac{\sqrt{KT}}{p} + 2.8\sqrt{\frac{T}{K \log K}} + 2.8\sqrt{\frac{T}{e}}.$$

Hence, in comparison to ClusUCB-AE and ClusUCB-CE, the elimination error contribution to regret is lower in ClusUCB when  $\sqrt{\log K} \leq p \leq \frac{K}{2}$ .

Thus, we observe that clustering in conjunction with improved exploration via  $\rho_a, \rho_s$  and  $\psi$  helps in reducing the constant associated with the factor  $\sqrt{KT}$  for the gap-independent regret bound for ClusUCB.

## 4 Proof of Theorem 1

*Proof.* Let  $A' = \{i \in A, \Delta_i > b\}$ ,  $A'' = \{i \in A, \Delta_i > 0\}$ ,  $A'_{s_k} = \{i \in A_{s_k}, \Delta_i > b\}$  and  $A''_{s_k} = \{i \in A_{s_k}, \Delta_i > 0\}$ .  $C_g$  is the cluster set containing max payoff arm from each cluster in  $g$ -th round. The arm having the highest payoff in a cluster  $s_k$  is denote

by  $a_{\max_{s_k}}$ . Let for each sub-optimal arm  $i \in A$ ,  $m_i = \min \{m | \sqrt{\rho_a \epsilon_m} < \frac{\Delta_i}{2}\}$  and let for each cluster  $s_k \in S$ ,  $g_{s_k} = \min \{g | \sqrt{\rho_s \epsilon_g} < \frac{\Delta_{a_{\max_{s_k}}}}{2}\}$ . Let  $\check{A} = \{i \in A' | i \in s_k, \forall s_k \in S\}$ .

The analysis proceeds by considering the contribution to the regret in each of the following cases:

**Case a:** Some sub-optimal arm  $i$  is not eliminated in round  $\max(m_i, g_{s_k})$  or before, with the optimal arm  $*$   $\in C_{\max(m_i, g_{s_k})}$ .

We consider an arbitrary sub-optimal arm  $i$  and analyze the contribution to the regret when  $i$  is not eliminated in the following exhaustive sub-cases:

**Case a1:** In round  $\max(m_i, g_{s_k})$ ,  $i \in s^*$ .

Similar to case (a) of [4], observe that when the following two conditions hold, arm  $i$  gets eliminated:

$$\hat{r}_i \leq r_i + c_{m_i} \text{ and } \hat{r}^* \geq r^* - c_{m_i}, \quad (1)$$

where  $c_{m_i} = \sqrt{\frac{\rho_a \log(\psi T \epsilon_{m_i}^2)}{2n_{m_i}}}$ . The arm  $i$  gets eliminated because

$$\begin{aligned} \hat{r}_i + c_{m_i} &\leq r_i + 2c_{m_i} < r_i + \Delta_i - 2c_{m_i} = r^* - 2c_{m_i} \\ &\leq \hat{r}^* - c_{m_i}. \end{aligned}$$

In the above, we have used the fact that

$$c_{m_i} = \sqrt{\rho_a \epsilon_{m_i+1}} < \frac{\Delta_i}{4}, \text{ since } n_{m_i} = \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}} \text{ and } \rho_a \in (0, 1].$$

From the foregoing, we have to bound the events complementary to that in (1) for an arm  $i$  to not get eliminated. Considering Chernoff-Hoeffding bound this is done as follows:

$$\begin{aligned} \mathbb{P}(\hat{r}^* \leq r^* - c_{m_i}) &\leq \exp(-2c_{m_i}^2 n_{m_i}) \\ &\leq \exp(-2 * \frac{\rho_a \log(\psi T \epsilon_{m_i}^2)}{2n_{m_i}} * n_{m_i}) \\ &\leq \frac{1}{(\psi T \epsilon_{m_i}^2)^{\rho_a}} \end{aligned}$$

Along similar lines, we have  $\mathbb{P}(\hat{r}_i \geq r_i + c_{m_i}) \leq \frac{1}{(\psi T \epsilon_{m_i}^2)^{\rho_a}}$ . Thus, the probability that a sub-optimal arm  $i$  is not eliminated in any round on or before  $m_i$  is bounded above by  $\left(\frac{2}{(\psi T \epsilon_{m_i}^2)^{\rho_a}}\right)$ . Summing up over all arms in  $A'_{s^*}$  in conjunction with a simple bound of  $T\Delta_i$  for each arm, we obtain

$$\begin{aligned} \sum_{i \in A'_{s^*}} \left( \frac{2T\Delta_i}{(\psi T \epsilon_{m_i}^2)^{\rho_a}} \right) &\leq \sum_{i \in A'_{s^*}} \left( \frac{2T\Delta_i}{(\psi T \frac{\Delta_i^4}{16\rho_a^2})^{\rho_a}} \right) \\ &= \sum_{i \in A'_{s^*}} \left( \frac{C_1(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right), \text{ where } C_1(x) = \frac{2^{1+4x}x^{2x}}{\psi^x} \end{aligned}$$

**Case a2:** In round  $\max(m_i, g_{s_k})$ ,  $i \in s_k$  for some  $s_k \neq s^*$ .

Following a parallel argument like in Case a1, we have to bound the following two events of arm  $a_{\max_{s_k}}$  not getting eliminated on or before  $g_{s_k}$ -th round,

$$\hat{r}_{a_{\max_{s_k}}} \geq r_{a_{\max_{s_k}}} + c_{g_{s_k}} \text{ and } \hat{r}^* \leq r^* - c_{g_{s_k}}$$

We can prove using Chernoff-Hoeffding bounds and considering independence of events mentioned above, that for  $c_{g_{s_k}} = \sqrt{\frac{\rho_s \log(\psi T \epsilon_{g_{s_k}}^2)}{2n_{g_{s_k}}}}$  and  $n_{g_{s_k}} = \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}}$  the probability of the above two events is bounded by  $\left(\frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}}\right)$ . Now, for any round  $g_{s_k}$ , all the elements of  $C_{\max(m_i, g_{s_k})}$  are the respective maximum payoff arms of their cluster  $s_k, \forall s_k \in S$ , and since all the surviving arms are pulled equally in each round and since clusters are fixed so we can bound the maximum probability that a sub-optimal arm  $i \in A'$  and  $i \in s_k$  such that  $a_{\max_{s_k}} \in C_{g_{s_k}}$  is not eliminated on or before the  $g_{s_k}$ -th round by the same probability as above.

Summing up over all  $p$  clusters and bounding the regret for each arm  $i \in A'_{s_k}$  trivially by  $T\Delta_i$ ,

$$\begin{aligned} \sum_{k=1}^p \sum_{i \in A'_{s_k}} \left( \frac{2T\Delta_i}{(\psi T \frac{\Delta_i^4}{16\rho_s^2})^{\rho_s}} \right) &= \sum_{i \in A'} \left( \frac{2T\Delta_i}{(\psi T \frac{\Delta_i^4}{16\rho_s^2})^{\rho_s}} \right) \\ &\leq \sum_{i \in A'} \left( \frac{2^{1+4\rho_s} \rho_s^{2\rho_s} T^{1-\rho_s}}{\psi^{\rho_s} \Delta_i^{4\rho_s-1}} \right) = \sum_{i \in A'} \frac{C_1(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \end{aligned}$$

Summing the bounds in Cases a1 – a2 and observing that the bounds in the aforementioned cases hold for any round  $C_{\max\{m_i, g_{s_k}\}}$ , we obtain the following contribution to the expected regret from case a:

$$\sum_{i \in A_{s^*}} \frac{C_1(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \sum_{i \in A'} \left( \frac{C_1(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right)$$

**Case b:** For each arm  $i$ , either  $i$  is eliminated in round  $\max(m_i, g_{s_k})$  or before or there is no optimal arm  $*$  in  $C_{\max(m_i, g_{s_k})}$ .

**Case b1:**  $*$   $\in C_{\max(m_i, g_{s_k})}$  for each arm  $i \in A'$  and cluster  $s_k \in \check{A}$ .

The condition in the case description above implies the following:

- (i) each sub-optimal arm  $i \in A'$  is eliminated on or before  $\max(m_i, g_{s_k})$  and hence pulled not more than  $n_{m_i}$  number of times.
- (ii) each sub-optimal cluster  $s_k \in \check{A}$  is eliminated on or before  $\max(m_i, g_{s_k})$  and hence pulled not more than  $n_{g_{s_k}}$  number of times.

Hence, the maximum regret suffered due to pulling of a sub-optimal arm or a sub-optimal cluster is no more than the following:

$$\begin{aligned}
 & \sum_{i \in A'} \Delta_i \left\lceil \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}} \right\rceil + \sum_{k=1}^p \sum_{i \in A'_{s_k}} \Delta_i \left\lceil \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}} \right\rceil \\
 & \leq \sum_{i \in A'} \Delta_i \left( 1 + \frac{32 \rho_a \log \left( \psi T \left( \frac{\Delta_i}{2\sqrt{\rho_a}} \right)^4 \right)}{\Delta_i^2} \right) \\
 & \quad + \sum_{i \in A'} \Delta_i \left( 1 + \frac{32 \rho_s \log \left( \psi T \left( \frac{\Delta_i}{2\sqrt{\rho_s}} \right)^4 \right)}{\Delta_i^2} \right) \\
 & \leq \sum_{i \in A'} \left\{ 2\Delta_i + \frac{32 \rho_a \log(\psi T \frac{\Delta_i^4}{16 \rho_a^2})}{\Delta_i} \right. \\
 & \quad \left. + \frac{32 \rho_s \log(\psi T \frac{\Delta_i^4}{16 \rho_s^2})}{\Delta_i} \right\}
 \end{aligned}$$

In the above, the first inequality follows since  $\sqrt{\rho_a \epsilon_{m_i}} < \frac{\Delta_i}{2}$  and  $\sqrt{\rho_s \epsilon_{g_{s_k}}} < \frac{\Delta_{a_{\max s_k}}}{2}$ .

**Case b2:**  $*$  is eliminated by some sub-optimal arm in  $s^*$ . Optimal arm  $a^*$  can get eliminated by some sub-optimal arm  $i$  only if arm elimination condition holds, i.e.,

$$\hat{r}_i - c_{m_i} > \hat{r}^* + c_{m_i},$$

where, as mentioned before,  $c_{m_i} = \sqrt{\frac{\rho_a \log(\psi T \epsilon_{m_i}^2)}{2n_{m_i}}}$ . From analysis in Case a1, notice that, if (1) holds in conjunction with the above, arm  $i$  gets eliminated. Also, recall from Case a1 that the events complementary to (1) have low-probability and can be upper bounded by  $\frac{2}{(\psi T \epsilon_{m_*}^2)^{\rho_a}}$ . Moreover, a sub-optimal arm that eliminates  $*$  has to survive until round  $m_*$ . In other words, all arms  $j \in s^*$  such that  $m_j < m_*$  are eliminated on or before  $m_*$  (this corresponds to case b1). Let, the arms surviving till  $m_*$  round be denoted by  $A'_{s^*}$ . This leaves any arm  $a_b$  such that  $m_b \geq m_*$  to still survive and eliminate arm  $*$  in round  $m_*$ . Let, such arms that survive  $*$  belong to  $A''_{s^*}$ . Also maximal regret per step after eliminating  $*$  is the maximal  $\Delta_j$  among the remaining arms in  $A''_{s^*}$  with  $m_j \geq m_*$ . Let  $m_b = \min\{m_j | \sqrt{\rho_a \epsilon_{m_j}} < \frac{\Delta_b}{2}\}$ . Let  $C_2(x) = \frac{2^{2x+\frac{3}{2}} x^{2x}}{\psi^x}$ . Hence, the maximal regret after eliminating the arm  $*$  is upper bounded by,

$$\begin{aligned}
 & \max_{j \in A'_{s^*}} m_j \\
 & \sum_{m_*=0} \sum_{\substack{i \in A''_{s^*}: \\ m_i \geq m_*}} \left( \frac{2}{(\psi T \epsilon_{m_*}^2)^{\rho_a}} \right) \cdot T \max_{\substack{j \in A''_{s^*}: \\ m_j \geq m_*}} \Delta_j
 \end{aligned}$$

$$\begin{aligned}
 & \leq \sum_{m_*=0}^{\max_{j \in A'_{s^*}} m_j} \sum_{i \in A''_{s^*}: m_i \geq m_*} \left( \frac{2}{(\psi T \epsilon_{m_*}^2)^{\rho_a}} \right) \cdot T \cdot 2\sqrt{\rho_a \epsilon_{m_*}} \\
 & \leq \sum_{m_*=0}^{\max_{j \in A'_{s^*}} m_j} \sum_{i \in A''_{s^*}: m_i \geq m_*} 4 \left( \frac{T^{1-\rho_a}}{\psi \rho_a \epsilon_{m_*}^{2\rho_a-\frac{1}{2}}} \right) \\
 & \leq \sum_{i \in A''_{s^*}: m_i \geq m_*} \sum_{m_*=0}^{\min\{m_i, m_b\}} \left( \frac{4T^{1-\rho_a}}{\psi \rho_a 2^{-(2\rho_a-\frac{1}{2})m_*}} \right) \\
 & \leq \sum_{i \in A'_{s^*}} \frac{4T^{1-\rho_a}}{\psi \rho_a 2^{-(2\rho_a-\frac{1}{2})m_*}} + \sum_{i \in A''_{s^*} \setminus A'_{s^*}} \frac{4T^{1-\rho_a}}{\psi \rho_a 2^{-(2\rho_a-\frac{1}{2})m_b}} \\
 & \leq \sum_{i \in A'_{s^*}} \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a+\frac{3}{2}}}{\psi \rho_a \Delta_i^{4\rho_a-1}} + \sum_{i \in A''_{s^*} \setminus A'_{s^*}} \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a+\frac{3}{2}}}{\psi \rho_a b^{4\rho_a-1}} \\
 & = \sum_{i \in A'_{s^*}} \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \sum_{i \in A''_{s^*} \setminus A'_{s^*}} \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}}
 \end{aligned}$$

**Case b3:**  $s^*$  is eliminated by some sub-optimal cluster.

Let  $C'_g = \{a_{\max s_k} \in A' | \forall s_k \in S\}$  and  $C''_g = \{a_{\max s_k} \in A'' | \forall s_k \in S\}$ . A sub-optimal cluster  $s_k$  will eliminate  $s^*$  in round  $g_*$  only if the cluster elimination condition of Algorithm 1 holds, which is the following when  $* \in C_{g_*}$ :

$$\hat{r}_{a_{\max s_k}} - c_{g_*} > \hat{r}^* + c_{g_*}. \quad (2)$$

Notice that when  $* \notin C_{g_*}$ , since  $r_{a_{\max s_k}} > r^*$ , the inequality in (2) has to hold for cluster  $s_k$  to eliminate  $s^*$ . As in case b2, the probability that a given sub-optimal cluster  $s_k$  eliminates  $s^*$  is upper bounded by  $\frac{2}{(\psi T \epsilon_{g_*}^2)^{\rho_s}}$  and all sub-optimal clusters with  $g_{s_j} < g_*$  are eliminated before round  $g_*$ .

This leaves any arm  $a_{\max s_b}$  such that  $g_{s_b} \geq g_*$  to still survive and eliminate arm  $*$  in round  $g_*$ . Let, such arms that survive  $*$  belong to  $C''_g$ . Hence, following the same way as case b2, the maximal regret after eliminating  $*$  is,

$$\begin{aligned}
 & \max_{a_{\max s_j} \in C'_g} g_{s_j} \\
 & \sum_{g_*=0} \sum_{\substack{a_{\max s_k} \in C''_g: \\ g_{s_k} \geq g_*}} \left( \frac{2}{(\psi T \epsilon_{g_*}^2)^{\rho_s}} \right) T \max_{\substack{a_{\max s_j} \in C''_g: \\ g_{s_j} \geq g_*}} \Delta_{a_{\max s_j}}
 \end{aligned}$$

Using  $A' \supset C'_g$  and  $A'' \supset C''_g$ , we can bound the regret contribution from this case in a similar manner as Case b2 as follows:

$$\begin{aligned}
 & \sum_{i \in A' \setminus A'_{s^*}} \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s+\frac{3}{2}}}{\psi \rho_s \Delta_i^{4\rho_s-1}} + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s+\frac{3}{2}}}{\psi \rho_s b^{4\rho_s-1}} \\
 & = \sum_{i \in A' \setminus A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}}
 \end{aligned}$$

**Case b4:**  $*$  is not in  $C_{\max(m_i, g_{s_k})}$ , but belongs to  $B_{\max(m_i, g_{s_k})}$ .

In this case the optimal arm  $*$   $\in s^*$  is not eliminated, also  $s^*$  is not eliminated. So, for all sub-optimal arms  $i$  in  $A'_{s^*}$  which gets eliminated on or before  $\max\{m_i, g_{s_k}\}$  will get pulled no less than  $\left\lceil \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}} \right\rceil$  number of times, which leads to the following bound the contribution to the expected regret, as in Case b1:

$$\sum_{i \in A'_{s^*}} \left\{ \Delta_i + \frac{32 \rho_a \log(\psi T \frac{\Delta_i^4}{16 \rho_a^2})}{\Delta_i} \right\}$$

For arms  $a_i \notin s^*$ , the contribution to the regret cannot be greater than that in Case b3. So the regret is bounded by,

$$\sum_{i \in A' \setminus A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}}$$

The main claim follows by summing the contributions to the expected regret from each of the cases above.  $\square$

**Proposition 1.** *The regret  $R_T$  for ClusUCB-AE satisfies*

$$\begin{aligned} \mathbb{E}[R_T] \leq & \sum_{\substack{i \in A \\ \Delta_i > b}} \left\{ \frac{C_1(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \Delta_i + \frac{32 \rho_a \log(\frac{\psi T \Delta_i^4}{16 \rho_a^2})}{\Delta_i} \right. \\ & \left. + \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right\} + \sum_{\substack{i \in A \\ 0 < \Delta_i \leq b}} \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}} + \max_{\substack{i \in A \\ \Delta_i \leq b}} \Delta_i T, \end{aligned}$$

for all  $b \geq \sqrt{\frac{e}{T}}$ . In the above,  $C_1, C_2$  are as defined in Theorem 1.

*Proof.* See Appendix A.  $\square$

**Proposition 2.** *The regret  $R_T$  for ClusUCB-CE satisfies,*

$$\begin{aligned} \mathbb{E}[R_T] \leq & \sum_{\substack{i \in A \\ \Delta_i > b}} \left\{ \frac{C_1(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \frac{64 \rho_s \log(\psi T \frac{\Delta_i^4}{16 \rho_s^2})}{\Delta_i} \right. \\ & \left. + 2\Delta_i \right\} + \sum_{\substack{i \in A \setminus A_{s^*} \\ \Delta_i > b}} \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \\ & + \sum_{\substack{i \in A \setminus A_{s^*} \\ 0 < \Delta_i \leq b}} \frac{2C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}} + \max_{i \in A: \Delta_i \leq b} \Delta_i T, \end{aligned}$$

for all  $b \geq \sqrt{\frac{e}{T}}$ , with  $C_1$  and  $C_2$  as defined in Theorem 1.

*Proof.* See Appendix B.  $\square$

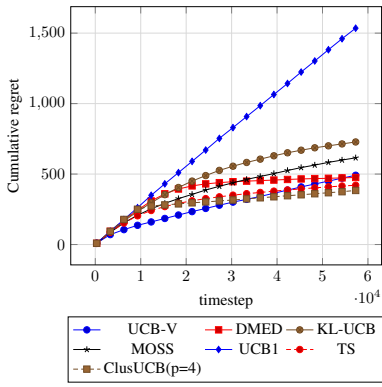
## 5 Simulation experiments

For the purpose of performance comparison using cumulative regret as the metric, we implement the following algorithms: KL-UCB[11], DMED[12], MOSS[2], UCB1[5], UCB-Improved[4], Median Elimination[10], Thompson Sampling(TS)[1] and UCB-V[3]<sup>2</sup>. The parameters of ClusUCB algorithm for all the experiments are set as follows:  $\psi = \log T$ ,  $\rho_s = \frac{1}{2^{2m+1}}$  and  $\rho_a = \frac{1}{2^{4m+1}}$  for  $m = 0, 1, \dots, \lfloor \frac{1}{2} \log_2 \frac{T}{e} \rfloor$ . So in all the experiments  $\rho_a$  and  $\rho_s$  are initialized to 1 and then reduced after every round. By this definition of  $\rho_a, \rho_s$  we have made sure that their value always remain bounded  $\in (0, 1]$ . When  $K$  is large and  $p$  is small it is advantageous to run  $\rho_a < \rho_s$  (see Corollary 2) because this will aggressively eliminate arms within each cluster while full cluster elimination will be more conservative. The intuition here is that since each cluster contains a large number of arms it should be eliminated less aggressively.

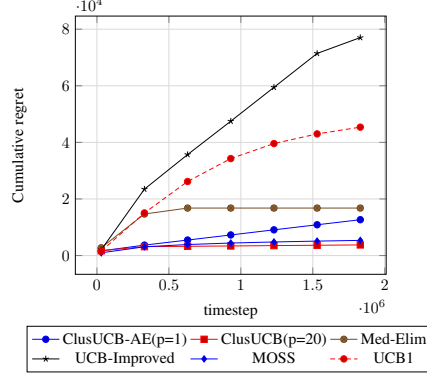
The first experiment is conducted over a testbed of 20 arms for the test-cases involving Bernoulli reward distribution with expected rewards of the arms  $r_{i \neq *}=0.07$  and  $r^*=0.1$ . These type of cases are frequently encountered in web-advertising domain. The horizon  $T$  is set to 60000. After limited exploratory experimentation the number of clusters  $p$  for ClusUCB is set to 4. The regret is averaged over 100 independent runs and is shown in Figure 1a. ClusUCB, MOSS, UCB1, UCB-V, KL-UCB, TS and DMED are run in this experimental setup and we observe that ClusUCB performs better than all the aforementioned algorithms. Because of the short horizon  $T$ , we do not implement UCB-Improved and Median Elimination on this test-case. We also observe that in this case the cumulative regret of ClusUCB and TS are very similar to each other with ClusUCB being slightly better.

The second experiment is conducted over a testbed of 100 arms involving Gaussian reward distribution with expected rewards of the arms  $r_{i \neq *:1-33}=0.01$ ,  $r_{i \neq *:34-99}=0.06$  and  $r_{i=100}^*=0.1$  with variance set at  $\sigma=0.3, \forall i \in A$ . The horizon  $T$  is set for a large duration of  $2 \times 10^6$  and the number of clusters  $p=20$ . The regret is averaged over 100 independent runs and is shown in Figure 1b. In this case, in addition to ClusUCB, we also show the performance of no-clustering version of ClusUCB algorithm (i.e.,  $p=1$ ). From the results in Figure 1b, we observe that ClusUCB with  $p=20$  outperforms ClusUCB with  $p=1$  as well as MOSS, UCB1, UCB-Improved and Median-Elimination( $\epsilon=0.03, \delta=0.1$ ). We also observed that the ClusUCB variant that uses only the arm elimination condition in Algorithm 1 performs worse than the variant that employs both cluster and arm elimination conditions. Also UCB- Improved performs the worst and it confirms

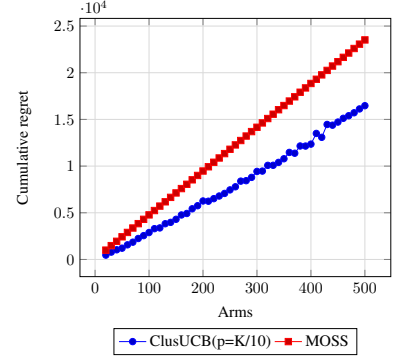
<sup>2</sup>The implementation for KL-UCB and DMED were taken from [9]



(a) Experiment 1: 20 Bernoulli-distributed arms with  $r_{i \neq *}$  = 0.07 and  $r^*$  = 0.1.

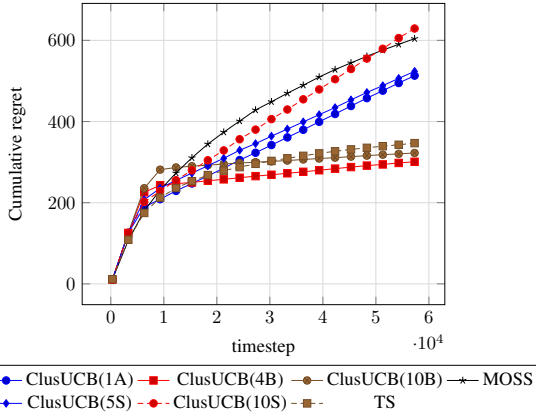


(b) Experiment 2: 100 Gaussian-distributed arms with  $r_{i \neq *; 1-33}$  = 0.01,  $r_{i \neq *; 34-99}$  = 0.06 and  $r_{i=100}^*$  = 0.1.



(c) Experiment 3: 20 to 500 Bernoulli-distributed arms with  $r_{i \neq *}$  = 0.05 and  $r^*$  = 0.1.

Figure 1: Cumulative regret for various bandit algorithms on three stochastic K-armed bandit environments.



(a) Experiment 4: Cumulative regret for ClusUCB variants: 1, 4, 5, 10 correspond to number  $p$  of clusters and A, S, B correspond to having only arm elimination, only cluster elimination and having both in Algorithm 1.

our assumption that it spends too much pulls in initial exploration.

The third experiment is conducted over a testbed of 20 – 500 (interval of 10) arms with Bernoulli reward distribution, where the expected rewards of the arms are  $r_{i \neq *} = 0.05$  and  $r^* = 0.1$ . The horizon  $T$  is set to  $10^5 + K^2 \times 10^4$  and the number of arms are increased from  $K = 20$  to 500. The proposed algorithm ClusUCB is run with  $p = K/10$ . The regret is averaged over 500 independent runs and is shown in Figure 1c. We report the performance of MOSS and ClusUCB only over this setup. From the results in Figure 1c, it is evident that the growth of regret for ClusUCB is lower than that of MOSS.

The fourth experiment is performed over a testbed having 20 Bernoulli-distributed arms with  $r_{i \neq *} = 0.06, \forall i \in A$  and  $r^* = 0.1$ . In Figure 2a, we report the results with  $T = 60000$  averaged over 100 independent runs for ClusUCB

with  $p = \{1, 4, 10\}$ . Also, in this experiment we take  $\psi = 1$  to make sure that the exploration is low. Under these conditions, exploration is low and the elimination parameters are decreased very fast. This leads to a greater number of errors committed by ClusUCB-AE that is ClusUCB( $p = 1$ ). Two other benchmark algorithms are considered here, MOSS which is one of our main competitors and TS which has performed near equivalently in experiment 1 (see Fig 1a). In this case we see that, since  $\rho_a$  is decreased very fast, the optimal arm  $a^*$  gets eliminated most of the time for no clustering  $p = 1$ . While a balance of  $p, \rho_a$  and  $\rho_s$  gives a much better result. ClusUCB with  $p = 4$  and 10 perform better than MOSS, while  $p = 1$  with just arm elimination does not converge and  $p = 5, 10$  with just Cluster elimination and no arm elimination also does not converge. In this test-bed we see that ClusUCB( $p = 4$ ) beats TS. As proved in Proposition 2, regret for using just cluster elimination is higher than using just arm elimination. A balance of cluster and arm elimination works best. For using just cluster elimination in ClusUCB( $\sqrt{\log K} < p \leq \frac{K}{2}$ ) we stop when we are left with one cluster and output the max payoff arm of that cluster.

## 6 Conclusions and future work

From a theoretical viewpoint, we conclude that the gap-dependent regret bound of ClusUCB is lower than MOSS and UCB-Improved. From the numerical experiments on settings with small gaps between optimal and sub-optimal mean rewards, we observed that ClusUCB outperforms several popular bandit algorithms. While we exhibited better regret bounds for ClusUCB, it would be interesting future research to improve the theoretical analysis of ClusUCB to achieve the gap-independent regret bound of MOSS and possibly also the gap-dependent bound con-



tured in Section 2.4.3 of [7].

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## Appendix

### A Proof of Proposition 1

*Proof.* Let  $p = 1$  whereby we put all the arms in set  $A$  into one cluster, that is we have one UCB-Improved running throughout. So, for each sub-optimal arm  $i$ ,  $m_i = \min \{m | \sqrt{\rho_a \epsilon_m} < \frac{\Delta_i}{2}\}$  be the first round when  $\sqrt{\rho_a \epsilon_m} < \frac{\Delta_i}{2}$ . Also in this proof, since the clusters are fixed, so throughout the rounds each  $m_i$  is tied to a single arm. We also take  $\rho_a \in (0, 1]$  as a constant in this proof whereby in Corollary 1 and 2 we use the different definitions. The theoretical analysis remains same as we have always bounded the values of  $\rho_a \in (0, 1]$ . Let  $A' = \{i \in A : \Delta_i > b\}$  and  $A'' = \{i \in A : \Delta_i > 0\}$ .

**Case a: Some sub-optimal arm  $i$  is not eliminated in round  $m_i$  or before and the optimal arm  $*$   $\in B_{m_i}$**

Following the steps of Theorem 1 Case a1, an arbitrary sub-optimal arm  $i \in A'$  can get eliminated only when the event,

$$\hat{r}_i \leq r_i + c_{m_i} \text{ and } \hat{r}^* \geq r^* - c_{m_i} \quad (3)$$

takes place. So to bound the regret we need to bound the probability of the complementary event of these two conditions.

Putting the value of  $n_{m_i} = \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}}$  in  $c_{m_i}$ ,  $c_{m_i} = \sqrt{\frac{\rho_a \epsilon_{m_i} \log(\psi T \epsilon_{m_i}^2)}{2 * 2 \log(\psi T \epsilon_{m_i}^2)}} = \frac{\sqrt{\rho_a \epsilon_{m_i}}}{2} = \sqrt{\rho_a \epsilon_{m_i+1}} < \frac{\Delta_i}{4}$ , as  $\rho_a \in (0, 1]$ .

Again, for  $i \in A'$ ,

$$\begin{aligned} \hat{r}_i + c_{m_i} &\leq r_i + 2c_{m_i} \\ &= \hat{r}_i + 4c_{m_i} - 2c_{m_i} \\ &< r_i + \Delta_i - 2c_{m_i} \\ &= r^* - 2c_{m_i} \\ &\leq \hat{r}^* - c_{m_i} \end{aligned}$$

Applying Chernoff-Hoeffding bound and considering independence of complementary of the two events in 3,

$$\begin{aligned} \mathbb{P}\{\hat{r}^* \leq r^* - c_{m_i}\} &\leq \exp(-2c_{m_i}^2 n_{m_i}) \\ &\leq \exp(-2 * \frac{\rho_a \log(\psi T \epsilon_{m_i}^2)}{2n_{m_i}} * n_{m_i}) \\ &\leq \frac{1}{(\psi T \epsilon_{m_i}^2)^{\rho_a}} \end{aligned}$$

Similarly,  $\mathbb{P}\{\hat{r}_i \geq r_i + c_{m_i}\} \leq \frac{1}{(\psi T \epsilon_{m_i}^2)^{\rho_a}}$

Summing, the two up, the probability that a sub-optimal arm  $i$  is not eliminated on or before  $m_i$ -th round is  $\left(\frac{2}{(\psi T \epsilon_{m_i}^2)^{\rho_a}}\right)$ .

Summing up over all arms in  $A$  and bounding the regret for each arm  $i \in A'$  trivially by  $T \Delta_i$ , we obtain

$$\begin{aligned} \sum_{i \in A'} \left( \frac{2T \Delta_i}{(\psi T \epsilon_{m_i}^2)^{\rho_a}} \right) &\leq \sum_{i \in A'} \left( \frac{2T \Delta_i}{(\psi T \frac{\Delta_i^4}{16 \rho_a^2})^{\rho_a}} \right) \\ &\leq \sum_{i \in A'} \left( \frac{2^{1+4\rho_a} T^{1-\rho_a} \rho_a^{2\rho_a} \Delta_i}{\psi \rho_a \Delta_i^{4\rho_a}} \right) \\ &\leq \sum_{i \in A'} \left( \frac{2^{1+4\rho_a} \rho_a^{2\rho_a} T^{1-\rho_a}}{\psi \rho_a \Delta_i^{4\rho_a-1}} \right) \end{aligned}$$

$$= \sum_{i \in A'} \left( \frac{C_1(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right), \text{ where } C_1(x) = \frac{2^{1+4x} x^{2x}}{\psi^x}$$

**Case b:** *Either an arm  $i$  is eliminated in round  $m_i$  or before or else there is no optimal arm  $*$   $\in B_{m_i}$*

**Case b1:**  *$*$   $\in B_{m_i}$  and each  $i \in A'$  is eliminated on or before  $m_i$*

Since we are eliminating a sub-optimal arm  $i$  on or before round  $m_i$ , it is pulled no longer than,

$$n_{m_i} = \left\lceil \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}} \right\rceil$$

So, the total contribution of  $i$  till round  $m_i$  is given by,

$$\begin{aligned} & \Delta_i \left\lceil \frac{2 \log(\psi T \epsilon_{m_i}^2)}{\epsilon_{m_i}} \right\rceil \\ & \leq \Delta_i \left\lceil \frac{2 \log(\psi T (\frac{\Delta_i}{2\sqrt{\rho_a}})^4)}{(\frac{\Delta_i}{2\sqrt{\rho_a}})^2} \right\rceil, \text{ since } \sqrt{\rho_a \epsilon_{m_i}} < \frac{\Delta_i}{2} \\ & \leq \Delta_i \left( 1 + \frac{32\rho_a \log(\psi T (\frac{\Delta_i}{2\sqrt{\rho_a}})^4)}{\Delta_i^2} \right) \\ & \leq \Delta_i \left( 1 + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i^2} \right) \end{aligned}$$

Summing over all arms in  $A'$  the total regret is given by,

$$\sum_{i \in A'} \Delta_i \left( 1 + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i^2} \right)$$

**Case b2:** *Optimal arm  $*$  is eliminated by a sub-optimal arm*

Firstly, if conditions of Case  $a$  holds then the optimal arm  $*$  will not be eliminated in round  $m = m_*$  or it will lead to the contradiction that  $r_i > r^*$ . In any round  $m_*$ , if the optimal arm  $*$  gets eliminated then for any round from 1 to  $m_j$  all arms  $j$  such that  $m_j < m_*$  were eliminated according to assumption in Case  $a$ . Let, the arms surviving till  $m_*$  round be denoted by  $A'$ . This leaves any arm  $a_b$  such that  $m_b \geq m_*$  to still survive and eliminate arm  $*$  in round  $m_*$ . Let, such arms that survive  $*$  belong to  $A'$ . Also maximal regret per step after eliminating  $*$  is the maximal  $\Delta_j$  among the remaining arms  $j$  with  $m_j \geq m_*$ . Let  $m_b = \min\{m | \sqrt{\rho_a \epsilon_m} < \frac{\Delta_b}{2}\}$ . Hence, the maximal regret after eliminating the arm  $*$  is upper bounded by,

$$\begin{aligned} & \sum_{m_*=0}^{\max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \left( \frac{2}{(\psi T \epsilon_{m_*}^2)^{\rho_a}} \right) \cdot T \max_{j \in A'' : m_j \geq m_*} \Delta_j \\ & \leq \sum_{m_*=0}^{\max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} \left( \frac{2}{(\psi T \epsilon_{m_*}^2)^{\rho_a}} \right) \cdot T \cdot 2\sqrt{\rho_a \epsilon_{m_*}} \\ & \leq \sum_{m_*=0}^{\max_{j \in A'} m_j} \sum_{i \in A'' : m_i > m_*} 4 \left( \frac{T^{1-\rho_a}}{\psi^{\rho_a} \epsilon_{m_*}^{2\rho_a - \frac{1}{2}}} \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{i \in A'' : m_i > m_*} \sum_{m_*=0}^{\min\{m_i, m_b\}} \left( \frac{4T^{1-\rho_a}}{\psi^{\rho_a} 2^{-(2\rho_a - \frac{1}{2})m_*}} \right) \\
 &\leq \sum_{i \in A'} \left( \frac{4T^{1-\rho_a}}{\psi^{\rho_a} 2^{-(2\rho_a - \frac{1}{2})m_*}} \right) + \sum_{i \in A'' \setminus A'} \left( \frac{4T^{1-\rho_a}}{\psi^{\rho_a} 2^{-(2\rho_a - \frac{1}{2})m_b}} \right) \\
 &\leq \sum_{i \in A'} \left( \frac{4\rho_a^{2\rho_a} T^{1-\rho_a} * 2^{2\rho_a - \frac{1}{2}}}{\psi^{\rho_a} \Delta_i^{4\rho_a - 1}} \right) + \sum_{i \in A'' \setminus A'} \left( \frac{4\rho_a^{2\rho_a} T^{1-\rho_a} * 2^{2\rho_a - \frac{1}{2}}}{\psi^{\rho_a} b^{4\rho_a - 1}} \right) \\
 &\leq \sum_{i \in A'} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a + \frac{3}{2}}}{\psi^{\rho_a} \Delta_i^{4\rho_a - 1}} \right) + \sum_{i \in A'' \setminus A'} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a + \frac{3}{2}}}{\psi^{\rho_a} b^{4\rho_a - 1}} \right) \\
 &= \sum_{i \in A'} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a - 1}} \right) + \sum_{i \in A'' \setminus A'} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a - 1}} \right), \text{ where } C_2(x) = \frac{2^{2x + \frac{3}{2}} x^{2x}}{\psi^x}
 \end{aligned}$$

Summing up **Case a** and **Case b**, the total regret till round  $m$  is given by,

$$\begin{aligned}
 R_T \leq & \sum_{i \in A : \Delta_i > b} \left\{ \left( \frac{C_1(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a - 1}} \right) + \left( \Delta_i + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \right) \right. \\
 & \left. + \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a - 1}} \right) \right\} + \sum_{i \in A : 0 \leq \Delta_i \leq b} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{\psi^{\rho_a} b^{4\rho_a - 1}} \right) + \max_{i \in A : \Delta_i \leq b} \Delta_i T
 \end{aligned}$$

□

**Corollary 3.** For  $\rho_a = 1$  in the result of proposition 1 for ClusUCB-AE, we get a regret bound of

$$\sum_{i \in A : \Delta_i > b} \left( \Delta_i + \frac{44}{\psi(\Delta_i)^3} + \frac{32 \log(\psi T \Delta_i^4)}{\Delta_i} \right) + \sum_{i \in A : 0 < \Delta_i \leq b} \frac{12}{\psi b^3}$$

*Proof.* In the result of Proposition 1 if we take  $\rho_a = 1$  then the regret bound becomes  $\sum_{i \in A : \Delta_i > b} \left( \Delta_i + \frac{44}{\psi(\Delta_i)^3} + \frac{32 \log(\psi T \Delta_i^4)}{\Delta_i} \right) + \sum_{i \in A : 0 \leq \Delta_i \leq b} \frac{12}{\psi b^3}$ . From the result we can see that for small  $\Delta_i$  and large  $K$ , the terms like  $\sum_{i \in A : \Delta_i > b} \left( \frac{44}{\psi(\Delta_i)^3} \right) + \sum_{i \in A : 0 < \Delta_i \leq b} \frac{12}{\psi b^3}$  can become the dominant term in the regret rather than  $\sum_{i \in A : \Delta_i > b} \frac{32 \log(\psi T \Delta_i^4)}{\Delta_i}$ . Intuitively, this actually suggests that the algorithm is trying to eliminate arms with too low exploration and so the probability of elimination is low and error(risk) is high. For this essentially we introduce  $\rho_a, \rho_s$  and  $\psi$  and by carefully defining their values enables us to eliminate arms and clusters aggressively and thereby reduce those two terms. □

## B Proof of Proposition 2

An illustrative diagram explaining Cluster Elimination is given in **Figure 3**. A slight modification to the algorithm allows us to do cluster elimination without any arm elimination. By taking  $p > 1$ , removing the arm elimination condition, stopping when we are just left with one cluster and pulling the  $\max\{\hat{r}_i\}$ , where  $i \in B_m$  we can achieve this. We also take  $\rho_s \in (0, 1]$  as a constant in this proof whereby in Corollary 1 and 2 we use the different definitions. The theoretical analysis remains same as we have always bounded the values of  $\rho_s \in (0, 1]$ .

*Proof.* Let  $C_{g_{s_k}} = \{\hat{r}_{a_{\max_{s_k}}} | \forall s_k \in S\}$ , that is let  $C_{g_{s_k}}$  be the set of all arms which has the maximum estimated payoff arms from their respective clusters in the  $g_{s_k}$ -th round. Let, for each sub-optimal cluster arm  $a_{\max_{s_k}} \in C_{g_{s_k}}$ ,  $g_{s_k} =$

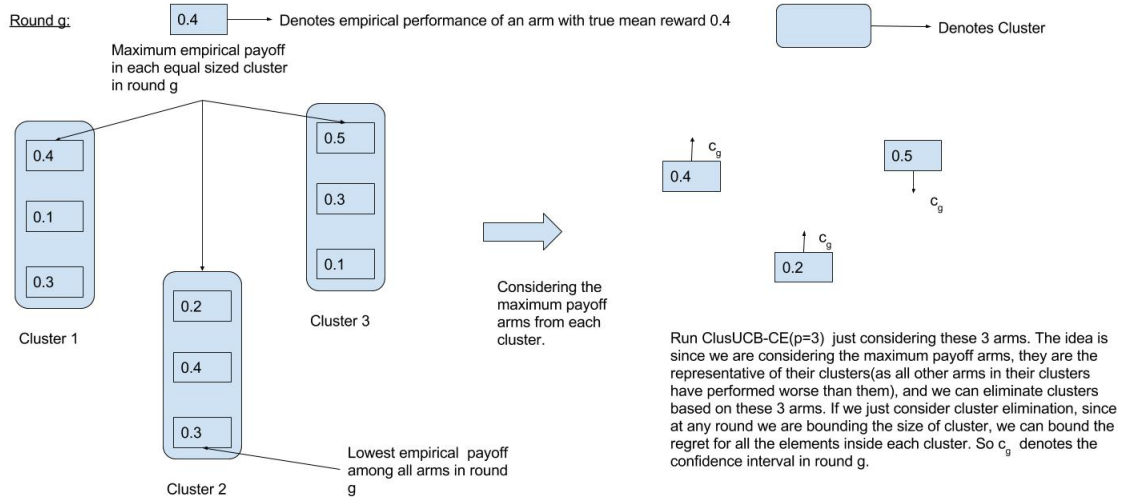


Figure 3: Cluster Elimination

$\min \{g | \sqrt{\rho_s \epsilon_g} < \frac{\Delta_{a_{\max_{s_k}}}}{2}\}$ . So,  $g_{s_k}$  be the first round when  $\sqrt{\rho_s \epsilon_{g_{s_k}}} < \frac{\Delta_{a_{\max_{s_k}}}}{2}$  where  $a_{\max_{s_k}} \in C_{g_{s_k}}$  is the maximum payoff arm in cluster  $s_k$ . Here,  $a_{\max_{s_k}}$  is called cluster arm. Here,  $A_{s_k}$  denotes the arm set in the cluster  $s_k$ . Let  $A'_{s_k} = \{i \in A_{s_k} : \Delta_i > b\}$ ,  $A''_{s_k} = \{i \in A_{s_k} : \Delta_i > 0\}$ ,  $A' = \{i \in A : \Delta_i > b\}$  and  $A'' = \{i \in A : \Delta_i > 0\}$ .

**Case a: Some sub-optimal cluster arm  $a_{\max_{s_k}}$  is not eliminated in round  $g_{s_k}$  or before with  $*$   $\in C_{g_{s_k}}$**

Following the steps of Theorem 1 Case a2, an arbitrary sub-optimal arm  $i \in A'$  can get eliminated only when the event,

$$\hat{r}_{a_{\max_{s_k}}} \leq r_{a_{\max_{s_k}}} + c_{m_i} \text{ and } \hat{r}^* \geq r^* - c_{m_i} \quad (4)$$

takes place. So to bound the regret we need to bound the probability of the complementary event of these two conditions.

Putting the value of  $n_{g_{s_k}} = \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}}$  in  $c_{g_{s_k}}$  we get,

$$\begin{aligned} c_{g_{s_k}} &= \sqrt{\frac{\rho_s * \epsilon_{g_{s_k}} \log(\psi T \epsilon_{g_{s_k}}^2)}{2 * 2 \log(\psi T \epsilon_{g_{s_k}}^2)}} \\ &= \sqrt{\frac{\rho_s \epsilon_{g_{s_k}}}{2}} \\ &= \sqrt{\rho_s \epsilon_{g_{s_k}+1}} < \frac{\sqrt{\rho_s} \Delta_{a_{\max_{s_k}}}}{4} < \frac{\Delta_{a_{\max_{s_k}}}}{4} \end{aligned}$$

Again, for  $a_{\max_{s_k}}, * \in C_{g_{s_k}}$ ,

$$\begin{aligned} \hat{r}_{a_{\max_{s_k}}} + c_{g_{s_k}} &\leq r_{a_{\max_{s_k}}} + 2c_{g_{s_k}} = \hat{r}_{a_{\max_{s_k}}} + 4c_{g_{s_k}} - 2c_{g_{s_k}} \\ &< r_{a_{\max_{s_k}}} + \Delta_{a_{\max_{s_k}}} - 2c_{g_{s_k}} \\ &= r^* - 2c_{g_{s_k}} \\ &\leq \hat{r}^* - c_{g_{s_k}} \end{aligned}$$

Applying Chernoff-Hoeffding bound and considering independence of complementary of the two events in 4,

$$\begin{aligned}
 \mathbb{P}\left\{\hat{r}^* \leq r^* - c_{g_{s_k}}\right\} &\leq \exp(-2c_{g_{s_k}}^2 n_{g_{s_k}}) \\
 &\leq \exp(-2 * \frac{\rho_s \log(\psi T \epsilon_{g_{s_k}}^2)}{2n_{g_{s_k}}} * n_{g_{s_k}}) \\
 &\leq \frac{1}{(\psi T \epsilon_{g_k}^2)^{\rho_s}}
 \end{aligned}$$

Similarly,  $\mathbb{P}\left\{\hat{r}_{a_{max_{s_k}}} \geq r_{a_{max_{s_k}}} + c_{g_{s_k}}\right\} \leq \frac{1}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}}$

Summing, the two up, the probability that a sub-optimal cluster arm  $a_{max_{s_k}} \in C_{g_{s_k}}$  is not eliminated in  $g_{s_k}$ -th round is  $\left(\frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}}\right)$ . Now, for each round  $g_{s_k}$ , all the elements of  $C_{g_{s_k}}$  are the respective max payoff arms of their cluster  $s_k$ , that is all the other arms in their respective clusters have performed worse than them. Hence, since  $A'_{s_k} \supset C_{g_{s_k}}$ , we are pulling all the surviving arms equally in each round and since clusters are fixed so we can bound the maximum probability that a sub-optimal arm  $j \in A'_{s_k}$  and  $j \in s_k$  such that  $a_{max_{s_k}} \in C_{g_{s_k}}$  is not eliminated on or before the  $g_{s_k}$ -th round by the same probability of

$$\left(\frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}}\right)$$

Summing up over all arms in  $s_k$  and bounding the regret trivially by  $T\Delta_i$ ,

$$\sum_{i \in A'_{s_k}} \left(\frac{2T\Delta_i}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}}\right)$$

Summing up over all  $p$  clusters and bounding the regret for each arm  $i \in A'_{s_k}$  trivially by  $T\Delta_i$ ,

$$\begin{aligned}
 \sum_{k=1}^p \sum_{i \in A'_{s_k}} \left(\frac{2T\Delta_i}{(\psi T \frac{\Delta_i^4}{16\rho_s^2})^{\rho_s}}\right) &= \sum_{i \in A'} \left(\frac{2T\Delta_i}{(\psi T \frac{\Delta_i^4}{16\rho_s^2})^{\rho_s}}\right) \\
 &\leq \sum_{i \in A'} \left(\frac{2^{1+4\rho_s} T^{1-\rho_s} \rho_s^{2\rho_s} \Delta_i}{\psi^{\rho_s} \Delta_i^{4\rho_s}}\right) \\
 &\leq \sum_{i \in A'} \left(\frac{2^{1+4\rho_s} \rho_s^{2\rho_s} T^{1-\rho_s}}{\psi^{\rho_s} \Delta_i^{4\rho_s-1}}\right) \\
 &= \sum_{i \in A'} \left(\frac{C_1(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}}\right), \text{ where } C_1(x) = \frac{2^{1+4x} x^{2x}}{\psi^x}
 \end{aligned}$$

**Case b:** For each arm  $i$ , either  $i$  is eliminated in round  $g_{s_k}$  or before or there is no optimal arm  $*$  in  $C_{g_{s_k}}$

**Case b1:**  $*$   $\in C_{g_{s_k}}$  for each arm  $i \in A'$  and cluster  $s_k$  eliminated on or before  $g_{s_k}$

Again, in the  $g_{s_k}$ -th round, the maximum total elements in the cluster  $s_k$  can be no more than  $\ell = \left\lceil \frac{K}{p} \right\rceil$ .

Also, since we are eliminating a sub-optimal cluster arm  $a_{max_{s_k}} \in C_{g_{s_k}}$  on or before round  $g_{s_k}$ , it is pulled (along with all the other arms in that cluster) no longer than,

$$n_{g_{s_k}} = \left\lceil \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}} \right\rceil$$

So, the total contribution of  $a_{\max_{s_k}}$  along with all the other arms in the cluster till round  $g_{s_k}$  is given by,

$$\begin{aligned}
 & \sum_{i \in A_{s_k}} \Delta_i \left\lceil \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}} \right\rceil \\
 & \leq \sum_{i \in A'_{s_k}} \Delta_i \left\lceil \frac{2 \log(\psi T (\frac{\Delta_i}{2\sqrt{\rho_s}})^4)}{(\frac{\Delta_i}{2\sqrt{\rho_s}})^2} \right\rceil, \text{ since } \sqrt{\rho_s \epsilon_{g_{s_k}}} < \frac{\Delta_{a_{\max_{s_k}}}}{2} < \frac{\Delta_i}{2}, \text{ as } r_{a_{\max_{s_k}}} > r_i, \forall i \in s_k \\
 & \leq \sum_{i \in A'_{s_k}} \Delta_i \left( 1 + \frac{32 * \rho_s * \log(\psi T (\frac{\Delta_i}{2\sqrt{\rho_s}})^4)}{\Delta_i^2} \right) \\
 & \leq \sum_{i \in A'_{s_k}} \Delta_i \left( 1 + \frac{32 \rho_s \log(\psi T \frac{\Delta_i^4}{16 \rho_s^2})}{\Delta_i^2} \right)
 \end{aligned}$$

Summing over all  $p$  clusters the total regret is given by,

$$\begin{aligned}
 & \sum_{k=1}^p \sum_{i \in A'_{s_k}} \Delta_i \left( 1 + \frac{32 \rho_s \log(\psi T \frac{\Delta_i^4}{16 \rho_s^2})}{\Delta_i^2} \right) \\
 & \leq \sum_{i \in A'} \Delta_i \left( 1 + \frac{32 \rho_s \log(\psi T \frac{\Delta_i^4}{16 \rho_s^2})}{\Delta_i^2} \right)
 \end{aligned}$$

**Case b2:  $s^*$  is eliminated by some sub-optimal cluster.**

Let  $C'_g = \{a_{\max_{s_k}} \in A' | \forall s_k \in S\}$  and  $C''_g = \{a_{\max_{s_k}} \in A'' | \forall s_k \in S\}$ . Firstly, if conditions of case b1 holds then the optimal arm  $*$   $\in C_{g_{s_k}}$  will not be eliminated in round  $g_{s_k} = g_*$  or it will lead to the contradiction that  $r_{a_{\max_{s_k}}} > r^*$  where  $a_{\max_{s_k}}, * \in C_{g_{s_k}}$ . In any round  $g_*$ , if the optimal arm  $*$  gets eliminated then for any round from 1 to  $g_{s_j}$  all arms  $a_{\max_{s_j}} \in C_{g_{s_k}}, \forall s_j \neq s^*$  such that  $g_{s_j} < g_*$  were eliminated according to assumption in Case a. Let, the arms surviving till  $g_*$  round be denoted by  $C'_g$ . This leaves any arm  $a_{s_b}$  such that  $g_{s_b} \geq g_*$  to still survive and eliminate arm  $*$  in round  $g_*$ . Let, such arms that survive  $*$  belong to  $C''_g$ . Also maximal regret per step after eliminating  $*$  is the maximal  $\Delta_{a_{\max_{s_j}}}$  among the remaining arms  $a_{\max_{s_j}} \in C_{g_{s_j}}$  with  $g_{s_j} \geq g_*$ . Hence, the maximal regret after eliminating the arm  $*$  is upper bounded by,

$$\begin{aligned}
 & \max_{g_*=0} \sum_{a_{\max_{s_j}} \in C'_g} \sum_{\substack{a_{\max_{s_k}} \in C''_g: \\ g_{s_k} \geq g_*}} \left( \frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}} \right) \cdot T \max_{\substack{a_{\max_{s_j}} \in C''_g: \\ g_{s_j} \geq g_*}} \Delta_{a_{\max_{s_j}}}
 \end{aligned}$$

But, we know that for any round  $g$ , elements of  $C_g$  are the best performers in their respective clusters. So, taking that into account and  $A' \supset C'_g$  and  $A'' \supset C''_g$  the regret can be bounded by,

$$\begin{aligned}
 & \max_{g_*=0} \sum_{j \in A' \setminus A'_{s^*}} \sum_{i \in A'' \setminus A'_{s^*}: g_{s_k} > g_*} \left( \frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}} \right) \cdot T \max_{j \in A'': g_{s_j} \geq g_*} \Delta_j \\
 & \leq \sum_{g_*=0} \sum_{i \in A'' \setminus A'_{s^*}: g_{s_k} > g_*} \left( \frac{2}{(\psi T \epsilon_{g_{s_k}}^2)^{\rho_s}} \right) \cdot T \cdot 2\sqrt{\rho_s \epsilon_{g_{s_j}}}
 \end{aligned}$$



$$\begin{aligned}
 &\leq \sum_{g_*=0}^{\max_{j \in A' \setminus A'_{s^*}} g_{sj}} \sum_{i \in A'' \setminus A'_{s^*} : g_{s_k} > g_*} \left( \frac{4T^{1-\rho_s}}{\psi^{\rho_s} \epsilon_{g_{s_k}}^{2\rho_s - \frac{1}{2}}} \right) \\
 &\leq \sum_{i \in A'' \setminus A'_{s^*} : g_{s_k} > g_*} \sum_{g_*=0}^{\min\{g_{s_k}, g_{s_b}\}} \left( \frac{4T^{1-\rho_s}}{\psi^{\rho_s} 2^{(2\rho_s - \frac{1}{2})g_*}} \right) \\
 &\leq \sum_{i \in A' \setminus A'_{s^*}} \left( \frac{4T^{1-\rho_s}}{\psi^{\rho_s} 2^{(2\rho_s - \frac{1}{2})g_*}} \right) + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \left( \frac{4T^{1-\rho_s}}{\psi^{\rho_s} 2^{(2\rho_s - \frac{1}{2})g_{s_b}}} \right) \\
 &\leq \sum_{i \in A' \setminus A'_{s^*}} \left( \frac{4\rho_s^{2\rho_s} T^{1-\rho_s} * 2^{2\rho_s - \frac{1}{2}}}{\psi^{\rho_s} \Delta_i^{4\rho_s - 1}} \right) + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \left( \frac{4\rho_s^{2\rho_s} T^{1-\rho_s} * 2^{2\rho_s - \frac{1}{2}}}{\psi^{\rho_s} b^{4\rho_s - 1}} \right) \\
 &\leq \sum_{i \in A' \setminus A'_{s^*}} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s + \frac{3}{2}}}{\psi^{\rho_s} \Delta_i^{4\rho_s - 1}} \right) + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s + \frac{3}{2}}}{\psi^{\rho_s} b^{4\rho_s - 1}} \right) \\
 &= \sum_{i \in A' \setminus A'_{s^*}} \left( \frac{C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s - 1}} \right) + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \left( \frac{C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s - 1}} \right), \text{ where } C_2(x) = \frac{2^{2x + \frac{3}{2}} x^{2x}}{\psi^x}
 \end{aligned}$$

**Case b3:**  $*$  is not in  $C_{g_{s_k}}$ , but belongs to  $B_{g_{s_k}}$

In this case the optimal arm  $* \in s^*$  is not eliminated, also  $s^*$  is not eliminated. So, for all sub-optimal arms  $i$  in  $A'$  which gets eliminated on or before  $g_{s_k}$  will get pulled no less than  $\left\lceil \frac{2 \log(\psi T \epsilon_{g_{s_k}}^2)}{\epsilon_{g_{s_k}}} \right\rceil$  number of times, which leads to the following bound the contribution to the expected regret, as in Case b1:

$$\sum_{i \in A'} \left\{ \Delta_i + \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right\}$$

For arms  $a_i \notin s^*$ , the contribution to the regret cannot be greater than that in Case b2. So the regret is bounded by,

$$\sum_{i \in A' \setminus A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s - 1}} + \sum_{i \in A'' \setminus A' \cup A'_{s^*}} \frac{C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s - 1}}$$

Summing up **Case a** and **Case b**, the total regret till round  $g$  is given by,

$$\begin{aligned}
 R_T &\leq \sum_{i \in A : \Delta_i > b} \left\{ \underbrace{\left( \frac{C_1(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s - 1}} \right)}_{\text{case a}} + \underbrace{\left( 2\Delta_i + \frac{64\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right)}_{\text{case b1+b3}} \right\} \\
 &\quad + \underbrace{\sum_{i \in A \setminus A_{s^*} : \Delta_i > b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s - 1}} \right)}_{\text{case b2}} + \underbrace{\sum_{i \in A \setminus A_{s^*} : 0 < \Delta_i \leq b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s - 1}} \right)}_{\text{case b2}} + \max_{i \in A : \Delta_i \leq b} \Delta_i T
 \end{aligned}$$

□

## C Proof of Corollary 1

*Proof.* Here we take  $\psi = \frac{T}{\log(KT)}$ ,  $\rho_a = \frac{1}{2}$  and  $\rho_s = \frac{1}{2}$ . Taking into account Theorem 1 below for all  $b \geq \sqrt{\frac{e}{T}}$

$$\begin{aligned}
 \mathbb{E}[R_T] \leq & \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \left\{ \frac{C_1(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \Delta_i + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \right\} + \sum_{\substack{i \in A, \\ \Delta_i > b}} \left\{ 2\Delta_i + \frac{C_1(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right. \\
 & + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} + \left. \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right\} + \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \frac{C_2(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \sum_{\substack{i \in A_{s^*}, \\ 0 < \Delta_i \leq b}} \frac{C_2(\rho_a)T^{1-\rho_a}}{b^{4\rho_a-1}} \\
 & + \sum_{i \in A \setminus A_{s^*}: \Delta_i > b} \frac{2C_2(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \sum_{i \in A \setminus A_{s^*}: 0 < \Delta_i \leq b} \frac{2C_2(\rho_s)T^{1-\rho_s}}{b^{4\rho_s-1}} + \max_{i: \Delta_i \leq b} \Delta_i T
 \end{aligned}$$

and putting the parameter values in the above Theorem 1 result,

$$\sum_{i \in A_{s^*}: \Delta_i > b} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{1+4\rho_a}}{\psi^{\rho_a} \Delta_i^{4\rho_a-1}} \right) = \sum_{i \in A_{s^*}: \Delta_i > b} \left( \frac{T^{1-\frac{1}{2}} \frac{1}{2} 2^{*\frac{1}{2}} 2^{1+4*\frac{1}{2}}}{\left(\frac{T}{\log(KT)}\right)^{\frac{1}{2}} \Delta_i^{4*\frac{1}{2}-1}} \right) = \sum_{i \in A_{s^*}: \Delta_i > b} \frac{4\sqrt{\log(KT)}}{\Delta_i}$$

Similarly for the term,

$$\sum_{i \in A: \Delta_i > b} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{1+4\rho_s}}{\psi^{\rho_s} \Delta_i^{4\rho_s-1}} \right) = \sum_{i \in A: \Delta_i > b} \frac{4\sqrt{\log(KT)}}{\Delta_i}$$

For the term involving arm pulls,

$$\sum_{i \in A: \Delta_i > b} \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} = \sum_{i \in A: \Delta_i > b} \frac{16 \log(T^2 \frac{\Delta_i^4}{4\log(KT)})}{\Delta_i} \approx \sum_{i \in A: \Delta_i > b} \frac{32 \log(T \frac{\Delta_i^2}{\sqrt{\log(KT)}})}{\Delta_i}$$

Similarly the term,

$$\sum_{i \in A: \Delta_i > b} \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \approx \sum_{i \in A: \Delta_i > b} \frac{32 \log(T \frac{\Delta_i^2}{\sqrt{\log(KT)}})}{\Delta_i}$$

Lastly we can bound the error terms as,

$$\sum_{i \in A_{s^*}: 0 < \Delta_i \leq b} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a+\frac{3}{2}}}{\psi^{\rho_a} \Delta_i^{4\rho_a-1}} \right) = \sum_{i \in A_{s^*}: 0 < \Delta_i \leq b} \frac{2.8\sqrt{\log(KT)}}{\Delta_i}$$

Similarly for the term,

$$\sum_{i \in A \setminus A_{s^*}: 0 < \Delta_i \leq b} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s+\frac{3}{2}}}{(\psi^{\rho_s}) \Delta_i^{4\rho_s-1}} \right) = \sum_{i \in A \setminus A_{s^*}: 0 < \Delta_i \leq b} \frac{2.8\sqrt{\log(KT)}}{\Delta_i}$$

So, the total gap dependent regret bound for using both arm and cluster elimination comes of as

$$\sum_{i \in A_{s^*}: \Delta_i > b} \left\{ \frac{4\sqrt{\log(KT)}}{\Delta_i} + \Delta_i + \frac{32 \log(T \frac{\Delta_i^2}{\sqrt{\log(KT)}})}{\Delta_i} \right\} + \sum_{i \in A: \Delta_i > b} \left\{ \frac{4\sqrt{\log(KT)}}{\Delta_i} + 2\Delta_i + \frac{64 \log(T \frac{\Delta_i^2}{\sqrt{\log(KT)}})}{\Delta_i} \right\}$$

$$\begin{aligned}
 & + \sum_{i \in A_{s^*} : \Delta_i > b} \frac{2.8\sqrt{\log(KT)}}{\Delta_i} + \sum_{i \in A_{s^*} : 0 < \Delta_i \leq b} \frac{2.8\sqrt{\log(KT)}}{\Delta_i} \\
 & + \sum_{i \in A \setminus A_{s^*} : \Delta_i > b} \frac{5.6\sqrt{\log(KT)}}{\Delta_i} + \sum_{i \in A \setminus A \cup A_{s^*} : 0 < \Delta_i \leq b} \frac{5.6\sqrt{\log(KT)}}{\Delta_i} + \max_{i \in A : \Delta_i \leq b} \Delta_i T
 \end{aligned}$$

□

## D Proof of Corollary 2

*Proof.* As stated in [4], we can have a bound on regret of the order of  $\sqrt{KT \log K}$  in non-stochastic setting. This is shown in Exp4([6]) algorithm. Similarly, by choosing  $\Delta_i = \Delta = \sqrt{\frac{K \log K}{T}}$  for all  $i : i \neq * \in A$ , in the bound of UCB1([5]) we get,

$$\sum_{i: r_i < r^*} \text{const} \frac{\log T}{\Delta_i} = \frac{\sqrt{KT} \log T}{\sqrt{\log K}}$$

So, this bound is worse than the non-stochastic setting and is clearly improvable and an upper bound regret of  $\sqrt{KT}$  is possible as shown in [2] for MOSS which is consistent with the lower bound as proposed by Mannor and Tsitsiklis([16]).

Hence, we take  $b \approx \sqrt{\frac{K \log K}{T}} > \sqrt{\frac{e}{T}}$  (the minimum value for  $b$ ),  $\psi = K^2 T$ ,  $\rho_a = \frac{1}{4}$  and  $\rho_s = \frac{1}{2}$ .

Taking into account Theorem 1 below,

$$\begin{aligned}
 \mathbb{E}[R_T] \leq & \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \left\{ \frac{C_1(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \Delta_i + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \right\} + \sum_{\substack{i \in A, \\ \Delta_i > b}} \left\{ 2\Delta_i + \frac{C_1(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right. \\
 & + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} + \left. \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right\} + \sum_{\substack{i \in A_{s^*}, \\ \Delta_i > b}} \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} + \sum_{\substack{i \in A_{s^*}, \\ 0 < \Delta_i \leq b}} \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}} \\
 & + \sum_{i \in A \setminus A_{s^*} : \Delta_i > b} \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} + \sum_{i \in A \setminus A_{s^*} : 0 < \Delta_i \leq b} \frac{2C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}} + \max_{i: \Delta_i \leq b} \Delta_i T
 \end{aligned}$$

and putting the parameter values in the above Theorem 1 result,

$$\sum_{i \in A_{s^*} : \Delta_i > b} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{1+4\rho_a}}{\psi^{\rho_a} \Delta_i^{4\rho_a-1}} \right) = \left( K \frac{T^{1-\frac{1}{4}} \frac{1}{4}^{2\frac{1}{4}} 2^{1+4\frac{1}{4}}}{p(T)^{\frac{1}{4}} \Delta_i^{4\frac{1}{4}-1}} \right) = 2 \frac{\sqrt{KT}}{p}$$

Similarly, for the term,

$$\sum_{i \in A : \Delta_i > b} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{1+4\rho_s}}{\psi^{\rho_s} \Delta_i^{4\rho_s-1}} \right) = 4 \sqrt{\frac{T}{K \log K}}$$

For the term regarding number of pulls,

$$\sum_{i \in A : \Delta_i > b} \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} = \frac{32K \sqrt{T}^{\frac{1}{2}} \log(T^2 \frac{K^4 (\log K)^2}{T^2})}{\sqrt{K \log K}}$$

$$\begin{aligned}
 &\leq \frac{32\sqrt{KT} \log(K^2(\log K))}{\sqrt{\log K}} \\
 &= 64\sqrt{KT \log K} + \frac{32\sqrt{KT} \log(\log K)}{\sqrt{\log K}}
 \end{aligned}$$

Similarly for the term,

$$\sum_{i \in A: \Delta_i > b} \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} = 32\sqrt{KT \log K} + \frac{16\sqrt{KT} \log(\log K)}{\sqrt{\log K}}$$

Lastly we can bound the error terms as,

$$\sum_{i \in A_{s^*}: 0 \leq \Delta_i \leq b} \left( \frac{T^{1-\rho_a} \rho_a^{2\rho_a} 2^{2\rho_a + \frac{3}{2}}}{\psi^{\rho_a} \Delta_i^{4\rho_a - 1}} \right) = \frac{K}{p} \left( \frac{T^{1-\frac{1}{4}} \frac{1}{4}^{\frac{1}{4}} 2^{2\frac{1}{4} + \frac{3}{2}}}{(T)^{\frac{1}{4}} (\Delta_i)^{4 \cdot \frac{1}{4} - 1}} \right) = \frac{2\sqrt{KT}}{p}$$

Similarly for the term,

$$\sum_{i \in A \setminus A_{s^*}: \Delta_i > b} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s + \frac{3}{2}}}{(\psi^{\rho_s}) \Delta_i^{4\rho_s - 1}} \right) < 2.8 \sqrt{\frac{T}{K \log K}}$$

Also, for all  $b \geq \sqrt{\frac{e}{T}}$ ,

$$\sum_{i \in A \setminus A_{s^*}: 0 < \Delta_i \leq b} \left( \frac{T^{1-\rho_s} \rho_s^{2\rho_s} 2^{2\rho_s + \frac{3}{2}}}{(\psi^{\rho_s}) b^{4\rho_s - 1}} \right) < 2.8 \sqrt{\frac{T}{e}}$$

So, the total bound for using both arm and cluster elimination cannot be worse than,

$$\begin{aligned}
 &2\frac{\sqrt{KT}}{p} + 32\frac{\sqrt{KT \log K}}{p} + \frac{16\sqrt{KT} \log(\log K)}{p\sqrt{\log K}} + 4\sqrt{\frac{T}{K \log K}} + 96\sqrt{KT \log K} \\
 &+ \frac{48\sqrt{KT} \log(\log K)}{\sqrt{\log K}} + 4\frac{\sqrt{KT}}{p} + 5.6\sqrt{\frac{T}{K \log K}} + 5.6\sqrt{\frac{T}{e}} \\
 &= 6\frac{\sqrt{KT}}{p} + 32\frac{\sqrt{KT \log K}}{p} + \frac{16\sqrt{KT} \log(\log K)}{p\sqrt{\log K}} + 4\sqrt{\frac{T}{K \log K}} + 96\sqrt{KT \log K} \\
 &+ \frac{48\sqrt{KT} \log(\log K)}{\sqrt{\log K}} + 5.6\sqrt{\frac{T}{K \log K}} + 5.6\sqrt{\frac{T}{e}}
 \end{aligned}$$

□

## E Why Clustering?

In this section we want to specify the apparent use of clustering. The regret bound for both arm elimination and pulls and error bound when a sub-optimal arm or a sub-optimal cluster eliminates  $*$  or  $s^*$  is given in Table 2, 3 respectively.

From Table 2 we can see that from the definition of  $\rho_s, \rho_a$  we can have a regret bound for jointly doing the arm and cluster elimination of the same order as doing arm elimination or cluster elimination alone. While looking at the error term for the 3 cases we see that using just Cluster elimination the error term is same as using just arm elimination while we can achieve a balance between the two by using both arm and cluster elimination simultaneously. In the experiments when we reduce

Table 2: Regret Bound on Arm Elimination and Pulls

Elim Type	Regret Bound on Arm Elimination and Pulls	Remarks
Only Arm Elimination (ClusUCB-AE)	$\sum_{i \in A: \Delta_i > b} \left\{ \underbrace{\left( \frac{C_1(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right)}_{\text{Case a1, Proposition 1}} + \underbrace{\left( \Delta_i + \frac{32\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \right)}_{\text{case b1, Proposition 1}} \right\}$	For $\rho_a = \frac{1}{4}$ and $\psi = K^2T$ this gives $\frac{2\sqrt{KT}}{16 \log(\log K)} + \frac{32\sqrt{KT \log K}}{16 \log(\log K)}$ . Hence the order is given by $O(\sqrt{KT \log K})$ .
Only Cluster Elimination (ClusUCB-CE)	$\sum_{i \in A: \Delta_i > b} \left\{ \underbrace{\left( \frac{C_1(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right)}_{\text{Case a, Proposition 2}} + \underbrace{\left( 2\Delta_i + \frac{64\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right)}_{\text{Case b1+b3, Proposition 2}} \right\}$	With $\rho_s = \frac{1}{4}$ and $\psi = K^2T$ this gives $\frac{2\sqrt{KT}}{32 \log(\log K)} + \frac{64\sqrt{KT \log K}}{32 \log(\log K)}$ . Hence, this is larger bound than using only arm elimination though the order is same $O(\sqrt{KT \log K})$ .
Arm & Cluster Elimination (ClusUCB)	$\sum_{i \in A_s^*: \Delta_i > b} \underbrace{\left( \frac{C_1(\rho_a)T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right)}_{\text{Case a1, Arm Elim, Theorem 1}} + \sum_{i \in A: \Delta_i > b} \left\{ \underbrace{\left( \frac{C_2(\rho_s)T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right)}_{\text{Case a2, Clus Elim, Theorem 1}} + \underbrace{\left( \Delta_i + \frac{64\rho_a \log(\psi T \frac{\Delta_i^4}{16\rho_a^2})}{\Delta_i} \right)}_{\text{Case b1+b4, Arm Elim, Theorem 1}} + \underbrace{\left( \Delta_i + \frac{32\rho_s \log(\psi T \frac{\Delta_i^4}{16\rho_s^2})}{\Delta_i} \right)}_{\text{Case b1, Clus Elim, Theorem 1}} \right\}$	With $\rho_a = \frac{1}{4}, \rho_s = \frac{1}{2}$ and $\psi = K^2T$ this gives $\left\{ \frac{2\sqrt{KT}}{p} + 4\sqrt{\frac{T}{K \log K}} + \frac{128\sqrt{KT \log K}}{64 \log(\log K)} \right\}$ . This is larger than the previous 2 bounds though the order is same $O(\sqrt{KT \log K})$ .

Table 3: Error Bound

Elim Type	Error Bound	Remarks
Only Arm Elimination (ClusUCB-AE)	$\underbrace{\sum_{i \in A: \Delta_i > b} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right)}_{\text{Case b2, Proposition 1}} + \underbrace{\sum_{i \in A: 0 \leq \Delta_i \leq b} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}} \right)}_{\text{Case b2, Proposition 1}}$	With $\rho_a = \frac{1}{4}$ , and $\psi = K^2 T$ this gives $2\sqrt{KT}$ . Hence, this has an order of $O(\sqrt{KT})$ .
Only Cluster Elimination (ClusUCB-CE)	$\underbrace{\sum_{i \in A \setminus A_{s^*}: \Delta_i > b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right)}_{\text{Case b2+b3, Proposition 2}} + \underbrace{\sum_{i \in A \setminus A_{s^*}: 0 \leq \Delta_i \leq b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}} \right)}_{\text{Case b2+b3, Proposition 2}}$	With $\rho_s = \frac{1}{4}$ and $\psi = K^2 T$ this gives $4\sqrt{KT}$ . This is more than the bound using only arm elimination and has an order of $O(\sqrt{KT})$ .
Arm & Cluster Elimination (ClusUCB)	$\underbrace{\sum_{i \in A_{s^*}: \Delta_i > b} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{\Delta_i^{4\rho_a-1}} \right) + \sum_{i \in A_{s^*}: 0 \leq \Delta_i \leq b} \left( \frac{C_2(\rho_a) T^{1-\rho_a}}{b^{4\rho_a-1}} \right)}_{\text{Case b2, Arm Elim, Theorem 1}} + \underbrace{\sum_{i \in A \setminus A_{s^*}: \Delta_i > b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{\Delta_i^{4\rho_s-1}} \right) + \sum_{i \in A \setminus A_{s^*}: 0 \leq \Delta_i \leq b} \left( \frac{2C_2(\rho_s) T^{1-\rho_s}}{b^{4\rho_s-1}} \right)}_{\text{Case b3+b4, Clus Elim, Theorem 1}}$	With $\rho_a = \frac{1}{4}$ , $\rho_s = \frac{1}{2}$ and $\psi = K^2 T$ this gives $4 \frac{\sqrt{KT}}{p} + 5.6 \sqrt{\frac{T}{K \log K}} + 5.6 \sqrt{\frac{T}{e}}$ , for $b \geq \sqrt{\frac{e}{T}}$ . So we can reduce the error bound taking $\sqrt{\log K} \leq p \leq \frac{K}{2}$ .

$\rho_a, \rho_s$  such that  $\rho_a \ll \rho_s$  we will see a significant reduction in cumulative regret for the clustering approach. From Table 3, we can see that the error term for using both arm and cluster elimination can become low depending on how we choose  $p$  since  $|A_{s^*}| \leq \lceil \frac{A}{p} \rceil$ . An intuitive justification is that since we are reducing  $\rho_a$  and  $\rho_s$  very fast, and if we partition the action space into clusters and for arm elimination only consider each cluster individually, the probability of eliminating  $*$  reduces.