Bandit Problem and UCB

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Overview

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- After say pulling each arm once we are presented with an *exploration-exploitation* problem, that is whether to continue to pull the arm for which we have observed the highest estimated reward till now(exploitation) or to explore a new arm(exploration).
- If we become too greedy and always exploit we may miss the chance of actually finding the optimal arm and get stuck with a sub-optimal arm.

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- Bandits allows us to study this behavior in a more formal way giving us strict guarantees regarding the performance of our algorithm.
- They form the linking pieces of a larger problem.
- They are easy to implement.

Stochastic Multi-Armed Bandit Problem

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- The distributions for each of the arms are fixed throughout the time horizon.

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- $\Delta_i = r^* - r_i$ denotes the gap between the means of the optimal arm and of the i-th arm.

UCB 1 Algorithm

Algorithm 1 UCB1

1: Pull each arm once
2: **for** $t = K + 1, ..., T$ **do**
3: Pull the arm such that $\max_{i \in A} \left\{ \hat{\mu} + \sqrt{\frac{2 \log t}{n_i}} \right\}$
4: **end for**

[Auer et al.(2002a)Auer, Cesa-Bianchi, and Fischer]

Concentration Bounds

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- Let $X_1, ..., X_n$ be random variables with common range [0, 1] and such that $E[X_t | X_1, ..., X_{t-1}] = \mu$. Let $\bar{S}_n = \dfrac{X_1 +, ..., +X_n}{n}$. Then for all $a \geq 0$,

$$\mathbb{P}\{\bar{S}_n \geq \mu + a\} \leq e^{-2a^2 n}$$
$$\mathbb{P}\{\bar{S}_n \leq \mu - a\} \leq e^{-2a^2 n}$$

UCB1 Theorem on Regret Bound

Theorem

For all $K > 1$, if policy UCB1 is run on K arms having arbitrary reward distributions $P_1, ..., P_K$ with support in $[0, 1]$, then its expected regret after any number n plays is at most,

$$\mathbb{E}[R_n] \leq \sum_{i \in A} \frac{8 \log n}{\Delta_i} + \sum_{i \in A} \Delta_i \left(1 + \frac{\pi^2}{3}\right)$$

where $\mu_1, ..., \mu_K$ are the expected values of $P_1, ..., P_K$.

UCB1 Proof

- The main goal is to bound the number of pulls ($T_i(n)$) of the sub-optimal arm i till the n-th timestep.

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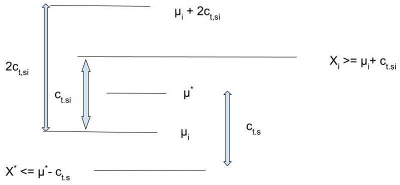
$$T_i(n) \leq \ell + \sum_{t=K+1}^{n} \{I_t = i, T_i(t-1) \geq \ell\}$$

- But, this is nothing but the probability that how many times after the ℓ pulls the UCB of $*$ is less than the UCB of i which will have the highest UCB among all arms in A.

$$T_i(n) \leq \ell + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{s_i=\ell}^{t-1} \{\bar{X}_s^* + c_{t,s} \leq \bar{X}_{i,s_i} + c_{t,s_i}\}$$

UCB1 Proof

- The main argument lies in this,



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$$\bar{X}_s^* \leq \mu^* - c_{t,s}$$
$$\bar{X}_{i,s_i} \geq \mu + c_{t,s_i}$$
$$\mu^* < \mu_i + 2c_{t,s_i}$$

UCB1 Proof

- Now, we get the value of confidence interval $c_{t,s_i} = \sqrt{\dfrac{2 \ln t}{s_i}}$ by plugging its value in the below equations,
- $\mathbb{P}\{\bar{X}_s^* \leq \mu^* - c_{t,s}\} \leq e^{-4 \log t} = t^{-4}$

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- And by plugging $\ell = \left\lceil \frac{8 \log n}{\Delta_i^2} \right\rceil$ in,

$$\mu^* - \mu_i - 2c_{t,s_i} = \mu^* - \mu_i - 2\sqrt{\frac{2 \log t}{s_i}} \geq \mu^* - \mu_i - \Delta_i = 0$$

we get $\mu^* - \mu_i - 2c_{t,s_i} \geq 0$. So for any pulls greater than ℓ, μ^* will surely be atleast $2c_{t,s_i}$ more than μ_i and one of the rest two events will occur with high probability.

UCB1 Proof

- Summing everything up, any sub-optimal arm i will get pulled atleast ℓ times and then the two events $\bar{X}^*_s \leq \mu^* - c_{t,s}$ and $\bar{X}_{i,s_i} \geq \mu + c_{t,s_i}$ will occur with atmost t^{-4} probability.

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- $$\begin{aligned}\mathbb{E}[T_i(n)] &\leq \left\lceil \frac{8 \log n}{\Delta_i^2} \right\rceil + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{s_i=\ell}^{t-1} 2t^{-4} \\ &\leq \frac{8 \log n}{\Delta_i^2} + 1 + \frac{\pi^2}{3}, \text{by Bazel's equation}\end{aligned}$$

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- So finally the cumulative regret is,

$$\mathbb{E}[R_n] \leq \sum_{i \in A} \mathbb{E}[T_i(n)]\Delta_i \leq \frac{8 \log n}{\Delta_i} + \Delta_i \left(1 + \frac{\pi^2}{3}\right)$$

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- We are interested in finding an arm such that it's ϵ close to the optimal arm and we can guarantee this with $1-\delta$ probability.
- Note, that ϵ and δ are given as input and the main aim is to *minimize the number of pulls of an arm i so that it is ϵ close to the optimal arm with $1-\delta$ probability*. This is called Sample complexity.

Algorithm 2 Naive Algorithm

1: Input: $\epsilon > 0$, $\delta > 0$
2: Output: An arm
3: **for** each arm $i \in A$ **do**
4: Sample it for $\ell = \frac{4}{\epsilon^2} \log \frac{2K}{\delta}$
5: Let $\bar{X}_i$ be the average reward of arm i
6: **end for**
7: Output $argmax_{i \in A}\{\bar{X}_i\}$

[Even-Dar et al.(2006)Even-Dar, Mannor, and Mansour]

Sample Complexity of Naive Algorithm

Theorem

The sample complexity of Naive Algorithm for a set of arms K is given by,

$$O\left(\frac{K}{\epsilon^2}\log\left(\frac{K}{\delta}\right)\right)$$

[Even-Dar et al.(2006)Even-Dar, Mannor, and Mansour]

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- So, we only need to bound the probability of,

$$\mathbb{P}\{\bar{X}_i > \bar{X}^*\} \leq \mathbb{P}\left\{\bar{X}_i > \mu_i + \frac{\epsilon}{2}\right\} + \mathbb{P}\left\{\bar{X}^* < \mu^* - \frac{\epsilon}{2}\right\}$$
$$\leq 2\exp\left(-2\big(\frac{\epsilon}{2}\big)^2 \ell\right) \leq 2\exp\left(-2\frac{\epsilon^2}{4}.\frac{4}{\epsilon^2}\log\frac{2K}{\delta}\right) \leq \frac{\delta}{K}$$

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- Summing over all the arms K we get, $\frac{(K-1)\delta}{K} < \delta$

Intuition about Median Elimination

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- One simple way to modify Naive Algorithm is to divide the time horizon into phases.
- In each phase pull all the arms equal number of times.
- After that eliminate half the arms with a high guarantee that they are surely not ϵ-optimal arms.

Median Elimination

Algorithm 3 Median Elimination

Input: $\epsilon > 0$, $\delta > 0$

Output: An arm

Set $S_1 = A$, $\epsilon_1 = \epsilon/4$, $\delta_1 = \delta/2$ and $\ell = 1$

for Repeat till $|S_\ell| = 1$ **do**

  Sample every arm in S_ℓ for $\frac{4}{\epsilon_\ell^2}\log(\frac{3}{\delta_\ell})$ times and let $\bar{X}_i$ denote the average estimated payoff of i.

  Find median m_ℓ of all the surviving arms based on their $\bar{X}_i, \forall i \in A$

  Eliminate all arms from S_ℓ such that $\bar{X}_i < m_\ell$ and create $S_{\ell+1}$.

  Reset Parameters: $\epsilon_{\ell+1} = \frac{3}{4}\epsilon$; $\delta_{\ell+1} = \frac{1}{2}\delta$; $\ell = \ell + 1$

end for

[Even-Dar et al.(2006)Even-Dar, Mannor, and Mansour]

Some Other Bandits and Applications

- Adversarial Bandits : Used in Investment in Stock Markets
- Contextual Bandits : Used in online Advertisement/news article selection
- Budgeted Bandits : Used in Clinical trials
- Distributed Bandits : Used in packet routing through a network

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Thank You