Thresholding Bandits with Augmented UCB

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Overview

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- The distributions for each of the arms are fixed throughout the time horizon denoted by T.

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- At the end of the given T timesteps the learner must recommend a set of arms which (according to it) are the arms having reward mean above τ.

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- Let $\hat{S}_\tau$ denote the recommendation of a learning algorithm after T time units of exploration, while $\hat{S}_\tau^c$ denotes its complement.
- The goal of the learning agent is to minimize the expected loss:

$$\mathbb{E}[\mathcal{L}(T)] = \mathbb{P}\Big(\underbrace{\{S_\tau \cap \hat{S}_\tau^c \neq \emptyset\}}_{\textbf{Rejected good arms}} \cup \underbrace{\{\hat{S}_\tau \cap S_\tau^c \neq \emptyset\}}_{\textbf{Accepted bad arms}} \Big)$$

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- Higher variance of the arms' $\Rightarrow$ harder the problem.

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- In anomaly detection and classification (see Locatelli et al. (2016)).

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- It is the first variance-based arm elimination algorithm for the considered TBP settings.
- It addresses an open problem discussed in [Auer and Ortner(2010)] of designing an algorithm that can eliminate arms based on variance estimates.
- We also define a new problem complexity which uses empirical variance estimates along with arm's mean for giving the theoretical bound.

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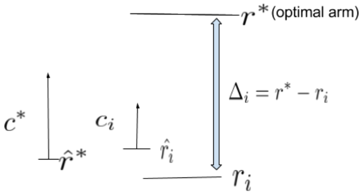
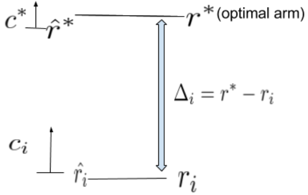
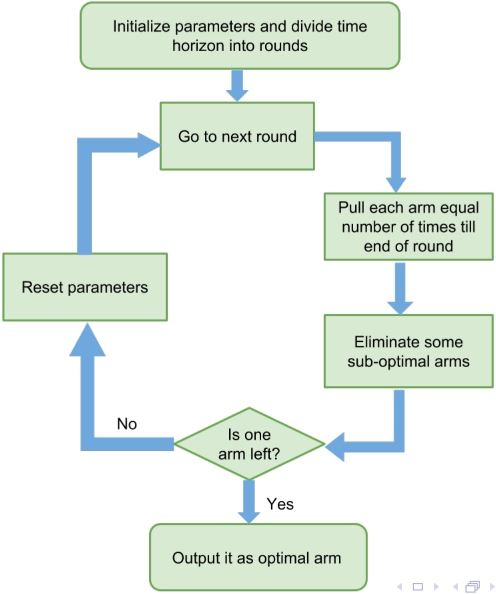
- c_i ensures that the arm i is properly explored and is gradually reduced with time as one pulls the arm i more.
- At every timestep pull arm that has the maximum value of $\hat{r}_i + c_i$ and this will ensure that proper exploration is done.

The UCB Approach

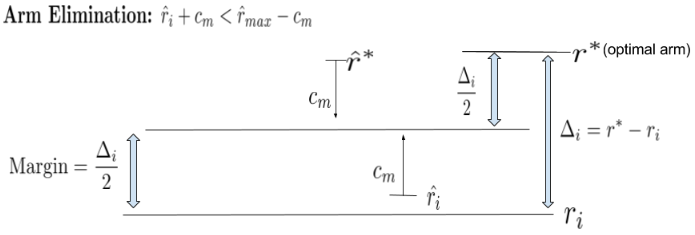




Approach of UCB-Improved (UCB-Imp)



Intuition of UCB-Improved (UCB-Imp)



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- This algorithm uses only mean estimation to find the S_τ.
- Theoretically they proved this algorithm to be almost optimal when only mean estimation is used as a metric of comparison.
- Empirically it outperformed other state-of-the-art algorithms which were modified to perform in the TBP setting.

APT Algorithm

Algorithm 1 APT

Input: Time horizon T, threshold τ, tolerance factor $\epsilon \geq 0$

Pull each arm once

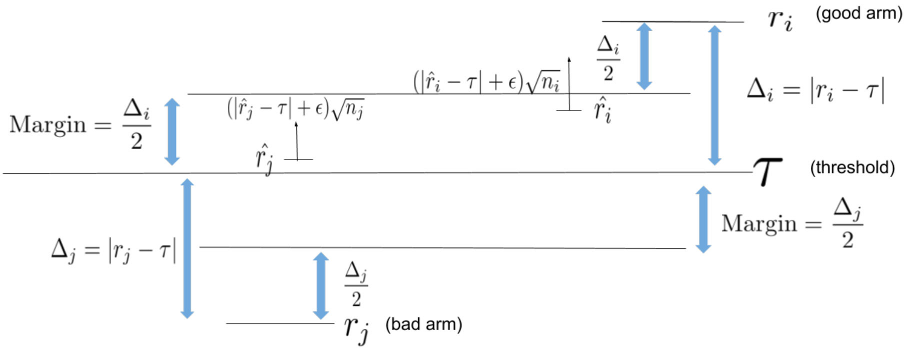
for $t = K + 1, .., T$ **do**

Pull arm $j \in \arg\min_{i \in A} \left\{ (|\hat{r}_i - \tau| + \epsilon) \sqrt{n_i} \right\}$ and observe the reward for arm j.

end for

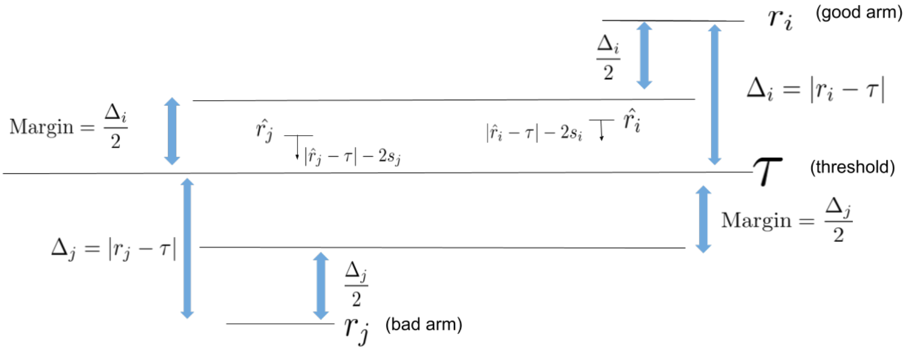
Output: $\hat{S}_\tau = \{i : \hat{r}_i \geq \tau\}$.

Intuition of APT



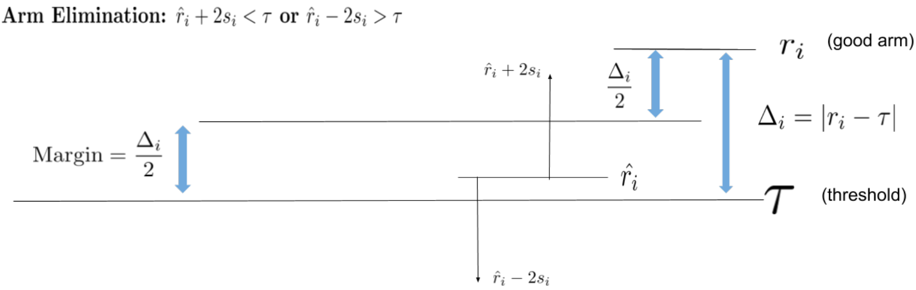
AugUCB algorithm (Intuition, Arm pulling)

- Like UCB-Imp, AugUCB also divides the time budget T into rounds.
- At every timestep we pull arm j s.t. $j \in \arg\min_{i \in B_m} \left\{ |\hat{r}_i - \tau| - 2s_i \right\}$ (like APT).



AugUCB algorithm (Intuition, Arm Elimination)

- It is risky to eliminate arm i while $\hat{r}_i$ is inside *Margin*.
- Confidence interval s_i will make sure arm i is not eliminated while inside Margin with a high probability.



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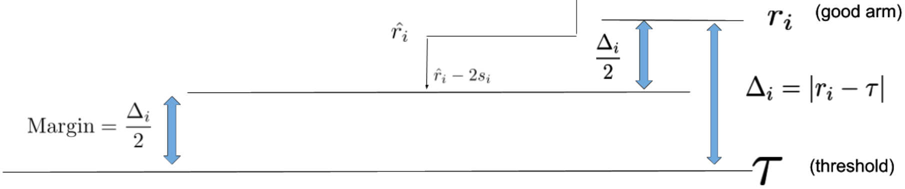
Arm Elimination: $\hat{r}_i + 2s_i < \tau$ **or** $\hat{r}_i - 2s_i > \tau$



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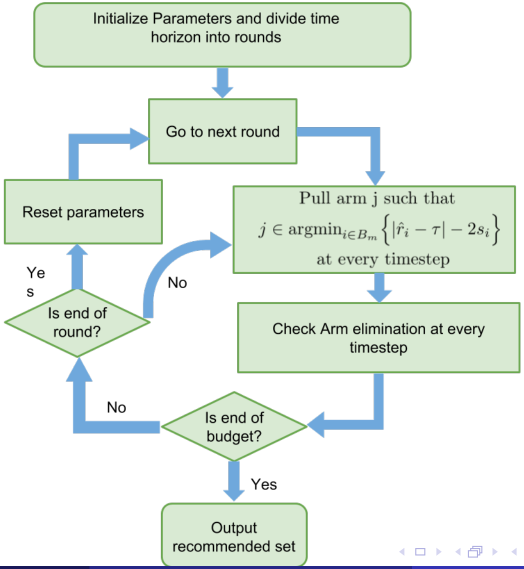
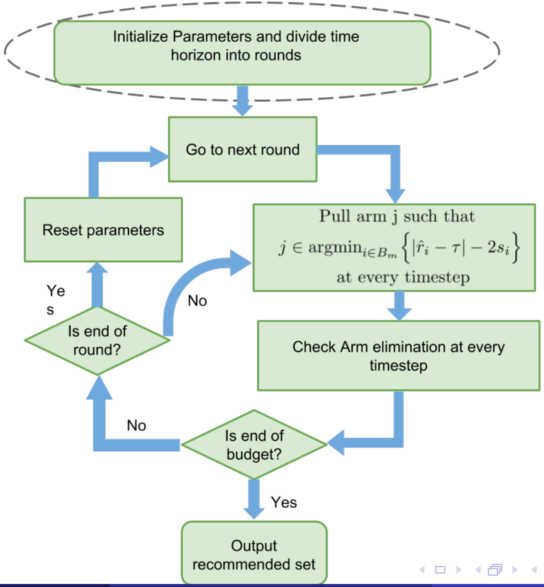


AugUCB parameter initialization



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- We define $\ell_0 = \left\lceil \frac{2\psi_0 \log(T\epsilon_0)}{\epsilon_0} \right\rceil$ as the budget allocated to each arm in a round.

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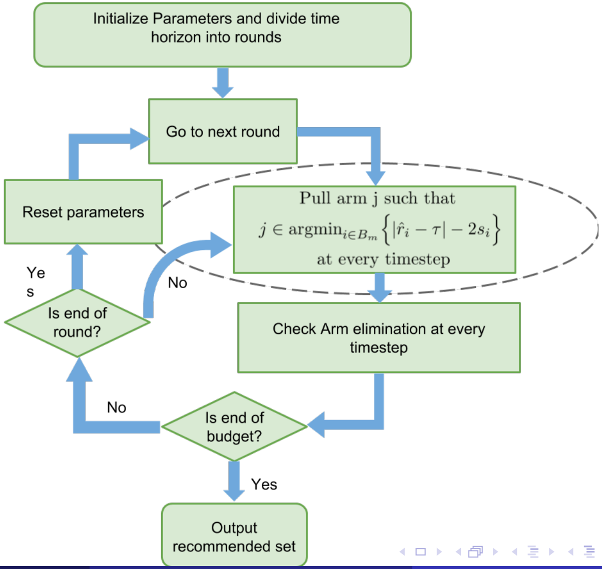
- The first round gets divided into $N_0 = K\ell_0$ timesteps.
- We define a large exploration regulatory factor $\psi_0 = \frac{T\epsilon_0}{128\left(\log(\frac{3}{16}K\log K)\right)^2}$ which controls exploration.

AugUCB arm pull



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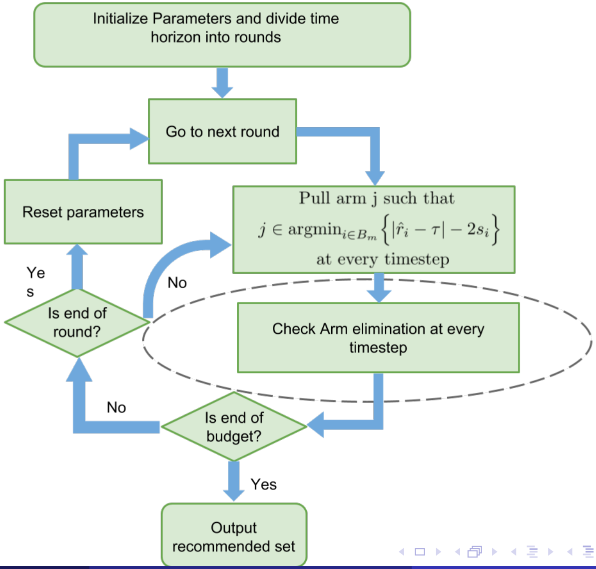
- s_i decreases with more n_i and ψ_m and ρ ensures that it decreases at a correct rate.
- Note that $\hat{v}_i$ estimated variance in s_i makes the algorithm pull the arm which shows more variance.

AugUCB arm elimination



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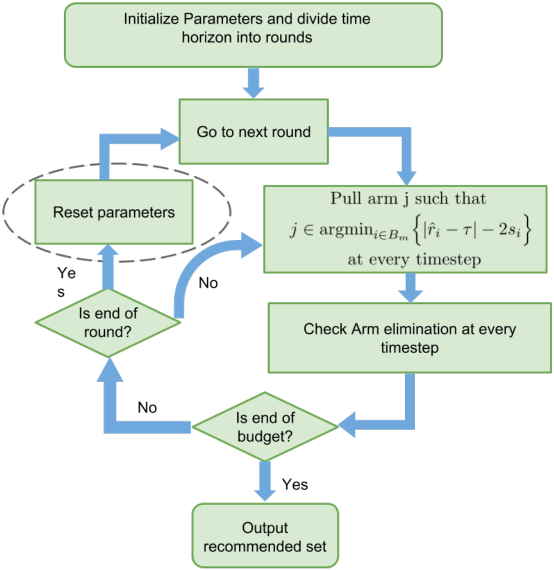
AugUCB parameter reset



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- The relationship between H_1 and H_2 can be derived as,

$$H_2 \leq H_1 \leq \log(2K) H_2$$

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- For a variance aware algorithm we define $H_{\sigma,1}$ (as in Gabillon et al. (2011)) that incorporates reward variances into its expression as:

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- Finally, analogous to H_2, we introduce $H_{\sigma,2}$, such that $H_{\sigma,2} = \max_{i \in \mathcal{A}} \frac{i}{\tilde{\Delta}_{(i)}^2}$, where $\tilde{\Delta}_i^2 = \frac{\Delta_i^2}{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}$, $(\tilde{\Delta}_{(i)})$ is an increasing ordering of $(\tilde{\Delta}_i)$.

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- Note that H_1, H_2 and $H_{\sigma,1}, H_{\sigma,2}$ are not directly comparable to each other except in a special case when variances and gaps (Δ_i) are very low we can say that $H_{\sigma,1} < H_1$.

Expected Loss of AugUCB

Theorem

For $K \geq 4$ and $\rho = 1/3$, the expected loss of the AugUCB algorithm is given by,

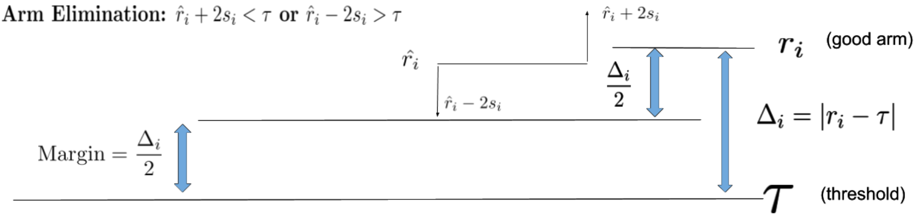
$$\mathbb{E}[\mathcal{L}(T)] \leq 2KT \exp\left(-\frac{T}{4096 \log(K \log K) H_{\sigma,2}}\right).$$

Table: AugUCB vs. State of the art

Algorithm	Upper Bound on Expected Loss
AugUCB	$\exp\left(-\frac{T}{4096 \log(K \log K) H_{\sigma,2}} + \log(2KT)\right)$
UCBEV	$\exp\left(-\frac{1}{512} \frac{T-2K}{H_{\sigma,1}} + \log(6KT)\right)$
APT	$\exp\left(-\frac{T}{64 H_1} + 2\log((\log(T)+1)K)\right)$
CSAR	$\exp\left(-\frac{T-K}{72 \log(K) H_{CSAR,2}} + 2\log(K)\right)$

Sketch of the proof

Figure: AugUCB arm elimination



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- APT, AugUCB, CSAR, UA do not require access to H_1 or $H_{\sigma,1}$.
- UCBE, UCBEV, CSAR and UA come from the pure exploration lineage and are modified suitably to perform in TBP setting.

Experimental Setup

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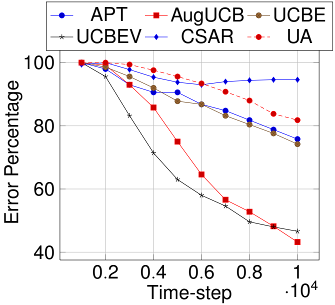
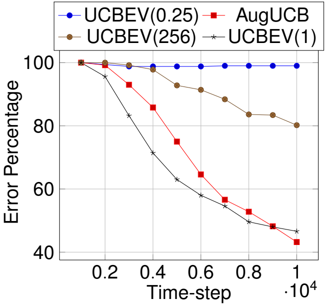
- The first 10 arms partitioned into two groups; the respective means are $r_{1:5} = 0.45$, $r_{6:10} = 0.55$.
- The means of arms $i = 11 : 100$ are chosen same as $r_{11:100} = 0.4$.
- Variances are set as $\sigma^2_{1:5} = 0.3$ and $\sigma^2_{6:10} = 0.8$; $\sigma^2_{11:100}$ are independently and uniformly chosen in the interval $[0.2, 0.3]$.

Experimental Results



(a) Expt-1: Two Group Setting (Advance)



(b) Expt-2: Two Group Setting (Advance)

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- A more detailed analysis of the non-uniform arm selection and parameter selection is also required.

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Thank You