Finite-time Analysis of Frequentist Strategies for Multi-armed Bandits

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January 8, 2018

Overview

SMAB Setting (Part 1)	TBP Setting (Part 2)
• Problem Definition • Contributions • EUCBV algorithm • Theory • Experiments	• Problem Definition • Contributions • AugUCB algorithm • Theory • Experiments

Stochastic Multi-Armed Bandit Problem (SMAB) (Chapter 2)

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- The learner does not know the mean $r_i, \forall i \in \mathbb{A}$ of the distribution or the variance σ_i^2.

Problem Definition of SMAB (Chapter 2)

- **Primary aim:** Minimize the cumulative regret by quickly identifying the arm whose expected mean is r^* such that $r^* > r_i, \forall i \in \mathbb{A}$.

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- **Condition:** This has to be achieved within a finite T timesteps.
- The expected regret of an algorithm after T timesteps is give by,

$$\mathbb{E}[R_T] = \sum_{i=1}^{K} \mathbb{E}[z_i(T)]\Delta_i,$$

where $\Delta_i = r^* - r_i$ is the gap.

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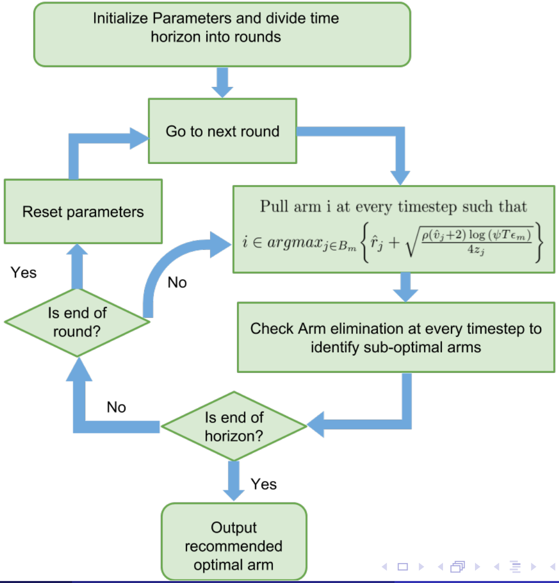
- Theoretically it achieves an order-optimal regret bound, the first for an arm elimination algorithm in SMAB setting.
- Empirically, it outperforms all the state-of-the-art algorithms for the considered environments.


Initialize Parameters and divide time horizon into rounds
Go to next round
Pull arm i at every timestep such that
$i \in argmax_{j \in B_m} \left\{ \hat{r}_j + \sqrt{\frac{\rho(\hat{v}_j+2)\log(\psi T \epsilon_m)}{4z_j}} \right\}$
Check Arm elimination at every timestep to identify sub-optimal arms
Is end of horizon?
Yes
Output recommended optimal arm
No
Is end of round?
No
Yes
Reset parameters

Expected Regret of EUCBV (Chapter 3)

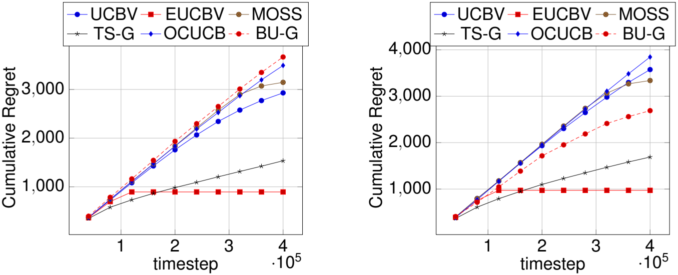
Corollary (***Gap-Independent Bound***)

The regret of EUCBV is upper bounded by the following gap-independent expression:

$$\mathbb{E}[R_T] \leq \frac{C_3 K^5}{T^{\frac{1}{4}}} + 80\sqrt{KT}.$$

Algorithm	GD Bound	GI Bound	Var
EUCBV	$O\left(\frac{K\sigma_{\max}^2 \log(\frac{T\Delta^2}{K})}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$	Yes
UCBV	$O\left(\frac{K\sigma_{\max}^2 \log T}{\Delta}\right)$	$O\left(\sqrt{KT \log T}\right)$	Yes
MOSS	$O\left(\frac{K^2 \log(T\Delta^2/K)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$	No
OCUCB	$O\left(\frac{K \log(T/H_i)}{\Delta}\right)$	$O\left(\sqrt{KT}\right)$	No

Experiments in SMAB (Chapter 3)



(a) Expt-3: Failure of TS (b) Expt-4: 3 Group Variance

- **Setting:** The Thresholding Bandit Problem (TBP) is a special case of SMAB setting.
- **Primary aim:** Identify all the arms whose mean of the reward distribution (r_i) is above a particular threshold τ given as input.

Problem Definition of TBP (Chapter 4)

- **Setting:** The Thresholding Bandit Problem (TBP) is a special case of SMAB setting.
- **Primary aim:** Identify all the arms whose mean of the reward distribution (r_i) is above a particular threshold τ given as input.
- **Condition:** This has to be achieved within T timesteps of exploration and this is termed as a fixed-budget problem.

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- Let $\hat{S}_\tau$ denote the recommendation of a learning algorithm after T time units of exploration, while $\hat{S}_\tau^c$ denotes its complement.
- The goal of the learning agent is to minimize the expected loss:

$$\mathbb{E}[\mathcal{L}(T)] = \mathbb{P}\Big(\underbrace{\{S_\tau \cap \hat{S}_\tau^c \neq \emptyset\}}_{\textbf{Rejected good arms}} \cup \underbrace{\{\hat{S}_\tau \cap S_\tau^c \neq \emptyset\}}_{\textbf{Accepted bad arms}} \Big)$$

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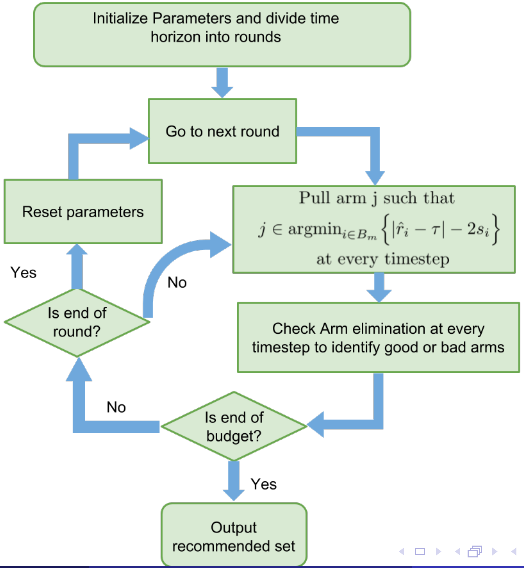
- We also define a new problem complexity which uses empirical variance estimates along with arm's mean for giving the theoretical bound.
- Empirically, in the considered environments AugUCB outperforms all the algorithms that rely only on mean estimation to conduct exploration.

AugUCB Algorithm for TBP (Chapter 5)



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- We define $H_{\sigma,1}$ (as in Gabillon et al. (2011)) that incorporates variances as:

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- Finally, analogous to H_2, we introduce $H_{\sigma,2}$, such that $H_{\sigma,2} = \max_{i \in \mathcal{A}} \frac{i}{\tilde{\Delta}_{(i)}^2}$, where $\tilde{\Delta}_i^2 = \frac{\Delta_i^2}{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}$, $(\tilde{\Delta}_{(i)})$ is an increasing ordering of $(\tilde{\Delta}_i)$.

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- From Audibert and Bubeck (2010), we can show that

$$H_{\sigma,2} \leq H_{\sigma,1} \leq \log(2K) H_{\sigma,2}.$$

Expected Loss of AugUCB (Chapter 5)

Theorem

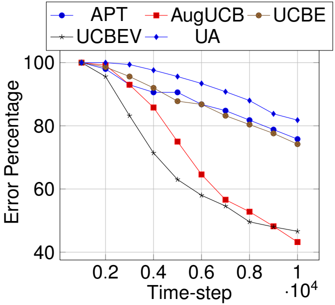
For $K \geq 4$ and $\rho = 1/3$, the expected loss of the AugUCB algorithm is given by,

$$\mathbb{E}[\mathcal{L}(T)] \leq 2KT \exp\left(- \frac{T}{4096 \log(K \log K) H_{\sigma,2}} \right).$$

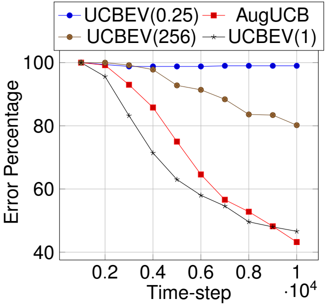
Table: AugUCB vs. State of the art

Algorithm	Upper Bound on Expected Loss	Oracle
AugUCB	$\exp\left(- \frac{T}{4096 \log(K \log K) H_{\sigma,2}} + \log(2KT) \right)$	No
UCBEV	$\exp\left(- \frac{1}{512} \frac{T-2K}{H_{\sigma,1}} + \log(6KT) \right)$	Yes
APT	$\exp\left(- \frac{T}{64H_1} + 2\log((\log(T) + 1)K) \right)$	No
UCBE	$\exp\left(- \frac{T-K}{18H_1} - 2\log(\log(T)K \right)$	Yes

Experiments in TBP (Chapter 5)



(c) Expt-1: Two Group Setting (Advance)



(d) Expt-2: Two Group Setting (Advance)

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- Further studies are required to establish a lower bound on the expected loss of AugUCB.
- A more detailed analysis of the non-uniform arm selection and parameter selection is also required for both AugUCB and EUCBV.

Papers based on Thesis

- Subhojyoti Mukherjee, K.P. Naveen, Nandan Sudarsanam, and Balaraman Ravindran, "*Thresholding Bandit with Augmented UCB*", *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017,2515-2521*.
- Subhojyoti Mukherjee, K.P. Naveen, Nandan Sudarsanam, and Balaraman Ravindran, "*Efficient UCBV: An Almost Optimal Algorithm using Variance Estimates*", *To appear in Proceedings of the Thirty-Second Association for the Advancement of Artificial Intelligence, AAAI 2018, New Orleans, Louisiana, USA, February 2-7*.

Thank You