## Thresholding Bandits with Augmented UCB

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#### Overview

- Stochastic Multi-Armed Bandit Problem
- **Problem Definition**
- Contribution
- **Previous Works**
- **AugUCB**
- Theoretical Analysis
- **Experiments**
- Conclusion



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- A finite set of actions or arms belonging to set A such that |A| = K.
- The rewards for each of the arms are identical and independent random variables drawn from distribution specific to the arm.
- The learner does not know the mean  $r_i$ ,  $\forall i \in A$  of the distribution or the variance  $\sigma_i^2$ .

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- The more we pull arm i the closer  $\hat{r}_i$  gets to  $r_i$ .
- Due to the uncertainty in  $\hat{r}_i$  we have carefully conduct exploration.

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- Condition: This has to be achieved within T timesteps of exploration and this is termed as a fixed-budget problem.
- At the end of the given T timesteps the learner must recommend a set of arms which (according to it) are the arms having reward mean above  $\tau$ .

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- Let  $\hat{S}_{\tau}$  denote the recommendation of a learning algorithm after Ttime units of exploration, while  $\hat{S}_{\tau}^{c}$  denotes its complement.
- The goal of the learning agent is to minimize the expected loss:

$$\mathbb{E}[\mathcal{L}(T)] = \mathbb{P}\big(\underbrace{\{\mathcal{S}_{\tau} \cap \hat{\mathcal{S}}_{\tau}^{\textit{C}} \neq \emptyset\}}_{\text{Rejected good arms}} \ \cup \underbrace{\{\hat{\mathcal{S}}_{\tau} \cap \mathcal{S}_{\tau}^{\textit{C}} \neq \emptyset\}}_{\text{Accepted bad arms}}\big)$$

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- Higher variance of the arms' ⇒ harder the problem.

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- In anomaly detection and classification (see Locatelli et al. (2016)).

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- It is the first variance-based arm elimination algorithm for the considered TBP settings.
- It addresses an open problem discussed in Auer and Ortner (2010) of designing an algorithm that can eliminate arms based on variance estimates.
- We also define a new problem complexity which uses empirical variance estimates along with arm's mean for giving the theoretical bound

## The Upper Confidence Bound (UCB) Approach

• Since there is an initial uncertainty in the estimated mean  $(\hat{r}_i)$  introduce a confidence interval term  $c_i$ .

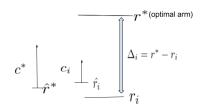
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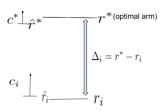
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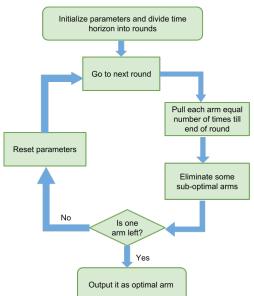
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- c<sub>i</sub> ensures that the arm i is properly explored and is gradually reduced with time as one pulls the arm i more.
- At every timestep pull arm  $j \in \arg\max_{i \in A} \{\hat{r}_i + c_i\}$  and this will ensure that proper exploration is done.

# The UCB Approach





# Approach of UCB-Improved (UCB-Imp)



# Intuition of UCB-Improved (UCB-Imp)

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- Theoretically they proved this algorithm to be almost optimal when only mean estimation is used as a metric of comparison.
- Empirically it outperformed other state-of-the-art algorithms which were modified to perform in the TBP setting.

#### **APT Algorithm**

#### Algorithm 1 APT

**Input:** Time horizon T, threshold  $\tau$ , tolerance factor  $\epsilon \geq 0$  Pull each arm once

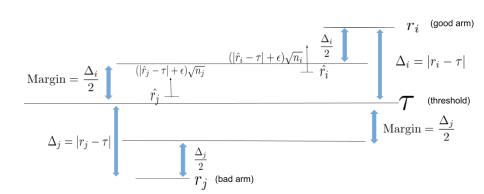
for 
$$t = K + 1, ..., T$$
 do

Pull arm  $j \in \arg\min_{i \in A} \left\{ (|\hat{r}_i - \tau| + \epsilon) \sqrt{n_i} \right\}$  and observe the reward for arm j.

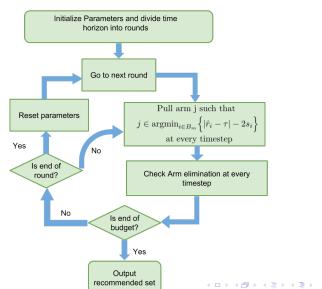
end for

**Output:**  $\hat{S}_{\tau} = \{i : \hat{r}_i \geq \tau\}.$ 

#### Intuition of APT

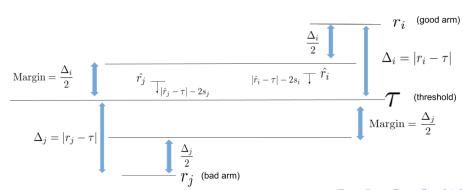


### AugUCB algorithm



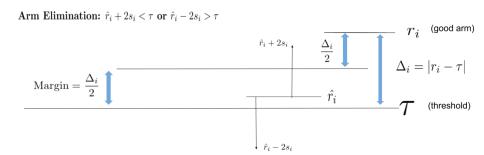
# AugUCB algorithm (Intuition, Arm pulling)

- Like UCB-Imp, AugUCB also divides the time budget *T* into rounds.
- At every timestep we pull arm j s.t.  $j \in \arg\min_{i \in B_m} \left\{ |\hat{r}_i \tau| 2s_i \right\}$  (like APT).

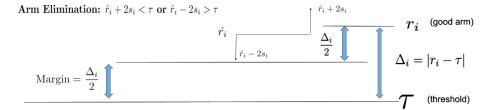


### AugUCB algorithm (Intuition, Arm Elimination)

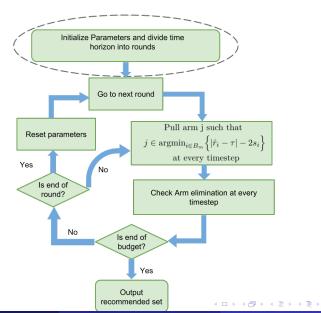
- It is risky to eliminate arm i while  $\hat{r}_i$  is inside Margin.
- Confidence interval  $s_i$  will make sure arm i is not eliminated while inside Margin with a high probability.



# AugUCB algorithm (Intuition, Arm Elimination)



# AugUCB parameter initialization



#### Parameter initialization

• We define  $\ell_0 = \left\lceil \frac{2\psi_0 \log(T\epsilon_0)}{\epsilon_0} \right\rceil$  as the budget allocated to each arm in a round.

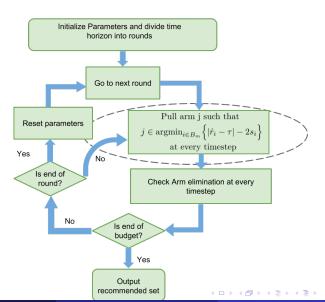
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- We define a large exploration regulatory factor  $\psi_0 = \frac{\tau_{\epsilon_0}}{{}_{128} \Big(\log(\frac{3}{16}K\log K)\Big)^2} \text{ which controls exploration.}$

### AugUCB arm pull



ullet We pull the arm that minimizes  $j\in rg \min_{i\in B_m}\left\{|\hat{r}_i- au|-2s_i
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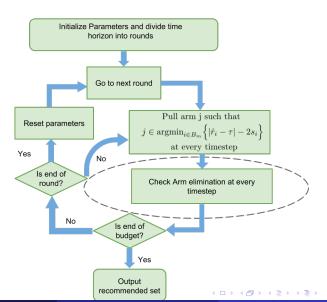


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- Note that  $\hat{v}_i$  estimated variance in  $s_i$  makes the algorithm pull the arm which shows more variance.

### AugUCB arm elimination



#### Arm elimination

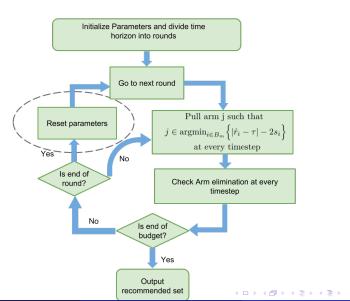
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- Re-allocates the remaining budget for surviving arms.



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- We define  $H_1 = \sum_{i=1}^K \frac{1}{\Delta_i^2}$  and  $H_2 = \min_{i \in \mathcal{A}} \frac{i}{\Delta_{(i)}^2}$  where  $\Delta_{(i)}$  is an increasing ordering of  $\Delta_i$ .

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- From Audibert and Bubeck (2010) the relationship between H<sub>1</sub> and H<sub>2</sub> can be derived as,

$$H_2 \leq H_1 \leq \log(2K)H_2$$



• For a variance aware algorithm we define  $H_{\sigma,1}$  (as in Gabillon et al. (2011)) that incorporates reward variances into its expression as:

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• Note that  $H_1$ ,  $H_2$  and  $H_{\sigma,1}$ ,  $H_{\sigma,2}$  are not directly comparable to each other except in a special case when variances and gaps  $(\Delta_i)$  are very low we can say that  $H_{\sigma,1} < H_1$ .

# **Expected Loss of AugUCB**

#### Theorem

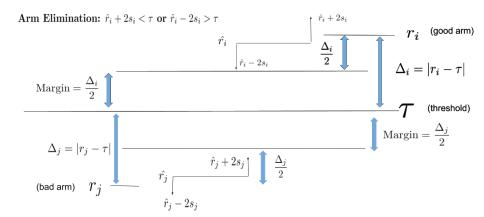
For  $K \ge 4$  and  $\rho = 1/3$ , the expected loss of the AugUCB algorithm is given by.

$$\mathbb{E}[\mathcal{L}(T)] \leq 2KT \exp\bigg(-\frac{T}{4096\log(K\log K)H_{\sigma,2}}\bigg).$$

Table: AugUCB vs. State of the art

Algorithm	Upper Bound on Expected Loss		Oracle
AugUCB	exp (	$\left(-\frac{T}{4096\log(K\log K)H_{\sigma,2}} + \log\left(2KT\right)\right)$	No
UCBEV	exp (	$\left(-\frac{1}{512}\frac{T-2K}{H_{\sigma,1}}+\log\left(6KT\right)\right)$	Yes
APT	exp (	$\left(-\frac{T}{64H_1} + 2\log((\log(T) + 1)K)\right)$	No
UCBE	exp (	$\left(-\frac{T-K}{18H_1}-2\log(\log(T)K)\right)$	Yes

# Sketch of the proof



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- APT, AugUCB, UA do not require access to  $H_1$  or  $H_{\sigma,1}$ .
- UCBE, UCBEV, CSAR and UA come from the pure exploration lineage and are modified suitably to perform in TBP setting.

### **Experimental Setup**

• This setup involves Gaussian reward distributions with  $K=100,\,T=10000$  and  $\tau=0.5$  with the reward means set in two groups.

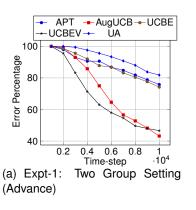
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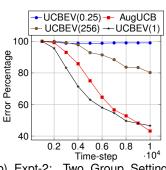
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- The means of arms i = 11 : 100 are chosen same as  $r_{11:100} = 0.4$ .
- Variances are set as  $\sigma_{1:5}^2=0.3$  and  $\sigma_{6:10}^2=0.8$ ;  $\sigma_{11:100}^2$  are independently and uniformly chosen in the interval [0.2, 0.3].

#### **Experimental Results**





(b) Expt-2: Two Group Setting (Advance)

#### Conclusion

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- This work has been accepted in the proceedings of IJCAI 2017.
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- A more detailed analysis of the non-uniform arm selection and parameter selection is also required.

# Thank You