

Finite-time Analysis of Frequentist Strategies for Multi-armed Bandits

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CS15S300

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SMAB Setting (Part 1)	TBP Setting (Part 2)
<ul style="list-style-type: none">• Problem Definition• Contributions• EUCEV algorithm• Theory• Experiments	<ul style="list-style-type: none">• Problem Definition• Contributions• AugUCB algorithm• Theory• Experiments

Stochastic Multi-Armed Bandit Problem (SMAB)

(Chapter 2)

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- The rewards for each of the arms are i.i.d random variables drawn from distribution specific to the arm which are fixed throughout the time horizon denoted by T .
- The learner does not know the mean $r_i, \forall i \in \mathbb{A}$ of the distribution or the variance σ_i^2 .

Problem Definition of SMAB (Chapter 2)

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- **Condition:** This has to be achieved within a finite T timesteps.
- The expected regret of an algorithm after T timesteps is give by,

$$\mathbb{E}[R_T] = \sum_{i=1}^K \mathbb{E}[z_i(T)] \Delta_i,$$

where $\Delta_i = r^* - r_i$ is the gap.

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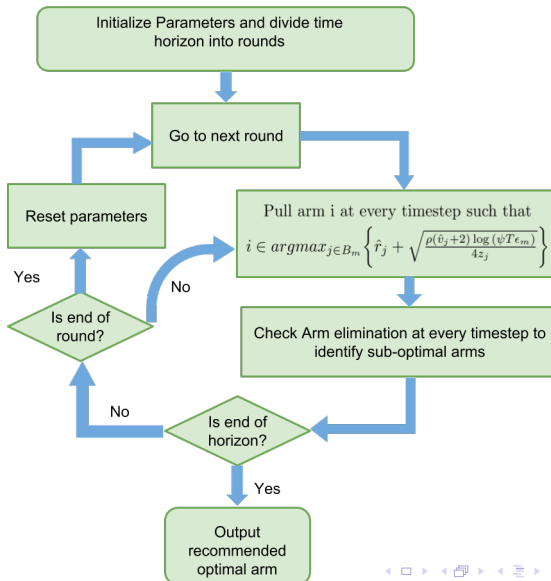
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- Theoretically it achieves an order-optimal regret bound, the first for an arm elimination algorithm in SMAB setting.
- Empirically, it outperforms all the state-of-the-art algorithms for the considered environments.

EUCBV Algorithm for SMAB (Chapter 3)



Expected Regret of EUCBV (Chapter 3)

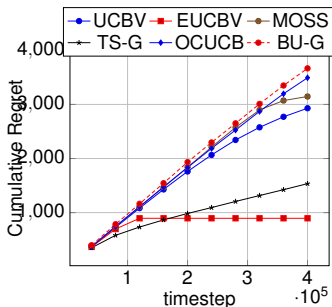
Corollary (*Gap-Independent Bound*)

The regret of EUCBV is upper bounded by the following gap-independent expression:

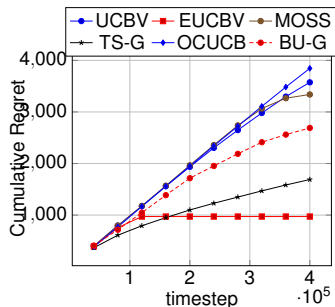
$$\mathbb{E}[R_T] \leq \frac{C_3 K^5}{T^{\frac{1}{4}}} + 80\sqrt{KT}.$$

Algorithm	GD Bound	GI Bound	Var
EUCBV	$O\left(\frac{K\sigma_{\max}^2 \log(\frac{T\Delta^2}{K})}{\Delta}\right)$	$O(\sqrt{KT})$	Yes
UCBV	$O\left(\frac{K\sigma_{\max}^2 \log T}{\Delta}\right)$	$O(\sqrt{KT \log T})$	Yes
MOSS	$O\left(\frac{K^2 \log(T\Delta^2/K)}{\Delta}\right)$	$O(\sqrt{KT})$	No
OCUCB	$O\left(\frac{K \log(T/H_i)}{\Delta}\right)$	$O(\sqrt{KT})$	No

Experiments in SMAB (Chapter 3)



(a) Expt-3: Failure of TS



(b) Expt-4: 3 Group Variance

Problem Definition of TBP (Chapter 4)

- **Setting:** The Thresholding Bandit Problem (TBP) is a special case of SMAB setting.
- **Primary aim:** Identify all the arms whose mean of the reward distribution (r_i) is above a particular threshold τ given as input.

Problem Definition of TBP (Chapter 4)

- **Setting:** The Thresholding Bandit Problem (TBP) is a special case of SMAB setting.
- **Primary aim:** Identify all the arms whose mean of the reward distribution (r_i) is above a particular threshold τ given as input.
- **Condition:** This has to be achieved within T timesteps of exploration and this is termed as a fixed-budget problem.

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- Let \hat{S}_τ denote the recommendation of a learning algorithm after T time units of exploration, while \hat{S}_τ^c denotes its complement.
- The goal of the learning agent is to minimize the expected loss:

$$\mathbb{E}[\mathcal{L}(T)] = \mathbb{P}\left(\underbrace{\{S_\tau \cap \hat{S}_\tau^c \neq \emptyset\}}_{\text{Rejected good arms}} \cup \underbrace{\{\hat{S}_\tau \cap S_\tau^c \neq \emptyset\}}_{\text{Accepted bad arms}} \right)$$

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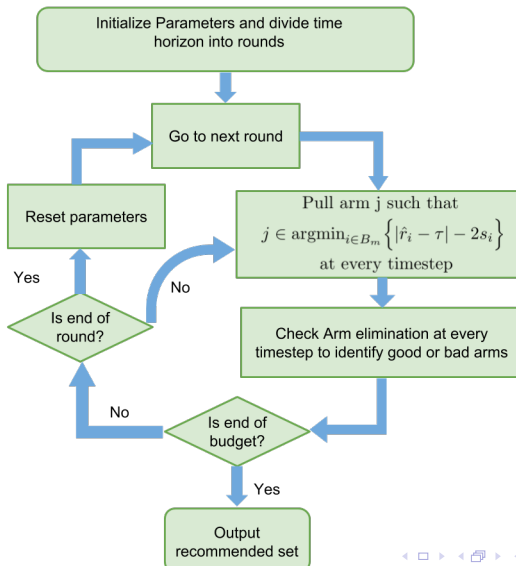
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- We also define a new problem complexity which uses empirical variance estimates along with arm's mean for giving the theoretical bound.
- Empirically, in the considered environments AugUCB outperforms all the algorithms that rely on estimation of mean to conduct exploration.

AugUCB Algorithm for TBP (Chapter 5)



Problem Complexity in TBP (Chapter 5)

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- We define $H_{\sigma,1}$ (as in Gabillon et al. (2011)) that incorporates variances as:

$$H_{\sigma,1} = \sum_{i=1}^K \frac{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}{\Delta_i^2}.$$

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- Finally, analogous to H_2 , we introduce $H_{\sigma,2}$, such that $H_{\sigma,2} = \max_{i \in \mathcal{A}} \frac{i}{\tilde{\Delta}_{(i)}^2}$, where $\tilde{\Delta}_i^2 = \frac{\Delta_i^2}{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}$, $(\tilde{\Delta}_{(i)})$ is an increasing ordering of $(\tilde{\Delta}_i)$.

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- From Audibert and Bubeck (2010), we can show that

$$H_{\sigma,2} \leq H_{\sigma,1} \leq \log(2K)H_{\sigma,2}.$$

Expected Loss of AugUCB (Chapter 5)

Theorem

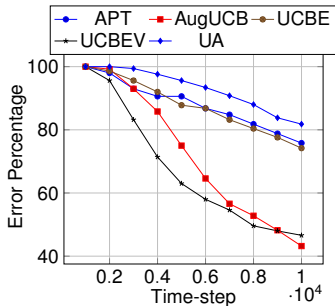
For $K \geq 4$ and $\rho = 1/3$, the expected loss of the AugUCB algorithm is given by,

$$\mathbb{E}[\mathcal{L}(T)] \leq 2KT \exp \left(- \frac{T}{4096 \log(K \log K) H_{\sigma,2}} \right).$$

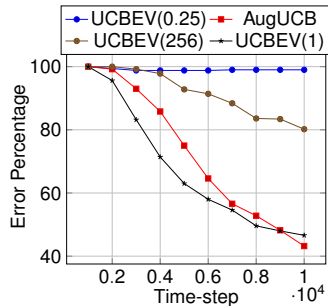
Table: AugUCB vs. State of the art

Algorithm	Upper Bound on Expected Loss	Oracle
AugUCB	$\exp \left(- \frac{T}{4096 \log(K \log K) H_{\sigma,2}} + \log(2KT) \right)$	No
UCBEV	$\exp \left(- \frac{1}{512} \frac{T-2K}{H_{\sigma,1}} + \log(6KT) \right)$	Yes
APT	$\exp \left(- \frac{T}{64H_1} + 2 \log((\log(T) + 1)K) \right)$	No
UCBE	$\exp \left(- \frac{T-K}{18H_1} - 2 \log(\log(T)K) \right)$	Yes

Experiments in TBP (Chapter 5)



(c) Expt-1: Two Group Setting (Advance)



(d) Expt-2: Two Group Setting (Advance)

Conclusion (Chapter 6)

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Conclusion (Chapter 6)

- We proposed the EUCBV algorithm for the SMAB setting which uses variance and mean estimation along with arm elimination to give an order-optimal theoretical guarantees.
- We proposed the AugUCB algorithm for the fixed budget TBP which uses variance estimation and arm elimination to give an improved theoretical and experimental guarantees than mean estimation based algorithms.
- Further studies are required to establish a lower bound on the expected loss of AugUCB.
- A more detailed analysis of the non-uniform arm selection and parameter selection is also required for both AugUCB and EUCBV.

- Subhojyoti Mukherjee, K.P. Naveen, Nandan Sudarsanam, and Balaraman Ravindran, "*Thresholding Bandit with Augmented UCB*", *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, 2515-2521.
- Subhojyoti Mukherjee, K.P. Naveen, Nandan Sudarsanam, and Balaraman Ravindran, "*Efficient UCBV: An Almost Optimal Algorithm using Variance Estimates*", *To appear in Proceedings of the Thirty-Second Association for the Advancement of Artificial Intelligence, AAAI 2018, New Orleans, Louisiana, USA, February 2-7*.

Thank You