Thresholding Bandits with Augmented UCB

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Abstract

In this paper we propose the Augmented UCB (AugUCB) algorithm for the fixed-budget setting of a specific combinatorial, pure-exploration, stochastic multi-armed bandit setup called the thresholding bandit problem. Our algorithm is based on arm elimination, employing both mean and variance estimates. Theoretically, our algorithm provides a weaker guarantee (in terms of an upper bound on the expected loss) than UCBEV, a variant of GapE-V [Gabillon *et al.*, 2011] algorithm, modified for thresholding bandit problem. However, UCBEV requires access to the problem complexity, while AugUCB requires no such complexity parameters as input. Through simulation experiments we establish that our algorithm, owing to its utilization of variance estimates in arm elimination, performs significantly better than state-of-the-art APT and CSAR algorithms, particularly when a large number of arms with different means and variances are involved.

1 Introduction

In this paper we study the fixed-budget setting of a specific combinatorial pure-exploration problem, called the thresholding bandit problem (TBP), in the context of stochastic multi-armed bandit (MAB) setting. MAB problems are instances of the classic sequential decision-making scenario; specifically an MAB problem comprises of a learner and a collection of actions (or arms), denoted $\mathcal{A}$. In each trial the learner plays (or pulls) an arm $i \in \mathcal{A}$ which yields independent and identically distributed (i.i.d.) reward samples from a distribution (corresponding to arm i), whose expectation is denoted by r_i. The learner's objective is to identify an arm corresponding to the maximum expected reward, denoted r^*. Thus, at each time-step the learner is faced with the *exploration vs. exploitation dilemma*, where it can pull an arm which has yielded the highest mean reward (denoted $\hat{r}_i$) thus far (*exploitation*) or continue to explore other arms with the prospect of finding a better arm whose performance is yet not observed sufficiently (*exploration*).

Pure-exploration problems are unlike their traditional (exploration vs. exploitation) counterparts where the objective is to minimize the cumulative regret, which is the total loss incurred by the learner for not playing the optimal arm throughout the time horizon T. Instead, in the pure exploration setup the learning algorithm is provided with a threshold τ, and the objective, after exploring for T rounds, is to output all arms i whose r_i is above τ. Thus, the learning algorithm, until time T, can invest entirely on exploring the arms without being concerned about the loss incurred while exploring. It is important to emphasize that the *thresholding* bandit problem is different from the *threshold* bandit setup studied in [Abernethy *et al.*, 2016], where the learner receives an unit reward whenever the value of an observation is above a threshold.

Formally, the problem we consider is the following. First, we define the set $S_\tau = \{i \in \mathcal{A} : r_i \geq \tau\}$. Note that, S_τ is the set of all arms whose reward mean is greater than τ. Let S_τ^c denote the complement of S_τ, i.e., $S_\tau^c = \{i \in \mathcal{A} : r_i < \tau\}$. Next, let $\hat{S}_\tau = \hat{S}_\tau(T) \subseteq \mathcal{A}$ denote the recommendation of the learning algorithm after T time units of exploration, while $\hat{S}_\tau^c$ denotes its complement. The performance of the learning agent is measured by the accuracy with which it can classify the arms into S_τ and S_τ^c after time horizon T. Equivalently, using $\mathbb{I}(E)$ to denote the indicator of an event E, the *loss* $\mathcal{L}(T)$ is defined as

$$\mathcal{L}(T) = \mathbb{I}\big(\{S_\tau \cap \hat{S}_\tau^c \neq \emptyset\} \cup \{\hat{S}_\tau \cap S_\tau^c \neq \emptyset\}\big).$$

Finally, the goal of the learning agent is to minimize the expected loss:

$$\mathbb{E}[\mathcal{L}(T)] = \mathbb{P}\big(\{S_\tau \cap \hat{S}_\tau^c \neq \emptyset\} \cup \{\hat{S}_\tau \cap S_\tau^c \neq \emptyset\}\big).$$

Note that the expected loss is simply the *probability of error*, that occurs either if a good arm is rejected or a bad arm is accepted as a good one.

The above TBP formulation has several applications, for instance, from areas ranging from anomaly detection and classification [Locatelli *et al.*, 2016] to industrial application. Particularly in industrial applications a learners objective is to choose (i.e., keep in operation) all machines whose productivity is above a threshold. Similarly, TBP finds applications in mobile communications [Audibert and Bubeck, 2010] where the users are to be allocated only those channels whose quality is above an acceptable threshold.

1.1 Related Work

Significant amount of literature is available on the stochastic MAB setting with respect to minimizing the cumulative

regret. While the seminal work of [Robbins, 1952], [Thompson, 1933], and [Lai and Robbins, 1985] prove asymptotic lower bounds on the cumulative regret, the more recent work of [Auer *et al.*, 2002] propose the UCB1 algorithm that provides finite time-horizon guarantees. Subsequent work such as [Audibert and Bubeck, 2009] and [Auer and Ortner, 2010] have improved the upper bounds on the cumulative regret. The authors in [Auer and Ortner, 2010] have proposed a *round-based*[1] version of the UCB algorithm, referred to as UCB-Improved. Of special mention is the work of [Audibert *et al.*, 2009] where the authors have introduced a *variance-aware* UCB algorithm, referred to as UCB-V; it is shown that the algorithms that take into account variance estimation along with mean estimation tends to perform better than the algorithms that solely focuses on mean estimation, for instance, such as UCB1. For a more detail survey of literature on UCB algorithms, we refer the reader to [Bubeck and Cesa-Bianchi, 2012].

In this work we are particularly interested in *pure-exploration MABs*, where the focus in primarily on simple regret rather than the cumulative regret. The relationship between cumulative regret and simple regret is proved in [Bubeck *et al.*, 2011] where the authors prove that minimizing the simple regret necessarily results in maximizing the cumulative regret. The pure exploration problem has been explored mainly under the following two settings:

1. Fixed Budget setting: Here the learning algorithm has to suggest the best arm(s) within a fixed time-horizon T, that is usually given as an input. The objective is to maximize the probability of returning the best arm(s). This is the scenario we consider in our paper. In [Audibert and Bubeck, 2010] the authors propose the UCBE and the Successive Reject (SR) algorithm, and prove simple-regret guarantees for the problem of identifying the single best arm. In the combinatorial fixed budget setup [Gabillon *et al.*, 2011] propose the GapE and GapE-V algorithms that suggest, with high probability, the best m arms at the end of the time budget. Similarly, [Bubeck *et al.*, 2013] introduce the Successive Accept Reject (SAR) algorithm, which is an extension of the SR algorithm; SAR is a round based algorithm whereby at the end of each round an arm is either accepted or rejected (based on certain confidence conditions) until the top m arms are suggested at the end of the budget with high probability. A similar combinatorial setup was explored in [Chen *et al.*, 2014] where the authors propose the Combinatorial Successive Accept Reject (CSAR) algorithm, which is similar in concept to SAR but with a more general setup.

2. Fixed Confidence setting: In this setting the learning algorithm has to suggest the best arm(s) with a fixed confidence (given as input) with as fewer number of attempts as possible. The single best arm identification has been studied in [Even-Dar *et al.*, 2006], while for the combinatorial setup [Kalyanakrishnan *et al.*, 2012] have proposed the LUCB algorithm which, on termination, returns m arms which are at least ϵ close to the true top-m arms with probability at least $1-\delta$. For a detail survey of this setup we refer the reader to [Jamieson and Nowak, 2014].

Apart from these two settings some unified approaches has also been suggested in [Gabillon *et al.*, 2012] which proposes the algorithms UGapEb and UGapEc which can work in both the above two settings. The thresholding bandit problem is a specific instance of the pure-exploration setup of [Chen *et al.*, 2014]. In the latest work of [Locatelli *et al.*, 2016] Anytime Parameter-Free Thresholding (APT) algorithm comes up with an improved anytime guarantee than CSAR for the thresholding bandit problem.

[1]An algorithm is said to be *round-based* if it pulls all the arms equal number of times in each round, and then proceeds to eliminate one or more arms that it identifies to be sub-optimal.

1.2 Our Contribution

In this paper we propose AugUCB, which is an arm-elimination based algorithm for the considered thresholding bandit problem. AugUCB essentially combines the approach of UCB-Improved, CCB [Liu and Tsuruoka, 2016] and APT algorithms. Our algorithm takes into account the empirical variances of the arms; to the best of our knowledge this is the first variance-aware algorithm for the considered TBP. Thus, we also address an open problem discussed in [Auer and Ortner, 2010] of designing an algorithm that can eliminate arms based on variance estimates. In this regard, note that both CSAR and APT are not variance-aware algorithms.

In Table 1 we compare the upper bound on expected loss incurred by the various algorithms. The terms H_1, H_2, $H_{CSAR,2}, H_{\sigma,1}$ and $H_{\sigma,2}$ represent various problem complexities, and are as defined in Section 3. From Section 3 we note that, for all $K \geq 8$, we have

$$\log(K \log K)\, H_{\sigma,2} > \log(2K) H_{\sigma,2} \geq H_{\sigma,1}.$$

Thus, we find that the upper bound for AugUCB is weaker than that for UCBEV. However, UCBEV algorithm requires the complexity factor $H_{\sigma,1}$ as input, which is not realistic in practice. In contrast, our AugUCB requires no such complexity factor as input.

Empirically we show that for a large number of arms when the variance of the arms lying above τ are high, our algorithm performs better than all other algorithms, except the algorithm UCBEV which has access to the underlying problem complexity and also is a variance-aware algorithm. AugUCB requires one input parameter and the exact choice for the parameter is derived in Theorem 3.1. Also, unlike SAR or

Table 1: AugUCB vs. State of the art

Algorithm	Upper Bound on Expected Loss
APT	$\exp\left(-\frac{T}{64H_1} + 2\log((\log(T)+1)K)\right)$
CSAR	$K^2 \exp\left(-\frac{T-K}{72\log(K)H_{CSAR,2}}\right)$
UCBEV	$\exp\left(-\frac{1}{512}\frac{T-2K}{H_{\sigma,1}} + \log(KT)\right)$
AugUCB	$\exp\left(-\frac{T}{4096\log(K\log K)H_{\sigma,2}} + \log(KT)\right)$

CSAR, AugUCB does not have explicit accept or reject sets rather the arm elimination condition simply removes arm(s) if it is sufficiently sure that the mean of the arms are very high or very low about the threshold based on mean and variance estimation thereby re-allocating the remaining budget among the surviving arms. This although is a tactic similar to SAR or CSAR, but here at any round, an arbitrary number of arms can be accepted or rejected thereby improving upon SAR and CSAR which accepts/rejects one arm in every round. Also their round lengths are non-adaptive and they pull all the arms equal number of times in each round.

The remainder of the paper is organized as follows. In section 2 we present our AugUCB algorithm. Section 3 contains our main theorem on expected loss, while section 4 contains simulation experiments. We finally draw our conclusions in section 5.

2 Augmented-UCB Algorithm

Notations and assumptions: $\mathcal{A}$ denotes the set of arms, and $|\mathcal{A}| = K$ is the number of arms in $\mathcal{A}$. For arm $i \in \mathcal{A}$, we use r_i to denote the true mean of the distribution from which the rewards are sampled, while $\hat{r}_i(t)$ denotes the estimated mean at time t. Formally, using $n_i(t)$ to denote the number of times arm i has been pulled until time t, we have $\hat{r}_i(t) = \frac{1}{n_i(t)} \sum_{z=1}^{n_i(t)} X_{i,z}$, where $X_{i,z}$ is the reward sample received when arm i is pulled for the z-th time. Similarly, we use σ_i^2 to denote the true variance of the reward distribution corresponding to arm i, while $\hat{v}_i(t)$ is the estimated variance, i.e., $\hat{v}_i(t) = \frac{1}{n_i(t)} \sum_{z=1}^{n_i(t)} (X_{i,z} - \hat{r}_i)^2$. Whenever there is no ambiguity about the underlaying time index t, for simplicity we neglect t from the notations and simply use $\hat{r}_i$, $\hat{v}_i$, and n_i, to denote the respective quantities. Let $\Delta_i = |\tau - r_i|$ denote the distance of the true mean from the threshold τ.

Finally, we assume that all the reward distributions are 1-sub-Gaussian (note that, 1-sub-Gaussian includes Gaussian distributions with variance less than 1, distributions supported on an interval of length less than 2, etc). Further, the rewards are assumed to take values in the interval $[0, 1]$.

The Algorithm: The Augmented-UCB (AugUCB) algorithm is presented in Algorithm 1. AugUCB is essentially based on the arm elimination method of the UCB-Improved [Auer and Ortner, 2010], but adapted to the thresholding bandit setting proposed in [Locatelli *et al.*, 2016]. However, unlike the UCB improved (which is based on mean estimation) our algorithm employs *variance estimates* (as in [Audibert *et al.*, 2009]) for arm elimination; to the best of our knowledge this is the first variance-aware algorithm for the thresholding bandit problem. Further, we allow for arm-elimination at each time-step, which is in contrast to the earlier work (e.g., [Auer and Ortner, 2010; Chen *et al.*, 2014]) where the arm elimination task is deferred to the end of the respective exploration rounds. The finer details of the algorithm are presented below.

The active set B_0 is initialized with all the arms from $\mathcal{A}$. We divide the entire budget T into rounds/phases like in UCB-Improved, CCB, SAR and CSAR. At every time-step AugUCB checks for arm elimination conditions, while updating parameters at the end of each round. As suggested by

Algorithm 1 AugUCB

Input: Time budget T; parameter ρ; threshold τ
Initialization: $B_0 = \mathcal{A}$; $m = 0$; $\epsilon_0 = 1$;

$$M = \left\lfloor \frac{1}{2} \log_2 \frac{T}{e} \right\rfloor; \quad \psi_0 = \frac{T\epsilon_0}{\left(\log(\frac{3}{16} K \log K)\right)^2};$$

$$\ell_0 = \left\lceil \frac{2\psi_0 \log(T\epsilon_0)}{\epsilon_0} \right\rceil; \quad N_0 = K\ell_0$$

for $t = K+1, .., T$ **do**
  Pull arm $j \in \arg\min_{i \in B_m} \left\{ |\hat{r}_i - \tau| - 2s_i \right\}$
  $t \leftarrow t + 1$
  for $i \in B_m$ **do**
    if $(\hat{r}_i + s_i < \tau - s_i)$ or $(\hat{r}_i - s_i > \tau + s_i)$ **then**
      $B_m \leftarrow B_m \backslash \{i\}$ (Arm deletion)
    end if
  end for
  if $t \geq N_m$ and $m \leq M$ **then**
    Reset Parameters
    $\epsilon_{m+1} \leftarrow \frac{\epsilon_m}{2}$
    $B_{m+1} \leftarrow B_m$
    $\psi_{m+1} \leftarrow \frac{T\epsilon_{m+1}}{(\log(\frac{3}{16} K \log K))^2}$
    $\ell_{m+1} \leftarrow \left\lceil \frac{2\psi_{m+1} \log(T\epsilon_{m+1})}{\epsilon_{m+1}} \right\rceil$
    $N_{m+1} \leftarrow t + |B_{m+1}| \ell_{m+1}$
    $m \leftarrow m + 1$
  end if
end for
Output: $\hat{S}_\tau = \{i : \hat{r}_i \geq \tau\}$.

[Liu and Tsuruoka, 2016] to make AugUCB to overcome too much early exploration, we no longer pull all the arms equal number of times in each round. Instead, we choose an arm in the active set B_m that minimizes $(|\hat{r}_i - \tau| - 2s_i)$ where

$$s_i = \sqrt{\frac{\rho \psi_m (\hat{v}_i + 1) \log(T\epsilon_m)}{4n_i}}$$

with ρ being the arm elimination parameter and ψ_m being the exploration regulatory factor. The above condition ensures that an arm closer to the threshold τ is pulled; parameter ρ can be used to fine tune the elimination interval. The choice of exploration factor, ψ_m, comes directly from [Audibert and Bubeck, 2010] and [Bubeck *et al.*, 2011] where it is stated that in pure exploration setup, the exploring factor must be linear in T (so that an exponentially small probability of error is achieved) rather than being logarithmic in T (which is more suited for minimizing cumulative regret).

3 Theoretical Results

Let us begin by recalling the following definitions of the *problem complexity* as introduced in [Locatelli *et al.*, 2016]:

$$H_1 = \sum_{i=1}^{K} \frac{1}{\Delta_i^2} \text{ and } H_2 = \min_{i \in \mathcal{A}} \frac{i}{\Delta_{(i)}^2}$$

where $(\Delta_{(i)} : i \in \mathcal{A})$ is obtained by arranging $(\Delta_i : i \in \mathcal{A})$ in an increasing order. Also, from [Chen *et al.*, 2014] we have

$$H_{CSAR,2} = \max_{i \in \mathcal{A}} \frac{i}{\Delta^2_{(i)}}.$$

$H_{CSAR,2}$ is the complexity term appearing in the bound for the CSAR algorithm. The relation between the above complexity terms are as follows (see [Locatelli *et al.*, 2016]):

$$H_1 \leq \log(2K) H_2 \text{ and } H_1 \leq \log(K) H_{CSAR,2}.$$

As ours is a variance-aware algorithm, we require H_1^σ (as defined in [Gabillon *et al.*, 2011]) that incorporates reward variances into its expression as given below:

$$H_{\sigma,1} = \sum_{i=1}^{K} \frac{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}{\Delta_i^2}.$$

Finally, analogous to $H_{CSAR,2}$, in this paper we introduce the complexity term $H_{\sigma,2}$, which is given by

$$H_{\sigma,2} = \max_{i \in \mathcal{A}} \frac{i}{\tilde{\Delta}^2_{(i)}}$$

where $\tilde{\Delta}_i^2 = \frac{\Delta_i^2}{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}$, and $(\tilde{\Delta}_{(i)})$ is an increasing ordering of $(\tilde{\Delta}_i)$. Following the results in [Audibert and Bubeck, 2010], we can show that

$$H_{\sigma,2} \leq H_{\sigma,1} \leq \overline{\log}(K) H_{\sigma,2} \leq \log(2K) H_{\sigma,2}.$$

Our main result is summarized in the following theorem where we prove an upper bound on the expected loss.

Theorem 3.1. *For $K \geq 4$ and $\rho = 1/3$, the expected loss of the AugUCB algorithm is given by,*

$$\mathbb{E}[\mathcal{L}(T)] \leq 2KT \exp\left(- \frac{T}{4096 \log(K \log K) H_{\sigma,2}} \right).$$

Proof. The proof comprises of two modules. In the first module we investigate the necessary conditions for arm elimination within a specified number of rounds, which is motivated by the technique in [Auer and Ortner, 2010]. Bounds on the arm-elimination probability is then obtained; however, since we use variance estimates, we invoke the Bernstein inequality (as in [Audibert *et al.*, 2009]) rather that the Chernoff-Hoeffding bounds (which is appropriate for the UCB-Improved [Auer and Ortner, 2010]). In the second module, as in [Locatelli *et al.*, 2016], we first define a favourable event that will yield an upper bound on the expected loss. Using union bound, we then incorporate the result from module-1 (on the arm elimination probability), and finally derive the result through a series of simplifications. The details are as follows.

Arm Elimination: Recall the notations used in the algorithm, Also, for each arm $i \in \mathcal{A}$, define $m_i = \min\left\{m | \sqrt{\rho \epsilon_m} < \frac{\Delta_i}{2}\right\}$. In the m_i-th round, whenever $n_i = \ell_{m_i} \geq \frac{2\psi_{m_i} \log(T\epsilon_{m_i})}{\epsilon_{m_i}}$, we obtain (as $\hat{v}_i \in [0,1]$)

$$s_i \leq \sqrt{\frac{\rho(\hat{v}_i + 1)\epsilon_{m_i}}{8}} \leq \frac{\sqrt{\rho \epsilon_{m_i}}}{2} < \frac{\Delta_i}{4}. \tag{1}$$

First, let us consider a bad arm $i \in \mathcal{A}$ (i.e., $r_i < \tau$). We note that, in the m_i-th round whenever $\hat{r}_i \leq r_i + 2s_i$, then arm i is eliminated as a bad arm. This is easy to verify as follows: using (1) we obtain,

$$\begin{aligned} \hat{r}_i &\leq r_i + 2s_i \\ &= r_i + 4s_i - 2s_i \\ &< r_i - \Delta_i - 2s_i \\ &= \tau - 2s_i \end{aligned}$$

which is precisely one of the elimination conditions in Algorithm 1. Thus, the probability that a bad arm is not eliminated correctly in the m_i-th round (or before) is given by

$$\mathbb{P}(\hat{r}_i > r_i + 2s_i) \leq \mathbb{P}\left(\hat{r}_i > r_i + 2\bar{s}_i\right) + \mathbb{P}\left(\hat{v}_i \geq \sigma_i^2 + \sqrt{\rho \epsilon_{m_i}}\right) \tag{2}$$

where

$$\bar{s}_i = \sqrt{\frac{\rho \psi_{m_i} (\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} + 1) \log(T \epsilon_{m_i})}{4 n_i}}$$

Note that, substituting $n_i = \ell_{m_i} \geq \frac{2\psi_{m_i} \log(T\epsilon_{m_i})}{\epsilon_{m_i}}$, $\bar{s}_i$ can be simplified to obtain,

$$2\bar{s}_i \leq \frac{\sqrt{\rho \epsilon_{m_i} (\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} + 1)}}{2} \leq \sqrt{\rho \epsilon_{m_i}}. \tag{3}$$

The first term in the LHS of (2) can be bounded using the Bernstein inequality as below:

$$\begin{aligned} &\mathbb{P}\left(\hat{r}_i > r_i + 2\bar{s}_i\right) \\ &\leq \exp\left(- \frac{(2\bar{s}_i)^2 n_i}{2\sigma_i^2 + \frac{4}{3}\bar{s}_i} \right) \\ &\leq \exp\left(- \frac{\rho \psi_{m_i} (\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} + 1) \log(T\epsilon_{m_i})}{2\sigma_i^2 + \frac{2}{3}\sqrt{\rho \epsilon_{m_i}}} \right) \\ &\overset{(a)}{\leq} \exp\left(- \frac{3\rho T \epsilon_{m_i}}{256 a^2} \left(\frac{\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} + 1}{3\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}}} \right) \log(T\epsilon_{m_i}) \right) \\ &:= \exp(-Z_i) \end{aligned} \tag{4}$$

where, for simplicity, we have used α_i to denoted the exponent in the inequality (a). Also, note that (a) is obtained by using $\psi_{m_i} = \frac{T\epsilon_{m_i}}{128 a^2}$, where $a = (\log(\frac{3}{16} K \log K))$.

The second term in the LHS of (2) can be simplified as follows:

$$\begin{aligned} &\mathbb{P}\left\{ \hat{v}_i \geq \sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} \right\} \\ &\leq \mathbb{P}\left\{ \frac{1}{n_i} \sum_{t=1}^{n_i} (X_{i,t} - r_i)^2 - (\hat{r}_i - r_i)^2 \geq \sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} \right\} \\ &\leq \mathbb{P}\left\{ \frac{\sum_{t=1}^{n_i} (X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} \right\} \\ &\overset{(a)}{\leq} \mathbb{P}\left\{ \frac{\sum_{t=1}^{n_i} (X_{i,t} - r_i)^2}{n_i} \geq \sigma_i^2 + 2\bar{s}_i \right\} \\ &\overset{(b)}{\leq} \exp\left(- \frac{3\rho \psi_{m_i}}{2} \left(\frac{\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}} + 1}{3\sigma_i^2 + \sqrt{\rho \epsilon_{m_i}}} \right) \log(T\epsilon_{m_i}) \right) \end{aligned}$$

$$= \exp(-Z_i) \quad (5)$$

where inequality (a) is obtained using (3), while (b) follows from the Bernstein inequality.

Thus, using (4) and (5) in (2) we obtain $\mathbb{P}(\hat{r}_i > r_i + 2s_i) \leq 2\exp(-Z_i)$. Proceeding similarly, for a good arm $i \in \mathcal{A}$, the probability that it is not correctly eliminated in the m_i-th round (or before) is also bounded by $\mathbb{P}(\hat{r}_i < r_i - 2s_i) \leq 2\exp(-Z_i)$. In general, for any $i \in \mathcal{A}$ we have

$$\mathbb{P}(|\hat{r}_i - r_i| > 2s_i) \leq 4\exp(-Z_i). \quad (6)$$

Favourable Event: Following the notation in [Locatelli *et al.*, 2016] we define the event

$$\xi = \left\{ \forall i \in \mathcal{A}, \forall t = 1, 2, .., T : |\hat{r}_i - r_i| \leq 2s_i \right\}.$$

Note that, on ξ each arm $i \in \mathcal{A}$ is eliminated correctly in the m_i-th round (or before). Thus, it follows that $\mathbb{E}[\mathcal{L}(T)] \leq P(\xi^c)$. Since ξ^c can be expressed as an union of the events $(|\hat{r}_i - r_i| > 2s_i)$ for all $i \in \mathcal{A}$ and all $t = 1, 2, \cdots, T$, using union bound we can write

$$\begin{aligned}
&\mathbb{E}[\mathcal{L}(T)] \\
&\leq \sum_{i\in\mathcal{A}} \sum_{t=1}^{T} \mathbb{P}(|\hat{r}_i - r_i| > 2s_i) \\
&\leq \sum_{i\in\mathcal{A}} \sum_{t=1}^{T} 4\exp(-Z_i) \\
&\leq 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{3\rho T \epsilon_{m_i}}{256a^2} \left(\frac{\sigma_i^2 + \sqrt{\rho\epsilon_{m_i}} + 1}{3\sigma_i^2 + \sqrt{\rho\epsilon_{m_i}}} \right) \log(T\epsilon_{m_i}) \right) \\
&\overset{(a)}{\leq} 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{3T\Delta_i^2}{4096a^2} \left(\frac{4\sigma_i^2 + \Delta_i + 4}{12\sigma_i^2 + \Delta_i} \right) \log(\frac{3}{16} T\Delta_i^2) \right) \\
&\overset{(b)}{\leq} 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{12T\Delta_i^2}{(12\sigma_i + 12\Delta_i)} \frac{\log(\frac{3}{16}K\log K)}{4096a^2} \right) \\
&\overset{(c)}{\leq} 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{T\Delta_i^2 \log(\frac{3}{16}K\log K)}{4096(\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i})a^2} \right) \\
&\overset{(d)}{\leq} 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{T\log(\frac{3}{16}K\log K)}{4096\tilde{\Delta}_i^{-2}a^2} \right) \\
&\overset{(e)}{\leq} 4T \sum_{i\in\mathcal{A}} \exp\left(-\frac{T\log(\frac{3}{16}K\log K)}{4096\max_j (j\tilde{\Delta}_{(j)}^{-2})(\log(\frac{3}{16}K\log K))^2} \right) \\
&\overset{(f)}{\leq} 4KT\exp\left(-\frac{T}{4096 H_{\sigma,2}(\log(K\log K))} \right).
\end{aligned}$$

The justification for the above simplifications are as follows:

- (a) is obtained by noting that in round m_i we have
$$\frac{\Delta_i}{4} \leq \sqrt{\epsilon_{m_i}\rho} < \frac{\Delta_i}{2}.$$
- For (b), we note that the function $x \mapsto x\exp(-Cx^2)$, where $x \in [0, 1]$, is decreasing on $[1/\sqrt{2C}, 1]$ for any $C > 0$ (see [Bubeck *et al.*, 2011; Auer and Ortner, 2010]). Thus, using $C = \lfloor \sqrt{e/T} \rfloor$ and $\min_{j\in\mathcal{A}} \Delta_j = \Delta = \sqrt{\frac{K\log K}{T}} > \sqrt{\frac{e}{T}}$, we obtain (b).
- To obtain (c) we have used the inequality $\Delta_i \leq \sqrt{\sigma_i^2 + (16/3)\Delta_i}$ (which holds because $\Delta_i \in [0, 1]$).
- (d) is obtained simply by substituting $\tilde{\Delta}_i = \frac{\Delta_i^2}{\sigma_i + \sqrt{\sigma_i^2 + (16/3)\Delta_i}}$ and $a = \log(\frac{3}{16}K\log K)$.
- Finally, to obtain (e) and (f), note that
$$\tilde{\Delta}_i^{-2} \leq i\tilde{\Delta}_i^{-2} \leq \max_{j\in\mathcal{A}} j\Delta_{(j)}^{-2} = H_{\sigma,2}.$$

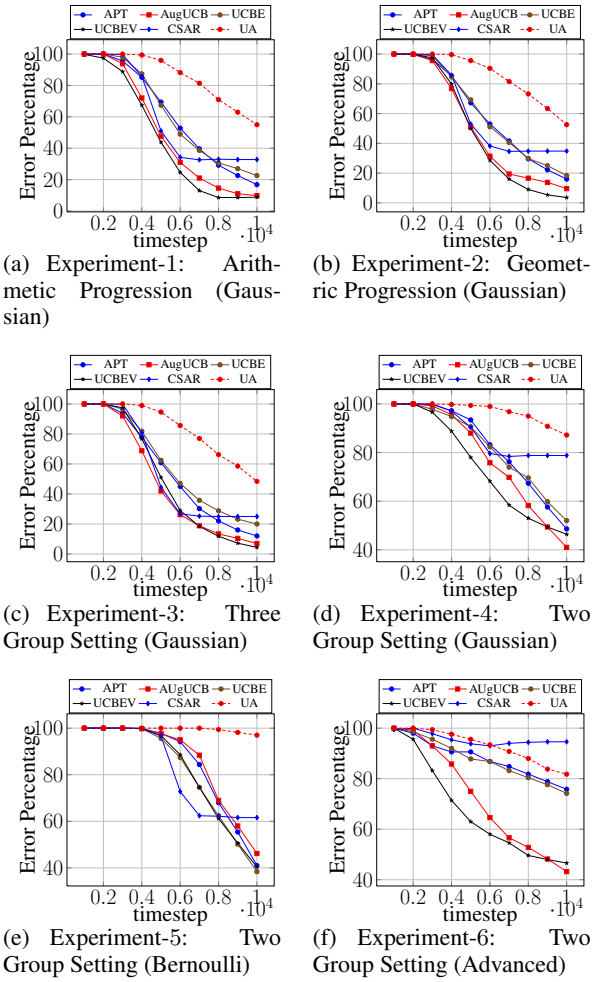
□

4 Numerical Experiments

In this section, we compare the empirical performance of AugUCB against the performances of the APT, CSAR, UCBE and UCBEV algorithms. We also implement the uniform-allocation (labeled UA) strategy, where at each time-step an arm is sampled uniformly from the set of all arms; UA is known to be optimal if all arms are equally difficult to classify. APT (proposed in [Locatelli *et al.*, 2016]) is run with $\epsilon = 0.05$, which denotes the precision with which the algorithm suggests the set of good arms. CSAR is modified for the TBP setting such that it behaves as a Successive Reject algorithm whereby it rejects the arm farthest from τ after each round. Similarly we modify the UCBE [Audibert *et al.*, 2009] and UCBEV [Gabillon *et al.*, 2011] algorithms (originally proposed for single best arm and TopM identification problems, respectively) to suit the TBP setting. Following [Locatelli *et al.*, 2016] the exploration parameter a in UCBE is set to $a = \frac{T-K}{H_1}$, while for UCBEV we set $a = \frac{T-2K}{H_{\sigma,1}}$. Then, at each time-step $t = 1, 2, .., T$ we pull the arm that minimizes $\{|\hat{r}_i - \tau| - \sqrt{\frac{a}{n_i}}\}$, where a is set as mentioned above for UCBE and UCBEV respectively. Finally, for AugUCB we take $\rho = \frac{1}{3}$ as in Theorem 3.1.

In total we conduct a set of six experiments with different reward means and variances. However, in all the experiments, the threshold τ is set to 0.5. Also, the number of arms in each experiment is $K = 100$ (indexed $i = 1, 2, \cdots, 100$), of which $\{6, 7, 8, 9, 10\}$ arms have their reward means above τ. In all the experiments, each algorithm is run independently for 10000 time-steps, and the output set of arms suggested by each algorithm at every time-step is recorded. The experiment is repeated for 500 independent iterations, and the average error percentage is plotted against the 10000 time-steps.

Experiment-1: Here, we consider Gaussian reward distributions, with expected rewards of the arms being $r_{1:4} = 0.2 + (0:3) \cdot 0.05$, $r_5 = 0.45$, $r_6 = 0.55$, $r_{7:10} = 0.65 + (0:3) \cdot 0.05$ and $r_{11:100} = 0.4$; note that, the means of first 10 arms follow an arithmetic progression. The corresponding variances are $\sigma_{1:5}^2 = 0.5$ and $\sigma_{6:10}^2 = 0.6$, while $\sigma_{11:100}^2$ is chosen independently and uniform in the interval $[0.38, 0.42]$. The means in the testbed are chosen in such a way that any algorithm has to spend a significant amount of budget to explore all the arms and variance is chosen in such a way that the arms above τ have high variance whereas arms below τ have lower variance. The result is shown in Figure 1(a), where we see that UCBEV, which has access to the problem complexity



(a) Experiment-1: Arithmetic Progression (Gaussian)

(b) Experiment-2: Geometric Progression (Gaussian)

(c) Experiment-3: Three Group Setting (Gaussian)

(d) Experiment-4: Two Group Setting (Gaussian)

(e) Experiment-5: Two Group Setting (Bernoulli)

(f) Experiment-6: Two Group Setting (Advanced)

Figure 1: Experiments with thresholding bandit

and is a variance-aware algorithm, outperforms all the algorithm (including UCBE which also has access to the problem complexity but does not take into account the variances of the arms). AugUCB, being variance-aware, outperforms UCBE, APT and the other non variance-aware algorithms that we have considered.

Experiment-2: In this experiment also we use Gaussian for the reward distribution, however, here the means of first 10 arms is set as a Geometric Progression. Formally, the reward means are $r_{1:4} = 0.4 - (0.2)^{1:4}$, $r_5 = 0.45$, $r_6 = 0.55$, $r_{7:10} = 0.6 + (0.2)^{5-(1:4)}$ and $r_{11:100} = 0.4$; the variances of all the arms are as set similarly as in experiment 1. The corresponding results are shown in Figure 1(b). Again we see that AugUCB outperforms the other algorithms, except UCBEV.

Experiment-3: In this experiment, the means of the first 10 arms are set in three groups. The testbed again involves Gaussian reward distributions, however with expected rewards being $r_{1:3} = 0.1$, $r_{4:7} = \{0.35, 0.45, 0.55, 0.65\}$, $r_{8:10} = 0.9$ and $r_{11:100} = 0.4$. The variances of all the arms are set in the same way as in experiment 1. The results from this experiment are presented Figure 1(c). The observations are inline with the observations made in the previous experiments.

Experiment-4: Here, the means of first 10 arms are set in two groups. The testbed again involves Gaussian reward distributions with expected rewards of the arms set to $r_{1:5} = 0.45$, $r_{6:10} = 0.55$ and $r_{11:100} = 0.4$. The variances are set in the same way as in experiment 1. The results are shown in Figure 1(d), from where we again observe the good performance of AugUCB.

Experiment-5: This setting is similar to that considered in Experiment-4, but with the reward distributions being Bernoulli instead of Gaussian. The results for this case is shown in Figure 1(e). Here, we observe that UCBE and UCBEV beating outperforms AugUCB, while the performance of AugUCB is comparable with that achieved by APT.

Experiment-6: This is again the two group setting involving Gaussian reward distributions. The reward means are as in Experiment-4, while the variances are set as $\sigma^2_{1:5} = 0.3$, $\sigma^2_{6:10} = 0.8$ and $\sigma^2_{11:100}$ are independently and uniformly chosen in the interval $[0.2, 0.3]$. The corresponding results are shown in Figure 1(f). We refer to this setup as *Advanced* because here the chosen variance values are such that only variance-aware algorithms will perform well. Hence, we see that UCBEV performs very well in comparison with the other algorithms. However, it is interesting to note that the performance of AugUCB catches-up with UCBEV as the time-step increases.

Finally, note that in all the experiments although CSAR performs well initially, but it quickly exhausts its budget and always saturates at a higher error percentage. This is because it pulls all arms equally in each round, where the round lengths are non-adaptive.

5 Conclusions and Future work

From a theoretical viewpoint we conclude the expected loss AugUCB is more than UCBEV (which has access to problem complexity). From the numerical experiments on settings with large number of arms with different mean and variance, we observed that AugUCB outperforms all the non-variance aware algorithms. It would be interesting future research to come up with an anytime version of AugUCB algorithm. This is also the first paper to apply elimination by variance estimation in the TBP problem by modifying UCB-Improved and CCB algorithms.

References

[Abernethy *et al.*, 2016] Jacob D Abernethy, Kareem Amin, and Ruihao Zhu. Threshold bandits, with and without censored feedback. In *Advances In Neural Information Processing Systems*, pages 4889–4897, 2016.

[Audibert and Bubeck, 2009] Jean-Yves Audibert and Sébastien Bubeck. Minimax policies for adversarial and stochastic bandits. In *COLT*, pages 217–226, 2009.

[Audibert and Bubeck, 2010] Jean-Yves Audibert and Sébastien Bubeck. Best arm identification in multi-armed bandits. In *COLT-23th Conference on Learning Theory-2010*, pages 13–p, 2010.

[Audibert *et al.*, 2009] Jean-Yves Audibert, Rémi Munos, and Csaba Szepesvári. Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. *Theoretical Computer Science*, 410(19):1876–1902, 2009.

[Auer and Ortner, 2010] Peter Auer and Ronald Ortner. Ucb revisited: Improved regret bounds for the stochastic multi-armed bandit problem. *Periodica Mathematica Hungarica*, 61(1-2):55–65, 2010.

[Auer *et al.*, 2002] Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

[Bubeck and Cesa-Bianchi, 2012] Sébastien Bubeck and Nicolo Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *arXiv preprint arXiv:1204.5721*, 2012.

[Bubeck *et al.*, 2011] Sébastien Bubeck, Rémi Munos, and Gilles Stoltz. Pure exploration in finitely-armed and continuous-armed bandits. *Theoretical Computer Science*, 412(19):1832–1852, 2011.

[Bubeck *et al.*, 2013] Sébastien Bubeck, Tengyao Wang, and Nitin Viswanathan. Multiple identifications in multi-armed bandits. In *ICML (1)*, pages 258–265, 2013.

[Chen *et al.*, 2014] Shouyuan Chen, Tian Lin, Irwin King, Michael R Lyu, and Wei Chen. Combinatorial pure exploration of multi-armed bandits. In *Advances in Neural Information Processing Systems*, pages 379–387, 2014.

[Even-Dar *et al.*, 2006] Eyal Even-Dar, Shie Mannor, and Yishay Mansour. Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *The Journal of Machine Learning Research*, 7:1079–1105, 2006.

[Gabillon *et al.*, 2011] Victor Gabillon, Mohammad Ghavamzadeh, Alessandro Lazaric, and Sébastien Bubeck. Multi-bandit best arm identification. In *Advances in Neural Information Processing Systems*, pages 2222–2230, 2011.

[Gabillon *et al.*, 2012] Victor Gabillon, Mohammad Ghavamzadeh, and Alessandro Lazaric. Best arm identification: A unified approach to fixed budget and fixed confidence. In *Advances in Neural Information Processing Systems*, pages 3212–3220, 2012.

[Jamieson and Nowak, 2014] Kevin Jamieson and Robert Nowak. Best-arm identification algorithms for multi-armed bandits in the fixed confidence setting. In *Information Sciences and Systems (CISS), 2014 48th Annual Conference on*, pages 1–6. IEEE, 2014.

[Kalyanakrishnan *et al.*, 2012] Shivaram Kalyanakrishnan, Ambuj Tewari, Peter Auer, and Peter Stone. Pac subset selection in stochastic multi-armed bandits. In *Proceedings of the 29th International Conference on Machine Learning (ICML-12)*, pages 655–662, 2012.

[Lai and Robbins, 1985] Tze Leung Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. *Advances in applied mathematics*, 6(1):4–22, 1985.

[Liu and Tsuruoka, 2016] Yun-Ching Liu and Yoshimasa Tsuruoka. Modification of improved upper confidence bounds for regulating exploration in monte-carlo tree search. *Theoretical Computer Science*, 2016.

[Locatelli *et al.*, 2016] Andrea Locatelli, Maurilio Gutzeit, and Alexandra Carpentier. An optimal algorithm for the thresholding bandit problem. *arXiv preprint arXiv:1605.08671*, 2016.

[Robbins, 1952] Herbert Robbins. Some aspects of the sequential design of experiments. In *Herbert Robbins Selected Papers*, pages 169–177. Springer, 1952.

[Thompson, 1933] William R Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, pages 285–294, 1933.