The Irrationality of $\sqrt{2}$

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1 Introduction

In this short paper we will show that $\sqrt{2}$ is an irrational number. In Section 2 we will introduce the concepts of rational and irrational numbers. In Section 3,

we will provide a proof of the fact that $\sqrt{2}$ is irrational. Finally, in 4, we will discuss what we know about other numbers that are known to be irrational.

2 Preliminaries

First, we define the rational numbers.

Definition 1. A *rational number* is a real number that can be expressed in the form $\dfrac{m}{n}$ where m and n are integers, and $n \neq 0$.
Rational numbers such as 5 or 9.2 can be expressed as $\frac{10}{2}$ and $\frac{92}{10}$ respectively.

Definition 2. An *irrational number* is any real number that cannot be expressed as the quotient of two integers. Irrational numbers such as π can be expressed as an infinitely expanding decimal with no regularly repeating groups of digits.

3 The Proof

Theorem 3. *The real number $\sqrt{2}$ is irrational.*

Proof of Theorem 3.

We will prove the statement by contradiction. First, let $\alpha = \sqrt{2}$ and, for the sake of contradiction, we assume that α is a rational number in lowest terms. Meaning m and q are integer numbers and have no common factors other than 1. Then, we can write $\alpha = \frac{m}{n}$.

Where

$$\alpha^2 = \left(\frac{m}{n}\right)^2 = 2$$

Since $\left(\frac{m}{n}\right)^2 = 2$, by isolating m it is shown that

$$m^2 = 2n^2.$$

The expression $m^2 = 2n^2$ implies that the value of m^2 must be an even number since it is equal to 2 multiplied by some number q^2. Since m^2 is even, it is additionally implied that m is even and can be divided by 2.
The fact that m is even while not trivial, can be proven with the contrapositive. [3]

m can now be represented as

$$m = 2m'$$

where m' is some other whole number.

Substituting $m = 2m'$ into equation $m^2 = 2n^2$, it is observed that

$$2m^2 = n^2$$

Repeating this process now for equation $n^2 = 2m^2$ can make the observation that both m and n share a common factor of 2, thus implying a contradiction.

□

4 Irrational Numbers

Proof by contradiction is one of several ways to prove the irrationality of $\sqrt{2}$. For example, an additional analytic method used to prove $\sqrt{2}$'s irrationality is via $Bezout's\ Lemma$. [1].

The set of irrational numbers is a higher order of infinity to that of rational numbers, and among these infinite irrationals the irrational number π seems to be dealt with most in intermediate mathematics.
While it is possible to prove $\pi's$ irrationality via contradiction [2], it can be helpful to couple the proper proof with this visual (see here).The visualization gives an intuitive understanding of irrationality by giving a visual of infinite non-repeating decimal expansion. If π were rational there would be no infinite pattern, instead the same pattern would repeat over and over.

It is additionally worth mentioning that the above proof by contradiction method used to prove the irrationality of $\sqrt{2}$ will not work to prove the irrationality of π.
Proof by contradiction of $\sqrt{2}$ works because $\sqrt{2}$ is a solution of a quadratic polynomial for example $x^2 - 2 = 0$. However, π is transcendental[4], meaning that there is no polynomial equation that it can satisfy.

References

[1] Alexander Bogomolny. “Cut-The-Knot”. Published 2020.
Link to source: (see here)

[2] Wikipedia. ”Irrationality of pi”. Last Modified 21 June 2025.
Link to source: (see here)

[3] Chili Math. ”Prove: Suppose n is an integer. If n^2 is even, then n is even.”. Date of access 10 September 2025.
Link to source: (see here)

[4] Nasa.gov. ”The Transcendentality of π.” Date of access 10 September 2025.
Link to source: (see here)