ORDER and CHAOS Assignment

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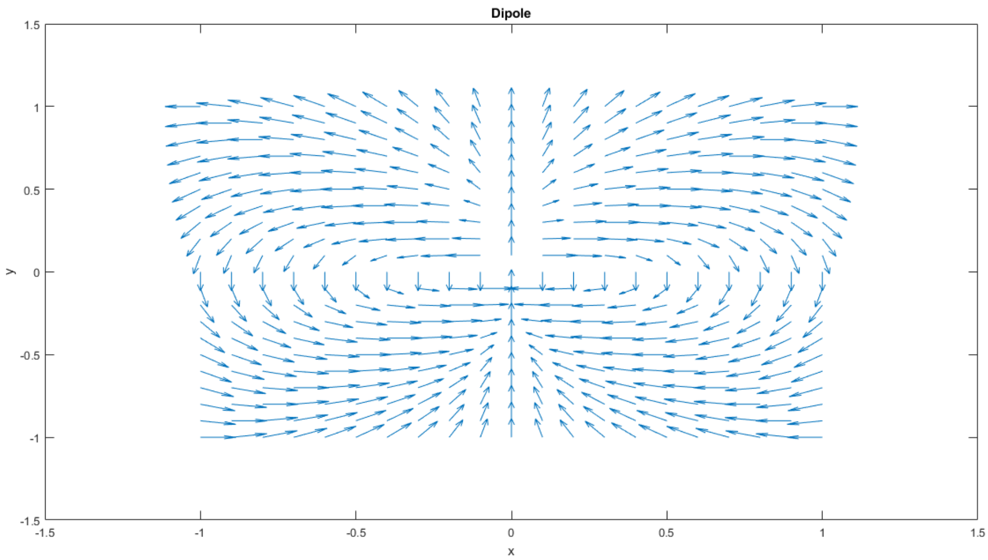
All Plots have been made in Matlab

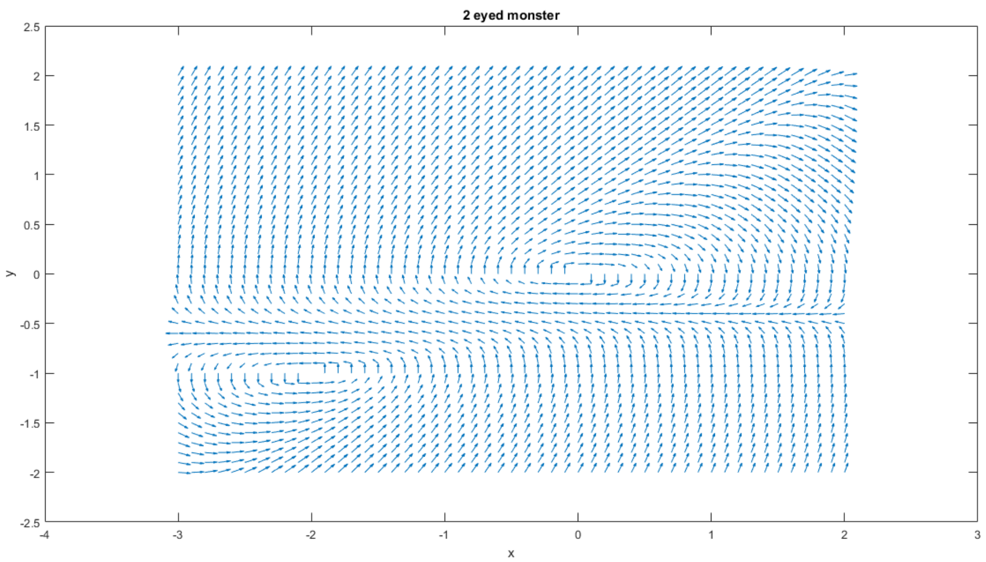
6.1.9

Dipole Fixed Point Plot. All vectors have been normalized for better clarity



6.1.10

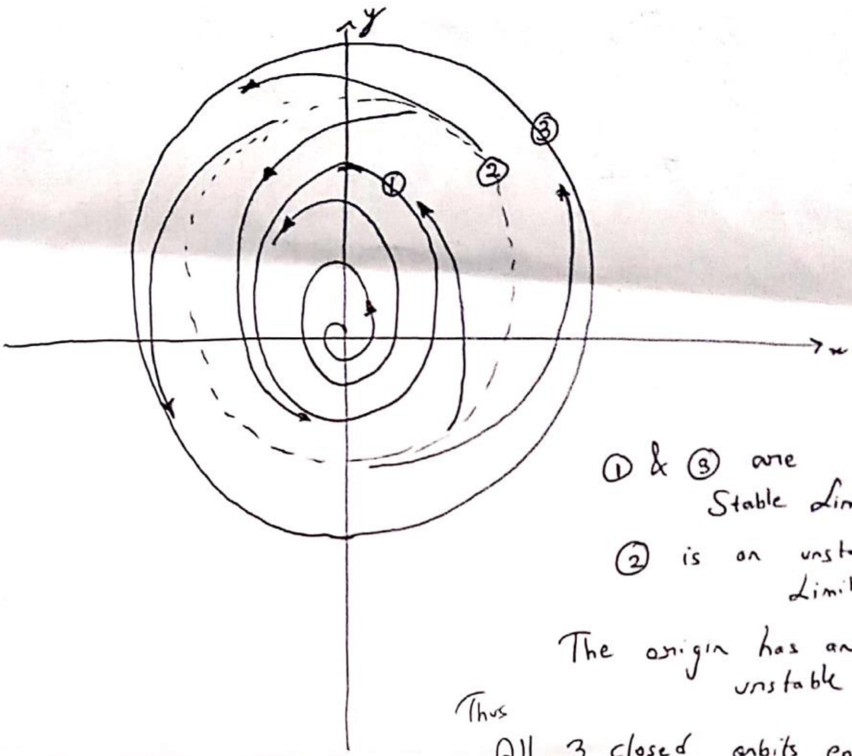
Two eyed monster



6.1.10 and 6.3.1

① H-W Assignment

6.1.13



① & ③ are Stable Limit Cycles

② is an unstable Limit Cycle

The origin has an unstable spiral

Thus

All 3 closed orbits enclose a fixed pt with index +1.

6.3.1

$\dot{x} = x - y$

$\dot{y} = x^2 - 4$

fixed pts $x = y$ & $x = \pm 2$

$\Rightarrow$ fixed pts $\longrightarrow$ $(2,2) \mid (-2,-2)$

Linearising near fixed points, the Jacobian is

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$$

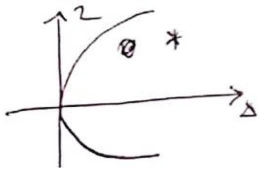
$$J_{2,2} = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$$

$\tau = 1$

$\Delta = 4$

$\tau^2 - 4\Delta = 1 - 16 = -15$

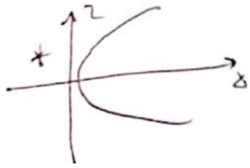
$\Rightarrow$ Unstable Spiral



$$J_{-2,-2} = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$$

$\tau = 1$ $\rightarrow$ Saddle point

$\Delta = -4$

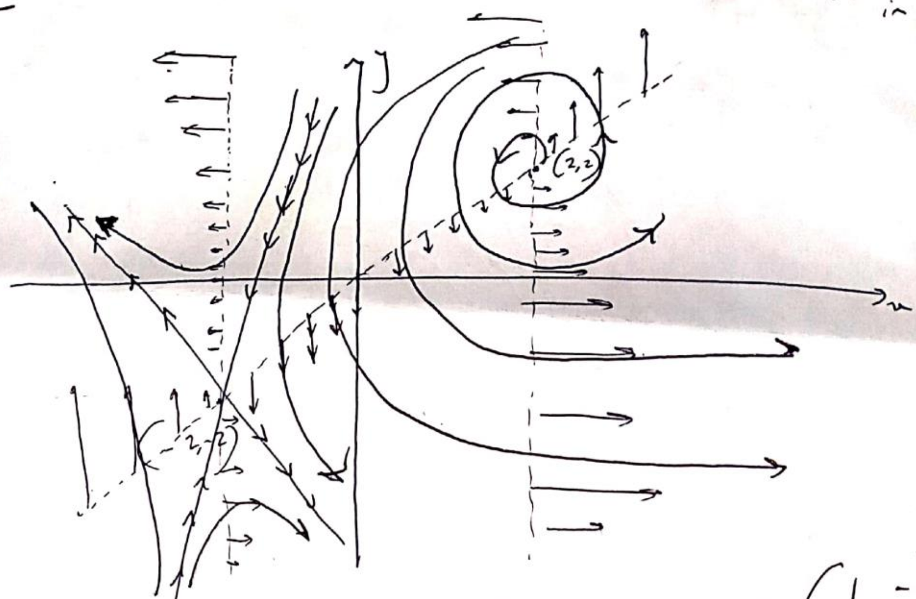


6.3.1

②

6.3.1

(Null clines in dotted lines)



for $(-3\ -2)$

$$\begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$$

To find eigenvectors → $\det\begin{pmatrix} 1-\lambda & -1 \\ -4 & -\lambda \end{pmatrix} = 0$

$$\Rightarrow (1-\lambda)(-\lambda) - 4 = 0$$

$$\Rightarrow -\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - \lambda - 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+16}}{2}$$

$$\lambda = \frac{1 \pm \sqrt{17}}{2}$$

for $\left(\lambda = \frac{1+\sqrt{17}}{2}\right)$ $\begin{pmatrix} 1-\lambda & -1 \\ -4 & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-4a - b\lambda = 0$$

$$\boxed{b = -\frac{4}{\lambda} a} \qquad \begin{pmatrix} 1 \\ -4/\lambda \end{pmatrix}$$

for $\lambda = \frac{1+\sqrt{17}}{2}$, eigenvec $\approx \begin{pmatrix} 1 \\ -\frac{8}{1+\sqrt{17}} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -1.6 \end{pmatrix}$

$\lambda > 0$ ⇒ growth

for $\lambda = \frac{1-\sqrt{17}}{2}$ eigenvec $\approx \begin{pmatrix} 1 \\ -\frac{8}{1-\sqrt{17}} \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2.5616 \end{pmatrix}$

$\lambda < 0$

⇒ decay

6.3.9 and 6.8.5

③

6.3.9

(a) $\dot{x} = y^3 - 4x$

$\dot{y} = y^3 - y - 3x$

fixed pts

$y^3 = 4x$ & $y^3 - y - 3x = 0$

$y^3 - y - 3\left(\frac{y^3}{4}\right) = 0$

$\Rightarrow \frac{y^3}{4} - y = 0$

$\Rightarrow y\left(\frac{y^2}{4} - 1\right) = 0$

$\Rightarrow y\left(\frac{(y-2)(y+2)}{4}\right) = 0$

for $y = 0$, $x = 0$

for $y = 2$, $x = 2$

for $y = -2$, $x = -2$

$\Rightarrow$ fixed pts

$(0,0)$ | $(2,2)$ | $(-2,-2)$

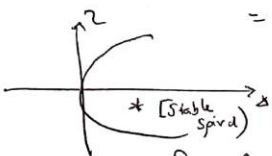
$$J = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}$$

$$J_{(0,0)} = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$$

$\tau = -5$, $\Delta = 4$

$\tau^2 - 4\Delta = 25 - 16$

$= 9$



$(0,0)$ Stable Spiral

$$J_{(2,2)} = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \quad J_{(-2,-2)} = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$$

for $(2,2)$ & $(-2,-2)$

$\tau = 7$, $\Delta = -44 + 36$

$= -8$

$\Rightarrow$ Saddle point.

$(2,2)$ & $(-2,-2)$ are saddle points

(b) On the line $x = y$,

$\dot{x} = x^3 - 4x$

$\dot{y} = x^3 - 4x$

$\Rightarrow$ $\dot{x}$ & $\dot{y}$ have the same value & they point along

X dot and y dot have same value and they point along the same direction. Hence x=y is an invariant line. All vectors lie along the line itself

along the line itself.

(c) Let $z = x - y$

$\Rightarrow \dot{z} = \dot{x} - \dot{y} = (\dot{x-y}) = +y - x$

$\Rightarrow \boxed{(\dot{x-y}) = -(x-y)}$ $\Rightarrow \dot{z} = -z$

6.3.4

$\dot{z} = -z$

This indicates a decaying trajectory for any value of z. ~~the~~ Starting at any point we will decay to the origin

$\Rightarrow z = 0 \Rightarrow x(t) \to y(t)$ as $t \to \infty$ from any starting pt.

6.8.5

To find index of fixed pts

$\dot{x} = xy$

$\dot{y} = x + y$

fixed pt $\to (0,0)$

~~Jacobian~~ $\cancel{J = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}}$

$\dot{x} = 0$ on y & x-axis

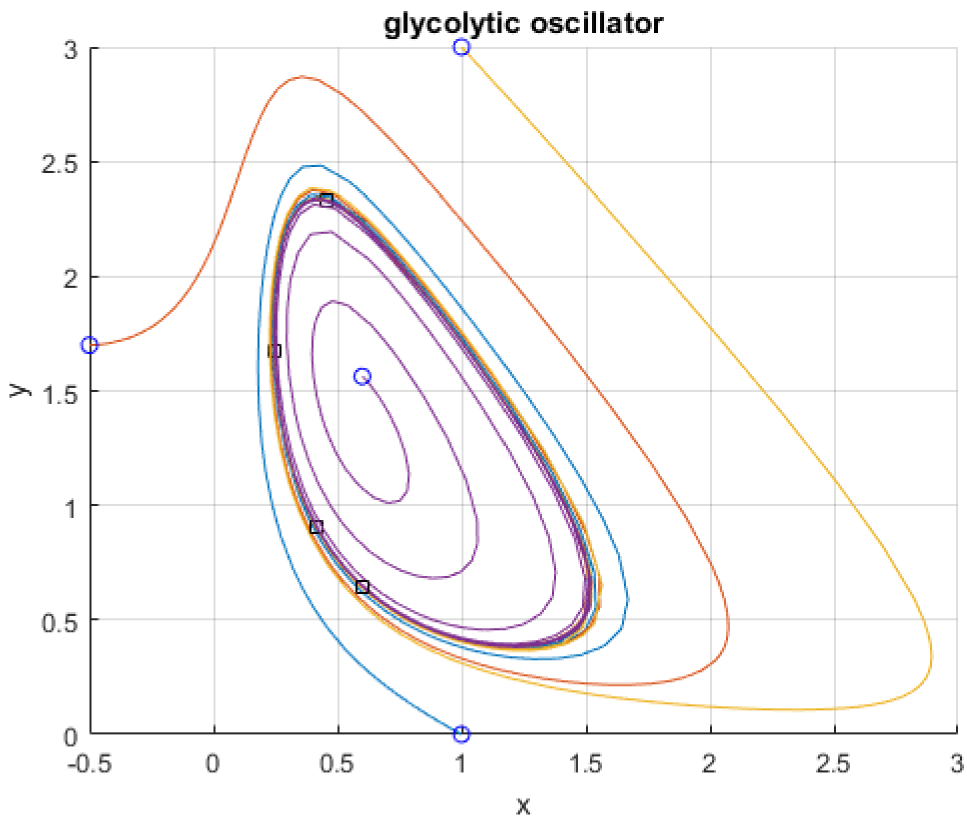
& $\dot{y} = 0$ on $x = -y$

$\Rightarrow \underline{\left(\dot{x} = -x^2\right)}$ on $x = -y$

Along the circle, the vectors first move clockwise then anticlockwise, $\Rightarrow$ (Index = 0)

Homework on Glycolytic Oscillator a=0.08 b = 0.6

In the plot below the blue circles indicate starting points and boxed indicate end points



Time evolution of Volume in Phase Space and stability of origin in Lorenz equations

~~Chaos- Ch- 9~~

~~H. W~~

~~From Continuity,~~ $\cancel{\frac{\partial c}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$

~~c is the concentration of the phase components~~

~~$\vec{J}$ indicates the flow incoming into the volume~~

H.W. Chap- 9

Considering a volume in phase plane,

$\dot{x} = f_1(x,y,z)$
$\dot{y} = f_2(x,y,z)$
$\dot{z} = f_3(x,y,z)$

~~Let volume be dx dy~~ dz

In the eq^n $\dot{\bar{x}} = \bar{f}(\bar{x})$

$\bar{f}$ indicates the instantaneous velocity of the components.

Let an infinitesimal volume be $\delta V = \delta x\, \delta y\, \delta z$

$\Rightarrow$ $\cancel{\frac{d\delta V}{dt} = \frac{d\delta x}{\delta t}\delta y\,\delta z + \frac{d\delta y}{\delta t}\delta x\,\delta z}$

$$\Rightarrow \frac{d}{dt}(\delta V) = \frac{d}{dt}(\delta x)\,\delta y\,\delta z + \frac{d}{dt}(\delta y)\,\delta x\,\delta z + \frac{d}{dt}(\delta z)\,\delta x\,\delta y$$

$$= f_x\,\delta y\,\delta z + f_y\,\delta x\,\delta z + f_z\,\delta x\,\delta y$$

$$\frac{d(\delta V)}{dt} = \vec{f}\cdot(\hat{n}\,dA) \Rightarrow \boxed{\dot{V} = \int \vec{f}\cdot\hat{n}\,dA = \int \nabla\cdot\vec{f}\,dV}$$

From divergence theorem

H.W

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-\sigma & \sigma\\ r & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$\tau = -(\sigma+1) < 0$ for origin fixed pt

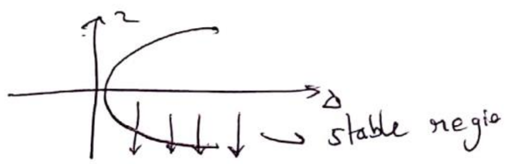
$\Delta = \sigma(1-r)$

~~for~~ if $r<1$

$\Rightarrow (1-r) > 0$

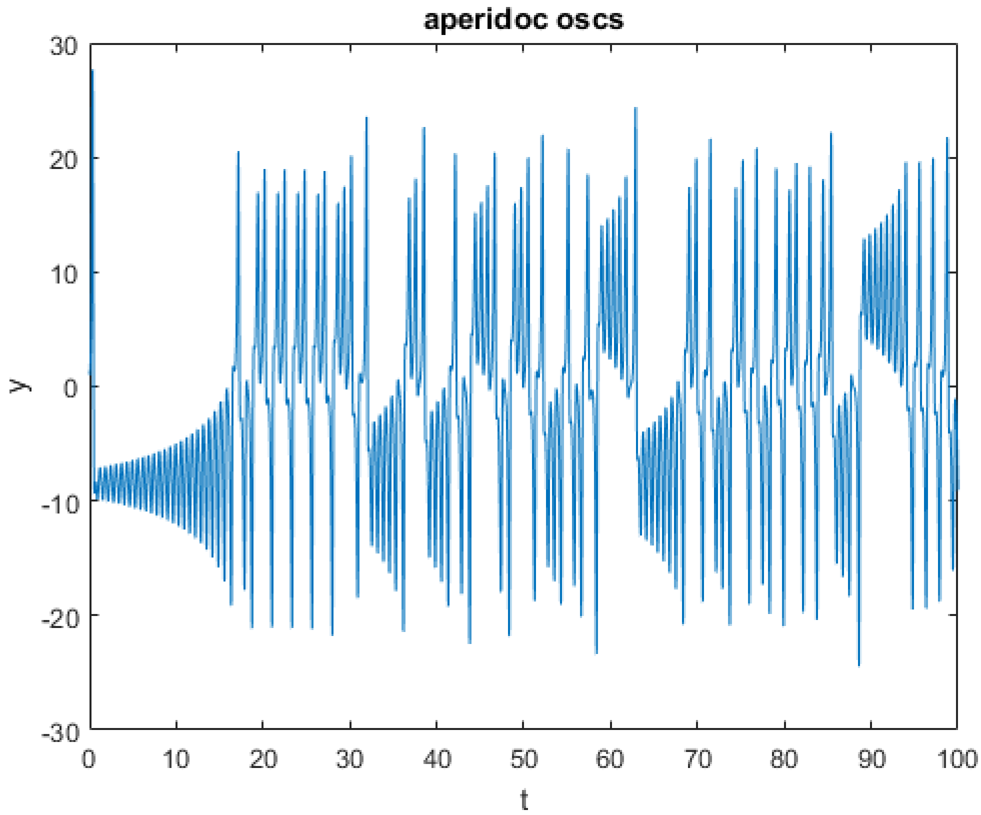
$\Rightarrow \Delta > 0$ $\Rightarrow$ for $r<1$ the origin is globally stable (spiral / node)

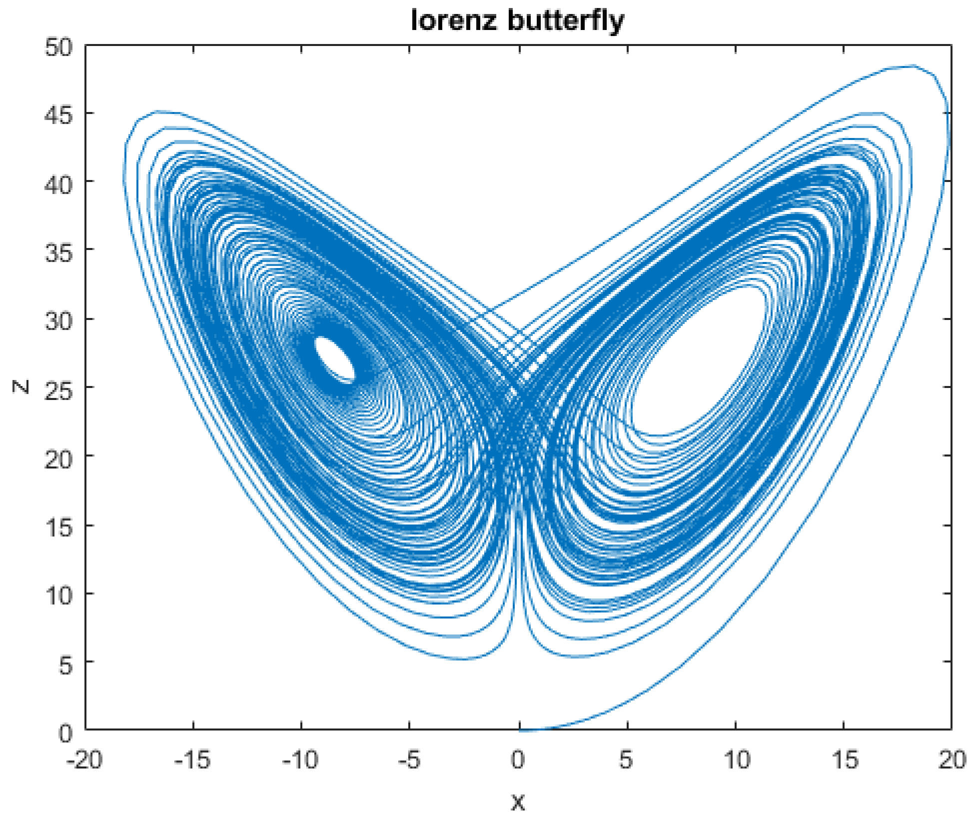
$\Rightarrow$



Chaos on Strange attractor (fig 9.3.1 and Butterfly pattern)

Fig 9.3.1

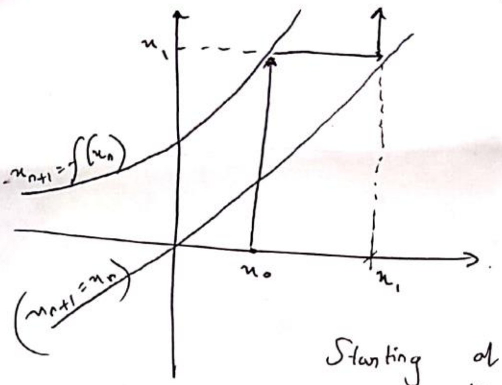




Homework on Cobweb construction Fig 10.1.1 and Fig 10.1.2

Cobweb Homework.

A cobweb is a useful scheme for geometrically iterating a map of the form $x_{n+1} = f(x_n)$



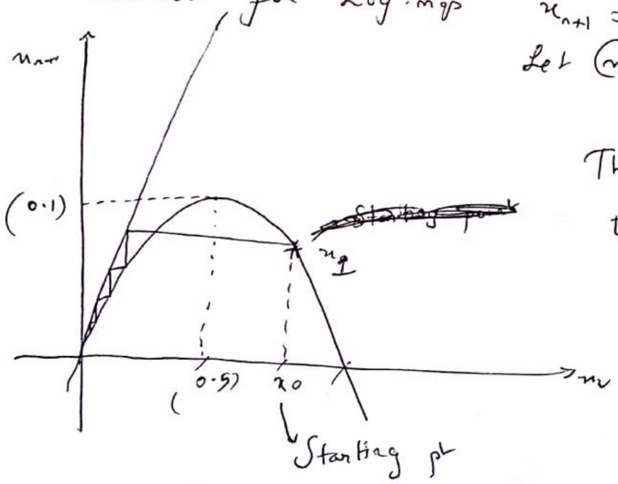
Starting at pt x_0, we go to x_1 vertically following the curve $x_1 = f(x_0)$

Now we go horizontally till we touch the line $(x_{n+1} = x_n)$. Thus we reach the pt (x_1, x_1) Again we move vertically to hit the curve at (x_2). This process is the cobweb constructn of geometrically iterating a 1-D map

Logistic - map

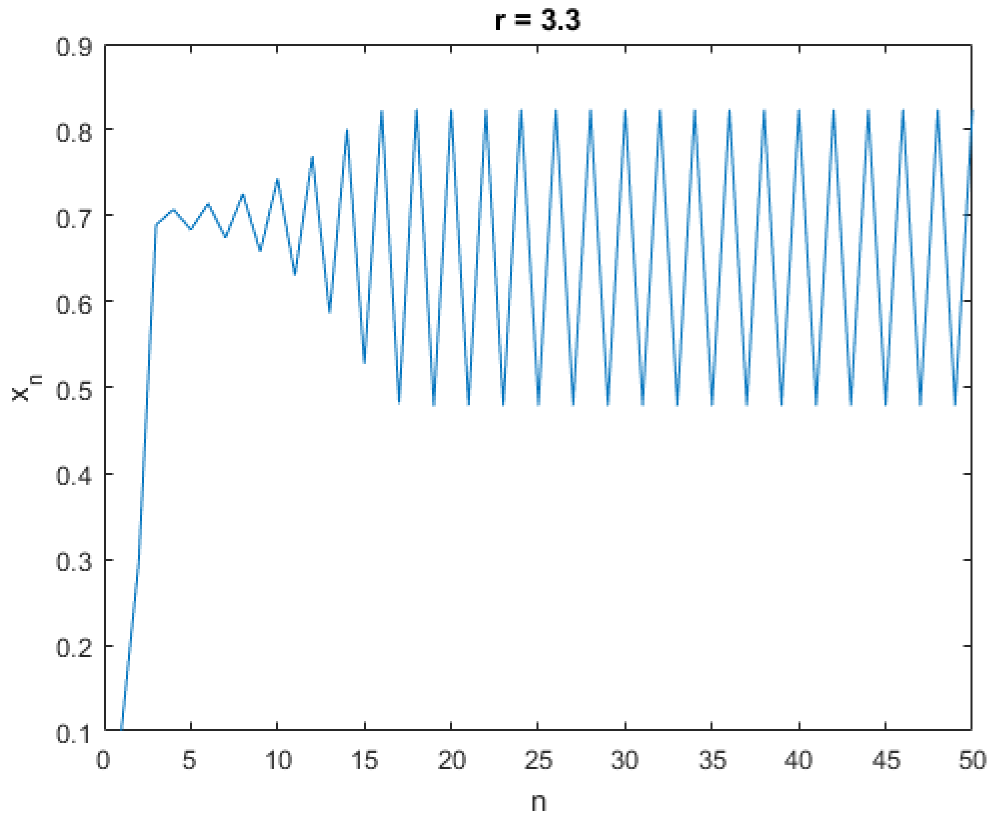
Cobweb for Log. map $\quad x_{n+1} = r\, x_n(1-x_n)$

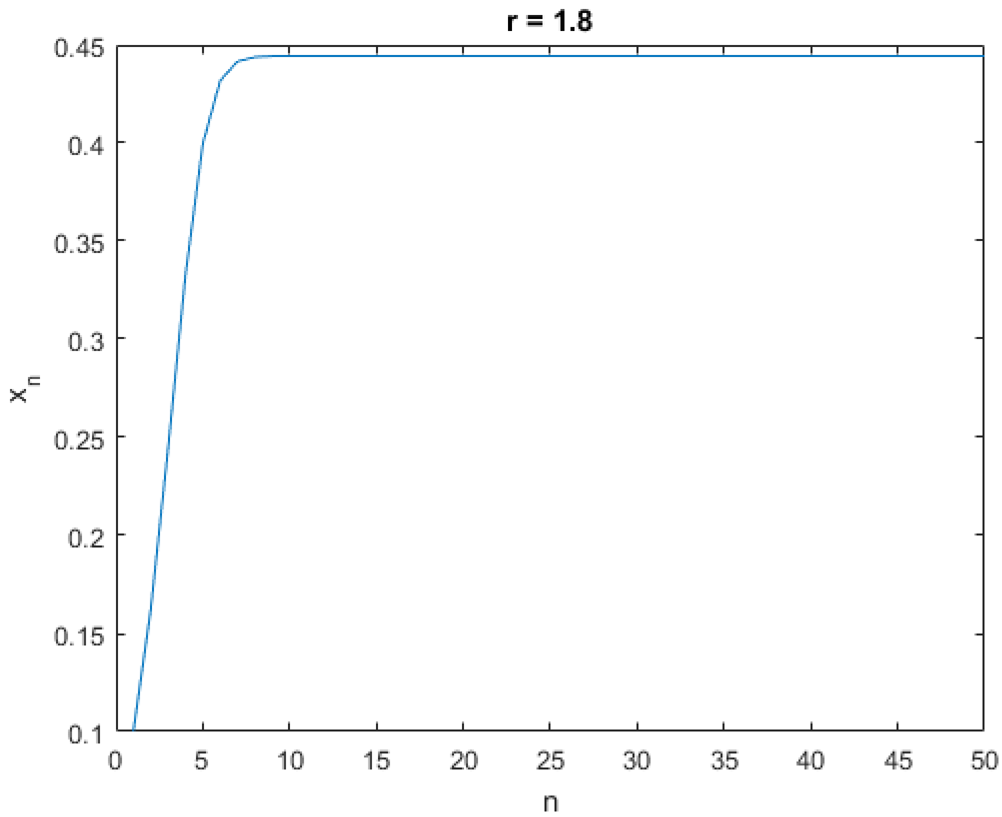
Let $(r = 0.4)$

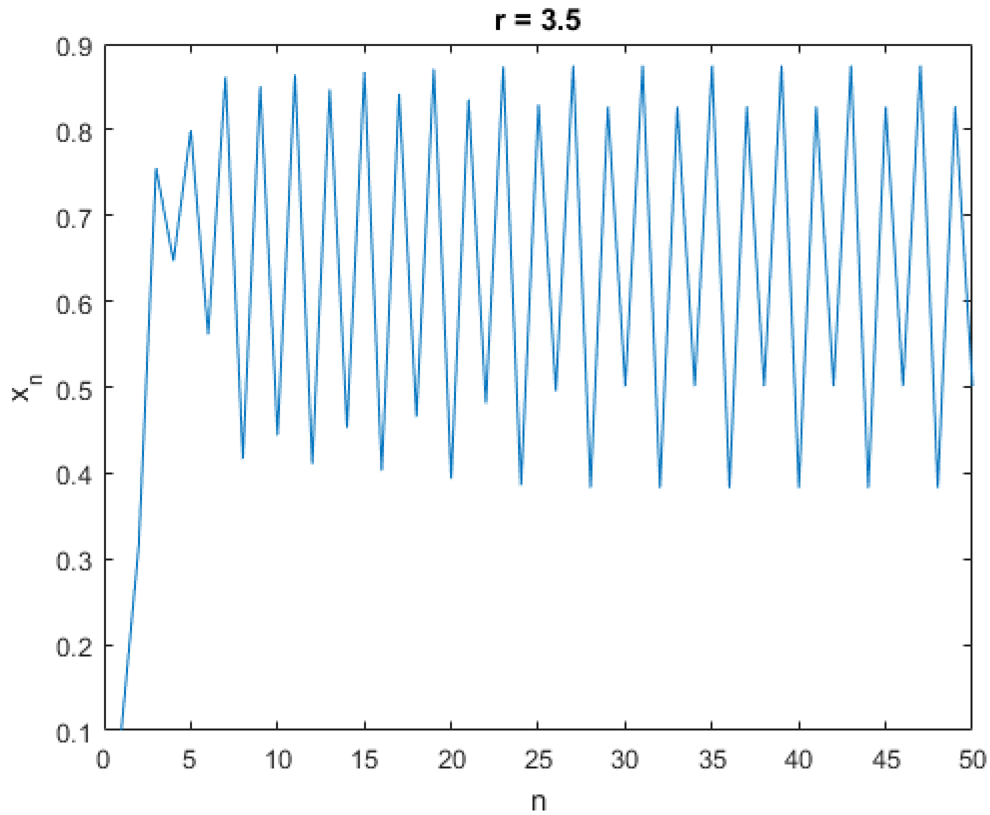


The cobweb indicates for the given 'r' value $x_n \to 0$ as $n \to \infty$ from any arbitrary starting point

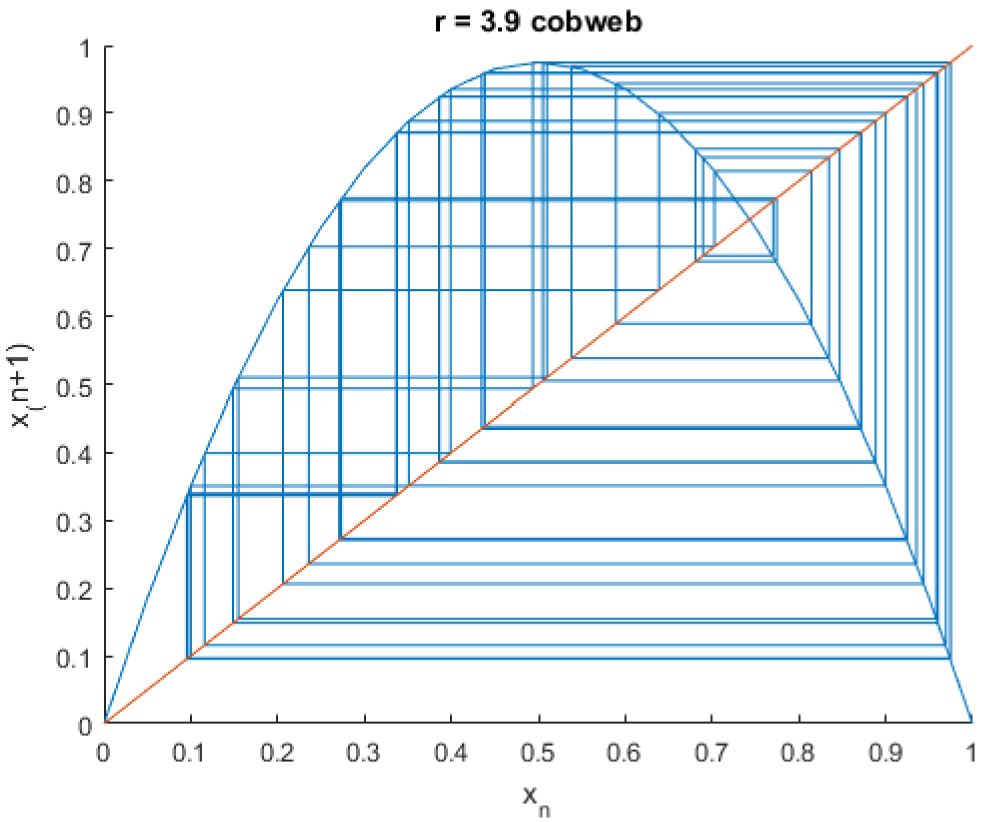
Plots on the Logistic maps for various values of r as indicated in plots

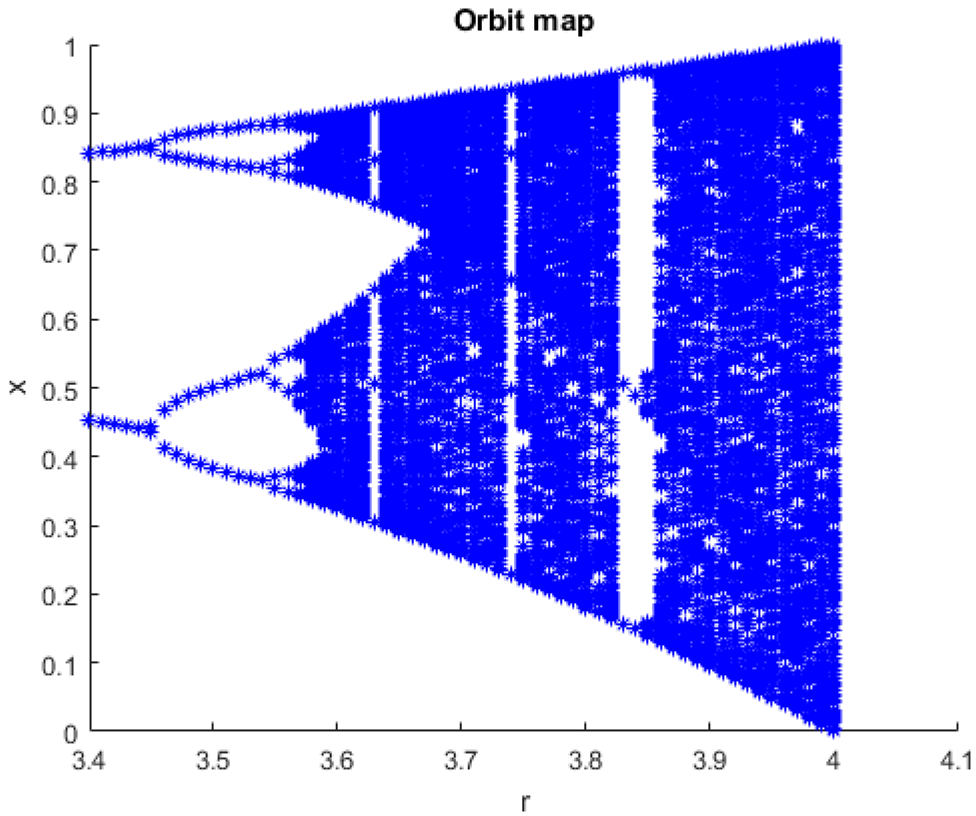






Homework on Cobweb construction Fig 10.2.6 and 10.2.7





Orbit map
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x
3.4
3.5
3.6
3.7
3.8
3.9
4
4.1
r

Homework on values of r for 2 cycle iterations of Xn

 $x = (1 - \frac{1}{r})$

Ex- 10.3.2

We need to find sol^ns of the 4th order equations, $f(f(x)) = x$ →

$$f(x) = rx(1-x)$$

$$\Rightarrow f(f(x)) = r[rx(1-x)][1-(rx(1-x))]$$

Eq^n :- $f(f(x)) - x = 0$

$$\Rightarrow r^2 \cancel{x}(1-x)[1 - rx(1-x)] - \cancel{x} = 0$$ (Since x is a root)

$$\Rightarrow r^2(1-x) - r^3 x(1-x)^2 - 1 = 0$$

$x = 1 - \frac{1}{r}$ is another root we need to factor it out

Let $(y = 1 - x - \frac{1}{r})$

We can rewrite the eq^n as:-

$$r^2\left(y + \frac{1}{r}\right) - r^3\left(-y + 1 - \frac{1}{r}\right)\left(y + \frac{1}{r}\right)^2 - 1 = 0$$

$$\Rightarrow r^2 y + \cancel{r} - r^3\left[-y^3 - \frac{2y^2}{r} - \frac{y}{r^2} + y^2 + \frac{2y}{r} + \cancel{\frac{1}{r^2}} - \frac{y^2}{r} - \frac{2y}{r^2} - \cancel{\frac{1}{r^3}}\right] \cancel{-1} = 0$$

$$\Rightarrow r^2 y + (ry)^3 + 2(yr)^2 + (ry) - r^3y^2 - 2r^2y + (ry)^2 + 2ry$$

$$\Rightarrow -r^2 y + 3(ry)^2 + 3(ry) - r^3y^2 + (ry)^3 = 0$$

Factoring out y

$$\Rightarrow -r^2 + r^2 y + r - r^3 y - r^3 y^2 = 0$$

$$- r^2 - r = 0$$

$$\Rightarrow -r^3 y^2 + y(r^2 - r^3)$$

$$\Rightarrow -r^3\left[\left(1-\frac{1}{r}\right) - x\right]^2 + \left[1 - \frac{1}{r} - x\right]\left[r^2 - r^3\right] - r^2 - r = 0$$

$$\Rightarrow -r^3\left[\left(1-\frac{1}{r}\right)^2 + x^2 - 2x\left(1-\frac{1}{r}\right)\right] + r^2 - r^3 - r + \cancel{r^2} - xr^2 + xr^3 + \cancel{r^2} - r = 0$$

$$\Rightarrow -r^3\left(1 + \frac{1}{r^2} - \frac{2}{r}\right) - r^3x^2 + 2x(r^3 - r^2) + r^2 - r^3 - 2r + r^3x - r^2x = 0$$

Factoring out y

$$r^3y^2 + y\left[3r^2 - r^3\right] + 3r - r^2 = 0$$

$$r^3\left[\left(1-\frac{1}{r}\right) - x\right]^2 + (ry+1)\left(3r-r^2\right) = 0$$

$$\Rightarrow r^3\left[\left(1-\frac{1}{r}\right)^2 + x^2 - 2x\left(1-\frac{1}{r}\right)\right] + \left[r\left(1-x-\frac{1}{r}\right)+1\right]\left[3r-r^2\right] = 0$$

$$\Rightarrow r^3\left[1-\frac{2}{r}+\frac{1}{r^2}+x^2-2x+\frac{2x}{r}\right] + \left[r(1-x)\right]\left[3r-r^2\right] = 0$$

$$\Rightarrow \cancel{r^3} - 2r^2 + r + r^3x^2 - 2xr^3 + 2xr^2 + 3r^2 - \cancel{r^3} - 3r^2x + r^3x = 0$$

$$\Rightarrow r^3x^2 + x\left[-r^3 - r^2\right] + r^2 + r = 0$$

$$\Rightarrow r^2x^2 - x\left(r^2+r\right) + r+1 = 0$$

$$\Rightarrow x = \frac{\left(r^2+r\right) \pm \sqrt{\left(r^2+r\right)^2 - 4r^2(r+1)}}{2r^2}$$

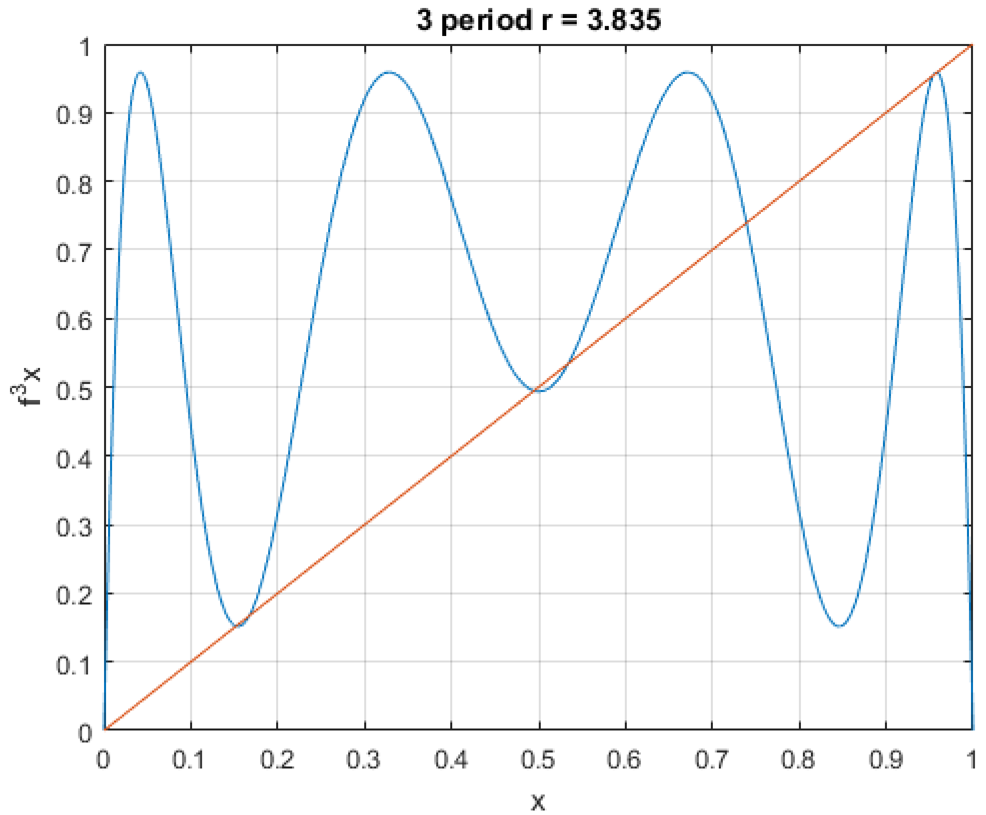
$$= \frac{\cancel{r}\left[(1+r) \pm r\sqrt{(1+r)^2 - 4(1+r)}\right]}{2r^{\cancel{2}}}$$

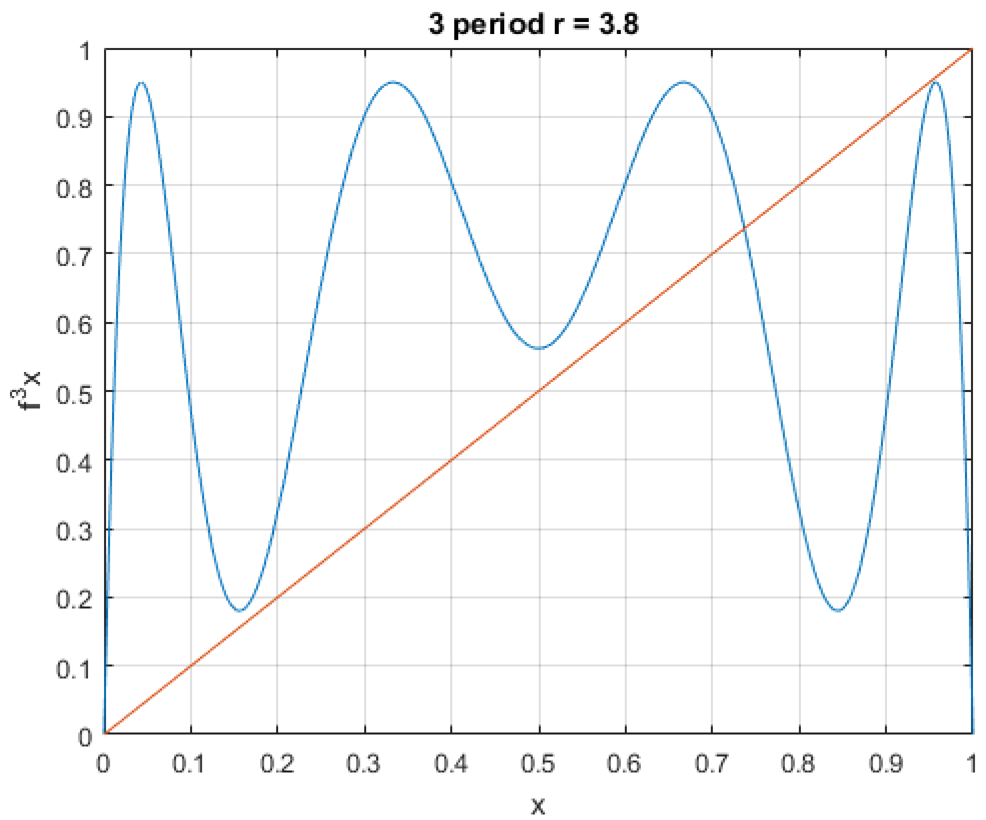
$$\Rightarrow x = (p/q) = \frac{1+r \pm \sqrt{(r-3)(r+1)}}{2r}$$

Thus logistic map has 2-cycle for $r>3$

Homework on 3 period window

The point of tangential bifurcation lies somewhere between r = 3.8 and 3.8.5 as indicated in plots





Homework on Liapunov exponent for Tent map

Liapunov Exponent

$$f^n(x) = \underbrace{f(f(f\cdots(f(x)))\cdots)}_{n \text{ times operated.}}$$

We know $f^n(x_0) = x_n$

$$\frac{d}{dx} f^n(x)\Big|_{x=x_0} = \frac{d}{dx} f(f^{n-1}(x))\Big|_{x=x_0} \frac{d}{dx}\left(f(f^{n-2}(x))\right)\Big|_{x=x_0} \cdots\cdots \frac{d}{dx} f(x)\Big|_{x=x_0}$$

$$= \frac{d}{dx} f(x_{n-1}) \times \frac{d}{dx}\left(f(x_{n-2})\right) \cdots\cdots \frac{d}{dx}\left(f(x_0)\right)$$

$$\Rightarrow \frac{d}{dx} f^n(x)\Big|_{x=x_0} = \prod_{i=0}^{n-1} f'(x_i)$$

H.W.

$$f(x) = \begin{cases} rx, & 0 \leq x \leq 1/2 \\ r - rx, & \frac{1}{2} < x \leq 1 \end{cases}$$

for $0 \leq r \leq 2$ & $0 \leq x \leq 1$

$$\lambda = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right\}$$

$|f'(x)| = r$ for any value of (x)

$$\Rightarrow \lambda = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln r \right] = \ln r$$

(ii) Logistic Map :- $f(x) = rx(1-x)$

$$f'(x) = r - 2rx$$

Result in plots :

$$\lambda = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |r - 2rx_i| \right\}$$

Homework on Liapunov exponent for Logistic Map

R lies between 3 and 4