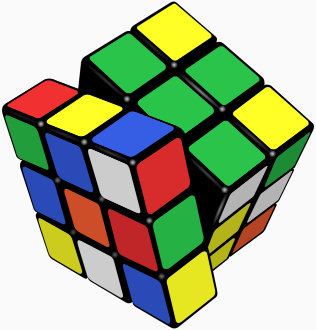
Minimum bases in permutation groups

Lawrence Chen

October 22, 2022

Honours presentation



Contents

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One answer: using permutations and *computational group theory*!

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One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

Definition (permutation)

Permutation of Ω is bijection $\sigma : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

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Cycle notation: $\sigma = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \sigma & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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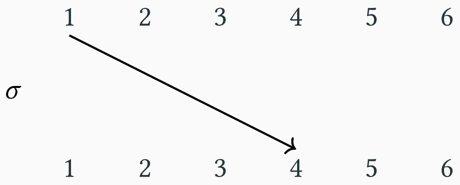

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$$\begin{tikzcd}
{} \arrow[d, phantom, "\sigma"] & 1 \arrow[drrr] & 2 & 3 & 4 \arrow[dr] & 5 & 6 \\
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It means $1^\sigma = 4,\ 4^\sigma = 5,$

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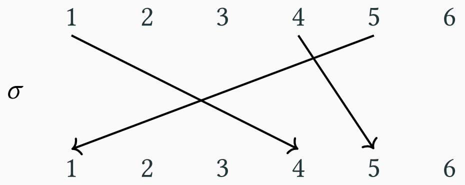

It means $1^\sigma = 4,\ 4^\sigma = 5,\ 5^\sigma = 1,$

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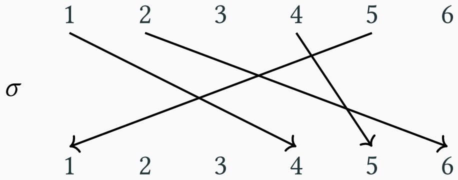

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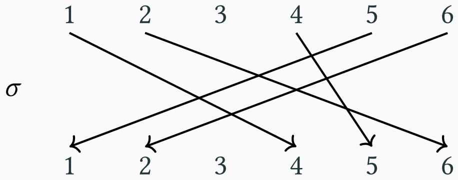

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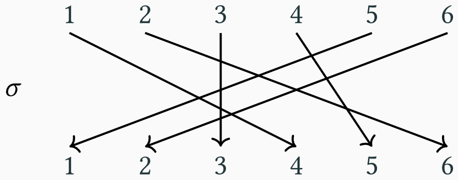

It means $1^\sigma = 4,\ 4^\sigma = 5,\ 5^\sigma = 1,\ 2^\sigma = 6,\ 6^\sigma = 2,\ 3^\sigma = 3.$

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Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means "first σ, then τ", so $i^{\sigma\tau} = (i^{\sigma})^{\tau}$.

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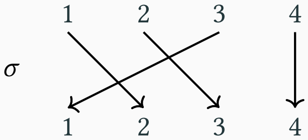

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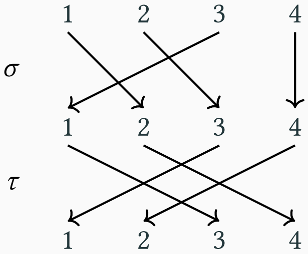

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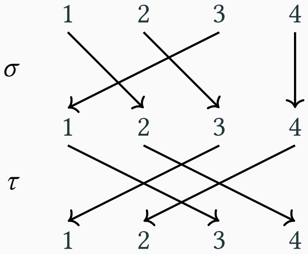

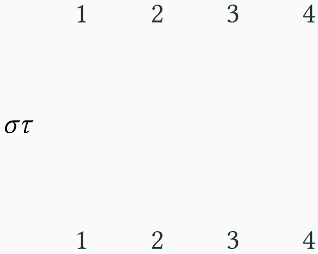

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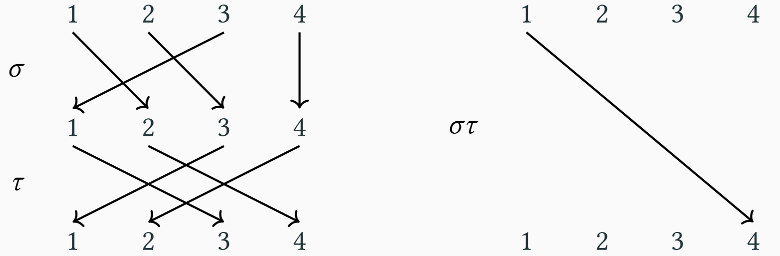

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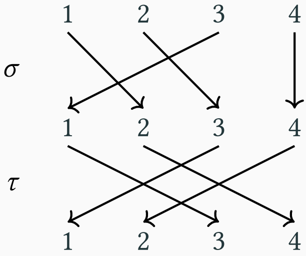

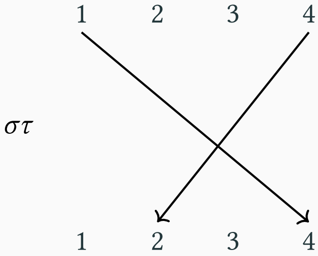

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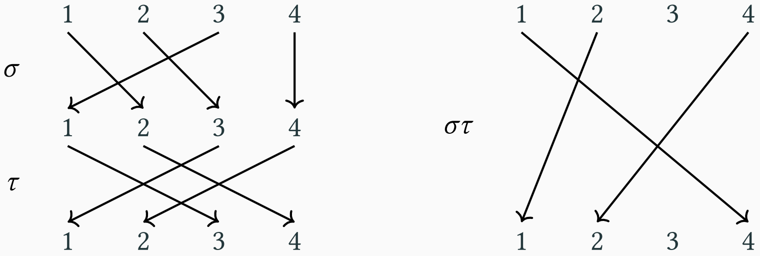

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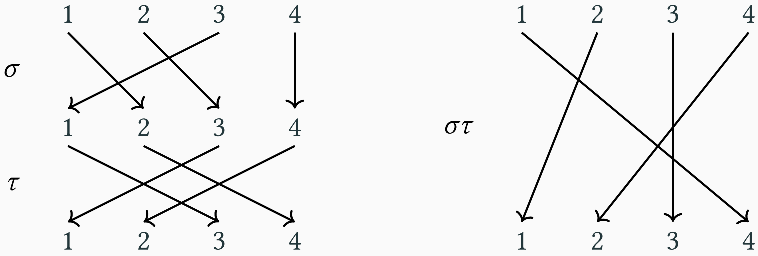

$$\sigma\tau = (1, 2, 3)(1, 3)(2, 4) = (1, 4, 2) \in \mathrm{Sym}(4).$$

Note: here, $\sigma\tau \neq \tau\sigma$, since $1^{\sigma\tau} = 4$ but $1^{\tau\sigma} = (1^\tau)^\sigma = 3^\sigma = 1$. Identity $1 = ()$ satisfies $1\sigma = \sigma 1 = \sigma$ for $\sigma \in \mathrm{Sym}(n)$.

Permutation groups

Note: for $g, h, k \in \mathrm{Sym}(\Omega)$, (i) $gh \in \mathrm{Sym}(\Omega)$, (ii) $1 = () \in \mathrm{Sym}(\Omega)$, (iii) $g^{-1} \in \mathrm{Sym}(\Omega)$, (iv) $(gh)k = g(hk)$. If true for subset:

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Alternating group $\mathrm{Alt}(3) = \{(), (1,2,3), (1,3,2)\} < \mathrm{Sym}(3)$.

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Example (alternating group)

Alternating group $\mathrm{Alt}(3) = \{(), (1,2,3), (1,3,2)\} < \mathrm{Sym}(3)$.
In general, $\mathrm{Alt}(n)$ is all *even* permutations of $[n]$ (product of even # of *transpositions* (i, j), e.g. $(1,2,3) = (1,2)(1,3) \in \mathrm{Sym}(n)$).

Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**; write $G = \langle X \rangle$.

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Example (symmetric group)

Consider $\mathrm{Sym}(3) = \{(), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$.
Not cyclic, but $\mathrm{Sym}(3) = \langle (1,2), (2,3) \rangle$ (adjacent swaps).
Also, $\mathrm{Sym}(3) = \langle (1,2), (1,2,3) \rangle$, e.g. $(2,3) = (1,2,3)(1,2)$.

Group actions

Definition (group action)

For (perm) group G and set $\Omega \neq \emptyset$, a G-**action** is map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$. **Degree** of action is $|\Omega|$.

Idea: $\alpha \in \Omega$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

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$G \leq \mathrm{Sym}(\Omega)$ acts on Ω by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

Example (right regular action)

Perm group G acts on $\Omega = G$ (itself) via $\alpha^g := \alpha g$ for $\alpha, g \in G$. (*Check:* $\alpha^1 = \alpha 1 = \alpha$ and $\alpha^{gh} = \alpha(gh) = (\alpha g)h = (\alpha^g)^h$.)

Example (dihedral group)

Let $r = (1, 2, 3, 4), s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** is $D_8 := \langle r, s\rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, *"symmetries of square"*.

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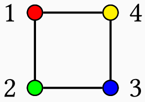

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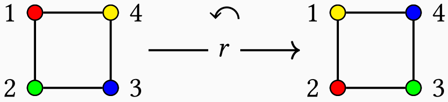

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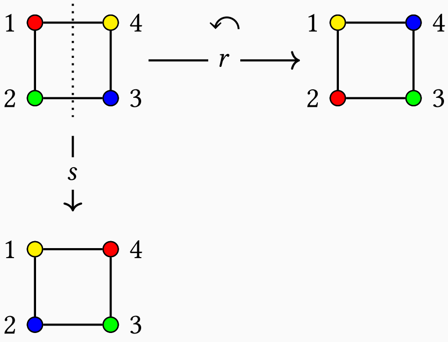

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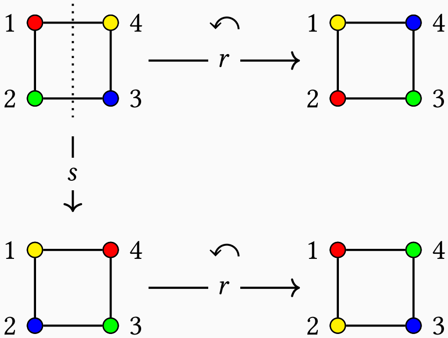

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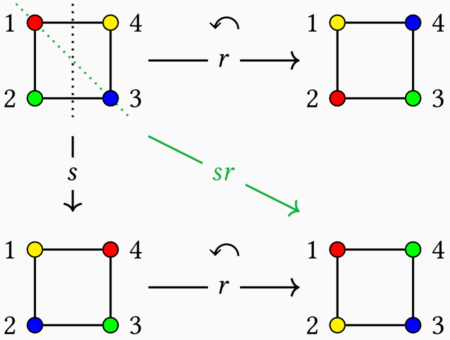

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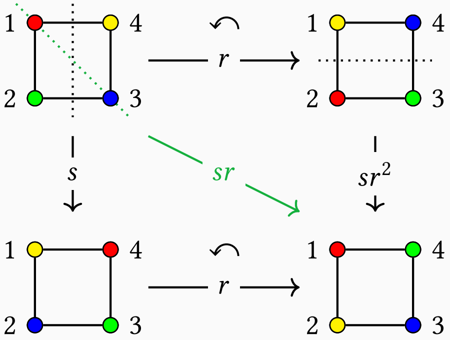

Orbits and stabilisers

Definition (orbit)

If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

Idea: states $\alpha^g \in \Omega$ reachable from fixed $\alpha \in \Omega$ by moves $g \in G$.

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One orbit only: **transitive** action.

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Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in \Omega$, $|\alpha^G||G_\alpha| = |G|$.

Blocks and primitivity

Definition (block)

If G acts transitively on Ω and $\Delta \subseteq \Omega$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \Omega$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

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For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions Ω); then G acts on Σ; if Δ is *maximal*, then acts primitively.

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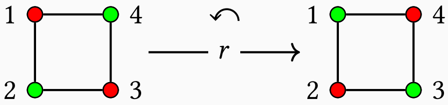
Block is $\Delta = \{1,3\}$ (nontrivial) with block system $\Sigma = \{\{1,3\},\{2,4\}\}$ (opposite vertices stay opposite):



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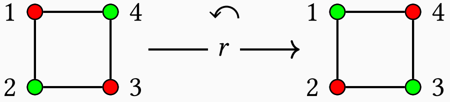


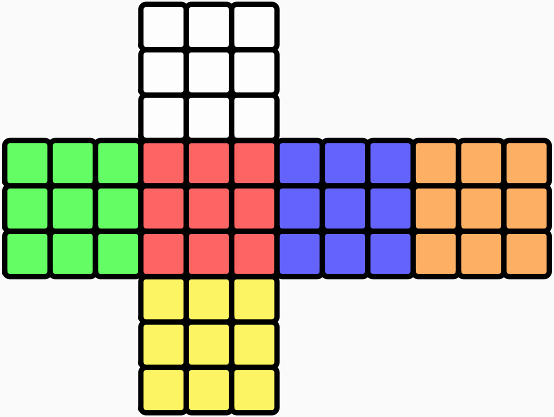
e.g. $\Delta^r = \{2,4\}$, $\Delta^s = \{4,2\}$, $\Delta^{sr} = \{1,3\} = \Delta$.

D_8 acts imprimitively on $[4]$ but primitively on Σ (degree 2).

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.



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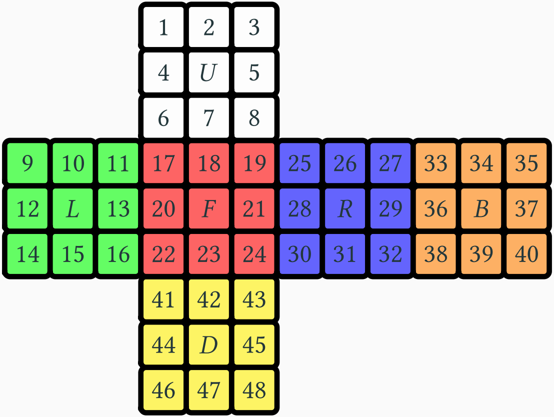
In **solved state** 1, label stickers (except each centre) using $[48]$:



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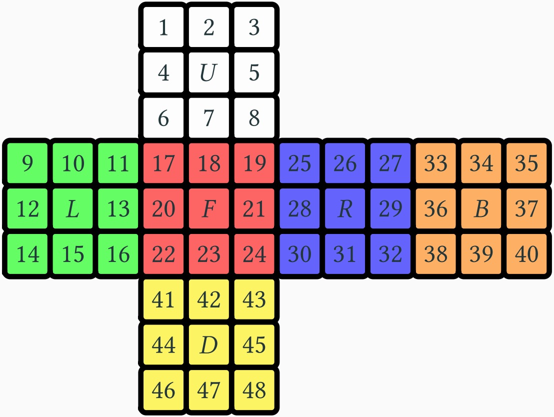

6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

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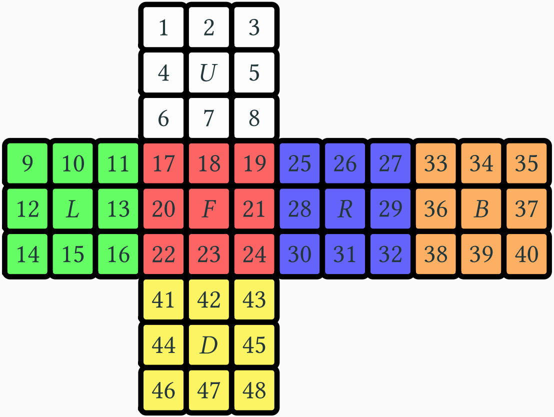
From *solved state* 1, consider F which rotates front face clockwise:



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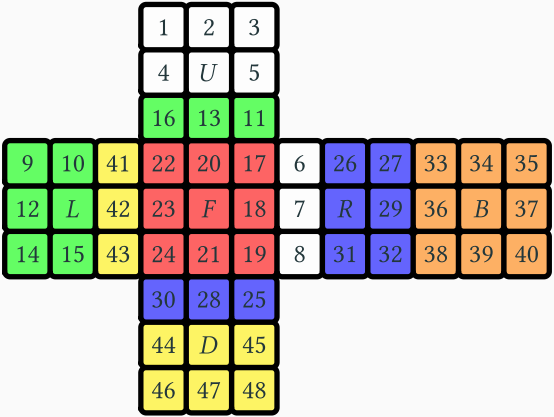

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels [48]:

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- $L = (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$
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Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as `G`):

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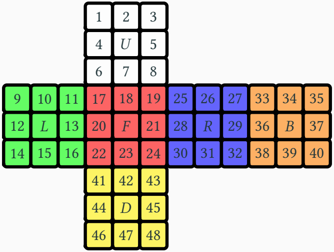
Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group

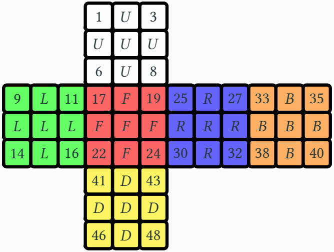



```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Transitive action on corners

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



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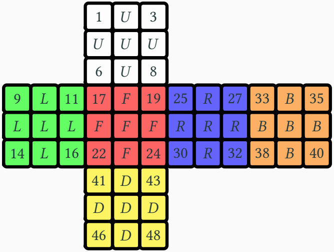
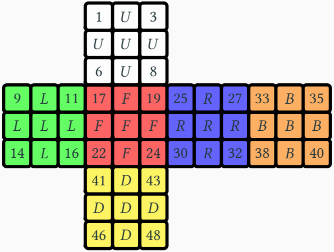


$$\Sigma = \{\overbrace{\{1, 35, 9\}}^{UBL}, \overbrace{\{6, 11, 17\}}^{ULF}, \overbrace{\{40, 46, 14\}}^{BDL}, \overbrace{\{27, 3, 33\}}^{RUB},$$
$$\underbrace{\{8, 25, 19\}}_{URF}, \underbrace{\{16, 41, 22\}}_{LDF}, \underbrace{\{32, 48, 38\}}_{RDB}, \underbrace{\{24, 43, 30\}}_{FDR}\}$$

is block system for *maximal* block $\{8, 25, 19\}$ (URF corner).

Transitive action on corners (ii)



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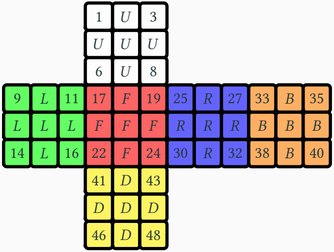

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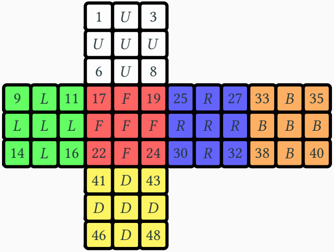
$$F \mapsto (\underbrace{FUL}_{FDL^F}, \underbrace{FUR}_{FUL^F}, \underbrace{FDR}_{FUR^F}, \underbrace{FDL}_{FDR^F}) \in \mathrm{Sym}(\Sigma).$$

Transitive action on corners (ii)



$\mathcal{G}$ acts primitively on Σ (degree 8); $g \in \mathcal{G}$ induces perm of Σ, e.g.

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$\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ "is" *primitive* quotient of $\mathcal{G}$).

Bases and stabiliser chains

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(\Omega)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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Theorem (Blaha, 1992)

Problem of finding minimum base for G is NP-complete, even for cyclic groups (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).

Bases and stabiliser chains (ii)

Example (Rubik's group)

Using `BaseOfGroup` cmd in `GAP`, base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

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Theorem

For Rubik's group $\mathcal{G}$, $b(\mathcal{G}) = 18$.

Bases and stabiliser chains (iii)

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(*Application*: check if restickering of Rubik's cube is valid state.)

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(Compared to arbitrary nonredundant base, with size $O(b(G) \log n)$*.)*

What is the size of the Rubik's group?

Theorem (size of perm group)

If $B = [\beta_1, \ldots, \beta_r]$ *is base for* $G \leq \mathrm{Sym}(n)$ *with stabiliser chain* $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, *then*

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Corollary
$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}.$

Base sizes of primitive groups

Affine groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

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Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

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Perm group G of degree n is **large base** if

$$\mathrm{Alt}(m)^r \trianglelefteq G \leq \mathrm{Sym}(m) \wr \mathrm{Sym}(r)$$

for some m, r, k, where $\mathrm{Sym}(m)$ acts on $\binom{[m]}{k}$ and $n = \binom{m}{k}^r$.

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Theorem (Liebeck, 1984)

For primitive perm group G of degree n, either:

(i) *G is large base; or*

(ii) *$b(G) < 9 \log n$.*

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"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

Large base permutation groups (ii)

Theorem (Moscatiello & Roney-Dougal, 2021)

For primitive perm group G of degree n, and G is non-large base:

(i) *G is the Mathieu group M_{24} (degree 24); or*

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Moreover, if $b(G) = \log n + 1$ then $G \leq \mathrm{AGL}_d(2)$ with $n = 2^d$.

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Question (Moscatiello & Roney-Dougal, 2021)

Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* d *with natural action on* K^d *with* $b(G) = d + 1$. *(Then* G *is perm group of degree* $n = 2^d$.*)*

(i) *For* $d = 1$, *there is no such* G.

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(i) *For* $d = 1$, *there is no such* G.

(ii) *For odd* $3 \leq d \leq 9$ *and* $d = 2$, *then* G *is* $\mathrm{AGL}_d(2)$.

Main result in thesis

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ be primitive for some d with natural action on K^d with $b(G) = d + 1$. (Then G is perm group of degree $n = 2^d$.)

(i) *For $d = 1$, there is no such G.*

(ii) *For odd $3 \leq d \leq 9$ and $d = 2$, then G is $\mathrm{AGL}_d(2)$.*

(iii) *For even $4 \leq d \leq 10$, then G is $\mathrm{AGL}_d(2)$ or $2^d : \mathrm{Sp}_d(2)$.*

Proof (idea).

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- Otherwise, recursively check for each representative M.

Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. $\square$

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

Main result in thesis (iii)

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

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Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

- $n = 2^d$ with $d \geq 2$, and G is $\mathrm{AGL}_d(2)$; or
- $n = 2^d$ with $d \geq 4$, and G is $2^d : \mathrm{Sp}_d(2)$.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP: https://www.gap-system.org/Doc/Examples/rubik.html
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