

# Minimum bases in permutation groups

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Honours presentation



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- How can we better *understand* operations of a cube?

*One answer: using permutations and computational group theory!*

**(J. A. Paulos, Innumeracy)**

*Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

## Some basic group theory

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# Permutations

## **Definition (permutation)**

**Permutation** of  $\Omega$  is bijection  $g : \Omega \rightarrow \Omega$ .

**Symmetric group**  $\text{Sym}(\Omega)$  is set of permutations of  $\Omega$ .

(For  $\Omega = [n] := \{1, \dots, n\}$ , write  $\text{Sym}(n)$ .)

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$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ g & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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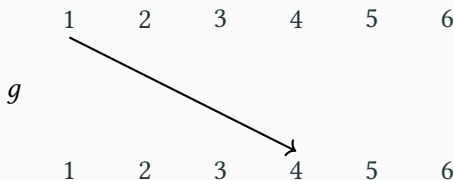
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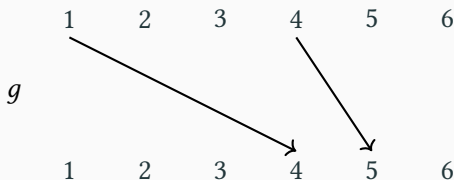
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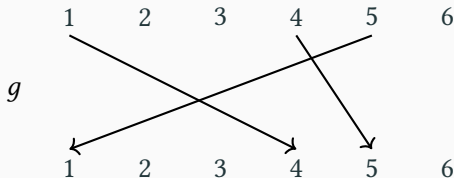
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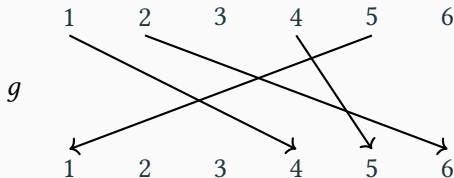
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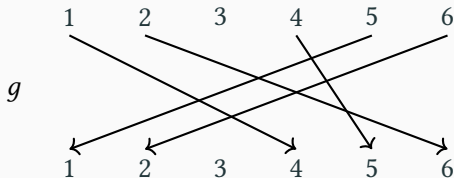
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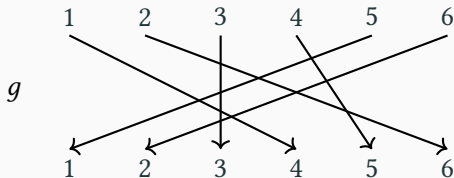
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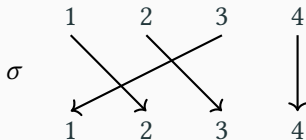
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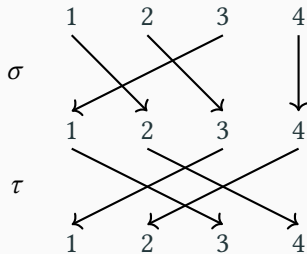
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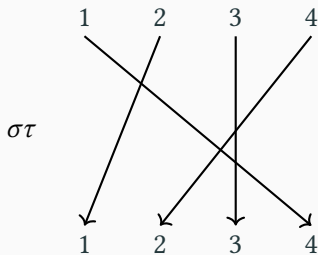
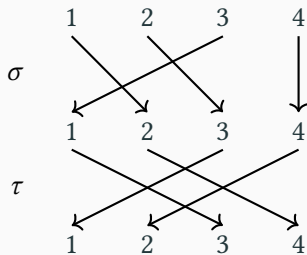
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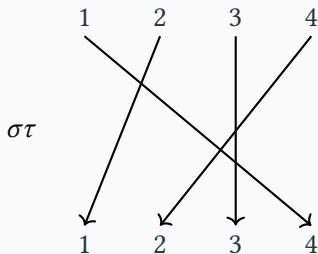
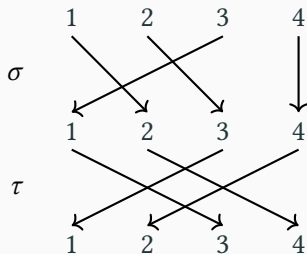
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*Note:* here,  $gh \neq hg$ , since  $1^{gh} = 4$  but  $1^{hg} = (1^h)^g = 3^g = 1$ . Identity  $1 = ()$  satisfies  $1g = g1 = g$  for  $g \in \text{Sym}(\Omega)$ .

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**Perm group** on  $\Omega$  (of deg  $n$ ) is subset  $G \leq \text{Sym}(\Omega)$  ( $|\Omega| = n$ ) s.t.

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Set  $X$  **generates**  $G$  if every  $g \in G$  is  $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$  for some  $r \in \mathbb{N}$ ,  $x_i \in X$  **generators**; write  $G = \langle X \rangle$ .

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## Example (dihedral group)

Let  $r = (1, 2, 3, 4), s = (1, 4)(2, 3) \in \text{Sym}(4)$ . **Dihedral group** is  $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ , “symmetries of square”.

## Definition (group action)

For  $G \leq \text{Sym}(\Omega)$  and  $\mathcal{S} \neq \emptyset$ , a  **$G$ -action** is map  $\mathcal{S} \times G \rightarrow \mathcal{S}$ ,  
 $(\alpha, g) \mapsto \alpha^g$  s.t.  $\alpha^1 = \alpha$  and  $\alpha^{gh} = (\alpha^g)^h$  for  $\alpha \in \mathcal{S}$  and  $g, h \in G$ .

**Degree** of action is  $|\mathcal{S}|$ .

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## Example (natural action)

$G \leq \text{Sym}(\Omega)$  acts on  $\mathcal{S} = \Omega$  by  $\alpha^g := \alpha^g$  (image) for  $\alpha \in \Omega$ ,  $g \in G$ .

### Example (dihedral group)

Recall  $D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$  acts naturally on  $[4]$ .

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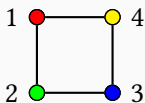
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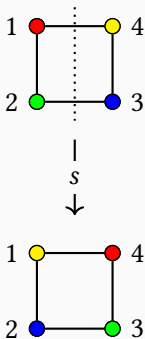


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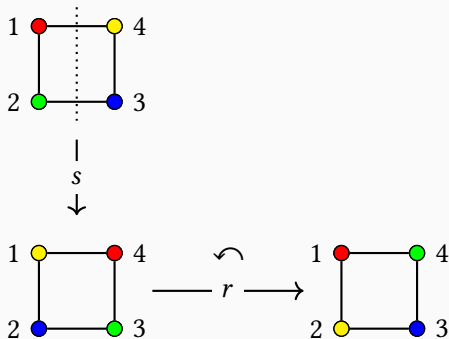


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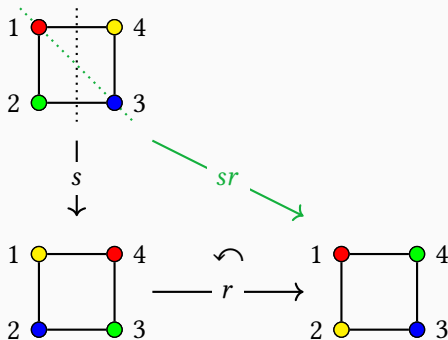


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## Definition (orbit)

If  $G$  acts on  $\mathcal{S}$ , then **orbit** of  $\alpha \in \mathcal{S}$  is  $\alpha^G := \{\alpha^g : g \in G\}$ .

*Idea:* states  $\alpha^g \in \mathcal{S}$  reachable from fixed  $\alpha \in \mathcal{S}$  by moves  $g \in G$ .

One orbit only: **transitive** action.

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## Definition (stabiliser)

If  $G$  acts on  $\mathcal{S}$ , then **stabiliser** of  $\alpha \in \mathcal{S}$  is  $G_\alpha := \{g \in G : \alpha^g = \alpha\}$ .

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## Orbits and stabilisers (ii)

Orbit  $\alpha^G$ : states  $\alpha^g \in \mathcal{S}$  reachable from fixed  $\alpha$  by moves  $g \in G$ .

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Orbit of 1:  $1^1 = 1$ ,  $1^r = 2$ ,  $1^{r^2} = 3$ ,  $1^{r^3} = 4$ , so  $1^G = [4]$  (transitive).

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Stabiliser of 1:  $sr = (2, 4)$ ,  $sr^2 = (1, 2)(3, 4)$ ,  $sr^3 = (1, 3)$ , so  $G_1 = \{(), (2, 4)\} = \{1, sr\}$ .

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### Theorem (orbit-stabiliser)

*If  $G$  acts on  $\mathcal{S}$ , then for  $\alpha \in \mathcal{S}$ ,  $|\alpha^G||G_\alpha| = |G|$ .*

## Definition (block)

If  $G$  acts transitively on  $\mathcal{S}$  and  $\Delta \subseteq \mathcal{S}$ , let  $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$ .

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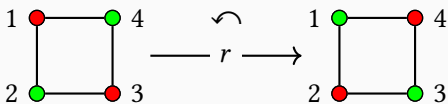
For block  $\Delta$ , define **block system**  $\Sigma = \{\Delta^g : g \in G\}$  (partitions  $\Omega$ ); then  $G$  acts on  $\Sigma$ ; if  $\Delta$  is *maximal*, then acts primitively.

## Blocks and primitivity (ii)

### Example (dihedral group)

Recall  $G = D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \text{Sym}(4)$  where  $r = (1, 2, 3, 4)$ ,  $s = (1, 4)(2, 3)$ ,  $sr = (2, 4)$ .

Block is  $\Delta = \{1, 3\}$  (nontrivial) with block system  $\Sigma = \{\{1, 3\}, \{2, 4\}\}$  (opposite vertices stay opposite):



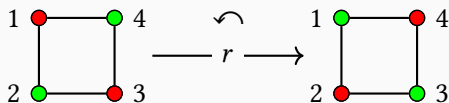
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e.g.  $\Delta^r = \{2, 4\}$ ,  $\Delta^s = \{4, 2\}$ ,  $\Delta^{sr} = \{1, 3\} = \Delta$ .

$D_8$  acts imprimitively on  $[4]$  but primitively on  $\Sigma$  (degree 2).

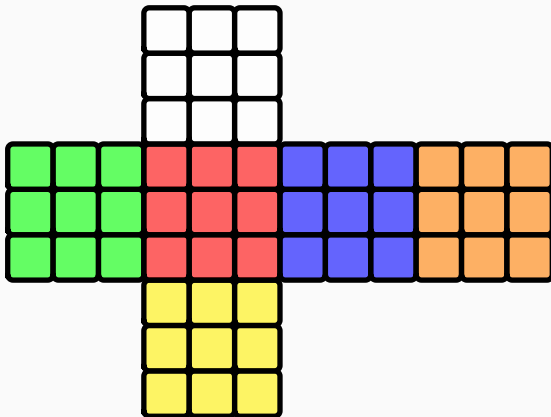
# The Rubik's group

---



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Rubik's cube has 6 faces, each with  $3 \times 3$  small *stickers*.



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			1	2	3							
			4	U	5							
			6	7	8							
9	10	11	17	18	19	25	26	27	33	34	35	
12	L	13	20	F	21	28	R	29	36	B	37	
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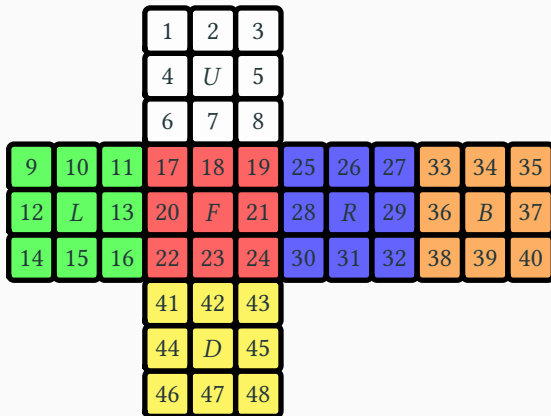
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6 **generators** (moves in CC):  $U, L, F, R, B, D$  (rot. clockwise).

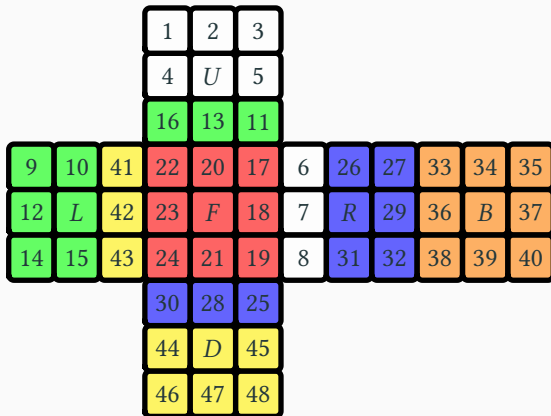
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$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$

$$(7, 28, 42, 13)(8, 30, 41, 11) \in \text{Sym}(48).$$

# The Rubik's group of permutations

Generators as permutations of labels [48]:

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**Definition (Rubik's group)**

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \text{Sym}(48)$  is permutation group of degree 48, called **Rubik's group**.

Clearly  $\mathcal{G}$  is finite, but what is  $|\mathcal{G}|$ ?

## The Rubik's group of permutations (ii)

GAP code to define generators and  $\mathcal{G} = \langle U, L, F, R, B, D \rangle$  (as G):

```
1 U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)
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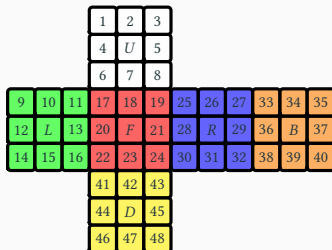
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Order cmd:  $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$ . *How?*

# Orbits in the Rubik's group



```

1 gap> Orbit( G, 1 );
2 [ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
   14, 19, 9, 22, 38 ]
3 gap> Orbit( G, 2 );
4 [ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
   15, 31, 18, 23, 39 ]

```

Two  $\mathcal{G}$ -orbits: corner stickers  $1^{\mathcal{G}}$ , edge stickers  $2^{\mathcal{G}}$ .

## Transitive action on corners

$\mathcal{G}$  acts transitively on corner stickers  $1^{\mathcal{G}}$ . In this action:

			1	U		3			
			U	U		U			
			6	U		8			
9	L	11	17	F	19	25	R	27	33
L	L	L	F	F	F	R	R	R	B
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$$\begin{array}{cccc}
 \text{UBL} & \text{ULF} & \text{BDL} & \text{RUB} \\
 \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \Sigma = \{ \{1, 35, 9\}, \{6, 11, 17\}, \{40, 46, 14\}, \{27, 3, 33\}, \\
 \{8, 25, 19\}, \{16, 41, 22\}, \{32, 48, 38\}, \{24, 43, 30\} \} \\
 \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \text{URF} & \text{LDF} & \text{RDB} & \text{FDR}
 \end{array}$$

is block system for *maximal* block  $\{8, 25, 19\}$  (URF corner).

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			U	U	U						
			6	U	8						
9	L	11	17	F	19	25	R	27	33	B	35
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$\mathcal{G}$  induces every perm of  $\Sigma$  (so  $\text{Sym}(8)$  “is” *primitive* quotient of  $\mathcal{G}$ ).

## **Bases and stabiliser chains**

---

## Definition (Base, stabiliser chain)

If  $G \leq \text{Sym}(\Omega)$ , distinct elts  $B = [\beta_1, \dots, \beta_r] \subseteq \Omega$  is **base** for  $G$  if  $G_{\beta_1, \dots, \beta_r} = 1$ . (Recall:  $G_{\beta_1, \dots, \beta_r} = \{g \in G : \beta_1^g = \beta_1, \dots, \beta_r^g = \beta_r\}$ .)

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Corresponding **stabiliser chain** is

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Base  $B$  contains elts of  $\Omega$  such that only  $1 \in G$  fixes every  $\beta_i \in B$ .  
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## Theorem (Blaha, 1992)

*Problem of finding minimum base for  $G$  is NP-complete (if  $P \neq NP$ , then no polynomial time algorithm).*

### Example (Rubik's group)

Using BaseOfGroup cmd in GAP, base of  $\mathcal{G}$  of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$



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### Theorem

For Rubik's group  $\mathcal{G}$ ,  $b(\mathcal{G}) = 18$ .

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(*Alternative:* random product of generators in  $X$ ; transitive Markov chain on Cayley graph (find mixing time?); distribution?)

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Generate uniformly random element of  $G$ .

(*Alternative:* random product of generators in  $X$ ; transitive Markov chain on Cayley graph (find mixing time?); distribution?)

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## Bases and stabiliser chains (iii)

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(*Application:* check if restickering of Rubik's cube is valid state.)



# What is the size of the Rubik's group?

## Theorem (size of perm group)

If  $B = [\beta_1, \dots, \beta_r]$  is base for  $G \leq \text{Sym}(\Omega)$  with stabiliser chain  $G = G^0 \geq G^1 \geq \dots \geq G^r = 1$ , then

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## Corollary

$$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}.$$

## Base sizes of primitive groups

---

## Definition

Let  $K$  be field. **Affine transformation** of  $K^d$  is map

$$t_{a,v} : K^d \rightarrow K^d, \quad u \mapsto ua + v$$

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Interested in  $q = 2$ , i.e. field  $\mathbb{F}_2 = \{0, 1\}$  with  $1 + 1 = 0$ ,  $1 \cdot 1 = 1$ , etc.

# Large base permutation groups

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Perm group  $G$  of degree  $n$  is **large base** if

$$\text{Alt}(m)^r \trianglelefteq G \leq \text{Sym}(m) \wr \text{Sym}(r)$$

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## Theorem (Liebeck, 1984)

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“Remarkable” proof used *classification of finite simple groups*,  
*O’Nan-Scott theorem* (classifies primitive groups).

### **Theorem (Moscatiello & Roney-Dougal, 2021)**

*For primitive perm group  $G$  of degree  $n$ , and  $G$  is non-large base:*

- (i)  $G$  is the Mathieu group  $M_{24}$  (degree 24); or*
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### **Question (Moscatiello & Roney-Dougal, 2021)**

Which primitive groups  $G \leq \text{Sym}(n)$  satisfy  $b(G) = \log n + 1$ ?

## Theorem

*Let  $G \leq \text{AGL}_d(2)$  be primitive for some  $d$  with natural action on  $K^d$  with  $b(G) = d + 1$ . (Then  $G$  is perm group of degree  $n = 2^d$ .)*

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- (i) For  $d = 1$ , there is no such  $G$ .*
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- (i) For  $d = 1$ , there is no such  $G$ .*
- (ii) For odd  $3 \leq d \leq 9$  and  $d = 2$ , then  $G$  is  $\text{AGL}_d(2)$ .*
- (iii) For even  $4 \leq d \leq 10$ , then  $G$  is  $\text{AGL}_d(2)$  or  $2^d : \text{Sp}_d(2)$ .*

### Proof (idea).

- Find representatives  $M$  of conjugacy classes of primitive maximal subgroups of  $\text{AGL}_d(2)$ .

## Main result in thesis (ii)

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- Otherwise, recursively check for each representative  $M$ .

Every primitive  $G \leq \text{AGL}_d(2)$  with  $b(G) = d + 1$  is found by process (plus perhaps false positives), up to conjugacy.  $\square$

Greedy base algorithm performed better than BaseOfGroup in testing; found no false positives.

From above theorem, we conjecture the following:

### Conjecture

Primitive group  $G \leq \text{Sym}(n)$  satisfies  $b(G) = \log n + 1$  iff:

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### Conjecture

Primitive group  $G \leq \text{Sym}(n)$  satisfies  $b(G) = \log n + 1$  iff:

- $n = 2^d$  with  $d \geq 2$ , and  $G$  is  $\text{AGL}_d(2)$ ; or
- $n = 2^d$  with  $d \geq 4$ , and  $G$  is  $2^d : \text{Sp}_d(2)$ .



## Concluding remarks

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## References and resources

- Analyzing Rubik's cube with GAP:  
<https://www.gap-system.org/Doc/Examples/rubik.html>
- J. A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Blaha — *Minimum bases for permutation groups: The greedy approximation*, 1992:  
[https://doi:10.1016/0196-6774\(92\)90020-D](https://doi:10.1016/0196-6774(92)90020-D)
- Liebeck — *On minimal degrees and base sizes of primitive permutation groups*, 1984: <https://doi.org/10.1007/bf01193603>
- Moscattiello and Roney-Dougal: *Base sizes of primitive permutation groups*, 2021: <https://doi.org/10.1007/s00605-021-01599-5>