# Rubik's cubes and permutation group theory

**Lawrence Chen**

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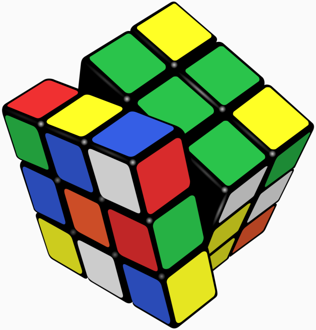
**Honours presentation**



# Contents

# Some basic group theory

# What is a group? i

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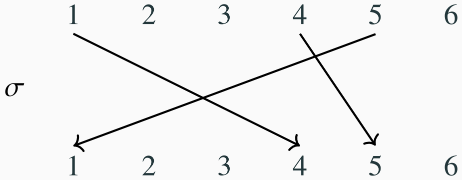


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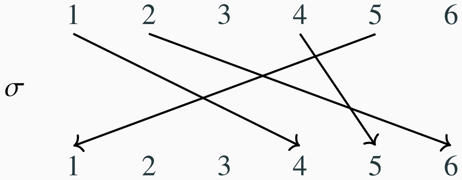


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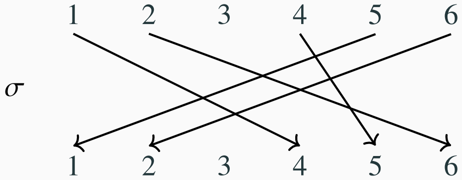


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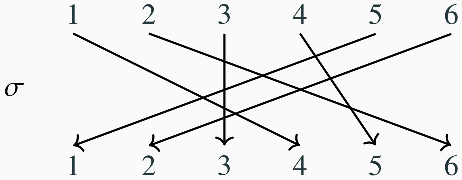


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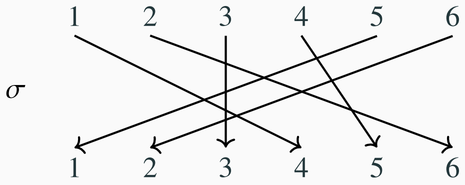

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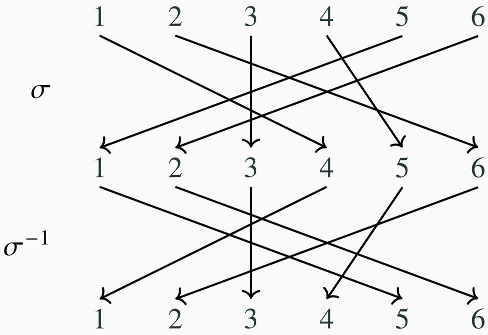


Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \mathrm{Sym}(6)$.

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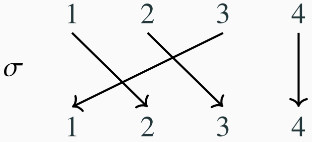

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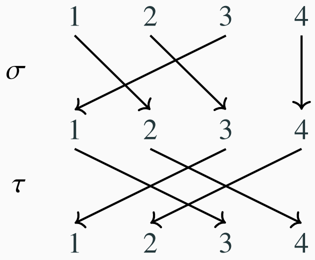

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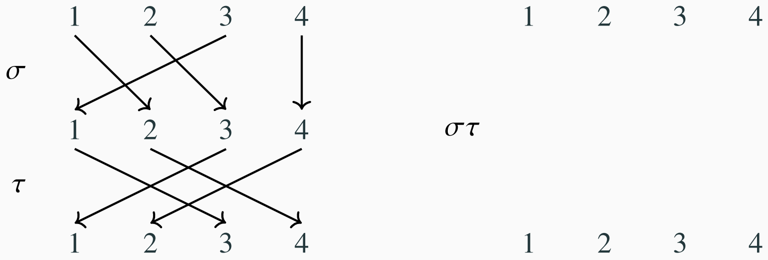


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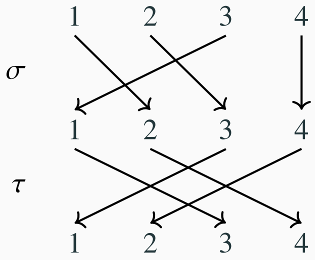


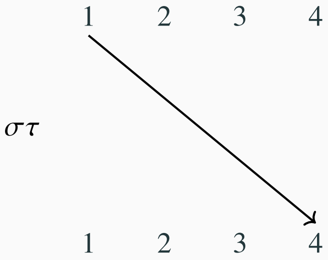


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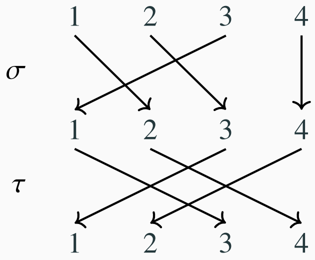


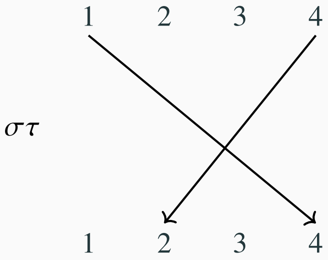


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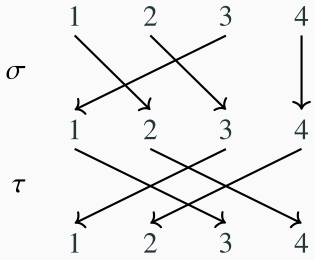


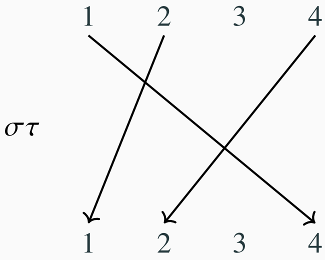


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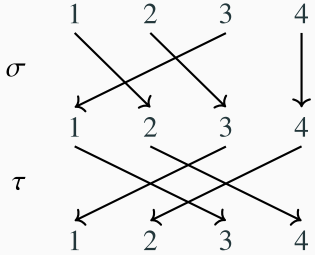


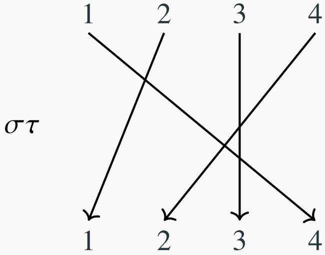


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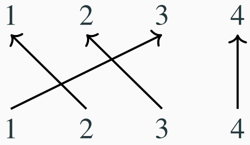

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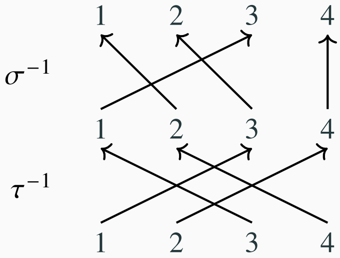

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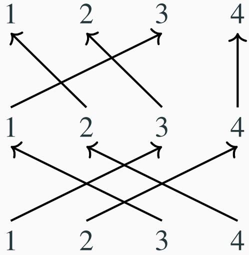


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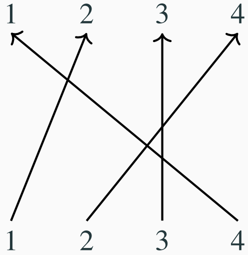

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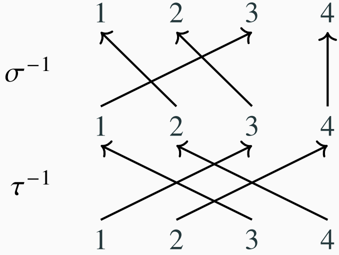


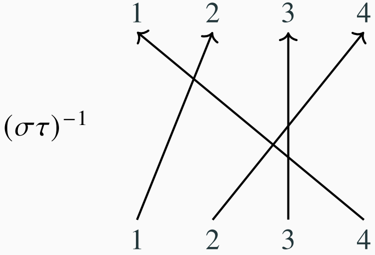


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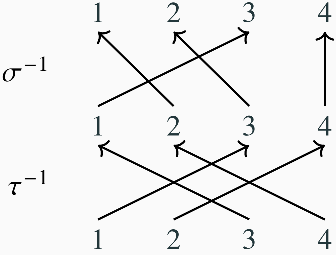


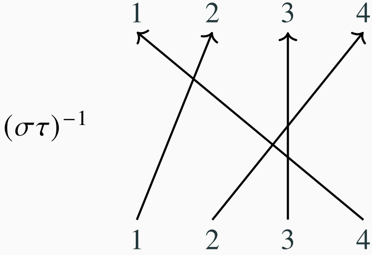


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## Permutations v

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Subset $H$ of group $G$ is **subgroup** if it is group under same operation; write $H \leq G$. (Need to check: nonempty, closure, inverses.)

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**Definition (permutation group)**

A **permutation group** of *degree* $n$ is a subgroup of $\mathrm{Sym}(n)$.

**Definition (group action)**

If $G$ is group and $\Omega \neq \emptyset$ is set, a $G$**-action** is a map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

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$\mathbb{Z}$ acts on $\Omega = \{12{:}00, 1{:}00, \ldots, 11{:}00\}$ by $(\alpha{:}00)^k = [\alpha + k]_{12}{:}00$ for $\alpha{:}00 \in \Omega$ and $k \in \mathbb{Z}$.

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**Example (right regular action)**

Group $G$ acts on $\Omega = G$ (itself) via $\alpha^g = \alpha g$ for $\alpha, g \in G$.

# Group actions ii

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If $G$ acts on $\Omega$, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

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**Example (Natural action)**

$G = \{(), (1,2,4), (1,4,2)\} \leq \mathrm{Sym}(4)$ acts on $\Omega = [4]$ naturally.
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Note: $|1^G||G_1| = 3 \cdot 1 = 3$

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**Theorem (orbit-stabiliser)**

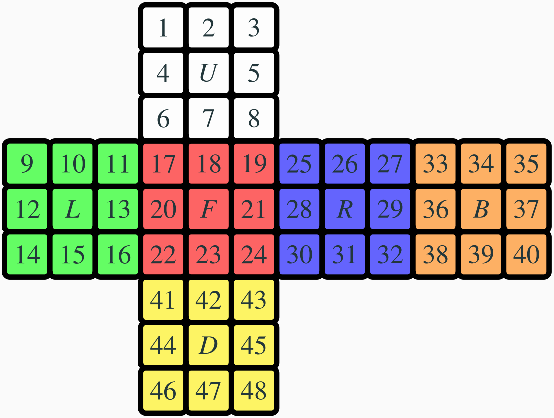
*If $G$ acts on $\Omega$, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.*

# The Rubik’s group

# Representing the cube i

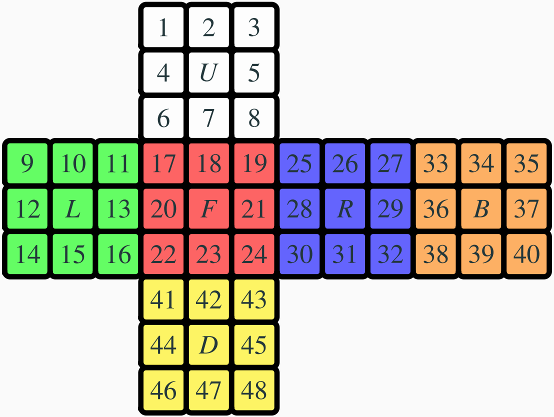
A Rubik's cube has 6 large faces (each with $3 \times 3$ smaller faces).

Label all smaller faces except centre on each side, using [48]:




6 *elementary moves* (**generators**): $U, L, F, R, B, D$ (rotate clockwise).

Consider move $F$ which rotates front face clockwise:



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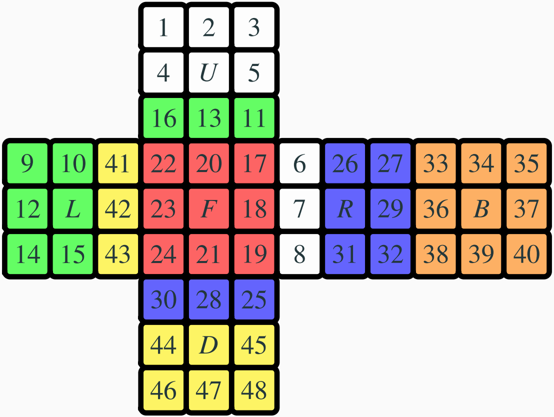
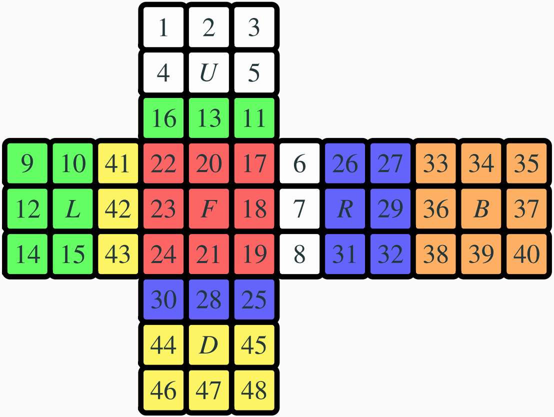



Under $F$: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

# Representing the cube ii

Consider move $F$ which rotates front face clockwise:




Under $F$: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

*Empty move* is $1 = ()$. In cubing community, *inverse elementary moves* usually denoted $U', L', F', R', B', D'$ (instead of $U^{-1}$, etc.); powers usually denoted $U2, R2$ etc. (instead of $U^2, R^2$).

*Valid move* is sequence of elementary moves (product of generators). E.g. $RUR^{-1}U^{-1}$, $URU^{-1}L^{-1}UR^{-1}U^{-1}L$, $RUR^{-1}URU^2R^{-1}U^2$.

# Moves vs states for Rubik's cube ii

Moves don't generally commute: $RU \neq UR$ since

$$19^{RU} = (19^R)^U = 3^U = 8 \quad \text{but} \quad 19^{UR} = (19^U)^R = 11^R = 11.$$

Let $\mathcal{S}$ be valid **states**; can represent state $x \in \mathcal{S}$ as element of $\mathrm{Sym}(48)$ giving permutation of labels from solved state $1 = ()$.

Let $\mathcal{G}$ be valid **moves**; can represent move $\sigma \in \mathcal{G}$ as element of $\mathrm{Sym}(48)$ giving corresponding permutation of labels.

- State $x \in \mathcal{S}$ corresponds to move $x \in \mathcal{G}$ required to get solved state $1 = ()$ into state $x$.
- Move $\sigma \in \mathcal{G}$ corresponds to state $\sigma \in \mathcal{S}$ reached by applying move $\sigma$ to solved state $1 = ()$.

So moves $\leftrightarrow$ states for Rubik's cube; as sets, $\mathcal{S} = \mathcal{G}$.

Set of moves $\mathcal{G}$ forms group: composition of valid moves is valid move; identity move $1 = () \in \mathcal{G}$, inverse moves exist; associative.

**Definition (Rubik's group)**
$\mathcal{G} \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called the **Rubik's group**; it acts naturally on $[48]$. Note: $G = \langle U, L, F, R, B, D \rangle$.

For move $\sigma \in \mathcal{G}$ and state $x \in \mathcal{S}$, applying $\sigma$ to $x$ gives state $x^{\sigma} = x\sigma \in \mathcal{S}$. This is regular action of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

TODO: orbits, stabilisers (corner pieces/edge pieces), GAP code?

# Orders of moves i

TODO

Order of generators: all 4

Order of commutator $RUR^{-1}U^{-1}$ is 6 (write out, get Wes video)

Order of $URU^{-1}L^{-1}UR^{-1}U^{-1}L$ is 3 (write out, get Wes video) – last layer corner permutation (3 states)

Order of $RUR^{-1}URU^2R^{-1}U^2$ is 3 (write out, get Wes video) – last layer edge permutation (3 states)

Order of $RU$ is 105, order of Clayton's move $UL'$ is 63

What is element of order 5? $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

**Theorem (Jake Vandenberg's conjecture)**
*There is no Rubik's cube move that cycles through all states.*

Recall: states $\leftrightarrow$ moves. Rubik's group $\mathcal{G}$ acts on states by applying move $\sigma \in \mathcal{G}$ to state $x \in \mathcal{G}$ to get state $x^\sigma = x\sigma \in \mathcal{G}$.

Equivalent question: for starting state, WLOG $1 = ()$, is there $\sigma \in \mathcal{G}$ with $\{1^{\sigma^k} : k \in \mathbb{Z}\} = \{1\sigma^k : k \in \mathbb{Z}\} = \{\sigma^k : k \in \mathbb{Z}\} = \mathcal{G}$? In group theory language:

**Theorem (Jake Vandenberg's conjecture)**
*The Rubik's group $\mathcal{G}$ is not cyclic. (I.e. no $\sigma \in \mathcal{G}$ with $\mathcal{G} = \langle \sigma \rangle$.)*

**Proof.**
If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. $\square$

**Theorem (Jake Vandenberg's theorem)**
*There is no Rubik's cube move that when repeated, if starting from the solved state, never returns to the solved state.*

A $k$-fold repetition of move $\sigma \in G$, applied to solved state $1 = ()$, gives $1^{\sigma^k} = 1\sigma^k = \sigma^k$. Returning to solved state: $\sigma^k = 1$ (for $k > 0$).

Equivalent question: does any $\sigma \in G$ have infinite order?

**Proposition**
*If $G$ is finite group and $g \in G$, then $g^{|G|} = 1$.*

**Corollary (Jake Vandenberg's theorem)**
*There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).*

# Analysing the Rubik’s group

# Bases and stabiliser chains i

TODO

# How many valid states are there? i

TODO

# Can this restickering be solved? i

TODO

# Generating random Rubik's cube states i

TODO

# Solving a Rubik's cube... i

TODO

# Concluding remarks

# References i

- TODO