Rubik's cubes and permutation group theory

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Honours presentation

Contents

- How can we represent *move sequences* and *states* of a cube?
- How can we tell *how many* states a Rubik's cube can take?
- If we repeat a move, do we eventually *get back to the start*?
- If a Rubik's cube is *restickered*, is it *solvable*?
- How can we use maths to *solve* a Rubik's cube?

Answer: using permutations and *permutation group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

Definition (group)

A **group** is a set $G \neq \emptyset$ with operation $G \times G \to G$, $(g, h) \mapsto gh$,

  (i) **(identity)** there is $1 \in G$ with $1g = g1 = g$ for all $g \in G$;

 (ii) **(inverses)** for all $g \in G$, there is $g^{-1} \in G$ with $g^{-1}g = gg^{-1} = 1$;

(iii) **(associative)** $(gh)k = g(hk)$ for all $g, h, k \in G$.

Example (Integers under addition)

The integers $(\mathbb{Z}, +)$ form an **abelian** group: identity 0, inverses $-k$ for $k \in \mathbb{Z}$, associative.

Example (Cyclic group)

The set $C_n = \{a^0, a^1, a^2, \ldots, a^{n-1}\}$ with rules $a^k a^\ell = a^{k+\ell}$, $a^n = a^0$ forms group: identity $1 = a^0$, inverses a^{-k} for $a^k \in C_n$, associative.

Definition (order)

Order of $g \in G$ is least $k \in \mathbb{Z}_+$ with $g^k = g \cdots g = 1$ (otherwise ∞).

Example (Cyclic group)

Consider group $C_4 = \{1, a, a^2, a^3\}$: order of 1 is 1, order of a is 4, order of a^2 is 2, order of a^3 is 4.

Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**; write $G = \langle X \rangle$. (If $|X| = 1$, G is **cyclic**.)

Example (Cyclic group)

Consider group $C_6 = \{1, a, a^2, a^3, a^4, a^5\}$: $C_6 = \langle a \rangle$. If $b = a^2$, $c = a^3$ then $C_6 = \langle b, c \rangle$ since $a = cb^{-1}$ so $a^k = cb^{-1} \cdots cb^{-1} = c^k b^{-k}$.
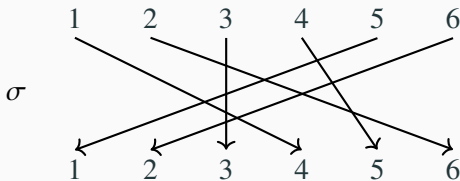
Definition (permutation)

Permutation of $[n] := \{1, \ldots, n\}$ is bijection $\sigma : [n] \to [n]$.

Write $1 = ()$ for identity. Write i^σ not $\sigma(i)$ for *image*.

Cycle notation: $\sigma = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:

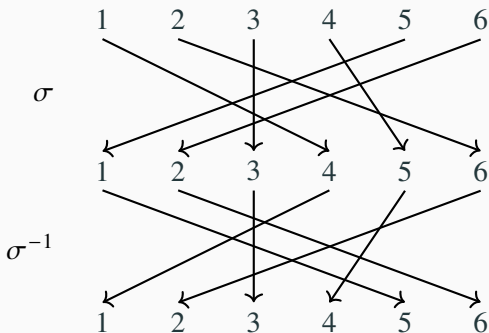$$\sigma \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

It means

$$1^\sigma = 4, \ 4^\sigma = 5, \ 5^\sigma = 1, \ 2^\sigma = 6, \ 6^\sigma = 2, \ 3^\sigma = 3.$$
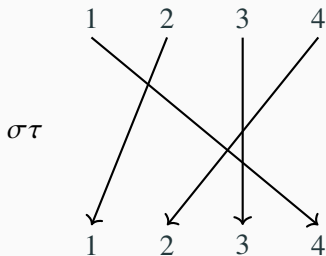
Inverses: For $\sigma = (1, 4, 5)(2, 6) \in \text{Sym}(6)$:

Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \text{Sym}(6)$.

Permutations iii

Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means "first σ, then τ", so $i^{\sigma\tau} = (i^\sigma)^\tau$. E.g. $\sigma = (1, 2, 3), \tau = (1, 3)(2, 4) \in \mathrm{Sym}(4)$,
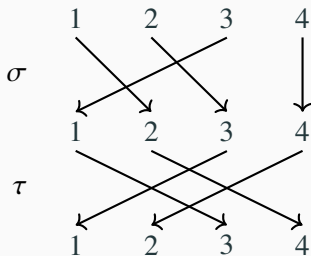
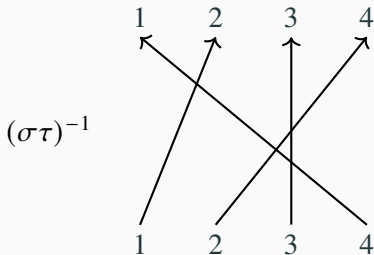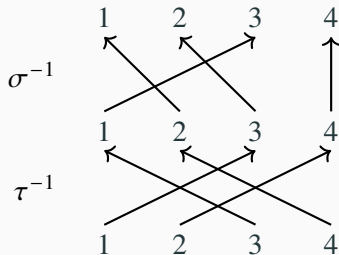$$\sigma\tau = (1, 2, 3)(1, 3)(2, 4) = (1, 4, 2) \in \mathrm{Sym}(4).$$

Inverse of product: Is $(\sigma\tau)^{-1} = \sigma^{-1}\tau^{-1}$?

$\sigma^{-1} = (1, 3, 2)$, $\tau^{-1} = (1, 3)(2, 4)$, $(\sigma\tau)^{-1} = (1, 2, 4)$.

$\sigma^{-1}\tau^{-1} = (1, 3, 2)(1, 3)(2, 4) = (2, 3, 4) \neq (\sigma\tau)^{-1}$,

$\tau^{-1}\sigma^{-1} = (1, 3)(2, 4)(1, 3, 2) = (1, 2, 4) = (\sigma\tau)^{-1}$.

Permutations v

Set of permutations under *product* is **symmetric group** Sym(n):
identity $1 = ()$, inverses (since bijection), associative.

What is size of Sym(n)? *Answer: $n!$*

Example (Order of permutation)

Consider $\sigma = (1, 4, 5)(2, 6) \in \text{Sym}(6)$. Then $1^{\sigma^3} = 4^{\sigma^2} = 5^{\sigma} = 1$,
$4^{\sigma^3} = 4, 5^{\sigma^3} = 5, 2^{\sigma^2} = 2, 6^{\sigma^2} = 6$ so $\sigma^6 = () = 1$; order of σ is 6.

Fact: order of $\sigma \in \text{Sym}(n)$ is lcm of cycle lengths.

Definition (subgroup)

Subset H of group G is **subgroup** if it is group under same operation;
write $H \leq G$. (Need to check: nonempty, closure, inverses.)

Definition (permutation group)

A **permutation group** of *degree n* is a subgroup of Sym(n).

Definition (group action)

If G is group and $\Omega \neq \emptyset$ is set, a G-**action** is a map $\Omega \times G \rightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

Idea: $\alpha \in \Omega$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects group operation.

Example (adding time)

$\mathbb{Z}$ acts on $\Omega = \{12{:}00, 1{:}00, \ldots, 11{:}00\}$ by $(\alpha{:}00)^k = [\alpha + k]_{12}{:}00$ for $\alpha{:}00 \in \Omega$ and $k \in \mathbb{Z}$.

E.g. 5:00 plus 9 hrs is $(5{:}00)^9 = [5 + 9]_{12}{:}00 = 2{:}00$.

Example (natural action)

$G \leq \mathrm{Sym}(n)$ acts on $\Omega = [n]$ by $\alpha^g = \alpha^g$ (image) for $\alpha \in [n]$, $g \in G$.

Example (right regular action)

Group G acts on $\Omega = G$ (itself) via $\alpha^g = \alpha g$ for $\alpha, g \in G$.

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Definition (orbit)

If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

Idea: states $\alpha^g \in \Omega$ reachable from fixed $\alpha \in \Omega$ by moves $g \in G$.

Definition (stabiliser)

If G acts on Ω, then **stabiliser** of $\alpha \in \Omega$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.

Idea: moves $g \in G$ that fix given $\alpha \in \Omega$.

Example (Adding time)

$\mathbb{Z}$-orbit of 11:00 is $\Omega = \{12{:}00, \ldots, 11{:}00\}$ (e.g. $(11{:}00)^{-2} = 9{:}00$).

$\mathbb{Z}$-stabiliser of 11:00 is $12\mathbb{Z} = \{12k : k \in \mathbb{Z}\}$ (add multiples of 12 hrs).

Example (right regular action)

G acts on $\Omega = G$ via $\alpha^g = \alpha g$ for $\alpha, g \in G$. Orbit of $\alpha \in G$ is $\Omega = G$

$(\alpha^{\alpha^{-1}\beta} = \beta \in G)$; stabiliser of α is $\{1\} = 1$ ($\alpha g = \alpha \implies g = 1$).

Definition (orbit, stabiliser (again))

If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$ and **stabiliser** of $\alpha \in \Omega$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.

Note: stabiliser G_α is subgroup of G. (So G_α acts on Ω.)

Example (Natural action)

$G = \{(), (1,2,4), (1,4,2)\} \le \text{Sym}(4)$ acts on $\Omega = [4]$ naturally. Orbit of 1 is $1^G = \{1, 2, 4\}$, stabiliser of 1 is $G_1 = \{()\} = 1$. Orbit of 3 is $3^G = \{3\}$, stabiliser of 3 is $G_3 = G$.

Note: $|1^G||G_1| = 3 \cdot 1 = 3 = |G|$, $|3^G||G_3| = 1 \cdot 3 = 3 = |G|$.

Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

A Rubik's cube has 6 large faces (each with 3×3 smaller faces).
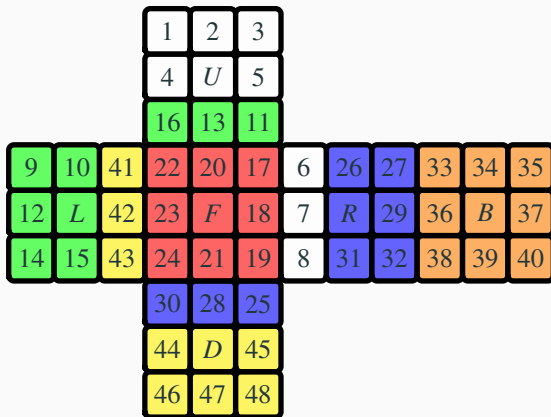
In **solved state** 1, label smaller faces (except each centre) using [48]:
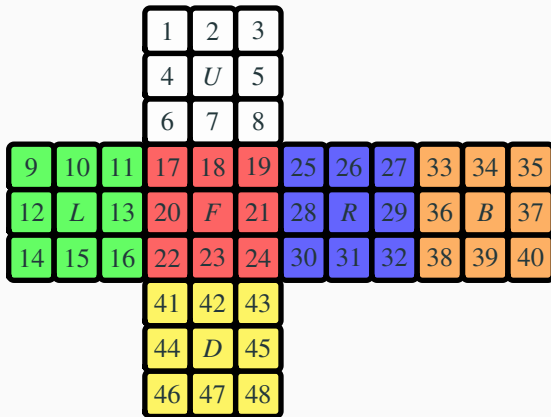
6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

From *solved state* 1, consider F which rotates front face clockwise:

Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$ 13

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

(Valid) move is sequence of generators and inverses. E.g.
$RUR^{-1}U^{-1}$, $URU^{-1}L^{-1}UR^{-1}U^{-1}L$, $RUR^{-1}URU^2R^{-1}U^2$.

Empty move is $1 = ()$ (valid: $1 = RR^{-1}$).

Solving is applying valid move to get to solved state 1.

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In cubing community: moves called *move sequences*. Generators called *moves. Inverse elementary moves* written U', L', F', R', B', D' (instead of U^{-1} etc.); powers written $U2$, $R2$ etc. (instead of U^2, R^2).

Recall: $\sigma = \tau$ in $\mathrm{Sym}(n)$ iff $i^\sigma = i^\tau$ for all $i \in [n]$.

Moves *don't generally commute*: $RU \neq UR$ since

- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$

$$19^{RU} = (19^R)^U = 3^U = 8 \quad \text{but} \quad 19^{UR} = (19^U)^R = 11^R = 11.$$

(Valid) state is result of applying *valid move* to *solved state* 1.

This new state is valid, as result of applying *F* to solved state.

Moves vs states for Rubik's cube ii

Restickering is valid state iff it can be *solved*. How to check?

Let $\mathcal{S}$ be valid **states**; let state $x \in \mathcal{S}$ be element of $\text{Sym}(48)$ giving permutation of labels to solved state $1 \in \mathcal{S}$.

(I.e. i^x is label at x-position of i in solved state 1.)

Let $\mathcal{G}$ be valid **moves**; let move $\sigma \in \mathcal{G}$ be element of $\text{Sym}(48)$ giving corresponding permutation of labels.

(I.e. i^σ is label at position σ maps i into.)

- State $x \in \mathcal{S}$ corresponds to move $x \in \mathcal{G}$ required to get solved state 1 into state x.
- Move $\sigma \in \mathcal{G}$ corresponds to state $\sigma \in \mathcal{S}$ reached by applying move σ to solved state 1.

So moves $\leftrightarrow$ states; as sets, $\mathcal{S} = \mathcal{G}$. *Solved state* is $1 = () \in \text{Sym}(48)$.

The Rubik's group of permutations i

Set of moves $\mathcal{G}$ forms group: composition of valid moves is valid move; identity move is $1 = () \in \mathcal{G}$, inverse moves exist (undo elementary moves/inverses); associative.

Definition (Rubik's group)
$\mathcal{G} \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called the **Rubik's group**; it acts naturally on $[48]$. *Note:* $G = \langle U, L, F, R, B, D \rangle$.

For *move* $\sigma \in \mathcal{G}$ and *state* $x \in \mathcal{S}$, applying σ to x gives *state* $x^\sigma = x\sigma \in \mathcal{S}$. This is *regular action* of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
1 U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)
     (11,35,27,19);
2 L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)(
     6,22,46,35);
3 F := (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)(
     8,30,41,11);
4 R := (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)(
     8,33,48,24);
5 B := (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)(
     1,14,48,27);
6 D := (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)
     (16,24,32,40);
7 G := Group( U, L, F, R, B, D );
```

Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

```
1  gap> Orbit( G, 1 );
2  [ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
       30, 14, 19, 9, 22, 38 ]
3  gap> Orbit( G, 2 );
4  [ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
       37, 15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.
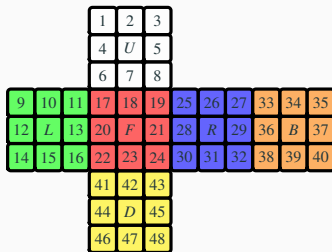
Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners $1, 3, 6, 8$.

```
gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
<permutation group of size 317842469683200 with 12 generators>
gap> Orbit( G_1368, 17 );
[ 17 ]
gap> Orbit( G_1368, 24 );
[ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
true
```

Some $\mathcal{H}$-orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Use GAP to compute products, order (using `Order` cmd).

```
1  gap> R*U*R^(-1)*U^(-1);
2  (1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
3  gap> Order( last );
4  6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect? I.e. for state $x \in \mathcal{S}$, smallest $k \in \mathbb{Z}_+$ with $x^{\sigma^k} = x\sigma^k = x \iff \sigma^k = 1$.
Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length 4, 4, 4, 4, 4: order is $\mathrm{lcm}(4, 4, 4, 4, 4) = 4$.
- *Commutator* $RUR^{-1}U^{-1}$

  $= (1, 27, 35, 33, 9, 3)(2, 21, 5)(8, 30, 25, 43, 19, 24)(26, 34, 28)$:

  order is $\mathrm{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

    $= (1, 9, 35)(2, 5, 7)(3, 33, 27)(8, 25, 19)(18, 34, 26)$:

  order is $\text{lcm}(3, 3, 3, 3, 3) = 3$.

- *Lawrence's move RU* has cycles of length $15, 7, 3, 7$: order is $\text{lcm}(15, 7, 3, 7) = 105$.

- *Clayton's move UL'* has cycles of length $9, 7, 9, 7$: order is $\text{lcm}(9, 7, 9, 7) = 63$.

Watch video demonstration by my friend Wes :D

Move of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

What is smallest $k \in \mathbb{Z}_+$ with no move of that order?

Theorem (Jake Vandenberg's conjecture)

There is no Rubik's cube move that cycles through all states.

Recall: states $\leftrightarrow$ moves. Rubik's group $\mathcal{G}$ acts on states by applying move $\sigma \in \mathcal{G}$ to state $x \in \mathcal{G}$ to get state $x^\sigma = x\sigma \in \mathcal{G}$.

Equivalent question: for starting state, WLOG $1 = ()$, is there $\sigma \in \mathcal{G}$ with $\{1^{\sigma^k} : k \in \mathbb{Z}\} = \{1\sigma^k : k \in \mathbb{Z}\} = \{\sigma^k : k \in \mathbb{Z}\} = \mathcal{G}$? In group theory language:

Theorem (Jake Vandenberg's conjecture)

The Rubik's group $\mathcal{G}$ is not cyclic. (I.e. no $\sigma \in \mathcal{G}$ with $\mathcal{G} = \langle \sigma \rangle$.)

Proof.

If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. $\quad\square$

Theorem (Jake Vandenberg's theorem)
There is no Rubik's cube move that when repeated, if starting from the solved state, never returns to the solved state.

k-fold repetition of move $\sigma \in G$, applied to solved state $1 = ()$, gives $1^{\sigma^k} = 1\sigma^k = \sigma^k$. Returning to solved state: $\sigma^k = 1$ (for $k \in \mathbb{Z}_+$).

Equivalent question: does any $\sigma \in G$ have infinite order?

Proposition (corollary of Lagrange's theorem)
If G is finite group and $g \in G$, then order of g divides $|G|$. So $g^{|G|} = 1$.

Corollary (Jake Vandenberg's theorem)
There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).

Analysing the Rubik's group

Bases and stabiliser chains i

Definition (Base, stabiliser chain)
If $G \leq \text{Sym}(n)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq [n]$ is **base** for G if $G_{\beta_1, \ldots, \beta_r} = 1$. (*Recall:* $G_{\beta_1, \ldots, \beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

Corresponding **stabiliser chain** is

$$G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$$

where $G^i = G^{i-1}_{\beta_i} = G_{\beta_1, \ldots, \beta_i}$.

Base B contains elts of $[n]$ such that only $1 \in G$ fixes every $\beta_i \in B$. (Short base desirable: how to compute minimum base?)

Stabiliser chain can be implemented computationally; useful in algorithms (membership testing, random element generation, factorisation into generators).

Example (Rubik's group)

Using GAP:

```
1 gap> BaseOfGroup( G );
2 [ 1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21,
     23, 24, 29, 31 ]
3 gap> Size( last );
4 18
```

Base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

If move $\sigma \in \mathcal{G}$ fixes every $\beta_i \in B$ then $\sigma = 1$ is empty move.

Theorem (size of perm group)

If $B = [\beta_1, \ldots, \beta_r]$ is base for $G \leq \mathrm{Sym}(n)$ with stabiliser chain $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, then

$$|G| = |\beta_1^{G^0}||\beta_2^{G^1}| \cdots |\beta_r^{G^{r-1}}|.$$

Proof.

OST implies $|G^{i-1}| = |\beta_i^{G^{i-1}}||G_{\beta_i}^{i-1}| = |\beta_i^{G^{i-1}}||G^i|$ for each $i = 1, \ldots, r$, i.e. $|G^{i-1}|/|G^i| = |\beta_i^{G^{i-1}}|$ and $|G^r| = 1$, so

$$|G| = |G^0| = \frac{|G^0|}{|G^1|}\frac{|G^1|}{|G^2|} \cdots \frac{|G^{r-1}|}{|G^r|} = |\beta_1^{G^0}||\beta_2^{G^1}| \cdots |\beta_r^{G^{r-1}}|. \quad \square$$

How many valid states are there? ii

Theorem (size of perm group)

If $B = [\beta_1, \ldots, \beta_r]$ is base for $G \leq \text{Sym}(n)$ with stabiliser chain $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, then

$$|G| = |\beta_1^{G^0}||\beta_2^{G^1}| \cdots |\beta_r^{G^{r-1}}|.$$

Example (rotation group of cube)

Compute order of rotation group $G \leq \text{Sym}(8)$ for cube: base of adjacent vertices say $1, 2$ (once fixed, can't rotate, so $G_{1,2} = 1$).

$|1^G| = 8$ (all vertices); in G_1, $|2^{G_1}| = 3$ (vertices adjacent to 1); so

$$|G| = |1^G||2^{G_1}| = 8 \cdot 3 = 24.$$

Theorem (size of perm group)
If $B = [\beta_1, \ldots, \beta_r]$ is base for $G \leq \mathrm{Sym}(n)$ with stabiliser chain
$G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, then

$$|G| = |\beta_1^{G^0}||\beta_2^{G^1}| \cdots |\beta_r^{G^{r-1}}|.$$

Orbits and stabilisers can be easily computed (e.g. using GAP).
Implementing base and stabiliser chain for Rubik's group $\mathcal{G}$ (using
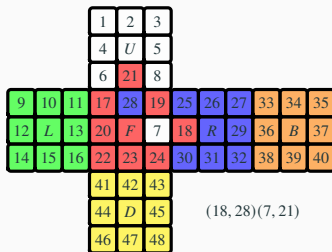BaseOfGroup and StabChain cmds), GAP computes:

Corollary
$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.
(*Note:* $|\mathcal{G}| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11$. Thus no move of order 13.)

Theorem (Wes's conjecture)

"I'm 99% sure you can't swap two [adjacent] edge pieces without affecting another piece?!"

(18, 28)(7, 21)

WLOG consider solved state. *Equivalent question:* does only restickering two adjacent edge pieces give solvable state?

By symmetry, just check one pair, say red/white (18/7) and red/blue (21/28). Four ways: given by where red pieces go (2 options each).

Can this restickering be solved? ii

These restickerings should be invalid states. In group theory language:

Theorem (Wes's conjecture)
$(18, 21)(7, 28) \notin \mathcal{G}$, $(18, 28)(7, 21) \notin \mathcal{G}$, $(18, 21, 7, 28) \notin \mathcal{G}$, and $(18, 28, 7, 21) \notin \mathcal{G}$.

Proof.
By GAP:

```
1 gap> (18,21)(7,28) in G or (18,28)(7,21) in G or
       (18,21,7,28) in G or (18,28,7,21) in G;
2 false
```

(GAP uses stabiliser chains to verify membership!) □

Can generalise to any two edge pieces (more cases)!

We can use GAP to solve Rubik's cube state:

```
1 gap> H := FreeGroup("u","l","f","r","b","d");
2 <free group on the generators [ u, l, f, r, b, d ]>
3 gap> h := GroupHomomorphismByImages( H, G, GeneratorsOfGroup( H ),
       GeneratorsOfGroup( G ) );
4 [ u, l, f, r, b, d ] -> [ (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)
       (11,35,27,19),
5   (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12),
6     (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,
       22)(18,21,23,20), (3,38,43,19)(5,36,45,21)(8,33,48,24)
       (25,27,32,30)(26,29,31,28),
7   (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36),
       (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,
8     46)(42,45,47,44) ]
```

($F = \langle u, \ell, f, r, b, d \rangle$ is free group on 6 generators. Then $f : F \to \mathcal{G}$
is hom given by $u \mapsto U, l \mapsto L, f \mapsto F, r \mapsto R, b \mapsto B, d \mapsto D$.)

Solving a Rubik's cube... ii

To simulate scramble, use `GAP` to generate random state $x \in \mathcal{G}$:

```
1 gap> x := Random( G );
2 (1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)(3,38,17,24,46,41,9)
      (5,26)(7,44,39,23,45,34,21,20,12)(11,30,40,16,35,33,48)(29,36)
```

$$x = (1, 27, 32, 6, 43, 14, 22)(2, 28, 13, 37, 18, 15, 47, 42, 31)$$
$$(3, 38, 17, 24, 46, 41, 9)(5, 26)(7, 44, 39, 23, 45, 34, 21, 20, 12)$$
$$(11, 30, 40, 16, 35, 33, 48)(29, 36)$$

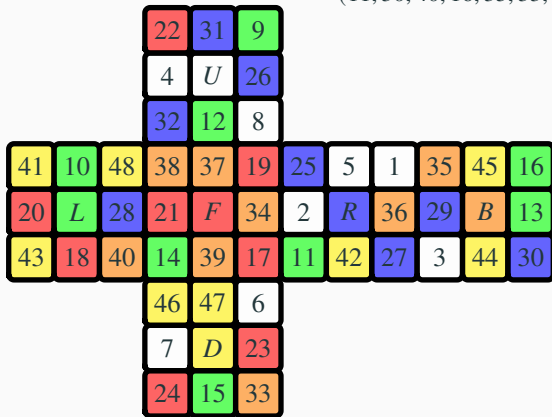Uniform distribution on $\mathcal{G}$ (w.p. $1/|\mathcal{G}| \approx 2.3 \cdot 10^{-20}$).

(*Note:* `GAP` uses stabiliser chain, not sequence of generators.)

$x = (1, 27, 32, 6, 43, 14, 22)(2, 28, 13, 37, 18, 15, 47, 42, 31)$

$(3, 38, 17, 24, 46, 41, 9)(5, 26)(7, 44, 39, 23, 45, 34, 21, 20, 12)$

$(11, 30, 40, 16, 35, 33, 48)(29, 36)$

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
1 gap> PreImagesRepresentative( h, x );
2 l*f^-1*l^-1*f*u*f*u^-1*f^2*l*f*l^-1*u^-1*l^-1*u*l*u^-1*l*u*f*u^-1*f
     ^-1*l^-2*u*l*f^-1*l*f*(l^-1*u)^2*b^-1*u*b*l*u*l^-1*f^-1*l^-1*f*l
     ^2*u*l^-1*u*l*b^-1*u^-1*b*l*d*f^2\
3 *d^-1*l*f^-1*u*l^-1*f*u^-1*l*d^-1*l*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
     u*d^-2
4 gap> Length( last );
5 78
```

$$x = LF^{-1}L^{-1}FUFU^{-1}F^2LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U$$
$$LF^{-1}LF(L^{-1}U)^2B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^2UL^{-1}ULB^{-1}U^{-1}BL$$
$$DF^2D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}$$

(GAP uses stabiliser chains to factorise almost instantly!)

Check this is correct:

```
1 gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
       L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
       B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
       ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
       *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
2 true
```

To solve state x, apply move $x^{-1} \in \mathcal{G}$ since $x^{x^{-1}} = xx^{-1} = 1$:

$$x^{-1} = D^2 U^{-1} F R^{-1} U B^{-1} R B U^2 D^{-1} B^{-1} L^{-1} D L^{-1} U F^{-1} L U^{-1} F L^{-1} D F^{-2} D^{-1}$$
$$L^{-1} B^{-1} U B L^{-1} U^{-1} L U^{-1} L^{-2} F^{-1} L F L U^{-1} L^{-1} B^{-1} U^{-1} B (U^{-1} L)^2 F^{-1} L^{-1} F$$
$$L^{-1} U^{-1} L^2 F U F^{-1} U^{-1} L^{-1} U L U^{-1} L U L F^{-1} L^{-1} F^{-2} U F^{-1} U^{-1} F^{-1} L F L^{-1}$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Not very efficient, since it solves one piece in base B at a time (proceeding up stabiliser chain)... but it works!

Concluding remarks

- Analyzing Rubik's cube with GAP: https://www.gap-system.org/Doc/Examples/rubik.html
- J.A. Paulos – Innumeracy (book)