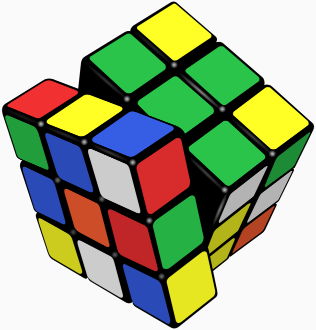
Rubik’s cubes and permutation group theory

Lawrence Chen

October 8, 2022

Honours presentation



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(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

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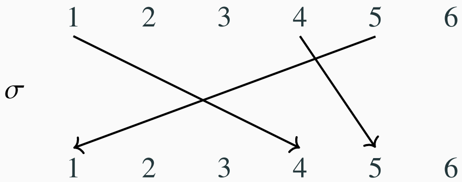

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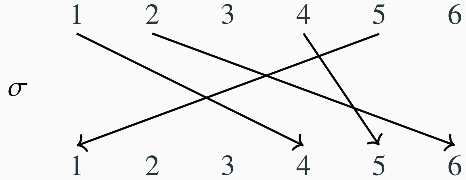

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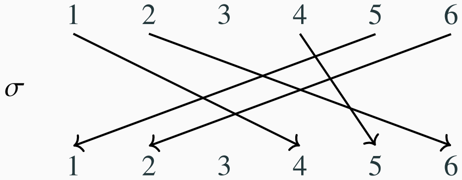

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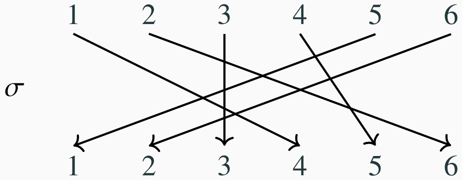

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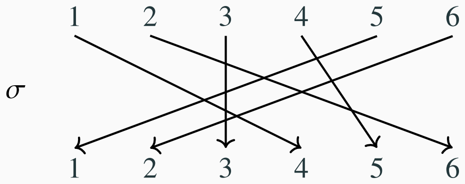

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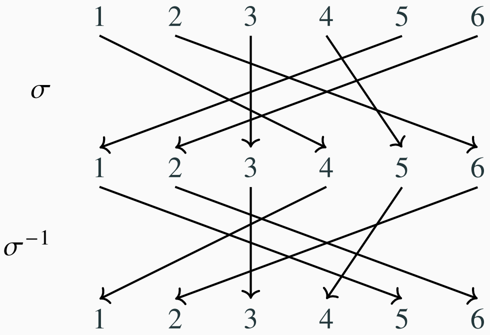

Inverse is $\sigma^{-1} = (1,5,4)(2,6) \in \mathrm{Sym}(6)$.

Permutations iii

Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means “first σ, then τ”, so $i^{\sigma\tau} = (i^{\sigma})^{\tau}$.

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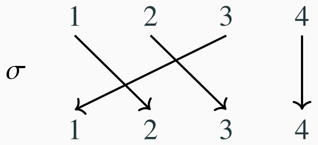

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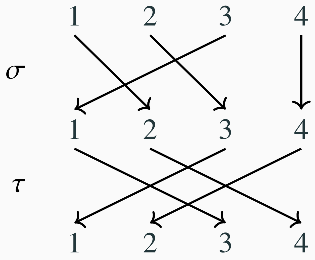

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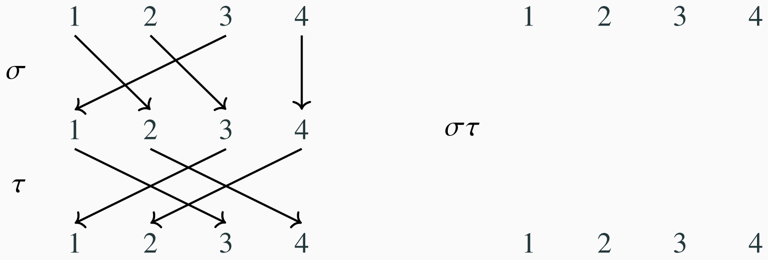

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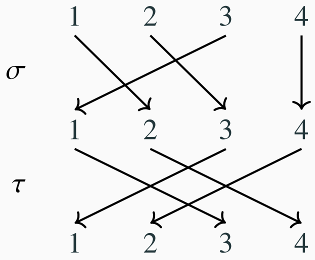

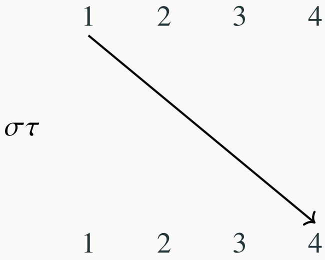

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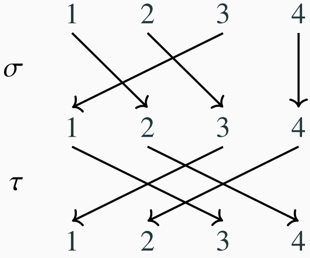

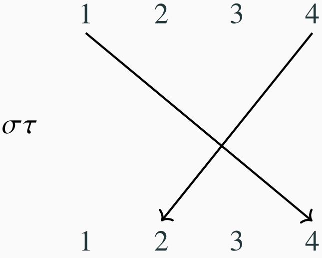

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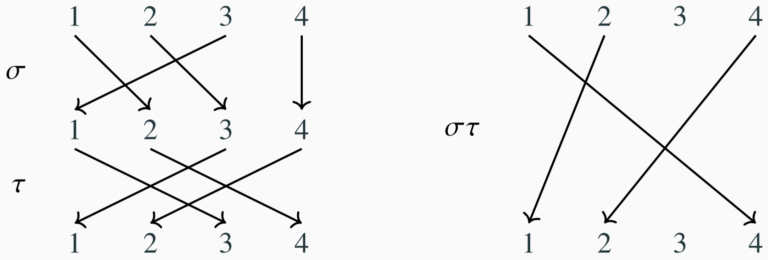

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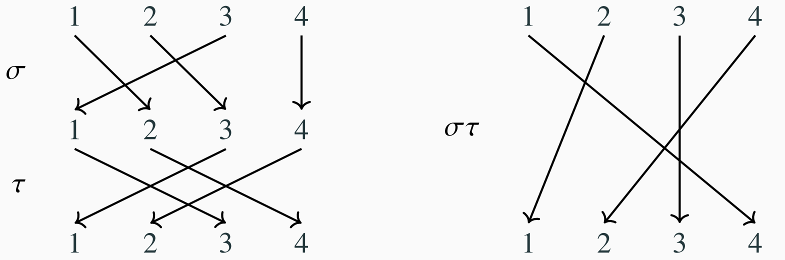
Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means "first σ, then τ", so $i^{\sigma\tau} = (i^{\sigma})^{\tau}$. E.g. $\sigma = (1,2,3), \tau = (1,3)(2,4) \in \mathrm{Sym}(4)$,



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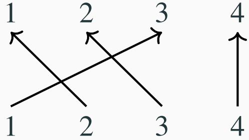

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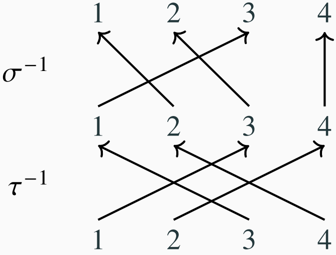

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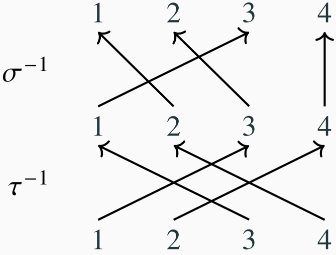

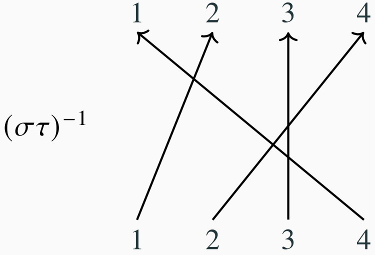

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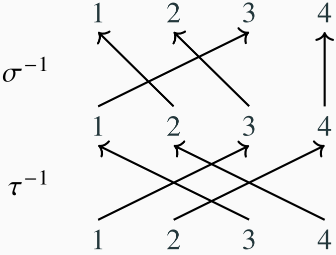

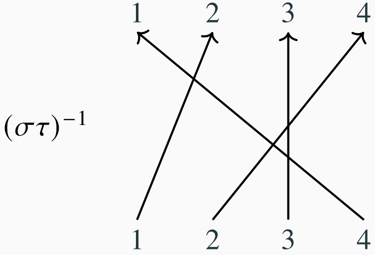

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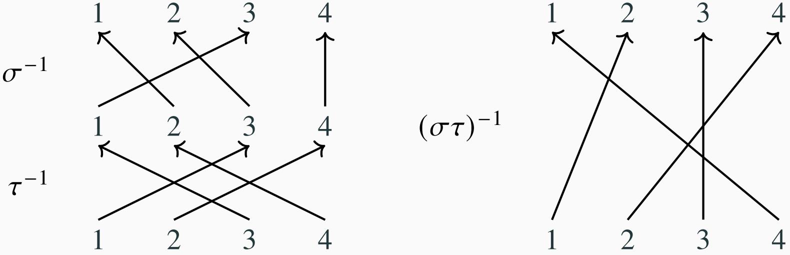

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Permutations v

Set of permutations under *product* is **symmetric group** $\mathrm{Sym}(n)$: identity $1 = ()$, inverses (since bijection), associative.

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Consider $\sigma = (1,4,5)(2,6) \in \mathrm{Sym}(6)$. Then $1^{\sigma^3} = 4^{\sigma^2} = 5^{\sigma} = 1$,

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Subset H of group G is **subgroup** if it is group under same operation; write $H \leq G$. (Need to check: nonempty, closure, inverses.)

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Definition (permutation group)

A **permutation group** of *degree* n is a subgroup of $\mathrm{Sym}(n)$.

Definition (group action)

If G is group and $\Omega \neq \emptyset$ is set, a G**-action** is a map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

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Group actions i

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$\mathbb{Z}$ acts on $\Omega = \{12{:}00, 1{:}00, \ldots, 11{:}00\}$ by $(\alpha{:}00)^k = [\alpha + k]_{12}{:}00$ for $\alpha{:}00 \in \Omega$ and $k \in \mathbb{Z}$.

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Example (right regular action)

Group G acts on $\Omega = G$ (itself) via $\alpha^g = \alpha g$ for $\alpha, g \in G$.

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If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

Idea: states $\alpha^g \in \Omega$ reachable from fixed $\alpha \in \Omega$ by moves $g \in G$.

Group actions ii

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Note: stabiliser G_α is subgroup of G. (So G_α acts on Ω.)

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$G = \{(), (1, 2, 4), (1, 4, 2)\} \leq \mathrm{Sym}(4)$ acts on $\Omega = [4]$ naturally.
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Group actions iii

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Group actions iii

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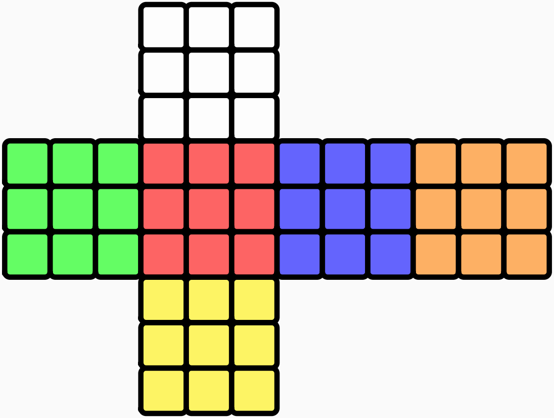
Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its moves i

A Rubik's cube has 6 large faces (each with 3×3 smaller faces).



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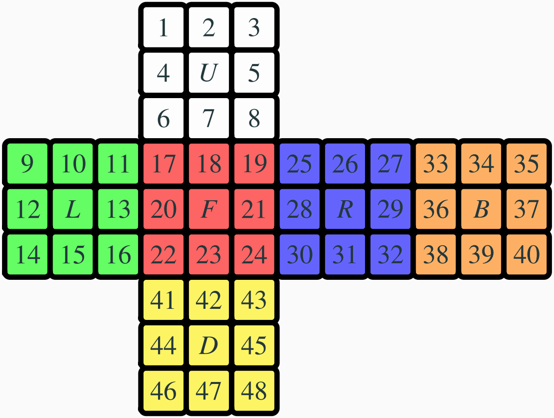
In **solved state** 1, label smaller faces (except each centre) using $[48]$:



Representing the cube and its moves i

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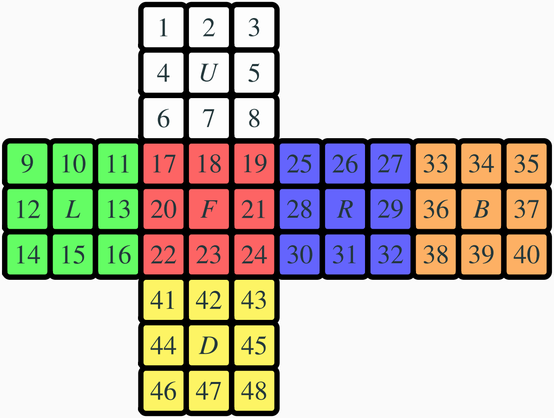
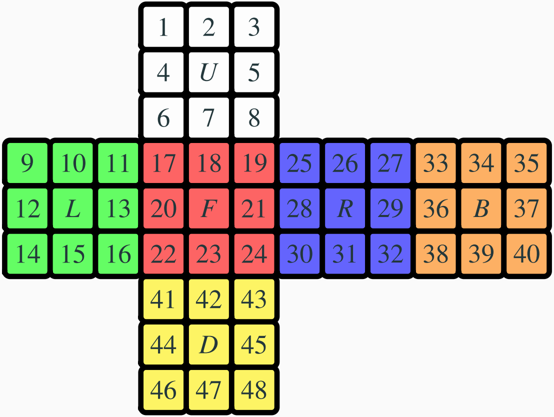


6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its moves ii

From *solved state* 1, consider F which rotates front face clockwise:



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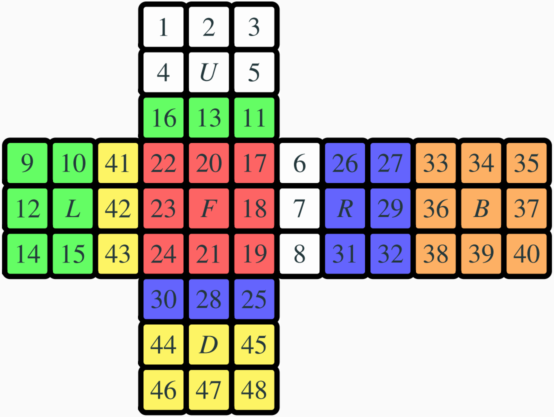

Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

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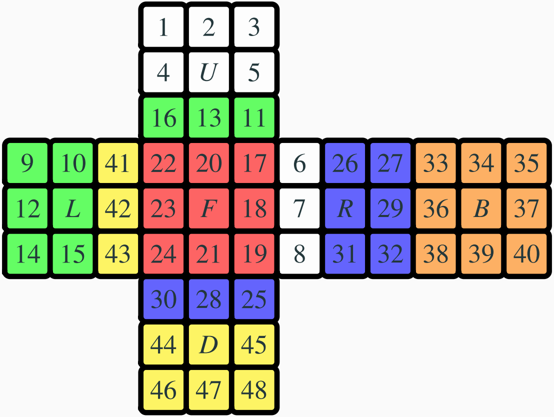

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$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Representing the cube and its moves iii

Generators as permutations of labels [48]:

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Solving is applying valid move to get to solved state 1.

Representing the cube and its moves iv

In cubing community: moves called *move sequences*. Generators called *moves*. *Inverse generators* written U', L', F', R', B', D' (instead of U^{-1} etc.); powers written $U2$, $R2$ etc. (instead of U^2, R^2).

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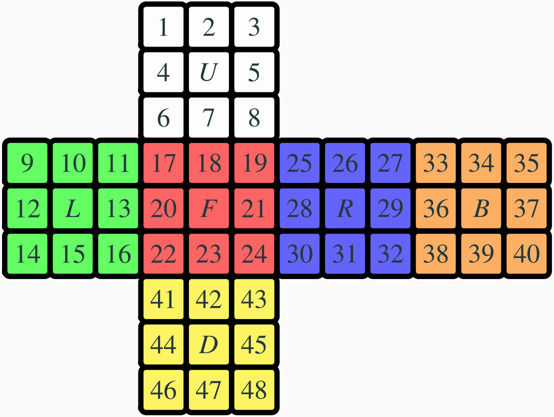
Moves vs states for Rubik's cube i

(Valid) state is result of applying *valid move* to *solved state* 1.



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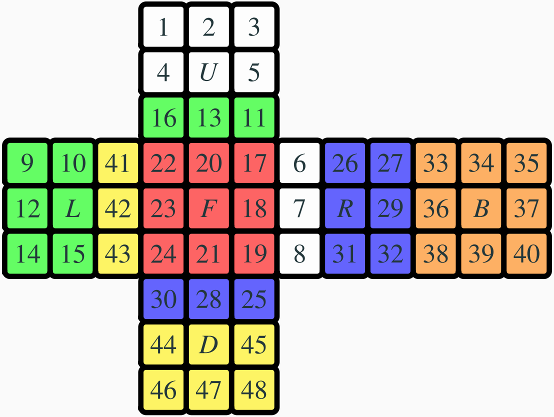

This new state is valid, as result of applying F to solved state.

Moves vs states for Rubik's cube ii

Restickering is valid state iff it can be *solved*. How to check?

Let $\mathcal{S}$ be valid **states**; let state $x \in \mathcal{S}$ be element of $\mathrm{Sym}(48)$ giving permutation of labels to solved state $1 \in \mathcal{S}$.
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So moves $\leftrightarrow$ states; as sets, $\mathcal{S} = \mathcal{G}$. *Solved state* is $1 = () \in \mathrm{Sym}(48)$.

Set of moves $\mathcal{G}$ forms group: composition of valid moves

The Rubik's group of permutations i

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For *move* $\sigma \in \mathcal{G}$ and *state* $x \in \mathcal{S}$, applying σ to x gives *state* $x^{\sigma} = x\sigma \in \mathcal{S}$. This is *regular action* of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

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Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);
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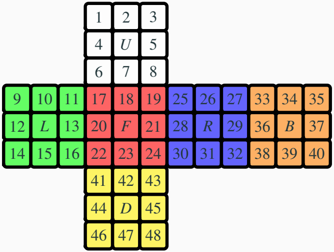
Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

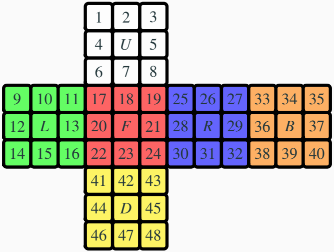
Orbits and stabilisers i




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
      30, 14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
      37, 15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.

Orbits and stabilisers i



Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners $1, 3, 6, 8$.

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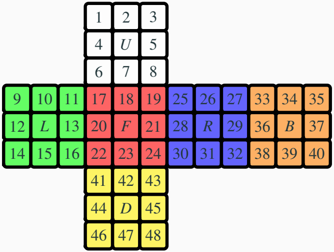

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```
gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
<permutation group of size 317842469683200 with 12 generators>
gap> Orbit( G_1368, 17 );
[ 17 ]
gap> Orbit( G_1368, 24 );
[ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
true
```

Some $\mathcal{H}$-orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Orders of moves i

Use GAP to compute products, order (using `Order` cmd).

```
gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
gap> Order( last );
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How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect?

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Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length $4, 4, 4, 4, 4$: order is $\operatorname{lcm}(4, 4, 4, 4, 4) = 4$.

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- *Commutator* $RUR^{-1}U^{-1}$

  $$= (1, 27, 35, 33, 9, 3)(2, 21, 5)(8, 30, 25, 43, 19, 24)(26, 34, 28):$$

  order is $\text{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

$$= (1, 9, 35)(2, 5, 7)(3, 33, 27)(8, 25, 19)(18, 34, 26):$$

order is $\mathrm{lcm}(3, 3, 3, 3, 3) = 3$.

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Move of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

What is smallest $k \in \mathbb{Z}_+$ with no move of that order?

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There is no Rubik's cube move that cycles through all states.

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If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian.

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Proof.
If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. □

Jake’s theorems ii

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If G is finite group and $g \in G$, then order of g divides $|G|$. So $g^{|G|} = 1$.

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Corollary (Jake Vandenberg's theorem)
There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).

Analysing the Rubik’s group

Bases and stabiliser chains i

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(n)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq [n]$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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Corresponding **stabiliser chain** is

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Base B contains elts of $[n]$ such that only $1 \in G$ fixes every $\beta_i \in B$.

(Short base desirable: how to compute minimum base?)

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Stabiliser chain can be implemented computationally; useful in algorithms (membership testing, random element generation, factorisation into generators).

Example (Rubik's group)

Using GAP:

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gap> BaseOfGroup( G );
[ 1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21,
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18
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Bases and stabiliser chains ii

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If move $\sigma \in \mathcal{G}$ fixes every $\beta_i \in B$ then $\sigma = 1$ is empty move.

How many valid states are there? i

Theorem (size of perm group)
If $B = [\beta_1, \ldots, \beta_r]$ *is base for* $G \leq \mathrm{Sym}(n)$ *with stabiliser chain* $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, *then*

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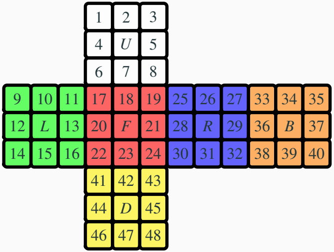
Corollary
$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.
(*Note:* $|\mathcal{G}| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11$. Thus no move of order 13.)

Can this restickering be solved? i

Theorem (Wes's conjecture)

"I'm 99% sure you can't swap two [adjacent] edge pieces without affecting another piece?!"



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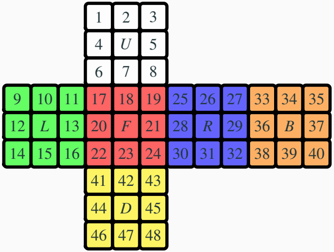

WLOG consider solved state. *Equivalent question:* does only restickering two adjacent edge pieces give solvable state?

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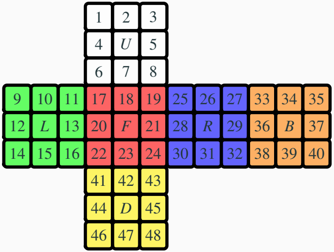

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By symmetry, just check one pair, say red/white (18/7) and red/blue (21/28).

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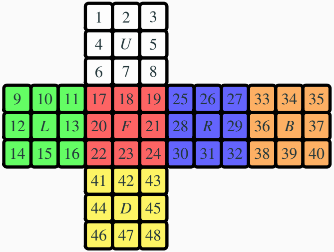

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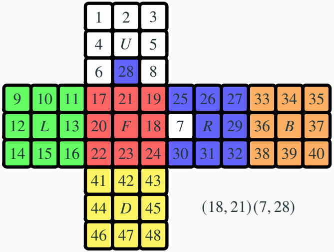

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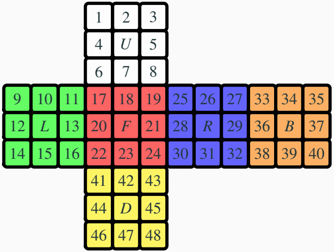

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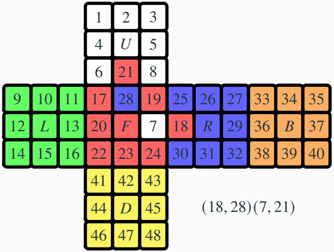

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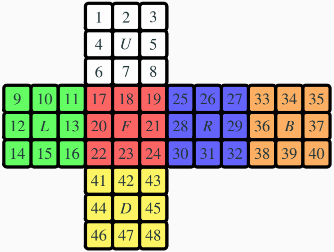

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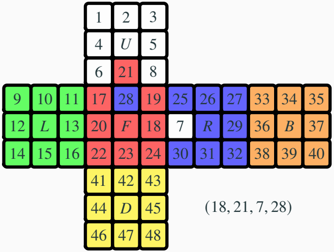

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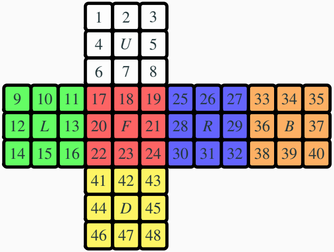

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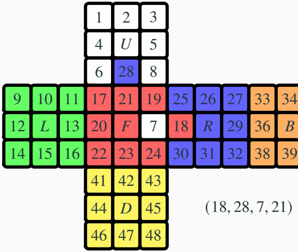

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$(18, 21)(7, 28) \notin \mathcal{G}$, $(18, 28)(7, 21) \notin \mathcal{G}$, $(18, 21, 7, 28) \notin \mathcal{G}$, *and* $(18, 28, 7, 21) \notin \mathcal{G}$.

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Proof.
By GAP:

```
gap> (18,21)(7,28) in G or (18,28)(7,21) in G or
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(GAP uses stabiliser chains to verify membership!) □

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Can generalise to any two edge pieces (more cases)!

Solving a Rubik's cube... i

We can use `GAP` to solve Rubik's cube state:

Solving a Rubik's cube... i

We can use GAP to solve Rubik's cube state:

```
gap> H := FreeGroup("u","l","f","r","b","d");
<free group on the generators [ u, l, f, r, b, d ]>
gap> h := GroupHomomorphismByImages( H, G, GeneratorsOfGroup( H ),
     GeneratorsOfGroup( G ) );
[ u, l, f, r, b, d ] -> [ (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)
     (11,35,27,19),
  (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12),
     (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,
    22)(18,21,23,20), (3,38,43,19)(5,36,45,21)(8,33,48,24)
     (25,27,32,30)(26,29,31,28),
  (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36),
     (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,
    46)(42,45,47,44) ]
```

($F = \langle u, \ell, f, r, b, d\rangle$ is free group on 6 generators. Then $f : F \longrightarrow \mathcal{G}$ is hom given by $u \mapsto U$, $l \mapsto L$, $f \mapsto F$, $r \mapsto R$, $b \mapsto B$, $d \mapsto D$.)

To simulate scramble, use `GAP` to generate random state $x \in \mathcal{G}$:

Solving a Rubik's cube... ii

To simulate scramble, use `GAP` to generate random state $x \in \mathcal{G}$:

```
gap> x := Random( G );
(1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)(3,38,17,24,46,41,9)
     (5,26)(7,44,39,23,45,34,21,20,12)(11,30,40,16,35,33,48)(29,36)
```

$$\begin{aligned} x = {} & (1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31) \\ & \quad (3,38,17,24,46,41,9)(5,26)(7,44,39,23,45,34,21,20,12) \\ & \qquad (11,30,40,16,35,33,48)(29,36). \end{aligned}$$

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$$(3,38,17,24,46,41,9)(5,26)(7,44,39,23,45,34,21,20,12)$$
$$(11,30,40,16,35,33,48)(29,36).$$

Uniform distribution on $\mathcal{G}$ (w.p. $1/|\mathcal{G}| \approx 2.3 \cdot 10^{-20}$).

(*Note:* `GAP` uses stabiliser chain, not sequence of generators.)

$$x = (1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)$$
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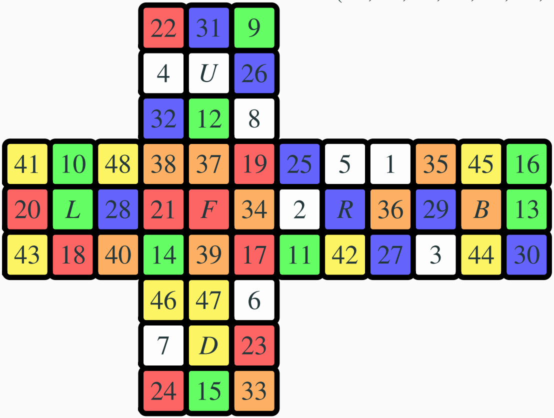

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

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Factorisation into 78 generators and inverses:

```
gap> PreImagesRepresentative( h, x );
l*f^-1*l^-1*f*u*f*u^-1*f^2*l*f*l^-1*u^-1*l^-1*u*l*u^-1*l*u*f*u^-1*f
    ^-1*l^-2*u*l*f^-1*l*f*(l^-1*u)^2*b^-1*u*b*l*u*l^-1*f^-1*l^-1*f*l
    ^2*u*l^-1*u*l*b^-1*u^-1*b*l*d*f^2\
*d^-1*l*f^-1*u*l^-1*f*u^-1*l*d^-1*l*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
    u*d^-2
gap> Length( last );
78
```

$$
\begin{aligned}
x = {} & LF^{-1}L^{-1}FUFU^{-1}F^{2}LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\
& LF^{-1}LF(L^{-1}U)^{2}B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^{2}UL^{-1}ULB^{-1}U^{-1}BL \\
& DF^{2}D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}.
\end{aligned}
$$

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
gap> PreImagesRepresentative( h, x );
l*f^-1*l^-1*f*u*f*u^-1*f^2*l*f*l^-1*u^-1*l^-1*u*l*u^-1*l*u*f*u^-1*f
    ^-1*l^-2*u*l*f^-1*l*f*(l^-1*u)^2*b^-1*u*b*l*u*l^-1*f^-1*l^-1*f*l
    ^2*u*l^-1*u*l*b^-1*u^-1*b*l*d*f^2\
*d^-1*l*f^-1*u*l^-1*f*u^-1*l*d^-1*l*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
    u*d^-2
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```

$$
\begin{aligned}
x = {} & LF^{-1}L^{-1}FUFU^{-1}F^2LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\
& LF^{-1}LF(L^{-1}U)^2B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^2UL^{-1}ULB^{-1}U^{-1}BL \\
& DF^2D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}.
\end{aligned}
$$

(GAP uses stabiliser chains to factorise almost instantly!)

Solving a Rubik’s cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
```

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
```

To solve state x, apply move $x^{-1} \in \mathcal{G}$ since

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gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
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     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
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To solve state x, apply move $x^{-1} \in \mathcal{G}$ since $x^{x^{-1}} = xx^{-1} = 1$:

$$
\begin{aligned}
x^{-1} = {} & D^2U^{-1}FR^{-1}UB^{-1}RBU^2D^{-1}B^{-1}L^{-1}DL^{-1}UF^{-1}LU^{-1}FL^{-1}DF^{-2}D^{-1} \\
& L^{-1}B^{-1}UBL^{-1}U^{-1}LU^{-1}L^{-2}F^{-1}LFLU^{-1}L^{-1}B^{-1}U^{-1}B(U^{-1}L)^2F^{-1}L^{-1}F \\
& L^{-1}U^{-1}L^2FUF^{-1}U^{-1}L^{-1}UL^{-1}U^{-1}LULF^{-1}L^{-1}F^{-2}UF^{-1}U^{-1}F^{-1}LFL^{-1}.
\end{aligned}
$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
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& L^{-1}U^{-1}L^2FUF^{-1}U^{-1}L^{-1}UL^{-1}U^{-1}LULF^{-1}L^{-1}F^{-2}UF^{-1}U^{-1}F^{-1}LFL^{-1}.
\end{aligned}
$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Not very efficient, since it solves one piece in base B at a time (proceeding up stabiliser chain)... but it works!

Concluding remarks

References i

- Analyzing Rubik's cube with GAP: `https://www.gap-system.org/Doc/Examples/rubik.html`
- J.A. Paulos — Innumeracy (book)