

Rubik's cubes and permutation group theory

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Honours presentation



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(J. A. Paulos, Innumeracy)

Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.

Some basic group theory

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$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \sigma & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

It means

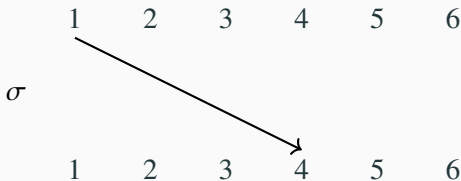
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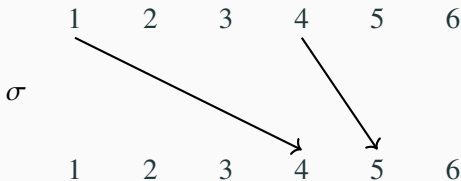
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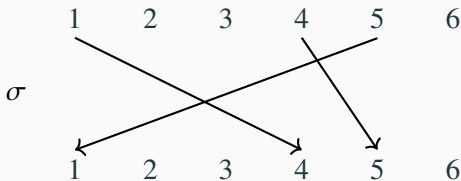
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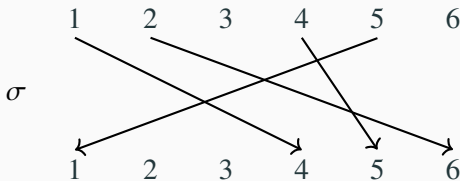
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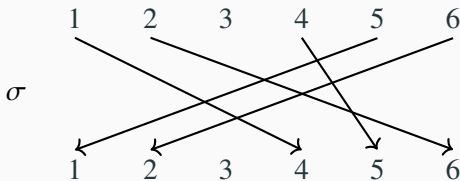
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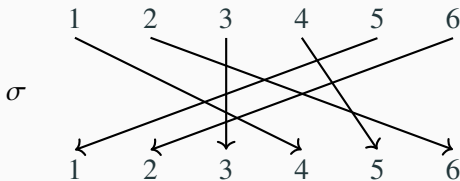
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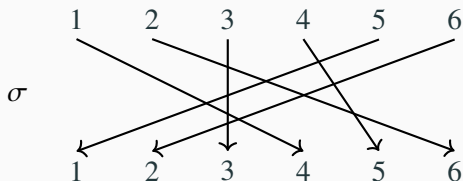


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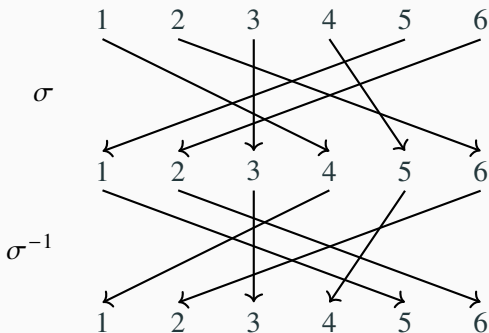
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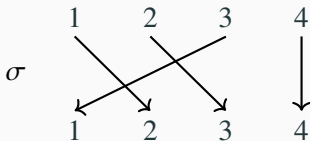


Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \text{Sym}(6)$.

Product/composition: for $\sigma, \tau \in \text{Sym}(n)$, $\sigma\tau$ means “first σ , then τ ”,
so $i^{\sigma\tau} = (i^\sigma)^\tau$.

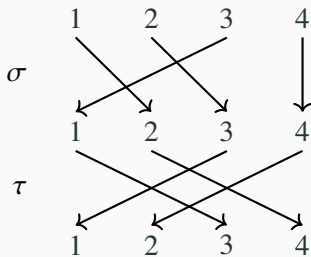
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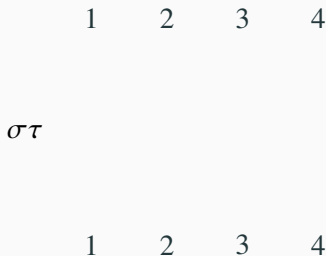
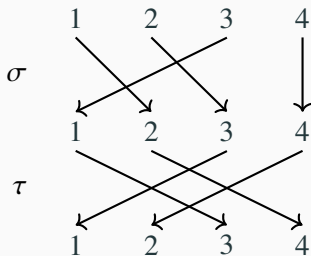
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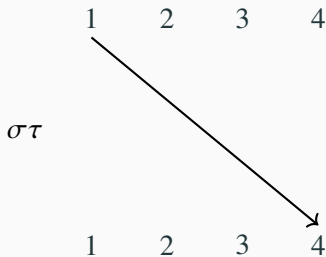
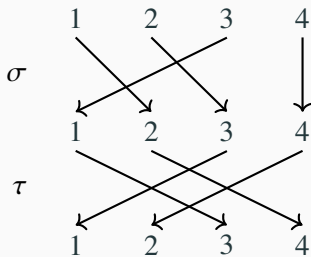
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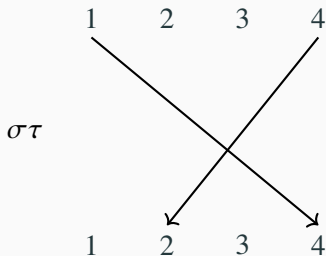
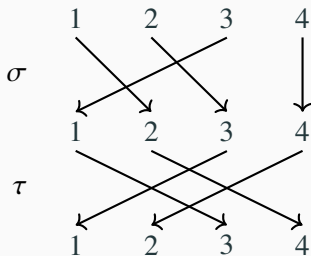
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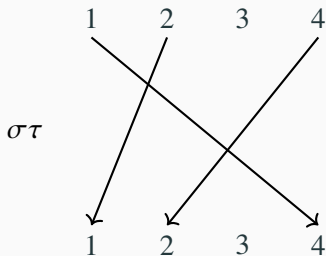
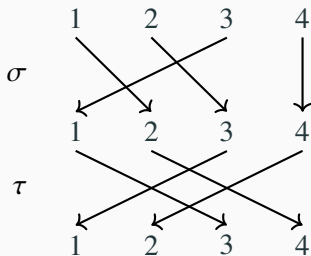
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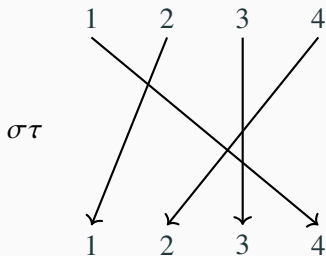
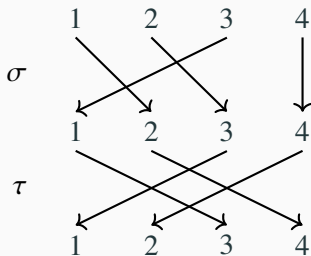
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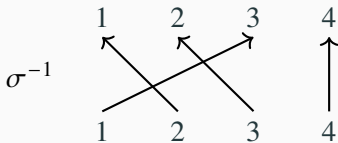
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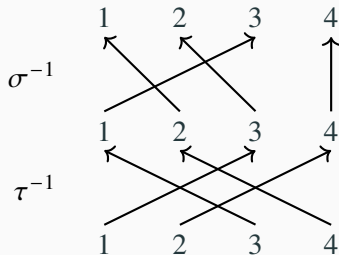
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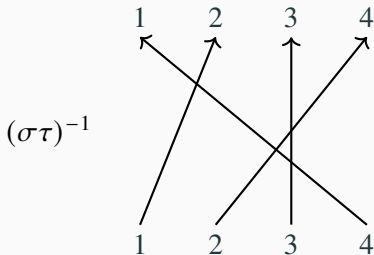
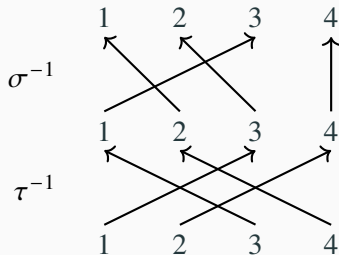
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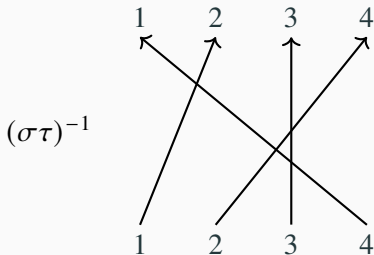
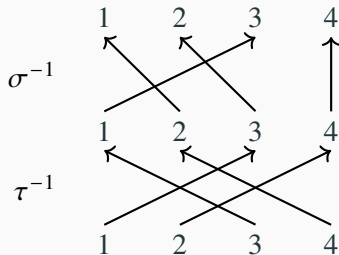
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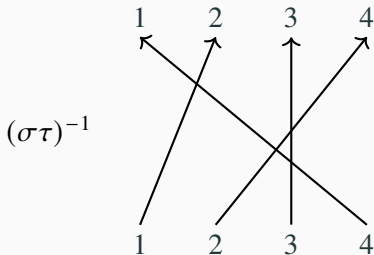
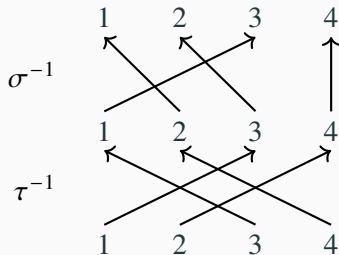


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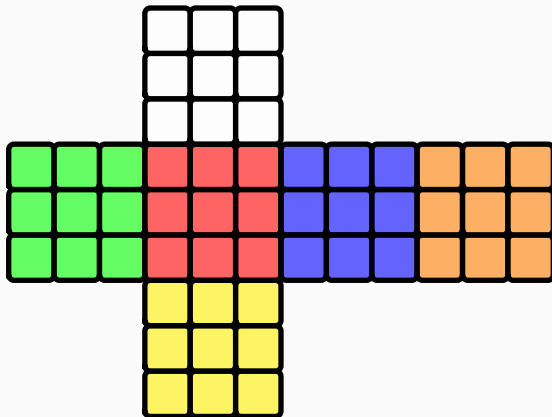
Theorem (orbit-stabiliser)

If G acts on Ω , then for $\alpha \in \Omega$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its moves i

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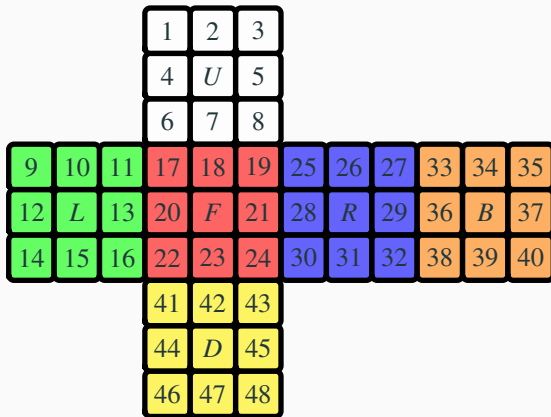
In **solved state 1**, label smaller faces (except each centre) using [48]:

			1	2	3							
			4	<i>U</i>	5							
			6	7	8							
9	10	11	17	18	19	25	26	27	33	34	35	
12	<i>L</i>	13	20	<i>F</i>	21	28	<i>R</i>	29	36	<i>B</i>	37	
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6 generators (moves in CC): U, L, F, R, B, D (rot. clockwise).

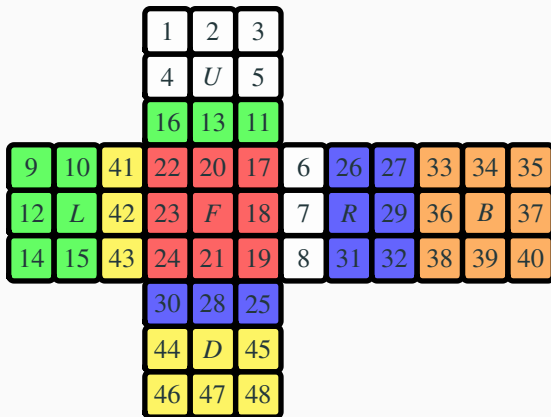
Representing the cube and its moves ii

From *solved state* 1, consider F which rotates front face clockwise:

			1	2	3							
			4	U	5							
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9	10	11	17	18	19	25	26	27	33	34	35	
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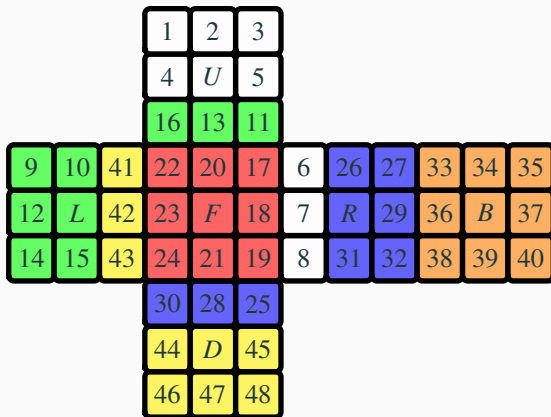
From *solved state 1*, consider F which rotates front face clockwise:



Under F : $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

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From *solved state 1*, consider F which rotates front face clockwise:



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$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \text{Sym}(48).$$

Representing the cube and its moves iii

Generators as permutations of labels [48]:

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(Valid) move is sequence of generators and inverses. E.g.

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Solving is applying valid move to get to solved state 1.

Representing the cube and its moves iv

In cubing community: moves called *move sequences*. Generators called *moves*. *Inverse generators* written U', L', F', R', B', D' (instead of U^{-1} etc.); powers written $U2, R2$ etc. (instead of U^2, R^2).

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Moves vs states for Rubik's cube i

(Valid) state is result of applying *valid move* to *solved state* 1.

			1	2	3							
			4	<i>U</i>	5							
			6	7	8							
9	10	11	17	18	19	25	26	27	33	34	35	
12	<i>L</i>	13	20	<i>F</i>	21	28	<i>R</i>	29	36	<i>B</i>	37	
14	15	16	22	23	24	30	31	32	38	39	40	
			41	42	43							
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14	15	43	24	21	19	8	31	32	38	39	40		
			30	28	25								
			44	<i>D</i>	45								
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This new state is valid, as result of applying *F* to solved state.

Moves vs states for Rubik's cube ii

Restickering is valid state iff it can be *solved*. How to check?

Let \mathcal{S} be valid **states**; let state $x \in \mathcal{S}$ be element of $\text{Sym}(48)$ giving permutation of labels to solved state $1 \in \mathcal{S}$.

(I.e. i^x is label at x -position of i in solved state 1.)

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So moves \leftrightarrow states; as sets, $\mathcal{S} = \mathcal{G}$. *Solved state* is $1 = () \in \text{Sym}(48)$.

The Rubik's group of permutations i

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$\mathcal{G} \leq \text{Sym}(48)$ is permutation group of degree 48, called the **Rubik's group**; it acts naturally on $[48]$.

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For move $\sigma \in \mathcal{G}$ and state $x \in \mathcal{S}$, applying σ to x gives state $x^\sigma = x\sigma \in \mathcal{S}$. This is *regular action* of \mathcal{G} . (Consider states $x \in \mathcal{G}$.)

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Clearly \mathcal{G} finite (states \leftrightarrow moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
1 U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)
      (11,35,27,19);
2 L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)(
      6,22,46,35);
3 F := (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)(
      8,30,41,11);
4 R := (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)(
      8,33,48,24);
5 B := (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)(
      1,14,48,27);
6 D := (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)
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7 G := Group( U, L, F, R, B, D );
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Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits and stabilisers i

			1	2	3						
			4	<i>U</i>	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	<i>L</i>	13	20	<i>F</i>	21	28	<i>R</i>	29	36	<i>B</i>	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	<i>D</i>	45						
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```
1 gap> Orbit( G, 1 );
2 [ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
   30, 14, 19, 9, 22, 38 ]
3 gap> Orbit( G, 2 );
4 [ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
   37, 15, 31, 18, 23, 39 ]
```

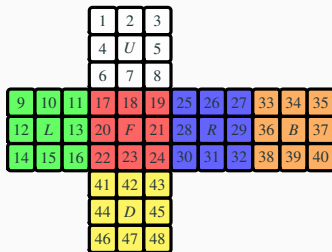
Two \mathcal{G} -orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.

Orbits and stabilisers i

		1	2	3					
		4	U	5					
		6	7	8					
9	10	11	17	18	19	25	26	27	33
12	L	13	20	F	21	28	R	29	36
14	15	16	22	23	24	30	31	32	38
			41	42	43				
			44	D	45				
			46	47	48				

Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners 1, 3, 6, 8.

Orbits and stabilisers i



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```

1 gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
2 <permutation group of size 3178424696832000 with 12 generators>
3 gap> Orbit( G_1368, 17 );
4 [ 17 ]
5 gap> Orbit( G_1368, 24 );
6 [ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
7 gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
8 true

```

Some \mathcal{H} -orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Orders of moves i

Use GAP to compute products, order (using Order cmd).

```
1 gap> R*U*R^(-1)*U^(-1);  
2 (1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)  
3 gap> Order( last );  
4 6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect?

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Recall: order of σ is lcm of cycle lengths.

- Any generator (U, L, F, R, B, D) has cycles of length 4, 4, 4, 4, 4:
order is $\text{lcm}(4, 4, 4, 4, 4) = 4$.

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 $= (1, 27, 35, 33, 9, 3)(2, 21, 5)(8, 30, 25, 43, 19, 24)(26, 34, 28):$
order is $\text{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

$$= (1, 9, 35)(2, 5, 7)(3, 33, 27)(8, 25, 19)(18, 34, 26):$$

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- *Lawrence's move* RU has cycles of length 15, 7, 3, 7: order is $\text{lcm}(15, 7, 3, 7) = 105$.

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Watch video demonstration by my friend Wes :D

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Move of order 5?

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Move of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

What is *smallest* $k \in \mathbb{Z}_+$ with no move of that order?

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There is no Rubik's cube move that cycles through all states.

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Proof.

If \mathcal{G} is cyclic, then \mathcal{G} is abelian.

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Proof.

If \mathcal{G} is cyclic, then \mathcal{G} is abelian. But \mathcal{G} is not abelian: $RU \neq UR$. \square

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If G is finite group and $g \in G$, then order of g divides $|G|$. So $g^{|G|} = 1$.

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Corollary (Jake Vandenberg's theorem)

There is no $\sigma \in \mathcal{G}$ with infinite order (since \mathcal{G} is finite).

Analysing the Rubik's group

Definition (Base, stabiliser chain)

If $G \leq \text{Sym}(n)$, distinct elts $B = [\beta_1, \dots, \beta_r] \subseteq [n]$ is **base** for G if $G_{\beta_1, \dots, \beta_r} = 1$. (Recall: $G_{\beta_1, \dots, \beta_r} = \{g \in G : \beta_1^g = \beta_1, \dots, \beta_r^g = \beta_r\}$.)

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Corresponding **stabiliser chain** is

$$G = G^0 \geq G^1 \geq \dots \geq G^r = 1$$

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Bases and stabiliser chains i

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Base B contains elts of $[n]$ such that only $1 \in G$ fixes every $\beta_i \in B$.

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Stabiliser chain can be implemented computationally; useful in algorithms (membership testing, random element generation, factorisation into generators).

Example (Rubik's group)

Using GAP:

```
1 gap> BaseOfGroup( G );
2 [ 1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21,
   23, 24, 29, 31 ]
3 gap> Size( last );
4 18
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If move $\sigma \in \mathcal{G}$ fixes every $\beta_i \in B$ then $\sigma = 1$ is empty move.

How many valid states are there? i

Theorem (size of perm group)

If $B = [\beta_1, \dots, \beta_r]$ is base for $G \leq \text{Sym}(n)$ with stabiliser chain $G = G^0 \geq G^1 \geq \dots \geq G^r = 1$, then

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$$|G| = |1^G| |2^{G_1}| = 8 \cdot 3 = 24.$$

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Implementing base and stabiliser chain for Rubik's group \mathcal{G} (using `BaseOfGroup` and `StabChain` cmds), GAP computes:

Corollary

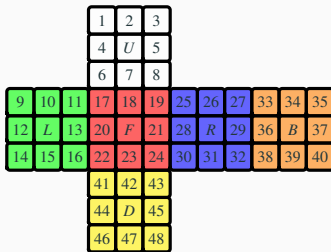
$$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}.$$

(Note: $|\mathcal{G}| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11$. Thus no move of order 13.)

Can this restickering be solved? i

Theorem (Wes's conjecture)

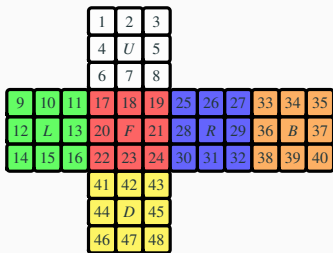
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By symmetry, just check one pair, say red/white (18/7) and red/blue (21/28).

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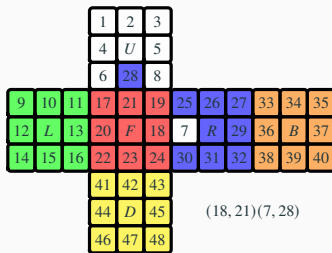
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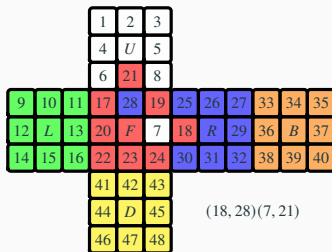
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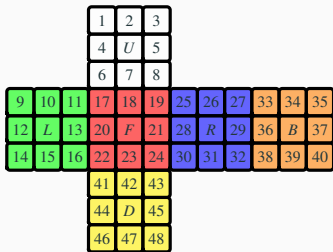
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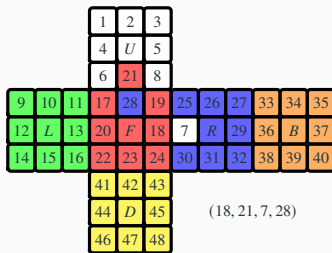
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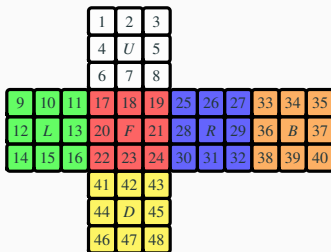
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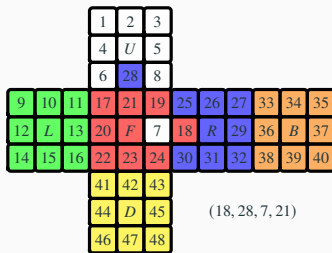
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$(18, 21)(7, 28) \notin \mathcal{G}$, $(18, 28)(7, 21) \notin \mathcal{G}$, $(18, 21, 7, 28) \notin \mathcal{G}$, and $(18, 28, 7, 21) \notin \mathcal{G}$.

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Proof.

By GAP:

```
1 gap> (18,21)(7,28) in G or (18,28)(7,21) in G or  
      (18,21,7,28) in G or (18,28,7,21) in G;  
2 false
```

(GAP uses stabiliser chains to verify membership!)

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Can generalise to any two edge pieces (more cases)!

Solving a Rubik's cube... i

We can use GAP to solve Rubik's cube state:

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```
1 gap> H := FreeGroup("u","l","f","r","b","d");
2 <free group on the generators [ u, l, f, r, b, d ]>
3 gap> h := GroupHomomorphismByImages( H, G, GeneratorsOfGroup( H ),
    GeneratorsOfGroup( G ) );
4 [ u, l, f, r, b, d ] -> [ (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)
    (11,35,27,19),
5 (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12),
    (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,
6 22)(18,21,23,20), (3,38,43,19)(5,36,45,21)(8,33,48,24)
    (25,27,32,30)(26,29,31,28),
7 (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36),
    (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,
8 46)(42,45,47,44) ]
```

$(F = \langle u, \ell, f, r, b, d \rangle)$ is free group on 6 generators. Then $f : F \rightarrow \mathcal{G}$ is hom given by $u \mapsto U, l \mapsto L, f \mapsto F, r \mapsto R, b \mapsto B, d \mapsto D.$)

To simulate scramble, use GAP to generate random state $x \in \mathcal{G}$:

Solving a Rubik's cube... ii

To simulate scramble, use GAP to generate random state $x \in \mathcal{G}$:

```
1 gap> x := Random( G );  
2 (1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)(3,38,17,24,46,41,9)  
   (5,26)(7,44,39,23,45,34,21,20,12)(11,30,40,16,35,33,48)(29,36)
```

$$x = (1, 27, 32, 6, 43, 14, 22)(2, 28, 13, 37, 18, 15, 47, 42, 31) \\ (3, 38, 17, 24, 46, 41, 9)(5, 26)(7, 44, 39, 23, 45, 34, 21, 20, 12) \\ (11, 30, 40, 16, 35, 33, 48)(29, 36).$$

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Uniform distribution on \mathcal{G} (w.p. $1/|\mathcal{G}| \approx 2.3 \cdot 10^{-20}$).

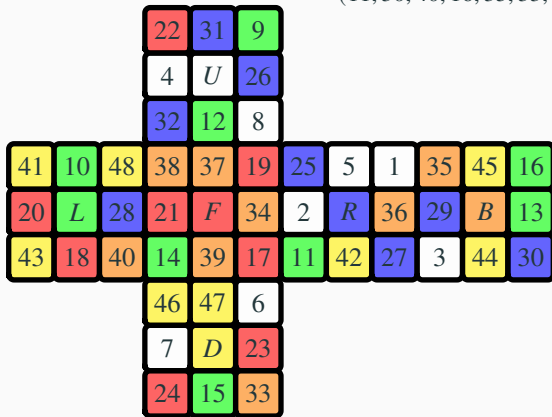
(Note: GAP uses stabiliser chain, not sequence of generators.)

Solving a Rubik's cube... iii

$$x = (1, 27, 32, 6, 43, 14, 22)(2, 28, 13, 37, 18, 15, 47, 42, 31)$$

$$(3, 38, 17, 24, 46, 41, 9)(5, 26)(7, 44, 39, 23, 45, 34, 21, 20, 12)$$

$$(11, 30, 40, 16, 35, 33, 48)(29, 36)$$



Factorisation into 78 generators and inverses:

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
1 gap> PreImagesRepresentative( h, x );
2 1*f^-1*1^-1*f*u*f*u^-1*f^2*1*f*1^-1*u^-1*1^-1*u*1*u^-1*1*u*f*u^-1*f
   ^-1*1^-2*u*1*f^-1*1*f*(1^-1*u)^2*b^-1*u*b*1*u*1^-1*f^-1*1^-1*f*1
   ^2*u*1^-1*u*1*b^-1*u^-1*b*1*d*f^2\
3 *d^-1*1*f^-1*u*1^-1*f*u^-1*1*d^-1*1*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
   u*d^-2
4 gap> Length( last );
5 78
```

$$x = LF^{-1}L^{-1}FUFU^{-1}F^2LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\ LF^{-1}LF(L^{-1}U)^2B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^2UL^{-1}ULB^{-1}U^{-1}BL \\ DF^2D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}.$$

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
1 gap> PreImagesRepresentative( h, x );
2 1*f^-1*1^-1*f*u*f*u^-1*f^2*1*f*1^-1*u^-1*1^-1*u*1*u^-1*1*u*f*u^-1*f
   ^-1*1^-2*u*1*f^-1*1*f*(1^-1*u)^2*b^-1*u*b*1*u*1^-1*f^-1*1^-1*f*1
   ^2*u*1^-1*u*1*b^-1*u^-1*b*1*d*f^2\
3 *d^-1*1*f^-1*u*1^-1*f*u^-1*1*d^-1*1*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
   u*d^-2
4 gap> Length( last );
5 78
```

$$x = LF^{-1}L^{-1}FUFU^{-1}F^2LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\ LF^{-1}LF(L^{-1}U)^2B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^2UL^{-1}ULB^{-1}U^{-1}BL \\ DF^2D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}.$$

(GAP uses stabiliser chains to factorise almost instantly!)

Solving a Rubik's cube... v

Check this is correct:

```
1 gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*  
L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*  
B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U  
^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D  
*U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);  
2 true
```

Solving a Rubik's cube... v

Check this is correct:

```
1 gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
  L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
  B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
  ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
  *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
2 true
```

To solve state x , apply move $x^{-1} \in \mathcal{G}$ since

Solving a Rubik's cube... v

Check this is correct:

```
1 gap> x = L * F ^ (-1) * L ^ (-1) * F * U * F * U ^ (-1) * F ^ 2 * L * F * L ^ (-1) * U ^ (-1) * L ^ (-1) * U *  
    L * U ^ (-1) * L * U * F * U ^ (-1) * F ^ (-1) * L ^ (-2) * U * L * F ^ (-1) * L * F * (L ^ (-1) * U) ^ 2 *  
    B ^ (-1) * U * B * L * U * L ^ (-1) * F ^ (-1) * L ^ (-1) * F * L ^ 2 * U * L ^ (-1) * U * L * B ^ (-1) * U  
    ^ (-1) * B * L * D * F ^ 2 * D ^ (-1) * L * F ^ (-1) * U * L ^ (-1) * F * U ^ (-1) * L * D ^ (-1) * L * B * D  
    * U ^ (-2) * B ^ (-1) * R ^ (-1) * B * U ^ (-1) * R * F ^ (-1) * U * D ^ (-2);  
2 true
```

To solve state x , apply move $x^{-1} \in \mathcal{G}$ since $x x^{-1} = x x^{-1} = 1$:

$$\begin{aligned} x^{-1} = & D^2 U^{-1} F R^{-1} U B^{-1} R B U^2 D^{-1} B^{-1} L^{-1} D L^{-1} U F^{-1} L U^{-1} F L^{-1} D F^{-2} D^{-1} \\ & L^{-1} B^{-1} U B L^{-1} U^{-1} L U^{-1} L^{-2} F^{-1} L F L U^{-1} L^{-1} B^{-1} U^{-1} B (U^{-1} L)^2 F^{-1} L^{-1} F \\ & L^{-1} U^{-1} L^2 F U F^{-1} U^{-1} L^{-1} U L^{-1} U^{-1} L U L F^{-1} L^{-1} F^{-2} U F^{-1} U^{-1} F^{-1} L F L^{-1}. \end{aligned}$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Solving a Rubik's cube... v

Check this is correct:

```
1 gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
  L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
  B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
  ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
  *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
2 true
```

To solve state x , apply move $x^{-1} \in \mathcal{G}$ since $x^{x^{-1}} = xx^{-1} = 1$:

$$\begin{aligned} x^{-1} = & D^2 U^{-1} F R^{-1} U B^{-1} R B U^2 D^{-1} B^{-1} L^{-1} D L^{-1} U F^{-1} L U^{-1} F L^{-1} D F^{-2} D^{-1} \\ & L^{-1} B^{-1} U B L^{-1} U^{-1} L U^{-1} L^{-2} F^{-1} L F L U^{-1} L^{-1} B^{-1} U^{-1} B (U^{-1} L)^2 F^{-1} L^{-1} F \\ & L^{-1} U^{-1} L^2 F U F^{-1} U^{-1} L^{-1} U L^{-1} U^{-1} L U L F^{-1} L^{-1} F^{-2} U F^{-1} U^{-1} F^{-1} L F L^{-1}. \end{aligned}$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Not very efficient, since it solves one piece in base B at a time (proceeding up stabiliser chain)... but it works!

Concluding remarks

- Analyzing Rubik's cube with GAP: <https://www.gap-system.org/Doc/Examples/rubik.html>
- J.A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Orders of elements in Rubik's group (1260 largest, 13 smallest without, 11 rarest, 60 most common, median 67.3, 73 options):
<https://www.jaapsch.net/puzzles/cubic3.htm#p34>
- Thistlethwaite's 52 move algorithm (using group theory):
<https://www.jaapsch.net/puzzles/thistle.htm>