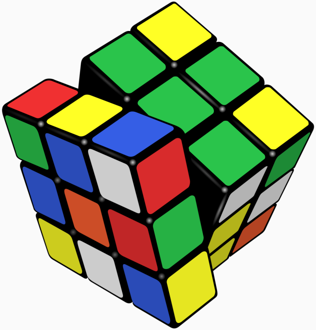
Rubik's cubes and permutation group theory

Lawrence Chen

October 7, 2022

Honours presentation



Contents

Some basic group theory

What is a group? i

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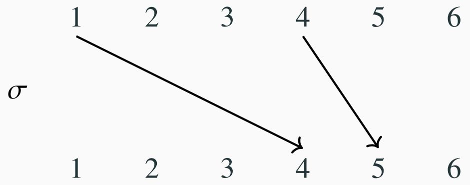

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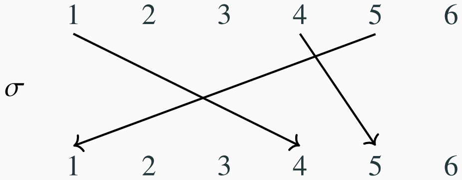

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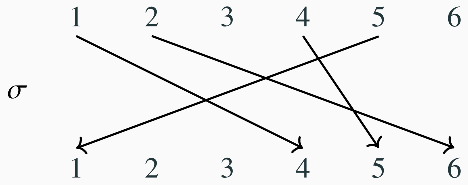

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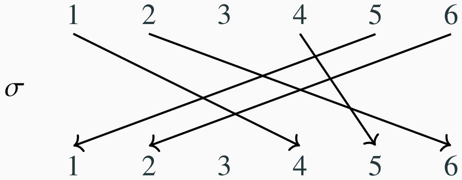

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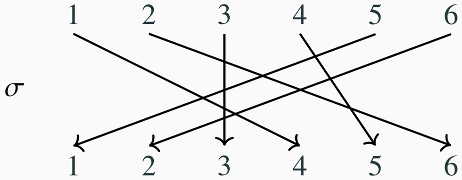

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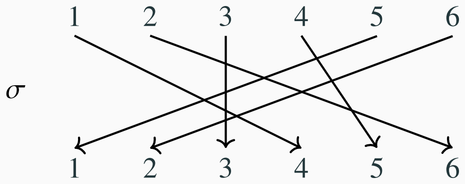

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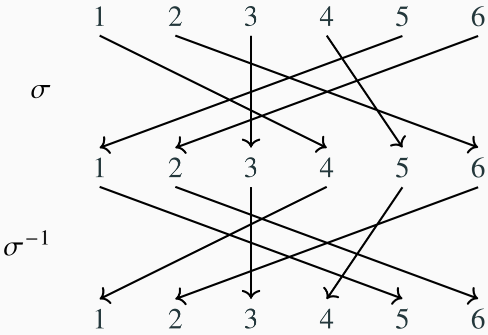

Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \mathrm{Sym}(6)$.

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Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means "first σ, then τ", so $i^{\sigma\tau} = (i^{\sigma})^{\tau}$.

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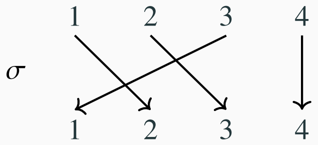

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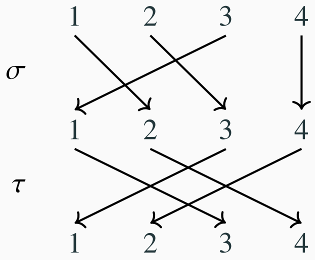

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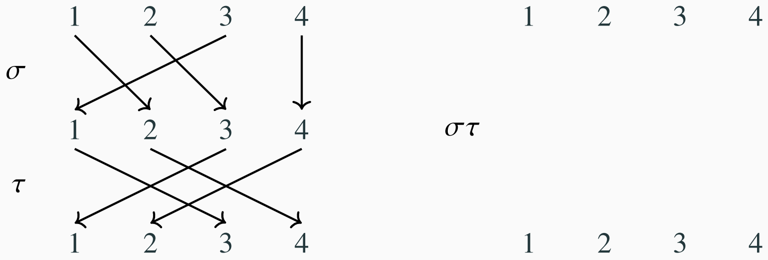

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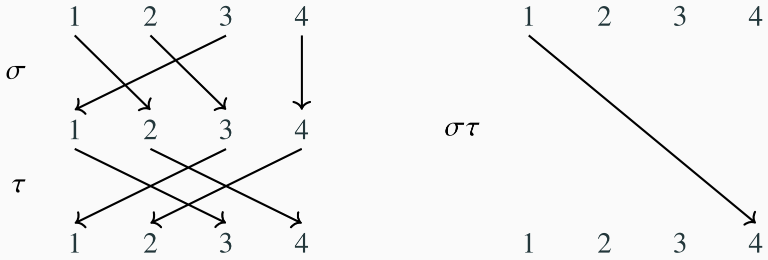

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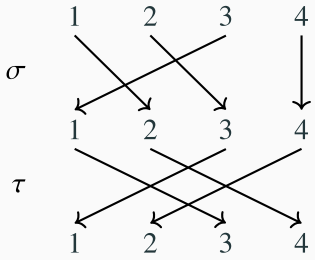

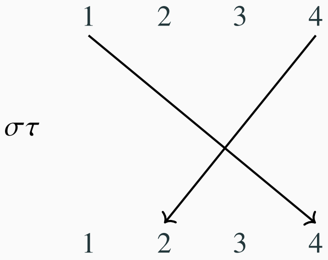

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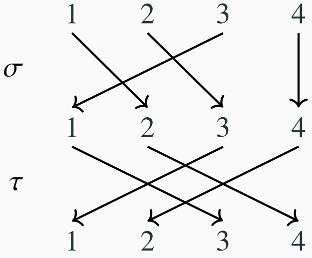

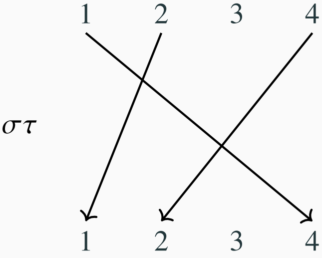

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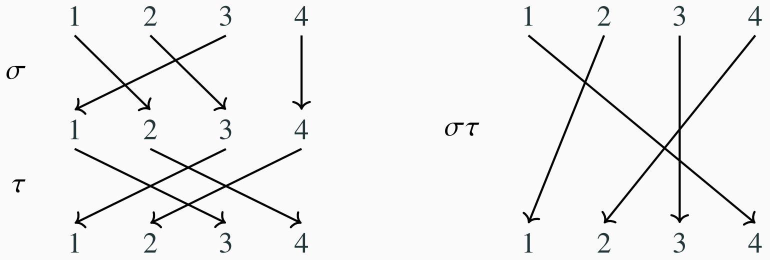

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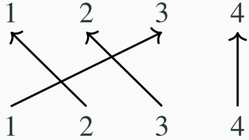

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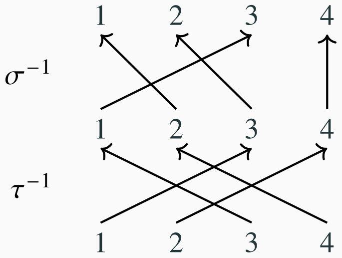

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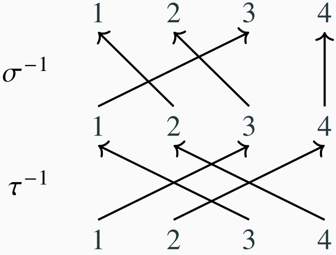

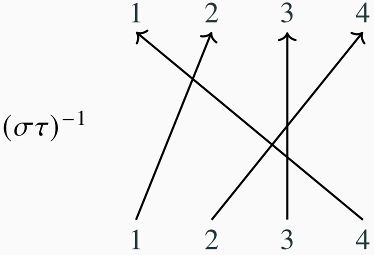

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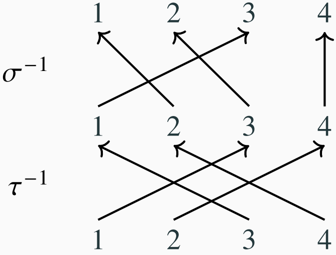

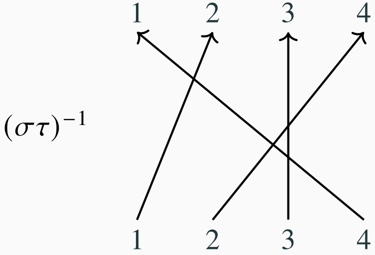

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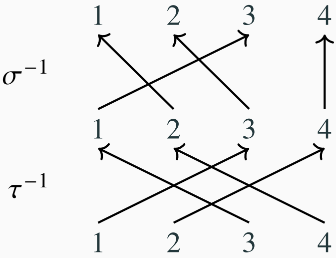

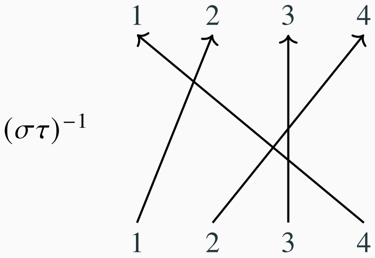

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If G is group and $\Omega \neq \emptyset$ is set, a G**-action** is a map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

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Group actions ii

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Group actions iii

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$G = \{(), (1,2,4), (1,4,2)\} \leq \mathrm{Sym}(4)$ acts on $\Omega = [4]$ naturally.

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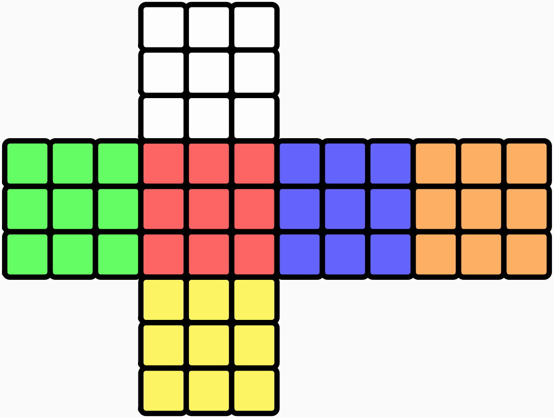
Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its moves i

A Rubik's cube has 6 large faces (each with 3×3 smaller faces).



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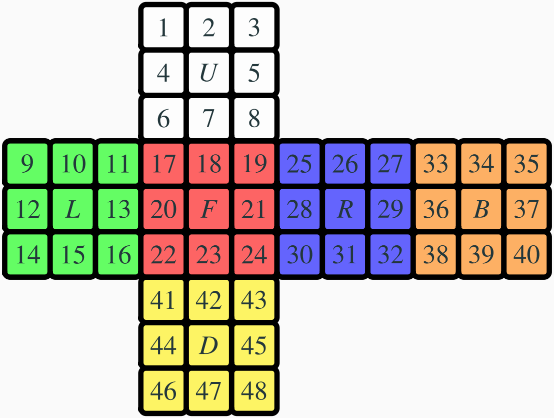

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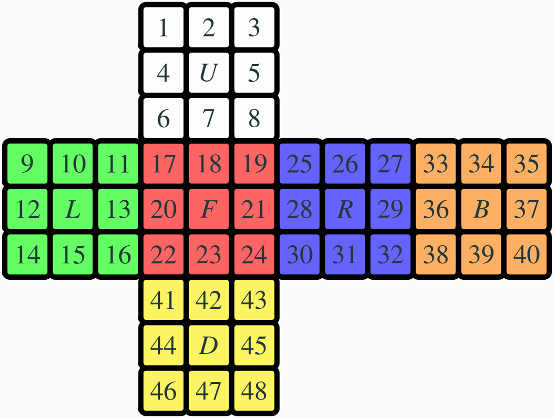

6 **elementary moves (generators)**: U, L, F, R, B, D (rot. *clockwise*).

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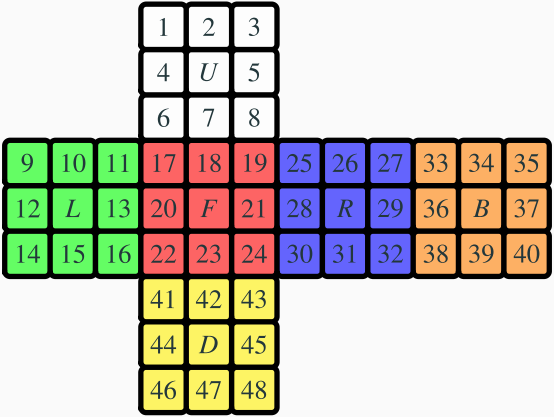

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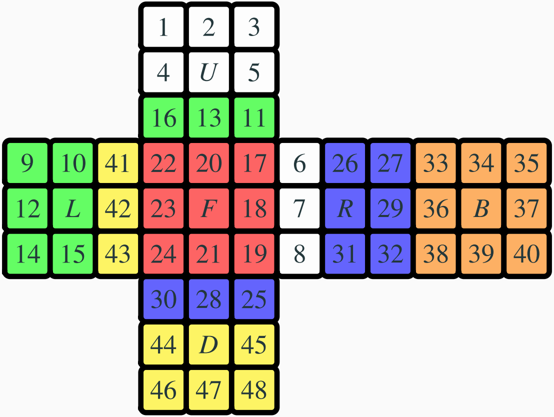
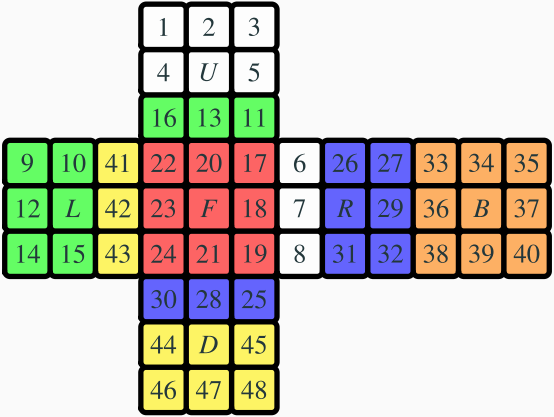


Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

Representing the cube and its moves ii

From *solved state*, consider F which rotates front face clockwise:



Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Representing the cube and its moves iii

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
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- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

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Solving is applying valid move to get to solved state 1.

Representing the cube and its moves iv

In cubing community: inverse elementary moves written U', L', F', R', B', D' (instead of U^{-1} etc.); powers written $U2$, $R2$ etc. (instead of U^2, R^2).

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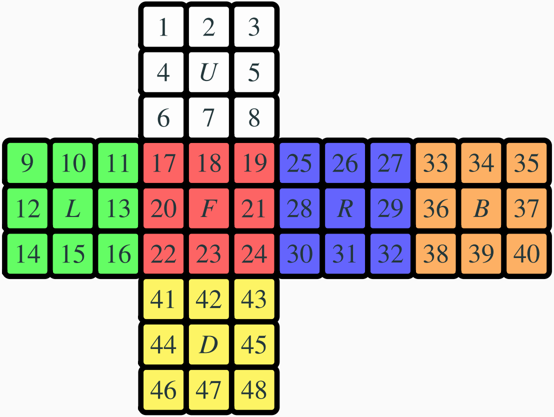
Moves vs states for Rubik's cube i

(Valid) state is result of applying *valid move* to *solved state* $1 = ()$.



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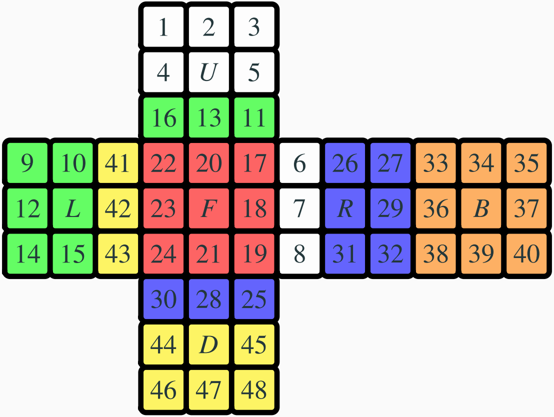

This new state is valid, as result of applying F to solved state.

Moves vs states for Rubik's cube ii

Restickering is valid state iff it can be *solved*. How to check?

Let $\mathcal{S}$ be valid **states**; can represent state $x \in \mathcal{S}$ as element of Sym(48) giving permutation of labels to solved state $1 = ()$. (I.e. i^x is label at x-position of i in solved state 1.)

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So moves $\leftrightarrow$ states for Rubik's cube; as sets, $\mathcal{S} = \mathcal{G}$.

The Rubik's group of permutations i

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Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);
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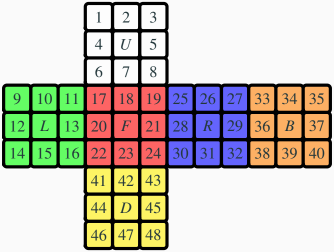
`Order` cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

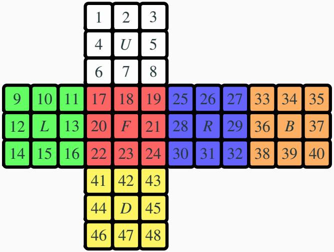
Orbits and stabilisers i




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
      30, 14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
      37, 15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.

Orbits and stabilisers ii



Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners 1, 3, 6, 8.

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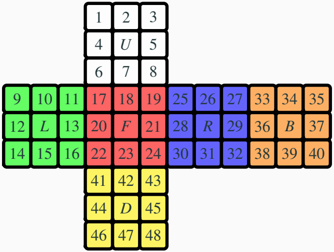

Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners 1, 3, 6, 8.

```
gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
<permutation group of size 317842469683200 with 12 generators>
gap> Orbit( G_1368, 17 );
[ 17 ]
gap> Orbit( G_1368, 24 );
[ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
true
```

Some $\mathcal{H}$-orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Orders of moves i

Use GAP to compute products, order (using `Order` cmd).

```
gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
gap> Order( last );
6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect?

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Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length $4, 4, 4, 4, 4$: order is $\operatorname{lcm}(4, 4, 4, 4, 4) = 4$.

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6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect? I.e. for state $x \in \mathcal{S}$, smallest $k \in \mathbb{Z}_+$ with $x^{\sigma^k} = x\sigma^k = x \iff \sigma^k = 1$.

Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length $4, 4, 4, 4, 4$: order is $\operatorname{lcm}(4, 4, 4, 4, 4) = 4$.
- *Commutator* $RUR^{-1}U^{-1}$

  $$= (1, 27, 35, 33, 9, 3)(2, 21, 5)(8, 30, 25, 43, 19, 24)(26, 34, 28):$$

  order is $\operatorname{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

$$= (1, 9, 35)(2, 5, 7)(3, 33, 27)(8, 25, 19)(18, 34, 26):$$

order is $\operatorname{lcm}(3, 3, 3, 3, 3) = 3$.

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Element of order 5?

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Element of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

Jake's theorems i

Theorem (Jake Vandenberg's conjecture)
There is no Rubik's cube move that cycles through all states.

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If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian.

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Proof.
If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. $\square$

Jake's theorems ii

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Corollary (Jake Vandenberg's theorem)
There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).

Analysing the Rubik's group

Bases and stabiliser chains i

TODO

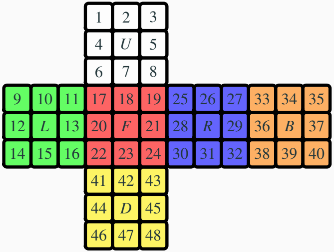
How many valid states are there? i

TODO

Can this restickering be solved? i

Theorem (Wes's conjecture)

"I'm 99% sure you can't swap two [adjacent] edge pieces without affecting another piece?!"



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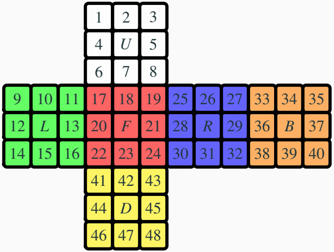

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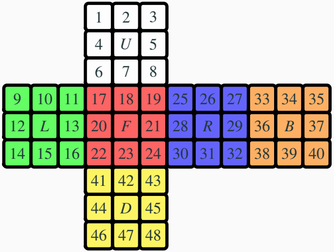

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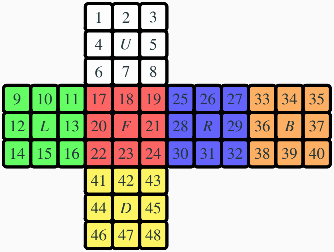

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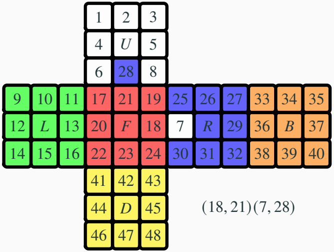

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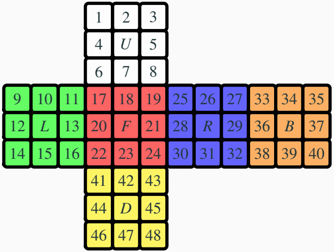

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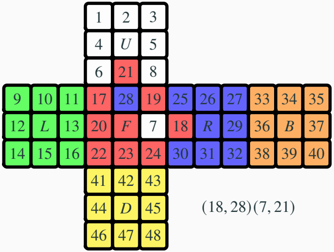

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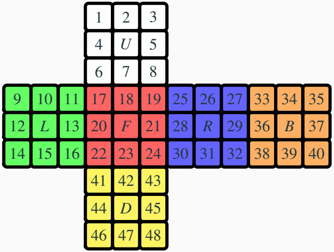

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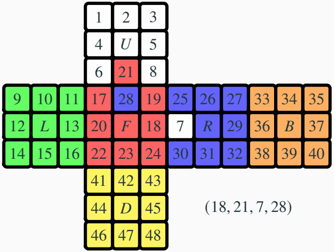

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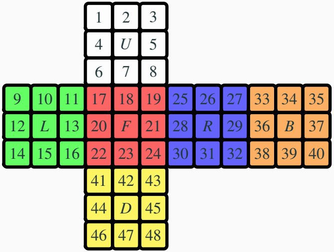

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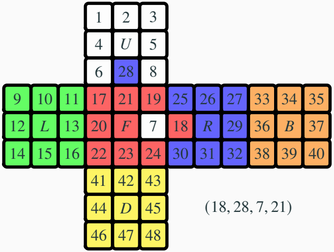

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Can this restickering be solved? ii

These restickerings should be invalid states. In group theory language:

Theorem (Wes's conjecture)
$(18,21)(7,28) \notin \mathcal{G}$, $(18,28)(7,21) \notin \mathcal{G}$, $(18,21,7,28) \notin \mathcal{G}$, *and* $(18,28,7,21) \notin \mathcal{G}$.

Proof.
By GAP:

```
gap> (18,21)(7,28) in G or (18,28)(7,21) in G or
    (18,21,7,28) in G or (18,28,7,21) in G;
false
```

(GAP uses bases and stabiliser chains to verify membership!) □

Can generalise to any two edge pieces (more cases)!

Solving a Rubik's cube... i

TODO

Concluding remarks

References i

- Analyzing Rubik's cube with GAP: `https://www.gap-system.org/Doc/Examples/rubik.html`