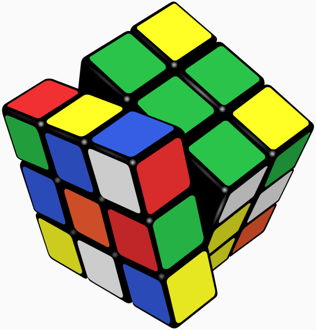
Minimum bases in permutation groups

Lawrence Chen

October 24, 2022

Honours presentation
Monash University
Supervised by A/Prof. Heiko Dietrich and
Dr Santiago Barrera Acevedo



Contents

Aim: analyse Blaha's 1992 paper on NP-completeness of min base problem, and recent results for primitive perm groups.

Motivation: understanding the Rubik's cube

- How can we represent *operations* of a cube?

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One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were* ***more than three billion*** *possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold* ***more than 120*** *hamburgers.*

Some basic group theory

Definition (permutation)

Permutation of Ω is bijection $g : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

(For $\Omega = [n] := \{1, \ldots, n\}$, write $\mathrm{Sym}(n)$.)

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Cycle notation: $g = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ g & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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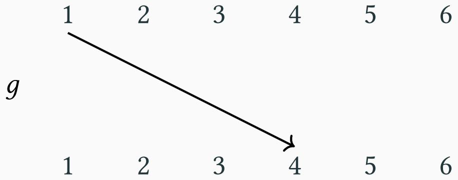

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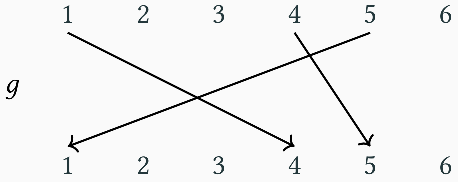

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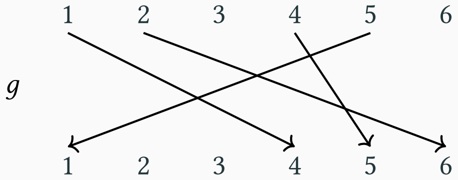

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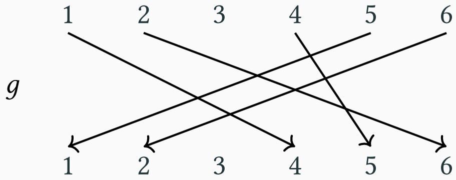

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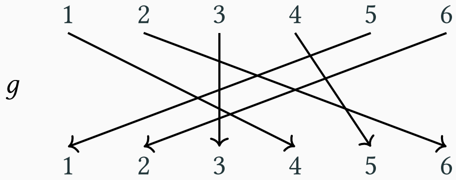

It means $1^g = 4,\ 4^g = 5,\ 5^g = 1,\ 2^g = 6,\ 6^g = 2,\ 3^g = 3$.

Permutations (ii)

Product/composition: for $g, h \in \mathrm{Sym}(\Omega)$, gh means "first g, then h", so $\alpha^{gh} = (\alpha^g)^h$.

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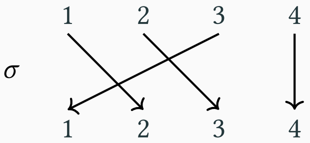

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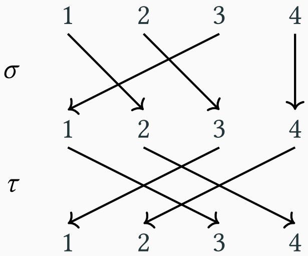

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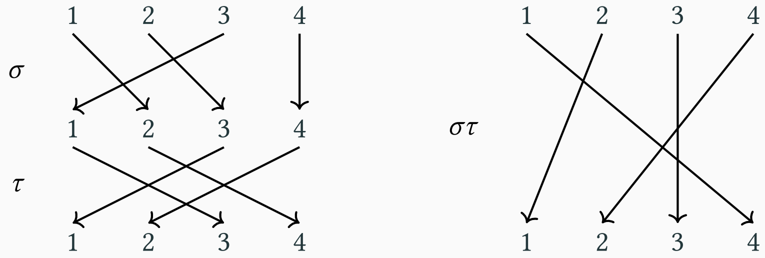

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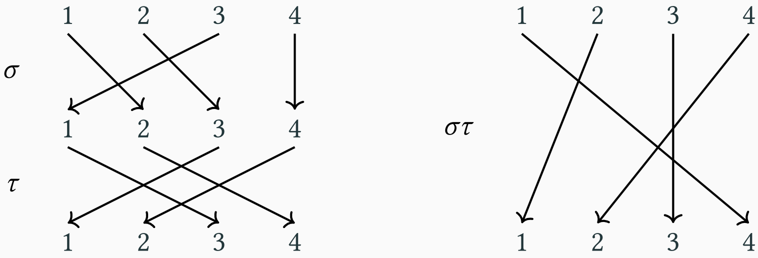

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Note: here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \mathrm{Sym}(\Omega)$.

Permutation groups

Definition (permutation group)

Perm group on Ω (of deg n) is subset $G \leq \mathrm{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

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Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**, $\varepsilon_i \in \{\pm 1\}$; write $G = \langle X \rangle$.

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Example (dihedral group)

Let $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** of order 8 is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ (e.g. $srs^{-1}r^2 = r$), *"symmetries of square"* .

Group actions

Definition (group action)
For $G\,\mathrm{Sym}(\Omega)$ and $\mathcal{S} \neq \emptyset$, a G-**action** is map $\mathcal{S} \times G \to \mathcal{S}$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \mathcal{S}$ and $g, h \in G$.
Degree of action is $|\mathcal{S}|$.

Idea: $\alpha \in \mathcal{S}$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

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Example (natural action)
$G \leq \mathrm{Sym}(\Omega)$ acts on $\mathcal{S} = \Omega$ by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

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Recall $D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ acts naturally on $[4]$.

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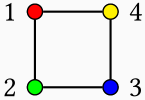

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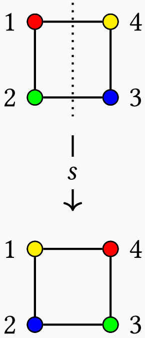

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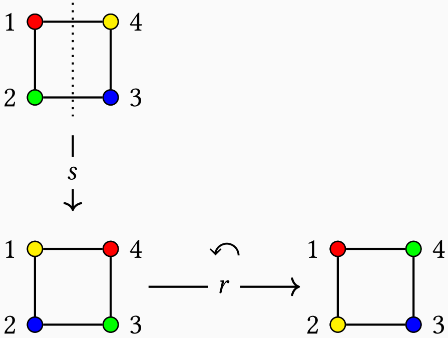

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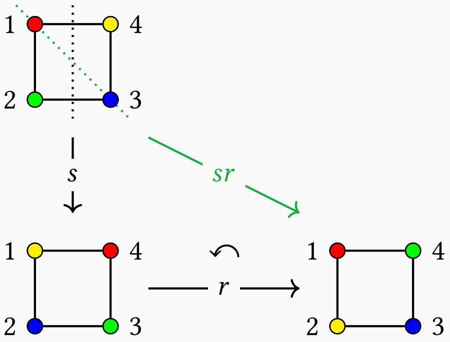

Definition (orbit)

If G acts on $\mathcal{S}$, then **orbit** of $\alpha \in \mathcal{S}$ is $\alpha^G := \{\alpha^g : g \in G\}$.
Idea: states $\alpha^g \in \mathcal{S}$ reachable from fixed $\alpha \in \mathcal{S}$ by moves $g \in G$.

One orbit only: **transitive** action.

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Definition (stabiliser)

If G acts on $\mathcal{S}$, then **stabiliser** of $\alpha \in \mathcal{S}$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.
Idea: moves $g \in G$ that fix given $\alpha \in \mathcal{S}$.

Orbits and stabilisers (ii)

Orbit α^G: states $\alpha^g \in \mathcal{S}$ reachable from fixed α by moves $g \in G$.

Stabiliser G_α: moves $g \in G$ that fix given α.

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Recall $G = D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \mathrm{Sym}(4)$ where $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$.

Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (*transitive*).

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Stabiliser of 1: $sr = (2, 4)$, $sr^2 = (1, 2)(3, 4)$, $sr^3 = (1, 3)$, so $G_1 = \{(), (2, 4)\} = \{1, sr\}$.

Note: $|1^G||G_1| = 4 \cdot 2 = 8 = |G|$. Coincidence?

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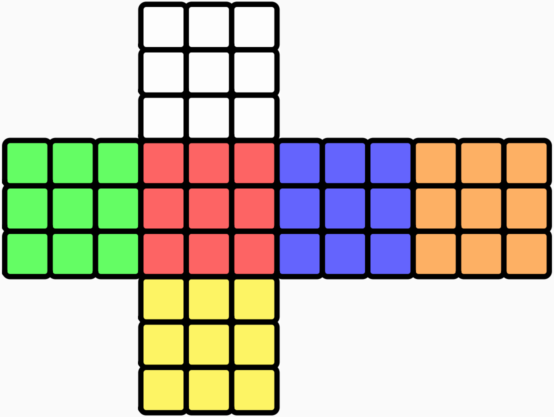
Theorem (orbit-stabiliser)

If G acts on $\mathcal{S}$, then for $\alpha \in \mathcal{S}$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.



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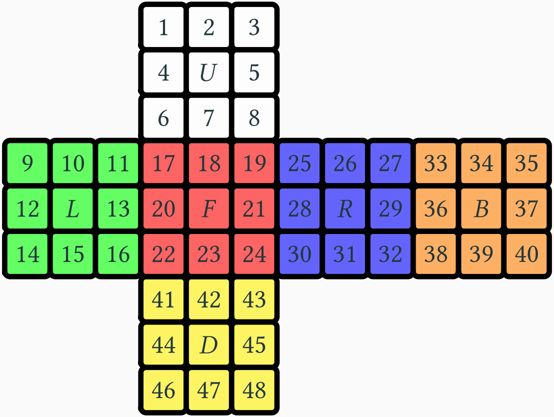
In **solved state** 1, label stickers (except each centre) using $[48]$:



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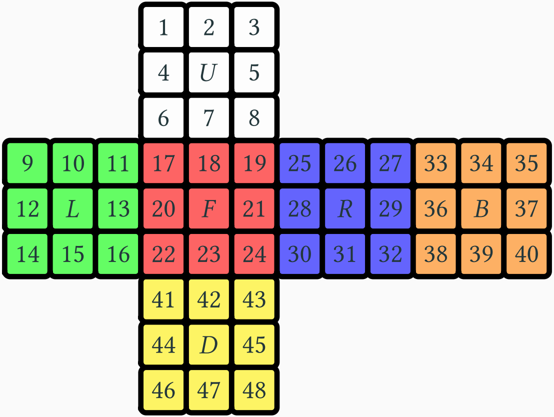
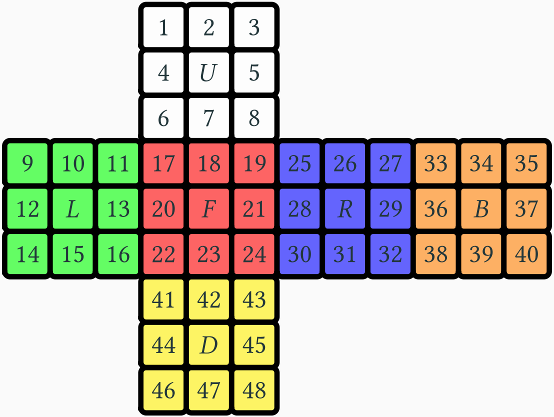


6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

From *solved state* 1, consider F which rotates front face clockwise:



Representing the cube and its operations (ii)

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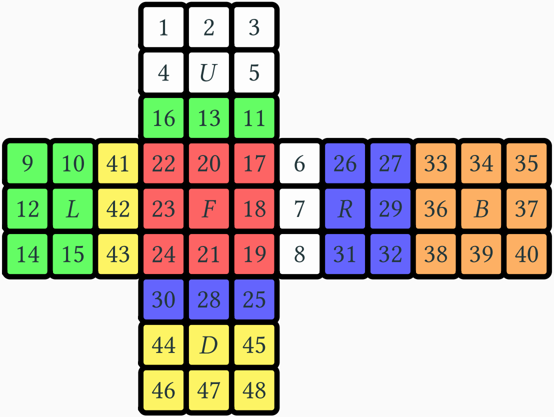

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

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Operation is sequence of generators and inverses. E.g. $RUR^{-1}U^{-1}$,

The Rubik's group of permutations

Generators as permutations of labels $[48]$:

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Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D\rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

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U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);;
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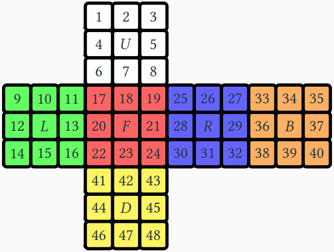
Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Transitive action on corners

Definition (block)

If G acts transitively on $\mathcal{S}$ and $\Delta \subseteq \mathcal{S}$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \mathcal{S}$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

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Examples of blocks: singletons, $\mathcal{S}$, orbits.

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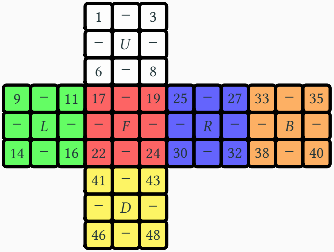
For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions $\mathcal{S}$); then G acts on Σ; if Δ is *maximal*, then acts primitively.

Transitive action on corners (ii)

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



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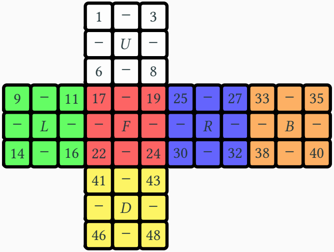
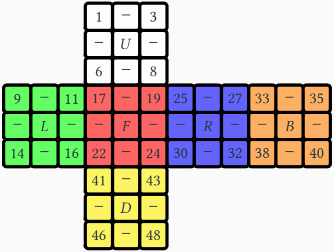


$$\Sigma = \{\overbrace{\{1,35,9\}}^{UBL}, \overbrace{\{6,11,17\}}^{ULF}, \overbrace{\{40,46,14\}}^{BDL}, \overbrace{\{27,3,33\}}^{RUB},$$

$$\underbrace{\{8,25,19\}}_{URF}, \underbrace{\{16,41,22\}}_{LDF}, \underbrace{\{32,48,38\}}_{RDB}, \underbrace{\{24,43,30\}}_{FDR}\}$$

is block system for *maximal* block $\{8, 25, 19\}$ (URF corner); corner stickers stay together.

Transitive action on corners (iii)



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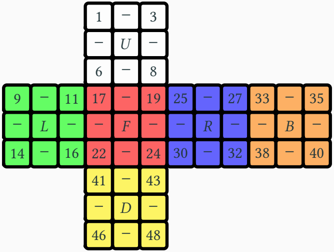

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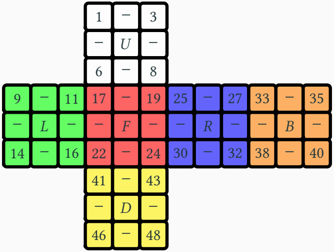

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$\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ "is" *primitive* quotient of $\mathcal{G}$).

Bases and stabiliser chains

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If $G \leq \mathrm{Sym}(\Omega)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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Theorem (Blaha, 1992)

Problem of finding minimum base for G is NP-complete (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).

Bases and stabiliser chains (ii)

Example (Rubik's group)

Using `BaseOfGroup` cmd in `GAP`, base of $\mathcal{G}$ of size 18 is

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Theorem

For Rubik's group $\mathcal{G}$, $b(\mathcal{G}) = 18$.

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Generate uniformly random element of G.

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(*Alternative:* random product of generators in X — Markov chain; mixing time/distribution?)

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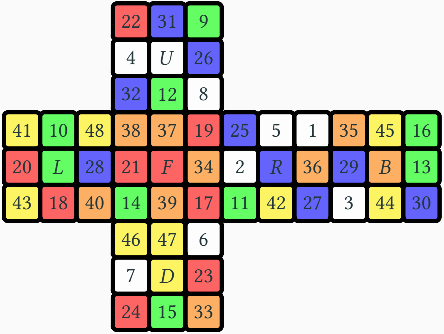
(*Application*: check if restickering of Rubik's cube is valid state.)



What is the size of the Rubik's group?

Theorem (size of perm group)

If $B = [\beta_1, \ldots, \beta_r]$ *is base for* $G \leq \mathrm{Sym}(\Omega)$ *with stabiliser chain* $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, *then*

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Corollary

For Rubik's group $\mathcal{G}$, $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Base sizes of primitive groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

Note: $t_{a,v} \in \mathrm{Sym}(K^d)$ (bijection).

Affine groups

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Definition

Affine group $\mathrm{AGL}_d(K) \leq \mathrm{Sym}(K^d)$ of dim d is affine transfs of K^d.
For $K = \mathbb{F}_q$ finite field, write $\mathrm{AGL}_d(q)$ (perm group of deg q^d).

Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

Non-large base permutation groups

Theorem (Liebeck, 1984)

For primitive perm group G of degree n, either:

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Previous best (Babai, 1981): $b(G) = O(\sqrt{n})$ if not containing $\mathrm{Alt}(n)$.

"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

Non-large base permutation groups (ii)

Theorem (Moscatiello & Roney-Dougal, 2021)
For primitive perm group G of degree n, and G is non-large base:

(i) *G is the Mathieu group M_{24} (degree 24); or*

(ii) $b(G) \leq \lceil \log n \rceil + 1$.

Moreover, if $b(G) = \log n + 1$ then $G \leq \mathrm{AGL}_d(2)$ with $n = 2^d$.

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Question (Moscatiello & Roney-Dougal, 2021)
Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Main result in thesis

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ be primitive for some $d \leq 10$ with natural action on K^d with $b(G) = d + 1$. (Then G is perm group of degree $n = 2^d$.) Then:

(i) *G is $\mathrm{AGL}_d(2)$ with $d \geq 2$; or*

(ii) *G is $\mathrm{Sp}_d(2) \ltimes C_2^d$ with $d \geq 4$ even.*

Proof (idea).

- Find representatives M of conjugacy classes of primitive maximal subgroups of $\mathrm{AGL}_d(2)$.

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- Use *greedy base algorithm* to find base for M; if base of length at most d is found then $b(M) \leq d$ and discard.
- Otherwise, recursively check for each representative M.

Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. $\square$

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff $n = 2^d$ and:

- G is $\mathrm{AGL}_d(2)$ with $d \geq 2$; or
- G is $\mathrm{Sp}_d(2) \ltimes C_2^d$ with $d \geq 4$ even.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP: https://www.gap-system.org/Doc/Examples/rubik.html
- J. A. Paulos — *Innumeracy* (book)
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- Moscatiello and Roney-Dougal: *Base sizes of primitive permutation groups*, 2021: https://doi.org/10.1007/s00605-021-01599-5