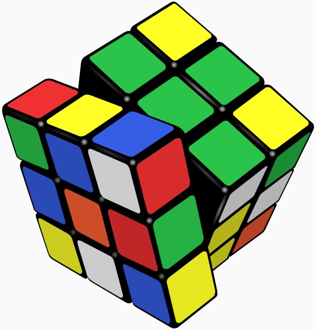
# Minimum bases in permutation groups

**Lawrence Chen**

October 24, 2022

**Honours presentation**
**Monash University**
Supervised by A/Prof. Heiko Dietrich and
Dr Santiago Barrera Acevedo



# Contents



*Aim:* analyse Blaha's 1992 paper on NP-completeness of min base problem, and recent results for primitive perm groups.

## Motivation: understanding the Rubik's cube

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**(J. A. Paulos, Innumeracy)**

> *Ideal Toy Company stated on the package of the original Rubik cube that there were **more than three billion** possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold **more than 120** hamburgers.*

# Some basic group theory

# Permutations

**Definition (permutation)**

**Permutation** of $\Omega$ is bijection $g : \Omega \to \Omega$.

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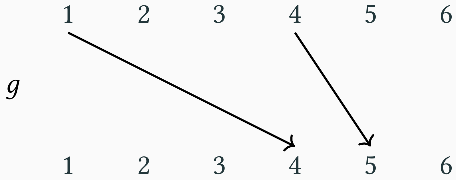


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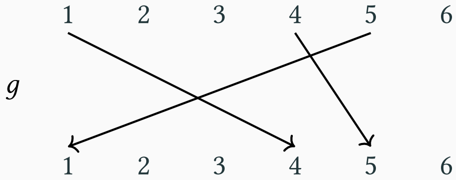


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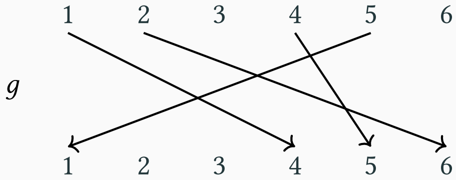


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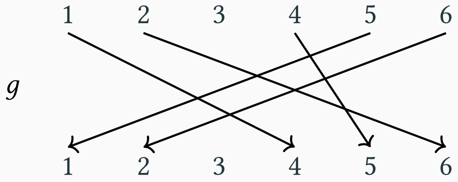


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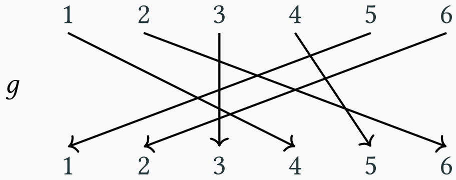


It means $1^g = 4,\ 4^g = 5,\ 5^g = 1,\ 2^g = 6,\ 6^g = 2,\ 3^g = 3.$

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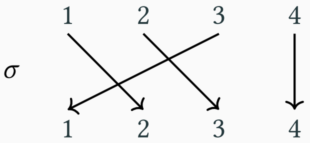

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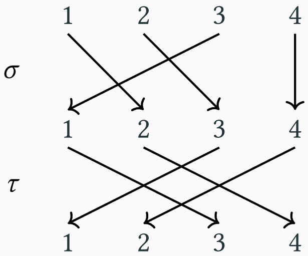

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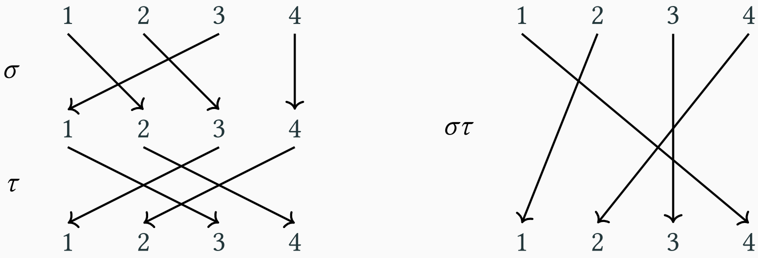


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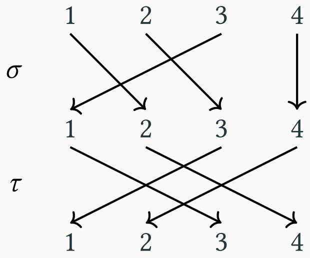


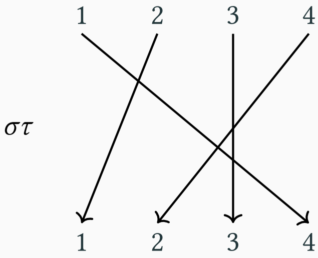


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*Note:* here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \mathrm{Sym}(\Omega)$.

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Set $X$ **generates** $G$ if every $g \in G$ is $g = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**, $\varepsilon_i \in \{\pm 1\}$; write $G = \langle X \rangle$.

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**Example (dihedral group)**

Let $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** of order 8 is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ (e.g. $srs^{-1}r^2 = r$), *"symmetries of square"* .

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**Definition (group action)**

For $G\,\mathrm{Sym}(\Omega)$ and $\mathcal{S} \neq \emptyset$, a $G$-**action** is map $\mathcal{S} \times G \to \mathcal{S}$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \mathcal{S}$ and $g, h \in G$.
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**Example (natural action)**

$G \leq \mathrm{Sym}(\Omega)$ acts on $\mathcal{S} = \Omega$ by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

### Example (dihedral group)

Recall $D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ acts naturally on $[4]$.

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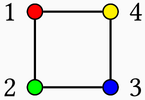

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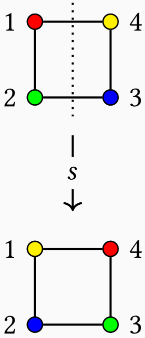

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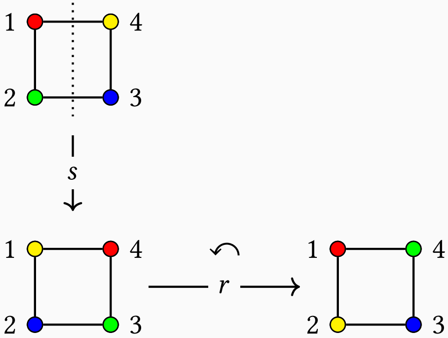

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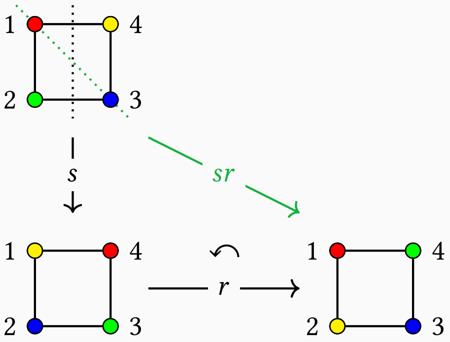

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If $G$ acts on $\mathcal{S}$, then **orbit** of $\alpha \in \mathcal{S}$ is $\alpha^G := \{\alpha^g : g \in G\}$.
*Idea:* states $\alpha^g \in \mathcal{S}$ reachable from fixed $\alpha \in \mathcal{S}$ by moves $g \in G$.

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**Definition (stabiliser)**

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Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (*transitive*).

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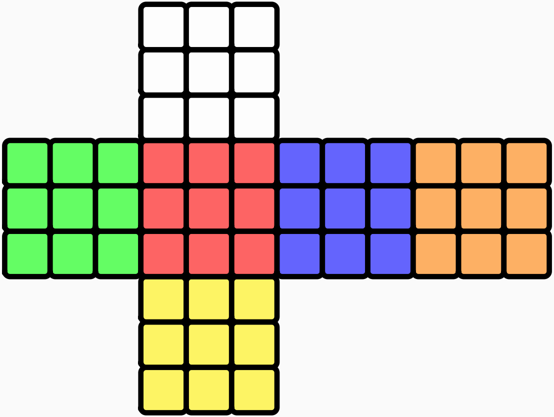
**Theorem (orbit-stabiliser)**

*If $G$ acts on $\mathcal{S}$, then for $\alpha \in \mathcal{S}$, $|\alpha^G||G_\alpha| = |G|$.*

# The Rubik's group

# Representing the cube and its operations

Rubik's cube has 6 faces, each with $3 \times 3$ small *stickers*.



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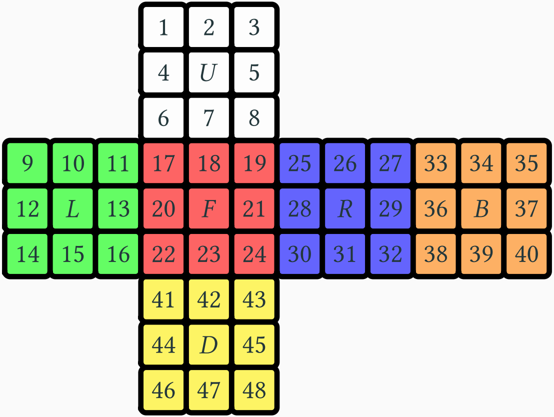

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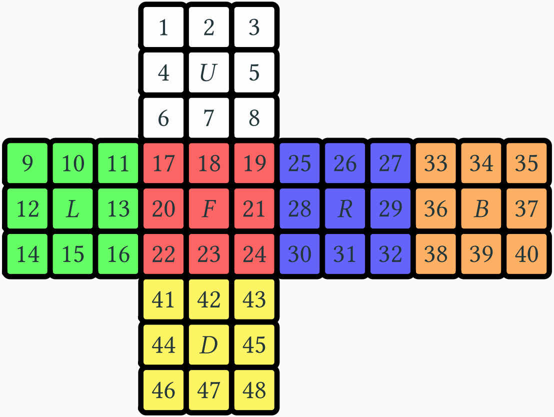


6 **generators** (*moves* in CC): $U, L, F, R, B, D$ (rot. *clockwise*).

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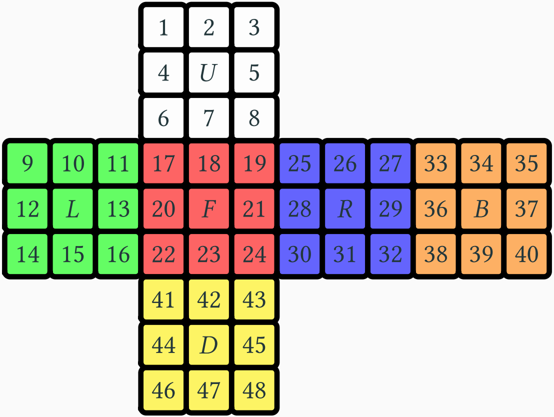

# Representing the cube and its operations (ii)

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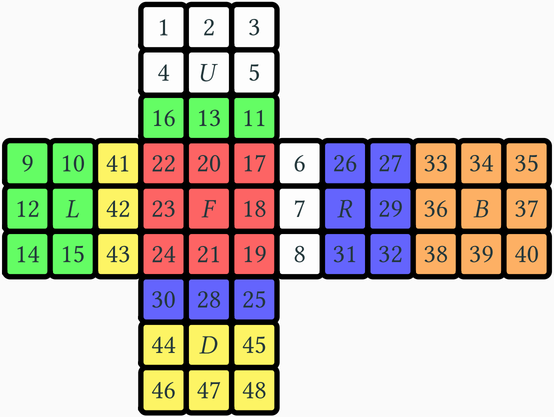


$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

# The Rubik's group of permutations

Generators as permutations of labels $[48]$:

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**Definition (Rubik's group)**

$\mathcal{G} = \langle U, L, F, R, B, D\rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

# The Rubik’s group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

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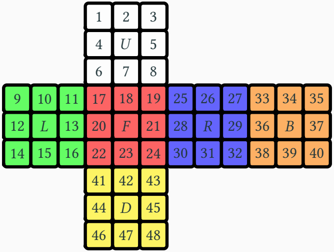
`Order` cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

# Orbits in the Rubik's group




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

# Transitive action on corners

**Definition (block)**

If $G$ acts transitively on $\mathcal{S}$ and $\Delta \subseteq \mathcal{S}$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

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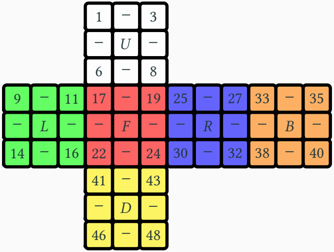
For block $\Delta$, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions $\mathcal{S}$); then $G$ acts on $\Sigma$; if $\Delta$ is *maximal*, then acts primitively.

# Transitive action on corners (ii)

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



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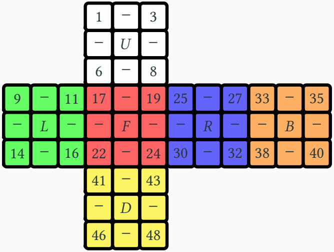
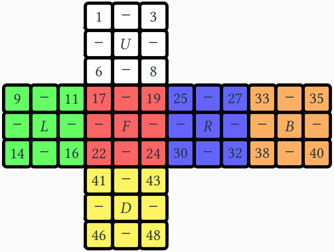



$$\Sigma = \{\overbrace{\{1, 35, 9\}}^{UBL}, \overbrace{\{6, 11, 17\}}^{ULF}, \overbrace{\{40, 46, 14\}}^{BDL}, \overbrace{\{27, 3, 33\}}^{RUB},$$

$$\underbrace{\{8, 25, 19\}}_{URF}, \underbrace{\{16, 41, 22\}}_{LDF}, \underbrace{\{32, 48, 38\}}_{RDB}, \underbrace{\{24, 43, 30\}}_{FDR}\}$$

is block system for *maximal* block $\{8, 25, 19\}$ ($URF$ corner); corner stickers stay together.

# Transitive action on corners (iii)

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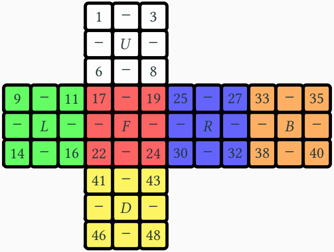


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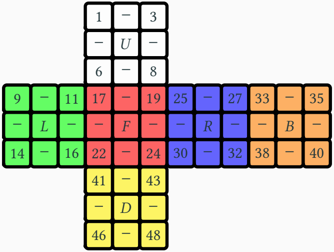


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$\mathcal{G}$ induces every perm of $\Sigma$ (so $\mathrm{Sym}(8)$ “is” *primitive* quotient of $\mathcal{G}$).

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**Theorem (Blaha, 1992)**

*Problem of finding minimum base for $G$ is NP-complete (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).*

**Example (Rubik's group)**

Using `BaseOfGroup` cmd in `GAP`, base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

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**Theorem**

*For Rubik's group $\mathcal{G}$, $b(\mathcal{G}) = 18$.*

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(*Alternative:* random product of generators in $X$ — Markov chain; mixing time/distribution?)

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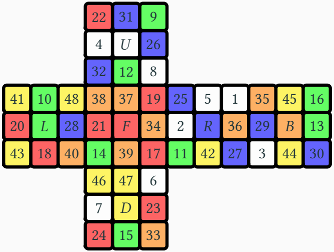
(*Application*: check if restickering of Rubik's cube is valid state.)



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# Base sizes of primitive groups

# Affine groups

**Definition**

Let $K$ be field. **Affine transformation** of $K^d$ is map

$$t_{a,v} : K^d \longrightarrow K^d, \quad u \mapsto ua + v$$

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Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

# Non-large base permutation groups

**Theorem (Liebeck, 1984)**

*For primitive perm group $G$ of degree $n$, either:*

(i) *$G$ is "large base"; or*

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(ii) $b(G) < 9 \log n$.

*Previous best (Babai, 1981):* $b(G) = O(\sqrt{n})$ if not containing $\mathrm{Alt}(n)$.

"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

# Non-large base permutation groups (ii)

**Theorem (Moscatiello & Roney-Dougal, 2021)**

*For primitive perm group $G$ of degree $n$, and $G$ is non-large base:*

(i) *$G$ is the Mathieu group $M_{24}$ (degree 24); or*

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*Moreover, if $b(G) = \log n + 1$ then $G \leq \mathrm{AGL}_d(2)$ with $n = 2^d$.*

**Question (Moscatiello & Roney-Dougal, 2021)**
Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

# Main result in thesis

**Theorem**

*Let* $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* $d \leq 10$ *with natural action on* $K^d$ *with* $b(G) = d + 1$*. (Then* $G$ *is perm group of degree* $n = 2^d$*.) Then*

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(i) $G$ *is* $\mathrm{AGL}_d(2)$ *with* $d \geq 2$, *or*

(ii) $G$ *is* $2^d : \mathrm{Sp}_d(2) = \mathrm{Sp}_d(2) \ltimes C_2^d$ *with* $d \geq 4$ *even.*

## Main result in thesis (ii)

**Proof (idea).**

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- Otherwise, recursively check for each representative $M$.

Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. $\square$

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

# Main result in thesis (iii)

From above theorem, we conjecture the following:

**Conjecture**

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

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From above theorem, we conjecture the following:

**Conjecture**

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

- $n = 2^d$ with $d \geq 2$, and $G$ is $\mathrm{AGL}_d(2)$; or
- $n = 2^d$ with $d \geq 4$, and $G$ is $2^d : \mathrm{Sp}_d(2)$.

# Concluding remarks

# References and resources


- Analyzing Rubik's cube with GAP: `https://www.gap-system.org/Doc/Examples/rubik.html`
- J. A. Paulos — *Innumeracy* (book)
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