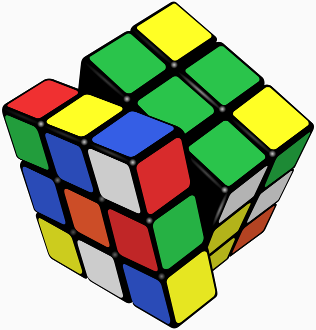
Rubik's cubes and permutation group theory

Lawrence Chen

October 7, 2022

Honours presentation



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Some basic group theory

What is a group? i

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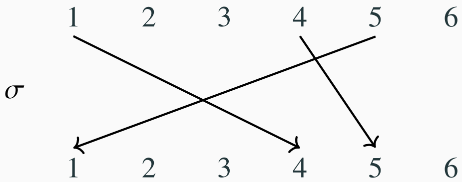

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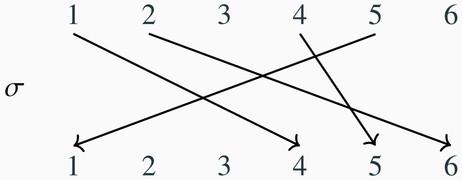

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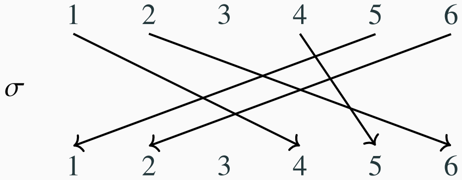

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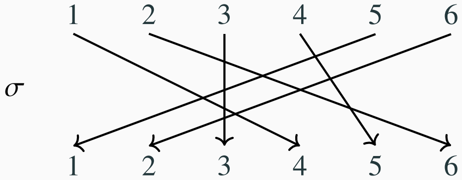

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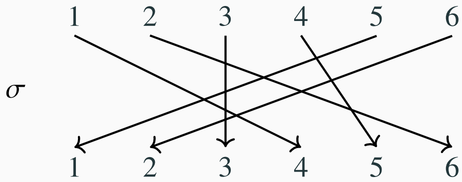

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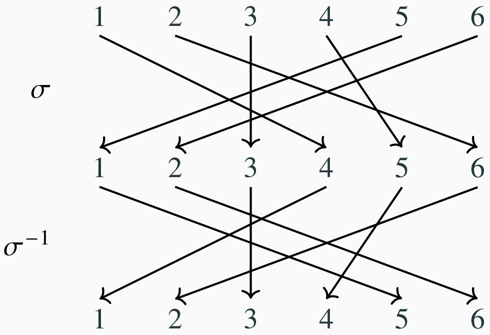

Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \mathrm{Sym}(6)$.

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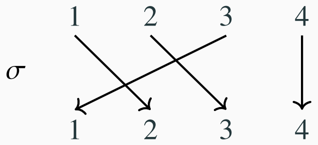

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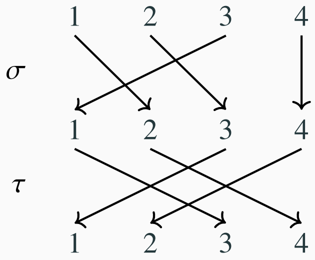

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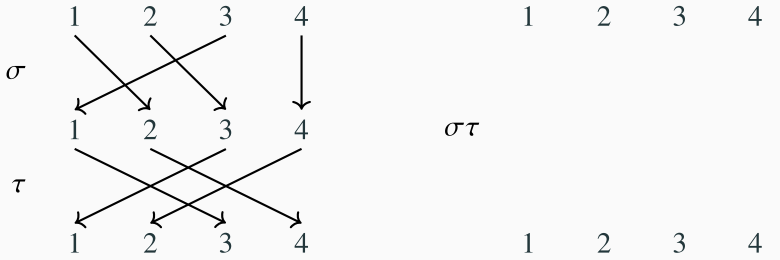

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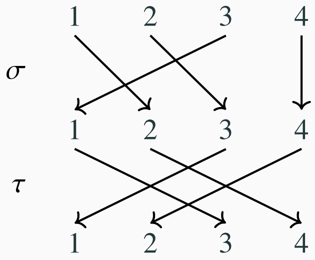

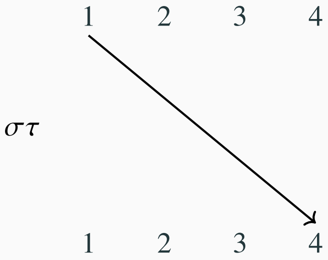

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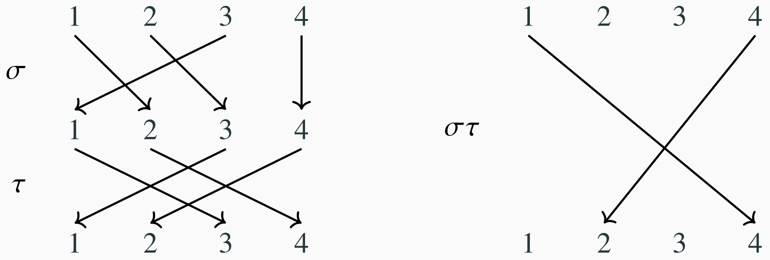

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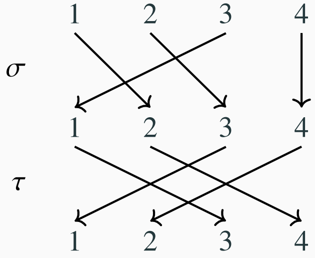

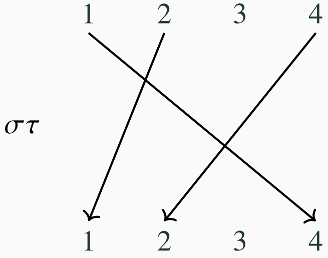

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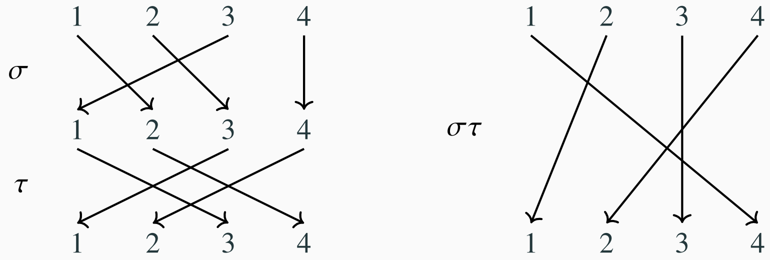

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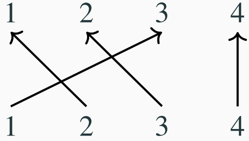

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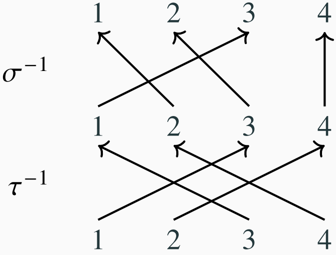

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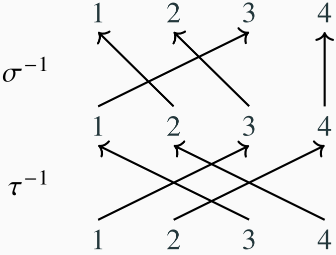

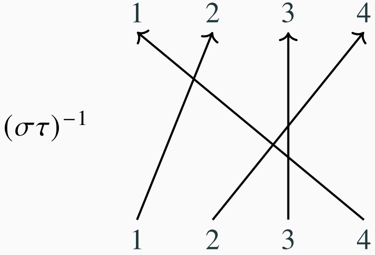

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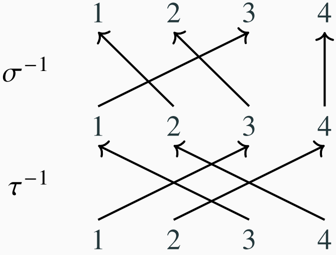

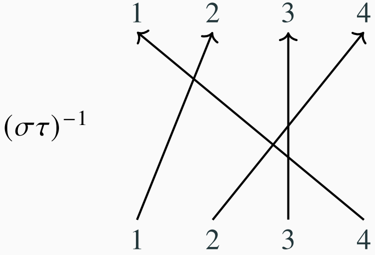

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Permutations iv

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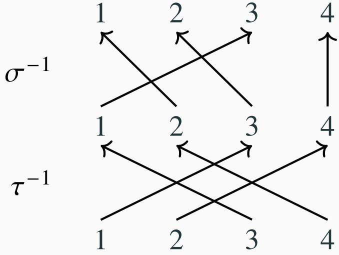

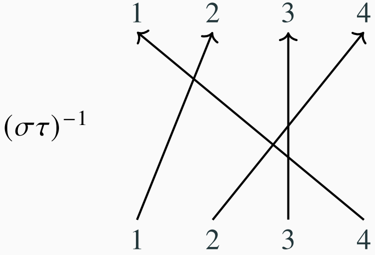

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A **permutation group** of *degree* n is a subgroup of $\mathrm{Sym}(n)$.

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If G is group and $\Omega \neq \emptyset$ is set, a G**-action** is a map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

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$\mathbb{Z}$ acts on $\Omega = \{12{:}00, 1{:}00, \ldots, 11{:}00\}$ by $(\alpha{:}00)^k = [\alpha + k]_{12}{:}00$ for $\alpha{:}00 \in \Omega$ and $k \in \mathbb{Z}$.

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Group G acts on $\Omega = G$ (itself) via $\alpha^g = \alpha g$ for $\alpha, g \in G$.

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If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

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Note: stabiliser G_α is subgroup of G. (So G_α acts on Ω.)

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$G = \{(), (1,2,4), (1,4,2)\} \leq \mathrm{Sym}(4)$ acts on $\Omega = [4]$ naturally.
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Group actions iii

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Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Analysing the Rubik’s group

Concluding remarks

References i

- TODO