

Minimum bases in permutation groups

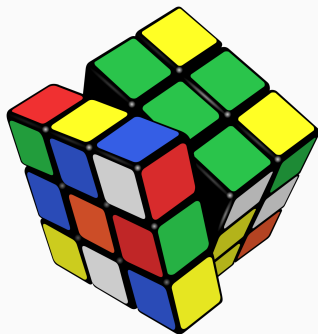
Lawrence Chen

October 24, 2022

Honours presentation

Monash University

Supervised by A/Prof. Heiko Dietrich and
Dr Santiago Barrera Acevedo



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Some basic group theory

- Permutations

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- Group actions

- Orbits and stabilisers

The Rubik's group

- Representing the cube and its operations

- The Rubik's group of permutations

- Orbits in the Rubik's group

- Transitive action on corners

Bases and stabiliser chains

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- What is the size of the Rubik's group?

Base sizes of primitive groups

- Affine groups

- Non-large base permutation groups

- Main result in thesis

Aim: analyse Blaha's 1992 paper on NP-completeness of min base problem, and recent results for primitive perm groups.

Motivation: understanding the Rubik's cube

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One answer: using permutations and *computational group theory*!

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One answer: using permutations and computational group theory!

(J. A. Paulos, Innumeracy)

*Ideal Toy Company stated on the package of the original Rubik cube that there were **more than three billion** possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold **more than 120** hamburgers.*

Some basic group theory

Permutations

Definition (permutation)

Permutation of Ω is bijection $g : \Omega \rightarrow \Omega$.

Symmetric group $\text{Sym}(\Omega)$ is set of permutations of Ω .

(For $\Omega = [n] := \{1, \dots, n\}$, write $\text{Sym}(n)$.)

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Cycle notation: $g = (1, 4, 5)(2, 6) \in \text{Sym}(6)$ is:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ g & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

It means

Permutations

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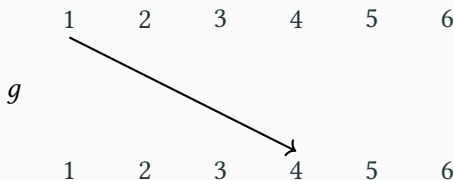
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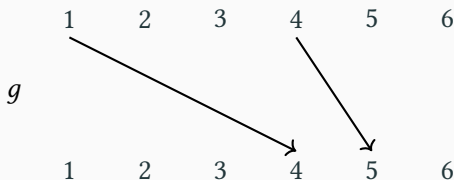
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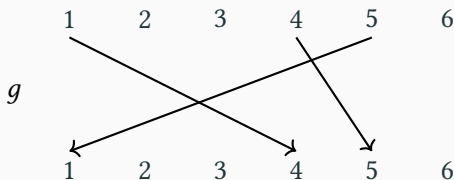
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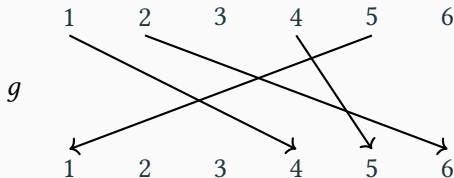
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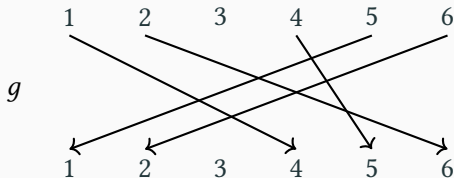
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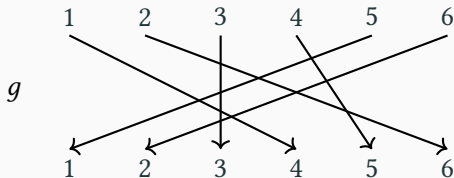
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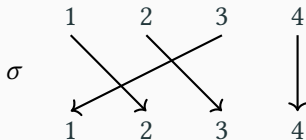
It means $1^g = 4$, $4^g = 5$, $5^g = 1$, $2^g = 6$, $6^g = 2$, $3^g = 3$.

Permutations (ii)

Product/composition: for $g, h \in \text{Sym}(\Omega)$, gh means “first g , then h ”,
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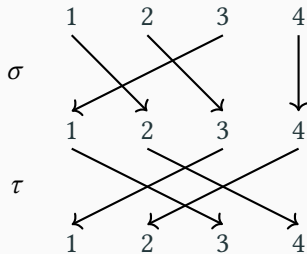
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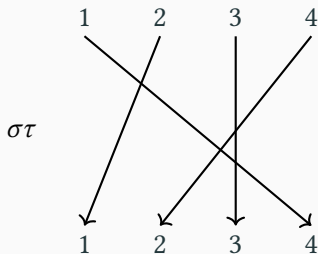
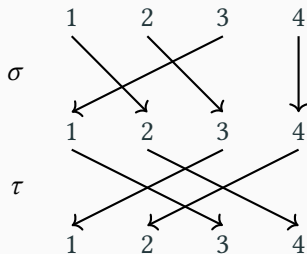
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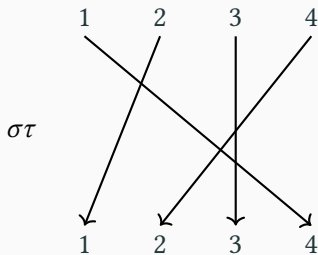
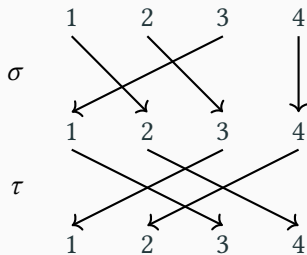
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Note: here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \text{Sym}(\Omega)$.

Permutation groups

Definition (permutation group)

Perm group on Ω (of deg n) is subset $G \leq \text{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

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Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**, $\varepsilon_i \in \{\pm 1\}$; write $G = \langle X \rangle$.

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Example (dihedral group)

Let $r = (1, 2, 3, 4), s = (1, 4)(2, 3) \in \text{Sym}(4)$. **Dihedral group** of order 8 is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ (e.g. $srs^{-1}r^2 = r$), “symmetries of square”.

Definition (group action)

For $G \leq \text{Sym}(\Omega)$ and $\mathcal{S} \neq \emptyset$, a G -**action** is map $\mathcal{S} \times G \rightarrow \mathcal{S}$,
 $(\alpha, g) \mapsto \alpha^g \in \mathcal{S}$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \mathcal{S}$ and $g, h \in G$.
Degree of action is $|\mathcal{S}|$.

Idea: $\alpha \in \mathcal{S}$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

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Example (natural action)

$G \leq \text{Sym}(\Omega)$ acts on $\mathcal{S} = \Omega$ by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

Example (dihedral group)

Recall $D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ acts naturally on $[4]$.

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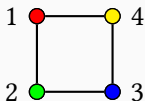
Note: $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$, $sr = (2, 4)$. Visualise D_8 -action by labelling vertices of square by $[4]$: $g \in D_8$ sends vertex at i to i^g .

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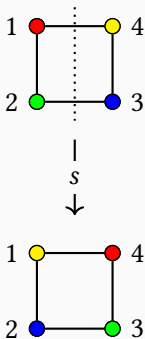


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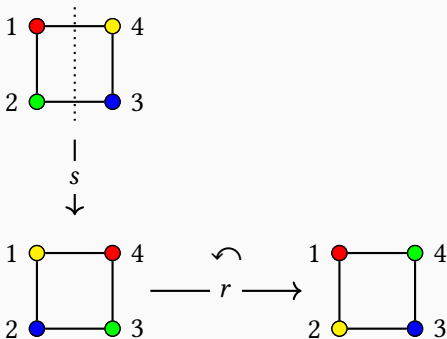


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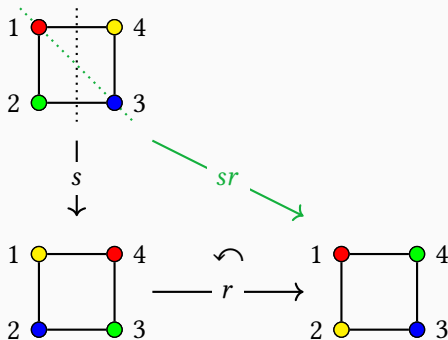


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Definition (orbit)

If G acts on \mathcal{S} , then **orbit** of $\alpha \in \mathcal{S}$ is $\alpha^G := \{\alpha^g : g \in G\}$.

Idea: states $\alpha^g \in \mathcal{S}$ reachable from fixed $\alpha \in \mathcal{S}$ by moves $g \in G$.

One orbit only: **transitive** action.

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Definition (stabiliser)

If G acts on \mathcal{S} , then **stabiliser** of $\alpha \in \mathcal{S}$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.

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Orbits and stabilisers (ii)

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Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (*transitive*).

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Stabiliser of 1: $sr = (2, 4)$, $sr^2 = (1, 2)(3, 4)$, $sr^3 = (1, 3)$, so $G_1 = \{(), (2, 4)\} = \{1, sr\}$.

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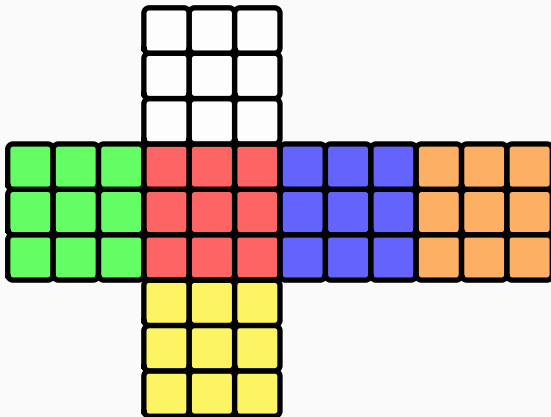
Theorem (orbit-stabiliser)

If G acts on \mathcal{S} , then for $\alpha \in \mathcal{S}$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.



Representing the cube and its operations

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In **solved state 1**, label stickers (except each centre) using [48]:

			1	2	3							
			4	U	5							
			6	7	8							
9	10	11	17	18	19	25	26	27	33	34	35	
12	L	13	20	F	21	28	R	29	36	B	37	
14	15	16	22	23	24	30	31	32	38	39	40	
			41	42	43							
			44	D	45							
			46	47	48							

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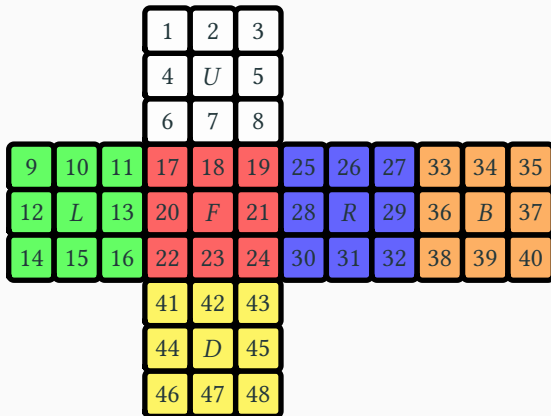
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6 **generators** (moves in CC): U, L, F, R, B, D (rot. *clockwise*).

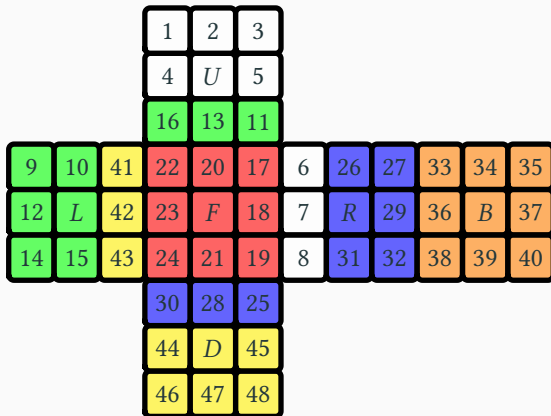
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Representing the cube and its operations (ii)

From *solved state 1*, consider F which rotates front face clockwise:



$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$

$$(7, 28, 42, 13)(8, 30, 41, 11) \in \text{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

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Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \text{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly \mathcal{G} is finite, but what is $|\mathcal{G}|$?

The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
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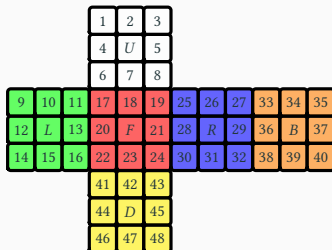
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Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group



```

1 gap> Orbit( G, 1 );
2 [ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
   14, 19, 9, 22, 38 ]
3 gap> Orbit( G, 2 );
4 [ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
   15, 31, 18, 23, 39 ]

```

Two \mathcal{G} -orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Definition (block)

If G acts transitively on \mathcal{S} and $\Delta \subseteq \mathcal{S}$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \mathcal{S}$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

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For block Δ , define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions \mathcal{S}); then G acts on Σ ; if Δ is *maximal*, then acts primitively.

Transitive action on corners (ii)

\mathcal{G} acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:

			1	-	3						
			-	<i>U</i>	-						
			6	-	8						
9	-	11	17	-	19	25	-	27	33	-	35
-	<i>L</i>	-	-	<i>F</i>	-	-	<i>R</i>	-	-	<i>B</i>	-
14	-	16	22	-	24	30	-	32	38	-	40
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$$\begin{array}{cccc}
 \text{UBL} & \text{ULF} & \text{BDL} & \text{RUB} \\
 \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \Sigma = \{ \{1, 35, 9\}, \{6, 11, 17\}, \{40, 46, 14\}, \{27, 3, 33\}, \\
 \{8, 25, 19\}, \{16, 41, 22\}, \{32, 48, 38\}, \{24, 43, 30\} \} \\
 \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \text{URF} & \text{LDF} & \text{RDB} & \text{FDR}
 \end{array}$$

is block system for *maximal* block $\{8, 25, 19\}$ (URF corner); corner stickers stay together.

Transitive action on corners (iii)

			1	–	3							
			–	U	–							
			6	–	8							
9	–	11	17	–	19	25	–	27	33	–	35	
–	L	–	–	F	–	–	R	–	–	B	–	
14	–	16	22	–	24	30	–	32	38	–	40	
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\mathcal{G} induces every perm of Σ (so $\text{Sym}(8)$ “is” *primitive* quotient of \mathcal{G}).

Bases and stabiliser chains

Definition (Base, stabiliser chain)

If $G \leq \text{Sym}(\Omega)$, distinct elts $B = [\beta_1, \dots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1, \dots, \beta_r} = 1$. (Recall: $G_{\beta_1, \dots, \beta_r} = \{g \in G : \beta_1^g = \beta_1, \dots, \beta_r^g = \beta_r\}$.)

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Theorem (Blaha, 1992)

Problem of finding minimum base for G is NP-complete (if $P \neq NP$, then no polynomial time algorithm).

Example (Rubik's group)

Using BaseOfGroup cmd in GAP, base of \mathcal{G} of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

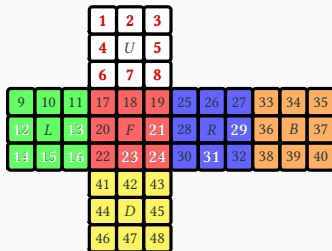
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Contains: 7 corner stickers (from 7 of 8 corners), 11 edge stickers (from 11 of 12 edges).



Stabiliser chain implemented in GAP; useful in algorithms.

Bases and stabiliser chains (iii)

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Let $G = \langle X \rangle \leq \text{Sym}(\Omega)$ have base B and stabiliser chain

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(*Alternative: random product of generators in X — Markov chain; mixing time/distribution?*)

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(Application: check if restickering of Rubik's cube is valid state.)

What is the size of the Rubik's group?

Theorem (size of perm group)

If $B = [\beta_1, \dots, \beta_r]$ is base for $G \leq \text{Sym}(\Omega)$ with stabiliser chain $G = G^0 \geq G^1 \geq \dots \geq G^r = 1$, then

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Corollary

For Rubik's group \mathcal{G} , $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Base sizes of primitive groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \rightarrow K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

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Definition

Affine group $\mathrm{AGL}_d(K) \leq \mathrm{Sym}(K^d)$ of dim d is affine transfs of K^d .
For $K = \mathbb{F}_q$ finite field, write $\mathrm{AGL}_d(q)$ (perm group of deg q^d).

Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

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For primitive perm group G of degree n , either:

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- (ii) $b(G) < 9 \log n$.*

Previous best (Babai, 1981): $b(G) = O(\sqrt{n})$ if not containing $\text{Alt}(n)$.

*“Remarkable” proof used classification of finite simple groups,
O’Nan-Scott theorem (classifies primitive groups).*

Non-large base permutation groups (ii)

Theorem (Moscatiello & Roney-Dougal, 2021)

For primitive perm group G of degree n , and G is non-large base:

- (i) G is the Mathieu group M_{24} (degree 24); or*
- (ii) $b(G) \leq \lceil \log n \rceil + 1$.*

Moreover, if $b(G) = \log n + 1$ then $G \leq \text{AGL}_d(2)$ with $n = 2^d$.

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Question (Moscatiello & Roney-Dougal, 2021)

Which primitive groups $G \leq \text{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Theorem

Let $G \leq \text{AGL}_d(2)$ be primitive for some $d \leq 10$ with natural action on K^d with $b(G) = d + 1$. (Then G is perm group of degree $n = 2^d$.) Then:

- (i) G is $\text{AGL}_d(2)$ with $d \geq 2$; or*
- (ii) G is $\text{Sp}_d(2) \ltimes C_2^d$ with $d \geq 4$ even.*

Proof (idea).

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- Use *greedy base algorithm* to find base for M ; if base of length at most d is found then $b(M) \leq d$ and discard.
- Otherwise, recursively check for each representative M .

Every primitive $G \leq \text{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. \square

Greedy base algorithm performed better than BaseOfGroup in testing; found no false positives.

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \text{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff $n = 2^d$ and:

- G is $\text{AGL}_d(2)$ with $d \geq 2$; or
- G is $\text{Sp}_d(2) \ltimes C_2^d$ with $d \geq 4$ even.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP:
<https://www.gap-system.org/Doc/Examples/rubik.html>
- J. A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Blaha — *Minimum bases for permutation groups: The greedy approximation*, 1992:
[https://doi:10.1016/0196-6774\(92\)90020-D](https://doi:10.1016/0196-6774(92)90020-D)
- Liebeck — *On minimal degrees and base sizes of primitive permutation groups*, 1984: <https://doi.org/10.1007/bf01193603>
- Moscatiello and Roney-Dougal: *Base sizes of primitive permutation groups*, 2021: <https://doi.org/10.1007/s00605-021-01599-5>

References and resources (ii)

The **order** of $g \in G \leq \text{Sym}(\Omega)$ is smallest $k \in \mathbb{Z}_+$ such that $g^k = 1$.

Fact: order of g is lcm of cycle lengths; it divides $|G|$.

Note: for Rubik's group, R has order 4, $RUR^{-1}U^{-1}$ has order 6, RU has order 105 (GAP). Order 7?

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$$|\mathcal{G}| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11.$$

- *Bonus:* Orders of elements in Rubik's group (1260 largest, 13 smallest without, 11 rarest, 60 most common, median 67.3, 73 options):
<https://www.jaapsch.net/puzzles/cubic3.htm#p34>
- *Bonus:* Thistlethwaite's 52 move algorithm (using group theory):
<https://www.jaapsch.net/puzzles/thistle.htm>

Definition

Perm group G of degree n is **large base** if

$$\text{Alt}(m)^r \trianglelefteq G \leq \text{Sym}(m) \wr \text{Sym}(r)$$

for some m, r, k , where $\text{Sym}(m)$ acts on $\binom{[m]}{k}$, and if $r > 1$ then wreath product has *product action* of degree $n = \binom{m}{k}^r$.