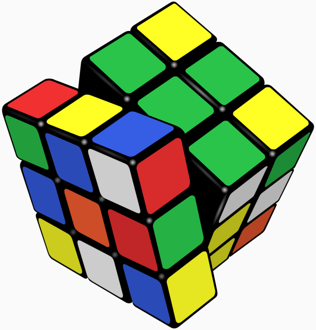
Minimum bases in permutation groups

Lawrence Chen

October 24, 2022

Honours presentation



Contents

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One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

Permutations

Definition (permutation)

Permutation of Ω is bijection $g : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

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Cycle notation: $g = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ g & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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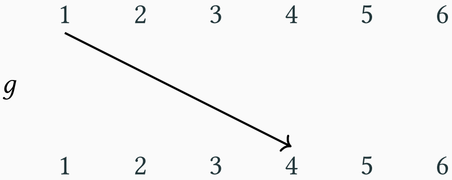

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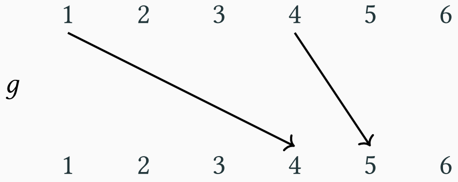

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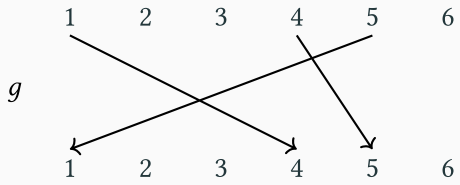

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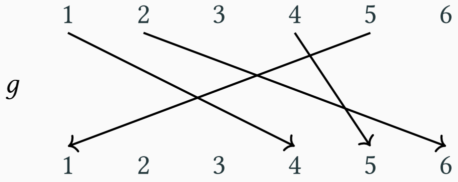

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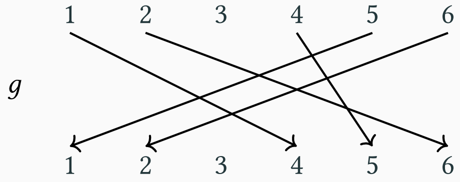

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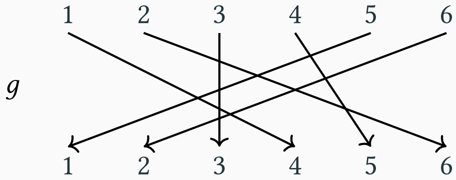

It means $1^g = 4,\ 4^g = 5,\ 5^g = 1,\ 2^g = 6,\ 6^g = 2,\ 3^g = 3.$

Permutations (ii)

Product/composition: for $g, h \in \mathrm{Sym}(\Omega)$, gh means “first g, then h”, so $\alpha^{gh} = (\alpha^g)^h$.

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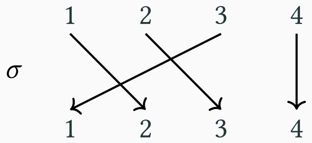

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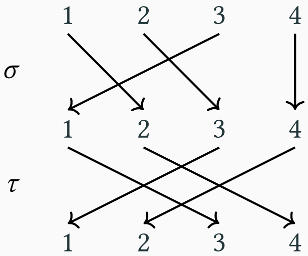

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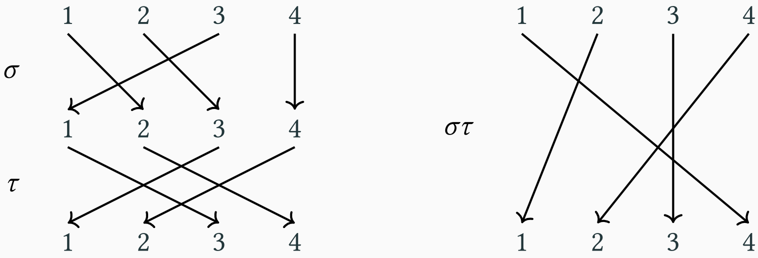

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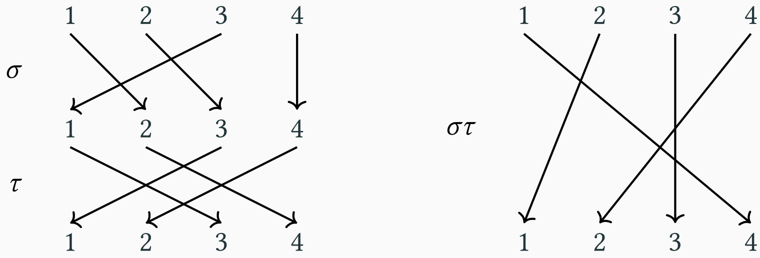

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Note: here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \mathrm{Sym}(\Omega)$.

Permutation groups

Definition (permutation group)

Perm group on Ω (of deg n) is subset $G \leq \mathrm{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

(i) **(closure)** $gh \in G$ for $g, h \in G$;

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Set X **generates** G if every $g \in G$ is $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**; write $G = \langle X \rangle$.

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Example (dihedral group)

Let $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, *"symmetries of square"* (e.g. $srsr^2 = r$).

Group actions

Definition (group action)

For $G\ \mathrm{Sym}(\Omega)$ and $\mathcal{S} \neq \emptyset$, a G-**action** is map $\mathcal{S} \times G \to \mathcal{S}$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \mathcal{S}$ and $g, h \in G$. **Degree** of action is $|\mathcal{S}|$.

Idea: $\alpha \in \mathcal{S}$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

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Example (natural action)

$G \leq \mathrm{Sym}(\Omega)$ acts on $\mathcal{S} = \Omega$ by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

Group actions (ii)

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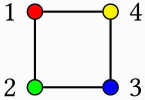

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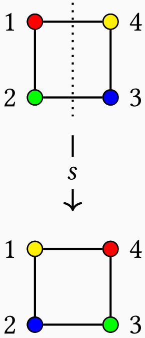

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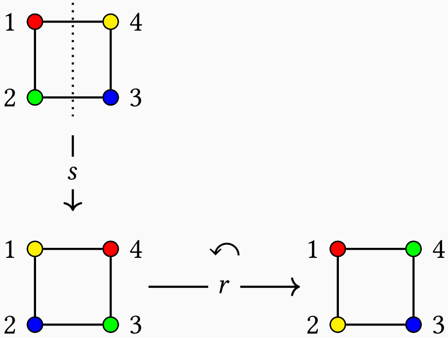

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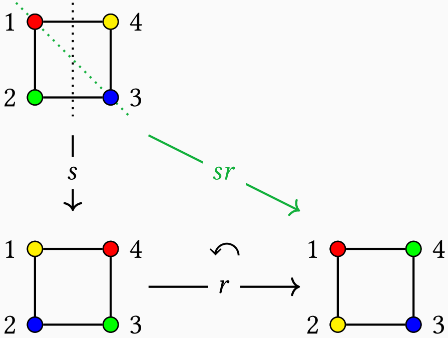

Orbits and stabilisers

Definition (orbit)

If G acts on $\mathcal{S}$, then **orbit** of $\alpha \in \mathcal{S}$ is $\alpha^G := \{\alpha^g : g \in G\}$.
Idea: states $\alpha^g \in \mathcal{S}$ reachable from fixed $\alpha \in \mathcal{S}$ by moves $g \in G$.

One orbit only: **transitive** action.

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If G acts on $\mathcal{S}$, then **stabiliser** of $\alpha \in \mathcal{S}$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.
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Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (*transitive*).

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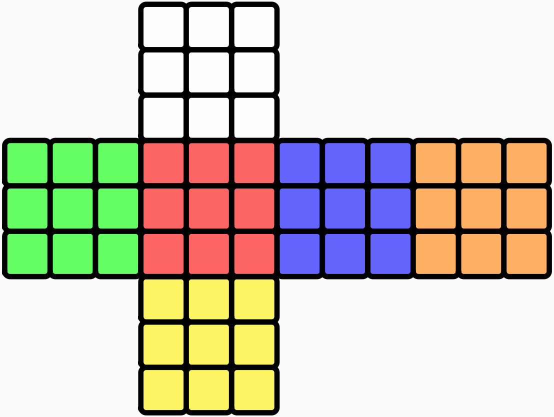
Theorem (orbit-stabiliser)

If G acts on $\mathcal{S}$, then for $\alpha \in \mathcal{S}$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.



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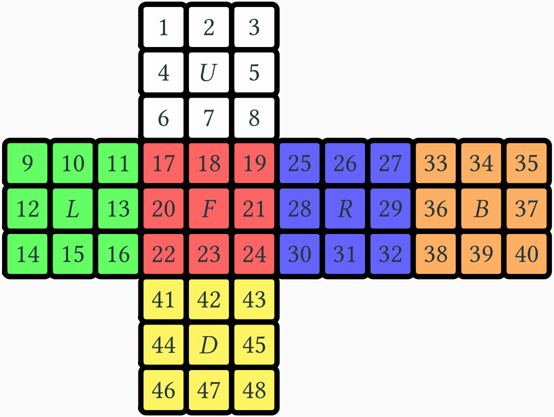
In **solved state** 1, label stickers (except each centre) using $[48]$:



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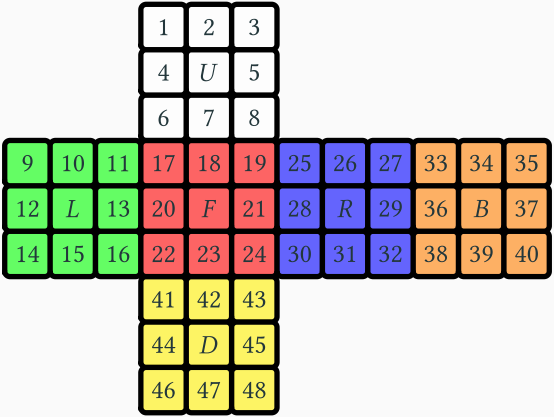
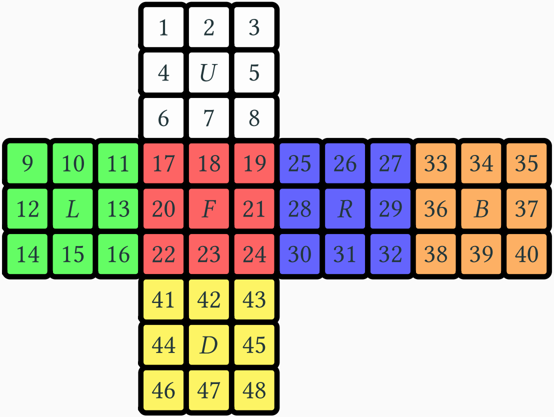


6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its operations (ii)

From *solved state* 1, consider F which rotates front face clockwise:



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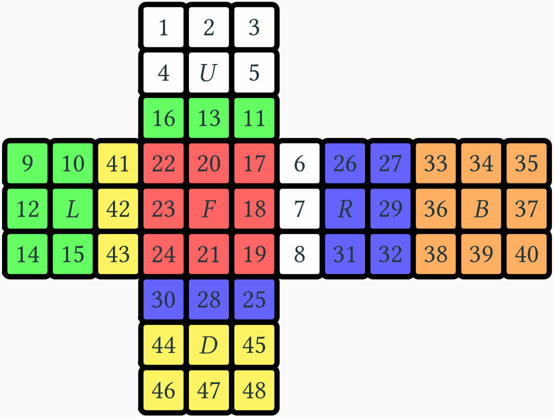

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

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Generators as permutations of labels $[48]$:

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Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D\rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

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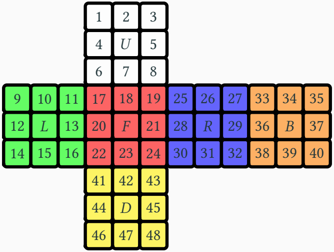
`Order` cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Definition (block)

If G acts transitively on $\mathcal{S}$ and $\Delta \subseteq \mathcal{S}$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \mathcal{S}$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

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A *transitive* G-action is **primitive** if there are no nontrivial blocks; otherwise it is **imprimitive**.

If G is perm group with primitive natural action, G is **primitive**.

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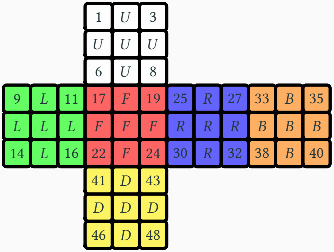
For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions $\mathcal{S}$); then G acts on Σ; if Δ is *maximal*, then acts primitively.

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



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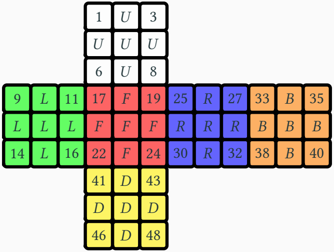
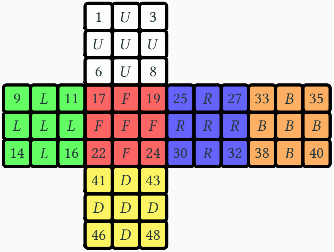


$$\Sigma = \{\overbrace{\{1,35,9\}}^{UBL}, \overbrace{\{6,11,17\}}^{ULF}, \overbrace{\{40,46,14\}}^{BDL}, \overbrace{\{27,3,33\}}^{RUB},$$
$$\underbrace{\{8,25,19\}}_{URF}, \underbrace{\{16,41,22\}}_{LDF}, \underbrace{\{32,48,38\}}_{RDB}, \underbrace{\{24,43,30\}}_{FDR}\}$$

is block system for *maximal* block $\{8, 25, 19\}$ (URF corner); corner stickers stay together.

Transitive action on corners (iii)



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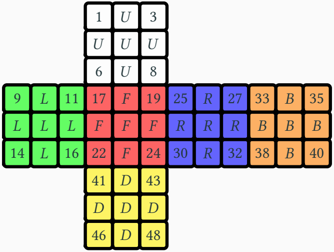

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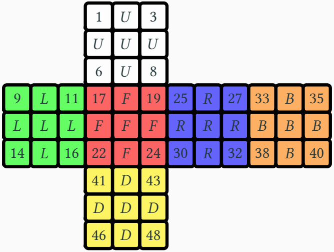

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$\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ "is" *primitive* quotient of $\mathcal{G}$).

Bases and stabiliser chains

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(\Omega)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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Theorem (Blaha, 1992)

Problem of finding minimum base for G is NP-complete (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).

Example (Rubik's group)

Using `BaseOfGroup` cmd in `GAP`, base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

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Theorem

For Rubik's group $\mathcal{G}$, $b(\mathcal{G}) = 18$.

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Generate uniformly random element of G.

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(*Alternative:* random product of generators in X — Markov chain; mixing time/distribution?)

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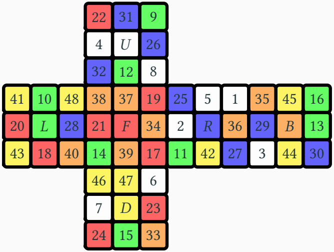
(*Application*: check if restickering of Rubik's cube is valid state.)



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Corollary

For Rubik's group $\mathcal{G}$, $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Base sizes of primitive groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

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Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

Non-large base permutation groups

Theorem (Liebeck, 1984)

For primitive perm group G of degree n, either:

(i) *G is "large base"; or*

(ii) $b(G) < 9 \log n$.

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Previous best result (Babai, 1981): $b(G) < O(\sqrt{n})$ if not containing $\mathrm{Alt}(n)$.

"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

Non-large base permutation groups (ii)

Theorem (Moscatiello & Roney-Dougal, 2021)
For primitive perm group G of degree n, and G is non-large base:

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Question (Moscatiello & Roney-Dougal, 2021)

Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* d *with natural action on* K^d *with* $b(G) = d + 1$*. (Then* G *is perm group of degree* $n = 2^d$*.)*

(i) *For* $d = 1$*, there is no such* G*.*

Theorem

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(i) *For $d = 1$, there is no such G.*

(ii) *For odd $3 \leq d \leq 9$ and $d = 2$, then G is $\mathrm{AGL}_d(2)$.*

Main result in thesis

Theorem

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(i) *For $d = 1$, there is no such G.*

(ii) *For odd $3 \leq d \leq 9$ and $d = 2$, then G is $\mathrm{AGL}_d(2)$.*

(iii) *For even $4 \leq d \leq 10$, then G is $\mathrm{AGL}_d(2)$ or $2^d : \mathrm{Sp}_d(2)$.*

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Proof (idea).

- Find representatives M of conjugacy classes of primitive maximal subgroups of $\mathrm{AGL}_d(2)$.

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Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. □

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

Main result in thesis (iii)

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

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Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

- $n = 2^d$ with $d \geq 2$, and G is $\mathrm{AGL}_d(2)$; or
- $n = 2^d$ with $d \geq 4$, and G is $2^d : \mathrm{Sp}_d(2)$.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP:
  https://www.gap-system.org/Doc/Examples/rubik.html
- J. A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Blaha — *Minimum bases for permutation groups: The greedy approximation*, 1992:
  https://doi:10.1016/0196-6774(92)90020-D
- Liebeck — *On minimal degrees and base sizes of primitive permutation groups*, 1984: https://doi.org/10.1007/bf01193603
- Moscatiello and Roney-Dougal: *Base sizes of primitive permutation groups*, 2021: https://doi.org/10.1007/s00605-021-01599-5