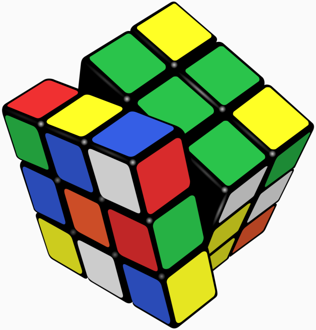
Rubik's cubes and permutation group theory

Lawrence Chen

October 6, 2022

Honours presentation



Contents

Some basic group theory

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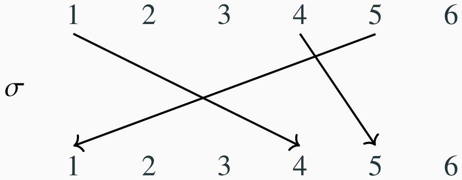

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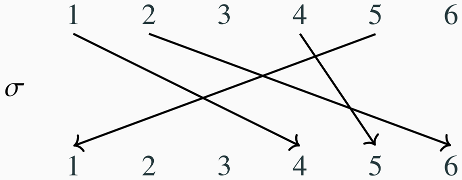

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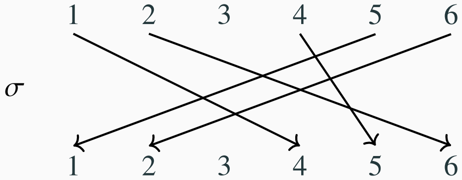

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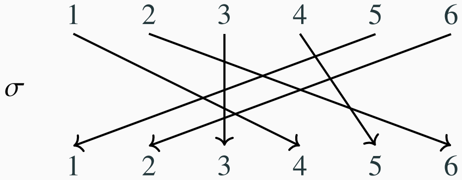

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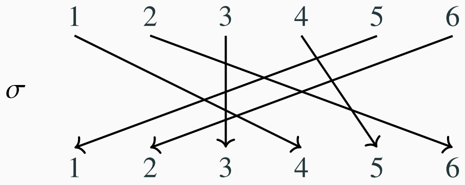

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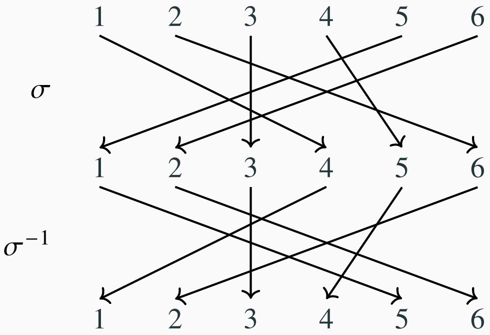

Inverse is $\sigma^{-1} = (1, 5, 4)(2, 6) \in \mathrm{Sym}(6)$.

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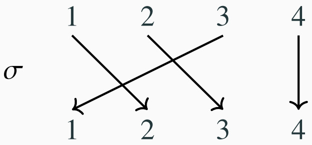

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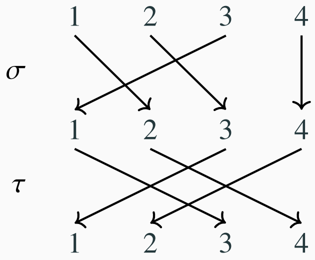

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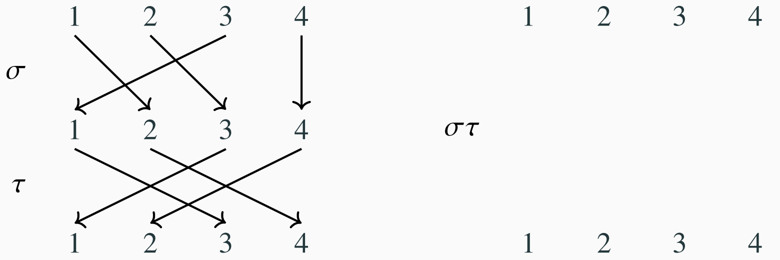

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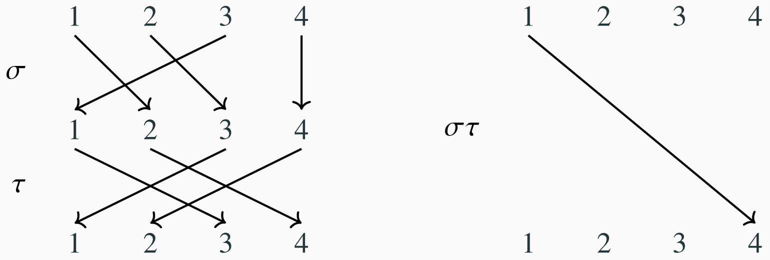

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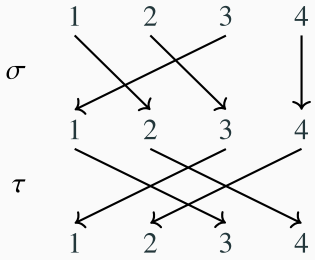

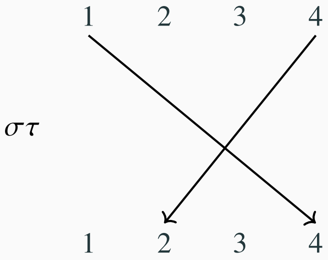

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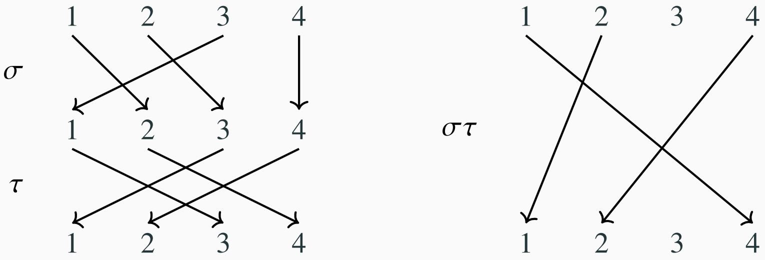

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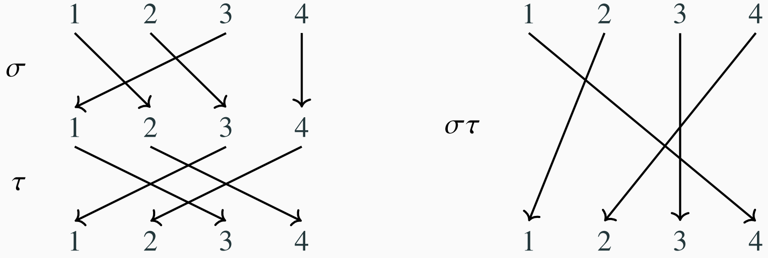

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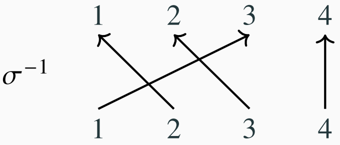

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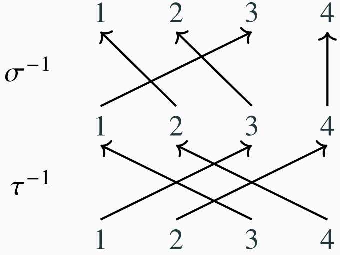

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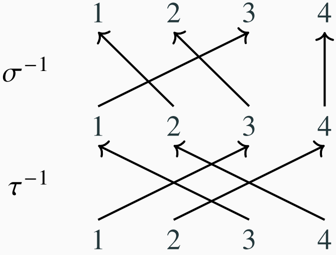

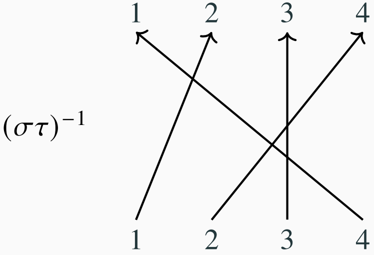

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$\sigma^{-1} = (1,3,2)$, $\tau^{-1} = (1,3)(2,4)$, $(\sigma\tau)^{-1} = (1,2,4)$.

σ^{-1}

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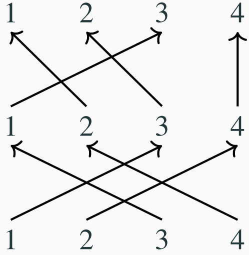
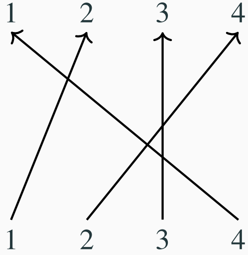


$(\sigma\tau)^{-1}$



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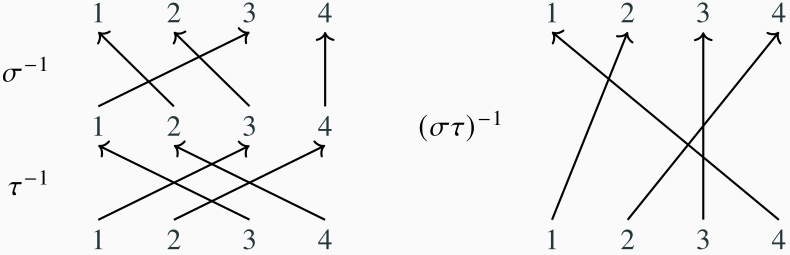

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Permutations v

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Subset H of group G is **subgroup** if it is group under same operation; write $H \leq G$. (Need to check: nonempty, closure, inverses.)

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Definition (permutation group)

A **permutation group** of *degree* n is a subgroup of $\mathrm{Sym}(n)$.

Definition (group action)

If G is group and $\Omega \neq \emptyset$ is set, a G**-action** is a map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.

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$\mathbb{Z}$ acts on $\Omega = \{12{:}00, 1{:}00, \ldots, 11{:}00\}$ by $(\alpha{:}00)^k = [\alpha + k]_{12}{:}00$ for $\alpha{:}00 \in \Omega$ and $k \in \mathbb{Z}$.

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Example (right regular action)

Group G acts on $\Omega = G$ (itself) via $\alpha^g = \alpha g$ for $\alpha, g \in G$.

Group actions ii

Definition (orbit)

If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

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$G = \{(), (1,2,4), (1,4,2)\} \le \mathrm{Sym}(4)$ acts on $\Omega = [4]$ naturally.

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Group actions iii

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Theorem (orbit-stabiliser)

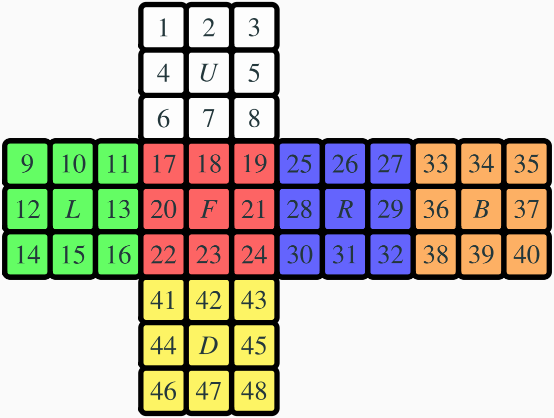
If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube i

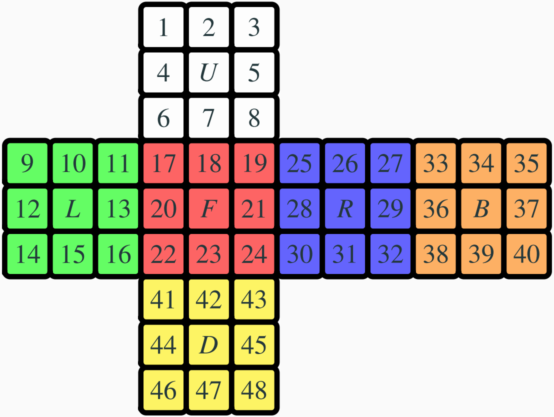
A Rubik's cube has 6 large faces (each with 3×3 smaller faces).

Label all smaller faces except centre on each side, using [48]:



6 *elementary moves* (**generators**): U, L, F, R, B, D (rotate clockwise).

Consider move F which rotates front face clockwise:



Representing the cube ii

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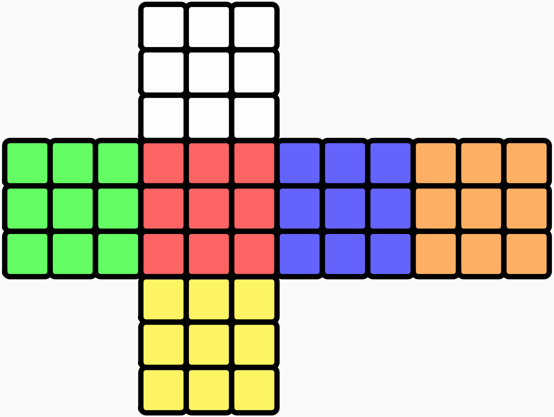

Moves vs states for Rubik's cube i

6 special *elementary moves* called **generators**: U, L, F, R, B, D. As permutations of labels $[48]$:

- $U =$
- $L =$
- $F =$
- $R =$
- $B =$
- $D =$

Empty move is $1 = ()$. In cubing community, *inverse elementary moves* usually denoted U', L', F', R', B', D' (instead of U^{-1}, etc.); powers usually denoted $U2, R2$ etc. (instead of U^2, R^2).

Valid move is sequence of elementary moves (product of generators). E.g. $RUR^{-1}U^{-1}$, $URU^{-1}L^{-1}UR^{-1}U^{-1}L$, $RUR^{-1}URU^2R^{-1}U^2$.

Moves vs states for Rubik's cube ii

Moves don't generally commute:

- $RU =$ (look at image of 1 point that doesn't match)
- $UR =$

Let $\mathcal{S}$ be valid **states**; can represent state $x \in \mathcal{S}$ as element of Sym(48) giving permutation of labels from solved state $1 = ()$.

Let $\mathcal{G}$ be valid **moves**; can represent move $\sigma \in \mathcal{G}$ as element of Sym(48) giving corresponding permutation of labels.

- State $x \in \mathcal{S}$ corresponds to move $x \in \mathcal{G}$ required to get solved state $1 = ()$ into state x.
- Move $\sigma \in \mathcal{G}$ corresponds to state $\sigma \in \mathcal{S}$ reached by applying move σ to solved state $1 = ()$.

So moves $\leftrightarrow$ states for Rubik's cube; as sets, $\mathcal{S} = \mathcal{G}$.

Set of moves $\mathcal{G}$ forms group: composition of valid moves is valid move; identity move $1 = () \in \mathcal{G}$, inverse moves exist; associative.

Definition (Rubik's group)
$\mathcal{G} \le \mathrm{Sym}(48)$ is permutation group of degree 48, called the **Rubik's group**; it acts naturally on $[48]$. Note: $G = \langle U, L, F, R, B, D \rangle$.

For move $\sigma \in \mathcal{G}$ and state $x \in \mathcal{S}$, applying σ to x gives state $x^{\sigma} = x\sigma \in \mathcal{S}$. This is regular action of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \le 48!$). But what is $|\mathcal{G}|$?

TODO: orbits, stabilisers (corner pieces/edge pieces), GAP code?

TODO

Order of generators: all 4

Order of commutator $RUR^{-1}U^{-1}$ is 6 (write out, get Wes video)

Order of $URU^{-1}L^{-1}UR^{-1}U^{-1}L$ is 3 (write out, get Wes video) – last layer corner permutation (3 states)

Order of $RUR^{-1}URU^2R^{-1}U^2$ is 3 (write out, get Wes video) – last layer edge permutation (3 states)

Order of RU is 105, order of Clayton's move UL' is 63

What is element of order 5? $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

Jake's theorems i

Theorem (Jake Vandenberg's conjecture)
There is no Rubik's cube move that cycles through all states.

Recall: states $\leftrightarrow$ moves. Rubik's group $\mathcal{G}$ acts on states by applying move $\sigma \in \mathcal{G}$ to state $x \in \mathcal{G}$ to get state $x^\sigma = x\sigma \in \mathcal{G}$.

Equivalent question: for starting state, WLOG $1 = ()$, is there $\sigma \in \mathcal{G}$ with $\{1^{\sigma^k} : k \in \mathbb{Z}\} = \{1\sigma^k : k \in \mathbb{Z}\} = \{\sigma^k : k \in \mathbb{Z}\} = \mathcal{G}$? In group theory language:

Theorem (Jake Vandenberg's conjecture)
The Rubik's group $\mathcal{G}$ is not cyclic. (I.e. no $\sigma \in \mathcal{G}$ with $\mathcal{G} = \langle\sigma\rangle$.)

Proof.
If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. $\square$

Theorem (Jake Vandenberg's theorem)
There is no Rubik's cube move that when repeated, if starting from the solved state, never returns to the solved state.

A k-fold repetition of move $\sigma \in G$, applied to solved state $1 = ()$, gives $1^{\sigma^k} = 1\sigma^k = \sigma^k$. Returning to solved state: $\sigma^k = 1$ (for $k > 0$).

Equivalent question: does any $\sigma \in G$ have infinite order?

Proposition
If G is finite group and $g \in G$, then $g^{|G|} = 1$.

Corollary (Jake Vandenberg's theorem)
There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).

Analysing the Rubik's group

Bases and stabiliser chains i

TODO

How many valid states are there? i

TODO

Can this restickering be solved? i

TODO

Generating random Rubik's cube states i

TODO

Solving a Rubik's cube... i

TODO

Concluding remarks

References i

- TODO