Minimum bases in permutation groups

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Honours presentation
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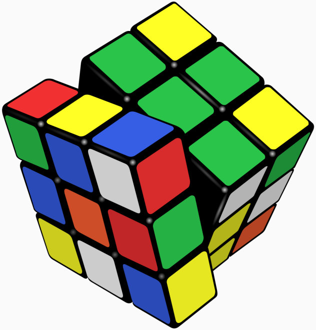


Contents

Aim: analyse Blaha's 1992 paper on NP-completeness of min base problem, and recent results for primitive perm groups.

Motivation: understanding the Rubik's cube

- How can we represent *operations* of a cube?
- *How many* states does a Rubik's cube have?
- How can we better *understand* operations of a cube?

One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were* ***more than three billion*** *possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold* ***more than 120*** *hamburgers.*

Some basic group theory

Permutations

Definition (permutation)

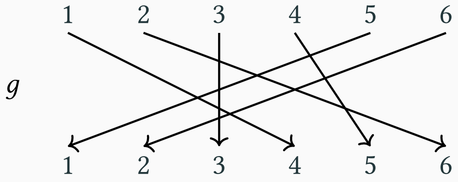
Permutation of Ω is bijection $g : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

(For $\Omega = [n] := \{1, \ldots, n\}$, write $\mathrm{Sym}(n)$.)

Write $1 = ()$ for identity. Write i^g instead of $g(i)$ for *image*.

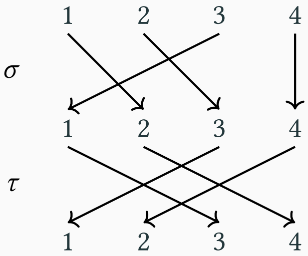
Cycle notation: $g = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:

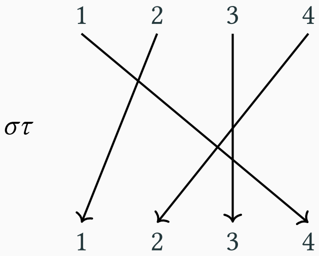


It means $1^g = 4,\ 4^g = 5,\ 5^g = 1,\ 2^g = 6,\ 6^g = 2,\ 3^g = 3.$

Permutations (ii)

Product/composition: for $g, h \in \mathrm{Sym}(\Omega)$, gh means “first g, then h”, so $\alpha^{gh} = (\alpha^g)^h$. E.g. $g = (1,2,3) \in \mathrm{Sym}(4)$, $h = (1,3)(2,4) \in \mathrm{Sym}(4)$,





$$gh = (1,2,3)(1,3)(2,4) = (1,4,2) \in \mathrm{Sym}(4).$$

Note: here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \mathrm{Sym}(\Omega)$.

Definition (permutation group)

Perm group on Ω (of deg n) is subset $G \leq \mathrm{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

(i) **(closure)** $gh \in G$ for $g, h \in G$;

(ii) **(identity)** $1 = () \in G$;

(iii) **(inverses)** $g^{-1} \in G$ for $g \in G$.

Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**, $\varepsilon_i \in \{\pm 1\}$; write $G = \langle X \rangle$.

Example (dihedral group)

Let $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** of order 8 is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ (e.g. $srs^{-1}r^2 = r$), *"symmetries of square"* .

Definition (group action)

For $G \leq \mathrm{Sym}(\Omega)$ and $\mathcal{S} \neq \emptyset$, a G-**action** is map $\mathcal{S} \times G \to \mathcal{S}$, $(\alpha, g) \mapsto \alpha^g \in \mathcal{S}$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \mathcal{S}$ and $g, h \in G$.
Degree of action is $|\mathcal{S}|$.

Idea: $\alpha \in \mathcal{S}$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

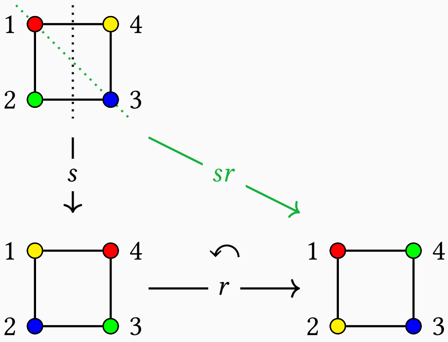
Example (natural action)

$G \leq \mathrm{Sym}(\Omega)$ acts on $\mathcal{S} = \Omega$ by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

Example (dihedral group)

Recall $D_8 = \langle r, s\rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ acts naturally on $[4]$.

Note: $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$, $sr = (2, 4)$. Visualise D_8-action by labelling vertices of square by $[4]$: $g \in D_8$ sends vertex at i to i^g.



Orbits and stabilisers

Definition (orbit)

If G acts on $\mathcal{S}$, then **orbit** of $\alpha \in \mathcal{S}$ is $\alpha^G := \{\alpha^g : g \in G\}$.
Idea: states $\alpha^g \in \mathcal{S}$ reachable from fixed $\alpha \in \mathcal{S}$ by moves $g \in G$.

One orbit only: **transitive** action.

Definition (stabiliser)

If G acts on $\mathcal{S}$, then **stabiliser** of $\alpha \in \mathcal{S}$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.
Idea: moves $g \in G$ that fix given $\alpha \in \mathcal{S}$.

Orbits and stabilisers (ii)

Orbit α^G: states $\alpha^g \in \mathcal{S}$ reachable from fixed α by moves $g \in G$.
Stabiliser G_α: moves $g \in G$ that fix given α.

Example (dihedral group)

Recall $G = D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \mathrm{Sym}(4)$ where $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$.

Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (*transitive*).

Stabiliser of 1: $sr = (2, 4)$, $sr^2 = (1, 2)(3, 4)$, $sr^3 = (1, 3)$, so $G_1 = \{(), (2, 4)\} = \{1, sr\}$.

Note: $|1^G||G_1| = 4 \cdot 2 = 8 = |G|$. Coincidence?

Theorem (orbit-stabiliser)

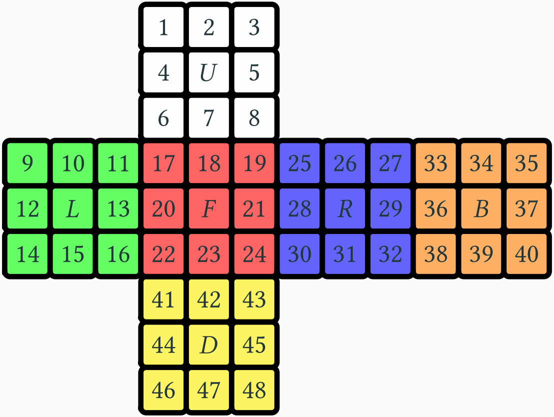
If G acts on $\mathcal{S}$, then for $\alpha \in \mathcal{S}$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers.*

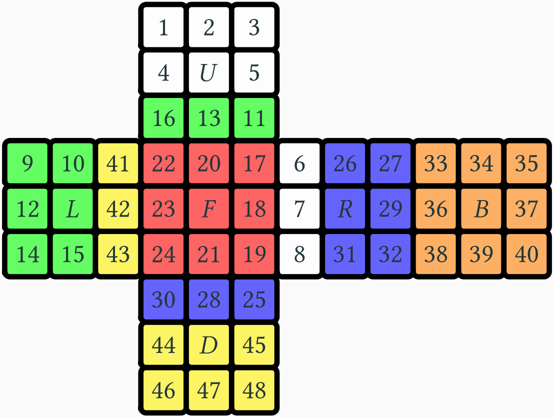
In **solved state** 1, label stickers (except each centre) using $[48]$:



6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its operations (ii)

From *solved state* 1, consider F which rotates front face clockwise:



$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Generators as permutations of labels $[48]$:

- $U = (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)$
- $L = (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$
- $F = (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)$
- $R = (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)$
- $B = (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)$
- $D = (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)$

Operation is sequence of generators and inverses. E.g. $RUR^{-1}U^{-1}$, $URU^{-1}L^{-1}UR^{-1}U^{-1}L$, $RUR^{-1}URU^2R^{-1}U^2$, $1 = ()$.

Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

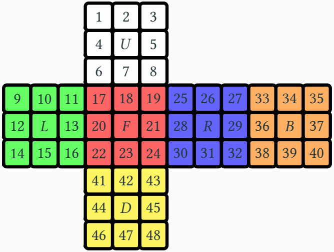
The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);;
L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35);;
F := (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11);;
R := (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24);;
B := (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27);;
D := (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40);;
G := Group( U, L, F, R, B, D );
```

Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Transitive action on corners

Definition (block)

If G acts transitively on $\mathcal{S}$ and $\Delta \subseteq \mathcal{S}$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \mathcal{S}$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

Examples of blocks: singletons, $\mathcal{S}$, orbits.

Block is **nontrivial** if $|\Delta| > 1$ and $\Delta \neq \mathcal{S}$.

Definition (primitivity)

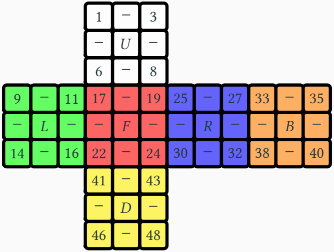
A *transitive* G-action is **primitive** if there are no nontrivial blocks; otherwise it is **imprimitive**.

If G is perm group with primitive natural action, G is **primitive**.

For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions $\mathcal{S}$); then G acts on Σ; if Δ is *maximal*, then acts primitively.

Transitive action on corners (ii)

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:

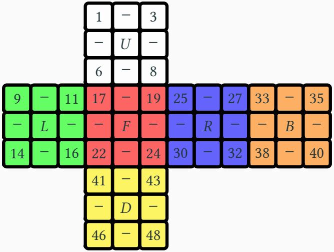


$$\Sigma = \{\overbrace{\{1, 35, 9\}}^{UBL}, \overbrace{\{6, 11, 17\}}^{ULF}, \overbrace{\{40, 46, 14\}}^{BDL}, \overbrace{\{27, 3, 33\}}^{RUB},$$

$$\underbrace{\{8, 25, 19\}}_{URF}, \underbrace{\{16, 41, 22\}}_{LDF}, \underbrace{\{32, 48, 38\}}_{RDB}, \underbrace{\{24, 43, 30\}}_{FDR}\}$$

is block system for *maximal* block $\{8, 25, 19\}$ (URF corner); corner stickers stay together.

Transitive action on corners (iii)



$\mathcal{G}$ acts primitively on Σ (degree 8); $g \in \mathcal{G}$ induces perm of Σ, e.g.

$$F \mapsto (\underbrace{\{6, 11, 17\}}_{ULF}, \underbrace{\{8, 25, 19\}}_{URF}, \underbrace{\{24, 43, 30\}}_{FDR}, \underbrace{\{16, 41, 22\}}_{LDF}) \in \mathrm{Sym}(\Sigma).$$

$\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ "is" *primitive* quotient of $\mathcal{G}$).

Bases and stabiliser chains

Bases and stabiliser chains

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(\Omega)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

Corresponding **stabiliser chain** is

$$G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$$

where $G^i = G^{i-1}_{\beta_i} = G_{\beta_1,\ldots,\beta_i}$.

Base B contains elts of Ω such that only $1 \in G$ fixes every $\beta_i \in B$.
(Short base desirable: how to compute **min base** of length $b(G)$?)

Theorem (Blaha, 1992)

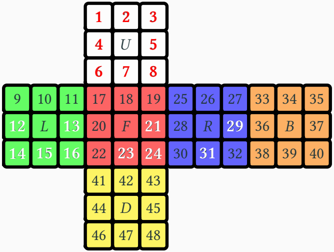
Problem of finding minimum base for G is NP-complete (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).

Example (Rubik's group)

Using `BaseOfGroup` cmd in GAP, base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

Contains: 7 corner stickers (from 7 of 8 corners), 11 edge stickers (from 11 of 12 edges).



Bases and stabiliser chains (iii)

Stabiliser chain implemented in GAP; useful in algorithms.

Let $G = \langle X \rangle \leq \mathrm{Sym}(\Omega)$ have base B and stabiliser chain

$$G = G^0 \geq G^1 \geq \cdots \geq G^r = 1.$$

Problem (random element generation)

Generate uniformly random element of G.

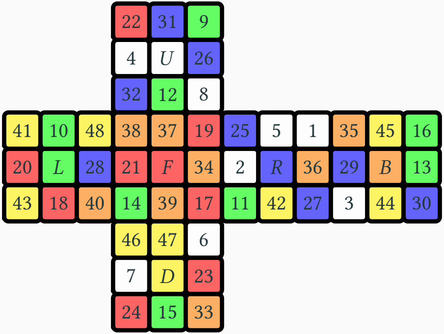
(*Alternative:* random product of generators in X — Markov chain; mixing time/distribution?)

Stabiliser chain implemented in GAP; useful in algorithms.

Problem (membership testing)

For $g \in \mathrm{Sym}(\Omega)$, test if $g \in G$.

(*Application*: check if restickering of Rubik's cube is valid state.)



What is the size of the Rubik's group?

Theorem (size of perm group)
If $B = [\beta_1, \ldots, \beta_r]$ is base for $G \leq \mathrm{Sym}(\Omega)$ with stabiliser chain $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, then

$$|G| = |\beta_1^{G^0}||\beta_2^{G^1}| \cdots |\beta_r^{G^{r-1}}|.$$

Orbits and stabilisers can be easily computed (e.g. using `GAP`).

Implementing base and stabiliser chain for Rubik's group $\mathcal{G}$ (using `BaseOfGroup` and `StabChain` cmds), `GAP` computes:

Corollary
For Rubik's group $\mathcal{G}$, $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Base sizes of primitive groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

Note: $t_{a,v} \in \mathrm{Sym}(K^d)$ (bijection).

Definition

Affine group $\mathrm{AGL}_d(K) \le \mathrm{Sym}(K^d)$ of dim d is affine transfs of K^d.
For $K = \mathbb{F}_q$ finite field, write $\mathrm{AGL}_d(q)$ (perm group of deg q^d).

Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

Theorem (Liebeck, 1984)

For primitive perm group G of degree n, either:

(i) *G is "large base"; or*

(ii) $b(G) < 9 \log n$.

Previous best (Babai, 1981): $b(G) = O(\sqrt{n})$ if not containing $\mathrm{Alt}(n)$.

"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

Non-large base permutation groups (ii)

Theorem (Moscatiello & Roney-Dougal, 2021)
For primitive perm group G of degree n, and G is non-large base:

(i) *G is the Mathieu group M_{24} (degree 24); or*

(ii) $b(G) \leq \lceil \log n \rceil + 1$.

Moreover, if $b(G) = \log n + 1$ then $G \leq \mathrm{AGL}_d(2)$ with $n = 2^d$.

Question (Moscatiello & Roney-Dougal, 2021)
Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Main result in thesis

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* $d \leq 10$ *with natural action on* K^d *with* $b(G) = d + 1$. *(Then* G *is perm group of degree* $n = 2^d$.*) Then:*

(i) G *is* $\mathrm{AGL}_d(2)$ *with* $d \geq 2$*; or*

(ii) G *is* $\mathrm{Sp}_d(2) \ltimes C_2^d$ *with* $d \geq 4$ *even.*

Main result in thesis (ii)

Proof (idea).

- Find representatives M of conjugacy classes of primitive maximal subgroups of $\mathrm{AGL}_d(2)$.
- Use *greedy base algorithm* to find base for M; if base of length at most d is found then $b(M) \leq d$ and discard.
- Otherwise, recursively check for each representative M.

Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. $\square$

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

Main result in thesis (iii)

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff $n = 2^d$ and:

- G is $\mathrm{AGL}_d(2)$ with $d \geq 2$; or
- G is $\mathrm{Sp}_d(2) \ltimes C_2^d$ with $d \geq 4$ even.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP:
  https://www.gap-system.org/Doc/Examples/rubik.html
- J. A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Blaha — *Minimum bases for permutation groups: The greedy approximation*, 1992:
  https://doi:10.1016/0196-6774(92)90020-D
- Liebeck — *On minimal degrees and base sizes of primitive permutation groups*, 1984: https://doi.org/10.1007/bf01193603
- Moscatiello and Roney-Dougal: *Base sizes of primitive permutation groups*, 2021: https://doi.org/10.1007/s00605-021-01599-5

References and resources (ii)

The **order** of $g \in G \leq \mathrm{Sym}(\Omega)$ is smallest $k \in \mathbb{Z}_+$ such that $g^k = 1$.
Fact: order of g is lcm of cycle lengths; it divides $|G|$.

Note: for Rubik's group, R has order 4, $RUR^{-1}U^{-1}$ has order 6, RU has order 105 (GAP). Order 7? $(RU)^{15}$. Order 13? None;

$$|\mathcal{G}| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11.$$

- *Bonus:* Orders of elements in Rubik's group (1260 largest, 13 smallest without, 11 rarest, 60 most common, median 67.3, 73 options): https://www.jaapsch.net/puzzles/cubic3.htm#p34
- *Bonus:* Thistlethwaite's 52 move algorithm (using group theory): https://www.jaapsch.net/puzzles/thistle.htm

Large base definition

Definition

Perm group G of degree n is **large base** if

$$\mathrm{Alt}(m)^r \trianglelefteq G \leq \mathrm{Sym}(m) \wr \mathrm{Sym}(r)$$

for some m, r, k, where $\mathrm{Sym}(m)$ acts on $\binom{[m]}{k}$, and if $r > 1$ then wreath product has *product action* of degree $n = \binom{m}{k}^r$.