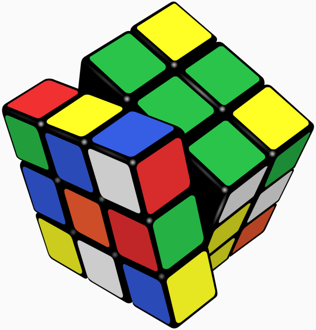
Minimum bases in permutation groups

Lawrence Chen

October 23, 2022

Honours presentation



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- Blocks and primitivity

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- Transitive action on corners

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Motivation: understanding the Rubik's cube

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One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

Definition (permutation)

Permutation of Ω is bijection $g : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

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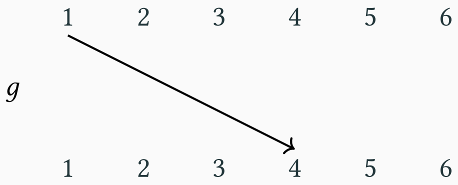

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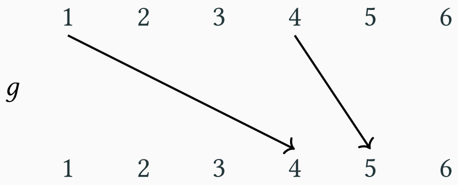

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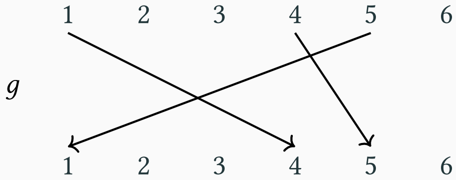

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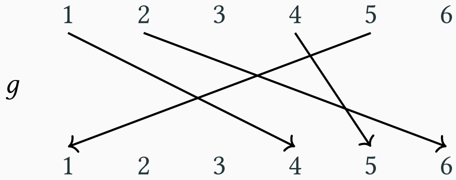

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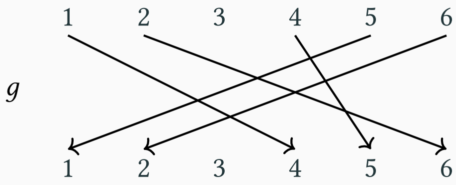

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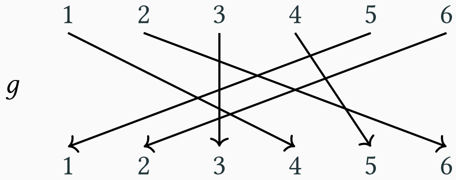

It means $1^g = 4,\ 4^g = 5,\ 5^g = 1,\ 2^g = 6,\ 6^g = 2,\ 3^g = 3$.

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Product/composition: for $g, h \in \mathrm{Sym}(\Omega)$, gh means "first g, then h", so $\alpha^{gh} = (\alpha^g)^h$.

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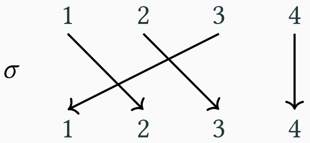

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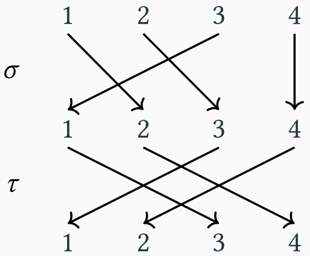

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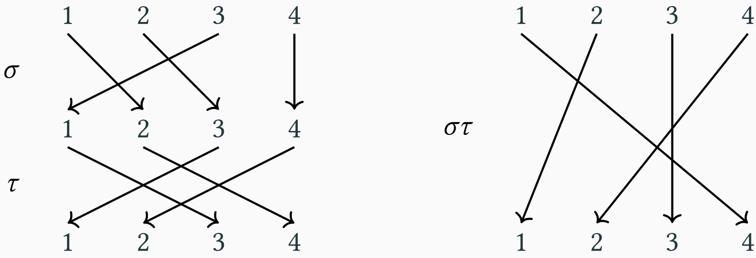

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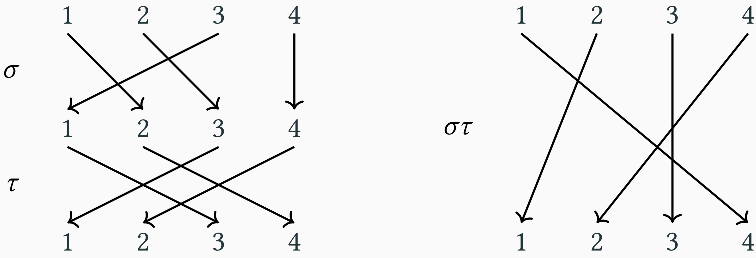

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Note: here, $gh \neq hg$, since $1^{gh} = 4$ but $1^{hg} = (1^h)^g = 3^g = 1$. Identity $1 = ()$ satisfies $1g = g1 = g$ for $g \in \mathrm{Sym}(\Omega)$.

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Perm group on Ω (of deg n) is subset $G \leq \mathrm{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

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Set X **generates** G if every $g \in G$ is $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**; write $G = \langle X \rangle$.

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Example (dihedral group)

Let $r = (1, 2, 3, 4), s = (1, 4)(2, 3) \in \mathrm{Sym}(4)$. **Dihedral group** is $D_8 := \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, *"symmetries of square"*.

Definition (group action)
For (perm) group G and set $\Omega \neq \emptyset$, a G-**action** is map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.
Degree of action is $|\Omega|$.

Idea: $\alpha \in \Omega$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

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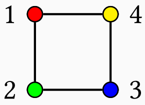

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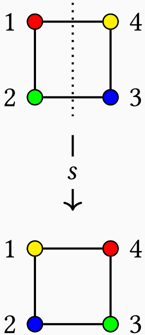

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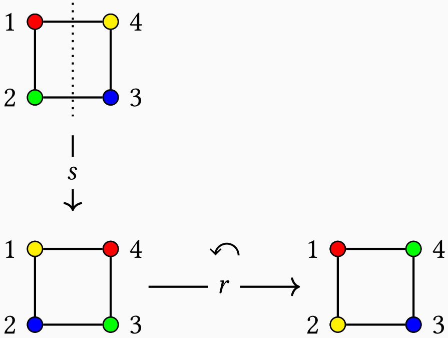

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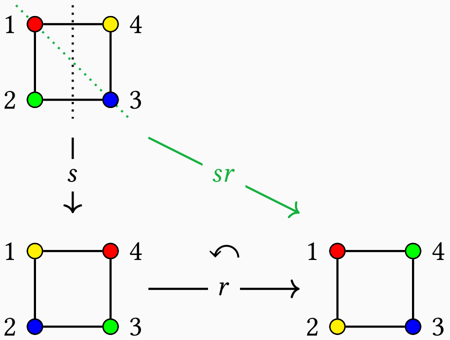

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If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.
Idea: states $\alpha^g \in \Omega$ reachable from fixed $\alpha \in \Omega$ by moves $g \in G$.

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Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in \Omega$, $|\alpha^G||G_\alpha| = |G|$.

Blocks and primitivity

Definition (block)

If G acts transitively on Ω and $\Delta \subseteq \Omega$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.

A **block** is $\Delta \subseteq \Omega$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

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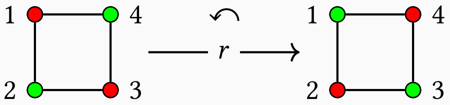
For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions Ω); then G acts on Σ; if Δ is *maximal*, then acts primitively.

Example (dihedral group)

Recall $G = D_8 = \langle r, s\rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \mathrm{Sym}(4)$ where $r = (1,2,3,4)$, $s = (1,4)(2,3)$, $sr = (2,4)$.

Block is $\Delta = \{1,3\}$ (nontrivial) with block system $\Sigma = \{\{1,3\},\{2,4\}\}$ (opposite vertices stay opposite):



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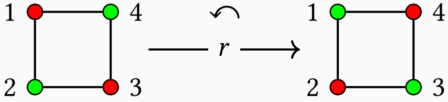


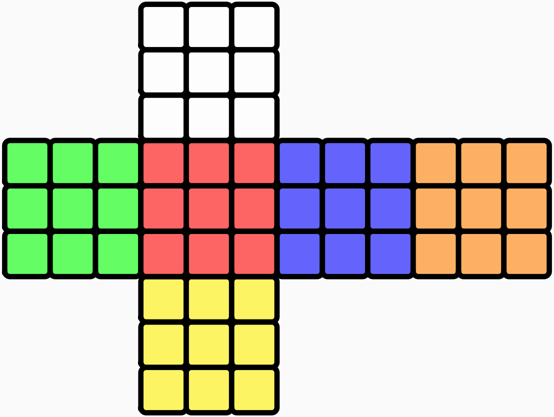
e.g. $\Delta^r = \{2,4\}$, $\Delta^s = \{4,2\}$, $\Delta^{sr} = \{1,3\} = \Delta$.

D_8 acts imprimitively on $[4]$ but primitively on Σ (degree 2).

The Rubik's group

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.



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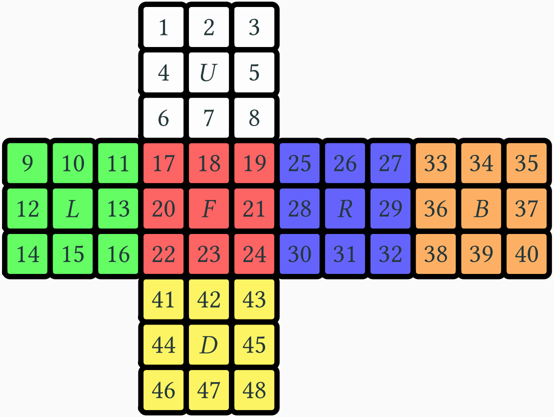

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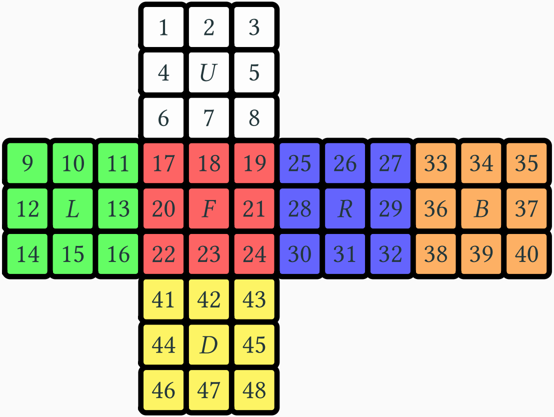
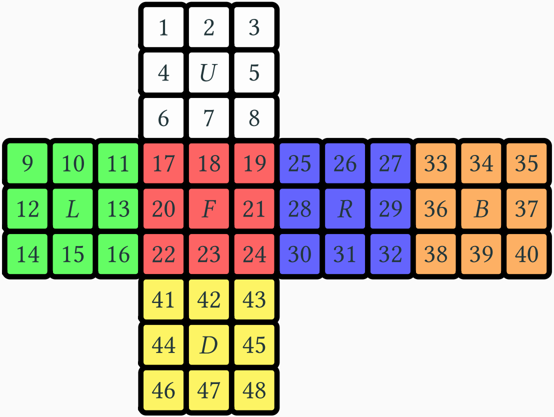


6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its operations (ii)

From *solved state* 1, consider F which rotates front face clockwise:



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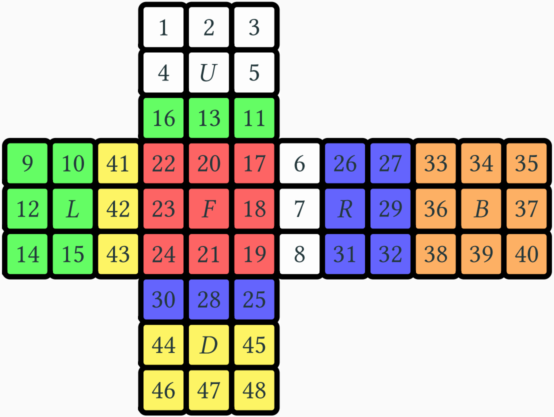

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels [48]:

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Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)
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G := Group( U, L, F, R, B, D );
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The Rubik's group of permutations (ii)

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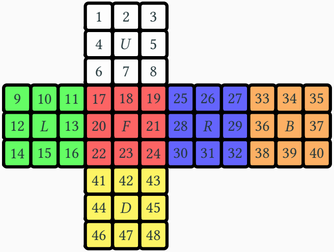
Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

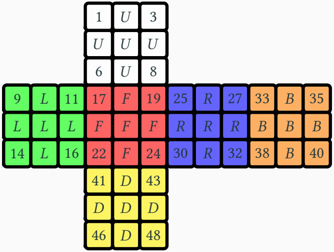



```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Transitive action on corners

$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



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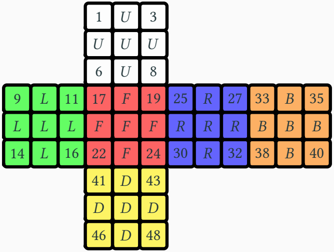
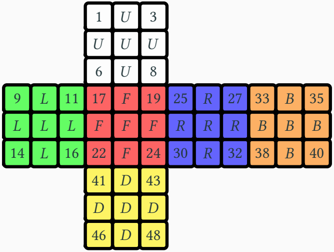


$$\Sigma = \{\overbrace{\{1,35,9\}}^{UBL}, \overbrace{\{6,11,17\}}^{ULF}, \overbrace{\{40,46,14\}}^{BDL}, \overbrace{\{27,3,33\}}^{RUB},$$
$$\underbrace{\{8,25,19\}}_{URF}, \underbrace{\{16,41,22\}}_{LDF}, \underbrace{\{32,48,38\}}_{RDB}, \underbrace{\{24,43,30\}}_{FDR}\}$$

is block system for *maximal* block $\{8,25,19\}$ (URF corner).

Transitive action on corners (ii)



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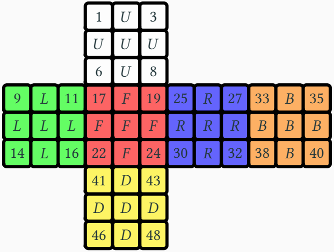

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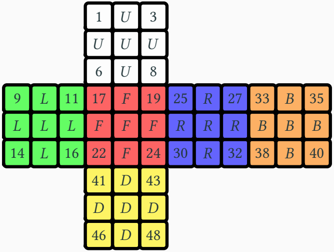

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$\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ “is” *primitive* quotient of $\mathcal{G}$).

Bases and stabiliser chains

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(\Omega)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq \Omega$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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$$G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$$

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Theorem (Blaha, 1992)

Problem of finding minimum base for G is NP-complete (if $\mathrm{P} \neq \mathrm{NP}$, then no polynomial time algorithm).

Bases and stabiliser chains (ii)

Example (Rubik's group)

Using `BaseOfGroup` cmd in `GAP`, base of $\mathcal{G}$ of size 18 is

$$B = [1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21, 23, 24, 29, 31].$$

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Theorem

For Rubik's group $\mathcal{G}$, $b(\mathcal{G}) = 18$.

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(*Application*: check if restickering of Rubik's cube is valid state.)

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If $B = [\beta_1, \ldots, \beta_r]$ *is base for* $G \leq \mathrm{Sym}(n)$ *with stabiliser chain* $G = G^0 \geq G^1 \geq \cdots \geq G^r = 1$, *then*

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Corollary

$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Base sizes of primitive groups

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

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Affine group $\mathrm{AGL}_d(K) \leq \mathrm{Sym}(K^d)$ of dim d is affine transfs of K^d.
For $K = \mathbb{F}_q$ finite field, write $\mathrm{AGL}_d(q)$ (perm group of deg q^d).

Definition

Let K be field. **Affine transformation** of K^d is map

$$t_{a,v} : K^d \to K^d, \quad u \mapsto ua + v$$

for $a \in \mathrm{GL}_d(K)$ and $v \in K^d$. (Treat u, v as row vectors.)

Note: $t_{a,v} \in \mathrm{Sym}(K^d)$ (bijection).

Definition

Affine group $\mathrm{AGL}_d(K) \leq \mathrm{Sym}(K^d)$ of dim d is affine transfs of K^d.
For $K = \mathbb{F}_q$ finite field, write $\mathrm{AGL}_d(q)$ (perm group of deg q^d).

Interested in $q = 2$, i.e. field $\mathbb{F}_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 \cdot 1 = 1$, etc.

Definition

Perm group G of degree n is **large base** if

$$\mathrm{Alt}(m)^r \trianglelefteq G \leq \mathrm{Sym}(m) \wr \mathrm{Sym}(r)$$

for some m, r, k, where $\mathrm{Sym}(m)$ acts on $\binom{[m]}{k}$ and $n = \binom{m}{k}^r$.

Large base permutation groups

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Theorem (Liebeck, 1984)

For primitive perm group G of degree n, either:

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"Remarkable" proof used *classification of finite simple groups*, *O'Nan-Scott theorem* (classifies primitive groups).

Theorem (Moscatiello & Roney-Dougal, 2021)

For primitive perm group G of degree n, and G is non-large base:

(i) *G is the Mathieu group M_{24} (degree 24); or*

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Large base permutation groups (ii)

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Moreover, if $b(G) = \log n + 1$ then $G \leq \mathrm{AGL}_d(2)$ with $n = 2^d$.

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Question (Moscatiello & Roney-Dougal, 2021)

Which primitive groups $G \leq \mathrm{Sym}(n)$ satisfy $b(G) = \log n + 1$?

Main result in thesis

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* d *with natural action on* K^d *with* $b(G) = d + 1$*. (Then* G *is perm group of degree* $n = 2^d$*.)*

(i) *For* $d = 1$*, there is no such* G*.*

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ *be primitive for some* d *with natural action on* K^d *with* $b(G) = d + 1$. *(Then* G *is perm group of degree* $n = 2^d$.*)*

(i) *For* $d = 1$, *there is no such* G.

(ii) *For odd* $3 \leq d \leq 9$ *and* $d = 2$, *then* G *is* $\mathrm{AGL}_d(2)$.

Main result in thesis

Theorem

Let $G \leq \mathrm{AGL}_d(2)$ be primitive for some d with natural action on K^d with $b(G) = d + 1$. (Then G is perm group of degree $n = 2^d$.)

(i) *For $d = 1$, there is no such G.*

(ii) *For odd $3 \leq d \leq 9$ and $d = 2$, then G is $\mathrm{AGL}_d(2)$.*

(iii) *For even $4 \leq d \leq 10$, then G is $\mathrm{AGL}_d(2)$ or $2^d : \mathrm{Sp}_d(2)$.*

Main result in thesis (ii)

Proof (idea).

- Find representatives M of conjugacy classes of primitive maximal subgroups of $\mathrm{AGL}_d(2)$.

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- Find representatives M of conjugacy classes of primitive maximal subgroups of $\mathrm{AGL}_d(2)$.
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- Otherwise, recursively check for each representative M.

Every primitive $G \leq \mathrm{AGL}_d(2)$ with $b(G) = d + 1$ is found by process (plus perhaps false positives), up to conjugacy. □

Greedy base algorithm performed better than `BaseOfGroup` in testing; found no false positives.

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

Main result in thesis (iii)

From above theorem, we conjecture the following:

Conjecture

Primitive group $G \leq \mathrm{Sym}(n)$ satisfies $b(G) = \log n + 1$ iff:

- $n = 2^d$ with $d \geq 2$, and G is $\mathrm{AGL}_d(2)$; or
- $n = 2^d$ with $d \geq 4$, and G is $2^d : \mathrm{Sp}_d(2)$.

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP: https://www.gap-system.org/Doc/Examples/rubik.html
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