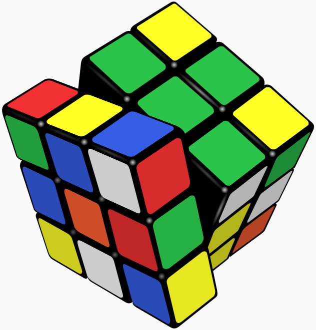
Rubik's cubes and permutation group theory

Lawrence Chen

October 7, 2022

Honours presentation



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Some basic group theory

The Rubik’s group

Representing the cube and its moves i

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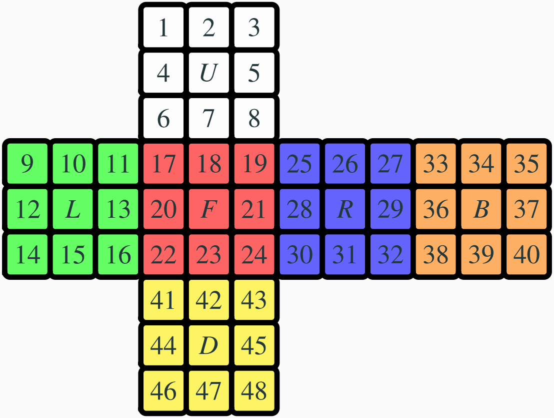

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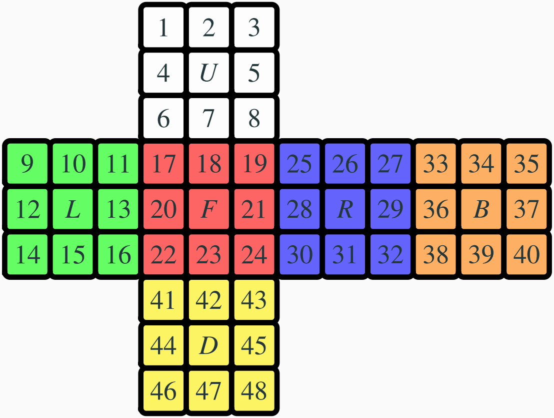
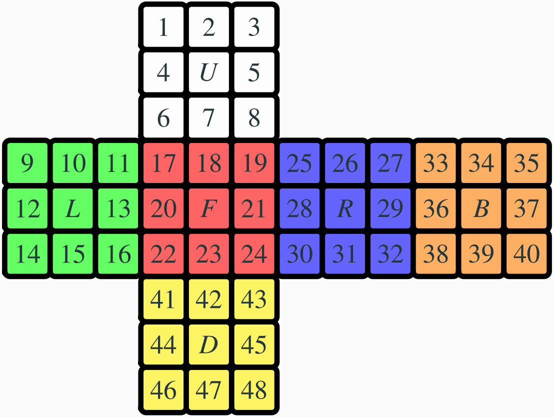


6 **elementary moves (generators)**: U, L, F, R, B, D (rot. *clockwise*).

From *solved state*, consider F which rotates front face clockwise:



Representing the cube and its moves ii

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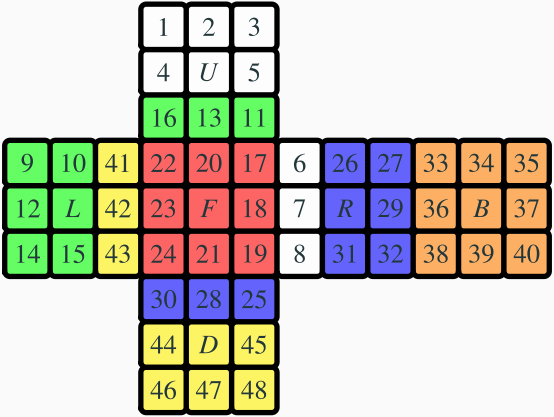

Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

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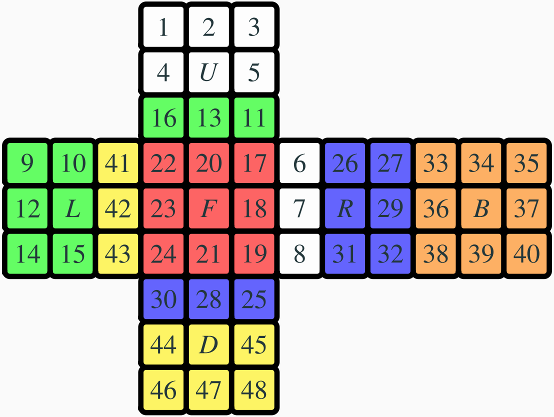

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$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Representing the cube and its moves iii

Generators as permutations of labels [48]:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
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In cubing community: inverse elementary moves written U', L', F', R', B', D' (instead of U^{-1} etc.); powers written $U2$, $R2$ etc. (instead of U^2, R^2).

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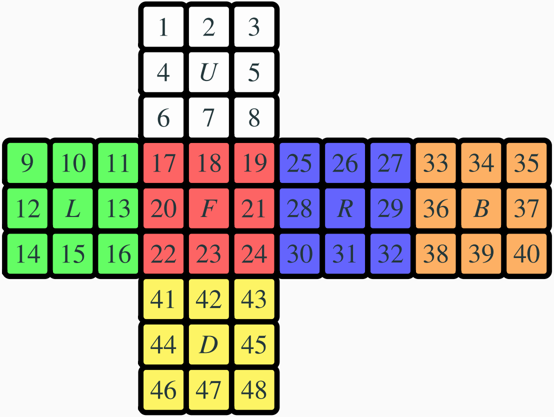
Moves vs states for Rubik's cube i

(Valid) state is result of applying *valid move* to *solved state* $1 = ()$.



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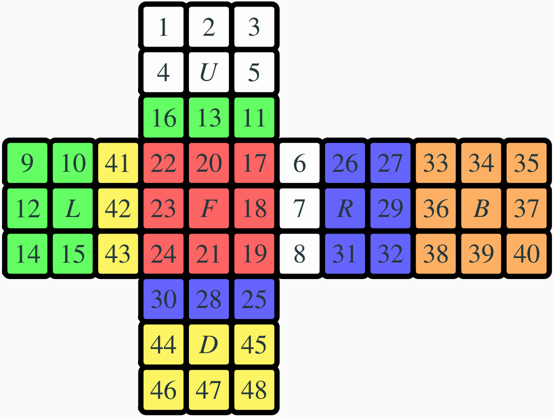

This new state is valid, as result of applying F to solved state.

Moves vs states for Rubik's cube ii

Let $\mathcal{S}$ be valid **states**; can represent state $x \in \mathcal{S}$ as element of Sym(48) giving permutation of labels to solved state $1 = ()$. (I.e. i^x is label at x-position of i in solved state 1.)

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So moves $\leftrightarrow$ states for Rubik's cube; as sets, $\mathcal{S} = \mathcal{G}$.

The Rubik's group of permutations i

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For *move* $\sigma \in \mathcal{G}$ and *state* $x \in \mathcal{S}$, applying σ to x gives *state* $x^{\sigma} = x\sigma \in \mathcal{S}$. This is *regular action* of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

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Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);
L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35);
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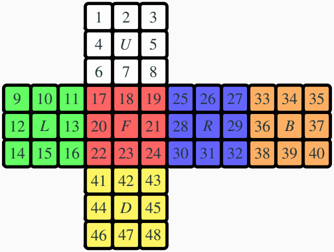
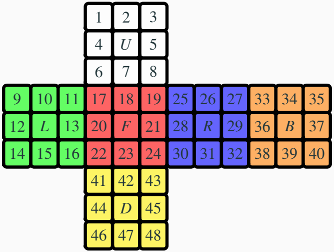
`Order` cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits and stabilisers i



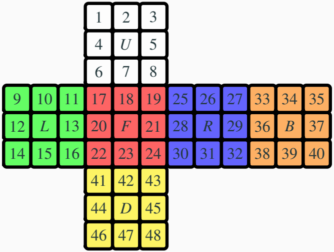

```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
      30, 14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
      37, 15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.



Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners $1, 3, 6, 8$.

Orbits and stabilisers ii



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```
gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
<permutation group of size 317842469683200 with 12 generators>
gap> Orbit( G_1368, 17 );
[ 17 ]
gap> Orbit( G_1368, 24 );
[ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
true
```

Some $\mathcal{H}$-orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Orders of moves i

Use GAP to compute products, order (using Order cmd).

```
gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
gap> Order( last );
6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect?

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```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect? I.e. for state $x \in \mathcal{S}$, smallest $k \in \mathbb{Z}_+$ with $x^{\sigma^k} = x\sigma^k = x \iff \sigma^k = 1$.

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Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length $4, 4, 4, 4, 4$: order is $\operatorname{lcm}(4, 4, 4, 4, 4) = 4$.

Orders of moves i

Use GAP to compute products, order (using Order cmd).

```
gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
gap> Order( last );
6
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  $= (1, 27, 35, 33, 9, 3)(2, 21, 5)(8, 30, 25, 43, 19, 24)(26, 34, 28)$:

  order is $\mathrm{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

$$= (1,9,35)(2,5,7)(3,33,27)(8,25,19)(18,34,26):$$

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Element of order 5?

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Element of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

Jake's theorems i

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There is no Rubik's cube move that cycles through all states.

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Proof.
If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian. But $\mathcal{G}$ is not abelian: $RU \neq UR$. □

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Analysing the Rubik’s group

Concluding remarks

References i

- TODO