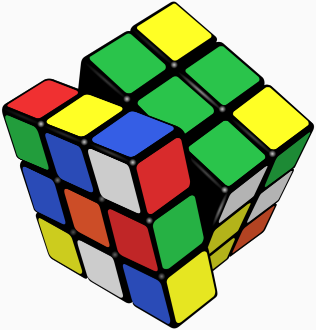
Rubik's cubes and permutation group theory

Lawrence Chen

October 8, 2022

Honours presentation



Contents

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(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic group theory

What is a group? i

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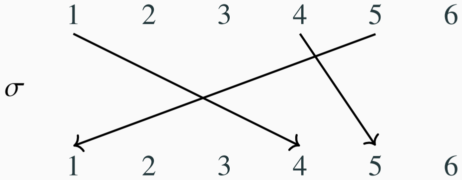

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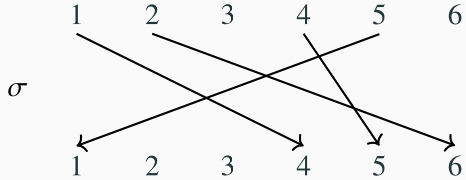

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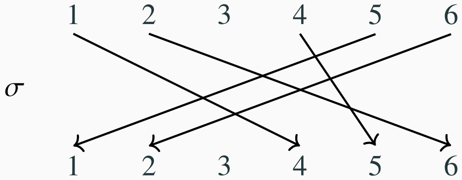

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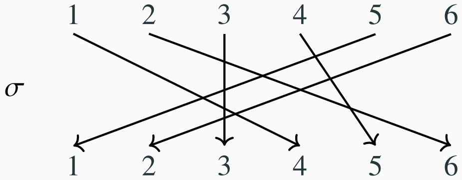

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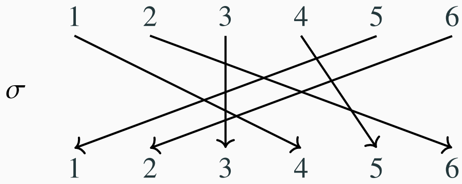
Inverses: For $\sigma = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$:



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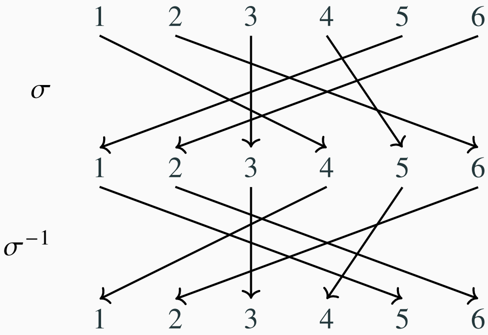

Inverse is $\sigma^{-1} = (1,5,4)(2,6) \in \mathrm{Sym}(6)$.

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Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means “first σ, then τ”, so $i^{\sigma\tau} = (i^{\sigma})^{\tau}$.

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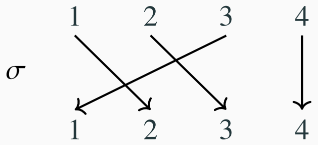

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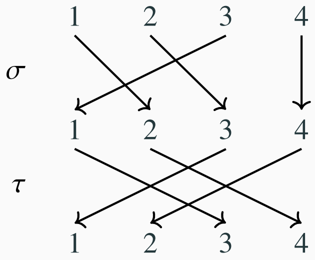

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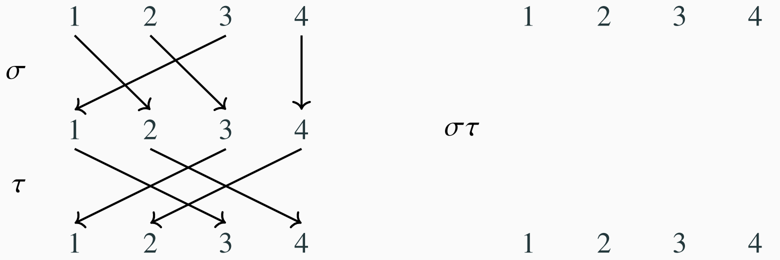

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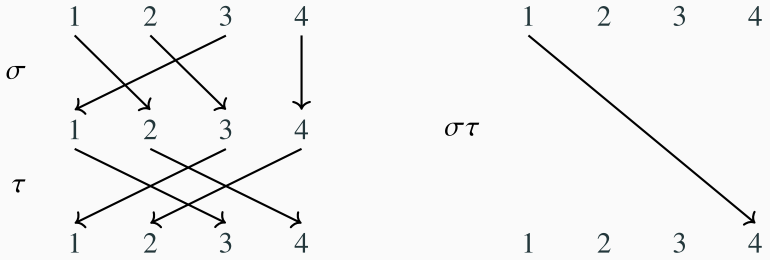

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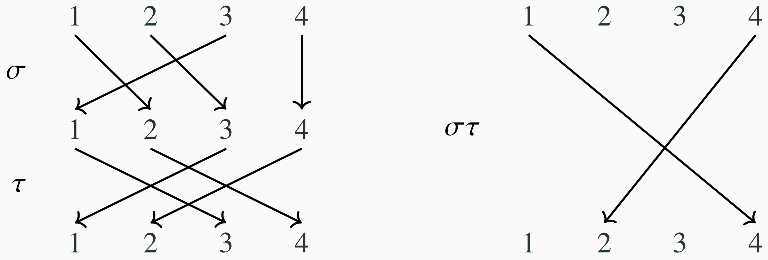

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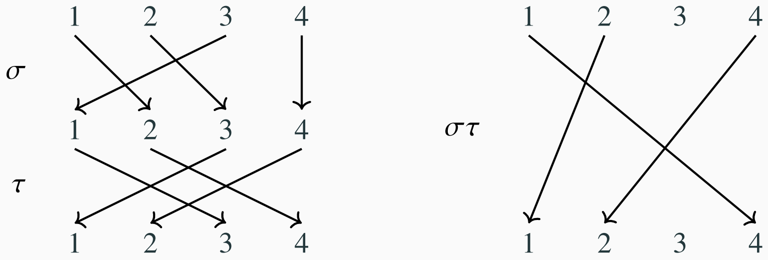

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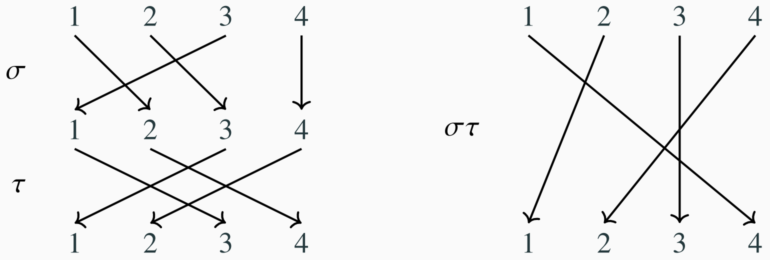

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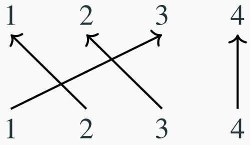

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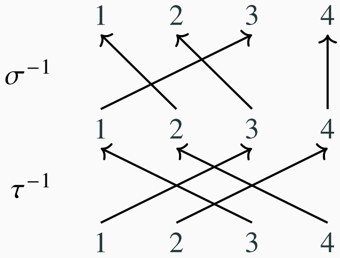

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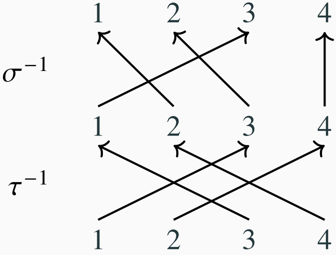

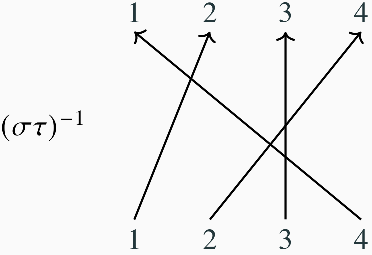

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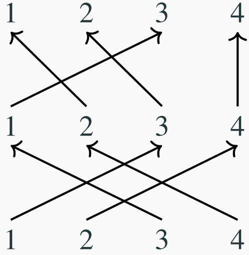

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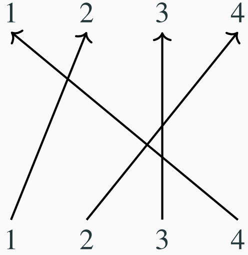

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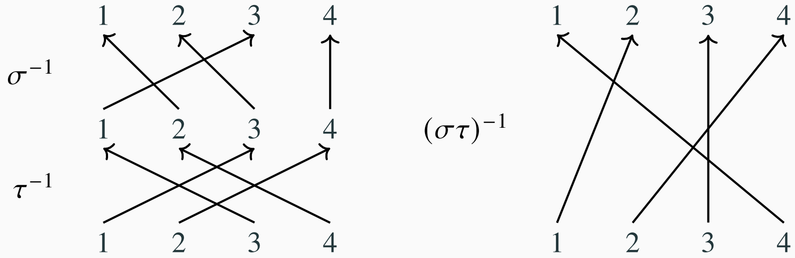

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Definition (permutation group)

A **permutation group** of *degree* n is a subgroup of $\mathrm{Sym}(n)$.

Group actions i

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If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

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Group actions iii

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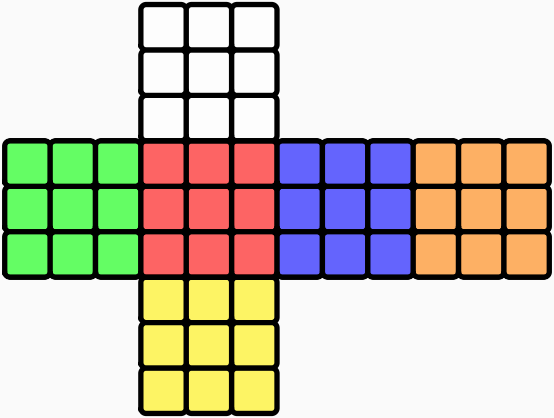
Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in G$, $|\alpha^G||G_\alpha| = |G|$.

The Rubik's group

Representing the cube and its moves i

A Rubik's cube has 6 large faces (each with 3×3 smaller faces).



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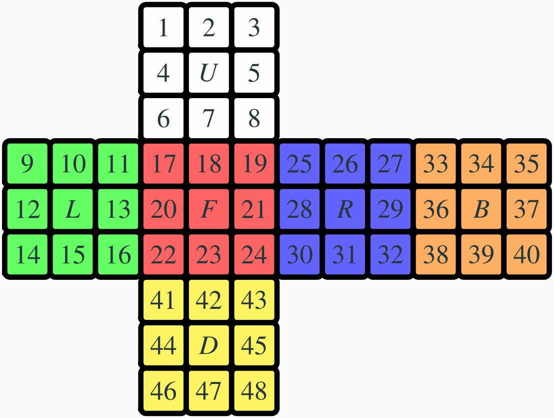
In **solved state** 1, label smaller faces (except each centre) using $[48]$:



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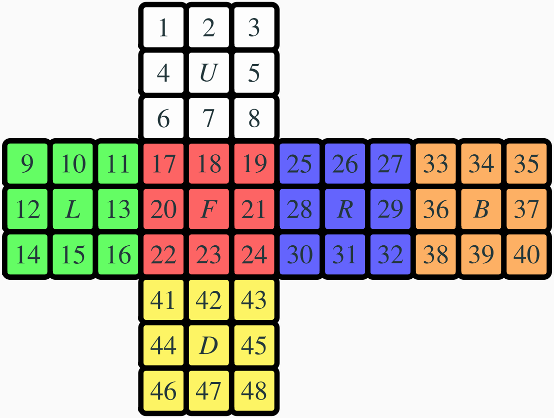
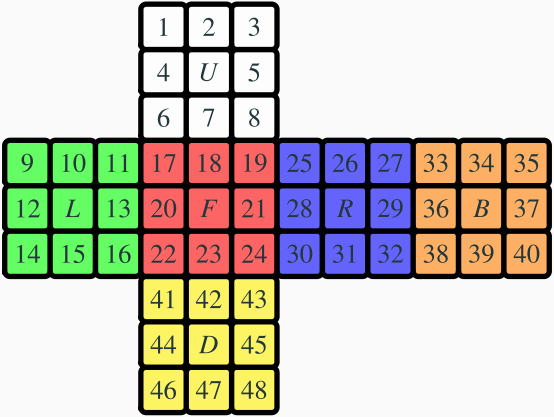


6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its moves ii

From *solved state* 1, consider F which rotates front face clockwise:



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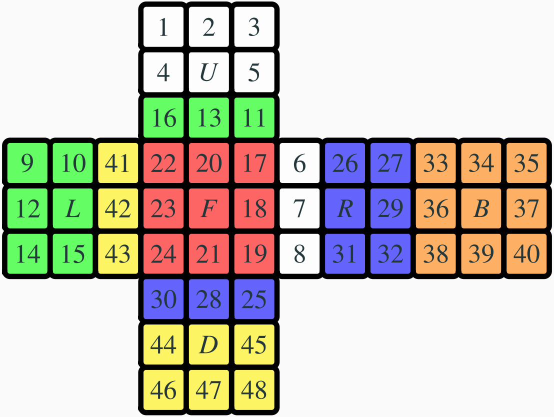

Under F: $17 \mapsto 19 \mapsto 24 \mapsto 22 \mapsto 17$, $18 \mapsto 21 \mapsto 23 \mapsto 20 \mapsto 18$, $6 \mapsto 25 \mapsto 43 \mapsto 16 \mapsto 6$, $7 \mapsto 28 \mapsto 42 \mapsto 13 \mapsto 7$, $8 \mapsto 30 \mapsto 41 \mapsto 11 \mapsto 8$, else fixed. So

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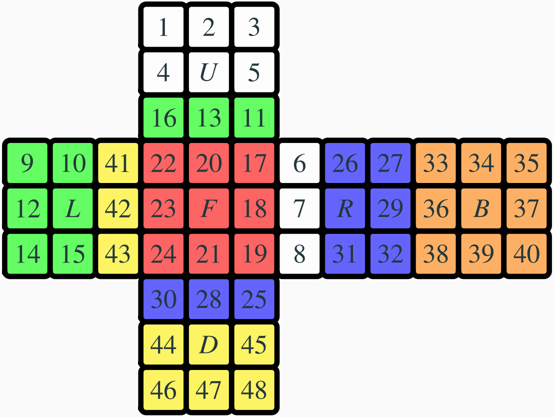

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$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

Representing the cube and its moves iii

Generators as permutations of labels [48]:

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Solving is applying valid move to get to solved state 1.

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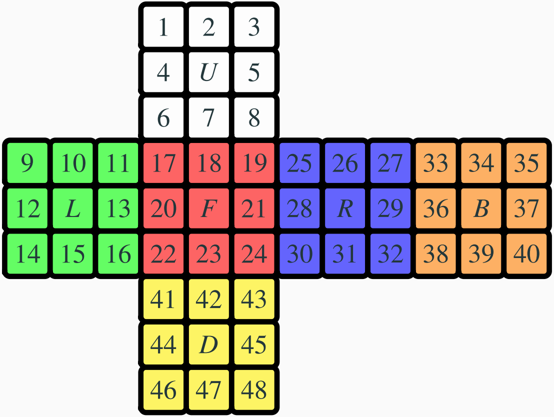
(Valid) state is result of applying *valid move* to *solved state* 1.



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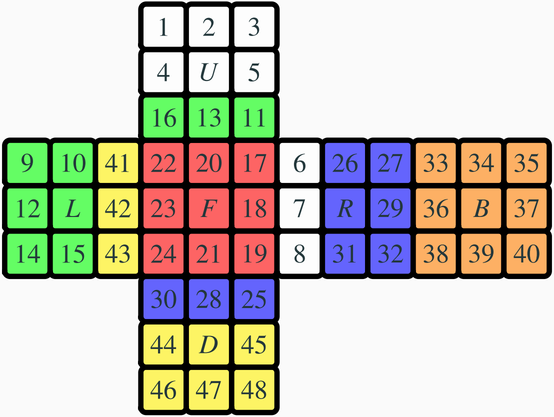

This new state is valid, as result of applying F to solved state.

Moves vs states for Rubik's cube ii

Restickering is valid state iff it can be *solved*. How to check?

Let $\mathcal{S}$ be valid **states**; let state $x \in \mathcal{S}$ be element of $\mathrm{Sym}(48)$ giving permutation of labels to solved state $1 \in \mathcal{S}$.
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So moves $\leftrightarrow$ states; as sets, $\mathcal{S} = \mathcal{G}$. *Solved state* is $1 = () \in \mathrm{Sym}(48)$.

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For *move* $\sigma \in \mathcal{G}$ and *state* $x \in \mathcal{S}$, applying σ to x gives *state* $x^\sigma = x\sigma \in \mathcal{S}$. This is *regular action* of $\mathcal{G}$. (Consider states $x \in \mathcal{G}$.)

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Clearly $\mathcal{G}$ finite (states $\leftrightarrow$ moves; also $|\mathcal{G}| \leq 48!$). But what is $|\mathcal{G}|$?

The Rubik's group of permutations ii

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)
    (11,35,27,19);
L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)(
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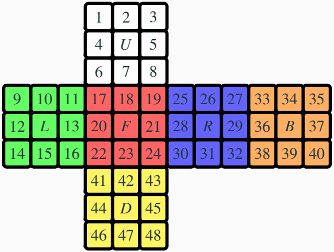
Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits and stabilisers i




```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17,
      30, 14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21,
      37, 15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner pieces $1^{\mathcal{G}}$, edge pieces $2^{\mathcal{G}}$.

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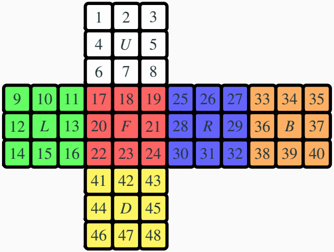

Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners 1, 3, 6, 8.

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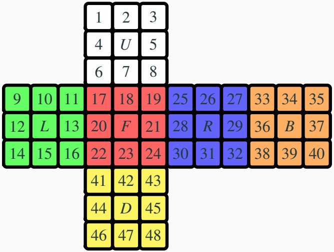

Moves in $\mathcal{H} = \mathcal{G}_{1,3,6,8} = (((\mathcal{G}_1)_3)_6)_8$ fix white corners $1, 3, 6, 8$.

```
gap> G_1368 := Stabilizer( G, [ 1, 3, 6, 8 ], OnTuples );
<permutation group of size 317842469683200 with 12 generators>
gap> Orbit( G_1368, 17 );
[ 17 ]
gap> Orbit( G_1368, 24 );
[ 24, 30, 43, 32, 38, 46, 48, 40, 14, 41, 16, 22 ]
gap> Set( Orbit( G_1368, 2 ) ) = Set( Orbit( G, 2 ) );
true
```

Some $\mathcal{H}$-orbits: $17^{\mathcal{H}} = \{17\}$, bottom corner pieces $24^{\mathcal{H}}$, edge pieces $2^{\mathcal{H}} = 2^{\mathcal{G}}$.

Orders of moves i

Use GAP to compute products, order (using `Order` cmd).

```
gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
gap> Order( last );
6
```

How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect?

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How many times must we repeat move $\sigma \in \mathcal{G}$ to have no effect? I.e. for state $x \in \mathcal{S}$, smallest $k \in \mathbb{Z}_+$ with $x^{\sigma^k} = x\sigma^k = x \iff \sigma^k = 1$.

Orders of moves i

Use `GAP` to compute products, order (using `Order` cmd).

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gap> R*U*R^(-1)*U^(-1);
(1,27,35,33,9,3)(2,21,5)(8,30,25,43,19,24)(26,34,28)
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Recall: order of σ is lcm of cycle lengths.

- Any *generator* (U, L, F, R, B, D) has cycles of length $4, 4, 4, 4, 4$: order is $\mathrm{lcm}(4, 4, 4, 4, 4) = 4$.

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  order is $\mathrm{lcm}(6, 3, 6, 3) = 6$.

Orders of moves ii

- *Sune* $RUR^{-1}URU^2R^{-1}U^2$

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Element of order 5? *Answer:* $(RU)^{21}$ since $((RU)^{21})^5 = (RU)^{105} = 1$.

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There is no Rubik's cube move that cycles through all states.

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If $\mathcal{G}$ is cyclic, then $\mathcal{G}$ is abelian.

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Jake's theorems ii

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There is no $\sigma \in \mathcal{G}$ with infinite order (since $\mathcal{G}$ is finite).

Analysing the Rubik’s group

Definition (Base, stabiliser chain)

If $G \leq \mathrm{Sym}(n)$, distinct elts $B = [\beta_1, \ldots, \beta_r] \subseteq [n]$ is **base** for G if $G_{\beta_1,\ldots,\beta_r} = 1$. (*Recall:* $G_{\beta_1,\ldots,\beta_r} = \{g \in G : \beta_1^g = \beta_1, \ldots, \beta_r^g = \beta_r\}$.)

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Stabiliser chain can be implemented computationally; useful in algorithms (membership testing, random element generation, factorisation into generators).

Bases and stabiliser chains ii

Example (Rubik's group)

Using GAP:

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gap> BaseOfGroup( G );
[ 1, 3, 6, 8, 2, 4, 5, 7, 12, 13, 14, 15, 16, 21,
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If move $\sigma \in \mathcal{G}$ fixes every $\beta_i \in B$ then $\sigma = 1$ is empty move.

How many valid states are there? i

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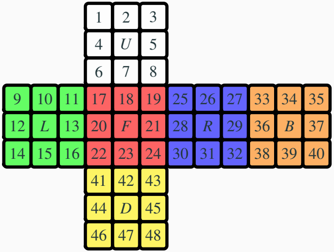
Corollary
$|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$.

Can this restickering be solved? i

Theorem (Wes's conjecture)

"I'm 99% sure you can't swap two [adjacent] edge pieces without affecting another piece?!"



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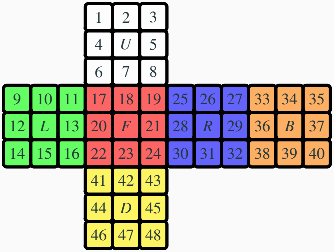

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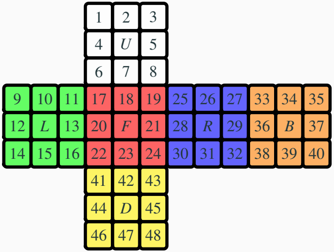

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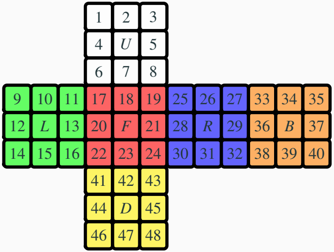

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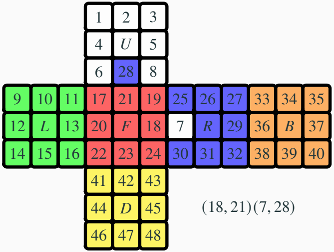

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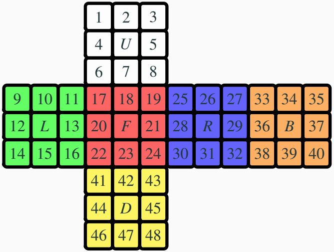

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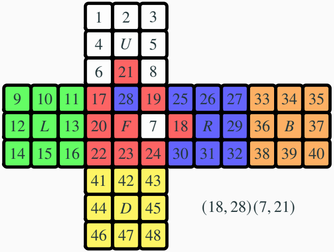

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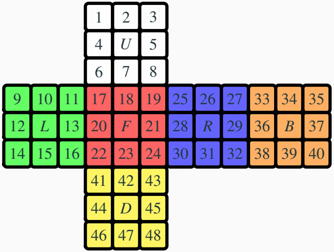

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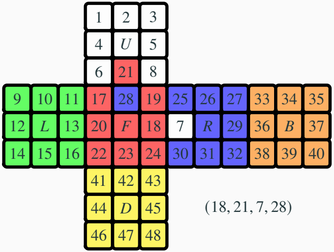

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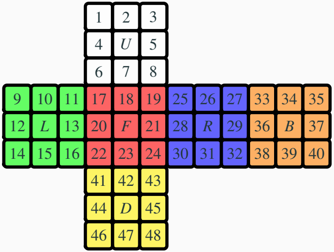

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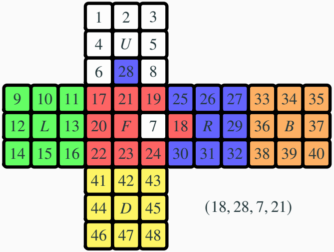
By symmetry, just check one pair, say red/white (18/7) and red/blue (21/28). Four ways: given by where red pieces go (2 options each).

Can this restickering be solved? i

Theorem (Wes's conjecture)
"I'm 99% sure you can't swap two [adjacent] edge pieces without affecting another piece?!"



$(18, 28, 7, 21)$

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By symmetry, just check one pair, say red/white (18/7) and red/blue (21/28). Four ways: given by where red pieces go (2 options each).

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$(18,21)(7,28) \notin \mathcal{G}$, $(18,28)(7,21) \notin \mathcal{G}$, $(18,21,7,28) \notin \mathcal{G}$, *and* $(18,28,7,21) \notin \mathcal{G}$.

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Proof.
By GAP:

```
gap> (18,21)(7,28) in G or (18,28)(7,21) in G or
    (18,21,7,28) in G or (18,28,7,21) in G;
false
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(GAP uses stabiliser chains to verify membership!) □

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Can generalise to any two edge pieces (more cases)!

Solving a Rubik's cube... i

We can use `GAP` to solve Rubik's cube state:

Solving a Rubik's cube... i

We can use GAP to solve Rubik's cube state:

```
gap> H := FreeGroup("u","l","f","r","b","d");
<free group on the generators [ u, l, f, r, b, d ]>
gap> h := GroupHomomorphismByImages( H, G, GeneratorsOfGroup( H ),
     GeneratorsOfGroup( G ) );
[ u, l, f, r, b, d ] -> [ (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)
     (11,35,27,19),
  (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12),
     (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,
    22)(18,21,23,20), (3,38,43,19)(5,36,45,21)(8,33,48,24)
     (25,27,32,30)(26,29,31,28),
  (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36),
     (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,
    46)(42,45,47,44) ]
```

($F = \langle u, \ell, f, r, b, d \rangle$ is free group on 6 generators. Then $f : F \longrightarrow \mathcal{G}$ is hom given by $u \mapsto U$, $l \mapsto L$, $f \mapsto F$, $r \mapsto R$, $b \mapsto B$, $d \mapsto D$.)

Solving a Rubik's cube... ii

Use `GAP` to generate random state $x \in \mathcal{G}$ (uses stabiliser chain):

Solving a Rubik's cube... ii

Use GAP to generate random state $x \in \mathcal{G}$ (uses stabiliser chain):

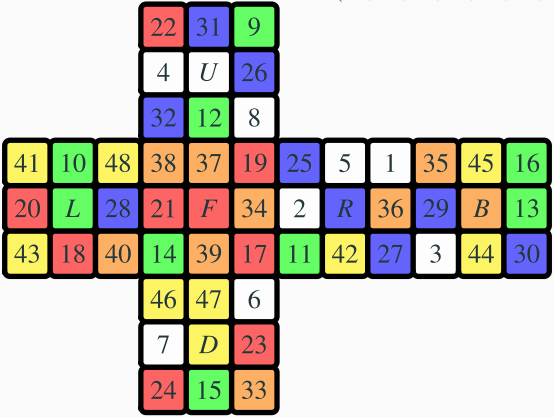
```
gap> x := Random( G );
(1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)(3,38,17,24,46,41,9)
     (5,26)(7,44,39,23,45,34,21,20,12)(11,30,40,16,35,33,48)(29,36)
```

Randomly generated state (uniform distribution on $\mathcal{G}$):

$$x = (1,27,32,6,43,14,22)(2,28,13,37,18,15,47,42,31)$$
$$(3,38,17,24,46,41,9)(5,26)(7,44,39,23,45,34,21,20,12)$$
$$(11,30,40,16,35,33,48)(29,36)$$

Solving a Rubik's cube... iii

$$x = (1, 27, 32, 6, 43, 14, 22)(2, 28, 13, 37, 18, 15, 47, 42, 31)$$
$$(3, 38, 17, 24, 46, 41, 9)(5, 26)(7, 44, 39, 23, 45, 34, 21, 20, 12)$$
$$(11, 30, 40, 16, 35, 33, 48)(29, 36)$$



Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
gap> PreImagesRepresentative( h, x );
l*f^-1*l^-1*f*u*f*u^-1*f^2*l*f*l^-1*u^-1*l^-1*u*l*u^-1*l*u*f*u^-1*f
     ^-1*l^-2*u*l*f^-1*l*f*(l^-1*u)^2*b^-1*u*b*l*u*l^-1*f^-1*l^-1*f*l
     ^2*u*l^-1*u*l*b^-1*u^-1*b*l*d*f^2\
*d^-1*l*f^-1*u*l^-1*f*u^-1*l*d^-1*l*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
     u*d^-2
gap> Length( last );
78
```

$$
\begin{aligned}
x = {} & LF^{-1}L^{-1}FUFU^{-1}F^2LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\
& LF^{-1}LF(L^{-1}U)^2B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^2UL^{-1}ULB^{-1}U^{-1}BL \\
& DF^2D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}
\end{aligned}
$$

Solving a Rubik's cube... iv

Factorisation into 78 generators and inverses:

```
gap> PreImagesRepresentative( h, x );
l*f^-1*l^-1*f*u*f*u^-1*f^2*l*f*l^-1*u^-1*l^-1*u*l*u^-1*l*u*f*u^-1*f
     ^-1*l^-2*u*l*f^-1*l*f*(l^-1*u)^2*b^-1*u*b*l*u*l^-1*f^-1*l^-1*f*l
     ^2*u*l^-1*u*l*b^-1*u^-1*b*l*d*f^2\
*d^-1*l*f^-1*u*l^-1*f*u^-1*l*d^-1*l*b*d*u^-2*b^-1*r^-1*b*u^-1*r*f^-1*
     u*d^-2
gap> Length( last );
78
```

$$
\begin{aligned}
x = {} & LF^{-1}L^{-1}FUFU^{-1}F^{2}LFL^{-1}U^{-1}L^{-1}ULU^{-1}LUFU^{-1}F^{-1}L^{-2}U \\
& LF^{-1}LF(L^{-1}U)^{2}B^{-1}UBLUL^{-1}F^{-1}L^{-1}FL^{2}UL^{-1}ULB^{-1}U^{-1}BL \\
& DF^{2}D^{-1}LF^{-1}UL^{-1}FU^{-1}LD^{-1}LBDU^{-2}B^{-1}R^{-1}BU^{-1}RF^{-1}UD^{-2}
\end{aligned}
$$

(GAP uses stabiliser chains to factorise!)

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
```

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
```

To solve state x, apply move $x^{-1} \in \mathcal{G}$ since

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
true
```

To solve state x, apply move $x^{-1} \in \mathcal{G}$ since $x^{x^{-1}} = xx^{-1} = 1$:

$$\begin{aligned}x^{-1} = {} & D^2U^{-1}FR^{-1}UB^{-1}RBU^2D^{-1}B^{-1}L^{-1}DL^{-1}UF^{-1}LU^{-1}FL^{-1}DF^{-2}D^{-1} \\ & L^{-1}B^{-1}UBL^{-1}U^{-1}LU^{-1}L^{-2}F^{-1}LFLU^{-1}L^{-1}B^{-1}U^{-1}B(U^{-1}L)^2F^{-1}L^{-1}F \\ & L^{-1}U^{-1}L^2FUF^{-1}U^{-1}L^{-1}UL^{-1}U^{-1}LULF^{-1}L^{-1}F^{-2}UF^{-1}U^{-1}F^{-1}LFL^{-1}\end{aligned}$$

(Just invert each term in factorisation above and reverse, thus 78 steps.)

Solving a Rubik's cube... v

Check this is correct:

```
gap> x = L*F^(-1)*L^(-1)*F*U*F*U^(-1)*F^2*L*F*L^(-1)*U^(-1)*L^(-1)*U*
     L*U^(-1)*L*U*F*U^(-1)*F^(-1)*L^(-2)*U*L*F^(-1)*L*F*(L^(-1)*U)^2*
     B^(-1)*U*B*L*U*L^(-1)*F^(-1)*L^(-1)*F*L^2*U*L^(-1)*U*L*B^(-1)*U
     ^(-1)*B*L*D*F^2*D^(-1)*L*F^(-1)*U*L^(-1)*F*U^(-1)*L*D^(-1)*L*B*D
     *U^(-2)*B^(-1)*R^(-1)*B*U^(-1)*R*F^(-1)*U*D^(-2);
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(Just invert each term in factorisation above and reverse, thus 78 steps.)

Not very efficient, since it solves one piece in base B at a time (proceeding up stabiliser chain)... but it works!

Concluding remarks

References i

- Analyzing Rubik's cube with GAP: `https://www.gap-system.org/Doc/Examples/rubik.html`
- J.A. Paulos – Innumeracy (book)