Minimum bases in permutation groups

Lawrence Chen

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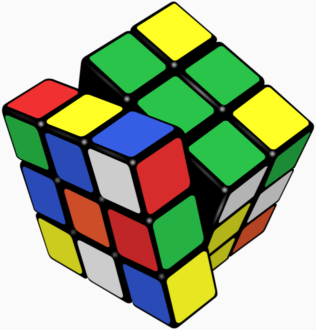
Honours presentation



Contents

Motivation: understanding the Rubik's cube

- How can we represent *operations* of a cube?
- How can we tell *how many* states a Rubik's cube can take?
- How can we better *understand* operations of a cube?

One answer: using permutations and *computational group theory*!

(J. A. Paulos, Innumeracy)

> *Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.*

Some basic (permutation) group theory

Permutations

Definition (permutation)

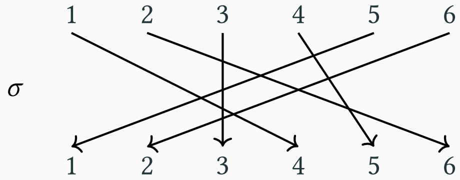
Permutation of Ω is bijection $\sigma : \Omega \to \Omega$.

Symmetric group $\mathrm{Sym}(\Omega)$ is set of permutations of Ω.

(For $\Omega = [n] := \{1, \ldots, n\}$, write $\mathrm{Sym}(n)$.)

Write $1 = ()$ for identity. Write i^σ not $\sigma(i)$ for *image*.

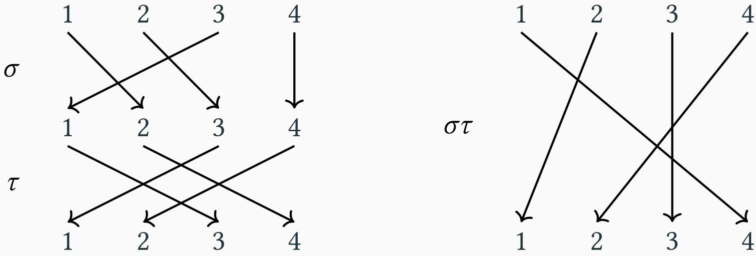
Cycle notation: $\sigma = (1, 4, 5)(2, 6) \in \mathrm{Sym}(6)$ is:



It means $1^\sigma = 4,\ 4^\sigma = 5,\ 5^\sigma = 1,\ 2^\sigma = 6,\ 6^\sigma = 2,\ 3^\sigma = 3.$

Permutations (ii)

Product/composition: for $\sigma, \tau \in \mathrm{Sym}(n)$, $\sigma\tau$ means "first σ, then τ", so $i^{\sigma\tau} = (i^\sigma)^\tau$. E.g. $\sigma = (1,2,3)$, $\tau = (1,3)(2,4) \in \mathrm{Sym}(4)$,



$$\sigma\tau = (1,2,3)(1,3)(2,4) = (1,4,2) \in \mathrm{Sym}(4).$$

Note: here, $\sigma\tau \neq \tau\sigma$, since $1^{\sigma\tau} = 4$ but $1^{\tau\sigma} = (1^\tau)^\sigma = 3^\sigma = 1$. Identity $1 = ()$ satisfies $1\sigma = \sigma 1 = \sigma$ for $\sigma \in \mathrm{Sym}(n)$.

Permutation groups

Note: for $g, h, k \in \mathrm{Sym}(\Omega)$, (i) $gh \in \mathrm{Sym}(\Omega)$, (ii) $1 = () \in \mathrm{Sym}(\Omega)$, (iii) $g^{-1} \in \mathrm{Sym}(\Omega)$, (iv) $(gh)k = g(hk)$. If true for subset:

Definition (permutation group)

Perm group on Ω (of deg n) is subset $G \leq \mathrm{Sym}(\Omega)$ ($|\Omega| = n$) s.t.

(i) **(closure)** $gh \in G$ for $g, h \in G$;

(ii) **(identity)** $1 = () \in G$;

(iii) **(inverses)** $g^{-1} \in G$ for $g \in G$.

Example (alternating group)

Alternating group $\mathrm{Alt}(3) = \{(), (1,2,3), (1,3,2)\} < \mathrm{Sym}(3)$.
In general, $\mathrm{Alt}(n)$ is all *even* permutations of $[n]$ (product of even # of *transpositions* (i, j), e.g. $(1,2,3) = (1,2)(1,3) \in \mathrm{Sym}(n)$).

Generating a group

Definition (generator)

Set X **generates** G if every $g \in G$ is $g = x_1^{\pm 1} \cdots x_r^{\pm 1}$ for some $r \in \mathbb{N}$, $x_i \in X$ **generators**; write $G = \langle X \rangle$.

(If $G = \langle X \rangle$ for some X with $|X| = 1$, G is **cyclic**.)

Example (cyclic group)

Consider $\mathrm{Alt}(3) = \{(), (1,2,3), (1,3,2)\}$: $(1,2,3)^2 = (1,3,2)$, $(1,2,3)^3 = ()$, so $\mathrm{Alt}(3) = \langle (1,2,3) \rangle$ is cyclic (only for $n = 3$).

Example (symmetric group)

Consider $\mathrm{Sym}(3) = \{(), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$.
Not cyclic, but $\mathrm{Sym}(3) = \langle (1,2), (2,3) \rangle$ (adjacent swaps).
Also, $\mathrm{Sym}(3) = \langle (1,2), (1,2,3) \rangle$, e.g. $(2,3) = (1,2,3)(1,2)$.

Group actions

Definition (group action)

For (perm) group G and set $\Omega \neq \emptyset$, a G-**action** is map $\Omega \times G \longrightarrow \Omega$, $(\alpha, g) \mapsto \alpha^g$ s.t. $\alpha^1 = \alpha$ and $\alpha^{gh} = (\alpha^g)^h$ for $\alpha \in \Omega$ and $g, h \in G$.
Degree of action is $|\Omega|$.

Idea: $\alpha \in \Omega$ is *state*, apply *move* $g \in G$ to get state $\alpha^g \in \Omega$, in way that respects permutation product.

Example (natural action)

$G \leq \mathrm{Sym}(\Omega)$ acts on Ω by $\alpha^g := \alpha^g$ (image) for $\alpha \in \Omega$, $g \in G$.

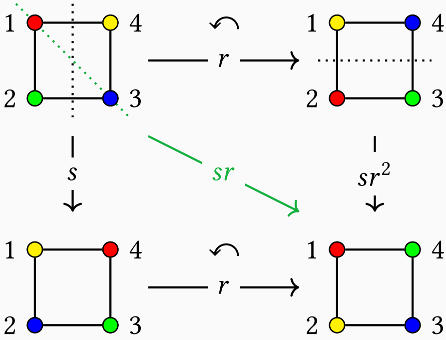
Example (right regular action)

Perm group G acts on $\Omega = G$ (itself) via $\alpha^g := \alpha g$ for $\alpha, g \in G$.
(*Check:* $\alpha^1 = \alpha 1 = \alpha$ and $\alpha^{gh} = \alpha(gh) = (\alpha g)h = (\alpha^g)^h$.)

Example (dihedral group)

Let $r = (1,2,3,4)$, $s = (1,4)(2,3) \in \mathrm{Sym}(4)$. **Dihedral group** is $D_8 := \langle r, s\rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, *"symmetries of square"*.

Note: $sr = (2,4)$, $sr^2 = (1,2)(3,4)$. Action of D_8 on vertices of square (labelled by $[4]$): $g \in D_8$ sends vertex at i to i^g.



Orbits and stabilisers

Definition (orbit)

If G acts on Ω, then **orbit** of $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$.

Idea: states $\alpha^g \in \Omega$ reachable from fixed $\alpha \in \Omega$ by moves $g \in G$.

Definition (stabiliser)

If G acts on Ω, then **stabiliser** of $\alpha \in \Omega$ is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$.

Idea: moves $g \in G$ that fix given $\alpha \in \Omega$.

Example (natural action)

$G = \mathrm{Alt}(3) = \{(), (1, 2, 3), (1, 3, 2)\}$ acts on $\Omega = [3]$ naturally.

Orbit of 1 is $1^G = \{1, 2, 3\} = [3]$; stabiliser of 1 is $G_1 = \{()\} = 1$.

One orbit only: **transitive** action.

Orbits and stabilisers (ii)

Orbit α^G: states $\alpha^g \in \Omega$ reachable from fixed α by moves $g \in G$.
Stabiliser G_α: moves $g \in G$ that fix given α.

Example (dihedral group)

Recall $G = D_8 = \langle r, s\rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \mathrm{Sym}(4)$ where $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$.

Orbit of 1: $1^1 = 1$, $1^r = 2$, $1^{r^2} = 3$, $1^{r^3} = 4$, so $1^G = [4]$ (transitive).

Stabiliser of 1: $sr = (2, 4)$, $sr^2 = (1, 2)(3, 4)$, $sr^3 = (1, 3)$, so $G_1 = \{(), (2, 4)\} = \{1, sr\}$.

Note: $|1^G||G_1| = 4 \cdot 2 = 8 = |G|$. Coincidence?

Theorem (orbit-stabiliser)

If G acts on Ω, then for $\alpha \in \Omega$, $|\alpha^G||G_\alpha| = |G|$.

Blocks and primitivity

Definition (block)

If G acts transitively on Ω and $\Delta \subseteq \Omega$, let $\Delta^g := \{\alpha^g : \alpha \in \Delta\}$.
A **block** is $\Delta \subseteq \Omega$ with $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for all $g \in G$.

Block is **nontrivial** if $|\Delta| > 1$ and $\Delta \neq \Omega$.

Examples of blocks: singletons, Ω, orbits.

Definition (primitivity)

A *transitive* G-action is **primitive** if there are no nontrivial blocks; otherwise it is **imprimitive**.

If G is perm group with primitive natural action, G is **primitive**.

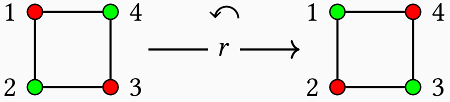
For block Δ, define **block system** $\Sigma = \{\Delta^g : g \in G\}$ (partitions Ω); then G acts on Σ; if Δ is *maximal*, then acts primitively.

Blocks and primitivity (ii)

Example (dihedral group)

Recall $G = D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\} \leq \mathrm{Sym}(4)$ where $r = (1, 2, 3, 4)$, $s = (1, 4)(2, 3)$, $sr = (2, 4)$.

Block is $\Delta = \{1, 3\}$ (nontrivial) with block system $\Sigma = \{\{1, 3\}, \{2, 4\}\}$ (opposite vertices stay opposite):



e.g. $\Delta^r = \{2, 4\}$, $\Delta^s = \{4, 2\}$, $\Delta^{sr} = \{1, 3\} = \Delta$.

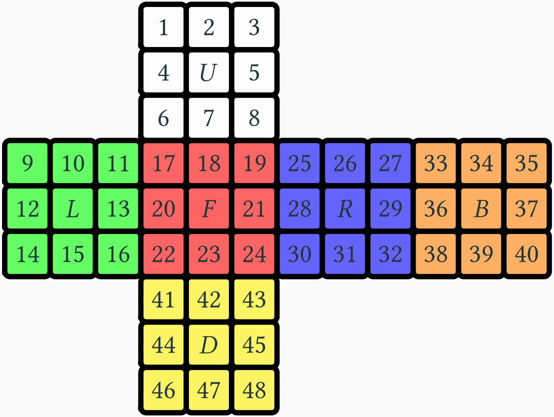
D_8 acts imprimitively on $[4]$ but primitively on Σ (degree 2).

The Rubik's group (an application)

Representing the cube and its operations

Rubik's cube has 6 faces, each with 3×3 small *stickers*.

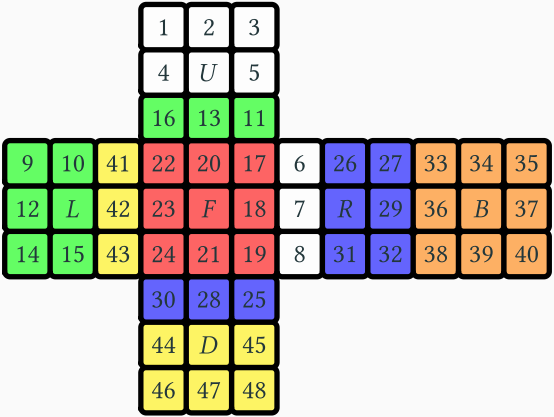
In **solved state** 1, label stickers (except each centre) using $[48]$:



6 **generators** (*moves* in CC): U, L, F, R, B, D (rot. *clockwise*).

Representing the cube and its operations (ii)

From *solved state* 1, consider F which rotates front face clockwise:



$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)$$
$$(7, 28, 42, 13)(8, 30, 41, 11) \in \mathrm{Sym}(48).$$

The Rubik's group of permutations

Generators as permutations of labels $[48]$:

- $U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)$
- $L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35)$
- $F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$
- $R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$
- $B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$
- $D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

Operation is sequence of generators and inverses. E.g. $RUR^{-1}U^{-1}$, $URU^{-1}L^{-1}UR^{-1}U^{-1}L$, $RUR^{-1}URU^2R^{-1}U^2$, $1 = ()$.

Definition (Rubik's group)

$\mathcal{G} = \langle U, L, F, R, B, D \rangle \leq \mathrm{Sym}(48)$ is permutation group of degree 48, called **Rubik's group**.

Clearly $\mathcal{G}$ is finite, but what is $|\mathcal{G}|$?

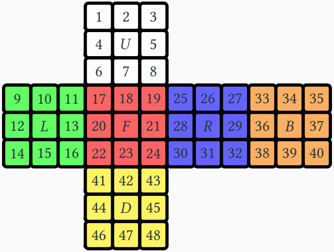
The Rubik's group of permutations (ii)

GAP code to define generators and $\mathcal{G} = \langle U, L, F, R, B, D \rangle$ (as G):

```
U := ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);;
L := ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35);;
F := (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11);;
R := (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24);;
B := (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27);;
D := (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40);;
G := Group( U, L, F, R, B, D );
```

Order cmd: $|\mathcal{G}| = 43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$. *How?*

Orbits in the Rubik's group

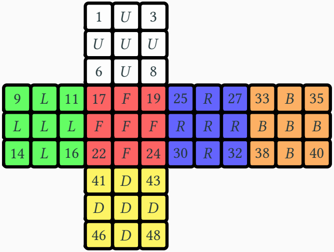



```
gap> Orbit( G, 1 );
[ 1, 6, 40, 27, 8, 35, 16, 41, 32, 25, 48, 3, 11, 24, 46, 33, 43, 17, 30,
      14, 19, 9, 22, 38 ]
gap> Orbit( G, 2 );
[ 2, 5, 12, 7, 36, 10, 47, 4, 28, 45, 34, 13, 29, 44, 20, 42, 26, 21, 37,
      15, 31, 18, 23, 39 ]
```

Two $\mathcal{G}$-orbits: corner stickers $1^{\mathcal{G}}$, edge stickers $2^{\mathcal{G}}$.

Transitive action on corners

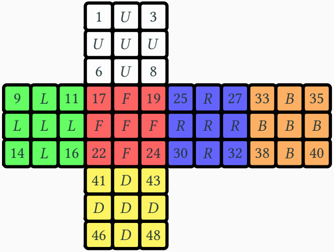
$\mathcal{G}$ acts transitively on corner stickers $1^{\mathcal{G}}$. In this action:



$$\Sigma = \{\overbrace{\{1,35,9\}}^{UBL}, \overbrace{\{6,11,17\}}^{ULF}, \overbrace{\{40,46,14\}}^{BDL}, \overbrace{\{27,3,33\}}^{RUB},$$
$$\underbrace{\{8,25,19\}}_{URF}, \underbrace{\{16,41,22\}}_{LDF}, \underbrace{\{32,48,38\}}_{RDB}, \underbrace{\{24,43,30\}}_{FDR}\}$$

is block system for *maximal* block $\{8,25,19\}$ (URF corner).

Transitive action on corners (ii)



$\mathcal{G}$ acts primitively on Σ (degree 8); $g \in \mathcal{G}$ induces perm of Σ, e.g.

$$F \mapsto (\underbrace{FUL}_{FDL^F}, \underbrace{FUR}_{FUL^F}, \underbrace{FDR}_{FUR^F}, \underbrace{FDL}_{FDR^F}) \in \mathrm{Sym}(\Sigma).$$

Overall, $\mathcal{G}$ induces every perm of Σ (so $\mathrm{Sym}(8)$ "is" quotient of $\mathcal{G}$).

Bases and stabiliser chains

Primitive subgroups of affine groups

Affine groups

Definition

Large base permutation groups

Definition

Large base permutation groups (ii)

Liebeck

Moscatiello, Roney-Dougal

Main result

Statement

Main result (ii)

Approach (dot points/observations)

Main result (iii)

Conjecture

Concluding remarks

References and resources

- Analyzing Rubik's cube with GAP:
  `https://www.gap-system.org/Doc/Examples/rubik.html`
- J.A. Paulos — *Innumeracy* (book)
- Holt — *Handbook of Computational Group Theory* (textbook)
- Dixon and Mortimer — *Permutation Groups* (textbook)
- Orders of elements in Rubik's group (1260 largest, 13 smallest without, 11 rarest, 60 most common, median 67.3, 73 options):
  `https://www.jaapsch.net/puzzles/cubic3.htm#p34`
- Thistlethwaite's 52 move algorithm (using group theory):
  `https://www.jaapsch.net/puzzles/thistle.htm`