1 Introduction to Groups

Keywords / Concepts:

- Describe and Draw the 8 Motions of a Square, Important Note? Order of the Dihedral Group?
- Square Motion Function Composition Example
- What is the Cayley Table? Closure Property, Identity & Inverse Existence? What's the Identity?
- What does it mean to be an Abelian / Commutative Group? Is D_4 Abelian?
- What are the 4 Conditions of a group?
- How to prove a group is associative?
  -- EOF Symmetries of a Square --
- Dihedral Group of Order $2n$, Give Example
- What is plane symmetry?
- What is symmetry group?
- What is a Reflection Across Line L? Give picture
  -- EOF The Dihedral Groups --

Symmetries of a Square

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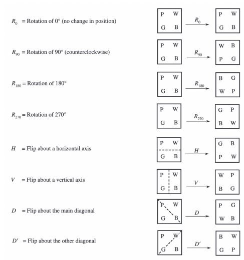
8 Motions of a Square
The eight motions together with the composition operation forms a Dihedral Group of Order 8 denoted by D_4.

- Note: Rotation is counter clockwise



Square Motion Function Composition
Write $HR_{90} = D$ as applying 90 degree rotation first then applying horizontal flip.
This is because $f \circ g$ function composition reads applying g then f.

Cayley Table For D_4

- Closure Property: It says that if $A, B \in D_4$ then so is AB.
- Identity Existence: Notice $AR_0 = R_0 A = A$. R_0 is the identity.
- Inverse Existence: For each element $A \in D_4$ there is exactly one element $B \in D_4$ such that $AB = BA = R_0$. B is the inverse of A and vice versa.

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Abelian Groups
If $\forall a, b \in G$, $ab = ba$, then we say that G is an Abelian (or commutative) group, else the group is non-Abelian.

- Note that D_4 is non-Abelian since $HD \neq DH$.

4 Conditions of a Group

- Closure
- Identity
- Inverse
- Associativity: If $\forall a, b, c \in G$, $(ab)c = a(bc)$ then G satisfies associativity. To prove, we can just say since function composition is associative, this property holds.

The Dihedral Groups

Dihedral Group of Order $2n$
This is applicable to any regular n-gon where $n \geq 3$. D_n is called the Dihedral Group of Order $2n$.

- This is often called the group of symmetries of a regular n-gon

Plane Symmetry
A function $f : F \to F$ where F is a plane and f is a function that preserves distances meaning for each point p, q, the plane, the distance between the images of p and q is the same as the distance between p and q.

- TLDR: $\sqrt{(f(q)_{coord} - f(p)_{coord})^2} = \sqrt{(q-p)^2}\ \forall p, q \in F$

Symmetry Group
The symmetry group of a plane figure is the set of all symmetries of the figure.
TBD: This is the set of all motions, eg. reflections / rotations.

Reflection Across a Line L
A function f that leaves points in line L fixed, but takes any other point q to q' where L is a perpendicular bisector of the line segment qq'.

Figure 1.4

Note: Reflections cannot be achieved by physical motion

Cyclic Rotation Group of Order n
A cyclic rotation group of order n is a symmetry group consisting of only rotational symmetries, and is denoted $\langle R_{360/n} \rangle$.

- Eg. $0^\circ, (\frac{360^\circ}{n}), 2(\frac{360^\circ}{n}), \ldots, (n-1)(\frac{360^\circ}{n})$