Exercise 1
Modelling the Dragon Capsule as a Pendulum System
AE332 - Modelling and Analysis Lab

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This is a report based on the modelling of the SpaceX Dragon Capsule as a simple pendulum as a part of an exercise to study, model and analyse the A-Frame Crane used by SpaceX for the Dragon Capsule retrieval. To solve the governing differential equations for the simple pendulum model of the system, 'odeint' function of the Scipy Library is used and implemented using python. The variation in efficiency of the 'odeint' solver with variations in its parameters, 'atol' and 'rtol' is first studied. Then the governing differential equations are numerically solved in two perpendicular directions with physical constraints. The problem is then analysed in two different constraints and conditions. Firstly, the physical constraints are relaxed followed by relaxing the condition on the simple pendulum, where its length is maintained constant.

OBJECTIVE & PROBLEM DESCRIPTION

Our main objective is to model the oscillations by the Dragon Capsule as a simple pendulum. The important assumptions in a simple pendulum are as follows:

1. The string is weightless.
2. $sin\theta \approx \theta$ i.e. small angle assumption.
3. No damping.
4. The bob of the pendulum is a point mass.
5. The length of the string is constant and there is no extension in the string.

The Dragon Capsule is retrieved from the ocean using a crane called the A-Frame with a cord/rope. The A-Frame is positioned on the Go-Navigator. The frame bends upto a certain angle towards the ocean, the capsule is then attached to the rope after which the A-Frame comes back to its position lifting the capsule and placing it on the Go-Navigator. During this period the Capsule is free to move and oscillate. The A-Frame has side supports which limit the motion of the capsule in the plane of the A-Frame as the Capsule must not be damaged in the process of retrieval. It is this constrained motion that we would be trying to model.

VARIATIONS IN RESULTS DUE TO 'ATOL' AND 'RTOL'

In the odeint solver the input parameters rtol and atol determine the error control performed by the solver. These parameters are optional and their values are taken to be 1.49012 x 10^{-8} by default.[1]
The solver will control the vector, e, of estimated local errors in y, according to an inequality of the form max-norm of $(e/ewt) <= 1$, where ewt is a vector of positive error weights computed as ewt = rtol * abs(y) + atol.[2]
The differential equations for a simple pendulum with the assumption that $sin(\theta) \approx \theta$ was solved. With this assumption we can also obtain the exact solution for the problem. The difference in the numerical and exact solutions was obtained for values of atol and rtol ranging from orders of 1×10^{-11} to 1×10^{0}.
For values of atol and rtol from 1×10^{-11} to 1×10^{-5} there is no visible deviations from the exact solution. This is evident from the Figs. 1 and 2.
The deviation of the numerical solution from the exact

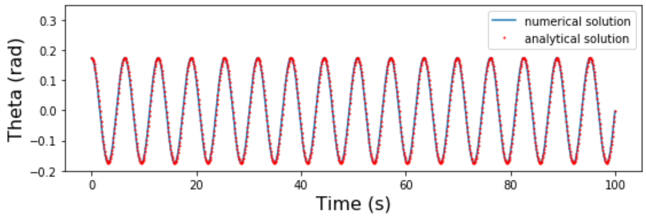

FIG. 1: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-8}.

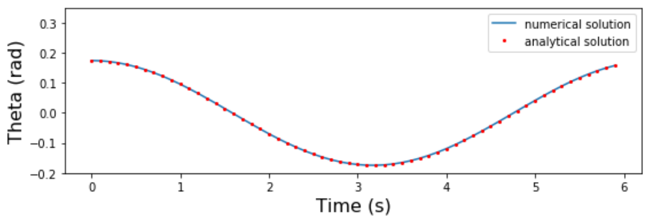

FIG. 2: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-5}.

solution become visible when atol and rtol are set to a value of $1 \times 10{-}4$. This is evident from the Fig. 3.
In the Fig. 3 we can see that the deviation is visible only in the later period and at initial times there is no much deviation visible.
But when tolerance values are set to $1 \times 10{-}2$ the deviations exist at initial times also and the numerical solution obtained does not match with the exact solution which

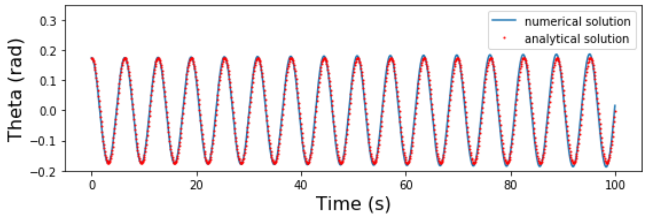

FIG. 3: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-4}.

is shown in Figs. 4 and 5.

In Fig. 5 even though the numerical solution deviated

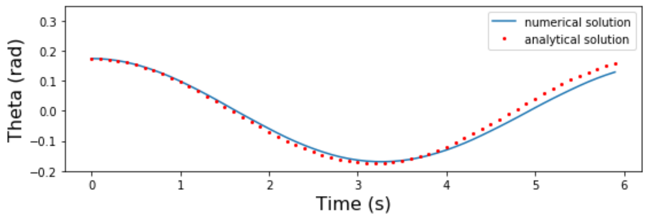

FIG. 4: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-2}.

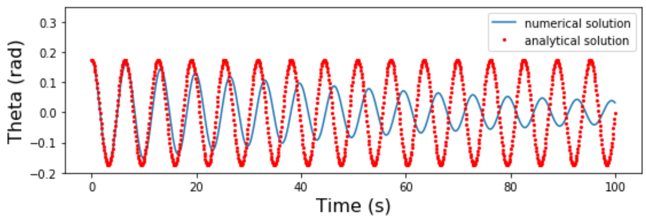

FIG. 5: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-2}.

largely from the exact solution, it remains sinusoidal. When the tolerance is set to $1 \times 10{-}1$ the numerical solution is not sinusoidal. This can be seen in the Fig. 6.

The values of the maximum error(difference between the numerical and exact solution) for different values of atol and rtol are tabulated in Tab. I.

MODELLING THE MOTION AS A SIMPLE PENDULUM

Governing Equations and Initial Conditions

The motion of the bob would be modelled as motion in two perpendicular directions, assuming that the motion in one of these directions dosen't influence or get influenced by the motion in the other direction.
Tab. II provides a list of the important parameters relevant to the problem.[3]

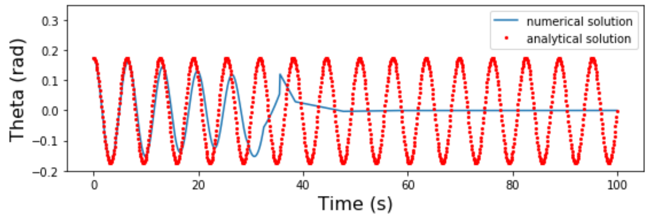

FIG. 6: Numerical and Exact solutions for simple pendulum for values of atol and rtol of 1×10^{-1}.

TABLE I: DIFFERENCE BETWEEN THE NUMERICAL AND EXACT SOLUTION FOR DIFFERENT VALUES OF ATOL AND RTOL

Tolerance Values (atol and rtol)	Maximum Error
1×10^{-11}	1.36×10^{-10}
1×10^{-10}	1.51×10^{-8}
1×10^{-9}	2.37×10^{-7}
1×10^{-8}	1.10×10^{-6}
1×10^{-7}	2.02×106^{-5}
1×10^{-6}	5.00×10^{-5}
1×10^{-5}	1.2×10^{-3}
1×10^{-4}	2.20×10^{-2}
1×10^{-3}	1.55×10^{-1}
1×10^{-2}	2.34×10^{-1}
1×10^{-1}	2.34×10^{-1}
1×10^{0}	3.04×10^{-1}
1×10^{1}	3.15

Here we have assumed that the entire dragon module is

TABLE II: SPACEX DRAGON2 & A-FRAME SPECIFICATIONS

Mass of the capsule	2394.12 kg
Location of COM of the capsule	3.9 m from the top
Height of the capsule	5.2 m
Width of the A-Frame	7.62 m
Height of the A-Frame	9.14m

of uniform mass density and hence the Center of Mass of the capsule is estimated as the COM of a uniformly dense cone. Given the fact that the height of the A-frame and the height of the capsule is known, we can estimate the length of the string to be $3.94m$. But since in the simple pendulum we assume that the mass is a point mass thus we consider the capsule to be a point mass and is placed at its Center of Mass. Due to this the length of the string to be considered while modelling must also include the distance of the center of mass from the point of attachment. Hence the string length used for modelling would be $7.84m$.

The governing differential equations to be solved is :

$$\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{g}{L}sin\theta = 0 \quad (1)$$

Here θ is the angle made by the string with the vertical, b is the damping constant, g is the acceleration due to gravity and L is the length of the string.
To solve the Eqn. 1 numerically we need to decompose the equation which is a second order equation into two first order equations as follows:

$$\frac{d\omega}{dt} + b\omega + \frac{g}{L}sin\theta = 0 \quad (2)$$

$$\frac{d\theta}{dt} = \omega \quad (3)$$

Here ω is the angular velocity of the bob.
To solve the system we also need to provide the initial conditions for the governing differential equations.

1. In the plane of the A-Frame(which will be henceforth reffered to as the X-Plane and denoted by X) the initial $\theta(\theta_{0x})$ ideally has to be zero as the bob is right below the point where the rope is attached to the frame. But since the capsule is lifted from the ocean which would have waves and hence the capsule can't be stationed exactly where it is expected, thus θ_{0x} can be assumed to be a small value, which we asusume to be 5°.

2. In the X-plane the initial angular velocity(ω_{0x}) is provided by the disturbances caused by waves in the ocean. At an initial level we will not model the waves and disturbances caused by it. Instead we will iterate through different values of ω_{0x} so that the maximum displacement in the X-plane is such that the Bob will not hit the side-frame. Thus we can obtain a maximum value of ω_{0x}, such that if its value is above a limit then the capsule would get damaged.

3. In the plane perpendicular to the plane of the A-Frame(which will be henceforth reffered to as the Y-Plane and denoted by Y), by the arguments made for the X-Plane we can assume the initial angular displacement in the Y-Plane (θ_{0y}) to be 5°.

4. In the Y-Plane the initial angular velocity(ω_{0y}) will have two contributions one from the disturbances caused by waves and the other due to the motion of the A-Frame i.e. lifting of the capsule. The former contribution will be taken to the limiting value obtained from motion in the X-Plane. The latter contribution can be estimated in the following way. The time taken by the A-Frame crane to lift and place the capsule on the Go-Naviagtor is about $40sec$ and the angle through which it moves is 80°. This gives an angular velocity of $0.035rad/s$, assuming it to be a constant.

Here we have assumed both the initial angular displacement and angular velocity in both directions to be positive as this would lead to the maximum possible amplitude under the given constraints and conditions.

RESULTS & DISCUSSIONS

Based on the above mentioned initial conditions the Eqns. 2 and 3 were solved and the results obtained are discussed below.
First the differential equations are solved for b=0 i.e. damping is absent.
Here we obtain the critical ω_{0x} to be 0.49 rad/s. This means that if the initial disturbances provide an angular velocity more than 0.49 then the capsule would be damaged as it would hit the side frame.
Figure 7 shows the variation of θ_x and θ_y with time. Here we can observe the effect of the contribution given by the motion of the A-frame to the ω_{0y}. This has also has introduced a phase difference with the amplitude along the X-Plane.
If we consider that air-damping is present, for $b = 1 \times 10^{-3}$ there is no visible difference in the plots of angular displacement vs time for the case of damping as compared to that of no-damping for 100s.

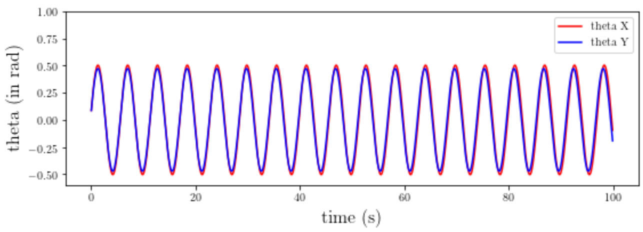

FIG. 7: Variation of θ_x and θ_y with time considering no damping.

If the damping constant is set to 1×10^{-2}, we can see the amplitude decreasing with time. This can be seen in the Fig. 8.
Even if the damping constant becomes more significant,

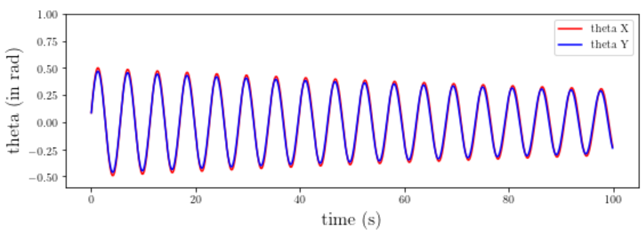

FIG. 8: Variation of θ_x and θ_y with time considering damping(b=0.01).

with respect to the problem it is benificial for us as the oscillations die out after some time.
But even without damping for disturbances that provide initial angular velocity below 0.49 rad/s the capsule is not damaged.
Figures 9 and 10 show us the variation of Kinetic and Potential Energy in the X-Plane and Y-Plane. We can see that the total energy seems to be conserved, which is true and the total energy in the X-Plane is 22958 J and in the Y-Plane is 20215.4 J. Since our governing equations are non-linear we can observe that the time taken for first oscillation is about 2.5s, for the second it is a little over 2.5s and this increases steadily.

Figures 11 and 12 show the variation of Kinetic and Potential Energies in the two planes seperately. We can see that there exists a phase difference between the energies in the X and the Y plane which is expected due to the difference in the initial angular velocities imparted to the bob in these directions.

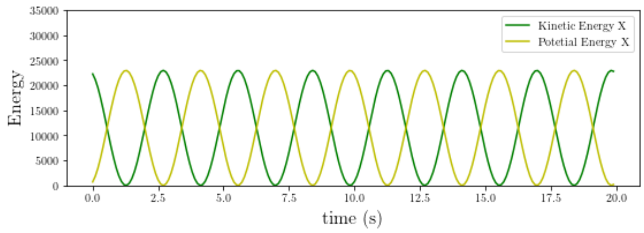

FIG. 9: Variation of Kinetic and Potential Energy in the X-Plane with time.

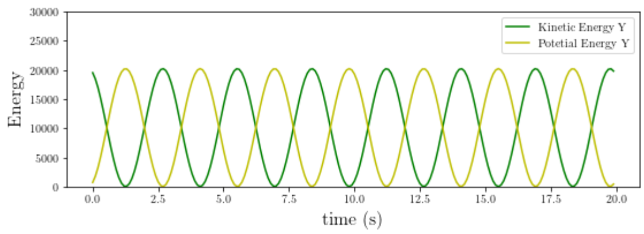

FIG. 10: Variation of Kinetic and Potential Energy in the Y-Plane with time.

As we could see in Figs. 9 and 10 the Total Energy appears constant which is confirmed from Fig. 13. If the numerical solution is calculated for a longer time, say 100s we can observe a variation in energy of about 0.1 J and also if we solve for 1000s then the variation in energy is about 0.4 J. This may be due to the rounding off errors which become visible after large number of iterations.

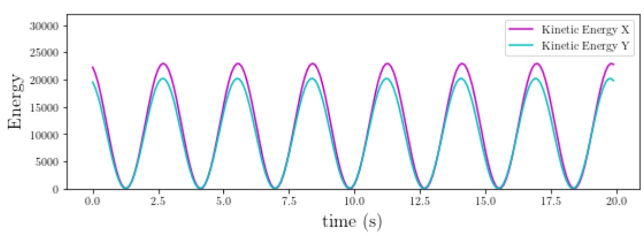

FIG. 11: Variation of Kinetic Energy in the X and Y Plane with time.

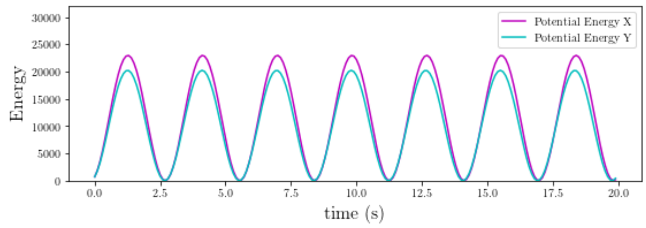

FIG. 12: Variation of Potential Energy in the X and Y Plane with time.

If we consider damping then the Total Energy would decrase with time owing to the dissipiation due to damping which can be seen in Fig. 14.

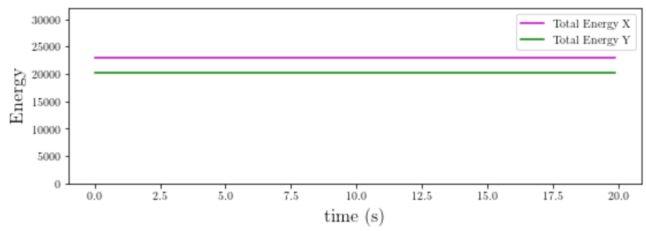

FIG. 13: Variation of Total Energy in the X and Y Plane with time.

One of the assumption of the simple pendulum is that the tension remains constant. To check its validity, the plots of tension in the rope for oscillations in X and Y directions are shown in Fig. 15. In the plot we can observe that tension in the string exhibits a periodic variation with a phase difference between the two planes which increases over time.
As we can see in Tab. III that the variation is more than 30% thus the assumption that tension remains constant can't be considered valid here.

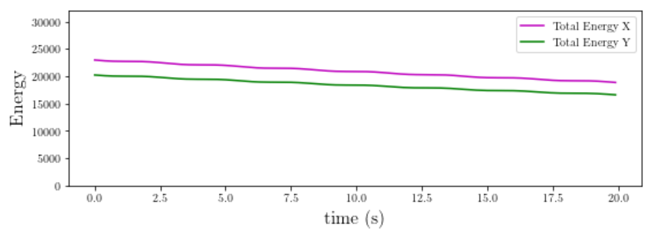

FIG. 14: Variation of Total Energy in the X and Y Plane with time considering damping to exist.

TABLE III: Mean and Range of tension in the rope

	Mean Value (N)	Range of variation (N)	Range of variation(%)
X-Plane	24950.4	8784.8	35.2
Y-Plane	24775.2	7734.7	31.2

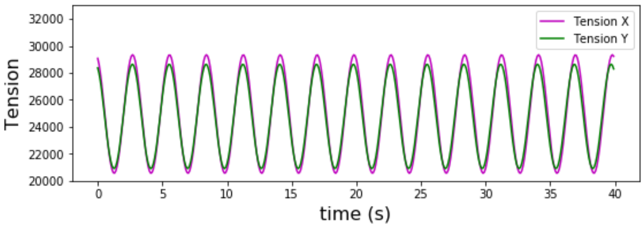

FIG. 15: Variation of Tension in the string for oscillations in the X and Y Plane with time.

Now we try to estimate the extension caused in the rope due to the tension in it. The rope used to retrieve the capsule is considered to be nylon. The Modulus of Elasticity of Nylon is $5 \times 10^9 N/m^2$. The capsule has a mass of about $2400kg$ for which based on the safety factor a nylon rope of diameter $44mm$ can be used. Such a rope would have a Breaking strength of $249kN$ and mass density of $1.16kg/m$.[4] Thus the rope is 0.37% of the mass of the capsule, so the assumption of weightless string can be considered valid.

$$\frac{\delta l}{l} = \frac{Tension}{AE} \tag{4}$$

Here A is the cross-sectional area of the rope and E is its modulus of elasticity.
Using these information and Eqn. 4 the plots of extension in the rope(in %) are shown in the Fig. 16. The % elongation ranges from 0.28 to 0.38, which is quite small and thus the assumption that the length of the string remains constant is valid in this analysis of the problem. From the figure we can also observe that the extension for the oscillation in X-Plane is slightly more than that in the Y-Plane which is due to the fact that the initial angular velocity provided in the X-Plane is more than that in the Y-Plane.

EXTENSION 1

In the original problem there was a physical constraint of the width of the A-Frame, thus there was a limit to the initial angular velocity that could be provided to the bob to ensure its safety. Now we analyse the possibilities if this constraint is removed.
Initially let us consider a simple pendulum system shown in Fig. 17 Now applying energy conservation at the two instants shown we get Eqn. 5.

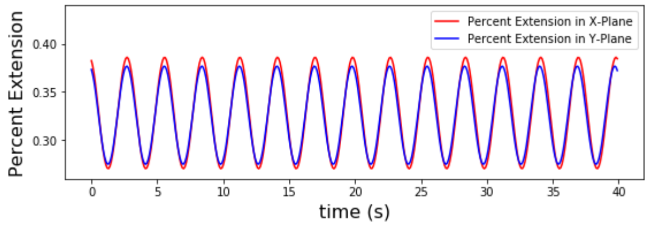

FIG. 16: Variation of Percent Extension in the string for oscillations in the X and Y Plane with time.

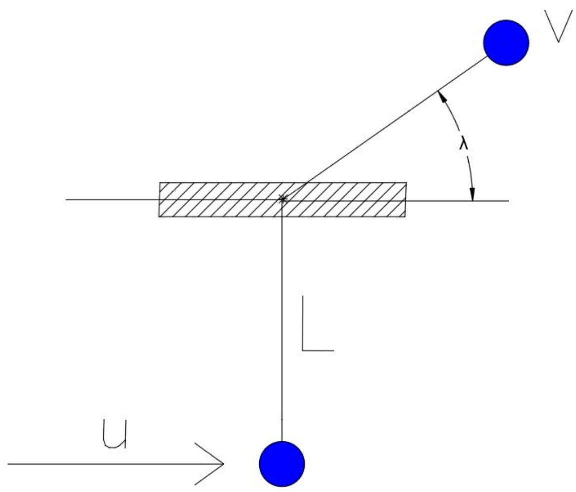

FIG. 17: A simple pendulum system.

$$\frac{1}{2}mu^2 = mgL(1 + sin\lambda) + \frac{1}{2}mv^2 \tag{5}$$

Writing the equation for centripetal acceleration we get Eqn. 6.

$$T = \frac{mv^2}{L} - mgsin\lambda \tag{6}$$

Now the different possibilities can be considered as follows:

1. The pendulum executes circular motion.
   For the pendulum to be able to execute circular motion, the string must not become slack even at the top most point(corresponds to $\lambda = 90$ in the Fig. 17). Thus in the limiting case if the Tension in the string becomes zero at the topmost point then the pendulum will complete a circle. Substituting T = 0 in Eqns. 5 and 6 we get $u = \sqrt{5gL}$. This means that if the initial velocity u is more than $\sqrt{5gL}$ then the pendulum will execute circular motion.

2. The pendulum oscillates.
   For the pendulum to oscillate, v must become zero

before the pendulum reaches the topmost point. Substituting $v = 0$ in the Eqns. 5 and 6 we get an expression for T. Applying the inequality $T \geq 0$ we get $u = \sqrt{2gL}$ Thus if the initial velocity is less than this we see that λ turns out to be negative which implies that the bob stops before the string becomes vertical and if the initial velocity is greater than $\sqrt{2gL}$ then the bob stops after the string becomes vertical i.e. λ is positive.

3. The string breaks.
For the string to break, the tension in the string must be greater than the breaking strength of the string. In the limiting case $\frac{T}{A} = \sigma_b$ where σ_b is the breaking stress of the rope and A the cross-section area of the string. Substituting this and eliminating v we get Eqn. 7.

$$u = \sqrt{\frac{\sigma_b AL}{m} + gL(1 + 2sin\lambda)} \tag{7}$$

Substituting the values from our problem we get
$v_1 = \sqrt{3gL} = 15.19m/s$
$v_2 = \sqrt{5gL} = 19.61m/s$
and the corresponding initial angular velocities become
$\omega_1 = \frac{1}{L}\sqrt{3gL} = 1.93rad/s$
$\omega_2 = \frac{1}{L}\sqrt{5gL} = 2.50rad/s$
Figure 18 shows the plot of maximum angular displacement with the initial angular velocities. In the plot we can observe that the curve rises suddenly at a point between $2rad/s$ and $2.5rad/s$.

In the Fig. 19 we can see that the maximum angular

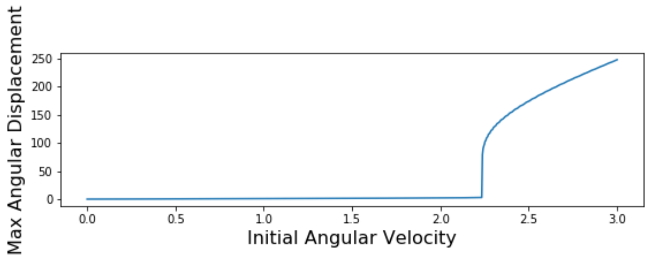

FIG. 18: Maximum angular displacement (in radians) with the initial angular velocities (in rad/s).

displacements go upto 3 rads which is the case where the bob comes to rest before reaching the topmost point. Thus from the graph we obtain values of ω_1 and ω_2 as 1.58 rad/s and 2.23rad/s respectively. In the Fig. 20 the maximum angular displacements goes upto 250 and it is almost linear. This corresponds to the case of the bob executing circular motion. This is confirmed by increasing the analysis time which leads to an increase in the maximum angular displacement.

Now we verify these behaviour by plotting angular displacement vs time for initial angular velocities that lie in different ranges pertaining to different behaviour of the oscillations. The plots in Figs. 21 and 22 show

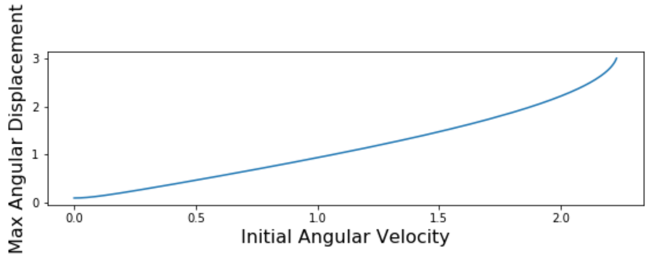

FIG. 19: Maximum angular displacement (in radians) with the initial angular velocities (in rad/s).

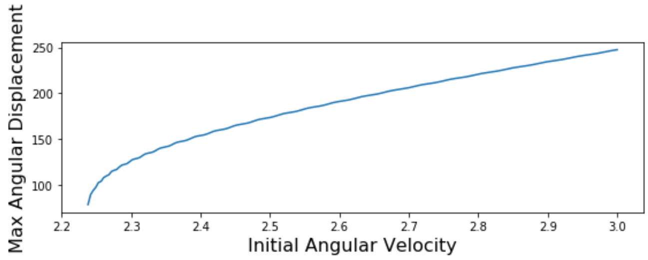

FIG. 20: Maximum angular displacement (in radians) with the initial angular velocities (in rad/s).

the variation of angular displacement with time for different initial angular velocities, we have considered 3 angular velocities 1.4 rad/s, 1.9 rad/s and 2.4 rad/s. 1.4 rad/s falls in between 0 and ω_1 and thus we expect the amplitude to be less than 90° i.e. $1.57rad$. $1.9rad/s$ falls between ω_1 and ω_2 thus we expect its amplitude A_{max} to be $1.54 < A_{max} < 3.14$. $2.4rad/s$ is more than ω_2 and thus we expect to see circular motion of the pendulum and hence a linear variation between angular displacement and time. The expected trends are seen in Fig. 22. The first two cases are shown seperately in Fig. 21.

From Eqn. 7 subustituting relevant values, the

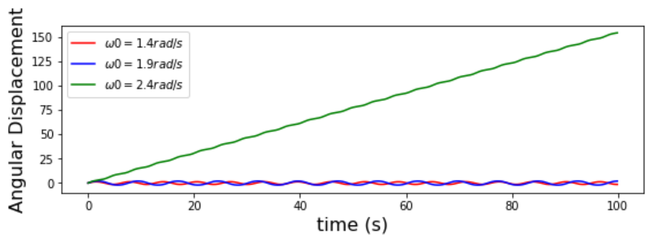

FIG. 21: Angular displacement (in radians) with time for different initial angular velocities.

minimum value of u for which the rope would break is $29.76m/s$ which corresponds to an initial angular velocity of $3.8rad/s$. Thus if the initial angular velocity exceeds this value then the tension exceeds the breaking strength and the rope breaks.

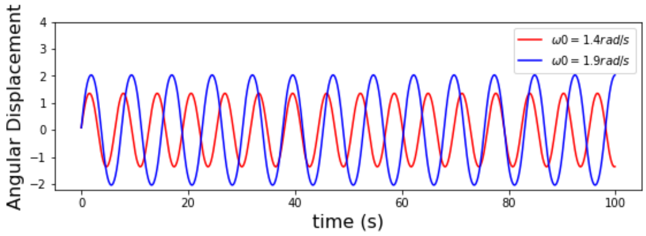

FIG. 22: Angular displacement (in radians) with time for different initial angular velocities.

EXTENSION 2

As an extension to the problem, instead of keeping the length of the string constant we now model it as changing with time. In the actual scenario too when the capsule is lifted from the sea, initially the rope is a bit long and as it is pulled up the rope is reeled back thus decreasing its length to place it on the deck of Go-Navigator.

The A-Frame can be bent upto 10° with the deck surface from its vertical position. The time taken to pull the dragon capsule back is about 40s.

We assume that the rope is initially 10% longer than its actual length($7.84m$) and then during the retrieval it is brought back to its actual length. We assume the length variation to be of the form $L = L_0 - Bt$ where L_0 is the initial length and B is the rate of decrease in the length. We assume two cases, one where the A-Frame is rotated through constant angular velocity and the other when it is rotated with constant angular acceleration.

The additional differential equations to be solved here are :

$$\frac{d\phi}{dt} = \psi \tag{8}$$

$$\frac{d\psi}{dt} = \alpha \tag{9}$$

$$\frac{dL}{dt} = -B \tag{10}$$

Rotation through a Constant Angular Velocity

First we assume that the A-Frame is rotated through a constant angular velocity of $\omega = 0.035rad/s$ with zero angular acceleration. This value is arrived upon based on the fact that the time taken to lift the Capsule is considered to be $40s$ and the frame moves through an angle of 80°.

Figures 23 and 24 show the variation of angular displacement in the X and Y-Plane with time considering constant and changing length. We can observe that the angular displacement is slightly lower for the case of changing length as compared to the constant length condition which is due to the increase in length. If the time required to lift the capsule is increased to about $100s$ and decreased to $10s$ there is no considerable difference in the angular dispacements for the two cases (constant and varying length).

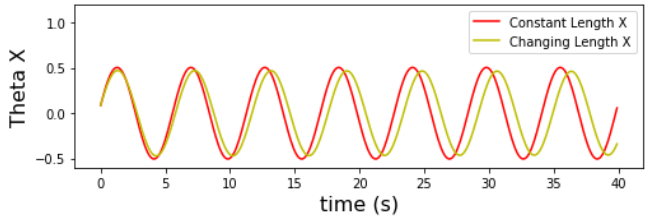

FIG. 23: Variation of $theta_x$ with time for constant length and varying length of the pendulum when the A-Frame is lifted at a constant Angular velocity.

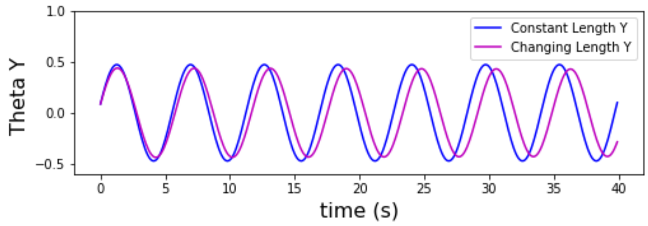

FIG. 24: Variation of $theta_y$ with time for constant length and varying length of the pendulum when the A-Frame is lifted at a constant Angular velocity.

Rotation through a Constant Angular Acceleration

Now we assume that the A-Frame is rotated through a constant angular acceleration of $\alpha = 1rad/s^2$. Figure 25 and 26 show the variation of angular displacement in the X and Y-Plane with time considering constant and changing length. Here the plots are shown for a time of 100s but the interval of interest is upto 40s since it is the time taken by the crane to lift the capsule. There is no considerable change when the angular acceleration is increased by a factor of 10.

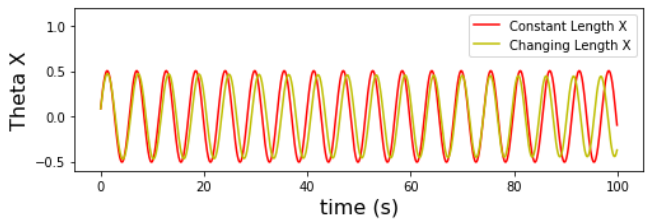

FIG. 25: Variation of $theta_y$ with time for constant length and varying length of the pendulum when the A-Frame is lifted at a constant Angular acceleration.

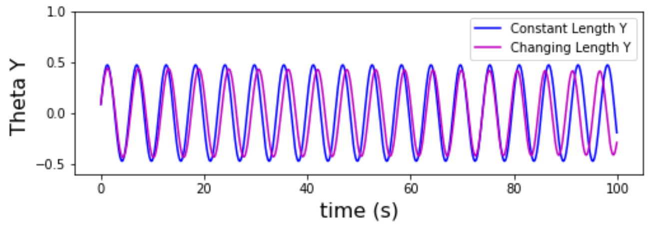

FIG. 26: Variation of $theta_y$ with time for constant length and varying length of the pendulum when the A-Frame is lifted at a constant Angular acceleration.

APPENDIX

Odeint Function

Odeint is one of the 3 modules available in `SciPy` for integrating ODE's. It uses the `LSODA` algorithm (Livermore Solver for Ordinary Differential Equations). This solver adapts the step size, by estimating the integration error, such that the estimated error is below a user-defined threshold specified by the arguments 'rtol' and 'atol'. These arguments allow the user to control precision.[5]

[1] https://stackoverflow.com/questions/49799535/how-does-odeint-from-scypy-python-module-work accessed on 01 September 2020 10:10 pm.

[2] https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html#scipy-integrate-odeint; accessed on 26 August 2020 04:27 pm.

[3] https://moodle.iist.ac.in/pluginfile.php/6718/mod_forum/post/95/goamerica.pdf; accessed on 26 August 2020 06:05 pm.

[4] https://www.engineeringtoolbox.com/nylon-rope-strength-d_1513.html; accessed on 01 September 2020 09:48 pm.

[5] https://stackoverflow.com/questions/49799535/how-does-odeint-from-scypy-python-module-work; accessed on 12 September 2020 11:05 am.