Assignment 1
Principal Values and Principal Directions of a Stress Distribution
AE314 - Theory of Elasticity

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This is a report based on the coding exercise given to find the principal values and principal directions for a given stress distribution.

PROBLEM DESCRIPTION & METHODOLOGY OF SOLUTION

Given a stress distribution

$$\sigma = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

which is symmetric i.e. $\tau_{12} = \tau_{21}$, $\tau_{32} = \tau_{23}$ and $\tau_{13} = \tau_{31}$. We can find the Prinicpal values and Principal directions corresponding to the given stress distribution.
This can be done in the following way[1, 2]:

1. Find the three invariants given by
   $I_1 = tr(\sigma)$
   $I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$
   $I_3 = \det(\sigma)$
2. The principal values can be found by solving the Equation $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$
3. Corresponding to each Principal value(σ_i) there exists a principal direction given by

$$\begin{bmatrix} \sigma_{11} - \sigma_i & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} - \sigma_i & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} - \sigma_i \end{bmatrix} \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Here l_i, m_i and n_i are direction cosines of the prinicpal direction corresponding to σ_i.
5. The cubic equation to find the principal values is solved by first, determining 2 intermediate quantities

$$Q = \frac{3I_2 - I)1^2}{9}$$

$$R = \frac{2I_1^3 - 9I_1I_2 + 27I3}{54}$$

$$\theta = cos^{-1}(R/\sqrt{-Q^3})$$

.

6. Subsequently the principal values are given by

$$\sigma_1 = 2\sqrt{-Qcos\left(\frac{\theta}{3}\right)} + \frac{I_1}{3}$$

$$\sigma_2 = 2\sqrt{-Qcos\left(\frac{\theta + 2\pi}{3}\right)} + \frac{I_1}{3}$$

$$\sigma_3 = 2\sqrt{-Qcos\left(\frac{\theta + 4\pi}{3}\right)} + \frac{I_1}{3}$$

By convention the principal values are denoted so that $\sigma_1 > \sigma_2 > \sigma_3$

RESULTS

The procedure for finding out the principal values and directions as explained in the previous section is implemented in python and verified for 2 test cases.

Case 1

$$\sigma = \begin{bmatrix} 50 & -20 & 10 \\ -20 & 30 & 40 \\ 10 & 40 & 20 \end{bmatrix}$$

Hand calculations give us $\sigma_1 = 70$, $\sigma_2 = 51.4$ and $\sigma_3 = -21.4$. From the code written we obtain $\sigma_1 = 70.0$, $\sigma_2 = 51.40054944640259$ and $\sigma_3 = -21.400549446402593$. These values match with the hand calculations. The Eigenvectors or principal directions with the components rounded off to the 3^{rd} decimal place for the system are

$$\hat{n_1} = \begin{bmatrix} 0.485 \\ -0.728 \\ -0.485 \end{bmatrix}$$

$$\hat{n_2} = \begin{bmatrix} 0.829 \\ 0.203 \\ 0.523 \end{bmatrix}$$

$$\hat{n_3} = \begin{bmatrix} 0.282 \\ 0.655 \\ -0.701 \end{bmatrix}$$

To verify these principal directions we find the dot product between these directional cosines.

$$\hat{n_1} \cdot \hat{n_2} = 1.11 \times 10^{-16}$$

$$\hat{n_2} \cdot \hat{n_3} = 0.0$$

$$\hat{n_3} \cdot \hat{n_1} = -5.55 \times 10^{-17}$$

Case 2

$$\sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

Hand calculations give us $\sigma_1 = 5$, $\sigma_2 = -5$ and $\sigma_3 = 2$. From the code written we obtain $\sigma_1 = 5.000000000000001$, $\sigma_2 = 2.0000000000000013$ and $\sigma_3 = -5.0$. These values match with the hand calculations. The Eigenvectors or principal directions with the components rounded off to the 3^{rd} decimal place for the system are

$$\hat{n_1} = \begin{bmatrix} 0.0 \\ -0.447 \\ 0.894 \end{bmatrix}$$

$$\hat{n_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{n_3} = \begin{bmatrix} 0.0 \\ 0.894 \\ 0.447 \end{bmatrix}$$

To verify these principal directions we find the dot product between these directional cosines.

$$\hat{n_1} \cdot \hat{n_2} = 0.0$$

$$\hat{n_2} \cdot \hat{n_3} = 0.0$$

$$\hat{n_3} \cdot \hat{n_1} = -1.665 \times 10^{-16}$$

CODE IMPLEMENTED IN PYTHON

Below is the code in python written and used to find the principal values and directions for reference only. The code can be found in .py and .txt format for execution here since the code given below is not properly indented.

```
from numpy import sqrt,arccos,cos,pi
def dot(a,b):
return a[0]*b[0] + a[1]*b[1] + a[2]*b[2]
"'s11 = input('Enter sigmaxx : ')
s12 = input('Enter sigmayx : ')
s13 = input('Enter sigmazx : ')
s21 = input('Enter sigmaxy : ')
s22 = input('Enter sigmayy : ')
s23 = input('Enter sigmazy : ')
s31 = input('Enter sigmaxz : ')
s32 = input('Enter sigmayz : ')
s33 = input('Enter sigmazz : ')"'
s11 = 2
s12 = 0
s13 = 0
s21 = 0
s22 = 3
s23 = 4
s31 = 0
s32 = 4
s33 = -3
I1 = s11 + s22 + s33
I2 = s11*s22 + s22*s33 + s33*s11 - s21*s12 -s31*s13 - s23*s32
I3 = (s11*(s22*s33 - s23*s32)-s12*(s21*s33 - s31*s23)+ s13*(s21*s32 - s22*s31))
Q = (3*I2 - I1**2)/9
```

```
R = (2*I1**3 - 9*I1*I2 +27*I3)/54
theta = arccos(R/sqrt(-Q**3))
r1 = 2*sqrt(-Q)*cos(theta/3) + I1/3
r2 = 2*sqrt(-Q)*cos((theta + 2*pi)/3) + I1/3
r3 = 2*sqrt(-Q)*cos((theta + 4*pi)/3) + I1/3
r = [r1,r2,r3]
eigen_vec = []
for i in range(0,len(r)):
if abs(r[i] - s11) ¡ 0.00001:
l1 = 1
m1 = 0
n1 = 0
elif abs(r[i] - s22) ¡ 0.00001:
l1 = 0
m1 = 1
n1 = 0
elif abs(r[i] - s33 ) ¡ 0.00001:
l1 = 0
m1 = 0
n1 = 1
else:
a1 = s11-r[i]
a2 = s12
a3 = s13
b1 = s21
b2 = s22-r[i]
b3 = s23
if a2 == 0 and a3 == 0:
l1s = 0
m1s = b2
n1s = b3
else:
l1s = 1
m1s = (a1*b3 - a3*b1)/(b2*a3 - a2*b3)
n1s = (a1*b2 - a2*b1)/(a2*b3 - b2*a3)
mag = sqrt(l1s**2 + m1s**2 + n1s**2)
l1 = l1s/mag
m1 = m1s/mag
n1 = n1s/mag
e1 = [l1,m1,n1]
eigen_vec.append(e1)
E1 = eigen_vec[0]
E2 = eigen_vec[1]
E3 = eigen_vec[2]
dot_prod1 = dot(E1,E2)
dot_prod2 = dot(E2,E3)
dot_prod3 = dot(E3,E1)
print(dot_prod1)
print(dot_prod2)
print(dot_prod3)
```

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[1] https://mathworld.wolfram.com/CubicFormula.html
[2] http://www.continuummechanics.org/principalstress.html