Details of the numerical simulation through ~~Circuit~~ Qiskit

As we are dealing with qutrits, but to simulate in Qiskit, we need to encode this using qubits so we shall use 2 qubits to encode a qutrit where the basis state

2 Qubits		Qutrit
$\|00\rangle$	$\mapsto$	$\|0\rangle$
$\|01\rangle$	$\mapsto$	$\|1\rangle$
$\|11\rangle$	$\mapsto$	$\|2\rangle$

For demonstration, let us consider the unitary operator

$$U = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$\omega \rightarrow$ cube root of unity

$$\omega = e^{2\pi i/3}$$

So the eigenvalues are $e^{2\pi i/3}$ & $e^{2\pi i(2/3)}$

So that our φ which is $1/3$ (or) $2/3$ with eigenvectors $|0\rangle$ & $|1\rangle$ can be represented exactly in the form $b/3^t$

where $t = 1$

So our 1^{st} register has 1 qutrit (or) 2 qubits

Here we first choose eigenvector as $|0\rangle$

And while applying Controlled - U gates in qutrits

when control is $|0\rangle$, target not activated

Control is $|1\rangle$, target activated & U is executed once

Control is $|2\rangle$, target activated & U is executed twice

To mimic this in qiskit we apply controlled unitaries in both the qubits, so that

$|00\rangle \mapsto |0\rangle$: U not applied on $|u\rangle$

$|01\rangle \mapsto |1\rangle$: U applied only once to $|u\rangle$

$|11\rangle \mapsto |2\rangle$: U applied twice to $|u\rangle$

And the 1^{st} register is initialized in the state

$$\underbrace{\frac{1}{\sqrt{3}}\left\{|00\rangle + |01\rangle + |11\rangle\right\}}_{2\text{ Qubits}} \mapsto \underbrace{\frac{1}{\sqrt{3}}\left\{|0\rangle + |1\rangle + |2\rangle\right\}}_{\text{Qutrit}}$$

Coming to Inverse Quantum Fourier Transform stage,

As there is only one qutrit in 1^{st} register, QFT has only a Hadamard gate, hence IQFT (Qutrit) is $H^{\dagger}$

$$H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \qquad H^{\dagger} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

But to execute this $H^{\dagger}$ in 2 qubit encoding

$$H' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & \omega^2 & 0 & \omega \\ 0 & 0 & \sqrt{3} & 0 \\ 1 & \omega & 0 & \omega^2 \end{bmatrix}$$

Additional entries are added to making it Compatible to be applied to 2 qubits in the 1^{st} register, and to ~~the~~ Keep $|10\rangle$ undisturbed as we do not need this basis state

Hence the second stage is set

Now comes measurement in computational basis

On initializing the $|u\rangle$ with $|0\rangle$

we get measurement outcomes as $01 \mapsto 1$

$$\downarrow$$
$$\varphi_1$$

$$\therefore \quad \underline{\varphi = 0.\varphi_1 = \varphi_1/3 = 1/3}$$

$$\text{eigenvalue} = e^{2\pi i/3}$$

On initializing the $|u\rangle$ with $|1\rangle$

we get outcome as $11 \mapsto 2$

$$\downarrow$$
$$\varphi_1$$

$$\therefore \quad \underline{\varphi = 2/3}$$

$$\text{eigenvalue} = e^{4\pi i/3}$$