Theoretical Fundamentals of AI and Data Science
First Order Logic Exercises

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1 Propositional Logic

1.1 DeMorgan

Prove the propositional **DeMorgan laws**:

1. $\neg(P \land Q) \leftrightarrow \neg P \lor \neg Q$
2. $\neg(P \lor Q) \leftrightarrow \neg P \land \neg Q$

1.2 Formal proofs

(**Language, Proof and Logic**, Barwise and Etchemendy 2011, 19.11-13, p.555)

Give formal proofs of the following arguments:

1.

$$\begin{array}{r|l} 1 & P \to Q \\ 2 & \neg P \to Q \\ \hline 3 & Q \end{array}$$

2.

$$\begin{array}{r|l} 1 & (P \to Q) \to R \\ \hline 2 & \neg P \to R \end{array}$$

3.

$$\begin{array}{r|l} 1 & (P \to Q) \to R \\ \hline 2 & Q \to R \end{array}$$

1.3 Translation exercises

(**Language, Proof and Logic**, Barwise and Etchemendy 2011, 8.3, p.205)

Translate and prove the following argument:

- The unicorn, if it is not mythical, is a mammal, but if it is mythical, it is immortal.
- If the unicorn is either immortal or a mammal, it is horned.
- The unicorn, if horned, is magical.

Therefore

- The unicorn is magical.

1.4 Translation exercises

(**Language, Proof and Logic**, Barwise and Etchemendy 2011, 8.4, p.205)

Translate and prove the following argument:

- The unicorn, if horned, is elusive and dangerous.
- If elusive or mythical, the unicorn is rare.
- If a mammal, the unicorn is not rare.

Therefore

- The unicorn, if horned, is not a mammal.

1.5 Patterns of inference

(**Language, Proof and Logic**, Barwise and Etchemendy 2011, 8.18-25, p.213)

Which of the following patterns of inference are valid? Give formal proofs of the valid patterns and construct counterexamples for the invalid ones.

1. **Affirming the Consequent:** From $A \to B$ and B, infer A.
2. **Strengthening the Antecedent:** From $B \to C$, infer $(A \land B) \to C$.
3. **Strengthening the Consequent:** From $A \to B$, infer $A \to (B \land C)$.
4. **Constructive Dilemma:** From $A \lor B$, $A \to C$, and $B \to D$, infer $C \lor D$.
5. **Modus Tollens:** From $A \to B$ and $\neg B$, infer $\neg A$.
6. **Weakening the Antecedent:** From $B \to C$, infer $(A \lor B) \to C$.
7. **Weakening the Consequent:** From $A \to B$, infer $A \to (B \lor C)$.
8. **Transitivity of the Biconditional:** From $A \leftrightarrow B$ and $B \leftrightarrow C$, infer $A \leftrightarrow C$.

1.6 Remark

The proof checker from `https://proofs.openlogicproject.org/` also supports additional pattern of inference, e.g.:

- from $A \lor B$ and $\neg A$, infer B
- from $A \to B$ and $\neg A \to B$, infer B

The proof checker *Pithos ND* also supports

- $P \lor \neg P$ is a logical truth (*EM = Law of Excluded Middle*)

1.7 Translation exercises

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 8.7, p.205)

- b is small unless it's a square.
- If c is small, then either d or e is too.
- If d is small, then c is not.
- If b is a square, then e is not small.

Therefore

- If c is small, then so is b.

Translate the argument to FOL and prove it's validity or give a counterexample.

1.8 Translation exercises

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 8.6, p.205)

- a is a large triangle or a small square.
- b is not small.
- If a is a triangle or a square, then b is large or small.
- a is a triangle only if b is medium.

Therefore

- a is small and b is large.

Translate the argument to FOL and prove it's validity or give a counterexample.

1.9 Translation exercises

Which of the following arguments are valid?

1. The serum must arrive in 24 hours or Sherlock will die. If the expert on jungle diseases is available, and the serum arrives, then Sherlock will live. The expert will be available. However, if the expert is available, the serum will arrive in time. So, Sherlock will live
2. If the temperature does not rise tonight, we can go skating tomorrow. If either the temperature rises or the ice is rough, then we cannot skate. The ice will not be rough. Therefore, we can go skating tomorrow.
3. Unless Emily gets a bicycle for her birthday she will be unhappy. So, if she gets a dress, she will be unhappy.

(From **The Language of Logic**, Guttenplan 1986)

1.10 Scheffers Stroke

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 7.29, p.197)

All boolean connectives can be expressed using only the **Sheffer stroke** (also known as *nand*). Its truth table is:

P	Q	$P \vert Q$
F	F	T
F	T	T
T	F	T
T	T	F

1. Show how to express $\neg P$, $P \wedge Q$, and $P \vee Q$ using the Sheffer stroke.
2. Suggest introduction and elimination rules for $|$.

2 Quantifiers

2.1 Translation exercises

Translate the following sentences into everyday English:

`∀x∃y Likes(y, x)`

`∃x∀y Likes(y, x)`

The order of the quantifiers is important!

2.2 Translation exercises

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 11.16, p.308)

1. Every square is to the left of every triangle.
2. Every small square is in back of a large square.
3. Some square is in front of every triangle.
4. A large square is in front of a small square.
5. Nothing is larger than everything.
6. Every square in front of every triangle is large.
7. Everything to the right of a large square is small.
8. Nothing in back of a square and in front of a square is large.
9. Anything with nothing in back of it is a square.
10. Every pentagon is smaller than some triangle.

Translate the statements to FOL and build a world in which they are all true.

2.3 Translation exercises

(**Language, Proof and Logic**, Barwise and Etchemendy 2011, 11.23, p.313)

Translate the following into FOL. Explain the meanings of the names, predicates, and function symbols you use, and comment on any shortcomings in your translations.

1. There's a sucker born every minute.
2. Whither thou goest, I will go.
3. Soothsayers make a better living in the world than truthsayers.
4. To whom nothing is given, nothing can be required.
5. If you always do right, you will gratify some people and astonish the rest.

2.4 Translation exercise

Every person who owns a book reads it.

2.5 Formal proofs

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 13.24-25, p.368)

Prove the following arguments or give a counterexample.

$$\begin{array}{r|l} 1 & \exists x(Square(x) \lor Small(x)) \\ \hline 2 & \exists x Square(x) \lor \exists x Small(x) \end{array}$$

$$\begin{array}{r|l} 1 & \exists x Square(x) \lor \exists x Small(x) \\ \hline 2 & \exists x(Square(x) \lor Small(x)) \end{array}$$

2.6 Formal proofs

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 19.16, p.555)

Give a formal proof of the following argument:

$$\begin{array}{r|l} 1 & \forall x\ (P(x) \to Q(x)) \\ 2 & \forall x \forall y\ (x = y) \\ \hline 3 & \exists x\ P(x) \to \forall x\ Q(x) \end{array}$$

2.7 Formal proofs

(Adapted from **Language, Proof and Logic**, Barwise and Etchemendy 2011, 13.20-22, p.367-8)

Give a formal proof of the valid arguments, and counter-examples for the invalid ones.

1.

$$\begin{array}{r|l} 1 & \forall x\ [(B(x) \lor T(x)) \to (M(x) \land G(x))] \\ 2 & \forall y\ [(S(y) \lor M(y)) \to T(y)] \\ 3 & \exists x\ S(x) \\ \hline 4 & \exists x\ [S(x) \land M(x)] \end{array}$$

2.

$$\begin{array}{r|l} 1 & \forall x\ [B(x) \to (M(x) \land S(x))] \\ 2 & \forall y\ [S(y) \lor M(y)) \to T(y)] \\ 3 & \forall x\ [T(x) \to (O(x, b) \land B(x))] \\ \hline 4 & \forall z\ [B(z) M(z)] \end{array}$$

3.

$$
\begin{array}{r|l}
1 & \forall x\ [(B(x) \land T(x)) \to M(x)] \\
2 & \forall y\ [(T(y) \lor M(y)) \to S(y)] \\
3 & \exists x\ B(x) \land \exists x\ T(x) \\
\hline
4 & \exists z\ S(z)
\end{array}
$$

2.8 Formal proofs

Translate the **Barber Paradox**: There once was a small town where there was a barber who shaved all and only the men of the town who did not shave themselves.

Formalize this sentence and discuss its possible truth value.

2.9 Formal proofs

The Drinker's Paradox: In every (non-empty) pub there is a person such that, if that person drinks, then everybody drinks.

Formalize this sentence and prove it's validity.

Sources: The exercises were originally collected or created by Prof. Ionescu and adapted by Prof. Mayer.