Discrete Random Variables
Class 4a, TF, AID-M
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1 Learning Goals

1. Know the definition of a discrete random variable.
2. Know the Bernoulli, binomial, and geometric distributions and examples of what they model.
3. Be able to describe the probability mass function and cumulative distribution function using tables and formulas.
4. Be able to construct new random variables from old ones.

2 Random Variables

This topic is largely about introducing some useful terminology, building on the notions of sample space and probability function. The key words are

1. Random variable
2. Probability mass function (pmf)
3. Cumulative distribution function (cdf)

2.1 Recap

A discrete sample space Ω is a finite or listable set of outcomes $\{\omega_1, \omega_2 \ldots\}$. The probability of an outcome ω is denoted $P(\omega)$.

An event E is a subset of Ω. The probability of an event E is $P(E) = \sum_{\omega \in E} P(\omega)$.

2.2 Random variables as payoff functions

Example 1. A game with 2 dice.

Roll a die twice and record the outcomes as (i, j), where i is the result of the first roll and j the result of the second. We can take the sample space to be

$$\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,6)\} = \{(i,j) \,|\, i, j = 1, \ldots 6\}.$$

The probability function is $P(i, j) = 1/36$.

In this game, you win \$500 if the sum is 7 and lose \$100 otherwise. We give this payoff function the name X and describe it formally by

$$X(i,j) = \begin{cases} 500 & \text{if } i+j=7 \\ -100 & \text{if } i+j \neq 7. \end{cases}$$

Example 2. We can change the game by using a different payoff function. For example

$$Y(i,j) = ij - 10.$$

In this example if you roll $(6,2)$ then you win \$2. If you roll $(2,3)$ then you win -\$4 (i.e., lose \$4).

Question: Which game is the better bet?

Solution: We will come back to this once we learn about expectation.

These payoff functions are examples of random variables. A random variable assigns a number to each outcome in a sample space. More formally:

Definition: Let Ω be a sample space. A discrete random variable is a function

$$X : \Omega \to \mathbf{R}$$

that takes a discrete set of values. (Recall that $\mathbf{R}$ stands for the real numbers.)

Why is X called a random variable? It's 'random' because its value depends on a random outcome of an experiment. And we treat X like we would a usual variable: we can add it to other random variables, square it, and so on.

2.3 Events and random variables

For any value a we write $X = a$ to mean the event consisting of all outcomes ω with $X(\omega) = a$.

Example 3. In Example 1 we rolled two dice and X was the random variable

$$X(i,j) = \begin{cases} 500 & \text{if } i+j=7 \\ -100 & \text{if } i+j \neq 7. \end{cases}$$

The event $X = 500$ is the set {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}, i.e. the set of all outcomes that sum to 7. So $P(X = 500) = 1/6$.

We allow a to be any value, even values that X never takes. In Example 1, we could look at the event $X = 1000$. Since X never equals 1000 this is just the empty event (or empty set)

$$\text{`}X = 1000\text{'} \;=\; \{\} \;= \emptyset \qquad P(X = 1000) = 0.$$

2.4 Probability mass function and cumulative distribution function

It gets tiring and hard to read and write $P(X = a)$ for the probability that $X = a$. When we know we're talking about X we will simply write $p(a)$. If we want to make X explicit we will write $p_X(a)$. We spell this out in a definition.

Definition: The probability mass function (pmf) of a discrete random variable is the function $p(a) = P(X = a)$.

Note:
1. We always have $0 \leq p(a) \leq 1$.
2. We allow a to be any number. If a is a value that X never takes, then $p(a) = 0$.

Example 4. Let Ω be our earlier sample space for rolling 2 dice. Define the random variable M to be the maximum value of the two dice, i.e.

$$M(i, j) = \max(i, j).$$

For example, the roll (3,5) has maximum 5, i.e. $M(3, 5) = 5$.

We can describe a random variable by listing its possible values and the probabilities associated to these values. For the above example we have:

value a:	1	2	3	4	5	6
pmf $p(a)$:	1/36	3/36	5/36	7/36	9/36	11/36

For example, $p(2) = 3/36$.

Question: What is $p(8)$? **Solution:** $p(8) = 0$.

Think: What is the pmf for $Z(i, j) = i + j$? Does it look familiar?

2.5 Events and inequalities

Inequalities with random variables describe events. For example $X \leq a$ is the set of all outcomes ω such that $X(w) \leq a$.

Example 5. If our sample space is the set of all pairs of (i, j) coming from rolling two dice and $Z(i, j) = i + j$ is the sum of the dice then

$$Z \leq 4 \; = \; \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$$

2.6 The cumulative distribution function (cdf)

Definition: The cumulative distribution function (cdf) of a random variable X is the function F given by $F(a) = P(X \leq a)$. We will often shorten this to distribution function.

Note well that the definition of $F(a)$ uses the symbol less than **or equal to**. This will be important for getting your calculations exactly right.

Example. Continuing with the example M, we have

value a:	1	2	3	4	5	6
pmf $p(a)$:	1/36	3/36	5/36	7/36	9/36	11/36
cdf $F(a)$:	1/36	4/36	9/36	16/36	25/36	36/36

$F(a)$ is called the cumulative distribution function because $F(a)$ gives the total probability that accumulates by adding up the probabilities $p(b)$ as b runs from $-\infty$ to a. For example, in the table above, the entry 16/36 in column 4 for the cdf is the sum of the values of the pmf from column 1 to column 4. In notation:

As events: '$M \le 4$' $= \{1, 2, 3, 4\}$; $\quad F(4) = P(M \le 4) = 1/36+3/36+5/36+7/36 = 16/36$.

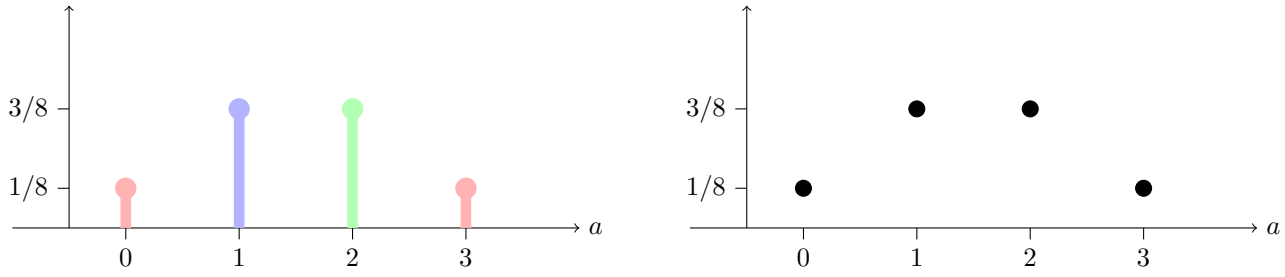
Just like the probability mass function, $F(a)$ is defined for all values a. In the above example, $F(8) = 1$, $F(-2) = 0$, $F(2.5) = 4/36$, and $F(\pi) = 9/36$.

2.7 Graphs of $p(a)$ and $F(a)$

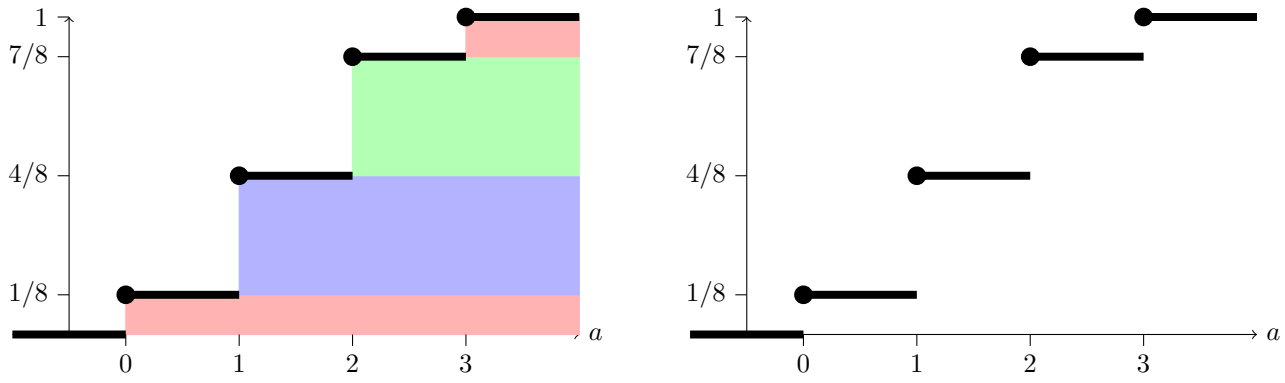
We can visualize the pmf and cdf with graphs. For example, let X be the number of heads in 3 tosses of a fair coin:

value a:	0	1	2	3
pmf $p(a)$:	1/8	3/8	3/8	1/8
cdf $F(a)$:	1/8	4/8	7/8	1

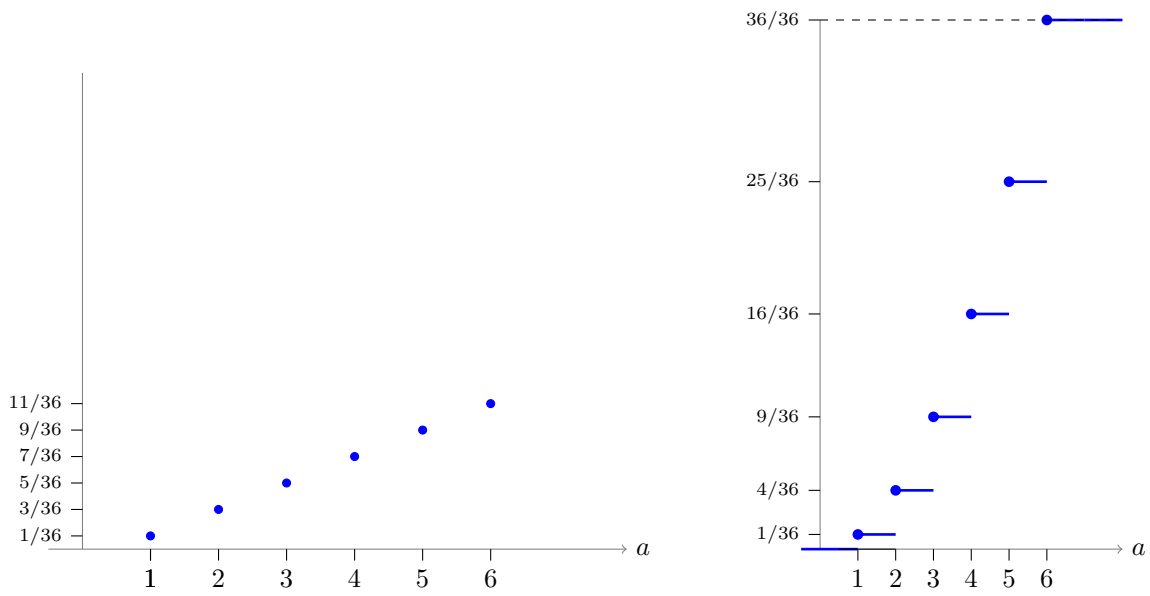
The colored graphs show how the cumulative distribution function is built by accumulating probability as a increases. The black and white graphs are the more standard presentations.



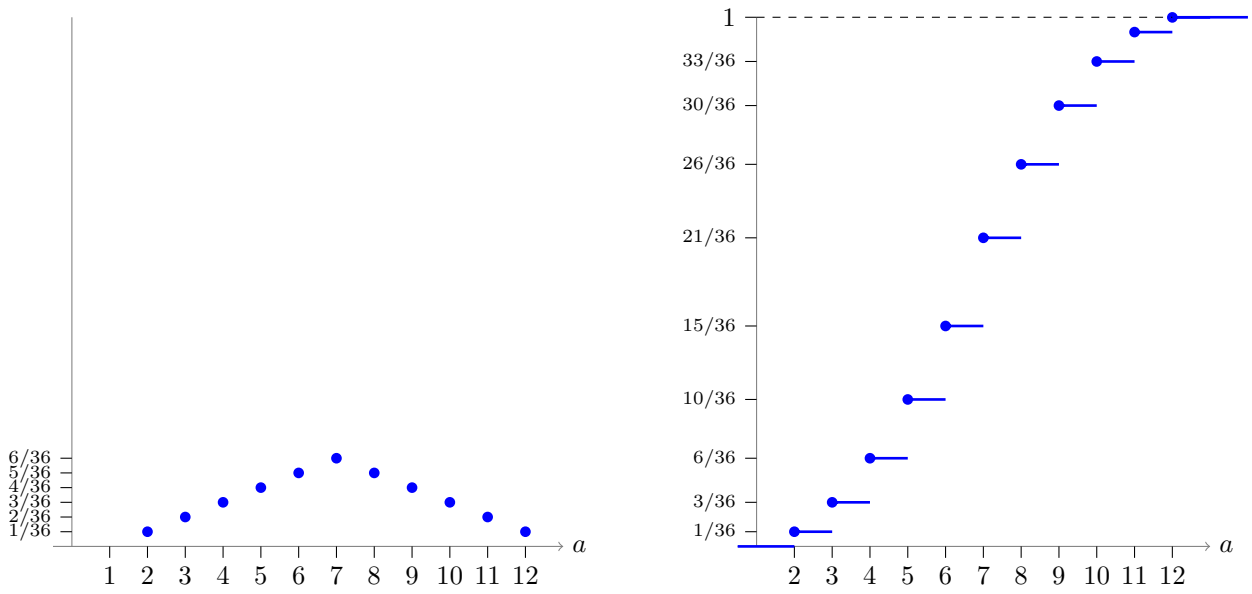
Probability mass function for X



Cumulative distribution function for X



pmf and cdf for the maximum of two dice (Example 4)



pmf and cdf for the sum of two dice

Histograms: Later we will see another way to visualize the pmf using histograms. These require some care to do right, so we will wait until we need them.

2.8 Properties of the cdf F

The cdf F of a random variable satisfies several properties:

1. F is non-decreasing. That is, its graph never goes down, or symbolically if $a \leq b$ then

$F(a) \le F(b)$.

2. $0 \le F(a) \le 1$.
3. $\lim_{a\to\infty} F(a) = 1, \quad \lim_{a\to-\infty} F(a) = 0$.

In words, (1) says the cumulative probability $F(a)$ increases or remains constant as a increases, but never decreases; (2) says the accumulated probability is always between 0 and 1; (3) says that as a gets very large, it becomes more and more certain that $X \le a$ and as a gets very negative it becomes more and more certain that $X > a$.

Think: Why does a cdf satisfy each of these properties?

3 Specific Distributions

3.1 Bernoulli Distributions

Model: The Bernoulli distribution models one trial in an experiment that can result in either success or failure This is the most important distribution and is also the simplest. A random variable X has a Bernoulli distribution with parameter p if:

1. X takes the values 0 and 1.
2. $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

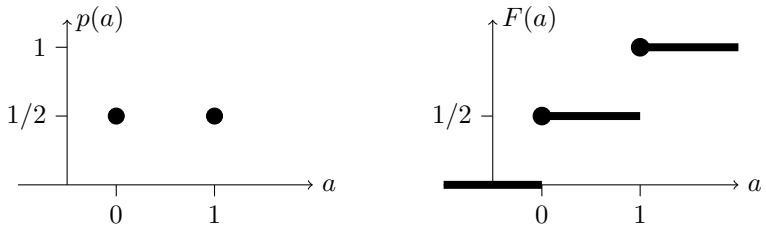
We will write $X \sim$ Bernoulli(p) or Ber(p), which is read "X follows a Bernoulli distribution with parameter p" or "X is drawn from a Bernoulli distribution with parameter p".

A simple model for the Bernoulli distribution is to flip a coin with probability p of heads, with $X = 1$ on heads and $X = 0$ on tails. The general terminology is to say X is 1 on success and 0 on failure, with success and failure defined by the context.

Many decisions can be modeled as a binary choice, such as votes for or against a proposal. If p is the proportion of the voting population that favors the proposal, than the vote of a random individual is modeled by a Bernoulli(p).

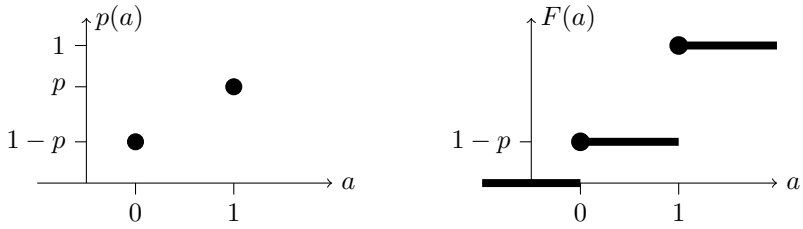
Here are the table and graphs of the pmf and cdf for the Bernoulli(1/2) distribution and below that for the general Bernoulli(p) distribution.

value a:	0	1
pmf $p(a)$:	1/2	1/2
cdf $F(a)$:	1/2	1



Table, pmf and cmf for the Bernoulli(1/2) distribution

values a:	0	1
pmf $p(a)$:	1-p	p
cdf $F(a)$:	1-p	1



Table, pmf and cmf for the Bernoulli(p) distribution

3.2 Binomial Distributions

The binomial distribution Binomial(n,p), or Bin(n,p), models the number of successes in n independent Bernoulli(p) trials.

There is a hierarchy here. A single Bernoulli trial is, say, one toss of a coin. A single binomial trial consists of n Bernoulli trials. For coin flips the sample space for a Bernoulli trial is $\{H, T\}$. The sample space for a binomial trial is all sequences of heads and tails of length n. Likewise a Bernoulli random variable takes values 0 and 1 and a binomial random variables takes values 0, 1, 2, ..., n.

Example 6. Binomial(1,p) is the same as Bernoulli(p).

Example 7. The number of heads in n flips of a coin with probability p of heads follows a Binomial(n, p) distribution.

We describe $X \sim \text{Binomial}(n, p)$ by giving its values and probabilities. For notation we will use k to mean an arbitrary number between 0 and n.

We remind you that 'n choose k = $\binom{n}{k}$ = ${}_nC_k$ is the number of ways to choose k things out of a collection of n things and it has the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{1}$$

(It is also called a binomial coefficient.) Here is a table for the pmf of a Binomial(n, k) random variable. We will explain how the binomial coefficients enter the pmf for the binomial distribution after a simple example.

values a:	0	1	2	$\cdots$	k	$\cdots$	n
pmf $p(a)$:	$(1-p)^n$	$\binom{n}{1}p^1(1-p)^{n-1}$	$\binom{n}{2}p^2(1-p)^{n-2}$	$\cdots$	$\binom{n}{k}p^k(1-p)^{n-k}$	$\cdots$	p^n

Example 8. What is the probability of 3 or more heads in 5 tosses of a fair coin?

Solution: The binomial coefficients associated with $n = 5$ are

$$\binom{5}{0} = 1, \quad \binom{5}{1} = \frac{5!}{1!\,4!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = 5, \quad \binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \frac{5\cdot4}{2} = 10,$$

and similarly

$$\binom{5}{3} = 10, \quad \binom{5}{4} = 5, \quad \binom{5}{5} = 1.$$

Using these values we get the following table for $X \sim$ Binomial(5,p).

values a:	0	1	2	3	4	5
pmf $p(a)$:	$(1-p)^5$	$5p(1-p)^4$	$10p^2(1-p)^3$	$10p^3(1-p)^2$	$5p^4(1-p)$	p^5

We were told $p = 1/2$ so

$$P(X \geq 3) = 10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^5 = \frac{16}{32} = \frac{1}{2}.$$

Think: Why is the value of 1/2 not surprising?

3.3 Explanation of the binomial probabilities

For concreteness, let $n = 5$ and $k = 2$ (the argument for arbitrary n and k is identical.) So $X \sim$ binomial$(5, p)$ and we want to compute $p(2)$. The long way to compute $p(2)$ is to list all the ways to get exactly 2 heads in 5 coin flips and add up their probabilities. The list has 10 entries:

HHTTT, HTHTT, HTTHT, HTTTH, THHTT, THTHT, THTTH, TTHHT, TTHTH, TTTHH

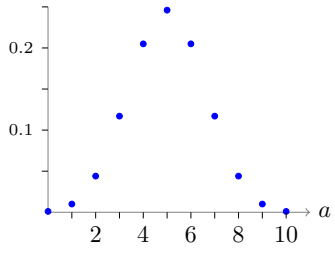
Each entry has the same probability of occurring, namely

$$p^2(1-p)^3.$$

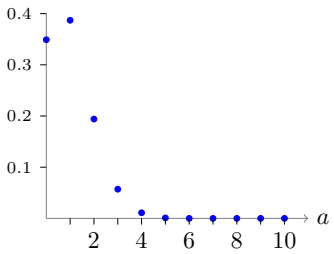
This is because each of the two heads has probability p and each of the 3 tails has probability $1-p$. Because the individual tosses are independent we can multiply probabilities. Therefore, the total probability of exactly 2 heads is the sum of 10 identical probabilities, i.e. $p(2) = 10p^2(1-p)^3$, as shown in the table.

This guides us to the shorter way to do the computation. We have to count the number of sequences with exactly 2 heads. To do this we need to choose 2 of the tosses to be heads and the remaining 3 to be tails. The number of such sequences is the number of ways to choose 2 out of 5 things, that is $\binom{5}{2}$. Since each such sequence has the same probability, $p^2(1-p)^3$, we get the probability of exactly 2 heads $p(2) = \binom{5}{2}p^2(1-p)^3$.

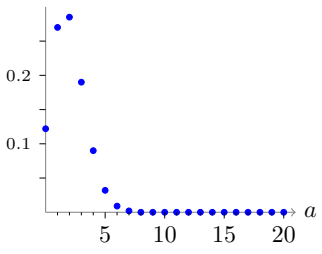
Here are some binomial probability mass functions:



Binomial(10, 0.5)



Binomial(10, 0.1)



Binomial(20, 0.1)

3.4 Geometric Distributions

A geometric distribution models the number of tails before the first head in a sequence of coin flips (Bernoulli trials).

Example 9. (a) Flip a coin repeatedly. Let X be the number of tails before the first heads. So, X can equal 0, i.e. the first flip is heads, 1, 2, ... In principle it takes any nonnegative integer value.

(b) Give a flip of tails the value 0, and heads the value 1. In this case, X is the number of 0's before the first 1.

(c) Give a flip of tails the value 1, and heads the value 0. In this case, X is the number of 1's before the first 0.

(d) Call a flip of tails a success and heads a failure. So, X is the number of successes before the first failure.

(e) Call a flip of tails a failure and heads a success. So, X is the number of failures before the first success.

You can see this models many different scenarios of this type. The most neutral language is the number of tails before the first head.

Formal definition. The random variable X follows a geometric distribution with parameter p if

- X takes the values 0, 1, 2, 3, ...
- its pmf is given by $p(k) = P(X = k) = (1-p)^k p$.

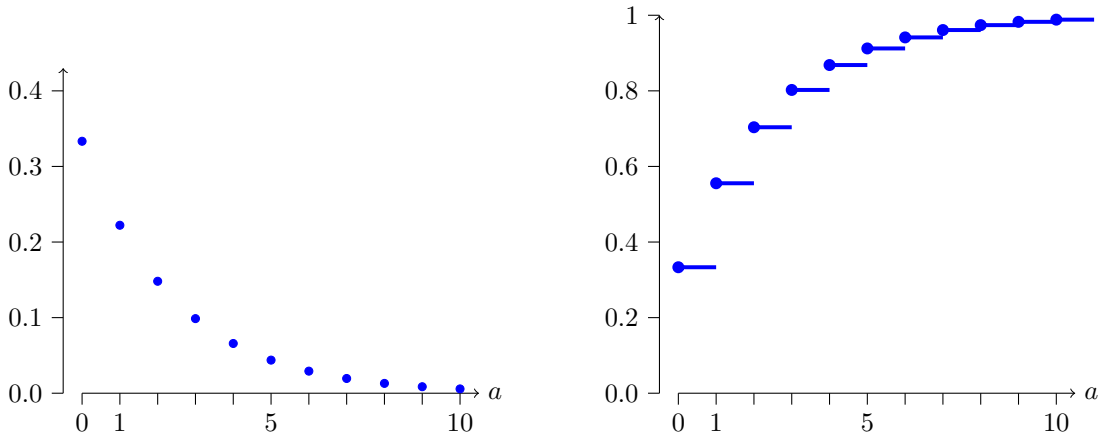
We denote this by $X \sim$ geometric(p) or geo(p). In table form we have:

value	a:	0	1	2	3	...	k	...
pmf	$p(a)$:	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	...	$(1-p)^kp$	...

Table: $X \sim$ geometric(p): $X =$ the number of 0s before the first 1.

We will show how this table was computed in an example below.

The geometric distribution is an example of a discrete distribution that takes an infinite number of possible values. Things can get confusing when we work with successes and failure since we might want to model the number of successes before the first failure or we might want the number of failures before the first success. To keep straight things straight you can translate to the neutral language of the number of tails before the first heads.



pmf and cdf for the geometric(1/3) distribution

Example 10. Computing geometric probabilities. Suppose that the inhabitants of an island plan their families by having babies until the first girl is born. Assume the probability of having a girl with each pregnancy is 0.5 independent of other pregnancies, that all babies survive and there are no multiple births. What is the probability that a family has k boys?

Solution: In neutral language we can think of boys as tails and girls as heads. Then the number of boys in a family is the number of tails before the first heads.

Let's practice using standard notation to present this. So, let X be the number of boys in a (randomly-chosen) family. So, X is a geometric random variable. We are asked to find $p(k) = P(X = k)$. A family has k boys if the sequence of children in the family from oldest to youngest is

$$BBB \dots BG$$

with the first k children being boys. The probability of this sequence is just the product of the probability for each child, i.e. $(1/2)^k \cdot (1/2) = (1/2)^{k+1}$. (Note: The assumptions of equal probability and independence are simplifications of reality.)

Think: What is the ratio of boys to girls on the island?

More geometric confusion. Another common definition for the geometric distribution is the number of tosses until the first heads. In this case X can take the values 1, i.e. the first flip is heads, 2, 3, ... This is just our geometric random variable plus 1. The methods of computing with it are just like the ones we used above.

3.5 Uniform Distribution

The uniform distribution models any situation where all the outcomes are equally likely.

$$X \sim \text{uniform}(N).$$

X takes values $1, 2, 3, \dots, N$, each with probability $1/N$. We have already seen this distribution many times when modeling to fair coins ($N = 2$), dice ($N = 6$), birthdays ($N = 365$), and poker hands ($N = \binom{52}{5}$).

3.6 Discrete Distributions Applet

The applet at https://mathlets.org/mathlets/probability-distributions/ gives a dynamic view of some discrete distributions. The graphs will change smoothly as you move the various sliders. Try playing with the different distributions and parameters.

This applet is carefully color-coded. Two things with the same color represent the same or closely related notions. By understanding the color-coding and other details of the applet, you will acquire a stronger intuition for the distributions shown.

3.7 Other Distributions

There are a million other named distributions arising is various contexts. We don't expect you to memorize them (we certainly have not!), but you should be comfortable using a resource like Wikipedia to look up a pmf. For example, take a look at the info box at the top right of https://en.wikipedia.org/wiki/Hypergeometric_distribution. The info box lists many (surely unfamiliar) properties in addition to the pmf.

4 Arithmetic with Random Variables

We can do arithmetic with random variables. For example, we can add subtract, multiply or square them.

There is a simple, but extremely important idea for counting. It says that if we have a sequence of numbers that are either 0 or 1 then the sum of the sequence is the number of 1s.

Example 11. Consider the sequence with five 1s

$$1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0.$$

It is easy to see that the sum of this sequence is 5 the number of 1s.

We illustrate this idea by counting the number of heads in n tosses of a coin.

Example 12. Toss a fair coin n times. Let X_j be 1 if the jth toss is heads and 0 if it's tails. So, X_j is a Bernoulli(1/2) random variable. Let X be the total number of heads in the n tosses. Assuming the tosses are independent we know $X \sim \text{binomial}(n, 1/2)$. We can also write

$$X = X_1 + X_2 + X_3 + \ldots + X_n.$$

Again, this is because the terms in the sum on the right are all either 0 or 1. So, the sum is exactly the number of X_j that are 1, i.e. the number of heads.

The important thing to see in the example above is that we've written the more complicated binomial random variable X as the sum of extremely simple random variables X_j. This will allow us to manipulate X algebraically.

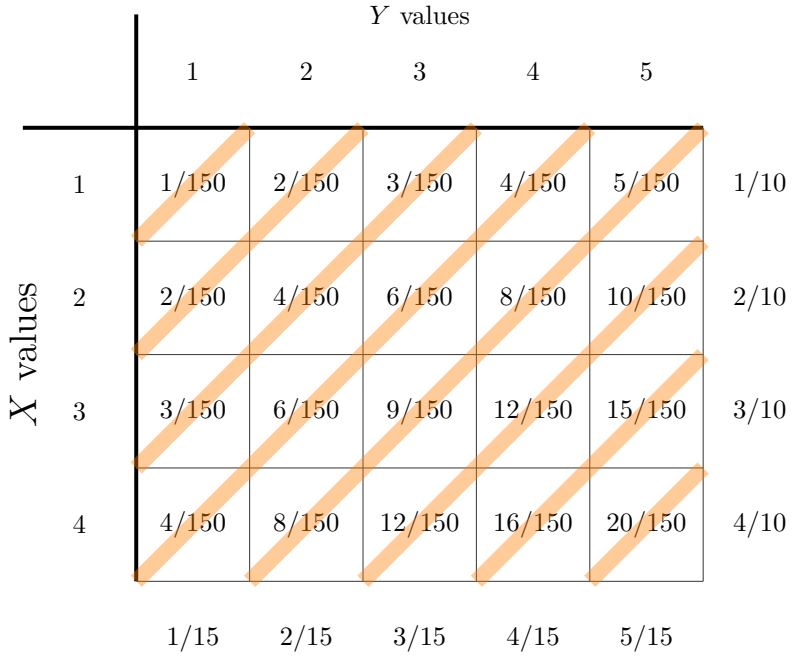
Think: Suppose X and Y are independent and $X \sim \text{binomial}(n, 1/2)$ and $Y \sim \text{binomial}(m, 1/2)$. What kind of distribution does $X + Y$ follow? (**Answer:** binomial$(n + m, 1/2)$. Why?)

Example 13. Suppose X and Y are independent random variables with the following tables.

Values of X	x:	1	2	3	4	
pmf	$p_X(x)$:	1/10	2/10	3/10	4/10	
Values of Y	y:	1	2	3	4	5
pmf	$p_Y(y)$:	1/15	2/15	3/15	4/15	5/15

Check that the total probability for each random variable is 1. Make a table for the random variable $X + Y$.

Solution: The first thing to do is make a two-dimensional table for the product sample space consisting of pairs (x, y), where x is a possible value of X and y one of Y. To help do the computation, the probabilities for the X values are put in the far right column and those for Y are in the bottom row. Because X and Y are independent the probability for (x, y) pair is just the product of the individual probabilities.



The diagonal stripes show sets of squares where $X + Y$ is the same. All we have to do to compute the probability table for $X + Y$ is sum the probabilities for each stripe.

$X+Y$ values:	2	3	4	5	6	7	8	9
pmf:	1/150	4/150	10/150	20/150	30/150	34/150	31/150	20/150

When the tables are too big to write down we'll need to use purely algebraic techniques to compute the probabilities of a sum. We will learn how to do this in due course.

Note by Prof. Mayer: This script is the work of the original authors (Jeremy Orloff and Jonathan Bloom), which appear in the title and all praise for this work must be to them. The only modifications done to fit it to "Theoretical fundamentals of AI and Data Science" (DIT, Faculty AI, Study course AIN-B) are leaving out certain topics and changing the class numbering.

The complete original script (and more: slides, exams, question sheets, and solutions) can be downloaded at MIT

Prof. Mayer thanks Jeremy Orloff and Jonathan Bloom not only for making the course material open to the public, but also for giving him access to the original LaTeX sources to tailor the material better to the DITs needs. Without their help, this course would by far not have the quality of its current state.