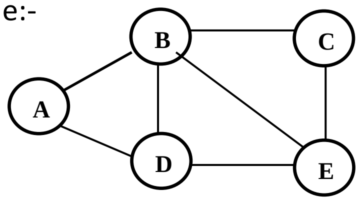
Graph Prerequisites

Graph is a nonlinear data structure, consists of a set of nodes and a set of links. A node is called **vertex**. A link is called the **edge** that connects two vertices. In Graph, relationship among the nodes is less restricted. It means a node can have multiple predecessors.

Definition: A Graph G= (V, E) consist of two sets : A set of V of vertices and set E of edges.

Example:-



Graph G=(V,E)
set V ={A,B,C,D,E}
set E = { (A,B), (A,D), (B,C), (B,D), (B,E), (C,E),(D,E) }

[Figure 1: Graph]

Graph Terminology

Directed graph and Un-directed graph:

A graph is called directed graph if each edge is identified by an ordered pair of vertices (vi,vj), such that the edge (vi,vj) connects vertex vi to vertex vj. A directed graph G also called digraph. The graph, in which no such ordered edge exists, is called undirected Graph. So the pair (vi,vj) is equal to (vj,vi).

Ex :-

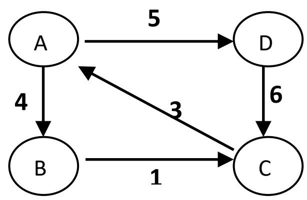
[Figure 2: Directed Graph and Undirected Graph]

Adjacent/Incident:

Let there is an edge(E_i) between two vertices V_i and V_j in graph then the vertices V_i and V_j are called adjacent and edge E_i called incident on vertices V_i and V_j.

Weighted Graph:

A weighted graph is a graph in which edges are assigned with some weights. Weight may be considered as cost or time to travel from one vertex to another.



[Figure 3: Weighted Graph]

Degree , Indegree , Outdegree:

The **indegree** of a vertex x is the number of edge with x as their end vertex.

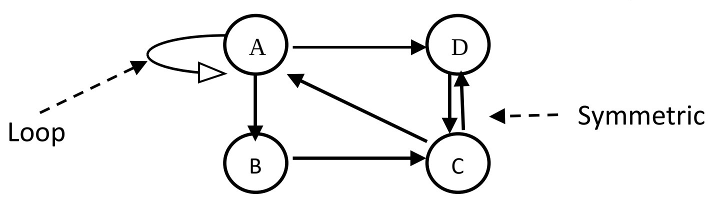
The **out degree** of vertex x is the number of edges whose start vertex is x .

The **degree o**f a vertex is the number of edges incident on this vertex. The degree of a vertex x is indegree of x + outdegree of x

Example : In figure 3, indegree(A)=1, outdegree(A)=2, degree(A)=3

Loop/self loop : An edge e is called a loop if its two end points are same. If an edge connected two vertices Vi and Vj where i=j, then the edge is called the Loop.

Symmetric: A directed graph is called symmetric if the existence of an edge from V_i to V_j implies that there must also be an edge directed from V_j to V_i.



[Figure 4: Graph having Loop and symmetric]

Parallel edge: If there is more than one edge exist between same pair of vertices, then they are called parallel edge.

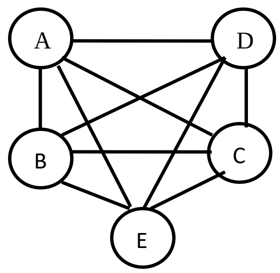
Path: Path is a sequence of vertices in graph in which each vertex is adjacent to next vertex. The length of such path is the number of edges in that path.

Cycle: A path in which the start vertex and end vertex are the same called cycle. A cycle consist of minimum of 3 vertices.

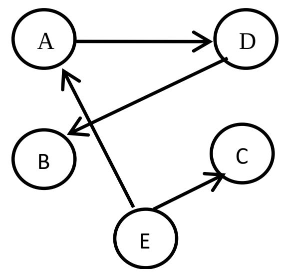
Directed Acyclic graph (DAG): A directed graph having no cycle is called DAG or Tree

Tree: It can be defined as a connected graph without any cycle.

Complete Graph: A graph is said to be complete if each and every pair of vertices are adjacent. For a complete graph with N number of vertices, number of edge= (N× (N-1))/2



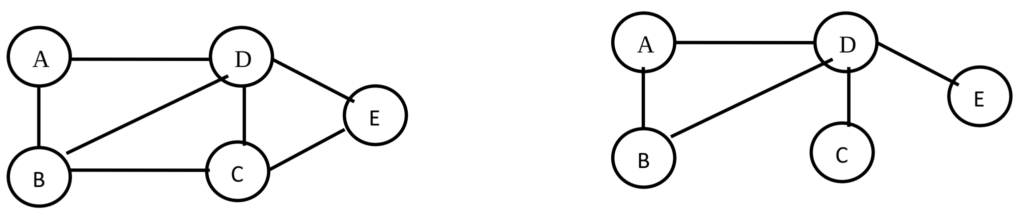
[Figure 5: Complete Graph]



[Figure 6: DAG]

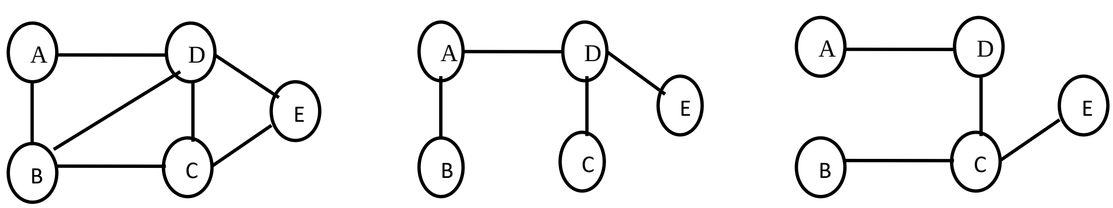
Sub Graph

A graph $G' = (V', E')$ is called sub graph of G= (V , E) if $V' \in$ is V and $E' \in$ E. For G' to be a sub graph of G, all the edges and vertices of G' should be in G.



[Figure 7: The Graph G and its sub graph]

Spanning tree: A sub graph of a graph (G) which is a tree containing all the vertices of G is called spanning tree. The spanning tree, where the sum of weight of all the edges is minimum, it is called minimal cost of spanning tree.



[Figure 8: The Graph G and its two Spanning Trees]

Representation of Graph

A. Sequential Representation : 1. Adjacency Matrix
2. Incidence Matrix
B. Linked List Representation : 3. Adjacency List

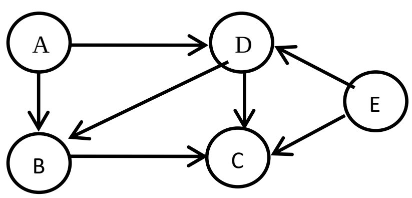
1. Adjacency Matrix:

Suppose G is a simple directed graph with N vertices and the vertices have been ordered as v1, v2,…, vn. Then the adjacency matrix A is an N X N matrix, where each element A_{ij} is defined as follows:

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge } (v_i,v_j) \text{ from vertex } v_i \text{ to vertex } v_j \\ 0 & \text{otherwise} \end{cases}$$

If the graph is undirected, then the adjacency matrix is symmetric, i,e Aij = Aji

Example:



A=

	A	B	C	D	E
A	0	1	0	1	0
B	0	0	1	0	0
C	0	0	0	0	0
D	0	1	1	0	0
E	0	0	1	1	0

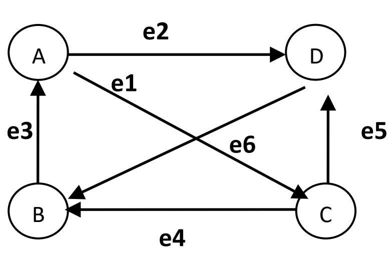
[Figure 9: The Directed graph and its adjacency matrix A]

Incidence matrix:

Incidence matrix (I) for a Graph G=(V,E) is a |V| x |E| matrix whose elements are 0,1 or -1. An edge Ek between V_i to V_j means

$I[V_i,E_k]=1$, $I[V_j,E_k]=0$ and rest of the cells in E_k column are set to 0.

Example:



	e1	e2	e3	e4	e5	e6
A	1	1	-1	0	0	0
B	0	0	1	-1	0	-1
C	-1	0	0	1	1	0
D	0	-1	0	0	-1	1

[Figure 10: The directed Graph and its incidence Matrix]

Path Matrix/ Reachability Matrix:

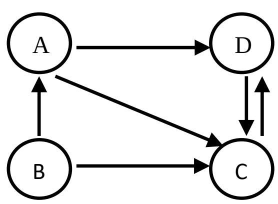
Let G be a simple directed graph with M nodes, v1,v2,…,vm. The path matrix or reachability matrix of G is the M X M matrix P where Pij defined as follows:

$$
\mathbf{P_{ij}} = \begin{cases} \mathbf{1} & \textbf{if there is a path from } \mathbf{v_i} \textbf{ to } \mathbf{v_j} \\ \mathbf{0} & \textbf{Otherwise} \end{cases}
$$

Path Matrix is computed as follows:

Let A be the adjacent matrix of a graph, then A_{ij}^{k} that is, the entry in ith row and jth column of Matrix A_k gives the number of path of path length k from V_i to V_j.

Example: For following Graph We need to compute Path matrix



	A	B	C	D
A	0	0	1	1
B	1	0	1	0
C	0	0	0	1
D	0	0	1	0

A: Path Matrix of path length 1
(Same as the adjacency Matrix A)

	A	B	C	D
A	0	0	1	1
B	0	0	1	2
C	0	0	1	0
D	0	0	0	1

A^2: Path Matrix of path length 2

	A	B	C	D
A	0	0	1	1
B	0	0	2	1
C	0	0	0	1
D	0	0	1	0

A^3: Path Matrix of path length 3

	A	B	C	D
A	0	0	1	1
B	0	0	1	2
C	0	0	1	0
D	0	0	0	1

A^4: Path Matrix of path length 4

Let the Matrix Bk = A + A^2 +A^3 + A^4 , then $B^k[i,j]$ gives number of path of length k or less from node Vi to Vj.

Then The Path Matrix can be defined as

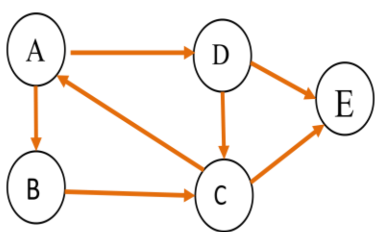
$$P_{ij} = \begin{cases} 1, & \text{if } B^k[i,j] >= 1 \\ 0, & \text{if } B^k[i,j] = 0 \end{cases}$$

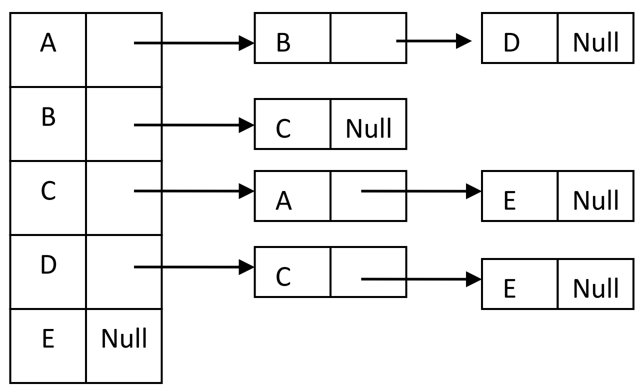
P=

	A	B	C	D
A	0	0	1	1
B	0	0	1	1
C	0	0	1	0
D	0	0	0	1

3. Adjacency List

Adjacency List consists of |V| number of linked list, one list for each vertex. For each vertex V, the adjacency list Adj[U] contains all vertices V, such that there is an edge (u,v) ∈ Set E





For a directed graph, number of nodes = |V| + |E|
For a Undirected graph, number of nodes = |V| + 2 * |E|