MATH1231_Assignment

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Assignment

MATH1231/1241 ASSIGNMENT
2020 T1

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1. Linear Maps

The function $T:\mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-5x_2\\3x_1+2x_2\\-4x_1\end{pmatrix} \quad \text{for all} \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2.$$

Show that T is linear.

By the definition of a linear map:

Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T\colon V \to W$ is called a ***linear map*** *or* ***linear transformation*** *if the following two conditions are satisfied.*

Addition Condition. $T(\mathbf{v_1}+\mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2})$ for all $\mathbf{v_1}, \mathbf{v_2} \in V$

Scalar Multiplication Condition. $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{F}$ and $\mathbf{v} \in$ V

The function defined above has a domain of $\mathbb{R}^2$ and a co-domain of $\mathbb{R}^3$ which are both known vector spaces. Now we will check that the addition (I) and scalar multiplication (II) conditions are preserved.

(I) Addition

$$\text{Let } \mathbf{v_1} = \begin{pmatrix}x_1\\x_2\end{pmatrix}, \mathbf{v_2} = \begin{pmatrix}y_1\\y_2\end{pmatrix}, \ \mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2,$$

$$T(\mathbf{v_1}+\mathbf{v_2}) = T\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}$$

$$= \begin{pmatrix}-5(x_2+y_2)\\3(x_1+y_1)+2(x_2+y_2)\\-4(x_1+y_1)\end{pmatrix}$$

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$$= \begin{pmatrix}-5x_2-5y_2\\3x_1+3y_1+2x_1+2y_2\\-4x_1-4y_1\end{pmatrix}$$

$$= \begin{pmatrix}-5x_2-5y_2\\3x_1+2x_2+3y_1+2y_2\\-4x_1-4y_1\end{pmatrix}.$$

Similarly,

$$T(\mathbf{v_1})+T(\mathbf{v_2}) = T\begin{pmatrix}x_1\\x_2\end{pmatrix} + T\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$$= \begin{pmatrix}-5(x_2)\\3(x_1)+2(x_2)\\-4x_1\end{pmatrix} + \begin{pmatrix}-5(y_2)\\3(y_1)+2(y_2)\\-4y_1\end{pmatrix}$$

$$= \begin{pmatrix}-5x_2-5y_2\\3x_1+2x_2+3y_1+2y_2\\-4x_1-4y_1\end{pmatrix}$$

$$= T(\mathbf{v_1}+\mathbf{v_2}).$$

Hence $T(\mathbf{v_1})+T(\mathbf{v_2}) = T(\mathbf{v_1}+\mathbf{v_2})$, thus satisfying the addition condition.

(II) Scalar Multiplication

$$\text{Let } \mathbf{v_1} = \begin{pmatrix}x_1\\y_1\end{pmatrix}, \ \mathbf{v_1} \in \mathbb{R}^2 \text{ and } \lambda \in \mathbb{R}$$

$$T(\lambda\mathbf{v_1}) = T\begin{pmatrix}\lambda x_1\\\lambda y_1\end{pmatrix}$$

$$= \begin{pmatrix}-5(\lambda y_1)\\3(\lambda x_1)+2(\lambda y_1)\\-4(\lambda x_1)\end{pmatrix}$$

$$= \begin{pmatrix}-5\lambda y_1\\3\lambda x_1+2\lambda y_1\\-4\lambda x_1\end{pmatrix}.$$

Furthermore,

$$\lambda T(\mathbf{v_1}) = \lambda T\begin{pmatrix}x_1\\y_1\end{pmatrix}$$

$$= \begin{pmatrix}-5(\lambda y_1)\\3(\lambda x_1)+2(\lambda y_1)\\-4(\lambda x_1)\end{pmatrix}$$

$$= \lambda\begin{pmatrix}-5y_1\\3x_1+2y_1\\-4x_1\end{pmatrix}$$

$$= \begin{pmatrix}-5\lambda y_1\\3\lambda x_1+2\lambda y_1\\-4\lambda x_1\end{pmatrix}$$

$$= T(\lambda\mathbf{v_1}).$$

Thus $T(\lambda\mathbf{v_1}) = \lambda T(\mathbf{v_1})$, therefore satisfying the scalar multiplication condition.

Since the domain and co-domain of T are both known vector spaces, and preserves addition and scalar multiplication, T is a linear map.

2. Prove a hyperplane is a subspace

Show that

$$S = \{x \in \mathbb{R}^3 : 5x_1 - 7x_2 - 4x_3 = 0\}$$

is a subspace.

Let this condition be denoted as $f(x)$, such that $f(x) = 5x_1 - 7x_2 - 4x_3$.

The Subspace Theorem states:

A subset $\mathbb{S}$ of a vector space $\mathbb{V}$ over a field $\mathbb{F}$, under the same rules for addition and multiplication by scalars, is a subspace of $\mathbb{V}$ if and only if

I) *The vector **0** in $\mathbb{V}$ also belongs to $\mathbb{S}$*

II) *$\mathbb{S}$ is closed under vector addition*

III) *$\mathbb{S}$ is closed under multiplication by scalars from $\mathbb{F}$*

We are given $\mathbb{S}$, a set of vectors of the known vector space $\mathbb{V} \in \mathbb{R}^3$. We will prove that $\mathbb{S}$ is a subspace of $\mathbb{V}$.

I) The vector 0 in $\mathbb{V}$ also belongs to $\mathbb{S}$.

The zero vector in $\mathbb{R}^3$ is given by:

$$\mathbf{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Also,

$$f(0) = 5(0) - 7(0) - 4(0)$$
$$= \mathbf{0}.$$

Hence, the zero vector **0** in $\mathbb{V}(\mathbb{R}^3)$ also belongs to $\mathbb{S}$.

II) **$\mathbb{S}$ is closed under vector addition.**

For S to be closed under addition, the sum of **u** and **v** must be of the form **u**+**v** = 0.

$$\text{Let } \mathbf{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}, \ \mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}, \ \mathbf{u}, \mathbf{v} \in \mathbb{S},$$

Where,

$$5(u_1) - 7(u_2) - 4(u_3) = 0$$

and

$$5(v_1) - 7(v_2) - 4(v_3) = 0.$$

Now,

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}.$$

Furthermore,

$$f(\mathbf{u}+\mathbf{v}) = 5(u_1+v_1) - 7(u_2+v_2) - 4(u_3+v_3)$$
$$= 5u_1 + 5v_1 - 7u_2 - 7v_2 - 4u_3 - 4v_3$$
$$= 5u_1 - 7u_2 - 4u_3 + 5v_1 - 7v_2 - 4v_3$$
$$= (5u_1 - 7u_2 - 4u_3) + (5v_1 - 7v_2 - 4v_3)$$
$$= 0 + 0 = 0.$$

Thus $\mathbf{u}+\mathbf{v} \in \mathbb{S}$, therefore $\mathbb{S}$ is closed under vector addition.

III) **$\mathbb{S}$ is closed under multiplication by scalars from $\mathbb{F}$**

Let $\gamma \in \mathbb{F}$ where $\mathbb{F}$ represents all scalars.

We have,

$$\gamma\mathbf{v} = \begin{pmatrix}\gamma v_1\\\gamma v_2\\\gamma v_3\end{pmatrix}.$$

Such that,

$$f(\gamma\mathbf{v}) = 5(\gamma v_1) - 7(\gamma v_2) - 4(\gamma v_3)$$

$$= \gamma(5v_1 - 7v_2 - 4v_3)$$

$$= \gamma\mathbf{0}$$

$$= \mathbf{0}.$$

Thus, $\gamma\mathbf{0} \in \mathbb{S}$, therefore $\mathbb{S}$ is closed under scalar multiplication.

Due to the zero vector in $\mathbb{V}$ also belonging to $\mathbb{S}$ as well as $\mathbb{S}$ being closed under vector addition and under scalar multiplication, $\mathbb{S}$ is therefore a subspace.

3. Prove a set of polynomials is a subspace

Let $\mathbb{P}_n$ be the set of real polynomials of degree at most n.

Show that

$$\mathbb{S} = \{p \in \mathbb{P}_4 : x^2 - x - 3, \text{ is a factor of } p(x)\}$$

is a subspace of $\mathbb{P}_4$.

Since $\mathbb{S}$ is a subset of a known vector space, $\mathbb{P}_4$, to prove that $\mathbb{S}$ is a subspace of $\mathbb{P}_4$, we must show that the conditions of the Subspace Theorem are satisfied.

I) *The vector **0** in $\mathbb{S}$*

First, we will need to prove the existence of the zero polynomial in $\mathbb{S}$.

Let $f(x) \in \mathbb{S}$; such that,

$$f(x) = (x^2 - x - 3)g(x) \text{ for some } g(x) \in \mathbb{P}_2.$$

Since $g(x) \in \mathbb{P}_2$, we can write the function $g(x)$ as,

$$g(x) = ax^2 - bx - c, \text{ where } a, b, c \in \mathbb{R}.$$

Now, we can express $f(x)$ as:

$$f(x) = (x^2 - x - 3)(ax^2 - bx - c).$$

The zero polynomial in $\mathbb{P}_2$:

$$p(\mathbf{0}) = 0x^2 - 0x - 0.$$

Since $g(x)$ is an arbitrary polynomial, then $g(x)$ is defined at the zero polynomial. Therefore, at the zero polynomial,

$$f(x) = (x^2 - x - 3)(0x^2 - x - 0) = 0.$$

Therefore, the zero polynomial is in $\mathbb{S}$.

II) *$\mathbb{S}$ is closed under vector addition*

Let $f(x), q(x) \in \mathbb{S}$, such that :

$$f(x) = (x^2 - x - 3)g(x) \text{ and } q(x) = (x^2 - x - 3)h(x), \text{ where } g(x), h(x) \in \mathbb{P}_2.$$

S will be closed under addition if $(f+q)(x) \in \mathbb{S}$.

By the distributive law $(f+q)(x) = f(x) + q(x)$. Therefore,

$$(f+q)(x) = (x^2 - x - 3)g(x) + (x^2 - x - 3)h(x).$$

$$(f+q)(x) = (x^2 - x - 3)(g(x) + h(x)); \text{ by the distributive law.}$$

Let, $g(x) = a_1x^2 - b_1x - c_1$ and $h(x) = a_2x^2 - b_2x - c_2$ where, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.

Now, rewriting $(f+q)(x)$;

$$(f+q)(x) = (x^2 - x - 3)(a_1x^2 - b_1x - c_1 + a_2x^2 - b_2x - c_2).$$

$$(f+q)(x) = (x^2 - x - 3)((a_1 + a_2)x^2 - (b_1 + b_2)x - (c_1 + c_2)), \text{ by the distributive law.}$$

Since $(g(x) + h(x)) \in \mathbb{P}_2$, and that $(x^2 - x - 3)$ is a factor of $(f+q)(x)$, it is clear that $(f+q)(x) \in \mathbb{S}$. Therefore, $\mathbb{S}$ is closed under addition.

III) *$\mathbb{S}$ is closed under multiplication by scalars.*

$\mathbb{S}$ will be closed under scalar multiplication if $\lambda f(x) \in \mathbb{S}$.

Let $f(x) \in \mathbb{S}$ and $\lambda \in \mathbb{R}$.

As above in I, take $f(x) = (x^2 - x - 3)g(x), g(x) \in \mathbb{P}_2$ and $g(x) = ax^2 - bx - c$, where $a, b, c \in \mathbb{R}$.

Now,

$$\lambda f(x) = \lambda(x^2 - x - 3)g(x),$$

$$\lambda f(x) = \lambda(x^2 - x - 3)(ax^2 - bx - c),$$

$$\lambda f(x) = (x^2 - x - 3)(\lambda ax^2 - \lambda bx - \lambda c).$$

Because a, b, c are arbitrary coefficients and λ is an arbitrary scalar, then $\lambda a, \lambda b, \lambda c$ are also arbitrary coefficients, where $g(x)$ is still an arbitrary function in $\mathbb{P}_2$. Since $(x^2 - x - 3)$ is a factor of $\lambda f(x)$, then $\lambda f(x) \in \mathbb{S}$. Therefore, $\mathbb{S}$ is closed under scalar multiplication.

By showing that the zero polynomial belongs to $\mathbb{S}$, and proving that $\mathbb{S}$ is closed under both addition and scalar multiplication, we can say that $\mathbb{S}$, by the Subspace Theorem, is a subspace of $\mathbb{P}_4$.

4. Modelling with a first order ODE

The air in a 79 cubic metre kitchen is initially clean, but when David burns his toast while making breakfast, smoke is mixed with the room's air at a rate of 0.07 mg per second. An air conditioning system exchanges the mixture of air and smoke with clean air at a rate of 7 cubic metres per minute. Assume that the pollutants are mixed uniformly throughout the room and that burnt toast is taken outside after 37 seconds. Let $S(t)$ be the amount of smoke in mg in the room at time t (in seconds) after the toast first began to burn.

(a) Find a differential equation obeyed by $S(t)$.

Let $S(t)$ = smoke S at time t. The rate of smoke saturating the room's air is given by the differential equation

$$\frac{dS(t)}{dt} = S_{in} - S_{out}.$$

Where S_{in} represents the smoke entering the room, and S_{out} represents the smoke exiting the room.

S_{in} is equal to the rate at which the smoke is mixing with the room's air, which is 0.07mg per second, this gives,

$$S_{in} = \frac{7}{100}.$$

S_{out} is equal to the rate at which the smoke is exiting the room. This is 7 cubic metres per minute, or, per 60 seconds. However, we must account for the total volume of the air in the room, which is 79 cubic metres, this gives,

$$S_{out} = \frac{7m^3}{60s} \cdot \frac{S(t)}{79m^3}.$$

Through simplification,

$$S_{out} = \frac{7}{4740}S(t).$$

Hence the differential equation obeyed by $S(t)$ becomes,

$$\frac{dS(t)}{dt} = \frac{7}{100} - \frac{7}{4740}S(t).$$

(b) Find S(t) for $0 \le t \le 37$ by solving the differential equation in (a) with an appropriate initial condition.

We have

$$\frac{dS(t)}{dt} = \frac{7}{100} - \frac{7}{4740}S(t).$$

This is a linear Ordinary Differential Equation (ODE), a method for solving this linear ODE is given below.

When we have the ODE in the format,

$$\frac{dy}{dx} + p(x)y = q(x).$$

We can find what is called an integrating factor "$R(x)$" such that,

$$R(x) = e^{\int p(x)dx}.$$

Note: We do not include a constant of integration in this step.

We can now begin solving this differential equation by writing it as,

$$y = \frac{1}{R(x)}\int R(x)q(x)dx.$$

Rearranging the differential equation found in part (a) we have,

$$\frac{dS(t)}{dt} + \frac{7}{4740} \cdot S(t) = \frac{7}{100}.$$

Such that,

$$p(t) = \frac{7}{4740} \text{ and } q(t) = \frac{7}{100}$$

Now, finding the integrating factor,

$$R(t) = e^{\int p(t)dt}$$

$$= e^{\int \frac{7}{4740}dt}$$

$$= e^{\frac{7}{4740}t}.$$

We now have,

$$s(t) = \frac{1}{e^{\frac{7}{4740}t}}\int e^{\frac{7}{4740}t}\frac{7}{100}dt$$

$$= e^{-\frac{-7}{4740}t}\Big(\frac{7}{100}\int e^{\frac{7}{4740}t}dt\Big)$$

$$= e^{-\frac{-7}{4740}t}\frac{7}{100}\Big(\frac{e^{\frac{7}{4740}t}}{\frac{7}{4740}} + C\Big)$$

$$= e^{-\frac{-7}{4740}t}\frac{7}{100}\Big(\frac{4740}{7}e^{\frac{7}{4740}t} + C\Big)$$

$$= e^{-\frac{-7}{4740}t}\Big(\frac{237}{5}e^{\frac{7}{4740}t} + C\Big)$$

$$s(t) = \frac{237}{5} + Ce^{\frac{-7}{4740}t}.$$

We now utilise the initial conditions such that when $t = 0$, $s(t) = 0$.

Now,

$$0 = \frac{237}{5} + Ce^{\frac{-7}{4740}(0)}.$$

Which simplifies to,

$$C = -\frac{237}{5}.$$

So the solution to our ODE for $0 \le t \le 37$ is given by,

$$s(t) = \frac{237}{5} - \frac{237}{5}e^{\frac{-7}{4740}t}.$$

(c) What is the level of pollution in mg per cubic meter after 37 seconds?

Substituting $t = 37$ into our equation

$$s(37) = \frac{237}{5} - \frac{237}{5}e^{\frac{-7}{4740}(37)}$$
$$s(37) = 2.520510854.$$

However this is the total saturation of smoke in the room, so the level of pollution is given by

$$pollution(mg/m^3) = \frac{2.520510854}{79}$$
$$= 0.03190520068.$$

Hence there is 0.03190520068 mg/m^3 in the room after 37 seconds.

(d) How long does it take for the level of pollution to fall to 0.004 mg per cubic metre after the toast is taken outside?

The toast has now been taken out of the room. This means $S_{in} = 0$, therefore the differential equation becomes

$$\frac{dS(t)}{dt} = \frac{-7}{4740}(S(t)).$$

This is a separable ODE, so rearranging this equation gives,

$$\frac{dS(t)}{S(t)} = \frac{-7}{4740}(dt).$$

Now through integrating both sides we have,

$$\int\frac{dS(t)}{S(t)} = \int\frac{-7}{4740}(dt).$$

Solving this we get,

$$\ln(S) = \frac{-7}{4740}(t) + C.$$

We also know that when $t = 0$, that is, zero seconds from when the toast was removed from the room, $S = 0.03190520068$. Using these new initial conditions we have,

$$\ln(0.03190520068) = \frac{-7}{4740}(0) + C$$

$$C = \ln(0.03190520068).$$

We now have ,

$$\ln(S) = \frac{-7}{4740}(t) + \ln(0.03190520068).$$

Now we need to find the time t such that $S(t) = 0.004mg/m^3$, so we have,

$$\frac{7}{4740}(t) = \ln(0.03190520068) - \ln(0.004)$$

$$t = \frac{4740}{7}\Big(\ln\Big(\frac{0.03190520068}{0.004}\Big)\Big)$$

$$t = 1406.069988 \text{ seconds}$$

Therefore, the time taken for the pollution level in the room to fall to $0.004mg/m^3$ is 1406 seconds after the toast has been removed from the room.