**Question 1)**

**Show that for all integers $n$:**

a) **If $d$ is an integer such that $d \mid n+5$ and $d \mid n^2+9$, then $d \mid 34$.**

<u>Proof:</u>

Let $d$ be an integer.

Suppose that $d \mid n+5$ and $d \mid n^2+9$.

Since $d \mid n+5$ is true, then we know that $d \mid (n+5)^2$ must also be true.

Therefore, we can add $d \mid (n+5)^2$ to $d \mid n^2+9$ to get $d \mid (n+5)^2 + n^2 + 9$.

By expanding the RHS gives:

$$d \mid 2n^2 + 10n + 34 .$$

Given $n$ is an integer so by definition, $n \in \mathbb{Z}$, therefore we know that $2n(n+5)$ is a multiple of $(n+5)$.

This means $d \mid 2n(n+5)$ must also be true.

Thus, by subtracting $d \mid 2n(n+5)$ from $d \mid 2n^2+10n+34$ gives:

$$d \mid 2n^2 + 10n + 34 - 2n(n+5)$$

$$d \mid 2n^2 + 10n + 34 - 2n^2 - 10n$$

$$d \mid 34.$$

Hence,

$$d \mid 34.$$

This completes the proof.

b) **If $n$ is a multiple of $34$ then $n+5$ and $n^2+9$ are coprime.**

<u>Proof:</u>

Let $n$ be a multiple of 34.

Let $d = \gcd(n^2+9, n+5)$.

First we can rewrite $n^2+9$ as a multiple of $(n+5)$ and $(n-5)$ meaning we can deduce:

$$\gcd\big((n-5)(n+5)+34, n+5\big).$$

Assume $x, y, z$ are non negative integers. Therefore we can let $x = n-5, y = n+5,\ z = 34$.

We now have:

$$\gcd(xy+z, y).$$

Thus, following the division algorithm, $\gcd(y, xy + z)$ can be expressed as

$$\gcd(a, y)$$

where $a$ is an integer such that $a = xy + z$.

Therefore, $\gcd(a, y)$ can also be expressed as:

$$\gcd(y, z).$$

Substitute $y = n + 5,\ z = 34$ in to get:

$$\gcd(n + 5, 34).$$

We can then substitute $n$ with $34k$, where $k$ is a non negative integer to get:

$$\gcd(34k + 5, 34).$$

Now applying the property of division, we can derive:

$$\gcd(34, 5)$$

$$= 1$$

Hence when $n$ is a multiple of 34, we see that $n + 5$ and $d \mid n^2 + 9$ are coprime as $\gcd(n^2 + 9, n + 5) = 1$.

This completes the proof.

**Question 2)**

**A relation $\preceq$ is defined on $\mathbb{Z}^2$ by**

$(\boldsymbol{x_1}, \boldsymbol{x_2}) \preceq (\boldsymbol{y_1}, \boldsymbol{y_2})$ **if and only if** $\boldsymbol{x_1} \geq \boldsymbol{y_1}$ **and** $\boldsymbol{x_1} + \boldsymbol{x_2} \leq \boldsymbol{y_1} + \boldsymbol{y_2}$**.**

**Prove that $\preceq$ is a partial order.**

Proof:

In order for a relation to be a partial order, by definition it must be reflexive, antisymmetric and transitive.

First we prove reflexivity:

Given that $x_1 \geq y_1$ and $x_1 + x_2 \leq y_1 + y_2$, this means $x_2 \leq y_2$. Therefore, as we know $(x_1, x_2), (x_1, x_2)\ ) \in \mathbb{Z}^2$, hence it is reflexive.

To prove antisymmetric:

Given that $x_1 \geq y_1$ and $x_1 + x_2 \leq y_1 + y_2$, then $x_2 \leq y_2$. In this case, by definition, $(x_1, x_2) \preceq (y_1, y_2)$. Thus we can see that $x_1 = y_1$ and $x_2 = y_2$ and proves it is antisymmetric.

To show that it is transitive:

Suppose we have $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{Z}^2$ where $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (z_1, z_2)$.

Then we know that $x_1 + x_2 \leq y_1 + y_2$ and $y_1 + y_2 \leq z_1 + z_2$.

This also means $x_1 \geq y_1$ and $y_1 \geq z_1$ as $x_2 \leq y_2$ and $y_2 \leq z_2$.

We can then deduce that $x_1 \geq z_1$ and $x_2 \leq z_{2.}$

Therefore we can also see that $x_1 + x_2 \leq z_1 + z_2$ proving that $(x_1, x_2) \preccurlyeq (z_1, z_2)$ and showing it is transitive.

This completes the proof.

**Question 3)**

**Prove that $7 \mid 13^{4n} + 8^{3n+3} + 5$ for all $n \in \mathbb{N}$.**

Let $f(n) = 13^{4n} + 8^{3n+3} + 5$.

When $n = 0$,

$$f(0) = 13^{4\times 0} + 8^{3\times 0+3} + 5$$
$$= 1 + 512 + 5$$
$$= 518$$
$$= 74 \times 7$$

Therefore we can see that when $f(n)$ when $n = 0$ is a multiple of 7.

Now we test for values of $n$ up to some $k$ where $k > 0$ and $k \in \mathbb{N}$.

First, let $n = k + 1$.

Now the function $f(k + 1)$ becomes

$$f(k+1) = 13^{4(k+1)} + 8^{3(k+1)+3} + 5.$$

Expanding the LHS, we get

$$f(k+1) = 13^{4k+4} + 8^{3k+6} + 5$$
$$f(k+1) = 13^{4k} \times 13^4 + 8^{3k+3} \times 8^3 + 5.$$

From this we can derive

$$f(k+1) = 13^{4k} + 8^{3k+3} + 5 + 13^{4k} \times 28561 + 8^{3k+3} \times 512$$
$$f(k+1) = f(k) + 13^{4k} \times 28561 + 8^{3k+3} \times 512.$$

Therefore we have

$$7 \mid f(k), 8^{3k+3} \times 512$$

and

$$7 \mid 13^{4k} \times 28561$$

meaning $f(k+1)$ ia a multiple of 7.

Theus, we see that as it holds for $n = k + 1$ we can therefore assume this holds for all $n \in \mathbb{N}$.