MATH 1081 – Discrete Mathematics

Assignment 1 (draft)

Q1. Consider the following sets:

$$\boldsymbol{A = \{60n - 31 \mid n \in \mathbb{Z}\}}$$

$$\boldsymbol{B = \{12n + 5 \mid n \in \mathbb{Z}\}}$$

$$\boldsymbol{C = \{10n - 1 \mid n \in \mathbb{Z}\}}$$

(a) Show that $\boldsymbol{A}$ is a proper subset of $\boldsymbol{B}$.
(b) Show that $\boldsymbol{A}$ is a proper subset of $\boldsymbol{C}$.
(c) Show that there is no containment relation between $\boldsymbol{B}$ and $\boldsymbol{C}$.

(a) We begin by assuming that $x \in B$. This means that we can write

$$x = 12n + 5, \text{ for some } n \in \mathbb{Z}.$$

We will prove that $x \in A$, that is, we will show that there exists some integer $k \in \mathbb{Z}$, such that $x = 60k - 31$.

$$12n + 5 = 12(5k - 3) + 5, \text{ where } n = 5k - 3$$

$$= 60k - 36 + 5$$

$$= 60k - 31.$$

Hence, we have proven that for every element $x \in B$, x can be represented as $60k - 31$, and thus $x \in A$. This means that $A \subseteq B$.

For A to be a proper subset of B, $A \subseteq B$, but $B \nsubseteq A$, or $A \subset B$. There must be some $x \in B$, such that $x \notin A$.

We know $5 \in B$ as $12(0) + 5 = 5$, and $0 \in \mathbb{Z}$, but for $5 \in A$,

$$5 = 60n - 31,$$

$$60n = 36,$$

$$n = \frac{36}{60} = \frac{3}{5} \notin \mathbb{Z},$$

thus $5 \notin A$.

Hence $A \subset B$.

(b) We begin by assuming that $x \in C$. This means that we can write

$$x = 10n - 1, \text{ for some } n \in \mathbb{Z}.$$

We will prove that $x \in A$, that is, we will show that there exists some integer $k \in \mathbb{Z}$, such that $x = 60k - 31$.

$$10n - 1 = 10(6k - 3) - 1, \text{ where } n = 6k - 3$$

$$= 60k - 30 - 1$$

$$= 60k - 31.$$

Hence, we have proven that for every element $x \epsilon C$, x can be represented as $60k - 31$, and thus $x \in A$. This means that $A \subseteq C$.

For A to be a proper subset of C, $A \subseteq C$, but $C \nsubseteq A$, or $A \subset C$. There must be some $x \in C$, such that $x \notin A$.

We know $-1 \in C$ as $10(0) - 1 = -1$, and $0 \in \mathbb{Z}$, but for $-1 \in A$,

$$-1 = 60n - 31,$$

$$60n = 30,$$

$$n = \frac{30}{60} = \frac{1}{2} \notin \mathbb{Z},$$

thus, $-1 \notin A$.

Hence $A \subset C$.

(c) Containment relations are $\subseteq, \supseteq, =$.

$B \subseteq C$:

For element x to be in B, $x = 12n + 5$, $n \in \mathbb{Z}$ and for x to be in A, $x = 10k + 1$, $k \in \mathbb{Z}$, but $12n + 5$ can only be expressed as $10k + 1$ if $n = \frac{10k-1}{12}$, such that

$$12\left(\frac{10k-1}{12}\right) = 10k - 1,$$

but $n = \frac{10k-1}{12} \notin \mathbb{Z}$, when $k \in \mathbb{Z}$.

Thus $B \nsubseteq C$.

$B \supseteq C$ or $C \subseteq B$:

For element x to be in C, $x = 10n - 1$, $n \epsilon \mathbb{Z}$ and for x to be in B, $x = 12k + 5$, $k \in \mathbb{Z}$, but $10n - 1$ can only be expressed as $12k + 5$ if $n = \frac{12k+5}{10}$, such that

$$10\left(\frac{12k+5}{10}\right) = 12k + 5,$$

but $n = \frac{12k+5}{10} \notin \mathbb{Z}$, when $k \in \mathbb{Z}$.

Thus $B \nsupseteq C$ or $C \nsubseteq B$.

$B = C$:

Since we have proven that $B \nsubseteq C$ and $C \nsubseteq B$, $B \neq C$.

Thus, since $B \nsubseteq C$, $C \nsubseteq B$, and $B \neq C$, there is no containment relation between the sets B and C.

Q2. A relation $\preccurlyeq$ is defined on $\mathbb{R}$ by

$\boldsymbol{x \preccurlyeq y}$ **if and only if** $\boldsymbol{x = 7^k y}$ **for some non-negative integer** $\boldsymbol{k}$**.**

Prove that $\preccurlyeq$ is a partial order.

We know from the definition of a partial order that for a relation to be a partial order, the relation must be reflexive, antisymmetric and transitive. We will prove these properties for the relation $\preccurlyeq$.

Reflexive:

We can see that

$$x = 7^0 x,$$

and since 0 is a non-negative number, $x \preccurlyeq x$.

Thus, $(x, x) \in \preccurlyeq$ or relation $\preccurlyeq$ is reflexive.

Antisymmetric:

We know that if $x \preccurlyeq y$,

$$x = 7^k y, \tag{1}$$

and if $y \preccurlyeq x$,

$$y = 7^l x,$$

for non-negative integers k and l.

From (1),

$$y = 7^l 7^k y,$$

$$y = 7^{l+k} y,$$

thus, $7^{l+k} = 1$, so $l + k = 0$, but we know from the definition of the relation that l and k are non-negative, thus $l = k = 0$.

From (1), $x = 7^0 y$, and so $x = y$.

Hence, we have proven that if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$, so the relation $\preccurlyeq$ is said to be antisymmetric.

Transitive:

We know that if $x \preccurlyeq y$,

$$x = 7^k y, \tag{2}$$

and if $y \preccurlyeq z$,

$$y = 7^l z,$$

for non-negative integers k and l.

From (2),

$$x = 7^k 7^l z,$$

$$x = 7^{k+l} z,$$

and since k and l are non-negative integers, $k + l$ is also a non-negative integer, thus we can say $x \preccurlyeq z$ or that if $x \preccurlyeq y$ and $y \preccurlyeq z$, we have proven that $x \preccurlyeq z$. Hence the relation $\preccurlyeq$ is transitive.

Since relation $\preccurlyeq$ is reflexive, antisymmetric, and transitive, $\preccurlyeq$ is a partial order.

Q3. Prove $\sqrt[3]{\mathbf{56}}$ is irrational.

Let us assume that $\sqrt[3]{56}$ is rational.

The definition of a rational number implies that $\sqrt[3]{56}$ can be expressed as a ratio of integers p and q, where $q \neq 0$, and $GCD(p,q) = 1$, or

$$\sqrt[3]{56} = \frac{p}{q}, \quad p, q \in \mathbb{Z}, \;\; q \neq 0,$$

$$56 = \frac{p^3}{q^3},$$

$$56q^3 = p^3. \tag{1}$$

Since 56 is a multiple of 7, p^3, and therefore p must also be a multiple of 7, or $7 \mid p,$ and thus p may be written as $7r$, where r is an arbitrary integer.

(1) now becomes

$$56q^3 = 7^3 r^3,$$

$$8q^3 = 7^2 r^3,$$

This implies that $7 \mid 8q^3$, and since $7 \nmid 8,\; 7 \mid q^3$, and therefore, $7 \mid q$. Thus, we know that $7 \mid p$ and $7 \mid q$. This contradicts our initial assumption that the $GCD(p,q) = 1$ as 7 is a common divisor of p and q and $7 > 1$. Since the $GCD(p,q) > 1$, $\sqrt[3]{56}$ cannot be expressed as a rational number.

Hence, $\sqrt[3]{56}$ must be irrational.