MATH1231 Mathematics 1B

and

MATH1241 Higher Mathematics 1B

PAST EXAM PAPERS

AND

SOLUTIONS

Contents

The 2016 to 2019 exam papers will be provided on Moodle.

PAST EXAM PAPERS

The four in questions in each paper were of equal value. Until 2013, Question 1 was a common question between MATH1231 and MATH1241. Since 2013 Questions 1 and Question 2 are in common between MATH1231 and MATH1241.

The final results for MATH1231 and MATH1241 will still be moderated so that similar performance in the common tasks will, on average, lead to similar results for students obtaining a pass or credit.

The emphasis on topics in the syllabus can change a little from year to year.

Important Note: The algebra syllabus has changed for 2012 onwards. The changes are minimal and effect Chapter 9 Statistics and Probability. New sections on continuous random variables and the Normal Distribution have been added. For MATH1241 there is also a new section on the Exponential Distribution, while joint discrete distributions has been removed.

The solutions to the examination papers contained here have been written by many members of staff of the School of Mathematics.

While we aim to provide perfectly correct solutions, we cannot guarantee that the solutions are error-free. Please report any serious errors to the Director of First Year Mathematics.

The format of this terms exam may be different. Details of this terms exam will be provided on Moodle.

MATH1231 NOVEMBER 2010

1. i) Show that the set

$$S = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 x_2 \geq 0 \right\}$$

is not a vector subspace of the vector space $\mathbb{R}^2$.

ii) The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 4 \\ -21 \\ 25 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

with the set D given by $D = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and the matrix A defined in the Maple session below.

You may use the following Maple session to assist in answering the questions below.

```
> with(LinearAlgebra):
> A:= <<1,3,-2>|<2,-5,7>|<4,-21,25>>;
```

$$A := \begin{bmatrix} 1 & 2 & 4 \\ 3 & -5 & -21 \\ -2 & 7 & 25 \end{bmatrix}$$

```
> M:=<A|<a,b,c>>;
```

$$M := \begin{bmatrix} 1 & 2 & 4 & a \\ 3 & -5 & -21 & b \\ -2 & 7 & 25 & c \end{bmatrix}$$

```
> GaussianElimination(M);
```

$$\begin{bmatrix} 1 & 2 & 4 & a \\ 0 & -11 & -33 & b-3a \\ 0 & 0 & 0 & b+c-a \end{bmatrix}$$

a) Is D a linearly independent set? Give reasons.

b) State the condition(s) for the vector $\mathbf{w}$ to belong to span(D).

c) Determine all possible real scalars α_1, α_2 and α_3 such that

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3.$$

d) Find a basis for the kernel of A.

e) State the rank of the matrix A.

f) Find a basis for the image of A.

iii) Find each of the following integrals:

a) $I_1 = \displaystyle\int \cosh^3 x \, dx$;

b) $I_2 = \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}} \, dx$.

iv) Find the general solution to the differential equation

$$y'' - 5y' + 4y = 6e^{2x} + 12x.$$

2. i) For $n = 0, 1, 2, \ldots$, let $I_n = \int_0^{\pi/2} \sin^n x\, dx$. You are given that

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{for } n \geq 2.$$

Use this recurrence to calculate I_6.

ii) Determine whether each of the following **sequences** converges or diverges, giving reasons for your answers. Find the limit of each sequence that converges.

a) $\left\{ \dfrac{\ln n}{\sqrt{n}} \right\} \quad n = 1, 2, 3, \ldots$

b) $\left\{ (-1)^n \left(1 - \dfrac{1}{n} \right) \right\} \quad n = 1, 2, 3, \ldots$

iii) By using an appropriate test, determine whether each of the following series converges or diverges:

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$;

b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2+1}$;

c) $\displaystyle\sum_{k=1}^{\infty} \frac{e^k}{k!}$.

iv) Let $A = \begin{pmatrix} 5 & -8 \\ 1 & -1 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.

b) Write down matrices P and D such that $A = PDP^{-1}$.

c) Hence evaluate $A^8 P$.

v) Suppose that the fixed vector $\mathbf{b}$ in $\mathbb{R}^3$ is given by

$$\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

You are given that the function $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{b} \quad \text{for } \mathbf{x} \in \mathbb{R}^3$$

is a linear map.
Find a matrix B which transforms the vector $\mathbf{x}$ into $T(\mathbf{x})$.

3. i) Team A has played all its matches in the home and away season and its qualification for the finals depends on the results of three remaining matches. Teams B, C and D each have one match against three other teams and team A will make the finals if team B loses its last match and either of teams C or D also lose their last match. Sports analysts have estimated that:

- team B will lose with probability 0.8;
- team C will lose with probability 0.3 and
- team D will lose with probability 0.6.

The result of any of these matches is independent of the other teams' results and drawn matches are not possible. Based on these data, calculate the probability that team A will make the finals. Show all of your working.

ii) Let X be a discrete random variable with the following probability distribution:

k:	0	1	2	3	4
$P(X=k)$:	0.2	0.3	0.1	0.2	0.2

a) Determine $E(X)$ and $\mathrm{Var}[(]X)$.

b) A random variable Y is defined by $Y = -2X + 3$. Determine $E(Y)$ and $\mathrm{Var}[(]Y)$.

iii) In 1995 half of the women in Australia were taller that 164cm, and half were shorter. In 2010, clothing manufacturers need to know whether women are in general taller than in 1995, and so fifteen women were chosen at random and their heights recorded:

162	165	168	171	170	166	165	162
151	166	172	178	168	166	167	

a) Assume that in 2010 Australian women were equally likely to be taller or shorter that 164cm. What distribution could be used to model the number of women in a random sample of 15 who were taller than 164cm?

b) Using the assumption made in part (a), determine a tail probability that measures how unusual it would be to observe as many women taller than 164cm as was observed in this sample.

c) From the results of part b), is there evidence that in 2010 more than half of women in Australia are taller than 164cm?

iv) A matrix A has eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$, respectively.

Determine $A\mathbf{b}$ for the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. i) Solve the initial value problem

$$(\cos^2 x)\frac{dy}{dx} + y = 1 \quad \text{where } y(0) = 3.$$

ii) The point P with coordinates $(2, 1, 7)$ lies on the surface S given by $z = x^2 + xy^2 + y^4$.

a) Find the Cartesian form of the equation of the plane tangent to the surface S at the point P.

b) Write down a vector normal to the surface S at the point P.

iii) Suppose that $f(x) = \dfrac{1}{x}$. Find the Taylor polynomial of degree 4 for f about $x = 1$.

iv) The position $(x(t), y(t))$ of a particle P at time t is given by

$$x(t) = t^2, \quad y(t) = t^3 \quad \text{where } t \geq 0.$$

a) Find the speed, v, of the particle as a function of t.

b) Hence, or otherwise, find the length of the path travelled by the particle between times $t = 0$ and $t = 1$.

v) A tank has a total volume of 200 litres. Initially it holds 40 litres of pure water. Brine containing 2 grams of salt per litre is run into the tank at the rate of 3 litres per minute and the mixture is stirred continuously so that the concentration of the dissolved salt is uniform throughout the tank. At the same time as the brine starts to flow into the tank the mixture is removed from the tank at the rate of 1 litre per minute.
Let t denote time, measured in minutes, from when the brine started to enter the tank and let $x(t)$ denote the mass of salt, in grams, present in the tank after t minutes.

Set up a first order differential equation in x and t which models this system up until the time the tank is full.

MATH1231 NOVEMBER 2011

1. i) Consider a set of vectors
$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subset \mathbb{R}^3$$

   a) Can S be a spanning set for $\mathbb{R}^3$? Give a reason for your answer.

   b) Will all such sets S be spanning sets? Give a reason for your answer.

   c) Suppose S consists of the vectors
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

   Determine a subset of S that forms a basis for $\mathbb{R}^3$.

ii) Suppose A is a fixed matrix in $\mathbb{M}_{m,n}$. Apply the subspace theorem to show that
$$S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$
is a subspace of $\mathbb{R}^n$.

iii) A linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has rank k. State the value of the nullity of T.

iv) The following MAPLE session may assist you with (b) below.

```
>  f := (x^2-8*x-11)/((x+3)*(x^2+2)):

> convert(f, parfrac,x);
```

$$\frac{-x-5}{x^2+2} + 2\ (x+3)^{-1}$$

Evaluate each integral:

   a) $I_1 = \displaystyle\int \sin^3\theta\ \cos^5\theta\, d\theta$

   b) $I_2 = \displaystyle\int \frac{x^2-8x-11}{(x+3)(x^2+2)}\, dx$

v) a) Determine whether the sequence $a_n = \dfrac{2n+\ln n}{1+3n}$ converges or diverges as $n \to \infty$. If it converges, find its limit.

   b) Does the series
$$\sum_{n=2}^{\infty} (-1)^n\ \frac{2n+\ln n}{1+3n}$$
   converge? Give a reason for your answer.

vi) By using an appropriate test, determine whether the series
$$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$
converges or diverges.

2. i) a) Find the general solution of $y'' + 4y = \cos x$.

b) If the right-hand side is changed to $\cos 2x$, write down the *form* of the particular solution you would try. (Do not attempt to find the unknown coefficient(s).)

ii) In a direct-current circuit, the total resistance z, (measured in ohms), produced by two parallel resistors with resistances x and y ohms, is given by

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}.$$

The values of x and y are measured to be 6 ohms and 12 ohms respectively and each of the measurements is made with an error whose absolute value is at most 0.1 ohms.

a) Use the given measurements (ignoring measurement error) to calculate the total resistance z.

b) Show that $\dfrac{\partial z}{\partial x} = \dfrac{z^2}{x^2}$ and $\dfrac{\partial z}{\partial y} = \dfrac{z^2}{y^2}$.

c) Use the total differential approximation to estimate the maximum absolute error in the calculated value of the total resistance z, correct to three significant figures.

iii) Show that the map $T : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$T(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ -x_2 \end{pmatrix}, \quad \text{for } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

is **not** linear.

iv) Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 2 & 5 \end{pmatrix}.$$

The mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x} - 2\mathbf{x}, \qquad \text{for } \mathbf{x} \in \mathbb{R}^3,$$

is linear. (You are not required to prove this.)

a) Find the matrix B for T, such that $T(\mathbf{x}) = B\mathbf{x}$.

b) Explain why $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ belongs to the kernel of T.

c) Write down the rank and nullity of T.

v) Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}.$$

a) Verify that A has eigenvalues 3 and -1 with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ respectively.

b) Hence determine the solution to the system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

3. i) Consider the matrix

$$A = \begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}.$$

a) Find $A\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

b) Find a basis for the kernel of A.

c) Find a basis for the image of A.

d) Given that $A^2 = A$, prove that if $\mathbf{b}$ belongs to the image of A, then $A\mathbf{b} = \mathbf{b}$.

e) Hence, (or otherwise), write down the eigenvalues and corresponding eigenvectors of A.

ii) It has been determined that 80% of supporters of the Liverpool Football Club wear red shirts on a day Liverpool plays a game. On those days 30% of non-supporters will wear a red shirt. If 5% of all people support Liverpool, what is the exact probability a person wearing a red shirt on a game day is a Liverpool supporter?

iii) Fred recorded the average maximum monthly temperatures at his home each month in 2009 and then again in 2010. These temperatures are shown in degrees centigrade in the following table:

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2009	26	31	27	24	26	19	21	15	23	28	31	30
2010	24	30	28	26	26	22	22	19	24	29	32	26

a) What distribution could be used to model the number of times there was an increase in maximum temperature rather than a decrease from one year to the next?

b) Calculate a tail probability that measures how unusual it would be to observe as many months with such an increase in temperature from 2009 to 2010 as was observed, if increases and decreases are equally likely.

iv) A fair coin is tossed four times and the **absolute value of the difference** between the number of heads and tails is recorded as a random variable X. (Thus, if $X = 4$, then either exactly 4 heads appeared, or exactly 4 tails appeared.) A probability distribution table is partially completed below.

x	0	2	4
$P(X = x)$		$\frac{1}{2}$	

a) Explain why $P(X = 4) = \frac{1}{8}$

b) Find $P(X = 0)$.

c) Find the expected value $E(X)$ of X.

d) What is the probability that X exceeds $E(X)$?

4. i) For $n \geq 1$, let $I_n = \displaystyle\int_0^1 \frac{dx}{(1+x^2)^n}$.

You are given that

$$I_{n+1} = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n} I_n, \quad \text{for } n \geq 1.$$

Use this reduction formula to calculate I_2.

ii) A ship of mass m starts from rest under the force of a propeller thrust of L Newtons. It experiences a resistance in Newtons given by kv, where v is the velocity of the ship in metres/sec, and k is a positive constant. We assume that no other forces are in play.

Newton's second law tells us that the force F acting on the ship may be written as $F = m\frac{dv}{dt}$.

a) By considering the forces acting on the ship, briefly explain why

$$\frac{dv}{dt} + \frac{k}{m} v = \frac{L}{m}.$$

b) By treating this as a linear equation, find the integrating factor and hence solve the equation for v as a function of t.

c) Explain why the speed of the ship cannot exceed $\frac{L}{k}$.

d) At what time will the ship reach half this speed?

iii) The Maclaurin series for $\tan^{-1} x$, valid for $|x| < 1$, is given by

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

a) Using an appropriate convergence test, show that the Maclaurin series for $\tan^{-1} x$ also converges at $x = 1$.

b) Hence find the value of the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

iv) In this question you may use the Fundamental Theorem of Calculus. That is, for a continuous function f on $[a, b]$,

$$\frac{d}{dt}\Big(\int_a^t f(u)\, du\Big) = f(t),$$

for each $t \in (a, b)$.

The curve in Figure 1 is defined by the parametric equations

$$x(t) = \int_1^t \frac{\cos u}{u}\, du \quad \text{and} \quad y(t) = \int_1^t \frac{\sin u}{u}\, du, \qquad t \geq 1,$$

and it has many vertical tangents. A curve in parametric form has a vertical tangent when $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$.

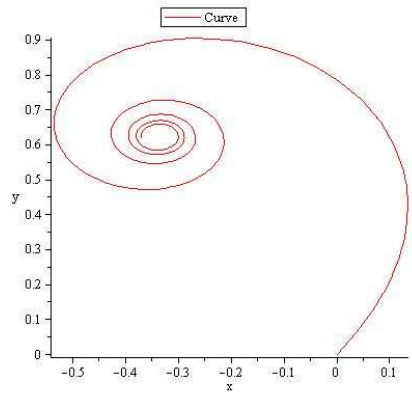


Figure 1

a) Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

b) Show that the curve defined above has a vertical tangent at the point where $t = t_1 = \frac{\pi}{2}$.

c) State the next value, t_2, of t, at which the curve has a vertical tangent.

d) Find, in simplest form, the length of the arc of the curve from t_1 to t_2.

MATH1231 FEBRUARY 2012

1. i) Let $S = \{\mathbf{x} \in \mathbb{R}^2 : x_1 x_2 \geq 0\}$.

   a) Show that S is closed under multiplication by a scalar.

   b) Show that S is a not a subspace.

   ii) Consider the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ of vectors in $\mathbb{R}^4$ given by

$$S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 8 \\ 7 \end{pmatrix} \right\},$$

   and let $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -10 \\ -7 \end{pmatrix}$.

   a) Does $\mathbf{b}$ belong to span(S)? Give reasons.

   b) Does S span $\mathbb{R}^4$? Give reasons.

   c) Write down a basis for span(S).

   iii) A mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T(\mathbf{x}) = \begin{pmatrix} x_1 - x_3 \\ 2x_1 + x_2 \end{pmatrix}.$$

   a) Show that T is linear.

   b) Write down a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

   iv) The linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ has matrix

$$A = \begin{pmatrix} 1 & 3 & -3 & 1 & 4 \\ -1 & 3 & -9 & 2 & -1 \\ 2 & -4 & 14 & 3 & -3 \\ -1 & -3 & 3 & -1 & -4 \\ 0 & 6 & -12 & 3 & 3 \end{pmatrix}.$$

   Using the Maple below, or otherwise, answer the following questions, giving brief reasons.

   a) Find rank(T).

   b) Find nullity(T).

   c) Find a basis for ker(T).

   d) Write down conditions on $\mathbf{b} \in \mathbb{R}^5$ for $\mathbf{b}$ to be in im(T).

   e) Find a basis for $\mathbb{R}^5$ containing as many columns of A as possible.

```
> with(LinearAlgebra):
> A:=<<1,-1,2,-1,0>|<3,3,-4,-3,6>|<-3,-9,14,3,-12>|
> <1,2,3,-1,3>|<4,-1,-3,-4,3>>:
> Id5:=IdentityMatrix(5):
> AI:=<A|Id5>;
```

$$AI := \begin{bmatrix} 1 & 3 & -3 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -9 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 14 & 3 & -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & -3 & 3 & -1 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 6 & -12 & 3 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> ReducedRowEchelonForm(AI);
```

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 & 0 & 0 & \frac{1}{12} & -\frac{5}{6} & -\frac{13}{36} \\ 0 & 1 & -2 & 0 & 1 & 0 & 0 & -\frac{1}{12} & -\frac{1}{6} & \frac{1}{36} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{5}{18} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

2. i) Let

$$A = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix}.$$

a) Find the eigenvalues and corresponding eigenvectors of A.

b) Hence find the general solution to the system of linear differential equations

$$\begin{aligned} \frac{dy_1}{dt} &= 4y_1 - 2y_2 \\ \frac{dy_2}{dt} &= 7y_1 - 5y_2. \end{aligned}$$

ii) You are given the following information about two events A and B.

$$P(B) = 0.65, \quad P(B|A) = 0.95, \quad P(B^c|A^c) = 0.85.$$

Using a tree diagram, or otherwise, find:

a) the exact value of $P(A)$,

b) the value of $P(A|B)$ correct to two decimal places.

iii) A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \alpha(1 - x^2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the value of α.

b) Find $E(X)$.

iv) A six-sided die, which is believed to be biased, is rolled 720 times and shows a '6' 100 times.

a) Write down the formula for the tail probability of getting 100 or less 6's in 720 rolls of a fair die.

b) Using the normal approximation to the binomial distribution, calculate the probability in (i), giving your answer to 3 decimal places.

c) Giving reasons, do you think that this die is biased?

v) You are given that a 2×2 matrix B satisfies,

$$B^2 = 3B - 2I.$$

Giving reasons, find the eigenvalues of B.

3. i) Consider the surface $\mathcal{S}$, given by $z = \sqrt{x^2 + y^2}$, at the point

$$\mathbf{x}_0 = \begin{pmatrix} 8 \\ 6 \\ 10 \end{pmatrix}.$$

a) Determine a vector $\mathbf{n}$ normal to the surface $\mathcal{S}$ at the point $\mathbf{x}_0$.

b) Determine the equation of the tangent plane to the surface $\mathcal{S}$ at the point $\mathbf{x}_0$.

c) Using your result from part (ii) find an approximation to the value of $\sqrt{8.02^2 + 5.97^2}$.

ii) Consider the function $z = xf(xy)$, where f is a differentiable function of one variable. Prove that

$$z = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\,.$$

iii) Evaluate each integral:

a) $I_1 = \displaystyle\int \cos^3\theta \sin^4\theta \, d\theta,$

b) $I_2 = \displaystyle\int \frac{x^2 + x}{(x-2)(x^2+2)}\, dx.$

iv) Solve the differential equation defined, for $x > 0$, by

$$\frac{dy}{dx} - \frac{3}{x}y = \frac{1}{x^4}, \qquad y(1) = 4.$$

v) Consider the differential equation $2xy + (x^2 + y^3)\dfrac{dy}{dx} = 0$.

a) Show that the differential equation is exact.

b) Hence solve the differential equation.

4. i) A tank can hold 1000 litres. Initially it holds 250 litres of pure water. Brine, which contains 5 grams of salt per litre, is run into the tank at the rate of 6 litres per minute. The mixture, which is stirred continuously, is run off at a rate of 1 litre per minute. Let $M(t)$ denote the mass of salt (measured in grams) present in the tank after t minutes, where $0 \le t \le 150$.

Determine (with reasoning) a first order differential equation for $M(t)$.

[Note: You are **NOT** being asked to solve the differential equation.]

ii) Determine the general solution of $y'' - 4y' + 4y = x$.

iii) By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n^2}\right)$,

b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 3n}{n^2}$.

iv) Determine the open interval of convergence for $\displaystyle\sum_{n=2}^{\infty}\frac{(x-1)^n}{\ln n}$.

v) Consider the Maclaurin series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, x \in (-1, 1].$$

a) By differentiating the above series for $\ln(1+x)$, find the Maclaurin series for $\dfrac{1}{1+x}$.

b) Hence write down the Taylor Polynomial $P_6(x)$ about 0 for $\dfrac{1}{1+x^2}$.

c) Using your result from the previous part, find a rational approximation to $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$.

(Your answer should be written as a fraction.)

vi) The area A of the surface of revolution about the y-axis of the curve $\mathcal{C}$, described parametrically by $\mathcal{C} = \{(x(t), y(t)) \in \mathbb{R}^2 : a \le t \le b\}$, is given by

$$A = \int_a^b 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt\,.$$

(You are NOT required to show this.)

a) The curve $\mathcal{C}$ is described using polar coordinates by

$$r = r(\theta), \quad \theta_0 \le \theta \le \theta_1\,.$$

Use the change of variables $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$ to show that the area A of the surface of revolution of $\mathcal{C}$ about the y-axis is given by

$$A = \int_{\theta_0}^{\theta_1} 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,.$$

b) Determine the area of the surface formed when the polar curve $r = 1 + \sin\theta$, where $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, is rotated about the y-axis.

MATH1231 NOVEMBER 2012

1. i) Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Find a basis for span(S).

ii) Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 6 & 2 & 2 & 1 \end{pmatrix}.$$

a) Find a basis for the kernel of A.

b) Hence state the nullity(A).

iii) The vector space $\mathbb{P}_2$ consists of all polynomials, with real coefficients, of degree at most 2. Suppose W is the subset of $\mathbb{P}_2$ defined by

$$W = \{p \in \mathbb{P}_2 : p'(0)p''(0) = 0\},$$

where $'$ denotes differentiation.

a) Show that W is closed under scalar multiplication.

b) Determine, with reasons, whether or not W is a subspace of $\mathbb{P}_2$.

iv) a) Find constants b and c such that

$$\frac{3x+2}{(x^2+2x+2)(x-1)} = \frac{1}{x-1} + \frac{bx+c}{x^2+2x+2}.$$

b) Hence find

$$J = \int \frac{3x+2}{(x^2+2x+2)(x-1)}\,dx.$$

v) Solve the initial value problem

$$\frac{dy}{dx} - \frac{1}{(x-1)}y = 2, \quad y(2) = 3, \quad x > 1.$$

vi) The following MAPLE session may assist you with this question.

```
> x := t -> t-sin(t):
> y := t -> 1-cos(t):
> xp := diff(x(t),t):
> yp := diff(y(t),t):
> A := simplify(sqrt(xp^2 + yp^2));
```

$$\sqrt{2-2\cos(t)}$$

```
> int(A*(1-cos(t)),t=0..2Pi);
```

$$\frac{32}{3}$$

a) Write down the integral representing the area S of the surface of revolution formed when the curve with parametric equation
$x = x(t), y = y(t)$, between $t = t_0$ and $t = t_1$, is rotated about the x-axis.

b) Use the MAPLE output above to find the area of the surface of revolution when the curve given by

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \leq t \leq 2\pi,$$

is rotated about the x-axis.

2. i) For a gas confined in a container, the ideal gas law states that the pressure P is related to the volume V and the temperature T by

$$P = k\frac{T}{V},$$

where k is a positive constant.

a) Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$.

b) The volume V is increased by 4% and the temperature T is decreased by 3%.
Use the total differential to estimate the percentage increase or decrease in the pressure P.

ii) Determine, with reasons, whether or not the following sequences converge and find the limit if it exists:

a) $\{a_n\}$ where $a_n = \dfrac{n}{\sqrt{4n^2 - 1}}$,

b) $\{b_n\}$ where $b_n = \dfrac{n!}{n^n}$.

iii) Consider the ordinary differential equation

$$(2x + 3y)\ dx + (3x + 2y)\ dy = 0.$$

a) Show that the equation is exact.

b) Find the general solution of the given equation.

iv) Consider the map $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(\mathbf{x}) = \begin{pmatrix} x_2 \\ 2x_3 - x_1 \\ x_1 + x_2 + 5x_4 \\ 6x_2 - 3x_4 \end{pmatrix}.$$

You may assume (without proof) that T is linear.
Determine a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

v) Let A be the matrix $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors for A.

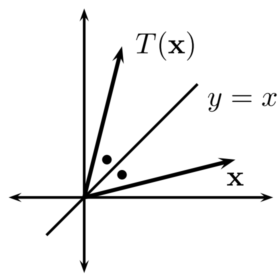
b) Hence write down matrices P and D, where D is diagonal and $P^{-1}AP = D$.

vi) Suppose $\mathbf{b}$ is a non-zero vector in $\mathbb{R}^3$ and consider the projection map $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(\mathbf{x}) = \dfrac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$

a) Show that T is linear.

b) Describe geometrically the kernel of T.

3. i) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ about the line $y = x$ as shown in the diagram.



Explain why $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of T and give their corresponding eigenvalues.

ii) In Sydney, 85% of workers use public transport (bus, train, or light-rail) to get to work. On a particular day, 50% of Sydney workers use at least the bus in their travels, 40% use at least the train, 30% use at least light-rail, 20% use at least bus and train, 10% use at least bus and light-rail, 8% use at least train and light-rail. What percentage of Sydney workers used all three modes of public transport to get to work on that day?

iii) Suppose X has probability density function given by

$$f(x) = \begin{cases} 1 + 4x, & x \in [0, a] \\ 0, & x \notin [0, a] \end{cases} \qquad \text{for some } a > 0.$$

a) Show that $a = \frac{1}{2}$.

b) Find, as a fraction, the value of $E(X)$, the expected value of X.

c) Find, as a fraction, the value of $\mathrm{Var}(X)$, the variance of X.

iv) A six-sided die, with faces numbered 1 to 6, is suspected of being unfair so that the number 6 will occur more frequently than should happen by chance. During 300 test rolls of the die, the number 6 occurred 68 times.

a) Write down an expression for a tail probability that measures the chance of rolling a 6 at least 68 times.

b) Use the normal approximation to the binomial to estimate this probability.

c) Is this evidence that the die is unfair?

v) Suppose A is a 3×2 matrix and let A^T denote its transpose.

a) Prove that the column space of the matrix AA^T is a subset of the column space of A and deduce that $\mathrm{rank}(AA^T) \leq \mathrm{rank}(A)$.

b) Deduce that nullity$(AA^T) \geq 1$, and explain why the matrix AA^T can never be the identity.

4. i) Find the open interval of convergence $I = (a, b)$ for the power series

$$\sum_{k=2}^{\infty} \frac{(x-3)^k}{2^k \log k}.$$

ii) Determine whether each of the following series converges or diverges, stating any tests you use.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n)}{(n^3+1)^{\frac{1}{2}}}$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

iii) You are given that

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \qquad \text{for } -1 < x < 1.$$

a) Find the Maclaurin series for $\log(1+x)$ that is valid for $-1 < x < 1$.

b) Given that $\displaystyle\int \log x \, dx = x \log x - x + C$, simplify the infinite series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{(n-1)n} = \frac{x^2}{1 \times 2} - \frac{x^3}{2 \times 3} + \frac{x^4}{3 \times 4} - \cdots$$

where $-1 < x < 1$.

iv) A fixed proportion k_1 of a certain isotope, A, decays each hour into isotope B. Also, a fixed proportion k_2 of the isotope B decays each hour into isotope C. Let $a(t)$ and $b(t)$ be the amounts of isotopes A and B respectively at time t, in hours. The rates of change of these quantities, can be modelled by the differential equations

$$\frac{da}{dt} = -k_1 a \tag{1}$$

$$\frac{db}{dt} = k_1 a - k_2 b. \tag{2}$$

a) By differentiating both sides of equation (2), show that b satisfies the differential equation

$$\frac{d^2b}{dt^2} + (k_1 + k_2)\frac{db}{dt} + k_1 k_2 b = 0. \tag{3}$$

b) Assuming that $k_1 \neq k_2$, find the general solution of the differential equation (3).

c) Suppose now that $k_1 = k_2$. Write down the general solution of the differential equation (3), given then $b(0) = 0$.

d) Under the conditions in part (c), find an expression for the time it takes for $b(t)$ to reach its maximum value.

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1. i) Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

   a) Prove that S is closed under scalar multiplication.
   b) Show that S is **not** a subspace of $\mathbb{R}^3$.

   ii) Suppose that

$$A = \begin{pmatrix} -23 & 3 & -5 & 37 \\ -49 & 8 & -10 & 73 \\ -87 & 9 & -15 & 129 \\ -26 & 3 & -5 & 40 \end{pmatrix}.$$

   Using the following Maple output, answer the questions below.

```
> with(LinearAlgebra):
> A := <<-23,-49,-87,-26>|<3,8,9,3>|<-5,-10,-15,-5>
     |<37,73,129,40>>;
```

$$A := \begin{bmatrix} -23 & 3 & -5 & 37 \\ -49 & 8 & -10 & 73 \\ -87 & 9 & -15 & 129 \\ -26 & 3 & -5 & 40 \end{bmatrix}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 5 & 2 & 2 \\ 3 & 0 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

   a) Without performing any further computations, explain why the matrix A has four linearly independent eigenvectors.
   b) Write down the general solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}.$$

   iii) A collection of discs consists of DVDs and Blu-ray discs. In the collection, 75 % are DVDs and 25 % are Blu-ray discs. Among the DVDs, 60 % are movies. Among the Blu-ray discs, 90 % are movies.

   a) Find the probability that a disc chosen randomly from the collection is a movie.
   b) Find the probability that a randomly chosen disc from the collection is a Blu-ray disc given that it is a movie.

iv) Evaluate each of the following integrals:

a) $I_1 = \displaystyle\int \frac{5x^2 - 6x + 4}{x(x-1)(x-2)}\, dx.$

b) $I_2 = \displaystyle\int_0^{\pi/2} \sin^3\theta \cos^2\theta \; d\theta.$

v) Solve the initial value problem

$$\frac{dy}{dx} = \frac{\sin x}{y^2}, \quad y(0) = 1.$$

vi) Let $f(x) = \ln(1+4x^2)$. The following MAPLE session may assist you with this question.

```
> f := x -> log(4*x^2+1):
> taylor(f(x), x = 0, 6);
```

$$4x^2 - 8x^4 + O(x^6)$$

Using the MAPLE output above or otherwise, write down the values of $f''(0)$ and $f'''(0)$.

2. i) Suppose that f is a differentiable function of one variable, and $F(x, y)$ is defined by $F(x, y) = f(2x - 3y^2)$.

a) Show that F satisfies the partial differential equation

$$3y\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 0.$$

b) Given that $F(x, 0) = \cos(x)$ for all x, write down a formula for $F(x, y)$.

ii) The curve C is given parametrically by $x = t^3, y = 2t^2$. Find the arc length of the curve C between $t = 0$ and $t = 1$.

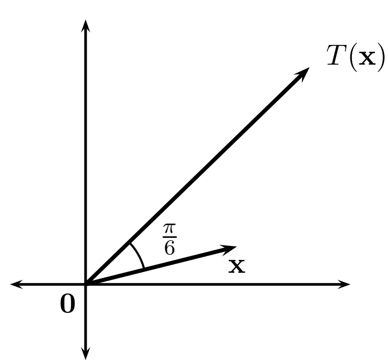
iii) a) Find the general solution $y(x)$ for the following ordinary differential equation:

$$y'' - 5y' + 6y = 0.$$

b) Find the general solution of the following equation:

$$y'' - 5y' + 6y = e^{2x}.$$

iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector $\mathbf{x}$ about the origin through $\frac{\pi}{6}$ anticlockwise and doubles its length, as shown in the diagram.



a) Show that $T(\mathbf{e}_1) = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

v) The probability density function f of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}(2-x) & \text{for } 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find $E(X)$ and $\mathrm{Var}(X)$.

b) The **median** of a distribution is defined to be the real number m such that $P(X \le m) = \frac{1}{2}$. Find the median of the above distribution.

vi) Let V be a vector space.

a) Let S be the set $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ of vectors in V.
State the condition for S to be a linearly dependent set.

b) Suppose that $\mathbf{v}_1,\ \mathbf{v}_2 \in V$ and $\mathbf{0}$ is the zero vector of V. Prove that the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{0}\}$ is linearly dependent.

3. i) The polynomials $p_1,\ p_2,\ p_3,\ p_4$ and p_5 are defined, for all $x \in \mathbb{R}$, by

$$\begin{aligned} p_1(x) &= x^3 + 3x^2 + 2x - 1 \\ p_2(x) &= 2x^3 - 3x^2 + 4x + 2 \\ p_3(x) &= 11x^3 - 3x^2 + 22x + 5 \\ p_4(x) &= 2x^3 - 21x^2 + 4x + 3 \\ p_5(x) &= 9x^3 + 18x^2 + 18x + 2 \end{aligned}$$

Let $S = \{p_1,\ p_2,\ p_3,\ p_4,\ p_5\}$.

a) Explain why S cannot be a basis for $\mathbb{P}_3(\mathbb{R})$. Using the following Maple output,

answer the questions below.

```
> with(LinearAlgebra):
> A := <<-1,2,3,1>|<2,4,-3,2>|<5,22,-3,11>|<3,4,-21,2>
     |<2,18,18,9>|<1,0,0,0>|<0,1,0,0>|<0,0,1,0>
     |<0,0,0,1>>;
```

$$A := \begin{bmatrix} -1 & 2 & 5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 2 & 4 & 22 & 4 & 18 & 0 & 1 & 0 & 0 \\ 3 & -3 & -3 & -21 & 18 & 0 & 0 & 1 & 0 \\ 1 & 2 & 11 & 2 & 9 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} -1 & 2 & 5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 8 & 32 & 10 & 22 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{63}{4} & \frac{63}{4} & \frac{9}{4} & -\frac{3}{8} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

b) Find a basis for span(S).

c) Find a basis for $\mathbb{P}_3(\mathbb{R})$ which contains as many of the polynomials of S as possible.

ii) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 3 \\ 1 & 3 & 4 & 9 \end{pmatrix}.$$

a) Find a basis for ker(A).

b) Hence state the value of nullity(A). Give a reason.

iii) Let

$$C = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}.$$

a) Find the eigenvalue(s) of C and for each eigenvalue find the corresponding eigenvectors.

b) Is C diagonalisable? Give a reason for your answer.

iv) A manufacturer claims that only 20 % of the one-litre size soft drinks produced by his factory contain less than 1.05 litres. Consumer Watchdog examined a random sample of 100 bottles of one-litre soft drinks produced by this manufacturer and found that 30 % of the sample contain less than 1.05 litres. To test whether the manufacturer's claim is true, we shall use a binomial distribution $B(n, p)$ to model the number, X, of bottles containing less than 1.05 litres in a random sample of 100.

a) State the values of n and p.

b) Find an expression for $P(X \geq 30)$ in terms of binomial coefficients.

c) Use the normal approximation to the binomial distribution to estimate the probability in the previous part.

d) Based on this probability, is there evidence that the manufacturer has made a false claim? (Give a reason.)

4. i) Use appropriate tests to determine whether each of the following series converges or diverges:

a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k + 1}{3^k}$,

b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k(k+1)}}$.

ii) The Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

Do not prove this result.

a) Write down the Maclaurin series for $\sin(x^2)$.

b) Find the Maclaurin series for $\cos(x)$.

c) Hence derive the identity

$$\frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}.$$

iii) Determine, with reasons, whether or not each of the following sequences converges and find each limit if it exists:

a) $\{a_n\}$ where $a_n = \dfrac{\ln n}{n}$,

b) $\{b_n\}$ where $b_n = \sqrt[n]{n}$.

iv) A rumour is spreading through an isolated population. Modelling suggests that $r(t)$, the **proportion** of the population who have heard the rumour after t days, is governed by a differential equation of the form

$$\frac{dr}{dt} = kr(1-r), \tag{4}$$

where k is a constant.

a) Show that the general solution to (4) is given implicitly by

$$\frac{r}{1-r} = \beta\, e^{kt},$$

with $\beta > 0$. (You may assume that $0 < r < 1$.)

b) At time $t = 0$, 1% of the population have heard the rumour. After 5 days, 20% of the population have heard the rumour.

A) Find β and k. (Give k correct to 4 decimal places.)

B) Hence, estimate how long it will take until 80% of the population have heard the rumour. (Give your answer correct to 2 decimal places.)

c) How does $r(t)$ behave as $t \to \infty$?

MATH1231 NOVEMBER 2014

1. i) By expanding $\sin(A+B) + \sin(A-B)$, or otherwise, find $I_1 = \displaystyle\int \sin(5x)\cos(x)\,dx$.

   ii) Evaluate the integral $I_2 = \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

   iii) Use appropriate tests to determine whether each of the following series converges or diverges

   a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln\ n)^3}$,

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n+3^n}$.

   iv) Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 \right\}$.

   a) Prove that S is closed under scalar multiplication.

   b) Prove that S is **not** a subspace of $\mathbb{R}^3$.

   v) Let $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors for the matrix A.

   b) Write down an invertible matrix M and a diagonal matrix D such that

$$D = M^{-1}AM.$$

   vi) Let

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{pmatrix}$$

   Using the MAPLE output below, find a basis for $\ker(A)$.

```
> with(LinearAlgebra):
> A := <<1,2,3,1>|<2,4,-3,2>|<2,10,-3,1>|<-1,-44,24,6>>;
```

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. i) Consider the initial value problem $\dfrac{dy}{dx}+(2+\dfrac{1}{x})y=\dfrac{2}{x}$, with $y(1)=0$, defined for $x>0$.

   a) Show that an integrating factor for this equation is xe^{2x}.

   b) Hence solve the initial value problem.

   ii) Find the general solution to $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2y=20e^{2x}$.

   iii) Consider the MAPLE session:

```
> a:=n->n^n/n!*(x-1)^n;
```

$$a := n \to \frac{n^n(x-1)^n}{n!}$$

```
> a(n+1);
```

$$\frac{(n+1)^{(n+1)}(x-1)^{(n+1)}}{(n+1)!}$$

```
> limit(a(n+1)/a(n),n=infinity);
```

$$ex-e$$

Using MAPLE session above, or otherwise, find the open interval of convergence $I=(a,b)$ for the power series

$$\sum_{n=1}^{\infty}\frac{n^n(x-1)^n}{n!}.$$

   iv) Consider the set S consisting of the vectors $\mathbf{v_1}=\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\mathbf{v_2}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$ from $\mathbb{R}^3$ and let $\mathbf{u}=\begin{pmatrix}1\\-1\\12\end{pmatrix}$.

   a) Find scalars λ and μ such $\mathbf{u}=\lambda\mathbf{v_1}+\mu\mathbf{v_2}$.

   b) A linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ has $\mathbf{v_1},\mathbf{v_2}$ as eigenvectors with eigenvalues 2 and -1, respectively.

      α) Find $T(\mathbf{u})$ as a linear combination of $\mathbf{v_1},\mathbf{v_2}$.

      β) Denote $T(T(\mathbf{u}))$ by $T^2(\mathbf{u})$, $T(T(T(\mathbf{u})))$ by $T^3(\mathbf{u})$, and so on. Express $T^n(\mathbf{u})$ as a linear combination of $\mathbf{v_1},\mathbf{v_2}$, where n is a positive integer.

v) The two most popular soft drinks in Old South Wales are AppleAde and BananAde. Assume that no-one in Old South Wales likes both of these drinks equally (that is, everyone has a preference for one or the other). Past statistics show that 50 % of the population prefer AppleAde.

Last month the manufacturer advertised AppleAde on television for a week. After that, a survey was conducted by taking a random sample of 100 people. Of the 100 people sampled, 60 preferred AppleAde and 40 preferred BananAde.

a) Assuming that the advertising had **no effect** on people's preferences, write down an expression for the tail probability that 60 or more people preferred AppleAde in a sample of 100.

b) Use the normal approximation to the binomial to calculate the tail probability in (a), giving your answer to 3 decimal places.

c) Giving reasons, is there evidence that the advertising campaign increased the percentage of the population that prefer AppleAde?

3. i) Let $\mathbb{P}_1$ denote the space of all polynomials of degree less or equal to 1. Prove that function $T : \mathbb{R}^3 \to \mathbb{P}_1$ defined by

$$T(\mathbf{v}) = 2a + (b - c)x, \quad \text{for all } \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

is a linear transformation.

ii) Let S be the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ given by

$$S = \left\{ \begin{pmatrix} 21 \\ 12 \\ -6 \\ -31 \\ -12 \end{pmatrix}, \begin{pmatrix} 11 \\ 20 \\ -14 \\ -29 \\ -21 \end{pmatrix}, \begin{pmatrix} 18 \\ -72 \\ 60 \\ 50 \\ 78 \end{pmatrix}, \begin{pmatrix} 29 \\ -52 \\ 46 \\ 21 \\ 57 \end{pmatrix}, \begin{pmatrix} 16 \\ -17 \\ -31 \\ 28 \\ -21 \end{pmatrix} \right\}.$$

Using the following MAPLE output, answer the questions below. Give reasons.

```
> with(LinearAlgebra):
> M := <<21, 12, -6, -31, -12>|<11, 20, -14, -29, -21>
           |<18, -72, 60, 50, 78>|<29, -52, 46, 21, 57>
           |<16, -17, -31, 28, -21>>;
```

$$\begin{bmatrix} 21 & 11 & 18 & 29 & 16 \\ 12 & 20 & -72 & -52 & -17 \\ -6 & -14 & 60 & 46 & -31 \\ -31 & -29 & 50 & 21 & 28 \\ -12 & -21 & 78 & 57 & -21 \end{bmatrix}$$

```
> GaussianElimination(M);
```

$$\begin{bmatrix} 21 & 11 & 18 & 29 & 16 \\ 0 & \frac{96}{7} & -\frac{576}{7} & -\frac{480}{7} & -\frac{183}{7} \\ 0 & 0 & 0 & 0 & -\frac{377}{8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find a basis for span(S).
b) Write down the dimension of span(S).

iii) Employment data at a large company reveal that 40 % of the employees are thirty years of age or younger; 60 % are older than thirty. All employees belong to exactly one of the types: full-time, part-time or casual. Among those who are thirty years of age or younger, 25 % are full-time; 15 % are part-time; the others are casual. Among those who are older than thirty, 45 % are full-time; 20 % are part-time; the others are casual.

a) What proportion of the employees are full-time?
b) What is the probability that a randomly chosen employee is older than thirty given that the employment type of the employee is full-time?

iv) The discrete random variable X can only take the values $-2, -1, 1, 2, 5$.
The probability distribution for X is given in the following table:

x	-2	-1	1	2	5
$P(X = x)$	0.3	$4c$	0.35	$6c$	0.05

Another random variable Y is defined by $Y = X^2$.

a) Find the value of c.
b) Calculate $P(Y = 4)$.

v) The density function f for a random variable X is defined by

$$f(x) = \begin{cases} 3x^2 & \text{for } \quad 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the cumulative probability density function F corresponding to f.
b) Calculate $E(X)$ for the probability density function f.
c) Calculate $\mathrm{Var}(X)$.

vi) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three non-zero vectors in $\mathbb{R}^3$.

a) State the definition for the set B to be a linearly independent set.

b) Prove that if B is an orthogonal set then B is linearly independent.

c) Hence explain why any orthogonal set of 3 non-zero vectors in $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$.

4. i) A right circular cone with base radius r cm and a perpendicular height h cm is formed by rotating the line segment given by

$$y = \frac{r}{h}\,x, \qquad 0 \le x \le h,$$

about the x-axis.

a) Use calculus to show that the surface area S of the curved surface of the cone is given by

$$S = \pi r\sqrt{r^2 + h^2}.$$

b) Find the partial derivatives $\dfrac{\partial S}{\partial r}$ and $\dfrac{\partial S}{\partial h}$.

c) If the values of r and h are measured to be 3.0 and 4.0 cms respectively and each of these measurements is made with an error whose absolute value is at most 0.05 cm, then use the total differential approximation of S to estimate the maximum absolute error in the measured value of the total surface area S.

ii) During the winter the daytime temperature in the Physics Theatre is maintained at 20°C. The heating is turned off at 10pm and turned on again at 6am. On a certain day, the temperature inside the Theatre at 11pm was found to be 18°C. The outside temperature was found to be constant throughout the night at 10°C. Let $P(t)$ be the temperature (in °C) in the Physics Theatre at time t (in hours), from 10pm. The rate of cooling of the air inside the Theatre can be modelled by the differential equation

$$\frac{dP}{dt} = -k(P - 10),$$

where k is a positive constant. (Do not prove this.)

a) By first solving the differential equation, show that $k = \ln(5/4)$.

b) Find, to two decimal places, the temperature inside the Physics Theatre when the heating was turned on at 6am.

iii) The n^{th} term of the sequence $\{a_n\}$ is given by

$$a_n := \begin{cases} n^{1/n}/2, & \text{for } n \text{ odd,} \\ \\ 1/2, & \text{for } n \text{ even.} \end{cases}$$

a) Explain briefly why

$$\frac{1}{2} \leq a_n \leq \frac{n^{1/n}}{2}, \text{ for } n \geq 1.$$

b) Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

c) Hence find

$$\lim_{n\to\infty} a_n.$$

Give reasons for your answer.

d) Does the series

$$\sum_{n=2}^{\infty} (-1)^n a_n$$

converge or diverge? Name any test that you use.

iv) Let $f(x)$ be a function satisfying

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -2 \;\text{ and }\; |f'''(x)| \leq 6, \text{ for } 0 \leq x \leq 1.$$

a) Write down the second Taylor polynomial $p_2(x)$ of $f(x)$ about $x = 0$.

b) The function $f(x)$ is approximated by $p_2(x)$, for x in the interval $0 \leq x \leq 1$.
Using Taylor's Theorem with Lagrange remainder formula, show that

$$\frac{1}{8} \leq f(1/2) \leq \frac{3}{8}.$$

c) Using (a), or otherwise, find

$$\lim_{x\to 0^+} \frac{x - f(x)}{x^2}.$$

PAST HIGHER EXAM PAPERS

MATH1241 NOVEMBER 2010

2. i) A surface with height $z = F(x, y)$ is defined by

$$z = 1 + \frac{25}{x^2 + y^2} \quad \text{for } 1 \le x^2 + y^2 \le 25.$$

Find the equation of the plane tangent to the surface at the point $(1, 2, 6)$ on the surface.

ii) A function $u(x, t)$ is defined implicitly by the equation

$$u(x, t) = f(x + tu(x, t)),$$

where f is a differentiable function of one variable. Show that $u(x, t)$ is a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}.$$

iii) Find the integral

$$I = \int \frac{6x^2 - 11x + 9}{(x-1)(x^2 - 2x + 2)}\, dx.$$

iv) Let $\mathbb{P}_3(\mathbb{R})$ denote the vector space of polynomials of degree 3 or less with real coefficients and let S be given by

$$S = \{p \in \mathbb{P}_3(\mathbb{R}) \,:\, (x-3)p'(x) - 2p(x) = 0 \quad \text{for all } x \in \mathbb{R}\}.$$

Show that S is a subspace of $\mathbb{P}_3(\mathbb{R})$.

v) On the basis of the health records of a particular group of people, an insurance company accepted 60% of the group for a 10 year life insurance policy and rejected the others. Ten years later the company examined the survival rates for the whole group and found that 80% of those accepted for the policy had survived the 10 years, while 40% of those rejected had survived the 10 years. If a person from the original group did survive 10 years, what is the probability that they had been refused a life insurance policy?

vi) Suppose that the vectors $\mathbf{v}_j \in \mathbb{R}^n$, for $j = 1, 2, \ldots, n$, are mutually orthogonal, *i.e.* $\mathbf{v}_j \cdot \mathbf{v}_k = 0$ for $j \ne k$ and $j, k = 1, 2, \ldots, n$.
Let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^n$.

a) Given that the set $G = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, find an expression for $\mathbf{x}$ in terms of the vectors in the basis G.

b) Suppose that the linear map $T : \mathbb{R}^n \to \mathbb{R}$ has the property that

$$T(\mathbf{v}_j) = \gamma_j(\mathbf{v}_j \cdot \mathbf{v}_j) \quad \text{for } j = 1, 2, \ldots, n,$$

where the $\gamma_j \in \mathbb{R}$ are given constants.
Find an expression for $T(\mathbf{x})$ in terms of $\mathbf{x}$, the constants γ_j and the vectors $\mathbf{v}_j$.

3. i) Let $A = \begin{pmatrix} 5 & -8 \\ 1 & -1 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.

b) Write down matrices P and D such that $A = PDP^{-1}$.

c) Hence evaluate $A^8 P$.

ii) According to the Bureau of Meteorology, Sydney experienced rain on 103 out of the 365 days in 2009. Let us make the simple assumptions that

- the probability p of rain on any given day in Sydney is constant; (5)
- the weather on any given day is independent of the weather on other days. (6)

a) Under assumptions (1) and (2), use the stated data to estimate the probability p that a given day in Sydney is rainy.

b) Under assumptions (1) and (2), estimate the probability that precisely 5 out of 10 given days are rainy in Sydney.

c) Suppose that Sydney will experience rain on 1 January, 2011. Using the assumptions (1) and (2), estimate the probability that the 6th of January 2011 is the first dry day of the new year.

iii) Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the ordered basis $B = (e^{-x}, \sin(x), \cos(x))$. Define the map $T : S \to S$ by

$$T(f) = f' - 2f \qquad \text{where} \quad f'(x) = \frac{df}{dx}\,.$$

a) Show that the map T is linear.

b) Calculate the matrix C which represents T with respect to the basis B.

c) What is the rank of the matrix C found in part (b)?

d) From part (c), what can be deduced about the solutions $y \in S$ of

$$y' - 2y = g$$

where g is a given function in S?

e) Use your results from parts (b) and (d) to find all non-zero solutions $y \in S$ satisfying

$$y' - 2y = e^{-x}\,.$$

4. i) Find the general solution of the ordinary differential equation

$$(x^2 - xy)\frac{dy}{dx} - 2y^2 + 5xy = 3x^2.$$

ii) Suppose that $a_k = \dfrac{k}{\ln(k!)}$ for $k = 2, 3, 4, \ldots$. In the questions below you may use the inequality

$$\ln(k!) - \ln(k) + k - 1 < k\ln(k) < \ln(k!) + k - 1 \quad \text{for } k = 2, 3, 4, \ldots.$$

a) Find $\lim\limits_{k\to\infty} a_k$, clearly stating your reasons.

b) Does the series $\sum_{k=2}^{\infty} a_k$ converge? Give reasons for your answer.

iii) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n}}(x-1)^n.$$

iv) a) Write down the first three non-zero terms in the Taylor series of $\sinh x$ about $x = 0$, including an expression for the Lagrange form of the remainder.

b) Hence, or otherwise, write down the first three non-zero terms in the Taylor series, about $x = 0$, of the function

$$\operatorname{Shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt.$$

v) A curve C in the xy-plane is defined parametrically by

$$x(t) = a\cos t, \quad y(t) = b\sin t, \quad \text{for } 0 \leq t \leq \pi, \text{ where } 0 < a < b.$$

Find the surface area of the ellipsoid defined by rotating the curve C about the x-axis.

vi) A tank has a total volume of 200 litres. Initially it holds 40 litres of pure water. Brine containing 2 grams of salt per litre is run into the tank at the rate of 3 litres per minute and the mixture is stirred continuously so that the concentration of the dissolved salt is uniform throughout the tank. At the same time as the brine starts to flow into the tank the mixture is removed from the tank at the rate of 1 litre per minute.
Let t denote time, measured in minutes, from when the brine started to enter the tank and let $x(t)$ denote the mass of salt, in grams, present in the tank after t minutes.

Set up a first order differential equation in x and t which models this system up until the time the tank is full.

MATH1241 NOVEMBER 2011

2. i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be a point in $\mathbb{R}^2$.

   a) Carefully write down the definition of the derivative $f_y(x_0, y_0)$.

   b) If $f(x, y) := x^{1/3}y^{2/3}$ find the value $f_y(0, 0)$.

   ii) In a direct-current circuit, the total resistance z, (measured in ohms), produced by two parallel resistors with resistances x and y ohms, is given by

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}.$$

   The values of x and y are measured to be 6 ohms and 12 ohms respectively and each of the measurements is made with an error whose absolute value is at most 0.1 ohms.

   a) Use the given measurements (ignoring the measurement error) to calculate the total resistance z.

   b) Show that $\dfrac{\partial z}{\partial x} = \dfrac{z^2}{x^2}$ and $\dfrac{\partial z}{\partial y} = \dfrac{z^2}{y^2}$.

   c) Use the total differential approximation to estimate the maximum absolute error in the calculated value of the total resistance z, correct to three significant figures.

   iii) Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathbf{u} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}.$$

   a) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? If so, give reasons; if not, then find condition(s) on a, b, c such that $\mathbf{w}$ belongs to $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

   b) Determine whether or not $\mathbf{u}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

   iv) One semester, 65% of the students studying MATH1231/41 attended all their tutorials, and of these students, 45% were awarded credit (CR) or higher in the final exam. In contrast, only 30% of the students who did not attend all their tutorials were awarded credit (CR) or higher in the final exam.

   a) What percentage of the students studying MATH1231/41 were **not** awarded credit (CR) or higher in the final exam?

   b) Of the students who were awarded credit (CR) or higher in the final exam, what percentage, correct to 1 decimal place, attended all their tutorials ?

3. i) Consider the ordinary differential equation (ODE)

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}.$$

   a) Determine the general solution $y = y(x)$ to the above ODE.

   b) Find the particular solution that passes through the point $(1, 1)$.

   ii) a) Determine whether the following series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\ln(k!)}{k^3}.$$

b) Does the following series converge absolutely or conditionally? Give reasons for your answers.

$$\sum_{k=2}^{\infty} \frac{(-1)^k \log k}{\sqrt{k^2+1}};$$

iii) Consider the ODE

$$\frac{dy}{dx} = e^{x^2} y, \qquad y(0) = 1.$$

a) Write down the Maclaurin series for e^{x^2}.
b) Hence, or otherwise, find a solution (involving a series) to the above initial value problem.
c) Carefully determine the values of x for which the solution is defined.

iv) A sequence of positive numbers $\{a_k\}_{k=1}^{\infty}$ is defined recursively by

$$a_{k+1} = \sqrt{1+a_k}, \qquad a_1 = 1.$$

a) Prove that a_k is strictly increasing.
b) Prove that a_k is bounded above by 3.
c) Hence, explain why there is a number L such that

$$\lim_{k\to\infty} a_k = L.$$

d) Determine the value of L.

4. i) Let $A = \begin{pmatrix} 22 & -100 \\ 5 & -23 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.
b) Write down matrices P and D such that $A = PDP^{-1}$.
c) Hence evaluate $A^n P$, for any positive integer n.

ii) A discrete random variable X has the probability distribution given by $p_k = \dfrac{c}{2^k}$ for $k = 0, 1, 2, 3, 4$, where c is a constant.

a) Find the value of the constant c.
b) Calculate $P(X = 2)$.
c) Calculate $P((X-2)^2 < 4)$.

iii) Let V be the vector space $M_{22}(\mathbb{R})$ of all 2×2 real matrices.
Define the linear map $T : M_{22} \to M_{22}$ by $T(A) = A^T$, where A^T denotes the transpose of the matrix A.

a) Show that for any non-zero $n \times n$ matrix C, if $C^T = \lambda C$ then $\lambda = \pm 1$.
b) Find the matrix representation B for T with respect to the ordered basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

c) State the rank of T.
d) Explain why the eigenvalues of T are ± 1.
e) Find bases for each of the eigenspaces of T.

MATH1241 NOVEMBER 2012

2. i) For any given differentiable function $f : \mathbb{R} \to \mathbb{R}$, let u be defined by

$$u(x,y) = f(s), \quad \text{where } s = \frac{x}{y}.$$

Determine the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

ii) Consider the function $u(x,y) = \ln(x^2 + y^2)$. Show that u satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

iii) a) Show that the area A of the planar region between the lines $x = -\dfrac{a}{2}$ and $x = \dfrac{a}{2}$ bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b > 0$, has the form Kab, and find the value of the constant K.

b) If the ellipse is deformed in such a way that a is increased by 4% and b is decreased by 1%, use the total differential approximation to show that the approximate relative change of area $\dfrac{\Delta A}{A}$ is independent of a and b and calculate its value.

iv) Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ 7 \\ 11 \\ 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Let S be the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

a) Find a subset of S that is linearly independent, containing as many vectors from S as possible.

b) Give conditions on $\mathbf{a}$ such that $\mathbf{a} \in \text{span}(S)$.

c) Find all ways of representing $\mathbf{a}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$.

d) Describe $\text{span}(S)$ geometrically.

v) Suppose that a continuous random variable T has an exponential distribution with parameter λ.

a) Find $P(4 - 2T \geq 0)$.

b) Find $E(T^2)$.

3. i) A six-sided die, with faces numbered 1 to 6, is suspected of being unfair so that the number 6 will occur more frequently than should happen by chance. During 300 test rolls of the die, the number 6 occurred 68 times.

a) Write down an expression for a tail probability that measures the chance of getting a 6 at least 68 times.

b) Use the normal approximation to the binomial to estimate this probability.

c) Is this evidence that the die is unfair?

ii) We roll a die successively, until we roll six consecutive sixes, in which case we stop rolling the die.

a) What is the probability that we stop after the thirteenth throw?

b) What is the probability that we roll the die at least ten times?

iii) Let A be a 3×2 matrix and let A^T denote its transpose.

a) Prove that the column space of the matrix AA^T is a subset of the column space of A and deduce that $\operatorname{rank}(AA^T) \leq \operatorname{rank}(A)$.

b) Is it possible for AA^T to be the identity matrix? Give examples or reasons for your answer.

iv) Let M be a 3×3 matrix and M^T be its transpose. Assume that M is orthogonal, that is, $M^T M = I$. Let $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ be the three columns of M.

a) Explain why $\det M = \pm 1$.

b) Explain why $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are mutually perpendicular vectors of length 1.

c) Show that for any square matrix A,

$$(A\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (A^T \mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, where $\cdot$ denotes the dot product.

d) Deduce that $|M\mathbf{v}| = |\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^3$.

e) Explain why M must have at least one real eigenvalue and at least one real eigenvector $\mathbf{v} \in \mathbb{R}^3$.

f) Show that if $\mathbf{u} \in \mathbb{R}^3$ is an eigenvector of M, then the corresponding eigenvalue is 1 or -1.

4. i) a) Write down the general solution to the second order differential equation $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$, $y(0) = 0$.

b) For what value of x does the solution have a maximum?

ii) a) Consider the solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 - a^2}, \qquad a > 0 \tag{7}$$

which passes through the point $(x_0, y_0) = (2a, a)$.
Find the equation of the normal to this solution at (x_0, y_0).

b) Obtain the **general** solution of the differential equation (7) and interpret it geometrically for $|x| < a$ and $|x| > a$.

iii) a) Sketch the graph of the function

$$f(x) = \frac{x}{x^2 - 99}$$

for $x \geq 0$.

b) Carefully investigate the (conditional) convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2 - 99},$$

justifying the steps in your proof.

iv) a) State the Maclaurin series of $f(t) = 1 - \cos t$.

b) Let $g(t) = \begin{cases} \frac{1-\cos t}{t} & t \neq 0 \\ 0 & t = 0. \end{cases}$

Find the Maclaurin series of the function

$$\mathrm{Cin}(x) = \int_0^x g(t)dt,$$

and state where the series converges.

v) Consider the sequence defined by $a_1 = \frac{1}{4}$, and, for $n \geq 1$,

$$a_{n+1} = 2a_n(1 - a_n).$$

a) Show that the sequence is bounded above by $\frac{1}{2}$.

b) Show that the sequence is strictly increasing.

c) Explain why the sequence is convergent.

d) Find $\lim_{n\to\infty} a_n$.

MATH1241 NOVEMBER 2013

3. i) Suppose that a continuous random variable T has an exponential distribution with parameter λ, $\lambda > 0$.

   a) Find $\mathrm{Var}(8 - 2T)$.

   b) Find $P(\lambda T \geq 1)$.

   ii) An industrial machine produces large numbers of light bulbs of which 2% are defective. The defects occur randomly during production. The bulbs are packaged into boxes of 100.

   By approximating the binomial distribution with the normal distribution, estimate the probability that a box contains more than 3 defective bulbs.

   iii) An urn contains a red balls and b black balls, where $a, b \geq 1$. Draw a ball randomly from the urn, replace it with c, $c \geq 1$, balls of the same colour, and repeat this procedure n, $n \geq 1$, times. Let R_i denote the event that a red ball is drawn on the ith draw.

   a) Briefly explain why $P(R_1) = \dfrac{a}{a+b}$.

   b) Show that $P(R_2) = P(R_1)$.

   iv) A real $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and can be expressed as $M^{-1}DM$, where M is an invertible matrix and D is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as its diagonal entries.

   a) Show that $A^k = M^{-1}D^kM$.

   b) Explain why D^k is a diagonal matrix with diagonal entries $\lambda_1^k, \ldots, \lambda_n^k$.

   c) Write the characteristic polynomial $p(x)$ of A as

   $$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\,.$$

   By replacing the variable x by the matrix D, and a_0 with a_0I, where I is the $n \times n$ identity matrix, prove that $p(D) = \mathbf{0}$ where $\mathbf{0}$ is the $n \times n$ zero matrix.

   d) Hence or otherwise, prove that $p(A) = \mathbf{0}$.

   e) Use the result in (d) to show that the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ satisfies $A^2 = A + 6I$.

4. i) Using the substitition $y(x) = xv(x)$, solve the differential equation

   $$x^2\frac{dy}{dx} = 4x^2 + xy + y^2.$$

   ii) Find the interval of convergence of the power series, (making sure to check the end points),

   $$\sum_{n=1}^{\infty} \frac{2^n n^2}{n^3+1}(x-1)^n.$$

   iii) A certain chemical reaction creates a substance S from two other chemicals A and B. Let X be the amount in grams of the substance S at a given time t. It can be shown that

   $$\frac{dX}{dt} = k(k_1 - X)(k_2 - X),$$

where the constant k depends only on the reaction conditions, while the constants k_1 and k_2 depend on the amount of the original chemicals A and B present at the start of the reaction.

a) The reaction is run with initial conditions set so that $X(0) = 0$ and $k_1 = k_2 = 2\,\mathrm{g}$. If there is 1 gram of S present after 1 hour, show it will take a total of 9 hours to create $1.8\,\mathrm{g}$ of S.

b) Chemical A is a lot cheaper than chemical B, so the reaction is re-run under the same conditions, that is, with the same value of k, but with an excess of chemical A available. If we take $X(0) = 0$, $k_1 = 20\,\mathrm{g}$ and $k_2 = 2\,\mathrm{g}$, how long will it now take to create $1.8\,\mathrm{g}$ of S? (Give your answer correct to 2 decimal places.)

iv) Let $f : [a, b] \to [a, b]$ be a continuous function such that $f(x) \geq x$ for all $x \in [a, b]$. Let $c_1 \in [a, b)$ and define a sequence $\{c_k\}_{k=1}^{\infty} \subset [a, b]$ by

$$c_{k+1} = f(c_k),$$

for $k \geq 1$.

a) Prove that the sequence is convergent.

b) Let $L = \lim_{n\to\infty} c_n$. Prove that $f(L) = L$.

MATH1241 NOVEMBER 2014

1. i) By expanding $\sin(A+B) + \sin(A-B)$, or otherwise, find $I_1 = \displaystyle\int \sin(5x)\cos(x)\,dx$.

   ii) Evaluate the integral $I_2 = \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

   iii) Use appropriate tests to determine whether each of the following series converges or diverges

   a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln\ n)^3}$,

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n+3^n}$.

   iv) Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 \right\}$.

   a) Prove that S is closed under scalar multiplication.

   b) Prove that S is **not** a subspace of $\mathbb{R}^3$.

   v) Let $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors for the matrix A.

   b) Write down an invertible matrix M and a diagonal matrix D such that

$$D = M^{-1}AM.$$

   vi) Let

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{pmatrix}$$

   Using the MAPLE output below, find a basis for $\ker(A)$.

```
> with(LinearAlgebra):
> A := <<1,2,3,1>|<2,4,-3,2>|<2,10,-3,1>|<-1,-44,24,6>>;
```

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. i) Consider the initial value problem $\dfrac{dy}{dx} + (2 + \dfrac{1}{x})y = \dfrac{2}{x}$, with $y(1) = 0$, defined for $x > 0$.

 a) Show that an integrating factor for this equation is xe^{2x}.
 b) Hence solve the initial value problem.

 ii) Find the general solution to $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 20e^{2x}$.

 iii) Consider the MAPLE session:

```
> a:=n->n^n/n!*(x-1)^n;
```

$$a := n \to \frac{n^n (x-1)^n}{n!}$$

```
> a(n+1);
```

$$\frac{(n+1)^{(n+1)}(x-1)^{(n+1)}}{(n+1)!}$$

```
> limit(a(n+1)/a(n),n=infinity);
```

$$ex - e$$

Using MAPLE session above, or otherwise, find the open interval of convergence $I = (a, b)$ for the power series

$$\sum_{n=1}^{\infty} \frac{n^n (x-1)^n}{n!}.$$

 iv) Consider the set S consisting of the vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ from $\mathbb{R}^3$ and let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$.

 a) Find scalars λ and μ such $\mathbf{u} = \lambda \mathbf{v_1} + \mu \mathbf{v_2}$.
 b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $\mathbf{v_1}, \mathbf{v_2}$ as eigenvectors with eigenvalues 2 and -1, respectively.
 α) Find $T(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$.
 β) Denote $T(T(\mathbf{u}))$ by $T^2(\mathbf{u})$, $T(T(T(\mathbf{u})))$ by $T^3(\mathbf{u})$, and so on. Express $T^n(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, where n is a positive integer.

v) The two most popular soft drinks in Old South Wales are AppleAde and BananAde. Assume that no-one in Old South Wales likes both of these drinks equally (that is, everyone has a preference for one or the other). Past statistics show that 50 % of the population prefer AppleAde.

Last month the manufacturer advertised AppleAde on television for a week. After that, a survey was conducted by taking a random sample of 100 people. Of the 100 people sampled, 60 preferred AppleAde and 40 preferred BananAde.

a) Assuming that the advertising had **no effect** on people's preferences, write down an expression for the tail probability that 60 or more people preferred AppleAde in a sample of 100.

b) Use the normal approximation to the binomial to calculate the tail probability in (a), giving your answer to 3 decimal places.

c) Giving reasons, is there evidence that the advertising campaign increased the percentage of the population that prefer AppleAde?

3. i) Prove that the function $T : \mathbb{P}(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}, \quad \text{for all polynomials } p \in \mathbb{P}(\mathbb{R}),$$

is a linear transformation.

ii) Given a square matrix A, the *matrix exponential* e^A is defined by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots .$$

(You may assume without proof that this series always converges.)

Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

a) Find N^2.

b) Hence or otherwise, calculate the matrix exponential e^N.

c) Prove that if a matrix P is idempotent (that is, $P^2 = P$), then $e^P = I + (e-1)P$.

iii) Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the **ordered** basis $\mathcal{B} = \{\cos(x), \sin(x)\}$. Define the linear map $T : S \to S$ by

$$T(f) = f - 2f' \qquad \text{where} \quad f' = \frac{df}{dx}.$$

a) Calculate the matrix C that represents T with respect to the basis $\mathcal{B}$.

b) State the rank of the matrix C found in part (a).

c) From part b), what can be deduced about the solutions $y \in S$ of

$$y - 2y' = g$$

where g is a given function in S?

d) Using parts a) and c), or otherwise, find all non-zero solutions $y \in S$ satisfying

$$y - 2y' = \cos x\,.$$

A company finds that on average 0.05 of its customers return a particular product as faulty within the six year warranty period. Assuming that the number of returned products follows an exponential distribution, find, to 3 decimal places, the probability that for a given product the first claim occurs within the warranty period.

iv) Two (not necessarily distinct) vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are chosen at random from the (finite) set of vectors

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \; : \; x, y \in \{-1, 0, 1\} \right\} .$$

Define A to be the event that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

a) Show that $P(A) = \frac{16}{27}$.

b) Define the discrete random variable $X = \dim(\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\})$.
Copy and complete the following table for the probability distribution $p_k = P(X = k)$.

k	0	1	2
p_k			

c) Calculate $E(X)$.

d) If the vector $\mathbf{b}$ is chosen randomly from S, state the value of $P(B|A)$, where B is the event that $\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

4. i) Show that the equation of the tangent plane to the paraboloid S given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

at the point $P(x_0, y_0, z_0)$ on S is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{z + z_0}{2c}\,.$$

ii) Current observations of the Universe suggest that the reciprocal, u, of the matter density satisfies a differential equation that can be written as

$$\left(\frac{du}{dt}\right)^2 = k^2(2u + \epsilon u^2),$$

where k is a positive constant, $\epsilon = \pm 1$ and t is time. Which of the two values ϵ takes is still a matter of debate, but u is never negative. The radius of the universe is a positive fractional power of u, so if $u \to 0$ the universe is crushed to a point.

Take time $t = 0$ to be the present and assume that $u(0) = 1$. Measurements tell us that $\dfrac{du}{dt}$ is currently positive.

a) Solve the differential equation in the $\epsilon = 1$ case and show that in this case there is no value of $t > 0$ for which $u = 0$. (In this case the universe grows without bound.

b) Repeat the previous part with $\epsilon = -1$ and show that in this case there **is** a $t > 0$ for which $u = 0$. (In this case the universe is said to undergo a Big Crunch).

iii) The following MAPLE session may assist you with this question.

```
> x:=-8/3*t^3 + 12*t^2 +2:
> y:=(t+1)^2*(t-3)^2:
> factor( diff(x,t)^2 + diff(y,t)^2 );
```

$$16(t-3)^2(t^2+1)^2$$

Let C be the curve given parametrically by

$$\left(-\frac{8}{3}t^3 + 12t^2 + 2, (t+1)^2(t-3)^2\right).$$

a) A parametric curve $\big(x(t), y(t)\big)$ has vertical tangent when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$. Show that the curve C has exactly one vertical tangent.

b) Find the arc length of the curve C from $t = 0$ to $t = 3$.

iv) Suppose that y satisfies the initial value problem

$$\frac{dy}{dx} + y^2 = \cos(x) \quad \text{with} \quad y(0) = 0\,.$$

Using implicit differentiation, or otherwise, find the first two non-zero terms of the Maclaurin series of y.

v) Let c be a positive real number and define the sequence $\{a_n\}$ for $n \geq 1$ by

$$a_n = n(c^{1/n} - 1).$$

Find, with reasons, $\lim\limits_{n\to\infty} a_n$.

PAST EXAM SOLUTIONS

MATH1231 November 2010 Solutions

1. i) Take $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$. Both vectors lie in S, but their sum, $\mathbf{x}+\mathbf{y} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, does NOT lie in S. Hence S is not closed under vector addition.

ii) a) No, since $2\mathbf{v_1} - 3\mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$.

b) $\mathbf{w} \in \text{span}(D)$ if and only if $b + c - a = 0$.

c) We obtain $\mathbf{u}$ from $\mathbf{w}$ by putting $a = 3, b = 1, c = 2$. This gives the system

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & -11 & -33 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Put $\alpha_3 = \lambda$, then backsubstitution gives $\alpha_2 = -3\lambda + \frac{8}{11}, \alpha_1 = 2\lambda + \frac{17}{11}$.

d) Write the solution to (c) as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{17}{11} \\ \frac{8}{11} \\ 0 \end{pmatrix}.$$

Then the kernel of A is the homogenous solution which is $\text{span}\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}$. This vector is a basis for the kernel.

e) $\text{rank}(A) = 3 - \text{nullity}(A) = 2$.

f) $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} \right\}$.

iii) a)
$$\begin{aligned} I_1 &= \int \cosh^3 x \, dx \\ &= \int \cosh x (1 + \sinh^2 x) \, dx \\ &= \sinh x + \frac{\sinh^3 x}{3} + C. \end{aligned}$$

b) Put $x = 2\sin\theta$, then
$$\begin{aligned} I_2 &= \int_0^{\pi/6} \frac{4\sin^2\theta . 2\cos\theta}{2\cos\theta} \, d\theta \\ &= 4\int_0^{\pi/6} \sin^2\theta \, d\theta \\ &= 2\int_0^{\pi/6} 1 - \cos 2\theta \, d\theta \\ &= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} \\ &= \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{2}. \end{aligned}$$

iv) The corresponding homogeneous equation is $y'' - 5y' + 4y = 0$, which has characteristic equation $\lambda^2 - 5\lambda + 4 = 0$, whose roots are $\lambda = 1, 4$.

Hence the homogeneous solution is $y_h = Ae^x + Be^{4x}$.
We propose a particular solution of the form $y_p = Ce^{2x} + Dx + E$ (none of whose components is a solution of the homogeneous equation).
Substituting into the differential equation we obtain:

$$-2Ce^{2x} + 4Dx + (4E - 5D) \equiv 6e^{2x} + 12x.$$

Equating coefficients and solving, we have $C = -3, D = 3, E = \frac{15}{4}$.
Hence the general solution is $y = Ae^x + Be^{4x} - 3e^{2x} + 3x + \frac{15}{4}$.

2. i) $I_6 = \frac{5}{6}I_4 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2}I_0 = \frac{5\pi}{32}$.

ii) a) $\frac{\ln n}{\sqrt{n}} \to 0$ as $n \to \infty$, since $\ln n$ grows more slowly than $\sqrt{n}$.

b) $1-\frac{1}{n}$ approaches 1 as n goes to infinity, but $(-1)^n$ oscillates, hence the given sequence does not have a limit.

iii) a) No, since as $k \to \infty, a_k = \frac{k}{k+1} \to 1 \neq 0$.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2+1} < \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}$ and this series converges by the p-series test. Hence the given series converges by comparison.

c) With $a_k = \frac{e^k}{k!}$, we have $\frac{a_{k+1}}{a_k} = \frac{e}{k+1} \to 0$ as $k \to \infty$. Since this limit is less than 1, the series converges by the ratio test.

iv) a) Set $\det(A - \lambda I) = 0$, so

$$\left|\begin{pmatrix} 5-\lambda & -8 \\ 1 & -1-\lambda \end{pmatrix}\right| = 0.$$

Expanding out this gives $\lambda^2 - 4\lambda + 3 = 0$ with roots $\lambda = 3, 1$. These are the eigenvalues.

For $\lambda = 3$, the kernel of $(A - 3I) = \ker\begin{pmatrix} 2 & -8 \\ 1 & 4 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$

For $\lambda = 1$, the kernel of $(A - I) = \ker\begin{pmatrix} 4 & -8 \\ 1 & -2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.

Thus $\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ are the corresponding eigenvectors.

b) Put $P = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, then $A = PDP^{-1}$.

c) Hence, $A^8 = PD^8P^{-1}$ so $A^8P = PD^8 = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3^8 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 \times 3^8 & 2 \\ 3^8 & 1 \end{pmatrix}$.

v) The columns of B are the images of the standard basis vectors of $\mathbb{R}^3$.

$$T(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 1\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

$$T(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = 3\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

$$T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 5\begin{pmatrix}1\\3\\5\end{pmatrix}.$$

Hence $B = \begin{pmatrix}1 & 3 & 5\\3 & 9 & 15\\5 & 15 & 25\end{pmatrix}$.

3. i) Define the following events:
A - 'Team A wins the final', B- 'Team B loses the final', C- 'Team C loses the final', D- 'Team D loses the final'.
Then $P(A) = P(B \cap (C \cup D)) = P(B) \times P(C \cup D)$
$= P(B) \times (P(C) + P(D) - P(C \cup D)) = 0.8 \times (0.3 + 0.6 - 0.3 \times 0.6) = 0.567$.

ii) a) $E(X) = (0 \times 0.2) + (1 \times 0.3) + (2 \times 0.1) + (3 \times 0.2) + (4 \times 0.2) = 1.9$
$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 5.7 - 1.9^2 = 2.09$

b) $\mathrm{VAr}(Y) = (-2)^2\mathrm{Var}(Y) = 8.36$.

iii) a) $\mathrm{Binomial}(15, \frac{1}{2}, k)$.

b) $P(X \geq 12) = \sum_{k \geq 12} \mathrm{Binomial}(15, \frac{1}{2}, k)$
$= \frac{1}{2^{15}}\left[\binom{15}{12} + \binom{15}{13} + \binom{15}{14} + \binom{15}{15}\right] = \frac{9}{512} \approx 0.0176.$

c) Yes, as 0.0176 is small (less than 0.05).

iv) We can write $\mathbf{b} = \begin{pmatrix}1\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\-1\end{pmatrix}$. Hence

$$A\mathbf{b} = A(\frac{2}{3}\mathbf{v_1} - \frac{1}{3}\mathbf{v_2}) = 2\mathbf{v_1} - \frac{1}{3}\mathbf{v_2} = \frac{1}{3}\begin{pmatrix}11\\1\end{pmatrix}.$$

4. i) Re-write the equation in standard linear form:

$$\frac{dy}{dx} + \sec^2 xy = \sec^2 x. \qquad (*)$$

The integrating factor is $\phi(x) = e^{\int \sec^2 x\, dx} = e^{\tan x}$.
Multiplying the differential equation (*) by the integrating factor, we obtain

$$\frac{d}{dx}\left(ye^{\tan x}\right) = \sec^2 xe^{\tan x}.$$

Integrating we have

$$ye^{\tan x} = e^{\tan x} + C \Rightarrow y = 1 + Ce^{-\tan x}.$$

Finally, we apply the initial condition, $y(0) = 3$ to obtain $C = 2$ and so the solution is $y = 1 + 2e^{-\tan x}$.

ii) a) The formula for the tangent plane to the surface $z = F(x,y)$ is given by $z = z_0 + F_x(x - x_0) + F_y(y - y_0)$. Now $F_x = 2x + y^2 = 5$ at P, $F_y = 2xy + 4y^3 = 8$ at P. Substitution gives:

$$z = 7 + 5(x-2) + 8(y-1) \Rightarrow 5x + 8y - z = 11.$$

b) The vector normal to the surface is the vector normal to the tangent plane which we can read off as $\begin{pmatrix} 5 \\ 8 \\ -1 \end{pmatrix}$.

iii)

$$\begin{array}{ll} f(x) = \frac{1}{x} & f(1) = 1 \\ f'(x) = \frac{-1}{x^2} & f'(1) = -1 \\ f''(x) = \frac{2}{x^3} & f''(1) = 2 \\ f'''(x) = \frac{-6}{x^4} & f'''(1) = -6 \\ f^4(x) = \frac{24}{x^5} & f^4(1) = 24 \end{array}$$

Hence the Taylor polynomial is

$$P_4(x) = 1-(x-1)+\frac{2}{2!}(x-1)^2-\frac{6}{3!}(x-1)^3+\frac{24}{4!}(x-1)^4 = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4.$$

iv) a) The velocity vector is given by $\mathbf{v} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$ so the speed is $|\mathbf{v}| = \sqrt{4t^2 + 9t^4} = t\sqrt{9t^2+4}$.

b) The arc length is the given by $\displaystyle\int |\mathbf{v}|\,dt = \int_0^1 t\sqrt{9t^2+4}\,dt = \left[\frac{1}{27}(9t^2+4)^{\frac{3}{2}}\right]_0^1 = \frac{1}{27}(13\sqrt{13} - 4)$.

v) The volume in the tank satisfies the equation $\frac{dV}{dt} = 3 - 1 = 2$ and the initial condition $V = 40$ at $t = 0$. The volume of liquid in the tank at time t is therefore $V = 40 + 2t$, hence the ratio of salt to liquid in the tank at time t is $\frac{x(t)}{40+2t}$. The inflow rate of salt is 6gm/min. The outflow rate of salt is $\frac{x(t)}{40+2t} \times 1$.
Hence the differential equation that models this situation is given by:

$$\begin{aligned} \frac{dx}{dt} &= \text{inflow rate of salt} - \text{outflow rate of salt}, \\ \frac{dx}{dt} &= 6 - \frac{x(t)}{40+2t}. \end{aligned}$$

MATH1231 November 2011 Solutions

1. i) a) Yes, since it contains more than 3 vectors in $\mathbb{R}^3$.
   b) No, at least three vectors in S must be linearly independent.
   c) Let $A = (\mathbf{v_1}|\mathbf{v_2}|\mathbf{v_3}|\mathbf{v_4}|)$ then row-reducing,

$$A \to \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

   Hence $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$ forms a basis for $\mathbb{R}^3$.

   ii) $\mathbf{0} \in S$ since $A\mathbf{0} = \mathbf{0}$.
   If $\mathbf{x}, \mathbf{y} \in S$ then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. Hence $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0}$.
   Also, $A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda\mathbf{0} = \mathbf{0}$. Hence S is a subspace of $\mathbb{R}^n$.

   iii) Since rank(T)+ nullity(T) $= m$, nullity(T) $= m - k$.

   iv) a) Let $u = \sin\theta$, then $\frac{du}{d\theta}\cos\theta$. Hence

$$I_1 = \int u^3(1-u^2)^2\, du = \int u^3 - 2u^5 + u^7\, du - \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C.$$

   b) From the MAPLE,

$$I_2 = \int \frac{-2}{x^2+1} - \frac{5}{x^2+2} + \frac{2}{x+3}\, dx = -\frac{1}{2}\log(x^2+2) - \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + 2\log|x+3| + C.$$

   v) a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2+\frac{\ln n}{n}}{3+\frac{1}{n}} = \dfrac{2}{3}$ so the sequence converges.

   b) Since $\lim\limits_{n\to\infty} a_n \neq 0$ the series $\sum^{\infty}(-1)^n a_n$ does not converge.

   vi) Using the ratio test with $b_n = \dfrac{5^n}{n!}$, we consider

$$\frac{b_{n+1}}{b_n} = \frac{5}{n+1} \to 0$$

   as $n \to \infty$. Since this limit is less than 1, the series converges by the ratio test.
   Alternatively, one might observe that the series equals $e^5 - 1$.

2. i) a) Try $y_H = e^{\lambda x}$. The characteristic equation is $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2\imath$.
   Therefore

$$y_H = A\cos 2x + B\sin 2x\,.$$

   To find a particluar solution, try $y_P = C\cos x + D\sin x$ (OR $y_P = C\cos x$ since there is no y' term in the DE). Therefore $y_P{}'' = -C\cos x - D\sin x$.
   Thus

$$y_P{}'' + 4y_P = 3C\cos x + 3D\sin x = \cos x\,.$$

   Hence $C = \frac{1}{3}$ and $D = 0$ (as expected).
   The general solution is this $y(x) = A\cos 2x + B\sin 2x + \frac{1}{3}\cos x\,.$

b) The forcing function $\cos 2x$ is contained in the homogeneous solution $y_H = A\cos 2x + B\sin 2x$. Hence we try the particular solution

$$y_P = Cx\cos 2x + Dx\sin 2x\,.$$

ii) a) $x = 6\Omega$, $y = 12\Omega$. Therefore $\frac{1}{z} = \frac{1}{6} + \frac{1}{12} \Rightarrow z = 4\Omega$.

b)

$$\frac{\partial}{\partial x}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x} + \frac{1}{y}\right) = -\frac{1}{x^2}\,.$$

Thus

$$-\frac{1}{z^2}\frac{\partial z}{\partial x} = \frac{1}{x^2} \Rightarrow \frac{\partial z}{\partial x} = \frac{z^2}{x^2}\,.$$

$\dfrac{\partial z}{\partial y} = \dfrac{z^2}{y^2}$ since the expression for z is symmetric in x and y.

c)

$$\begin{aligned} |\Delta z| &\le \left|\frac{\partial z}{\partial x}\right| |\Delta x| + \left|\frac{\partial z}{\partial y}\right| |\Delta y| \\ &= \frac{4^2}{6^2}0.1 + \frac{4^2}{12^2}0.1 \\ &= \left(\frac{4}{9} + \frac{1}{9}\right)0.1 = \frac{5}{90} \\ &= \frac{1}{18}\Omega \approx 0.0556\Omega \end{aligned}$$

iii) If a transformation is linear it should satisfy $T(\mathbf{0}) = \mathbf{0}$, $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ and $T(\lambda\mathbf{x}) = \lambda(\mathbf{x})$. If it fails any of these conditions then it is NOT a linear transformation.

Consider $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\1\\-1\end{pmatrix}$.

Now $T\left(\begin{pmatrix}2\\2\end{pmatrix}\right) = \begin{pmatrix}4\\4\\-2\end{pmatrix} \neq 2\times T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}4\\2\\-2\end{pmatrix}$

Therefore the transformation is NOT linear.

iv) a) $T(\mathbf{x}) = A\mathbf{x} - 2\mathbf{x} = A\mathbf{x} - 2I_3\mathbf{x} = (A - 2I_3)\mathbf{x} = B\mathbf{x}$. Therefore

$$B = A - 2I_3 = \begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix}.$$

b) For a vector $\mathbf{x}$ to be an element of $\ker(T)$ it must satisfy $T(\mathbf{x}) = \mathbf{0}$. Thus

$$T\left(\begin{pmatrix}-3\\0\\1\end{pmatrix}\right) = B\begin{pmatrix}-3\\0\\1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}-3\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \mathbf{0}\,.$$

Hence $\begin{pmatrix}-3\\0\\1\end{pmatrix} \in \ker(T)$.

c) Note the matrix B is row equivalent to the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$\begin{aligned} \#\text{ of non-leading columns in row equivalent matrix for } B &= \text{nullity}(T) = 2 \\ \#\text{ of leading columns in row equivalent matrix for } B &= \text{rank}(T) = 1 \\ &\equiv \dim(\mathbb{R}^3) - \text{nullity}(T) \\ &= 3 - 2 = 1 \end{aligned}$$

v) a) Use $A\mathbf{v} = \lambda\mathbf{v}$. Thus

$$\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Hence the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has associated eigenvalue 3. Similarly

$$\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} .$$

Hence the eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ has associated eigenvalue -1.

b) The general form of the solution is given by

$$\mathbf{x}(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}, \quad \alpha, \beta \in \mathbb{R} .$$

The initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ together with the general solution satisfy

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \end{pmatrix} .$$

Solving for α and β yields $\beta = -\frac{1}{2} \Rightarrow \alpha = \frac{5}{2}$. Overall

$$\mathbf{x}(t) = \frac{5}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} - \frac{1}{2} \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} .$$

3. i) a)

$$A \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix} .$$

b) We solve

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Row reducing A gives the matrix

$$\begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix},$$

so $x = 2y$, where $y \in \mathbb{R}$. Thus a basis for $\ker A$ is $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

c) The image of A is the column space of A, and the two columns of A are multiples of each other. Thus a basis for the kernel of A is $\left\{\begin{pmatrix}5\\2\end{pmatrix}\right\}$.

d) If $\mathbf{b}$ belongs to the image of A, then $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$. Hence

$$A\mathbf{b} = A^2\mathbf{x} = A\mathbf{x} = \mathbf{b}.$$

Alternatively,

$$A\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}5 & -10\\2 & -4\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix},$$

and so

$$A\lambda\begin{pmatrix}5\\2\end{pmatrix} = \lambda\begin{pmatrix}5 & -10\\2 & -4\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix} = \lambda\begin{pmatrix}5\\2\end{pmatrix},$$

that is,

$$A\mathbf{b} = \mathbf{b}$$

for any vector in the image of A, by part (c).
Alternatively, from part (c), if $\mathbf{b}$ is in the image of A, then $2b_1 = 5b_2$, and so

$$A\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}5 & -10\\2 & -4\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}5b_1 - 10b_2\\2b_1 - 4b_2\end{pmatrix} = \begin{pmatrix}5b_1 - 4b_1\\5b_2 - 4b_2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\end{pmatrix},$$

as required.

e) If $\mathbf{b} \in \ker A$, then $A\mathbf{b} = \mathbf{0} = 0\mathbf{b}$, and so $\mathbf{b}$ is an eigenvector with eigenvalue 0. From (b), we may take $\mathbf{b} = \begin{pmatrix}5\\2\end{pmatrix}$.
From (d), if $\mathbf{b} \in$ image A, then $A\mathbf{b} = 1\mathbf{b}$, and so $\mathbf{b}$ is an eigenvector with eigenvalue 1. From (c), we may take $\mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}$.

ii) We may present the information using a tree diagram or a table. We write L for Liverpool supporters, and $\bar{L}$ for nonsupporters; we also write R for red shirt wearers and $\bar{R}$ for the others.

	L	$\bar{L}$
R	0.05×0.80	0.95×0.30
$\bar{R}$	0.05×0.20	0.95×0.70

With our notation

$$P(L|R) = \frac{P(L \cap R)}{P(R)} = \frac{0.05 \times 0.80}{0.05 \times 0.80 + 0.95 \times 0.30} = \frac{0.04}{0.325} = \frac{8}{65}.$$

iii) a) A binomial distribution could be used, with $p = \frac{1}{2}, n = 11$.

b) There are 8 months in which the temperature increased, 3 months in which the temperature decreased, and one month with no change. We ignore the month in which there is no change.
Let the random variable X be the number of months in which the temperature increases. Then

$$P(X \geq 8) = \sum_{k=8}^{11}\binom{11}{k}\frac{1}{2^k}\frac{1}{2^{11-k}} = \frac{1}{2^{11}}\sum_{k=8}^{11}\binom{11}{k} = 0.113\ldots.$$

iv) a) Of the 16 possible outcomes of four letters, each of which is H or T, two lead to $X = 4$, namely, $HHHH$ and $TTTT$. Thus

$$P(X=4) = \frac{2}{16} = \frac{1}{8}.$$

Alternate solution: if $X = 4$, then the possibilities were $HHHH$ and $TTTT$. The first toss can give any outcome, and the subsequent tosses then have to match the first toss; since the match occurs with probability $\frac{1}{2}$ and there are 3 subsequent tosses,

$$P(X=4) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$$

b) Since $P(X=0) + P(X=2) + P(X=4) = 1$,

$$P(X=0) = 1 - \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

Alternative solution: there are $\binom{4}{2}$ (that is, 6) outcomes that lead to $X = 0$, hence

$$P(X=0) = \frac{6}{16} = \frac{3}{8}.$$

c)

$$E(X) = \sum_x xP(X=x) = 0 + 2 \times \frac{1}{2} + 4 \times \frac{1}{8} = \frac{3}{2}.$$

d) If $X > E(X)$, then either $X = 2$ or $X = 4$. Hence

$$P(X > E(X)) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$

4. i) Application of the reduction formula for $n = 1$ yields

$$I_2 = \frac{1}{4} + \frac{1}{2} I_1.$$

On the other hand,

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}$$

so that

$$I_2 = \frac{1}{4} + \frac{\pi}{8}.$$

ii) a) The total force acting on the ship is given by

$$F = F_{\text{thrust}} + F_{\text{resistance}} = L - kv.$$

Newton's second law states that

$$m\frac{dv}{dt} = F,$$

leading to the equation of motion

$$m\frac{dv}{dt} = L - kv.$$

b) The differential equation in a) has standard form so that the integrating factor is given by

$$\mathrm{IF} = \exp\left(\int \frac{k}{m}\,dt\right) = \exp\left(\frac{k}{m}t\right)$$

without loss of generality. By definition of the integrating factor, multiplication by IF produces

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = \frac{L}{m}e^{\frac{k}{m}t}.$$

Intergration of both sides yields

$$e^{\frac{k}{m}t}v = \frac{L}{m}\frac{m}{k}e^{\frac{k}{m}t} + C$$

so that

$$v = \frac{L}{k} + Ce^{-\frac{k}{m}t}.$$

Since the ship is initially at rest, we are led to the condition

$$0 = v(0) = \frac{L}{k} + C \quad \Rightarrow \quad C = -\frac{L}{k}$$

and, hence,

$$v = \frac{L}{k}\left(1 - e^{-\frac{k}{m}t}\right).$$

c) Since the exponential function is strictly positiv, we conclude that

$$1 - e^{-\frac{k}{m}t} < 1$$

so that $v < L/k$.

d) If T denotes the time at which the speed of the ship reaches $L/(2k)$ then

$$\frac{1}{2}\frac{L}{k} = v(T) = \frac{L}{k}\left(1 - e^{-\frac{k}{m}T}\right).$$

Division by L/k and rearrangement of terms yield

$$e^{-\frac{k}{m}T} = \frac{1}{2}$$

so that application of the logarithmic function produces

$$-\frac{k}{m}T = \ln\frac{1}{2}.$$

Hence,

$$T = -\frac{m}{k}\ln\frac{1}{2} = \frac{m}{k}\ln 2.$$

iii) a) At $x = 1$, we obtain the alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n, \quad a_n = \frac{1}{2n+1}.$$

It is evident that the sequence (a_n) is non-negativ, non-decreasing and approaches 0 for large n, that is,

$$\begin{aligned} &a_n \geq 0 \\ &a_{n+1} \leq a_n \\ &a_n \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

The alternating series test (Leibniz test) then implies that the alternating series converges.

b) The Maclaurin series for the function $\tan^{-1}$ converges at 1, Thus,

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \tan^{-1} 1 = \frac{\pi}{4}.$$

(Note: Technically Abel's theorem is required here which implies that the limit coincides with $\tan^{-1} x$ at $x = 1$. This is not in the course, but may be tacitly assumed here.)

iv) a) The Fundamental Theorem of Calculus implies that

$$\frac{dx}{dt} = \frac{d}{dt}\left(\int_1^t \frac{\cos u}{u} du\right) = \frac{\cos t}{t}.$$

Similarly,

$$\frac{dy}{dt} = \frac{\sin t}{t}.$$

b) At $t = t_1 = \pi/2$, we see that

$$\frac{dx}{dt} = \frac{\cos(\pi/2)}{\pi/2} = 0, \qquad \frac{dy}{dt} = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi} \neq 0.$$

Accordingly, the curve has a vertical tangent at $\big(x(\pi/2), y(\pi/2)\big)$.

c) The next value of t for which the reasoning of b) may be repeated is $t = t_2 = 3\pi/2$.

d) By definition of arc length,

$$\begin{aligned} L &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{\cos t}{t}\right)^2 + \left(\frac{\sin t}{t}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \frac{1}{t} dt \\ &= \ln t\Big|_{t_1}^{t_2} \qquad (t > 0) \\ &= \ln t_2 - \ln t_1 \\ &= \ln \frac{t_2}{t_1} \\ &= \ln 3. \end{aligned}$$

MATH1231 February 2012 Solutions

1. i) a) Suppose $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S$ and $\lambda \in \mathbb{R}$, then $\lambda\mathbf{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$.
Now $\lambda x_1 . \lambda x_2 = \lambda^2 x_1 x_2 \geq 0$, since $x_1 x_2 \geq 0$. Hence $\lambda\mathbf{x} \in S$.

b) For example, $\begin{pmatrix} -1 \\ -3 \end{pmatrix} \in S$, and $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \in S$ but $\mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \notin S$.

ii) a) Consider the system of equationa $\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \lambda_3 \mathbf{v_3} + \lambda_4 \mathbf{v_4} = \mathbf{b}$.
Putting this into matrix form and reducing to echelon form we have:

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & -2 \\ 0 & 0 & 2 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since the $(A|\mathbf{b})$ system has solution, we conclude that $\mathbf{b}$ belong to span(S).

b) No, since the row echelon form of A, has a non-leading column, the system $(A|\mathbf{b})$, for arbitrary $\mathbf{b}$ will not always have a solution.

c) Using the row-echelon form above, we see that a basis for span(S) is:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 8 \\ 7 \end{pmatrix} \right\} = \{\mathbf{v_1}, \mathbf{v_3}, \mathbf{v_4}\}.$$

iii) a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, we have

$$T(\mathbf{x}+\mathbf{y}) = \begin{pmatrix} (x_1+y_1) - (x_3+y_3) \\ 2(x_1+y_1) + (x_2+y_2) \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ 2x_1 + x_2 \end{pmatrix} + \begin{pmatrix} y_1 - y_3 \\ 2y_1 + y_2 \end{pmatrix} = T(\mathbf{x}) + T(\mathbf{y}).$$

Also

$$T(\lambda\mathbf{x} = \begin{pmatrix} \lambda x_1 - \lambda x_3) \\ 2\lambda x_1 + \lambda x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 - x_3 \\ 2x_1 + x_2 \end{pmatrix} = \lambda T(\mathbf{x}).$$

b)

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

iv) a) rank$(T) = 3$ since there are 3 leading columns.

b) nullity$(T) = 2$ since $5 - 3 = 2$.

c) Putting $x_5 = \lambda$ gives $x_4 = \lambda$ and putting $x_3 = \mu$ we have $x_2 = 2\mu - \lambda$ and $x_1 = -3\mu - 2\lambda$. Hence a basis for the kernel of T is

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

d) $b_1 + b_4 = 0, b_2 = b_4 - b_5 = 0.$

e) One possible basis is:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -4 \\ -3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

2. i) a) Set $\det(A - \lambda I) = 0$, then

$$\begin{vmatrix} 4-\lambda & -2 \\ 7 & -5-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 3) = 0.$$

Hence $\lambda = 2$ or $\lambda = -3$.

For $\lambda = 2$, $\ker(A - 2I) = \ker \begin{pmatrix} 2 & -2 \\ 7 & -7 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -3$, $\ker(A + 3I) = \ker \begin{pmatrix} 7 & -2 \\ 7 & -2 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

So we take $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ as the eigenvectors.

b) Since the system can be written as $\frac{d}{dt}\mathbf{y} = A\mathbf{y}$, the general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

ii) a) Let $P(A) = x$ then we can draw up the following diagram.
diagram
Hence

$$P(B) = x \times 0.95 + (1 - x) \times 0.15 = 0.65.$$

Solving we obtain $x = \frac{5}{8} = 0.625$.

b)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.95 \times \frac{5}{8}}{0.65} = 0.913.$$

iii) a) Since f is a density function,

$$1 = \int_0^1 \alpha(1 - x^2)\, dx = \frac{2}{3}\alpha,$$

hence $\alpha = \frac{3}{2}$.

b)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{3}{2} \int_0^1 x(1 - x^2)\, dx = \frac{3}{8}.$$

iv) a) Let X be the number of sixes in 720 rolls of a fair die, the X follows a binomial distribution, $B(720, \frac{1}{6})$ and so

$$P(X \le 100) = \sum_{k=0}^{100} \binom{720}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{720-k}.$$

b) X can be approximated by the normal random variable $Y \sim N(\mu, \sigma^2)$, where $\mu = np = 120$ and $\sigma^2 = npq = 100$.
Then, using the standard normal distribution,

$$P(X \leq 100) \approx P(Y \leq 100.5) = P(Z \leq \frac{100.5 - 120}{10} = P(Z \leq -1.95) \approx 0.026,$$

using the Normal table.

c) Since $0.026 < 0.05$ the tail probability is significantly low. Hence there is good evidence that the die is biased.

v) Let $\mathbf{v}$ be an eigenvalue for B with eigenvalue λ, then since $B\mathbf{v} = \lambda\mathbf{v}$, we have $B^2 = B\lambda\mathbf{v} = \lambda^2\mathbf{v}$. Thus, multiplying both sides of $B^2 = 3B - 2I$ by $\mathbf{v}$ we have

$$B^2\mathbf{v} = 3B\mathbf{v} - 2I\mathbf{v} \Rightarrow \lambda^2\mathbf{v} = 3\lambda\mathbf{v} - 2\mathbf{v}.$$

Since $\mathbf{v} \neq \mathbf{0}$, we conclude that $\lambda^2 - 3\lambda + 2 = 0$ and so $\lambda = 1$ or $\lambda = 2$.

3. i) a) We have the surface $z = \sqrt{x^2 + y^2}$. Using the formula for the normal we find (at $\mathbf{x_0}$)

$$\mathbf{n} = \begin{pmatrix} z_x \\ z_y \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{8}{10} \\ \frac{6}{10} \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}.$$

b) The generic plane perpedicular to $\mathbf{n}$ has the form

$$4x + 3y - 5z = d$$

substituting x_0 gives

$$32 + 18 - 50 = 0 = d.$$

So the equation is $4x + 3y - 5z = 0$.

c) $x = 8, y = 6, \Delta x = 0.02, \Delta y = -0.03$.

$$\begin{aligned} z &= \sqrt{8.02^2 + 5.87^2} \\ &\approx z_0 + \frac{4}{5}0.02 + \frac{3}{5}(-0.03) \\ &= 0.998 \end{aligned}$$

ii) $z = xf(xy)$

$$\begin{aligned} \frac{\partial z}{\partial x} &= f(xy) + xyf'(xy) \\ \frac{\partial z}{\partial y} &= x^2 f'(xy) \end{aligned}$$

therefore

$$\begin{aligned} & x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} \\ &= xf + x^2yf' - x^2yf'(xy) \\ &= xf \\ &= z \end{aligned}$$

iii) a)

$$\begin{aligned} I_1 &= \int \cos^3(\theta)\sin^4(\theta)\mathrm{d}\theta \\ &= \int \cos(\theta)(1-\sin^2(\theta)\sin^4(\theta)\mathrm{d}\theta \\ &= \int \cos(\theta)\sin^4(\theta) - \cos(\theta)\sin^6(\theta)\mathrm{d}\theta \\ &= \frac{\sin^5(\theta)}{5} - \frac{\sin^7(\theta)}{7} + C. \end{aligned}$$

b) We begin by considering a partial fraction expansion of the form

$$\frac{x^2+x}{(x-2)(x^2+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$

which implies

$$x^2 + x = A(x^2+2) + (Bx+C)(x-2).$$

Considering $x = 2$ gives $A = 1$. Comparing x^2 terms, yields $B = 0$. And comparing the constant term gives $2 - 2C = 0 \implies C = 1$.
So

$$\begin{aligned} I_2 &= \int \frac{1}{x-2} + \frac{1}{x^2+2}\mathrm{d}x \\ &= \ln|x-2| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C. \end{aligned}$$

iv) We can calculate and integrating factor

$$F(x) = \exp\left(-3\int \frac{1}{x}\mathrm{d}x\right) = \exp(-3\ln(x)) = \frac{1}{x^3}.$$

So therefore

$$\begin{aligned} \frac{d}{dx}\left(y\frac{1}{x^3}\right) &= \frac{1}{x^7}, \\ y\frac{1}{x^3} &= \frac{-1}{6x^6} + C, \\ y &= \frac{-1}{6x^3} + Cx^3. \end{aligned}$$

And $y(1) = 4 \implies C = \frac{25}{6}$.

v) a) Calculating the appropriate partial derivatives gives

$$\begin{aligned} M &= 2xy \\ N &= x^2 + y^3 \end{aligned}$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

So therefore the equation is exact.

b) Solution is $F(x,y)=C$ where

$$\frac{\partial F}{\partial x}=2xy, \quad \frac{\partial F}{\partial y}=x^2+y^3.$$

This implies

$$F=x^2y+\phi(y)$$
$$F=x^2y+\frac{y^4}{4}+\psi(y)$$

comparing the equations above gives the solution

$$x^2y+\frac{y^4}{4}=C.$$

4. i) Let $M(t)$ denote the mass of salt in the tank after t minutes for $0\leq t\leq 150$. *Intially tank contains 250L of water.
* 5 g/L and Brine is run into the tank at 6L/m
* The mixed liquid in the tank is constantly run off at 1L/m

Note that $\frac{dM}{dt}$ = rate of change of $M(t)$ = rate of salt in − rate of salt out. Now

$$\text{Rate of salt in} = 5\times 6 = 30.$$

Hence, after any time t, the amount of salt in the tank is given by

$$\frac{M(t)}{250+6t-t}.$$

Multiply by the rate of outflow yields that

$$\text{rate of salt out} = \frac{M(t)}{250+5t}\times 1.$$

Thus, we have

$$\frac{dM}{dt}=30-\frac{M(t)}{250+5t}.$$

ii) Assume that $y=e^{\lambda x}$ is a solution then $\lambda^2-4\lambda+4=0$, so $\lambda=2$.
Thus, $y_1(x)=e^{2x}$ and $y_2(x)=xe^{2x}$.
So the solution to the homogeneous equation is $y_H(x)=Ay_1(x)+By_2(x)$. Now we guess the particular solution

$$y_p(x)=\alpha x+\beta.$$

Substituting we have $-4\alpha+4\alpha x+4\beta=x$. This implies

$$\alpha=1/4=\beta.$$

Therefore,

$$y_G(x)=y_H(x)+y_p(x)=Ae^{2x}+Bxe^{2x}+\frac{1}{4}(x+1)$$

is the general solution.

iii) a) Since $a_n := (-1)^n \cos(\pi/n^2) \not\to 0$ then the series diverges by the k^{th} term test.

b) Let $a_n := \frac{\sin^2(3n)}{n^2}$ and see that $a_n \leq 1/n^2$ for all $n \geq 1$. If we consider the series $\sum_{n=1}^{\infty} 1/n^2$ then since $\sum_{n=1}^{\infty} 1/n^2$ converges by the $p-series$ test then $\sum a_n$ converges.

iv) Let $a_n := \frac{(x-1)^n}{\ln n}$. If we consider

$$\lim_{n\to\infty} |a_{n+1}|/|a_n|,$$

then we have that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-1| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = |x-1|.$$

Thus for convergence of the series via the ratio test, we require $|x-1| < 1$. This implies the interval of convergence is $(0, 2)$

v) a)

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}, \quad x \in (-1, 1].$$

b) Note that from i) that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n}, \quad x \in (-1, 1].$$

Thus, $P_6(x) = 1 - x^2 + x^4 - x^6$.

c) By the previous part, it follows that

$$\int_0^1 \frac{1}{1+x^2}\, dx = \int_0^1 1 - x^2 + x^4 - x^6\, dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = 76/105.$$

vi) a) By substituting, $\theta = t$, we have that

$$x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta,$$

and

$$y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta.$$

It now follows that

$$\begin{aligned} [x'(\theta)]^2 + [y'(\theta)]^2 &= [r'(\theta)\cos\theta - r(\theta)\sin\theta]^2 + [r'(\theta)\sin\theta + r(\theta)\cos\theta]^2 \\ &= [r'(\theta)]^2 + [r(\theta)]^2 \qquad\qquad (\cos^2\theta + \sin^2\theta = 1) \end{aligned}$$

By changing variables, the boundary transforms to $\theta_1 \leq \theta \leq \theta_2$ and the desired formula is obtained.

b) Here $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$ and $r = 1 + \sin\theta$. By using the above formula and $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}
\int_{\theta_1}^{\theta_2} 2\pi r(\theta)\cos\theta\sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2\pi(1+\sin\theta)\cos\theta\sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\ d\theta, \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2^{3/2}\pi(1+\sin\theta)\cos\theta\sqrt{(1+\sin\theta}\ d\theta \\
&= 2^{3/2}\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\theta(1+\sin\theta)^{3/2}\ d\theta \\
&= 2^{3/2}\pi\left[\frac{2}{5}(1+\sin\theta)^{5/2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{2^5}{5}\pi = \frac{32}{5}\pi.
\end{aligned}$$

MATH1231 November 2012 Solutions

1. i) Putting the vectors into a matrix, we have

$$\begin{pmatrix} 1 & 3 & 0 & 3 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 2 \end{pmatrix}$$

after row reducing. Hence $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for span(S).

ii) a) We solve $A\mathbf{x} = \mathbf{0}$. So $(A|\mathbf{0}) =$

$$\sim \left(\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right).$$

Hence $x_4 = 0$ and putting $x_2 = 3\alpha, x_3 = 3\beta$, we have $x_1 = -\alpha - \beta$ and so the kernel is $\alpha \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$. Hence a basis for the kernel is

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

b) nullity$(A) = 2$.

iii) a) Suppose $p \in W$, then $p'(0)p''(0) = 0$. Now $(\lambda p)'(0)(\lambda p)''(0) = \lambda^2 p'(0)p''(0) = 0$, hence $\lambda p \in W$.

b) The polynomials p, q where $p(x) = x, q(x) = x^2$ belong to W but it is easy to check that their sum $p + q$, where $(p + q)(x) = x + x^2$ does not.

iv) a) $b = -1, c = 0$.

b) We can hence write

$$J = \int \frac{1}{x-1}\,dx - \int \frac{x+1}{x^2+2x+2} + \int \frac{1}{(x+1)^2+1}$$

$$= \log|x-1| - \frac{1}{2}\log(x^2+2x+1) + \tan^{-1}(x+1) + C.$$

v) The equation is linear. The integrating factor is $e^{-\log(x-1)} = \frac{1}{x-1}$. Multiplying by the integrating factor we have,

$$\frac{d}{dx}\left(\frac{y}{x-1}\right) = \frac{2}{x-1}$$

and so $y = 2(x-1)\log|x-1| + C(x-1)$.
Now $y(2) = 3 \Rightarrow C = 3$ and so $y = y = 2(x-1)\log|x-1| + 3(x-1)$.

vi) a)

$$S = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

b)

$$S = 2\pi \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}\, dt = 2\pi \frac{32}{3} = \frac{64\pi}{3}.$$

2. i) a)

$$\frac{\partial P}{\partial V} = -k\,\frac{T}{V^2}\,, \qquad \frac{\partial P}{\partial T} = k\,\frac{1}{V}\,.$$

b) Given in the problem the information $\dfrac{\Delta V}{V} = +4\%$ and $\dfrac{\Delta T}{T} = -3\%$. The change in the pressure is approximately given by

$$\begin{aligned}\Delta P &\approx \frac{\partial P}{\partial V}\Delta V + \frac{\partial P}{\partial T}\Delta T \\ &= -k\,\frac{T}{V^2}\Delta V + k\,\frac{1}{V}\Delta T \\ &= -k\,\frac{T}{V}\frac{\Delta V}{V} + k\,\frac{T}{V}\frac{\Delta T}{T} \\ &= -P\frac{\Delta V}{V} + P\frac{\Delta T}{T}\,.\end{aligned}$$

Thus

$$\frac{\Delta P}{P} \approx -\frac{\Delta V}{V} + \frac{\Delta T}{T} = -(+4\%) + (-3\%) = -7\%$$

Hence the pressure <u>*decreases*</u> by 7%.

ii) a) Consider the function $f(x) = \dfrac{x}{\sqrt{4x^2 - 1}}$ which is continuous on the interval $(\frac{1}{2}, \infty)$. We now investigate the limit as x tends to infinity for function f.

$$\lim_{x\to\infty} \frac{x}{\sqrt{4x^2 - 1}} = \lim_{x\to\infty} \frac{x}{2x\sqrt{1 - \frac{1}{4x^2}}} = \frac{1}{2}\lim_{x\to\infty} \frac{1}{\sqrt{1 - \frac{1}{4x^2}}} = \frac{1}{2} \times \frac{1}{\sqrt{1 - 0}} = \frac{1}{2}\,.$$

Since the limit of the continuous function exists and is $\dfrac{1}{2}$ then the limit of the sequence exists and is also equal to $\dfrac{1}{2}$, i.e.,

$$\lim_{n\to\infty} \frac{n}{\sqrt{4n^2 - 1}} = \frac{1}{2}\,.$$

b) Note this is an example from the notes. Consider

$$b_n = \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\cdots\frac{n}{n} \le \frac{1}{n}\frac{n}{n}\cdots\frac{n}{n} = \frac{1}{n}$$

whenever $n \ge 1$. On the other hand, b_n is always positive and thus $0 \le b_n \le \dfrac{1}{n}$. Hence by the pinching (squeeze, sandwich) theorem $b_n \to 0$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0\,.$$

iii) a) Set $F = 2x + 3y, G = 3x + 2y$ then $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = 3$. Hence the equation is exact.

b) Let the solution be $u(x, y) = c$ then,

$$\begin{array}{ccc} \dfrac{\partial u}{\partial x} = 2x + 3y & \Rightarrow & u(x,y) = x^2 + 3xy + f(y) \\ & & \Downarrow \\ \dfrac{\partial u}{\partial y} = 3x + 2y & \Leftrightarrow & \dfrac{\partial u}{\partial y} = 3x + f'(y) \end{array}$$

Therefore $f'(y) = 2y$ and thus $f(y) = y^2 -$ constant. Overall

$$u(x, y) = x^2 + 3xy + y^2 - \text{constant} = 0$$

and therefore $x^2 + 3xy + y^2 =$ constant is the general solution.

iv) The matrix A is constructed by first calculating the transformation on each of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ of $\mathbb{R}^4$. The resulting vectors form the columns of the matrix A, i.e.,

$$\begin{aligned} A &= \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \end{pmatrix} \\ &= \begin{pmatrix} T\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) & T\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right) & T\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) & T\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 5 \\ 0 & 6 & 0 & -3 \end{pmatrix} \end{aligned}$$

v) a) The eigenvalues are calculated by solving $|A - \lambda I| = 0$, i.e.,

$$\begin{aligned} |A - \lambda I| = 0 = \begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} &= (6-\lambda)(3-\lambda) + 2 \\ &= \lambda^2 - 9\lambda + 20 \\ &= (\lambda - 4)(\lambda - 5) \Rightarrow \lambda = 4, 5\,. \end{aligned}$$

Check: Trace$(A) = 6 + 3 = 9 = 4 + 5 =$ sum of eigenvalues.

- $\underline{\lambda = 4}$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right) \Rightarrow \mathbf{v}_{\lambda=4} = t\begin{pmatrix}1\\2\end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}$$

- $\underline{\lambda = 5}$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \Rightarrow \mathbf{v}_{\lambda=5} = t\begin{pmatrix}1\\1\end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}$$

Check: $A\mathbf{v} = \lambda\mathbf{v}$

b)

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$$

OR

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

vi) a) For a transformation to be linear it must satisfy

$$T(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda T(\mathbf{x}) + \mu T(\mathbf{y})$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$.
Consider

$$\begin{aligned} T(\lambda\mathbf{x} + \mu\mathbf{y}) &= \frac{(\lambda\mathbf{x} + \mu\mathbf{y}) \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\ &= \frac{\lambda\mathbf{x} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} + \frac{\mu\mathbf{y} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\ &= \lambda\frac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} + \mu\frac{\mathbf{y} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\ &= \lambda T(\mathbf{x}) + \mu T(\mathbf{y}) \end{aligned}$$

Hence the transformation T is *linear*.

b) In addition to the zero vector, the $\ker(T)$ in this case is the set of vectors perpendicular (normal, orthogonal) to the vector $\mathbf{b}$ since $\mathbf{x} \cdot \mathbf{b}$ will be equal to zero and hence $T(\mathbf{x}) = \mathbf{0}$. Thus geometrically the kernel of T is a plane through the origin with a normal in the direction of the vector $\mathbf{b}$.

3. i) The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ lies on the line $y = x$, so it does not change under T, that is,

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence this vector is an eigenvector and the eigenvalue is 1.

The vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is perpendicular to the line $y = x$, so it reflects across the line, that is

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence this vector is an eigenvector and the eigenvalue is -1.

ii) Let B, T and L be the events that a worker used the bus, the train or the light rail respectively. By the inclusion-exclusion formula,

$$\begin{aligned} &P(B \cup T \cup L) \\ &\quad = P(B \cap T \cap L) - P(B \cup T) - P(B \cup L) - P(T \cup L) + P(B) + P(T) + P(L), \end{aligned}$$

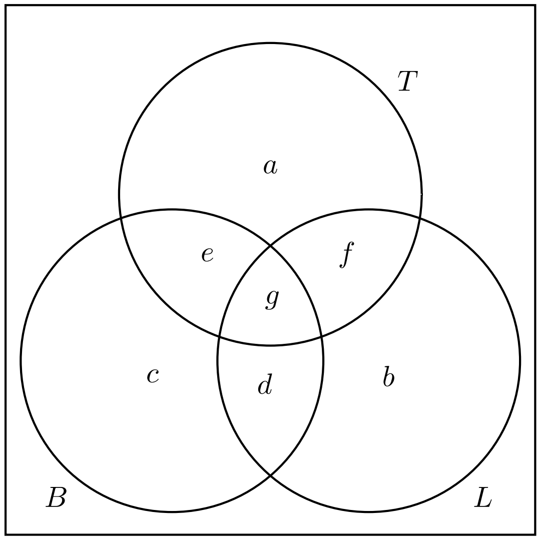


Figure 1: Three sets intersecting

and hence

$$P(B \cap T \cap L) = 0.85 + 0.20 + 0.10 + 0.08 - 0.50 - 0.40 - 0.30 = 0.03.$$

Alternative Solution
This problem can be solved by setting up a Venn diagram, as in Figure 1. Consider the regions with probabilities a, b, c, $\ldots$, as shown. Then we are told that

$$a + b + c + d + e + f + g = 0.85 \tag{8}$$
$$c + d + e + g = 0.50 \tag{9}$$
$$a + e + f + g = 0.40 \tag{10}$$
$$b + d + f + g = 0.30 \tag{11}$$
$$e + g = 0.20 \tag{12}$$
$$d + g = 0.10 \tag{13}$$
$$f + g = 0.08. \tag{14}$$

Solving these equations, by subtracting equations (2) to (4) from (1) and then adding (5) to (7), we find that $g = 0.03$.

Essentially the same solution was also presented just working with the Venn diagram, and putting numbers instead of the letters a, b, and so on.

iii) a) Since f is a pdf,

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_0^a (1 + 4x)\,dx = (x + 2x^2)\big|_0^a = a + 2a^2.$$

Hence $(2a - 1)(a + 1) = 0$, but since $a > 0$, the only relevant solution is $a = 1/2$.

b)

$$E(X) = \int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{1/2} x(1+4x)\,dx = \left(\frac{x^2}{2} + \frac{4x^3}{3}\right)\Bigg|_0^{1/2} = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}\,.$$

c)

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^{1/2} x^2(1+4x)\,dx = \left(\frac{x^3}{3} + x^4\right)\Bigg|_0^{1/2} = \frac{1}{24} + \frac{1}{16} = \frac{5}{48}\,.$$

Hence

$$\mathrm{Var}[(]X) = E(X^2) - E(X)^2 = \frac{11}{576}\,.$$

iv) a) Let X be the number of the rolls that yielded a 6.
Then

$$P(X \geq 68) = \sum_{k=68}^{300} \binom{300}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{300-k}.$$

b) *Assuming that the die is fair*, we see that $\mu = 50$ and $\sigma^2 = 300 \times \frac{1}{6} \times \frac{5}{6} = \frac{125}{3} = 41.666\ldots$, so $\sigma \approx 6.455$. Now, set

$$Z = \frac{X - 50}{6.455}\,.$$

Then $P(X \geq 68) = P(Z \geq 2.788\ldots)$, and Z is approximately normal, and hence (from the tables)

$$P(X \geq 68) \approx 0.003.$$

c) Since this probability is much less than 0.05, we may conclude that it is likely that the die is biased.

v) a) By definition,

$$\begin{aligned} col(AA^T) &= \{AA^T\vec{x} : \vec{x} \in \mathbb{R}^3\} \\ &= \{A\vec{y} : \vec{y} \in col(A^T)\} \\ &\subseteq \{A\vec{y} : \vec{y} \in \mathbb{R}^3\} \\ &= col(A). \end{aligned}$$

Now $\mathrm{rank}(AA^T) \leq \mathrm{rank}(A)$, since $\mathrm{rank}(A) = \dim(col(A)$.

b) First, $col(A) = span\left\{A\begin{pmatrix}1\\0\end{pmatrix}, A\begin{pmatrix}0\\1\end{pmatrix}\right\}$, and so $\mathrm{rank}(A) \leq 2$. It follows that $\mathrm{rank}(AA^T) \leq 2$.
By the rank-nullity theorem,

$$\mathrm{nullity}(AA^T) = 3 - \mathrm{rank}(AA^T) \geq 3 - 2 = 1.$$

The nullity of the identity matrix is 0 and hence AA^T cannot be the identity matrix.

4. i) The series may be formulated as

$$\sum_{k=2}^{\infty} \frac{(x-3)^k}{2^k \log k} = \sum_{k=2}^{\infty} a_k(x) = \sum_{k=2}^{\infty} b_k (x-3)^k.$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{2^{k+1}\log(k+1)} \frac{2^k \log k}{(x-3)^k} \right| \\
&= \lim_{k\to\infty} \frac{|x-3|}{2} \frac{\log k}{\log(k+1)} \\
&= \frac{|x-3|}{2}.
\end{aligned}$$

Therefore, according to the ratio test, the series $\sum_{k=2}^{\infty} |a_k(x)|$ and, hence, the series

$$\sum_{k=2}^{\infty} a_k(x)$$

converges if

$$\frac{|x-3|}{2} < 1.$$

Accordingly, the open interval of convergence is $I = (1, 5)$.
Alternatively, the radius of convergence may be obtained from

$$R = \lim_{k\to\infty} \left| \frac{b_k}{b_{k+1}} \right| = \lim_{k\to\infty} 2 \frac{\log(k+1)}{\log k} = 2$$

since the above limit exists. The open interval of convergence is then given by $I = (3 - R, 3 + R)$ which is in agreement with the previous result.

ii) a) Since

$$\left| \frac{\cos 2n}{(n^3+1)^{\frac{1}{2}}} \right| \leq \frac{1}{(n^3+1)^{\frac{1}{2}}} < \frac{1}{n^{\frac{3}{2}}}$$

and the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

converges, the comparison test implies that the series

$$\sum_{n=1}^{\infty} \frac{\cos 2n}{(n^3+1)^{\frac{1}{2}}}$$

converges absolutely and therefore in the normal sense.

b) Application of the ratio test yields

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^4}{(n+1)!}\frac{n!}{n^4}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4\frac{1}{n+1}=0<1$$

and, hence, the series

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n^4}{n!}$$

converges.

iii) a) Evaluation of

$$\int\frac{1}{1+x}\,dx=\int(1-x+x^2-x^3+\cdots)\,dx$$

produces

$$\log(1+x)+D=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

so that, at $x=0$,

$$\log(1+0)+D=0\quad\Rightarrow\quad D=0.$$

b) Integration of the above result, that is,

$$\int\log(1+x)\,dx=\int\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)dx,$$

leads to

$$(x+1)\log(x+1)-(x+1)+C=\frac{x^2}{2}-\frac{x^3}{2\times 3}+\frac{x^4}{3\times 4}-\frac{x^5}{4\times 5}+\cdots.$$

Evaluation at $x=0$ then produces

$$(0+1)\log(0+1)-(0+1)+C=0\quad\Rightarrow\quad C=1.$$

iv) a) Differentation of

$$\frac{db}{dt}=k_1a-k_2b$$

produces

$$\frac{d^2b}{dt^2}=k_1\frac{da}{dt}-k_2\frac{db}{dt}=-k_1^2a-k_2(k_1a-k_2b)$$

by virtue of

$$\frac{da}{dt}=-k_1a.$$

Hence, the expression

$$\frac{d^2b}{dt^2}+(k_1+k_2)\frac{db}{dt}+k_1k_2b$$

vanishes.

b) The characteristic equation associated with the differential equation

$$\frac{d^2b}{dt^2} + (k_1 + k_2)\frac{db}{dt} + k_1k_2b = 0$$

is given by

$$0 = \lambda^2 + (k_1 + k_2)\lambda + k_1k_2 = (\lambda + k_1)(\lambda + k_2).$$

Hence, if $k_1 \neq k_2$ then the general solution is given by

$$b = Ce^{-k_1t} + De^{-k_2t}.$$

c) If $k_1 = k_2 = k$ then $\lambda = -k$ constitutes a double root and

$$b = Ce^{-kt} + Dte^{-kt}.$$

The initial condition $b(0) = 0$ then produces

$$0 = Ce^0 \quad \Rightarrow \quad C = 0$$

so that

$$b = Dte^{-kt}.$$

d) Differentiation of b as given above leads to

$$\frac{db}{dt} = D(1 - kt)e^{-kt}$$

so that the condition

$$\frac{db}{dt} = 0$$

is equivalent to

$$1 - kt = 0 \quad \Rightarrow \quad t = \frac{1}{k}.$$

MATH1231 November 2013 Solutions

1. i) a) Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in S$ then $a^3 + b^3 + c^3 = 0$. Thus $\lambda^3 a^3 + \lambda^3 b^3 + \lambda^3 c^3 = 0$ so $\begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix} \in S$.

b) For example the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ lie in S, but their sum $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ does not. Hence S is not closed under addition and so does not form a subspace.

ii) a) The MAPLE output shows that A has 4 distinct eigenvalues and hence it has four linearly independent eigenvectors.

b)

$$\mathbf{y}(\mathbf{t}) = Ae^{2t}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 2 \\ 5 \\ 0 \\ 1 \end{pmatrix} + Ce^{5t}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} + D\begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

iii) a) Pr(disc is a movie)$= \frac{3}{4} \times \frac{3}{5} + \frac{1}{4} \times \frac{9}{10} = \frac{27}{40}$. (A tree diagram can also be used).

b) For sets A and B, PR$(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$ and so $\Pr(BR|\text{movie}) = \frac{9}{40}/\frac{27}{40} = \frac{1}{3}$.

iv) a) Using partial fractions,

$$\int \frac{5x^2 - 6x + 4}{x(x-1)(x-2)}\,dx = \int \frac{2}{x} - \frac{3}{x-1} + \frac{6}{x-2}\,dx = 2\log|x| - 3\log|x-1| + 6\log|x-2| + C.$$

b) $I_2 =$

$$\int_0^{\pi/2} \sin^3\theta \cos^2\theta \; d\theta = \int_0^{\pi/2} \sin\theta \times (1 - \cos^2\theta)\cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta\cos^2\theta - \sin\theta\cos^4\theta \, d\theta = -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\Big|_0^{\frac{\pi}{2}} = \frac{2}{15}.$$

v) Separating and integrating

$$\int y^2\,dy = \int \sin x\,dx \Rightarrow y^3 = C - 3\cos\ x.$$

Apppying the initial condition, we find $C = 4$.

vi) $f''(0) = 8$ and $f'''(0) = 0$.

2. i) a) Let $u(x,y) = 2x - 3y^2$ and hence $F(x,y) = f\left(u(x,y)\right)$. Thus

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{df}{du}\frac{\partial u}{\partial x} = f' \times (2) = 2f' \\ \frac{\partial F}{\partial y} &= \frac{df}{du}\frac{\partial u}{\partial y} = f' \times (-6y) = -6yf'\end{aligned}$$

such that $f' \equiv \dfrac{df}{du}$. Therefore

$$3y\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 3y \times 2f' - 6yf' = 6yf' - 6yf' = 0$$

as required.

b) Given $F(x,0) = \cos(x)$ for all x. Note

$$F(x,0) = \cos(x) = \cos\left(\frac{1}{2}(2x)\right) = f(2x)$$

and therefore $f(u) = \cos\left(\frac{1}{2}u\right)$. Hence

$$F(x,y) = f(2x - 3y^2) = \cos\left(\frac{1}{2}(2x - 3y^2)\right).$$

ii) The arc length l of the curve C, given parametrically by $x = t^3$ and $y = 2t^2$ between $t = 0$ and $t = 1$, is given by the integral

$$\begin{aligned}l = \int_{t=0}^{t=1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_{t=0}^{t=1} \sqrt{(3t^2)^2 + (4t)^2}\, dt \\ &= \int_{t=0}^{t=1} \sqrt{16t^2 + 9t^4}\, dt \\ &= \int_{t=0}^{t=1} t\sqrt{16 + 9t^2}\, dt \\ &= \frac{1}{18}\frac{2}{3}\left(16 + 9t^2\right)^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{1}{27}(125 - 64) \\ &= \frac{61}{27}.\end{aligned}$$

The arc length of C is $\frac{61}{27}$ units of length.

iii) a) Try $y_H = e^{\lambda x}$. Characteristic equation $\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 3$. Therefore

$$y_H(x) = Ae^{2x} + Be^{3x}.$$

b) Try $y_P = Cxe^{2x}$ since the forcing function e^{2x}, i.e., RHS of the DE, is proportional to the homogeneous solution $y_H(x)$ (calculated in the previous part). Thus

$$y_P^{'} = Ce^{2x} + 2Cxe^{2x} \quad \text{and} \quad y_P^{''} = 4Ce^{2x} + 4Cxe^{2x}\,.$$

Substituting into the DE with $y_P, y_P^{'}$ and $y_P^{''}$ yields

$$\begin{aligned} y_P^{''} - 5y_P^{'} + 6y_P = 4Ce^{2x} + 4Cxe^{2x} - 5\left(Ce^{2x} + 2Cxe^{2x}\right) + 6Cxe^{2x} &= -Ce^{2x} \\ &= e^{2x}\,. \end{aligned}$$

Therefore $C = -1$ and $y_P(x) = -xe^{2x}$. The general solution $y_G(x)$ is the addition of the homogeneous solution $y_H(x)$ (from the previous part) and the particular solution $y_P(x)$, i.e.,

$$\begin{aligned} y_G(x) &= y_H(x) + y_P(x) \\ &= Ae^{2x} + Be^{3x} - xe^{2x}\,. \end{aligned}$$

iv) a) The vector $\mathbf{e}_1$ has length 1 and thus when it is rotated the transformed vector will have length 2. Thus

$$\begin{aligned} T(\mathbf{e}_1) &= T\left(\begin{pmatrix} 1\cos(0) \\ \\ 1\sin(0) \end{pmatrix}\right) \\ &= \begin{pmatrix} 2\cos\left(0 + \dfrac{\pi}{6}\right) \\ \\ 2\sin\left(0 + \dfrac{\pi}{6}\right) \end{pmatrix} = \begin{pmatrix} 2\cos\dfrac{\pi}{6} \\ \\ 2\sin\dfrac{\pi}{6} \end{pmatrix} = \begin{pmatrix} 2\dfrac{\sqrt{3}}{2} \\ \\ 2\dfrac{1}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ \\ 1 \end{pmatrix} \end{aligned}$$

as required.

b) The matrix A is constructed by first calculating the transformation on each of the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of $\mathbb{R}^2$. Note

$$\begin{aligned} T(\mathbf{e}_2) &= T\left(\begin{pmatrix} 1\cos\left(\dfrac{\pi}{2}\right) \\ \\ 1\sin\left(\dfrac{\pi}{2}\right) \end{pmatrix}\right) \\ &= \begin{pmatrix} 2\cos\left(\dfrac{\pi}{2} + \dfrac{\pi}{6}\right) \\ \\ 2\sin\left(\dfrac{\pi}{2} + \dfrac{\pi}{6}\right) \end{pmatrix} = \begin{pmatrix} 2\cos\dfrac{2\pi}{3} \\ \\ 2\sin\dfrac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} 2\left(-\dfrac{1}{2}\right) \\ \\ 2\dfrac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ \\ \sqrt{3} \end{pmatrix}. \end{aligned}$$

The resulting vectors form the columns of the matrix A, i.e.,

$$A = \left(T(\mathbf{e}_1) \quad T(\mathbf{e}_2)\right) = \left(T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right) = \begin{pmatrix} \sqrt{3} & -1 \\ \\ 1 & \sqrt{3} \end{pmatrix}.$$

v) a)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{2} \frac{x}{2}(2-x)\, dx \;=\; \frac{1}{2}\int_{0}^{2} \left(2x - x^2\right)\, dx$$
$$= \frac{1}{2}\left(x^2 - \frac{1}{3}x^3 \Big|_0^2\right)$$
$$= \frac{2}{3}\,.$$

Similarly

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{0}^{2} \frac{x^2}{2}(2-x)\, dx \;=\; \frac{1}{2}\int_{0}^{2} \left(2x^2 - x^3\right)\, dx$$
$$= \frac{1}{2}\left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \Big|_0^2\right)$$
$$= \frac{2}{3}\,.$$

Therefore

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}\,.$$

b) Note $P(X \le m) = \int_{0}^{m} f(x)\, dx$.

$$\frac{1}{2} = \frac{1}{2}\int_{0}^{m} (2-x)\, dx \quad \Rightarrow \quad 1 = 2m - \frac{1}{2}m^2 \;\Rightarrow\; m^2 - 4m + 2 = 0$$
$$\Rightarrow\; m = 2 \pm \sqrt{2}\,.$$

Since the median must lie in the interval $[0, 2]$ the median value is $m = 2 - \sqrt{2}$.

vi) a) For a set S of n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to be linearly dependent there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all of which are zero, such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}\,.$$

Note another way to say this is at least one of the vectors in S can be written as a linear combination of the others.

b) The vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{0}$ are linearly dependent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{0} = \mathbf{0}$$

is true with not all the λ_is equal to zero.

3. i) Every basis for $\mathbb{P}_3(\mathbb{R})$ has 4 elements and the set S has five elements. Hence S is not a basis for $\mathbb{P}_3(\mathbb{R})$.

   a)

   b) The matrix A consists of the coordinate vectors in $\mathbb{R}^4$ for the polynomials in S and the standard basis elements of $\mathbb{P}_3(\mathbb{R})$. Taking the elements corresponding to leading columns in the part of the row-echelon form matrix corresponding to polynomials in S, the basis is $B = \{p_1, p_2, p_4\}$.

   c) The next leading column in the full row echelon form matrix is the one corresponding to the coordinate vector $\mathbf{b}_2$, that this the polynomial $e(x) = x$. Thus a basis for $\mathbb{R}^4$ can be formed by $B' = \{p_1, p_2, p_4, e\}$.

ii) a) $\boldsymbol{x}$ is in $\ker(A)$ if $A\boldsymbol{x} = 0$. This is solved using Gaussian elemination on A:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 3 \\ 1 & 3 & 4 & 9 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -5 \\ 0 & 1 & 1 & 5 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T$. As the third and fourth columns are non-leading, let $x_3 = \lambda_1$ and $x_4 = \lambda_2$. Then $-x_2 = x_3 + 5x_4 = \lambda_1 + 5\lambda_2$ so $x_2 = -\lambda_1 - 5\lambda_2$. From the first row

$$x_1 = -2x_2 - 3x_3 - 4x_3 = 2\lambda_1 + 10\lambda_2 - 3\lambda_1 - 4\lambda_2 = -\lambda_1 + 6\lambda_2.$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 6 \\ -5 \\ 0 \\ 1 \end{pmatrix}.$$

Thus a basis for the kernel is

$$B = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

   b) As the basis has two elements, the nullity of A, which is the dimension of the kernel, is 2. (Alternatively, the row-echelon form for A has two non-leading columns.)

iii) a) The characteristic equation for C is

$$p(\lambda) = \det(C - \lambda I) = (1 - \lambda)(5 - \lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus the only eigenvalue is $\lambda = 3$.
$C - 3I = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$, so $(C - 3I)\mathbf{b} = 0$ has solution $\mathbf{b} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In particular, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue 3.

   b) If $C = MDM^{-1}$ for some invertible matrix M and diagonal matrix D then the columns of M consist of linearly independent eigenvectors of C. Since C does not have two linearly independent eigenvectors, no such M can exist and hence C is not diagonalizable.

iv) a) $n = 100$ and $p = 0.2$

b) The probability of getting greater than or equal to 30 bottles containing less than 1.05 litres is

$$P(X \leq 30) = \sum_{k=30}^{100} \binom{100}{k} 0.2^k \, 0.8^{100-k}.$$

c) X is approximately normally distributed with mean $\mu = np = 20$ and variance $\sigma^2 = np(1-p) = 16$. Thus, if $Z \sim N(0,1)$,

$$\begin{aligned} P(X \geq 30) &= P\left(\frac{X-\mu}{\sigma} \geq \frac{30-20}{4}\right) \\ &\approx P(Z \geq 2.5) \\ &= 1 - P(Z \leq 2.5) \\ &= 1 - 0.9938 = 0.0062. \end{aligned}$$

d) As $P(X \geq 30) \approx 0.0062 \ll 0.05$, there is strong evidence that the Manufacturer's claim is false.

4. i) a) Let $a_k = \dfrac{2^k+1}{3^k}$. Then

$$\frac{a_{k+1}}{a_k} = \frac{2^{k+1}+1}{3^{k+1}} \times \frac{3^k}{2^k+1} = \frac{1}{3} \times \frac{2+2^{-k}}{1+2^{-k}} \to \frac{2}{3} \quad \text{as } k \to \infty.$$

By the ratio test, the series $\sum_{k=0}^{\infty} a_k$ converges.

b) Let $a_k = \dfrac{1}{\sqrt{k(k+1)}}$. Then

- $\{a_k\}$ is decreasing,
- $a_k \to 0$ as $k \to \infty$.

By Leibniz' test, the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

ii) a) By replacing x in the given formula by x^2 we obtain

$$\sin(x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}.$$

b) By differentiating both sides of the given formula with respect to x, noting that we can differentiate term by term of the power series on the right hand side in its interval of convergnence, we obtain

$$\cos x = \sum_{k=0}^{\infty}(-1)^k (2k+1) \frac{x^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}.$$

c) Letting $x = \pi/4$ in the above formula we deduce

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k}}{4^{2k}(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}.$$

iii) a) By using L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Hence $\lim_{n\to\infty} a_n = 0$.

b) By using the above result and the continuity of the exponential function we have

$$b_n = \exp(\ln(n^{1/n})) = \exp\left(\frac{\ln n}{n}\right) \to \exp(0) = 1.$$

iv) a) Equation (1) is separable, thus

$$\int \frac{dr}{r(1-r)} = \int k\ dt.$$

Partial fractions give

$$\frac{1}{r(1-r)} = \frac{1}{r} + \frac{1}{1-r},$$

so that

$$\int \left(\frac{1}{r} + \frac{1}{1-r}\right) dr = \int k\ dt.$$

Hence

$$\ln r - \ln(1-r) = kt + c,$$

or

$$\ln \frac{r}{1-r} = kt + c,$$

so that

$$\frac{r}{1-r} = e^{kt+c} = \beta e^{kt},$$

where $\beta = e^c > 0$.

b) A) When $t = 0$ we have

$$\beta = \frac{0.01}{1-0.01} = \frac{1}{99}.$$

When $t = 5$ (days) we have

$$\frac{1}{99} e^{5k} = \frac{0.2}{1-0.2} = \frac{1}{4}.$$

Hence

$$k = \frac{1}{5}\ln(99/4) = 0.6418.$$

B) At time t when $r = 80$ % there holds

$$e^{kt} = \frac{1}{\beta} \times \frac{0.8}{1-0.8} = 4 \times \frac{1}{\beta},$$

so that

$$kt = \ln \frac{4}{\beta}.$$

Hence

$$t = \frac{1}{k} \ln \frac{4}{\beta} = 9.32 \text{ days}.$$

c) When $t \to \infty$

$$\frac{r}{1-r} \to \infty$$

so $r \to 1$, i.e., the whole population have heard the rumour.

MATH1231 November 2014 Solutions

1. i) $\sin(A+B) = \sin A\cos B + \cos A \sin B$ and $\sin(A-B) = \sin A\cos B - \cos A\sin B$.
Adding these we have $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$.
Put $A = 5x, B = x$ we have

$$I_1 = \int \sin(5x)\cos(x)\,dx = \int \sin 6x + \sin 4x\,dx = -\frac{1}{12}\cos(6x) - \frac{1}{8}\cos(4x) + C.$$

ii) Let $x = 2\sin\theta$ then $\frac{dx}{d\theta} = 2\cos\theta$. Now $x = 1 \Rightarrow \theta = \frac{\pi}{6}$ and $x = 0 \Rightarrow \theta = 0$. Substituting and simplifying gives

$$I_2 = 4\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta = \int_0^{\frac{\pi}{6}} 2 - 2\cos\theta\,d\theta = \frac{\pi}{3} - \frac{\sqrt{3}}{2}.$$

iii) a) Using the integral test, we consider $\displaystyle\int_3^\infty \frac{dx}{x(\ln x)^3}\,dx$. This converges to $\frac{1}{2(\ln 3)^2}$ which is finite and so the series converges by the integral test.

b) For $n \geq 1, 0 \leq \dfrac{2}{2^n+3^n} < \dfrac{2}{2^n+2^n} = \dfrac{1}{2^n}$. Now $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series which converges to 1 and so the given series converges by comparison test.

iv) a) Suppose $\mathbf{x} \in S$ and λ is a scalar, then $x^2+y^2 = z^2$. Consider the components of $\lambda\mathbf{x}$. We have

$$(\lambda x)^2 + (\lambda y)^2 = \lambda^2(x^2+y^2) = (\lambda z)^2$$

so $\lambda\mathbf{x} \in S$.

b) S is not closed under addition since for example $\begin{pmatrix}0\\2\\2\end{pmatrix} \in S$ and $\begin{pmatrix}2\\0\\2\end{pmatrix} \in S$ but their sum $\begin{pmatrix}2\\2\\4\end{pmatrix} \notin S$, since $2^2+2^2 \neq 4^2$.

v) a) Set $\det(A-\lambda I) = 0$ giving $\lambda^2 - 3\lambda + 2 = 0$, which has solutions $\lambda = 1, 2$. These are the eigenvalues.
For $\lambda = 1$ the eigenspace is the kernel of $A - I = \ker\begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix} = \alpha\begin{pmatrix}1\\-1\end{pmatrix}$ so we take $\begin{pmatrix}1\\-1\end{pmatrix}$ as an eigenvector.
For $\lambda = 2$ the eigenspace is the kernel of $A - 2I = \ker\begin{pmatrix}-2 & -1\\ 2 & 1\end{pmatrix} = \alpha\begin{pmatrix}1\\-2\end{pmatrix}$ so we take $\begin{pmatrix}1\\-2\end{pmatrix}$ as an eigenvector.

b) Tkae $M = \begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$ and $D = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$.

vi) To find the kernel we solve $A\mathbf{x} = \mathbf{0}$. Using the output from the MAPLE session, we put $x_4 = \lambda$ then $x_3 = 7\lambda, x_2 = -4\lambda$ and $x_1 = -5\lambda$. Hence $\ker(A) = \left\{ \lambda \begin{pmatrix} -5 \\ -4 \\ 7 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$ and so $\left\{ \begin{pmatrix} -5 \\ -4 \\ 7 \\ 1 \end{pmatrix} \right\}$ is a basis for the kernel of A.

2. i) a) The integrating factor for the first order linear ODE $\dfrac{dy}{dx} + \left(2 + \dfrac{1}{x}\right) y = \dfrac{2}{x}, x > 0$ is given by

$$e^{\int \left(2+\frac{1}{x}\right) dx} = e^{2x+\ln|x|} = |x|e^{2x} = xe^{2x} \quad \text{since } x > 0.$$

b) Multiplying the ODE by the integrating factor yields

$$\begin{aligned} \frac{d}{dx}\left(xe^{2x}y\right) &= 2e^{2x} \\ \Rightarrow \quad xe^{2x}y &= e^{2x} + C \\ \Rightarrow \quad y &= \frac{1}{x}\left(1 + Ce^{-2x}\right) . \end{aligned}$$

Applying the initial condition $y(1) = 0$ determines the value of C:

$$y(1) = 0 = \frac{1}{1} + \frac{C}{1}e^{-2} \quad \Rightarrow \quad C = -e^2 .$$

Thus the solution to the first order ODE is $y(x) = \dfrac{1}{x}\left(1 - e^{-2(x-1)}\right)$.

ii) Homogeneous solution $y'' + 2y' + 2y = 0$.
Try $y_H = e^{\lambda x}$. Characteristic equation $\lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 1 = 0 \Rightarrow \lambda = -1 \pm \imath$.
Therefore

$$y_H(x) = Ae^{-x}\cos x + Be^{-x}\sin x .$$

Particluar solution $y'' + 2y' + 2y = 20e^{2x}$.
Try $y_P = Ce^{2x}$. Thus

$$y'_P = 2Ce^{2x} \quad \text{and} \quad y''_P = 4Ce^{2x} .$$

Substituting into the DE with y_P, y'_P and y''_P yields

$$\begin{aligned} y''_P + 2y'_P + 2y_P = 4Ce^{2x} + 4Ce^{2x} + 2Ce^{2x} &= 10Ce^{2x} \\ &= 20e^{2x} . \end{aligned}$$

Therefore $C = 2$ and $y_P(x) = 2e^{2x}$. The general solution $y_G(x)$ is the addition of the homogeneous solution $y_H(x)$ and the particular solution $y_P(x)$, i.e.,

$$\begin{aligned} y_G(x) &= y_H(x) + y_P(x) \\ &= Ae^{-x}\cos x + Be^{-x}\sin x + 2e^{2x} . \end{aligned}$$

iii) Applying the Ratio Test to the power series $\sum_{n=1}^{\infty} \frac{n^n (x-1)^n}{n!}$ along with the Maple output yields

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}(x-1)^{n+1}}{(n+1)!}}{\frac{n^n(x-1)^n}{n!}} \right| = e|x-1|\,.$$

For the power series to be convergent $e|x-1| < 1$. Thus the interval of convergence is $I = \left(1 - e^{-1}, 1 + e^{-1}\right)$.

iv) a) To find scalars λ and μ such that $\mathbf{u} = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$ we construct an augmented matrix and use Gaussian elimination:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & -1 & 12 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right).$$

Using back substitution, $\mu = -2$ and $\lambda = 5$ and hence $\begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

b) [α)] Since T is a linear transformation then $T(\mathbf{u}) = -2T(\mathbf{v}_1) + 5T(\mathbf{v}_2)$.
But $T(\mathbf{v}_1) = 2\mathbf{v}_1$ and $T(\mathbf{v}_2) = -\mathbf{v}_2$ since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of T with eigenvalues 2 and -1 respectively. Thus $T(\mathbf{u}) = -2T(\mathbf{v}_1) + 5T(\mathbf{v}_2) = -4\mathbf{v}_1 - 5\mathbf{v}_2$.
[β)] Consider $T^2(\mathbf{u}) = T(-4\mathbf{v}_1 - 5\mathbf{v}_2) = -4T(\mathbf{v}_1) - 5T(\mathbf{v}_2) = -8\mathbf{v}_1 + 5\mathbf{v}_2$ and $T^3(\mathbf{u}) = T(-8\mathbf{v}_1 + 5\mathbf{v}_2) = -8T(\mathbf{v}_1) + 5T(\mathbf{v}_2) = -16\mathbf{v}_1 - 5\mathbf{v}_2$. The pattern for positive integer n is

$$T^n(\mathbf{u}) = -2^{n+1}\mathbf{v}_1 + (-1)^{n+1}(5\mathbf{v}_2)\,.$$

To formally prove this pattern you would use the Principle of Mathematical Induction.

v) a) If advertising has no effect on people's preference then $P(\text{AppleAde}) = 50\% = \frac{1}{2}$ and $P(\text{BananAde}) \equiv P(\text{AppleAde})^c = \frac{1}{2}$. Hence the tail probability for 60 or more people preferring AppleAde is given by the binomial distribution, i.e.,

$$P(X \geq 60) = \sum_{k=60}^{100} \binom{100}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{100-k}.$$

where X is the binomial random variable for the number of people who prefer AppleAde.

b)

$$\begin{aligned} P(X \geq 60) \approx P(Y > 59.5) &= P\left(Z > \frac{59.5 - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \\ &= P(Z > 1.9) \\ &= 1 - P(Z < 1.9) \\ &= 1 - 0.971 \\ &= 0.029 \end{aligned}$$

where Y is the normal random variable approximating X and Z is the standard normal variable.
Note the answer without the correction factor is $P(Z > 2) = 0.023$ or 2.3%.

c) There is a 2.90% chance a group of 60 or more people would prefer AppleAde assuming no effect of advertising. This percentage is lower than our 5% threshold and since 60 out of 100 DID prefer AppleAde we conclude there is some evidence that the advertising campaign increased the percentage of the population who prefer AppleAde.

3. i) Define

$$T(\mathbf{v}) := 2a + (b - c)x \quad \text{for all} \quad \mathbf{v} = (a, b, c).$$

Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ where

$$\mathbf{v} := (a, b, c), \quad \mathbf{w} := (d, e, f).$$

We see that by applying T, we have

$$T(\mathbf{v}+\mathbf{w}) = 2(a+d)+(b+e-(c+f))x = 2a+(b-c)x+2d+(e-f)x = T(\mathbf{v})+T(\mathbf{w}).$$

Also, suppose $\lambda \in \mathbb{R}$ then

$$T(\lambda\mathbf{v}) = 2\lambda a + (\lambda b - \lambda c)x = \lambda T(\mathbf{v}).$$

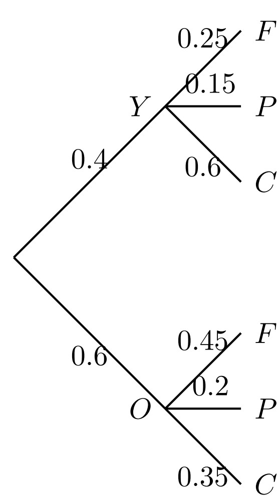
This proves that T is a linear map.

ii) a) The three vectors from the given maple output are $\{\mathbf{v}_1, \mathbf{v}_i, \mathbf{v}_5\}$ where $i = 2, 3, 4$, for example:

$$\begin{pmatrix} 21 \\ 12 \\ -6 \\ -31 \\ -12 \end{pmatrix}, \begin{pmatrix} 11 \\ 20 \\ -14 \\ -29 \\ -21 \end{pmatrix}, \begin{pmatrix} 16 \\ -17 \\ -31 \\ 28 \\ -21 \end{pmatrix}$$

b) The dimension of the span(S) is 3.

iii) a) The tree diagram is:



From the tree diagram,

$$P(fulltime) = 0.4 \times 0.25 + 0.6 \times 0.45 = 37/100.$$

b) Using the conditional probability formula,

$$P(O|F) = \frac{P(O \cap F)}{P(F)}.$$

Hence,

$$P(O|F) = \frac{27}{37}.$$

iv) a) The sum of all the probabilities is 1:

$$1 = P(all\ events) = 0.3 + 4c + 0.35 + 6c + 0.05.$$

Hence, $c = 0.03$.

b) $P(Y = 4) = P(X^2 = 4) = 0.3 + 0.18 = 0.48$.

v) a) The cumulative density function is:

$$F(x) := \begin{cases} 0 & \text{for} \quad x < 0 \\ x^3 & \text{for} \quad 0 \leq x \leq 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

b) The expected value for the continuous function is

$$E(X) = \int_0^\infty x f(x)\ dx = \int_0^1 3x^3\ dx = \frac{3}{4}.$$

c) Using

$$Var(X) = E(X^2) - [E(X)]^2,$$

the variance is

$$Var(X) = 3/5 - 9/16 = 3/80.$$

vi) a) If $\alpha, \beta, \gamma \in \mathbb{R}$ then the set B is linearly independent if and only if the only solution to

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 = \mathbf{0}$$

is $\alpha = \beta = \gamma = 0$.

b) Suppose

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 = \mathbf{0}.$$

Taking the dot product of both sides with $\mathbf{v}_1$ gives $\alpha|\mathbf{v}_1|^2 = 0$ and since $\mathbf{v}_1$ is not the zero vector, we have $\alpha = 0$. Repeating the process with $\mathbf{v}_2$ and $\mathbf{v}_3$, we find $\beta = \gamma = 0$ also. Hence the vectors are linearly independent.

c) By the above, the 3 vectors are linearly independent and since $\mathbb{R}^3$ has dimension 3, the three vectors also span $\mathbb{R}^3$ and so form a basis for $\mathbb{R}^3$.

4. i) a) $S = \int_0^h 2\pi y(x)\sqrt{1+y'(x)^2}\,dx$. Now $y(x) = \dfrac{r}{h}x$ so $\sqrt{1+y'(x)^2} = \sqrt{1+\dfrac{r^2}{h^2}}$ and so

$$S = 2\pi \int_0^h \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\int_0^h x\,dx = 2\pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\frac{h^2}{2} = \pi r\sqrt{r^2+h^2}.$$

b) $\dfrac{\partial S}{\partial r} = \pi\sqrt{r^2+h^2} + \pi r \cdot \dfrac{2r}{2\sqrt{r^2+h^2}} = \dfrac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$.
$\dfrac{\partial S}{\partial h} = \pi r \cdot \dfrac{2h}{2\sqrt{r^2+h^2}} = \dfrac{\pi r h}{\sqrt{r^2+h^2}}$.

c) Using the total differential approximation, the maximum absolute error is

$$\begin{aligned}
|\Delta S| &\approx \left|\frac{\partial S}{\partial r}\Delta r + \frac{\partial S}{\partial h}\Delta h\right| \\
&= \left|\frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\Delta r + \frac{\pi r h}{\sqrt{r^2+h^2}}\Delta h\right| \\
&= \left|\frac{34\pi}{5}\Delta r + \frac{12\pi}{5}\Delta h\right| \\
&\le \frac{34\pi}{5}(0.05) + \frac{12\pi}{5}(0.05) \\
&= \frac{23\pi}{50} \approx 1.45\text{cm}^2.
\end{aligned}$$

ii) a) This equation is separable: $\dfrac{dP}{P-10} = -k\,dt$, so

$$\int \frac{dP}{P-10} = \ln|P-10| = -\int k\,dt = -kt + C.$$

Thus (if $P > 10$)

$$P = 10 + \exp(-kt + C) = 10 + ce^{-kt}$$

for some positive constant c.
Now $P(0) = 20 = 10 + c$, so $c = 10$. Also $P(1) = 18$ so

$$18 = 10 + 10e^{-k}$$

so

$$-k = \ln\frac{8}{10} = \ln\frac{4}{5} \qquad \text{or} \qquad k = \ln\frac{5}{4}.$$

b) 6am corresponds to $t = 8$.

$$P(8) = 10 + 10e^{-8\ln(5/4)} = 11.68^\circ C.$$

iii) a) For all n, $n^{1/n} \ge 1$ and so $\dfrac{1}{2} \le \dfrac{n^{1/n}}{2}$. Each term a_n is equal to either the lower value or the upper value.

b) Let $b_n = \ln n^{1/n} = \dfrac{\ln n}{n}$. By L'Hôpital's Rule (formally applied to $f(x) = \dfrac{\ln x}{x}$, $x \geq 1$)

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

and so, by the continuity of the exponential function,

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{b_n} = \exp(\lim_{n\to\infty} b_n) = e^0 = 1.$$

c) From (a), $\dfrac{1}{2} \leq a_n \leq \dfrac{n^{1/n}}{2}$. Now, from (b)

$$\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n^{1/n}}{2} = \frac{1}{2}.$$

The Pinching Theorem then implies that $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$ too.

d) If $\sum x_n$ converges then $x_n \to 0$ as $n \to \infty$ (the n-th term test). Here $(-1)^n a_n$ doesn't converge at all and hence $\sum(-1)^n a_n$ doesn't converge.

iv) a) The second Taylor polynomial about 0 is

$$\begin{aligned} p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= x - x^2 \end{aligned}$$

from the given data.

b) The remainder is given by

$$R_3(x) = f(x) - p_2(x) = \frac{f'''(c)}{3!}x^3$$

for some c between 0 and x. In particular, if $x = \frac{1}{2}$ then, using the bound on f''',

$$|R_3(1/2)| \leq \frac{6}{3!}\left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Now $p_2(1/2) = \frac{1}{4}$ so this implies that

$$-\frac{1}{8} \leq f(1/2) - p_2(1/2) = f(1/2) - \frac{1}{4} \leq \frac{1}{8}$$

or $\dfrac{1}{8} \leq f(1/2) \leq \dfrac{3}{8}$

c) For any $x > 0$,

$$\frac{x - f(x)}{x^2} = \frac{x - (p_2(x) + R_2(x))}{x^2} = \frac{x^2 + R_3(x)}{x^2} = 1 + \frac{f'''(c_x)x^3}{6x^2} = 1 + \frac{f'''(c_x)}{6}x.$$

Since $|f'''(c_x)/6| \leq 1$, the Pinching Theorem tells us that $\dfrac{x - f(x)}{x^2} \to 1$ as $x \to 0^+$.

PAST HIGHER EXAM SOLUTIONS

MATH1241 November 2010 Solutions

2. i) Let $P = (1, 2, 6)$. Then
$F_x = -50\frac{x}{(x^2+y^2)^2} = -2$ at P
$F_y = -50\frac{y}{(x^2+y^2)^2} = -4$ at P
The equation of the tangent plane is given by $z = 6 + F_x(x-1) + F_y(y-2) = 6 - 2(x-1) - 4(y-2)$ so the desired equation is $2x + 4y + z = 16$.

ii) Using the chain rule,

$$\frac{\partial u}{\partial t} = f'(u + t\frac{\partial u}{\partial t}) \Rightarrow \frac{\partial u}{\partial t}(1 - tf') = uf'.$$

Also

$$\frac{\partial u}{\partial x} = f'(1 + t\frac{\partial u}{\partial x}) \Rightarrow \frac{\partial u}{\partial x}(1 - tf') = f'.$$

Hence $\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}$.

iii) Using the method of partial fractions,

$$\frac{6x^2 - 11x + 9}{(x-1)(x^2-2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-2x+2}.$$

$$\Rightarrow A(x^2 - 2x + 2) + (Bx + C)(x - 1) \equiv 6x^2 - 11x + 9.$$

Putting $x = 1$, gives $A = 4$; equating the coefficients of x^2 gives $B = 2$ and finally by considering the constant terms we have $C = -1$.
Hence

$$I = \int \frac{4}{x-1} + \frac{2x-1}{x^2-2x+2}\,dx = \int \frac{4}{x-1} + \frac{2x-2}{x^2-2x+2} + \frac{1}{(x-1)^2+1}\,dx$$

$$= 4\ln|x-1| + \ln(x^2 - 2x + 2) + \tan^{-1}(x-1) + C.$$

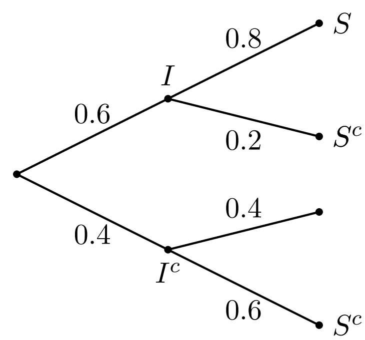
iv) The zero polynomial satifies the differential equation and so belongs to S. Now suppose $p, q \in S$ and consider the polynomial $\lambda p + \mu q$.

$$(x-3)(\lambda p + \mu q)' - 2(\lambda p + \mu q) = (x-3)\lambda p' + (x-3)\mu q' - 2\lambda p - 2\mu q$$

$$= \lambda[(x-3)p' - 2p] + \mu[(x-3)q' - 2q] = 0 + 0 = 0$$

hence $\lambda p + \mu q \in S$ and so S is a subspace of $\mathbb{P}_3(\mathbb{R})$

v) Let I be the event 'a particular person is injured', and S be the event 'a particular person survives 10 years'. We draw a weighted tree diagram as follows.



We want $P(I^c|S) = \frac{P(I^c \cap S)}{P(S)} = \frac{0.16}{0.48+0.16} = 25\%$.

vi) a) since G is a basis,

$$\mathbf{x} = \lambda_1 \mathbf{v_1} + ... + \lambda_n \mathbf{v_n}.$$

Now for $i = 1, 2, ..., n$, by orthogonality, $\mathbf{x v_i} = \lambda_i \mathbf{v_i} . \mathbf{v_i}$ so $\lambda_i = \frac{\mathbf{x}.\mathbf{v_i}}{\mathbf{v_i}.\mathbf{v_i}}$. Hence

$$\mathbf{x} = \frac{\mathbf{x}.\mathbf{v_1}}{\mathbf{v_1}.\mathbf{v_1}} \mathbf{v_1} + ... + \frac{\mathbf{x}.\mathbf{v_n}}{\mathbf{v_n}.\mathbf{v_n}} \mathbf{v_n}.$$

b) Since T is linear,

$$\begin{aligned} T(\mathbf{x}) &= \sum_{i=1}^{n} \frac{\mathbf{x}.\mathbf{v_i}}{\mathbf{v_i}.\mathbf{v_i}} T(\mathbf{v_i}) \\ &= \sum_{i=1}^{n} \frac{\mathbf{x}.\mathbf{v_i}}{\mathbf{v_i}.\mathbf{v_i}} \gamma_i (\mathbf{v_i}.\mathbf{v_i}) \\ &= \sum_{i=1}^{n} \gamma_i (\mathbf{x}.\mathbf{v_i}). \end{aligned}$$

3. i) a) Set $\det(A - \lambda I) = 0$, so

$$\left| \begin{pmatrix} 5-\lambda & -8 \\ 1 & -1-\lambda \end{pmatrix} \right| = 0.$$

Expanding out this gives $\lambda^2 - 4\lambda + 3 = 0$ with roots $\lambda = 3, 1$. These are the eigenvalues.

For $\lambda = 3$, the kernel of $(A - 3I) = \ker \begin{pmatrix} 2 & -8 \\ 1 & 4 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$

For $\lambda = 1$, the kernel of $(A - I) = \ker \begin{pmatrix} 4 & -8 \\ 1 & -2 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

Thus $\left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ are the corresponding eigenvectors.

b) Put $P = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, then $A = PDP^{-1}$.

c) Hence, $A^8 = PD^8P^{-1}$ so $A^8P = PD^8 = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^8 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 \times 3^8 & 2 \\ 3^8 & 1 \end{pmatrix}$.

ii) a) $p = \frac{103}{365}$.

b) binomial$(10, p, 5) = \binom{10}{5} p^5(1-p)^5 \approx 8.59\%$

c) geometric$(1-p, 5) = (1-(1-p))^4(1-p) = p^4(1-p) = 0.0455\%$

iii) a) For $f, g \in S$ and $\lambda, \mu \in \mathbb{R}$,

$$T(\lambda f + \mu g) = (\lambda f + \mu g)' - 2(\lambda f + \mu g)$$

$$= \lambda(f' - 2f) + \mu(g' - 2g) = \lambda T(f) + \mu T(g)$$

and so T is linear.

b) $T(e^{-x}) = -3e^{-x}$, $T(\sin x) = -2\sin x + \cos x$, and $T(\cos x) = -\sin x - 2\cos x$. Hence the matrix representing the linear map with respect to the basis B is

$$C = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

c) By inspection the rank of C is 3.

d) Since the matrix C is invertible, so T is a bijection and has an inverse. Hence $y' - 2y = g$ has a unique inverse solution $(y = T^{-1}(g))$.

e) The coordinate vector of e^{-x} with respect to the basis above is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $C^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ so $C^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 0 \\ 0 \end{pmatrix}$. Hence the solution is $y = -\frac{1}{3}e^{-x}$. (Note: We have only found the entries in C^{-1} that were needed to perform the desired calculation.)

4. i) The equation can be written as

$$\frac{dy}{dx} = \frac{3x^2 - 5xy + 2y^2}{x^2 - xy}$$

and so is a homogeneous equation. We put $y = xu(x)$. This gives

$$u + xu' = \frac{3x^2 - 5x^2u + 2u^2x^2}{x^2 - x^2u} = \frac{3 - 5u + 2u^2}{1 - u}.$$

This equation is now separable. After a little algebra we can separate the variables and obtain

$$\int \frac{1-u}{3u^2 - 6u + 3}\, du = \int \frac{dx}{x}$$

$$-\frac{1}{3}\int \frac{1}{u-1}\, du = \int \frac{dx}{x}$$

$$\ln|u-1| = \ln|x^{-3}| + K$$

Exponentiating both sides and replacing u with $\frac{y}{x}$ we obtain the solution, $y = x + \frac{A}{x^2}$, where A is a constant.

ii) a) $0 \leq \dfrac{k}{\ln k!} < \dfrac{k}{k\ln k - k + 1} = \dfrac{1}{\ln k - 1 + \frac{1}{k}} \to 0$ as $k \to \infty$. Hence the limit is 0 by the pinching theorem.

b) Note the $k! = k(k-1)...1 < \underbrace{k.k.....k}_{k \text{ times}} = k^k$.

Hence $\dfrac{k}{\ln k!} > \dfrac{k}{\ln k^k} = \dfrac{1}{\ln k}$. So,

$$\sum_{k=2}^{\infty} \frac{k}{\ln k!} > \sum_{k=2}^{\infty} \frac{1}{\ln k} > \sum_{k=2}^{\infty} \frac{1}{k}$$

and the latter series diverges. Hence $\displaystyle\sum_{k=2}^{\infty} a_k$ diverges by comparison.

iii) With $a_n = \frac{1}{2^n\sqrt{n}}$, the radius of convergence is given by

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}} = 2$$

iv) a)

$$\begin{array}{ll} f(x) = \sinh x & f(0) = 0 \\ f'(x) = \cosh x & f'(0) = 1 \\ f''(x) = \sinh x & f''(0) = 0 \\ f'''(x) = \cosh x & f'''(0) = 1 \end{array}$$

and the pattern repeats.
Hence the Taylor polynomial is

$$P_4(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5.$$

The remainder term is given by $R_6(x) = \frac{\sinh c}{6!}x^6$.

b)

$$\int_0^x \frac{\sinh t}{t}\,dt = \int_0^x \frac{t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5}{t}\,dt$$

$$= \int_0^x 1 + \frac{1}{3!}t^2 + \frac{1}{5!}t^4\,dt = x + \frac{x^3}{18} + \frac{x^5}{600}.$$

v) The surface area formula is given by

$$S = \int 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Substitution yields,

$$S = 2\pi b\int_0^{\pi} \sin t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$= 2\pi b \int_0^{\pi} \sin t \sqrt{a^2 + (b^2 - a^2)\cos^2 t}\, dt.$$

We can make the change of variable $u = \cos t$ to give

$$S = 2\pi b \int_{-1}^{1} \sqrt{a^2 + (b^2 - a^2)u^2}\, du$$

$$= 2\pi b\sqrt{b^2 - a^2} \int_{-1}^{1} \sqrt{u^2 + \frac{a^2}{b^2 - a^2}}\, du.$$

This integral is hard to do and wise students moved on to the next question at this stage. For completeness of the solutions, I continue.

The integral $J = \displaystyle\int \sqrt{u^2 + k^2}\, du$ can be done as follows: Put $u = k\sinh\theta$. This gives

$$J = \int k^2 \cosh^2\theta\, d\theta = \frac{k^2}{2}\int 1 + \cosh 2\theta\, d\theta$$

$$= \frac{k^2}{2}\left(\theta + \frac{1}{2}\sinh 2\theta\right) = \frac{k^2}{2}\left(\sinh^{-1}\frac{u}{k} + \frac{u}{k}\sqrt{1 + \left(\frac{u}{k}\right)^2}\right)$$

using $\sinh 2\theta = 2\sinh\theta\cosh\theta$.

Hence

$$\int_{-1}^{1} \sqrt{u^2 + k^2}\, du = k^2 \sinh^{-1}\frac{1}{k} + \sqrt{k^2 + 1}.$$

Using this result, with $k^2 = \frac{a^2}{b^2 - a^2}$, and noting that $1 + k^2 = \frac{b^2}{b^2 - a^2}$, we have

$$S = 2\pi b\sqrt{b^2 - a^2}\left(\frac{a^2}{b^2 - a^2}\sinh^{-1}\frac{\sqrt{b^2 - a^2}}{a} + \frac{b}{\sqrt{b^2 - a^2}}\right)$$

$$= 2\pi b\left(\frac{a^2}{\sqrt{b^2 - a^2}}\sinh^{-1}\frac{\sqrt{b^2 - a^2}}{a} + b\right).$$

vi) The volume of liquid in the tank at time t is $V = 40 + 2t$, hence the ratio of salt to liquid in the tank at time t is $\frac{x(t)}{40+2t}$. The inflow rate of salt is 6gm/min.

The outflow rate of salt is $\frac{x(t)}{40+2t} \times 1$.

Hence the differential equation that models this situation is given by:

$$\frac{dx}{dt} = \text{inflow rate of salt} - \text{outflow rate of salt},$$

$$\frac{dx}{dt} = 6 - \frac{x(t)}{40 + 2t}.$$

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2. i) a)

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

whenever the limit exists

b) Using the above formula, we see that

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Alternatively, noticing that $f(0,y) = 0$ then it follows that $f_y(0,0) = 0$.

ii) a) Total resistance, $\frac{1}{z_0} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$, so $z_0 = 4$.

b) Differentiating both sides of

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

with respect to x yields,

$$\frac{-1}{z^2}\frac{dz}{dx} = \frac{-1}{x^2}$$

and from here by rearranging, the desired result follows.

c) Let $x_0 = 6$, $y_0 = 12$. To compute the maximum absolute error in the calculated value of the total resistance z; $|\Delta z|$, consider

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

By taking absolute values and using the triangle inequality, we have

$$|\Delta z| \le |\frac{\partial z}{\partial x}\Delta x| + |\frac{\partial z}{\partial y}\Delta y|.$$

Since $|\Delta x| \le 0.1$ and $|\Delta y| \le 0.1$; we have

$$|\Delta z| \le |\frac{z_0^2}{x_0^2}|0.1 + |\frac{z_0^2}{y_0^2}|0.1 = \frac{1}{10}(\frac{16}{36} + \frac{16}{144}) = 0.0556.$$

iii) a) Since there are 3 vectors (same dimension as $\mathbb{R}^3$), it suffices to determine if the vectors are linearly independent. To save time, consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = w.$$

This can be written as the augmented matrix;

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & a \\ -1 & 1 & 2 & b \\ -3 & 2 & 3 & c \end{array}\right).$$

By row-reducing this matrix, we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & a \\ 0 & 3 & 9 & a+b \\ 0 & 0 & 0 & a-8b+3c \end{array}\right).$$

If $w = 0$ then this implies that the vectors; v_1, v_2, v_3 are linearly dependent and thus $\{v_1, v_2, v_3\}$ does not span $\mathbb{R}^3$. Alternatively, at this stage; we see that there are only 2 "leading" columns resulting in the same conclusion.
The condition such that w belongs to the span(v_1, v_2, v_3) is $a - 8b + 3c = 0$.

b) No, since $5 - 48 + 0 = -43 \neq 0$.

iv) a) Let CR be the event that a student gets a CR or higher,

$$P(CR) = 0.65 \times 0.45 + 0.35 \times 0.30 = 0.3975.$$

Thus,

$$P(CR^c) = 0.6025.$$

b)

$$P(A|CR) = \frac{P(A \cap CR)}{P(CR)} = \frac{0.65 \times 0.45}{0.3975} = 0.736.$$

Thus as a percentage, we have the final answer to be 73.6

3. i) a) The differential equation is homogeneous since

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

meaning we can use the substitution

$$y = xu, \qquad \frac{dy}{dx} = u + x\frac{du}{dx}.$$

The resulting differential equation is separable:

$$u + x\frac{du}{dx} = u - u^2$$

$$\int -\frac{1}{u^2} du = \int \frac{1}{x} dx$$

and has solution $\frac{1}{u} = \ln|x| + c$. Hence

$$\frac{x}{y} = \ln|x| + c$$

$$y = \frac{x}{\ln|x| + c}.$$

b) If $(x, y) = (1, 1)$, then $1 = \frac{1}{0 + c}$ and $c = 1$. Hence the particular solution is

$$y = \frac{x}{\ln|x| + 1}.$$

ii) a)

b) We use the comparison test to show convergence. First, we use a comparison to eliminate the factorial. Since $k! < k^k$,

$$\frac{\ln k!}{k^3} < \frac{\ln k^k}{k^3} = \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}.$$

Now we use a comparison to elimate the logarithm. For any $\alpha > 0$, $\ln k < k^\alpha$ when k is sufficiently large. For our purposes, choose $\alpha = 1/2$, so for k sufficiently large,

$$\frac{\ln k}{k^2} < \frac{k^{1/2}}{k^2} = \frac{1}{k^{3/2}}.$$

Finally, since $\sum \frac{1}{k^{3/2}}$ converges, $\sum_{k=2}^{\infty} \frac{\ln k!}{k^3}$ converges via the comparison test.

c) First, we show the alternating series converges using the Leibniz Test. To see that the sequence converges to zero, notice that for k sufficiently large,

$$a_k = \frac{\ln k}{\sqrt{k^2+1}} < \frac{k^{1/2}}{\sqrt{k^2+1}} < \frac{k^{1/2}}{\sqrt{k^2}} = \frac{1}{k^{1/2}}$$

so via the Pinching Theorem, $\lim_{k\to\infty} a_k = 0$. Next, we need the sequence to be (eventually) decreasing. Let

$$f(x) = \frac{\ln k}{\sqrt{k^2+1}}$$

$$f'(x) = \frac{\frac{1}{k}\sqrt{k^2+1} - \ln k \frac{k}{\sqrt{k^2+1}}}{k^2+1} = \frac{1 - k^2(\ln k - 1)}{k(k^2+1)^{3/2}}$$

which is negative for k sufficiently large ($k \geq 4$), which shows the sequence is (eventually) decreasing. Hence, $\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{\sqrt{k^2+1}}$ converges by the Leibniz Test. [Note, the fact that the sequence might not be initially decreasing does not prohibit us from using the Leibniz Test since convergence depends only on the tail of the series and not a finite number of initial terms.]

We now show the convergence is not absolute using the comparison test. For $k \geq 3$,

$$\frac{\ln k}{\sqrt{k^2+1}} > \frac{1}{\sqrt{k^2+1}} > \frac{1}{\sqrt{k^2+3k^2}} = \frac{1}{2k}.$$

Since $\sum \frac{1}{2k}$ diverges, the series $\sum_{k=2}^{\infty} \frac{\ln k}{\sqrt{k^2+1}}$ also diverges. We conlude that $\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{\sqrt{k^2+1}}$ converges conditionally.

iii) a) We use the Maclaurin series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} \cdots$$

b) The ODE is separable:

$$\int \frac{1}{y} dy = \int e^{x^2} dx = \int \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \right) dx.$$

The Maclaurin series can be integrated term by term:

$$\int \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \right) dx = \sum_{k=0}^{\infty} \left(\int \frac{x^{2k}}{k!} dx \right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!} + c.$$

Hence, the general solution is $\ln|y| = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!} + c$. By letting $x = 0$ and $y = 1$, we get $0 = 0 + c$ so the particular solution is $\ln|y| = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!}$.

c) We use the ratio test to see when the series is convergent:

$$\begin{aligned}
\lim_{k\to\infty} \left[\frac{|x|^{2(k+1)+1}}{(2(k+1)+1)(k+1)!} \times \frac{(2k+1)k!}{|x|^{2k+1}} \right] &= \lim_{k\to\infty} \left(|x|^2 \times \frac{2k+1}{2k+3} \times \frac{1}{k} \right) \\
&= |x|^2 \times \left(\lim_{k\to\infty} \frac{2k+1}{2k+3} \right) \times \left(\lim_{k\to\infty} \frac{1}{k} \right) \\
&= |x|^2 \times 1 \times 0 \\
&= 0
\end{aligned}$$

which is less than 1 for every x. Hence the solution is defined for all real x.

iv) a) We proceed by induction to show $a_{k+1} > a_k$. The base case is $k = 1$:

$$a_2 = \sqrt{1 + a_1} = \sqrt{2} > 1 = a_1.$$

Now assume the hypothesis for $k = n$: i.e. $a_{n+1} > a_n$. We will verify it for $k = n+1$:

$$a_{n+2} = \sqrt{1 + a_{n+1}} > \sqrt{1 + a_n} = a_{n+1}.$$

Hence, $a_{k+1} > a_k$ is true for all $k \geq 1$.

b) We show $a_k < 3$ by induction. The base case is $k = 1$, and $a_1 = 1 < 3$. Now assume $a_n < 3$. Then $a_{n+1} = \sqrt{1 + a_n} < \sqrt{1 + 3} = 2 < 3$. Hence $\{a_k\}_{k=1}^{\infty}$ is bounded above by 3.

c) Since the sequence is increasing and bounded above, it must converge: i.e. there is a number L such that $\lim_{k\to\infty} a_k = L$.

d) Since the sequence converges, $a_{k+1} = \sqrt{1 + a_k} \approx a_k$ when k is large. Thus we solve for L in the equation $L = \sqrt{1 + L}$. Equivalently, it is the positive solution of $L^2 - L - 1 = 0$. Hence $L = \frac{1+\sqrt{5}}{2}$, which is the Golden Ratio.

4. i) a) The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned} p_A(x) = \det(A - xI) &= \begin{vmatrix} 22-x & -100 \\ 5 & -23-x \end{vmatrix} \\ &= (22-x)(-23-x) - (-100)5 \\ &= x^2 + x - 6 \\ &= (x-2)(x+3)\,, \end{aligned}$$

that is, $\lambda = -3, 2$.
Let us now find the eigenvectors corresponding to $\lambda = -3$;
these are the non-zero vectors in $\ker(A + 3I)$:

$$\begin{pmatrix} 25 & -100 \\ 5 & -20 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}$$

Thus, the eigenvectors corresponding to $\lambda = -3$ are $t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ for all $t \neq 0$.
Let us now find the eigenvectors corresponding to $\lambda = 2$;
these are the non-zero vectors in $\ker(A - 2I)$:

$$\begin{pmatrix} 20 & -100 \\ 5 & -25 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -5 \\ 0 & 0 \end{pmatrix}$$

Thus, the eigenvectors corresponding to $\lambda = 2$ are $t\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ for all $t \neq 0$.

b) $D = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$, or $D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ and $P = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$,
or these matrices with columns scaled by non-zero constants.

c)

$$\begin{aligned} A^n P = PD^n = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}^n &= \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-3)^n & 0 \\ 0 & 2^n \end{pmatrix} \\ &= \begin{pmatrix} 4(-3)^n & 5 \times 2^n \\ (-3)^n & 2^n \end{pmatrix} \end{aligned}$$

ii) a) Since

$$1 = \sum_{k=0}^{4} p_k = \sum_{k=0}^{4} \frac{c}{2^k} = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)c = \frac{31}{16}c\,,$$

we see that $c = \dfrac{16}{31}$.

b) $P(X = 2) = \dfrac{16}{31 \times 2^2} = \dfrac{4}{31}$.

c)

$$\begin{aligned} P((X-2)^2 < 4) &= P(X=1) + P(X=2) + P(X=3) \\ &= \frac{16}{31 \times 2} + \frac{16}{31 \times 4} + \frac{16}{31 \times 8} \\ &= \frac{14}{31}\,. \end{aligned}$$

iii) a) Suppose that non-zero matrix $C \in M_{22}$ satisfies $C^T = \lambda C$.
Then $C^T = \lambda C = \lambda (C^T)^T = \lambda(\lambda C)^T = \lambda^2 C^T$.
Since C^T is non-zero, $\lambda^2 = 1$, so $\lambda = \pm 1$.

b) Since

$$T(\mathbf{v}_1) = \mathbf{v}_1 \qquad T(\mathbf{v}_2) = \mathbf{v}_3 \qquad T(\mathbf{v}_3) = \mathbf{v}_2 \qquad T(\mathbf{v}_4) = \mathbf{v}_4$$

we see that

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) $\operatorname{rank} T = \operatorname{rank} B = 4$.

d) Let λ be an eigenvalue of T.
Then $C^T = T(C) = \lambda C$ for some matrix $C \in M_{22}$, so by part a), $\lambda = \pm 1$.
Conversely, $\lambda = -1$ and $\lambda = 1$ are both eigenvalues of T:

$$T(I) = I = 1 \times I \qquad \text{and} \qquad T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (-1) \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[Part b) could also have been used here.]

e) A basis for the eigenspace of T corresponding to $\lambda = -1$ is $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

A basis for the eigenspace of T corresponding to $\lambda = 1$ is
$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

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2. i)

$$\frac{\partial u}{\partial x} = f'(s)\frac{1}{y}, \frac{\partial u}{\partial y} = -f'(s)\frac{x}{y^2}.$$

ii)

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}.$$

iii) a) area $A =$

$$= \int_{-a/2}^{a/2} \frac{2b}{a}\sqrt{a^2-x^2}dx \tag{15}$$

$$= \frac{2b}{a}\int_{\pi/2}^{2\pi/3} a^2\sin^2(\theta)d\theta, \quad x = a\cos(\theta) \tag{16}$$

$$= 2ab\int_{\pi/3}^{2\pi/3} \frac{1-\cos(2\theta)}{2}d\theta = ab\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right) \tag{17}$$

b) $\Delta A = \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial b}\Delta b$. $\frac{\Delta A}{A} = \frac{\Delta a}{a} + \frac{\Delta b}{b} = 0.04 - 0.01 = 3\%$.

iv) a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & a \\ 2 & 3 & 1 & 7 & b \\ 1 & 4 & -2 & 11 & c \\ 3 & 1 & 5 & 0 & d \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & a \\ 0 & 1 & -1 & 3 & 2a-b \\ 0 & 0 & 0 & 0 & 5a-2b-c \\ 0 & 0 & 0 & 0 & 7a-5b+d \end{array}\right) \tag{18}$$

b) $5a - 2b - c = 0$ and $7a - 5b + d = 0$

c) $\mathbf{a} = (-3a + 2b + \gamma)\mathbf{v_1} + (2a - b - 3\gamma)\mathbf{v_2} + \gamma\mathbf{v_3}$ for some parameter γ.

d) A hyper plane in $\mathbb{R}^4$

v) a)

$$P(4-2T \geq 0) = P(T \leq 2) = \int_{-\infty}^{2} f(x,\lambda)dx = [-\exp(-\lambda x)]_{x=0}^{x=2} = 1 - \exp(-2\lambda) \tag{19}$$

b)

$$E(T^2) = \int_{-\infty}^{0} x^2\lambda\exp(-\lambda x)dx = \lim_{R\to\infty}\left[-x^2\exp(\lambda x)\right]_{x=0}^{x=R} - \int_0^\infty (-2x)\exp(-\lambda x)dx \tag{20}$$

$$= \frac{2}{\lambda^2} \tag{21}$$

3. i) a) Let X be the number of times a 6 is rolled out of 300 rolls. Assuming the die is unbiased, $X \sim \text{Bin}(300, \frac{1}{6})$ and the tail probability that measures the chance of getting a 6 at least 68 times is $P(X \geq 68)$.

b) We have $\text{E}(X) = 600 \cdot \frac{1}{6} = 50$ and $\text{Var}(X) = 600 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{125}{3}$ and so $X \overset{\text{approx.}}{\sim} \text{N}\left(50, \frac{125}{3}\right)$. Thus

$$\begin{aligned} P\left(X > 67.5\right) &= P\left(Z > \frac{67.5 - 50}{\sqrt{\frac{125}{3}}}\right) \\ &= P\left(Z > 2.71\right) \\ &= 1 - 0.9966 \\ &= 0.0026 \end{aligned}$$

c) Yes, since $0.0026 < 0.05$.

ii) a) P(stop after 13-th roll)= P(don't stop after 6 rolls)$\times$ P(not roll a six on 7-th roll)$\times P$(roll six 6's in a row) $= \left(1 - \left(\frac{1}{6}\right)^6\right) \cdot \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^6$.

b) P(roll at least ten times) $= 1 - P$(stop after 6,7,8 or 9 times). Now

$$\begin{aligned} P(\text{stop after 6 rolls}) &= \left(\frac{1}{6}\right)^6 \\ P(\text{stop after 7(or 8 or 9) rolls}) &= P(\text{don't roll 6 on 1st (2nd or 3rd) roll }) \\ &\quad \times P(\text{ roll six 6's in a row}) \\ &= \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right)^6 \end{aligned}$$

$$P(\text{roll at least ten times}) = 1 - \left(\left(\tfrac{1}{6}\right)^6 + \left(\tfrac{5}{6}\right) \cdot \left(\tfrac{1}{6}\right)^6 \cdot 3\right)$$

iii) a) Note: column space and image mean the same thing, when speaking about matrices. $\mathbf{v} \in \text{col}(AA^T)$. That is, there exists $\mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v} = AA^T\mathbf{w}$. Let $\mathbf{u} = A^T\mathbf{w}$. Then $\mathbf{v} = A\mathbf{u}$ and so we see that $\mathbf{v} \in \text{col}(A)$, which implies $\text{col}(\text{AA}^\text{T}) \subseteq \text{col}(\text{A})$. Since $\text{rank}(A) = \dim \text{col}(A)$ we get $\text{rank}(AA^T) \leq \text{rank}(A)$.

b) Note that AA^T is a 3×3 matrix. Suppose $AA^T = I$ then $\text{rank}(AA^T) = 3$. However, since A is 3×2, $\text{rank}(A) \leq 2$ and so (a) implies $\text{rank}(AA^T) \leq 2$. This is a contradiction, and so $AA^T \neq I$.

iv) a) If $MM^T = I$ then $\det(MM^T) = 1$. Since $\det M^T = \det M$ and $\det(MM^T) = \det M \cdot \det M^T$ we get $(\det M)^2 = 1$ and so $\det M = \pm 1$.

b) Note that $\mathbf{m}_i = M\mathbf{e}_i$ and recall that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $\mathbf{v}.\mathbf{u} = \mathbf{v}^T\mathbf{u}$. So, $\mathbf{m}_i.\mathbf{m}_j = M\mathbf{e}_i.M\mathbf{e}_j = (M\mathbf{e}_i)^T M\mathbf{e}_j = \mathbf{e}_i^T M^T M\mathbf{e}_j = \mathbf{e}_i^T I\mathbf{e}_j = \mathbf{e}_i^T\mathbf{e}_j = \mathbf{e}_i.\mathbf{e}_j$ which is zero if $i \neq j$ and one, otherwise.

c) $(A\mathbf{v}).\mathbf{w} = (A\mathbf{v})^T\mathbf{w} = \mathbf{v}^T A^T\mathbf{w} = \mathbf{v}.(A^T\mathbf{w})$.

d) $|M\mathbf{v}|^2 = M\mathbf{v}.M\mathbf{v} = \mathbf{v}.M^T M\mathbf{v} = \mathbf{v}.I\mathbf{v} = \mathbf{v}.\mathbf{v} = |\mathbf{v}|^2$. Thus $|M\mathbf{v}| = |\mathbf{v}|$.

e) Since M is 3×3 the characteristic polynomial of M is real and of degree 3. Any real polynomial of degree 3 must have at least one real root and thus M must have at least one real eigenvalue. Furthermore a real eigenvalue corresponds to a real eigenvector and vice versa.

f) If $\mathbf{u}$ is a real eigenvector then it corresponds to a real eigenvalue λ. We have $M\mathbf{u} = \lambda\mathbf{u}$ and so $|M\mathbf{u}| = |\lambda||\mathbf{u}|$. However, part (d) implies $|M\mathbf{u}| = |\mathbf{u}|$ and so $|\lambda||\mathbf{u}| = |\mathbf{u}|$. Since λ is real, we conclude that $\lambda = 1$ or -1.

4. i) a) The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \iff \quad (\lambda+3)^2 = 0 \quad \iff \quad \lambda = -3, -3.$$

So the general solution takes the form $y = Ae^{-3x} + Bxe^{-3x}$. Substituting $y(0) = 0$ yields $A = 0$. Hence the general solution is

$$y = Bxe^{-3x}.$$

b)

$$\frac{dy}{dx} = Be^{-3x} - 3Bxe^{-3x} = Be^{-3x}(1-3x) = 0 \quad \iff \quad x = \frac{1}{3}$$

ii) a)

$$\left.\frac{dy}{dx}\right|_{(2a,a)} = \frac{2a \cdot a}{(2a)^2 - a^2} = \frac{2}{3}$$

Thus the gradient of normal at $(2a, a)$ is $-3/2$. The equation of normal is therefore

$$y - a = -\frac{3}{2}(x - 2a) \quad \iff \quad y = -\frac{3}{2}x + 4a.$$

b) This is a separable equation.

$$\begin{aligned}\int \frac{1}{y}\,dy = \int \frac{x}{x^2-a^2}\,dx \quad &\iff \quad \ln|y| = \frac{1}{2}\ln|x^2-a^2| + c \\ &\iff \quad y^2 = A|x^2-a^2|, \quad A \geq 0.\end{aligned}$$

When $|x| > a$, we obtain $y^2 = A(x^2 - a^2)$, $A \geq 0$, which is a hyperbola. When $|x| < a$, we obtain $y^2 = -A(x^2 - a^2)$, $A \geq 0$, which is an ellipse.

iii) a) Sketch: $f(0) = 0$, $f(9) = -1/2$, $f(10) = 10$. $f(x) \to -\infty$ as $x \to (\sqrt{99})^-$, $f(x) \to \infty$ as $x \to (\sqrt{99})^+$, and $f(x) \to 0^+$ as $x \to \infty$.

b)

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2 - 99} = \sum_{n=0}^{9} \frac{(-1)^n n}{n^2-99} + \sum_{n=10}^{\infty} \frac{(-1)^n n}{n^2-99}$$

For $n \geq 10$, we see from the previous sketch that $a_n = n/(n^2 - 99)$ is decreasing and bounded below by 0. By Leibniz's alternating series test, $\sum_{n=10}^{\infty} \frac{(-1)^n n}{n^2-99}$ converges, and so does $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2-99}$.

iv) a)

$$f(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} t^{2k}}{(2k)!}$$

b)

$$\mathrm{Cin}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \int_0^x t^{2k-1}\, dt}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)!\,(2k)}$$

The series converges for all x.

v) a) Note that

$$x(1-x) = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} < \frac{1}{4} \quad \text{for} \quad x \neq \frac{1}{2}.$$

We have $a_1 = 1/4 < 1/2$. Suppose that $a_k < 1/2$ for some $k \geq 1$. Then

$$a_{k+1} = 2a_k(1-a_k) < 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

b) Using $a_n < 1/2$, we obtain

$$a_{n+1} - a_n = 2a_n(1-a_n) - a_n = a_n - 2a_n^2 = a_n(1-2a_n) > a_n \cdot 0 = 0.$$

Hence the sequence is strictly increasing

c) Since the sequence is strictly increasing and bounded above, by the monotone convergence theorem it converges.

d) Suppose $\ell = \lim_{n\to\infty} a_n$. Solving $\ell = 2\ell(1-\ell)$ gives $\ell = 0, 1/2$. Since the limit cannot be 0, we conclude that $\lim_{n\to\infty} a_n = 1/2$.

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3. i) a) The variance of the exponential distribution is $\frac{1}{\lambda^2}$. Hence $\text{Var}(8-2T) = 2^2\text{Var}(T) = \frac{4}{\lambda^2}$.

   b) $P(\lambda T \geq 1) = P(T \geq \frac{1}{\lambda}) = 1 - P(T < \frac{1}{\lambda}) = 1 - (1 - e^{-1}) = \frac{1}{e}$.

   ii) Let X = the number of defective bulbs in a box. Then $E(X) = 2$ and $\text{Var}(X) = 2 \times 0.98 = 1.96$. We can approximate the binomial by $X \sim B(100, 0.02)$ by the normal $X \sim N(2, 1.96)$. So

$$P(X > 3.5) = P(Z > \frac{3.5-2}{\sqrt{1.96}}) = P(Z > 1.07) = 1 - P(Z \leq 1.07) \approx 0.1423.$$

   iii) a) There are a red balls out of a total of $a+b$ balls.

   b) $P(R_2) = \frac{a}{a+b} \times \frac{a+c-1}{a+b+c-1} + \frac{b}{a+b} \times \frac{a}{a+b+c-1} = \frac{a}{a+b} = P(R_1)$.

   iv) a) One can either give an inductive proof or perhaps write

$$A^k = \underbrace{A....A}_{k \text{ times}} = \underbrace{M^{-1}DM \times M^{-1}DM...M^{-1}DM}_{k \text{ times}} = M^{-1}D^kM^k.$$

   b) Best shown by induction.

   c) By (ii) the general entry in the matrix sum $D^n + a_{n-1}D^{n-1} + ... + a_0I$ has the form $\lambda_i^n + a_n\lambda_i^{n-1} + ... + a_0$, which is zero, since λ_i is a root of the characteristic polynomial. The result follows.

   d) $P(A) = M^{-1}P(D)M = \mathbf{0})$.

   e) The characteristic polynomial is $x^2 - x - 6$ and so by (iii), $A^2 - A + 6I = 0$ and the result follows.

4. i) Setting $y = xv$, we get

$$x^2\frac{d}{dx}(xv) = 4x^2 + x(xv) + (xv)^2.$$

By the product rule,

$$x^2(v + x\frac{dv}{dx}) = 4x^2 + x^2v + x^2v^2,$$

or

$$x\frac{dv}{dx} = 4 + v^2$$

(assuming $x \neq 0$). Then separating variables, we have

$$\int \frac{1}{x}d = \int \frac{1}{4+v^2}dv.$$

Integrating, we have

$$\frac{1}{2}\tan^{-1}(\frac{v}{2}) = \ln|x| + C,$$

or

$$y = 2x\tan(2ln|x| + C).$$

ii) By the ratio test, the radius of convergence is

$$\lim_{n\to\infty} \frac{\frac{2^n n^2}{n^3+1}}{\frac{2^{n+1}(n+1)^2}{(n+1)^3+1}} = \frac{1}{2},$$

so the power series converges converges on the open interval

$$\{|x| : |x-1| < \frac{1}{2}\} = (\frac{1}{2}, \frac{3}{2}).$$

At the endpoints, the series becomes

$$\sum_{n=1}^{\infty} (\mp 1)^n \frac{n^2}{n^3+1},$$

which converges in the "−" case by the alternating series test, and diverges in the "+" case by the integral test. Therefore the interval of convergence is $[\frac{1}{2}, \frac{3}{2})$.

iii) a) (We use "X" units in g and "t" units in hours.)
(a)
Given $k_1 = k_2 = 2$, we have after separating variables

$$\int (2-X)^{-2} dx = \int k dt.$$

Integrating,

$$(2-X)^{-1} = kt + C.$$

The initial condition $X(0) = 0$ gives $c = \frac{1}{2}$, and then $X(1) = (1)$ gives $k = \frac{1}{2}$. Finally, setting $X = 1.8$ gives $t = 9$ as desired.

b) Using $k = \frac{1}{2}$ from part (a) and $k_1 = 20, k_2 = 2$, we have after separating variables

$$\int (2-X)^{-1}(20-X)^{-1} dx = \int \frac{1}{2} dt.$$

Integrating, we get

$$\frac{1}{18} \ln(\frac{20-X}{2-X}) = \frac{1}{2} t + C.$$

The initial condition $X(0) = 0$ gives $C = \frac{\ln(10)}{18}$, and then setting $X = 1.8$ gives $t = \frac{1}{9} \ln(\frac{91}{10}) \approx .25$.

iv) a) Since f is nondecreasing, we have $c_{k+1} = f(c_k) \geq c_k$ for all k; therefore the sequence is monotone. Since $f([a,b]) \subseteq [a,b]$, we have $a \leq c_k \leq b$ for all k (by induction), so the sequence is bounded. Therefore the sequence converges by the monotone convergence theorem.

b) We have

$$f(L) = f(\lim_{k\to\infty} c_k) = \lim_{k\to\infty} f(c_k) = \lim_{k\to\infty} c_{k+1} = L,$$

where interchanging the function and the limit is allowed because f is continuous.

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3. i) For $q, h \in \mathbb{P}(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$,

$$T(\lambda q + \mu h) = \begin{pmatrix} \lambda q(0) + \mu h(0) \\ \lambda q(1) + \mu h(1) \end{pmatrix} = \lambda \begin{pmatrix} q(0) \\ q(1) \end{pmatrix} + \mu \begin{pmatrix} h(0) \\ h(1) \end{pmatrix} = \lambda T(q) + \mu T(h).$$

ii) a) N^2 is the 2×2 zero matrix.

b) By (a), $e^N = I + N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

c) If $P^2 = P$, then $P^n = P$ for all $n \geq 1$. Hence

$$\begin{aligned} e^P &= I + P + \frac{1}{2!}P + \frac{1}{3!}P + ... = I - P + P(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...) \\ &= I - P + eP = I + (e-1)P. \end{aligned}$$

iii) a) $T(\cos x) = \cos x + 2\sin x$ and $T(\sin x) = \sin x - 2\cos x$, These have coordinates $(1, 2)$ and $(-2, 1)$ respectively with respect to the given basis, so $C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

b) Since the determinant of C is not 0, the columns are linearly independent, and so the rank of T equals the rank of C which is 2.

c) By the rank-nullity theorem, the nullity of T is 0 and hence the map T is invertible and so the d.e. has a unique solution in S.

d) The solution is $y = T^{-1}(\cos x)$. Now the coordinates of the y w.r.t. the basis above are given by

$$C^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence $y = \frac{1}{5}\cos x - \frac{2}{5}\sin x$.

iv) Let $p = 0.05$ and let X be the number of warranty periods when the first claim occurs. Then from $\lambda = \ln \frac{1}{1-p}$ we have $\lambda \approx 0.5129$. Hence $P(X \leq 1) = \int_0^1 \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda} = p = 0.05$.

v) a) The sample space is $S \times S$ which has 81 elements. The event $A = \{(\mathbf{v}_1, \mathbf{v}_2) \in S \times S : \mathbf{v_1}, \mathbf{v_2}$ are linearly independent $\}$. Since there are 8 choices for $\mathbf{v_1}$ and 6 choices for $\mathbf{v_2}$ it follows that $|A| = 48$. Hence $P(A) = \frac{48}{81} = \frac{16}{27}$.

b)

k	0	1	2
p_k	$\frac{1}{81}$	$\frac{32}{81}$	$\frac{16}{27}$

c) $E(X) = \frac{128}{81}$.

d) Since the event $\mathbf{b} \in \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$ is statistically independent of the selection of $\{(\mathbf{v}_1, \mathbf{v}_2)\} \in A$, we have

$$P(B|A) = \frac{P(B)P(A)}{P(A)} = P(B) = 1.$$

4. i) The equation for tangent plane at a point $P(x_0, y_0, z_0)$ on a surface $S \subset \mathbb{R}^3$ given by an equation of the form $f(x,y) = z$ is

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) = (z - z_0).$$

In this question

$$f(x,y) = c(\frac{x^2}{a^2} + \frac{y^2}{b^2}),$$

and therefore

$$\begin{aligned} f_x &= \frac{2cx}{a^2} \\ f_y &= \frac{2cy}{b^2}. \end{aligned}$$

From the above the equation of the tangent plane at P is

$$\frac{2cx_0(x - x_0)}{a^2} + \frac{2cy_0(y - y_0)}{b^2} = (z - z_0).$$

The left hand side can be re-arranged as

$$\begin{aligned} & -2c(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}) + 2c(\frac{xx_0}{a^2} + \frac{yy_0}{b^2}) \\ = \; & -2z_0 + 2c(\frac{xx_0}{a^2} + \frac{yy_0}{b^2}). \end{aligned}$$

Therefore the equation of the tangent is

$$-2z_0 + 2c(\frac{xx_0}{a^2} + \frac{yy_0}{b^2}) = (z - z_0)$$

which after adding $2z_0$ to both sides and then dividing both sides by $2c$ becomes

$$(\frac{xx_0}{a^2} + \frac{yy_0}{b^2}) = \frac{z + z_0}{2c}.$$

ii) a) Since $\frac{du}{dt} > 0$ it is not equal to zero and therefore

$$\frac{dt}{du} = (\frac{du}{dt})^{-1}.$$

So

$$t = \frac{1}{k} \int \frac{1}{\sqrt{2u + \epsilon u^2}} du.$$

If $\epsilon = 1$, then we need to evaluate

$$\int \frac{1}{\sqrt{2u + u^2}} du.$$

We make the substitution $u + 1 = \cosh(\theta)$ to get

$$du = \sinh(\theta) d\theta.$$

Then completing the square gives

$$\begin{aligned}
\int \frac{du}{\sqrt{2u+\epsilon u^2}} &= \int \frac{du}{\sqrt{(u+1)^2-1}} \\
&= \int \frac{\sinh(\theta)d\theta}{\sqrt{\cosh^2(\theta)-1}} \\
&= \int \frac{\sinh(\theta)d\theta}{\sqrt{\sinh^2(\theta)}} \\
&= \int d\theta \\
&= \cosh^{-1}(u+1)+C.
\end{aligned}$$

So

$$\begin{aligned}
kt &= \cosh^{-1}(u+1)+C \\
kt-C &= \cosh^{-1}(u+1) \\
u+1 &= \cosh(kt-C) \\
u &= \cosh(kt-C)-1.
\end{aligned}$$

Since $u(0)=1$ we have $2=\cosh(-C)$ i.e. $4=e^{-C}+e^{C}$. Re-arranging we get $e^{2C}-4e^{C}+1=0$ and so

$$e^C=\frac{4\pm\sqrt{12}}{2}$$

i.e.

$$C=\ln(2\pm\sqrt{3}).$$

To determine which value C takes we observe that $\frac{du}{dt}>0$ and

$$\begin{aligned}
\frac{du}{dt}(0) &= \sinh(-C) \\
&= \frac{1}{2}\left(\frac{1}{2\pm\sqrt{3}}-2\pm\sqrt{3}\right)
\end{aligned}$$

is greater than zero only if $C=\ln(2-\sqrt{3})$.
Now if $u(t)=0$ then

$$\cosh(kt-C)=1$$

so

$$e^{kt-C}+e^{-kt+C}=2$$

and if $x=e^{kt-C}$ we have

$$x^2-2x+1=0$$

i.e.

$$x=1$$

or

$$1=e^{kt-C}$$

which implies that
$$kt = C = \ln(2-\sqrt{3}) < 0$$
which is impossible as k and t are ≥ 0.

b) We need to evaluate
$$\int \frac{1}{\sqrt{2u-u^2}}du.$$
We make the substitution $u-1 = \sin(\theta)$ to get (following the same reasoning as above)
$$\int \frac{1}{\sqrt{2u-u^2}}du = \sin^{-1}(u-1)+C.$$
Therefore
$$u = \sin(kt-C)+1.$$
Now $u(0)=1$ so
$$\sin(-C) = 0$$
i.e.
$$C = m\pi$$
where $m \in \mathbb{Z}$.
Note that since
$$\sin(x-2\pi) = \sin(x)$$
we have $m = 0, 1$, i.e. $C = 0, \pi$. Moreover since u is assumed to be positive (initially, once $u = 0$ it becomes negative, at which point the mathematical model is no longer modeling the physical universe) we must have $m = 0$. That is $C = 0$. If $u(t) = 0$ then
$$\sin(kt) = -1$$
which happens whenever kt is a multiple of $3\pi/2$, and which will happen first when $t = \frac{3\pi}{2k}$.
Note: (this was not needed to answer the question, but it is worth stating) The question states that current observations indicate that $\frac{du}{dt} > 0$; however if $\epsilon = -1$, then once $t > \frac{3\pi}{2k}$ we have $\frac{du}{dt} < 0$, in which case present observations imply that $0 < t < \frac{3\pi}{2k}$.

iii) a)

$$\begin{aligned}
\frac{dx}{dt} &= -8t^2+24t \\
\frac{dy}{dt} &= 2(t+1)(t-3)^2+2(t+1)^2(t-3) \\
&= 2(t+1)(t-3)(t-3+t+1) \\
&= 2(t+1)(t-3)(2t-2).
\end{aligned}$$

Now
$$\frac{dx}{dt} = 0$$

means

$$8t(t-3)=0$$

so $t=0$ or $t=3$, so there are at most 2 vertical tangents at $t=0$ or $t=3$. If $t=0$ then

$$\frac{dy}{dt}=2\times 1\times(-3)\times(-2)\neq 0$$

and therefore there is a vertical tangent at $t=0$. If $t=3$ then

$$\frac{dy}{dt}=0$$

and so there is no vertical tangent at $t=3$.

b) Recall the arc-length formula for a parametric curve from $t=0$ to $t=3$ is

$$\int_0^3\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}dt.$$

Now the maple output tells us that

$$(\frac{dx}{dt})^2+(\frac{dy}{dt})^2=16(t-3)^2(t^2+1)^2$$

and the *positive* squareroot of this last expression *for values of* $t\in[0,3)$ is

$$4(3-t)(t^2+1)$$

so the arc-length is given by

$$\begin{aligned}
& 4\int_0^3(3-t)(t^2+1)dt \\
= & \ 4\int_0^3(-t^3+3t^2-t+3)dt \\
= & \ 4[-t^4/4+t^3-t^2/2+3t]_0^3 \\
= & \ 4(-81/4+27-9/2+9) \\
= & \ (-81+108-18+36) \\
= & \ 45.
\end{aligned}$$

iv) The Maclaurin series for y has the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

and so

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+\cdots.$$

We are told that

$$y(0)=0$$

so

$$a_0 = 0.$$

Now the Maclaurin series for $\cos(x)$ is

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

Expanding y^2 we get

$$y^2 = a_1^2 x^2 + \cdots$$

Substituting the two expressions for y and $\frac{dy}{dx}$ into the equation

$$\frac{dy}{dx} + y^2 = \cos(x)$$

we therefore obtain

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (a_1^2 x^2 + \cdots) = 1 - \frac{x^2}{2!} + \cdots.$$

Comparing co-efficients of powers of x we therefore get

$$a_1 = 1$$

$$a_2 = 0$$

$$3a_3 + 1 = -1/2.$$

Therefore the first two terms of the Maclaurin series for y are given by

$$y = x - \frac{1}{2}x^3 + \cdots.$$

The above argument assumes that we may differentiate the Maclaurin series of y to obtain that for $\frac{dy}{dx}$ and that the Maclaurin series of y^2 is just the square of the Macclaurin series of y. We can reason alternatively as follows:
The Macclaurin series is given by

$$y = y(0) + y'(0)x + y''(0)x^2/2! + y'''(0)x^3/3! + \cdots$$

so we in fact want to find the first two non-zero derivatives of y (since $y(0) = 0$).
From the equation

$$\frac{dy}{dx} + y^2 = \cos(x)$$

we see that

$$y'(0) = 1.$$

Now implicit differentiation, gives

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}y = -\sin(x)$$

and therefore

$$y''(0) = 0.$$

Implicit differentiating one more time gives

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}y + 2(\frac{dy}{dx})^2 = -\cos(x)$$

and so

$$y'''(0) = -3$$

which yields the same result.

v) We use that fact that

$$L = \lim_{n\to\infty} n(c^{1/n} - 1) = \lim_{x\to\infty} x(c^{1/x} - 1).$$

Define a continuous function

$$f(x) = (c^{1/x} - 1)$$

then

$$f'(x) = \frac{-\ln(c)}{x^2}c^{1/x}.$$

Then L is given by

$$\lim_{x\to\infty} xf(x)$$

and can be re-written as

$$L = \lim_{x\to\infty} \frac{f(x)}{\frac{1}{x}}$$

and as the numerator and denominator both approach zero as x approaches infinity, we can apply l'Hôpital's rule to get

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{\frac{-\ln(c)}{x^2}c^{1/x}}{-1/x^2} \\ &= \lim_{x\to\infty} \ln(c)c^{1/x} \\ &= \ln(c). \end{aligned}$$

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$