MATH1081

Discrete Mathematics

Tutorial Problems

and

Past Exam Papers

With Solutions

PROBLEM SETS

Recommended Problems: It is strongly recommended that you attempt all questions marked by †. You should regard these questions as the minimum that you should attempt if you are to pass this course. However, the more practice in solving problems you get the better you are likely to do in class tests and exams, and so you should aim to solve as many of the problems on this sheet as possible. Ask your tutor about any problems you cannot solve.

Problems marked by a star ($*$) are more difficult, and should only be attempted after you are sure you can do the unstarred problems.

Problem Set 0 is for the first tutorial. A tutorial by tutorial schedule of suggested problems will be provided on Moodle.

PROBLEM SET 0

Introductory Problems

The following eight problems will be addressed in your first tutorial in order to introduce just some of the problems that you will learn how to solve during this course. You are not expected to be able to solve them at this point but feel free to see them as challenges and try to solve them anyway. Your tutor will provide some hints and might initiate class discussion on how to solve these problems but will not necessarily provide any full answers in this full tutorial. Most of the answers, and all of the methods required, will be given later throughout the course.

IP1. Without using a calculator, find a simple way to calculate the value of

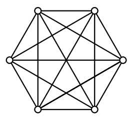
$$(3 \times 3 + 4)^2 + (3 \times 4 + 4)^2 + \cdots + (3 \times 22 + 4)^2 .$$

(*Hint*: See Topic 1 Problem 39)

IP2. Find the last two digits of 7^{1001}.
(*Hint*: Use Topic 2 methods to calculate modulo 100.)

IP3. Prove that $\log_2 7$ is irrational.
(*Hint*: See Topic 3 Problem 19c)

IP4. Six points are pairwise connected by lines:

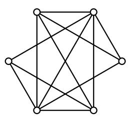


Each of the lines is now coloured either red or blue. Prove that it is possible to find three points that are joined by three lines of the same colour, regardless of how the lines were coloured.
(*Hint*: See Topic 3 Problem 32)

IP5. Prove that it is possible to find two different sets A and B, each containing 10 distinct positive integers no greater than 100, for which the sum of integers in A is equal the sum of integers in B.
(*Hint*: Use Topic 4 methods.)

IP6. We have a $2 \times n$ rectangular board and some 1×2 tiles, some 2×1 tiles, and some some 2×2 tiles. Let a_n be the number of ways in which we can tile the board using these tiles. Prove that, for $n > 2$, $a_n = a_{n-1} + 2a_{n-2}$, and use this identity to find a closed formula for a_n.
(*Hint*: See Topic 4 Problem 37.)

IP7. Is it possible to draw continuously from A to B along the lines, passing each of the 14 lines exactly once?

A  B

(*Hint*: Use Topic 5 methods.)

IP8. My partner and I recently attended a party at which there were four other couples. Various handshakes took place. No-one shook hands with themselves, nor with their partner, and no-one shook hands with the same person more than once. Afterwards, I asked each person, including my partner, how many people they had shaken hands with. To my surprise they each gave a different answer. How many hands did my partner shake?
(*Hint*: Use Topic 5 methods or otherwise.)

PROBLEM SET 1

Basic Set Theory

1. Are any of the sets $A = \{1, 1, 2, 3\}$, $B = \{3, 1, 2, 2\}$, $C = \{1, 2, 1, 2, 4, 2, 3\}$ equal?

†2. Show that
$$A = \{x \in \mathbb{R} \mid \cos x = 1\}$$
is a subset of
$$B = \{x \in \mathbb{R} \mid \sin x = 0\}.$$
Is the first a proper subset of the second? Give reasons.

3. a) List all the subsets of the set $A = \{a, b, c\}$.
   b) List all the elements of $A \times B$ where $A = \{a, b, c\}$ and $B = \{1, 2\}$.

4. Given the sets $X = \{24k + 7 \,|\, k \in \mathbb{Z}\}$, $Y = \{4n + 3 \,|\, n \in \mathbb{Z}\}$, $Z = \{6m + 1 \,|\, m \in \mathbb{Z}\}$, prove that $X \subseteq Y$ and $X \subseteq Z$ but $Y \not\subseteq Z$.

†5. If $S = \{0, 1\}$, find
   a) $|P(S)|$,
   b) $\big|P\big(P(S)\big)\big|$,
   c) $\Big|P\Big(P\big(P(S)\big)\Big)\Big|$.

†6. Determine whether the following are true or false
   a) $a \in \{a\}$,
   b) $\{a\} \in \{a\}$,
   c) $\{a\} \subseteq \{a\}$,
   d) $a \subseteq \{a\}$,
   e) $\{a\} = \{a, \{a\}\}$,
   f) $\{a\} \in \{a, \{a\}\}$,
   g) $\{a\} \subseteq \{a, \{a\}\}$,

7. If A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.

8. Is it true that if $P(A) = P(B)$ for two sets A, B then $A = B$?

Set Operations and Algebra

9. If $A = \{$ letters in the word *mathematics* $\}$ and

   $B = \{$ letters in the words *set theory* $\}$, list the elements of the sets

   a) $A \cup B$,
   b) $A \cap B$,
   c) $A - B$,
   d) $B - A$.

10. Define the sets R, S and T by

$$\begin{aligned} R &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\} \\ S &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 3\} \\ T &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 6\}. \end{aligned}$$

   a) Is $S = T$?
   b) Is $R \subseteq T$?
   c) Is $T \subseteq R$?
   d) Is $T \subseteq S$?
   e) Find $R \cap S$.

†11. In a class of 40 people studying music: 2 play violin, piano and recorder, 7 play at least violin and piano, 6 play at least piano and recorder, 5 play at least recorder and violin, 17 play at least violin, 19 play at least piano, and 14 play at least recorder. How many play none of these instruments?

†12. Prove the following statements if they are true and give a counter-example if they are false.

   a) For all sets A, B and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ then $A \subseteq B$.
   b) For all sets A, B and C, $(A \cup B) \cap C = A \cup (B \cap C)$.

†13. Let A and B be general sets. Determine the containment relation ($\subseteq, \supseteq, =$, none) that holds between

   a) $P(A \cup B)$ and $P(A) \cup P(B)$,
   b) $P(A \cap B)$ and $P(A) \cap P(B)$.

†14. Let A and B be general sets. Determine the containment relation ($\subseteq, \supseteq, =$, none) that holds between

$$P(A \times B) \quad \text{and} \quad P(A) \times P(B)$$

15. Show that $A - B = A \cap B^c$ and hence simplify the following using the laws of set algebra.

   a) $A \cap (A - B)$.
   b) $(A - B) \cup (A \cap B)$.
   c) $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$.

†16. Use the laws of set algebra to simplify

$$(A - B^c) \cup (B \cap (A \cap B)^c).$$

†17. Simplify

$$[A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)]\,,$$

and hence simplify

$$[A \cup (A \cup B^c)] \cap [(A \cup B) \cap (B \cup A^c)]\,.$$

18. Draw a Venn diagram for the general situation for three sets A, B and C. Use it to answer the following:

a) If $B - A = C - A$, what subregions in your diagram must be empty?

b) Prove or produce a counter-example to the following statement:

$$\text{If } B - A = C - A \quad \text{then} \quad B = C.$$

19. Use the laws of set algebra to prove that

$$(R - P) - Q = R - (P \cup Q)$$

for any sets R, P and Q.

*20. Define the *symmetric difference* $A \oplus B$ of two sets A and B to be

$$A \oplus B = (A - B) \cup (B - A).$$

a) Draw a Venn diagram illustrating $A \oplus B$.

b) If $A = \{$ even numbers strictly between 0 and 20 $\}$,

$B = \{$ multiples of 3 strictly between 0 and 20 $\}$, write down the set $A \oplus B$.

c) Explain why $A \oplus B$ can also be written as $(A \cup B) - (A \cap B)$.

d) Suppose A, B and C are sets such that $A \oplus C = B \oplus C$. Prove that $A = B$.

(Hint: You may use a Venn diagram to assist your argument.)

†21. Prove if true or give a counter example if false:

For all sets A, B and C, $\quad A \times (B \cup C) = (A \times B) \cup (A \times C)$.

†22. Let

$$A_k = \{n \in \mathbb{N} \mid k \le n \le k^2 + 5\}$$

for $k = 1, 2, 3, \ldots.$ Find

a) $\displaystyle\bigcup_{k=1}^{4} A_k$;

b) $\displaystyle\bigcap_{k=10}^{90} A_k$;

c) $\displaystyle\bigcap_{k=1}^{\infty} A_k$

23. Repeat the previous question if

$$A_k = \{x \in \mathbb{R} \mid 1 - \frac{1}{k} < x \le k\}.$$

Functions

24. Which of the following are functions?
    a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{x^2 - 1}$.
    b) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x + 1$.
    c) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$.
    d) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = q$ where $x = \frac{p}{q}$, p, q integers.
    e) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = q$ where $x = \frac{p}{q}$, p, q integers with $q > 0$ and no common factor except 1.

25. Recall from lectures that $\lfloor x \rfloor$ is the largest integer less than or equal to x, and that $\lceil x \rceil$ is the smallest integer greater than or equal to x. Evaluate
    a) $\lfloor \pi \rfloor$,
    b) $\lceil \pi \rceil$,
    c) $\lfloor -\pi \rfloor$,
    d) $\lceil -\pi \rceil$.

*26. Prove that if n is an integer, then

$$n - \left\lfloor \frac{1}{3} n \right\rfloor - \left\lfloor \frac{2}{3} n \right\rfloor$$

equals either 0 or 1.

(Hint: Write n as $3k, 3k + 1$ or $3k + 2$, where k is an integer.)

†27. Determine which of the following functions are one-to-one, which are onto, and which are bijections.
    a) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x$.
    b) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = 2x + 3$.
    c) $f : \mathbb{R} \to \mathbb{Z} \quad f(x) = \lceil x \rceil$.
    *d) $f : \mathbb{R} \to \mathbb{R} \quad f(x) = x - \lfloor x \rfloor$.

†28. a) Let S be the set $\{ n \in \mathbb{N} \mid 0 \le n \le 11 \}$ and define $f : S \to S$ by letting $f(n)$ be the remainder when $5n + 2$ is divided by 12. Is f one-to-one? Is f onto?
    b) Repeat part (a) with $5n + 2$ replaced by $4n + 2$.

†29. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 - 4x + 6.$$

    a) What is the range of f?
    b) Is f onto? Explain.
    c) Is f one-to-one? Explain.

30. Repeat the above question with $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = x^3 - x + 1$$

and $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = x^3 + x + 1$$

(Hint: You may use differentiation.)

†31. Let $\mathbb{Z}^+$ be the set of all positive integers and $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined by $f(m,n) = mn$ for all $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Determine whether f is one-to-one or onto.

32. Find $g \circ f$ for each of the following pairs of functions

a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x - 3 \qquad g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \sqrt{x^2 + 2}$,

b) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x + 1 \qquad g : \{\text{odd integers}\} \to \mathbb{Z}, \quad g(x) = \frac{x-1}{2}$.

†33. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

a) Show that if f and g are both onto then $g \circ f$ is also onto.

b) Is it true that if f and g are both one-to-one then $g \circ f$ is also one-to-one?

*34. If $f, g\ : \mathbb{N} \to \mathbb{N}$ are functions defined by $f(n) = 2n$ and

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

show that $g \circ f = \iota$ but $f \circ g \neq \iota$ where ι is the identity function.

35. For each of the following bijections find the inverse and the domain and range of the inverse.

a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 5x + 3$.

b) * $g : \mathbb{Z} \to \mathbb{N}$

$$g(x) = \begin{cases} 2|x| - 1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

†36. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x^2 - 1$. Find

a) $f(A)$ if $A = \{x \mid -2 \leq x \leq 3\}$,

b) $f^{-1}(B)$ if $B = \{y \mid 1 \leq y \leq 17\}$.

†37. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function given by

$$f(m,n) = m^2 - n^2.$$

a) Show that f is not onto.

b) Find $f^{-1}(\{8\})$.

†38. Suppose f is a function from X to Y and A, B are subsets of X, and suppose that S, T are subsets of Y.

a) What containment relation (if any) is there between

i) $f(A) \cup f(B)$ and $f(A \cup B)$,

ii) $f(A) \cap f(B)$ and $f(A \cap B)$?

b) Show that $f^{-1}(S) \cup f^{-1}(T) = f^{-1}(S \cup T)$.

c) Show that $f^{-1}(S) \cap f^{-1}(T) = f^{-1}(S \cap T)$.

Sequences and Summation

†39. Use the formulae

$$\sum_{k=1}^{N} k = \frac{N}{2}(N+1) \quad \text{and} \quad \sum_{k=1}^{N} k^2 = \frac{N}{6}(N+1)(2N+1)$$

to evaluate

$$\sum_{k=3}^{22} (3k+4)^2.$$

40. a) Make use of changes of summation index to show that

$$\sum_{k=1}^{N} (k+1)^3 - \sum_{k=1}^{N} (k-1)^3 = (N+1)^3 + N^3 - 1.$$

b) Hence show that

$$\sum_{k=1}^{N} k^2 = \frac{N}{6}(N+1)(2N+1).$$

†41. Show that

$$\frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$$

and hence, that for $N \geq 1$

$$\sum_{k=1}^{N} \frac{2}{k(k+2)} = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}.$$

†42. Show that

$$\frac{5k-2}{k(k-1)(k-2)} = \frac{4}{k-2} - \frac{3}{k-1} - \frac{1}{k}$$

and hence evaluate

$$\sum_{k=3}^{n} \frac{5k-2}{k(k-1)\,(k-2)}.$$

43. By writing out the terms, show that for $N > 0$

$$\prod_{k=1}^{N} \frac{k}{k+2} = \frac{2}{(N+1)(N+2)}.$$

44. Using the fact that

$$1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2}$$

find an expression for $\displaystyle\prod_{k=2}^{N} \left(1 - \frac{1}{k^2}\right)$ in terms of N.

PROBLEM SET 2

Integers and Modular Arithmetic

†1. Find the quotient and (non-negative) remainder when

a) 19 is divided by 7,
b) -111 is divided by 11,
c) 1001 is divided by 13.

2. Are the following true or false?

$7 \mid 161, \quad 7 \mid 162, \quad 17 \mid 68, \quad 17 \mid 1001.$

3. Which of the following are prime?

17, 27, 37, 111, 1111, 11111.

†4. Find the prime factorization of the following

117, 143, 3468, 75600.

†5. Find the gcd and lcm of the following pairs

a) $2^2 \cdot 3^5 \cdot 5^3$ and $2^5 \cdot 3^3 \cdot 5^2$,
b) $2^2 \cdot 3 \cdot 5^3$ and $3^2 \cdot 7$,
c) 0 and 3.

†6. Evaluate 13 **mod** 3, 155 **mod** 19, (-97) **mod** 11.

†7. Prove for $a, b, c, d \in \mathbb{Z}$, $k, m \in \mathbb{Z}^+$ that

a) if $a \mid c$ and $b \mid d$ then $ab \mid cd$,
b) if $ab \mid bd$ and $b \neq 0$, then $a \mid d$,
c) if $a \mid b$ and $b \mid c$ then $a \mid c$,
d) if $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $a - b \equiv c - d \pmod{m}$,
e) if $k \mid m$ and $a \equiv b \pmod{m}$ then $a \equiv b \pmod{k}$,
f) if $d = \gcd(a, b)$ then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$,
*g) if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

†8. a) Find the least positive integer n for which

$$3^n \equiv 1 \pmod{7}.$$

Hence evaluate $3^{100} \pmod{7}$.

b) Find the least positive integer n for which

$$5^n \equiv 1 \pmod{17} \quad \text{or} \quad 5^n \equiv -1 \pmod{17}.$$

Hence evaluate $5^{243} \pmod{17}$.

c) Evaluate $2^4 \pmod{18}$ and hence evaluate $2^{300} \pmod{18}$.

†9. For each of the following, use the Euclidean Algorithm to find $d = \gcd(a, b)$ and $x,\ y \in \mathbb{Z}$ with $d = ax + by$

a) $\gcd(12, 18)$,
b) $\gcd(111, 201)$,
c) $\gcd(13, 21)$,
d) $\gcd(83, 36)$,
e) $\gcd(22, 54)$,
f) $\gcd(112, 623)$.

†10. Solve, or prove there are no solutions. Give your answer in terms of the original modulus and also, where possible, in terms of a smaller modulus.

a) $151x - 294 \equiv 44 \pmod{7}$,
b) $45x + 113 \equiv 1 \pmod{20}$,
c) $25x \equiv 7 \pmod{11}$,
d) $2x \equiv 3 \pmod{1001}$,
e) $111x \equiv 75 \pmod{321}$,
f) $1215x \equiv 560 \pmod{2755}$,
g) $182x \equiv 481 \pmod{533}$.

*11. Let a, b be integers, not both zero, let S be the set of integers defined by

$$S = \{ax + by \mid x, y \in \mathbb{Z}\},$$

and let d_0 be the smallest positive integer in the set S.

The aim of this question is to use the Division Algorithm and the definition of greatest common divisor (gcd) to show that $d_0 = \gcd(a, b)$.

Prove the following

a) If $s \in S$, then d_0 is a divisor of s. (Hint: Write $s = d_0 q + r$ and show $r \in S$).
b) d_0 is a divisor of both a and b.
c) If d is a divisor of both a and b, then d is a divisor of d_0.
d) $d_0 = \gcd(a, b)$, and hence there exist integers x, y such that $ax + by = \gcd(a, b)$.

12. a) Prove that if a, b and m are integers with the properties $\gcd(a, b) = 1$ and $a \mid m$ and $b \mid m$, then $ab \mid m$. (Hint: use 11(d).)
b) Prove that if a and b are coprime integers and $a \mid bc$, then $a \mid c$. (Hint: use 11(d).)

Relations

†13. List the ordered pairs in the relations R_i, for $i = 1, 2, 3$, from $A = \{2, 3, 4, 5\}$ to $B = \{2, 4, 6\}$ where

a) $(m, n) \in R_1$ iff $m - n = 1$,

b) $(m, n) \in R_2$ iff $m \mid n$,

c) $(m, n) \in R_3$ iff $\gcd(m, n) = 1$.

†14. Represent each relation R_i of Question 13 by:

a) an arrow diagram,

b) a matrix M_{R_i}.

15. Construct arrow diagrams representing relations on $\{a, b, c\}$ that have the following properties.

a) Reflexive, but neither transitive nor symmetric.

b) Symmetric, but neither transitive nor reflexive.

c) Transitive, but neither symmetric nor reflexive.

d) Symmetric and transitive, but not reflexive.

e) Transitive and reflexive, but not symmetric.

f) Symmetric and antisymmetric and reflexive.

16. A relation R on $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is defined by $a\,R\,b$ iff $a - b$ has either 2 or 3 as a divisor. Show that R is reflexive and symmetric, but not transitive.

†17. Define an equivalence relation $\sim$ on the set $S = \{0, 1, 2, 3, 4, 5, 6\}$ by $x \sim y$ if and only if $x \equiv y \pmod 3$. Partition S into equivalence classes.

†18. Define a relation $\sim$ on the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ by $x \sim y$ if and only if $x^2 \equiv y^2 \pmod 5$.

a) Prove that $\sim$ is an equivalence relation on S.

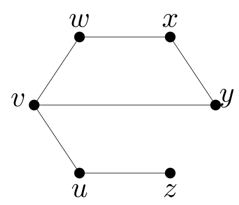
b) Partition S into equivalence classes.

19. a) Let $A_1 = \{0\}$, $A_2 = \{1, 2\}$, $A_3 = \{3, 4\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Define a relation $\sim$ on the set S by $x \sim y$ if and only if $x, y \in A_i$ for some $i \in \{1, 2, 3\}$. Show that $\sim$ is an equivalence relation on S.

b) Let $B_1 = \{0\}$, $B_2 = \{1, 2\}$, $B_3 = \{3\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Explain why the relation $\sim$ on S defined by $x \sim y$ if and only if $x, y \in B_i$ for some $i \in \{1, 2, 3\}$ is **not** an equivalence relation on S.

c) Let $C_1 = \{0, 1\}$, $C_2 = \{1, 2\}$, $C_3 = \{3, 4\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Explain why the relation $\sim$ on S defined by $x \sim y$ if and only if $x, y \in C_i$ for some $i \in \{1, 2, 3\}$ is **not** an equivalence relation on S.

20. The following diagram represents a set $V = \{u, v, x, y, z\}$ of six cities and direct flights between them.



a) Define a relation $\sim$ on V by $a \sim b$ if and only if it is possible to fly from a to b using an even number of flights (including 0 flights).

i) Prove that $\sim$ is an equivalence relation on V.

ii) Partition the set V into equivalence classes.

b) Define a relation R on V by aRb if and only if it is possible to fly from a to b using an odd number of flights. Prove that R is *not* an equivalence relation.

†21. Consider the set $S = \{0, 1, 2, \ldots, 11\}$ of integers modulo 12. Define the relation $\sim$ on S by $x \sim y$ iff $x^2 \equiv y^2 \bmod 12$. Given that $\sim$ is an equivalence relation, partition S into equivalence classes.

†22. Let a and b be two fixed real numbers. Define a relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ iff $ax_1 + by_1 = ax_2 + by_2$. Prove that $\sim$ is an equivalence relation and give a geometric description of the equivalence class of $(1, 1)$.

23. Answer the following questions for the Poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$:

a) Draw a Hasse diagram for the Poset.

b) Find the maximal elements.

c) Find the minimal elements.

d) Is there a greatest element?

e) Is there a least element?

f) Find all upper bounds of $\{\{2\}, \{4\}\}$.

g) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

h) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.

i) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ if it exists.

24. Answer the following questions for the Poset $(\{2, 4, 6, 9, 12, 27, 36, 54, 60, 72\}, |)$:

a) Draw a Hasse diagram for the Poset.

b) Find the maximal elements.

c) Find the minimal elements.

d) Is there a greatest element?

e) Is there a least element?

f) Find the set of upper bounds of $\{2, 9\}$.

g) Find the least upper bound of $\{2, 9\}$, if it exists.

h) Find the set of lower bounds of $\{60, 72\}$.

i) Find the greatest lower bound of $\{60, 72\}$ if it exists.

25. Give an example of a poset that

    a) Has a minimal element but no least element.
    b) Has a maximal element but no greatest element.

†26. A relation $|$ is defined on $A = \{1, 2, 4, 6, 8, 9, 12, 18, 36, 72, 108\}$ by $a \mid b$ iff a divides b.

    a) Show that $(A, |)$ is a poset.
    b) Construct its Hasse diagram.
    c) Which members of A are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?

†27. Let $S = \{0, 1, 2, 3\}$ and $P(S)$ denote the power set of S.
Define the relation $A \preceq B$ for $A,\ B \in P(S)$ by $A \preceq B$ iff $A \subseteq B$.

    a) Show that $\preceq$ is a partial order.
    b) Construct the Hasse diagram of $\big(P(S),\ \preceq\big)$.
    c) Which members of $P(S)$ are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?

PROBLEM SET 3

Note that the subject matter of this section of the course is mathematical proof itself, and not the particular results proved in classes or posed in the following problem set. You should be prepared to prove results from any area of school or first-year university mathematics.

In this problem set, F&D refers to the book by Franklin and Daoud, and the problems have been printed here with the permission of the author. Note the solutions to some of the F&D questions are available in the book "Introduction to Proofs in Mathematics" by J. Franklin and A. Daoud.

Basic Proofs

1. (F&D Chapter 1 Q1) Show that: $\dfrac{1}{1,000} - \dfrac{1}{1,002} < \dfrac{2}{1,000,000}$

†2. (F&D Chapter 1 Q3) Show that: $\sqrt{1,001} - \sqrt{1,000} < \dfrac{1}{2\sqrt{1,000}}$
Hint: multiply $\sqrt{1,001} - \sqrt{1,000}$ by $\dfrac{\sqrt{1,001} + \sqrt{1,000}}{\sqrt{1,001} + \sqrt{1,000}}$

†3. (F&D Chapter 1 Q9) Prove that: $\sqrt[7]{7!} > \sqrt[6]{6!}$

†4. [V] (F&D Chapter 1 Q12) Show that: $\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}} < 2\sqrt{2}$

5. (F&D Chapter 1 Q14)

   a) Prove that: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

   b) Hence show that $\cos\dfrac{2\pi}{3}$ is a root of the equation $4x^3 - 3x - 1 = 0$

6. a) Suppose that n is a positive integer. Use the Binomial theorem and appropriate inequalities to prove that
$$0 < \left(1 + \frac{1}{n}\right)^n < 3.$$

   b) (F&D Chapter 1 Q20) Prove that: $99^{100} > 100^{99}$

7. (F&D Chapter 1 Q21) Show that: $\dfrac{\pi}{4} = \tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right)$

Generalisation; "all" statements

†8. Prove by exhaustion of cases that for any real number x we have
$$|x(x-2)| \leq 2x^2 + 1\ .$$

†9. (F&D Chapter 2 Q3) Prove that the product of any two odd numbers is an odd number.

†10. (F&D Chapter 2 Q11) Prove that $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$ for any positive integer n (imitate the proof in Chapter 1).

11. (F&D Chapter 2 Q12) Prove that the product of three consecutive integers, of which the middle one is odd, is divisible by 24.

12. (F&D Chapter 2 Q19) Find a generalisation of:

$$\frac{1}{1000} - \frac{1}{1002} < \frac{2}{(1000)^2}$$

and prove it.

13. (F&D Chapter 2 Q21) Prove that for all positive integers $n, (n+1)(n+2)\ldots(2n-1)(2n) = 2^n.1.3.5\ldots(2n-1)$

†14. (F&D Chapter 5 Q9) Consider the following statement concerning a positive integer $x \geq 2$.

"If x is not divisible by any positive integer n satisfying $2 \leq n \leq \sqrt{x}$ then x is a prime number."

a) Show that the above statement is true.

b) Is the statement still true if the condition on n is replaced by $2 \leq n < \sqrt{x}$?

*15. a) Let a and n be integers greater than 1. Prove that $a^n - 1$ is prime only if $a = 2$ and n is prime. Is the converse of this statement true?

b) [V] Show that $2^n + 1$ is prime only if n is a power of 2.

*16. Consider the following statement concerning a positive integer n:

"For all $a, b \in \mathbb{Z}$, if $n \mid ab$ then either $n \mid a$ or $n \mid b$."

a) Prove that if n is prime then the statement is true.

b) Prove that the statement is false when $n = 30$.

c) Prove that if n is composite then the statement is false.

17. a) Use the result of part (a) of the previous question to prove that if p is prime, a is an integer and $p \mid a^2$, then $p \mid a$.

*b) Are there any integers n other than primes for which it is true that for all integers a, if $n \mid a^2$ then $n \mid a$? If so, describe all such n.

*c) Prove that if p is prime then $\sqrt{p}$ is irrational.

18. a) Show that 2 is a multiplicative inverse for 4 (mod 7) and 3 is a multiplicative inverse for 5 (mod 7). Hence determine the value of 5! (mod 7)

*b) [V] Prove that if p is prime then

$$(p-2)! \equiv 1 \pmod{p}.$$

Hint: What is the multiplicative inverse of $p - 1 \pmod{p}$ when p is prime?

Writing proofs

†19. In the following questions you are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow; and any necessary algebraic details.

a) **Theorem**. If $x^3 + x^2 + x + 2 \ = 0 \mod 5$ then $x \ = 1 \mod 5$.
Basic ideas: if $x \ = 0, 2, 3, 4 \mod 5$ then $x^3 + x^2 + x + 2 \ = 2, 1, 1, 1 \mod 5$.

b) **Theorem**. Let x, y and m be integers. If $m \mid (4x + y)$ and $m \mid (7x + 2y)$ then $m \mid x$ and $m \mid y$.
Basic ideas: $2(4x + y) - (7x + 2y) = x$ and $m \mid \text{LHS}$.

c) **Theorem**. $\log_2 7$ is irrational.
Basic ideas: if $\log_2 7 = p/q$ then $2^p = 7^q$, but then LHS is even and RHS is odd.

d) **Theorem**. If n is a non–negative integer then 11 is a factor of $2^{4n+3} + 3 \times 5^n$.
Basic ideas: if $2^{4n+3} = 11k - 3 \times 5^n$ then $2^{4n+7} = 11(16k - 3 \times 5^n) - 3 \times 5^{n+1}$.

Converse; if and only if

†20. Prove the following statement, then write down its converse. Is the converse true or false? Prove your answer.

"For all $x \in \mathbb{Z}$, if $x \equiv -1 \pmod 7$ then $x^3 \equiv -1 \pmod 7$."

21. a) Prove that the following proposition is true.

$$\text{If } x \equiv 3 \pmod 4 \text{ then } x^3 + 2x - 1 \equiv 0 \pmod 4.$$

b) Write down the contrapositive of the proposition in part (a). Is it true? Explain.

c) Write down the converse of the proposition in part (a). Is it true? Explain.

†22. (F&D Chapter 3 Q6) Prove that an integer is odd if and only if its square is odd.

†23. (F&D Chapter 3 Q10) Prove that a triangle is isosceles if and only if two of its angles are equal. (An isosceles triangle is by definition a triangle with two equal sides.)

24. (F&D Chapter 3 Q13) Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

25. (F&D Chapter 3 Q18) Prove that a real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

*26. For integers x and y, show that $7|x^2 + y^2$ if and only if $7|x$ and $7|y$.

27. A parallelogram is defined to be a quadrilateral with both pairs of opposite sides parallel. Use properties of congruent and similar triangles to prove that a quadrilateral is a parallelogram if and only if two opposite sides are equal and parallel.

"Some" statements

†28. (F&D Chapter 4 Q11) Show that there is a solution of, $x^{100} + 5x - 2 = 0$ between $x = 0$ and $x = 1$.

29. [V] (F&D Chapter 4 Q4) A perfect number is one which equals the sum of its factors (counting 1 as a factor, but not the number itself). Show that there exists a perfect number.

†30. (F&D Chapter 4 Q13) Consider the infinite geometric progression,

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \ldots$$

Prove that there exists an integer N such that the sum of the first N terms of the above series differs from 1 by less than 10^{-6}.

31. (F&D Chapter 4 Q16) A formula for e^x is,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

Show that $e^3 = 20.1$, correct to 1 decimal place, by showing:

a) For any $n \geq 4$,

$$\frac{3^n}{n!} < \frac{9}{2}\left(\frac{3}{4}\right)^{n-3}$$

b) Hence, that there exists N such that,

$$\frac{3^{N+1}}{(N+1)!} + \frac{3^{N+2}}{(N+2)!} + \cdots < \frac{1}{20}$$

c) Hence, that $e^3 = 20.1$ correct to 1 decimal place.

*32. (F&D Chapter 5 Q11) Every one of six points is joined to every other one by either a red or a blue line. Show that there exist three of the points joined by lines of the same colour.

Multiple quantifiers

*33. [V] (F&D Chapter 5 Q8) Prove that between any two irrational numbers there is a rational number.

†34. Show that the sequence $\{u_n\}$ given by

$$u_n = n^2 \quad \text{for all } n \in \mathbb{N}$$

diverges to infinity, by showing that

$$\forall M \in \mathbb{N} \quad \exists N \in \mathbb{N} \quad \forall n > N \quad u_n > M.$$

35. A function $f(x)$ is called continuous at $x = a$ iff:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in (a - \delta, a + \delta) \quad |f(x) - f(a)| < \epsilon.$$

a) Complete: "A function $f(x)$ is not continuous at $x = a$ iff: ......"

b) * Show that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 3$.

†36. The definition of $\lim_{x\to\infty} f(x) = \ell$ is

$$\text{“}\forall\epsilon > 0 \;\; \exists M \in \mathbb{R} \;\; \forall x > M \;\; |f(x) - \ell| < \epsilon.\text{”} \qquad (**)$$

a) Write down the negation of $(**)$ (that is, $\lim_{x\to\infty} f(x) \neq \ell$) and simplify it so that the negation symbol does not appear.

b) By working directly from the definition prove that $\lim_{x\to\infty} \frac{2x^2+1}{x^2+2} = 2$.

Negation; proof by contradiction

†37. (F&D Chapter 6 Q16) Prove that each of the following is irrational:

a) $\sqrt[3]{4}$

b) $1 + \sqrt{3}$

c) $\sqrt{3} - \sqrt{2}$

†38. Prove that $\log_{10} 3$ is irrational.

39. Prove that $\log_5 15$ is irrational.

†40. (F&D Chapter 6 Q1) What is the common feature in showing:

a) a “not all” statement to be true?

b) an “all” statement to be false?

Give an example in each case.

41. (F&D Chapter 6 Q11) Are the following statements true or false? Prove your answers.

a) Let a, b, c be three integers. If a divides c and b divides c, then either a divides b or b divides a.

b) If a, b, c, d are real numbers with $a < b$ and $c < d$, then $ac < bd$.

42. (F&D Chapter 5 Q5)

a) Prove that if a and b are rational numbers with $a \neq b$ then,

$$a + \frac{1}{\sqrt{2}}(b - a)$$

is irrational.

b) Hence prove that between any two rational numbers there is an irrational number.

43. (F&D Chapter 7 Q14) A set of real numbers is called bounded if it does not “go to infinity”. More precisely, S is bounded if there exist real numbers M, N such that for all $s \in S, M < s < N$.
(For example, the set of real numbers x such that $1 < x^3 < 2$ is bounded, since for all $x \in S, 1 < x < 1.5$.)

a) Give an example of a set which is not bounded.

b) Prove that if S is bounded and $T \subset S$, then T is bounded.

c) Prove that any finite set of real numbers is bounded.

d) Prove that if T is not bounded and $T \subset S$, then S is not bounded.

e) Prove that if S and T are bounded, then $S \cap T$ is bounded.

f) If S and T are bounded, is $S \cup T$ always bounded? Prove your answer.

g) Let S and T be bounded, Let,

$$U = \{u \in \mathbb{R} : u = s + t \quad \text{for some} \quad s \in S, t \in T\}$$

Show that U is bounded. It might help to calculate some examples first, say,

$$S = [0, 1], T = [2, 3]$$

(The set U is sometimes denotes $S + T$, since it is the set of all sums of something in S with something in T.)

44. (F&D Chapter 7 Q15) A region in the plane is called convex if the line segment joining any two points in the region lies wholly inside the region. For example, an ellipse, a parallelogram, a triangle and a straight line are convex, but on annulus and a star-shaped region are not. In symbols, R is convex if, for all (x_1, y_1) and (x_2, y_2) in R, $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in R$ for all $\lambda \in [0, 1]$.

a) Prove that if R and S are convex, then $R \cap S$ is convex.

b) If R and S are convex, is $R \cup S$ always convex? Prove your answer.

c) Prove that if R is convex, then the reflection of R in the x-axis is convex.

d) If R is convex, is the set,

$$2R = \{(x, y) : (x, y) = (2x', 2y') \text{ for some } (x', y') \in R\}$$

always convex? Prove your answer and illustrate with a diagram.

†45. (F&D Chapter 6 Q24) Show that there do not exist three consecutive integers such that the cube of the greatest equals the sum of the cubes of the other two.

Mathematical induction

46. (F&D Chapter 8 Q2) Show that, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots + (-1)^{n-1}\frac{1}{n}$ is always positive.

†47. (F&D Chapter 8 Q5)

a) Prove, by mathematical induction, that if n is a positive integer then,

$$n^3 + 3n^2 + 2n \quad \text{is divisible by 6.}$$

b) Prove the same result without mathematical induction by first factorising $n^3 + 3n^2 + 2n$.

†48. (F&D Chapter 8 Q7)

a) Calculate,

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

for a few small values of n.

b) Make a conjecture about a formula for this expression.

c) Prove your conjecture by mathematical induction.

49. (F&D Chapter 8 Q8) The following is a famous fallacy that uses the method of mathematical induction. Explain what is wrong with it.
Theorem
Everything is the same colour.
Proof
Let $P(n)$ be the statement, "In every set of n things, all the things have the same colour".
We will show that $P(n)$ is true for all $n = 1, 2, 3, \ldots$, so that every set consists of things of the same colour.
Now, $P(1)$ is true, since in every set with only one thing in it, everything is obviously of the same colour.
Now, suppose $P(n)$ is true.
Consider any set of $n + 1$ things.
Take an element of the set, a. The n things other than a form a set of n things, so they are all the same colour (since $P(n)$ is true).
Now take a set of n things out of the $n + 1$ which does include a.
These are also all the same colour, so a is the same colour as the rest.
Therefore $P(n + 1)$ is true.

50. Show that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ whenever $n \in \mathbb{Z}^+$.

51. Show that $1^2 + 3^2 + \cdots + (2n + 1)^2 = \frac{1}{3}(n + 1)(2n + 1)(2n + 3)$ whenever $n \in \mathbb{N}$.

†52. Show that for all $n \in \mathbb{N}$, $64 \bigm| 7^{2n} + 16n - 1$.

53. Prove that for all $n \in \mathbb{Z}^+$, $21 \mid 4^{n+1} + 5^{2n-1}$.

*54. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \frac{2}{3} n\sqrt{n}.$$

55. Prove by induction that if the sequence (u_n) is defined by

$$\begin{cases} u_0 &= 0 \\ u_1 &= 1 \\ u_n &= u_{n-1} + u_{n-2} \text{ for all } n \geq 2 \end{cases}$$

then

$$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for all $n \geq 0$.

†56. Prove, using induction, that if the sequence $\{u_n\}$ is defined by:

$$\begin{cases} u_1 &= 12 \\ u_2 &= 30 \\ u_n &= 5u_{n-1} - 6u_{n-2} \quad \text{for } n \geq 3, \end{cases}$$

then $u_n = 3 \times 2^n + 2 \times 3^n$, for all $n \geq 1$.

57. a) Prove by induction that for $n \in \mathbb{N}$, the nth derivative of e^{x^2} is a polynomial times e^{x^2}. Can you say anything about these polynomials?
(**Note:** By convention, the $0'$th derivative of $f(x)$ is just $f(x)$.)

b) Guess and prove a similar result concerning the derivatives of the function $e^{1/x}$.

†58. [V] Prove that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have $3^n < n!$.

Logic and truth tables

59. Which of the following are propositions?

    a) The moon is made of green cheese.
    b) Read this carefully.
    c) Two is a prime number.
    d) Will the game be over soon?
    e) Next year interest rates will rise.
    f) Next year interest rates will fall.
    g) $x^2 - 4 = 0$

60. Using letters for the component propositions, translate the following compound statements into symbolic notation:

    a) If prices go up, then housing will be plentiful and expensive; but if housing is not expensive, then it will still be plentiful.
    b) Either going to bed or going swimming is a sufficient condition for changing clothes; however, changing clothes does not mean going swimming.
    c) Either it will rain or it will snow but not both.
    d) If Janet wins or if she loses, she will be tired.
    e) Either Janet will win or, if she loses, she will be tired.

61. Write a statement that represents the negation of each of the following:

    a) If the food is good, then the service is excellent.
    b) Either the food is good or the service is excellent.
    c) Either the food is good and the service is excellent, or else the price is high.
    d) Neither the food is good nor the service excellent.
    e) If the price is high, then the food is good and the service is excellent.

†62. Construct a truth table for each of the following propositions.

a) $\sim \big((p \wedge q) \to (p \wedge q)\big)$;

b) $p \to (p \to q)$;

c) $\sim p \to (p \to q)$;

d) $\big(p \to (q \to r)\big) \to \big((p \to q) \to (p \to r)\big)$.

Which of the above propositions are tautologies? Are any contradictions?

†63. Show, by using truth tables, that the following pairs of propositions are logically equivalent.

a) $\sim (p \vee q),\ \sim p \wedge \sim q$;

b) $\sim p \vee q,\ p \to q$;

c) $\sim p \to (q \vee r),\ \sim q \to (\sim r \to p)$;

d) $p \vee (p \wedge q), p$.

64. Use standard logical equivalences to simplify each of the following logical expressions.

a) $(p \vee \sim q) \wedge \sim (p \wedge q)$

b) $[p \to (q \vee \sim p)] \to (p \wedge q)$

c) $(p \wedge q) \wedge \ \sim (\sim p \wedge q)\ \wedge\ (q \wedge r)$

d) $(p \leftrightarrow q)\ \vee\ (\sim p \wedge q)$

65. Use standard logical equivalences to show that $\sim (p \vee \sim q) \to (q \to r)$ is logically equivalent to each of the following propositions

a) $q \to (p \vee r)$;

b) $(q \to p) \vee (q \to r)$;

c) $\sim (q \to p) \to (\sim q \vee r)$.

Valid and invalid arguments

66. Suppose that

> “If I do not do my homework, I will not pass.
> If I study hard, I will pass.
> I passed.”

Did I do my homework or not? Did I study hard or not? Explain.

†67. “Watson, I have uncovered the following facts:

- If Mrs Smith is lying then Moriarty has not escaped.
- Either Moriarty is dead or he is really Jones.
- If Moriarty is really Jones then he has escaped.
- I am convinced Mrs Smith is lying.”

“Good Lord Holmes,”replied Dr Watson, “what can you make of all this?”
“Elementary my dear Watson, Moriarty is dead!”

Is Holmes correct? Justify your answer.

†68. Let e denote "Einstein is right", b denote "Bohr is right" q denote "Quantum mechanics is right", w denote "The world is crazy" and consider the sentences:

(1) "Either Einstein or Bohr is right, but they are not both right".
(2) "Einstein is right only if quantum mechanics is wrong, and the world is crazy if quantum mechanics is right".

a) Write the two sentences (1) and (2) as symbolic expressions involving e, b, q and w.
b) Suppose the world is not crazy. From the truth of (1) and (2) is it valid to deduce that Einstein is right? Explain.

69. Consider the following two statements:

(1) "If either Peta or Queenie has passed then either Roger and Peta have both passed, or Roger and Queenie have both passed".

and

(2) "If either Peta has passed or Queenie has failed then Roger has passed."

a) Write the two statements (1) and (2) in symbolic form by letting p stand for "Peta has passed," q stand for "Queenie has passed" and r stand for "Roger has passed".
b) Suppose that the statement (1) is false. Deduce that Roger has failed.
c) Suppose that the statement (1) is false and that the statement (2) is true. Decide whether or not Peta has passed.

PROBLEM SET 4

Enumeration and Probability

†1. How many strings of six non-zero decimal digits

a) begin with two odd digits?
b) consist of one even digit, followed by two odd digits, followed by three digits less than 7?
c) have no digit occurring more than once?
d) contain exactly three nines?
e) contain fewer than three nines?
f) contain exactly three nines, with no other digit repeated?
g) have their last digit equal to twice their first digit?

†2. a) Express in terms of factorials $P(21,8)$, $C(21,8)$, $\binom{321}{123}$, $C(2n,n)$.

b) Calculate explicitly $P(7,4)$, $\binom{7}{4}$, $P(6,3)$, $C(4,2)$, $C(201,199)$.

†3. How many seven-letter words can be made from the English alphabet which contain

a) exactly one vowel,
b) exactly two vowels,
c) exactly three vowels,
d) at least three vowels?

4. Repeat the previous question if repeated letters are not permitted.

†5. a) A set of eight Scrabble® tiles can be arranged to form the word SATURDAY. How many three-letter "words" can be formed with these tiles if no tile is to be used more than once?

b) How many ten-letter words can be formed from the letters of PARRAMATTA? How many nine-letter words? *How many eight-letter words?

c) How many four-letter "words" come before "UNSW" if all four-letter words are listed in alphabetical order?

†6. How many eight-letter words constructed from the English alphabet have

a) exactly two Ls?
b) at least two Ls?

7. Consider the following.

"*Problem*: How many eight-card hands chosen from a standard pack have at least one suit missing?

Solution: Throw out one entire suit (4 possibilities), then select 8 of the remaining 39 cards. The number of hands is $4C(39,8)$."

a) What is wrong with the given solution?
b) Solve the problem correctly.

†8. a) Find the number of positive integers less than or equal to 200 which are divisible *neither* by 6 *nor* by 7.

b) Find the number of positive integers less than or equal to 200 which are divisible neither by 6, nor by 7, nor by 8.

9. What is the probability that a hand of eight cards dealt from a shuffled pack contains:

a) exactly three cards of the same value and the remaining cards all from the remaining suit (*for example,* $\heartsuit 4, \diamondsuit 4, \spadesuit 4$ *and five clubs not including the* $\clubsuit 4$);

b) exactly three cards in at least one of the suits;

c) exactly three cards in exactly one of the suits. (*Hint.* First find the number of ways in which five cards can be chosen from three specified suits, with none of the three represented by three cards.)

10. For any positive integer n we define $\phi(n)$ to be the number of positive integers which are less than or equal to n and relatively prime to n. Given that p, q, r are primes, all different, use the inclusion/exclusion principle to find $\phi(pq)$, $\phi(p^2q)$ and $\phi(pqr)$.

†11. A hand of thirteen cards is dealt from a shuffled pack. Giving reasons for your answers, determine which of the following statements are *definitely true* and which are *possibly false*.

a) The hand has at least four cards in the same suit.

b) The hand has exactly four cards in some suit.

c) The hand has at least five cards in some suit.

d) The hand has *exactly one* suit containing four or more cards.

†12. a) Prove that if nine rooks (castles) are placed on a chessboard in any position whatever, then at least two of the rooks attack each other.

b) Prove that if fifteen bishops are placed on a chessboard then two of them attack each other.

†13. Let A be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Prove that if 5 integers are selected at random from A, then at least one pair of these integers has sum 9.

14. In a square with sides of length 3cm, 10 points are chosen at random. Prove that there must be at least two of these points whose distance apart is less than or equal to $\sqrt{2}$ cm.

†15. If 31 cards are chosen from a pack, prove that there must be at least 3 of the same value, and there must be at least 8 in the same suit.

†16. a) To each integer n we assign an ordered pair $p(n)$ whose members are the remainders when n is divided by 3 and 4 respectively. For example, $p(5)$ and $p(17)$ are both equal to $(2, 1)$. If ten thousand integers are chosen at random, how many can you say for certain must have the same value for p?

*b) Repeat part (a) with the divisors 3 and 4 replaced by 4 and 6.

17. Let S and T be finite sets with $|S| > |T|$, and let f be a function from S to T. Show that f is not one-to-one.

†18. Prove that there were two people in Australia yesterday who met exactly the same number of other people in Australia yesterday.

19. Twenty hotel management students all guess the answers on the final examination, so it can be taken that all orders of students on the list of results are equally likely. The top student is given a mark of 100, the next 95, and so on, down to 5 for the last student and no two students get the same mark. Find the probability that Polly gets an HD, both Manuel and Sybil get a CR or better, and Basil fails.

†20. A die is rolled 21 times. Find the probability of obtaining a 1, two 2s, ... and six 6s.

†21. For this problem assume that the 365 dates of the year are equally likely as birthdays.

  a) Find the probability that two people chosen at random have the same birthday.
  b) Find the probability that in a group of n people, at least two have the same birthday.
  c) How large does n have to be for the probability in (b) to be greater than $\frac{1}{2}$?
  d) Criticise the assumption made at the beginning of this question.

†22. Twenty cars to be bought by a company must be selected from up to four specific models. In how many ways may the purchase be made if

  a) no restrictions apply?
  b) at least two of each model must be purchased?
  c) at most three different models must be purchased?

23. How many outcomes are possible from the roll of four dice

  a) if the dice are distinguishable (for example, they are of different colours)?
  b) if the dice are not distinguishable?

†24. a) Find the coefficient of $w^2x^5y^7z^9$ in $(w + x + y + z)^{23}$.
  b) When $(w + x + y + z)^{23}$ is expanded and terms collected, how many different terms will there be?

†25. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40$$

if $x_1, x_2, x_3, x_4, x_5, x_6$ are non-negative integers,

  a) with no further assumptions?
  b) with each $x_j \geq 3$?
  *c) with each $x_j \leq 10$?
  d) with each $x_j \leq 8$?
  e) if each x_j is even?
  f) if at least one x_j is odd?
  *g) if every x_j is odd?
  *h) with $x_1 \leq 9$, $5 \leq x_2 \leq 14$ and $10 \leq x_3 \leq 19$?

*26. By counting in two ways the number of non-negative integer solutions of the inequality $x_1 + x_2 + \cdots + x_r \leq n$, prove that

$$\binom{n+r}{r} = \binom{n+r-1}{r-1} + \binom{n+r-2}{r-1} + \cdots + \binom{r}{r-1} + \binom{r-1}{r-1}.$$

Interpret this result in Pascal's triangle.

Recurrence Relations

27. Write down the first four terms of the sequences defined recursively by

    a) $a_n = \frac{3}{2}a_{n-1}, \quad a_0 = 16;$
    b) $a_n = 3a_{n-1} - 2a_{n-2}, \quad a_0 = 2, \quad a_1 = 5;$
    c) $a_n = 2(a_{n-1} + a_{n-2} + \cdots + a_1 + a_0), \quad a_0 = 1.$

28. Write down a recurrence and initial conditions to describe each of the following sequences.

    a) $\{2, 4, 8, 16, ...\}$
    b) $\{1, 3, 5, 7, ...\}$
    c) $\{3, -6, 12, -24, ...\}$
    d) $\{1, 3, 6, 10, 15, 21, 28, ...\}$
    e) $\{2, 2, 2, 2, 2, ...\}$
    f) $\{1, 2, 3, 5, 8, 13, 21, 34, ...\}$

†29. Let a_n be the number of ways to climb n steps, if the person climbing the stairs can only take two steps or three steps at a time.

    a) Write down a recurrence relation with initial conditions for a_n.
    b) Find a_4 and a_8.

†30. Let a_n denote the number of bit strings of length n where no two consecutive zeros are allowed.

    a) Find a_1, a_2, a_3.
    b) Show that a_n satisfies the recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

31. A bank lends me \$50,000 at 18% per year interest, compounded monthly, and I pay back \$900 per month. (So at the end of each month the amount I owe is increased by $\left(\frac{18}{12}\right)\%$ and then reduced by \$900.)

    If $\$u_n$ is the amount still owing after n months, write down a recurrence relation for u_n.

†32. a) Find the general solution of the first order recurrence $a_n = 5a_{n-1}$.
    b) Find the solution of the first order recurrence $a_n + 4a_{n-1} = 0$ subject to the initial condition $a_1 = -12$.

33. Find the general solution of the following recurrence relations, each holding for $n \geq 2$.

    a) $a_n + a_{n-1} - 6a_{n-2} = 0.$
    b) $a_n = 3a_{n-1} - 2a_{n-2}.$
    c) $a_n = 6a_{n-1} - 9a_{n-2}.$
    d) $a_n - 2a_{n-1} - 4a_{n-2} = 0.$

†34. Find the solution of the following recurrence relations (defined for $n \geq 2$) subject to the given initial conditions

    a) $a_n + 2a_{n-1} - 15a_{n-2} = 0, \qquad a_0 = 7, \quad a_1 = -3.$
    b) $a_n = 5a_{n-1} - 6a_{n-2}, \qquad a_0 = 5, \quad a_1 = 13.$

c) $a_n + 4a_{n-1} + 4a_{n-2} = 0, \qquad a_0 = 2, \quad a_1 = 4.$

d) $a_n - 4a_{n-1} - 6a_{n-2} = 0, \qquad a_0 = 2, \quad a_1 = 4.$

†35. Find the general solution of the following recurrence relations (defined for $n \geq 2$).

a) $a_n + a_{n-1} - 6a_{n-2} = 4^n$

b) $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$

c) $a_n = 6a_{n-1} - 9a_{n-2} + 8n + 4$

d) $a_n = 6a_{n-1} - 9a_{n-2} + 3^n$

36. Find the solution of the recurrence

$$a_n - 3a_{n-1} - 4a_{n-2} = 5(-1)^n \quad \text{for } n \geq 2, \text{ given that } \ a_0 = 1, \quad a_1 = 8.$$

†37. Suppose we wish to tile a $2 \times n$ rectangular board with smaller tiles of size 1×2 and 2×2.

Let a_n be the number of ways in which this can be done.

a) Show that, for $n > 2$, $\quad a_n = a_{n-1} + 2a_{n-2}$ and find a_1, a_2.

b) Solve the recurrence to find a closed formula for a_n.

*38. An n-digit *quaternary sequence* is a string of n digits chosen from the numbers $0, 1, 2, 3$.

Let a_n be the number n–digit of quaternary sequences with an even number of 0's.

a) Show that, for $n \geq 1$,

$$a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$$

and

$$a_0 = 1, a_1 = 3.$$

b) Find a closed formula for a_n.

†39. Define a set S of words on the alphabet $\{x, y, z\}$ by

(B) $\quad x, y \in S$

(R) $\quad$ If $w \in S$ then $wx, wy, wzx, wzy \in S$.

Let a_n be the number of words of length n in S.

a) Find the first few values of a_n.

b) Find a recurrence relation for a_n.

c) Solve the recurrence to find a closed formula for a_n.

†40. We call a word on the alphabet $\{x, y, z\}$ *z-abundant* if the letter z appears at least once in any two successive letters. (So, for example, the empty word is z-abundant and so is $xzzxzy$ but $xzyxzy$ is not.)

Let a_n be the number of z-abundant words of length n.

a) Show that $a_1 = 3$ and that $a_2 = 5$.

b) Explain carefully why, for $n \geq 2$,

$$a_n = a_{n-1} + 2a_{n-2}.$$

(Hint: Consider the possible ways in which a z-abundant word can begin.)

c) Find an explicit formula for a_n.

*41. a) Show that the set $\{1, 2, 3, ..., n\}$ can be partitioned into two non-empty sets in precisely $2^{n-1} - 1$ ways.

b) Let s_n be the number of ways in which the set $\{1, 2, 3, ..., n\}$ can be partitioned into three non-empty sets.

Show that $s_1 = 0, \quad s_2 = 0, \quad s_3 = 1$ and write down the six such partitions for $n = 4$.

c) Show that, for $n \geq 2$, the sequence s_n defined above satisfies $s_n = 3s_{n-1} + 2^{n-2} - 1$ and find a closed formula for s_n.

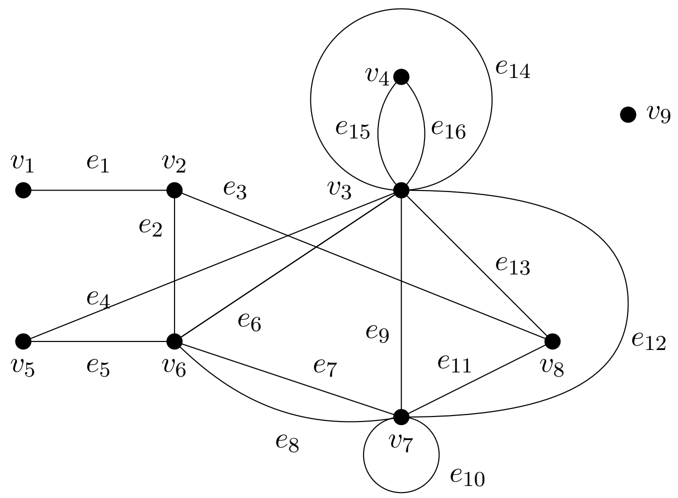
PROBLEM SET 5

Graphs

1. Draw the graph $G = (V, E, f)$ with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and edge–endpoint or incidence function $f : E \to \big\{\{x, y\} \,|\, x, y \in V\big\}$

e	$f(e)$
e_1	$\{v_3, v_4\}$
e_2	$\{v_1\}$
e_3	$\{v_1, v_5\}$
e_4	$\{v_2, v_5\}$
e_5	$\{v_2\}$
e_6	$\{v_5, v_2\}$
e_7	$\{v_4, v_2\}$

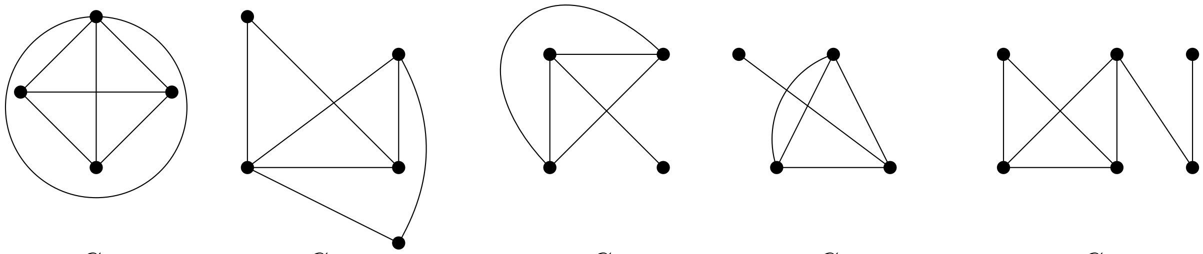
2. For the graph



Find

a) the edge–endpoint function (in a table),
b) the vertex degrees (in a table),
c) total vertex degree and total number of edges,
d) all loops,
e) all parallel edges,
f) edges incident on v_3,
g) vertices adjacent to v_8,
h) all isolated vertices.

†3. Determine with reasons which of the following graphs are simple.



G_1 $\quad$ G_2 $\quad$ G_3 $\quad$ G_4 $\quad$ G_5

†4. How many edges does a graph have if it has vertices of degree $4, 3, 3, 2, 2$? Draw a simple graph with these vertex degrees. (Is it the only example?)

†5. Determine whether or not there is a graph or simple graph for each of the following sequences of vertex degrees. Draw examples of those that exist. (Try to minimise the number of loops or parallel edges in non–simple examples.)

i) $4, 4, 3, 2, 2, 1$. ii) $4, 4, 3, 3, 2, 1$. iii) $5, 5, 3, 2, 2, 1$.

iv) $5, 4, 3, 2, 2$. v) $6, 5, 4, 4, 2, 2, 1$. vi) $6, 5, 4, 3, 2, 1, 1$.

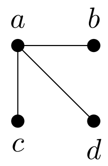
6. Draw the graphs

i) K_6. ii) $K_{2,3}$.

†7. Find the total number of vertices and edges for the special simple graphs

a) K_n
b) $K_{m,n}$
c) C_n, (cyclic graph with n vertices)
d) Q_n (n-cube graph)

†8. How many subgraphs are there of the graph G :

G: 

9. Recall that two vertices in the complement $\overline{G}$ are neighbours iff they are not neighbours in G. Find the following

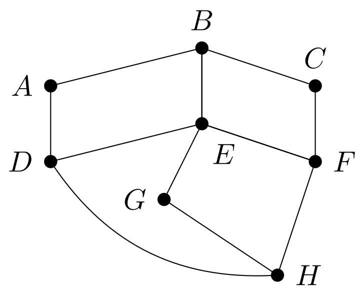
i) $\overline{K_n}$ ii) $\overline{K_{m,n}}$ iii) $\overline{C_n}$ for $n = 5, 6$

10. If a simple graph G has n vertices and m edges how many edges does $\overline{G}$ have?

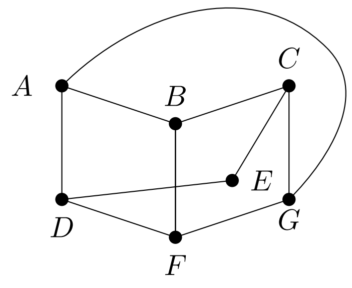
Bipartite Graphs

†11. Determine, with reasons, which of the following graphs is bipartite.

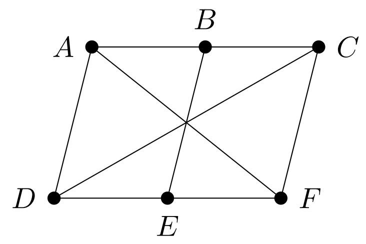
a)



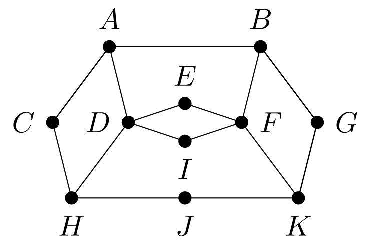
b)



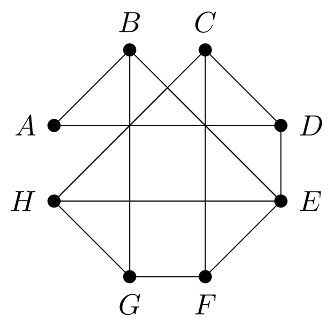
c)



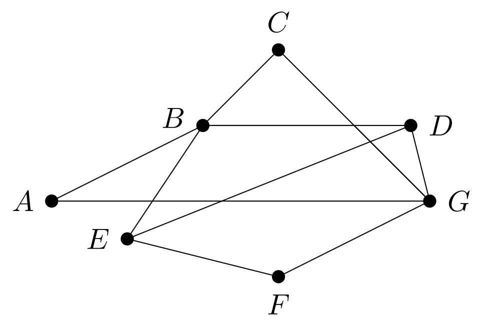
d)



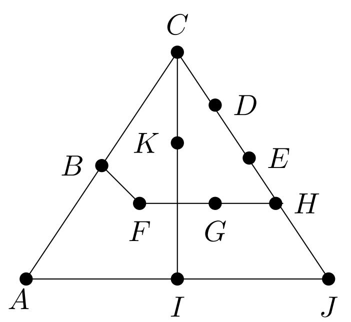
e)



f)



g)



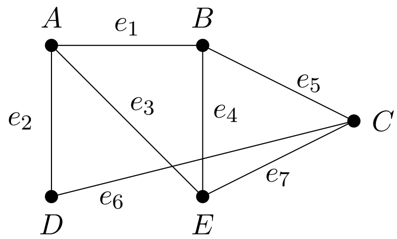
12. For what values of n are the following graphs bipartite?

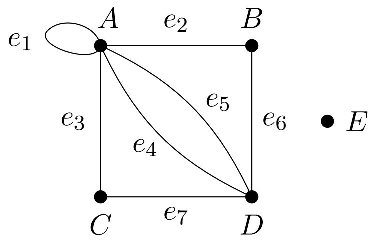
i) K_n ii) C_n iii) Q_n

Adjacency Matrices

†13. Represent the following graphs by adjacency matrices

a)





(Order vertices alphabetically)

b) K_4

c) $K_{2,3}$.

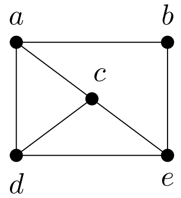
†14. Draw the graph with adjacency matrices

iv) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & 1 & 3 & 2 \end{bmatrix}$

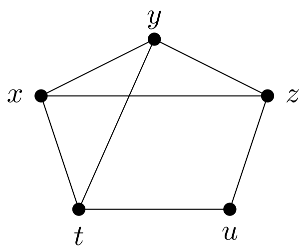
Isomorphism

†15. Determine, with reasons, whether or not the following pairs of graphs are isomorphic.

a)

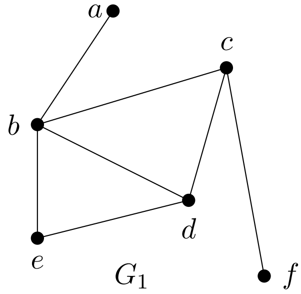


G_1

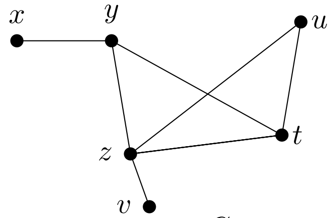


G_2

b)



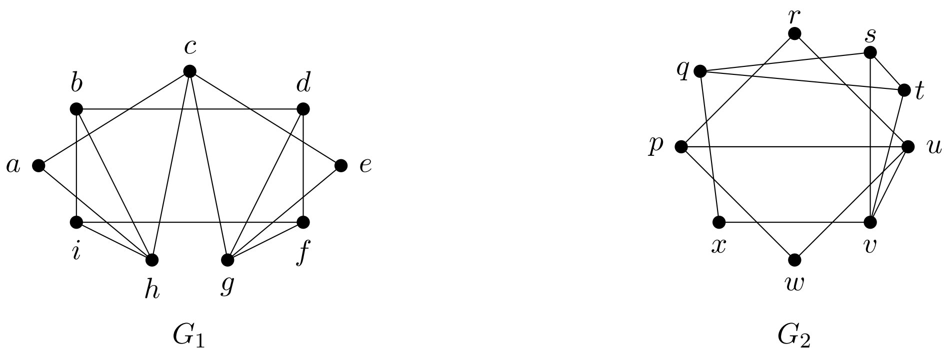
G_1



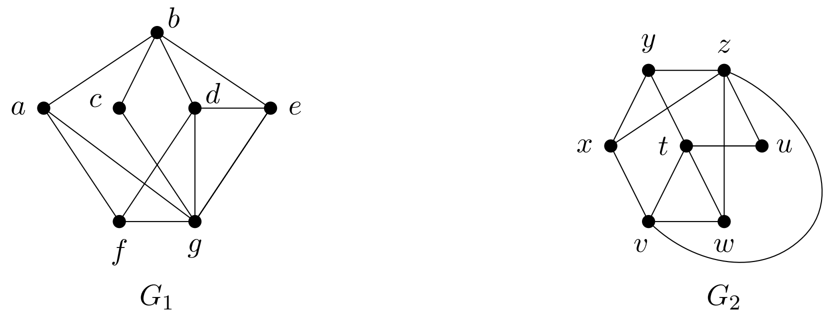
G_2

c)

d)



e)



f)

16. A simple graph G is *self–complementary* if it is isomorphic to its complement $\overline{G}$ (see Q. 9).

a) Show if G is self–complementary then G has $4k$ or $4k+1$ vertices for some integer k. Is the converse true?

*b) Find all self–complementary graphs with four or five vertices.

** c) (Challenge) Find a self–complementary graph with 8 vertices.

*17. Find all non–isomorphic simple graphs with

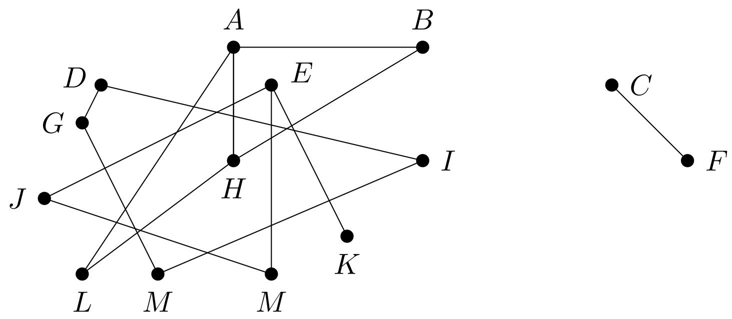
a) 2 vertices,

b) 3 vertices,

c) 4 vertices.

Connectivity

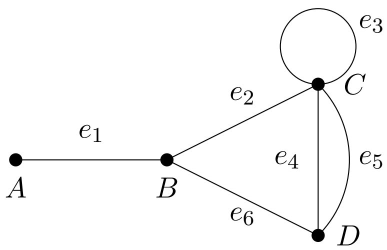
18. Find the connected components (draw separately) of the graph



Adjacency Matrix and Paths

19. What is the number of walks of length n between any two adjacent vertices of K_4 for $n = 2, 3, 4, 5$?

†20. a) Find the adjacency matrix M of the graph G



(ordering the vertices as A, B, C, D).

b) Find M^2, M^3.

c) How many walks of length 2 and 3 are there between B and C? Write them down.

d) Let $N = I_4 + M + M^2 + M^3$.

i) The $(3, 4)$ entry of N is 20. What does this mean in terms of walks?

ii) No entries of N are zero. What does this mean?

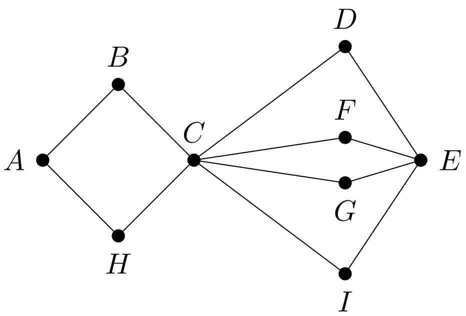
Euler and Hamilton Paths and Circuits.

†21. Determine, with reasons, whether or not there is

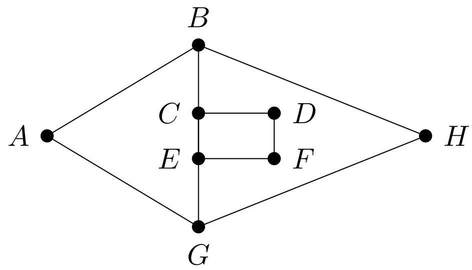
α) an Euler path, which is not a circuit,
β) an Euler circuit,
γ) a Hamilton path, which is not a circuit,
δ) a Hamilton circuit,

in the following graphs. Give an example, if one exists, in each case.

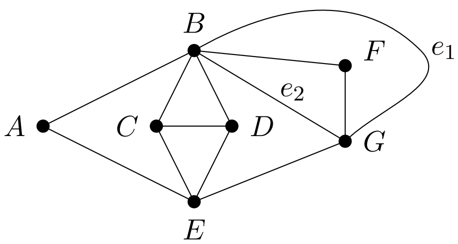
a)



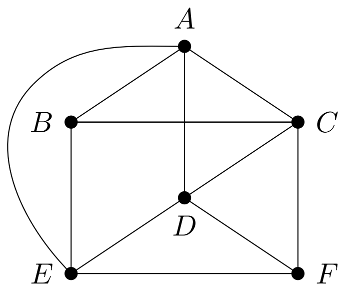
b)



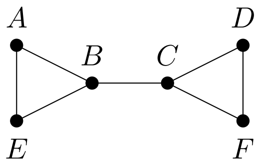
c)



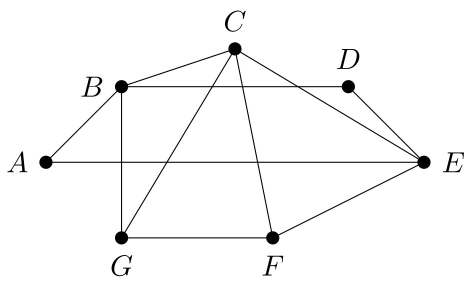
d)



e)



f)



†22. For what values of n does K_n, C_n, Q_n have an Euler circuit?

23. For what values of n does K_n, C_n, Q_n have an Euler path which is not a circuit?

†24. For what values of m, n does $K_{m,n}$ have

a) an Euler circuit,
b) an Euler path which is not a circuit,
c) a Hamilton circuit.

*25. Show each Q_n has a Hamilton circuit.

[Hint: Construct one recursively, constructing Hamiltonian paths between adjacent vertices of Q_n. Construct such a path for Q_{n+1} from one such path for Q_n].

*26. The **Knight's Tour Puzzle** asks if it is possible to find a sequence of 64 knight's moves so that a knight on a chessboard visits all the different squares and ends up on the starting point.

a) Formulate the problem in terms of graphs.

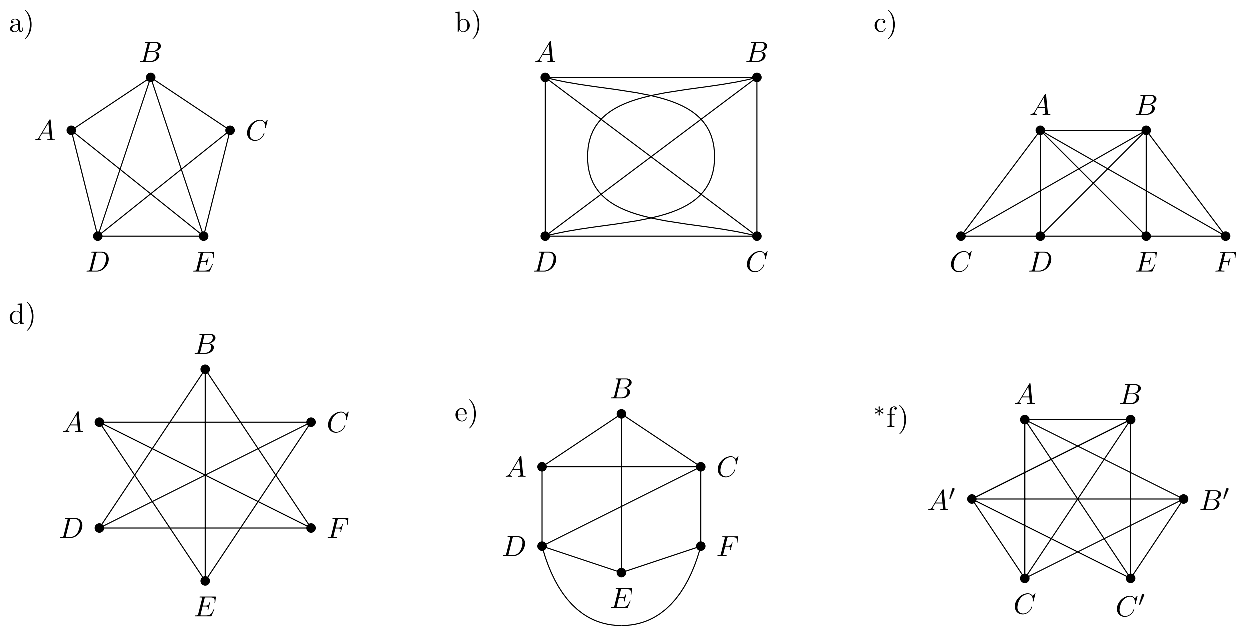
b) Can the puzzle be solved?

c) Can you find an explicit solution?

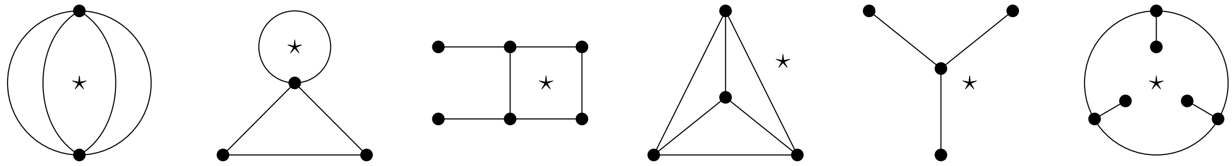
d) What happens on a 3×3 chessboard?

Planar Graphs

†27. Show that the following graphs are planar by redrawing them as planar maps.



†28. For each of



a) Find the degree of each of the regions indicated by an asterisk in each map.

b) What is the sum of the degrees of the regions in each map?

c) Give the dual of each of the planar maps, drawing it on a separate diagram.

d) Verify Euler's formula in the maps.

†29. A connected planar graph has 11 vertices; 5 have degree 1, 5 have degree 4, and 1 has degree 5.

a) How many edges are there?

b) How many regions are there?

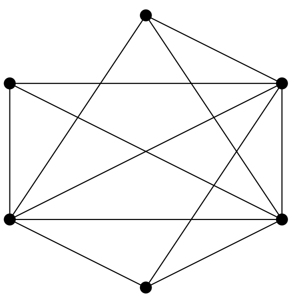
c) Give an example of such a graph, drawing it as a planar map.

30. a) Show if G is a connected planar simple graph with v vertices and e edges with $v \geq 3$ then $e \leq 3v - 6$.

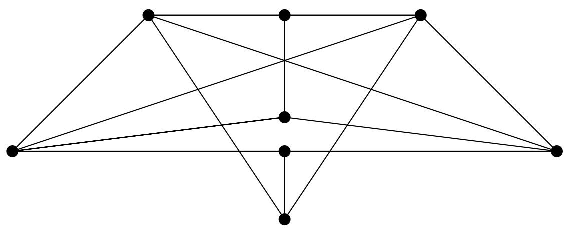
b) Further show if G has no circuits of length 3 then $e \leq 2v - 4$.

31. Apply the last question to show the following graphs are non–planar.

a)



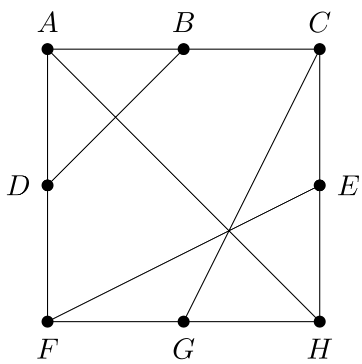
b)



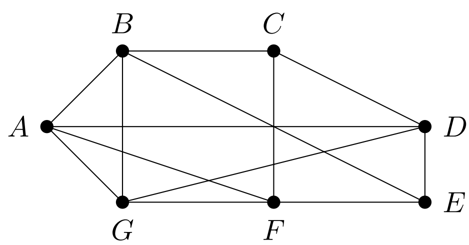
32. Use the results of Q30 to decide which Q_n are planar.

†33. Use Kuratowski's Theorem to show that the following graphs are not planar.

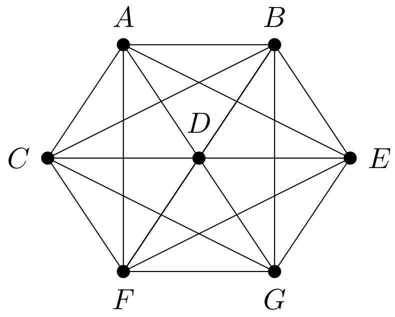
a)



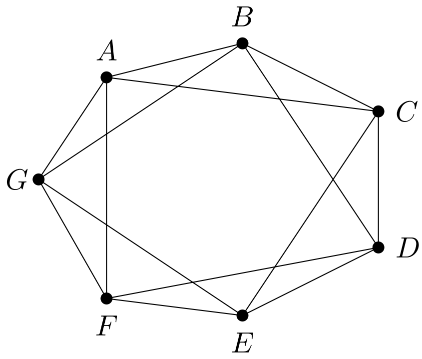
b)



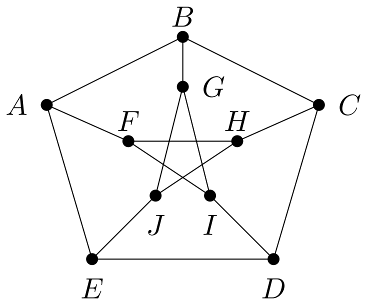
c)



d)

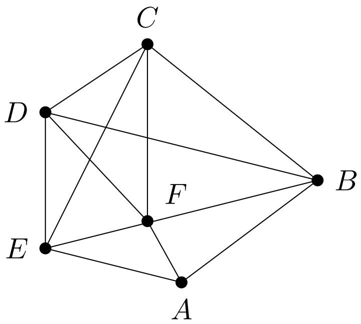


*e)

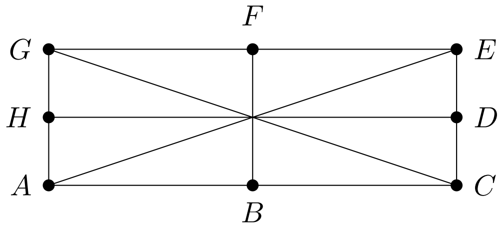


34. Show the converses of Q30 a) and b) are false by considering the following examples. (Hint: Kuratowski's Theorem.)

a)



b)



Trees

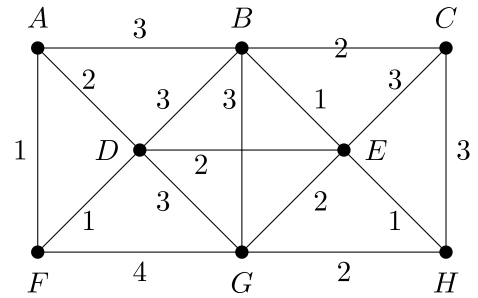
35. A tree T has 8 vertices, **at least** two of which have degree 3.

   a) How many edges are there?
   b) What are the possible vertex degrees for T in non–increasing order?
   c) What are the possible forms for T up to isomorphism?

*36. Prove that a tree has at least one vertex of degree 1.

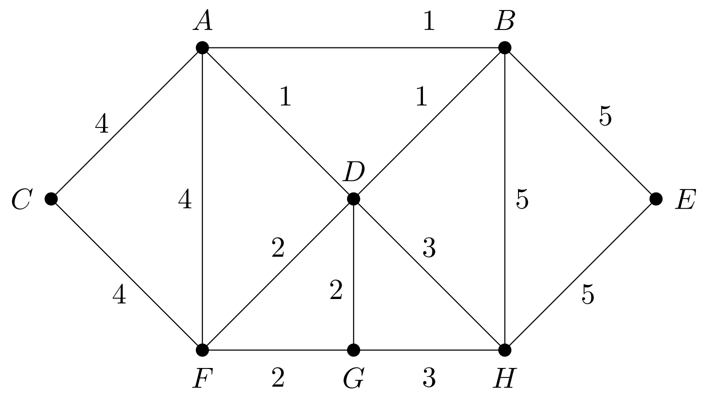
*37. Use Q36 to prove by induction that a tree with n vertices has $n-1$ edges.

†38. a) Use Kruskal's algorithm to find a minimal spanning tree for the following weighted simple graph.



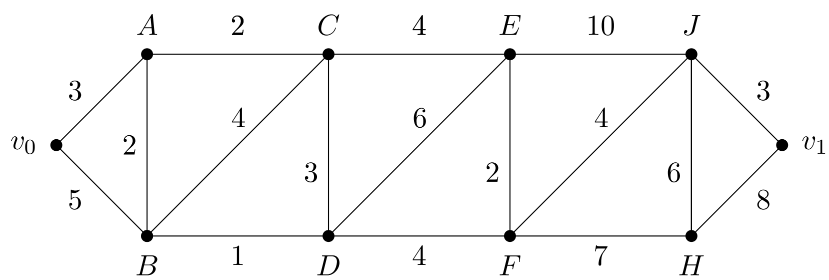
b) Use Dijkstra's algorithm to construct a tree giving shortest paths from A to each of the other vertices in the weighted graph in part (a).

39. a) Use Kruskal's algorithm to find a minimal spanning tree for the following weighted graph. Then find a second minimal spanning tree for the graph. How many other minimal spanning trees are there?



b) Use Dijkstra's algorithm to construct a tree giving shortest paths from A to each of the other vertices in the weighted graph in part (a).

40. Use Dijkstra's algorithm to find a shortest path from the vertex v_0 to the vertex v_1 in the following weighted graph.



ANSWERS TO SELECTED PROBLEMS

Important Note

Here are some answers (not solutions) and some hints to the problems. These are NOT intended as complete solutions and rarely are any reasons given. To obtain full marks in test and examination questions FULL reasoning must be given, and your work should be clearly and logically set out.

Problem Set 1.

1. $A = B$.

2. Yes.

5. a) 4; b) 16; c) 65536.

6. a) T; b) F; c) T; d) F; e) F; f) T; g) T

8. Yes.

10. a) No; b) No; c) Yes; d) Yes; e) T.

11. 6.

12. a) True; b) False.

13. a) $P(A) \cup P(B) \subseteq P(A \cup B)$; b) The sets are equal.

14. There is no containment relation.

15. a) $A - B$; b) A; c) U.

16. B.

17. $A \cup B$; $A \cap B$.

20. b) $\{2, 3, 4, 8, 9, 10, 14, 15, 16\}$.

22. a) $\{1, 2, \ldots, 21\}$; b) $\{90, \ldots, 105\}$; c) $\emptyset$.

23. a) $(0, 4]$; b) $(\frac{89}{90}, 10]$; c) $\{1\}$.

24. a) No; b) Yes; c) No; d) No; e) Yes.

25. a) 3; b) 4; c) -4; d) -3.

27. a) $1-1$, not onto; b) bijection; c) onto, not $1-1$; d) not $1-1$ and not onto.

28. a) One-to-one and onto; b) Neither one-to-one nor onto.

29. a) $f(x) \geq 2$; b) No; c) No, eg. $f(0) = f(4) = 6$.

31. f is not $1-1$ but is onto.

32. a) $\sqrt{4x^2 - 12x + 11}$; b) $g \circ f(x) = x$.

33. b) Yes.

35. a) $f^{-1}(x) = \frac{x-3}{5}$.

36. a) $[-1, 17]$; b) $[-3, -1] \cup [1, 3]$.

37. b) $(m, n) = (\pm 3, \pm 1)$.

38. a) i) The sets are equal. ii) $f(A \cap B) \subseteq f(A) \cap f(B)$.

39. 40430.

42. $\frac{9}{2} - \frac{4}{n-1} - \frac{1}{n}$.

44. $\frac{N+1}{2N}$.

Problem Set 2.

1. a) 2, 5; b) -11, 10; c) 77, 0.

2. T, F, T, F.

3. T, F, T, F, F, F.

4. $3^2 \cdot 13$, $11 \cdot 13$, $2^2 \cdot 3 \cdot 17^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 7$.

5. a) $2^2 \cdot 3^3 \cdot 5^2$, $2^5 \cdot 3^5 \cdot 5^3$; b) 3, $2^2 \cdot 3^2 \cdot 5^3 \cdot 7$; c) 3, not defined.

6. 1, 3, 2.

8. a) $n = 6$, 4; b) $5^8 \equiv -1$, 6; c) -2, 10.

9. a) $6 = -1 \cdot 12 + 1 \cdot 18$; b) $3 = 29 \cdot 11 - 16 \cdot 201$; c) $1 = -8 \cdot 13 + 5 \cdot 21$;

   d) $1 = -13 \cdot 83 + 30 \cdot 36$; e) $2 = 5 \cdot 22 - 2 \cdot 54$; f) $7 = 39 \cdot 112 - 7 \cdot 623$.

10. a) $x \equiv 4 \pmod 7$; b) no solution; c) $x \equiv 6 \pmod{11}$;

    d) $x \equiv 502 \pmod{1001}$; e)$x \equiv 99, 206, 313 \pmod{321}$ or $x \equiv 99 (\text{mod } 107)$

    f) $x \equiv 200, 751, 1302, 1853, 2404 \pmod{2755}$ or $x \equiv 200 (\text{mod } 551)$

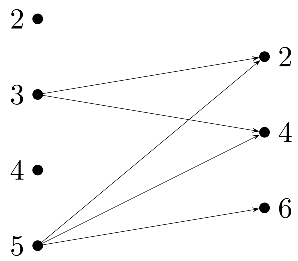
    g)$x \equiv 29, 70, 111, 152, 193, 234, 275, 316, 357, 398, 439, 480, 521 \pmod{533}$ or

    $x \equiv 29 \pmod{41}$

13. c) $R_3 = \{(3, 2),\ (3, 4),\ (5, 2),\ (5, 4),\ (5, 6)\}$.

14. R_3

a)



b) $M_{R_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

17. $\{\{0,3,6\},\{1,4\},\{2,5\}\}$

18. b) $\{\{0,5\},\{1,4,6\},\{2,3,7,8\}\}$

21. $S = \{0,6\} \cup \{1,5,7,11\} \cup \{2,4,8,10\} \cup \{3,9\}$.

24. b) 54, 60, 72 c) 2, 9 d) no e) no

f) $\{36,54,72\}$ g) none h) $\{2,4,6,12\}$ i) 12

Problem Set 3.

6. a) Hint: $k! \geqslant 2^{k-1}$ for $k \in \mathbb{Z}^+$.

15. a) Converse is false. It is known that $2^p - 1$ is prime for the primes $p = 2,3,5,7,13,17,19,31,61,89,107,127$ and more than 20 other (larger) values of p. These are called Mersenne primes.

20. Converse is false.

21. b) True.
    c) True.

59. a, c, e and f

60. a) $p =$"prices go up",
    $q =$"housing will be plentiful";
    $r =$"housing will be expensive"
    $[p \to q \wedge r] \wedge [\sim r \to q]$

    e) $p =$"Janet wins"; $q =$"Janet loses"; $r =$ "Janet will be tired"
    $p \vee (q \to r)$.

61. a) The food is good but the service is poor.
    b) The food is poor and so is the service.

62. a) Contradiction.
    b) Contingency.
    c) Tautology.
    d) Tautology

64. a) $\sim q$
    b) p
    c) $p \wedge q \wedge r$
    d) $\sim p \vee q$.

66. I did my homework, but there is no way of deciding whether or not I studied.

67. Holmes is correct.

68. We cannot deduce whether Einstein is right or wrong.

69. c) Peta must have failed.

Problem Set 4.

1. a) $5^2 \times 9^4$; b) $4 \times 5^2 \times 6^3$; c) $P(9,6) = 9 \times 8 \times 7 \times 6 \times 5 \times 4$; d) $C(6,3) \times 8^3$; e) $8^6 + 6 \times 8^5 + C(6,2)8^4$; f) $C(6,3)P(8,3)$; g) 4×9^4.

2. a) $21!/13!$, $21!/8!\,13!$, $321!/123!\,198!$, $(2n)!/(n!)^2$; b) 840, 35, 120, 6, 20100.

3. a) $7 \times 5 \times 21^6$; b) $C(7,2) \times 5^2 \times 21^5$; c) $C(7,3) \times 5^3 \times 21^4$; d) $26^7 - 21^7 - 7 \times 5 \times 21^6 - C(7,2) \times 5^2 \times 21^5$.

4. a) $C(7,1)P(5,1)P(21,6)$; b) $C(7,2)P(5,2)P(21,5)$; c) $C(7,3)P(5,3)P(21,4)$; d) $C(7,3)P(5,3)P(21,4) + C(7,4)P(5,4)P(21,3) + C(7,5)P(5,5)P(21,2)$.

5. a) 228; b) 37800; 37800; 22260; c) $20 \times 26^3 + 13 \times 26^2 + 18 \times 26 + 22 = 360798$.

6. a) $C(8,2) \times 25^6$; b) $26^8 - 25^8 - 8 \times 25^7$.

7. b) $4C(39,8) - 6C(26,8) + 4C(13,8)$.

8. a) 143; b) 128.

9. a) $13C(4,3)C(12,5)/C(52,8)$;
   b) $(4C(13,3)C(39,5) - 6C(13,3)^2C(26,2))/C(52,8)$;
   c) $4C(13,3) \times (C(39,5) - 3C(13,3)C(26,2))/C(52,8)$.

10. $\phi(pq) = (p-1)(q-1)$, $\phi(p^2q) = p(p-1)(q-1)$, $\phi(pqr) = (p-1)(q-1)(r-1)$;

16. a) 834; b) 834 again (NOT 417).

19. $4 \times 7 \times 6 \times 9 \times 16!/20!$.

20. $21!/(1!\,2!\,3!\,4!\,5!\,6!\,6^{21})$.

21. a) $\frac{1}{365}$; b) $1 - P(365,n)/365^n$; c) $n \geq 23$.

22. a) $C(23,3)$; b) $C(15,3)$; c) $C(23,3) - C(19,3)$.

23. a) 6^4; b) $C(9,5)$.

24. a) $23!/2!\,5!\,7!\,9!$; b) $C(26,3)$.

25. a) $C(45,5)$; b) $C(27,5)$;

    c) $C(45,5) - C(6,1)C(34,5) + C(6,2)C(23,5) - C(6,3)C(12,5) = C(25,5) - 6C(14,5)$

    d) $C(13,5)$; e) $C(25,5)$; f) $C(45,5) - C(25,5)$;

    g) $C(22,5)$; h) $C(30,5) - 3C(20,5) + 3C(10,5)$.

28. (There are other correct answers.)

    a) $a_n = 2a_{n-1},\ a_0 = 2$; b) $a_n = a_{n-1} + 2,\ a_0 = 1$; c) $a_n = -2a_{n-1},\ a_0 = 3$;
    d) $a_n = a_{n-1}+n,\ a_1 = 1$; e) $a_n = a_{n-1}, a_0 = 2$; f) $a_n = a_{n-1}+a_{n-2},\ a_0 = 1,\ a_1 = 2$.

29. a) $a_n = a_{n-2} + a_{n-3}$ for $n \geq 4$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$; b) 1, 4.

30. 2, 3, 5.

31. $u_n = \left(1 + \frac{18}{1200}\right) u_{n-1} - 900$, for $n \geq 1$, $u_0 = 50000$.

32. a) $a_n = A(5^n)$; b) $a_n = 3(-4)^n$.

33. a) $a_n = A(2^n) + B(-3)^n$; b) $a_n = A + B(2^n)$; c) $a_n = A(3^n) + Bn(3^n)$;
    d) $a_n = A(1 + \sqrt{5})^n + B(1 - \sqrt{5})^n$.

34. a) $a_n = 4(3^n) + 3(-5)^n$; b) $a_n = 2^{n+1} + 3^{n+1}$; c) $a_n = 2(-2)^n - 4n(-2)^n$;
    d) $a_n = (2 + \sqrt{10})^n + (2 - \sqrt{10})^n$.

35. a) $a_n = A(2^n) + B(-3)^n + \frac{8}{7}\,4^n$; b) $a_n = A + B(2^n) + n(2^{n+1})$;
    c) $a_n = A(3^n) + Bn(3^n) + 2n + 7$; d) $a_n = A(3^n) + Bn(3^n) + \frac{1}{2}\,n^2 3^n$.

36. $a_n = 2(4^n) + (-1)^{n+1} + n(-1)^n$.

37. a) $a_1 = 1,\ a_2 = 3$; b) $a_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$.

38. b) $a_n = 2^{n-1} + \frac{1}{2}\,4^n$

39. a) $0, 2, 4, 12, 32, 88$; b) $a_n = 2a_{n-1} + 2a_{n-2}$; c) $\frac{1}{\sqrt{3}}\left((1 + \sqrt{3})^n - (1 - \sqrt{3})^n\right)$.

40. c) $a_n = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right)$

41. c) $\frac{1}{2}\left(3^{n-1} + 1 - 2^n\right)$

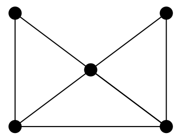
Problem Set 5.

2. c) Total Vertex degree = 32, total number of edges = 16.
   d) Loops are e_{10}, e_{14}.
   e) Sets of parallel edges are $\{e_7, e_8\}$, $\{e_9, e_{12}\}$, $\{e_{15}, e_{16}\}$.

f) Edges incident to v_3 are $e_4, e_6, e_9, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.
g) Vertices adjacent to v_8 are v_2, v_3, v_7.
h) v_9 is the only isolated vertex.

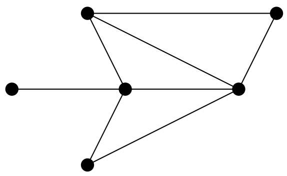
3. G_1, G_3, G_4 are nonsimple. The rest are simple.

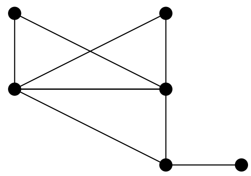
4. Number of edges $= 7$.



It is the only simple graph with these degrees. (The two degree 2 vertices cannot be adjacent.)

5. a) $\exists$ simple examples





b) There is no **graph** with these vertex degrees.

c) d) e) f) There is no **simple** graph with these vertex degrees.

7. $|V(G)|,\ |E(G)|$ are respectively

a) $n,\ \binom{n}{2}$

b) $m+n,\ mn$

c) $n,\ n$

d) $2^n,\ n2^{n-1}$

8. 35 $(= 15 + 12 + 6 + 1 + 1)$

9. a) $\overline{K}_n \cong E_n$, ie, the graph with n vertices and no edges.
b) $\overline{K}_{m,n} \cong K_m \oplus K_n$ (disjoint union)

10. $\dfrac{n(n-1)}{2} - m$

11. Y =Bipartite, N =Not Bipartite.

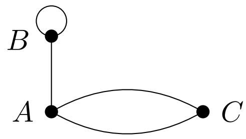
a) Y b) N c) Y d) N e) Y f) N g) N.

12. a) $n = 2$ b) $n \geq 3$ and even c) all $n \geq 0$.

13. a) $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

14. a)



15. a) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$f:$	a	b	c	d	e
	t	u	y	x	z

b) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$f:$	a	b	c	d	e	f
	v	z	y	t	u	x

c) Not isomorphic.

d) Not isomorphic.

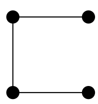
e) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$f:$	a	b	c	d	e	f	g
	y	t	u	v	w	x	z

f) Not isomorphic.

16. b) Up to isomorphism there is/are

α) Only 1 self–complementary graph on 4 vertices



β) 2 self–complementary graphs on 5 vertices

17. a) $n = 2 : 2$ possible

b) $n = 3 : 4$ possible

c) $n = 4 : 11$ possible

18. There are 4 components with vertex sets $\{A, B, H, L\}$, $\{C, F\}$, $\{D, G, I, M\}$, $\{E, J, K, N\}$.

19. a) 2 b) 7 c) 20 d) 61.

20. a) $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

b) $M^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 2 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 5 & 10 & 9 \\ 3 & 10 & 15 & 15 \\ 2 & 9 & 15 & 8 \end{pmatrix}$

c) 3 and 10 respectively.

Walks of length 2 B to C are

$$BDe_4C$$
$$BDe_5C$$
$$BCe_3C$$

Walks of length 3 B to C are

$$BABC$$
$$BDBC$$
$$BCBC$$
$$BCe_4De_4C$$
$$BCe_4De_5C$$
$$BCe_5De_4C$$
$$BCe_5De_5C$$
$$BDe_4Ce_3C$$
$$BDe_5Ce_3C$$
$$BCe_3Ce_3C$$

d) i) This means there are 20 walks of length ≤ 3 from C to D.
ii) This means G is connected.

21. a) α) N
β) Y eg. $ABCDEFCGEICHA$
γ) N
δ) N

b) α) N
β) N
γ) Y
δ) N

c) α) Y eg $CBAECDEGe_1BFGe_2BD$.
β) N
γ) Y eg $DCEABFG$
δ) N

d) α) Y eg $BAEBCADCFDEF$
β) N
γ) Y eg $ABCDFE$
δ) Y eg $ABCDFEA$.

e) α) Y eg $BAEBCDFC$
β) N
γ) Y eg $EABCDF$
δ) N

f) α) Y eg $GBAEDBCGFCEG$
β) N
γ) Y eg $ABDECGF$
δ) N.

22. a) $n \geq 1$ and odd
b) All $n \geq 3$
c) All $n \geq 0$ and even.

23. a) $n = 2$
b) Never
c) $n = 1$.

24. a) m, n both even
    b) $m = 2$ and n odd or $n = 2$ and m odd
    c) $m = n \geq 2$.

28. i) a) 2 b) 8
    ii) a) 1 b) 8
    iii) a) 4 b) 12
    iv) a) 3 b) 12
    v) a) 6 b) 6
    vi) a) 9 b) 12

29. a) 15 b) 6

32. Only for $n = 0, 1, 2, 3$.

36. a) 7 b) $3, 3, 3, 1, 1, 1, 1, 1$ and $3, 3, 2, 2, 1, 1, 1, 1$ c) There are six possible answers.