MEEN 672 **Fall Semester 2021**

An Introduction to the Finite Element Method

Professor J. N. Reddy **Tel:** 862 2417; Room: 401 MEOB (O)
e-mail: jnreddy@tamu.edu **Web:** http://www.tamu.edu/acml

ASSIGNMENT No. 8

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PART I: THEORETICAL FORMULATION

Problem 1: For axisymmetric flows of viscous incompressible fluids (i.e., flow field is independent of θ coordinate), the governing equations in the cylindrical coordinates (r, θ, z) are given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} + f_r = \rho\frac{\partial u}{\partial t} \tag{1.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} + f_z = \rho\frac{\partial w}{\partial t} \tag{1.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \tag{1.3}$$

where

$$\sigma_{rr} = -P + 2\mu\frac{\partial u}{\partial r}, \quad \sigma_{\theta\theta} = -P + 2\mu\frac{u}{r}, \quad \sigma_{zz} = -P + 2\mu\frac{\partial w}{\partial z}, \quad \sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{1.4}$$

(a) Develop the semidiscrete (mixed) finite element model of the equation by the pressure-velocity formulation, and (b) complete the element development by completing the time approximation step (i.e., full discretization) using the α-family of approximation (use what is already developed in Chapter 6).

Problem 2: The equations of motion of an **axisymmetric** elasticity problem are given by

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r &= \rho\frac{\partial^2 u_r}{\partial t^2}, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z &= \rho\frac{\partial^2 u_z}{\partial t^2}. \end{aligned} \tag{2.1}$$

where u_r and u_z are the displacement components along the r and z coordinates, respectively (due to symmetry about the θ-axis, the displacement component $u_\theta = 0$ and all variables are independent of θ). The stresses are related to the strains* (Hooke's law for an isotropic material)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}; \quad (\sigma_{11} = \sigma_{rr},\ \sigma_{12} = \sigma_{r\theta},\ \sigma_{33} = \sigma_{zz}, \quad \text{and so on}) \tag{2.2}$$

and the nonzero strains are related to the displacements (u_r, u_z) by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2.3}$$

Develop: (a) the weak form of the equations with displacements (u_r, u_z) as the unknowns, (b) the semidiscrete finite element model of the equations, and (c) the fully discretized finite element model by completing the time approximation step.

* Those who are not familiar with the index notation used in Eq. (2.2), the explicit stress-strain relations are

$$\begin{aligned} \sigma_{rr} &= 2\mu\varepsilon_{rr} + \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}\right); \quad \sigma_{\theta\theta} = 2\mu\varepsilon_{\theta\theta} + \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}\right) \\ \sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}\right); \quad \sigma_{r\theta} = 2\mu\varepsilon_{r\theta}, \quad \sigma_{rz} = 2\mu\varepsilon_{rz} \end{aligned} \tag{2.4}$$

Problem 3: For the plane elasticity problem shown in Figure 1, give (a) global stiffness coefficients K_{99}, $K_{9(32)}$, and $K_{(10)(28)}$; and (b) the specified boundary degrees of freedom and compute the contribution of the specified forces to the nodes.

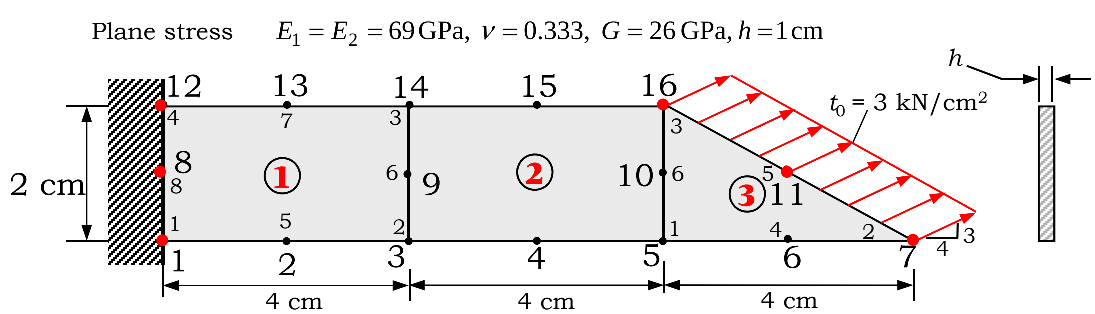

Figure 1: A plane elasticity problem.

PART II: COMPUTER SOLUTIONS USING FEM2DF21.EXE

Use FEM2DF21.EXE to solve the following two problems.

Problem 4: Verify the results of the *lid-driven cavity flow* problem of Example 11.6.3. Take viscosity μ to be equal to unity, and the penalty parameter to be $\gamma = 10^2$ and $\gamma = 10^8$. You must submit the output in the form of graphs and tables using the uniform

8×8 mesh of linear rectangular elements and 4×4 mesh of nine-node rectangular elements (see pages 692–697 for the discussion of the problem and results in Figures 11.6.8–11.6.10 and Table 11.6.4).

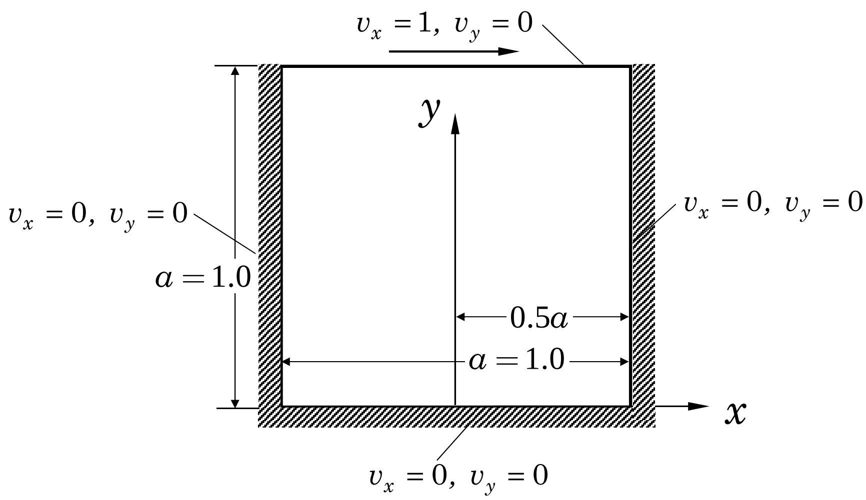

Figure 2: Lid-driven cavity problem.

Problem 5: Analyze the *plane elasticity problem* in Example 12.6.1 using meshes of linear (a) triangular and (b) rectangular elements given in Table 12.6.3. You must submit tabulated results of the *maximum* displacements and stresses for various meshes shown in Table 12.6.3 (i.e., duplicate the results of Table 12.6.3).

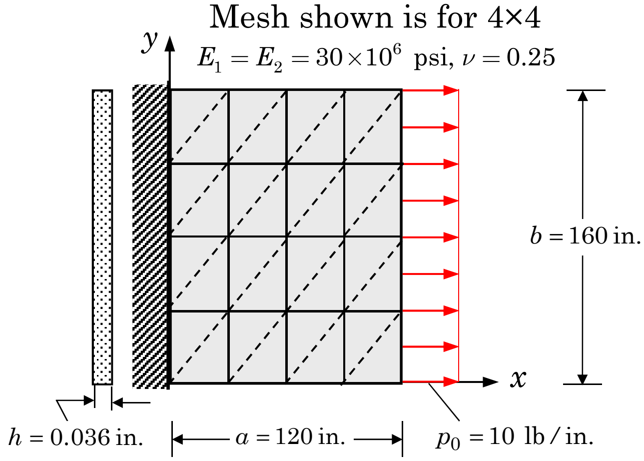

Figure 3: A plane elastic body.