Homework 5

1) $x = 2\cos t, y = \sin t, z = t$ at $t = \pi/2$

Let $f(t) = (2\cos t, \sin t, t)$

$$f'(t) = \left(\frac{d}{dt}(2\cos t), \frac{d}{dt}(\sin t), \frac{d}{dt} t\right)$$

$$= (-2\sin t, \cos t, 1)$$

Parametric eq. of the tangent line is given by the equation,

$$f(\pi/2) + t f'(\pi/2)$$

$$= \left(2\cos\frac{\pi}{2}, \sin\frac{\pi}{2}, \frac{\pi}{2}\right) + t\left(-2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1\right)$$

$$= (0, 1, \pi/2) + t(-2, 0, 1)$$

$$= (-2t, 1, \pi/2 + t)$$

$$\therefore l(t) = (-2t, 1, t + \pi/2).$$

2.a) $f(x,y) = x\ln(y^2 - x)$

$\ln(y^2 - x)$ is defined for $y^2 - x > 0 \Rightarrow y^2 > x$

$\therefore$ Domain of $f = \{(x,y) \in \mathbb{R}^2 : y^2 > x\}$

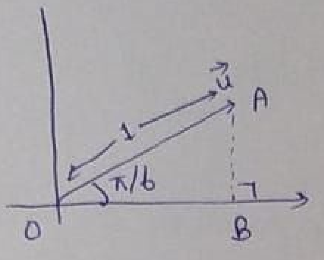
(b) $g(x,y) = \sqrt{4 - x^2 - y^2}$

$g(x,y)$ is a real number iff $4 - x^2 - y^2 \geq 0$

$$\Rightarrow x^2 + y^2 \leq 4$$

Domain of $g = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

3) $f(x,y) = x^3 - 3xy + 4y^2$.



Let $\vec{OA}$ be the vector representing $\vec{u}$.

Since u is a unit vector length of $OA = 1$.

$\cos\frac{\pi}{6} = \frac{OB}{OA} \Rightarrow \frac{\sqrt{3}}{2} = \frac{OB}{1}$

$\Rightarrow OB = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{6} = \frac{AB}{OA} \Rightarrow \frac{1}{2} = AB$

$\therefore$ The coordinates of A is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$u = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y & -3x + 8y \end{bmatrix}$$

$$Df(1,2) = [3 - 3.2 \quad -3.1 + 8.2]$$
$$= [-3 \quad 13]$$

$$D_u f(1,2) = Df(1,2) \cdot u$$
$$= [-3 \quad 13]\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = -3.\frac{\sqrt{3}}{2} + 13.\frac{1}{2}.$$

$$\Rightarrow D_u f(1,2) = \frac{13 - 3\sqrt{3}}{2}.$$

4) $f(x,y) = xe^y$, $P = (2,0)$, $Q = (12,2)$

(a) $\vec{PQ} = Q - P = (12,2) - (2,0) = (10,2)$

$Df = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] = \left[e^y \quad xe^y\right]$

$Df(P) = Df(2,0) = [1 \quad 2]$

rate of change of f at P in the direction of PQ

$= D_{PQ} f(P) = Df(P) . PQ$

In this case PQ must be a unit vector.

unit vector corresponding to $PQ = \left(\frac{10}{\sqrt{10^2+2^2}}, \frac{2}{\sqrt{10^2+2^2}}\right)$

$= \left(\frac{10}{\sqrt{104}}, \frac{2}{\sqrt{104}}\right) = \left(\frac{10}{2\sqrt{26}}, \frac{2}{2\sqrt{26}}\right)$

$= \left(\frac{5}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$

$\therefore$ rate of change along PQ at P $= [1 \quad 2] \begin{bmatrix} 5/\sqrt{26} \\ 1/\sqrt{26} \end{bmatrix}$

$= \frac{5}{\sqrt{26}} + 2 \cdot \frac{1}{\sqrt{26}} = \frac{7}{\sqrt{26}}$ units.

(b) f has maximum rate of change in the direction of $Df(P)$ at P.

$Df(P) = [1 \quad 2]$

let $\vec{v}$ be the unit vector along $Df(P)$.

$\vec{v} = \left(\frac{1}{\sqrt{1^2+2^2}}, \frac{2}{\sqrt{1^2+2^2}}\right) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

max rate of change $= D_v f(P)$

$= Df(P) . v = [1 \quad 2] \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$= \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \sqrt{5}$ units

Note that, corresponding to any point (a, b) the max rate of change in f occurs in the direction $Df(a, b)$.

Let $\vec{u}$ be the unit vector corresponding to $Df(a, b)$

$$Df(a, b) = (e^b, ae^b)$$

$$\vec{u} = \left(\frac{e^b}{\sqrt{e^{2b} + a^2 e^{2b}}}, \frac{ae^b}{\sqrt{e^{2b} + a^2 e^{2b}}}\right) = \left(\frac{e^b}{e^b\sqrt{a^2+1}}, \frac{ae^b}{e^b\sqrt{a^2+1}}\right)$$

$$= \left(\frac{1}{\sqrt{a^2+1}}, \frac{a}{\sqrt{a^2+1}}\right)$$

Max rate of change $= D_u f(a, b)$

$$= Df(a, b) \cdot u$$

$$= (e^b, ae^b) \cdot \left(\frac{1}{\sqrt{a^2+1}}, \frac{a}{\sqrt{a^2+1}}\right)$$

$$= \frac{e^b}{\sqrt{a^2+1}} + \frac{a^2 e^b}{\sqrt{a^2+1}}.$$

$$= e^b \frac{(a^2+1)}{\sqrt{a^2+1}} = e^b\sqrt{a^2+1}.$$

$\therefore$ Max rate of change of f at (a, b) occurs in the direction $\left(\frac{1}{\sqrt{a^2+1}}, \frac{a}{\sqrt{a^2+1}}\right)$ and has value $e^b\sqrt{a^2+1}$ units.

5) Distance of $(\cos t, \sin t, \sin(t/2))$ from the origin

$$= \sqrt{(\cos t - 0)^2 + (\sin t - 0)^2 + (\sin(t/2) - 0)^2}$$

$$= \sqrt{\cos^2 t + \sin^2 + \sin^2(t/2)} = \sqrt{1 + \sin^2(t/2)}$$

To find the farthest point on the curve we need to maximize the distance i.e. $\sqrt{1+\sin^2(t/2)}$

$\Rightarrow$ maximize $1+\sin^2(t/2)$.

Let $f(t) = 1+\sin^2(t/2)$. $f : \mathbb{R} \longrightarrow \mathbb{R}$

We shall find the critical points of f.

$f'(t) = 2\sin\left(\frac{t}{2}\right) \cdot \cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}$

$= \frac{1}{2}\sin t$

$f'(t) = 0 \Rightarrow \frac{1}{2}\sin t = 0 \Rightarrow \sin t = 0$.

$\Rightarrow t = 0, n\pi, n \in \mathbb{Z}$.

We shall apply the second derivative test

$f''(t) = \frac{1}{2}\cos t$.

$f''(0) = \frac{1}{2} > 0 \Rightarrow$ at $t=0$ a local minimum occurs.

$f''(n\pi) = \frac{1}{2}\cos(n\pi)$.

To maximize $f(t)$ we need, $f''(n\pi) < 0$.

$\Rightarrow \frac{1}{2}\cos(n\pi) < 0 \Rightarrow \cos(n\pi) < 0$.

$\cos n\pi < 0 \quad \forall\, n \in \mathbb{Z}$ s.t. n is odd.

$\therefore t = (2k+1)\pi, \quad k \in \mathbb{Z}$.

points farthest from the origin are $= \left(\cos(2k+1)\pi, \sin(2k+1)\pi, \sin(2k+1)\frac{\pi}{2}\right)$

$\forall\, k \in \mathbb{Z}$.

$f((2k+1)\pi) = 1+\sin^2\left((2k+1)\frac{\pi}{2}\right)$

$= 1+\sin^2\left(k\pi+\frac{\pi}{2}\right) = 1+\sin^2\left(\frac{\pi}{2}\right) = 2$

$\therefore$ maximum distance $= \sqrt{2}$ units

$\therefore \{(\cos((2k+1)\pi), \sin((2k+1)\pi), \sin\frac{\pi}{2}) : k \in \mathbb{Z}\}$

$= \{(\cos((2k+1)\pi), \sin((2k+1)\pi), 1) : k \in \mathbb{Z}\}$ is

the set of point farthest from the origin.

b) Let $f(x,y,z) = x^2 + xy + y^2 + yz + z^2$

eq of a sphere of radius 1 := $x^2+y^2+z^2 = 1$

$\therefore$ we have to,

$$\text{Max } f(x,y,z)$$
$$\text{s.t. } x^2+y^2+z^2 = 1$$

Let $g(x,y,z) = x^2+y^2+z^2-1$

Let the Lagrangian be,

$L(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$= x^2 + xy + y^2 + yz + z^2 + \lambda(x^2+y^2+z^2-1)$.

$\frac{\partial L}{\partial x} = 2x + y + 2\lambda x$

$\frac{\partial L}{\partial y} = x + 2y + z + 2\lambda y$

$\frac{\partial L}{\partial z} = y + 2z + 2\lambda z$

$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$

By the Lagrangian condition, $\nabla L = 0$ at the critical pt. P.

i.e. $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial z} = 0$, $\frac{\partial L}{\partial \lambda} = 0$.

$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x + y + 2\lambda x = 0$

$\Rightarrow 2x + 2\lambda x = -y$

$\Rightarrow 2x(1+\lambda) = -y \quad \ldots (i)$

$\frac{\partial L}{\partial y} = 0 \Rightarrow x + 2y + z + 2\lambda y = 0$

$\Rightarrow x + z + 2y(1+\lambda) = 0.$

$\Rightarrow 2y(1+\lambda) = -x - z \quad \ldots (ii)$

$\frac{\partial L}{\partial z} = 0 \Rightarrow y + 2z + 2\lambda z = 0$

$\Rightarrow y + 2z(1+\lambda) = 0$

$\Rightarrow 2z(1+\lambda) = -y \quad \ldots (iii)$

from (i) & (iii) we get,

$2x(1+\lambda) = 2z(1+\lambda)$

$\Rightarrow 2(1+\lambda)(x-z) = 0$

$\Rightarrow$ either $x = z$ or $\lambda = -1$.

If $\lambda = -1$, In eq (i) we get, $y = 0$.

and in eq (ii) we get, $-x - z = 0$

$\Rightarrow x = -z$.

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 + z^2 - 1 = 0$.

when $\lambda = -1$, $y = 0$, $x = -z$ so we have,

$x^2 + 0 + x^2 - 1 = 0$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$\therefore$ The critical points are $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

If $\lambda \neq -1$, from eq (i) we have,

$$x = \frac{-y}{2(1+\lambda)}$$

from eq (iii) we have, $z = \frac{-y}{2(1+\lambda)}$

Substituting these in eq (ii) we get,

$$2y(1+\lambda) = \frac{y}{2(1+\lambda)} + \frac{y}{2(1+\lambda)}$$

$$\Rightarrow 2y(1+\lambda) = \frac{\cancel{2}y}{\cancel{2}(1+\lambda)}$$

$$\Rightarrow 2y(1+\lambda) - \frac{y}{1+\lambda} = 0$$

$$\Rightarrow y\left[2(1+\lambda) - \frac{1}{1+\lambda}\right] = 0$$

$$\Rightarrow y = 0 \quad \text{or} \quad 2(1+\lambda) - \frac{1}{1+\lambda} = 0$$

If $y = 0$, $x = 0$, $z = 0$

but $(0,0,0)$ doesn't satisfy $x^2 + y^2 + z^2 = 1$.

$$\therefore 2(1+\lambda) = \frac{1}{1+\lambda} \Rightarrow (1+\lambda)^2 = \frac{1}{2}$$

$$\Rightarrow \lambda = -1 \pm \frac{1}{\sqrt{2}}$$

If $\lambda = -1 + \frac{1}{\sqrt{2}}$, $x = \frac{-y}{2\left(1 - 1 + \frac{1}{\sqrt{2}}\right)} = \frac{-y}{\sqrt{2}}$

$$z = \frac{-y}{2\left(1 - 1 + \frac{1}{\sqrt{2}}\right)} = \frac{-y}{\sqrt{2}}$$

$x^2+y^2+z^2=1$

$\Rightarrow \left(\frac{-y}{\sqrt{2}}\right)^2 + y^2 + \left(\frac{-y}{\sqrt{2}}\right)^2 = 1$

$\Rightarrow \frac{y^2}{2} + \frac{y^2}{2} + \frac{y^2}{2} = 1$

$\Rightarrow 2y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$.

$x = \frac{-1}{\sqrt{2}}\left(\pm\frac{1}{\sqrt{2}}\right) = \mp\frac{1}{2}$

$z = \frac{-1}{\sqrt{2}}\left(\pm\frac{1}{\sqrt{2}}\right) = \mp\frac{1}{2}$.

$\therefore$ The critical points are $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$

If $\lambda = -1-\frac{1}{\sqrt{2}}$, $x = \frac{-y}{2\left(1-1-\frac{1}{\sqrt{2}}\right)} = \frac{y}{\sqrt{2}}$.

$z = \frac{-y}{2\left(1-1-\frac{1}{\sqrt{2}}\right)} = \frac{y}{\sqrt{2}}$.

$x^2+y^2+z^2=1$

$\Rightarrow \left(\frac{y}{\sqrt{2}}\right)^2 + y^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$

$\Rightarrow \frac{y^2}{2} + y^2 + \frac{y^2}{2} = 1 \Rightarrow 2y^2 = 1$

$\Rightarrow y = \pm\frac{1}{\sqrt{2}}$.

$x = \frac{1}{\sqrt{2}}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$, $z = \frac{1}{\sqrt{2}}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$

$\therefore$ The critical points are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$

∴ All the critical points are :

$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$,

$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$

We shall evaluate f at each of the above critical points

$$f\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)\cdot 0 + 0^2 + 0\cdot\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)^2$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$f\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)\cdot 0 + 0^2 + 0\cdot\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$f(-1/2, 1/\sqrt{2}, -1/2) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2$$
$$= \frac{1}{4} - \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{\sqrt{2}}$$

$$f\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{4} - \frac{1}{2\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{4} = 1 - \frac{1}{\sqrt{2}}$$

$$f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{4} + \frac{1}{2\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} = 1 + \frac{1}{\sqrt{2}}$$

$$f\left(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2$$
$$= \frac{1}{4} + \frac{1}{2\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} = 1 + \frac{1}{\sqrt{2}}$$

$\therefore$ The max value of f is $1 + \frac{1}{\sqrt{2}}$ which occurs

at two points $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$

7) cost of 1 unit of product A = \$11

cost of 1 unit of product B = \$3

units of C produced = $g(x,y) = -3x^2 + 10xy - 3y^2$

Total cost of producing C $= 11 \times x + 3 \times y$

$= 11x + 3y$

we have,

Minimize $11x + 3y$

s.t. $-3x^2 + 10xy - 3y^2 = 80$

Let $f(x,y) = 11x + 3y$.

$\therefore L(x,y) = f(x,y) + \lambda(g(x,y) - 80)$

$= 11x + 3y + \lambda(-3x^2 + 10xy - 3y^2 - 80)$

$\frac{\partial L}{\partial x} = 11 + \lambda(-6x + 10y)$

$\frac{\partial L}{\partial y} = 3 + \lambda(10x - 6y)$

$\frac{\partial L}{\partial \lambda} = (-3x^2 + 10xy - 3y^2 - 80)$

By Lagrangian condition we have,

$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial \lambda} = 0$

$\frac{\partial L}{\partial x} = 0 \Rightarrow 11 - 6\lambda x + 10\lambda y = 0$.

$\Rightarrow 11 = 6\lambda x - 10\lambda y \quad \cdots$ (i)

$\frac{\partial L}{\partial y} = 0 \Rightarrow 3 + 10\lambda x - 6\lambda y = 0$

$\Rightarrow 3 = -10\lambda x + 6\lambda y$ ... (ii)

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow -3x^2 + 10xy - 3y^2 = 80$ ... (iii).

We need to solve eq (i) & (ii) for x and y.

$6 \times$ eq (i) $+ 10 \times$ eq (ii) gives,

$$\begin{aligned} 66 &= 36\lambda x - \cancel{60\lambda y} \\ +\ 30 &= -100\lambda x + \cancel{60\lambda y} \\ \hline 96 &= -64\lambda x \end{aligned}$$

$\Rightarrow \lambda x = -\frac{\cancel{96}^{3}}{\cancel{64}_{2}} \Rightarrow x = -\frac{3}{2\lambda}$

$10 \times$ eq (i) $+ 6 \times$ eq (ii) gives,

$$\begin{aligned} 110 &= \cancel{60\lambda x} - 100\lambda y \\ +\ 18 &= -\cancel{60\lambda x} + 36\lambda y \\ \hline 128 &= -64\lambda y \end{aligned}$$

$\Rightarrow \lambda y = -2 \Rightarrow y = -\frac{2}{\lambda}$.

$\therefore x = \frac{-3}{2\lambda}, \ y = -\frac{2}{\lambda}$, Substituting this in eq (iii),

$-3\left(-\frac{3}{2\lambda}\right)^2 + 10\left(-\frac{3}{2\lambda}\right)\left(-\frac{2}{\lambda}\right) - 3\left(-\frac{2}{\lambda}\right)^2 = 80$

$\Rightarrow -\frac{27}{4\lambda^2} + \frac{120}{4\lambda^2} - \frac{48}{4\lambda^2} = 80$.

$\Rightarrow \frac{45}{4\lambda^2} = 80 \Rightarrow \lambda^2 = \frac{\cancel{45}^{9}}{\cancel{320}_{64}} = \frac{9}{64}$

$\Rightarrow \lambda = \pm \frac{3}{8}$.

When $\lambda = \frac{3}{8}$, x & $y < 0$ but x & y are the units of products A and B bought hence cannot be negative.

$\therefore \lambda = -\frac{3}{8}$

$$x = \frac{+3}{2 * 3} + \frac{8}{3} = 4$$

$$y = \frac{-2}{-3} + 8 = \frac{16}{3}.$$

$\therefore$ 4 units of product A is used and $\frac{16}{3}$ units of B to produce 80 units of C.

Also cost of producing 80 units of C $= 11 \times 4 + 3 \times \left(\frac{16}{3}\right)$
$= \$60$.

8) Let $f(x, y) = \log(1 + xy)$.

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Taylor's formula for several variables is given by,

$$f(x+h, y+k) = f(x, y) + df(x, y) + \frac{d^2 f(x, y)}{2!} + \dots R_n$$

where $R_n = \frac{1}{(n-1)!} d^{n+1} f(x + \theta h, y + \theta k)$, $0 < \theta < 1$.

when h & k are small

at (0, 0) the order 2 Taylor formula is,

$$f(h, k) = f(0, 0) + df(0, 0) + \frac{d^2 f(0, 0)}{2!} + R_2$$

$$df(0,0) = \frac{\partial f}{\partial x}(0,0) \cdot h + \frac{\partial f}{\partial y}(0,0) \cdot k$$

and $d^2 f(0,0) = h^2 f_{xx}(0,0) + 2hk f_{xy}(0,0) + k^2 f_{yy}(0,0)$.

and $R_2 = d^3 f(\theta h, \theta k)$, $0 < \theta < 1$.

$$f_x = \frac{1}{1+xy} \cdot y = \frac{y}{1+xy} , \quad f_y = \frac{1}{1+xy} \cdot x = \frac{x}{1+xy}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{y}{1+xy}\right) = y \frac{(-1)}{(1+xy)^2} \cdot y = \frac{-y^2}{(1+xy)^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{y}{1+xy}\right) = \frac{(1+xy) \cdot 1 - y(x)}{(1+xy)^2} = \frac{1}{(1+xy)^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{x}{1+xy}\right) = x \frac{(-1)}{(1+xy)^2} \cdot x = \frac{-x^2}{(1+xy)^2}$$

$$\therefore df(0,0) = h \cdot 0 + k \cdot 0 = 0 .$$

$$d^2 f(0,0) = h^2 \cdot 0 + 2hk \cdot \frac{1}{1+0} + k^2 \cdot 0 .$$

$$\Rightarrow d^2 f(0,0) = 2hk .$$

$$\therefore f(h,k) = f(0,0) + 0 + \frac{2hk}{2!} + R_2 .$$

$$\Rightarrow \log(1+hk) = \log(1) + \frac{2hk}{2!}$$

$$\Rightarrow \log(1+hk) \approx \frac{2hk}{2!} = hk .$$

9.a) $f(x,y) = x^2 - 3xy + y^2$.

$\nabla f(x,y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$

$= (2x - 3y, -3x + 2y)$

To find the critical points $\nabla f = 0$

$\Rightarrow 2x - 3y = 0$ and $-3x + 2y = 0$.

$\Rightarrow 2x = 3y \Rightarrow x = 3y/2$

$-3x + 2y = 0 \Rightarrow -3 \cdot \frac{3y}{2} + 2y = 0$

$\Rightarrow -\frac{9y}{2} + 2y = 0$

$\Rightarrow y = 0$.

$\therefore x = 0, y = 0$.

critical points is $(0,0)$.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

$\det(H(0,0)) = 4 - 9 = -5 < 0$.

$\therefore (0,0)$ is a saddle point of $f(x,y)$

$f(0,0) = 0$.

9.b) $f(x,y) = x e^{-(x^2+y^2)/2}$

$$\nabla f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{bmatrix} 1 \cdot e^{-(x^2+y^2)/2} + x \cdot e^{-(x^2+y^2)/2} \cdot (-2x/2) \\ x \cdot e^{-(x^2+y^2)/2} \cdot (-2y/2) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-(x^2+y^2)/2} - x^2 e^{-(x^2+y^2)/2} \\ -xy\, e^{-(x^2+y^2)/2} \end{bmatrix}$$

To find the critical points we put $\nabla f = 0$

$$e^{-(x^2+y^2)/2} - x^2 e^{-(x^2+y^2)/2} = 0 \quad \ldots (i)$$

$$-xy\, e^{-(x^2+y^2)/2} = 0 \quad \ldots (ii)$$

From eq (ii) we have, $xy\, e^{-(x^2+y^2)/2} = 0$

$\Rightarrow$ either $x=0$ or $y=0$ $\quad (e^z \neq 0 \ \forall z \in \mathbb{R})$

From eq (i) we have,

$$e^{-(x^2+y^2)/2} (1-x^2) = 0$$

$\Rightarrow 1-x^2 = 0 \quad (e^z \neq 0 \ \forall z \in \mathbb{R})$

$\Rightarrow (1-x)(1+x) = 0 \Rightarrow x = \pm 1$

Since $x \neq 0$ to satisfy eq (ii) we must have $y = 0$.

$\therefore$ the critical points are $(-1,0)$ and $(1,0)$

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-(x^2+y^2)/2}\cdot(-2x) + e^{-(x^2+y^2)/2}(1-x^2)(-x) & (1-x^2)\cdot e^{-(x^2+y^2)/2}\cdot(-y) \\ -ye^{-(x^2+y^2)/2} + (-xy)\cdot e^{-(x^2+y^2)/2}\cdot(-x) & -x\cdot e^{-(x^2+y^2)/2} + (-xy)\,e^{-(x^2+y^2)/2}\cdot(-y) \end{bmatrix}$$

$$= \begin{bmatrix} -xe^{-(x^2+y^2)/2}\{2+(1-x^2)\} & -y(1-x^2)\,e^{-(x^2+y^2)/2} \\ -ye^{-(x^2+y^2)/2}(1-x^2) & -xe^{-(x^2+y^2)/2}(1-y^2) \end{bmatrix}$$

Evaluate the Hessian at the two critical points.

$$H(-1,0) = \begin{bmatrix} e^{-1/2}\cdot 2 & 0 \\ 0 & e^{-1/2} \end{bmatrix} \Rightarrow \det H(-1,0) = 2\cdot e^{-1} > 0$$
and $a = 2\cdot e^{-1/2} > 0$.

$\therefore$ $H(-1,0)$ is positive definite $\Rightarrow$ $(-1,0)$ is a point of relative minima for f.

$$f(-1,0) = -e^{-1/2}$$

$$H(1,0) = \begin{bmatrix} -e^{-1/2}\cdot 2 & 0 \\ 0 & -e^{-1/2} \end{bmatrix} \Rightarrow \det H(1,0) = 2\cdot e^{-1} > 0$$
and $a = -2e^{-1/2} < 0$

$\therefore$ $H(1,0)$ is a negative definite matrix
$\Rightarrow (1,0)$ is a point of relative maxima for f.

$$f(1,0) = e^{-1/2}$$