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ANALYSIS FINAL EXAM.

1) $f(x, y) = x^2y^3 - 4y$

equation of the tangent at (2, 1).

Let $z = x^2y^3 - 4y$

On differentiating the above eq w.r.t x we get,

$$\frac{\partial z}{\partial x} = 2xy^3 \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,1)} = 4$$

On differentiating w.r.t. y we get,

$$\frac{\partial z}{\partial y} = x^2 3y^2 - 4 \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 12 - 4 = 8.$$

Also, $f(2,1) = 4 - 4 = 0$

$\therefore$ Equation of the tangent plane at (2, 1) is,

$$(z - f(2,1)) = \left.\frac{\partial z}{\partial x}\right|_{(2,1)} \cdot (x-2) + \left.\frac{\partial z}{\partial y}\right|_{(2,1)} (y-1)$$

$$\Rightarrow z - 0 = 4(x-2) + 8(y-1)$$

$$\Rightarrow z = 4x - 8 + 8y - 8$$

$$\Rightarrow z = 4x + 8y - 16.$$

2) $f(x,y) = \frac{x^3}{3} - x - \left(\frac{y^3}{3} - y\right)$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$= \left(\frac{3x^2}{3} - 1, -\frac{3y^2}{3} + 1\right)$$

We shall first find the critical points of f by solving,

$$\nabla f(x,y) = 0.$$

$$\Rightarrow \left(\frac{3x^2}{3} - 1, -\frac{3y^2}{3} + 1\right) = 0.$$

$\Rightarrow x^2 - 1 = 0$ and $-y^2 + 1 = 0$.

$\Rightarrow (x-1)(x+1) = 0$ and $(-y+1)(+y+1) = 0$.

$\Rightarrow x = \pm 1$ and $y = \pm 1$.

$\therefore$ The critical points are,

$(-1,-1), (-1,1), (1,-1), (1,1)$.

To examine the nature of the critical points we need to look at the Hessian matrix at each of these points.

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_{xx} = \frac{\partial}{\partial x}(x^2 - 1) = 2x$$

$$f_{xy} = \frac{\partial}{\partial y}(x^2 - 1) = 0$$

$f_{yx} = \frac{\partial}{\partial x}(-y^2+1) = 0$

$f_{yy} = \frac{\partial}{\partial y}(-y^2+1) = -2y$

$\therefore H(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$

$H(-1,-1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det(H(-1,-1)) = -4 < 0$

Since the determinant is negative then $d^2f(-1,-1)$ is indefinite $\Rightarrow (-1,-1)$ is a saddle point.

$f(-1,-1) = -\frac{1}{3} - (-1) - \left(\frac{-1}{3} - (-1)\right)$

$= \cancel{-\frac{1}{3}} + \cancel{1} + \cancel{\frac{1}{3}} - \cancel{1} = 0.$

$H(-1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \det(H(-1,1)) = 4 > 0.$

Also, $a = -2 < 0$.

Since $ac - b^2 > 0$ and $a < 0$, $d^2f(-1,1)$ is negative definite $\Rightarrow (-1,1)$ is a point of local ~~minimum~~ maximum.

$f(-1,1) = -\frac{1}{3} - (-1) - \left(\frac{1}{3} - 1\right)$

$= -\frac{1}{3} + 1 - \frac{1}{3} + 1 = 2 - \frac{2}{3} = \frac{4}{3}$

$H(1,-1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det(H(1,-1)) = 4 > 0$

Also, $a = 2 > 0$.

Since, $ac - b^2 = 4 > 0$ and $a = 2 > 0$, $d^2 f(1,-1)$ is positive definite $\Rightarrow (1,-1)$ is a point of local minimum.

$$f(1,-1) = \frac{1}{3} - 1 - \left(-\frac{1}{3} - (-1)\right)$$

$$= \frac{1}{3} - 1 + \frac{1}{3} - 1 = \frac{2}{3} - 2 = -\frac{4}{3}$$

$$H(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \det H(1,1) = -4 < 0.$$

Since $ac - b^2 = -4 < 0 \Rightarrow d^2 f(1,1)$ is indefinite

$\Rightarrow (1,1)$ is a saddle point.

$$f(1,1) = \frac{1}{3} - 1 - \left(\frac{1}{3} - 1\right)$$

$$= \cancel{\frac{1}{3}} - \cancel{1} - \cancel{\frac{1}{3}} + \cancel{1} = 0.$$

$\therefore$ we have, $(-1,-1)$ and $(1,1)$ as the two saddle points s.t. $f(-1,-1) = f(1,1) = 0$.

$(-1,1)$ is the point of local maximum with the local maximum value $= f(-1,1) = \frac{4}{3}$.

$(1,-1)$ is the point of local minimum with the local minimum value $= f(1,-1) = -\frac{4}{3}$.

3) Minimize $f(x,y,z) = x^2+2y^2+z^2$

s.t. $x+2y+3z=1$

$x-2y+z=5$.

let $g_1(x,y,z) = x+2y+3z-1$

and $g_2(x,y,z) = x-2y+z-5$.

let $L(x,y,z) = f(x,y,z) + \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z)$

$= x^2+2y^2+z^2 + \lambda_1(x+2y+3z-1) + \lambda_2(x-2y+z-5)$

By the Lagrangian condition for optimality we have,

$\frac{\partial L}{\partial x}=0$, $\frac{\partial L}{\partial y}=0$, $\frac{\partial L}{\partial z}=0$, $\frac{\partial L}{\partial \lambda_1}=0$, $\frac{\partial L}{\partial \lambda_2}=0$.

$\frac{\partial L}{\partial x} = 2x+\lambda_1+\lambda_2 = 0$.

$\Rightarrow 2x = -\lambda_1-\lambda_2 \Rightarrow x = \frac{-\lambda_1-\lambda_2}{2}$ ...(i)

$\frac{\partial L}{\partial y} = 4y+2\lambda_1-2\lambda_2 = 0$

$\Rightarrow 2y = \lambda_2-\lambda_1 \Rightarrow y = \frac{\lambda_2-\lambda_1}{2}$ ...(ii)

$\frac{\partial L}{\partial z} = 2z+3\lambda_1+\lambda_2 = 0$

$\Rightarrow 2z = -3\lambda_1-\lambda_2 \Rightarrow z = \frac{-3\lambda_1-\lambda_2}{2}$ ...(iii)

$\frac{\partial L}{\partial \lambda_1} = x+2y+3z-1 = 0$ ...(iv)

$\frac{\partial L}{\partial \lambda_2} = x-2y+z-5 = 0$ ...(v)

Substituting the values of x, y, z from eq (i), (ii) & (iii) in eq (iv) we get,

$$-\frac{\lambda_1-\lambda_2}{2} + 2\left(\frac{\lambda_2-\lambda_1}{2}\right) + 3\left(\frac{-3\lambda_1-\lambda_2}{2}\right) = 1$$

$$\Rightarrow \frac{-\lambda_1-\lambda_2+2\lambda_2-2\lambda_1-9\lambda_1-3\lambda_2}{2} = 1$$

$$\Rightarrow -12\lambda_1 - 2\lambda_2 = 2 \quad \ldots (vi)$$

Similarly eq (v) becomes,

$$-\frac{\lambda_1-\lambda_2}{2} - 2\left(\frac{\lambda_2-\lambda_1}{2}\right) + \left(\frac{-3\lambda_1-\lambda_2}{2}\right) = 5$$

$$\Rightarrow \frac{-\lambda_1-\lambda_2-2\lambda_2+2\lambda_1-3\lambda_1-\lambda_2}{2} = 5$$

$$\Rightarrow -2\lambda_1 - 4\lambda_2 = 10 \quad \ldots (vii)$$

$2 \times$ eq (vi) $-$ eq (vii) gives,

$$\begin{array}{rrcl} & -24\lambda_1 - \cancel{4\lambda_2} & = & 4 \\ (-) & \overset{+}{-}2\lambda_1 \overset{(+)}{-} \cancel{4\lambda_2} & \overset{(-)}{=} & -10 \\ \hline & -22\lambda_1 & = & -6 \end{array}$$

$$\Rightarrow \lambda_1 = \frac{\cancel{6}^{3}}{\cancel{22}_{11}} = \frac{3}{11}.$$

$$\therefore -12\left(\frac{3}{11}\right) - 2\lambda_2 = 2$$

$$\Rightarrow -\frac{36}{11} - 2 = 2\lambda_2$$

$$\Rightarrow \frac{-\cancel{58}^{29}}{\cancel{22}\,11} = \lambda_2$$

$$\Rightarrow \lambda_2 = -\frac{29}{11}.$$

$$\therefore x = \frac{1}{2}\left(-\left(\frac{3}{11}\right) - \left(-\frac{29}{11}\right)\right)$$

$$= \frac{1}{2} \times \frac{26}{11} = \frac{13}{11}.$$

$$y = \frac{1}{2}\left(-\frac{29}{11} - \frac{3}{11}\right) = \frac{1}{2}\left(-\frac{32}{11}\right).$$

$$= -\frac{16}{11}$$

$$z = \frac{1}{2}\left(-3\cdot\frac{3}{11} - \left(-\frac{29}{11}\right)\right)$$

$$= \frac{1}{2}\left(-\frac{9}{11} + \frac{29}{11}\right) = \frac{10}{11}.$$

$\therefore$ the critical point satisfying the constraints is

$$\left(\frac{13}{11}, -\frac{16}{11}, \frac{10}{11}\right).$$

$$f\left(\frac{13}{11}, -\frac{16}{11}, \frac{10}{11}\right) = \left(\frac{13}{11}\right)^2 + 2\left(-\frac{16}{11}\right)^2 + \left(\frac{10}{11}\right)^2$$

$$= \frac{169}{121} + 2\left(\frac{256}{121}\right) + \frac{100}{121}$$

$$= \frac{\cancel{781}^{71}}{\cancel{121}\,11} = \frac{71}{11} = 6\frac{5}{11}$$

$\therefore$ The minimum value of the function occurs at $\left(\frac{13}{11}, -\frac{16}{11}, \frac{10}{11}\right)$ and is $6\frac{5}{11}$ w.r.t. the constraints.

4.a) P.T.P $\sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$ is a convergent series.

Let $f(x) = \frac{(\log x)^3}{x^2}$, $f'(x) = \frac{x^{\cancel{2}} \cdot 3(\log x)^2 \cdot \frac{1}{\cancel{x}} - (\log x)^3 \cdot 2x}{x^4}$.

$$\Rightarrow f'(x) = \frac{3x(\log x)^2 - 2x(\log x)^3}{x^4} = \frac{3(\log x)^2}{x^3} - \frac{2(\log x)^3}{x^3}$$

If $f'(x) > 0$

$$\Rightarrow \frac{3(\log x)^2}{x^3} - \frac{2(\log x)^3}{x^3} > 0.$$

$$\Rightarrow 3(\log x)^2 > 2(\log x)^3 \quad (x \geq 1)$$

$$\Rightarrow 3(\log x)^2 - 2(\log x)^3 > 0$$

$$\Rightarrow (\log x)^2 (3 - 2\log x) > 0.$$

$$\Rightarrow 3 > 2\log x \Rightarrow \log x < 2/3 \quad \left(\because (\log x)^2 \geq 0 \ \forall x \geq 1\right).$$

~~But for larger values of x~~,

$$\Rightarrow e^{\log x} < e^{2/3} \Rightarrow x < e^{2/3}$$

But for large values of x this is not true.

$\therefore$ $f'(x) \le 0 \Rightarrow f(x)$ is decreasing.

Also, $f(x) \geq 0 \ \forall \ x \geq 1$.

We shall apply the integral test to $f(x)$.

$\therefore$ $\displaystyle\int_1^{\infty} f(x)\, dx$.

$$= \int_1^{\infty} \frac{(\log x)^3}{x^2}\, dx.$$

Let $\log x = z \Rightarrow \frac{1}{x} dx = dz$.

$x = e^z$.

x	1	∞
z	0	∞

$$\therefore \int_0^{\infty} \frac{z^3}{e^z} \cdot dz.$$

$$= \int_0^{\infty} z^3 \cdot e^{-z}\, dz.$$

$$\int z^3 e^{-z}\, dz = z^3 \cdot \frac{e^{-z}}{-1} - \int 3z^2 \cdot (-e^{-z})\, dz$$

$$= -e^{-z} \cdot z^3 + 3 \int z^2 e^{-z}\, dz.$$

$$= -e^{-z} \cdot z^3 + 3 \left[z^2 \cdot \frac{e^{-z}}{-1} - \int 2z \cdot \frac{e^{-z}}{-1}\, dz \right]$$

$= -e^{-z} \cdot z^3 - 3z^2 \cdot e^{-z} + 6 \int z \cdot e^{-z} \, dz$

$= -e^{-z} \cdot z^3 - 3z^2 e^{-z} + 6\left[z \cdot \frac{e^{-z}}{-1} - \int 1 \cdot \frac{e^{-z}}{-1} \, dz \right]$

$= -z^3 e^{-z} - 3z^2 e^{-z} - 6z e^{-z} + 6 \int e^{-z} \, dz$.

$= -z^3 e^{-z} - 3z^2 e^{-z} - 6z e^{-z} - 6e^{-z}$.

$\therefore \int_0^\infty z^3 e^{-z} \, dz = \left[-e^{-z} \left(z^3 + 3z^2 + 6z + 6 \right) \right]_0^\infty$

$= 0 - \left(-e^0 \left(0 + 0 + 0 + 6 \right) \right)$

$= 6$

$\therefore \sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$ converges iff $\int_1^\infty f(x) \, dx$ converges.

$\Rightarrow$ the series is convergent since the integral exists.

b)

$$a_n = (-1)^n \frac{n^2}{n^2+1}.$$

$$\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{1}{1+1/n^2}.$$

$$= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \left(\frac{1}{1+1/n^2}\right) \quad \left[\because \lim a_n b_n = \lim a_n \cdot \lim b_n\right]$$

$$= \lim_{n\to\infty} (-1)^n \cdot 1. = \lim_{n\to\infty} (-1)^n$$

Now, $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}.$

$\therefore \lim_{n\to\infty} a_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}.$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$. $\Rightarrow$ The series is not convergent.

$$|a_n| = \frac{n^2}{n^2+1}$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} \quad \left(\text{On dividing by } n^2\right)$$

$$= 1.$$

$\Rightarrow \lim_{n\to\infty} |a_n| \neq 0 \Rightarrow$ The series is not absolutely convergent.

c) $a_n = \dfrac{\sin \frac{n\pi}{2}}{2^n}$.

$$\sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$$

$\therefore \sin \frac{n\pi}{2}$ takes values in $\{-1, 0, 1\}$

Let us apply the root test to find the radius of convergence.

$$\lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup \left| \frac{\sin \frac{n\pi}{2}}{2^n} \right|^{1/n}$$

$$= \lim_{n\to\infty} \sup \left| \frac{(\sin n\pi/2)^{1/n}}{2} \right|.$$

The maximum value of $\sin n\pi/2$ is 1

$$\therefore \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{2}. = \frac{1}{2}.$$

radius of convergence $= \dfrac{1}{\lim\limits_{n\to\infty} \sup |a_n|^{1/n}} = \dfrac{1}{1/2} = 2$

$\therefore$ radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{\sin \frac{n\pi}{2}}{2^n} x^n$ is 2.

5) $f(x,y,z) = xy^2 e^{z^2}$ at $a = (1,1,1)$.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

Taylor's formula for several variables is given by,

$$f(x+h, y+k, z+l) = f(x,y,z) + df(x,y,z) + \frac{d^2 f(x,y,z)}{2!} + \cdots R_n.$$

where $R_n = \frac{1}{(n-1)!} d^{n+1} f(x+\theta h, y+\theta k, z+\theta l)$

$0 < \theta < 1$

and h, k, l are small values.

$\therefore$ At $a = (1,1,1)$ Taylor's series of order 2 is,

$$f(1+h, 1+k, 1+l) = f(1,1,1) + df(1,1,1) + \frac{d^2 f(1,1,1)}{2!} + R_2.$$

$$R_2 = \frac{1}{(2-1)!} \cdot d^3 f(\cancel{1}+\theta h, 1+\theta k, 1+\theta l).$$

$$df(1,1,1) = \frac{\partial f}{\partial x}(1,1,1) \cdot h + \frac{\partial f}{\partial y}(1,1,1) \cdot k + \frac{\partial f}{\partial z}(1,1,1) \cdot l$$

$$d^2 f(1,1,1) = (d\overset{2}{f})^2 \Big|_{(1,1,1)}.$$

$$df = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k + \frac{\partial f}{\partial z} l.$$

$$(d)^2 f = \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k + \frac{\partial}{\partial z} l\right)^2 f$$

$$= h^2 f_{xx} + k^2 f_{yy} + l^2 f_{zz} + 2(hk\, f_{xy} + kl\, f_{yz} + lh\, f_{xz})$$

$$f_x = y^2 e^{z^2}, \quad f_y = x 2y e^{z^2}, \quad f_z = xy^2 e^{z^2} \cdot 2z.$$

$$f_{xx} = 0, \quad f_{xy} = 2y e^{z^2}$$

$$f_{yy} = 2x e^{z^2}$$

$$f_{yz} = 2xy e^{z^2} \cdot 2z = 4xyz e^{z^2}$$

$$f_{zz} = 2xy^2 (e^{z^2} + z \cdot e^{z^2} \cdot 2z).$$

$$f_{xz} = y^2 \cdot e^{z^2} \cdot 2z.$$

$$\therefore df(1,1,1) = \left(y^2 e^{z^2} \cdot h + 2xy e^{z^2} \cdot k + 2xy^2 z e^{z^2} \cdot l\right)\Big|_{(1,1,1)}$$

$$= e \cdot h + 2e \cdot k + 2e\, l.$$

$$\Rightarrow df(1,1,1) = eh + 2ek + 2el$$

$$d^2 f(1,1,1) = \left[h^2 \cdot 0 + k^2 \cdot 2x e^{z^2} + l^2 \cdot 2xy\left(e^{z^2} + 2z^2 e^{z^2}\right) + 2\left(hk\, 2y e^{z^2} + kl\, 4xyz e^{z^2} + lh\, 2y^2 z e^{z^2}\right)\right]_{(1,1,1)}$$

$$= 2k^2 e + l^2\, 2\,(e + 2e) + 2(hk\, 2e + kl\, 4e + lh\, 2e)$$

$$= 2k^2 e + 6el^2 + 4ehk + 8ekl + 4elh$$

~~$\therefore f(1+h, 1+k, 1+l)$~~

$$f(1,1,1) = 1 \cdot e^1 = e$$

$$\therefore f(1+h, 1+k, 1+l) = e + (eh + 2ek + 2el) + \frac{(2k^2 e + 6el^2 + 4ehk + 8ekl + 4elh)}{2!}$$

$$\Rightarrow f(1+h, 1+k, 1+l) = e + eh + 2ek + 2el + k^2 e + 3el^2 + 2ehk + 4ekl + 2elh$$

$$= e\left[1 + h + 2k + k^2 + 4kl + 3l^2 + 2l + 2hk + 2lh\right]$$

$$= e\left[k^2 + 3l^2 + 2hk + 4kl + 2lh + h + 2k + 2l + 1\right]$$

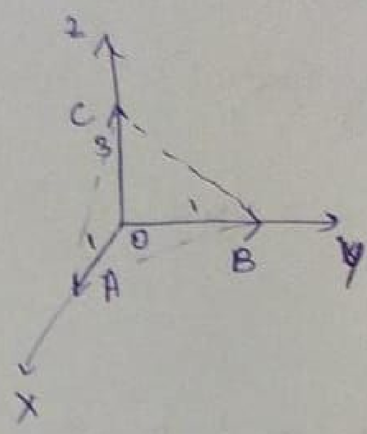
For general for (x,y,z) near $(1,1,1)$, $h = x-1$, $k = y-1$, $l = z-1$

$$\therefore f(x,y,z) = e\left[(y-1)^2 + 3(x-1)^2 + 2(x-1)(y-1) + 2(z-1)(x-1) + (x-1) + 2(y-1) + 2(z-1) + 1\right]$$

6) speed of A = 1cm/s

speed of B = 1cm/s

speed of C = 3cm/s



Since A is travelling along the positive x axis,

$$\frac{dx}{dt} = 1$$

B is travelling along the positive y axis, $\frac{dy}{dt} = 1$

C is travelling along the positive z axis, $\frac{dz}{dt} = 3$

area $= \frac{1}{2}\sqrt{x^2y^2 + y^2z^2 + z^2x^2} = u$ (say)

$\frac{du}{dt}$ is the rate of change of the area of the triangle

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{2}(x^2y^2 + y^2z^2 + z^2x^2)^{-1/2}\cdot(2xy^2 + 2xz^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{1}{2}(x^2y^2 + y^2z^2 + z^2x^2)^{-1/2}\cdot(2yz^2 + 2yx^2)$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}\cdot\frac{1}{2}(x^2y^2 + y^2z^2 + z^2x^2)^{-1/2}(2zy^2 + 2zx^2)$$

$$\therefore \frac{du}{dt} = \frac{1}{8u}\Big[(2xy^2 + 2xz^2)\frac{dx}{dt} + (2yz^2 + 2yx^2)\frac{dy}{dt} + (2zy^2 + 2zx^2)\frac{dz}{dt}\Big]$$

The coordinates of the triangle are,

A (2, 0, 0), B (0, 1, 0), C (0, 0, 1).

$\therefore (x, y, z) = (2, 1, 1)$.

$$\left.\frac{\partial u}{\partial x}\right|_{(2,1,1)} = \frac{1}{4}(4+1+4)^{-1/2}(4+4)$$

$$= \pm\frac{1}{\cancel{4}} \times \frac{1}{3} \times \cancel{8}^{2} = \pm \frac{2}{3}$$

$$\left.\frac{\partial u}{\partial y}\right|_{(2,1,1)} = \frac{1}{4}(4+1+4)^{-1/2}(2+8)$$

$$= \frac{1}{\cancel{4}_2}\left(\pm\frac{1}{3}\right)\cdot \cancel{10}^{5} = \pm\frac{5}{6}.$$

$$\left.\frac{\partial u}{\partial z}\right|_{(2,1,1)} = \frac{1}{4}(4+1+4)^{-1/2}(2+8)$$

$$= \frac{1}{4}\left(\pm\frac{1}{3}\right)\cdot 10 = \pm\frac{5}{6}.$$

$$\therefore \left.\frac{du}{dt}\right|_{(2,1,1)} = \pm\left(\frac{2}{3}\cdot 1 + \frac{5}{6}\cdot 1 + \frac{5}{6}\cdot 3\right)$$

$$= \pm\left(\frac{4}{6} + \frac{5}{6} + \frac{15}{6}\right) = \pm 4 \text{ cm}^2/\text{sec}$$

Note that since A, B, C are moving away from the origin in the positive direction, the triangle A, B, C is expanding $\Rightarrow$ area of $\triangle ABC$ is increasing

$$\Rightarrow \frac{du}{dt} > 0 \Rightarrow \left.\frac{du}{dt}\right|_{(2,1,1)} = 4 \text{ cm}^2/\text{sec}.$$

$\therefore$ The rate of change of the area of ΔABC is $4\ cm^2/sec$ when $A(2,0,0)$, $B(0,1,0)$ and $C(0,0,1)$ are the coordinates of the triangle.

7) $F(x,y,z,u,v) : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$

$$F(x,y,z,u,v) = \begin{pmatrix} xy^2 + xzu + yv - 3 \\ uyz + 2xu - u^2v^2 - 2 \end{pmatrix}$$

$$F(1,1,1,1,1) = (0,0)$$

Note that, $5-2=3$.

$\Rightarrow$ other two variables might ~~...~~ depend on the three variables.

Let $f_1(x,y,z,u,v) = xy^2 + xzu + yv - 3$

$f_2(x,y,z,u,v) = uyz + 2xu - u^2v^2 - 2$

$$\frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} xz & y \\ yz + 2x - 2uv^2 & -2u^2v \end{bmatrix}$$

$$\frac{\partial(F_1, F_2)}{\partial(u,v)}\Bigg|_{(1,1,1,1,1)} = \begin{bmatrix} 1 & 1 \\ 1+\cancel{x}-\cancel{x} & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det\left(\frac{\partial(f_1, f_2)}{\partial(u,v)}\Bigg|_{(1,1,1,1)} \right) = -2-1 = -3 < 0$$

$\therefore$ Since the determinant of the Jacobian is non zero and $F(1,1,1,1,1) = 0$ and both the components of F, f_1 and f_2, are polynomials hence continuously differentiable i.e. C^1 we can apply the Implicit function Theorem.

By the Implicit function theorem we know,

$\exists!\, g_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous

s.t. $g_1(x, y, z) = u$ and $g_2(x, y, z) = v$

and $g_1(1,1,1) = 1$ and $g_2(1,1,1) = 1$.

i.e. $\exists!\, g : B \longrightarrow \mathbb{R}^2$ where B is a neighbourhood of $(1,1,1)$ in $\mathbb{R}^3$ s.t.

$g(x, y, z) = (u, v)$ and $g(1,1,1) = (1,1)$.

$\therefore$ Yes, we can solve for u, v as functions of x, y, z near $(1,1,1)$.