27 November 2021 Time allowed: 3 hours

M.Sc. Data Science

Analysis - Final Exam

Note: Answers must be clear and complete to receive grades. Total: 55 points

1. (5 points) Let $f(x,y) = x^2y^3 - 4y$. Find the equation of the tangent plane at $(2,1)$.

2. (10 points) Find the local maximum, local minimum and saddle points (if any) of $f(x,y) = \dfrac{x^3}{3} - x - \left(\dfrac{y^3}{3} - y\right)$.

3. (10 points) Minimize the function $f(x,y,z) = x^2 + 2y^2 + z^2$ w.r.t. the constraints

$$\begin{aligned} x + 2y + 3z &= 1, \\ x - 2y + z &= 5. \end{aligned}$$

4. (5 points each)

   (a) Show that the series $\dfrac{(\log n)^3}{n^2}$ is convergent.

   (b) Is the series $(-1)^n \dfrac{n^2}{n^2+1}$ convergent? Is it absolutely convergent?

   (c) Find the radius of convergence of the series $\dfrac{\sin\left(\frac{n\pi}{2}\right)}{2^n} x^n$.

5. (5 points) Compute the second-order Taylor polynomial of $f(x,y,z) = xy^2e^{z^2}$ at the point $a = (1,1,1)$.

6. (5 points) Three ants A,B and C crawl along the positive x, y and z axes respectively. A and B are crawling at a constant speed of 1 cm/s, C is crawling at a constant speed of 3 cm/s and they are all traveling away from the origin. Find the rate of change of the area of triangle ABC when A is 2 cm away from the origin while B and C are 1 cm away from the origin. (The area of the triangle $A(x,0,0)$, $B(0,y,0)$, $C(0,0,z)$ is given by $\dfrac{1}{2}\sqrt{x^2y^2 + y^2z^2 + z^2x^2}$).

7. (5 points) Consider the map $F(x,y,z,u,v) : R^5 \to R^2$ given by:

$$F(x,y,z,u,v) = \begin{pmatrix} xy^2 + xzu + yv - 3 \\ uyz + 2xu - u^2v^2 - 2 \end{pmatrix}.$$

   Notice that $F(1,1,1,1,1) = (0,0)$. Can we solve for u, v as functions of x,y,z near $(1,1,1,1,1)$? Give reasons for your answer.