9 November 2021

Time allowed: 3 hours
Total: 40 points

M.Sc. Data Science
MMA - Midterm

Note: Answers must be clear and complete to receive grades. There is zero tolerance for copying.

1. You are given the function $f(x) = x^x$ for $x > 0$. Answer the following questions with proper analytical reasoning using the function definition, its derivative and second derivative. Supplement your answer with graphical reasoning:
   (a) (2 points) What are the first and second derivatives of f?
   (b) (2 points) What are the critical point(s) of f?
   (c) (2 points) Analyse the behaviour of f at these critical points giving the correct reasoning.
   (d) (2 points) In what intervals is the function (strictly) increasing? (Strictly) decreasing? What about the concavity of f in these regions?
2. (3 points) Let $f(x) = x\sin(1/x)$, for $x \neq 0$. Can you define $f(0)$ appropriately so that f becomes a continuous function? Your choice should be supported by proper reasoning.
3. (4 points) The cost of producing x units of a product is given by $f(x) = 10x^2 + 200x + 6000$. If the company increases the price, then fewer units are sold. The price per unit as a function of units sold is given by $p(x) = 1000 - 10x$.

   For what value of x will the profit be maximum? What is the maximum possible profit?
4. (3 points) Find $\displaystyle\int_0^\pi f(x)\,dx$, where the function f is defined by
$$f(x) = \begin{cases} \sin x, & \text{if } 0 \le x < \pi/2 \\ \cos x, & \text{if } \pi/2 \le x < \pi. \end{cases}$$
5. (4 points) Find the area under the curve $y = x\sin(x^2)$ from $x = 0$ to $x = \pi$.
6. (4 points) It is estimated that x years from now the value of an acre of farmland will be increasing at a rate of $\dfrac{0.4x^3}{\sqrt{0.2x^4 + 8000}}$ lakh rupees per acre.
   If the land is currently worth Rs.4.5 lakhs per acre, how much will it be worth in 10 years?
7. Recall that the probability density function associated with the probability $p(x)$ that a random variable X is between a and b is given by $\displaystyle\int_a^b p(x)\,dx$. Also recall that the expected value for a discrete random variable X that can take n values $\{x_1, \dots, x_n\}$ with probabilities $p(x_i)$ is given by $\displaystyle\sum_1^n x_i p(x_i)$.
   (a) (4 points) Set up Riemann sums to prove that if X is a random variable that takes infinitely many values between $(-\infty, \infty)$ with associated probability distributed function $p(x)$, then the expected value of X is given by $\displaystyle\int_{-\infty}^{\infty} xp(x)\,dx$ (whenever the integral converges).
   (b) (4 points) Use the above formula to calculate the expected value of the standard normal distribution $p(x) = \dfrac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$.

8. (a) (3 points) Express the function $f(x) = \dfrac{1}{1+x}$, $(|x| < 1)$ as a sum $p_{n-1}(x) + R_n(x)$, where $p_{n-1}(x)$ is the Taylor polynomial (around zero) of degree $(n-1)$ and $R_n(x)$ is the remainder term.

   (b) (3 points) Analyse the remainder term: either try to obtain a bound of the form $|R_n(x)| \leq M_n \frac{|x|^n}{n!}$ or give an expression for R_n and explain what happens as $n \to \infty$.