Homework 3

1) Chance of winning the lottery = 0.3

$X \rightarrow$ number of times the lottery will be won

(a) Note that,

Range $(X) = \{0, 1, 2, 3, 4, 5\}$.

$\therefore X$ is a discrete random variable.

$P(X=k)$ denotes the chance of winning k times in the next five days. If a person wins k times then he must lose 5-k times with the chance of winning each time being 0.3 and the chance of losing each time being 0.7. If a person wins k times then there are ^5C_k many possibilities.

$\therefore P(X=k) = {^5C_k} (0.3)^k (0.7)^{5-k}$.

$\Rightarrow X$ is a random variable following Binomial Distribution and the parameters are $n = 5$ & $p = 0.3$.

$X \sim B(5, 0.3)$

(b) $E(X) = np \quad (X \sim B(n, p))$

$= 5 \times 0.3 = 1.5$

(c) The most likely event is the event with the highest value of the probability mass function.

$P(X=0) = {^5C_0} (0.3)^0 (0.7)^5 \approx 0.17$

$P(X=1) = {^5C_1} (0.3)(0.7)^4 \approx 0.36$

$P(X=2) = {}^5C_2 (0.3)^2 (0.7)^3 \approx 0.31$

$P(X=3) = {}^5C_3 (0.3)^3 (0.7)^2 \approx 0.13$

$P(X=4) = {}^5C_4 (0.3)^4 (0.7) \approx 0.03$

$P(X=5) = {}^5C_5 (0.3)^5 (0.7)^0 \approx 0.002$

$P(X=1) = 0.36 \geq P(X=k) \ \forall \ k \in \{0, 1, \ldots 5\}$

$\Rightarrow$ The most likely event is the winning of the lottery exactly once.

(d) $P(X=1 \cup X=2) = P(X=1) + P(X=2)$

$= {}^5C_1 \, 0.3(0.7)^4 + {}^5C_2 (0.3)^2 (0.7)^3$

$\approx 0.36 + 0.31 = 0.67$

$\Rightarrow$ There's a 67% chance that the lottery will be won in either one or two of the next five days.

2) Probability of knowing the answer $= 0.88$

$Y \Rightarrow$ number of questions asked until the contestant does not know the correct answer.

(a) Range $(Y) = \{1, 2, \ldots\}$

Y is a discrete random variable.

$P(Y=k)$ denotes the chance that the contestant answers the question correctly $k-1$ times & does not know the answer the k^{th} time. The chance of answering the question correctly each time is 0.88.

$\therefore P(Y=k) = (0.88)^{k-1} \, (1-0.88)$

The probability mass fn of X is similar to that of a geometric distribution. $(1-p)^{k-1} \cdot p = P(Y=k)$.

where $p = 1-0.88$.

$\therefore$ Y is a random variable following Geometric Distribution with parameter $p = 0.12 \Rightarrow Y \sim \text{Geometric}(p)$

(b) $E(Y) = \frac{1}{p}$ (Y follows Geometric distribution)

$= \frac{1}{0.12} \approx 8.33$

(c) The mode shall be at the point having the highest value for probability mass function.

$P(Y=k) = (0.88)^{k-1} \cdot 0.12$.

$P(Y=k+1) = (0.88)^{k} \; 0.12$

$P(Y=k+1) < P(Y=k) \quad \forall \quad k \in \text{Range}(Y)$.

$\therefore$ The mode must occur at the first element of Range(Y).

$P(Y=1) = (0.88)^0 \cdot 0.12$

$= 0.12$.

The most likely number of questions that will be asked until the contestant doesn't know the correct answer is 1.

(d) The contestant answers all questions each with probability 0.88.

$\therefore$ The chance of answering 12 questions $= (0.88)^{12}$
≈ 0.22

3) Let X be the random variable denoting the the amount won by the player.

Range $(X) = \{2, 4, 6, 8, \ldots\}$

If $X = 2^n$ then that means that when the player flipped the coin n times, the player got a tail $(n-1)$ times and at the n^{th} flip the player got a head.

Since it is a fair coin, the probability of getting a head is equal to the probability of getting a tail $= \frac{1}{2}$

$$\therefore P(X = 2^n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n$$

The game shall become a fair game if the player has to give some amount, say c, on losing. The c should be such that the average chance of winning becomes 0 i.e. $E(X - c) = 0$.

$$\Rightarrow E(X) = c$$

$$E(X) = \sum_{k \in \text{Range}(X)} k \, P(X = k)$$

$$= \sum_{k = 2^i, i=1}^{\infty} k \, P(X = 2^i)$$

$$= \sum_{i=1}^{\infty} 2^i \cdot P(X = 2^i) = \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} = \sum 1 = \infty$$

$\Rightarrow c = \infty \Rightarrow$ paying any amount can't make this a fair game.

4) $E(X) = 0.P(X=0) + 1.P(X=1) + 2.P(X=2) + 3.P(X=3)$

$= 0 + 1\times 0.5 + 2\times 0.2 + 3\times 0.1$

$= 0.5 + 0.4 + 0.3 = 1.2$.

$$E(X^2) = \sum_{k \in \text{range}(X^2)} k\, P(X^2 = k)$$

X can take values in $\{0,1,2,3\}$ so X^2 can take values in $\{0,1,4,9\}$.

Also, $P(X^2 = k) = P(X = \sqrt{k})$.

$\therefore E(X^2) = 0.P(X=0) + 1.P(X=1) + 2^2.P(X=2) + 3^2 P(X=3)$

$= 0 + 1\times 0.5 + 4\times 0.2 + 9\times 0.1$

$= 0.5 + 0.8 + 0.9 = 2.2$

$Var(X) = E(X^2) - (E(X))^2$

$= 2.2 - (1.2)^2 = 0.76$

$SD(X) = \sqrt{Var(X)} = \sqrt{0.76} \approx 0.872$

$P(X > SD(X) + E(X)) = P(X > 0.872 + 1.2)$

$= P(X > 2.072) = P(X=3) = 0.1$

$\Rightarrow$ The probability that a random variable will produce a result more than one standard deviation from its expected value is 0.1.

5) (a) Let X denote the number of heads in 100 flips of a fair coin.

$\text{Range}(X) = \{0, 1, 2, \ldots 100\}$

$P(X=k)$ is the probability of obtaining k many heads out of the 100 flips. X is a Binomial Random Variable.

$\Rightarrow X \sim \text{Bin}(100, 0.5)$

$\text{Var}(X) = np(1-p)$

$= 100 \times 0.5 \times 0.5 = 25$

$\text{SD}.(X) = \sqrt{\text{Var}(X)} = 5$

(b) If $n = 400$, let $\tilde{X}$ be the number of heads in 400 flips.

$\tilde{X} \sim \text{Binomial}(400, 0.5)$

$\text{Var}(\tilde{X}) = 400 \times 0.5 \times 0.5$

$= 100$

$\text{SD}(\tilde{X}) = \sqrt{\text{Var}(\tilde{X})} = 10 = 2 \cdot \text{SD}(X)$.

$\Rightarrow$ Standard deviation has been doubled.

6) $X \to$ number of rolls before we see a 3 & the roll in which we see 3.

(a) Range $(X) = \{1, 2, \ldots\}$

$P(X=k)$ is the chance of getting a 3 in the k^{th} roll but some other numbers in the $(k-1)$ rolls.

$$P(X=k) = \left(1-\frac{1}{6}\right)^{k-1} \cdot \frac{1}{6}$$

$X \sim$ Geometric $(1/6)$

$$E(X) = \frac{1}{p} = \frac{1}{1/6} = 6.$$

(b) $Var(X) = \frac{1}{p^2} - \frac{1}{p} = \frac{1}{1/36} - \frac{1}{1/6} = 30.$

$SD(X) = \sqrt{30} \approx 5.48$

(c) Range of X in terms of $SD(X)$

$= (E(X) - SD(X),\ E(X) + SD(X))$

$= (0.523, 11.48)$

Since X has a maximum value of approximately 11, getting a 3 after nine rolls is not unlikely.

(d) $P(X>9) = P(X=10) + P(X=11) + \cdots$

$$= \left(1-\frac{1}{6}\right)^9 \cdot \frac{1}{6} + \left(1-\frac{1}{6}\right)^{10} \cdot \frac{1}{6} + \cdots$$

$$= \left(1-\frac{1}{6}\right)^9 \left[\frac{1}{6} + \left(1-\frac{1}{6}\right)\frac{1}{6} + \left(1-\frac{1}{6}\right)^2 \cdot \frac{1}{6} + \cdots\right]$$

$$= \left(1-\frac{1}{6}\right)^9 \sum_{k=1}^{\infty} \left(1-\frac{1}{6}\right)^{k-1} \cdot \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^9 \cdot \frac{1}{6} \cdot \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1}$$

$$= \left(\frac{5}{6}\right)^9 \cdot \frac{1}{6} \cdot \frac{1}{1-5/6} = \left(\frac{5}{6}\right)^9$$

$\Rightarrow P(X>9) \approx 0.194$

(e) $P(0.523 < X < 11.48)$

$$= P(1 \leq X \leq 11)$$

$$= \frac{1}{6} + \left(1-\frac{1}{6}\right) \cdot \frac{1}{6} + \left(1-\frac{1}{6}\right)^2 \frac{1}{6} + \cdots \left(1-\frac{1}{6}\right)^{10} \frac{1}{6}$$

$$= \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots \left(\frac{5}{6}\right)^{10}\right]$$

$$= \frac{1}{6} \cdot 1 \cdot \left[\frac{1-(5/6)^{11}}{1-5/6}\right] \approx 0.865$$