Homework 7

1)

	X = 0	X = 1	X = 2	
Y = 0	1/12	0	3/12	4/12
Y = 1	2/12	1/12	0	3/12
Y = 2	3/12	1/12	1/12	5/12
	6/12	2/12	4/12	

(a) To compute the marginal distribution of X, sum each row.

$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$

$= \frac{1}{12} + \frac{2}{12} + \frac{3}{12} = \frac{6}{12} = \frac{1}{2}$

$P(X=1) = P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$

$= 0 + \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$

$P(X=2) = P(X=2, Y=0) + P(X=2, Y=1) + P(X=2, Y=2)$

$= \frac{3}{12} + 0 + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$

Similarly the marginal distribution of Y can be calculated by adding each column.

$P(Y=0) = \frac{1}{12} + 0 + \frac{3}{12} = \frac{4}{12} = \frac{1}{3}$

$P(Y=1) = \frac{2}{12} + \frac{1}{12} + 0 = \frac{3}{12} = \frac{1}{4}$

$P(Y=2) = \frac{3}{12} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$.

(b) Range (X) = {0, 1, 2}.

$$P(X=0 \mid Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)}$$

$$= \frac{3/12}{5/12} = \frac{3}{5}$$

$$P(X=1 \mid Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)}$$

$$= \frac{1/12}{5/12} = \frac{1}{5}$$

$$P(X=2 \mid Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)}$$

$$= \frac{1/12}{5/12} = \frac{1}{5}$$

(c) Range (Y) = {0, 1, 2}

$$P(Y=0 \mid X=2) = \frac{P(Y=0, X=2)}{P(X=2)}$$

$$= \frac{3/12}{1/3} = \frac{3}{4}$$

$$P(Y=1 \mid X=2) = \frac{P(Y=1, X=2)}{P(X=2)}$$

$$= \frac{0}{1/3} = 0$$

$$P(Y=2 \mid X=2) = \frac{P(Y=2, X=2)}{P(X=2)}$$

$$= 1/12 / 1/3 = \frac{1}{4}$$

(d) From the table we have,

$$P(X=0, Y=0) = \frac{1}{12}$$

$$P(X=0)\,P(Y=0) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\frac{1}{12} \neq \frac{1}{6} \Rightarrow P(X=0, Y=0) \neq P(X=0, Y=0)$$

$\Rightarrow$ X & Y are not independent.

2) Range $(X) = \{0, 1\}$, $P(X=0) = \frac{1}{3}$, $P(X=1) = \frac{2}{3}$

Range $(Y) = \{0, 1, 2\}$, $P(Y=0) = \frac{1}{5}$, $P(Y=1) = \frac{1}{5}$, $P(Y=2) = \frac{3}{5}$

Since X & Y are independent,

$$P(X=x, Y=y) = P(X=x)\,P(Y=y) \quad \forall\, x \in \text{Range}(X), \ \forall\, y \in \text{Range}(Y)$$

$$P(X=0, Y=0) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

$$P(X=0, Y=1) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

$$P(X=0, Y=2) = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$$

$$P(X=1, Y=0) = 2/3 \cdot 1/5 = 2/15$$

$$P(X=1, Y=1) = 2/3 \cdot 1/5 = 2/15$$

$$P(X=1, Y=2) = 2/3 \cdot 3/5 = 2/5$$

	$X=0$	$X=1$
$Y=0$	$\frac{1}{15}$	$\frac{2}{15}$
$Y=1$	$\frac{1}{15}$	$\frac{2}{15}$
$Y=2$	$\frac{1}{5}$	$\frac{2}{5}$

3) $N \rightarrow$ number of earthquakes in a year

$M \rightarrow$ number of earthquakes in a year with magnitude at least 5.

$N \sim \text{Poisson}(\lambda)$, $(M \mid N=n) \sim \text{Binomial}(n, p)$

(a) $P(N=n) = e^{-\lambda} \dfrac{\lambda^n}{n!}$

$$P(M=k \mid N=n) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$$

$$P(M=k, N=n) = P(M=k \mid N=n)\, P(N=n)$$

$$= {}^{n}C_{k}\, p^{k} (1-p)^{n-k} \cdot e^{-\lambda} \frac{\lambda^{n}}{n!}$$

$$= \frac{\cancel{n!}}{k!\,(n-k)!} \cdot p^{k} (1-p)^{n-k} \cdot e^{-\lambda} \cdot \frac{\lambda^{n}}{\cancel{n!}}$$

$$= \frac{e^{-\lambda} \lambda^{n}}{k!\,(n-k)!}\, p^{k} (1-p)^{n-k}$$

(b) $P(M=m)$ =: number of earthquakes with magnitude at least 5 is m, $m > 0$.

$\Rightarrow$ At least m earthquakes have occured in a year.

$\Rightarrow n \geq m$.

$$P(M=m) = P(M=m, N=m) + P(M=m, N=m+1) + P(M=m, N=m+2) + \cdots$$

$$= \sum_{n=m}^{\infty} P(M=m, N=n)$$

$$= \sum_{n=m}^{\infty} \frac{e^{-\lambda} \lambda^{n}}{m!\,(n-m)!} p^{m} (1-p)^{n-m}$$

$$= \frac{p^{m} e^{-\lambda}}{m!} \sum_{n=m}^{\infty} \frac{\lambda^{n-m} \cdot \lambda^{m}}{(n-m)!} \cdot (1-p)^{n-m}$$

$$= \frac{e^{-\lambda}}{m!} (\lambda p)^{m} \sum_{n=m}^{\infty} \frac{\lambda^{n-m}}{(n-m)!} (1-p)^{n-m} \quad \ldots \text{(1)}$$

(c) In eq (1) substitute $n-m = k$.

n	m	∞
k	0	∞

$\therefore$ Range of k is from 0 to ∞.

$$\therefore P(M=m) = \frac{1}{m!} e^{-\lambda} (\lambda p)^{m} \sum_{n=0}^{\infty} \frac{(\lambda(1-p))^{k}}{k!}.$$

(d) $\sum_{k=0}^{\infty} \frac{(\lambda(1-p))^k}{k!}$

$= e^{\lambda(1-p)}$ $\quad \left(\because e^x = \sum \frac{x^k}{k!}\right).$

$\therefore$ In the equation from (c) we have,

$P(M=m) = \frac{1}{m!} e^{-\lambda} \cdot (\lambda p)^m \cdot e^{\lambda(1-p)}$

$= \frac{1}{m!} e^{-\lambda} \cdot e^{\lambda - \lambda p} \cdot (\lambda p)^m$

$= e^{-\lambda p} \frac{(\lambda p)^m}{m!}.$

$\Rightarrow M \sim \text{Poisson}(\lambda p)$

~~4) X ~ Bernoulli (p), Y ~ Bernoulli (q)~~

4) $X \sim \text{Geometric}(p)$, $Y \sim \text{Geometric}(p)$

$Z = X + Y$.

(a) $\text{Range}(X) = \{1, 2, 3, \ldots\}$

$\text{Range}(Y) = \{1, 2, 3, \ldots\}$

$\text{Range}(Z) = \{2, 3, 4, \ldots\}$

(b) $P(Z=n) = P(X+Y=n)$

$$= P\left(\bigcup_{i=1}^{n-1} (X=i, Y=n-i)\right)$$

$$= \sum_{i=1}^{n-1} P(X=i, Y=n-i)$$ (Mutually Exclusive)

$$= \sum_{i=1}^{n-1} P(X=i)\, P(Y=n-i)$$ (X & Y are independent)

$$= \sum_{i=1}^{n-1} p\cdot(1-p)^{i-1} \cdot p(1-p)^{n-i-1}$$

$$= \sum_{i=1}^{n-1} p^2 \cdot (1-p)^{\cancel{i}-1+n-\cancel{i}-1}$$

$$= \sum_{i=1}^{n-1} p^2 \cdot (1-p)^{n-2}$$

$$= p^2(1-p)^{n-2} \sum_{i=1}^{n-1} 1.$$

$$= (n-1)\, p^2 (1-p)^{n-2}$$

(c) $P(Z=2) = P(X+Y=2)$

$= P(X=1, Y=1)$

$= P(X=1)\, P(Y=1)$

$= p \cdot p = p^2$

$P(Z=3) = (3-1)\, p^2 (1-p)^1$

$= 2p^2(1-p).$

$$2p^2(1-p) > p^2$$

$$\Rightarrow p^2(2-2p-1) > 0$$

$p^2 > 0 \;\forall p \Rightarrow 2-2p-1 > 0$

$$\Rightarrow p < 1/2$$

$\therefore \forall p < 1/2$, $\quad P(Z=2) < P(Z=3)$

$\Rightarrow Z = 2$ is not the most likely outcome

5) X_1, X_2, X_3, X_4 iid Bernoulli(p).

$Y = X_1 + X_2$, $Z = X_3 + X_4$

$Y, Z \sim$ Binomial$(2, p)$

(a) Let $y \in$ Range(Y) & $z \in$ Range(Z).

$\therefore P(Y=y, Z=z) = P(X_1 + X_2 = y, X_3 + X_4 = z)$

$$= \sum_{\substack{x_1 \in \text{Range}(X_1) \\ x_3 \in \text{Range}(X_3)}} P(X_1 = x_1, X_2 = y - x_1, X_3 = x_3, X_4 = z - x_3)$$

$$= \sum_{\substack{x_1 \in \text{Range}(X_1) \\ x_3 \in \text{Range}(X_3)}} P(X_1 = x_1) P(X_2 = y - x_1) P(X_3 = x_3) P(X_4 = z - x_3)$$

(X_1, X_2, X_3, X_4 iid)

$$= \sum_{x_1 \in \text{Range}(X_1)} P(X_1 = x_1) P(X_2 = y - x_1) \cdot \sum_{x_3 \in \text{Range}(X_3)} P(X_3 = x_3) P(X_4 = z - x_3)$$

$= P(X_1 + X_2 = y) \cdot P(X_3 + X_4 = z)$

$= P(Y=y) P(Z=z)$

$\therefore Y$ & Z are independent.

Range $(X_i) = \{0,1\}$ $\forall i \in \{1,2,3,4\}$

Range $(Y) = \{0,1,2\}$ = Range (Z).

$$P(Y=y, Z=z) = P(Y=y)P(Z=z)$$
$$= {}^2C_y \cdot p^y (1-p)^{2-y} \cdot {}^2C_z p^z (1-p)^{2-z}$$
$$= {}^2C_y \, {}^2C_z \, p^{y+z} (1-p)^{4-(y+z)}$$

	$Y=0$	$Y=1$	$Y=2$
$Z=0$	$(1-p)^4$	$2 \cdot p(1-p)^3$	$p^2(1-p)^2$
$Z=1$	$2p(1-p)^3$	$4p^2(1-p)^2$	$2p^3(1-p)$
$Z=2$	$p^2(1-p)^2$	$2p^3(1-p)$	p^4

(b) $P(Z=1, Y=1) = 4p^2(1-p)^2$

$P(Z=1)P(Y=1) = {}^2C_1 \, p(1-p)^1 \cdot {}^2C_1 \, p(1-p)$

$= 4p^2(1-p)^2$.

Similarly for the other values it can be checked, Y & Z are independent.

(c) By the theorem we have if,

$X_{i,j}$ is an array of mutually independent R.V.

then $Y_j = f_j(X_{1,j}, X_{2,j}, \ldots X_{m_j,j})$ also gives

a collection $Y_1, Y_2, \ldots Y_n$ of mutually independent R.V.

Here, $Y_1 = Y = X_1 + X_2$.

$Y_2 = Z = X_3 + X_4$.

$\therefore Y_1$ & Y_2 are independent

$\Rightarrow Y$ & Z are independent R.V.

7) $SD[X|A] = 3, \quad SD[X|B] = 2, \quad SD[X|C] = 3$

$E[X|A] = 3, \quad E[X|B] = 1, \quad E[X|C] = -1.$

$P(A) = 0.1, \quad P(B) = 0.4, \quad P(C) = 0.5$

$Var[X|A] = 3^2 = 9,$

$Var[X|B] = 2^2 = 4$

$Var[X|C] = 3^2 = 9.$

$E[X] = E[X|A].P(A) + E[X|B]P(B) + E[X|C]P(C)$

$= 3 \times 0.1 + 1 \times 0.4 + (-1) \times 0.5$

$= 0.2.$

$$Var[X] = \sum_{i=1}^{\infty} \left(Var[X|B_i] + (E[X|B_i])^2 \right) P(B_i) - (E[X])^2$$

$= (Var[X|A] + (E[X|A])^2) P(A)$

$+ (Var[X|B] + (E[X|B])^2) P(B)$

$+ (Var[X|C] + (E[X|C])^2) P(C) - (E[X])^2$

$= (9 + 3^2) 0.1 + (4 + 1^2) 0.4 + (9 + (-1)^2) 0.5 - (0.2)^2$

$= 1.8 + 2 + 5 - 0.04 = 8.76.$

$\therefore Var[X] = 8.76.$

8) Let A be the event of choosing a standard bulb and B be the event of choosing a Super S-Lum bulb.

$P(A) = 0.9, \quad P(B) = 0.1$

Let X denote the lifetime of a bulb.

$E[X|A] = 4, \quad SD[X|A] = 1 \Rightarrow Var[X|A] = 1$

$E[X|B] = 8, \quad SD[X|B] = 3 \Rightarrow Var[X|B] = 9$

$E[X] = E[X|A]P(A) + E[X|B]P(B)$

$= 4 \times 0.9 + 8 \times 0.1$

$= 3.6 + 0.8 = 4.4$

$Var[X] = (Var[X|A] + (E[X|A])^2)P(A)$

$+ (Var[X|B] + (E[X|B])^2)P(B) - (E[X])^2$

$= (1 + 4^2)0.9 + (9 + 8^2)0.1 - (4.4)^2$

$= 15.3 + 7.3 - 19.36$

$= 3.24$

$\therefore Var[X] = 3.24$.

$SD[X] = \sqrt{Var[X]}$

$= 1.8$.