Homework 10

1.a) x # Edges

3 20

b) $G(10, x/6)$ has 10 vertices so the possible no. of edges is $^{10}C_2 = 45$.

$E \rightarrow$ no. of edges in a realisation of $G(10, x/6)$

i.e. E is the count of occurrence of ~~an~~ edges among the 45 possible edges, probability of success of each edge is $\frac{x}{6}$

$\therefore E \sim \text{Binomial}\left(45, \frac{x}{6}\right)$

pmf of E
$f(E=e) = \binom{45}{e}\left(\frac{x}{6}\right)^{e}\left(1-\frac{x}{6}\right)^{45-e}$

$\therefore L(x;E) = \binom{45}{E}\left(\frac{x}{6}\right)^{E}\left(1-\frac{x}{6}\right)^{45-E} \quad \ldots (i)$

c) On taking log of eq (i) we get,

$$\log L = \log\left(\binom{45}{E}\right) + E\log\left(\frac{x}{6}\right) + (45-E)\log\left(1-\frac{x}{6}\right)$$

To maximise L we need to maximise log L. We shall find the critical points of log L by taking the derivative of the above equation,

$$\frac{1}{L}\frac{dL}{dx} = 0 + E \cdot \frac{\cancel{6}}{x} \cdot \frac{1}{\cancel{6}} + (45-E) \cdot \frac{1}{1-\frac{x}{6}} \cdot \left(-\frac{1}{6}\right)$$

$$\Rightarrow 0 = \frac{E}{x} + \frac{45-E}{1-\frac{x}{6}}\left(\frac{-1}{6}\right)$$

$\Rightarrow \frac{E}{x} = \frac{45-E}{1-\frac{x}{6}} \cdot \frac{1}{6}$

$\Rightarrow E - \frac{Ex}{6} = \frac{(45-E)}{6} x$

$\Rightarrow E - \cancel{\frac{Ex}{6}} = \frac{45}{6} x - \cancel{\frac{Ex}{6}}$

$\Rightarrow x^* = \frac{6E}{45}$

let $T(x) = \frac{1}{L}\frac{dL}{dx} = \frac{E}{x} + \frac{45-E}{1-\frac{x}{6}} \cdot \left(-\frac{1}{6}\right)$

$\Rightarrow T''(x) = \frac{-E}{x^2} - \frac{(45-E)}{6} \cdot \frac{(-1/6)}{-(1-\frac{x}{6})^2}$

$T''(6E/45) = \frac{-E \times 45^2}{36 \times E^2} - \frac{45-E}{36} \cdot \frac{1}{(1-\frac{6E}{45})^2}$

since we know $E \leq 45$ and right hand side has the negative of the sum of two positive terms,

$T''(6E/45) < 0$.

$\Rightarrow 6E/45$ is the point of maxima for $L(x;E)$.

$x^* = \frac{6E}{45}$

d) $E = 20 \Rightarrow x^* = \frac{\overset{2}{\cancel{6}} \times \overset{4}{\cancel{20}}}{\cancel{45}\,\cancel{9}\,3} = \frac{8}{3} \simeq 2.67$

my chosen x was 3 which is close to 2.67

2) $X_1, X_2, \ldots X_n$ are iid Binomial (N, p)

Realisation = 8, 7, 6, 11, 8, 5, 3, 7, 6, 9.

Since there are two unknown parameters, $d = 2$

$$\mu_k(X_1, \ldots X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i^k \quad , k \in \{1, 2\}.$$

$$\therefore \mu_1(X_1, \ldots X_n) = \frac{1}{n}\sum X_i$$

$$= \frac{1}{10}(8+7+6+11+8+5+3+7+6+9)$$

$$= \frac{70}{10} = 7$$

$$\mu_2(X_1, \ldots X_n) = \frac{1}{n}\sum X_i^2$$

$$= \frac{1}{10}(8^2+7^2+6^2+11^2+8^2+5^2+3^2+7^2+6^2+9^2)$$

$$= \frac{534}{10} = 53.4$$

$X \sim$ Binomial (N, p)

$m_k(X) = E[X^k] \quad k \in \{1, 2\}$

$\therefore m_1(X) = E[X] = Np \quad \ldots (i) \; (X \sim \text{Binomial}(N, p))$

$m_2(X) = E[X^2]$

$Var(X) = E[(X - E(X))^2] = E[X^2] - (E(X))^2$

$\Rightarrow Np(1-p) = E[X^2] - (Np)^2$

$\Rightarrow E[X^2] = Np(1-p) + (Np)^2 \quad \ldots (ii)$

For the method of moments estimator we need to solve,

$\mu_1(X_1, \ldots X_n) = m_1(X)$

$\mu_2(X_1, \ldots X_n) = m_2(X)$

$\Rightarrow 7 = Np$ ... (iii)

and $53.4 = Np(1-p) + (Np)^2$ ... (iv)

Substituting eq (iii) in eq (iv) we get,

$53.4 = 7(1-p) + 7^2$

$\Rightarrow 4.4 = 7(1-p) \Rightarrow 0.6286 = 1-p$

$\Rightarrow p = 0.371$

From (iii), $7 = N \times 0.371$

$\Rightarrow N \approx 18.85$

N must be an integer so let us take $N = 19$.

$\therefore X_1, \ldots X_n \sim \text{Binomial}(19, 0.371)$

5) $X_1, X_2, \ldots X_n$ are ~~iid~~ Bernoulli(p)

$$f(X_i = x_i | p) = \begin{cases} p^{x_i}(1-p)^{1-x_i} & \text{if } x \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

Here only parameter p is unknown.

$$L(p | x_1, \ldots x_n) = \prod_{i=1}^{n} f(X_i = x_i | p)$$

$$= \prod_{i=1}^{n} f(x_i | p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^{\sum_{i=1}^{n}(1-x_i)}$$

$$= p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x_i}$$

$$\therefore L(p) = p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x_i}$$

On taking logarithm on both sides we get,

$$\log L(p) = \sum_{i=1}^{n} x_i \log p + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-p)$$

Note that $p \in (0,1)$ so $\log p$ & $\log(1-p)$ are defined.

Let $T(p) = \log L(p)$.

Case I: $\sum_{i=1}^{n} x_i = 0 \Rightarrow T(p) = n \log(1-p)$

$\log(1-p)$ is a decreasing fn. and attains its maximum value at $p = 0 \Rightarrow T(p)$ is maximum at $p = 0 = \frac{0}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$.

Case II: $\sum_{i=1}^{n} x_i = a \Rightarrow T(p) = a \log p + (n-a) \log(1-p)$

where $0 < a < n$.

We shall find the point of maxima for T.

$$T'(p) = \frac{a}{p} + (n-a)\frac{(-1)}{1-p} = 0$$

$$\Rightarrow \frac{a}{p} = \frac{n-a}{1-p} \Rightarrow a - \cancel{ap} = np - \cancel{ap}$$

$$\Rightarrow p = \frac{a}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Case <u>III</u>: $\sum_{i=1}^{n} x_i = n \Rightarrow T(p) = n \log p$

$\log p$ is an increasing function $\Rightarrow T(p)$ is also increasing and for a fixed n will attain its maximum at $p = 1 = \frac{n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$

$\therefore$ In all two cases $\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$

$\Rightarrow X_1, \ldots X_n \sim \text{Bernoulli}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right).$