Homework 5

1) $\lambda > 0$. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 < x \\ 0 & \text{otherwise} \end{cases}$

(a) R.T.P : f is a probability density function.

Note that, (i) for $x > 0$, $e^{-\lambda x} > 0$, since $\lambda > 0$

$\Rightarrow \lambda e^{-\lambda x} > 0 \quad \forall\ x > 0 \Rightarrow f(x) > 0 \ \forall x > 0$

Also, if $x \leq 0$, $f(x) = 0$

$\therefore f(x) \geq 0 \quad \forall\ x \in \mathbb{R}$.

(ii) exponential function is continuous

$\Rightarrow e^{-\lambda x}$ is a continuous function

$\Rightarrow \lambda e^{-\lambda x}$ is a continuous function

$\Rightarrow f(x)$ is a continuous function $\forall\ x > 0$.

$\forall\ x \leq 0$, $f(x) = 0$ is a constant function

$\Rightarrow f(x)$ is a continuous function $\forall\ x \leq 0$

$\therefore f$ is continuous in $(-\infty, 0]$ and $(0, \infty)$

$\Rightarrow f$ is piecewise continuous.

(iii) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= 0 + \lambda \int_{0}^{\infty} e^{-\lambda x}\,dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} = 1$$

1. b) Let $a > 0$.

$P((a,\infty)) = P(x > a) = 1 - P(X \leq a) = 1 - F(a)$

where F is the distribution function for f.

$$P(x > a) = \int_a^\infty f(x)\,dx$$

$$= \int_a^\infty \lambda e^{-\lambda x}\,dx = \frac{\lambda}{-\lambda}\left[e^{-\lambda x}\right]_a^\infty$$

$$= e^{-\lambda a}$$

$$\therefore P(x \in (a,\infty)) = e^{-\lambda a}$$

2) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 1/x^2 & 1 < x \\ 0 & \text{Otherwise} \end{cases}$.

(a) P.T.P. f is a probability density function.

(i) $x^2 > 0 \quad \forall\, x > 1$

$\Rightarrow 1/x^2 > 0 \quad \forall\, x > 1$

$\Rightarrow f(x) > 0 \quad \forall\, x > 1$

$\forall x \leq 1$, $f(x) = 0$

$\Rightarrow f(x) \geq 0 \quad \forall\, x \in \mathbb{R}$.

(ii) $\forall\, x > 1$, $\frac{1}{x^2}$ is a continuous function $(x \neq 0)$

$\Rightarrow \forall\, x > 1$, $f(x)$ is continuous.

$\forall x \leq 1$, $f(x) = 0$ is a constant function.

$\Rightarrow \forall\, x \leq 1$, $f(x)$ is continuous.

$\Rightarrow$ f is continuous on $(1, \infty)$ and $(-\infty, 1]$

$\Rightarrow$ f is piecewise continuous.

(iii) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$$

$$= 0 + \left[-\frac{1}{x}\right]_1^{\infty} = 1.$$

$\therefore$ f is a probability density function.

2.b) $a > 1$, find : $P((a, \infty))$

$P((a, \infty)) = F(x > a)$

where F is the distribution function for f.

$$f(x > a) = \int_a^{\infty} f(x)\,dx$$

$$= \int_a^{\infty} \frac{1}{x^2}\,dx \qquad (a > 1)$$

$$= \left[-\frac{1}{x}\right]_a^{\infty} = \frac{1}{a}.$$

$$\therefore P((a, \infty)) = \frac{1}{a}.$$

3) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{6} x^2 e^{-x} & 0 < x \\ 0 & \text{otherwise} \end{cases}$.

R.T.P: f is a probability density function.

(i) $\forall x \in \mathbb{R}$, $e^{-x} > 0$ and $x^2 > 0$.

$$\Rightarrow e^{-x} \cdot x^2 > 0$$

$\Rightarrow \forall x > 0$, $\frac{1}{6} e^{-x} \cdot x^2 > 0$

$$\Rightarrow f(x) > 0 \quad \forall x > 0.$$

$\forall x \leq 0$, $f(x) = 0$

$\therefore f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) e^{-x} and x^2 are continuous $\forall x \in \mathbb{R}$

$\Rightarrow e^{-x} \cdot x^2$ is continuous (product of continuous is continuous)

$\Rightarrow \frac{1}{6} e^{-x} \cdot x^2$ is continuous

$\Rightarrow f(x)$ is continuous $\forall x \in (0, \infty)$.

$f(x) = 0 \quad \forall x \leq 0 \Rightarrow f(x)$ is a constant function

$\Rightarrow f(x)$ is continuous $\forall x \leq 0$.

$\therefore f$ is continuous on $(0, \infty)$ and $(-\infty, 0]$

$\Rightarrow f$ is piecewise continuous.

(iii) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_0^\infty \frac{x^2 e^{-x}}{6}\,dx$$

$$\int x^2 e^{-x}\,dx = x^2\int e^{-x}\,dx - \int\left[\frac{d}{dx}(x^2)\cdot\int e^{-x}\,dx\right]dx$$

(By parts)

$$= -x^2 e^{-x} - \int 2x\cdot(-e^{-x})\,dx$$

$$= -x^2 e^{-x} + 2\left[x\int e^{-x}\,dx - \int\left(\frac{d}{dx}(x)\cdot\int e^{-x}\,dx\right)dx\right]$$

$$= -x^2\cdot e^{-x} + 2\left[-xe^{-x} - \int(-e^{-x})\,dx\right]$$

$$= -x^2\cdot e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 0 + \frac{1}{6}\left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

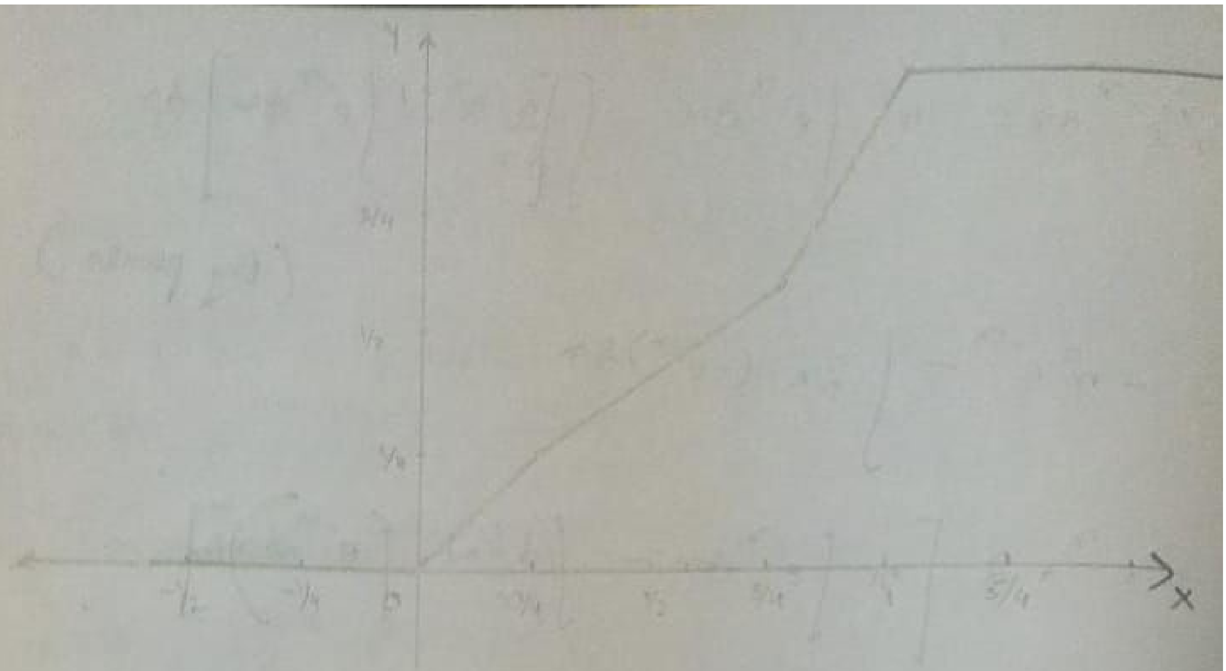
$$= \frac{1}{6}\times 2 = \frac{1}{3}$$

$$\int_{-\infty}^{\infty} f(x)\,dx \neq 1$$

$\Rightarrow$ f is not a probability density function.

4) $$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1/4 \\ \frac{x}{2} + \frac{1}{8} & 1/4 \le x \le 3/4 \\ 2x - 1 & 3/4 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

(a)



(b) $P([0, 1/4)) = P(0 \le X < 1/4)$

$= P(X < 1/4) - P(X < 0)$

$= F(1/4) - F(0)$

$= 1/4 - 0 = 1/4$

$P([1/8, 3/2]) = P(1/8 \le X \le 3/2)$

$= P(X \le 3/2) - P(X \le 1/8)$

$= P(1/8 \le X \le 1/4) + P(1/4 \le X \le 3/4) + P(3/4 \le X \le 1)$
$+ P(1 \le X \le 3/2)$

$= \cancel{F(1/4)} - F(1/8) + \cancel{F(3/4)} - \cancel{F(1/4)} + \cancel{F(1)} - \cancel{F(3/4)}$
$+ F(3/2) - \cancel{F(1)}$

$= F(3/2) - F(1/8) = 1 - \frac{1}{8} = \frac{7}{8}$

$P([\frac{3}{4}, \frac{7}{8}]) = P(3/4 \le X \le 7/8)$

$= F(7/8) - F(3/4)$

$= \left(2 \times \frac{7}{8} - 1\right) - \left(2 \times \frac{3}{4} - 1\right)$

$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

(c) We know, $F'(x) = f(x)$.

For, $x < 0$, $f(x) = \frac{d}{dx}(0) = 0$.

For, $0 < x < 1/4$, $f(x) = \frac{d}{dx}(x) = 1$

For, $(1/4 \le x < 3/4)$, $f(x) = \frac{d}{dx}\left(\frac{x}{2} + \frac{1}{8}\right) = \frac{1}{2}$.

For, $3/4 \le x < 1$, $f(x) = \frac{d}{dx}(2x - 1) = 2$

for, $x \geq 1$, $f(x) = \frac{d}{dx}(1) = 0$.

$$\therefore f(x) = \begin{cases} 1 & 0 < x < 1/4 \\ 1/2 & 1/4 \le x < 3/4 \\ 2 & 3/4 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$\forall x \in \mathbb{R}$, $f(x) \geq 0$.

Also, on $(0, 1/4)$, $[1/4, 3/4)$ and $[3/4, 1)$ f is constant,

hence continuous in each part.

$\Rightarrow f$ is piecewise continuous.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{1/4} 1\,dx + \int_{1/4}^{3/4} \frac{1}{2}\,dx + \int_{3/4}^{1} 2\,dx$$

$$= \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} = 1.$$

$\therefore f$ is the probability density function of F.

5) $f : \mathbb{R} \longrightarrow [0,1]$, $f(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & -R < x < R \\ 0 & \text{otherwise} \end{cases}$

$$F(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f(x)\, dx$$

if $\hat{x} \leq -R$, $f(\hat{x}) = 0$

$$\Rightarrow F(\hat{x}) = \int_{-\infty}^{\hat{x}} 0 = 0 .$$

if $-R < \hat{x} < R$, $f(\hat{x}) = \frac{2}{\pi R^2}\sqrt{R^2 - \hat{x}^2}$

$$\Rightarrow F(\hat{x}) = \int_{-\infty}^{-R} f(x)\, dx + \int_{-R}^{\hat{x}} f(x)\, dx$$

$$= 0 + \int_{-R}^{\hat{x}} \frac{2}{\pi R^2}\sqrt{R^2 - x^2}\, dx$$

$$= \frac{2}{\pi R^2}\left[\frac{x\sqrt{R^2 - x^2}}{2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]_{-R}^{\hat{x}}$$

$$= \frac{2}{\pi R^2}\left[\frac{\hat{x}\sqrt{R^2 - \hat{x}^2}}{2} + \frac{R^2}{2}\sin^{-1}\left(\frac{\hat{x}}{R}\right) + \frac{R\sqrt{R^2 - R^2}}{2} + \frac{R^2}{2}\sin^{-1}(1)\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{\hat{x}\sqrt{R^2 - \hat{x}^2}}{2} + \frac{R^2}{2}\sin^{-1}\left(\frac{\hat{x}}{R}\right) + \frac{R^2}{2} \times \frac{\pi}{2}\right]$$

$$\Rightarrow F(\hat{x}) = \frac{2}{\pi R^2}\left[\frac{\hat{x}\sqrt{R^2 - \hat{x}^2}}{2} + \frac{R^2}{2}\sin^{-1}\left(\frac{\hat{x}}{R}\right) + \frac{\pi R^2}{4}\right]$$

if $\hat{x} \geq R$, $f(\hat{x}) = 0$.

$$F(\hat{x}) = \int_{-\infty}^{\hat{x}} f(x)dx = \int_{-\infty}^{-R} f(x)dx + \int_{-R}^{R} f(x)dx + \int_{R}^{\hat{x}} f(x)dx$$

$$= 0 + \int_{-R}^{R} \frac{2}{\pi R^2}\sqrt{R^2-x^2}\,dx + 0$$

$$= \frac{2}{\pi R^2}\left[\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]_{-R}^{R}$$

$$= \frac{2}{\pi R^2}\left[0 + \frac{R^2}{2}\sin^{-1}(1) + 0 - \frac{R^2}{2}\sin^{-1}(-1)\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{\pi R^2}{4} + \frac{\pi R^2}{4}\right] = 1$$

$$\therefore F(x) = \begin{cases} 0 & x \leq -R \\ \frac{2}{\pi R^2}\left[\frac{x\sqrt{R^2-x^2}}{2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right) + \frac{\pi R^2}{4}\right] & -R < x < R \\ 1 & x \geq R \end{cases}$$

6) $F: \mathbb{R} \longrightarrow [0,1]$, $F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{\pi}\arcsin(\sqrt{x}) & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$

If f is the probability density function of X then,

$$F'(x) = f(x)$$

if $x \leq 0$, $f(x) = \frac{d}{dx}(0) = 0$

if $0 < x < 1$, $f(x) = \frac{d}{dx}\left(\frac{2}{\pi} \arcsin(\sqrt{x})\right)$

$$= \frac{\cancel{2}}{\pi} \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{\cancel{2}} \cdot x^{-1/2}$$

$$= \frac{1}{\pi\sqrt{x}(1+x)}.$$

if $x \geq 1$, $f(x) = \frac{d}{dx}(1) = 0$.

$$\therefore f(x) = \begin{cases} \frac{1}{\sqrt{x}(1+x)\pi} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

if $0 < x < 1$, $\sqrt{x}(1+x) > 0 \Rightarrow f(x) > 0$

otherwise, $f(x) = 0$.

$\Rightarrow f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$\frac{1}{\pi\sqrt{x}(1+x)}$ is continuous on $\mathbb{R} \setminus \{0, -1\}$

$\Rightarrow f(x)$ is continuous on $(0,1)$.

$f(x) = 0$ otherwise, $f(x)$ is constant, hence continuous $\forall x \in \mathbb{R} \setminus (0,1)$

$\therefore f$ is piecewise continuous.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{1} \frac{1}{\sqrt{x}(1+x)}\,dx + 0$$

$$= \left[\frac{2}{\pi} \arcsin x\right]_0^1$$

$$= \frac{2}{\pi}\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = 1.$$

$\therefore \int_{-\infty}^{\infty} f(x)\, dx = 1.$

$\therefore$ f is the probability density function of F.

7) (a) $X \sim$ uniform(a, b)

$$P(X < x) = F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

we need to find x s.t. $P(X < x) = 1/2 \neq 0, 1$

$\Rightarrow x \in [a, b] \Rightarrow F(x) = \frac{1}{2}$

$$\Rightarrow \frac{x-a}{b-a} = \frac{1}{2}$$

$\Rightarrow 2x - 2a = b - a \Rightarrow x = \frac{b+a}{2}$

$P(X > x) = 1 - P(X < x) = \frac{1}{2}$ if $x = \frac{b+a}{2}$

$\therefore$ median $= \frac{a+b}{2}$

(b) $Y \sim Exp(\lambda)$

$$P(X \le x) = F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & 0 \le x \end{cases}$$

We need to find x s.t. $P(X \le x) = 1/2 \neq 0$, $x \geq 0$

$\Rightarrow x \geq 0 \Rightarrow F(x) = 1/2$

$\Rightarrow 1 - e^{-\lambda x} = 1/2$

$\Rightarrow e^{-\lambda x} = \frac{1}{2}$

$\Rightarrow -\lambda x = \log(1/2)$

$\Rightarrow \lambda x = \log 2 \Rightarrow x = \frac{\log 2}{\lambda}$

$P(X > x) = 1 - P(X \le x)$ $\Bigg]$ $x = \frac{\log 2}{\lambda}$

$= 1 - 1/2 = 1/2$

$\therefore P(X < x) = P(X > x) = 1/2$ for $x = \frac{\log 2}{\lambda}$

median $= \frac{\log 2}{\lambda}$

(c) $Z \sim Normal(\mu, \sigma^2)$

R.T.P: $P(X > \mu) = P(X < \mu) = \frac{1}{2}$.

$$P(X < \mu) = \int_{-\infty}^{\mu} f(x)\, dx.$$

where, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$\therefore P(X<\mu) = \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$.

Let $y = \frac{x-\mu}{\sigma} \Rightarrow dy = \frac{dx}{\sigma}$

x	$-\infty$	μ
y	$-\infty$	0

. . (i)

$P(X<\mu) = \int_{-\infty}^{0} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-y^2/2} \cancel{\sigma} \, dy$

$= \int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy$

We have, $\int_{-\infty}^{\infty} f(x) \, dx = 1$ (f is a p.d.f)

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\ \sigma} \, dx = 1$.

From (i) we can substitute & get,

$\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy = 1$

$\Rightarrow \int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy + \int_{0}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy = 1$ . . . (ii)

Note that, $\int_{0}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy = \int_{0}^{-\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} (-dz)$ $\left(\begin{array}{l} z = -y \\ dz = -dy \end{array}\right)$

$= \int_{-\infty}^{0} \frac{e^{-z/2}}{\sqrt{2\pi}} \, dz = \int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy$

$\therefore$ In eq (ii) we get,

$$2\int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy = 1$$

$$\Rightarrow \int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy = \frac{1}{2}$$

$$\therefore P(X < \mu) = \frac{1}{2}$$

Also, $P(X > \mu) = 1 - P(X < \mu) = \frac{1}{2}$

$\therefore \mu$ is the median.