31 October 2021 Due on 14 November 2021

M.Sc. Data Science
Analysis - HW 3

Note: Copying will not be tolerated. You may discuss among yourselves but the final work should be your own.

1. Determine if the integral $\int_1^\infty e^{-x}\,dx$ exists. Find its value if it does.

2. (a) Let s be a number < 1. Show that the integral $\int_0^1 \frac{1}{x^s}\,dx$ exists.
   (b) If $s > 1$, show that the integral does not exist.
   (c) Does the integral exist when $s = 1$?

3. (a) If $s > 1$, show that the integral $\int_1^\infty \frac{1}{x^s}\,dx$ exists.
   (b) If $s < 1$, show that the integral does not exist.

4. Suppose you are given a continuous probability distribution function $f(x)$ over an interval $[a, b]$. Set up Riemann sums and the corresponding Riemann integral to calculate the mean and variance of the probability distribution.

5. The following functions are related to commonly used continuous probability distributions.
   (a) (Uniform distribution) Let
   $$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$
      i. Show that $f(x)$ is a probability distribution function (pdf).
      ii. Show that the corresponding expected value and variance are $E(x) = \frac{b+a}{2}$ and $Var(x) = \frac{(b-a)^2}{12}$ respectively.
   (b) (Gamma distribution) The Gamma function is defined as
   $$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,$$
   where α is a positive constant.
      i. Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.
      ii. Show that $\Gamma(\alpha) = \alpha!$ when α is a positive integer. (Correction: $\Gamma(\alpha + 1) = \alpha!$.)
      iii. Prove that $f(t) = \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}$, $(0 < t < \infty)$, is a pdf.

6. (Standard logistic function) Let $\phi(x) = \frac{1}{1+e^{-x}}$.
   (a) Is $\phi(x)$ differentiable? Find $\phi'(x)$.
   (b) Is $\phi(x)$ monotonic (i.e. either increasing or decreasing)?
   (c) Show that $\phi'(x) = \phi(x)(1 - \phi(x))$.
   (d) What is the range of ϕ?.
   (e) Sketch the graph on ϕ on the domain $[-10, 10]$. What is $\phi'(0)$?

7. (Hyperbolic tangent function) Let $f(x) = \tanh(x)$.
   (a) Is $f(x)$ differentiable? Find $f'(x)$.

(b) Is $f(x)$ monotonic?

(c) Show that $f'(x) = 1 - f(x)^2$.

(d) What is the range of f?

(e) Show that $e^x = \dfrac{1 + \tanh\left(\frac{x}{2}\right)}{1 - \tanh\left(\frac{x}{2}\right)}$

(f) Sketch the graph on ϕ on the domain $[-10, 10]$. What is $f'(0)$?

8. Show that the standard logistic function and the hyperbolic tangent function are related as follows:

$$\phi(x) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right)$$

Note: The standard logistic function, the hyperbolic tan function and the inverse tan function are some examples of nonlinear sigmoid (i.e. 'S' shaped) activation functions used in the last layer of neural networks (NN). These functions squeeze the output of the previous layer of the NN to give values lying in a smaller range. You can read more about activation functions online. Knowing these functions and the behaviour of their derivatives helps decide which activation function to use in different situations. The slopes (derivatives) help decide how the parameters of the NN should be adjusted.