Homework 6

1) Black → 1, White → 2, Green → 8, Red → 8, Blue → 8

A = {the ball is either black or green}

B = {the ball is either black or red}

C = {the ball is either black or blue}

(a) $P(A \cap B \cap C) = \dfrac{n(A \cap B \cap C)}{n(S)}$

$A \cap B \cap C$ is the event the ball is (black or green) and (black or red) and (black or blue).

A = Black ∪ Green

B = Black ∪ Red

C = Black ∪ Blue

$\therefore A \cap B \cap C = \big((\text{Black} \cup \text{Green}) \cap (\text{Black} \cup \text{Red})\big) \cap (\text{Black} \cup \text{Blue})$

$= \big(\text{Black} \cup (\text{Green} \cap \text{Red})\big) \cap (\text{Black} \cup \text{Blue})$

$= \text{Black} \cup (\text{Green} \cap \text{Red} \cap \text{Blue})$

$= \text{Black} \cup \phi$

$= \text{Black}$

$\therefore A \cap B \cap C$ is the event of drawing black balls,

$P(A \cap B \cap C) = \dfrac{1}{27}$

(b) $P(A)P(B)P(C) = P(\text{Black} \cup \text{Green}) \cdot P(\text{Black} \cup \text{Red}) \cdot P(\text{Black} \cup \text{Blue})$

$= \dfrac{\cancel{9}^{1}}{\cancel{27}_{3}} \times \dfrac{\cancel{9}^{1}}{\cancel{27}_{3}} \times \dfrac{\cancel{9}^{1}}{\cancel{27}_{3}} = \dfrac{1}{27}$

(c) We have $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Consider, $A \cap B \cap C^c$

$= (\text{Black} \cup \text{Green}) \cap (\text{Black} \cup \text{Red}) \cap (\text{White} \cup \text{Green} \cup \text{Blue})$

$= (\text{Black} \cup (\text{Green} \cap \text{Red})) \cap (\text{White} \cup \text{Green} \cup \text{Blue})$

$= (\text{Black} \cup \phi) \cap (\text{White} \cup \text{Green} \cup \text{Blue})$

$= \text{Black} \cap (\text{White} \cup \text{Green} \cup \text{Blue})$

$= \phi$

$\therefore P(A \cap B \cap C^c) = 0$.

$P(A)P(B)P(C^c) = \frac{\cancel{9}^{1}}{\cancel{27}_{3}} \times \frac{\cancel{9}^{1}}{\cancel{27}_{3}} \times \frac{\cancel{18}^{2}}{\cancel{27}_{3}}$

$= \frac{2}{27} > 0$.

$P(A \cap B \cap C^c) \neq P(A)P(B)P(C^c)$

$\Rightarrow A, B$ & C are not mutually independent.

2) Female $\rightarrow 90$, Pencil $\rightarrow 60$, Glasses $\rightarrow 30$

$A_1 = \{$the student is a female$\}$

$A_2 = \{$the student uses a pencil$\}$

$A_3 = \{$the student is wearing eye glasses$\}$

(a) Let $n(A_1 \cap A_2 \cap A_3) = x$.

$\therefore P(A_1 \cap A_2 \cap A_3) = \frac{x}{150}$

$$P(A_1)P(A_2)P(A_3) = \frac{\cancel{90}^{3}}{\cancel{150}_{5}} \times \frac{\cancel{60}^{2}}{\cancel{150}_{5}} \times \frac{\cancel{30}^{1}}{\cancel{150}_{5}}$$

$$= \frac{6}{125}$$

If A_1, A_2 & A_3 are mutually independent then,

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$$\Rightarrow \frac{x}{150} = \frac{6}{125}$$

$$\Rightarrow x = \frac{6 \times \cancel{150}^{\cancel{30}\,6}}{\cancel{125}_{\cancel{25}\,5}} = \frac{36}{5}$$

$x \notin \mathbb{N} \cup \{0\}$, which is not possible.

$\therefore A_1, A_2$ & A_3 can never be mutually independent.

(b) Let $n(A_1 \cap A_2) = 36$, $n(A_2 \cap A_3) = 12$,

$n(A_1 \cap A_3) = 18$.

$$\therefore P(A_1 \cap A_2) = \frac{36}{150} = \frac{6}{25}$$

$$P(A_1)P(A_2) = \frac{\cancel{90}^{3}}{\cancel{150}_{5}} \times \frac{\cancel{60}^{2}}{\cancel{150}_{5}} = \frac{6}{25}$$

$$P(A_2 \cap A_3) = \frac{12}{150} = \frac{2}{25}$$

$$P(A_2)P(A_3) = \frac{\cancel{60}^{2}}{\cancel{150}_{5}} \times \frac{\cancel{30}^{1}}{\cancel{150}_{5}} = \frac{2}{25}$$

$$P(A_1 \cap A_3) = \frac{18}{150} = \frac{3}{25}$$

$$P(A_1)P(A_3) = \frac{\cancel{90}^{3}}{\cancel{150}_{5}} \times \frac{\cancel{30}^{1}}{\cancel{150}_{5}} = \frac{3}{25}$$

$\therefore P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \quad \forall\ 1 \le i, j \le 3$

$\Rightarrow A_1, A_2, A_3$ are pairwise independent.

3) $X \sim \text{Binomial}(n, p)$, $Y \sim \text{Binomial}(m, p)$
$Z = X + Y$.

R.T.P: $Z \sim \text{Binomial}(m+n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(Y = k') = \binom{m}{k'} p^{k'} (1-p)^{m-k'}$$

Range $(X) = \{0, 1, \ldots n\}$
Range $(Y) = \{0, 1, \ldots m\}$
$\Rightarrow$ Range $(Z) = \{0, 1, 2, \ldots m+n\}$
~~let~~ $z \in$ Range (Z)

$$P(Z = z) = P(X + Y = z)$$

$$= P\left(\bigcup_{i=0}^{z} (X = i, Y = z - i)\right)$$

$$= \sum_{i=0}^{z} P(X = i, Y = z - i) \quad \text{(Mutually Exclusive)}$$

$$= \sum_{i=0}^{z} P(X = i) P(Y = z - i) \quad \text{(X \& Y are independent)}$$

$$= \sum_{i=0}^{z} \binom{n}{i} p^i (1-p)^{n-i} \cdot \binom{m}{z-i} p^{z-i} (1-p)^{m-z+i}$$

$$= \sum_{i=0}^{z} \binom{n}{i} \binom{m}{z-i} p^z \cdot (1-p)^{(m+n) - (z - i + i)}$$

$$= \sum_{i=0}^{z} \binom{n}{i}\binom{m}{z-i} \cdot p^{z}(1-p)^{m+n-z}$$

$$= p^{z}(1-p)^{m+n-z} \sum_{i=0}^{z} \binom{n}{i}\binom{m}{z-i}$$

$$= \binom{m+m}{z} p^{z}(1-p)^{m+n-z}$$

$\Rightarrow Z \sim$ Binomial $(m+n, p)$

4) $X \Rightarrow$ number of heads, $Y \rightarrow$ number of tails.

(a) when we flip a single fair coin the result can either be a head or a tail.

Let p be the probability of head.

Since fair coin, $P(\text{Tail}) = P(\text{Head}) = p$

$$p + p = 1 \Rightarrow p = \frac{1}{2}$$

$\text{Range}(X) = \{0, 1\} = \text{Range}(Y)$

$P(X=0) = \frac{1}{2} = P(X=1)$

$P(Y=0) = \frac{1}{2} = P(Y=1)$

$\therefore X, Y \sim$ Bernoulli$(1/2)$.

(b) $Z = X+Y$. $\text{Range}(Z) = \{0,1,2\}$.

$P(Z=z) = P(X+Y=z)$

$= P\left(\bigcup_{i=0}^{z}(X=i, Y=z-i)\right)$

$P(Z=1) = P(X+Y=1)$

$= P\big((X=0, Y=1) \cup (X=1, Y=0)\big)$

$= P(X=0, Y=1) + P(X=1, Y=0)$

$= P(X=0 \mid Y=1)\,P(Y=1) + P(X=1 \mid Y=0)\,P(Y=0)$

$X=0 \mid Y=1$ is the event that head does not occurs if tail appears. This event has propability 1.

Similarly, $P(X=1 \mid Y=0) = 1$ since given tail didn't appear, there is 100% chance that head will appear.

$\therefore P(Z=1) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1.$

(c) In Example 3.3.3, if $X \sim Bernoulli(p)$ & $Y \sim Bernoulli(p)$ and X and Y are independent then $Z = X+Y$ s.t.

$Z \sim Binomial(2, p)$

Here, $P(X=0, Y=1) = \frac{1}{2}$ & $P(X=0)P(Y=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Rightarrow X$ & Y are not independent

$\rightarrow$ the result of Example 3.3.3 might not hold.

5) $X \sim Geometric(p)$, $Y \sim Geometric(p)$.

$Z = X+Y$.

(a) $Range(X) = \{1, 2, 3, \ldots\}$

$Range(Y) = \{1, 2, 3, \ldots\}$

$Range(Z) = \{2, 3, 4, \ldots\}$

(b) $P(Z=n) = P(X+Y=n)$

$$= P\left(\bigcup_{i=1}^{n-1} (X=i, Y=n-i)\right)$$

$$= \sum_{i=1}^{n-1} P(X=i, Y=n-i) \quad \text{(Mutually Exclusive)}$$

$$= \sum_{i=1}^{n-1} P(X=i)\, P(Y=n-i) \quad \text{(X \& Y are Independent)}$$

$$= \sum_{i=1}^{n-1} p\cdot(1-p)^{i-1} \cdot p\,(1-p)^{n-i-1}$$

$$= \sum_{i=1}^{n-1} p^2 \cdot (1-p)^{\cancel{i}-1+n-\cancel{i}-1}$$

$$= \sum_{i=1}^{n-1} p^2 \cdot (1-p)^{n-2}$$

$$= p^2(1-p)^{n-2} \sum_{i=1}^{n-1} 1.$$

$$= (n-1)\, p^2(1-p)^{n-2}$$

(c) $P(Z=2) = P(X+Y=2)$

$= P(X=1, Y=1)$

$= P(X=1)\, P(Y=1)$

$= p \cdot p = p^2$

$P(Z=3) = (3-1)\, p^2(1-p)^1$

$= 2p^2(1-p)$

$2p^2(1-p) > p^2$

$\Rightarrow p^2(2-2p-1) > 0$

$p^2 > 0 \ \forall p \quad \Rightarrow 2-2p-1 > 0$

$\Rightarrow p < 1/2$

$\therefore \forall p < 1/2, \quad P(Z=2) < P(Z=3)$

$\Rightarrow Z = 2$ is not the most likely outcome

6) $X \sim$ Geometric(p) $\quad A \Rightarrow X \leq 3$

Range $(X) = \{1, 2, \ldots\}$.

Note that, $P(X=i \mid A) = \dfrac{P(X=i \cap (X \leq 3))}{P(A)} \ \forall i > 3$

Note that, $(X=i) \cap (X \leq 3) = \phi \quad \forall i > 3$

$\therefore P(X=i \mid A) = 0 \quad \forall i > 3$

$P(A) = P(X \leq 3)$

$= P(X=1) + P(X=2) + P(X=3)$

$= p + p(1-p) + p(1-p)^2$

$= p(1 + 1 - p + 1 - 2p + p^2)$

$= p(3 - 3p + p^2)$

$P(X=1 \mid A) = \dfrac{P(X=1 \cap (X \leq 3))}{P(A)}$

$= \dfrac{P(X=1)}{P(A)} = \dfrac{\cancel{p}}{\cancel{p}(3-3p+p^2)} = \dfrac{1}{p^2-3p+3}$

$$P(X=2|A) = \frac{P(X=2 \cap (X \le 3))}{P(A)}$$

$$= \frac{P(X=2)}{P(A)}$$

$$= \frac{\cancel{p}(1-p)}{\cancel{p}(p^2-3p+3)} = \frac{1-p}{p^2-3p+3}$$

$$P(X=3|A) = \frac{P(X=3 \cap (X \le 3))}{P(A)}$$

$$= \frac{P(X=3)}{P(A)}$$

$$= \frac{p(1-p)^2}{p(p^2-3p+3)} = \frac{(1-p)^2}{p^2-3p+3}$$

$$\therefore E[X|A] = \sum_{t \in Range(X)} t \; P(X=t|A)$$

$$= 1 \cdot \frac{1}{p^2-3p+3} + 2 \cdot \frac{1-p}{p^2-3p+3} + 3 \cdot \frac{(1-p)^2}{p^2-3p+3} + 0$$

$$= \frac{1}{p^2-3p+3} \left(1 + 2-2p + 3-6p+3p^2\right)$$

$$= \frac{6-8p+3p^2}{p^2-3p+3}$$

$$Var[X|A] = E[(X-E[X|A])^2|A]$$

$$= \left(1 - \frac{6-8p+3p^2}{p^2-3p+3}\right)^2 \cdot \frac{1}{p^2-3p+3} + \left(2 - \frac{6-8p+3p^2}{p^2-3p+3}\right)^2 \cdot \frac{1-p}{p^2-3p+3} + \left(3 - \frac{6-8p+3p^2}{p^2-3p+3}\right)^2 \frac{(1-p)^2}{p^2-3p+3} + 0$$

$$= \frac{1}{(p^2-3p+3)^3}\Big[(p^2-3p+3-6+8p-3p^2)^2 \cdot 1$$

$$+ (2p^2-6p+6-6+8p-3p^2)^2 \cdot (1-p)$$

$$+ (\cancel{3}p^2 - 9p + 9 - 6 + 8p \cancel{- 3p^2})^2 (1-p)^2\Big]$$

$$= \frac{1}{(p^2-3p+3)^3}\left[(-2p^2+5p-3)^2 + (1-p)(-p^2+2p)^2 + (1-p)^2(3-p)^2\right]$$

$$= \frac{(2p^2-5p+3)^2 + p^2(1-p)(p-2)^2 + (1-p)^2(3-p)^2}{(p^2-3p+3)^3}$$

$$= \frac{(p-1)^2(2p-3)^2 + p^2(p-2)^2(1-p) + (1-p)^2(3-p)^2}{(p^2-3p+3)^2}$$

$$= \frac{(1-p)^2}{(p^2-3p+3)^2}\left[(2p-3)^2 + \frac{p^2(p-2)^2}{1-p} + (3-p)^2\right].$$

7) $X \sim$ Uniform $(\{0,1,2\})$. $Y \to$ number of heads in X flips.

(a) Since Y is a variable dependent on the value of X, $\text{Cov}[X,Y] \neq 0$.

Also, since if we increase the number of flips the chances of getting more heads also increases. That is, if we increase the value of X, Y should also increase. So, according to me, X & Y are positively correlated.

(b) $\text{cov}[X,Y] = E[(X-E[X])(Y-E[Y])]$

$= E[XY] - E[X]E[Y]$

Range $(X) = \{0,1,2\}$

Range $(Y) = \{0,1,2\}$

$P(X=0, Y=0) = P(Y=0 \mid X=0)\,P(X=0)$

$= 1 \cdot \frac{1}{3}$

$P(X=0, Y=1) = 0 = P(X=0, Y=2)$

$P(X=1, Y=0) = P(Y=0 \mid X=1)\,P(X=1)$

$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$P(X=1, Y=1) = P(Y=1 \mid X=1)\,P(X=1)$

$= \frac{1}{6}$

$P(X=2, Y=0) = P(Y=0 \mid X=2)\,P(X=2)$

$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(X=2, Y=1) = P(Y=1 \mid X=2)\,P(X=2)$

$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(X=2, Y=2) = P(Y=2 \mid X=2)\,P(X=2)$

$= \frac{1}{9}$.

	$X=0$	$X=1$	$X=2$	sum	
$Y=0$	1/3	1/6	1/9	11/18	$P(Y=0)$
$Y=1$	0	1/6	1/9	5/18	$P(Y=1)$
$Y=2$	0	0	1/9	1/9	$P(Y=2)$
sum	1/3	1/3	1/3		
	$P(X=0)$	$P(X=1)$	$P(X=2)$		

Range $(XY) = \{0, 1, 2, 4\}$

$P(XY=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$

$+ P(X=1, Y=0) + P(X=2, Y=0)$

$= \frac{11}{18} + 0 = \frac{11}{18}$.

$P(XY=1) = P(X=1, Y=1)$

$= \frac{1}{6}$

$P(XY=2) = P(X=2, Y=1) + P(X=1, Y=2)$

$= \frac{1}{9}$

$P(XY=4) = P(X=2, Y=2)$

$= \frac{1}{9}$.

$\therefore E[XY] = 0 \cdot \frac{11}{18} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{9}$.

$= \frac{1}{6} + \frac{6}{9} = \frac{\cancel{15}^{5}}{\cancel{18}_{6}} = \frac{5}{6}$.

$E[X] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$

$E[Y] = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) + 2 \cdot P(Y=2)$

$= 0 \cdot \frac{11}{18} + 1 \cdot \frac{5}{18} + 1 \cdot \frac{1}{9}$

$= \frac{7}{18}$.

$\text{Cov}[X, Y] = \frac{5}{6} - \frac{7}{18} = \frac{8}{18} = \frac{4}{9}$.