# Solution Manual for *[Introduction to Analytic Number Theory, Apostol]*

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## Contents

# 1 Chapter 2: Arithmetical Functions and Dirichlet Multiplication

## Problem 1.1

Find all integers $n$ such that:

(a) $\varphi(n) = \frac{n}{2}$,

(b) $\varphi(n) = \varphi(2n)$,

(c) $\varphi(n) = 12$.

### Solution

1. a) Recall that Euler's totient function $\varphi(n)$ can be given by:

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product runs over all prime divisors $p$ of $n$.

Equating this with our hypothesis, we have:

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{n}{2}.$$

First, notice that when n is a power of two $n = 2^k$, $k \geq 1$ the equation is satisfied, since:

$$\varphi(2^k) = 2^k \cdot \left(1 - \frac{1}{2}\right) = \frac{2^k}{2}.$$

On the other hand, $n = 1$ does not work because $\varphi(1) = 1 \neq \frac{1}{2}$.

Now, consider $n > 2$ and $n$ not a power of two .

Case 1: $2 \mid n$
Since $n$ is not a power of two, there is a prime $p \neq 2$ that divides $n$ and in this case:

$$\varphi(n) = \frac{n}{2} \prod_{\substack{p|n \\ p\neq 2}} \left(1 - \frac{1}{p}\right).$$

Since $\prod_{\substack{p|n \\ p\neq 2}} \left(1 - \frac{1}{p}\right) < 1$, it follows that:

$$\varphi(n) < \frac{n}{2}.$$

Contradicting our hypothesis $\varphi(n) = \frac{n}{2}$. Hence, $2 \nmid n$.

Case 2: $2 \nmid n$
If $2 \nmid n$, then all prime divisors of $n$ are odd. For each prime divisor $p$, $p - 1$ is even. Let:

$$a = \prod_{p|n} (p-1), \quad b = \prod_{p|n} p.$$

Then:

$$n\frac{a}{b} = n\frac{\prod_{p|n}(p-1)}{\prod_{p|n} p} = n\prod_{p|n}\left(1-\frac{1}{p}\right) = \varphi(n) = n/2$$

Here, $a$ is even (product of even numbers) and $b$ is odd (product of odd primes). Therefore:

$$\frac{a}{b} = \frac{1}{2}.$$

This implies:

$$2a = b,$$

which is a contradiction because $b$ is odd.

Therefore the only solution are the powers of two, $n = 2^k$. ■

2. b) The canonical factorization fo $n$ can be written as: $n = \prod_{\substack{p|n \\ p\neq 2}}$