COL 106 – Data Structures and Algorithms

Asymptotic Analysis & Program Correctness

Announcements:

(Written)

- Quiz 1 on Friday (11 - 11:50 am)
- Venue: LH 325, LH 121
- Seating plan: on Piazza (by Wednesday night)
- Reminder: Lab Quiz 1 on Saturday. (9 - 11 am
  11 am - 1 pm)

Primality Testing

Input: $\boxed{n \in \mathbb{N}}$

Output: yes if n is prime

Factoring

Check-Prime(n)

1. if ($n \leq 1$) the return("not prime")
2. for ($i = 2$; $i < n$; i++)

   if (i divides n) then return("not prime")
3. return("prime")

$O(n)$ time algorithm

Check-Prime(n)

1. if ($n \leqslant 1$) then return("not prime")
2. set $i = 2$
3. while ($i \times i \leqslant n$)

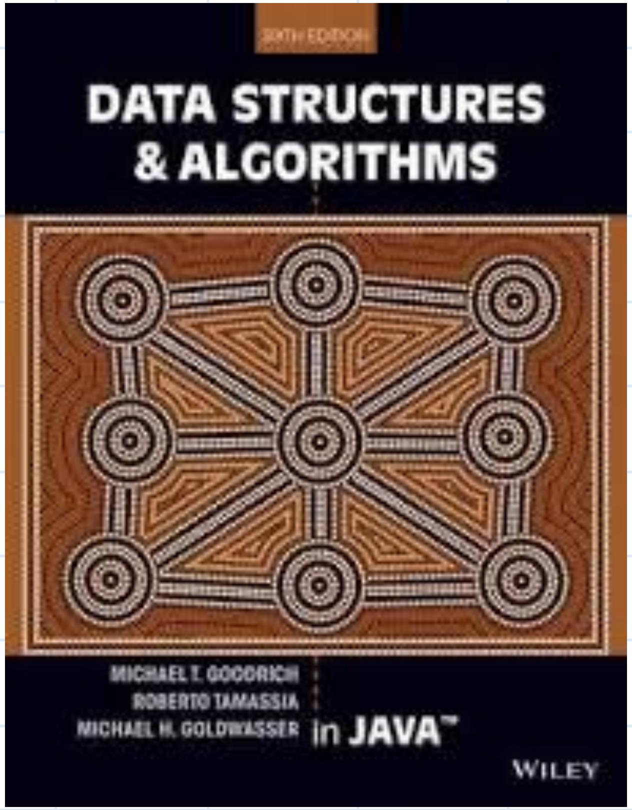
   if (i divides n) then return("not prime")

   $i = i + 1$
4. return("prime")

$O(\sqrt{n})$ algorithm

- What is the input size?
- What is the running time?

$(\lceil \log_2 n \rceil + 1)^2$ — bits



C-4.47 Communication security is extremely important in computer networks, and one way many network protocols achieve security is to encrypt messages. Typical *cryptographic* schemes for the secure transmission of messages over such networks are based on the fact that no efficient algorithms are known for factoring large integers. Hence, if we can represent a secret message by a large prime number p, we can transmit, over the network, the number $r = p \cdot q$, where $q > p$ is another large prime number that acts as the *encryption key*. An eavesdropper who obtains the transmitted number r on the network would have to factor r in order to figure out the secret message p.

Using factoring to figure out a message is hard without knowing the encryption key q. To understand why, consider the following naive factoring algorithm:

```
for (int p=2; p < r; p++)
  if (r % p == 0)
    return "The secret message is p!";
```

a. Suppose the eavesdropper's computer can divide two 100-bit integers in μs (1 millionth of a second). Estimate the worst-case time to decipher the secret message p if the transmitted message r has 100 bits.

b. What is the worst-case time complexity of the above algorithm? Since the input to the algorithm is just one large number r, assume that the input size n is the number of bytes needed to store r, that is, $n = \lfloor (\log_2 r)/8 \rfloor + 1$, and that each division takes time $O(n)$.

Greatest Common Divisors (Highest common factor)

$gcd(15, 12) = 3$

$gcd(11, 20) = 1$

$m = p \times q$

$n = p \times l$

$O(\sqrt{n})$ time

- How do you compute gcd?

input: $m, n \in \mathbb{N}$

output: $gcd(m, n)$

$O(\sqrt{a}) + O(\sqrt{b})$

$$\underbrace{\min(\log_2 x,\ \log_2 y)}$$

EuclideanGCD(a, b)

```
int  x, y, z;
x = a; y = b;
while (y > 0) {
    z = x mod y
    x = y
    y = z
}
return x;
```

$x = a$

$y = b$

input size? $\left(\lceil \log_2 a \rceil + 1 + \lceil \log_2 b \rceil + 1 \right)$

$x \geq y$

$a \quad b$

$\gcd(15, 12)$

$z = 15 \bmod 12 = 3$

$x = 12$

$y = 3$

$(x \bmod y,\ y)$

$\boxed{x \bmod y}$

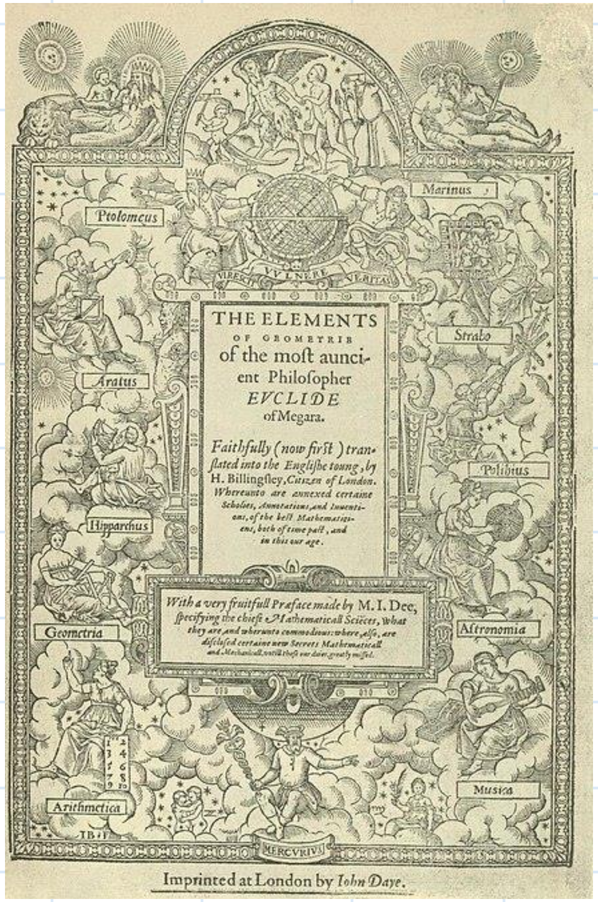
$y < x/2$

$\leq x/2$

$y \geq x/2$

$x \bmod y = (x - y)$

Running time: ?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Euclidean GCD(a,b)

```
int x, y, z;
x = a; y = b;
while (y > 0) {
    z = x mod y
    x = y
    y = z
}
return x;
```

gcd(x,y)

gcd(x,y) = gcd(a,b) = gcd(b, a mod b)

gcd(y, a mod b)

(y, x mod y)

x	a	b
y	b	a mod b

$\boxed{\log_2 a}$

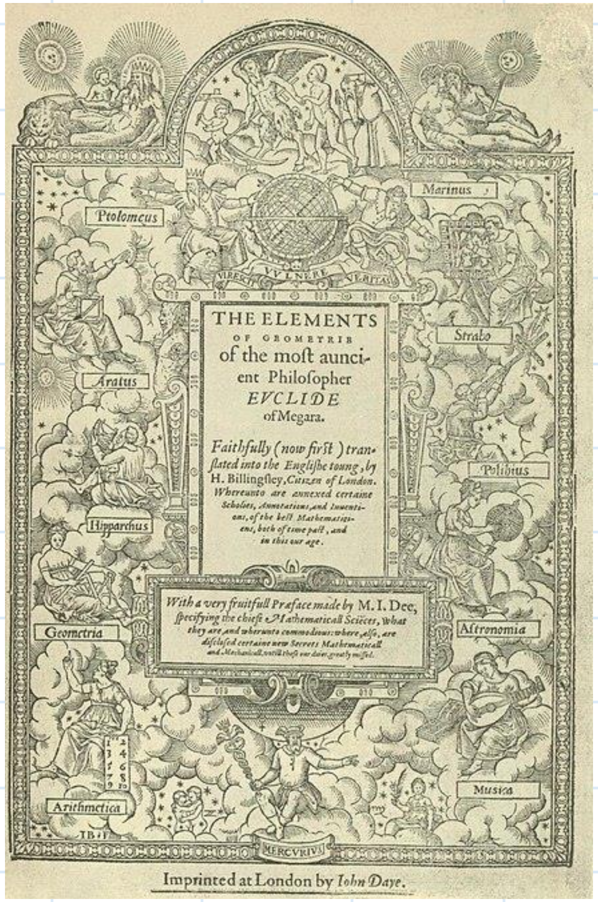
$O(\log_2 a)$ iterations

$a = lp$

$b = lq$

$a - b = l(p-q)$

Obs: If l divides a & b, then it also divides $a-b$ (assuming $a > b$)



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With a very fruitfull Praeface made by M. I. Dee, specifying the chiefe Mathematicall Sciences, what they are, and whereunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, untill these our daies, greatly missed.

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Idea: Instead of computing $\gcd(a, b)$ might as well compute $\gcd(b, a-b)$

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$\gcd(a-b, a-2b)$

$\vdots$

$\gcd(a,b) = \underbrace{\gcd(b, a \bmod b)}$ as long as $\neq 0$

$\boxed{\text{Theorem (Euclid):}}$ If $a > b > 0$ are positive integers, then if b divides a then $GCD(a,b) = b$.

Otherwise, $\gcd(a,b) = \gcd(a \bmod b, b)$

Idea: Instead of computing $gcd(a, b)$ might as well compute $gcd(b, a\ b)$

$gcd(a-b, a-2b)$

$\vdots$

$gcd(b, a \bmod b)$

as long as $\neq 0$

Correctness: Invariant condition of the algorithm:

$$gcd(x, y) = gcd(a, b)$$

Correctness via Invariants (Robert Floyd)

$P(1)$

$P(i) \Rightarrow P(i+1)$

① **Precondition:** Something that is true before the start of the loop.

$P(1)$

$P(2)$

② **Postcondition:** Something that is true after the end of the loop.

③ **Loop Invariant:** If true at step i, then is also true if the loop is executed once more.

Fend the largest element in an array

Algorithm arrayMax (A, n)

Input : array A with n elements

Output : largest element in the array

```
currentMax ← A[0];
for i ← 1 to n-1 do
    if currentMax < A[i] then
        currentMax ← A[i]
return currentMax
```

Fend the largest element in an array

i/p: Program (P, X)
o/p: Does P halt on x

Halting Problem

Algorithm arrayMax (A, n)
Input : array A with n elements
Output : largest element in the array

```
currentMax ← A[0];
for i ← 1 to n-1 do
    if currentMax < A[i] then
        currentMax ← A[i]
return currentMax
```

Precondition:
currentMax contains A[0]

Loop Invariant:
If currentMax contains maximum of A[0, ... i] & the loop is executed once more it contains max of A[0, ... i+1]

Postcondition: CurrentMax contains max of A

0, 1, 1, 2, 3, 5, 8, 13, - - - - .

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n > 1 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0\,. \end{cases}$$

Fibonacci Numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n > 1 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0\,. \end{cases}$$

```
function fib1(n)
if n = 0:  return 0
if n = 1:  return 1
return fib1(n-1) + fib1(n-2)
```

- Running time of fib1?

Fibonacci Numbers

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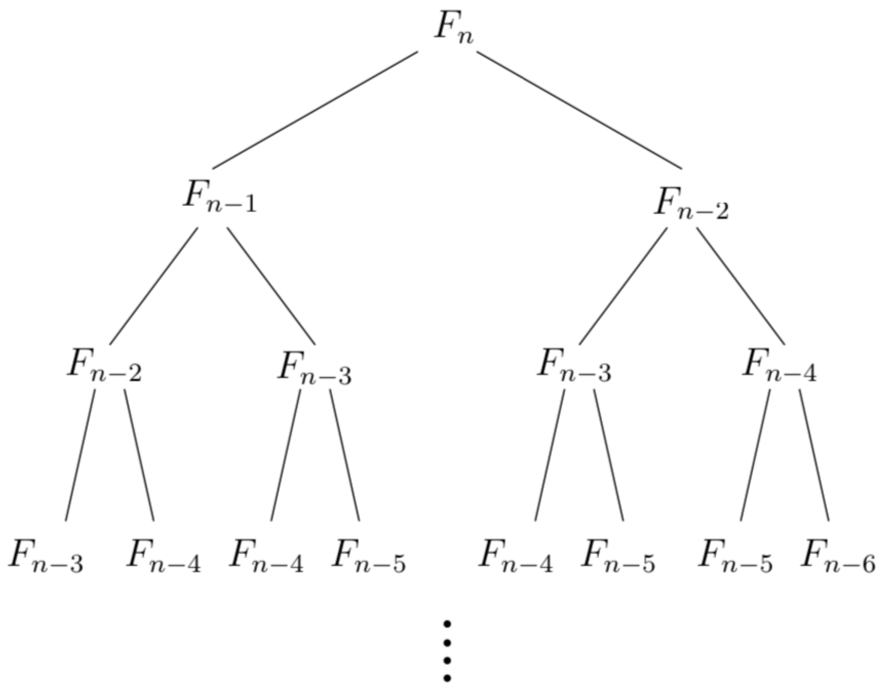


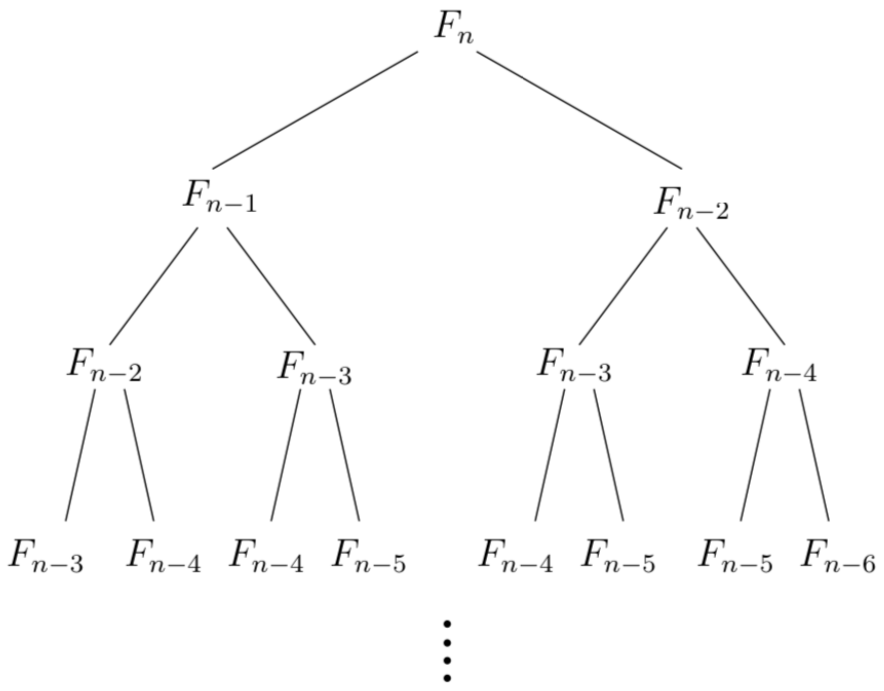
$$T(n) = T(n-1) + T(n-2)$$

- Running time of fib1?

How fast does $T(n)$ grow?

Fibonacci Numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n > 1 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0\,. \end{cases}$$




```
function fib1(n)
if n = 0:  return 0
if n = 1:  return 1
return fib1(n − 1) + fib1(n − 2)
```

$$T(n) = T(n-1) + T(n-2)$$

- Running time of fib1?

$$\left(\frac{3}{2}\right)^n \leq T(n) \leq 2^n$$

A better algorithm

```
function fib2(n)
if n = 0 return 0
create an array f[0...n]
f[0] = 0, f[1] = 1
for i = 2...n:
    f[i] = f[i-1] + f[i-2]
return f[n]
```

- Running time?
- Can you do better?

Question:

What is the largest Fibonacci number that can fit into a word in 64-bit machine?
128-bit

Algorithm Binary Search (A, n, T)

Inputs: A is an array of n sorted elements.
T is the element value we want to locate

Output: index of the array element with value T.
if it is present, else return NIL.

BINARY SEARCH

1	2	3	4	5	6	7	8	9	10	11
2	5	8	12	16	23	46	59	60	150	200
					23					

ITERATIVE-BINARY-SEARCH (A, v)

```
low = A[1]
high = A[A.length]
while low ≤ high
    mid = ⌊(low + high)/2⌋
    if v == A[mid]
        return mid
    elseif v > A[mid]
        low = mid + 1
    else
        high = mid - 1
return NIL
```

- What is the time Complexity of Binary Search? $\underline{O(\log n)}$
- Does it change with the data structure used to implement A?
- Loop Invariant?