1. Direct Grometric Model

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \tag{1}$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

2. Inverse Geometric Model

$$x^2 + y^2 = l_1^2 \cos^2\theta_1 + l_2^2 \cos^2(\theta_1 + \theta_2) + 2l_1 l_2 \cos\theta_1 \cos(\theta_1 + \theta_2) \tag{2}$$

$$+ l_1^2 \sin^2\theta_1 + l_2^2 \sin^2(\theta_1 + \theta_2) + 2l_1 l_2 \sin\theta_1 \sin(\theta_1 + \theta_2)$$

$$= l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2$$

$$\Rightarrow \cos\theta_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$\theta_2 = \cos^{-1}\left[\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right]$$

3. Jacobian

$$\dot{x} = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \tag{3}$$

$$\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{4}$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad \textit{where } J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \tag{5}$$

4. Probably don't need this

$$P = \begin{bmatrix} x \\ y \end{bmatrix} = l_1 \underline{u}_1 + l_1 \underline{u}_2 \tag{6}$$

$$\underline{u}_1 = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} \qquad \dot{\underline{u}}_1 = \dot{\theta} \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} \tag{7}$$

Where $\underline{v}_1 = \underline{u}_1^T$, or $\underline{v}_1 = E\underline{u}_1$ and E is a rotation matrix such that $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Similarly,

$$\underline{u}_2 = \begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix} \qquad \dot{\underline{u}}_2 = (\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -\sin(\theta_1 + \theta_2) \\ \cos(\theta_1 + \theta_2) \end{bmatrix} \tag{8}$$

Where $\underline{v}_2 = \underline{u}_2^T$, or $\underline{v}_2 = E\underline{u}_2$

5. Cartesian Stiffness Matrix

Prerequisites to finding Cartesian Stiffness Matrix:

- From Eqn(5) we have $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$
- If position is denoted by vector $\underline{p}$ and generalised displacement is $\underline{q} = \underline{\theta}$, we have $\delta\underline{p} = J\delta\underline{\theta} \Rightarrow \delta\underline{\theta} = \delta\underline{p}J^{-1}$
- Motor torque vector $\tau = K_\theta \delta\underline{\theta}$, where K_θ is the**joint stiffness matrix**.
- According to the Principle of Virtual Work, $\underline{W}^T\delta\underline{p} = \tau^T\delta\underline{\theta}$
- From the above equations, we have $\underline{W}^T\delta\underline{p} = \delta\underline{\theta}^T K_\theta^T \delta\underline{\theta}$
- Assuming that $\det(J) \neq 0$, we have $\underline{W} = J^{-T}K_\theta J^{-1}\delta\underline{p}$
- Where **Cartesian Stiffness Matrix** $K_x = J^{-T}K_\theta J^{-1}$
- And force applied on end effector $\underline{W} = K_x\delta\underline{p}$
- **If only stiffness in links is taken into account, then** $K_\theta = I_2$ **and** $\mathbf{K_x} = \mathbf{J^{-T}J^{-1}}$

We know from Eqn(5) that

$$J = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1 + \theta_2) & -l_2\sin(\theta_1 + \theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2) & l_2\cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\Rightarrow \quad J^T = \begin{bmatrix} -l1\sin\theta_1 - l_2\sin(\theta_1 + \theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2) \\ -l_2\sin(\theta_1 + \theta_2) & l_2\cos(\theta_1 + \theta_2) \end{bmatrix}$$