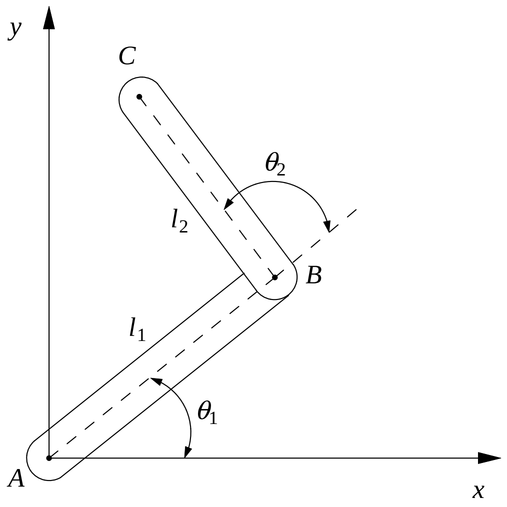
1. 2R Mechanism

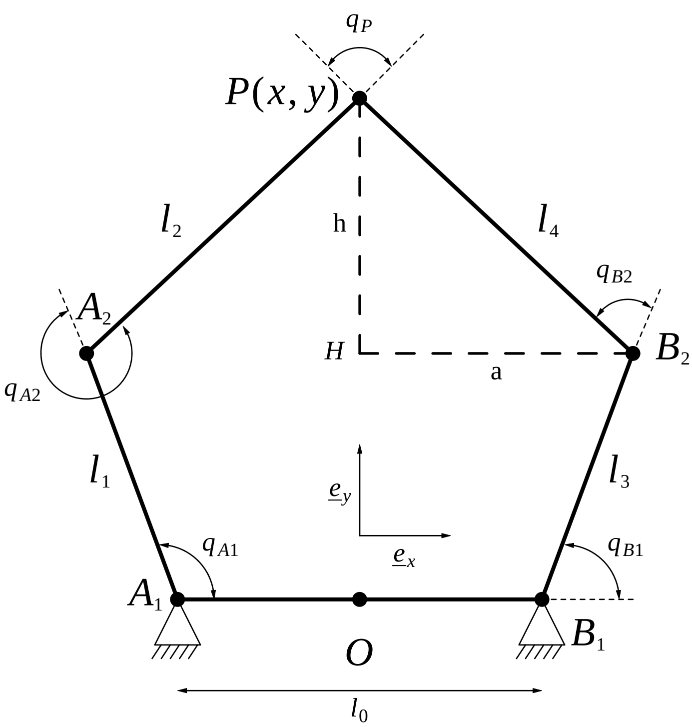


1.1. Direct Geometric Model

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \qquad y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \tag{1}$$

2. 5-bar Mechanism

2.1. Defining the problem



Consider the 5 bar mechanism shown above such that the viewer is seeing the top view of the mechanism. It consits of two active links of length l_1 and l_3, two passive links l_2 and l_4, as well as a fixed link l_0. These links are connected with revolute joints A_1, A_2, B_1, B_2, and P. The origin of the mechanism is at O and the end effector is assumed to be situated at P.

Also consider a point H positioned on the y-axis, and at the same height as A_1 and A_2. This point is part of a right angle triangle ΔPHB_2 of height h and base a.

2.2. Direct Geometric Model

The position $P(x, y)$ can be expressed in vector form as the sum of all the vectors from$O$ to P.

$$\overline{OP} = \overline{OB_1} + \overline{B_2H} + \overline{HP} \tag{2}$$

First, in order to determine the co-ordinates of point H,

$$\overline{OA_2} = l_1 \begin{bmatrix} \cos q_{A1} - \frac{d}{2} \\ \sin q_{A1} \end{bmatrix} \qquad \overline{OB_2} = l_3 \begin{bmatrix} \cos q_{B1} + \frac{d}{2} \\ \sin q_{B1} \end{bmatrix} \tag{3}$$

So the co-ordinates of H are:

$$\bar{H} = \frac{1}{2} \begin{bmatrix} l_1 \cos q_{A1} + l_3 \cos q_{B1} \\ l_1 \sin q_{A1} + l_3 \sin q_{B1} \end{bmatrix} \tag{4}$$

$\overline{B_2H}$ can be expressed as follows:

$$\overline{B_2H} = \begin{bmatrix} \frac{1}{2}\left(l_1 \cos q_{A1} + l_3 \cos q_{B1}\right) - l_3 \cos q_{B1} - \frac{lo}{2} \\ \frac{1}{2}\left(l_1 \sin q_{A1} + l_3 \sin q_{B1}\right) - l_3 \sin q_{B1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(l_1 \cos q_{A1} - l_3 \cos q_{B1}\right) - \frac{lo}{2} \\ \frac{1}{2}\left(l_1 \sin q_{A1} - l_3 \sin q_{B1}\right) \end{bmatrix} \tag{5}$$

$\overline{HP}$ can be found as:

$$\overline{HP} = \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\overline{B_2H} = \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} \frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{l_0}{2} \end{bmatrix} \tag{6}$$

$\overline{OB_1}$ and $\overline{B_1B_2}$ can be trivially found as,

$$\overline{OB_1} = \begin{bmatrix} \frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{B_1B_2} = \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} \tag{7}$$

Adding all of these equations, we obtain OP,

$$\overline{OP} = \begin{bmatrix} \frac{lo}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} - l_3 \cos q_{B1}) - \frac{lo}{2} \\ \frac{1}{2}(l_1 \sin q_{A1} - l_3 \sin q_{B1} \end{bmatrix} + \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} \frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{l_0}{2} \end{bmatrix}$$

$$\therefore \quad \boxed{\overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) + \tan^{-1}\left(\frac{h}{a}\right)\frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \sin q_{A1} + l_3 \sin q_{B1}) + \tan^{-1}\left(\frac{h}{a}\right)\left[\frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{lo}{2}\right] \end{bmatrix}} \tag{8}$$

2.3. Inverse Geometric Model

The Inverse Geometric Model can be found by considering each half of the 5 bar mechanism separately.

2.3.1. Left Half

First, let us find q_{A2} by considering the left part of the mechanism where,

$$\overline{OP} = \overline{OA_1} + \overline{A_1A_2} + \overline{A_2P} \tag{9}$$

$$\overline{OA_1} = \begin{bmatrix} -\frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{A_1A_2} = \begin{bmatrix} l_1 \cos q_{A1} \\ l_1 \sin q_{A1} \end{bmatrix} \qquad \overline{A_2P} = \begin{bmatrix} l_2 \cos(q_{A1} + q_{A2}) \\ l_2 \sin(q_{A1} + q_{A2}) \end{bmatrix} \tag{10}$$

$$\Rightarrow \overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{l_0}{2} + l_1 \cos q_{A1} + l_2 \cos(q_{A1} + q_{A2}) \\ l_1 \sin q_{A1} + l_2 \sin(q_{A1} + q_{A2}) \end{bmatrix} \tag{11}$$

Consider the first row of the matrix in the above equation,

$$\cos(q_{A1} + q_{A2}) = \frac{x - \frac{lo}{2} - l_1 \cos q_{A1}}{l_2} \tag{12}$$

Similary, the second row can be written as,

$$\sin(q_{A1} + q_{A2}) = \frac{y - l_1 \sin q_{A1}}{l_2} \tag{13}$$

Dividing these, we get:

$$\tan(q_{A1} + q_{A2}) = \frac{2y - 2l_1 \sin q_{A1}}{2x + l_0 - 2l_1 \cos q_{A1}} \tag{14}$$

$$\therefore \boxed{q_{A1} = \tan^{-1}\left[\frac{2y - 2l_1 \sin q_{A1}}{2x + l_0 - 2l_1 \cos q_{A1}}\right] - q_{A1}} \tag{15}$$

2.3.2. Right Half

Similarly, we can find q_{B2} by considering the right part of the mechanism where,

$$\overline{OP} = \overline{OB_1} + \overline{B_2H} + \overline{HP} \tag{16}$$

$$\overline{OB_1} = \begin{bmatrix} \frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{B_1B_2} = \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} \qquad \overline{B_2P} = \begin{bmatrix} l_4 \cos(q_{B1} + q_{B2}) \\ l_4 \sin(q_{B1} + q_{B2}) \end{bmatrix} \tag{17}$$

$$\overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{l_0}{2} + l_3 \cos q_{B1} + l_4 \cos(q_{B1} + q_{B2}) \\ l_3 \sin q_{B1} + l_4 \sin(q_{B1} + q_{B2}) \end{bmatrix} \tag{18}$$

Consider the first row ot eh matrix in the above equation,

$$\cos(q_{B1} + q_{B2}) = \frac{2x - l_0 - 2l_3 \cos q_{B1}}{l_4} \tag{19}$$

Similarly, the second row can be written as,

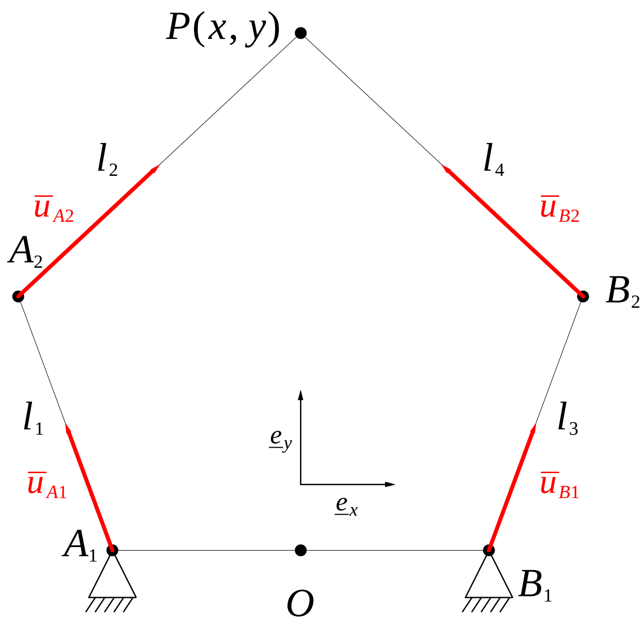
$$\sin(q_{B1} + q_{B2}) = \frac{y - l_3 \sin q_{B1}}{l_4} \tag{20}$$

Dividing these, we get:

$$\tan(q_{B1} + q_{B2}) = \frac{2y - 2l_3 \sin q_{B1}}{2x - l_0 - 2l_3 \cos q_{B1}} \tag{21}$$

$$\therefore \boxed{q_{B2} = \tan^{-1}\left[\frac{2y - 2l_3 \sin q_{B1}}{2x - l_0 - 2l_3 \cos q_{B1}}\right] - q_{B1}} \tag{22}$$

2.4. Jacobian Matrix



In order to write the Jacobian matrix, first let us rewrite the problem in terms of unit vectors $\bar{u}_{A1}$, $\bar{u}_{A2}$, $\bar{u}_{B1}$, and $\bar{u}_{B2}$. We can also rewrite $\overline{OP}$ as $\mathbf{p}$ and subsequently,

$$\overline{OP} = \overline{OA_1} + \overline{A_1A_2} + \overline{A_2P} \qquad \overline{OP} = \overline{OB_1} + \overline{B_2H} + \overline{HP} \tag{23}$$

$$\mathbf{p} = -\tfrac{l_0}{2}\,\bar{e}_x + l_1\bar{u}_{A1} + l_2\bar{u}_{A2} \qquad \mathbf{p} = \tfrac{l_0}{2}\,\bar{e}_x + l_3\bar{u}_{B1} + l_4\bar{u}_{B2} \tag{24}$$

Differentiating both of these equations with respect to time, we get

$$\dot{\mathbf{p}} = l_1\dot{q}_{A_1}E\bar{u}_{A1} + l_2\dot{q}_{A_2}E\bar{u}_{A2} \qquad \dot{\mathbf{p}} = l_3\dot{q}_{B_1}U\bar{u}_{B1} + l_4\dot{q}_{B_2}E\bar{u}_{B2} \tag{25}$$

Where $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the rotation matrix.

Since $\bar{u}_{A2}$ and $\bar{u}_{B2}$ are the two unit vectors associated with the passive links. Multiplying the above equations by the transpose of these vectors, we obtain:

$$\bar{u}_{A2}^T\dot{\mathbf{p}} = l_1\dot{q}_{A_1}\bar{u}_{A2}^TE\bar{u}_{A1} \;+\; l_2\dot{q}_{A_2}\bar{u}_{A2}^TE\bar{u}_{A2} \qquad \bar{u}_{B2}^T\dot{\mathbf{p}} = l_3\dot{q}_{B_1}\bar{u}_{B2}^TE\bar{u}_{B1} \;+\; l_4\dot{q}_{B_2}\bar{u}_{B2}^TE\bar{u}_{B2} \tag{26}$$

The product of a vector and its transpose is equal to 0. Hence the second terms in both of the above equations can be eliminated. By combining the remaining terms, we can form a the matrices:

$$\begin{bmatrix} \bar{u}_{A2}^T \\ \bar{u}_{B2}^T \end{bmatrix}\dot{\mathbf{p}} = \begin{bmatrix} l_1\bar{u}_{A2}^TE\bar{u}_{A1} & 0 \\ 0 & l_3\bar{u}_{B2}^TE\bar{u}_{B1} \end{bmatrix}\begin{bmatrix} \dot{q}_{A_1} \\ \dot{q}_{B_2} \end{bmatrix} \tag{27}$$

If we substitute $A = \begin{bmatrix} \bar{u}_{A2}^T \\ \bar{u}_{B2}^T \end{bmatrix}$ and $B = \begin{bmatrix} l_1\bar{u}_{A2}^TE\bar{u}_{A1} & 0 \\ 0 & l_3\bar{u}_{B2}^TE\bar{u}_{B1} \end{bmatrix}$ then we have,

$$A\dot{\mathbf{p}} = B\begin{bmatrix} \dot{q}_{A_1} \\ \dot{q}_{B_2} \end{bmatrix} \;\Rightarrow\; \dot{\mathbf{p}} = A^{-1}B\begin{bmatrix} \dot{q}_{A_1} \\ \dot{q}_{B_2} \end{bmatrix} \tag{28}$$

Where Jacobian Matrix $J = A^{-1}B$

2.4.1. Calculating the Inverse Jacobian Matrix

$$A = \begin{bmatrix} \cos(q_{A1}+q_{A2}) & \sin(q_{A1}+q_{A2}) \\ \cos(q_{B1}+q_{B2}) & \sin(q_{B1}+q_{B2}) \end{bmatrix} \tag{29}$$

$$\Rightarrow \det(A) = \sin(q_{B1}+q_{B2})\cos(q_{A1}+q_{A2}) - \cos(q_{B1}+q_{B2})\sin(q_{A1}+q_{A2}) \tag{30}$$

$$= sin(q_{B1}+q_{B2}-q_{A1}-q_{A2})$$

$$adj(A) = \begin{bmatrix} \sin(q_{B1}+q_{B2}) & -\sin(q_{A1}+q_{A2}) \\ -\cos(q_{B1}+q_{B2}) & \cos(q_{A1}+q_{A2}) \end{bmatrix} \tag{31}$$

$$A^{-1} = \frac{1}{\det(A)}\, adj(A)$$

$$\Rightarrow A^{-1} = \frac{1}{sin(q_{B1}+q_{B2}-q_{A1}-q_{A2})} \begin{bmatrix} \sin(q_{B1}+q_{B2}) & -\sin(q_{A1}+q_{A2}) \\ -\cos(q_{B1}+q_{B2}) & \cos(q_{A1}+q_{A2}) \end{bmatrix} \tag{32}$$

2.4.2. Calculating the Forward Jacobian Matrix

Now, we find the values of $l_1\bar{u}_{A2}^T E\bar{u}_{A1}$ and $l_3\bar{u}_{B2}^T E\bar{u}_{B1}$.

$$l_1\bar{u}_{A2}^T E\bar{u}_{A1} = l_1[\cos(q_{A1}+q_{A2}) \quad \sin(q_{A1}+q_{A2})] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos q_{A1} \\ \sin q_{A1} \end{bmatrix} \tag{33}$$

$$= l_1 \sin(q_{A1}+q_{A2}-q_{A1}) \;\; = l_1 \sin q_{A2}$$

$$l_3\bar{u}_{B2}^T Eub = l_3[\cos(q_{B1}+q_{B2}) \quad \sin(q_{B1}+q_{B2})] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos q_{B1} \\ \sin q_{B1} \end{bmatrix} \tag{34}$$

$$= l_3 \sin(q_{B1}+q_{B2}-q_{B1}) \;\; = l_3 \sin q_{B2}$$

$$\Rightarrow B = \begin{bmatrix} l_1 \sin q_{A2} & 0 \\ 0 & l_3 \sin q_{B2} \end{bmatrix} \tag{35}$$

2.4.3. Calculating the Jacobian Matrix

We know that $J = A^{-1}B$

$$\boxed{J = \frac{1}{sin(q_{B1} + q_{B2} - q_{A1} - q_{A2}} \begin{bmatrix} l_1 \sin q_{A2} \sin(q_{B1} + q_{B2}) & -l_3 \sin(q_{A1} + q_{A2}) \sin q_{B2} \\ -l_1 \sin q_{A2} \cos(q_{B1} + q_{B2}) & l_3 \cos(q_{A1} + q_{A2}) \sin q_{B2} \end{bmatrix}} \quad (36)$$

2.5. Stiffness Matrix

The Stiffness Matrix is usually written as $K_x = J^{-T} K_\theta J^{-1}$. However, since Joint Stiffness K_θ is being ignored, the Cartesian Stiffness Matrix can be written as $K_x = J^{-T} J^{-1}$. Where,

$$J = \frac{1}{sin(q_{B1} + q_{B2} - q_{A1} - q_{A2}} \begin{bmatrix} l_1 \sin q_{A2} \sin(q_{B1} + q_{B2}) & -l_3 \sin(q_{A1} + q_{A2}) \sin q_{B2} \\ -l_1 \sin q_{A2} \cos(q_{B1} + q_{B2}) & l_3 \cos(q_{A1} + q_{A2}) \sin q_{B2} \end{bmatrix} \quad (37)$$