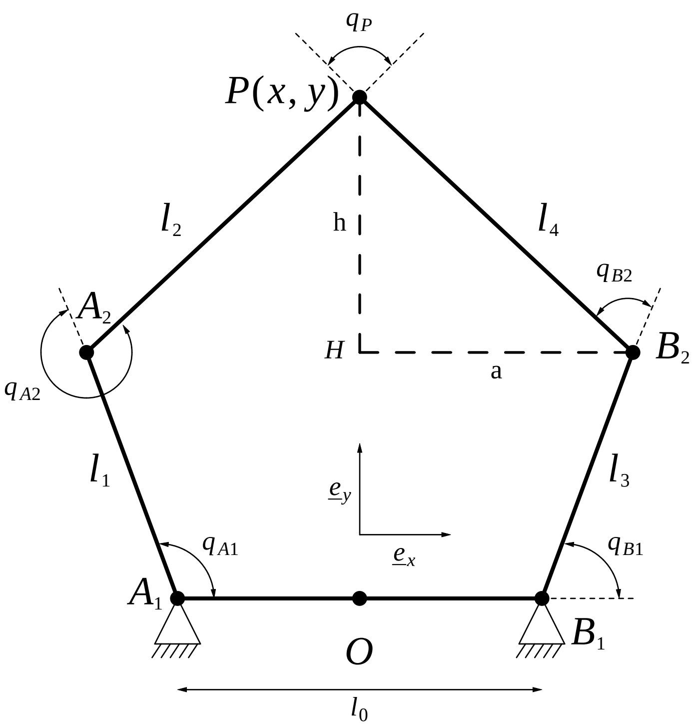
# 1. 2R Mechanism

# 2. 5-bar Mechanism

## 2.1. Defining the problem




Consider the 5 bar mechanism shown above such that the viewer is seeing the top view of the mechanism. It consits of two active links of length $l_1$ and $l_3$, two passive links $l_2$ and $l_4$, as well as a fixed link $l_0$. These links are connected with revolute joints $A_1$, $A_2$, $B_1$, $B_2$, and $P$. The origin of the mechanism is at $O$ and the end effector is assumed to be situated at $P$.

Also consider a point $H$ positioned on the y-axis, and at the same height as $A_1$ and $A_2$. This point is part of a right angle triangle $\Delta PHB_2$ of height $h$ and base $a$.

## 2.2. Direct Geometric Model

The position $P(x, y)$ can be expressed in vector form as the sum of all the vectors from$O$ to $P$.

$$\overline{OP} = \overline{OB_1} + \overline{B_2H} + \overline{HP} \tag{1}$$

First, in order to determine the co-ordinates of point $H$,

$$\overline{OA_2} = l_1 \begin{bmatrix} \cos q_{A1} - \frac{d}{2} \\ \sin q_{A1} \end{bmatrix} \qquad \overline{OB_2} = l_3 \begin{bmatrix} \cos q_{B1} + \frac{d}{2} \\ \sin q_{B1} \end{bmatrix} \tag{2}$$

So the co-ordinates of $H$ are:

$$\bar{H} = \frac{1}{2}\begin{bmatrix} l_1 \cos q_{A1} + l_3 \cos q_{B1} \\ l_1 \sin q_{A1} + l_3 \sin q_{B1} \end{bmatrix} \tag{3}$$

$\overline{B_2H}$ can be expressed as follows:

$$\overline{B_2H} = \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - l_3 \cos q_{B1} - \frac{lo}{2} \\ \frac{1}{2}(l_1 \sin q_{A1} + l_3 \sin q_{B1}) - l_3 \sin q_{B1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} - l_3 \cos q_{B1}) - \frac{lo}{2} \\ \frac{1}{2}(l_1 \sin q_{A1} - l_3 \sin q_{B1} \end{bmatrix} \tag{4}$$

$\overline{HP}$ can be found as:

$$\overline{HP} = \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\overline{B_2H} = \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} \frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{l_0}{2} \end{bmatrix} \tag{5}$$

$\overline{OB_1}$ and $B_1B_2$ can be trivially found as,

$$\overline{OB_1} = \begin{bmatrix} \frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{B_1B_2} = \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} \tag{6}$$

Adding all of these equations, we obtain$OP$,

$$\overline{OP} = \begin{bmatrix} \frac{lo}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} - l_3 \cos q_{B1}) - \frac{lo}{2} \\ \frac{1}{2}(l_1 \sin q_{A1} - l_3 \sin q_{B1} \end{bmatrix} + \tan^{-1}\left(\frac{h}{a}\right)\begin{bmatrix} \frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{l_0}{2} \end{bmatrix}$$

$$\therefore \boxed{\overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) + \tan^{-1}\left(\frac{h}{a}\right)\frac{1}{2}(l_3 \sin q_{B1} - l_1 \sin q_{A1}) \\ \frac{1}{2}(l_1 \sin q_{A1} + l_3 \sin q_{B1}) + \tan^{-1}\left(\frac{h}{a}\right)\left[\frac{1}{2}(l_1 \cos q_{A1} + l_3 \cos q_{B1}) - \frac{lo}{2}\right] \end{bmatrix}} \tag{7}$$

## 2.3. Inverse Geometric Model

The Inverse Geometric Model can be found by considering each half of the 5 bar mechanism separately.

### 2.3.1. Left Half

First, let us find $q_{A2}$ by considering the left part of the mechanism where,

$$\overline{OP} = \overline{OA_1} + \overline{A_1A_2} + \overline{A_2P} \tag{8}$$

$$\overline{OA_1} = \begin{bmatrix} -\frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{A_1A_2} = \begin{bmatrix} l_1 \cos q_{A1} \\ l_1 \sin q_{A1} \end{bmatrix} \qquad \overline{A_2P} = \begin{bmatrix} l_2 \cos(q_{A1} + q_{A2}) \\ l_2 \sin(q_{A1} + q_{A2}) \end{bmatrix} \tag{9}$$

$$\Rightarrow \overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{l_0}{2} + l_1 \cos q_{A1} + l_2 \cos(q_{A1} + q_{A2}) \\ l_1 \sin q_{A1} + l_2 \sin(q_{A1} + q_{A2}) \end{bmatrix} \tag{10}$$

Consider the first row of the matrix in the above equation,

$$\cos(q_{A1} + q_{A2}) = \frac{x - \frac{l_0}{2} - l_1 \cos q_{A1}}{l_2} \tag{11}$$

Similary, the second row can be written as,

$$\sin(q_{A1} + q_{A2}) = \frac{y - l_1 \sin q_{A1}}{l_2} \tag{12}$$

Dividing these, we get:

$$\tan(q_{A1} + q_{A2}) = \frac{2y - 2l_1 \sin q_{A1}}{2x + l_0 - 2l_1 \cos q_{A1}} \tag{13}$$

$$\therefore \boxed{q_{A1} = \tan^{-1}\left[\frac{2y - 2l_1 \sin q_{A1}}{2x + l_0 - 2l_1 \cos q_{A1}}\right] - q_{A1}} \tag{14}$$

### 2.3.2. Right Half

Similarly, we can find $q_{B2}$ by considering the right part of the mechanism where,

$$\overline{OP} = \overline{OB_1} + \overline{B_2H} + \overline{HP} \tag{15}$$

$$\overline{OB_1} = \begin{bmatrix} \frac{l_0}{2} \\ 0 \end{bmatrix} \qquad \overline{B_1B_2} = \begin{bmatrix} l_3 \cos q_{B1} \\ l_3 \sin q_{B1} \end{bmatrix} \qquad \overline{B_2P} = \begin{bmatrix} l_4 \cos(q_{B1} + q_{B2}) \\ l_4 \sin(q_{B1} + q_{B2}) \end{bmatrix} \tag{16}$$

$$\overline{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{l_0}{2} + l_3 \cos q_{B1} + l_4 \cos(q_{B1} + q_{B2}) \\ l_3 \sin q_{B1} + l_4 \sin(q_{B1} + q_{B2}) \end{bmatrix} \tag{17}$$

Consider the first row ot eh matrix in the above equation,

$$\cos(q_{B1} + q_{B2}) = \frac{2x - l_0 - 2l_3 \cos q_{B1}}{l_4} \tag{18}$$

Similarly, the second row can be written as,

$$\sin(q_{B1} + q_{B2}) = \frac{y - l_3 \sin q_{B1}}{l_4} \tag{19}$$

Dividing these, we get:

$$\tan(q_{B1} + q_{B2}) = \frac{2y - 2l_3 \sin q_{B1}}{2x - l_0 - 2l_3 \cos q_{B1}} \tag{20}$$

$$\therefore \quad \boxed{q_{B2} = \tan^{-1}\left[\frac{2y - 2l_3 \sin q_{B1}}{2x - l_0 - 2l_3 \cos q_{B1}}\right] - q_{B1}} \tag{21}$$

## 2.4. Jacobian Matrix