Optimal portfolios using linear programming models

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Christos Papahristodoulou
Mälardalen University

Optimal portfolios using Linear Programming models

Christos Papahristodoulou[1]
Mälardalen University, Västerås, Sweden
Erik Dotzauer
Mälardalen University, Västerås, Sweden

The classical Quadratic Programming (QP) formulation of the well-known portfolio selection problem has traditionally been regarded as cumbersome and time consuming. This paper formulates two additional models, (i) maximin, and (ii) minimization of mean absolute deviation. Data from 67 securities over 48 months are used to examine to what extent all three formulations provide similar portfolios. As expected, the maximin formulation yields the highest return and risk, while the QP formulation provides the lowest risk and return, which also creates the efficient frontier. The minimization of mean absolute deviation is close to the QP formulation. When the expected returns are confronted with the true ones at the end of a six months period, the maximin portfolios seem to be the most robust of all.

Keywords: Linear Programming; optimal portfolios; return and risk

[1] Correspondence: Christos Papahristodoulou, Mälardalens Business School, Mälardalen University. P.O. Box 883, SE-721 28 Västerås, Sweden.
E-mail: Christos.Papahristodoulou@mdh.se

Introduction

Expected return and risk are the most important parameters with regard to optimal portfolios. Two well-known approaches to formulate optimal portfolios are (i) risk minimization, given some minimum return, and (ii) return maximization, given the maximum risk investors wish to bear. These formulations might not necessarily lead to efficient portfolios though. It is possible to find other efficient portfolios that yield higher expected return for the same risk, or lower risk for the same expected return.

In this paper we will formulate the well-known portfolio problem as the classical (a) Quadratic Programming (QP), and compare it with (b) maximin, and (c) minimization of absolute deviation formulations. All formulations will then be tested using 67 stocks from the Swedish Stock Exchange. The models will also be compared with a simple utility function and with out of sample data as well to investigate their true performance.

(a) Markowitz Quadratic Programming

Harry Markowitz[1] was the first to apply variance or standard deviation as a measure of risk. His classical formulation is the following:

$$\mathbf{min} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

s.t. the following conditions:

$$\sum_{j=1}^{n} \bar{r}_j x_j \geq \alpha B, \qquad (a_1)$$

$$\sum_{j=1}^{n} x_j = B, \qquad (a_2)$$

$$0 \leq x_j \leq u_j, \; j = 1, ..., n, \qquad (a_3)$$

with i and j securities; (n is the number of securities).

$\sigma_{ij} = \frac{1}{T} \sum_{t=1}^{T} (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$, is the covariance of these securities;

r_{jt} is the per krona return, invested in security j over period t;

$\bar{r}_j$ is the average return in security j over the entire period T;

x_j is the portfolio allocation of security j. These are the variables of the problem and should not exceed an upper bound u_j;

α is the minimum (expected) return required by a particular investor;

B is the total budget that is invested in portfolio.

This classical model is always valid given two important assumptions: (a) the expected return is multivariate normally distributed; (b) the investor is risk averter and always prefers lower risk.

Notice that constraint (a_3) does not allow short selling of securities. If that constraint is excluded and short selling is allowed, a different solution will be obtained, increasing perhaps the number of securities in the portfolio.

Moreover, the following simple example shows that the minimization of the variance-covariance matrix might lead to inefficient portfolios, unless one sets explicit an expected return, high enough. Portfolio X, over a specific period, yields a return of either 8% (even quarters) or of 16% (odd quarters). Portfolio Y on the other hand, over the same period, yields returns of either 7% or 8%. Portfolio X should be not selected, if $\alpha < 7\%$, because Y has lower variance, despite the fact that X outperforms Y in all quarters!

Another problem with the Markowitz classical formulation is its complexity. Since the objective function is quadratic, it might take some time to find optimal solutions with a large number of securities. For instance, for 300 securities, we must calculate a variance-covariance matrix of $(n(n+1))/2$ = 44,850 combinations. The timing issue however, is not crucial anymore, because modern computers and global optimisation packages have shortened the solution time dramatically.

On the other hand, even if the calculations are not cumbersome, the implementation of the optimal solution is demanding. In reality, one is often satisfied with local minima, or sub optimal solutions. If for instance the number of securities is larger than 500, there might be up to 200 of them that will be included in the portfolio. That forces the investor to allocate a part of his budget into a large number of small blocks of shares. Given the transaction costs, it might be unprofitable to split the budget into many small blocks of shares. If we reformulate the problem as integer (such as a minimum block of 100 shares), the difficulty increases dramatically. Experts argue that Integer Quadratic Programs with more than 50 securities might be very difficult to solve! Linear Programming approximations on the other hand, are more suitable to treat such modifications.

(b) Maximin formulation

The standard return and risk formulation, as above, regards variance as a measure of risk volatility. There are many researchers (and traders as well) who question if the covariance matrix σ is an appropriate measure of risk. They assume that the normal investor's view regarding risk is not symmetric. Very often, a small loss is enough to make somebody very sad. On the other hand, the profit must be considerably high in order to make the investor very happy. This implies that the Markowitz classical model should be considered as an approximation to rather complex problems that all investors face.

An alternative formulation is to maximize the minimum return required by the investor. According to Young[2] such a formulation, based on monthly returns on the stock indices from eight countries, from January 1991 until December 1995, as well as from a simulation study, performs similarly with the classical Markowitz model.

In addition, Young argues that, when data is log-normally distributed, or skewed, the maximin formulation might be a more appropriate method, compared to the classical minimization of variance, which is optimal for normally distributed data. The maximin formulation might also be preferable, if the portfolio optimisation problem involves a large number of decision variables, including integer ones, or if the utility function is more risk averse than it is implied by the classical minimization of variance.

To formulate the problem as a maximin, we need an additional variable $Z \geq 0$, which is defined as the minimum return for every period. Notice that Z is not decided ex ante, but it will be a part of the optimal solution and might differ from the investor's explicit required return, α. The objective function is then to maximize that minimum return.

Regarding the constraints, one might assume that every period's return will be at least equal to Z. For period t, this constraint can be formulated as:

$$\sum_{j=1}^{n} r_{jt} x_j \geq Z \,, \qquad (b_1)$$

where, as before, r_{jt} is the return for security j over period t.

This is obviously possible if there were not additional constraints, such as (a_1), (a_2) and (a_3). Moreover, it is easy to show that it is also possible even if these constraints are included. If for instance, α is rather low, Z would be non-negative, and a feasible solution would be obtained.

To summarise, define the following Linear Program:

max Z

s.t. (a_1) - (a_3), (b_1) and $Z \geq 0$.

The main point with such a formulation is that all these constraints capture the interesting "downside" risk in portfolios volatility. We simply do not allow this "downside" risk to be below Z. "Upside" risk on the other hand, is not of interest and is free to vary.

This formulation would lead to an infeasible solution though, if the sum of the shares during a certain period yields a negative return. In that case, no portfolio with positive shares would exist to provide the minimum return during that period. One possibility to achieve a feasible solution then, would be to allow Z free, i.e. to accept negative minimum returns during a specific period as well, which might not be satisfactory. Another alternative to repair infeasibility is to apply fuzzy optimisation methods as proposed by León, Liern and Vercher[3].

(c) Mean absolute deviation minimization

Another alternative to simplify the Markowitz classic formulation is to use the absolute deviation as a risk measure. Konno & Yamazaki[4] and later Speranza[5], Mansini & Speranza[6] and Rudolf, Wolter & Zimmermann[7] formulated similar models.

According to Konno & Yamazaki, if the return is multivariate normally distributed, the minimization of the Mean Absolute Deviation (MAD) provides similar results as the classical Markowitz formulation. Also, according to Rudolf, Wolter & Zimmermann, minimization of MAD, or of the absolute downside deviation, is equivalent to expected utility maximization under risk aversion. Mansini & Speranza formulated a general mean semi-absolute deviation model, and developed a Mixed Integer LP algorithm to solve a relatively small problem (with less than 20 securities), traded at the Milan Stock Exchange.

As is known, MAD is defined as:

$$|\sigma| = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt}-\bar{r}_j)x_j\right| .$$

This function replaces the Markowitz variance-covariance objective function and will be minimized.

Since this objective function is not linear, we must first linearize it. We are going to follow Konno & Yamazaki in the transformation procedure.

We define first all $Y_t \geq 0$ variables, $t = 1, \dots, T$. These Y_t variables can be interpreted as linear mappings of the non-linear $\left|\sum_{j=1}^{n}(r_{jt}-\bar{r}_j)x_j\right|$. Notice that $(r_{jt}-\bar{r}_j)$ are the deviations parameters for share j.

Thus, the objective function is to minimize the average absolute deviation $\frac{1}{T}\sum_{t=1}^{T} Y_t$.

Now relate the objective variables Y_t, with those appeared in other constraints.

Consider the period t. Because Y_t is related to $\left|\sum_{j=1}^{n}(r_{jt}-\bar{r}_j)x_j\right|$, Y_t must fulfil

$$Y_t \geq -\sum_{j=1}^{n}\left(r_{jt} - \bar{r}_j\right)x_j \qquad (c_1)$$

and,

$$Y_t \geq \sum_{j=1}^{n}\left(r_{jt} - \bar{r}_j\right)x_j\,. \qquad (c_2)$$

Given constraints (a_1) - (a_3), constraint (c_1) ensures that Y_t will be positive if all deviations for that period are negative, and constraint (c_2) ensures the same, if all deviations are positive.

Finally, the Linear Program becomes:

$$\textbf{min}\ \frac{1}{T}\sum_{t=1}^{T} Y_t$$

s.t. (a_1) - (a_3), (c_1) - (c_2) and $Y_t \geq 0$.

In the case of mixed positive and negative deviations though, some optimal Y_t values might be zero too. However, when we continue with all remaining periods, some of Y_t will be positive, no matter if the signs for the same period alternate. This is ensured of course from the budget constraint.

This formulation has many advantages. For instance, one does not need to estimate the variance-covariance matrix and the optimal solution of this model is rather fast, even for large problems. In fact it is T that decides the number of constraints, no matter if the number of shares is very large. The number of constraints is simply $2T + 2$, if we count the budget and the return demand constraints. The optimal solution cannot include more than $2T + 2$ shares in portfolio, irrespectively if the number of shares is very large. One can therefore use T as a control variable, if one wishes to limit the number of shares in portfolio. The solution exists always, even if all possible shares happen to yield a negative return during the same period. Finally, this model can easily be reformulated as an Integer LP, to take into account fixed and variable costs or other decision variables, as recently proposed by Konno and Wijayanayake[8].

Data and computational results

All three models were tested, using monthly returns from 67 shares traded in the Stockholm Stock Exchange (SSE), between January 1997 and December 2000. In addition, six more observations (until June 2001) were left out, in order to test the true performance of these models. We stopped in December 2000, because over the last two years, as in many parts of the world, the returns fell very sharply. The average annual returns of SAX-All Share index over the selected period were 27, 10.8, 66.5 and –12 percent respectively, i.e. an average of 23%, which is considered as rather high historically. The 48 observations used might not be an adequate sample to construct optimal portfolios. Moreover, the trend in 1995 and 1996 was rather similar to 1997 and 1998. Thus, even if the portfolios with 72 observations might have been different, the returns and risks would be rather similar.

Although our ambition was to include all shares traded in Stockholm, we excluded almost 3/4 of shares which are listed nowadays, for the following reasons: (i) More than 100 companies were not listed the starting period, but entered the SSE at different dates afterwards. (ii) Since all portfolios tested are based on historical returns, 18 shares that experienced negative average returns over the examined period were also excluded. (iii) More than 50 companies left the SSE for various reasons over the examined period. (iv) The trade volume of about 10 shares was rather low, over long periods. (v) Finally, almost 20 shares are listed as A, B and sometimes, as C type. In these cases, only the most traded share was included (often the B type). It is important to mention though that none of the most traded shares was excluded.

Our investors plan to invest SEK 100,000, and request various monthly returns from, at a least 1% and up to 3%. The investors also wish that no share will receive more than 60%, of their budget. In addition, the risk-free interest rate is neglected and short selling is not allowed.

Table A1, in Appendix, depicts the descriptive statistics for 37 companies only, i.e. those that appear in portfolios at least once, as for instance the subsequent Tables A2a and A2b show. There are 27 companies that exhibit positive skewness, i.e. high upside potential, and 10 with negative skewness. Also, 25 companies had a leptokurtic distribution and only 12 were platykurtic. Thus, since the data are more skewed, we should expect the maximin model to be the most appropriate, as Young argues.

We solved first all models for portfolios consisting of 25, 50 and 67 shares to find out if the number of shares influences the speed of the solution procedure, particularly in model one (min σ). No such evidence was found. In fact, the optimal solution to that model was computed in less than 0.10 seconds, with both 25 and 67 shares. In addition, the solutions to that model were some tens of a second faster! Thus, the small number of shares and observations does not permit us to accept the common argument that the classical QP model is cumbersome and time consuming. For all these models though, as expected, the returns and/or risks were improved, when the number of shares increased.

Using all 67 shares then, we tested to what extent models two (maximin) and three (min $|\sigma|$) provide "similar" estimates to model one, as the proponents of these models argue. In Table 1, we summarize the expected returns and risks for various values of α. The weights of the shares in the optimal portfolios for four different α-values are presented in Appendix (Tables A2a and A2b). The largest number of shares (31) appears in model three for low α-values and the smallest (16) in model two for $\alpha = 3\%$.

Despite the fact that the correlation coefficients among these portfolios are very high, there are some differences. The solution to the classical QP model over a range of α-values yields the efficient frontier. That is shown clearly in Figure 1, where all portfolios are plotted. The degree of portfolio inefficiency is higher for the maximin model, since it leads to a σ that is 40-50% higher for the same return, while in the MAD model the σ is 8-13% higher. Thus, although the data favour the maximin formulation, the MAD model is closer to the QP model. Moreover, for low to moderate α-values, models two and three provide higher expected returns than required, and also higher risks. Notice that the portfolio in model two is very robust and remains unchanged, for α-values ranging from 1 to 2.25%.

Table 1 Monthly average expected returns ($\bar{R}$) and risks (σ)

α		min σ	maximin	min $\lvert\sigma\rvert$
1-1.25	$\bar{R}$	1.379	2.271	1.510
	σ	1.005	1.944	1.147
1.5	$\bar{R}$	1.500	2.271	1.510
	σ	1.010	1.944	1.147
1.75	$\bar{R}$	1.750	2.271	1.750
	σ	1.058	1.944	1.270
2	$\bar{R}$	2.000	2.271	2.000
	σ	1.158	1.944	1.348
2.25	$\bar{R}$	2.250	2.271	2.250
	σ	1.303	1.944	1.513
2.5	$\bar{R}$	2.500	2.500	2.500
	σ	1.484	2.148	1.766
2.75	$\bar{R}$	2.750	2.750	2.750
	σ	1.693	2.415	1.924
3	$\bar{R}$	3.000	3.000	3.000
	σ	1.931	2.712	2.102
	Variables	67	68	115
	Constraints	2	50	98
	Iterations	23-31	286-332	308-390

It is important to mention at this point though, that all portfolios perform better than Didner & Gerge equity fund, one of the best Swedish funds over the same period. Didner & Gerge achieved an average monthly return of 2.8% at a risk of 4.9%. A main explanation to that is the fact that Swedish equity funds are not allowed to allocate more than 10% into a single share, while in our portfolios, the weights for two or three shares exceeded that limit. Also all three portfolios perform much better than the market portfolio whose returns were slightly less than 2% and its risk 5.4%. The most striking result was perhaps the absence of some of the most known Swedish companies, like Volvo, Electrolux, SKF, Investor, SEB and Hennes & Mauritz, and the rather low weights of Ericsson, for all models and all α-values. It is hard to find a Swedish equity fund that does not include all these companies.

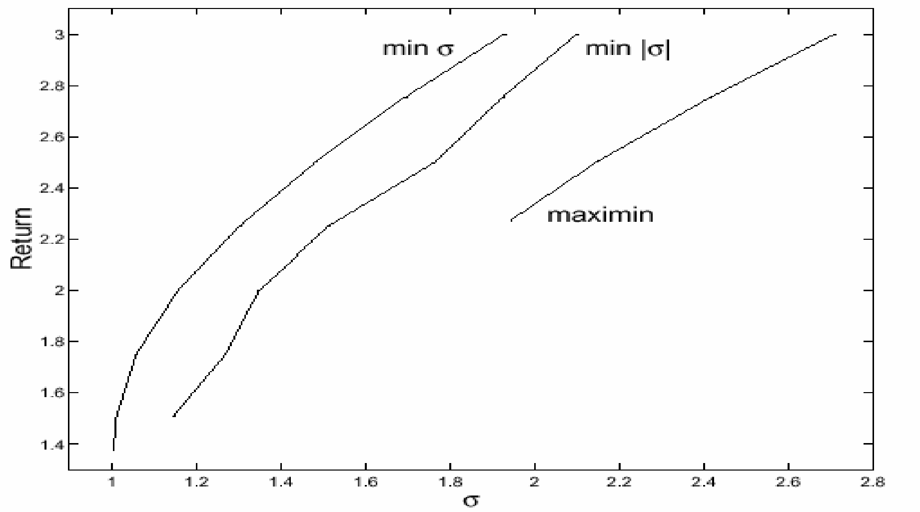

Figure 1 The frontiers of the three portfolios.

Risk aversion and choice of portfolios

A standard method to determine which portfolios will be selected for different risk measures is to assume various utility functions, see Sharpe, Alexander and Bailey[9]. The following simple form represents the investor's indifference curves in a mean-variance context.

$$U = \overline{R} - w\sigma^2 ,$$

where U is the level of utility, $\overline{R}$ is the expected return and w is a positive constant that indicates the investor's risk aversion. If $w = 0$, the investor is risk-neutral, because the utility level the specific portfolio provides is independent from its risk. If the value of w approaches infinity, the investor will never invest in risky assets, and prefer the risk-free interest rate.

Table 2 summarizes the utility levels for various values of w, for two different α-values.

Table 2 Utility levels for α = 1.25 and 2.5 percent per month

		$\alpha = 1.25$			$\alpha = 2.5$	
w	min σ	maximin	min $\|\sigma\|$	min σ	maximin	min $\|\sigma\|$
0	1.379	2.271	1.510	2.500	2.500	2.500
0.2	1.177	1.515	1.247	2.060	1.577	1.877
0.4	0.975	0.759	0.984	1.619	0.654	1.253
0.6	0.774	0.003	0.720	1.179	-0.268	0.630
0.8	0.572	-0.753	0.457	0.738	-1.191	0.006
1	0.370	-1.510	0.194	0.298	-2.114	-0.617

In general, for low degrees of risk aversion, the utility levels are higher with higher α-values. The Markowitz model provides the highest utility levels for α = 2.5% though, while the min $|\sigma|$ formulation seems to provide equally high utilities, but only for low α-values and low degrees of risk aversion. In addition, the utility levels in the Markowitz model do not fall at the same rate as in the other models, if we move to higher degrees of risk aversion. On the other hand, the maximin formulation yields high utility levels at very low degrees of risk aversion and risk neutrality. That finding contradicts Young´s argument that the maximin formulation is more suitable if the utility function is more risk averse than it is implied by the Markowitz formulation. Notice though that Young uses another utility function, characterized by an extreme form of risk aversion.

Out of sample performance of portfolios

In this section we examine the true performance of these portfolios during the next six months, despite the fact that none of these models claim to predict the future returns satisfactory. We assume that these portfolios were constructed at the end of December 2000, each with a value of SEK 100,000. If investors believed in these models, they should expect to receive a positive return, given the fact that all portfolios have very low risk. As is known, the decline in stock market that started in 2000, continued in 2001. Thus, the performances, as expected are poor. All models predicted positive expected returns, $\overline{R}$, but the true monthly returns were negative, ranging from –0.265%, for low α-values, to –2.70% for $\alpha = 3$.

Their performance is summarized in Table 3. The expected total returns of all portfolios at the end of this six months period are statistically significant from the true returns. The correlation coefficients were almost minus one, indicating that the more aggressive a portfolio is, the higher its true losses. One might argue that this is normal, because the risks are higher for the most aggressive portfolios. However, the true losses cannot be explained only by the higher risks of the aggressive portfolios. Notice that that the true portfolio values for the maximin model are higher compared to the other models, when the α-values are larger than 2%.

In addition, if we look at the Value-at-Risk criterion (VaR), Hull[10], Benninga and Wiener[11], i.e. the lowest quantile of the potential losses that can occur during a specific period, model two again seems to perform better for α-values ranging from 1.5 to 2.25 percent. For instance, for $\alpha = 1.75$, there was a 1% chance that more than SEK 5,409 would be lost by the end of June, or equivalently, with 99% probability, it would not decrease by more than that amount, leading to 100–VaR value of at least 94,591. The true portfolio value though, turned out to be 95,167, i.e. a slightly lower loss. On the contrary, model one and three lead to lower true values, than the predicted 100–VaR values for almost all α-values. Thus, if VaR is an accurate risk measure, the maximin portfolio true values are more close to its VaR predictions than the other two portfolios.

Again, all three portfolios performed better than the market portfolio during the same period. The SAX-All Share index portfolio at the end of June would be worth approximately 82,000 which is lower even from the most aggressive ones (with α = 3%). They even perform better than the Didner & Gerge equity fund for α-values less than 2.25%.

Table 3 Expected, 100-VaR and true values in June 2001

α	Value	min σ	maximin	min $\lvert\sigma\rvert$
1-1.25	Expected	108,565	114,423	109,409
	100-VaR	97,666	94,591	97,187
	True	98,408	95,167	95,431
1.5	Expected	109,344	114,423	109,409
	100-VaR	97,928	94,591	97,187
	True	97,498	95,167	95,431
1.75	Expected	110,970	114,423	110,970
	100-VaR	98,259	94,591	97,074
	True	95,627	95,167	92,782
2	Expected	112,616	114,423	112,616
	100-VaR	98,295	94,591	97,234
	True	93,559	95,167	91,443
2.25	Expected	114,283	114,423	114,283
	100-VaR	98,084	94,591	96,910
	True	90,709	95,167	89,727
2.5	Expected	115,969	115,969	115,969
	100-VaR	97,668	94,014	96,100
	True	87,397	90,521	88,305
2.75	Expected	117,677	117,677	117,677
	100-VaR	97,097	93,149	95,817
	True	85,357	87,992	87,310
3	Expected	119,405	119,405	119,405
	100-VaR	96,367	92,131	95,424
	True	83,784	85,497	84,507

Conclusions

It is often claimed that the Markowitz classical Quadratic Optimisation to the portfolio problem, is cumbersome and time consuming. Our data set was perhaps not large enough to test if that argument is still correct. In fact, the optimal solution to that model was computed equally fast with both 25 and 67 shares and some tens of a second faster than the solution of the other LP models. Modern computers and packages are extremely fast in solving complex and large problems.

The Markowitz model provides, as expected, the lowest risk, while the maximin formulation seems to yield higher returns and risks, with the minimization of MAD being close to Markowitz. All three formulations though, outperform the top equity fund portfolios in Sweden. In addition, all Markowitz portfolios that have been estimated seem to comprise the efficient frontier, followed by the MAD and the maximin. The Markowitz model yields higher utility levels for high α-values and higher degrees of risk aversion. Contrary to Young´s argument, the maximin formulation is more suitable than the Markowitz formulation, if the utility function is less and not more risk averse. The maximin portfolio has its merits though, on the actual performance, since it is relatively more robust to the true decline in stock prices. For most of the required returns, the maximin portfolios lose less than the other two at the end of a six months period. The true declines are also close to the predicted VaR values.

We plan to explore two routes in the future. The first is to reformulate this problem as an Integer Linear Programming (ILP), by taking into account fixed and variable costs associated with the purchase of shares. Very often, if the fixed costs are high, it is unprofitable to purchase small posts. Large ILP, although they are harder to solve, they are rather “easy” compared to Integer Quadratic Program (IQP) of the Markowitz type. We should expect, that our small data set might be large enough to make the IQP formulation very hard and time consuming. The second route is to transform this static formulation into a dynamic one, by rebalancing the portfolios at discrete time intervals.

Appendix

Table A1 Descriptive statistics for all 37 companies appeared in portfolios

Company (symbol)	**Mean**	**Standard Deviation**	**Min**	**Max**	**Skewness**	**Kurtosis**
ABB	2.3196	7.4854	-17.24	19.01	-0.207	2.773
ASSAB	4.4706	8.0715	-11.34	36.76	1.297	6.969
AZN	3.3066	9.6023	-12.34	20.17	-0.080	1.795
ATCOA	1.9116	8.7933	-15.21	28.45	0.419	3.323
ERICB	4.7766	14.2589	-23.56	35.23	0.118	2.433
FSPAA	1.9079	8.5067	-12.34	28.13	0.685	3.378
GAMBB	0.1987	11.7742	-32.81	33.95	0.387	4.323
HOLMB	1.4216	11.2653	-28.72	22.14	-0.499	3.011
INDUA	2.1925	7.8633	-16.72	18.97	-0.127	2.538
NOKI	6.7190	12.0553	-18.00	30.97	0.139	2.136
PHA	2.3513	9.3548	-22.07	20.48	-0.164	2.790
SHBA	2.1801	7.5477	-17.88	17.14	-0.075	2.713
SAND	0.7278	7.6648	-20.67	15.85	-0.113	2.922
SDIA	5.0904	12.2912	-24.72	52.76	0.940	6.428
SKAB	0.8744	8.3352	-20.97	19.85	0.167	3.484
SWMA	1.0643	6.0080	-8.71	19.49	0.871	3.883
ALLGB	2.2827	16.0188	-35.67	53.34	0.506	4.491
ANGPB	2.2916	12.4717	-25.67	35.28	0.543	3.463
BERGB	0.4133	7.6856	-11.23	21.56	0.929	3.437
BILIA	0.7771	7.8638	-13.76	18.34	0.305	2.364
BURE	0.5331	5.1299	-11.56	13.56	0.477	3.229
GUNN	0.9483	8.8635	-15.09	32.10	1.019	4.727
HLDX	0.1402	9.7212	-15.23	32.92	0.734	3.820
JM	1.3867	6.8599	-14.28	24.31	0.487	4.363
SCVB	1.0619	12.0900	-19.78	59.66	2.241	12.969
CUST	1.9475	6.2710	-17.58	12.63	-0.669	3.329
EURO	4.9017	14.0778	-23.61	41.23	0.348	3.143
RROS	0.6991	12.5024	-25.27	54.77	1.695	9.203
WIHL	0.3312	5.7515	-7.85	14.34	0.414	2.155
BEIAB	0.9852	7.3309	-10.23	28.56	1.313	5.467
BONG	2.2458	9.3065	-26.31	24.00	-0.184	3.857
CELL	0.7831	15.7250	-48.34	47.89	0.026	5.227
CCORB	1.4696	13.9395	-27.65	49.56	0.890	4.992
ELANB	2.5304	13.0258	-39.92	44.68	0.022	5.787
GRAN	0.2272	3.1850	-9.72	6.40	-0.355	3.863
NEAB	0.6908	7.9277	-18.41	19.44	0.565	3.625
PEABB	1.1933	7.4854	-11.16	20.00	0.582	3.035

Table A2a Optimal portfolios for α = 1.25 and 1.75 percent per month

Company (symbol)	min σ		maximin		min $\lvert\sigma\rvert$	
	α = 1.25%	α = 1.75%	α = 1.25%	α = 1.75%	α = 1.25%	α = 1.75%
ABB	7.3644	7.1126	6.6869	6.6869	7.5066	5.4210
ASSAB	5.6415	9.0891	18.2507	18.2507	6.0911	7.6288
AZN	0.3928	1.5656	2.5380	2.5380	0.9688	0.9688
ATCOA	0.4824	1.2286	0	0	1.2296	2.3295
ERICB	4.1129	4.2714	2.1373	2.1373	4.2714	4.6064
FSPAA	0.0308	0	0	0	0	0
GAMBB	0	0	0	0	0.0890	2.5735
HOLMB	4.1898	4.9038	5.7944	5.7944	2.4886	4.9275
INDUA	3.6117	3.3737	1.4765	1.4765	4.9165	5.1938
NOKI	0.2997	1.6542	2.6820	2.6820	2.1588	1.1487
PHA	3.4485	4.6613	6.7250	6.7250	2.0922	5.1569
SHBA	4.4627	6.0020	3.4418	3.4418	1.8355	5.1082
SAND	0	0	0	0	0	0.0601
SDIA	0	0	3.2794	3.2794	0	0
SKAB	0.2599	0	0	0	0	0.7249
SWMA	2.2121	3.1767	2.1744	2.1744	2.3896	1.1947
ALLGB	1.1100	0.8008	0	0	1.8149	0.6549
ANGPB	1.2407	1.2299	6.9980	6.9980	0	0
BERGB	0	0	0.4514	0.4514	0	0
BILIA	4.2373	3.4281	0	0	2.2784	1.9406
BURE	6.2844	3.4908	1.8726	1.8726	7.1241	5.2389
GUNN	0	0	0	0	0.9639	1.2567
HLDX	1.5855	0	0	0	1.2882	0.3525
JM	0.5141	0.9771	2.3292	2.3292	0.8429	0.1268
SCVB	1.1137	0.1119	0	0	0	0
CUST	0	0	0	0	0.0308	2.1224
EURO	0	0	0	0	0	2.1748
RROS	0	0	0	0	0.7167	2.4922
WIHL	12.5825	11.4160	4.6308	4.6308	12.8969	11.6611
BEIAB	6.3952	6.9161	0.2527	0.2527	7.1584	4.0773
BONG	1.3199	2.1886	0	0	0	1.5427
CELL	0	0	0.7629	0.7629	0	0
CCORB	0	0.3714	4.2893	4.2893	0.5698	0.1151
ELANB	0.2485	0.8853	1.9469	1.9469	0.6417	0.3482
GRAN	22.1018	15.7680	20.1274	20.1274	17.3054	8.8420
NEAB	3.0081	3.8043	0.8734	0.8734	7.0126	6.5126
PEABB	1.7491	1.5728	0.2789	0.2789	3.0385	3.4982
$\overline{R}$	1.379%	1.75%	2.271%	2.271%	1.51%	1.75%
σ	1.005	1.058	1.944	1.944	1.147	1.270

Table A2b Optimal portfolios for α = 2.5 and 3 percent per month

Company (symbol)	min σ		maximin		min $|\sigma|$	
	α = 2.5%	α = 3%	α = 2.5%	α = 3%	α = 2.5%	α = 3%
ABB	7.4827	6.9411	8.3542	9.5854	0	6.2223
ASSAB	15.7586	19.7756	19.8446	26.6429	17.5018	23.0312
AZN	2.8278	3.8838	2.4530	6.1486	0.1407	4.5540
ATCOA	2.1432	1.4734	0	0.5822	4.2880	0
ERICB	5.4953	7.1127	2.8130	0.4887	4.4709	5.4335
GAMBB	0.4734	1.2655	0	0	4.3338	4.0925
HOLMB	6.3474	5.9307	5.9638	5.7490	5.2866	2.6010
INDUA	3.0283	3.9335	1.7587	1.0637	7.2590	2.3286
NOKI	4.3203	6.7424	2.8092	5.1903	4.4490	8.9196
PHA	6.4666	7.6645	9.7105	13.8245	8.7389	5.0064
SHBA	9.7104	13.1372	6.1894	5.1658	13.2618	12.5157
SDIA	0.1023	1.0060	2.8280	2.2522	0	0
SKAB	0	0	0	0	0.7537	0
SWMA	3.0168	2.3806	0	0	0	0
ALLGB	0.7317	0.6417	0	0	0.7186	2.1173
ANGPB	2.0272	3.6976	7.6616	9.0465	2.6086	2.9725
BERGB	0	0	0.3140	6.5281	7.9017	6.4297
BILIA	1.0007	0	0	0	1.8209	0.4048
JM	1.3237	0.7336	1.6490	0	0	1.6768
CUST	1.0045	0.1257	0	0	0	0
WIHL	7.9067	3.4448	4.0443	0	2.3287	0.8963
BEIAB	5.9888	2.2741	0	0	0.1788	0
BONG	2.7003	2.3849	0	0	4.8382	2.9006
CELL	0	0	1.9118	1.1623	0	0
CCORB	1.2184	0.4954	5.6795	6.3156	0.8182	0
ELANB	1.9702	2.6308	1.5190	0.2544	1.3316	2.0404
GRAN	1.0034	0	14.2187	0	0	0
NEAB	4.3383	2.3244	0.2778	0	5.2640	5.8125
PEABB	1.4331	0	0	0	0.8772	0.0443
$\overline{R}$	2.5%	3%	2.5%	3%	2.5%	3%
σ	1.484	1.931	2.148	2.712	1.766	2.102

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