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Simulating Long-Term Hill Instability of the Circular-Restricted Three-Body Problem

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ABSTRACT

For this paper I use REBOUND, a numerical N-body solver, to observe Hill instability in a circular-restricted three-body system. In theory, it is known from previous literature that a circular-restricted three-body system is guaranteed to have Hill Stability, given the orbital separation parameter, Δ, between the two orbiting protoplanets is greater than a defined distance. By setting a critical orbital separation parameter Δ_c equal to the minimum separation distance, we can expect Hill instability to arise and as such close encounters will occur. I define a close encounter to be when one protoplanet is within the Hill sphere radius of the other. The initial conditions of the system involve two Jupiter-massed protoplanets which orbit around a central Sun-massed star. The semi-major axis of the first protoplanet is 1 AU, with the second protoplanet having a semi-major axis of $1 + \Delta_c$ AU. We integrate the system over a timescale of 7.5×10^3 years to observe the eccentricity and orbital separation evolution and run Lomb-Scargle periodograms for both protoplanets.

1. INTRODUCTION

Three-body problems have peaked the interests of mathematicians and astrophysicists since the 17th century, after Isaac Newton published his famous work on Classical Mechanics and Gravity. While the trajectories of two masses moving under Newtonian gravitational forces can be analytically determined in a general form, no such general closed form analytical solutions exist for three-body systems (Armitage 2020). However, there does exist a specific example of three-body motion for which we can easily derive a stability condition, known as the *circular-restricted three-body problem*. In this example, the system is a co-planar planetary system with a central Sun-massed star $M_\odot$ and two Jupiter-massed protoplanets M_J, where $\frac{M_J}{M_\odot} << 1$.

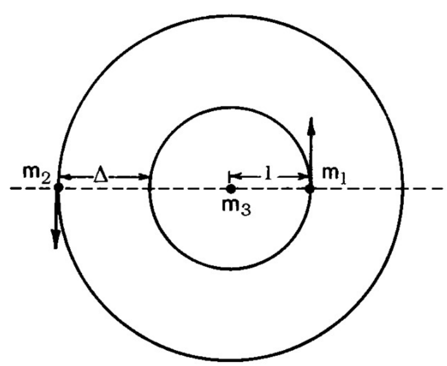

Figure 1. Circular-restricted three-body system where the initial argument of periapsis for m_1 is $\omega_1 = 0$ radians, and for m_2 is $\omega_2 = \pi$ radians (Gladman 1993)

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It must be noted that stability in this case refers to Hill stability, and not the stronger Langrangian stability criterion. Hill stability is achieved when a system has no close encounters, meaning that no protoplanets fall within the Hill sphere radius of the other. The Hill Sphere radius is approximated as:

$$r_H \approx a(1-e)\sqrt{\frac{m_p}{3M_*}}, \quad (1)$$

where $m_p = M_J$ is the mass of the protoplanet and $M_* = M_\odot$ is the mass of the central star. Using REBOUND, an N-body integrator that can integrate the motion of particles under the influence of gravity, I attempt to simulate *Hill instability* conditions and observe the eccentricity and orbital separation evolution, given the system has an initial small eccentricity.

2. THEORY

2.1. *Three-Body System*

Let us assume we have a test particle of negligible mass. Considering that the system rotates in a non-inertial frame of reference, co-rating with the orbital motion of the planets we can derive the equations of motion. Assuming the star and planet orbit in the x-y plane we have (Armitage 2020):

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -G\left(\frac{M_\odot(x + x_\odot)}{r_\odot^3} + \frac{M_J(x - x_J)}{r_J^3}\right), \tag{2}$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -G\left(\frac{M_\odot}{r_\odot^3} + \frac{M_J}{r_J^3}\right)y, \tag{3}$$

$$\ddot{z} = -G\left(\frac{M_\odot}{r_\odot^3} + \frac{M_J}{r_J^3}\right)z. \tag{4}$$

Here Ω is the protoplanet's angular velocity, $r_\odot$ is the instantaneous distance between the test particle and the central star, and r_J is the instantaneous distance between the test particle and the star. Defining the potential energy of the system, including acceleration due to fictitious forces as:

$$U = \frac{\Omega^2}{2}(x^2 + y^2) + \frac{GM_\odot}{r_\odot} + \frac{GM_J}{r_J}, \tag{5}$$

we can rewrite the e.o.m as:

$$\ddot{x} - 2\Omega\dot{y} = \frac{\partial U}{\partial x}, \tag{6}$$

$$\ddot{y} + 2\Omega\dot{x} = \frac{\partial U}{\partial y}, \tag{7}$$

$$\ddot{z} = \frac{\partial U}{\partial z}. \tag{8}$$

Canceling out the Coriolis terms, and then time-differentiating the equation we get:

$$\frac{d}{dt}\left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{y}^2)\right) = \frac{dU}{dt}. \tag{9}$$

Simply integrating with respect to time and we have found the equation of motion as:

$$C_J = 2U - v^2, \tag{10}$$

where C_J is the Jacobi constant. Using the Jacobi constant we can define zero-velocity surfaces where the particle can never cross. As such, a protoplanet's orbit is guaranteed to be stable (where stability is defined to never have a close encounter) if its Jacobi constant is such that a zero-velocity surface lies between between it and the other protoplanet. This can be written in terms of a minimum orbital separation, expressed with the zero-eccentricity orbital parameter Δ given q_1 and $q_2 << 1$, where $q = \frac{m_p}{M_*}$ (Gladman 1993):

$$\Delta > 2.40(q_1 + q_2)^{1/3}.$$

If the intial conditions include an initial (small) eccentricity, along with the fact that $q_1 = q_2$, the stability of the circular restricted three-body problem can be determined to be:

$$\Delta > \sqrt{\frac{8}{3}(e_1^2 + e_2^2) + 9q^{2/3}}.$$

For simplicity, we can define a critical separation where instability is guaranteed and $e_1 = e_2$:

$$\Delta_c = \sqrt{\frac{16e^2}{3} + 9q^{2/3}}.$$

2.2. *REBOUND N-Body Simulation*

For a classical N-body system, the Hamiltonian of the system would be:

$$H = \sum_{i=1}^{N} \frac{\vec{p_i}^2}{2m_i} - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{Gm_i m_j}{|\vec{q_i} - \vec{q_j}|}. \tag{11}$$

Here the canonical momentum and position are partial derivatives of the Hamiltonian:

$$\dot{q} = \frac{\partial H}{\partial \vec{p_i}}, \tag{12}$$

$$\dot{\vec{p_i}} = -\frac{\partial H}{\partial \vec{q_i}}. \tag{13}$$

As such, REBOUND is essential solving a system of differential equations numerically. The most common types of integrator methods are the Euler-Cromer method and the Runga-Kutta method. However, in this project I use a special integrator called IAS15 (Integrator with Adaptive Step-size Control, 15th Order). This algorithm mathematically tries to solve the an ODE in the form (Rein & Spiegel 2015):

$$\ddot{y} = F(\dot{y}, y, t), \tag{14}$$

where F is a function describing the gravitational force, which depends on $\dot{y}, y, t$. Expanding the equation, a constant term arises which would be the force at the beginning of a time-step:

$$\ddot{y} \approx \ddot{y}_0 + a_0 t + a_1 t^2 + ... + a_6 t^7. \tag{15}$$

Since the IAS15 integrator uses an adaptive time-step, it makes a prediction on the next value based on previous time-steps and predicts the length of the time-step for each sequential step. This process is looped until $y(t)$ converges.

3. METHODOLOGY AND RESULTS

Using SymPy, we solved for the maximum eccentricity needed for the small-eccentricity orbital parameter to give the same separation value has the zero-eccentricity orbital parameter. As such, an eccentricity, $e = 0.0161470323464714$ was used in the initial conditions of the circular-restricted three-body system.

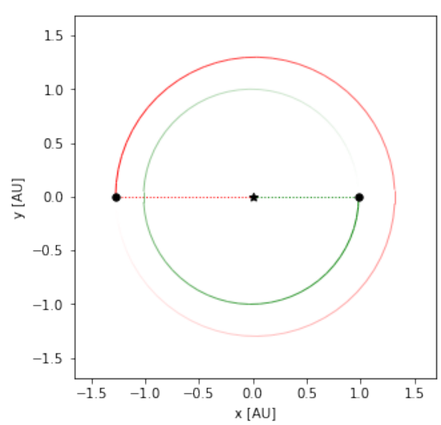

Figure 2. The first Jupiter-massed protoplanet has a semi-major axis of 1 AU, while the second protoplanet has a semi-major axis of $1 + \Delta_c$ AU and an argument of periapsis $\omega = \pi$ radians.

After this, I write a collision-resolve function that can count the number of close encounters. Then using the direct simulation type and integrating to 7.5×10^3 years, we can see that there were 798 times in which the protoplanets interacted within each other's Hill sphere radius leading to significant perturbations in orbital separation and eccentricity. While the eccentricity graph was extremely chaotic with very little even periodicity present, I still attemped a FFT-like analysis. Since the time series is not uniform, rather than using a Fast-Fourier Transform, I used a Lomb-Scargle periodogram.

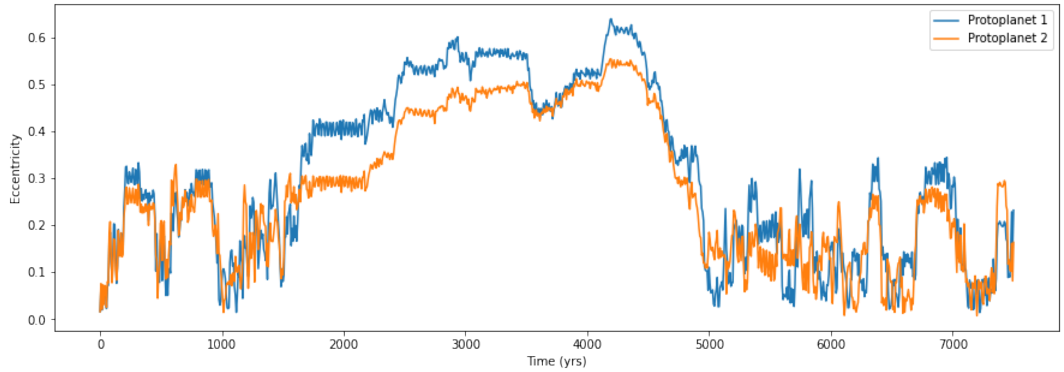

Figure 3. The eccentricities of both protoplanets become heavily chaotic, with almost no signs of even periodicity.

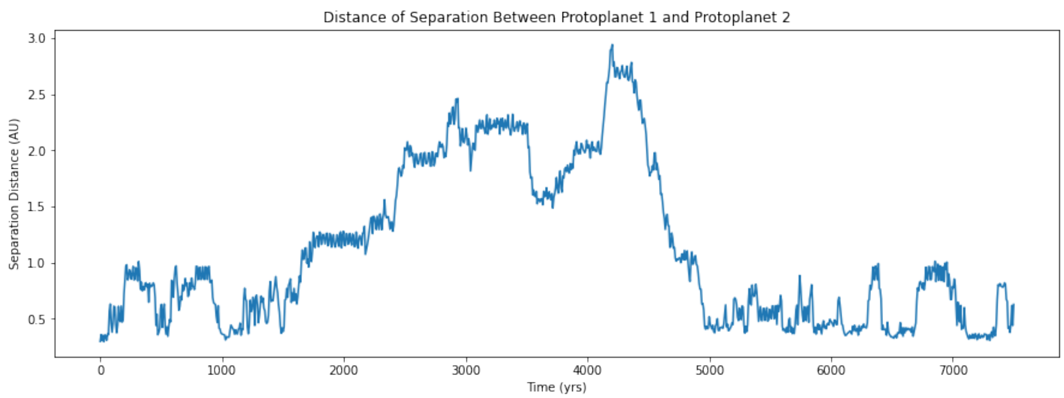

Figure 4. The orbital separation between the protoplanets become heavily chaotic as well, reaching a significant apex at around $4.1{\times}10^3$ years before decreasing and oscillating between 0.5 - 1.0 AU separation.

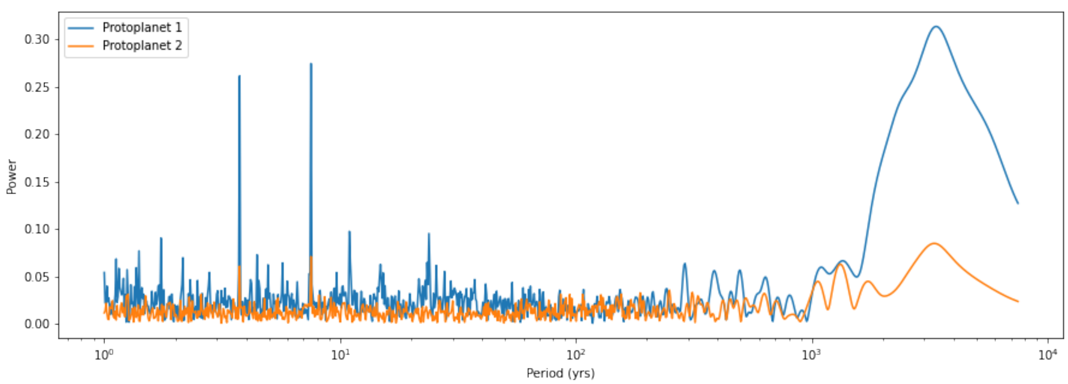

Figure 5. The chaotic eccentricity being transformed didn't give much information about the period/resonance of the system. However, there is a large signal that match on both protoplanets at about 2.6 and 8.5 years, suggesting a brief resonance between the two bodies.

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