15

Data envelopment analysis

The purpose of *data envelopment analysis* (DEA) is to compare the operating performance of a set of units such as companies, university departments, hospitals, bank branch offices, production plants, or transportation systems. In order for the comparison to be meaningful, the units being investigated must be homogeneous.

The performance of a unit can be measured on several dimensions. For example, to evaluate the activity of a production plant one may use quality indicators, which estimate the rate of rejects resulting from manufacturing a set of products, and also flexibility indicators, which measure the ability of a system to react to changes in the requirements with quick response times and low costs.

Data envelopment analysis relies on a productivity indicator that provides a measure of the efficiency that characterizes the operating activity of the units being compared. This measure is based on the results obtained by each unit, which will be referred to as *outputs*, and on the resources utilized to achieve these results, which will be generically designated as *inputs* or *production factors*. If the units represent bank branches, the outputs may consist of the number of active bank accounts, checks cashed or loans raised; the inputs may be the number of cashiers, managers or rooms used at each branch. If the units are university departments, it is possible to consider as outputs the number of active teaching courses and scientific publications produced by the members of each department; the inputs may include the amount of financing received by each department, the cost of teaching, the administrative staff and the availability of offices and laboratories.

Business Intelligence: Data Mining and Optimization for Decision Making   *Carlo Vercellis*
© 2009 John Wiley & Sons, Ltd. ISBN: 978-0-470-51138-1

https://E-next.in

15.1 Efficiency measures

In data envelopment analysis the units being compared are called *decision-making units* (DMUs), since they enjoy a certain decisional autonomy. Assuming that we wish to evaluate the efficiency of n units, let $\mathcal{N} = \{1, 2, \ldots, n\}$ denote the set of units being compared.

If the units produce a single output using a single input only, the *efficiency* of the jth decision-making unit DMU_j, $j \in \mathcal{N}$, is defined as

$$\theta_j = \frac{y_j}{x_j}, \tag{15.1}$$

in which y_j is the output value produced by DMU_j and x_j the input value used.

If the units produce multiple outputs using various input factors, the efficiency of DMU_j is defined as the ratio between a weighted sum of the outputs and a weighted sum of the inputs. Denote by $\mathcal{H} = \{1, 2, \ldots, s\}$ the set of production factors and by $\mathcal{K} = \{1, 2, \ldots, m\}$ the corresponding set of outputs. If x_{ij}, $i \in \mathcal{H}$, denotes the quantity of input i used by DMU_j and y_{rj}, $r \in \mathcal{K}$, the quantity of output r obtained, the efficiency of DMU_j is defined as

$$\theta_j = \frac{u_1 y_{1j} + u_2 y_{2j} + \ldots + u_m y_{mj}}{v_1 x_{1j} + v_2 x_{2j} + \ldots + v_s x_{sj}} = \frac{\sum_{r \in \mathcal{K}} u_r y_{rj}}{\sum_{i \in \mathcal{H}} v_i x_{ij}}, \tag{15.2}$$

for weights $u_1, u_2, \ldots, u_m$ associated with the outputs and $v_1, v_2, \ldots, v_s$ assigned to the inputs.

In this second case, the efficiency of DMU_j depends strongly on the system of weights introduced. At different weights, the efficiency value may undergo relevant variations and it becomes difficult to fix a single structure of weights that might be shared and accepted by all the evaluated units. In order to avoid possible objections raised by the units to a preset system of weights, which may privilege certain DMUs rather than others, data envelopment analysis evaluates the efficiency of each unit through the weights system that is best for the DMU itself – that is, the system that allows its efficiency value to be maximized. Subsequently, by means of additional analyses, the purpose of data envelopment analysis is to identify the units that are efficient in absolute terms and those whose efficiency value depends largely on the system of weights adopted.

15.2 Efficient frontier

The *efficient frontier*, also known as *production function*, expresses the relationship between the inputs utilized and the outputs produced. It indicates the

E-next THE NEXT LEVEL OF EDUCATION

https://E-next.in

maximum quantity of outputs that can be obtained from a given combination of inputs. At the same time, it also expresses the minimum quantity of inputs that must be used to achieve a given output level. Hence, the efficient frontier corresponds to *technically efficient* operating methods.

The efficient frontier may be empirically obtained based on a set of observations that express the output level obtained by applying a specific combination of input production factors. In the context of data envelopment analysis, the observations correspond to the units being evaluated. Most statistical methods of parametric nature, which are based for instance on the calculation of a regression curve, formulate some prior hypotheses on the shape of the production function. Data envelopment analysis, on the other hand, forgoes any assumptions on the functional form of the efficient frontier, and is therefore nonparametric in character. It only requires that the units being compared are not placed above the production function, depending on their efficiency value. To further clarify the notion of efficient frontier consider Example 15.1.

Example 15.1 – Evaluation of the efficiency of bank branches. A bank wishes to compare the operational efficiency of its nine branches, in terms of staff size and total value of savings in active accounts. Table 15.1 shows for each branch the total value of accounts, expressed in hundreds of thousands of euros, and the number of staff employed, with the corresponding efficiency values calculated based on definition (15.1). The graph shown in Figure 15.1 shows for each branch the number of employees on the horizontal axis and the value of accounts on the vertical axis. The slope of the line connecting each point to the origin represents the efficiency value associated with the corresponding branch. The line with the maximum slope, represented in Figure 15.1 by a solid line, is the efficient frontier for all branches being analyzed. The branches that are on this line correspond to efficient units, while the branches that are below the efficient frontier are inefficient units. The area between the efficient frontier and the positive horizontal semi-axis is called the *production possibility set*.

A possible alternative to the efficient frontier is the regression line that can be obtained based on the available observations, indicated in Figure 15.1 by a dashed line. In this case, the units that fall above the regression line may be deemed excellent, and the degree of excellence of each unit could be expressed by its distance from the line. However, it is appropriate to underline the difference that exists between the prediction line obtained using a regression model and the efficient frontier obtained using data envelopment analysis. The

E-next THE NEXT LEVEL OF EDUCATION

https://E-next.in

Table 15.1 Input and output values for the bank branches in Example 15.1

bank branch	staff size	accounts value	efficiency
A	3	2.5	0.733
B	2	1.0	0.500
C	5	2.7	0.540
D	3	3.0	1.000
E	7	5.0	0.714
F	5	2.3	0.460
G	4	3.2	0.700
H	5	4.5	0.900
I	6	4.5	0.633

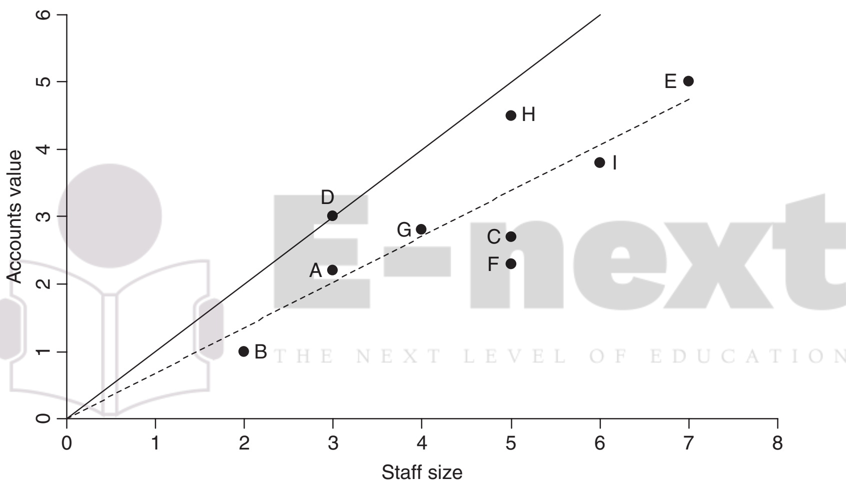

Figure 15.1 Evaluation of efficiency of bank branches

regression line reflects the average behavior of the units being compared, while the efficient frontier identifies the best behavior, and measures the inefficiency of a unit based on the distance from the frontier itself.

Notice also that the efficient frontier provides some indications for improving the performance of inefficient units. Indeed, it identifies for each input level the output level that can be achieved in conditions of efficiency. By the same token, it identifies for each output level the minimum level of input that should be used in conditions of efficiency. In particular, for each DMU_j, $j \in \mathcal{N}$, the *input-oriented* efficiency θ_j^I can be defined as the ratio between the ideal input quantity x^* that should be used by the unit if it were efficient and the actually used quantity x_j:

$$\theta_j^I = \frac{x^*}{x_j}. \tag{15.3}$$

https://E-next.in

Similarly, the *output-oriented* efficiency θ_j^O is defined as the ratio between the quantity of output y_j actually produced by the unit and the ideal quantity y^* that it should produce in conditions of efficiency:

$$\theta_j^O = \frac{y_j}{y^*}. \tag{15.4}$$

The problem of making an inefficient unit efficient is then turned into one of devising a way by which the inefficient unit can be brought close to the efficient frontier.

If the unit produces a single output only by using two inputs, the efficient frontier assumes the shape shown in Figure 15.2. In this case, the inefficiency of a given unit is evaluated by the length of the segment connecting the unit to the efficient frontier along the line passing through the origin of the axes. For the example illustrated in Figure 15.2, the efficiency value of DMU_A is given by

$$\theta_A = \frac{\overline{OP}}{\overline{OA}}, \tag{15.5}$$

where $\overline{OP}$ and $\overline{OA}$ represent the lengths of segments OP and OA, respectively. The inefficient unit may be made efficient by a displacement along segment OA that moves it onto the efficient frontier. Such displacement is tantamount to progressively decreasing the quantity of both inputs while keeping unchanged the quantity of output. In this case, the production possibility set is defined as the region delimited by the efficient frontier where the observed units being compared are found.

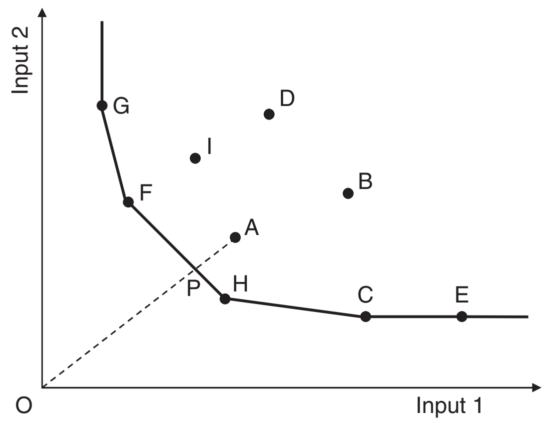

Figure 15.2 Efficient frontier with two inputs and one output

https://E-next.in

15.3 The CCR model

Using data envelopment analysis, the choice of the optimal system of weights for a generic DMU_j involves solving a mathematical optimization model whose decision variables are represented by the weights u_r, $r \in \mathcal{K}$, and v_i, $i \in \mathcal{H}$, associated with each output and input. Various formulations have been proposed, the best-known of which is probably the Charnes–Cooper–Rhodes (CCR) model. The CCR model formulated for DMU_j takes the form

$$\max \quad \vartheta = \frac{\sum_{r\in\mathcal{K}} u_r y_{rj}}{\sum_{i\in\mathcal{H}} v_i x_{ij}}, \tag{15.6}$$

$$\text{s.to} \quad \frac{\sum_{r\in\mathcal{K}} u_r y_{rj}}{\sum_{i\in\mathcal{H}} v_i x_{ij}} \leq 1, \qquad j \in \mathcal{N}, \tag{15.7}$$

$$u_r, v_i \geq 0, \qquad r \in \mathcal{K}, i \in \mathcal{H}. \tag{15.8}$$

The objective function involves the maximization of the efficiency measure for DMU_j. Constraints (15.7) require that the efficiency values of all the units, calculated by means of the weights system for the unit being examined, be lower than one. Finally, conditions (15.8) guarantee that the weights associated with the inputs and the outputs are non-negative. In place of these conditions, sometimes the constraints $u_r, v_i \geq \delta$, $r \in \mathcal{K}$, $i \in \mathcal{H}$ may be applied, where $\delta > 0$, preventing the unit from assigning a null weight to an input or output.

Model (15.6) can be linearized by requiring the weighted sum of the inputs to take a constant value, for example 1. This condition leads to an alternative optimization problem, the *input-oriented* CCR model, where the objective function consists of the maximization of the weighted sum of the outputs

$$\max \quad \vartheta = \sum_{r\in\mathcal{K}} u_r y_{rj}, \tag{15.9}$$

$$\text{s.to} \quad \sum_{i\in\mathcal{H}} v_i x_{ij} = 1, \tag{15.10}$$

$$\sum_{r\in\mathcal{K}} u_r y_{rj} - \sum_{i\in\mathcal{H}} v_i x_{ij} \leq 0, \qquad j \in \mathcal{N}, \tag{15.11}$$

$$u_r, v_i \geq 0, \qquad r \in \mathcal{K}, i \in \mathcal{H}. \tag{15.12}$$

Let ϑ^* be the optimum value of the objective function corresponding to the optimal solution $(\mathbf{v}^*, \mathbf{u}^*)$ of problem (15.9). DMU_j is said to be *efficient* if $\vartheta^* = 1$ and if there exists at least one optimal solution $(\mathbf{v}^*, \mathbf{u}^*)$ such that $\mathbf{v}^* > \mathbf{0}$ and $\mathbf{u}^* > \mathbf{0}$.

By solving a similar optimization model for each of the n units being compared, one obtains n systems of weights. The flexibility enjoyed by the

https://E-next.in

units in choosing the weights represents an undisputed advantage, in that if a unit turns out to be inefficient based on the most favorable system of weights, its inefficiency cannot be traced back to an inappropriate evaluation process. However, given a unit that scores $\vartheta^* = 1$, it is important to determine whether its efficiency value should be attributed to an actual high-level performance or simply to an optimal selection of the weights structure.

Dual of the CCR model

For the input-oriented CCR model, the following dual problem, which lends itself to an interesting interpretation, can be formulated:

$$\min \quad \vartheta, \tag{15.13}$$

$$\text{s.to} \quad \sum_{j \in \mathcal{N}} \lambda_j x_{ij} - \vartheta x_{ij} \le 0, \quad i \in \mathcal{H}, \tag{15.14}$$

$$\sum_{j \in \mathcal{N}} \lambda_j y_{rj} - y_{rj} \ge 0, \quad r \in \mathcal{K}, \tag{15.15}$$

$$\lambda_j \ge 0, \quad j \in \mathcal{N}. \tag{15.16}$$

Based on the optimum value of the variables λ_j^*, $j \in \mathcal{N}$, the aim of model (15.13) is to identify an ideal unit that lies on the efficient frontier and represents a term of comparison for DMU_j. Constraints (15.14) and (15.15) of the model require that this unit produces an output at least equal to the output produced by DMU_j, and uses a quantity of inputs equal to a fraction of the quantity used by the unit examined. The ratio between the input used by the ideal unit and the input absorbed by DMU_j is defined as the optimum value ϑ^* of the dual variable ϑ. If $\vartheta^* < 1$, DMU_j lies below the efficient frontier. In order to be efficient, this unit should employ $\vartheta^* x_{ij}$, $i \in \mathcal{H}$, of each input.

The quantity of inputs utilized by the ideal unit and the level of outputs to be produced are expressed as a linear combination of the inputs and outputs associated with the n units being evaluated:

$$x_i^{\text{ideal}} = \sum_{j \in \mathcal{N}} \lambda_j^* x_{ij}, \quad i \in \mathcal{H}, \tag{15.17}$$

$$y_r^{\text{ideal}} = \sum_{j \in \mathcal{N}} \lambda_j^* y_{rj}, \quad r \in \mathcal{K}. \tag{15.18}$$

For each feasible solution (ϑ, λ) to problem (15.13), the slack variables s_i^-, $i \in \mathcal{H}$, and s_r^+, $r \in \mathcal{K}$, can be defined, which represent respectively the quantity of input i used in excess by DMU_j and the quantity of output r produced in

https://E-next.in

shortage by the DMU_j with respect to the ideal unit:

$$s_i^- = \vartheta x_{ij} - \sum_{j \in \mathcal{N}} \lambda_j x_{ij}, \quad i \in \mathcal{H}, \tag{15.19}$$

$$s_r^+ = \sum_{j \in \mathcal{N}} \lambda_j y_{rj} - y_{rj}, \quad r \in \mathcal{K}. \tag{15.20}$$

As with the primal problem, it is possible also for the dual problem to provide a definition of efficiency. DMU_j is efficient if $\vartheta^* = 1$ and if the optimum value of the slack variables is equal to zero: $s_i^{-*} = 0$, $i \in \mathcal{H}$, and $s_r^{+*} = 0$, $r \in \mathcal{K}$. In other words, DMU_j is efficient if it is not possible to improve the level of an input used or the level of an output produced without a deterioration in the level of another input or of another output. If $\vartheta^* < 1$, DMU_j is said to be *technically inefficient*, in the sense that, in order to obtain the same output, the input quantities used could be simultaneously reduced in the same proportion. The maximum reduction allowed by the efficient frontier is defined by the value $1 - \vartheta^*$. If $\vartheta^* = 1$, but some slack variables are different from zero, DMU_j presents a *mix inefficiency* since, keeping the same output level, it could reduce the use of a few inputs without causing an increase in the quantity of other production factors used.

E-next
THE NEXT LEVEL OF EDUCATION

15.3.1 Definition of target objectives

In real-world applications it is often desirable to set improvement objectives for inefficient units, in terms of both outputs produced and inputs utilized. Data envelopment analysis provides important recommendations in this respect, since it identifies the output and input levels at which a given inefficient unit may become efficient. The efficiency score of a unit expresses the maximum proportion of the actually utilized inputs that the unit should use in conditions of efficiency, in order to guarantee its current output levels. Alternatively, the inverse of the efficiency score indicates the factor by which the current output levels of a unit should be multiplied for the unit to be efficient, holding constant the level of the productive inputs used. Based on the efficiency values, data envelopment analysis therefore gives a measure for each unit being compared of the savings in inputs or the increases in outputs required for the unit to become efficient.

To determine the target values, it is possible to follow an input- or output-oriented strategy. In the first case, the improvement objectives primarily concern the resources used, and the target values for inputs and outputs are given by

$$x_{ij}^{\text{target}} = \vartheta^* x_{ij} - s_i^{-*}, \quad i \in \mathcal{H}, \tag{15.21}$$

$$y_{rj}^{\text{target}} = y_{rj} + s_r^{+*}, \quad r \in \mathcal{K}. \tag{15.22}$$

https://E-next.in

In the second case, target values for inputs and outputs are given by

$$x_{ij}^{\text{target}} = x_{ij} - \frac{s_i^{-*}}{\vartheta^*}, \quad i \in \mathcal{H}, \tag{15.23}$$

$$y_{rj}^{\text{target}} = \frac{y_{rj} + s_r^{+*}}{\vartheta^*}, \quad r \in \mathcal{K}. \tag{15.24}$$

Other performance improvement strategies may be preferred over the proportional reduction in the quantities of inputs used or the proportional increase in the output quantities produced:

- priority order for the production factors – the target values for the inputs are set in such a way as to minimize the quantity used of the resources to which the highest priority has been assigned, without allowing variations in the level of other inputs or in the outputs produced;
- priority order for the outputs – the target values for the outputs are set in such a way as to maximize the quantity produced of the outputs to which highest priority has been assigned, without allowing variations in the level of other outputs or inputs used;
- preferences expressed by the decision makers with respect to a decrease in some inputs or an increase in specific outputs.

E-next THE NEXT LEVEL OF EDUCATION

15.3.2 Peer groups

Data envelopment analysis identifies for each inefficient unit a set of excellent units, called a *peer group*, which includes those units that are efficient if evaluated with the optimal system of weights of an inefficient unit. The peer group, made up of DMUs which are characterized by operating methods similar to the inefficient unit being examined, is a realistic term of comparison which the unit should aim to imitate in order to improve its performance.

The units included in the peer group of a given unit DMU_j may be identified by the solution to model (15.9). Indeed, these correspond to the DMUs for which the first and the second member of constraints (15.11) are equal:

$$E_j = \left\{ j : \sum_{r \in \mathcal{K}} u_r^* y_{rj} = \sum_{i \in \mathcal{H}} v_i^* x_{ij} \right\}. \tag{15.25}$$

Alternatively, with respect to formulation (15.13), the peer group consists of those units whose variable λ_j in the optimal solution is strictly positive:

$$E_j = \left\{ j : \lambda_j^* > 0 \right\}. \tag{15.26}$$

https://E-next.in

Notice that within a peer group a few excellent units more than others may represent a reasonable term of comparison. The relative importance of a unit belonging to a peer group depends on the value of the corresponding variable λ_j in the optimal solution of the dual model.

The analysis of peer groups allows one to differentiate between really efficient units and apparently efficient units for which the choice of an optimal system of weights conceals some abnormal behavior. In order to draw this distinction, it is necessary to consider the efficient units and to evaluate how often each belongs to a peer group. One may reasonably expect that an efficient unit often included in the peer groups uses for the evaluation of its own efficiency a robust weights structure. Conversely, if an efficient unit rarely represents a term of comparison, its own system of optimal weights may appear distorted, in the sense that it may implicitly reflect the specialization of the unit along a particular dimension of analysis.

15.4 Identification of good operating practices

By identifying and sharing *good operating practices*, one may hope to achieve an improvement in the performance of all units being compared. The units that appear efficient according to data envelopment analysis certainly represent terms of comparison and examples to be imitated for the other units. However, among efficient units some more than others may represent a target to be reached in improving the efficiency.

The need to identify the efficient units, for the purpose of defining the best operating practices, stems from the principle itself on which data envelopment analysis is grounded, since it allows each unit to evaluate its own degree of efficiency by choosing the most advantageous structure of weights for inputs and outputs. In this way, a unit might appear efficient by purposely attributing a non-negligible weight only to a limited subset of inputs and outputs. Furthermore, those inputs and outputs that receive greater weights may be less critical than other factors more intimately connected to the primary activity performed by the units being analyzed. In order to identify good operating practices, it is therefore expedient to detect the units that are really efficient, that is, those units whose efficiency score does not primarily depend on the system of weights selected. To differentiate these units, we may resort to a combination of different methods: *cross-efficiency analysis*, evaluation of *virtual inputs* and *virtual outputs*, and *weight restrictions*.

15.4.1 Cross-efficiency analysis

The analysis of *cross-efficiency* is based on the definition of the *efficiency matrix*, which provides information on the nature of the weights system adopted

https://E-next.in

by the units for their own efficiency evaluation. The square efficiency matrix contains as many rows and columns as there are units being compared. The generic element θ_{ij} of the matrix represents the efficiency of DMU_j evaluated through the optimal weights structure for DMU_i, while the element θ_{jj} provides the efficiency of DMU_j calculated using its own optimal weights. If DMU_j is efficient (i.e. if $\theta_{jj} = 1$), although it exhibits a behavior specialized along a given dimension with respect to the other units, the efficiency values in the column corresponding to DMU_j will be less than 1.

Two quantities of interest can be derived from the efficiency matrix. The first represents the average efficiency of a unit with respect to the optimal weights systems for the different units, obtained as the average of the values in the jth column. The second is the average efficiency of a unit measured applying its optimal system of weights to the other units. The latter is obtained by averaging the values in the row associated with the unit being examined. The difference between the efficiency score θ_{jj} of DMU_j and the efficiency obtained as the average of the values in the jth column provides an indication of how much the unit relies on a system of weights conforming with the one used by the other units in the evaluation process. If the difference between the two terms is significant, DMU_j may have chosen a structure of weights that is not shared by the other DMUs in order to privilege the dimensions of analysis on which it appears particularly efficient.

THE NEXT LEVEL OF EDUCATION

15.4.2 Virtual inputs and virtual outputs

Virtual inputs and virtual outputs provide information on the relative importance that each unit attributes to each individual input and output, for the purpose of maximizing its own efficiency score. Thus, they allow the specific competencies of each unit to be identified, highlighting at the same time its weaknesses.

The *virtual inputs* of a DMU are defined as the product of the inputs used by the unit and the corresponding optimal weights. Similarly, *virtual outputs* are given by the product of the outputs of the unit and the associated optimal weights.

Inputs and outputs for which the unit shows high virtual scores provide an indication of the activities in which the unit being analyzed appears particularly efficient. Notice that model (15.9) admits in general multiple optimal solutions, corresponding to which it is possible to obtain different combinations of virtual inputs and virtual outputs.

Two efficient units may yield high virtual values corresponding to different combinations of inputs and outputs, showing good operating practices in different contexts. In this case, it might be convenient for each unit to follow the principles and operating methods shown by the other, aiming at improving its own efficiency on a specific dimension.

https://E-next.in

15.4.3 Weight restrictions

To separate the units that are really efficient from those whose efficiency score largely depends on the selected weights system, we may impose some restrictions on the value of the weights to be associated with inputs and outputs. In general, these restrictions translate into the definition of maximum thresholds for the weight of specific outputs or minimum thresholds for the weight of specific inputs. Notice that, despite possible restrictions on the weights, the units still enjoy a certain flexibility in the choice of multiplicative factors for inputs and outputs. For this reason it may be useful to resort to the evaluation of virtual inputs and virtual outputs in order to identify the units with the most efficient operating practices with respect to the usage of a specific input resource or to the production of a given output.

15.5 Other models

Model (15.9) is based on the hypothesis that the units being compared operate with constant returns to scale. Recall that the returns to scale express the variation in the quantity of outputs in terms of variations in the quantity of inputs used. When the returns to scale are constant, if the inputs increase in a given proportion then the outputs also increase in the same proportion. The hypothesis of constant returns to scale leads to an efficient frontier like the one shown in Figure 15.1. In particular, if $\mathbf{X}$ denotes the matrix of inputs used by the n units and $\mathbf{Y}$ denotes the corresponding matrix of outputs, in the hypothesis of constant returns to scale we can express the production possibility set as

$$P = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \mathbf{X}\boldsymbol{\lambda}, \mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}. \tag{15.27}$$

This means that if the point $(\mathbf{x}, \mathbf{y})$ belongs to P, then any other point of the form $(k\mathbf{x}, k\mathbf{y})$, $k > 0$, will also belong to the production possibility set. If the hypothesis of constant returns to scale is not adequate, one may resort to formulations other than model (15.9). For example, the Banker–Charnes–Cooper model is based on the hypothesis of variable returns to scale, and takes the form

$$\min \quad \vartheta, \tag{15.28}$$

$$\text{s.to} \quad \sum_{j \in \mathcal{N}} \lambda_j x_{ij} - \vartheta x_{ij} \leq 0, \quad i \in \mathcal{H}, \tag{15.29}$$

$$\sum_{j \in \mathcal{N}} \lambda_j y_{rj} - y_{rj} \geq 0, \quad r \in \mathcal{K}, \tag{15.30}$$

https://E-next.in

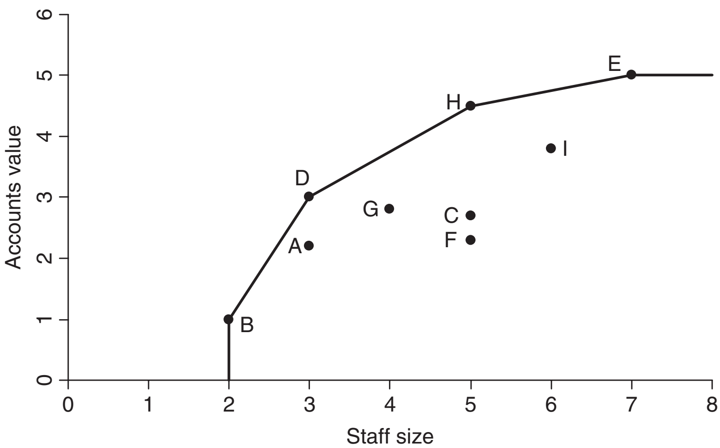

Figure 15.3 Efficient frontier for variable returns to scale

$$\sum_{j \in \mathcal{N}} \lambda_j = 1, \tag{15.31}$$

$$\lambda_j \geq 0, \qquad j \in \mathcal{N}. \tag{15.32}$$

In model (15.28) the production possibility set is defined as

$$P = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \mathbf{X}\boldsymbol{\lambda}, \mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda}, \mathbf{e}\boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{15.33}$$

where **e** is a unit vector. The only difference between model (15.28) and the dual of the CCR model stems from condition (15.31) which, together with constraints (15.32), specifies that the efficient frontier is convex. Figure 15.3 describes the production possibility set for the units of Example 15.1 in the hypothesis of variable returns to scale. The models described above do not exhaust the broad variety of formulations that have been proposed in the framework of data envelopment analysis, although they represent good examples. For more in-depth information, see the suggested references below.

15.6 Notes and readings

The CCR model was first proposed in Charnes *et al*. (1978), based on a previous contribution by Farrell (1957). For an extensive discussion of data envelopment analysis models the reader is referred to Cooper *et al*. (2000) and Thanassoulis (2001). More recent developments are described in Ray (2004). A description of production functions and returns to scale can be found in any microeconomics textbook.

https://E-next.in