Econometrics I, ECO341A, Summer 2023
Homework II (100 points)

Instructor: M.A. Rahman

Deadline: 4:00 pm, June 28, 2023.

Please read the instructions carefully and follow them while writing answers.

- *Solutions to homework should be written in A4 size loose sheets. If you are not comfortable writing on white sheets, please ask for biology paper in Tarun Book Store.*
- *Questions should be answered in order as they appear in the homework. Every new question should begin in a new page. Please number all the pages of your homework solution.*
- *Please leave a margin of one inch from top and one inch from left. Staple the sheets on the top-left.*
- *For Matlab assignments (if any), please answer the questions. Please do not dump the codes as an answer, they should be in the appendix.*
- *Please write your name and names of your group members on the first page of your answer script.*

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1. (20 points). Suppose $Y \sim \text{Beta}(\alpha, \delta)$ such that the *pdf* is given by,

$$f(y|\alpha, \delta) = \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)\Gamma(\delta)}\, y^{\alpha-1}(1-y)^{\delta-1}, \qquad \alpha > 0, \delta > 0, 0 < y < 1.$$

Construct the log-likelihood for a sample of n observations and find the Fisher's Information matrix. Hint: To simplify the derivations, please make use of the following notations:

$$\begin{aligned}
\Gamma'(\alpha) &= \frac{\partial \Gamma(\alpha)}{\partial \alpha}, \qquad \psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}, \qquad \psi'(\alpha) = \frac{\partial \psi(\alpha)}{\partial \alpha} \\
\Gamma'(\delta) &= \frac{\partial \Gamma(\delta)}{\partial \delta}, \qquad \psi(\delta) = \frac{\Gamma'(\delta)}{\Gamma(\delta)}, \qquad \psi'(\delta) = \frac{\partial \psi(\delta)}{\partial \delta}, \\
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\end{aligned}$$

2. (10 points) Consider the simple linear regression model $y_i = \beta_1 + x_{2i}\beta_2 + \epsilon_i$. Show that the matrix formulation of the OLS estimator $\hat{\beta} = (X'X)^{-1}(X'y)$ yields the following,

$$\begin{aligned}\hat{\beta}_1 &= \overline{y} - \hat{\beta}_2\overline{x}, \\ \hat{\beta}_2 &= \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2}.\end{aligned}$$

3. (10+10 = 20 points). Let $X_1, X_2, \ldots$ be i.i.d. random variable with exponential pdf $f(x) = \frac{1}{\lambda}\exp(-x/\lambda)$ if $x > 0$ (and $= 0$ otherwise), $\lambda > 0$.

(a) Show that $\dfrac{\sqrt{n}(\overline{X}_n - \lambda)}{\overline{X}_n} \xrightarrow{d} N(0,1)$ as $n \to \infty$.

(b) Let $\beta = 1/\lambda$. Show that $\sqrt{n}(\beta\overline{X}_n - 1) \xrightarrow{d} N(0,1)$ as $n \to \infty$.

4. (20 points) Consider the linear regression model $y = X\beta + \epsilon$. We know the least squares estimator $\hat{\beta} = (X'X)^{-1}(X'y)$ is BLUE, so let's prove it. Consider another linear and unbiased estimator,

$$\overline{\beta} = \big[(X'X)^{-1}X' + c\big]y,$$

where c is matrix of dimension $k \times n$. Show that $\Sigma_{\overline{\beta}} - \Sigma_{\hat{\beta}}$ is a positive definite matrix, where Σ_η denotes the covariance matrix of the estimator η.

5. (5+5+5+5+10=30 points) Consider the data presented in the file 'ProdFuncData.xlsx'. The data are statewide observations on SIC33, the primary metals industry in the US. There are 3 variables in the data, `value added` (which is the output, Y), `labor`, and `capital`. They were originally constructed by Hilderbrand and Liu (1957) and subsequently used by many authors.

(a) Estimate a Cobb-Douglas production function by regressing $\log Y$ on an intercept, $\log L$, and $\log K$, where Y, L, and K denote output, labor, and capital, respectively. Report the regression coefficients, standard errors, and t-statistics in a table. Also, report the covariance matrix of $\hat{\beta}$ and the R-square.

(b) The hypothesis of constant returns to scale is often tested in studies of production. This hypothesis is equivalent to the restriction that the two coefficients of Cobb-Douglas production function sum to 1 i.e., $H_0 : \beta_L + \beta_K = 1$, versus $H_1 : \beta_L + \beta_K \neq 1$. Test the hypothesis and report your inference at 95% confidence level.
(`Hint`: $Var(\hat{\beta}_L + \hat{\beta}_K) = \text{Var}(\hat{\beta}_L) + \text{Var}(\hat{\beta}_K) + 2 * \text{cov}(\hat{\beta}_L, \hat{\beta}_K)$.)

(c) A generalization of the Cobb-Douglas model is the *translog* model, which is

$$\log Y = \beta_1 + \beta_2 \log L + \beta_2 \log K + \beta_4\left(\frac{1}{2}\log^2 L\right) + \beta_5\left(\frac{1}{2}\log^2 K\right) + \beta_6(\log L * \log K) + \epsilon,. \quad (1)$$

Estimate the model and report the regression coefficients, standard errors, and t-statistics in a table. Also, report the covariance matrix of $\hat{\beta}$ and the R-square.

(d) Test the hypothesis that the Cobb-Douglas model is the appropriate model i.e., test $H_0 : \beta_4 = \beta_5 = \beta_6 = 0$, versus $H_1 : H_0$ `is not true`. Report the F-statistic and draw your inference at 95% confidence level.

(e) The hypothesis of constant returns to scale can also be tested from the translog model. In this case, the null hypothesis is $H_0 : \beta_2 + \beta_3 = 1$ and $\beta_4 + \beta_5 + 2\beta_6 = 0$, versus $H_1 : H_0$ `is not true`. Report your F-statistics and draw your inference at 95% confidence level.