Q1 Roll No : Sudha 2020900003

1 Problem Set for In-class Problem Solving

Answer with sufficient details with insightful discussions (not just factual answers) and submit by 11am. (details discussed in the last two lecture sessions)

1. Consider a linearly separable data

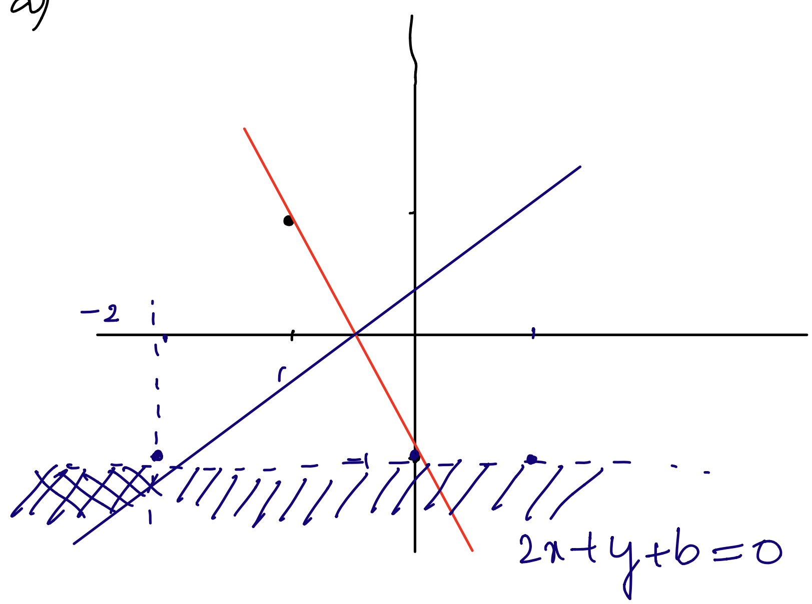
$$(-1,+1),(0,-1),(+1,-1)$$

   (a) Write the primal problem. Geometrically show the feasible region in a w, b 2D plane. Show the optimal w as -2 and b as -1. Geometrically validate the solution.

   (b) Write the dual objective $J(\alpha_1, \alpha_2, \alpha_3)$. And the constraint $\sum_i \alpha_i y_i = 0$. Show that the solution is $\alpha_1 = \alpha_2 = 2$ and $\alpha_3 = 0$. Which are the support vectors then?

   (c) Show that the dual solution also leads to the same w and b.

Sol a)



Primal equation

$\min \frac{1}{2}\|w\|$ subject to $y_i(w^T x_i + b) \geq 1$

$+ (wx + b) \geq 1$

$\boxed{-w + b \geq 1}$ $\quad -2x + -1 = 0, x = -\frac{1}{2}$

(b) Dual objective is $\sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j x_i^T x_j$

Take derivative of dual objective function

$$\alpha_1 * \alpha_2 * \alpha_3 - \frac{1}{2}\left(\alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3(-1)\right)$$

$$\boxed{\alpha_1 + \alpha_2 + \alpha_3 - \frac{1}{2}(\alpha_1 + \alpha_3)^2} \quad \text{---} \ (1)$$

Equate to 0

$$\sum \alpha_i y_i = 0 \Rightarrow \alpha_1 - \alpha_2 - \alpha_3 = 0$$

$$\alpha_2 = \alpha_1 - \alpha_3 \quad \text{---} \ (2)$$

$$\boxed{2\alpha_1 - \frac{1}{2}\left[\alpha_1 + \alpha_3\right]^2} \Rightarrow \alpha_3 = 0 \ \text{---}$$

$$\boxed{wx + b = +1}$$

$$2\alpha_1 - \frac{1}{2}\alpha_1^2$$

$$\boxed{\text{Max} \ -\frac{1}{2}(\alpha_1 - 2)^2 + 2} \quad \text{---} \ (3)$$

$\alpha_1 = 2$

(c)

$$\alpha_1 = 2 \quad \alpha_2 = 2 \quad \alpha_3 = 0$$

$$\boxed{w = \sum_i \alpha_i y_i x_i} \qquad \boxed{\begin{matrix} w = -2 \\ b = -1 \end{matrix}}$$

data points $(-1, +1)$ $(0, -1)$ $(1, -1)$

$w = \{ 2(-1)(+1) + 0 + 0 \}$

$w = -2$.

and $y(w^T x + b) \geq 1$

$b = -1$

Q2.

(a) $\sum_{i=1}^{N} \alpha_i - \sum_i \sum_j \alpha_i \alpha_j y_i y_j \boxed{K(x_i, x_j)}$

$$K = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ x_1 & 4 & 1 & 0 \\ x_2 & 1 & 1 & 1 \\ x_3 & 0 & 1 & 4 \end{array}$$

eg: $((-1)(-1)+1)^2 = 4$

substituted $-1, 0, 1$ as x_1, x_2, x_3

$K(p, q) = (p^T q + 1)^2$

Substitute $\alpha_i y_i = 0$

$\alpha_1(1) - \alpha_2 + \alpha_3 = 0$.

$$\boxed{\alpha_2 = \alpha_1 + \alpha_3}$$

(b) Eliminate α_2 in objective function

$\alpha_1 + \alpha_2 + \alpha_3 - \frac{1}{2}\left(4\alpha_1^2 + \alpha_2^2 + 4\alpha_3^2 - \alpha_1\alpha_2 + \alpha_2\alpha_3\right)$

$$2\alpha_1 + 2\alpha_3 - \left(2\alpha_1^2 + 2\alpha_3^2 - \frac{1}{2}(\alpha_1 + \alpha_3)^2\right)$$

(3) Take derivative with respect to α and equate to 0

w.r.t α_1

$$2 - (4\alpha_1 + \alpha_1 + \alpha_3) = 2 - 3\alpha_1 + \alpha_3 = 0$$

w.r.t α_3

$$2 + \alpha_1 - 3\alpha_3 = 0$$

$$\boxed{\alpha_1 = 1 \quad \alpha_3 = 1 \quad \alpha_2 = 2} \ * \ \text{——— (1)}$$

(4). $2x^2 - 1 = 0$

$$x^2 = 1/2$$

$$\text{sign}\left(\sum_i \alpha_i y_i K(x_i, x) + b\right)$$

$$= \alpha_1 y_1 K(x_1, x) + \alpha_2 y_2 K(x_2, x) + \alpha_3 y_3 K(x_3, x) - 1$$

From eqn (1).

$$= 1 \cdot 1 \cdot (-x+1)^2$$

$$2(-1) \cdot (0 \cdot x + 1)^2$$

$$1 \cdot (+1)(x \cdot 1 + 1)^2 - 1$$

$$K \quad \begin{array}{c|ccc} & x & x & x \\ \hline x_1 & & & \\ x_2 & & & \\ x_3 & & & \end{array}$$

$$(x_i \cdot x + 1)^2$$

$$= x^2 \cancel{2x} + 1$$

$$-2$$

$$x^2 + \cancel{2x} + 1 \quad - 1$$

$$(x_2 \cdot x + 1)^2$$

$$= 2x^2 - 1$$

Q3. $K(p, q) = \phi(x\ )$

	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
$(-1,1)$	9	1	1	1
$(-1,1)$	1	9	1	1
$(+1,-1)$	1	1	9	1
$(+1,+1)$	1	1	1	9

Eg. :- $K(x_i, x_j) = 1 - x_i x_j))^2$

$$= (1 + x_i^1 x_j^1 + x_i^2 x_j^2)^2$$

$$= 1 + (x_i^1)^2 (x_j^1)^2 + 2(x_i^1)(x_j^1)(x_i^2)(x_j^2)$$
$$+ (x_i^2)^2 (x_j^2)^2 + 2(x_i^1)(x_j^1) + 2(x_i^2)(x_j^2)$$

$— (1)$

Q13b. From eqn 1

$$\bar{L}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}(9\alpha_1^2$$
$$-2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 - 2\alpha_1\alpha_3 + 2\alpha_1\alpha_4$$
$$+9\alpha_2^2 + 2\alpha_2\alpha_3 - 2\alpha_2\alpha_4 + 9\alpha_3^2 - 2\alpha_3\alpha_4$$
$$+9\alpha_4^2$$

Q3(c) Differential w.r.t $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$\frac{\partial L}{\partial \alpha_1} = 1 - 9\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$$

$$\frac{\partial L}{\partial \alpha_2} = 1 + \alpha_1 - 9\alpha_2 - \alpha_3 + \alpha_4 = 0$$

$$\frac{\partial L}{\partial \alpha_3} = 1 + \alpha_1 - \alpha_2 - 9\alpha_3 + \alpha_4 = 0$$

$$\frac{\partial L}{\partial \alpha_4} = 1 - \alpha_1 + \alpha_2 + \alpha_3 - 9\alpha_4 = 0$$

Q d) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{8}$

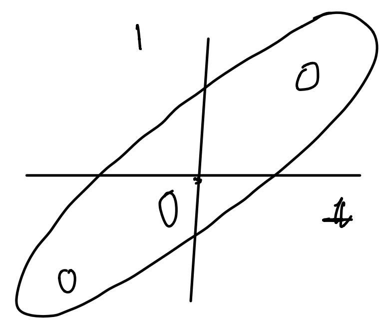
$\bar{L} = 1/4$

Yes, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the same.

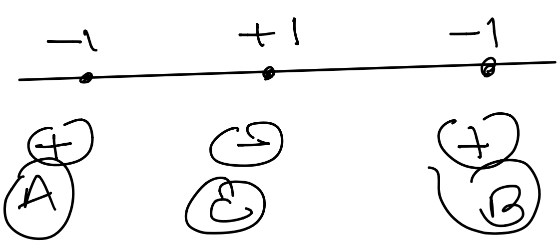
e) $\phi() = [1, x_1^2, \sqrt{2}x_1, x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2]^T$

$$w = \sum_{i=1}^{N} \alpha_i y_i \phi(x_i)$$

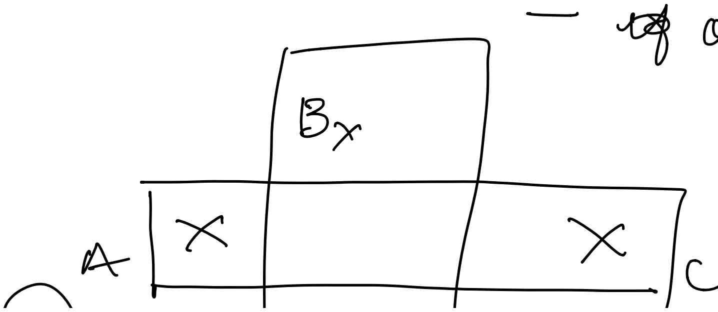
Because of $\sqrt{2}$, the data becomes linearly seperable.



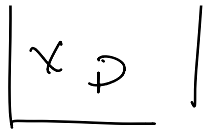
Q4

① 

② Vc dimension is $+$ if $x > \alpha$ → ①
$-$ ~~if~~ if otherwise

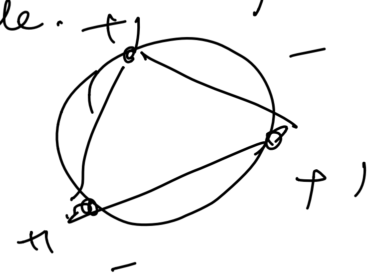


(3)

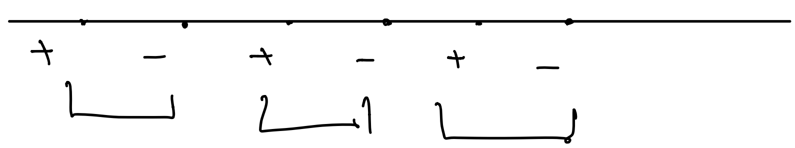


Any set of 5 cannot be seperated

(4) VC dimension of convex polynomial is ∞ eg. consider a circle.



Union $\bigcup [\alpha_i \; \alpha_j]$



Q5 $L(w,b,a,\varepsilon) = \frac{1}{2} w^T w + \frac{C}{2} \sum_{i=1}^{N} \varepsilon_i^2 - \sum_{i=1}^{N} \alpha_i \left[y_i (w^T x_i + b) - 1 + \varepsilon_i \right]$

L2 dual function for SVM

$$\frac{1}{2}\|w\|^2 + \frac{C}{2}\sum_{i=1}^{N} \xi_i^2 \quad \text{and } \alpha_i \geq 0$$

$$\forall_i \in \{1 \ldots n\} \quad \xi_i \geq 0$$