Damped oscillator (Oscillation under dissipative force)

$$\boxed{m\ddot{x} + m\gamma\dot{x} + kx = 0}$$

$\frac{k}{m} = \omega^2$

$\gamma >$ positive constant

$$\Rightarrow \ddot{x} + \gamma\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \boxed{\ddot{x} + \gamma\dot{x} + \omega^2 x = 0}$$

Then we solve this equation under different choices of parameters: Trial sol^n $\Rightarrow \boxed{x(t) = Ae^{-\lambda t}}$

$$\boxed{\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2} = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4\omega^2}}{2}}$$

~~Case 1: $\gamma^2 < 4\omega^2 \rightarrow$~~

$$x(t) = A_1 \exp\left[-\frac{\gamma}{2} + \frac{\sqrt{\gamma^2 - 4\omega^2}}{2}\right]t + A_2 \exp\left[-\frac{\gamma}{2} - \frac{\sqrt{\gamma^2 - 4\omega^2}}{2}\right]t$$

Where A_1 and A_2 are arbitary constants to be determined from the ~~intial~~ initial condition

(1) $\gamma^2 < 4\omega^2 \rightarrow$ (Small damping) — underdamped

$$4\omega^2 - \gamma^2 = 4\omega_s^2 \Rightarrow \boxed{\omega_s^2 = \omega^2 - \frac{\gamma^2}{4}} \Rightarrow \omega_s = \left(\omega^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

$\gamma = 0$ ($\omega_s = \omega$)

$$x(t) = e^{-\gamma/2\, t}\left[A_1 e^{i\omega_s t} + A_2 e^{-i\omega_s t}\right]$$

$$= e^{-\gamma/2\, t}\left[A_1 \cos\omega_s t + A_2 \cos\omega_s t + i(A_1 \sin\omega_s t + A_2 \sin\omega_s t)\right]$$

$\cancel{x(t) = A e^{-\gamma t/2} \cos\omega_s t}$ $\quad A_1 + A_2$

$\cancel{= A e}$ $\quad \underline{A_1 + A_2 = A\cos\delta} \quad = \underline{A\cos\delta}.$

$\underline{i(A_1 - A_2)} = A\sin\delta$

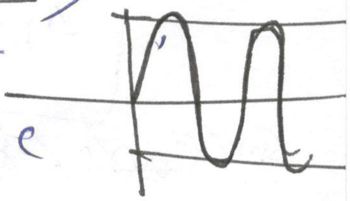
$$x(t) = A e^{-\frac{\gamma t}{2}} \left[\underline{\cos\omega_s t \cos\delta} + \underline{\sin\omega_s t \cdot \sin\delta} \right]$$

$$= \underline{A e^{-\gamma t/2} \cos(\omega_s t + \delta)}$$ ω_s is different from ω.

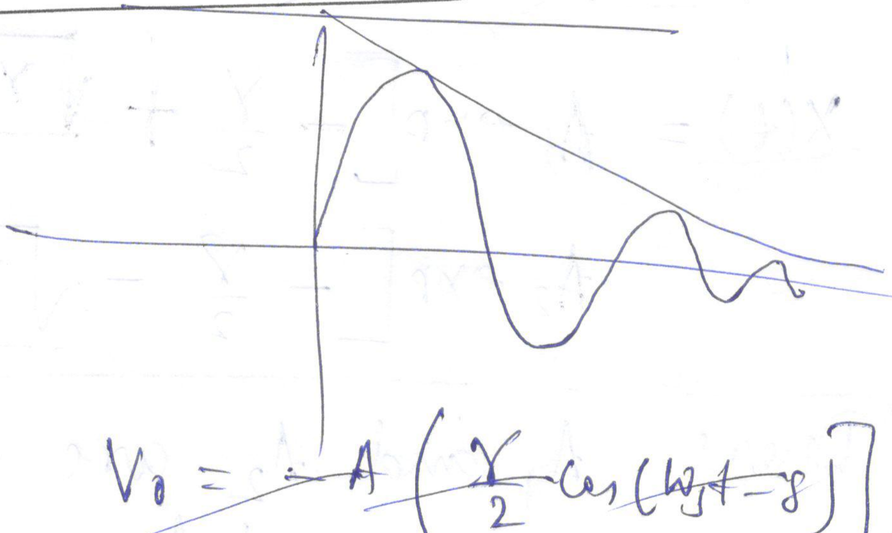
$\Downarrow$

Amplitude decays exponentially.

Not SHM — not periodic



$t = 0, \quad x = 0$
$t = 0, \quad \dot{x} = V_0$



$\cancel{0 = A\cos\delta}$ $\quad V_0 = \cancel{-A\left(\frac{\gamma}{2}\cos(\omega_s t - \delta\right)}$

$\cancel{V_0 = +A}\,[$ $\quad x(t) = A e^{-\frac{\gamma t}{2}} \left[\cos\omega_s t \cos\delta + \sin\omega_s t \sin\delta \right]$

$\cancel{x=0}$ — $\quad 0 = A * 1 \cdot [\cos\delta + 0]$

$\boxed{A\cos\delta = 0}$ $\quad V_0 = A[\omega_s \sin\delta]$

$\cancel{V_0 = -A\left[\frac{\gamma}{2}\cos\delta - \omega_s \sin\delta\right]}$ $\quad + A \cdot \frac{\gamma}{2}\cos\delta$

$\dot{x}(t) = A e^{-\gamma t/2}[-\omega_s \sin\omega_s t \cos\delta + \omega_s \cos\omega_s t \sin\delta] - A\frac{\gamma}{2} e^{-\gamma t}[\quad]$

$$\boxed{V_0 = A\left[\omega_s \sin\delta - \frac{\gamma}{2}\omega s\delta\right]}$$

$$V_0 = A\,\omega_s \sin\delta - \frac{\gamma}{2} A \cancel{\cos\delta}$$

$A\cos\delta = 0$

$A = 0$ X

$\cos\delta = 0$

$\Rightarrow \delta = \pi/2$

$$V_0 = A\,\omega_s \sin\delta \Rightarrow A\,\omega_s \sin\delta = V_0$$

$$\Rightarrow \boxed{A = \frac{V_0}{\omega_s \sin\delta}} \Rightarrow \boxed{A = \frac{V_0}{\omega_s \sin\delta}}$$

$$\delta = \pi/2 \Rightarrow \boxed{A = \frac{V_0}{\omega_s}}$$

$$V_0 = -A\left(\frac{\gamma}{2}\cos\delta - \omega_s \sin\delta\right)$$

$$0 = A\cos\delta \longrightarrow A = 0 \quad \delta = \pi/2$$

$$V_0 = -A\left[\frac{\gamma}{\cancel{2}} \times 0 - \omega_s \times 1\right]$$

$\cos(\pi/2 - \theta) = \sin\theta$

$\cos(-\theta) = \cos\theta$

$$\Rightarrow A \ast \omega_s = V_0 \Rightarrow \boxed{A = \frac{V_0}{\omega_s}}$$

$$X(t) = \frac{V_0}{\omega_s} e^{-\gamma t/2}\left[\cos(\omega_s t - \delta)\right]$$

γ^2

ω

$$= \frac{V_0}{\omega_s} e^{-\frac{\gamma t}{2}}\left[\cos\left(\omega_s t - \pi/2\right)\right]$$

$$X(t) = \frac{V_0}{\omega_s} e^{-\frac{\gamma t}{2}} \sin \omega_s t$$

$$\cancel{\dot{x}(t) = v_0 e^{-\gamma t/2} \left[\cos \omega_s t - \frac{\gamma}{\omega_s} \right.}$$

$$x(t) = \frac{v_0}{\omega_s} e^{-\gamma t/2} \sin \omega_s t$$

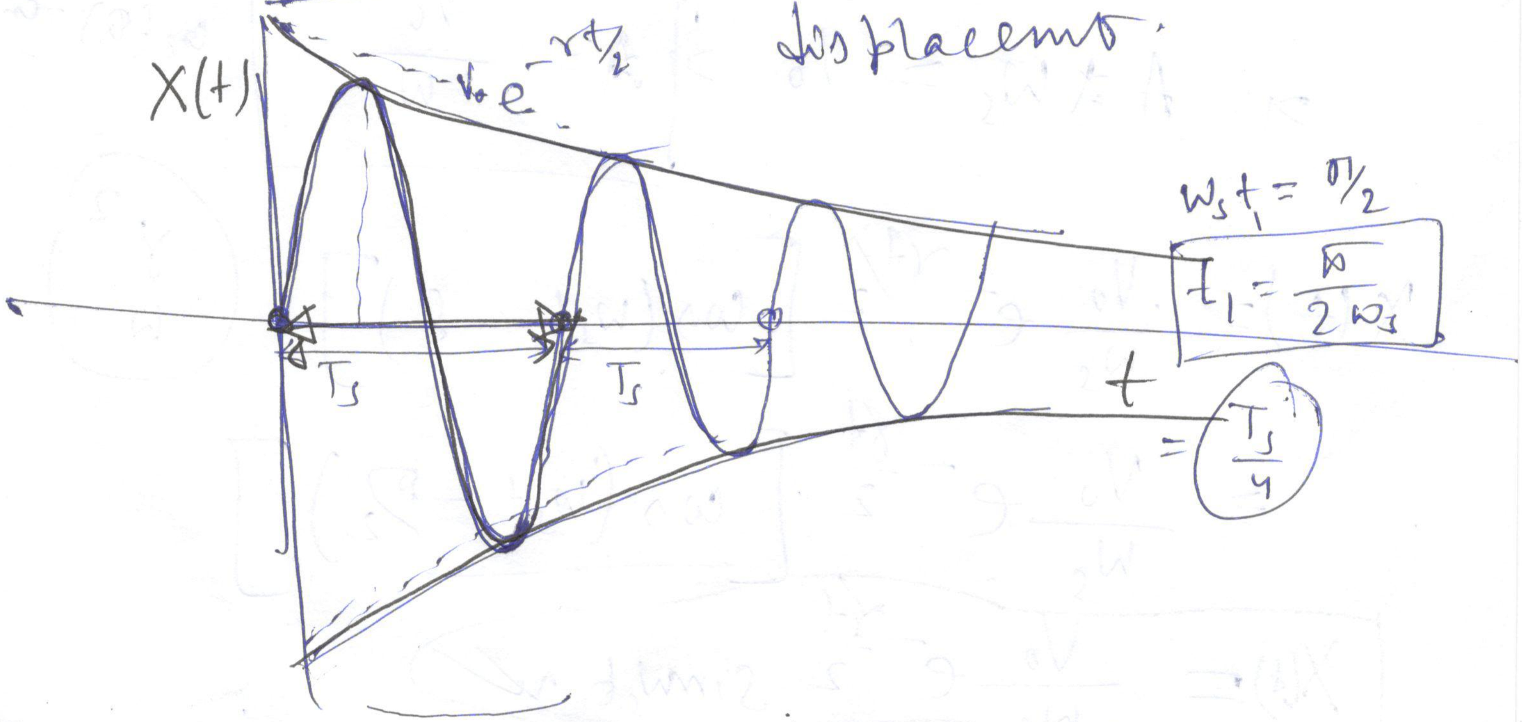
$$\dot{x}(t) = \frac{V_0}{\omega_s} \left[-\frac{\gamma}{2} e^{-\frac{\gamma t}{2}} \sin \omega_s t + \omega_s \cos \omega_s t \, e^{-\gamma t/2} \right]$$

$$= \frac{V_0}{\omega_s} e^{-\gamma t/2} \left[-\frac{\gamma}{2} \sin \omega_s t + \omega_s \cos \omega_s t \right]$$

$$= V_0 e^{-\frac{\gamma t}{2}} \left[-\frac{\gamma}{2\omega_s} \sin \omega_s t + \cos \omega_s t \right]$$

$$\boxed{\dot{x}(t) = V_0 e^{-\frac{\gamma t}{2}} \left[\cos \omega_s t - \frac{\gamma}{2\omega_s} \sin \omega_s t \right]}$$

$\boxed{T_s = \frac{2\pi}{\omega_s}}$ → time interval betn two alternative zeros of displacement.



$\omega_s t_1 = \pi/2$

$\boxed{t_1 = \frac{\pi}{2\omega_s}}$

$= \frac{T_s}{4}$

Energy of a weakly damped Oscillator

$\boxed{\gamma^2 < 4\omega^2 \rightarrow \text{weakly damped.}}$

K.E $\Rightarrow$ Instantaneous kinetic energy

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}mA^2e^{-\gamma t}\left[\omega_s\sin(\omega_s t-\delta) + \frac{\gamma}{2}\cos(\omega_s t-\delta)\right]^2$$

$$= \frac{1}{2}mA^2e^{-\gamma t}\left[\omega_s^2\sin^2(\omega_s t-\delta) + \frac{\gamma^2}{4}\cos^2(\omega_s t-\delta) + \omega_s\gamma\sin(\omega_s t-\delta)\cos(\omega_s t-\delta)\right]$$

Instantaneous potential energy

$$P.E. = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

$\omega^2 = \frac{k}{m}$

$k = m\omega^2$

$$= \frac{1}{2}mA^2\omega^2e^{-\gamma t}\cos^2(\omega_s t-\delta)$$

$\sin\theta\cos\theta$

$\sin 2\theta$
$= 2\sin\theta\cos\theta$

$$E(t) = K.E + P.E$$

$$= \frac{1}{2}mA^2\left[\omega_s^2\sin^2(\omega_s t-\delta) + \frac{\gamma^2}{4}\cos^2(\omega_s t-\delta) + \omega_s\gamma\sin(\omega_s t-\delta)\cos(\omega_s t-\delta) + \omega^2\cos^2(\omega_s t-\delta)\right]$$

$+ e^{-\gamma t}$

$$= \frac{1}{2}mA^2e^{-\gamma t}\left[\omega_s^2\sin^2(\omega_s t-\delta) + \left(\frac{\gamma^2}{4}+\omega^2\right)\cos^2(\omega_s t-\delta) + \frac{\omega_s\gamma}{2}\sin[2(\omega_s t-\delta)]\right]$$

$\gamma < 2\omega$

$$\left\langle E(t) \right\rangle = \underline{\frac{1}{2} m A^2 e^{-\gamma t}} \Big\{ \underline{\omega_s^2 \left\langle \sin^2(\omega_s t - \delta) \right\rangle}$$

$$+ \frac{\omega_s \gamma}{2} \underline{\left\langle \sin 2(\omega_s t - \delta) \right\rangle} + \left(\frac{\gamma^2}{4} + \omega^2\right) \left\langle \cos^2(\omega_s t - \delta) \right\rangle$$

$\underline{\underline{\left\langle \right\rangle}} \Rightarrow$ implies averaging over one
$\underline{\text{time period } T_s}$ $\qquad f(t) = \dfrac{\int_0^T f(t)\,dt}{\int_0^T dt} = \dfrac{1}{T}\int_0^T f(t)\,dt$

$\omega_s = \frac{2\pi}{T}$

$$\underline{\left\langle \sin^2(\omega_s t - \delta) \right\rangle} = \frac{1}{T_s} \int_0^{T_s} \sin^2\left(\frac{2\pi t}{T_s} - \delta\right) dt.$$

$$\boxed{\frac{2\pi t}{T_s} - \delta = \alpha} \qquad \underline{\frac{2\pi}{T} dt = d\alpha}.$$

$\omega_s t - \delta =$

$\frac{2\pi t}{T_s}$

$$= \frac{1}{T_s} \int_{-\delta}^{2\pi - \delta} \sin^2\alpha \cdot \left(\frac{T_s}{2\pi}\right) d\alpha \quad \Bigg| \quad 1 - \cos^2\alpha \quad 1 -$$

$$= \frac{1}{\cancel{T_s}} \times \left(\frac{\cancel{T_s}}{2\pi}\right) \int_{-\delta}^{2\pi - \delta} \sin^2\alpha \, . \, d\alpha \qquad 1 - \cos 2\alpha$$

$$= \frac{1}{2\pi} \int_{-\delta}^{2\pi - \delta} \sin^2\alpha \, d\alpha = \cancel{\frac{1}{2\pi}\left[\int_{-\delta}^{2\pi-\delta} (1 - \cos 2\alpha\right.}$$

$$= \frac{1}{2\pi}\int_{-\delta}^{2\pi-\delta} \sin^2\alpha \, d\alpha$$

$\cos 2\alpha$

$= \cos^2\alpha - \sin^2\alpha$

$= (1-\sin^2\alpha) - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$

$$= \frac{1}{4\pi}\int_{-\delta}^{2\pi-\delta} (1-\cos 2\alpha)\, d\alpha$$

$2\sin^2\alpha = 1 - \cos 2\alpha$

$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

$$= \frac{1}{4\pi}\left[\alpha\Big|_{-\delta}^{2\pi-\delta} + \frac{\sin 2\alpha}{2}\Big|_{-\delta}^{2\pi-\delta}\right]$$

$$= \frac{1}{4\pi}\left[(2\pi-\cancel{\delta}+\cancel{\delta}) + \frac{\sin 2\alpha}{2}\Big|_{-\delta}^{2\pi-\delta}\right]$$

$\int \cos\theta \, d\theta$

$= -\sin\theta$

$$= \frac{1}{4\pi} \times 2\pi \cancel{\times} + \cancel{\frac{\sin 2\alpha}{2}\Big|_{0}^{2\pi}}$$

$-\delta = 0$

$$= \frac{1}{2}$$

$$\boxed{\langle \cos^2(\omega_s t - \delta)\rangle = \frac{1}{2}}$$

$$\boxed{\langle \sin 2(\omega_s t - \delta)\rangle = 0}$$

$$\Rightarrow E(t) = \frac{1}{2} m\omega^2 A^2 e^{-\gamma t}\left(\cancel{\frac{1}{2} + \frac{1}{2} + 0}\right)$$

$$\left[\frac{1}{2} \times \omega_s^2 + \frac{1}{2}\left(\frac{\gamma^2}{4} + \omega^2\right)\right]$$

$$= \frac{1}{4} m A^2 e^{-\gamma t}\left(\omega_s^2 + \omega^2 + \frac{\gamma^2}{4}\right)$$

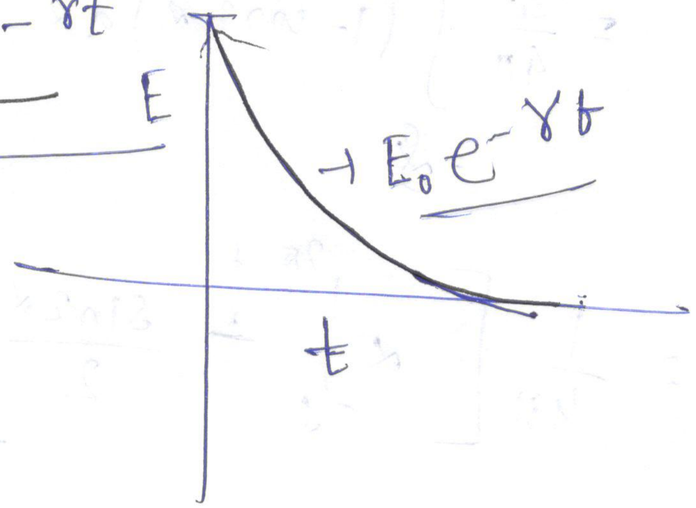
$$\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\Rightarrow \langle E(t)\rangle = \frac{1}{4} m A^2 e^{-\gamma t}\left[\omega^2 - \cancel{\frac{\gamma^2}{4}} + \cancel{\frac{\gamma^2}{4}} + \omega^2\right]$$

$$= \frac{1}{2} m \omega^2 A^2 e^{-\gamma t}$$

$$\boxed{\langle E(t)\rangle = E_0 e^{-\gamma t}}$$

$\gamma \neq 0$



Power dissipation

$$\langle P(t)\rangle = \text{rate of loss of energy}$$

$$= \frac{d}{dt}\langle E(t)\rangle$$

$$= \gamma \langle E(t)\rangle$$

Quality Factor: Rate at which energy of a damped oscillator decays. —

Q = quality factor $= \dfrac{\omega_s}{\gamma}$

Where $\boxed{\omega_s = \omega\left(1 - \dfrac{\gamma^2}{4\omega^2}\right)^{1/2}}$ → Angular frequency of the damped oscillator

Q: measures the rate of decay of energy

$$\frac{d}{dt}\langle E(t)\rangle = \langle P(t)\rangle = \underline{\gamma\langle E(t)\rangle}.$$

The average energy dissipated in time period

$$\left(T_s = \frac{2\pi}{\omega_s}\right)$$

$$\underline{\gamma T_s\langle E(t)\rangle} = \frac{2\pi\gamma}{\omega_s}\left(\langle E(t)\rangle\right)$$

$$= \cancel{\frac{2\pi\gamma}{\omega_s} * \omega\left(1 - \frac{\gamma^2}{4\omega^2}\right)^{1/2}}$$

$$= \frac{2\pi\gamma}{\omega_s}\langle E(t)\rangle = \frac{2\pi}{Q}\langle E(t)\rangle$$

$= \dfrac{2\pi}{Q} *$ average stored energy

$= 2\pi * \dfrac{\text{avg. energy stored in one cycle}}{\text{avg. energy lost in one cycle}}$

$\gamma < 2\omega_0$

$\boxed{\omega_s \simeq \omega}$

When γ^2 is very small compared to $4\omega^2$

$$\boxed{Q = \frac{\omega_s}{\gamma} = \frac{\omega_0}{\gamma}}$$

$$\omega_s = \left(\omega^2 - \frac{\gamma^2}{4}\right)^{\frac{1}{2}}$$

$$= \omega\left(1 - \frac{\gamma^2}{4\omega^2}\right)^{\frac{1}{2}}$$

$$\simeq \omega$$

$Q > 1 \rightarrow$ Oscillating system with small rates of dissipation of energy.

Lower the damping larger the Q value is.

$$\gamma \rightarrow 0 \quad \rightarrow \quad Q \rightarrow \infty$$

$$X = A e^{-\gamma t/2} \cos(\omega_s t - \delta)$$

$$= A e^{-\left(\omega_0 t / 2Q\right)} \cos(\omega_s t - \delta)$$

$$\langle E(t) \rangle = E_0 e^{-\frac{\omega_0 t}{Q}}$$

It may be noted that Q is closely related to the number of oscillations over which total energy fall to $\left(\frac{1}{e}\right)$ of its original value E_0

$\hookrightarrow t = \tau$

$$\frac{\omega_0 \tau}{Q} = 1$$

$$\frac{E_0}{e} = E_0 e^{-\frac{\omega_0 \tau}{Q}}$$

$$e = e^{\omega_0 \tau / Q}$$

$$\tau = \frac{Q}{\omega} = \frac{TQ}{2\pi}$$

$T \rightarrow$ period of oscillation

$$\frac{\omega \tau}{Q} = 1$$

$$\tau = (Q/\omega)$$

During τ

n = number of complete oscillation executed is given by

$$\Rightarrow n = \frac{\omega}{2\pi} * \tau = \left(\frac{Q}{2\pi}\right)$$

Forced Oscillator

$$\omega_0^2 = \frac{k}{m}$$

$$\omega_0 = \sqrt{k/m}$$

$$m\ddot{x} + \gamma m\dot{x} + k^2 x = 0$$

$$m\ddot{x} + \gamma m\dot{x} + kx = F_0 \cos\omega t$$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos\omega t}$$

In homogeneous second-order linear differential equation

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0(t)} \quad (1)$$

How $f(t)$ will change the motion of the oscillator, we shall choose $\boxed{f(t) = f_0 \cos\omega t}$

Next step: Solve the eq. (1)

$$\gamma^2 < 4\omega_0^2 \longrightarrow \boxed{\omega_s = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}}$$

$f(t)$: external force — will impose its own frequency ω on the oscillator.

→ The actual motion in this case is some sort of superposition of two oscillations

⇒ one with → ω_1

⇒ other one with → ω

$$\left\{ \begin{aligned} x_1 &= A_1 \cos(\omega t + \phi_1) \\ x_2 &= A_2 \cos(\omega t + \phi_2) \end{aligned} \right\}$$

↳ Principle of superposition — resultant displacement should be the sum of the individual displacements

$$\vec{X} = x_1 + x_2$$

$$= A_1 \cos(\omega t + \phi) + A_2 \cos(\omega t - \phi)$$

$$= \underline{(A_1 \cos\phi_1 + A_2 \cos\phi_2)} \cos\omega t + (A_1 \sin\phi_1 + A_2 \sin\phi_2) \sin\omega t$$

$$= A \cos\delta \cos\omega t - A \sin\delta \sin\omega t$$

$$\boxed{X = A\cos(\omega t + \delta)} \quad \Big| \quad \delta = \tan^{-1}\left[\frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}\right]$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$$

$$= A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_2 - \phi_1)$$

$$x_1 = A_1 \cos\omega_1 t \quad \Big| \quad x_2 = A_2 \cos\omega_2 t$$

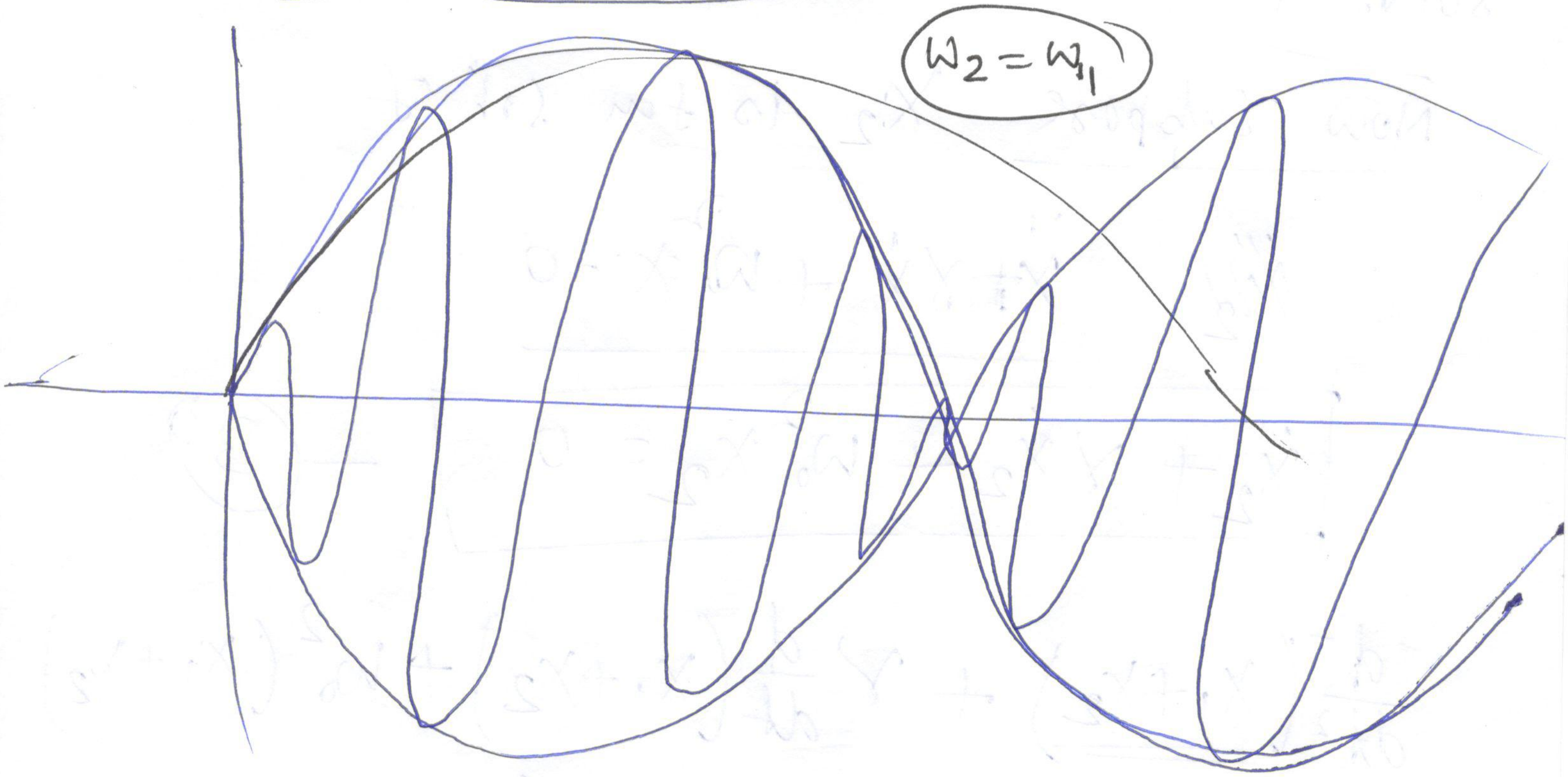
$$\left.\begin{aligned} x_1 &= A\cos\omega_1 t \\ x_2 &= A\cos\omega_2 t \end{aligned}\right\}$$

$$x = x_1 + x_2 = A\cos\omega_1 t + A\cos\omega_2 t$$

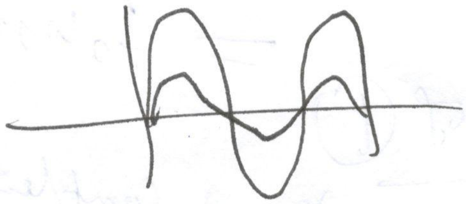
$$= A\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$= 2A\cos\left\{\left(\frac{\omega_1+\omega_2}{2}\right)t\right\}\cos\left\{\left(\frac{\omega_2-\omega_1}{2}\right)t\right\}$$

$$= \boxed{2A\cos\left(\frac{\omega_2-\omega_1}{2}\right)t} \cdot \cos\left(\frac{\omega_1+\omega_2}{2}\right)t$$



$\omega_2 = \omega_1$



$\omega_2 \simeq \omega_1$

$$m\ddot{x} + m\gamma\dot{x} + Kx = F_0 \cos\omega t$$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos\omega t} \quad \text{--- (1)}$$

$\gamma^2 < 4\omega_0^2$

ω_s $\qquad\qquad\qquad\qquad$ ω

~~trainsie~~ transient state

stady state

x_1 One of the sol^n of eq. 1

$$\boxed{\ddot{x}_1 + \gamma\dot{x}_1 + \omega_0^2 x_1 = f_0 \cos\omega t} \quad \text{--- (2)}$$

$\cancel{x_2}$

Now suppose x_2 is the sol^n of

$\cancel{x_2}$ $\quad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

$$\boxed{\ddot{x}_2 + \gamma\dot{x}_2 + \omega_0^2 x_2 = 0} \quad \text{--- (3)}$$

$$\frac{d^2}{dt^2}(x_1 + x_2) + \gamma\frac{d}{dt}(x_1 + x_2) + \omega_0^2(x_1 + x_2)$$

$$= f_0 \cos\omega t$$

$(x_1 + x_2)$ is a sol^n of of (1)

$x_1 \to$ particular solution $\qquad x_2 \Rightarrow$ complementary sol^n

$$\boxed{x_2 = A e^{-\gamma t/2} \cos(\omega_s t - \delta)}$$

↳ transient sol^n → Because of damping x_2 died out — the oscillator executes harmonic oscillations at frequency of the driving force. — The steady state —

In steady ~~sta~~ $x_1 = B\cos(\omega t - \phi)$

B and ϕ

$$X = x_1 + x_2 = A e^{-\gamma t/2} \cos(\omega_s t - \delta) + B\cos(\omega t - \phi)$$

$$\boxed{X = B\cos(\omega t - \phi)}$$

$$Z = B e^{i(\omega t - \phi)}$$

$$\ddot{Z} + \gamma \dot{Z} + \omega_0^2 Z = f_0 e^{i\omega t}$$

$$= f_0 e^{i(\omega t - \phi)} e^{i\phi}$$

$$B(\omega_0^2 - \omega^2 + i\omega\gamma) = f_0 e^{i\phi}$$

$$B = \frac{f_0 e^{i\phi}}{(\omega_0^2 - \omega^2 + i\omega\gamma)}$$

$$Be^{-i\phi} = \frac{f_0}{(\omega_0^2-\omega^2+i\omega\gamma)}$$

$$= \frac{f_0(\omega_0^2-\omega^2-i\omega\gamma)}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$B\cos\phi = \frac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$B\sin\phi = \frac{f_0\,\omega\gamma}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)}$$

$$B^2 = \frac{f_0^2(\omega_0^2-\omega^2)^2+f_0^2\omega^2\gamma^2}{\left((\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right)^2}$$

$$= \frac{f_0^2}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]}$$

$$\tan\phi = \frac{\omega\gamma}{(\omega_0^2-\omega^2)}$$

$$\phi = \tan^{-1}\left[\frac{\omega\gamma}{\omega_0^2-\omega^2}\right]$$

$$B = \frac{F_0/m}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^{1/2}}$$