Spatial Signal Processing (Beamforming)

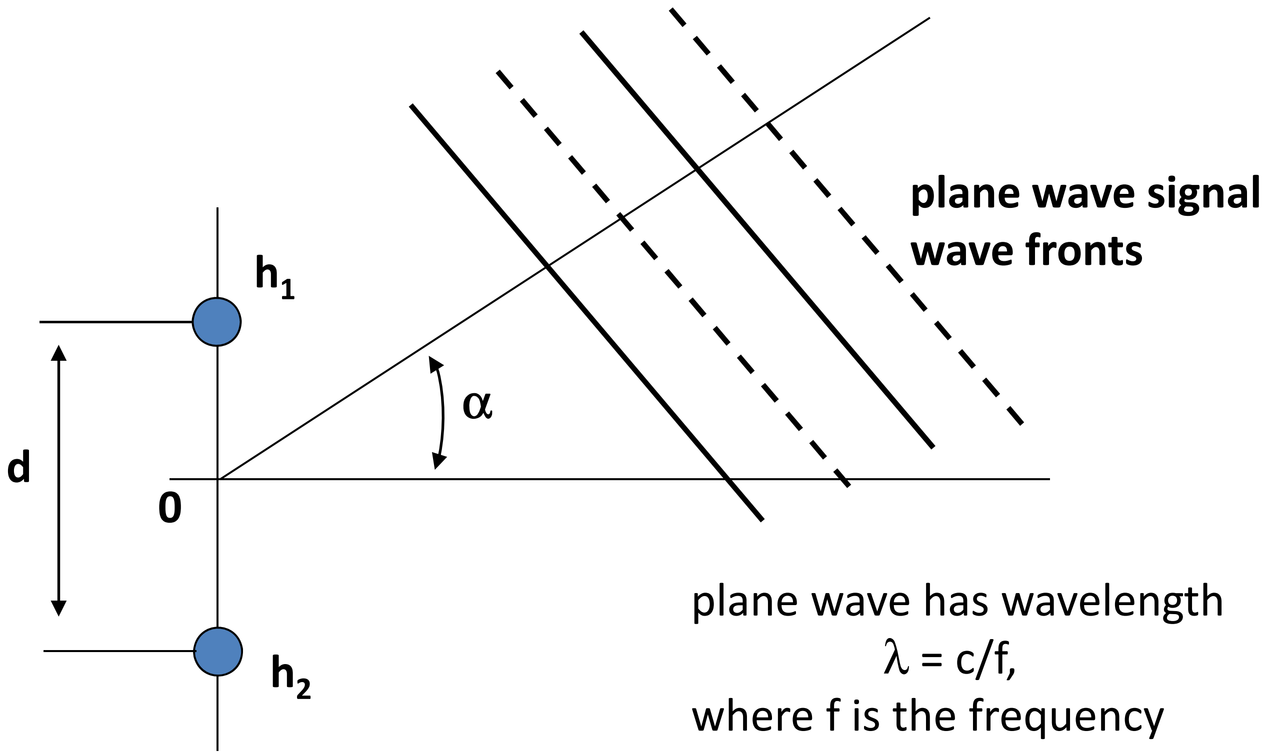
What Is Beamforming?

- Beamforming is spatial filtering, a means of transmitting or receiving sound preferentially in some directions over others.
- Beamforming is exactly analogous to frequency domain analysis of time signals.
- In time/frequency filtering, the frequency content of a time signal is revealed by its Fourier transform.
- In beamforming, the angular (directional) spectrum of a signal is revealed by Fourier analysis of the way sound excites different parts of the set of transducers.
- Beamforming can be accomplished physically (shaping and moving a transducer), electrically (analog delay circuitry), or mathematically (digital signal processing).

Beamforming Requirements

- Directivity – A beamformer is a spatial filter and can be used to increase the signal-to-noise ratio by blocking most of the noise outside the directions of interest.
- Side lobe control – No filter is ideal. Must balance main lobe directivity and side lobe levels, which are related.
- Beam steering – A beamformer can be electronically steered, with some degradation in performance.
- Beamformer pattern function is frequency dependent:
  - Main lobe narrows with increasing frequency
  - For beamformers made of discrete hydrophones, spatial aliasing (“grating lobes”) can occur when the the hydrophones are spaced a wavelength or greater apart.

A Simple Beamformer



h_1 h_1 are two omnidirectional hydrophones

plane wave has wavelength

$\lambda = c/f,$

where f is the frequency

c is the speed of sound

Analysis of Simple Beamformer

- **Given a signal incident at the center C of the array:**

$$s(t) = R(t) \cdot e^{i\omega(t)}$$

- **Then the signals at the two hydrophones are:**

$$s_i(t) = R(t) \cdot e^{i\omega(t)} e^{i\phi_i(t)}$$

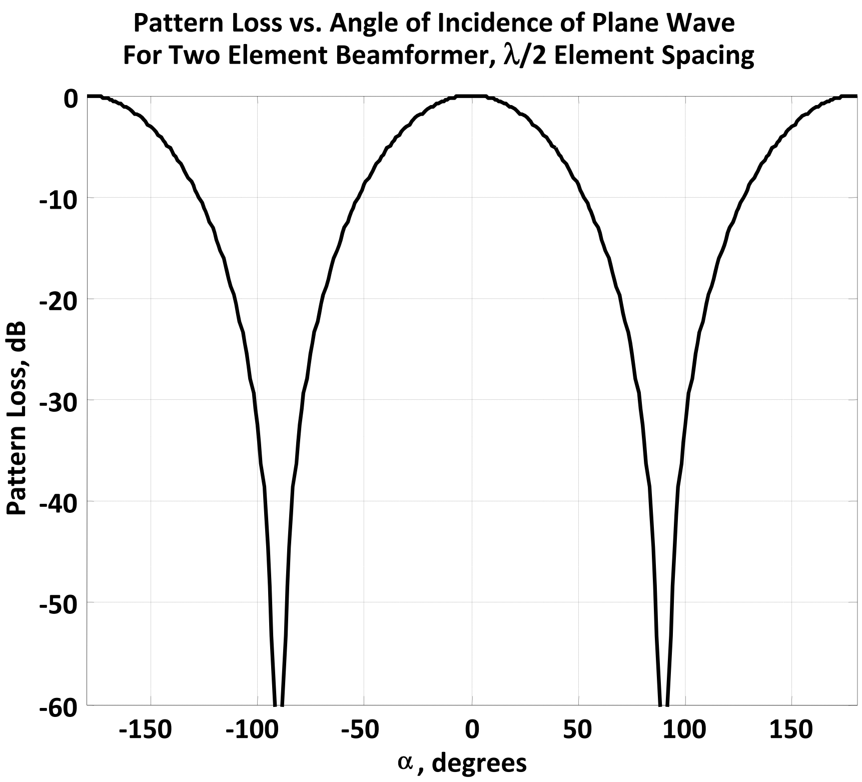
where

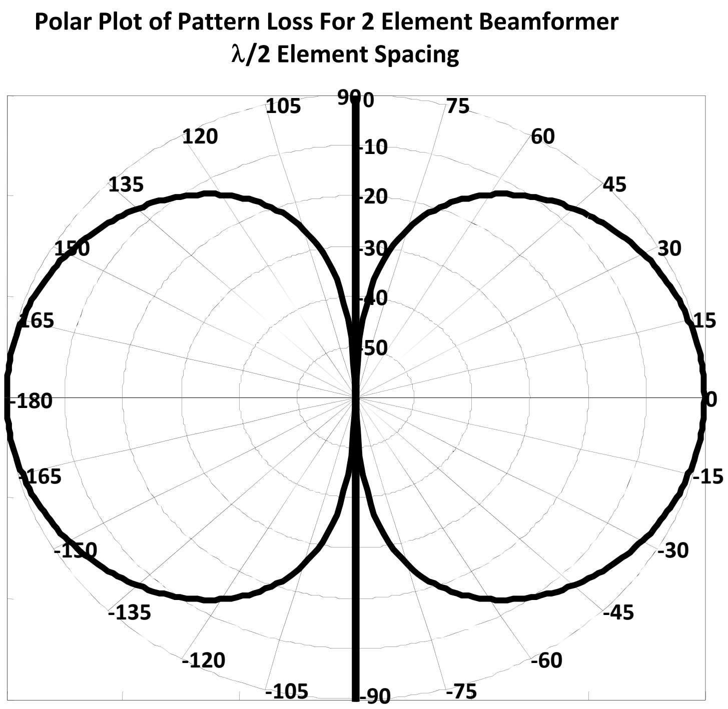
$$\phi_n = (-1)^n \frac{\pi d}{\lambda} \sin\alpha$$

- **The pattern function of the dipole is the normalized response of the dipole as a function of angle:**

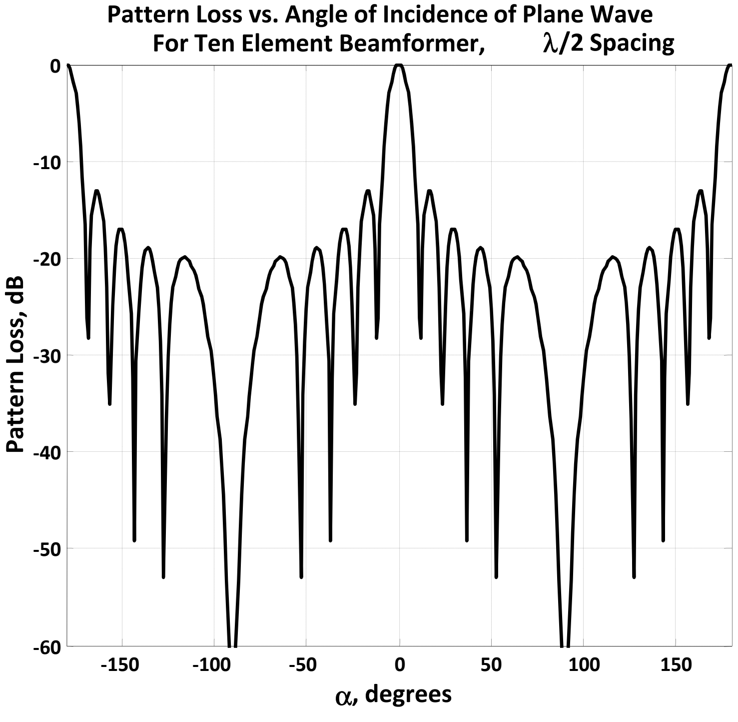
$$b(\alpha) = \frac{s_1 + s_2}{s} = \cos\left(\frac{\pi d}{\lambda} \sin\alpha\right)$$

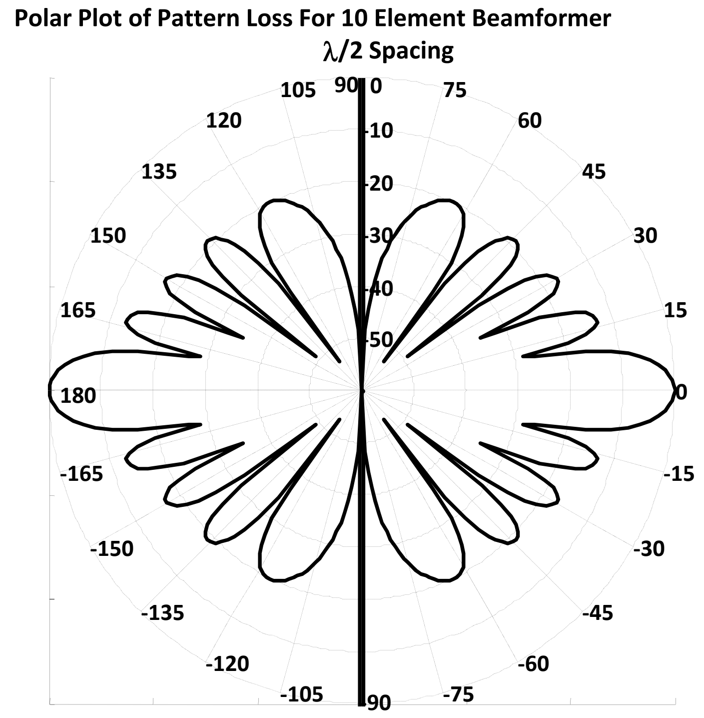
Beam Pattern of Simple Beamformer





Beam Pattern of a 10 Element Array

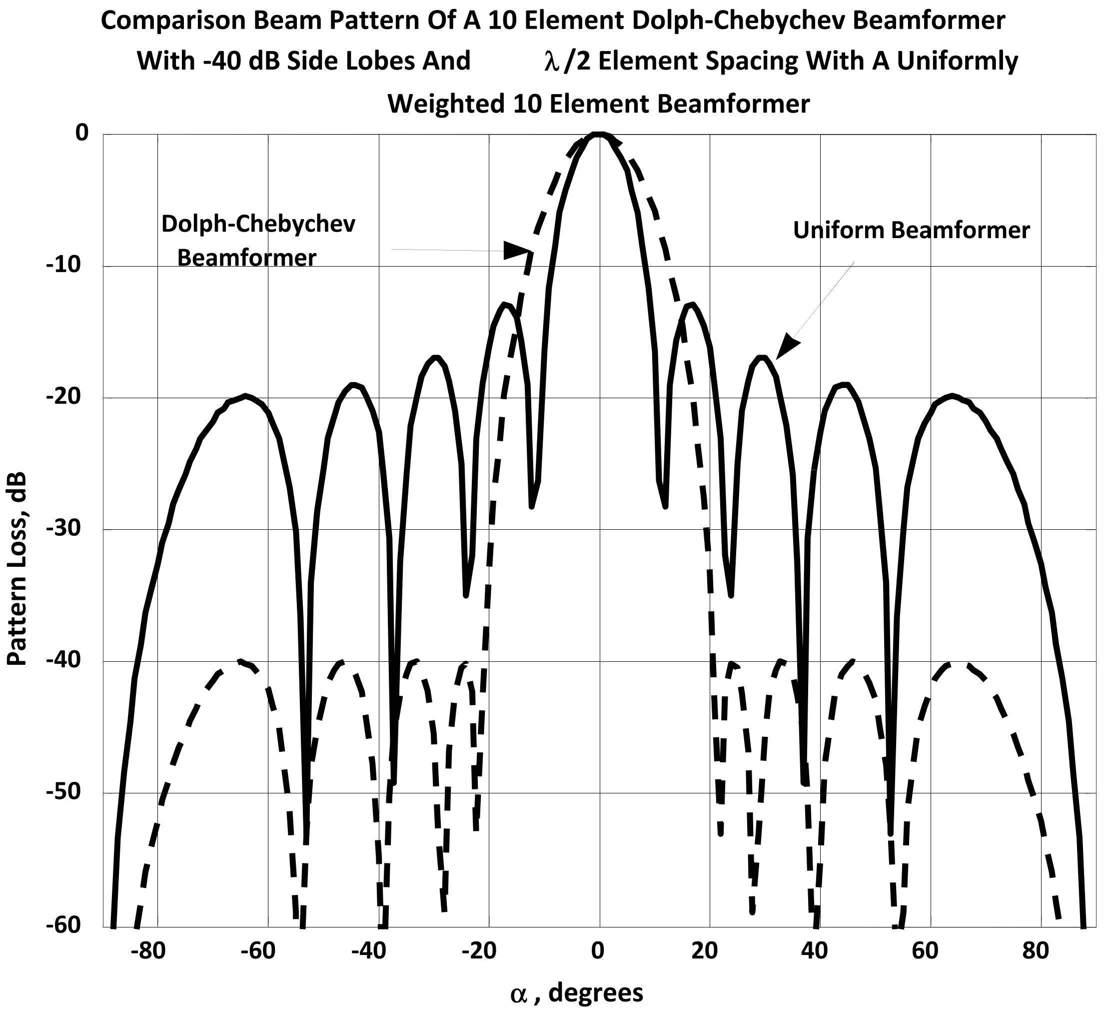




Beamforming – Amplitude Shading

- Amplitude shading is applied as a beamforming function.
- Each hydrophone signal is multiplied by a "shading weight"
- Effect on beam pattern:
  - Used to reduce side lobes
  - Results in main lobe broadening

Beam Pattern of a 10 Element Dolph-Chebychev Shaded Array



Analogy Between Spatial Filtering (Beamforming) and Time-Frequency Processing

Goals of Spatial Filtering:

1. **Increase SNR for plane wave signals in ambient ocean noise.**

2. **Resolve (distinguish between) plane wave signals arriving from different directions.**

3. **Measure the direction from which plane wave signals are arriving.**

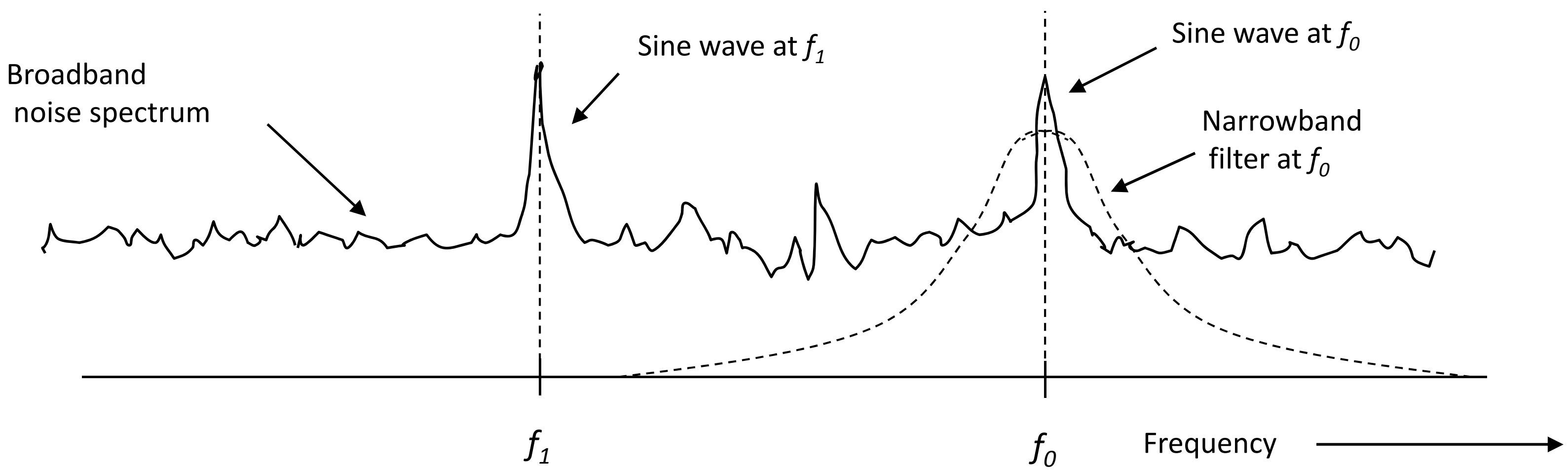
Goals of Time-Frequency Processing:

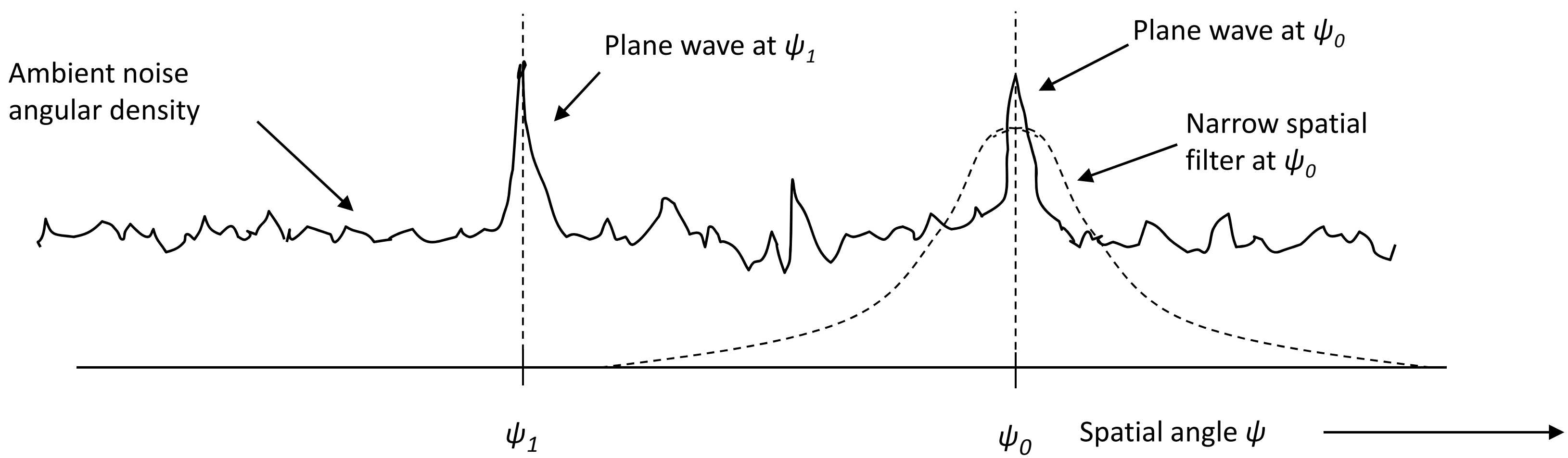
1. **Increase SNR for narrowband signals in broadband noise.**

2. **Resolve narrowband signals at different frequencies.**

3. **Measure the frequency of narrowband signals.**

Time-Frequency Filtering and Beamforming





SNR Calculation: Time-Frequency Filtering

Define

$$\alpha^2 \delta(f - f_0) \equiv \text{Signal power spectral density (W/Hz)}$$

$$|N(f)| \equiv \text{Noise power spectral density (W/Hz)}$$

$$|H(f)|^2 \equiv \text{Filter power response}$$

Signal Power Is:

$$P_s = \alpha^2 \int_{-\infty}^{\infty} \delta(f - f_0) |H(f)|^2 df = \alpha^2 |H(f_0)|^2 \quad \text{(watts)}$$

SNR Calculation: Time-Frequency Filtering (Cont'd)

If we assume

$$|H(f)|^2 = \begin{cases} 1, & -\dfrac{\beta}{2} \le f - f_0 \le \dfrac{\beta}{2} \\ 0, & \text{otherwise} \end{cases}$$

Idealized rectangular filter with bandwidth β

Then the noise power is:

$$P_N = \int_{-\infty}^{\infty} |N(f)| \cdot |H(f)|^2 df = N_0 \beta \quad \text{(watts)}$$

N_0 is the noise level in band

And SNR is:

$$SNR = \frac{P_s}{P_N} = \frac{\alpha^2}{N_0 \beta}$$

SNR Calculation: Spatial Filtering

Define

$$\alpha^2 \delta(\psi - \psi_0) \equiv \textbf{Signal power angular density (W/steradian)}$$

$$|N(\psi)| \equiv \textbf{Noise power angular density (W/steradian)}$$

$$|G(\psi)|^2 \equiv \textbf{Spatial filter angular power response}$$

Signal Power Is:

$$P_s = \alpha^2 \int_{4\pi} \delta(\psi - \psi_0) |G(\psi)|^2 df = \alpha^2 |G(\psi_0)|^2 \quad \textbf{(watts)}$$

SNR Calculation: Time-Frequency Filtering (Cont'd)

If we assume

$$|G(\psi)|^2 = \begin{cases} 1, & -\dfrac{\psi_\beta}{2} \le \psi - \psi_0 \le \dfrac{\psi_\beta}{2} \\ 0, & \text{otherwise} \end{cases}$$

Idealized "cookie cutter" beam pattern with width ψ_β

Then the noise power is:

$$P_N = \int_{4\pi} |N(\psi)| \cdot |G(\psi)|^2 d\Omega = \psi_\beta K \quad \text{(watts)}$$

K is the noise intensity in beam

And SNR is:

$$SNR = \frac{P_s}{P_N} = \frac{\alpha^2}{\psi_\beta K}$$

Array Gain and Directivity Calculations

<u>Define</u>

$$\textbf{Array Gain} = \frac{SNR_{Array}}{SNR_{OH}}$$

<u>Assume</u>

- plane wave signal
- arbitrary noise distribution

For the omnidirectional hydrophone,

$$\left|G(\Omega)\right|^2 = 1 \qquad \textbf{for all } \Omega$$

Then

$$SNR_{OH} = \frac{P_s}{P_N} = \frac{\alpha^2 \int_{4\pi} \delta(\Omega - \Omega_0)\left|G_{OH}(\Omega)\right|^2 d\Omega}{\int_{4\pi} \left|N(\Omega)\right| \cdot \left|G_{OH}(\Omega)\right|^2 d\Omega} = \frac{\alpha^2}{\int_{4\pi} \left|N(\Omega)\right| d\Omega}$$

Array Gain and Directivity Calculations (Cont'd)

For the array, assume it is steered in the direction of Ω_0 and that $\left|G^2(\Omega_0)\right|^2 = 1$

Then

$$SNR_{array} = \frac{P_s}{P_N} = \frac{\alpha^2 \int_{4\pi} \delta(\Omega - \Omega_0)\left|G_{array}(\Omega)\right|^2 d\Omega}{\int_{4\pi} \left|N(\Omega)\right| \cdot \left|G_{array}(\Omega)\right|^2 d\Omega} = \frac{\alpha^2}{\int_{4\pi} \left|N(\Omega)\right| \cdot \left|G_{array}(\Omega)\right|^2 d\Omega}$$

Putting these together yields

$$AG = \frac{\int_{4\pi} \left|N(\Omega)\right| d\Omega}{\int_{4\pi} \left|N(\Omega)\right| \cdot \left|G_{array}(\Omega)\right|^2 d\Omega}$$

Array Gain and Directivity Calculations (Cont'd)

If the noise is isotropic (the same from every direction)

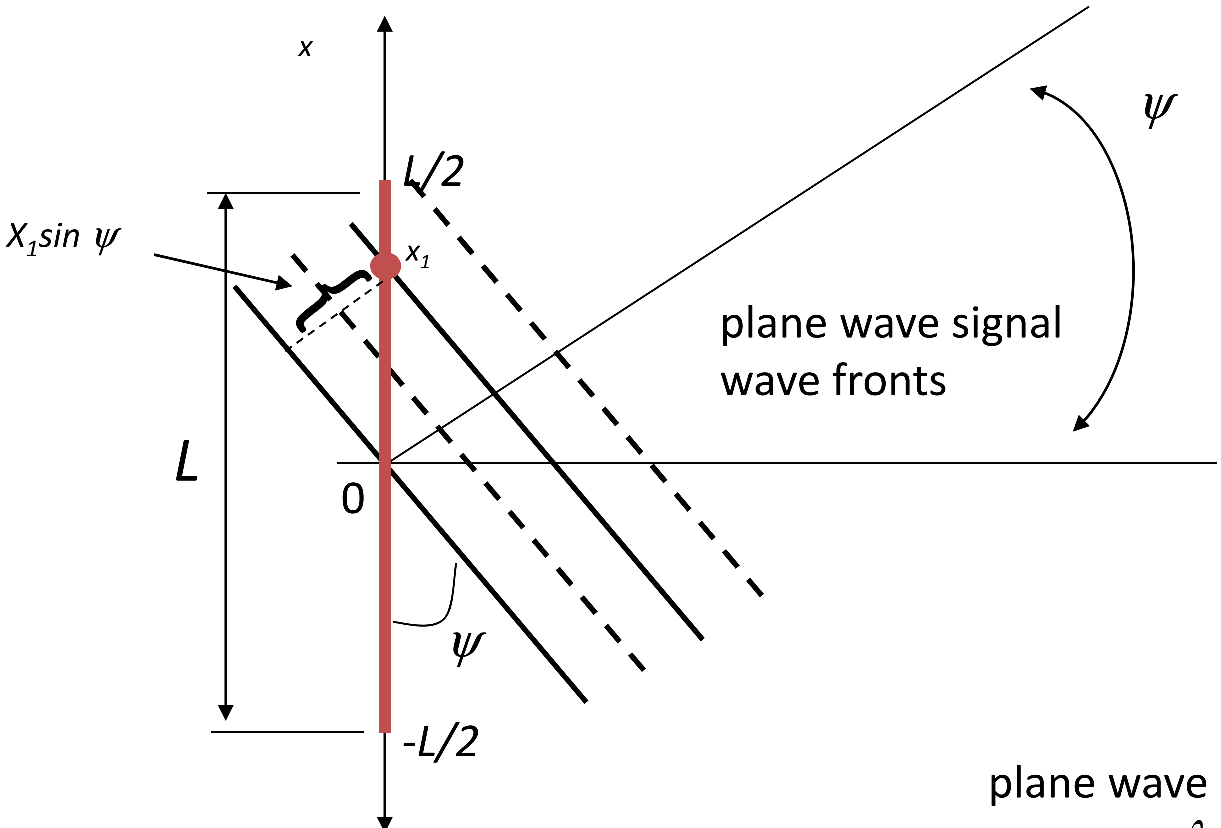
$$\left| N(\Omega) \right| = K$$

Then the Array Gain (AG) becomes the Directivity Index (DI), a performance index
For the array that is independent of the noise field.

$$DI = \frac{4\pi}{\int_{4\pi} \left| G_{array}(\Omega) \right|^2 d\Omega}$$

Array Gain and Directivity Index are usually expressed in decibels.

Line Hydrophone Spatial Response



plane wave has wavelength

$\lambda = c/f$,

where f is the frequency

c is the speed of sound

Line Hydrophone Spatial Response (Cont'd)

The received signal is

$$s(t) \qquad \text{at the origin}$$

$$s\left(t+\frac{x\sin\psi}{c}\right) \qquad \text{at point } x$$

Let the hydrophone's response or sensitivity at the point *x* be *g(x)*.
Then, the total hydrophone response is

$$s_{out}(t)=\int_{-L/2}^{L/2} g(x)s\left(t+\frac{x\sin\psi}{c}\right)dx$$

Line Hydrophone Spatial Response (Cont'd)

Using properties of the Fourier Transform:

$$S(f) = \int s(t)e^{-i2\pi ft}dt$$

And:

$$\int_{-\infty}^{\infty} s(t + \frac{x\sin\psi}{c})e^{-i2\pi ft}dt = exp(\frac{i2\pi fx\sin\psi}{c})S(f)$$

Or:

$$s(t + \frac{x\sin\psi}{c}) = \int_{-\infty}^{\infty} exp(\frac{i2\pi fx\sin\psi}{c} + i2\pi ft)S(f)df$$

Line Hydrophone Spatial Response (Cont'd)

Thus, the total hydrophone response can be written:

$$s_{out}(t) = \int_{-L/2}^{L/2} g(x) \int_{-\infty}^{\infty} exp\left(\frac{i2\pi fx \sin\psi}{c} + i2\pi ft \right) S(f)\,df\,dx$$

$$= \int_{-\infty}^{\infty} S(f) \left[\int_{-L/2}^{L/2} g(x)\, exp\left(\frac{i2\pi fx \sin\psi}{c} \right) dx \right] e^{i2\pi ft} df$$

$$\equiv \int_{-\infty}^{\infty} S(f)\, G\left(\frac{f \sin\psi}{c} \right) e^{i2\pi ft} df$$

Where

$$G\left(\frac{f \sin\psi}{c} \right) \equiv \int_{-L/2}^{L/2} g(x)\, exp\left[i2\pi \left(\frac{f \sin\psi}{c} \right) x \right] dx$$

We call *g(x)* the <u>aperture function</u> and *G((fsin ψ)/c)* the <u>pattern function</u>. They are a Fourier Transform pair.

Response To Plane Wave

An Example:

Unit Amplitude Plane Wave from direction ψ_0:

$$s(t) = e^{j2\pi f_0 t} \qquad\qquad S(f) = \delta(f - f_0)$$

The Line Hydrophone Response is:

$$s_{out}(t) = \int_{-\infty}^{\infty} \delta(f - f_0) G\left(\frac{f \sin \psi_0}{c}\right) e^{i2\pi ft} df$$

$$= G\left(\frac{f_0 \sin \psi_0}{c}\right) e^{j2\pi f_0 t}$$

Note that the output is the input signal modulated by the value of the pattern function at ψ_0

Response To Plane Wave (Cont'd)

The pattern function is the same as the angular power response defined earlier.

Sometimes we use electrical angle *u*:

$$u \equiv \frac{f \sin \psi}{c} = \frac{\sin \psi}{\lambda}$$

instead of physical angle ψ.

Uniform Aperture Function

Consider a uniform aperture function

$$g(x) = \begin{cases} \dfrac{1}{L}, & -\dfrac{L}{2} \leq x \leq \dfrac{L}{2} \\ \\ 0, & \textbf{otherwise} \end{cases}$$

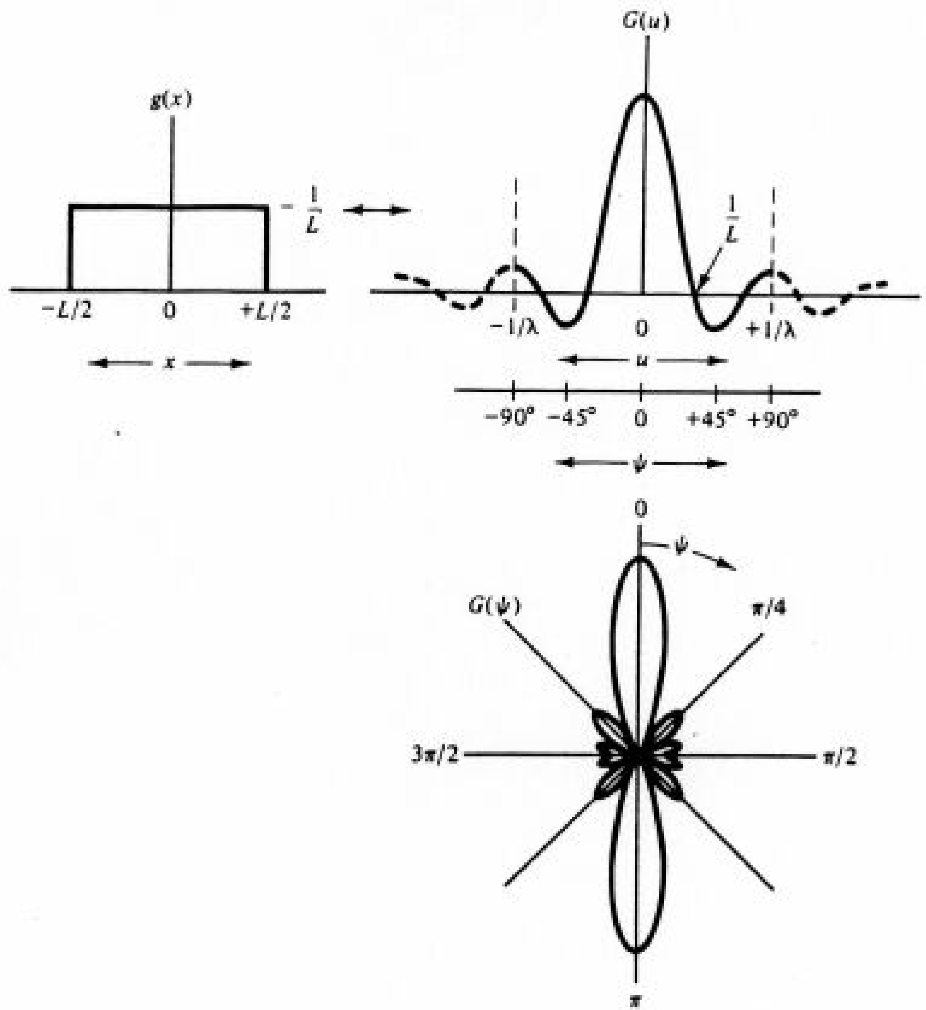
The pattern function is:

$$G(u) = \int_{-\infty}^{\infty} g(x) e^{j2\pi ux} dx = \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{e}^{\mathrm{j}2\pi \mathrm{ux}} dx \; = \frac{e^{j2\pi Lu/2} - e^{j2\pi Lu/2}}{j2\pi Lu}$$

$$= \frac{e^{j2\pi Lu/2} - e^{j2\pi Lu/2}}{j2\pi Lu} = \left. \frac{e^{j2\pi ux}}{j2\pi Lu} \right|_{-L/2}^{L/2} = \frac{\sin(\pi Lu)}{\pi Lu}$$

$$\equiv \mathrm{sinc}(Lu)$$

Rectangular Aperture Function and Pattern Function

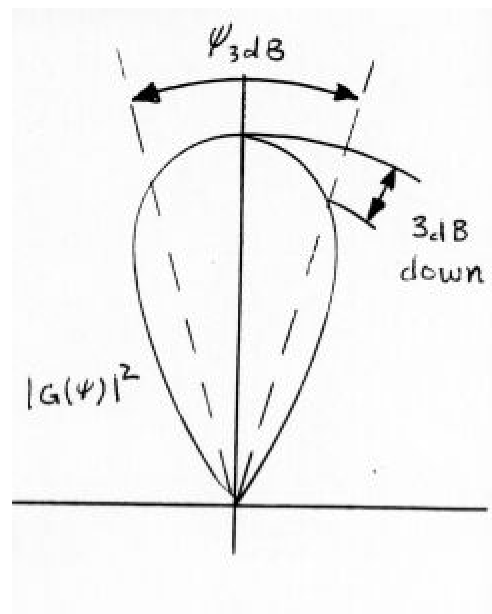


Array Main Lobe Width (Beamwidth)

3 dB (Half-Power) Beamwidth

To find it, solve

$$\left| G\left(\frac{\psi_{3dB}}{2} \right) \right|^2 = \frac{1}{2}$$



For

$$\psi_{3dB}$$

Beamwidth Calculation Example: Uniform Weighting

$$\frac{\sin\left(\frac{\pi L u_{3dB}}{2}\right)}{\frac{\pi L u_{3dB}}{2}} = \frac{1}{\sqrt{2}}$$

$$\frac{\pi L u_{3dB}}{2} = 1.39$$

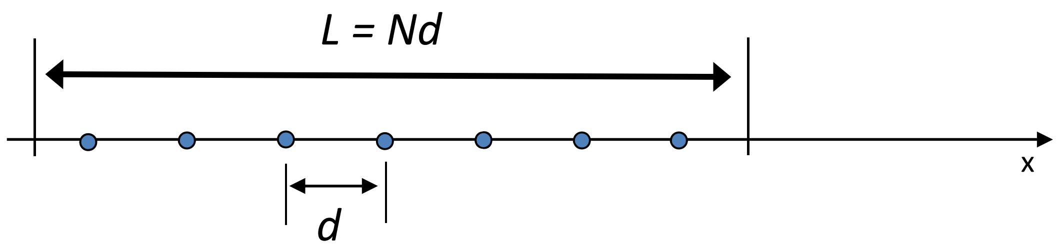
$$\psi_{3dB} = \sin^{-1}\left(\lambda u_{3dB}\right) = \sin^{-1}\left(\frac{2 \cdot 1.39 \cdot \lambda}{\pi L}\right)$$

$$= \sin^{-1}\left(.885\frac{\lambda}{L}\right) \approx .885\frac{\lambda}{L} \quad \text{radians}$$

$$\approx 50\frac{\lambda}{L} \quad \text{deg}$$

For $\lambda \langle\langle L$. Note the effect of increasing L.

Line Array of Discrete Elements



<u>Aperture Function:</u>

$$g(x) = \sum_{n=1}^{N} a_n \delta(x - x_n)$$

For $N = 2M + 1$ (Odd), and uniform spacing d,

$$g(x) = \sum_{n=-M}^{M} a_n \delta(x - nd)$$

<u>Pattern Function</u>: (N odd, uniform spacing)

$$G(u) = \int_{\infty}^{\infty} g(x) e^{j2\pi ux}\, dx$$

$$= \int_{\infty}^{\infty} \sum_{n=-M}^{M} a_n \delta(x - nd) e^{j2\pi ux}\, dx$$

$$= \sum_{n=-M}^{M} a_n e^{j2\pi ndu}$$

Uniformly Weighted Discrete Line Array

Assume $a_n = \dfrac{1}{N}$, uniform spacing, n odd.

Temporarily define :

$r \equiv e^{j2\pi du}$

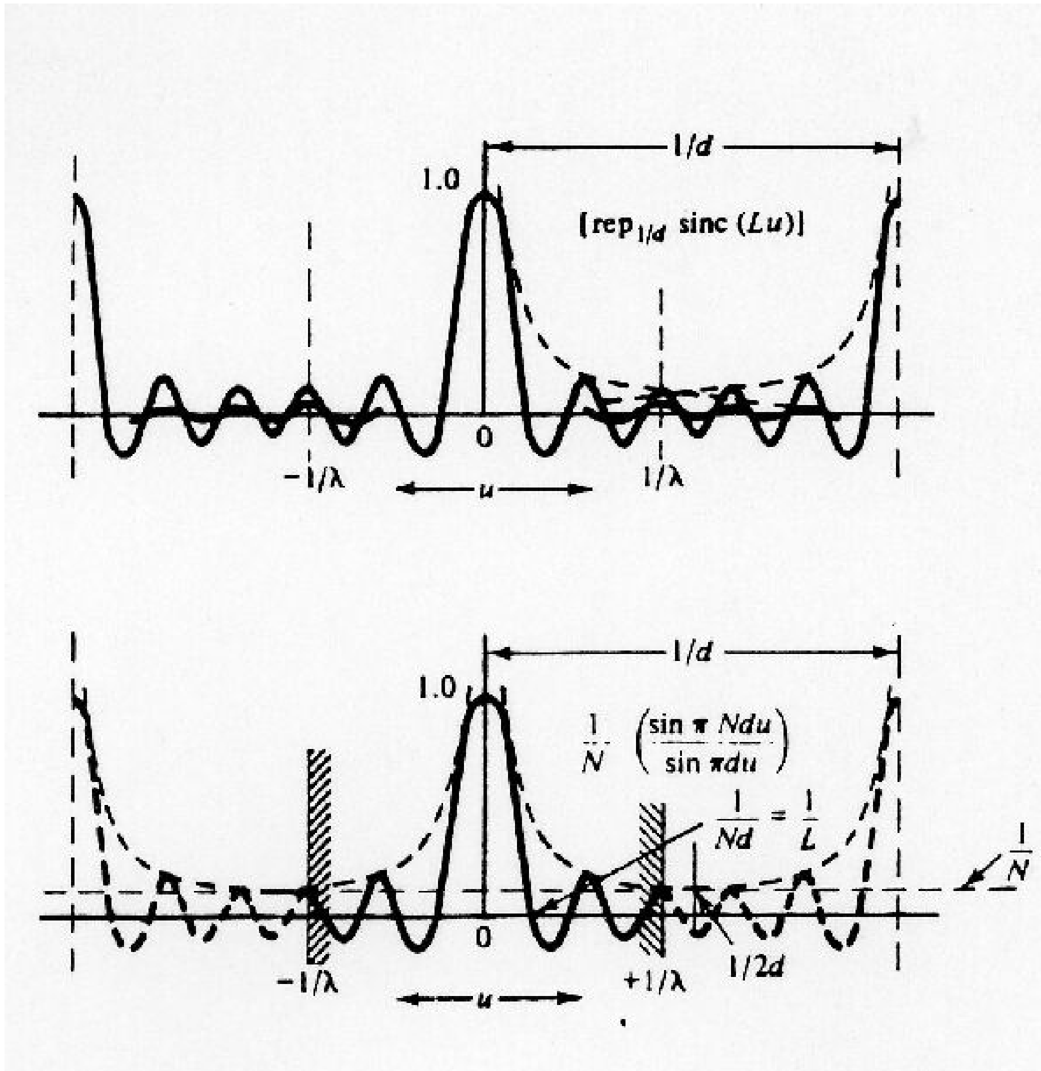
then

$$G(r) = \frac{1}{N}\sum_{n=-M}^{M} r^n = a_n \frac{1}{N}\frac{r^{N/2} - r^{-N/2}}{r^{1/2} - r^{-1/2}}$$

or

$$G(u) = \frac{1}{N}\frac{\sin(\pi u N d)}{\sin(\pi u d)}$$

Pattern Function For Uniform Discrete Line Array



Notes On Pattern Function For Discrete Line Array

- u Only has physical significance over the range $\left(-\frac{1}{\lambda},\frac{1}{\lambda}\right)$
- The region $-\frac{1}{\lambda} \le u \le \frac{1}{\lambda}$ may include more than

one main lobe if $d \ge \lambda$, which causes ambiguity (called grating lobes).

General trade-offs in array design:

1)Want *L* large so that beamwidth is small and resolution is good

2)Want $d \le \lambda$ to avoid grating lobes.

3)Since *L=Nd*, or *N=L/d*, increasing *L* and decreasing *d*
Both cause *N* to increase, which costs more money

Effects of Array Shading (Non-Uniform Aperture Function)

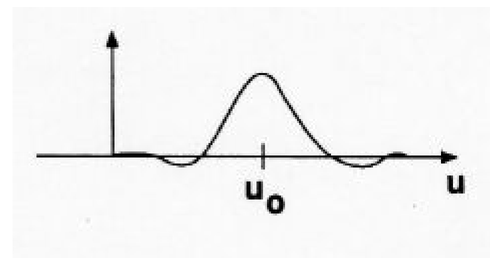
- Shading reduces sidelobe levels at the expense of widening the main lobe.
- For other aperture functions:

Aperture	ψ_{3dB} ,degrees	First sidelobe level, dB
Rectangular	$50\lambda/L$	-13.3
Circular	$58\lambda/L$	-17.5
Parabolic	$66\lambda/L$	-22.0
Triangular	$73\lambda/L$	-26.5

Beam Steering

Want to shift the peak of the pattern function from $u = 0$ to $u = u_0$.
What is the aperture function needed to accomplish this?

$$
\begin{aligned}
g(x) &= \int_{\infty}^{\infty} G(u)e^{-j2\pi ux}\,du \\
g'(x) &= \int_{\infty}^{\infty} G(u-u_0)e^{-j2\pi ux}\,du \\
&= e^{j2\pi u_0 x}\int_{\infty}^{\infty} G(p)e^{-j2\pi px}\,dp \\
&= e^{j2\pi u_0 x}\,g(x)
\end{aligned}
$$



$G(u- u_0)$

Beam steering is accomplished by multiplying the non-steered aperture function by a unit amplitude complex exponential, which is just a delay whose value depends on *x*.

Beam Steering For Discrete Arrays

The steered aperture function becomes

$$
\begin{aligned}
g(x) &= \sum_{n=1}^{N} g(nd)\,\delta\left(x - nd\right)e^{j2\pi ndu_0} \\
&= \ldots g(-d)e^{-j2\pi ndu_0}\,\delta\left(x - d\right) + g(0)\,\delta(x) + g(d)e^{j2\pi ndu_0}\,\delta\left(x + d\right) + \ldots
\end{aligned}
$$

The steered <u>physical angle</u> is $\psi_0 = \sin^{-1}(\lambda u_0).$

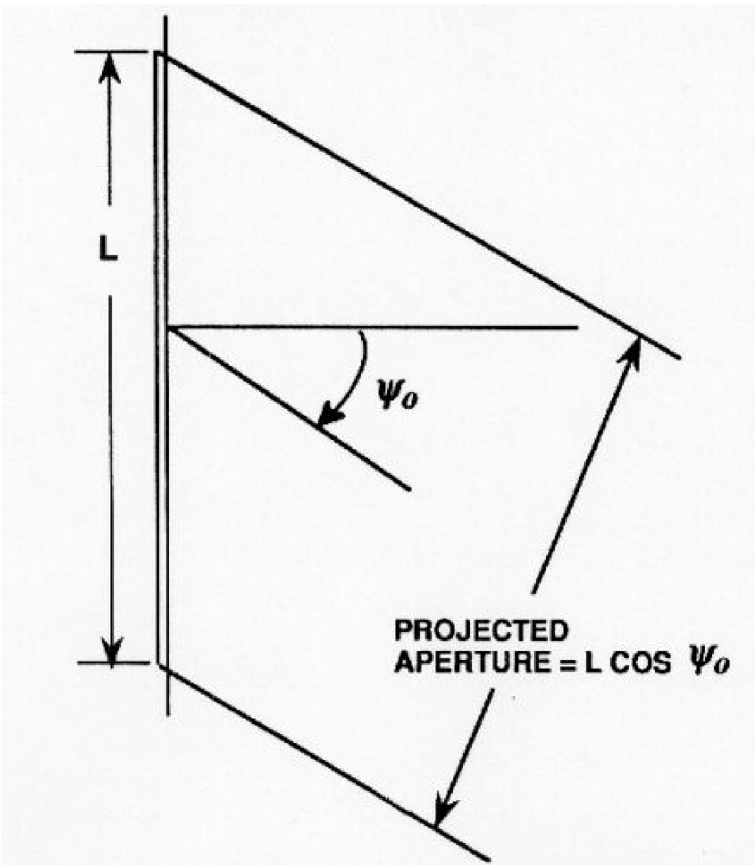
The phase shift at element n is equivalent to a time shift:

$$
2\pi ndu_0 = \frac{2\pi nd \sin\psi_0}{\lambda} = 2\pi f\,\frac{nd\sin\psi_0}{c} = 2\pi f\tau_n.
$$

where $\tau_n = \dfrac{nd\sin\psi_0}{c}$

Is the time shift which must be applied to the *nth* element.

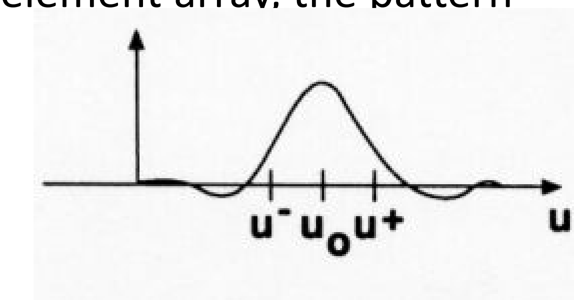
Effect Of Beam Steering On Main Lobe Width



Beam steering produces a projected aperture. Since reducing the aperture increases the beam width, beam steering causes the width of the (steered) main lobe to increase. The lobe distorts (fattens) more on the side of the beam toward which the beam is being steered,

Beam Steering: An Example

For a uniformly weighted, evenly spaced (d=λ/2), 8 element array, the pattern function is



$$G(u-u_0) = \frac{1}{N}\frac{\sin[\pi N d(u-u_0)]}{\sin[\pi d(u-u_0)]}$$

To find the beamwidth, set $\left|G(u^+ - u_0)\right|^2 = \left|G(u_0 - u^-)\right|^2 = \frac{1}{2}$

Which yields

$$\pi d(u^+ - u_0) = \pi d(u_0 - u^-) = 0.175$$

Using $d = \lambda/2$ **and** $u = \sin\psi/\lambda$ **the condition is**

$$\sin\psi^+ - \sin\psi_0 = \sin\psi_0 - \sin\psi^- = 0.1114$$

Beam Steering: An Example (Cont'd)

As an example, take $N=8$, $(d=\lambda/2)$.

$$\text{Case 1:}\quad \sin\psi_0 = 0 \qquad \sin\psi^+ = \sin\psi^- = 0.1114$$

$$\Rightarrow \psi^+ = 6.4^0 \qquad \psi^- = 6.4^0 \qquad \psi_{3dB} = 12.8^0$$

$$\text{Note:}\quad 50\frac{\lambda}{L} = 50\frac{\lambda}{nd} = \frac{50}{4} = 12.5^0$$

Beam Steering: An Example (Cont'd)

Case 2 : $\psi_0 = 45^0 \quad \sin\psi_0 = 0.707$

$$\psi^+ = \sin^{-1}(0.1114 + 0.707) = 54.9^0$$

$$\psi^- = \sin^{-1}(0.1114 - 0.707) = 36.6^0$$

$\psi_{3dB} = 18.3^0$ (wider)

But, beam is not symmetrical :

$$54.9^0 - 45^0 = 9.9^0$$

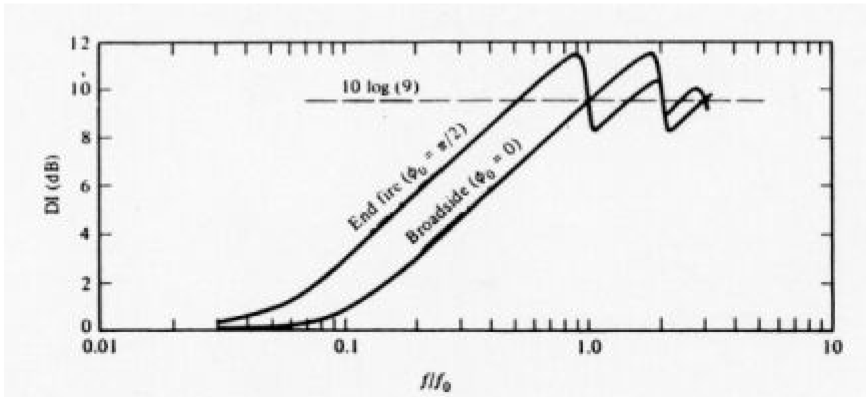
$$45^0 - 36.6^0 = 8.4^0$$

Notice that there is no grating lobe until $\psi_0 = 90^0$

at which point there is a grating lobe at $\psi = -90^0$

This is why $d = \frac{\lambda}{2}$ is optimal

Directivity Index For A Discrete Line Array



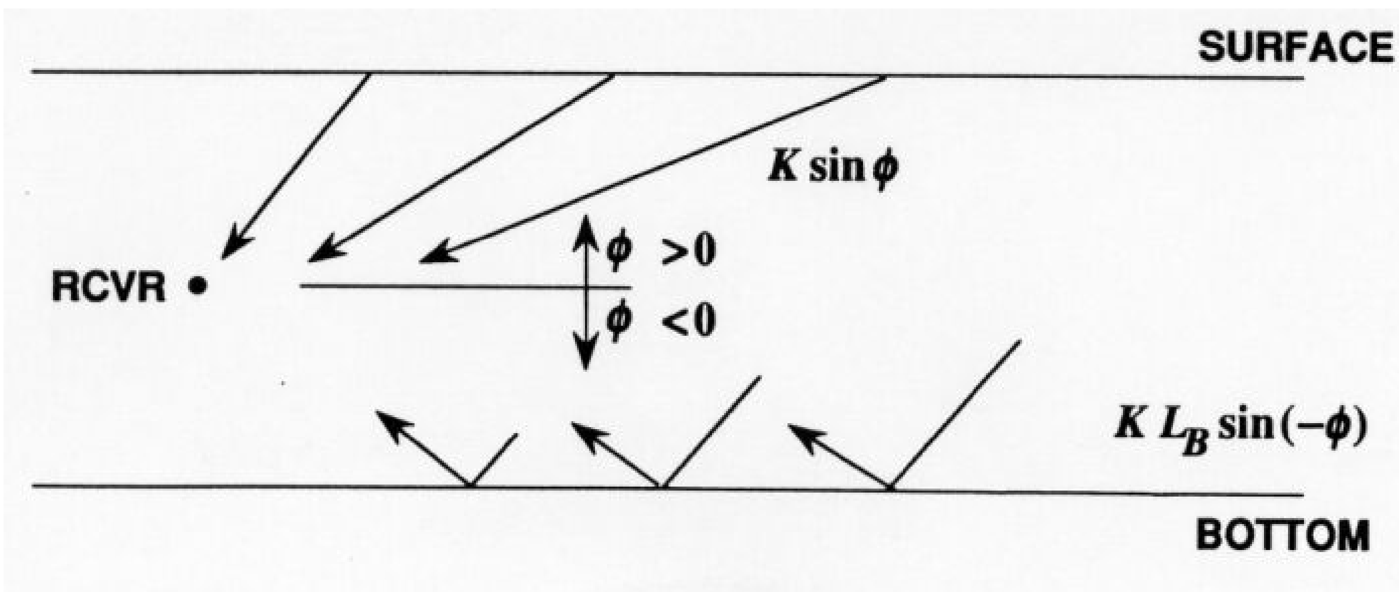
DI for a discrete line array, broadside and endfire with N=9.

- Uniform weighting
- f_0 is the frequency at which $d = \lambda / 2$
- When $f = f_0 \ (\frac{f}{f_0} = 1), \ DI = 10 \log N$
- When $f > f_0$, DI oscillates about $10 \log N$. However, grating lobes cause ambiguity.
- When $f < f_0$, DI drops at about 3dB per octave. End fire beam has higher DI, but broader beamwidth, hence poorer directional resolution.

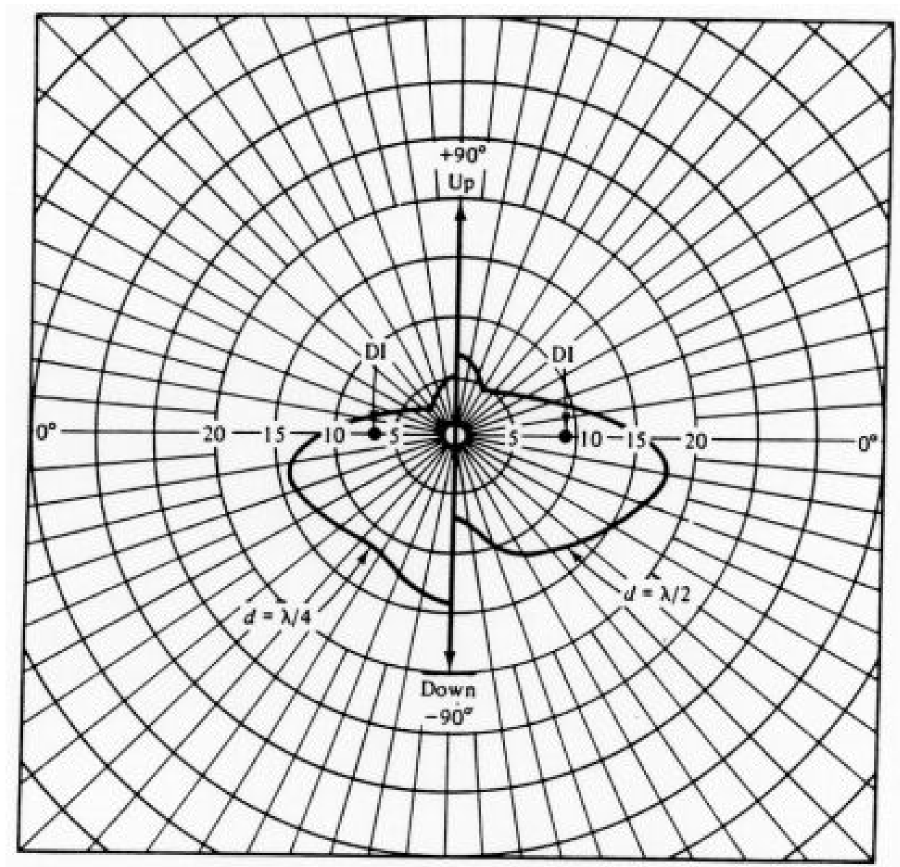
Array Gain: Discrete Line Array

Noise Model :

$$|N(\phi)| = \begin{cases} K\sin\phi, & 0^0 \le \phi \le 90^0 \\ KL_B \sin(-\phi), & -90^0 \le \phi \le 0^0 \end{cases} \qquad (L_B \approx 0.1)$$



Array Gain: Discrete Line Array (Cont'd)



Nine-element vertical line array gain versus elevation steering angle in a surface-generated ambient noise field