

## BINOMIAL THEOREM

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### 1. State and prove Binomial Theorem

$$\text{Statement: } (a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots + {}^n C_n b^n$$

Put  $n=1$

$$\text{LHS} = (a+b)^1 = \underline{\underline{a+b}}$$

$$\begin{aligned} \text{RHS} &= {}^1 C_0 a^1 + {}^1 C_1 a^{1-1} b \\ &= \underline{\underline{a+b}} \end{aligned}$$

$\therefore$  Result is true for  $\underline{\underline{n=1}}$

We assume the result is true for  $n=k$

$$(a+b)^k = {}^k C_0 a^k + {}^k C_1 a^{k-1} b + {}^k C_2 a^{k-2} b^2 + \dots + {}^k C_k b^k$$

Now we prove the result is true for  $n=k+1$

$$\begin{aligned} (a+b)^{k+1} &= (a+b)^k \cdot (a+b)^1 & [\because \rightarrow \text{Multiplication}] \\ &= (a+b) \cdot (a+b)^k \end{aligned}$$

$$= (a+b) \cdot [{}^k C_0 a^k + {}^k C_1 a^{k-1} b + {}^k C_2 a^{k-2} b^2 + \dots + {}^k C_k b^k]$$

Upon multiplication

$$\begin{aligned} & {}^k C_0 a^k a + {}^k C_1 a^{k-1} a b + {}^k C_2 a^{k-2} a b^2 + \dots + {}^k C_k a b^k + {}^k C_0 a^k b + \\ &= {}^k C_1 a^{k-1} b b + {}^k C_2 a^{k-2} b b^2 + \dots + {}^k C_k b^k b \\ &= {}^k C_0 a^{k+1} + \underline{{}^k C_1 a^k b} + \underline{{}^k C_2 a^{k-1} b^2} + \dots + \underline{{}^k C_k a b^k} + \underline{{}^k C_0 a^k b} + \underline{{}^k C_1 a^{k-1} b^2} + \\ &= {}^k C_0 a^{k-2} b^3 + \dots + {}^k C_k b^{k+1} \end{aligned}$$

$$\begin{aligned}
 &= \text{...} + \underline{kC_2 a^{k-2} b^3} + \dots + \underline{kC_k b^{k+1}} \\
 &= kC_0 a^{k+1} + a^k b [kC_1 + kC_0] + a^{k-1} b^2 [kC_2 + kC_1] + \dots + kC_k b^{k+1} \\
 &= (1) a^{k+1} + a^k b [k+1 C_1] + a^{k-1} b^2 [k+1 C_2] + \dots (1) b^{k+1} \\
 &= \cancel{k+1 C_0 a^{k+1}} + \cancel{k+1 C_1 a^k b} + \cancel{k+1 C_2 a^{k-1} b^2} + \dots \cancel{k+1 C_{k+1} b^{k+1}}
 \end{aligned}$$

∴ The result is true for  $n = k+1$

### PASCAL'S TRIANGLE

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1

2. Expand each of the following expressions

i.  $\left(\frac{x}{3} + \frac{1}{x}\right)^5 \rightarrow \text{formula: } (a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots + {}^n C_n b^n$

$n=5$

$\therefore (x + 1)^5 = {}^5 C_0 x^5 + {}^5 C_1 x^4 + {}^5 C_2 x^3 + {}^5 C_3 x^2 + {}^5 C_4 x + {}^5 C_5$

Note:- The values of  ${}^5C_0 \dots {}^5C_5$  are obtained using the Pascals triangle. This gives the value as

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$5c_0 \quad 5c_1 \quad 5c_2 \quad 5c_3 \quad 5c_4 \quad 5c_5$$


$$\text{ii. } \left(x + \frac{1}{x}\right)^n \Rightarrow \text{Formula :- } (a+b)^n = n c_0 a^n + n c_1 a^{n-1} b + n c_2 a^{n-2} b^2 + \dots + n c_n b^n$$

$$\left(x + \frac{1}{x}\right)^6 = {}^6C_0 x^6 + {}^6C_1(x)^{6-1}\left(\frac{1}{x}\right) + {}^6C_2(x)^{6-2}\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^{6-3}\left(\frac{1}{x}\right)^3 + \\ {}^6C_4(x)^{6-4}\left(\frac{1}{x}\right)^4 + {}^6C_5(x)^{6-5}\left(\frac{1}{x}\right)^5 + {}^6C_6(x)^{6-6}\left(\frac{1}{x}\right)^6$$

$$= (1)x^6 + 6x^4\left(\frac{1}{x}\right) + 15x^2\left(\frac{1}{x^2}\right) + 20x^0\left(\frac{1}{x^4}\right) + 15x^{-2}\left(\frac{1}{x^6}\right) + \\ 6x^{-4}\left(\frac{1}{x^8}\right) + (1)(1)\left(\frac{1}{x^6}\right)$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Note the values of  ${}^6C_0, {}^6C_1, \dots, {}^6C_6$  is obtained using the pascals triangle

$$n=6 \Rightarrow 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$\text{iii. } (1 - 2x)^5 \Rightarrow \text{Formula: } (a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_n b^n$$

$$(1 - 2x)^5 = {}^5C_0(1)^5 - {}^5C_1(1)^4(2x)^1 + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 +$$

$${}^5C_4(1)^1(2x)^4 - {}^5C_5(1)^0(2x)^5$$

$$= (1)(1) - 5(1)(2x) + 10(1)(4x^2) - 10(1)(8x^3) + 5(16x^4) - (1)(1)(32x^5)$$

$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

$$\text{iv. } \left(\frac{2}{x} - \frac{x}{2}\right)^5 \rightarrow \text{formula} \Rightarrow (a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots + {}^n C_n b^n$$

$$\begin{aligned} \left(\frac{2}{x} - \frac{x}{2}\right)^5 &= {}^5 C_0 \left(\frac{2}{x}\right)^5 - {}^5 C_1 \left(\frac{2}{x}\right)^4 \left(\frac{x}{2}\right)^1 + {}^5 C_2 \left(\frac{2}{x}\right)^3 \left(\frac{x}{2}\right)^2 - \\ &\quad {}^5 C_3 \left(\frac{2}{x}\right)^2 \left(\frac{x}{2}\right)^3 + {}^5 C_4 \left(\frac{2}{x}\right) \left(\frac{x}{2}\right)^4 - {}^5 C_5 \left(\frac{2}{x}\right)^0 \left(\frac{x}{2}\right)^5 \\ &= 1 \left(\frac{32}{x^5}\right) - 5 \left(\frac{16}{x^4}\right) \left(\frac{x}{2}\right) + 10 \left(\frac{8}{x^3}\right) \left(\frac{x^2}{4}\right) - 10 \left(\frac{4}{x^2}\right) \left(\frac{x^3}{8}\right) + \\ &\quad 5 \left(\frac{1}{x}\right) \left(\frac{x^4}{16}\right) - 1 (1) \left(\frac{x^5}{32}\right) \\ &= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32} \end{aligned}$$

3. Using Binomial Theorem indicate which number is larger  
 i.  $(1.1)^{10000}$  or 1000

$$\begin{aligned} (1.1)^{10000} &= (1+0.1)^{10000} \\ &= 10000 C_0 (1)^{10000} + 10000 C_1 (1)^{9999} (0.1) + \dots + 10000 (1) (0.1) + \dots + \text{+ve value} \\ &= (1) + 1000 + \dots + \text{+ve value} \\ &= 1001 + \dots + \text{values} > 1000 \\ \therefore (1.1)^{10000} &\text{ is greater} \end{aligned}$$

ii.  $(1.01)^{1000000}$  or 10000

$$\begin{aligned}(1.01)^{1000000} &= (1+0.01)^{1000000} \\&= 1000000 C_0 (1)^{1000000} + 1000000 C_1 (1)^{999999} (0.1) + \dots + \text{ve values} \\&= (1)(1) + 10000 + \dots + \text{ve values} \\&\approx 1 + 10,000 + \dots + \text{ve values} \\&= 10,001 + \dots + \text{ve values} > 10,000 \\&\approx \underline{\underline{(1.01)^{10,00,000}} > 10,000}\end{aligned}$$

4. Using binomial theorem evaluate each of the following

i.  $(101)^4$

$$\begin{aligned}(101)^4 &= (100+1)^4 \\&= 4C_0(100)^4 + 4C_1(100)^3(1) + 4C_2(100)^2(1)^2 + 4C_3(100)(1)^3 + 4C_4(100)^0(1)^4 \\&= (1)(100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + 1(1) \\&= 100000000 + 4(1000000) + 6(10000) + 400 + 1 \\&= 100000000 + 4000000 + 60000 + 400 + 1 \\&= \underline{\underline{104060401}}\end{aligned}$$

Note:- For the powers of 100  $\Rightarrow 2 \times 2 = 4 \Rightarrow$  Add 4 zeros to the front after 1st digit

$$(a+b)^n = nC_0 a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + nC_3 a^{n-3} b^3 + \dots + nC_n a^0 b^n$$

$$(a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + {}^n C_3 a^{n-3} b^3 + \dots {}^n C_n a^0 b^n$$

$$= {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + {}^n C_3 a^{n-3} b^3 + \dots {}^n C_n b^n$$

ii.  $(102)^5$

$$(102)^5 = (100+2)^5$$

$$= {}^5 C_0 (100)^5 + {}^5 C_1 (100)^4 (2) + {}^5 C_2 (100)^3 (2)^2 + {}^5 C_3 (100)^2 (2)^3 +$$

$$\quad {}^5 C_4 (100) (2)^4 + {}^5 C_5 (2)^5$$

$$= (1)(1000000000) + 5(100000000) (2) + 10(1000000) (4) +$$

$$\quad 10(1000) (8) + 5(100) (16) + (1)(32)$$

$$= 1000000000 + 1000000000 + 40000000 + 800000 + 32$$

$$= 11040800032$$

Note:-  $(a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots {}^n C_n b^n$

$n=5 \Rightarrow 1 \ 5 \ \underline{\quad 10 \quad 10 \quad 5 \quad 1} \rightarrow \text{Pascal's triangle}$

iii.  $(99)^5$

$$(99)^5 = (100-1)^5$$

$$= {}^5 C_0 (100)^5 - {}^5 C_1 (100)^4 (1) + {}^5 C_2 (100)^3 (1) - {}^5 C_3 (100)^2 (1) +$$

$$5C_4 (100)(1) - 5(5)(1)$$

$$= (1)(10000000000) - 5(100000000) + 10(1000000) - 10(10000) + 5(100) - (1)(1)$$

$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1 \\ = \underline{\underline{950\ 99\ 004\ 99}}$$

Note :-  $(a+b)^n = nC_0 a^n + nC_1 a^{n-1}b + nC_2 a^{n-2}b^2 + \dots + nC_n a^0 b^n$

$$= nC_0 a^n + nC_1 a^{n-1}b + nC_2 a^{n-2}b^2 + \dots + nC_n (1)b^n$$

$$= nC_0 a^n + nC_1 a^{n-1}b + nC_2 a^{n-2}b^2 + \dots + nC_n b^n$$

$n=5 \Rightarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \Rightarrow$  Pascal's triangle.

5. Find  $(a+b)^4 - (a-b)^4$  hence evaluate  $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$

$$(a+b)^4 = 4C_0 a^4 + 4C_1 a^3 b + 4C_2 a^2 b^2 + 4C_3 a b^3 + 4C_4 a^0 b^4$$

$$= (1)a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + (1)b^4$$

~~$$(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$~~

~~$$(a-b)^4 = a^4 - 4a^3 b + 6a^2 b^2 - 4ab^3 + b^4$$~~

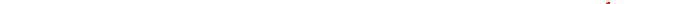
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$$(a+b)^4 - (a-b)^4 = 8a^3 b + 8ab^3$$

Similarly  $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8(\sqrt{3})^3(\sqrt{2}) +$

$$\begin{aligned}
 & \text{Similarly (iii) } (\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8(\sqrt{3})^3(\sqrt{2}) + \\
 & \quad 8(\sqrt{3})(\sqrt{2})^3 \\
 & = 8(3\sqrt{3})(\sqrt{2}) + 8(\sqrt{3})(2\sqrt{2}) \\
 & = 24\sqrt{6} + 16\sqrt{6} \\
 & = \underline{\underline{40\sqrt{6}}}
 \end{aligned}$$

$$\text{Note: } -(a+b)^n = nC_0 a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \dots + nC_n b^n$$

$$n=4 \quad 1 \quad 4 \quad 6 \quad 4 \quad 1$$


6. Find  $(x + 1)^6 + (x - 1)^6$  hence evaluate  $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

$$(x+1)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5(1) + {}^6C_2(x)^4(1)^2 + {}^6C_3(x)^3(1)^3 + \\ {}^6C_4(x)^2(1)^4 + {}^6C_5(x)(1)^5 + {}^6C_6(x)^0(1)^6$$

$$= (1) x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

$$(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

$$(x-1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

$$(x+1)^6 + (x-1)^6 = 2x^6 + 30x^4 + 30x^2 + 2$$

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$$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2(\sqrt{2})^6 + 30(\sqrt{2})^4 + 30(\sqrt{2})^2 + 2$$

$$\begin{aligned}
 (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 &= 2(\sqrt{2})^6 + 30(\sqrt{2})^4 + 30(\sqrt{2})^2 + 2 \\
 &= 2(8) + 30(4) + 30(2) + 2 \\
 &= 16 + 120 + 60 + 2 \\
 &= \underline{\underline{198}}
 \end{aligned}$$

7. Prove that  $\sum_{r=0}^n 3^r nC_r = 4^n$

$$\begin{aligned}
 \sum_{r=0}^n 3^r nC_r &= 3^0 nC_0 + 3^1 nC_1 + 3^2 nC_2 + \dots + 3^n nC_n \\
 &= (1)(1) + nC_1(3) + nC_2(9) + \dots + (1) 3^n
 \end{aligned}$$

and

$$\begin{aligned}
 4^n &= (1+3)^n \\
 &= nC_0 1^n + nC_1 1^{n-1}(3) + nC_2 1^{n-2}(3)^2 + \dots + nC_n 3^n \\
 &= (1)(1) + nC_1(3) + nC_2(9)
 \end{aligned}$$

$$\therefore \sum_{r=0}^n 3^r nC_r = \underline{\underline{4^n}}$$

## GENERAL TERM

$T_{r+1} = nC_r a^{n-r} b^r$  is the general term for  $(a+b)^n$

8. Write the general term expansion for the following  
 i.  $(x^2 - y)^6$

Compare with  $(a+b)^n$

$$\Rightarrow a = x^2 \quad b = -y \quad n = 6$$

$$\begin{aligned}\text{General term (G.T)} \quad T_{r+1} &= {}^n C_r a^{n-r} b^r \\ &= {}^6 C_r (x^2)^{6-r} (-y)^r \\ &= {}^6 C_r x^{12-2r} (-y)^r \\ &\quad \underline{\hspace{10em}}\end{aligned}$$

ii.  $(x^2 - yx)^{12}$

Compare with  $(a+b)^n$

$$\Rightarrow a = x^2 \quad b = -yx \quad n = 12$$

General Term (GT)  $\Rightarrow$

$$\begin{aligned}T_{r+1} &= {}^n C_r a^{n-r} b^r \\ &= {}^{12} C_r (x^2)^{12-r} (-yx)^r \\ &= {}^{12} C_r x^{24-2r} (-y)^r (x)^r = {}^{12} C_r x^{24-2r+r} (-y)^r \\ &= {}^{12} C_r x^{24-r} (-y)^r \\ &\quad \underline{\hspace{10em}}\end{aligned}$$

9. Find the fourth term in the expansion of  $(x - 2y)^{12}$

Compare with  $(a+b)^n$

$$\Rightarrow a = x \quad b = -2y \quad n = 12 \quad r = 3$$

General term

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

General term

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

$$T_{3+1} = {}^{12} C_3 x^{12-3} (-8y)^3$$

$$T_4 = {}^{12} C_3 x^9 (-8y^3)$$

$$T_4 = \frac{12!}{9! \cdot 3!} x^9 (-8y^3)$$

$$= \frac{\cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot 9!}{\cancel{9!} \times \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} x^9 (-8y^3)$$

$$= \underline{\underline{-1760 x^9 y^3}}$$

10. Find 13th term of the expansion  $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

$$a = 9x \quad b = -\frac{1}{3\sqrt{x}} \quad n = 18 \quad r = 12$$

General term (G.T)

$$\Rightarrow T_{r+1} = {}^n C_r a^{n-r} b^r$$

$$T_{12+1} = {}^{18} C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$T_{13} = \frac{18!}{6! \cdot 12!} (9x)^6 \left(\frac{1}{3^{12} (x^{1/2})^{12}}\right)$$

$$T_{13} = \frac{(18)(17)(16)(15)(14)(13) \cancel{12!}}{(6)(5)(4)(3)(2)(1) \cdot 12!} \left(\frac{3^{12} x^6}{x^6}\right) \left(\frac{1}{3^{12} x^6}\right)$$

$$T_{13} = (17)(2)(3)(7)(13)$$

$$T_{13} = (17)(2)(3)(7)(13) \\ = \underline{\underline{18564}}$$

4. Find the coefficient of  
i.  $x^5$  in  $(x + 3)^8$

$$a = x \quad b = 3 \quad n = 8$$

$$\text{GT :- } T_{r+1} = {}^n C_r a^{n-r} b^r$$

$$T_{r+1} = {}^8 C_r x^{(8-r)} (3)^r \rightarrow ①$$

Put  $8-r = 5$  ( $\because$  The value of  $x$  power is 5)

$$-r = 5 - 8$$

$$r = 3$$

$$\underline{\underline{r = 3}}$$

$\therefore ①$  becomes

$$T_{3+1} = {}^8 C_3 x^{8-3} (3)^3$$

$$T_4 = \frac{8!}{5! 3!} x^5 (27)$$

$$T_4 = \frac{(8)(7)(6)(5)!}{5!(3)(2)(1)} x^5 (27)$$

$$T_4 = \underline{\underline{1512x^5}}$$

Coefficient of  $x^5$  is 1512

ii.  $a^5 b^7$  in  $(a - 2b)^2$

$$a = a \quad b = -2b \quad n = 12$$

$$\text{G.T} \quad T_{r+1} = n C_r a^{n-r} b^r$$

$$T_{r+1} = 12 C_r a^{12-r} (-2b)^r$$

Put

$$12-r = 5 \quad (\because \text{The value of a power is 5})$$

$$-r = 5 - 12$$

$$-r = 7$$

$$\underline{\underline{r = 7}}$$

$$T_{7+1} = 12 C_7 a^{12-7} (-2b)^7 \\ = \frac{12!}{5! 7!} a^5 (-2)^7 b^7$$

$$= \frac{(12)(11)(10)(9)(8) \cancel{7!}}{(5)(4)(3)(2)(1) \cancel{7!}} a^5 (-128) b^7$$

$$= (11)(9)(8) a^5 (-128) b^7$$

$$= \underline{\underline{-101376 a^5 b^7}}$$

iii.  $x^6 y^3$  in  $(x + 2y)^9$

$$a=x \quad b=2y \quad n=9$$

$$\text{G.T} \quad T_{r+1} = {}^n C_r a^{n-r} b^r$$

$$T_{r+1} = {}^9 C_r x^{9-r} (2y)^r$$

$$\text{Put } 9-r=6$$

$$9-6=r$$

$$\underline{\underline{r=3}}$$

$$T_{3+1} = {}^9 C_3 x^{9-3} (2y)^3$$

$$T_4 = {}^9 C_3 x^6 (8)y^3$$

$$= \frac{9!}{(9-3)!3!} (8)x^6 y^3$$

$$= \frac{9!}{6! 3!} (8) x^6 y^3$$

$$= \frac{^3 (9)(8)(7) \cancel{6!}}{(8)(7)(6)(1) \cdot \cancel{6!}} (8) x^6 y^3$$

$$= \underline{\underline{672 x^6 y^3}}$$