BINOMIAL THEOREM

18 February 2024 20:00

1. State and prove Binomial Theorem

Statement :- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n b^n$

Put $n=1$

LHS $= (a+b)^1 = \underline{\underline{a+b}}$

RHS $= {}^1C_0 a^1 + {}^1C_1 a^{1-1} b$ $\quad [\because a^0 = 0,\ {}^1C_1 = 1]$

$= \underline{\underline{a+b}}$

$\therefore$ Result is true for $\underline{\underline{n=1}}$

We assume the result is true for $n=k$

$\underline{\underline{(a+b)^k = {}^kC_0 a^k + {}^kC_1 a^{k-1} b + {}^kC_2 a^{k-2} b^2 + \dots {}^kC_k b^k}}$

Now we prove the result is true for $n=k+1$

$(a+b)^{k+1} = (a+b)^k \cdot (a+b)^1$ $\quad [\cdot \rightarrow$ Multiplication$]$

$= (a+b)\cdot(a+b)^k$

$= (a+b)\cdot\left[{}^kC_0 a^k + {}^kC_1 a^{k-1} b + {}^kC_2 a^{k-2} b^2 \dots {}^kC_k b^k\right]$

Upon multiplication

$= {}^kC_0 a^k a + {}^kC_1 a^{k-1} a b + {}^kC_2 a^{k-2} a b^2 + \dots {}^kC_k a b^k + {}^kC_0 a^k b +$
${}^kC_1 a^{k-1} b \cdot b + {}^kC_2 a^{k-2} b \cdot b^2 + \dots {}^kC_k b^k \cdot b$

$= {}^kC_0 a^{k+1} + \underline{{}^kC_1 a^k} b + {}^kC_2 a^{k-1} b^2 + \dots {}^kC_k a b^k + \underline{{}^kC_0 a^k} b + {}^kC_1 a^{k-1} b^2 +$
${}^kC_2 a^{k-2} b^3 + \dots {}^kC_k b^{k+1}$

$$= {}^kC_2 a^{k-2} b^3 + \dots {}^kC_k b^{k+1}$$

$$= {}^kC_0 a^{k+1} + a^k b \left[{}^kC_1 + {}^kC_0\right] + a^{k-1} b^2 \left[{}^kC_2 + {}^kC_1\right] + \dots {}^kC_k b^{k+1}$$

$$= (1) a^{k+1} + a^k b \left[{}^{k+1}C_1\right] + a^{k-1} b^2 \left[{}^{k+1}C_2\right] + \dots (1) b^{k+1}$$

$$= {}^{k+1}C_0 a^{k+1} + {}^{k+1}C_1 a^k b + {}^{k+1}C_2 a^{k-1} b^2 + \dots {}^{k+1}C_{k+1} b^{k+1}$$

$\therefore$ The result is true for $n = k+1$

PASCAL'S TRIANGLE

$n=0$						1					
$n=1$					1		1				
$n=2$				1		2		1			
$n=3$			1		3		3		1		
$n=4$		1		4		6		4		1	
$n=5$	1		5		10		10		5		1

2. Expand each of the following expressions

i. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ $\Rightarrow$ formula :- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n b^n$

$n = 5$

$n = 5$

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^{5-1}\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^{5-2}\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^{5-3}\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)^{5-4}\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{x}{3}\right)^{5-5}\left(\frac{1}{x}\right)^5$$

$$= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\underbrace{\left(\frac{x}{3}\right)^0}_{\to 1}\left(\frac{1}{x}\right)^5$$

$$= (1)\left(\frac{x^5}{243}\right) + 5\left(\frac{x^{\cancel{4}\,3}}{81}\right)\left(\frac{1}{\cancel{x}}\right) + 10\left(\frac{x^{\cancel{3}\,1}}{27}\right)\left(\frac{1}{\cancel{x^2}}\right) + 10\left(\frac{\cancel{x^2}}{9}\right)\left(\frac{1}{x^{\cancel{3}\,1}}\right) + 5\left(\frac{\cancel{x}}{3}\right)\left(\frac{1}{x^{\cancel{4}\,3}}\right) + (1)(1)\left(\frac{1}{x^5}\right)$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Note:- The values of $5C_0 \ldots 5C_5$ are obtained using the Pascals triangle. This gives the value as

1	5	10	10	5	1
$5C_0$	$5C_1$	$5C_2$	$5C_3$	$5C_4$	$5C_5$

ii. $\left(x+\frac{1}{x}\right)^6 \Rightarrow$ Formula:- $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots {}^nC_nb^n$

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0\, x^6 + {}^6C_1 (x)^{6-1}\left(\frac{1}{x}\right) + {}^6C_2 (x)^{6-2}\left(\frac{1}{x}\right)^2 + {}^6C_3 (x)^{6-3}\left(\frac{1}{x}\right)^3 + {}^6C_4 (x)^{6-4}\left(\frac{1}{x}\right)^4 + {}^6C_5 (x)^{6-5}\left(\frac{1}{x}\right)^5 + {}^6C_6 (x)^{6-6}\left(\frac{1}{x}\right)^6$$

$$= (1)\, x^6 + 6\, x^{\cancel{5}\,4}\left(\frac{1}{\cancel{x}}\right) + 15 x^{\cancel{4}\,2}\left(\frac{1}{x^{\cancel{2}}}\right) + 20\,\cancel{x^3}\left(\frac{1}{\cancel{x^3}}\right) + 15\, \cancel{x^2}\left(\frac{1}{x^{\cancel{4}\,2}}\right) + 6\,\cancel{x}\left(\frac{1}{x^{\cancel{5}\,4}}\right) + (1)(1)\left(\frac{1}{x^6}\right)$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Note the values of ${}^6C_0, {}^6C_1 \ldots {}^6C_6$ is obtained using the pascals triangle

$n = 6 \Rightarrow 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$

iii. $(1-2x)^5$ $\Rightarrow$ Formula:- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots {}^nC_n b^n$

$$(1-2x)^5 = {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x)^1 + {}^5C_2 (1)^3 (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^1 (2x)^4 - {}^5C_5 (1)^0 (2x)^5$$

$$= (1)(1) - 5(1)(2x) + 10(1)(4x^2) - 10(1)(8x^3) + 5(16x^4) - (1)(1)(32x^5)$$

$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

iv. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ $\Rightarrow$ formula $\Rightarrow (a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots {}^nC_n b^n$

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)^1 + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 -$$

$${}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{2}{x}\right)^0\left(\frac{x}{2}\right)^5$$

$$= 1\left(\frac{32}{x^5}\right) - 5\left(\frac{\cancel{16}^{8}}{x^{\cancel{4}3}}\right)\left(\frac{\cancel{x}}{\cancel{2}_1}\right) + 10\left(\frac{\cancel{8}^{2}}{x^{\cancel{3}1}}\right)\left(\frac{\cancel{x^2}}{\cancel{4}_1}\right) - \cancel{10}^{5}\left(\frac{\cancel{4}^{1}}{\cancel{x^2}}\right)\left(\frac{x^{\cancel{3}1}}{\cancel{8}_{\cancel{2}1}}\right) +$$

$$5\left(\frac{\cancel{2}^{1}}{\cancel{x}}\right)\left(\frac{\cancel{x^4}^{3}}{\cancel{16}_8}\right) - 1(1)\left(\frac{x^5}{32}\right)$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32}$$

3. Using Binomial Theorem indicate which number is larger

i. $(1.1)^{10000}$ or 1000

$$(1.1)^{10000} = (1+0.1)^{10000}$$

$$= {}^{10000}C_0(1)^{10000} + {}^{10000}C_1(1)^{9999}(0.1) + \dots (1)(1) + 10000(1)(0.1) + \dots + \text{ve value}$$

$$= (1) + 1000 + \dots + \text{ve value}$$

$$= 1001 + \dots + \text{values} > 1000$$

$\therefore (1.1)^{10000}$ is greater

ii. $(1.01)^{1000000}$ or 10000

$(1.01)^{1000000} = (1+0.01)^{1000000}$

$= {}^{1000000}C_0 (1)^{1000000} + {}^{1000000}C_1 (1)^{99999} (0.1) + \dots +^{ve}$ values

$= (1)(1) + 10000 + \dots +^{ve}$ values

$= 1 + 10{,}000 + \dots +^{ve}$ values

$= 10{,}001 + \dots +^{ve}$ values $> 10{,}000$

$\therefore (1.01)^{10{,}00{,}000} > 10{,}000$

4. Using binomial theorem evaluate each of the following

i. $(101)^4$

$(101)^4 = (100+1)^4$

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(100)^0(1)^4$

$= (1)(100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + 1(1)$

$= 100000000 + 4(1000000) + 6(10000) + 400 + 1$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

Note:- For the powers of 100 $\Rightarrow 2 \times 2 = 4 \Rightarrow$ Add 4 zeros to the front after 1st digit

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots {}^nC_n a^0 b^n$

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots {}^nC_n a^0 b^n$$

$$= {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots {}^nC_n b^n$$

ii. $(102)^5$

$$(102)^5 = (100+2)^5$$

$$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)(2)^4 + {}^5C_5 (2)^5$$

$$= (1)(10000000000) + 5(100000000)(2) + 10(1000000)(4) + 10(10000)(8) + 5(100)(16) + (1)(32)$$

$$= 10000000000 + 1000000000 + 40000000 + 800000 + 32$$

$$= \underline{\underline{11040800032}}$$

Note:- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n b^n$

$n = 5 \Rightarrow \quad 1 \quad 5 \quad \underline{\underline{10 \quad 10 \quad 5}} \quad 1 \rightarrow$ Pascal's triangle

iii. $(99)^5$

$$(99)^5 = (100-1)^5$$

$$= {}^5C_0 (100)^5 - {}^5C_1 (100)^4 (1) + {}^5C_2 (100)^3 (1) - {}^5C_3 (100)^2 (1) +$$

$${}^5C_4(100)(1) - {}^5C_5(1)$$

$$= (1)(10000000000) - 5(100000000) + 10(1000000) - 10(10000) + 5(100) - (1)(1)$$

$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$$

$$= \underline{\underline{9509900499}}$$

Note :- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n a^0 b^n$

$$= {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n (1) b^n$$

$$= \underline{\underline{{}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n b^n}}$$

$n = 5 \Rightarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \Rightarrow$ Pascal's triangle.

5. Find $(a+b)^4 - (a-b)^4$ hence evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$

$$(a+b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 \; {}^4C_3 a b^3 + {}^4C_4 a^0 b^4$$

$$= (1)a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + (1) b^4$$

$$(a+b)^4 = \cancel{a^4} + 4a^3 b + \cancel{6a^2b^2} + 4ab^3 + \cancel{b^4}$$

$$(a-b)^4 = \cancel{a^4} - 4a^3 b + \cancel{6a^2b^2} - 4ab^3 + \cancel{b^4}$$

$$(-) \quad \cancel{(-)} \quad (+) \quad \cancel{(-)} \quad (+) \quad \cancel{(-)}$$

$$(a+b)^4 - (a-b)^4 = 8a^3 b + 8ab^3$$

Similarly (III^4) $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8(\sqrt{3})^3(\sqrt{2}) +$

Similarly (IIIly) $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4 = 8(\sqrt{3})^3(\sqrt{2})+ 8(\sqrt{3})(\sqrt{2})^3$

$$= 8(3\sqrt{3})(\sqrt{2})+8(\sqrt{3})(2\sqrt{2})$$

$$= 24\sqrt{6}+16\sqrt{6}$$

$$= \underline{\underline{40\sqrt{6}}}$$

Note:- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots {}^nC_n b^n$

$n=4 \quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

6. Find $(x+1)^6+(x-1)^6$ hence evaluate $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

$$(x+1)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5(1) + {}^6C_2(x)^4(1)^2 + {}^6C_3(x)^3(1)^3 + {}^6C_4(x)^2(1)^4 + {}^6C_5(x)(1)^5 + {}^6C_6(x)^0(1)^6$$

$$= (1)x^6+6x^5+15x^4+20x^3+15x^2+6x+1$$

$$(x+1)^6 = x^6+\cancel{6x^5}+15x^4+\cancel{20x^3}+15x^2+\cancel{6x}+1$$

$$(x-1)^6 = x^6-\cancel{6x^5}+15x^4-\cancel{20x^3}+15x^2-\cancel{6x}+1$$

$$(x+1)^6+(x-1)^6 = 2x^6+30x^4+30x^2+2$$

IIIly

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2(\sqrt{2})^6+30(\sqrt{2})^4+30(\sqrt{2})^2+2$$

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2(\sqrt{2})^6 + 30(\sqrt{2})^4 + 30(\sqrt{2})^2 + 2$$

$$= 2(8) + 30(4) + 30(2) + 2$$

$$= 16 + 120 + 60 + 2$$

$$= \underline{\underline{198}}$$

7. Prove that $\sum_{r=0}^{n} 3^r \, n_{C_r} = 4^n$

$$\sum_{r=0}^{n} 3^r \, {}^nC_r = 3^0 \, {}^nC_0 + 3 \, {}^nC_1 + 3^2 \, {}^nC_2 + \cdots 3^n \, {}^nC_n$$

$$= (1)(1) + {}^nC_1(3) + {}^nC_2(9) + \cdots (1)\,3^n$$

and

$$4^n = (1+3)^n$$

$$= {}^nC_0 \, 1^n + {}^nC_1 \, 1^{n-1}(3) + {}^nC_2 (1)^{n-2}(3)^2 + \cdots {}^nC_n \, 3^n$$

$$= (1)(1) + {}^nC_1(3) + {}^nC_2(9)$$

$$\therefore \underline{\underline{\sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n}}$$

GENERAL TERM

$T_{r+1} = {}^nC_r \, a^{n-r} b^r$ is the general term for $(a+b)^n$

8. Write the general term expansion for the following

i. $(x^2 - y)^6$

Compare with $(a+b)^n$

$\Rightarrow a = x^2 \quad b = -y \quad n = 6$

General term (G.T) $T_{r+1} = {}^nC_r a^{n-r} b^r$

$= {}^6C_r (x^2)^{6-r} (-y)^r$

$= {}^6C_r x^{12-2r} (-y)^r$

ii. $(x^2 - yx)^{12}$

Compare with $(a+b)^n$

$\Rightarrow a = x^2 \quad b = -yx \quad n = 12$

General Term (GT) $\Rightarrow$

$T_{r+1} = {}^nC_r a^{n-r} b^r$

$= {}^{12}C_r (x^2)^{12-r} (-yx)^r$

$= {}^{12}C_r x^{24-2r} (-y)^r (x)^r = {}^{12}C_r x^{24-2r+r} (-y)^r$

$= {}^{12}C_r x^{24-r} (-y)^r$

9. Find the fourth term in the expansion of $(x - 2y)^{12}$

compare with $(a+b)^n$

$\Rightarrow a = x \quad b = -2y \quad n = 12 \quad r = 3$

General term

$T_{r+1} = {}^nC_r a^{n-r} b^r$

General term

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

$$T_{3+1} = {}^{12}C_3\, x^{12-3} (-2y)^3$$

$$T_4 = {}^{12}C_3\, x^9 (-8y^3)$$

$$T_4 = \frac{12!}{9!\,3!} x^9 (-8y^3)$$

$$= \frac{\cancel{(12)}^{2}(11)(10)\cancel{9!}}{\cancel{9!} \times \cancel{(3)(2)(1)}} x^9 (-8y^3)$$

$$= -1760\, x^9 y^3$$

10. Find 13th term of the expansion $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

$a = 9x \quad b = \frac{-1}{3\sqrt{x}} \quad n = 18 \quad r = 12$

General term (G.T)

$$\Rightarrow T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

$$T_{12+1} = {}^{18}C_{12} (9x)^{18-12} \left(\frac{-1}{3\sqrt{x}}\right)^{12}$$

$$T_{13} = \frac{18!}{6!\cdot 12!} (9x)^6 \left(\frac{1}{3^{12}(x^{1/2})^{12}}\right)$$

$$T_{13} = \frac{\cancel{(18)}^{8}(17)\cancel{(16)}^{\cancel{4}\,2}\cancel{(15)}^{3}\cancel{(14)}(13)\cancel{12!}}{\cancel{(6)}\cancel{(5)}\cancel{(4)}\cancel{(3)}\cancel{(2)}(1)\cdot\cancel{12!}} \cancel{(3^{12}x^6)} \left(\frac{1}{\cancel{3^{12}x^6}}\right)$$

$$T_{13} = (17)(2)(3)(7)(13)$$

$$T_{13} = (17)(2)(3)(7)(13)$$

$$= \underline{\underline{18564}}$$

4. Find the coefficient of

i. x^5 in $(x+3)^8$

$a = x \quad b = 3 \quad n = 8$

$GT:- \; T_{r+1} = {}^nC_r \, a^{n-r} b^r$

$$T_{r+1} = {}^8C_r \, x^{(8-r)} (3)^r \longrightarrow 1$$

Put $8 - r = 5$

$$-r = 5 - 8$$

$$\cancel{-}r = \cancel{-}3$$

$$\underline{\underline{r = 3}}$$

$\therefore$ 1 becomes

$$T_{3+1} = {}^8C_3 \, x^{8-3} (3)^3$$

$$T_4 = \frac{8!}{5!\,3!} x^5 (27)$$

$$T_4 = \frac{(8)(7)\cancel{(6)}\cancel{5!}}{\cancel{5!}\,\cancel{(3)(2)(1)}} x^5 (27)$$

$$T_4 = \underline{\underline{1512 x^5}}$$

Coefficient of x^5 is $\underline{\underline{1512}}$