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JEE Main Mathematics

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Specially Prepared Questions for JEE Main with

Complete Theory • 2 Levels Exercises • Exams Questions

Prafful K Agarwal

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ARIHANT PRAKASHAN (Series), MEERUT

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PREFACE

In sync with the recent changes in the test pattern and format of **JEE Main** (Joint Engineering Entrance), it is my pleasure to introduce **Master Resource Book in Mathematics for JEE Main**, for the Students aspiring a seat in a reputed Engineering College. JEE Main is a gateway examination for candidates expecting to seek admission in Bachelor in Engineering (BE), Bachelor of Technology (B.Tech) and Bachelor of Architecture (B.Arch) at Indian Institutes of Information Technology (IIITs), National Institutes of Technology (NITs), Delhi Technological University and other Centrally Funded Technical Institutes (CFTIs).

JEE Main is also an examination which is like screening examination for **JEE Advanced**
(The gateway examination to India's most reputed Technical Institutes, Indian Institutes of Technology IITs). Only the top 2.2 lacs students passed in JEE Main will be able to attempt JEE Advanced. Gradually, the number of students aspiring for the seat in the Engineering College has increased rapidly in the last 5 Years or so. This year nearly 10 lacs students appeared for

JEE Main and only a few were able to reserve a seat in the college of their choice, so there is a cut throat competition among the aspirants. Thus, it calls for a systematic mastery of all the subjects of the test with paramount importance to problem-solving. Most of the books now in the market have become repetitive with scant respect to the needs of true and effective learning. This book has been designed to fulfill the perceived needs of the students as such.

- This book comprehensively covers all the topics of **JEE Main Mathematics syllabus**. The chapters have been sequenced according to the **syllabus of class 11th & 12th.** Each chapter has essential theoretical discussion of the related concepts with sufficient number of solved examples, practice problems and other solved problems. In each chapter previous years' questions of AIEEE and JEE Main have been included to help students know the difficulty levels and nature of questions asked in competitive exams at this level.
- All types of questions have been included in this book: **Single Correct Answer Types, Multiple Correct Answer Types, Reasoning Types, Matches, Passage-based Questions** etc.
- This is the only book which has its subject matter divided as per class 11th & 12th syllabus. It covers almost all questions of **NCERT Textbook & NCERT Exemplar problems**.

It is hoped this new effort will immensely benefit the students in their goal to secure a seat in the prestigious engineering college, and would be convenient to teachers in planning their teaching programmes. Suggestions for further improvement are welcome from the students and teachers.

Prafful K Agarwal

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SYLLABUS

UNIT 1 Sets, Relations and Functions

Sets and their representation; Union, intersection and complement of sets and their algebraic properties; Power set; Relation, Types of relations, equivalence relations, functions;. one-one, into and onto functions, composition of functions.

UNIT 2 Complex Numbers and Quadratic Equations

Complex numbers as ordered pairs of reals, Representation of complex numbers in the form $a+ib$ and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality, Quadratic equations in real and complex number system and their solutions. Relation between roots and co-efficients, nature of roots, formation of quadratic equations with given roots.

UNIT 3 Matrices and Determinants

Matrices, algebra of matrices, types of matrices, determinants and matrices of order two and three. Properties of determinants, evaluation of deter-minants, area of triangles using determinants. Adjoint and evaluation of inverse of a square matrix using determinants and elementary transformations, Test of consistency and solution of simultaneous linear equations in two or three variables using determinants and matrices.

UNIT 4 Permutations and Combinations

Fundamental principle of counting, permutation as an arrangement and combination as selection, Meaning of $P(n,r)$ and $C(n,r)$, simple applications.

UNIT 5 Mathematical Induction

Principle of Mathematical Induction and its simple applications.

UNIT 6 Binomial Theorem and its Simple Applications

Binomial theorem for a positive integral index, general term and middle term, properties of Binomial coefficients and simple applications.

UNIT 7 Sequences and Series

Arithmetic and Geometric progressions, insertion of arithmetic, geometric means between two given numbers. Relation between A.M. and G.M. Sum upto n terms of special series: $\sum n$, $\sum n^2$, $\sum n^3$. Arithmetico - Geometric progression.

UNIT 8 Limit, Continuity and Differentiability

Real valued functions, algebra of functions, polynomials, rational, trigonometric, logarithmic and exponential functions, inverse functions. Graphs of simple functions. Limits, continuity and differentiability.

Differentiation of the sum, difference, product and quotient of two functions. Differentiation of trigonometric, inverse trigonometric, logarithmic exponential, composite and implicit functions derivatives of order upto two. Rolle's and Lagrange's Mean Value Theorems. Applications of derivatives: Rate of change of quantities, monotonic - increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

UNIT 9 Integral Calculus

Integral as an anti - derivative. Fundamental integrals involving algebraic, trigonometric, exponential and

logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities.

Evaluation of simple integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \quad \int \frac{dx}{a^2 - x^2}, \quad \int \frac{dx}{\sqrt{a^2 - x^2}},$$

$$\int \frac{dx}{ax^2 + bx + c}, \quad \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \quad \int \frac{(px + q)\, dx}{ax^2 + bx + c},$$

$$\int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}, \quad \int \sqrt{a^2 \pm x^2}\, dx \quad \text{and} \quad \int \sqrt{x^2 - a^2}\, dx$$

Integral as limit of a sum. Fundamental Theorem of Calculus. Properties of definite integrals. Evaluation of definite integrals, determining areas of the regions bounded by simple curves in standard form.

UNIT 10 Differential Equations

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type $\frac{dy}{dx} + p(x)\, y = q(x)$

UNIT 11 Coordinate Geometry

Cartesian system of rectangular coordinates in a plane, distance formula, section formula, locus and its equation, translation of axes, slope of a line, parallel and perpendicular lines, intercepts of a line on the coordinate axes.

- **Straight lines**

  Various forms of equations of a line, intersection of lines, angles between two lines, conditions for concurrence of three lines, distance of a point from a line, equations of internal and external bisectors of angles between two lines, coordinates of centroid, orthocentre and circumcentre of a triangle, equation of family of lines passing through the point of intersection of two lines.

- **Circles, conic sections**

  Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent. Sections of cones, equations of conic sections (parabola, ellipse and hyperbola) in standard forms, condition for y = mx + c to be a tangent and point (s) of tangency.

UNIT 12 Three Dimensional Geometry

Coordinates of a point in space, distance between two points, section formula, direction ratios and direction cosines, angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms, intersection of a line and a plane, coplanar lines.

UNIT 13 Vector Algebra

Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.

UNIT 14 Statistics and Probability

Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data. Calculation of standard deviation, variance and mean deviation for grouped and ungrouped data.

Probability: Probability of an event, addition and multiplication theorems of probability, Baye's theorem, probability distribution of a random variate, Bernoulli trials and Binomial distribution.

UNIT 15 Trigonometry

Trigonometrical identities and equations. Trigonometrical functions. Inverse trigonometrical functions and their properties. Heights and Distances.

UNIT 16 Mathematical Reasoning

Statements, logical operations and implies, implied by, if and only if. Understanding of tautology, contradiction, converse and contra positive.

PART - I

Chapters from Class 11th Syllabus

1 Sets

JEE Main MILESTONE

- Sets
- Representation of Sets
- Power Set
- Venn Diagram
- Operations on Sets
- Laws of Algebra of Sets
- Cardinal Number of a Finite and Infinite Set

1.1 Sets

In Mathematical language all living and non-living things in universe are known as objects. The collection of well defined distinct objects is known as a set.

Well defined means in a given set, it must be possible to decide whether or not the object belongs to the set and by distinct means object should not be repeated. The object in the set is called its **member** or **element**. A set is represented by { }.

Generally, sets are denoted by capital letters $A, B, C, \ldots$ and its elements are denoted by small letters $a, b, c, \ldots\ldots$.

Let A is a non-empty set. If x is an element of A, then we write '$x \in A$' and read as 'x is an element of A' or 'x belongs to A'. If x is not an element of A, then we write '$x \notin A$' and read as x is not an element of A' or 'x does not belong to A'.

e.g., A = Set of all vowels in English alphabets.

In this set a, e, i, o and u are members.

The theory of sets was developed by German Mathematician Georg Cantor (1845-1918). *The concept of sets is widely used in the foundation of relations, functions, logic, probability theory, etc. According to Cantor 'A set is any collection into a whole of definite and distinct objects of our intuition or thought'.*

***Sample* Problem 1** *Which of the following is a correct set?*

(a) *The collection of all the months of a year beginning with the letter J.*
(b) *The collection of ten most talented writers of India.*
(c) *A team of eleven best cricket batsmen of the world.*
(d) *A collection of most dangerous animals of the world.*

Interpret (a)

(a) We are sure that members of this collection are January, June and July.
So, this collection is well defined and hence, it is a set.

(b) A writer may be most talented for one person and may not be for other. Therefore, it cannot be said accountely that which writers will be there in the relation.
So, this collection is not well defined. Hence, it is not a set.

(c) A batsman may be best for one person and may not be so for other. Therefore, it cannot be said accountely that which batsman will be there in our relation.
So, this collection is not well defined. Hence, it is not a set.

(d) The term most dangerous is not a clear term. An animal may be most dangerous to one person and may not be for the other.
So, it is not well defined, hence it is not a set.

1.2 Representation of Sets

We can use the following two methods to represent a set.

(i) **Listing Method** In this method, elements are listed and put within a braces { } and separated by commas.

This method is also known as **Tabular method** or **Roster method**.

e.g., A = Set of all prime numbers less than 11
$= \{2, 3, 5, 7\}$

(ii) **Set Builder Method** In this method, instead of listing all elements of a set, we list the property or properties satisfied by the elements of set and write it as

$$A = \{x : P(x)\} \text{ or } \{x \mid P(x)\}$$

It is read as "A is the set of all elements x such that x has the property $P(x)$." The symbol ':' or '|' stands for such that.

This method is also known as **Rule method** or **Property method.**

e.g., $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$= \{x : x \in N \text{ and } x \le 8\}$

Note

- The order of elements in a set has no importance *e.g.*, {1, 2, 3} and {3, 1, 2} are same sets.
- The repetition of elements in a set does not effect the set, *e.g.*, {1, 2, 3} and {1, 1, 2, 3} both are same sets.

Notation of Some Standard Sets

(i) Set of all natural numbers, $N = \{1, 2, 3, \ldots\}$

(ii) Set of all whole numbers, $W = \{0, 1, 2, 3, \ldots\}$

(iii) (a) Set of all integers, I or $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(b) Set of all positive or negative integers,

$$I^+ = \{1, 2, 3, \ldots \infty\}$$

or $$I^- = \{-1, -2, -3, \ldots \infty\}$$

(c) Set of all even (E) or odd (O) integers,

$$E = \{\ldots, -4, -2, 0, +2, +4, \ldots\}$$

or $$O = \{\ldots, -3, -1, 0, 1, 3, \ldots\}$$

(iv) (a) Set of all rational numbers,

$Q = \{p/q$, where p and q are integers and $q \ne 0\}$

(b) Set of all irrational numbers,

$IR = \{$which cannot be p and $I \in I, q \ne 0\}$

(c) Set of all real numbers,

$$R = \{x : -\infty < x < \infty\}$$

(v) Set of all complex numbers,

$$C = \{a + ib; a, b \in R \text{ and } i = \sqrt{-1}\}$$

***Sample* Problem 2** *The builder form of following set is* $A = \{3, 6, 9, 12\}$, $B = \{1, 4, 9, \ldots, 100\}$ **[NCERT]**

(*a*) $A = \{x : x = 3n, n \in N \text{ and } 1 \le n \le 5\}$,
$B = \{x : x = n^2, n \in N \text{ and } 1 \le n \le 10\}$

(*b*) $A = \{x : x = 3n, n \in N \text{ and } 1 \le n \le 4\}$,
$B = \{x : x = n^2, n \in N \text{ and } 1 \le n \le 10\}$

(*c*) $A = \{x : x = 3n, n \in N \text{ and } 1 \le n \le 4\}$,
$B = \{x : x = n^2, n \in N \text{ and } 1 < n < 10\}$

(*d*) *None of the above*

Interpret (*b*) Given, $A = \{3, 6, 9, 12\}$ and

$$= \{x : x = 3n, n \in N \text{ and } 1 \le n \le 4\}$$

and $B = \{1, 4, 9, \ldots, 100\} = \{x : x = n^2, n \in N \text{ and } 1 \le n \le 10\}$

Different Types of Sets

(i) Empty (Void/Null) Set

A set which has no element, is called an empty set.
It is denoted by ϕ or { }.

e.g., A = Set of all odd numbers divisible by 2

and $B = \{x : x \in N \text{ and } 5 < x < 6\}$

Such sets which have atleast one element, are called non-void set.

Note If ϕ represents a null set, then ϕ is never written with in braces *i.e.*, $\{\phi\}$ is not the null set.

(ii) Singleton Set

A set which have only one element, is called a singleton set.

e.g., $A = \{x : x \in N \text{ and } 3 < x < 5\}$

and $B = \{5\}$

(iii) Finite and Infinite Sets

A set in which the process of counting of elements surely comes to an end, is called a finite set. In other words 'A set having finite number of elements is called a finite set'. Otherwise it is called infinite set *i.e.*, if the process of

counting of elements does not come to an end in a set, then set is called an infinite set.

e.g., $A = \{x : x \in N \text{ and } x < 5\}$

and B = Set of all points on a plane

In above two sets A and B, set A is finite while set B is infinite. Since, in a plane any number of points are possible.

(iv) Equivalent Sets

Two finite sets A and B are said to be equivalent, if they have the same number of elements.

e.g., If $A = \{1, 2, 3\}$ and $B = \{3, 7, 9\}$

Number of elements in $A = 3$

and number of elements in $B = 3$

$\therefore$ A and B are equivalent sets.

(v) Equal Sets

If A and B are two non-empty sets and each element of set A is an element of set B and each element of set B is an element of set A, then sets A and B are called equal sets.

Symbolically, if $x \in A \Rightarrow x \in B$

and $x \in B \Rightarrow x \in A$

e.g., $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x \le 3\}$

Here, each element of A is an element of B, also each element of B is an element of A, then both sets are called equal sets.

Note Equal sets are equivalent sets while its converse need not to be true.

(vi) Subset and Superset

Let A and B be two non-empty sets. If each element of set A is an element of set B, then set A is known as subset of set B. If set A is a subset of set B, then set B is called the superset of A.

Also, if A is a subset of B, then it is denoted as $A \subseteq B$ and read as 'A is a subset of B'.

Thus, if $x \in A \Rightarrow x \in B$,

then $A \subseteq B$

If $x \in A \Rightarrow x \notin B$,

then $A \not\subseteq B$

and read as 'A is not a subset of B.'

e.g., If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

Here, each element of A is an element of B. Thus, $A \subseteq B$ *i.e.*, A is a subset of B and B is a superset of A.

Note
- Null set is a subset of each set.
- Each set is a subset of itself.
- If A has n elements, then number of subsets of set A is 2^n.

(vii) Proper Subset

If each element of A is in set B but set B has atleast one element which is not in A, then set A is known as proper subset of set B. If A is a proper subset of B, then it is written as '$A \subset B$' and read as A is a proper subset of B.

e.g., If $N = \{1, 2, 3, 4, ...\}$

and $I = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

then $N \subset I$

Note If A has n elements, then number of proper subsets is $2^n - 1$.

(viii) Comparability of Sets

Two sets A and B are said to be comparable, if either $A \subset B$ or $B \subset A$ or $A = B$, otherwise, A and B are said to be incomparable.

e.g., Suppose $A = \{1, 2, 3\}$, $B = \{1, 2, 4, 6\}$ and $C = \{1, 2, 4\}$

Since, $A \not\subset B$ or $B \not\subset A$ or $A \ne B$

$\therefore$ A and B are incomparable.

But $C \subset B$

$\therefore$ B and C are comparable sets.

(ix) Universal Set

If there are some sets under consideration, then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set and it is denoted by S or U.

This set can be chosen arbitrarily for any discussion of given sets but after choosing it is fixed.

e.g., Suppose $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ and $C = \{7, 8, 9\}$

$\therefore$ $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is universal set for all three sets.

***Sample* Problem 3** *Consider the following sets*

A = The set of lines which are parallel to the X-axis.

B = The set of letters in the English alphabet.

and C = The set of animals living on the earth. **[NCERT]**

Which of these is finite or infinite set?

(a) *Finite set* $\rightarrow A, B$, *Infinite set* $\rightarrow C$

(b) *Finite set* $\rightarrow B, C$, *Infinite set* $\rightarrow A$

(c) *Finite set* $\rightarrow A, C$, *Infinite set* $\rightarrow B$

(d) *None of the above*

Interpret (*b*)

A = Infinite lines can be drawn parallel to X-axis

B = There are finite 26 English alphabets

C = There are finite number of animals living on earth

***Sample* Problem 4** *Two finite sets have m and n elements, respectively. The total number of subsets of the first set is 56 more than the total number of subsets of second set. What are the values of m and n, respectively?*

(a) 7, 6 (b) 6, 3 (c) 5, 1 (d) 8, 7

Interpret (b) Since, total possible subsets of sets A and B are 2^m and 2^n, respectively.

According to given condition,

$$2^m - 2^n = 56$$

$$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^3 \times (2^3 - 1)$$

On comparing both sides, we get

$$2^n = 2^3 \text{ and } 2^{m-n} = 2^3$$

$$\Rightarrow \quad n = 3 \text{ and } m - n = 3$$

$$\Rightarrow \quad m = 6 \text{ and } n = 3$$

1.3 Power Set

Let A be a non-empty set, then collection of all possible subsets of set A is known as power set. It is denoted by $P(A)$.

e.g., Suppose $A = \{1, 2, 3\}$

$\therefore \quad P(A) = [\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}]$.

(a) $A \notin P(A)$ (b) $\{A\} \in P(A)$

Properties of Power Set

(i) Each element of a power set is a set.

(ii) If $A \subseteq B$, then $P(A) \subseteq P(B)$

(iii) Power set of any set is always non-empty.

(iv) If set A has n elements, then $P(A)$ has 2^n elements.

(v) $P(A) \cap P(B) = P(A \cap B)$

(vi) $P(A) \cup (B) \subseteq P(A \cup B)$

(vii) $P(A \cup B) \neq P(A) \cup P(B)$

***Sample* Problem 5** *If set $A = \{1, 3, 5\}$, then number of elements in $P\{P(A)\}$ is*

(a) 8 (b) 256 (c) 248 (d) 250

Interpret (b) Given, $A = \{1, 3, 5\}$

$$\therefore \quad n\{P(A)\} = 2^3 = 8$$

$$\therefore \quad n[P\{P(A)\}] = 2^8 = 256$$

***Sample* Problem 6** *Consider $A = \{1, 2\}$, $B = \{2, 3\}$. Then which of the following option is correct?*

(a) $P(A \cup B) \neq P(A) \cup P(B)$ (b) $P(A \cup B) = P(A) \cup P(B)$

(c) $P(A \cup B) = P(A) \cap P(B)$ (d) *None of these*

Interpret (a) Here $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$,

$$P(B) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$$

$$AB = \{1, 2, 3\}$$

$$\therefore \quad P(A \cup B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

$$P(A)\, P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$$

$$\therefore \quad P(A \cup B) \neq P(A) \cup P(B)$$

1.4 Venn Diagram

A Swiss Mathematician (1707-1783) Euler gave an idea to represent a set by the points in a closed curve. Later on British Mathematician John Venn (1834-1923) brought this idea to practice. So, the diagrams drawn to represent sets are called Venn Euler diagram or simply Venn diagram.

In Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set. Also, an element of a set A is represented by a point within the circle of set A.

e.g., If $U = \{1, 2, 3, 4, ..., 10\}$ and $A = \{1, 2, 3\}$

Then, its Venn diagram is as shown in the figure.

1.5 Operations on Sets

Now, we introduce some operations on sets to construct new sets from the given ones.

(i) Union of Two Sets

Let A and B be two sets, then union of A and B is a set of all those elements which are in A or in B or in both A and B. It is denoted by $A \cup B$ and read as 'A union B'.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Clearly, $x \in A \cup B$

$\Rightarrow \quad x \in A$ or $x \in B$

If $x \notin A \cup B$

$\Rightarrow \quad x \notin A$ and $x \notin B$

The venn diagram of $A \cup B$ is as shown in the figure and the shaded portion represents $A \cup B$.

$A \cup B$ (when $A \subseteq B$) | $A \cup B$ when neither $A \subseteq B$ nor $B \subseteq A$ | $A \cup B$ when A and B are disjoint sets

e.g., If $A = \{1, 2, 3, 4\}$

and $B = \{4, 8, 5, 6\}$

$\therefore \quad A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$.

General Form

The union of a finite number of sets $A_1, A_2, \ldots, A_n$ is represented by

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n \text{ or } \bigcup_{i=1}^{n} A_i$$

Symbolically, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for atleast one } i\}$

(ii) Intersection of Two Sets

If A and B are two sets, then intersection of A and B is a set of all those elements which are in both A and B. The intersection of A and B is denoted by $A \cap B$ and read as "A intersection B".

Symbolically,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $\quad x \in A \cap B \Rightarrow x \in A$ and $x \in B$

and if $\quad x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

The Venn diagram of $A \cap B$ is as shown in the figure and the shaded region represents $A \cap B$.

$A \cap B$ when $A \subseteq B$ or $A \cap B = A$ | $A \cap B$ when neither $A \subseteq B$ nor $B \subseteq A$ | $A \cap B = \phi$ (no shaded region)

e.g., If $\quad A = \{1, 2, 3, 4\}$ and $B = \{4, 3, 5, 6\}$

$\therefore \quad A \cap B = \{3, 4\}$

General Form

The intersection of a finite number of sets

$A_1, A_2, A_3, \ldots, A_n$ is represented by

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n \text{ or } \bigcap_{i=1}^{n} A_i$$

Symbolically, $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } i\}$

(iii) Disjoint of Two Sets

Two sets A and B are known as disjoint sets, if $A \cap B = \phi$ *i.e.*, if A and B have no common element.

The Venn diagram of disjoint sets as shown in the figure

$$A \cap B$$

e.g., If $\quad A = \{1, 2, 3\}$

and $\quad B = \{4, 5, 6\}$,

then $\quad A \cap B = \{\ \} = \phi$

$\therefore$ A and B are disjoint sets.

(iv) Difference of Two Sets

If A and B are two non-empty sets, then difference of A and B is a set of all those elements which are in A but not in B. It is denoted as $A - B$. If difference of two sets is $B - A$, then it is a set of those elements which are in B but not in A.

Hence, $\quad A - B = \{x : x \in A \text{ and } x \notin B\}$

and $\quad B - A = \{x : x \in B \text{ and } x \notin A\}$

If $\quad x \in A - B \Rightarrow x \in A$ but $x \notin B$

and if $\quad x \in B - A \Rightarrow x \in B$ but $x \notin A$

The Venn diagram of $A - B$ and $B - A$ are as shown in the figure and shaded region represents $A - B$ and $B - A$.

$A - B$ when $A \subseteq B$, *i.e.*, $(A - B = \phi)$ | $A - B$ when $B \subseteq A$

$A - B$ when neither $A \subseteq B$ nor $B \subseteq A$ | $A - B$ when A and B are disjoint sets. Clearly, $A - B = A$

e.g., If $\quad A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7, 8\}$

$\therefore \quad A - B = \{1, 2, 3\}$ and $B - A = \{5, 6, 7, 8\}$

Note

- $A - B \neq B - A$
- $A - B \subseteq A$ and $B - A \subseteq B$
- $A - \phi = A$ and $A - A = \phi$
- The sets $A - B$ and $B - A$ are disjoint sets.

(v) Symmetric Difference of Two Sets

If A and B are two sets, then set $(A - B) \cup (B - A)$ is known as symmetric difference of sets A and B and is denoted by $A \Delta B$.

$A \Delta B$

The Venn diagram of $A \Delta B$ is as shown in the figure and shaded region represents $A \Delta B$.

e.g., $\quad A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$,

then $\quad A \Delta B = (A - B) \cup (B - A)$

$= \{1, 2\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

Note

- Symmetric difference can also be written as
$$A \Delta B = (A \cup B) - (A \cap B)$$
- $A \Delta B = B \Delta A$ (commutative)

(vi) Complement of a Set

The complement of a set A is the set of all those elements which are in universal set but not in A. It is denoted by A' or A^c.

If U is a universal set and $A \subset U$,

then $A' = U - A = \{x : x \in U \text{ but } x \notin A\}$

i.e., $x \in A \Rightarrow x \notin A'$

The Venn diagram of complement of a set A is as shown in the figure and shaded portion represents A'.

e.g., If $U = \{1, 2, 3, 4, 5, ...\}$

and $A = \{2, 4, 6, 8, ...\}$

$\therefore$ $A' = U - A = \{1, 3, 5, 7, ...\}$

By Venn diagram, the operation between three sets can be represented given below.

Note

- $\phi = U'$
- $\phi' = U$
- $(A')' = A$
- $A \cup A' = U$
- $A \cap A' = \phi$

***Sample* Problem 7** *If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$, then $(A \cup B)'$, $(A' \cap B')$, $(A \Delta B)$ is equal to*

(a) $\{1, 9\}, \{2, 8\}, \{3, 4, 5, 6, 7, 8\}$
(b) $\{1, 9\}, \{1, 9\}\ \{3, 4, 5, 6, 7, 8\}$
(c) $\{1, 9\}, \{1, 9\}\ \{5, 6, 7, 8\}$
(d) *None of the above*

Interpret (b) Given sets are

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$

Now, $A' = U - A$

$= \{1, 2, 3, , 4, 5, 6, 7, 8, 9\} - \{2, 4, 6, 8\}$

$= \{1, 3, 5, 7, 9\}$

and $B' = U - B$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 3, 5, 7\}$

$= \{1, 4, 6, 8, 9\}$

(i) $A \cup B = \{2, 4, 6, 8\} \cup \{2, 3, 5, 7\}$

$= \{2, 3, 4, 5, 6, 7, 8\}$

$\therefore (A \cup B)' = U - A \cup B$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 3, 4, 5, 6, 7, 8\}$

$= \{1, 9\}$

(ii) $(A' \cap B') = \{1, 3, 5, 7, 9\} \cap \{1, 4, 6, 8, 9\}$

$= \{1, 9\}$

(iii) Now, $A - B = \{2, 4, 6, 8\} - \{2, 3, 5, 7\}$

$= \{4, 6, 8\}$

and $B - A = \{2, 3, 5, 7\} - \{2, 4, 6, 8\}$

$= \{3, 5, 7\}$

$\therefore$ $A \Delta B = (A - B) \cup (B - A)$

$= \{4, 6, 8\} \cup \{3, 5, 7\} = \{3, 4, 5, 6, 7, 8\}$

***Sample* Problem 8** *The shaded region in the given figure is*

(a) $B \cap (A \cup C)$
(b) $B \cup (A \cap C)$
(c) $B \cap (A - C)$
(d) $B - (A \cup C)$

Interpret (d) It is clear from the figure that set $A \cup C$ is not shading and set B is shading other than $A \cup C$.

i.e., $B - (A \cup C)$

***Sample* Problem 9** *If $U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\}$, $A = \{x : x^2 - 5x + 6 = 0\}$ and $B = \{x : x^2 - 3x + 2 = 0\}$ what is $(A \cap B)'$ equal to ?*

(a) $\{1, 3\}$
(b) $\{1, 2, 3\}$
(c) $\{0, 1, 3\}$
(d) $\{0, 1, 2, 3\}$

Interpret (c) $\because$ $U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\}$

$= \{0, 1, 2, 3\}$

$A = \{x : x^2 - 5x + 6 = 0\} = \{2, 3\}$

and $B = \{x : x^2 - 3x + 2 = 0\} = \{1, 2\}$

$\therefore$ $A \cap B = \{2\}$

Hence, $(A \cap B)' = U - (A \cap B)$

$= \{0, 1, 2, 3\} - \{2\} = \{0, 1, 3\}$

1.6 Laws of Algebra of Sets

If A, B and C are three non-empty sets, then

(i) Idempotent law

(a) $A \cup A = A$
(b) $A \cap A = A$

(ii) Identity law

(a) $A \cup \phi = A$
(b) $A \cap U = A$

(iii) Commutative law

(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$

(iv) Associative law

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) Distributive law

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) De-morgan's law

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(c) $A - (B \cup C) = (A - B) \cap (A - C)$

(d) $A - (B \cap C) = (A - B) \cup (A - C)$

Sample **Problem 10** *If A and B are two sets, then $A \cap (A \cup B)'$ is equal to*

(a) A (b) B

(c) ϕ (d) *None of these*

Interpret (c) $A \cap (A \cup B)' = A \cap (A' \cap B')$ ($\because$ by de-Morgan's law)

$= (A \cap A') \cap B'$ ($\because$ by associative law)

$= \phi \cap B'$ ($\because A \cap A' = \phi$)

$= \phi$

Sample **Problem 11** *If A and B are non-empty sets, then $(A \cap B) \cup (A - B)$ is equal to*

(a) B (b) A (c) A' (d) B'

Interpret (b) $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$

$[\because (A - B) = (A \cap B')]$

$= A \cap (B \cup B')$ ($\because$ by distributive law)

$= A \cap U$ ($\because B \cup B' = U$)

$= A$

1.7 Cardinal Number of a Finite and Infinite Set

The number of distinct elements in a finite set A is called cardinal number and it is denoted by $n(A)$. And if it is not finite set, then it is called infinite set.

e.g., If $A = \{-3, -1, 8, 10, 13, 17\}$,

then $n(A) = 6$

Properties

If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) If A and B are disjoint sets then, $n(A \cup B) = n(A) + n(B)$

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(iv) $n(A - B) = n(A) - n(A \cap B)$

(v) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$

(vi) $n(A') = n(U) - n(A)$

(vii) $n(A' \cup B') = n(U) - n(A \cap B)$

(viii) $n(A' \cap B') = n(U) - n(A \cup B)$

(ix) $n(A \cap B') = n(A) - n(A \cap B)$

Method to Find Common Roots

Sometimes the number of common elements cannot found easily y by solving the given sets. That type of problems can be solved by drawing a curves.

The intersection point of a curve is equal to the number of common elements in a set.

e.g., Consider the sets

$$A = \{(x, y) \mid y = \frac{1}{x}, 0 \neq x \in R\}$$

and $B = \{(x, y) \mid y = -x, x \in R\}$,

then determine $n(A \cap B)$.

Here, we see that, $\forall\ x$, we get infinite values of y. Hence, we find infinite sets A and B. And it is difficult to find the common elements between A and B.

Now, firstly we make the graph of given sets.

A = The set of all points on a curve $xy = 1$,

[$\because xy = c$ is a rectangular hyperbola curve]

and B = The set of all points on a curve $y = -x$.

[$\because y = -x$ is a straight which passes through origin]

Since, there is no intersection point on a curve. So, there is no common elements between two sets.

Sample **Problem 12** *In a town of 10000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all of three newspapers, then the number of families which buy A only, is*

(a) 4400 (b) 3300 (c) 2000 (d) 500

Interpret (b) $n(A) = 40\%$ of $10000 = 4000$, $n(B) = 2000$, $n(C) = 1000$, $n(A \cap B) = 500$, $n(B \cap C) = 300$, $n(C \cap A) = 400$, $n(A \cap B \cap C) = 200$

$\therefore\ n(A \cap \overline{B} \cap \overline{C}) = n\{A \cap (B \cup C)'\} = n(A) - n\{A \cap (B \cup C)\}$

$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$= 4000 - 500 - 400 + 200 = 3300$

WORKED OUT
Examples

Example 1 *If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, then $(A \cup D) \cap (B \cup C)$ is equal to*

(a) $\{5, 7, 9, 11, 15\}$ (b) $\{7, 9, 11, 15\}$
(c) $\{7, 9, 11, 13, 15\}$ (d) *None of these*

Solution (b) Given, $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$

Now, $A \cup D = \{3, 5, 7, 9, 11\} \cup \{15, 17\}$
$= \{3, 5, 7, 9, 11, 15, 17\}$

and $B \cup C = \{7, 9, 11, 13\} \cup \{11, 13, 15\} = \{7, 9, 11, 13, 15\}$

$\therefore$ $(A \cup D) \cap (B \cup C) = \{3, 5, 7, 9, 11, 15, 17\} \cap \{7, 9, 11, 13, 15\}$
$= \{7, 9, 11, 15\}$

Example 2 *The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to*

(a) $B \cap C'$ (b) $A \cap C$
(c) $B' \cap C'$ (d) *None of these*

Solution (a) $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$

$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C'$
$= (\phi \cup B \cup C) \cap C'$
$= (B \cup C) \cap C'$
$= (B \cap C') \cup \phi = B \cap C'$

Example 3 *Let A, B and C are subsets of universal set U. If $A = \{2, 4, 6, 8, 12, 20\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20\}$ and U is the set of all whole numbers. Then, the correct Venn diagram is* **[NCERT]**

(a)

(b)

(c)

(d) *None of these*

Solution (b) Given, $A = \{2, 4, 6, 8, 12, 20\}$

$B = \{3, 6, 9, 12, 15\}$

$C = \{5, 10, 15, 20\}$

Now, $A \cap B = \{2, 4, 6, 8, 12, 20\} \cap \{3, 6, 9, 12, 15\} = \{6, 12\}$

$B \cap C = \{3, 6, 9, 12, 15\} \cap \{5, 10, 15, 20\} = \{15\}$

$C \cap A = \{5, 10, 15, 20\} \cap \{2, 4, 6, 8, 12, 20\} = \{20\}$

and $A \cap B \cap C = \phi$

Example 4 *In a survey of 600 students in a school, 150 students were found to be taking tea and 225 taking coffee. 100 were taking both tea and coffee. Find how many students were taking neither tea nor coffee?* **[NCERT]**

(a) 310 (b) 320 (c) 327 (d) 325

Solution (d) Let C and T denote the students taking coffee and tea, respectively.

Here, $n(T) = 150, n(C) = 225, n(C \cap T) = 100$

Using the identity $n(C \cup T) = n(T) + n(C) - n(C \cap T)$, we have

$n(C \cup T) = 150 + 225 - 100 = 375 - 100$

$\Rightarrow$ $n(C \cup T) = 275$

Given, total number of students $= 600 = n(U)$

We are to find the number of students taking neither tea nor coffee *i.e.*, $n(C \cup T)'$.

$\therefore$ $n(C \cup T)' = n(U) - n(C \cup T) = 600 - 275 = 325$

Example 5 *If there are three atheletic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 15 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is*

(a) 42 (b) 43
(c) 45 (d) *None of these*

Solution (b) $\because n(B) = 21, n(H) = 26, n(F) = 29, n(H \cap B) = 14,$

$n(H \cap F) = 15, n(F \cap B) = 12, n(B \cap H \cap F) = 8$

$\therefore$ $n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(B \cap H)$
$- n(H \cap F) - n(B \cap F) + n(B \cap H \cap F)$
$= 21 + 26 + 29 - 14 - 15 - 12 + 8 = 43$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Sets and Their Representation

1. The set of intelligent students in a class is
(a) a null set
(b) a singleton set
(c) a finite set
(d) not a well defined collection

2. The following sets

$$A = \{x : x \text{ is an integer}, -\frac{1}{2} < x < \frac{9}{2}\}$$

and $B = \{x : x$ is a month of year not having 31 days$\}$ is equivalent to [NCERT]
(a) $A = \{0, 1, 2, 3\}$, $B = \{$February, April, June, September, November$\}$
(b) $A = \{0, 1, 2, 3, 4\}$, $B = \{$February, April, June, September, November$\}$
(c) $A = \{0, 3, 4\}$, $B = \{$April, June, September, November$\}$
(d) None of the above

3. Let A and B be two non-empty subsets of a set X such that A is not a subset of B, then
(a) A is always a subset of the complement of B
(b) B is always a subset of A
(c) A and B are always disjoint
(d) A and the complement of B are always non-disjoint

4. Which of the following is not null set? [NCERT]
(a) Set of odd natural numbers divisible by 2
(b) Set of even prime numbers
(c) $\{x : x$ is a natural number, $x < 5$ and $x > 7\}$
(d) $\{y : y$ is a point common to any two parallel lines$\}$

5. Which of the following is an equivalent set? [NCERT]
(a) $A = \{a, b, c, d\}$, $B = \{d, c, b, a\}$
(b) $A = \{4, 8, 12, 16\}$, $B = \{8, 4, 16, 18\}$
(c) $A = \{x : x$ is a multiple of $10\}$
$B = \{10, 15, 20, 25, 30, ...\}$
(d) None of the above

6. Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second set. The values of m and n are, respectively [NCERT Exemplar]
(a) 4, 7 (b) 7, 4 (c) 4, 4 (d) 7, 7

7. If $A = \{x : x = 4n + 1, 2 \le n \le 5\}$, then number of subsets of A is
(a) 16 (b) 15
(c) 4 (d) None of these

8. If $A = \{x : x = n^2, n = 1, 2, 3\}$, then number of proper subsets is
(a) 3 (b) 8
(c) 7 (d) None of these

9. If A and B are non-empty sets, then $P(A) \cup P(B)$ is equal to
(a) $P(A \cup B)$ (b) $P(A \cap B)$
(c) $P(A) = P(B)$ (d) None of these

Venn Diagram and Operations on Sets

10. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, \ldots, 18\}$ and N the set of natural numbers is the universal set, then $A' \cup \{(A \cup B) \cap B'\}$ is [NCERT Exemplar]
(a) ϕ (b) N (c) A (d) B

11. If $A = \{2, 3, 4, 8, 10\}$, $B = \{3, 4, 5, 10, 12\}$, $C = \{4, 5, 6, 12, 14\}$, then $(A \cap B) \cup (A \cap C)$ is equal to
(a) $\{3, 4, 10\}$ (b) $\{2, 8, 10\}$
(c) $\{4, 5, 6\}$ (d) $\{3, 5, 14\}$

12. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to
(a) A (b) B (c) ϕ (d) $A \cap B^c$

13. If $A = \{x : x$ is an odd natural number$\}$ and $B = \{x : x$ is a prime number$\}$, then $A \cap B$ is [NCERT]
(a) odd number (b) prime number
(c) odd prime number (d) None of these

14. Let F_1 be the set of parallelograms, F_2 be the set of rectangles, F_3 be the set of rhombuses, F_4 be the set of squares and F_5 be the set of trapeziums in a plane. Then, F_1 may be equal to [NCERT Exemplar]
(a) $F_2 \cap F_3$ (b) $F_3 \cap F_4$
(c) $F_2 \cup F_5$ (d) $F_2 \cup F_3 \cup F_4 \cup F_1$

15. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 3 elements and $B_1, B_2, \ldots, B_n$ are n sets each having 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each elements of S belongs to exactly 10 of A_i's and exactly 9 of B_j's. The value of n is equal to [NCERT Exemplar]
(a) 15 (b) 3
(c) 45 (d) None of these

16. If $A = \{x : x \text{ is a multiple of } 4\}$ and $B = \{x : x \text{ is a multiple of } 6\}$, then $A \cap B$ consists of all multiples of
(a) 16 (b) 12 (c) 8 (d) 4

17. Let S = set of points inside the square, T = the set of points inside the triangle and C = the set of points inside the circle. If the triangle and circle intersect each other and are contained in a square. Then, [NCERT Exemplar]
(a) $S \cap T \cap C = \phi$ (b) $S \cup T \cup C = C$
(c) $S \cup T \cup C = S$ (d) $S \cup T = S \cap C$

18. If $A = \{x : x^2 = 1\}$ and $B = \{x : x^4 = 1\}$, then $A \Delta B$ is equal to
(a) $\{i, -i\}$ (b) $\{-1, 1\}$
(c) $\{-1, 1, i, -1\}$ (d) None of these

19. Consider the following relations
I. $A - B = A - (A \cap B)$
II. $A = (A \cap B) \cup (A - B)$
III. $A - (B \cup C) = (A - B) \cup (A - C)$
Which of these is/are correct?
(a) I and III (b) Only II
(c) II and III (d) I and II

20. The set $(A \cap B')' \cup (B \cap C)$ is equal to [NCERT Exemplar]
(a) $A' \cup B \cup C$ (b) $A' \cup B$
(c) $A' \cup C'$ (d) $A' \cap B$

Cardinal Number of a Finite and Infinite Sets

21. Let X be the universal set for sets A and B. If $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to 300 provided $n(X)$ is equal to
(a) 600 (b) 700 (c) 800 (d) 900

22. Let $S = \{x : x \text{ is a positive multiple of 3 less than } 100\}$
$P = \{x : x \text{ is a prime number less than } 20\}$. Then, $n(S) + n(P)$ is [NCERT Exemplar]
(a) 34 (b) 41 (c) 33 (d) 30

23. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. The number of people speak atleast one of these two languages is [NCERT]
(a) 150 (b) 60
(c) 155 (d) None of these

24. In a class of 60 students, 25 students play Cricket and 20 students play Tennis, and 10 students play both the games. Then, the number of students who play neither is [NCERT Exemplar]
(a) 0 (b) 25 (c) 35 (d) 45

25. In a college of 300 students, every student reads 5 newspaper and every newspaper is read by 60 students. The number of newspaper is
(a) atleast 30 (b) atmost 20
(c) exactly 25 (d) None of these

26. In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. Then, the number of persons who read neither is [NCERT Exemplar]
(a) 210 (b) 290 (c) 180 (d) 260

Round II (Mixed Bag)

Only One Correct Option

1. Which of the following is a singleton set?
(a) $\{x : |x| < 1, x \in I\}$
(b) $\{x : |x| = 5, x \in I\}$
(c) $\{x : x^2 = 1, x \in I\}$
(d) $\{x : x^2 + x + 1 = 0, x \in R\}$

2. If $P(A) = P(B)$, then [NCERT]
(a) $A \not\subset B$ (b) $B \not\subset A$
(c) $A = B$ (d) None of these

3. Which sets are subsets of one and another?
$A = \{x : x \in R \text{ and } x \text{ satisfy } x^2 - 8x + 12 = 0\}$
$B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\}$, $D = \{6\}$ [NCERT]
(a) $D \subset A$, $D \subset B$ and $D \subset C$
(b) $A \subset D$, $B \subset D$ and $D \subset C$
(c) $D \subset A$, $B \subset D$ and $D \subset C$
(d) None of the above

4. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49n - 49 : n \in N\}$. Then, [NCERT Exemplar]
(a) $X \subset Y$ (b) $Y \subset X$ (c) $X \subseteq Y$ (d) $X \cap Y = \phi$

5. There are 100 families in a society, 40 families buy newspaper A, 30 families buy newspaper B, 30 families buy newspaper C, 10 families buy newspaper A and B, 8 families buy newspaper B and C, 5 families buy newspaper A and C, 3 families buy newspaper A, B and C, then the number of families who do not buy any newspaper, is [NCERT Exemplar]

(a) 20 (b) 80
(c) 0 (d) None of these

6. In a survey of 200 students of a school, it was found that 120 study Mathematics, 90 study Physics and 70 study Chemistry, 40 study Mathematics and Physics, 30 study Physics and Chemistry, 50 study Chemistry and Mathematics and 20 none of these subjects. The number of students who study all the three subjects is [NCERT Exemplar]

(a) 30 (b) 20
(c) 22 (d) 25

7. Let U be the set of all boys and girls in a school. G be the set of all girls in the school, B be the set of all boys in the school and S be the set of all students in the school who take swimming. Some but not all students in the school take swimming. [NCERT Exemplar]

(a)

(b)

(c)

(d) None of these

8. Out of 800 boys in a school, 224 played Cricket, 240 played Hockey and 336 played Basketball. Of the total, 64 played both Basketball and Hockey; 80 played Cricket and Basketball and 40 played Cricket and Hockey; 24 played all the three games. The number of boys who did not play any game is

(a) 128 (b) 216
(c) 240 (d) 160

9. If $aN = \{an : n \in N\}$ and $bN \cap cN = dN$, where $a, b, c \in N$ and b, c are coprime, then

(a) $b = cd$ (b) $c = bd$
(c) $d = bc$ (d) None of these

10. If $A = \left\{(x, y) : y = \frac{4}{x}, 0 \neq x \in R\right\}$

and $B = \{(x, y) : y = x, x > 0, x \in R\}$

(a) $A \cap B = \phi$ (b) $A \cap B$ is singleton set
(c) $A \cap B \neq 0$ (d) $A \cap B$ has two elements

11. If $A = \{(x, y) : y = e^{2x}, x \in R\}$

and $B = \{(x, y) : y = e^{-2x}, x \in R\}$, then

(a) $A \cap B = \phi$
(b) $A \cap B \neq \phi$
(c) $A \cap B$ is a singleton set
(d) None of the above

12. From 50 students taking examinations in Mathematics, Physics and Chemistry, each of the student has passed in atleast one of the subject, 37 passed Mathematics, 24 Physics and 43 Chemistry. Atmost 19 passed Mathematics and Physics, atmost 29 Mathematics and Chemistry and atmost 20 Physics and Chemistry. The largest possible number that could have passed all three examination is [NCERT Exemplar]

(a) 14 (b) 15
(c) 13 (d) 20

More Than One Correct Option

13. If $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then

(a) the smallest set X is $\{3, 5, 9\}$
(b) the smallest set X is $\{2, 3, 5, 9\}$
(c) the largest set X is $\{1, 2, 3, 5, 9\}$
(d) the largest set X is $\{2, 3, 4, 9\}$

14. In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows

French = 17, English = 13, Sanskrit = 15

French and English = 09, English and Sanskrit = 4

French and Sanskrit = 5, English, French and Sanskrit = 3. The number of students who study

(a) French only is 6
(b) Sanskrit only is 8
(c) French and Sanskrit but not English is 2
(d) atleast one of the three languages is 30

Comprehension Based Questions

Passage

Directions (Q. Nos. 15-17) *Out of* 100 *students;* 15 *passed in English,* 12 *passed in Mathematics,* 8 *in Science,* 6 *in English and Mathematics,* 7 *in Mathematics and Science,* 4 *in English and Science;* 4 *in all the three passed.*

15. The number of students passed in English and Mathematics but not in Science is

(a) 3 (b) 2 (c) 4 (d) 5

16. The number of students only passed in Mathematics is

(a) 5 (b) 4
(c) 3 (d) 2

17. The number of students only passed in more than one subject is

(a) 9 (b) 3
(c) 2 (d) 1

Assertion and Reason

Directions (Q. Nos. 18-21) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

18. **Statement I** If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$

Statement II $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

19. Suppose, A, B and C are three arbitrary sets and U is a universal set.

Statement I If $B = U - A$, then $n(B) = n(U) - n(A)$.

Statement II If $C = A - B$, then $n(C) = n(A) - n(B)$.

20. Suppose A be a non-empty set, then the collection of all possible subsets of set A is a power set $P(A)$.

Statement I $P(A) \cap P(B) = P(A \cap B)$

Statement II $P(A) \cup P(B) = P(A \cup B)$

21. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers.

Statement I The number of people who read exactly one newspaper is 30.

Statement II Exactly one of the newspaper read is

$$n(H) + n(T) + n(I) - 2\{n(H \cap I) + n(H \cap T) + n(T \cap I)\} + 3n(H \cap T \cap I)$$

22. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[AIEEE 2009]**

(a) $A = C$
(b) $B = C$
(c) $A \cap B = \phi$
(d) $A = B$

Answers

Round I

1. (d)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (b)
11. (a)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (a)	**19.** (d)	**20.** (b)
21. (b)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (b)				

Round II

1. (a)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (b)
11. (c)	**12.** (a)	**13.** (a, c)	**14.** (a,c,d)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (c)
21. (a)	**22.** (b)								

the Guidance

Round I

1. The set of intelligent students in a class is not a well defined collection, because intelligency is not defined for a student in a class.

2. Given, $A = \{x : x \text{ is an integer}, -\frac{1}{2} < x < \frac{9}{2}\} = \{0, 1, 2, 3, 4\}$

and $B = \{x : x \text{ is a month of year not having 31 days}\}$

$= \{\text{February, April, June, September, November}\}$

3. Since, A is not a subset of B.

$\therefore$ Some elements of A will not be elements of B. Hence, A and complement of B are always non-disjoint.

4. (a) There is no odd natural number divisible by 2, so there will be no element in this set, hence it is a null set.

(b) There is only one even prime number which is 2 *i.e.*, there is a element, so it is not a null set.

(c) There is no natural number which is less than 5 and greater than 7 *i.e.*, there is no element, so it is a null set.

(d) Since, parallel lines never intersect each other, so they have no common point *i.e.*, no element, so it is a null set. Hence, option (b) is correct.

5. (a) Here, all the elements of the sets A and B are same which are a, b, c and d. Thus, $A = B$.

(b) Here, 12 is an element of A but not the element of B and 18 is an element of B but not the element of A. Thus, all the elements of the sets A and B are not same. Hence, $A \neq B$.

(c) Here, $A = \{x : x \text{ is a multiple of } 10\}$

$\{10, 20, 30, 40, \ldots\}$ and $B = \{10, 15, 20, 25, 30\ldots\}$

Clearly, $A \neq B$ as all the elements of the sets A and B are not same.

6. We know, if any set has m elements, then the number of subset is 2^m.

$\therefore$ According to the given condition,

$$2^m = 112 + 2^n$$

$$\Rightarrow \quad 2^m - 2^n = 112 = 16 \times 7 = 112$$

$$\Rightarrow \quad 2^m - 2^n = 2^7 - 2^4$$

On comparing both sides, we get

$$2^m = 2^7 \text{ and } 2^n = 2^4$$

$$\Rightarrow \quad m = 7 \text{ and } n = 4$$

7. Given that, $A = \{x : x = 4n + 1, 2 \leq n \leq 5\}$

$= \{9, 13, 17, 21\}$

$\therefore$ Number of elements in A is 4.

So, number of subsets $= 2^4 = 16$

8. Given that, $A = \{x : x = n^2, n = 1, 2, 3\}$

$= \{1, 4, 9\}$

$\therefore$ Number of elements in A is 3.

So, number of proper subsets $= 2^3 - 1 = 7$

9. Let $A = \{1\}$, $B = \{2, 3\}$

$\therefore \quad A \cup B = \{1, 2, 3\}$

and $\quad A \cap B = \{\phi\}$

Now, $\quad P(A) = \{\phi, \{1\}\}, P(B) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$

$\therefore \quad P(A) \cup P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$

$P(A \cup B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$

and $\quad P(A \cap B) = \{\phi\}$

10. Given, $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, \ldots, 18\}$ and $U = N$ = The set of natural number

$= \{1, 2, 3, \ldots, \ldots 18\}$

$\therefore \quad B' = U - B$

$= \{1, 2, 3, \ldots\} - \{2, 4, \ldots, 18\}$

$= \{1, 3, 5, \ldots, 17\}$

Now, $(A \cup B) = \{1, 3, 5, 7, 9, 11, 13, 15, 17\} \cup \{2, 4, \ldots, 18\}$

$= \{1, 2, 3, 4, 5, \ldots, 18\}$

$\therefore (A \cup B) \cap B' = \{1, 2, 3, 4, \ldots, 18\} \cap \{1, 3, 5, \ldots, 17\}$

$= \{1, 3, 5, \ldots, 17\}$

$\therefore A' \cup [(A \cup B) \cap B'] = \{2, 4, 6, \ldots, 18\} \cup \{1, 3, 5, \ldots, 17\}$ $(\because A' = U - A)$

$= \{1, 2, 3, 4, \ldots, 18\} = N$

11. $A \cap B = \{2, 3, 4, 8, 10\} \cap \{3, 4, 5, 10, 12\} = \{3, 4, 10\}$

and $\quad A \cap C = \{2, 3, 4, 8, 10\} \cap \{4, 5, 6, 12, 14\}$

$= \{4\}$

$\therefore \quad (A \cap B) \cup (A \cap C) = \{3, 4, 10\} \cup \{4\}$

$= \{3, 4, 10\}$

Alternate Solution

$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

$= A \cap [\{3, 4, 5, 10, 12\} \cup \{4, 5, 6, 12, 14\}]$

$= A \cap \{3, 4, 5, 6, 10, 12, 14\}$

$= \{2, 3, 4, 8, 10\} \cap \{3, 4, 5, 6, 10, 12, 14\}$

$= \{3, 4, 10\}$

12. $A \cap (A \cap B)^c = A \cap (A^c \cup B^c)$ ($\because$ by De-Morgan's law)

$= (A \cap A^c) \cup (A \cap B^c)$

$= \phi \cup (A \cap B^c)$

$= A \cap B^c$

13. Given, $A = \{x : x \text{ is an odd natural number}\}$

$= \{1, 3, 5, 7, 9, 11, \ldots\}$

and $B = \{x : x \text{ is a prime number}\}$

$= \{2, 3, 5, 7, 11, \ldots\}$

$\therefore$ $A \cap B = \{1, 3, 5, 7, 9, 11, \ldots\} \cap \{2, 3, 5, 7, 11, \ldots\}$

$= \{3, 5, 7, 11, \ldots\}$

Hence, $A \cap B$ is an odd prime natural number.

14. Given, F_1 = The set of parallelograms

F_2 = The set of rectangles

F_3 = The set of rhombuses

F_4 = The set of squares

and F_5 = The set of trapeziums

By definition of parallelogram, opposite sides are equal and parallel.

In rectangles, rhombuses and squares all have opposite sides are equal and parallel, therefore

$$F_2 \subset F_1, F_3 \subset F_1, F_4 \subset F_1$$

$$\therefore \quad F_1 = F_1 \cup F_2 \cup F_3 \cup F_4$$

15. If elements are not repeated, then number of elements in $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{30}$ is 30×5. But each element is used 10 times, so

$$S = \frac{30 \times 5}{10} = 15 \quad \ldots\text{(i)}$$

Similarly, if elements in $B_1, B_2, \ldots, B_n$ are not repeated, then total number of elements is $3n$ but each element is repeated 9 times, so

$$S = \frac{3n}{9} \Rightarrow 15 = \frac{3n}{9} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad n = 45$$

16. Given that, $A = \{x : x \text{ is a multiple of } 4\}$

$= \{4, 8, 12, 16, 20, \ldots\}$

and $B = \{x : x \text{ is a multiple of } 6\}$

$= \{6, 12, 18, 24, \ldots\}$

$\therefore$ $A \cap B = \{12, 24, \ldots\} = \{x : x \text{ is a multiple of } 12\}$

17. Given, S = Set of points inside the square

T = Set of points inside the triangle

C = Set of points inside the circle.

According to the given condition, the Venn diagram is given below.

It is clear from the Venn diagram that $S \cup T \cup C = S$

18. Given that, $A = \{x : x^2 = 1\} = \{-1, 1\}$

and $B = \{x : x^4 = 1\} = \{-1, 1, i, -1\}$

Now, $A - B = \phi$ and $B - A = \{-i, i\}$

$\therefore$ $A \Delta B = (A - B) \cup (B - A) = \{-i, i\}$

19. Let us consider the sets

$A = \{1, 2, 4\}, B = \{2, 5, 6\}$ and $C = \{1, 5, 7\}$

I. $A - B = \{1, 4\}$ and $A - (A \cap B) = \{1, 2, 4\} - \{2\} = \{1, 4\}$

$\therefore$ $A - B = A - (A \cap B)$

II. $(A \cap B) \cup (A - B) = \{2\} \cup \{1, 4\}$

$= \{1, 2, 4\} = A$

III. $A - (B \cup C) = \{1, 2, 4\} - \{1, 2, 5, 6, 7\}$

$= \{4\}$

and $(A - B) \cup (A - C) = \{1, 4\} \cup \{2, 4\}$

$= \{1, 2, 4\}$

$\therefore$ $A - (B \cup C) \neq (A - B) \cup (A - C)$

20. $(A \cap B')' \cup (B \cap C)$

$= (A' \cup B'') \cup (B \cap C)$ ($\because$ by De-Morgan's law)

$= (A' \cup B) \cup (B \cap C)$ ($\because B'' = B$)

$= A' \cup B$

21. Given, $n(A) = 200, n(B) = 300, n(A \cap B) = 100$

We know that, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ $n(A \cup B) = 200 + 300 - 100 = 400$

Also, $n(A' \cap B') = n\{(A \cup B)'\}$

$= n(X) - n(A \cup B)$

$\Rightarrow$ $300 = n(X) - 400$

$\Rightarrow$ $n(X) = 700$

22. Given, $S = \{x : x \text{ is a positive multiple of 3 and less than 100}\}$

$= \{3, 6, 9, 12, \ldots 99\}$

$\Rightarrow$ $n(S) = 33$

and $P = \{x : x \text{ is a prime number less than 20}\}$

$= \{2, 3, 5, 7, 11, 13, 17, 19\}$

$\Rightarrow$ $n(P) = 8$

$\therefore$ $n(S) + n(P) = 33 + 8 = 41$

23. Given, $n(F) = 50, n(S) = 20, n(S \cap F) = 10$

Using the identity, $n(S \cup F) = n(S) + n(F) - n(S \cap F)$

$n(S \cup F) = 20 + 50 - 10$

$= 70 - 10 = 60$

24. Let C and T denote the students play Cricket and Tennis, respectively. And U denotes the total number of students in a class.

$\therefore \quad n(C) = 25, n(T) = 20$ and $n(U) = 60$

$\therefore \quad n(C \cup T) = n(C) + n(T) - n(C \cap T)$

$= 25 + 20 - 10$

$= 45 - 10 = 35$

$\therefore$ The number of students who play neither matches

$= n(C' \cap T') = n(U) - n(C \cup T)$

$= 60 - 35 = 25$

25. Let number of newspaper be x. If every student reads one newspaper, the number of students would be $60x$.

Since, every student reads 5 newspapers.

$\therefore \quad$ Number of students $= \dfrac{x \times 60}{5} = 300$

$\Rightarrow \quad x = 25$

26. Let H and E denote Hindi and English read newspaper.

Given, total number of persons, $n(U) = 840$

Total number of persons read Hindi newspaper,

$n(H) = 450$

Total number of persons read English newspaper,

$n(E) = 300$

and $\quad n(H \cap E) = 200$

$\because \quad n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$= 450 + 300 - 200$

$= 750 - 200 = 550$

$\therefore$ The number of persons who read neither of the newspaper

$= n(\overline{H} \cap \overline{E}) = n(U) - n(H \cup E)$

$= 840 - 550 = 290$

Round II

1. (a) $\{x : |x| < 1, x \in I\}$

$= \{x : -1 < x < 1, x \in I\} = \{0\}$

(b) $\{x : |x| = 5, x \in I\}$

$= \{x : x = \pm 5\} = \{\pm 5\}$

(c) $\{x : x^2 = 1, x \in I\}$

$= \{x : x = \pm 1\} = \{\pm 1\}$

(d) $\{x : x^2 + x + 1 = 0, x \in R\}$

$$= \left\{x : x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1}}{2(1)}, x \in R\right\}$$

$$= \left\{x : x = \frac{-1 \pm \sqrt{3}i}{2}, x \in R\right\} = \{\phi\}$$

($\because x$ is real but here x is a complex number)

2. Let x be an arbitrary element of A. Then, there exists a subset X of set A such that $x \in X$.

Now, $\quad x \in X \Rightarrow X \subset A$

$\Rightarrow \quad X \in P(A) \Rightarrow X \in P(B)$

$\Rightarrow \quad X \subset B \Rightarrow x \in B$

Hence, $\quad x \in A \Rightarrow x \in B$

$\therefore \quad A \subset B \quad$ ...(i)

Again, now let y be an arbitrary element of B.

Then, there exists a subset Y of set B such that $y \in Y$

Now, $\quad y \in Y \Rightarrow Y \subset B$

$\Rightarrow \quad Y \in P(B) \Rightarrow Y \in P(A) \quad [\because P(A) = P(B)]$

$\Rightarrow \quad Y \subset A \Rightarrow y \in A$

Hence, $\quad y \in B \Rightarrow y \in A$

$\therefore \quad B \subset A \quad$ ...(ii)

From Eqs. (i) and (ii), we have $A = B$

3. Given, $A = \{x : x \in R$ and x satisfy $x^2 - 8x + 12 = 0\}$

$\because \quad x^2 - 8x + 12 = 0$

$\Rightarrow \quad (x - 2)(x - 6) = 0$

$x = 2, 6$

$A = \{6, 2\}$

$B = \{2, 4, 6\}$

and $\quad C = \{2, 4, 6, 8\}$

$D = \{6\}$

Clearly, every element of A is in B and C.

$\therefore \quad A \subset B$ and $A \subset C$

Again, every element of B is in C

$\therefore \quad B \subset C$

Also, every element of D is in A, B and C.

$\therefore \quad D \subset A, D \subset B$ and $D \subset C$

4. Given, $X = \{8^9 - 7n - 1 : n \in N\}$ and $Y = \{49n - 49 : n \in N\}$

Now, $\quad 8^9 - 7n - 1 = (7 + 1)^n - 7n - 1$

$= 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \ldots + {}^nC_{n-2} 7^2$

$+ {}^nC_{n-1} 7 + {}^nC_n - 7n - 1$

[$\because$ by binomial expansion, $(x + 1)^n = x^n + {}^nC_1 x^{n-1}$

$+ {}^nC_2 x^{n-2} + \ldots + {}^nC_{n-2} x^2 + {}^nC_{n-1} x + {}^nC_n$]

$= {}^nC_n 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \ldots + {}^nC_2 7^2 + n7 + 1 - 7n - 1$

($\because {}^nC_n = 1, {}^nC_{n-1} = n, {}^nC_{n-2} = {}^nC_2$)

$= 7^n \, {}^nC_n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \ldots + {}^nC_2 7^2$

$= {}^nC_2 7^2 + \ldots + {}^nC_2 7^{n-2} + {}^nC_1 7^{n-1} + {}^nC_n 7^n$

$= 7^2[{}^nC_2 + \ldots + {}^nC_n 7^{n-2}]$

$= 49[{}^nC_2 + \ldots + {}^nC_n 7^{n-2}]$

$\therefore \quad 8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$

For $n = 1, 8^n - 7n - 1 = 8 - 7 - 1 = 0$

For $n = 2, 8^n - 7n - 1 = 64 - 14 - 1 = 49$

$\therefore 8^n - 7n - 1$ is a multiple of 49 for all $n \in N$.

X contains elements which are multiples of 49 and clearly Y contains all multiples of 49.

$\therefore \quad X \subseteq Y$

5. Given, $n(U) = 100, n(A) = 40, n(B) = 30, n(C) = 30$

$n(A \cap B) = 10, n(B \cap C) = 8, n(C \cap A) = 5, n(A \cap B \cap C) = 3$

Now, $n(A^c \cap B^c \cap C^c) = n(A \cup B \cup C)^c$

$= n(U) - [n(A \cup B \cup C)]$

$= 100 - [40 + 30 + 30 - 10 - 8 - 5 + 3] = 100 - 80 = 20$

6. Let M, P and C denote the students studying Mathematics, Physics and Chemistry.

Also, $n(U) = 200$, $n(M) = 120$, $n(P) = 90$, $n(C) = 70$, $n(M \cap P) = 40$, $n(P \cap C) = 30$, $n(M \cap C) = 50$, $n(M \cup P \cup C)' = 20$

$\because \quad n(M \cup P \cup C)' = 20$

$\therefore \quad n(M \cup P \cup C)' = n(U) - n(M \cup P \cup C)$

$\Rightarrow \quad 20 = 200 - n(M \cup P \cup C)$

$\Rightarrow \quad n(M \cup P \cup C) = 200 - 20 = 180$

$\because \quad n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(C \cap M) + n(C \cap M \cap P)$

$\therefore 180 = 120 + 90 + 70 - 40 - 30 - 50 + n(C \cap M \cap P)$

$\Rightarrow \quad 180 = 280 - 120 + n(C \cap M \cap P)$

$\Rightarrow \quad 180 + 120 - 280 = n(P \cap C \cap M)$

$\Rightarrow \quad n(P \cap C \cap M) = 300 - 280 = 20$

7. Given,

U = Set of all boys and girls in the school

G = Set of all girls in the school

B = Set of all boys in the school

S = Set of all students in the school.

According to the question,

Some boys student taking swimming *i.e.*, $B \cap S \neq \phi$

and some girls student taking swimming *i.e.*, $G \cap S \neq \phi$

8. Here, $n(C) = 224, n(H) = 240, n(B) = 336, n(H \cap B) = 64$

$n(B \cap C) = 80, n(H \cap C) = 40, n(C \cap H \cap B) = 24$

$\therefore \quad n(C^c \cap H^c \cap B^c) = n[(C \cup H \cup B)^c]$

$= n(U) - n(C \cup H \cup B)$

$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24]$

$= 800 - 640 = 160$

9. Given, $\quad aN = \{an : n \in N\}$

$\therefore \quad bN = \{bn : n \in N\}$ and $cN = \{cn : n \in N\}$

Also, given $bN \cap cN = dN$

$\therefore \quad bc \in bN \cap cN$ or $bc \in dN$

$\therefore \quad bc = d \quad (\because b$ and c are coprime)

[$\because$ coprime numbers are those numbers, whose common factor is 1. e.g., (i) 3 and 5 (ii) 5 and 9 etc.]

10. Given, $A = \{(x, y) : y = \frac{4}{x}, 0 \neq x \in R\} \Rightarrow A$ is the set of all points on the graph of $xy = 4$

and $\quad B = \{(x, y) : y = x, x > 0, x \in R\}$

$\Rightarrow B$ is the set of all points on the graph of $y = x$

Now, we plot the graph of given sets.

Since, the graph of $xy = 4$ and $y = x, x > 0$ intersect at one point.

$\therefore \quad A \cap B \neq \phi$ and $A \cap B$ is a singleton set.

11. Given, $A = \{(x, y) : y = e^{2x}, x \in R\}$

$\Rightarrow A$ is the set of all points on the graph of $y = e^{2x}$

and $\quad B = \{(x, y) : y = e^{-2x}, x \in R\}$

$\Rightarrow B$ is the set of all points on the graph of $y = e^{-2x}$

Now, we plot the graph of given sets

Since, the graph of $y = e^{2x}$ and $y = e^{-2x}$ intersect at one point.

$\therefore \quad A \cap B \neq \phi$ and $A \cap B$ is a singleton set.

12. Let M be the set of students passing in Mathematics.

P be the set of students passing in Physics.

C be the set of students passing in Chemistry.

Now, $n(M \cup P \cup C) = 50, n(M) = 37, n(P) = 24, n(C) = 43$

$n(M \cap P) \leq 19, n(M \cap C) \leq 29, n(P \cap C) \leq 20 \quad$ (given)

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C) \leq 50$

$\Rightarrow \quad 37 + 24 + 43 - 19 - 29 - 20 + n(M \cap P \cap C) \leq 50$

$\Rightarrow \quad n(M \cap P \cap C) \leq 50 - 36$

$\Rightarrow \quad n(M \cap P \cap C) \leq 14$

Thus, the largest possible number that could have passed all the three examinations is 14.

13. Given, $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$

The smallest of $X = \{1, 2, 3, 5, 9\} - \{1, 2\} = \{3, 5, 9\}$

The largest set of X = The set which contains atleast elements 3, 5, 9

$= \{1, 2, 3, 5, 9\}$

14. Given, $n(F) = 17, n(E) = 13, n(S) = 15, n(F \cap E) = 9, n(E \cap S) = 4,$

$n(F \cap S) = 5, n(E \cap F \cap S) = 3$

(a) Number of students study only French

$= n(F) - n(F \cap E) - n(F \cap S) + n(F \cap E \cap S)$

$= 17 - 9 - 5 + 3 = 6$

(b) Number of students study only Sanskrit

$= n(S) - n(S \cap F) - n(S \cap E) + n(F \cap E \cap S)$

$= 15 - 5 - 4 + 3 = 9$

(c) $n(F \cap S \cap \bar{E}) = n(F \cap S) - n(F \cap E \cap S)$

$= 5 - 3 = 2$

(d) $\therefore n(F \cup E \cup S) = n(F) + n(E) + n(S)$

$- n(F \cap E) - n(E \cap S) - n(S \cap F) + n(F \cap E \cap S)$

$= 17 + 13 + 15 - 9 - 4 - 5 + 3$

$= 30$

Solutions (Q. Nos. 15-17)

Let U, E, M and S be denote the total number of students passed in English, passed in Mathematics and passed in Science, respectively.

Here, $n(U) = 100, n(E) = 15, n(M) = 12, n(S) = 8, n(E \cap M) = 6,$ $n(M \cap S) = 7, n(E \cap S) = 4$ and $n(E \cap M \cap S) = 4$

15. $\therefore$ The number of students passed in English and Mathematics but not in Science

$= n(E \cap M \cap \bar{S})$

$= n(E \cap M) - n(E \cap M \cap S) = 6 - 4 = 2$

16. The number of students only passed in Mathematics

$= n(M \cap \bar{E} \cap \bar{S})$

$= n(M) - n(M \cap E) - n(M \cap S) + n(M \cap E \cap S)$

$= 12 - 6 - 7 + 4 = 16 - 13 = 3$

17. The number of students only passed in more than one subject

$= n(M \cap E) + n(M \cap S) + n(S \cap E) - 2n(M \cap E \cap S)$

$= 6 + 7 + 4 - 2(4) = 17 - 8 = 9$

18. We have, $B = B \cup (A \cap B)$

$= B \cup (A \cap C)$ $\quad (\because$ given $A \cap B = A \cap C)$

$= (B \cup A) \cap (B \cup C)$

$= (A \cup C) \cap (B \cup C)$ $\quad [\because$ given $(A \cup C) \cap (B \cup C)]$

$= (A \cap B) \cup C$

$= (A \cap C) \cup C = C$

Hence, using result given in statement II, we have proved that $B = C$. Hence, statement II is the correct explanation of statement I.

19. If U is universal set, then $B = U - A = A'$, for which $n(B) = n(A') = n(U) - n(A)$.

But for any three arbitrary sets A, B and C, we cannot always have $n(C) = n(A) - n(B)$, if $C = A - B$ as it is not specified here whether A is universal set or not. In case A is not universal set we cannot conclude

$n(C) = n(A) - n(B)$. Hence, statement I is true but statement II is false.

20. Let $X \in P(A \cap B)$

$\Leftrightarrow \quad X \subset A \cap B$

$\Leftrightarrow \quad X \subset A$ and $X \subset B$

$\Leftrightarrow \quad X \in P(A)$ and $X \in P(B)$

$\Leftrightarrow \quad X \in P(A) \cap P(B)$

Hence, $\quad P(A) \cap P(B) = P(A \cap B)$

Now, consider set $A = \{1\}$ and set $B = \{2\}$

$A \cup B = \{1, 2\}$

$P(A) = \{\phi, \{1\}\}, P(B) = \{\phi, \{2\}\}$

and $\quad P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Thus, $\quad P(A) \cup P(B) \neq P(A \cup B)$

Hence, statement I is true and statement II is false.

21. Given, $n(H) = 25, n(T) = 26, n(I) = 26,$

$n(H \cap I) = 9, n(H \cap T) = 11, n(T \cap I) = 8$

and $\quad n(H \cap T \cap I) = 3$

$\therefore$ Number of people who read exactly one newspaper

$= n(H) + n(T) + n(I)$

$- 2\{n(H \cap I) + n(H \cap T) + n(T \cap I)\} + 3n(H \cap T \cap I)$

$= 25 + 26 + 26 - 2(9 + 11 + 8) + 3 \times 3$

$= 77 - 2 \times 28 + 9 = 86 - 56 = 30$

22. Since, $A \cap B = A \cap C$ and $A \cup B = A \cup C$

$\therefore \quad B = C$

2 Fundamentals of Relations and Functions

JEE Main MILESTONE

- Ordered Pair
- Cartesian Product
- Relations
- Different Types of Relations
- Inverse Relation
- Composition of Relations
- Functions or Mappings
- Classification of Functions
- Algebra of Real Functions
- Inverse Function
- Composition of Functions

2.1 Ordered Pair

Two elements a and b listed in a specific order form an ordered pair, denoted by (a, b). In an ordered pair (a, b); a is regarded as the first element and b is the second element.

It is evident from the definition that

(i) $(a, b) \neq (b, a)$

(ii) $(a, b) = (c, d)$ iff $a = c, b = d$

In Mathematics we come across many relations such as number m is less than number n, line l is parallel to line m. In all these, a relation involves pairs of objects in certain order. Concept of function plays very important role in Mathematics, since it captures the idea of a mathematically precise correspondence between one quantity with the other.

Equality of Ordered Pair

Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff

$$a_1 = a_2 \text{ and } b_1 = b_2$$

i.e., $(a_1, b_1) = (a_2, b_2)$

$\Rightarrow$ $a_1 = a_2$ and $b_1 = b_2$

Thus, it is evident from the definition that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

***Sample* Problem 1** *If* $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, *find the values of x and y.* **[NCERT]**

(*a*) 2, 1 (*b*) 2, 2

(*c*) –2, 1 (*d*) 1, 1

Interpret (*a*) Given, $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$

$\Rightarrow$ $\frac{x}{3}+1=\frac{5}{3}$ and $y-\frac{2}{3}=\frac{1}{3}$

$\Rightarrow$ $\frac{x}{3}=\frac{5}{3}-\frac{1}{1}$ and $y=\frac{1}{3}+\frac{2}{3}$

$\Rightarrow \quad \frac{x}{3} = \frac{5-3}{3}$ and $y = \frac{1+2}{3}$

$\Rightarrow \quad \frac{x}{3} = \frac{2}{3}$ and $y = \frac{3}{3}$

$\Rightarrow \quad x = 2$ and $y = 1$

2.2 Cartesian Product

Let A and B be two non-empty sets. The cartesian product of A and B is denoted by $A \times B$ and is defined as the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

Symbolically, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If there are three sets A, B and C and $a \in A$, $b \in B$ and $c \in C$, then we form an ordered triplet (a, b, c). The set of all ordered triplets (a, b, c) is called the cartesian product of three sets A, B and C, *i.e.*,

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

An ordered pair and ordered triplet are also called 2-tuple and 3-tuple, respectively.

e.g., Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

$\therefore \quad A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

and $\quad B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

Note

- If $A \neq B$, then $A \times B \neq B \times A$
- If A has p elements and B has q elements, then $A \times B$ has pq elements.
- If $A = \phi$ and $B = \phi$, then $A \times B = \phi$.
- Cartesian product of n sets $A_1, A_2, A_3, \ldots, A_n$ is the set of all n-tuples $(a_1, a_2, a_3, \ldots, a_n)$, $a_i \in A_i$, $i = 1, 2, 3, \ldots, n$ and it is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or $\prod_{i=1}^{n} A_i$.

Properties of Cartesian Product

If A, B and C are three sets, then

(i) (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times (B - C) = (A \times B) - (A \times C)$

(iii) $A \times B = B \times A \Leftrightarrow A = B$

(iv) If $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$

(v) If $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

(vi) If $A \subseteq B$ and $C \subseteq D \Rightarrow A \times C \subseteq B \times D$

(vii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(viii) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(ix) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

(x) If A and B have n common elements, then $A \times B$ and $B \times A$ will have n^2 common elements.

***Sample* Problem 2** *If the set A has 3 elements and the set $B = \{3, 4, 5\}$, then find the number of elements in $(A \times B)$ is* **[NCERT]**

(a) 8 (b) 9

(c) 7 (d) 10

Interpret (b) Here, $n(A) = 3$, $B = \{3, 4, 5\} \Rightarrow n(B) = 3$

$\therefore \quad n(A \times B) = n(A) \times n(B) = 3 \times 3 = 9$

***Sample* Problem 3** *If A, B and C are any three sets, then $A \times (B \cup C)$ is equal to*

(a) $(A \times B) \cup (A \times C)$ (b) $(A \cup B) \times (A \cup C)$

(c) $(A \times B) \cap (A \times C)$ (d) *None of these*

Interpret (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by distributive law.

2.3 Relations

Let A and B be two non-empty sets. Then, a relation R from A to B is a subset of $A \times B$.

Thus, R is a relation from A to $B \Rightarrow R \subseteq A \times B$. If R is a relation from a non-empty set A to a non-empty set B and if $(a, b) \in R$, then we write aRb which is read as 'a is related to b by the relation R.' If $(a, b) \notin R$, then we write aRb and it is read as 'a is not related to b by the relation R'.

e.g., If R is a relation between two sets $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$ defined as "square root of".

Here, $1R1$, $2R4$, $3R9$.

$\therefore \quad R = \{(1, 1), (2, 4), (3, 9)\}$

Domain and Range of Relations

Let R be a relation from A to B. The domain of R is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in B$.

$\therefore$ Domain of $R = \{a \in A : (a, b) \in R, \forall b \in B\}$

and range of R is the set of all those elements $b \in B$ such that $(a, b) \in R$ for some $a \in A$.

$\therefore$ Range of $R = \{b \in B : (a, b) \in R, \forall a \in A\}$.

Here, B is called the codomain of R.

e.g., Let $A = \{1, 2, 3\}$

and $B = \{3, 5, 6\}$

Let $aRb \Rightarrow a < b$

Then, $R = \{(1, 5), (2, 5), (3, 5), (1, 6), (2, 6), (3, 6)\}$

$\therefore$ Domain of $R = \{1, 2, 3\}$, range of $R = \{5, 6\}$

and codomain of $R = \{3, 5, 6\}$

Note

- Let A and B be two non-empty finite sets having p and q elements respectively.
- Total number of relations form A to $B = 2^{pq}$.

***Sample* Problem 4** *Find the domain and range of the relation R defined by* $R = \{(x, x+5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

[NCERT]

(a) $\{0, 1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\}$
(b) $\{1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\}$
(c) $\{0, 1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}$
(d) *None of the above*

Interpret (a) Given, $R = \{(x, x+5) : x \in \{0, 1, 2, 3, 4, 5\}\}$ ...(i)

Putting $x = 0, 1, 2, 3, 4, 5$, we get

$\Rightarrow$ Domain $= \{0, 1, 2, 3, 4, 5\}$

After putting the value of x from Eq. (i), we get

$$y = x + 5 = 5, 6, 7, 8, 9, 10$$

$\Rightarrow$ Range $= \{5, 6, 7, 8, 9, 10\}$

2.4 Different Types of Relations

(i) Void Relation

Let A be a set and $\phi \subset A \times A$, it follows that ϕ is a relation on A which is called the empty or void relation.

(ii) Universal Relation

Let A be a set and $A \times A \subseteq A \times A$, it follows that $A \times A$ is a relation on A which is called the universal relation.

(iii) Identity Relation

The relation I_A on set A is called an identity relation, if every element of A is related to itself only. Symbolically, the relation

$$I_A = \{(a, a) : a \in A\}$$

is called an identity relation on A.

e.g., If $A = \{1, 2, 3\}$, then an identity relation on A is given by $I_A = \{(1, 1), (2, 2), (3, 3)\}$.

2.5 Inverse Relation

If R is a relation on set A, then the relation R^{-1} on A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$ is called an inverse relation to A.

Clearly, domain (R^{-1}) = range (R)

and range (R^{-1}) = domain (R)

e.g., Let $A = \{1, 2, 3\}$

and let $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$

Then, R being a subset of $A \times A$, it is a relation on A.

Clearly, $1R2$; $2R2$; $3R1$ and $3R2$.

Domain $(R) = \{1, 2, 3\}$ and range $(R) = \{2, 1\}$

Also, $R^{-1} = \{(2, 1), (2, 2), (1, 3), (2, 3)\}$

Domain $(R^{-1}) = \{2, 1\}$ and range $(R^{-1}) = \{1, 2, 3\}$

***Sample* Problem 5** *The relation R defined in* $A = \{1, 2, 3\}$ *by aRb, if* $|a^2 - b^2| \le 5$. *Which of the following is false?*

(a) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
(b) $R^{-1} = R$
(c) *Domain of* $R = \{1, 2, 3\}$
(d) *Range of* $R = \{5\}$

Interpret (b) $\because R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$R^{-1} = \{(y, x) : (x, y) \in R\}$

$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\}$

$= R$

Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$ and range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$.

2.6 Composition of Relations

Let $R \subseteq A \times B, S \subseteq B \times C$ be two relations. Then, composition of the relations R and S denoted by $SoR \subseteq A \times C$ and is defined by $(a, c) \in SoR$, iff $\exists b \in B$ s.t.

$$(a, b) \in R, (b, c) \in S$$

e.g., Let $A = \{1, 2, 3\}, B = \{a, b, c, d\}, C = \{\alpha, \beta, \gamma\}$

$R \subseteq (A \times B) = \{(1, a), (1, c), (2, d)\}$

$S \subseteq (B \times C) = \{(a, \alpha), (c, \gamma), (d, \beta)\}$

Then, $SoR \subseteq (A \times C) = \{(1, \alpha), (1, \gamma), (2, \beta)\}$

Note

- One should be careful in computing the relation RoS. Actually, SoR starts with R and RoS starts with S.
- In general, $SoR \neq RoS$
- $(SoR)^{-1} = R^{-1}oS^{-1}$, known as reversal rule.

Check Point 1

1. If the set A has 3 elements and the set B = {1, 3, 4, 5}, then the number of elements in (A × B) is
(a) 11 (b) 12
(c) 13 (d) 15

2. Let A = {1, 2}B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}
Following statements are given below :
I A × (B ∩ C) = (A × B) ∩ (A × C)
II A × C is a subset of B × D.
Which of the following statment is correct ?
(a) Only I (b) Only II
(c) Both I and II (d) None of these

3. Let R be the relation on Z defined by R = {(a, b) : a, b ∈ z, a – b is an interger}. Find the domain and Range of R
(a) z, z (b) z^+, z
(c) z, z^- (d) None of these

4. If A and B are two sets, then A × B = B × A. If and only if
(a) A ⊂ B
(b) B ⊂ A
(c) A = B
(d) None of the above

5. Let R be a relation from a set A to a set B, then
(a) R = A ∪ B (b) R = A ∩ B
(c) R ⊆ A × B (d) R ⊆ B × A

6. If the relation R : A → B, where A = {1, 2, 3, 4} and B = {1, 3, 5} is defined by R = {(x, y) : x < y, x ∈ A, y ∈ B}, then R^{-1}OR is
(a) {(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)}
(b) {(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)}
(c) {(3, 3), (3, 5), (5, 3), (5, 5)
(d) None of the above

7. Let a relation R be defined by R = {(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)}, then ROR is equal to
(a) {(1, 5), (1, 6), (3, 6)}
(b) {(1, 4), (1, 5), (3, 6)}
(c) {(1, 5), (1, 6), (3, 7)}
(d) {(1, 4), (1, 5), (3, 7)}

8. The relation R defined on the set A = {1, 2, 3, 4, 5} by R = {(x, y) : | $x^2 - y^2$ | < 16} is given by
(a) {(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)}
(b) {(2, 2), (3, 2), (4, 2), (2, 4)}
(c) {(3, 3), (4, 3), (5, 4), (3, 4)}
(d) None of the above

2.7 Functions or Mappings

Let A and B be two non-empty sets. Then, a function f from set A to B is a rule which associates elements of set A to elements of B such that all elements of set A are associated to elements of set B in unique way.

If f associates $x \in A$ to $y \in B$, then we say that y is the image of the element x and denote it by $f(x)$ and write as $y = f(x)$. The element x is called the pre-image or inverse image in B.

A function is denoted by $f : A \to B$ or $A \xrightarrow{f} B$.

Note

- There may exist some elements in set B which are not the images of any element in set A.
- To each and every independent element in A there corresponds one and only one image in B.
- Every function is a relation but every relation may or may not be a function.
- The number of functions from a finite set A into finite set B $= [n(B)]^{[n(A)]}$

Difference between a Relation and a Function

Geometrically, if we draw a vertical parallel line (VPL) *i.e.*, any line which is parallel to y-axis ($x = a$), then if this line intersects the graph of the expression in more than one point, then the expression is a relation else if it intersects at only one point, the expression is a function, *e.g.*,

(a) Function (b) Relation

Domain, Codomain and Range of a Function

Let $f : A \to B$, then A is known as domain of f while B is known as codomain of f.

Also, set $f(A) = \{f(x) : x \in A\}$ is known as range of f.

Clearly, $f(A) \subseteq B$

e.g., Let $A = \{1, -1, 2, -2\}$, $B = \{1, 4, 9\}$

$$f : A \xrightarrow{x^2} B \text{ i.e., } f(x) = x^2$$

From the figure, it is clear that domain of function $= \{1, -1, 2, -2\}$ and range of function $= \{1, 4\}$.

Also, codomain of function $= \{1, 4, 9\}$.

***Sample* Problem 6** *Find the domain and range of the function* $f(x) = \sqrt{9 - x^2}$. [NCERT]

(a) $[-3, 3]$, $[0, 3]$ (b) $[-3, 3)$, $[0, 3]$
(c) $(-3, 3)$, $(0, 3)$ (d) *None of these*

Interpret (a) $f(x) = \sqrt{9 - x^2}$

For function to be defined,

$$9 - x^2 \geq 0$$

$$\Rightarrow \quad x^2 - 9 \leq 0 \quad \ldots(i)$$

$$\Rightarrow \quad \{x - (-3)\}(x - 3) \leq 0$$

$$\Rightarrow \quad -3 \leq x \leq 3$$

$\{$if $a < b$ and $(x - a)(x - b) \leq 0 \Rightarrow a \leq x \leq b\}$

$\Rightarrow$ Domain $= [-3, 3]$

or Domain $= \{x : x \in R$ and $-3 \leq x \leq 3\}$

Again for range,

Let $f(x) = y$, then

$$\Rightarrow \quad y = \sqrt{9 - x^2}$$

On squaring both sides, we get

$$y^2 = 9 - x^2$$

$$\Rightarrow \quad x^2 = 9 - y^2 \quad \ldots(ii)$$

On putting the value of x^2 in Eq. (i), we get

$$\Rightarrow \quad 9 - y^2 - 9 \leq 0$$

$$\Rightarrow \quad -y^2 \leq 0$$

$$\Rightarrow \quad y^2 \geq 0$$

$$-\infty < y < \infty$$

But y cannot be negative $f(x) = \sqrt{9 - x^2}$ cannot contain negative values.

$$\therefore \quad y \geq 0$$

Also, from Eq. (ii), $x = \sqrt{9 - y^2}$

$$\Rightarrow \quad 9 - y^2 \geq 0 \quad (\because x^2 \geq 0)$$

$$\Rightarrow \quad y^2 - 9 \leq 0$$

$$\Rightarrow \quad -3 \leq y \leq 3$$

$\Rightarrow$ Range $= [0, 3]$ or $\{y : y \in R, 0 \leq y \leq 3\}$

***Sample* Problem 7** *Let* $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ *be a function from Z to Z defined by* $f(x) = ax + b$ *for some integers a and b. Find, a, b.*

(a) 1, 2 (b) 2, –1
(c) –1, –2 (d) *None of these*

Interpret (b) Given, $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$

and $\quad f(x) = ax + b$

or $\quad y = ax + b \quad \{\because y = f(x)\}$

At $x = 1, y = 1$

$$\Rightarrow \quad 1 = a + b \quad \ldots(i)$$

At $x = 2, y = 3$

$$\Rightarrow \quad 3 = 2a + b \quad \ldots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$$a = 2$$

On putting the value of a in Eq. (i), we get

$$b = 1 - 2 \quad \Rightarrow \quad b = -1$$

2.8 Classification of Functions

Constant Function

A function which does not change as its parameters vary *i.e.*, the function whose rate of change is zero. In short, a constant function is a function that always gives or returns the same value.

OR

Let k be a constant, then function $f(x) = k, \forall x \in R$ is known as constant function.

Domain of $f(x) = R$ and Range of $f(x) = \{k\}$

Polynomial Function

The function $y = f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are real coefficients and n is a non-negative integer, is known as a polynomial function. If $a_0 \neq 0$, then degree of polynomial function is n.

Domain of $f(x) = R$

On range varies from function to function.

Rational Function

If $P(x)$ and $Q(x)$ are polynomial functions, $Q(x) \neq 0$, then function $f(x) = \dfrac{P(x)}{Q(x)}$ is known as rational function.

Domain of $f(x) = R - \{x : Q(x) = 0\}$

and range varies from function to function.

Irrational Function

The function containing one or more terms having non-integral rational powers of x are called irrational function.

e.g., $$y = f(x) = \frac{5x^{3/2} - 7x^{1/2}}{x^{1/2} - 1}$$

Domain = varies from function to function.

Identity Function

Function $f(x) = x, \forall x \in R$ is known as identity function. It is straight line passing through origin and having slope unity.

Domain of $f(x) = R$ and Range of $f(x) = R$

Square Root Function

The function that associates every positive real number x to $+\sqrt{x}$ is called the square root function, *i.e.*, $f(x) = +\sqrt{x}$.

Range of $f(x) = [0, \infty)$.

Exponential Function

A function of the form $f(x) = a^x$, a is a positive real number, is an exponential function. The value of the function depends upon the value of a for $0 < a < 1$, function is decreasing and for $a > 1$, function is increasing.

Domain of $f(x) = R$

and Range of $f(x) = [0, \infty)$

Logarithmic Function

Function $f(x) = \log_a x, (x, a > 0)$ and $a \neq 1$, is known as logarithmic function.

Domain of $f(x) = (0, \infty)$

and Range of $f(x) = R$

Modulus Function

Function $y = f(x) = |x|$ is known as modulus function.

$$y = f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain of $f(x) = x \in R$

and Range of $f(x) = [0, \infty)$

Properties of Modulus Function

(i) For any real number x, $\sqrt{x^2} = |x|$.

(ii) If a and b are positive real numbers, then

(a) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$

(b) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

(c) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(iii) $|x \pm y| \leq |x| + |y|$

(iv) $|x + y| \geq ||x| - |y||$

Greatest Integer Function

For any real number x, the greatest integer function $[x]$ is equal to greatest integer less than or equal to x.

In general, if n is an integer and x is any number satisfying $n \leq x < n + 1$, then $[x] = n$. It is also known as integral part function.

e.g., If $2 \leq x < 3$, then $[x] = 2$

Domain = R

Range = I

Properties of Greatest Integer Function

If n is an integer and x is any real number between n and $n + 1$, then

(i) $[-n] = -[n]$

(ii) $[x + n] = [x] + n$

(iii) $[-x] = -[x] - 1$, x is not an integer.

(iv) $[x + y] \geq [x] + [y]$

(v) $[x] > n \Rightarrow x \geq n + 1$

(vi) $[x] < n \Rightarrow x < n$

(vii) $[x + y] = [x] + [y + x - [x]]$, for all $x, y \in R$

(viii) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$

Signum Function

The function defined by

$$f(x) = \frac{|x|}{x} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

is called the signum function.

Domain $= R$

Range $= \{-1, 0, 1\}$

***Sample* Problem 9** *The domain of the function* $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$ *is*

(a) $(-\infty, -2) \cup [4, \infty)$ (b) $(-\infty, -2] \cup [4, \infty)$

(c) $(-\infty, -2) \cup (4, \infty)$ (d) *None of these*

Interpret (a) $f(x)$ is defined, if $[x]^2 - [x] - 6 > 0$

$\Rightarrow$ $([x] - 3)([x] + 2) > 0$

$\Rightarrow$ $[x] < -2$ or $[x] > 3$

But $[x] < -2 \Rightarrow [x] = -3, -4, -5, \ldots$

$\therefore$ $x < -2$

Also, $[x] > 3 \Rightarrow [x] = 4, 5, 6, \ldots$

$\therefore$ $x \geq 4$

Domain of $f = (-\infty, -2) \cup [4, \infty)$

2.9 Algebra of Real Functions

Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$.

(i) **Addition of two real functions**

$$(f + g)(x) = f(x) + g(x)$$

(ii) **Subtraction of two real functions**

$$(f - g)(x) = f(x) - g(x)$$

(iii) **Multiplication by a scalar**

$$(\alpha f)(x) = \alpha f(x), \alpha, x \in R$$

(iv) **Multiplication of two real functions**

$$fg(x) = f(x)\, g(x)$$

(v) **Quotient of two real functions**

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

2.10 Inverse Function

Let f be defined a function from A to B such that for every element of B their exist a image. Let y be an arbitrary element of B. Then, f being onto, there exists an element $x \in A$ such that $f(x) = y$. Also, f being one-one, this x must be unique. Thus, for each $y \in B$, there exists a unique element $x \in A$ such that $f(x) = y$. So, we may define a function,

$$f^{-1} : B \to A$$

$\therefore$ $$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f.

Note

- Inverse of bijective function is also bijective function.
- If the inverse of f exist, then f is called an invertible function. *i.e.*, A function f is invertible if and only if f is one-one onto.

***Sample* Problem 10** *Let* $f : R \to R$ *be a function defined by* $f(x) = 2x - 3$. *Then,* $f^{-1}(x)$ *is equal to*

(a) $x + 3$ (b) $\frac{x+3}{2}$

(c) $\frac{x-3}{2}$ (d) *None of these*

Interpret (b) Let $y = 2x - 3 \Rightarrow 2x = y + 3$

$\Rightarrow$ $x = \frac{y+3}{2}$ $\therefore$ $f^{-1}(x) = \frac{x+3}{2}$

2.11 Composition of Functions

Let A, B and C be three non-empty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings (or functions), then $gof : A \to C$. This function is called the product or composite of f and g given by

$$gof(x) = g[f(x)] \; \forall x \in A$$

gof exists iff the range of f is a subset of domain of g. Similarly, fog exists if range of g is a subset of domain of f.

Properties of Composition of Function

(i) The composition of functions is not commutative.

i.e., $fog \neq gof$

(ii) The composition of functions is associative.

i.e., $fo(goh) = (fog)oh$

(iii) The composition of any function with the identity function is the function itself.

i.e., If $f : A \to B$, then $foI_A = I_Bof = f$.

***Sample* Problem 11** *If $f(x) = (ax^2 + b)^3$, then the function g such that $f\{g(x)\} = g\{f(x)\}$ is given by*

(a) $g(x) = \left(\dfrac{b - x^{1/3}}{a}\right)^{1/2}$ (b) $g(x) = \dfrac{1}{(ax^2 + b)^3}$

(c) $g(x) = (ax^2 + b)^{1/3}$ (d) $g(x) = \left(\dfrac{x^{1/3} - b}{a}\right)^{1/2}$

Interpret (*d*) Let $g(x) = \left(\dfrac{x^{1/3} - b}{a}\right)^{1/2}$

$$\therefore \quad f\{g(x)\} = f\left(\frac{x^{1/3} - b}{a}\right)^{1/2}$$

$$= \left[a\left(\frac{x^{1/3} - b}{a}\right) + b\right]^3$$

$$= (x^{1/3})^3 = x$$

and $g\{f(x)\} = g\{(ax^2 + b)^3\}$

$$= \left(\frac{ax^2 + b - b}{a}\right)^{1/2} = (x^2)^{1/2} = x$$

$$\Rightarrow \quad g\{f(x)\} = f\{g(x)\}$$

Thus, $g(x) = \left(\dfrac{x^{1/3} - b}{a}\right)^{1/2}$

Check Point 2

1. Find the range of the function $f(x) = x^2 + 2$
   (a) $(-2, 2)$ (b) $[2, \infty)$
   (c) $[3, \infty)$ (d) None of these

2. If the function $f(x) = \dfrac{a^x + a^{-x}}{2}$, $(a > 2)$, then $f(x + y) + f(x - y)$ is equal to
   (a) $2f(x)\,f(y)$ (b) $f(x)\,f(y)$
   (c) $\dfrac{f(x)}{f(y)}$ (d) None of these

3. Domain and range of $f(x) = \dfrac{|x - 3|}{x - 3}$ are respectively
   (a) R, $[-1, 1]$ (b) $R - \{3\}$, $\{1, -1\}$
   (c) R^T, R (d) None of these

4. The domain of the function $f(x) = \sqrt{(2 - 2x - x^2)}$ is
   (a) $-\sqrt{3} \le x \le \sqrt{3}$
   (b) $-1 - \sqrt{3} \le x \le -1 + \sqrt{3}$
   (c) $-2 \le x \le 2$
   (d) $-2 - \sqrt{3} \le x \le -2 + \sqrt{3}$

5. Which of the following function is invertible?
   (a) $f(x) = 2^x$ (b) $f(x) = x^3 - x$
   (c) $f(x) = x^2$ (d) None of these

6. If $f(x) = \sin^2 x$ and the composite function $g\{f(x)\} = |\sin x|$, then the function $g(x)$ is equal to
   (a) $\sqrt{x - 1}$ (b) $\sqrt{x}$ (c) $\sqrt{x + 1}$ (d) $-\sqrt{x}$

WORKED OUT Examples

Example 1 *Let* $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ *be a function from R to R. Determine the range of f.* **[NCERT]**

(a) [0, 1) (b) [0, 1]
(c) [0, 2) (d) *None of these*

Solution Given, $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ and $f : R \to R$

Let $y = \frac{x^2}{1+x^2}$

$\because$ $f(x)$ or y is positive for all values of x.

and $1 + x^2 > x^2$

$\Rightarrow$ $0 \le y < 1$

$\therefore$ Range of $f = \{y : y \in R \text{ and } y \in [0, 1)\}$

Example 2 *The simplified form of* $f(x) = |x-2| + |2-x|$, $-3 \le x \le 3$ *is* **[NCERT Exemplar]**

(a) $\begin{cases} 2x, & 2 \le x \le 3 \\ 4, & -2 \le x < 2 \\ -2x, & -3 \le x < -2 \end{cases}$ (b) $\begin{cases} -2x, & 2 \le x \le 3 \\ 4, & -2 \le x < 2 \\ 2x, & -3 \le x < -2 \end{cases}$

(c) $\begin{cases} 3x, & 2 \le x \le 3 \\ 4, & -2 \le x < 2 \\ -2x, & -3 \le x < -2 \end{cases}$ (d) *None of these*

Solution (a) $\because$ $|x-2| = \begin{cases} x-2, & x \ge 2 \\ -(x-2), & x < 2 \end{cases}$

and $|x+2| = \begin{cases} (x+2), & x \ge -2 \\ -(x+2), & x < -2 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} (x-2) + (2+x), & 2 \le x \le 3 \\ -(x-2) + (x+2), & -2 \le x < 2 \\ -(x-2) - (x+2), & -3 \le x < -2 \end{cases}$$

$$= \begin{cases} x - 2 + 2 + x, & 2 \le x \le 3 \\ -x + 2 + x + 2, & -2 \le x < 2 \\ -x + 2 - x - 2, & -3 \le x < -2 \end{cases}$$

$$= \begin{cases} 2x, & 2 \le x \le 3 \\ 4, & -2 \le x < 2 \\ -2x, & -3 \le x < -2 \end{cases}$$

Example 3 *Find the domain of the function* $f(x) = \frac{1}{\sqrt{x + |x|}}$. **[NCERT Exemplar]**

(a) R (b) R^+
(c) R^- (d) *None of these*

Solution (b) Given, $f(x) = \frac{1}{\sqrt{x+|x|}}$ for the function to be defined, the denominator should not be equal to 0)

$x + |x| > 0$ $(\because x + |x| \not< 0)$

$\Rightarrow$ $|x| > -x$

It is true for all R^+ (R^+ means set of all positive real numbers)

$\therefore$ Domain of the given function $= R^+$

Example 4 *If* $f(x) = \frac{x-1}{x+1}$, *then* $f\left(-\frac{1}{x}\right)$ *is equal to* **[NCERT]**

(a) $-\frac{1}{f(x)}$ (b) $\frac{1}{f(x)}$
(c) $f(x)$ (d) $-f(x)$

Solution (a) Replacing x by $\left(-\frac{1}{x}\right)$ in Eq. (i), we get

$$f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x} - 1}{-\frac{1}{x} + 1} = \frac{\frac{-1-x}{x}}{\frac{-1+x}{x}} = \frac{-(1+x)}{x-1} = -\frac{x+1}{x-1} = -\frac{1}{\left(\frac{x-1}{x+1}\right)}$$

$$\Rightarrow \quad f\left(\frac{-1}{x}\right) = -\frac{1}{f(x)}$$

Example 5 *Find the range of* $f(x) = 1 + 3\cos 2x$. **[NCERT Exemplar]**

(a) [2, 3] (b) [2, 4]
(c) [–2, 4] (d) *None of these*

Solution (c) $f(x) = 1 + 3\cos 2x$

$\because$ $-1 \le \cos 2x \le 1$

$\Rightarrow$ $-3 \le 3\cos 2x \le 3$

$\Rightarrow$ $1 - 3 \le 1 + 3\cos 2x \le 3 + 1$

$\Rightarrow$ $-2 \le f(x) \le 4$

$\therefore$ Range $= [-2, 4]$

Example 6 *Let A and B be two sets such that $n(A \times B) = 6$. If three elements of $A \times B$ are (3, 2), (7, 5), (8, 5), then*

(a) $A = \{3, 7, 8\}$ (b) $B = \{2, 5, 7\}$
(c) $B = \{3, 5\}$ (d) *None of these*

Solution (a) $\because (3, 2), (7, 5), (8, 5) \in A \times B$

$\therefore\ 3, 7, 8 \in A$ and $2, 5 \in B$

Also, $n(A \times B) = 6 = 3 \times 2$

$\therefore$ $A = \{3, 7, 8\}$ and $B = \{2, 5\}$

Example 7 *Let R be a relation in N defined by*

$$R = \{(1 + x, 1 + x^2) : x \le 5, x \in N\}.$$

Which of the following is false ?

(a) $R = \{(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)\}$
(b) *Domain of* $R = \{2, 3, 4, 5, 6\}$
(c) *Range of* $R = \{2, 5, 10, 17, 26\}$
(d) *Atleast one is false*

Solution (c) $R = \{(2, 2), (3, 5), (4, 10), (5, 17), (6, 26)\}$

Domain of $R = \{x : (x, y) \in R\} = \{2, 3, 4, 5, 6\}$

and range of $R = \{y : (x, y) \in R\} = \{2, 5, 10, 17, 26\}$

Example 8 *If $f(x) = \dfrac{x-1}{x+1}$, then $f(2x)$ is equal to*

(a) $\dfrac{3f(x)+1}{f(x)+3}$ (b) $\dfrac{f(x)+1}{f(x)+3}$

(c) $\dfrac{f(x)+3}{f(x)+1}$ (d) *None of these*

Solution (d)
$$\frac{3f(x)+1}{f(x)+3} = \frac{3\left(\dfrac{x-1}{x+1}\right)+1}{\dfrac{x-1}{x+1}+3}$$
$$= \frac{3x-3+x+1}{x-1+3x+3} = \frac{4x-2}{4x+2}$$
$$= f(2x)$$

Example 9 *Let $f : (4, 6) \to (6, 8)$ be a function defined by $f(x) = x + \left[\dfrac{x}{2}\right]$, where [.] denotes the greatest integer function, then $f^{-1}(x)$ is equal to*

(a) $x - \left[\dfrac{x}{2}\right]$ (b) $-x - 2$

(c) $x - 2$ (d) $\dfrac{1}{x + \left[\dfrac{\pi}{2}\right]}$

Solution (c) Since, $x \in (4, 6)$

$\therefore$ $f(x) = x + \left[\dfrac{x}{2}\right] = x + 2$

Let $f(x) = y$

$\therefore$ $y = x + 2$

$\Rightarrow$ $x = y - 2$

$\therefore$ $f^{-1}(x) = x - 2$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Cartesian Product

1. If two sets A and B are having 99 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are
(a) 2^{99} (b) 99^2
(c) 100 (d) 18

2. If $n(A) = 3$, $n(B) = 4$, then $n(A \times A \times B)$ is equal to
(a) 36 (b) 12
(c) 108 (d) None of these

3. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $(A \times B) \cap (B \times C)$ is equal to
(a) $\{(1, 4)\}$ (b) $\{(3, 4)\}$
(c) $\{(1, 4), (3, 4)\}$ (d) None of these

4. If $A = \{1, 2, 3\}$, $B = \{3, 8\}$, then $(A \cup B) \times (A \cap B)$ is equal to
(a) $\{(8, 3), (8, 2), (8, 1), (8, 8)\}$
(b) $\{(1, 2), (2, 2), (3, 3), (8, 8)\}$
(c) $\{(3, 1), (3, 2), (3, 3), (3, 8)\}$
(d) $\{(1, 3), (2, 3), (3, 3), (8, 3)\}$

5. Consider the following statements
I. If $A \cap B = \phi$, then either $A = \phi$ or $B = \phi$.
II. For $a \neq b$, $\{a, b\} = \{b, a\}$ and $(a, b) \neq (b, a)$.
III. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$.
IV. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
Which of these is/are correct?
(a) Only (II) (b) Only (I)
(c) Only (IV) (d) (II), (III) and (IV)

6. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is
(a) $\{(2, 4), (3, 4)\}$
(b) $\{(4, 2), (4, 3)\}$
(c) $\{(2, 4), (3, 4), (4, 4)\}$
(d) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

7. Let $A = \{1, 2, 3\}$. The total number of distinct relations that can be defined over A, is
(a) 2^9 (b) 6
(c) 8 (d) None of these

Relations

8. Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is [NCERT Exemplar]
(a) m^n (b) $n^m - 1$
(c) $mn - 1$ (d) $2^{mn} - 1$

9. If $R = \{(x, y) : x, y \in I, x^2 + y^2 \leq 4\}$ is a relation in I, then domain of R is
(a) $\{0, 1, 2\}$ (b) $\{-2, -1, 0\}$
(c) $\{-2, -1, 0, 1, 2\}$ (d) None of these

10. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is
(a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$
(c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$

11. The relation R defined on set $A = \{x : |x| < 3, x \in I\}$ by $R = \{(x, y) : y = |x|\}$ is
(a) $\{(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)\}$
(b) $\{(-2, -2), (-2, 2), (-1, 1), (0, 0), (1, -2), (1, 2), (2, -1), (2, -2)\}$
(c) $\{(0, 0), (1, 1), (2, 2)\}$
(d) None of the above

12. The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$, is given by
(a) $\{(1, 4), (2, 5), (3, 6), \ldots\}$
(b) $\{(4, 1), (5, 2), (6, 3), \ldots\}$
(c) $\{(1, 3), (2, 6), (3, 9), \ldots\}$
(d) None of the above

13. Let R be the relation from $A = \{2, 3, 4, 5\}$ to $B = \{3, 6, 7, 10\}$ defined by 'x divides y', then R^{-1} is equal to
(a) $\{(6, 2), (3, 3)\}$
(b) $\{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$
(c) $\{(6, 2), (10, 2), (3, 3), (6, 3)\}$
(d) None of the above

14. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. The relation R^{-1} is
(a) $\{(11, 8), (13, 10)\}$ (b) $\{(8, 11), (10, 13)\}$
(c) $\{(8, 11), (9, 12), (10, 13)\}$ (d) None of these

15. If R is a relation from a set A to the set B and S is a relation from B to C, then the relation SoR
(a) is from C to A (b) is from A to C
(c) does not exist (d) None of these

16. Let a relation R be defined by $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$. The relation $R^{-1}oR$ is given by
(a) $\{(1, 1), (4, 4), (7, 4), (4, 7), (7, 7)\}$
(b) $\{(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)\}$
(c) $\{(1, 5), (1, 6), (3, 6)\}$
(d) None of the above

Functions

17. Let $f(x) = \sqrt{1 + x^2}$, then [NCERT Exemplar]
(a) $f(xy) = f(x) \cdot f(y)$ (b) $f(xy) \geq f(x) \cdot f(y)$
(c) $f(xy) \leq f(x) \cdot f(y)$ (d) None of these

18. Domain of $\sqrt{a^2 - x^2}\,(a > 0)$ is [NCERT Exemplar]
(a) $(-a, a)$ (b) $[-a, a]$
(c) $[0, a]$ (d) $(-a, 0]$

19. If $f(x) = \dfrac{x}{x-1}$, $x \neq 1$, then $\underbrace{(fofo \ldots of)}_{19 \text{ times}}(x)$ is equal to
(a) $\dfrac{x}{x-1}$ (b) $\left(\dfrac{x}{x-1}\right)^{19}$
(c) $\dfrac{19x}{x-1}$ (d) x

20. If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(3) = 3$, then a and b are equal to [NCERT Exemplar]
(a) $a = -3, b = -1$ (b) $a = 2, b = -3$
(c) $a = 0, b = 2$ (d) $a = 2, b = 3$

21. If $f(x) = \dfrac{3x+2}{5x-3}$, then
(a) $f^{-1}(x) = f(x)$ (b) $f^{-1}(x) = -f(x)$
(c) $(fof)(x) = -x$ (d) $f^{-1}(x) = -\dfrac{f(x)}{19}$

22. Given, $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ and $g(x) = \dfrac{3x + x^3}{1 + 3x^2}$, then $fog(x)$ equals
(a) $-f(x)$ (b) $3f(x)$
(c) $[f(x)]^3$ (d) None of these

23. If $f : R \to R$ is given by

$$f(x) = \begin{cases} -1, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ if irrational} \end{cases}$$

then $(fof)(1 - \sqrt{3})$ is equal to
(a) 1 (b) -1
(c) $\sqrt{3}$ (d) 0

24. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$.

Which of the following relation is a function? [NCERT Exemplar]
(a) f (b) g
(c) f, g (d) None of these

25. The domain of the function f defined by $f(x) = \sqrt{4 - x} + \dfrac{1}{\sqrt{x^2 - 1}}$ is equal to [NCERT Exemplar]
(a) $(-\infty, -1) \cup (1, 4]$
(b) $(-\infty, -1] \cup (1, 4]$
(c) $(-\infty, -1) \cup [1, 4]$
(d) $(-\infty, -1) \cup [1, 4)$

26. The domain and range of the real function f defined by $f(x) = \dfrac{4-x}{x-4}$ is given by [NCERT Exemplar]
(a) Domain $= \mathrm{R}$, Range $= \{-1, 1\}$
(b) Domain $= \mathrm{R} - \{1\}$, Range $= R$
(c) Domain $= \mathrm{R} - \{4\}$, Range $= \{-1\}$
(d) Domain $= \mathrm{R} - \{-4\}$, Range $= \{-1, 1\}$

27. The domain of the function f given by $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$ [NCERT Exemplar]
(a) $\mathrm{R} - \{3, -2\}$ (b) $\mathrm{R} - \{-3, 2\}$
(c) $\mathrm{R} - [3, -2]$ (d) $\mathrm{R} - (3, -2)$

28. The domain and range of the function f given by $f(x) = 2 - |x - 5|$ is [NCERT Exemplar]
(a) Domain $= \mathrm{R}^+$, Range $= (-\infty, 1]$
(b) Domain $= \mathrm{R}$, Range $= (-\infty, 2]$
(c) Domain $= \mathrm{R}$, Range $= (-\infty, 2)$
(d) Domain $= \mathrm{R}^+$, Range $= (-\infty, 2]$

29. The domain for which the functions defined by $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal to [NCERT Exemplar]

(a) $\left\{-1, \frac{4}{3}\right\}$ (b) $\left\{-1, -\frac{4}{3}\right\}$

(c) $\left\{1, \frac{4}{3}\right\}$ (d) $\left\{1, -\frac{4}{3}\right\}$

30. Range of $f(x) = \dfrac{1}{1 - 2\cos x}$ is [NCERT Exemplar]

(a) $\left[\frac{1}{3}, 1\right]$ (b) $\left[-1, \frac{1}{3}\right]$

(c) $(-\infty, -1) \cup \left[\frac{1}{3}, \infty\right)$ (d) $\left[-\frac{1}{3}, 1\right]$

31. Let $f : [0, 1] \to [0, 1]$ defined by $f(x) = \dfrac{1-x}{1+x}, 0 \le x \le 1$ and let $g : [0, 1] \to [0, 1]$ be defined by $g(x) = 4x(1 - x)$, $0 \le x \le 1$, then fog and gof is

(a) $\dfrac{(2x-1)^2}{1+4x-4x^2}, \dfrac{8x(1-x)}{(1+x)^2}$ (b) $\dfrac{(2x-1)}{1+4x-4x^2}, \dfrac{8(1-x)x}{(1+x)^2}$

(c) $\dfrac{(2x+1)^2}{1+4x+4x^2}, \dfrac{8}{(1+x)^2}$ (d) $\dfrac{(2x+1)^2}{(1+x)^2}, \dfrac{8(1-x)}{(1+x)^2}$

32. If $f : R \to R$, $g : R \to R$ and $h : R \to R$ are such that $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$, then the value of $(ho(gof))(x)$, if $x = \sqrt{\frac{\pi}{4}}$ will be

(a) 0 (b) 1

(c) –1 (d) π

33. If $[x]^2 - 5[x] + 6 = 0$, where $[\cdot]$ denote the greatest integer function, then [NCERT Exemplar]

(a) $x \in [3, 4]$ (b) $x \in (2, 3]$

(c) $x \in [2, 3]$ (d) $x \in [2, 4)$

34. Let $f(x) = x^2 - x + 1$, $x \ge \frac{1}{2}$, then the solution of the equation $f(x) = f^{-1}(x)$ is

(a) $x = 1$ (b) $x = 2$

(c) $x = \frac{1}{2}$ (d) None of these

35. Let $f : R \to R$, $g : R \to R$ be two functions such that

$$f(x) = 2x - 3, \quad g(x) = x^3 + 5.$$

The function $(fog)^{-1}(x)$ is equal to

(a) $\left(\frac{x+7}{2}\right)^{1/3}$ (b) $\left(x - \frac{7}{2}\right)^{1/3}$

(c) $\left(\frac{x-2}{7}\right)^{1/3}$ (d) $\left(\frac{x-7}{2}\right)^{1/3}$

36. The inverse of the function $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is

(a) $\log_{10}(2 - x)$ (b) $\frac{1}{2}\log_{10}\frac{1+x}{1-x}$

(c) $\frac{1}{2}\log_{10}(2x - 1)$ (d) $\frac{1}{4}\log\left(\frac{2x}{2-x}\right)$

37. If $f(x) = \cos(\log x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(a) –1 (b) $\frac{1}{2}$

(c) –2 (d) 0

Previous Years' Questions

38. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where

$Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$.

If f is invertible, then its inverse is [AIEEE 2008]

(a) $g(y) = \frac{y-3}{4}$ (b) $g(y) = \frac{3y+4}{3}$

(c) $g(y) = 4 + \frac{y+3}{4}$ (d) $g(y) = \frac{y+3}{4}$

39. A real valued function $f(x)$ satisfies the functional equation

$$f(x - y) = f(x)f(y) - f(a - x)f(a + y)$$

where a is a given constant and $f(0) = 1$, $f(2a - x)$ is equal to [AIEEE 2005]

(a) $f(-x)$ (b) $f(a) + f(a - x)$

(c) $f(x)$ (d) $-f(x)$

40. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then [AIEEE 2004]

(a) $f(x + 2) = f(x - 2)$ (b) $f(2 + x) = f(2 - x)$

(c) $f(x) = f(-x)$ (d) $f(x) = -f(-x)$

41. If $f : R \to R$ satisfies $f(x + y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is [AIEEE 2003]

(a) $\frac{7n}{2}$

(b) $\frac{7(n+1)}{2}$

(c) $7n(n + 1)$

(d) $\frac{7n(n+1)}{2}$

Answers

Check Point 1

1. (b) **2.** (c) **3.** (a) **4.** (c) **5.** (c) **6.** (c) **7.** (a) **8.** (d)

Check Point 2

1. (b) **2.** (a) **3.** (b) **4.** (b) **5.** (a) **6.** (b)

Round I

1. (b)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (c)
11. (a)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (b)
31. (a)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (d)	**36.** (b)	**37.** (d)	**38.** (a)	**39.** (d)	**40.** (b)
41. (d)									

the Guidance

Round I

1. $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$

$= n(A \cap B) \times n(B \cap A) = 99 \times 99 = 99^2$

2. Given, $n(A) = 3, n(B) = 4$

$\therefore \quad n(A \times A \times B) = 3 \times 3 \times 4 = 36$

3. $A \times B = \{1, 2, 3\} \times \{3, 4\}$

$= \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

and $B \times C = \{3, 4\} \times \{4, 5, 6\}$

$= \{(3, 4), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6)\}$

$\therefore \quad (A \times B) \cap (B \times C) = \{(3, 4)\}$

4. $A \cup B = \{1, 2, 3\} \cup \{3, 8\} = 1, 2, 3, 8\}$

and $A \cap B = \{1, 2, 3\} \cap \{3, 8\} = \{3\}$

$\therefore \quad (A \cup B) \times (A \cap B) = \{1, 2, 3, 8\} \times \{3\}$

$= \{(1, 3), (2, 3), (3, 3), (8, 3)\}$

5. **Statement I** If $A \cap B = \phi$, then it is not necessary that $A = \phi$ or $B = \phi$.

Statement II It is true $\{a, b\} = \{b, a\}$ and $(a, b) \neq (b, a)$

Statement III By properties of cartesian product

If $A \leq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

Statement IV Also $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$ by properties of cartesian product.

6. Given, $A = \{x : x^2 - 5x + 6 = 0\}$

$= \{x : (x - 2)(x - 3) = 0\} = \{2, 3\}$

and $B = \{2, 4\}$ and $C = \{4, 5\}$

Now, $B \cap C = \{2, 4\} \cap \{4, 5\} = \{4\}$

$\therefore \quad A \times (B \cap C) = \{2, 3\} \times \{4\} = \{(2, 4), (3, 4)\}$

7. $\because n(A \times A) = n(A) = 3^2 = 9$. So, the total number of subsets of $A \times A$ is 2^9 and a subset of $A \times A$ is a relation over the set A.

8. Given, $n(A) = m$ and $n(B) = n$

$\therefore$ Total number of relations from A to $B = 2^{mn}$

$\therefore$ Total number of non-empty relations from A to $B = 2^{mn} - 1$

9. Given, $R = \{(x, y) : x, y \in I, x^2 + y^2 \leq 4\}$

$= \{(0, 0), (0, -1), (0, 1), (0, -2), \ldots, (-2, 0)\}$

$\therefore$ Domain of $R = \{x : (x, y) \in R\} = \{-2, -1, 0, 1, 2\}$

10. Given, $R = \{(x, y) : x + 2y = 8, x, y \in N\}$

$$x + 2y = 8 \Rightarrow y = \frac{8 - x}{2}$$

$\therefore \quad R = \{(2, 3), (4, 2), (6, 1)\}$

$\therefore$ Domain of $R = \{x : (x, y) \in R\} = \{2, 4, 6\}$

11. Given, set is $A = \{x : |x| < 3, x \in I\}$

$A = \{x : -3 < x < 3, x I\} = \{-2, -1, 0, 1\}$

Also, $R = \{(x, y) : y = |x|\}$

$\therefore \quad R = \{(-2, 2), (-1, 1), (1, 1), (0, 0), (2, 2)\}$

12. Given, $R = \{(a, b) : a - b = 3\} = \{(4, 1), (5, 2), (6, 3), \ldots\}$

13. Given, $A = \{2, 3, 4, 5\}$ and $B = \{3, 6, 7, 10\}$

$\therefore \quad R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$

$\Rightarrow \quad R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$

14. Let $A = \{11, 12, 13\}, B = \{8, 10, 12\}$

$\therefore \quad R = \{(11, 8), (13, 10)\}$

$R^{-1} = \{(8, 11), (10, 13)\}$

15. Given, $R \subseteq A \times B$ and $S \subseteq B \times C$, we have

$$SoR \subseteq A \times C$$

$\therefore$ SoR is a relation from A to C.

16. Given, $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$

$\therefore \quad R^{-1} = \{(5, 4), (4, 1), (6, 4), (6, 7), (7, 3)\}$

$\Rightarrow \quad R^{-1}oR = \{(4, 4), (1, 1), (4, 7), (7, 4), (7, 7), (3, 3)\}$

17. $f(x) = \sqrt{1 + x^2}$

Now, $\quad f(xy) = \sqrt{1 + x^2 y^2} \quad$ ...(i)

and $\quad f(x)f(x) = \sqrt{1 + x^2}\sqrt{1 + y^2}$

$\Rightarrow \quad f(x)f(y) = \sqrt{1 + x^2 y^2 + x^2 + y^2} \quad$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$$f(x)f(y) \geq f(xy)$$

18. Let $f(x) = \sqrt{a^2 - x^2}$

For $f(x)$ to be defined $a^2 - x^2 \geq 0 \Rightarrow x^2 \leq a \Rightarrow x \in [-a, a]$

19. $(fof)x = f\left(\dfrac{x}{x-1}\right) = \dfrac{\dfrac{x}{x-1}}{\left(\dfrac{x}{x-1}\right) - 1} = x$

$\Rightarrow \quad (fofof)x = f(fof)x = f(x) = \dfrac{x}{x-1}$

$\therefore \quad (fofof \ldots 19 \text{ times})(x) = \dfrac{x}{x-1}$

20. Given, $f(x) = ax + b$

Also, $\quad f(-1) = -5$ and $f(3) = 3$

$\therefore \quad -5 = -a + b$ and $3 = 3a + b$

On solving, we get

$$a = 2, b = -3$$

21. Since, $\quad y = \dfrac{3x + 2}{5x - 3} \Rightarrow x = \dfrac{3y + 2}{5y - 3} = f^{-1}(y)$

$\therefore \quad f^{-1}(x) = \dfrac{3x + 2}{5x - 3} = f(x)$

22. Since, $\quad g(x) = \dfrac{3x + x^3}{1 + 3x^2} = y$, (say) $\quad$ ...(i)

$\therefore \quad f[g(x)] = f(y) = \log\left(\dfrac{1 + y}{1 - y}\right)$

$\Rightarrow \quad f(g(x)) = \log\left\{\dfrac{1 + \dfrac{3x + x^3}{1 + 3x^2}}{1 - \dfrac{3x + x^3}{1 + 3x^2}}\right\}$

$\Rightarrow \quad f(g(x)) = \log\left(\dfrac{1 + x}{1 - x}\right)^3$

$\therefore \quad f(g(x)) = 3\log\left(\dfrac{1 + x}{1 - x}\right) = 3f(x)$

23. Given, $f(x) = \begin{cases} -1, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$

Now, $\quad (fof)(1 - \sqrt{3}) = f[f(1 - \sqrt{3})] = f(1) = -1$

24. Given, $\quad f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$

$\because \quad f(x) = x^2$ is well defined in $0 \leq x \leq 3$

and $\quad f(x) = 3x$ is also well defined in $3 \leq x \leq 10$

At $x = 3$, $\quad f(x) = x^2 \Rightarrow f(3) = 3^2 = 9$

At $x = 3$, $\quad f(x) = 3x \Rightarrow f(3) = 3 \times 3 = 9$

$\Rightarrow \quad f(x)$ is defined at $x = 3$. Hence, f is a function.

$$g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$$

$\because \quad g(x) = x^2$ is well defined in $0 \leq x \leq 2$.

and $\quad g(x) = 3x$ is also well defined in $2 \leq x \leq 10$

But at $x = 2$,

$$g(x) = x^2 \Rightarrow g(2) = 2^2 = 4$$

At $x = 2$,

$$g(x) = 3x \Rightarrow g(2) = 3 \times 2 = 6$$

Therefore, $g(x)$ is not defined at $x = 2$. Hence, g is not a function.

25. Given, $f(x) = \sqrt{4 - x} + \dfrac{1}{\sqrt{x^2 - 1}}$

For domain of $f(x)$,

$$4 - x \geq 0 \text{ and } x^2 - 1 > 0$$

$\Rightarrow \quad x \leq 4$ and $x^2 > 1$

$\Rightarrow \quad x \leq 4$ and $x \in (-\infty, -1) \cup (1, \infty)$

$\therefore \quad x \in (-\infty, -1) \cup (1, 4]$

26. Given, $\quad f(x) = \dfrac{4 - x}{x - 4} = -\dfrac{(x - 4)}{(x - 4)}$

$$f(x) = -1$$

$\therefore$ Range of $f(x)$ is $\{-1\}$

For domain of $f(x)$, $x - 4 \neq 0 \Rightarrow x \neq 4$

$$x \in R - \{4\}$$

27. Given, $\quad f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$

For domain of $f(x)$, $x^2 - x - 6 \neq 0$

$\therefore \quad (x - 3)(x + 2) \neq 0 \Rightarrow x \neq 3, -2$

$\therefore$ Domain of $f(x)$ is $R - \{3, -2\}$.

28. Given, $f(x) = 2 - |x - 5|$

Domain of $f(x)$ is defined for all real values of x.

Since, $\quad |x - 5| \geq 0$

$\Rightarrow \quad -|x-5| \le 0$

$\Rightarrow \quad 2-|x-5| \le 2$

$\Rightarrow \quad f(x) \le 2$

Hence, range of $f(x)$ is $(-\infty, 2]$

29. Given $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$

$\because \quad 3x^2 - 1 = 3 + x$

$\Rightarrow \quad 3x^2 - x - 4 = 0$

$\Rightarrow \quad 3x^2 - 4x + 3x - 4 = 0$

$\Rightarrow \quad x(3x-4) + 1(3x-4) = 0$

$\Rightarrow \quad (x+1)(3x-4) = 0 \Rightarrow x = -1, x = \frac{4}{3}$

30. Given, $f(x) = \frac{1}{1-2\cos x}$

we know, $\quad -1 \le \cos x \le 1$

$\Rightarrow \quad -2 \le 2\cos x \le 2$

$\Rightarrow \quad -2 \le -2\cos x \le 2$

$\Rightarrow \quad 1-2 \le 1-2\cos x \le 2+1$

$\Rightarrow \quad -1 \le 1-2\cos x \le 3 \Rightarrow -1 \le \frac{1}{1-2\cos x} \le \frac{1}{3}$

31. $fog = f[4x(1-x)] = \frac{1-4x(1-x)}{1+4x(1-x)} = \frac{1-4x+4x^2}{1+4x-4x^2} = \frac{(2x-1)^2}{1+4x-4x^2}$

and $\quad gof = g\left(\frac{1-x}{1+x}\right) = 4\left(\frac{1-x}{1+x}\right)\left(1-\frac{1-x}{1+x}\right)$

$= \frac{4(1-x)}{(1+x)}\left(\frac{1+x-1+x}{1+x}\right) = \frac{8x(1-x)}{(1+x)^2}$

32. $\{ho(gof)\}x = ho\{g(x^2)\} = ho(\tan x^2) = \log(\tan x^2)$

At $x = \sqrt{\frac{\pi}{4}}$, $\quad \{ho(gof)\}(x) = \log\left\{\tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right\}$

$= \log\left(\tan\frac{\pi}{4}\right) = \log 1 = 0$

33. Given $[x]^2 - 5[x] + 6 = 0$

$\Rightarrow \quad [x]^2 - 3[x] - 2[x] + 6 = 0$

$\Rightarrow \quad [x]([x]-3) - 2([x]-3) = 0$

$\Rightarrow \quad ([x]-3)([x]-2) = 0$

$\Rightarrow \quad [x] = 3$ or $[x] = 2$

$\Rightarrow \quad x \in [3, 4)$ or $x \in [2, 3)$

$\therefore \quad x \in [2, 4)$

34. Let $\quad y = x^2 - x + 1 \Rightarrow x = \frac{1 \pm \sqrt{1-4(1-y)}}{2}$

$\because \quad x > \frac{1}{2}$

$\therefore \quad x = \frac{1}{2} + \sqrt{y - \frac{3}{4}} \Rightarrow f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$

Now, $\quad x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$

$\Rightarrow \quad x = 1$ (by checking the given possibilities)

35. We have, $f(x) = 2x - 3$, $g(x) = x^3 + 5$

$(fog)\,x = f(x^3 + 5) = 2(x^3 + 5 - 3) = 2x^3 + 7$

Let $\quad y = (fog)x = 2x^3 + 7 \Rightarrow x = \left(\frac{y-7}{2}\right)^{1/3}$

$\therefore \quad (fog)^{-1}\,x = \left(\frac{x-7}{2}\right)^{1/3}$

36. Since, $\quad y = \frac{10^{2x} - 1}{10^{2x} + 1}$

$\Rightarrow \quad y10^{2x} + y = 10^{2x} - 1 \Rightarrow 10^{2x} = \frac{1+y}{1-y}$

$\therefore \quad 2x\log_{10} 10 = \log_{10}\frac{1+y}{1-y} \Rightarrow x = \frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right)$

Hence, the inverse of y is $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$.

37. We have, $f(x) = \cos(\log x)$

$\therefore \quad f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$= \cos(\log x)\cos(\log y) - \frac{1}{2}[2\cos(\log x)\cos(\log y)]$

$= \cos(\log x)\cos(\log y) - \cos(\log x)\cos(\log y) = 0$

38. Since, $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$

$\therefore \quad Y = \{7, 11, \ldots \infty\}$

Let $\quad y = 4x + 3 \Rightarrow x = \frac{y-3}{4}$

Inverse of $f(x)$ is $g(y) = \frac{y-3}{4}$.

39. Given, $\quad f(x-y) = f(x)\,f(y) - f(a-x)\,f(a+y)$

Let $\quad x = 0 = y$

$\Rightarrow \quad f(0) = \{f(0)\}^2 - \{f(a)\}^2$

$\Rightarrow \quad 1 = 1 - \{f(a)\}^2 \qquad [f(0) = 1, \text{ given}]$

$\Rightarrow \quad f(a) = 0$

$\therefore \quad f(2a - x) = f\{a - (x - a)\} = f(a)f(x-a) - f(a-a)f(x)$

$= 0 - f(x)\cdot 1 = -f(x)$

40. Since, graph is symmetrical about the line $x = 2$.

$\therefore \quad f(2 + x) = f(2 - x)$

41. $\sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \ldots + f(n)$

$= f(1) + 2f(1) + 3f(1) + \ldots + nf(1)$

[since, $f(x+y) = f(x) + f(y)$]

$= (1 + 2 + 3 + \ldots + n)f(1) = f(1)\Sigma n$

$= \frac{7n(n+1)}{2} \qquad [\because f(1) = 7, \text{ given}]$

3 Sequence and Series

JEE Main MILESTONE

- Sequence
- Series
- Progression
- Arithmetic Progression (AP)
- Insertion of Arithmetic Mean (AM) between Two Numbers
- Geometric Progression (GP)
- Insertion of Geometric Mean (GM) between Two Numbers
- Harmonic Progression (HP)
- Insertion of Harmonic Mean (HM) between Two Numbers
- Relation among Arithmetic Mean, Geometric Mean
- Arithmetico-Geometric Progression
- To Find nth Term by Difference Method
- Sum of n Terms of Special Series Σn, Σn^2 and Σn^3

3.1 Sequence

A set of numbers arranged in a definite order according to some definite rule is called a sequence.

OR

A sequence is a function from natural number N domain to real number (complex number) R/C (codomain)

e.g., 2, 4, 6, 8, ... is a sequence.

In Mathematics, the word 'sequence' is used in the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on.

3.2 Series

If $a_1, a_2, \ldots, a_n$ is a sequence, then the expression $a_1 + a_2 + \ldots + a_n + \ldots$ is called the series. The series is said to be finite or infinite according as the given series is finite or infinite.

e.g., (i) $2 + 4 + 6 + 8 + \ldots + 20$ is a finite series.

(ii) $1 + 3 + 5 + \ldots$ is an infinite series.

3.3 Progression

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicity formula for the nth term. Those sequences whose terms follow certain patterns are called progression.

OR

If the terms of a sequence are written under specific conditions, then the sequence is called progression.

3.4 Arithmetic Progression (AP)

A sequence is said to be an arithmetic progression, if the difference of a term and its previous term is always same.

i.e., $a_{n+1} - a_n = \text{Constant } (= d), \forall\, n \in N$

The constant difference, generally denoted by d, is called the common difference.

OR

An arithmetic progression (AP) is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference of the AP.

In other words, if $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$$

If a is the first term and d is the common difference, then AP can be written as $a, a + d, a + 2d, \ldots, [a + (n-1)d]$

e.g., (i) 1, 3, 5, 7, ...

(ii) 2, 4, 6, ...

The nth term of an AP

Let a be the first term d be the common difference and l be the last term of an AP, then nth term is given by

$$T_n = l = a + (n-1)d, \text{ where } d = T_n - T_{n-1}$$

The nth term from last is $T_n' = l - (n-1)d$

The sum of n terms of an AP

Suppose there are n terms of a sequence, whose first term is a, common difference is d and last term is l, then sum of n terms is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$$

***Sample* Problem 1** *The sum of all natural numbers lying between 100 and 1000, which are multiples of 5, is* [NCERT]

(a) 98400 (b) 98450

(c) 98436 (d) 98455

Interpret (b) The numbers are 105, 110, 115, ... , 995.

Here, first term $(a) = 105$

and common difference $(d) = 110 - 105 = 5$

Now, nth term, $T_n = a + (n-1)d$

$\Rightarrow$ $995 = 105 + (n-1)5$ $(\because a = 105, T_n = 995)$

$\Rightarrow$ $995 - 105 = (n-1)5$

$\Rightarrow$ $890 = (n-1)5$

$\Rightarrow$ $n - 1 = \frac{890}{5} = 178$

$\Rightarrow$ $n = 178 + 1 = 179$

Now, required sum, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$= \frac{179}{2}[2 \times 105 + (179-1)5]$$

$$= \frac{179}{2}(210 + 178 \times 5)$$

$$= \frac{179}{2}(210 + 890) = \frac{179}{2} \times 1100$$

$$= 179 \times 550 = 98450$$

***Sample* Problem 2** *The ratio of sums of m and n terms of an AP is $m^2 : n^2$. Then, the ratio of the mth and nth term is* [NCERT]

(a) $(2m+1):(2n+1)$ (b) $m:n$

(c) $(2m-1):(2n-1)$ (d) *None of these*

Interpret (c) Let a and d be the first term and common difference of given AP.

$\therefore$ $S_m = \frac{m}{2}[2a + (m-1)d]$

and $S_n = \frac{n}{2}[2a + (n-1)d]$

According to the question,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$\Rightarrow$ $\frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$

$\Rightarrow$ $2a(n-m) = d\{(n-1)m - (m-1)n\}$

$\Rightarrow$ $d = 2a$

Now, $\frac{T_m}{T_n} = \frac{a + (m-1)d}{a + (n-1)d}$

$$= \frac{a + (m-1)2a}{a + (n-1)2a} = \frac{2m-1}{2n-1}$$

Thus, $T_m : T_n = (2m-1):(2n-1)$

***Sample* Problem 3** *In a potato race 20 potatoes are placed in a line at intervals of 4 m with the first potato 24 m from the starting point. A contestant is required to being the potatoes back to the starting place on at a time. How far would he run in bringing back all the potatoes?* [NCERT Exemplar]

(a) 2485 *m* (b) 2480 *m*

(c) 2482 *m* (d) *None of these*

Interpret (b) According to the given condition, the sequence becomes,

$$24, 28, \ldots, 20\text{th term},$$

Here, $a = 24, d = 28 - 24 = 4, n = 20$

$\therefore$ $S_n = \frac{n}{2}[2a + (n-1)d]$

$$= \frac{20}{2}[2 \times 24 + (20-1)4]$$

$$= 10(48 + 76) = 10 \times 124 = 1240$$

The contestant is required to bring the potatoes back to the starting place $= 2S_n = 2 \times 1240 = 2480$ m

Important Results Related to AP

(i) If a fixed constant C is added to (or subtracted from) each term of a given AP, then the resulting sequence is also an AP with the same common difference as that of the given AP.

(ii) If each term of an AP is multiplied by (or divided by) a non-zero fixed constant C, then the resulting sequence is also an AP with common difference C times $\left(\text{or } \frac{1}{C} \text{ times}\right)$ the previous.

(iii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's, then $a_1 \pm b_1$, $a_2 \pm b_2, a_3 \pm b_3, \ldots$ is also an AP with common difference $d_1 \pm d_2$.

(iv) A sequence obtained by multiply or division of corresponding terms of two AP is not an AP.

(v) If in an AP, $S_p = q$ and $S_q = p$, then $S_{p+q} = -(p+q)$.

(vi) Three numbers in AP can be taken as $a-d, a, a+d$, five numbers in AP can be taken as $a-2d, a-d, a, a+d, a+2d$, where the common difference is d.

(vii) Four numbers in AP can be taken as $a-3d, a-d, a+d, a+3d$, where common difference is $2d$.

(viii) In an AP, the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term,

i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$

(ix) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it, *i.e.*,

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}), k < n$$

and for $k = 1, a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$

(x) $T_n = S_n - S_{n-1}\ (n \geq 2)$.

(xi) A sequence is an AP, if its nth term is of the form $An + B$, *i.e.*, a linear expression in n. In such a case the common difference is A *i.e.*, the coefficient of n.

(xii) If sum of n terms of any sequence is quadratic expression in n (*i.e.*, $S_n = an^2 + bn + c$), then sequence is AP.

(xiii) The common difference of an AP can be zero, positive or negative.

***Sample* Problem 4** *The sum of three numbers in AP is –3 and their product is 8. Then, sum of squares of the numbers is*

(a) 9 (b) 10
(c) 21 (d) 12

Interpret (c) Let the three numbers in AP are $a-d, a, a+d$.

$\therefore \quad (a-d) + a + (a+d) = -3$

$\Rightarrow \quad 3a = -3$

$\Rightarrow \quad a = -1$

and $\quad (a-d)(a)(a+d) = 8$

$\Rightarrow \quad (-1)\{(-1)^2 - d^2\} = 8$

$\Rightarrow \quad d^2 = 9$

$\Rightarrow \quad d = \pm 3$

If $d = 3$, then numbers are $-4, -1$ and 2.

If $d = -3$, then numbers are $2, -1$ and -4.

Hence, the sum of squares of numbers $= (-4)^2 + (-1)^2 + (2)^2$

$= 16 + 1 + 4 = 21$

3.5 Insertion of Arithmetic Mean (AM) between Two Numbers

Let $A_1, A_2, \ldots, A_n$; n arithmetic means are inserted between two numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ form an AP.

Here, total number of terms are $(n+2)$ and common difference be d (say).

$\therefore \ b = (n+2)$ th term $= a + (n+2-1)d$

$\Rightarrow \quad d = \dfrac{b-a}{n+1}$

$\therefore \quad A_1 = a + d = a + \left(\dfrac{b-a}{n+1}\right) = \dfrac{na+b}{n+1}$

and $\quad A_2 = a + 2d = a + 2\left(\dfrac{b-a}{n+1}\right)$

$= \dfrac{(n-1)a + 2b}{n+1}$

In general, $\quad A_r = a + r\left(\dfrac{b-a}{n+1}\right)$

$= \dfrac{(n-r+1)a + rb}{n+1}, \quad \forall\, r = 1, 2, \ldots, n$

(i) If $r = 1$ and $n = 1$, then

$$A_1 = \frac{(1-1+1)a + 1(b)}{1+1}$$

$\Rightarrow \quad A_1 = \dfrac{a+b}{2} = \text{Mean}$

Arithmetic Mean

If we insert one arithmetic mean between two numbers a and b, then

$$\text{Arithmetic mean} = \frac{a+b}{2}$$

Note

- The sum of n arithmetic means between a and b is n times of arithmetic mean of a and b

i.e., $\quad A_1 + A_2 + \ldots + A_n = n\left(\dfrac{a+b}{2}\right) = nA$

- The arithmetic mean of n positive numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$\text{AM} = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

***Sample* Problem 5** *If we insert five numbers between 8 and 26 such that the resulting sequence is an AP, then the resulting sequence is* [NCERT]

(a) 8, 11, 14, 16, 20, 23, 26 (b) 8, 11, 14, 17, 20, 23, 26
(c) 8, 11, 14, 17, 20, 22, 26 (d) *None of these*

Interpret (b) Let five numbers are A_1, A_2, A_3, A_4, A_5, then $8, A_1, A_2, A_3, A_4, A_5, 26$ are in AP.

$\because \quad T_n = a + (n-1)d$

$\therefore \quad 26 = 8 + (7-1)d$

($\because n = 7$ since, 2 terms and 5 numbers)

$\Rightarrow \quad 26 - 8 = 6d$

$\Rightarrow \quad 18 = 6d$

$\Rightarrow \quad d = \frac{18}{6}$

$\Rightarrow \quad d = 3$

Now, $A_1 = a + d = 8 + 3 = 11$

$\Rightarrow \quad A_2 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 8 + 3 \times 3 = 17$

$A_4 = a + 4d = 8 + 4 \times 3 = 20$

and $A_5 = a + 5d = 8 + 5 \times 3 = 23$

Hot Spot

Words Problem Based on AP

Word problems are directly asked in consecutive years of AIEEE (JEE Main) 2011, 2010. *To solve such types of problem following steps are given below.*

Step I Firstly, write all the given information in a order.

Step II Using the formula, sum of an AP, $S_n = \frac{n}{2}[2a + (n-1)d]$

Step III Use another condition, if it is given.

Step IV Simplify the relations and get the result.

***Sample* Problem 6** *A man arranges to pay off a debt of ₹ 3600 by 40 annual instalents which are in AP, when 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. Find the 8th instalment.*

(a) ₹ 65
(b) ₹ 60
(c) ₹ 70
(d) *None of the above*

Interpret (a) **Step I** Firstly, we write all the given information in a order.

Here, $S =$ ₹ 3600, $n = 40$

Let a and d be the first term and common difference of an AP.

Step II Using the formula of sum of an AP.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$3600 = \frac{40}{2}[2a + (40-1)d]$$

$\Rightarrow \quad 180 = 2a + 39d \quad \ldots(i)$

Step III Now, we write another given condition. After 30 instalments one-third of the debt is unpaid

$\therefore \quad \frac{3600}{3} = 1200$ is unpaid and 2400 is paid.

Step IV Again, using the formula of sum of an AP.

$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore \quad 2400 = \frac{30}{2}[2a + (30-1)d]$

$\Rightarrow \quad 160 = 2a + 29d \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$a = 51, d = 2$$

Step V Finding required results.

$\therefore$ 8th instalment,

$$T_8 = a + (8-1)d$$
$$= 51 + 7 \times 2$$
$$= 51 + 14 = ₹\ 65$$

Check Point 1

1. Which term of the AP, 19, $18\frac{1}{5}$, $18\frac{2}{5}$, ... is the first negative term?
(a) 24 (b) 25 (c) 26 (d) 23

2. The sum of all two digit numbers which when divided by 4, yield unity as remainder, is
(a) 1012 (b) 1201 (c) 1212 (d) 1210

3. If $\log_5 2$, $\log_5 (2^x - 3)$ and $\log_5\left(\frac{17}{2} + 2^{x-1}\right)$ are in AP, then the value of x is
(a) 0 (b) –1
(c) 3 (d) None of these

4. If S_1, S_2 and S_3 denote the sum of n_1, n_2 and n_3 terms respectively of an AP, then
$$\frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2)$$
is equal to
(a) 0 (b) 1 (c) $S_1S_2S_3$ (d) $n_1n_2n_3$

5. If a^2, b^2, c^2 are in AP, then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in
(a) AP (b) GP
(c) HP (d) None of these

6. If S_n denotes the sum of n terms of an AP, then $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$ is equal to
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

7. If $\frac{1}{p+q}, \frac{1}{q+r}, \frac{1}{r+p}$ are in AP, then
(a) p, q, r are in AP (b) q^2, p^2, r^2 are in AP
(c) p^2, q^2, r^2 are in AP (d) None of these

8. If 1, $\log_y x$, $\log_z y$, $-15 \log_x z$ are in AP, then
(a) $z^3 = x$ (b) $x = y^{-1}$
(c) $z^{-3} = y$ (d) All of these

9. A farmer buys a used tractor of ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalment of ₹ 500 plus 12% interest on the unpaid amount. The tractor cost for farmer is
(a) ₹ 16680 (b) ₹ 16670
(c) ₹ 16650 (d) None of these

10. A circle is completely divided into n sectors in such a way that the angles of the sectors are in AP. If the smallest of these angles is 8° and the largest is 72°, then the angle in the fifth sector is
(a) 40° (b) 35° (c) 42° (d) 43°

3.6 Geometric Progression (GP)

A sequence of non-zero numbers is called a geometric progression, if the ratio of a term and the term preceding to it is always a constant quantity. This constant factor is called the common ratio of the GP.

OR

A geometric progression (GP) is a sequence of numbers, whose first term is non-zero and each of the term is obtained by multiplying its just preceding term by a constant quantity. This constant quantity is called the common ratio of the GP.

In other words, if $a_1, a_2, a_3, \ldots, a_n$ are in GP, then
$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}} = r \quad \text{(say)}$$
r is known as common ratio of GP.

If a is the first term and r is the common ratio, then GP can be written as $a, ar, ar^2, \ldots, ar^{n-1}$. $(a \neq 0)$

e.g., 2, 4, 8, 16, ...

The nth term of a GP (General Term)

Let a be the first term, r be the common ratio and l be the last term of a GP, then nth term is given by
$$T_n = l = ar^{n-1}, \text{ where } r = \frac{T_n}{T_{n-1}}$$

The nth term from last is, $T_n' = \frac{l}{r^{n-1}}$

The sum of n terms of a GP

Suppose there are n terms of a sequence, whose first term is a, common ratio is r and last term is l, then sum of n terms is given by
$$S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1}, & \text{when } r > 1 \\ \frac{a(1 - r^n)}{1 - r}, & \text{when } r < 1 \\ \text{not define}, & \text{when } r = 1 \end{cases}$$

If, terms are infinite, then sum of GP is,
$$S_\infty = \frac{a}{1-r}, \text{ when } |r| < 1$$

Note If first term of a GP is positive *i.e.*, if $a > 0$, then GP is increasing, if $r > 1$ and decreasing, if $0 < r < 1$. And if $a < 0$, then GP is increasing, if $0 < r < 1$ and decreasing if $r > 1$.

- If a, b, c, d ... are in GP, they are also in continued proportion
i.e., $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots = \frac{1}{r}$ (say)
- If first term of a GP of n terms is a and last term is l. Then, the product of all the terms of the GP is $(al)^{n/2}$.
- If a, b, c are in AP, as well as in GP, then $a = b = c$.

Problem Solving Trick

Value of Recurring Decimal A number is of the form of $a \cdot bbb\ldots$ is recurring decimal.

e.g., Suppose any general terms of the form like $\underbrace{0 \cdot X}_{n \text{ terms}}\ \underbrace{Y}_{m \text{ terms}}$

where X is the non-recurring part and Y is the recurring part.

$$\therefore \qquad \text{Resultant} = \frac{XY - X}{10^{m+n} - 10^n}$$

***Sample* Problem 7** *The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, then number of bacteria will be present at the end of 2nd hour, 4th hour and nth hour will be* [NCERT]

(a) 120, 460, $30(2^n)$ (b) 120, 485, $30(2^{n-1})$

(c) 120, 480, $30(2^n)$ (d) *None of these*

Interpret (c) Number of bacteria present in the culture, form a GP whose first term $(a) = 30$ and common ratio $(r) = 2$

Bacteria present after 2nd hour $= ar^2 = 30 \times (2)^2 = 120$

$(\because T_n = ar^{n-1})$

Bacteria present after 4th hour

$= ar^4 = 30 \times (2)^4 = 30 \times 16 = 480$

Bacteria present after nth hour $= ar^n = 30 \times 2^n = (30) \cdot (2^n)$

***Sample* Problem 8** *The sum of the series* $5 + 55 + 555 + \ldots$ *to n terms is* [NCERT]

(a) $\frac{1}{80}(10^{n+1})$ (b) $\frac{5}{81}(10^{n+1} - 9n - 10)$

(c) $\frac{5}{81}(10^{n-1} - 8n - 1)$ (d) $\frac{5}{81}(10^{n+1} + 9n + 10)$

Interpret (b) $5 + 55 + 555 + \ldots$ to n terms

$= 5\{1 + 11 + 111 + \ldots \text{ to } n \text{ terms}\}$

$= \frac{5}{9}\{9 + 99 + 999 + \ldots \text{ to } n \text{ terms}\}$

(multiply numerator and denominator by 9)

$= \frac{5}{9}\{(10-1) + (100-1) + (1000-1) + \ldots \text{ to } n \text{ terms}\}$

$= \frac{5}{9}\{10 + 100 + 1000 + \ldots + n \text{ terms} - (1 + 1 + \ldots + n \text{ terms})\}$

$= \frac{5}{9}\left\{\frac{10(10^n - 1)}{10 - 1} - n\right\} = \frac{5}{81}\{10^{n+1} - 10 - 9n\}$

***Sample* Problem 9** *If a, b, c are pth, qth and rth terms of a GP, then* $(q - r)\log a + (r - p)\log b + (p - q)\log c$ *is equal to*

(a) $p + q + r$

(b) 1

(c) $-pqr$

(d) 0

Interpret (d) Let A and R be the first term and common ratio of the given GP. Then, $a = AR^{p-1}$

$\Rightarrow \qquad \log a = \log A + (p-1)\log R \qquad \ldots(i)$

Similarly, $\qquad \log b = \log A + (q-1)\log R \qquad \ldots(ii)$

and $\qquad \log c = \log A + (r-1)\log R \qquad \ldots(iii)$

Now, $\qquad (q - r)\log a + (r - p)\log b + (p - q)\log c$

$= (q - r)\{\log A + (p-1)\log R\}$

$+ (r - p)\{\log A + (q-1)\log R\}$

$+ (p - q)\{\log A + (r-1)\log R\}$

$= \log A\,[q - r + r - p + p - q]$

$+ \log R\,[p(q - r) + q(r - p) + r(p - q)$

$- (q - r) - (r - p) - (p - q)]$

$= \log A \cdot 0 + \log R \cdot 0 = 0$

Important Results Related to GP

(i) If each term of a GP is multiplied (or divided) by a non-zero constant $C\,(\neq 0)$, then the resulting sequence is also a GP with same common ratio.

(ii) If $a, ar, ar^2, \ldots$ and $a', a'r', a'r'^2, \ldots$ are two GP's, then the sequences aa', $aa'(rr')$, $aa'(rr')^2$, ... and $\frac{a}{a'}, \frac{a}{a'}\left(\frac{r}{r'}\right), \frac{a}{a'}\left(\frac{r}{r'}\right)^2, \ldots$ obtained by the products and quotients of the corresponding terms of the given GP's are also GP's with common ratios rr' and $\frac{r}{r'}$ respectively.

(iii) The reciprocals of the terms of a GP also form a GP.

(iv) If $a_1, a_2, a_3, \ldots$ is a GP of positive terms (*i.e.*, $a_i > 0, \ \forall\, i$), then $\log a_1, \log a_2, \log a_3, \ldots$ is an AP and the converse is also true in this case.

(v) The odd number of terms in a GP should be taken as ...

$ar^3, ar^2, ar, a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \ldots$ where common ratio is r while the even number of terms in a GP should be taken as ...

$ar^5, ar^3, ar, \frac{a}{r}, \frac{a}{r^3}, \frac{a}{r^5}, \ldots$ where common ratio is r^2.

(vi) If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then $a_1^k, a_2^k, a_3^k, \ldots, a_n^k$ will also be in GP whose common ratio is r^k.

(vii) If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then $a^{a_1}, a^{a_2}, a^{a_3}, \ldots, a^{a_n}$ will be in GP whose common ratio is a^d.

(viii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are in GP, then the sequence(s) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$, will not be a GP.

(ix) In a GP, the product of two terms equidistant from the beginning and end is a constant and is equal to the product of first term and last term, *i.e.*, if $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$ are in GP, then

$$a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = \ldots$$

***Sample* Problem 10** *Three numbers form an increasing GP. If the middle number is doubled, then the new numbers are in AP. The common ratio of the GP is*

(a) $2-\sqrt{3}$ (b) $2+\sqrt{3}$
(c) $\sqrt{3}-2$ (d) $3+\sqrt{2}$

Interpret (b) Let the three numbers in GP be $\frac{a}{r}, a, ar$. Also, the numbers form an increasing GP. So, $r > 1$. Now, $\frac{a}{r}, 2a, ar$ in AP.

$$\therefore \quad 4a = \frac{a}{r} + ar$$

$$\Rightarrow \quad r^2 - 4r + 1 = 0$$

$$\Rightarrow \quad r = 2 + \sqrt{3} \qquad (\because r \neq 2 - \sqrt{3})$$

***Sample* Problem 11** *If $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, then a, b, c, d are in*

(a) *AP* (b) *GP*
(c) *HP* (d) *None of these*

Interpret (b) Since, $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$. Therefore, when $ax^3 + bx^2 + cx + d$ is divided by $ax^2 + c$, the remainder is zero.

$$\therefore \quad d - \frac{bc}{a} = 0$$

$$\Rightarrow \quad bc = ad$$

$$\Rightarrow \quad \frac{b}{a} = \frac{d}{c}$$

$\Rightarrow$ a, b, c, d are in GP.

3.7 Insertion of Geometric Mean (GM) between Two Numbers

Let $G_1, G_2, \ldots, G_n$; n geometric means are inserted between two numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ form a GP.

Here, total number of terms are $(n + 2)$ and let common ratio is r.

$$\therefore \quad b = (n+2)\text{th term} = ar^{n+2-1}$$

$$\Rightarrow \quad r = \left(\frac{b}{a}\right)^{1/(n+1)}$$

$$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$$

In general, $G_k = a\left(\frac{b}{a}\right)^{k/(n+1)}, \forall\, k = 1, 2, 3, \ldots, n$

If $k = 1$ and $n = 1$, then

$$G_1 = a\left(\frac{b}{a}\right)^{1/(1+1)} = a\left(\frac{b}{a}\right)^{1/2} = \sqrt{ab}$$

Geometric Mean

If we insert one geometric mean between two numbers a and b, then

$$\text{Geometric mean} = \sqrt{ab}.$$

Note

- The product of n geometric means between a and b is equal to the nth power of geometric mean of a and b *i.e.*,
$$G_1 \cdot G_2 \cdot \ldots \cdot G_n = (ab)^{n/2} = G^n$$
- The product of n geometric means between a and $\frac{1}{a}$ is 1.
- The geometric mean of n positive numbers $a_1, a_2, a_3, \ldots, a_n$ is GM $= (a_1\, a_2\, a_3 \ldots a_n)^{1/n}$.
- If A and G be the AM and GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.
- If A and G are respectively AM and GM between two positive numbers a and b, then the quadratic equation having a, b as its roots is
$$x^2 - 2Ax + G^2 = 0$$

***Sample* Problem 12** *If we insert two numbers between 3 and 81 so that the resulting sequence is GP. Then, the numbers are*

(a) 9, 27 (b) 8, 27
(c) 9, 25 (d) *None of these*

Interpret (a) Let the two numbers are a and b, then $3, a, b, 81$ are in GP.

$$\because \quad n\text{th term } T_n = AR^{n-1}$$

$$\therefore \quad 81 = 3R^{4-1}$$

$$\Rightarrow \quad R^3 = \frac{81}{3} = 27$$

$$\Rightarrow \quad R^3 = 3^3$$

$$\Rightarrow \quad R = 3$$

$$\therefore \quad a = AR = 3 \times 3 = 9, b = AR^2 = 3 \times 3^2 = 27$$

Check Point 2

1. Let $a_1, a_2, a_3, \ldots$, be in AP and a_p, a_q, a_r be in GP, then $a_q : a_p$ is equal to
   (a) $\frac{r-p}{q-p}$ (b) $\frac{q-p}{r-q}$
   (c) $\frac{r-q}{q-p}$ (d) None of these

2. If $(1+x)(1+x^2)(1+x^4)\ldots(1+x^{128}) = \sum_{r=0}^{n} x^r$, then n is equal to
   (a) 255 (b) 127
   (c) 60 (d) None of these

3. Two numbers whose arithmetic mean is 34 and the geometric mean is 16, then the ratio of numbers is
   (a) 5 or $\frac{1}{5}$ (b) 4 or $\frac{1}{4}$ (c) 2 or $\frac{1}{2}$ (d) 16 or $\frac{1}{16}$

4. Let $< a_n >$ be a GP such that $\frac{a_4}{a_6} = \frac{1}{4}$ and $a_2 + a_5 = 216$. Then, a_1 is equal to
   (a) 12 or $\frac{108}{7}$ (b) 10
   (c) 7 or $\frac{54}{7}$ (d) None of these

5. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} (ab)^n$, where $|a|, |b| < 1$, then
   (a) $xyz = x + y + z$ (b) $xz + yz = xy + z$
   (c) $xy + yz = xz + y$ (d) $xy + xz = yz + x$

6. If a, b, c are in GP, then $\frac{b-a}{b-c} + \frac{b+a}{b+c}$ is equal to
   (a) $b^2 - c^2$ (b) ac
   (c) ab (d) 0

7. If the roots of equation $(b^2 + c^2)x^2 - 2(a+b)cx + (c^2 + a^2) = 0$ are equal, then
   (a) a, b, c are in GP (b) a, b, c are in AP
   (c) a, c, b are in GP (d) a, c, b are in AP

8. $4 + 44 + 444 + \ldots$ is equal to
   (a) $\frac{4}{81}[10^{n+1} - 10 - 9n]$ (b) $\frac{4}{9}[10^{n+1} - 10 - 9n]$
   (c) $\frac{4}{81}[10^{n+1} - 10 + 9n]$ (d) $\frac{4}{9}[10^{n+1} - 10 + 9n]$

9. The sum of sequence 0.15, 0.015, 0.0015, . . . , 20 terms is
   (a) $\frac{1}{6}[1-(0.1)^{20}]$ (b) $\frac{1}{6}[1+(0.1)^{20}]$
   (c) $\frac{1}{3}[1-(0.1)^{20}]$ (d) None of these

10. One side of an equilateral triangle is 24 cm. The mid-points of its sides are joined to form another triangle whose mid-points in term are joined to form still another triangle. The process continues indefinitely. The sum of perimeters of all the triangles is
    (a) 144 cm
    (b) 140 cm
    (c) 145 cm
    (d) None of the above

3.8 Harmonic Progression (HP)

A sequence is said be a harmonic progression, if reciprocal of terms of the sequence form an AP *i.e.*, $a_1, a_2, a_3, \ldots, a_n$ are in HP, if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

*n*th term of HP from beginning,

$$T_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2}{a_2 + (n-1)(a_1 - a_2)}$$

*n*th term of HP from end,

$$T_n' = \frac{1}{\frac{1}{a_n} - (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2 a_n}{a_1 a_2 - a_n(n-1)(a_1 - a_2)}$$

Note

- No one term of an HP can be zero.
- *n*th term of a HP is the reciprocal of the *n*th term of the corresponding AP.
- There is no formula for determining the sum of HP sequence.

***Sample* Problem 13** *If 20th term of a HP is 1 and 30th term is* $-\frac{1}{17}$, *then its largest term is*

(a) $\frac{1}{3}$ (b) 2
(c) 1 (d) *None of these*

Interpret (c) Here, $\frac{1}{a+19d} = 1$ and $\frac{1}{a+29d} = -\frac{1}{17}$

$\Rightarrow$ $a + 19d = 1$ ...(i)
and $a + 29d = -17$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a = \frac{176}{5} \text{ and } d = -\frac{9}{5}$$

Now, assume that its *n*th term is largest term.

$$T_n = \frac{1}{\frac{176}{5} + (n-1)\left(-\frac{9}{5}\right)} = \frac{5}{185 - 9n}$$

This term will be largest, if $185 - 9n$ is least. The least positive integer value is possible only for $n = 20$.

Hence, $T_n = \frac{5}{185 - 9n} = \frac{5}{185 - 180} = 1$

3.9 Insertion of Harmonic Mean (HM) between Two Numbers

Let $H_1, H_2, \ldots, H_n$; n harmonic means are inserted between two numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ form a HP.

Here, total number of terms is $(n+2)$. Also, $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_3}, \frac{1}{b}$ are in AP.

Let d be the common difference of this AP.

$$\therefore \quad \frac{1}{b} = \frac{1}{a} + (n+2-1)\,d$$

$$\Rightarrow \quad d = \frac{a-b}{(n+1)\,ab}$$

Now, $$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{(a-b)}{(n+1)\,ab}$$

$$\Rightarrow \quad \frac{1}{H_1} = \frac{bn+a}{(n+1)ab}$$

and $$\frac{1}{H_n} = \frac{1}{a} + nd = \frac{1}{a} + \frac{n(a-b)}{(n+1)\,ab}$$

$$\Rightarrow \quad \frac{1}{H_n} = \frac{an+b}{(n+1)\,ab}$$

If $r = 1$ and $n = 1$, then

$$\frac{1}{H_1} = \frac{a(1)+b}{(1+1)ab} = \frac{a+b}{2ab} \Rightarrow H_1 = \frac{2ab}{a+b}$$

i.e., If we insert one harmonic mean between two humbers, then

$$\text{HM} = \frac{2ab}{(a+b)}$$

Note

- The harmonic mean of n positive numbers $a_1, a_2, \ldots, a_n$ is $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)$.
- If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

3.10 Relation among Arithmetic Mean, Geometric Mean

Let a and b be two real positive and unequal numbers and A, G, H are arithmetic, geometric and harmonic mean respectively between them.

Then, $$A = \frac{a+b}{2}, G = \sqrt{ab}$$

Now, $$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2 > 0$$

$$\therefore \quad A - G > 0 \qquad (\because a \text{ and } b \text{ are positive})$$

$$\Rightarrow \quad A > G \qquad \ldots(\text{i})$$

Note

- If $a = b$, then $A = G = H$
- If $a \neq b$, then $A > G > H$
- $G^2 = AH$

***Sample* Problem 14** *Let $n \in N, n > 25$. Let A, G, H denote arithmetic mean, geometric mean and harmonic mean respectively of 25 and n. The least value of n for which $A, G, H \in \{25, 26, \ldots, n\}$ is*

(a) 49 (b) 81
(c) 169 (d) 225

Interpret (d) Here, $A = \frac{25+n}{2}, G = 5\sqrt{n}$

and $$H = \frac{50n}{25+n}$$

Also, A, G, H are natural numbers, so n must be an odd perfect square.

For $n = 225$, all of A, G and H are natural number.

3.11 Arithmetico-Geometric Progression (AGP)

A sequence whose each term is obtained by multiplying corresponding terms of AP and GP is called an arithmetico-geometric progression.

If $a, (a+d), (a+2d), \ldots, [a+(n-1)\,d]$ is an AP and $1, r, r^2, \ldots, r^{n-1}$ is a GP, then on multiplying corresponding terms of AP and GP, a progression $a, (a+d)\,r, (a+2d)\,r^2, \ldots, [a+(n-1)\,d]\,r^{n-1}$ is formed which is called an arithmetico-geometric progression.

Steps for Solving the AGP Series

Step I Multiply the common ratio of the corresponding GP.

Step II Shift the new obtained series in step I by one term.

Step III Subtract the given series from obtained series in step II.

Method of Solving AGP Series

Let AGP series

$$a + (a + d)r + (a + 2d)r^2 + \ldots + [a + (n-1)d]r^{n-1}$$

Types of problem can be solved by using the direct method given below.

The sum of n terms of arithmetico-geometric is

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)\,d]\,r^n}{1-r},\ r \neq 1$$

The sum of infinite terms of arithmetico-geometric progression,

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2},\ |r| < 1$$

***Sample* Problem 15** *The sum of the series*

$$1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + \ldots + 100\cdot 2^{99} \text{ is}$$

(a) $100\cdot 2^{100} + 1$ (b) $99\cdot 2^{100} + 1$

(c) $99\cdot 2^{99} - 1$ (d) $100\cdot 2^{100} - 1$

Interpret (b) Let

$$S = 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + \ldots + 100\cdot 2^{99} \quad \ldots\text{(i)}$$

It is an arithmetico-geometric series.

On multiplying Eq. (i) by 2 and then subtracting, we get

$$S = 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + \ldots + 100\cdot 2^{99}$$

$$2S = \quad 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \ldots + 99\cdot 2^{99} + 100\cdot 2^{100}$$

$$-S = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{99} - 100\cdot 2^{100}$$

$$\Rightarrow \quad -S = \frac{1(2^{100}-1)}{2-1} - 100\cdot 2^{100}$$

$$\Rightarrow \quad -S = 2^{100} - 1 - 100\cdot 2^{100}$$

$$\Rightarrow \quad -S = -1 - 99\cdot 2^{100}$$

$$\Rightarrow \quad S = 99\cdot 2^{100} + 1$$

3.12 To Find nth Term by Difference Method

If $T_1, T_2, \ldots, T_n$ are terms of any series and their difference $(T_2 - T_1), (T_3 - T_2), (T_4 - T_3), \ldots, (T_n - T_{n-1})$ are either in AP or in GP, then T_n and S_n of series may be found by the method of differences.

Let $\quad S_n = T_1 + T_2 + T_3 + \ldots + T_n$

Again, $\quad S_n = \quad T_1 + T_2 + \ldots + T_{n-1} + T_n$

$$S_n - S_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$$

$$\Rightarrow \quad T_n = T_1 + t_1 + t_2 + t_3 + \ldots + t_{n-1}$$

where $t_1, t_2, \ldots$ are terms of the new series.

***Sample* Problem 16** *The sum of the infinite series*

$$1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots \infty \text{ is}$$

(a) 4/2 (b) 9/2

(c) 4/9 (d) 1

Interpret (b) This is clearly not an arithmetico-geometric series, since 1, 4, 9, 16, … are not in AP. However, their successive differences $(4-1), (9-4), (16-9)$, … are in AP.

Let $\quad S_\infty = 1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots \infty$

$$\frac{1}{3}S_\infty = \quad \frac{1}{3} + \frac{4}{3^2} + \frac{9}{3^3} + \ldots \infty$$

On subtraction, we get

$$\frac{2}{3}S_\infty = 1 + \frac{3}{3} + \frac{5}{3^2} + \frac{7}{3^3} + \ldots \infty$$

$$\Rightarrow \quad \frac{1}{3}\cdot\frac{2}{3}S_\infty = \frac{1}{3} + \frac{3}{3^2} + \frac{5}{3^3} + \ldots \infty$$

$$\left(\text{multiplying both sides by } \frac{1}{3}\right)$$

On subtracting the two series, we get

$$\left(\frac{4}{9}\right)\cdot S_\infty = 1 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots \infty$$

$$= 1 + \frac{2}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots \infty\right) = 1 + \frac{2}{3} \times \frac{1}{\left(1 - \frac{1}{3}\right)} = 2$$

$$\therefore \quad S_\infty = \left(2 \times \frac{9}{4}\right) = \frac{9}{2}$$

3.13 Sum of n terms of Special Series Σn, Σn^2 and Σn^3

Generally, find the sum of n terms of any series, we use Σ before the nth term of the series.

Sum of some special series is given below.

(i) Sum of n natural numbers

$$= 1 + 2 + 3 + 4 + \ldots + n$$

$$= \Sigma n = \frac{n(n+1)}{2}$$

(ii) Sum of squares of n natural numbers

$$= 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$$

$$= \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cubes of n natural numbers

$$= 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$$= \Sigma n^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2 = (\Sigma n)^2$$

Note

- $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$
- $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$
- $1\cdot 2+2\cdot 3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$
- $1\cdot 2\cdot 3\cdot 4+2\cdot 3\cdot 4\cdot 5+\ldots+n(n+1)(n+2)(n+3)$
  $=\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$

***Sample* Problem 17** *The sum of the series*

$$3\times 1^2+5\times 2^2+7\times 3^2+\ldots \text{ is}$$ **[NCERT]**

(a) $\frac{n(n+1)(n^2-5n-1)}{6}$ (b) $\frac{n(n+1)(3n^2+5n+1)}{6}$

(c) $\frac{n(n-1)(3n^2-5n-1)}{6}$ (d) *None of these*

Interpret (*b*) Let given series is $S=3\times 1^2+5\times 2^2+7\times 3^2+\ldots$

First, we will split the given series into two parts which are 3, 5, 7, ... and $1^2, 2^2, 3^2, \ldots$ and find the *n*th term of each part separately to find the *n*th term of the given series.

$T_n=(n\text{th term of } 3,5,7,\ldots)\times(n\text{th term of } 1,2,3,\ldots)^2$

$=[3+(n-1)2][1+(n-1)1]^2=(3+2n-2)(n)^2=(2n+1)n^2=2n^3+n^2$

Now, $S=\Sigma T_n=2\Sigma n^3+\Sigma n^2$

$$=2\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{6}\quad\left[\because \Sigma n^3=\left\{\frac{n(n+1)}{2}\right\}^2,\ \Sigma n^2=\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\frac{n(n+1)}{2}\left[2\times\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n(n+1)+2n+1}{3}\right]$$

$$=\frac{n(n+1)}{6}\times(3n^2+3n+2n+1)=\frac{n(n+1)(3n^2+5n+1)}{6}$$

***Sample* Problem 18** *The sum of n terms of the series* $1^2+3^2+5^2+\ldots$ *to n terms, is*

(a) $4n^2-1$ (b) $4n^2+1$ (c) $\frac{n}{3}(4n^2-1)$ (d) $\frac{n}{3}(4n^2+1)$

Interpret (*c*) *n*th term, $t_n=[1+(n-1)\times 2]^2$
$=(2n-1)^2=(4n^2-4n+1)$

$$\therefore\quad S_n=\sum_{k=1}^{n}(4k^2-4k+1)$$

$$=\left[4\left(\sum_{k=1}^{n}k^2\right)-4\left(\sum_{k=1}^{n}k\right)+n\right]$$

$$=4\cdot\frac{1}{6}n(n+1)(2n+1)-4\cdot\frac{1}{2}n(n+1)+n$$

$$=\frac{n}{3}[2(n+1)(2n+1)-6(n+1)+3]=\frac{n}{3}(4n^2-1)$$

Check Point 3

1. The sum $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ to 16 terms is
   (a) 246 (b) 646 (c) 446 (d) 746
2. The sum $1+3+7+15+31+\ldots$ to n terms is
   (a) 2^n-2-n (b) $2^{n-1}-1-n$
   (c) $2^{n+1}-2-n$ (d) None of these
3. The sum of series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\infty$ is
   (a) $\frac{7}{16}$ (b) $\frac{5}{16}$ (c) $\frac{105}{64}$ (d) $\frac{35}{16}$
4. The sum of the series $3\cdot 6+4\cdot 7+5\cdot 8+\ldots$ upto $(n-2)$ terms is
   (a) n^3+n^2+n+2 (b) $\frac{1}{6}(2n^3+12n^2+10n-84)$
   (c) n^3+n^2+n (d) None of these
5. The sum to n terms of the series, whose nth term is (n^2+2^n), is
   (a) $\frac{n(n+1)(2n+1)+2(2^n-1)}{3}$ (b) $\frac{n(n+1)(2n+1)+2(2^n-1)}{6}$
   (c) $\frac{n(n+1)(2n+1)+(2^n-1)}{6}$ (d) None of these
6. The sum of n terms of the series $\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots$ to n terms is
   (a) $\frac{n}{2(3n+2)}$ (b) $\frac{n}{2(3n-2)}$
   (c) $\frac{n}{3(3n+4)}$ (d) None of these
7. For the series $S=1+\frac{1}{(1+3)}(1+2)^2+\frac{1}{(1+3+5)}(1+2+3)^2+\frac{1}{(1+3+5+7)}(1+2+3+4)^2+\ldots$ the 7th term is
   (a) 16 (b) 17
   (c) 15 (d) −16
8. If $b_i=1-a_i, n_a=\sum_{i=1}^{n}a_i, nb=\sum_{i=1}^{n}b_i$, then $\sum_{i=1}^{n}a_ib_i+\sum_{i=1}^{n}(a_i-a)^2$ is equal to
   (a) ab (b) −nab
   (c) (n + 1)ab (d) nab

WORKED OUT Examples

Example 1 *The sum of all two digit numbers which when divided by 4, yield as remainder is* [**NCERT**]

(*a*) 1211 (*b*) 1210
(*c*) 1200 (*d*) *None of these*

Solution (*b*) The two digit numbers which when divided by 4, leave 1 as remainder will be the number which are one more than the multiple of 4 *i.e.*,

$$13, 17, 21, \ldots, 97$$

Clearly, they form an AP, where

$$a = 13, d = 4, T_n = 97$$

$$\because \quad T_n = a + (n-1)d$$

$$\therefore \quad 97 = 13 + (n-1)4$$

$$\Rightarrow \quad 97 - 13 = (n-1)4$$

$$\Rightarrow \quad 84 = (n-1)4$$

$$\Rightarrow \quad 21 = n - 1$$

$$\Rightarrow \quad n = 21 + 1 = 22$$

Now, $$S_{22} = \frac{22}{2}[2 \times 13 + (22-1)4]$$

$$= 11(26 + 84)$$

$$= 11 \times 110 = 1210$$

Example 2 *If the sum of three numbers in AP is 24 and their product is 440, then numbers are* [**NCERT**]

(*a*) 4, 7, 10 (*b*) 5, 8, 11
(*c*) 6, 9, 12 (*d*) *None of these*

Solution (*b*) Let the three numbers are $a - d, a, a + d$.

Given, sum of the numbers = 24

$$\therefore \quad a - d + a + a + d = 24$$

$$\Rightarrow \quad 3a = 24 \quad \Rightarrow \quad a = 8$$

And product of the numbers = 440

$$\therefore \quad (a-d)a(a+d) = 440$$

$$\Rightarrow \quad a(a^2 - d^2) = 440$$

$$\Rightarrow \quad 8(64 - d^2) = 440$$

$$\Rightarrow \quad 64 - d^2 = 55$$

$$\Rightarrow \quad d^2 = 64 - 55 = 9 \quad \Rightarrow \quad d = \pm 3$$

When, $a = 8$ and $d = 3$, then numbers are

$$a - d = 8 - 3 = 5$$

and $$a = 8$$

and $$a + d = 8 + 3 = 11$$

$$\Rightarrow \quad 5, 8, 11$$

When $a = 8$ and $d = -3$, then numbers are

$$a - d = 8 + 3 = 11$$

$$a = 8$$

$$a + d = 8 - 3 = 5$$

$$\Rightarrow \quad 11, 8, 5$$

Hence, the numbers are 5, 8, 11 or 11, 8, 5.

Example 3 *Let the sum of $n, 2n, 3n$ terms of an AP be S_1, S_2 and S_3 respectively, then $3(S_2 - S_1)$ is* [**NCERT**]

(*a*) $2S_3$ (*b*) S_3
(*c*) $4S_3$ (*d*) *None of these*

Solution (*b*) Let a and d be the first term and common difference respectively of an AP.

Given, $$S_1 = \text{Sum of } n \text{ terms} = \frac{n}{2}[2a + (n-1)d] \quad \ldots(i)$$

$$S_2 = \text{Sum of } 2n \text{ terms} = \frac{2n}{2}[2a + (2n-1)d] \quad \ldots(ii)$$

and $$S_3 = \text{Sum of } 3n \text{ terms} = \frac{3n}{2}[2a + (3n-1)d] \quad \ldots(iii)$$

$$\therefore \quad 3(S_2 - S_1)$$

$$= 3\left[\frac{2n}{2}\{2a + (2n-1)d\} - \frac{n}{2}\{2a + (n-1)d\}\right]$$

[from Eqs. (i) and (ii)]

$$= \frac{3n}{2}[2(2a + (2n-1)d\} - \{2a + (n-1)d\}]$$

$$= \frac{3n}{2}[4a + 2(2n-1)d - 2a - (n-1)d]$$

$$= \frac{3n}{2}[(4a - 2a) + d(4n - 2 - n + 1)]$$

$$= \frac{3n}{2}[2a + (3n-1)d] = S_3 \quad \text{[from Eq. (iii)]}$$

Example 4 *The sum of the first four terms of an AP is 56. The sum of last four terms is 112. If its first term is 11, then the number of terms are* [**NCERT**]

(*a*) 10 (*b*) 11
(*c*) 12 (*d*) 13

Solution (*b*) Let AP series is $a, a+d, a+2d, \ldots$

Given, sum of first four terms = 56

i.e., $T_1 + T_2 + T_3 + T_4 = 56$

$\Rightarrow \quad a + a + d + a + 2d + a + 3d = 56$

$\Rightarrow \quad 4a + 6d = 56$

$\Rightarrow \quad 4 \times 11 + 6d = 56 \qquad (\because a = 11)$

$\Rightarrow \quad 6d = 56 - 44 = 12$

$\Rightarrow \quad d = \frac{12}{6} \Rightarrow d = 2$

If last term is T_n, then sum of last four terms = 112

$T_n + T_{n-1} + T_{n-2} + T_{n-3} = 112 \qquad [\because T_n = a + (n-1)d]$

$\therefore \quad a + (n-1)d + a + (n-1-1)d + a + (n-2-1)d + a + (n-3-1)d = 112$

$\Rightarrow \quad 4a + d(n-1+n-2+n-3+n-4) = 112$

$\Rightarrow \quad 4 \times 11 + 2(4n - 10) = 112 \qquad (\because a = 11, d = 2)$

$\Rightarrow \quad 44 + 8n - 20 = 112$

$\Rightarrow \quad 8n = 112 - 44 + 20 = 132 - 44 = 88$

$\Rightarrow \quad n = \frac{88}{8} = 11$

Hence, total numbers of terms in the series = 11

Example 5 *Shamshad Ali buys a scooter for ₹ 22000. He pays ₹ 4000 each and agrees to pay the balance in annual instalments of ₹ 1000 plus 10% interest on the unpaid amount. How much will scooter cost him?* **[NCERT]**

(*a*) ₹ 39150 (*b*) ₹ 39100
(*c*) ₹ 39200 (*d*) *None of these*

Solution (b) Scooter cost = ₹ 22000

Down payment = ₹ 4000, Balance payment = ₹ 18000

Now, interest on Ist instalment $= \frac{18000 \times 10 \times 1}{100} =$ ₹ 1800

$\left(\because I = \frac{P \times R \times T}{100}\right)$

Unpaid amount = 18000 − 1000 = 17000

Interest on IInd instalment $= \frac{17000 \times 10 \times 1}{100} =$ ₹ 1700

Unpaid amount = 17000 − 1000 = 16000

Interest on IInd instalment $= \frac{16000 \times 10 \times 1}{100} =$ ₹ 1600

.....

.....

∴ Total interest paid by him = 1800 + 1700 + 1600 + … + 18 terms which is an AP with $a = 1800, d = 1700 - 1800 = -100$

Therefore, total interest $= \frac{18}{2}[2 \times 1800 + (18-1)(-100)]$

$[\because \text{sum of AP}, = S_n = \frac{n}{2}\{2a + (n-1)d\}]$

$= 9(3600 - 1700) = 9 \times 1900 = 17100$

Hence, total amount or actual cost = 22000 + 17100
= ₹ 39100

Example 6 *If $a_1, a_2, a_3, \ldots, a_{2n}$ are in AP, then the value of $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$ is equal to*

(*a*) $\frac{n}{2n-1}(a_1^2 - a_{2n}^2)$ (*b*) $\frac{n}{2n+1}(a_1^2 - a_{2n}^2)$

(*c*) $\frac{n}{2n-1}(a_1^2 + a_{2n}^2)$ (*d*) *None of these*

Solution (a) $\because a_1, a_2, a_3, \ldots, a_{2n}$ are in AP.

$\therefore \quad a_2 - a_1 = a_3 - a_2 = \ldots = d$

$\therefore \quad a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$

$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \ldots + (a_{2n-1} - a_{2n})(a_{2n-1} + a_n)$

$= (-d)(a_1 + a_2) + (-d)(a_3 + a_4) + \ldots + (-d)(a_{2n-1} + a_{2n})$

$= (-d)(a_1 + a_2 + \ldots + a_{2n}) = (-d)\left[\frac{2n}{2}(a_1 + a_{2n})\right]$

$= -\left(\frac{a_{2n} - a_1}{2n-1}\right)[n(a_1 + a_{2n})] \qquad \left(\because d = \frac{a_{2n} - a_1}{2n-1}\right)$

$= \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$

Example 7 *If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in AP, then their common difference will be*

(*a*) ± 1 (*b*) ± 2 (*c*) ± 3 (*d*) ± 4

Solution (*c*) Since, the given equation is cubic, therefore we take three roots.

Let the roots be $a - d, a, a + d$.

Sum of three numbers in AP = $3a = 12$

$\Rightarrow \quad a = 4$ is a root.

∴ The given equation $x^3 - 12x^2 + 39x - 28 = 0$ can be rewritten as

$(x-4)(x^2 - 8x + 7) = 0$

$\therefore \quad x = 1, 4, 7$ or $7, 4, 1$

$\therefore \quad d = \pm 3$

Example 8 *If α and β are roots of the equation $x^2 - 3x + a = 0$ and γ and δ are roots of the equation $x^2 - 12x + b = 0$ and α, β, γ, δ form an increasing GP, then the values of a and b are respectively*

(*a*) 2, 16 (*b*) 4, 8
(*c*) 2, 32 (*d*) *None of these*

Solution (*c*) $\because \alpha, \beta, \gamma, \delta$ are in GP.

Let $\alpha = A, \beta = Ar, \gamma = Ar^2, \delta = Ar^3$

$\because$ α and β are the roots of the equation $x^2 - 3x + a = 0$, then

$\alpha + \beta = 3 \Rightarrow A(1 + r) = 3 \qquad \ldots$ (i)

γ and δ are the roots of the equation $x^2 - 12x + b = 0$, then

$$\gamma + \delta = 12$$

$$\Rightarrow \quad Ar^2(1 + r) = 12 \quad \ldots \text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$A = 1, r = 2$$

$$\Rightarrow \quad \alpha = 1, \beta = 2, \gamma = 4, \delta = 8$$

$$\therefore \quad a = \alpha\beta = 1 \times 2 = 2$$

and $\quad b = \gamma\delta = 4 \times 8 = 32$

Example 9 *The sum of n terms of the series* $1.4 + 3.04 + 5.004 + 7.0004 + \ldots$ *is*

(a) $n^2 + \frac{4}{9}\left(1 + \frac{1}{10^n}\right)$ (b) $n^2 + \frac{4}{9}\left(1 - \frac{1}{10^n}\right)$

(c) $n + \frac{4}{9}\left(1 - \frac{1}{10^n}\right)$ (d) *None of these*

Solution (b) $1.4 + 3.04 + 5.004 + \ldots$

$$= (1 + 3 + 5 + \ldots) + (0.4 + 0.04 + 0.004 + \ldots)$$

$$= \frac{n}{2}[2 + (n-1) \cdot 2] + \left[\frac{4}{10} + \frac{4}{10^2} + \ldots \text{upto } n \text{ terms}\right]$$

$$= n^2 + \frac{4}{10}\left[\frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}}\right]$$

$$= n^2 + \frac{4}{9}\left(1 - \frac{1}{10^n}\right)$$

Example 10 *If the ratio of HM and GM of two quantities is* $12 : 13$, *then the ratio of the number is*

(a) $1 : 2$ (b) $2 : 3$

(c) $3 : 4$ (d) *None of these*

Solution (d) Given, $\frac{\text{HM}}{\text{GM}} = \frac{12}{13} \Rightarrow \frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{12}{13}$

or $$\frac{a+b}{2\sqrt{ab}} = \frac{13}{12}$$

$$\Rightarrow \quad \frac{(a+b) + 2\sqrt{ab}}{(a+b) - 2\sqrt{ab}} = \frac{13+12}{13-12} = \frac{25}{1}$$

$$\Rightarrow \quad \frac{(\sqrt{a} + \sqrt{b})^2}{(\sqrt{a} - \sqrt{b})^2} = \frac{5^2}{1} \Rightarrow \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{5}{1}$$

$$\Rightarrow \quad \frac{(\sqrt{a} + \sqrt{b}) + (\sqrt{a} - \sqrt{b})}{(\sqrt{a} + \sqrt{b}) - (\sqrt{a} - \sqrt{b})} = \frac{5+1}{5-1}$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{6}{4}$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{1/2} = \frac{6}{4}$$

$$\Rightarrow \quad a : b = 9 : 4$$

Example 11 *The sum of three numbers in GP is 56. If we subtract the 1, 7, 21 from these numbers in that order, we obtain an Arithmetic Progression (AP). Find the numbers.* **[NCERT]**

(a) 32, 16, 8 (b) 30, 14, 6

(c) 33, 17, 9 (d) *None of these*

Solution (a) Let three numbers in GP are a, ar, ar^2.

Given, $$a + ar + ar^2 = 56 \quad \ldots\text{(i)}$$

Again, $a - 1, ar - 7, ar^2 - 21$ are in AP.

$$\Rightarrow \quad 2(ar - 7) = (a - 1) + (ar^2 - 21)$$

($\because$ if a, b, c are in AP, then $2b = a + c$)

$$\Rightarrow \quad 2ar - 14 = a + ar^2 - 22$$

$$\Rightarrow \quad a + ar^2 - 2ar = 8 \quad \ldots\text{(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{a + ar + ar^2}{a + ar^2 - 2ar} = \frac{56}{8}$$

$$\Rightarrow \quad \frac{1 + r + r^2}{1 + r^2 - 2r} = \frac{7}{1}$$

$$\Rightarrow \quad 1 + r + r^2 = 7 + 7r^2 - 14r$$

$$\Rightarrow \quad 6r^2 - 15r + 6 = 0 \Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow \quad (r - 2)(2r - 1) = 0 \Rightarrow r = 2, \frac{1}{2}$$

If $r = 2$, then from Eq.(i),

$$a + 2a + 4a = 56 \Rightarrow a = 8$$

Then, numbers are

$$a = 8, ar = 8 \times 2 = 16,$$

or $$ar^2 = 8 \times 4 = 32$$

Hence, numbers are 8, 16, 32.

If $r = \frac{1}{2}$, then from Eq. (i),

$$\frac{a}{1} + \frac{a}{2} + \frac{a}{4} = 56$$

$$\Rightarrow \quad \frac{7a}{4} = 56 \Rightarrow a = 32$$

Then, numbers are

$$a = 32,$$

$$ar = 32 \times \frac{1}{2} = 16 \text{ or } ar^2 = 32 \times \frac{1}{4} = 8$$

So, numbers are 32, 16, 8.

Hence, required numbers are 8, 16, 32 or 32, 16, 8.

Example 12 *If a be the AM between b and c and GM's are* G_1 *and* G_2, *then* $G_1^3 + G_2^3$ *is equal to*

(a) abc (b) $2abc$

(c) $3abc$ (d) $4abc$

Solution (b) $\because b, a, c$ are in AP.

$$\therefore \quad 2a = b + c \quad \ldots\text{(i)}$$

and b, G_1, G_2, c are in GP.

$\therefore \quad G_1 = br, \quad G_2 = br^2$ and $c = br^3$,

where r be the common ratio of GP.

Now, $G_1^3 + G_2^3 = (br)^3 + (br^2)^3$

$$= b^3r^3 + b^3r^6 = b^3\left(\frac{c}{b}\right) + b^3\left(\frac{c}{b}\right)^2$$

$$= b^2c + bc^2 = bc(b + c) = 2abc$$

$\therefore \quad G_1^3 + G_2^3 = 2abc$

Example 13 *A person writes a letter to four of his friends. He asks each one of them, to copy the letter and mail to four different persons with instructions that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. When the 8th set of letter is mailed, then the amount on postage will be* [**NCERT**]

(a) ₹ 42780 (b) ₹ 43690

(c) ₹ 43680 (d) *None of these*

Solution (b) Firstly, one person post four letters to four of his friends. Then, those four persons post 4-4 letters *i.e.*, a total $4 \times 4 = 16$ letters.Similarly, in next step $4 \times 4 \times 4 = 64$ letters are posted.

Thus, there will be a GP series

i.e., 4, 16, 64, 256, ...

where, $a = 4$ and $r = 4$

Now, total number of letters till 8th set

$$S_n = \frac{a(r^n - 1)}{r - 1} \Rightarrow S_8 = \frac{4(4^8 - 1)}{4 - 1} \Rightarrow S_8 = \frac{4}{3}(4^8 - 1)$$

$$\left[\because S_n = \frac{a(r^n - 1)}{(r - 1)}, r > 1\right]$$

Cost for one letter = ₹ 0.50

Hence, total cost $= \frac{4}{3}(4^8 - 1) \times 0.50 = \frac{4}{3} \times (65536 - 1) \times 0.50$

$$= \frac{4}{3} \times 65535 \times 0.50 = ₹\ 43690$$

Example 14 *If a, b, c are in HP, b, c, d are in GP and c, d, e are in AP, then $\frac{ab^2}{(2a - b)^2}$ is equal to*

(a) b (b) a

(c) e (d) d

Solution (c) $\because a, b, c$ are in HP.

$$\therefore \quad b = \frac{2ac}{a + c} \quad \text{...(i)}$$

b, c, d are in GP.

$$\therefore \quad c^2 = bd \quad \text{...(ii)}$$

and c, d, e are in AP.

$$\therefore \quad d = \frac{c + e}{2} \quad \text{...(iii)}$$

From Eq. (i), $ab + bc = 2ac$

$$\Rightarrow \quad c = \frac{ab}{2a - b} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$d = \frac{1}{2}\left(\frac{ab}{2a - b} + e\right) \quad \text{...(v)}$$

On putting the value of c and d from Eqs. (iv) and (v) in Eq. (ii), we get

$$\frac{a^2b^2}{(2a - b)^2} = \frac{b}{2}\left[\frac{ab}{2a - b} + e\right]$$

$$\therefore \quad \frac{2a^2b}{(2a - b)^2} - \frac{ab}{2a - b} = e$$

$$\Rightarrow \quad e = \frac{2a^2b - 2a^2b + ab^2}{(2a - b)^2}$$

Hence, $e = \frac{ab^2}{(2a - b)^2}$

Example 15 *The nth term of the series $1 + 2 + 5 + 12 + 25 + \ldots$ is*

(a) $(n - 1)(n - 2)$ (b) $\frac{1}{3}n(n - 1)(n - 2) + n$

(c) n (d) *None of these*

Solution (b) Let nth term of the series is T_n and sum is S, then

$$S = 1 + 2 + 5 + 12 + 25 + 46 + 77 + \ldots + T_n$$

$$S = \quad 1 + 2 + 5 + 12 + 25 + 46 + \ldots + T_{n-1} + T_n$$

On subtraction, we get

$$0 = 1 + 1 + 3 + 7 + 13 + 21 + 31 + \ldots + (T_n - T_{n-1}) - T_n$$

$\therefore \quad T_n = 1 + \{1 + 3 + 7 + 13 + 21 + \ldots + \text{upto } (n - 1)\}$

Let $(n - 1)$th term and S of the series $1 + 3 + 7 + 13 + \ldots$ are t_{n-1} and S' respectively, then

$$S' = 1 + 3 + 7 + 13 + 21 + \ldots + t_{n-1}$$

$$S' = \quad 1 + 3 + 7 + 13 + \ldots + t_{n-2} + t_{n-1}$$

On subtraction, we get

$$0 = 1 + 2 + 4 + 6 + 8 + \ldots + (t_{n-1} - t_{n-2}) - t_{n-1}$$

$\therefore \quad t_{n-1} = 1 + 2\{1 + 2 + 3 + 4 + \ldots \text{upto } (n - 2)\}$

$$= 1 + 2 \cdot \frac{1}{2}(n - 2)(n - 1) = n^2 - 3n + 3$$

$\Rightarrow \quad t_n = (n + 1)^2 - 3(n + 1) + 3$

$$= n^2 - n + 1$$

$\therefore \quad T_n = 1 + \{1 + 3 + 7 + \ldots \text{upto } (n - 1)\}$

$$= 1 + \sum_1^{n-1}(n^2 - n + 1) = 1 + \sum_1^{n-1} n^2 - \sum_1^{n-1} n + \sum_1^{n-1} 1$$

$$= 1 + \frac{1}{6}n(n - 1)(2n - 1) - \frac{1}{2}n(n - 1) + (n - 1)$$

$$= \frac{1}{3}n(n - 1)(n - 2) + n$$

Hence, $T_n = \frac{1}{3}n(n - 1)(n - 2) + n$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Arithmetic Progression

1. $\{a_n\}$ and $\{b_n\}$ be two sequences given by $a_n = (x)^{\frac{1}{2^n}} + (y)^{\frac{1}{2^n}}$ and $b_n = (x)^{\frac{1}{2^n}} - (y)^{\frac{1}{2^n}}$ for all $n \in N$, then $a_1 a_2 a_3 \ldots a_n$ is equal to

(a) $x - y$ (b) $\frac{x+y}{b_n}$

(c) $\frac{x-y}{b_n}$ (d) $\frac{xy}{b_n}$

2. If the sum of n terms of an AP is given by $S_n = 3n + 2n^2$, then the common difference of the AP is [NCERT Exemplar]

(a) 3 (b) 2 (c) 6 (d) 4

3. If 9 times the 9th term of an AP is equal to 13 times the 13th term, then the 22nd term of the AP is [NCERT Exemplar]

(a) 0 (b) 22 (c) 220 (d) 198

4. In an AP, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. The 20th term is [NCERT]

(a) 112 (b) –112 (c) 114 (d) –114

5. If a, b, c, d, e, f are in AP, then the value of $e - c$ will be

(a) $2(c - a)$ (b) $2(f - d)$

(c) $2(d - c)$ (d) $d - c$

6. Let S_n denote the sum of the first n terms of an AP. If $S_{2n} = 3S_n$, then $S_{3n} : S_n$ is equal to [NCERT Exemplar]

(a) 4 (b) 6 (c) 8 (d) 10

7. If the angles of a quadrilateral are in AP, whose common difference is 10°, then the angles of the quadrilateral are

(a) 65°, 85°, 95°, 105° (b) 75°, 85°, 95°, 105°

(c) 65°, 75°, 85°, 95° (d) 65°, 95°, 105°, 115°

8. The interior angles of a polygon are in AP. If the smallest angle be 120° and the common difference be 5, then the number of side is

(a) 8 (b) 10 (c) 9 (d) 6

9. In an AP, if the pth term is $\frac{1}{q}$ and qth term is $\frac{1}{p}$. Then, the sum of first pq terms is [NCERT]

(a) $(pq + 1)$ (b) $\frac{1}{2}(pq + 1)$

(c) $\frac{1}{2}(pq - 1)$ (d) None of these

10. If the sum of a certain n number of terms of the AP, 25, 22, 19, … is 116, then the last term is [NCERT]

(a) 4 (b) 3 (c) 2 (d) –4

11. If $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ be consecutive terms of an AP, then $(b-c)^2, (c-a)^2, (a-b)^2$ will be in

(a) GP (b) AP

(c) HP (d) None of these

12. If $\log 2, \log(2^n - 1)$ and $\log(2^n + 3)$ are in AP, then n is equal to

(a) $\frac{5}{2}$ (b) $\log_2 5$ (c) $\log_3 5$ (d) $\frac{3}{2}$

13. If the arithmetic mean of a and b is $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$, then the value of n is [NCERT]

(a) –1 (b) 0

(c) 1 (d) None of these

14. Sum of the first p, q and r terms of an AP are a, b and c respectively. Then,

$$\frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) \text{ is equal to}$$ [NCERT]

(a) pq (b) 0

(c) rp (d) 3

15. If the ratio of the sum of n terms of two AP's be $(7n+1):(4n+27)$, then the ratio of their 11th terms will be
(a) 2 : 3 (b) 3 : 4 (c) 4 : 3 (d) 5 : 6

16. Given that n AM's are inserted between two sets of numbers $a, 2b$ and $2a, b$ where $a, b \in R$. Suppose further that mth mean between these sets of numbers is same, then the ratio $a:b$ is equal to
(a) $(n-m+1):m$ (b) $(n-m+1):n$
(c) $n:(n-m+1)$ (d) $m:(n-m+1)$

17. A man saved ₹ 66000 in 20 yr. In each succeeding year after the first year he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year? [NCERT Exemplar]
(a) ₹ 1450 (b) ₹ 1400
(c) ₹ 1470 (d) ₹ 1480

18. A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job? [NCERT Exemplar]
(a) 11 (b) 10 (c) 12 (d) 14

19. The sum of n terms of three AP's whose first term is 1 and common differences are 1, 2, 3 respectively are S_1, S_2, S_3 respectively. The true relation is
(a) $S_1 + S_3 = S_2$ (b) $S_1 + S_3 = 2S_2$
(c) $S_1 + S_2 = 2S_3$ (d) $S_1 + S_2 = S_3$

20. If the first term of an AP is a and the sum of the first p terms is zero, then the sum of its next q terms is [NCERT Exemplar]
(a) $\frac{a(p+q)q}{p-1}$ (b) $\frac{-a(p+q)q}{p-1}$
(c) $\frac{a(p-q)q}{p-1}$ (d) None of these

21. In a cricket tournament 16 school terms participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last placed team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, amount will the first place team received is [NCERT Exemplar]
(a) ₹ 720 (b) ₹ 725 (c) ₹ 735 (d) ₹ 780

22. If a, b, c are in AP, then the straight line $ax + by + c = 0$ will always pass through the point
(a) $(-1, -2)$ (b) $(1, -2)$ (c) $(-1, 2)$ (d) $(1, 2)$

23. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP, whose common difference is d, then
$\sin d(\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n)$
is equal to [NCERT Exemplar]
(a) $\tan\theta_n - \tan\theta_2$ (b) $\tan\theta_n + \tan\theta_1$
(c) $\tan\theta_n - \tan\theta_1$ (d) None of these

24. If $a_1, a_2, a_3, \ldots, a_{24}$ are in arithmetic progression and $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \ldots + a_{23} + a_{24}$ is equal to
(a) 909 (b) 75
(c) 750 (d) 900

25. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, then value of $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{a_{n-1}+\sqrt{a_n}}$ is equal to [NCERT Exemplar]
(a) $\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ (b) $\frac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$
(c) $\frac{n-1}{\sqrt{a_1}-\sqrt{a_n}}$ (d) $\frac{n+1}{\sqrt{a_1}-\sqrt{a_n}}$

26. Let the sequence, $a_1, a_2, a_3, .., a_{2n}$, form an AP, then $a_1^2 - a_2^2 + a_3^2 - \ldots + a_{2n-1}^2 - a_{2n}^2$ is equal to
(a) $\frac{n}{2n-1}(a_1^2 - a_{2n}^2)$ (b) $\frac{2n}{n-1}(a_{2n}^2 - a_1^2)$
(c) $\frac{n}{n+1}(a_1^2 + a_{2n}^2)$ (d) None of these

27. If $a_1, a_2, a_3, \ldots, a_{4001}$ are terms of an AP such that $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{4000}a_{4001}} = 10$ and $a_2 + a_{4000} = 50$, then $|a_1 - a_{4001}|$ is equal to
(a) 20 (b) 30
(c) 40 (d) None of these

28. If $a_1, a_2, \ldots, a_n$ are in AP with common difference d, then the sum of the series
$\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$ is
(a) $\sec a_1 - \sec a_n$ (b) $\cot a_1 - \cot a_n$
(c) $\tan a_1 - \tan a_n$ (d) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$

29. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n, the area of S_n less than 1 sq cm ?
(a) 7 (b) 6
(c) 9 (d) None of these

30. If a, b, c are in AP, then $10^{ax+10}, 10^{bx+10}, 10^{cx+10}$ $(x \neq 0)$ are in
(a) AP (b) GP only when $x > 0$
(c) GP for all x (d) GP only when $x < 0$

31. The solution of the equation
$(x+1) + (x+4) + (x+7) + \ldots + (x+28) = 155$ is
(a) 1 (b) 2
(c) 3 (d) 4

32. Consider the following statements

I. If a_n denotes the nth term of an AP, then

$$a_n = \frac{a_{n+k} + a_{n-k}}{2}$$

II. In an AP, if the sum of m terms is equal to the sum of n terms, then the sum of $(m + n)$ terms is always zero.

Which of the statement is given above is/are correct ?

(a) Only I (b) Only II
(c) Both I and II (d) None of these

33. The sum of all two digit numbers which, when divided by 4, yield unity as a remainder is

(a) 1190 (b) 1197
(c) 1210 (d) None of these

34. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(a) 3000 (b) 3050
(c) 4050 (d) None of these

35. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is

(a) 2489 (b) 4735 (c) 2317 (d) 2632

36. If a^2, b^2, c^2 are in AP, then which of the following is also in AP ?

(a) $\sin A, \sin B, \sin C$ (b) $\tan A, \tan B, \tan C$
(c) $\cot A, \cot B, \cot C$ (d) None of these

37. Insert three arithmetic means between 3 and 19

(a) 6, 10, 14 (b) 7, 11, 15
(c) 8, 12, 16 (d) None of these

Geometric Progression

38. If a, b, c are in GP and $\log a - \log 2b$, $\log 2b - \log 3c$ and $\log 3c - \log a$ are in AP, then a, b, c are the length of the sides of a triangle which is

(a) acute angled (b) obtuse angled
(c) right angled (d) equilateral

39. If the 4th, 10th and 16th terms of a GP are x, y and z respectively, then x, y, z are in [NCERT]

(a) AP (b) GP
(c) AGP (d) HP

40. The sum of n terms of the sequence 8, 88, 888, 8888,... is [NCERT]

(a) $\frac{80}{81}(10^n - 1) - \frac{8n}{9}$ (b) $\frac{80}{81}(10^n + 1) + \frac{8n}{9}$
(c) $\frac{80}{81}(10^n - 1) - \frac{8n^2}{9}$ (d) None of these

41. Find four numbers forming a GP in which the third term is greater than the first term by 9 and the second term is greater than 4th by 18. [NCERT]

(a) 2, –6, 12, –24 (b) 3, 6, 12, 24
(c) 3, –6, 12, –24 (d) None of these

42. The number which should be added to the numbers 2, 14, 62, so that the resulting numbers may be in GP, is

(a) 1 (b) 2
(c) 3 (d) 4

43. Let a_n be nth term of the GP of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$
(c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

44. If the pth, qth, rth terms of a GP are a, b and c respectively, then $a^{q-r}b^{r-p}c^{p-q}$ is equal to [NCERT]

(a) 0 (b) 2 (c) 1 (d) 4

45. If a, b, c and d are in GP, then $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$ is equal to [NCERT]

(a) $(ab + bc + cd)^2$ (b) $(ab - bc - cd)^2$
(c) $(ab + bc + cd)$ (d) None of these

46. The third term of GP is 4. The product of its first 5 terms is [NCERT Exemplar]

(a) 4^3 (b) 4^4
(c) 4^5 (d) None of these

47. If $x, 2y$ and $3z$ are in AP, where the distinct numbers x, y, z are in GP, then the common ratio of the GP is [NCERT Exemplar]

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

48. The length of three unequal edges of a rectangular solid block are in GP. The volume of the block is 216 cm^3 and the total surface area is 252 cm^2. The length of the longest edge is [NCERT Exemplar]

(a) 12 cm (b) 6 cm (c) 18 cm (d) 3 cm

49. If a, b, c and d are in GP, then $a^n + b^n, b^n + c^n, c^n + d^n$ are in [NCERT]

(a) AP (b) GP
(c) HP (d) None of these

50. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$ is equal to [NCERT Exemplar]

(a) $2A$ (b) A
(c) $3A$ (d) None of these

51. If the pth and qth terms of a GP are q and p, respectively, then $(p-q)$th term is [NCERT Exemplar]

(a) $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$ (b) $\left(\frac{q^q}{p^p}\right)^{\frac{1}{p-q}}$

(c) $\left(\frac{p^q}{q^p}\right)^{\frac{1}{p-q}}$ (d) None of these

52. If a, b, c, d and p are different real numbers such that $(a^2+b^2+c^2)\ p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$, then a, b, c, d are in

(a) AP (b) GP

(c) HP (d) $ab = cd$

53. $(\underset{n \text{ digits}}{666\ldots6})^2 + (\underset{n \text{ digits}}{888\ldots8})$ is equal to

(a) $\frac{4}{9}(10^n - 1)$ (b) $\frac{4}{9}(10^{2n} - 1)$

(c) $\frac{4}{9}(10^n - 1)^2$ (d) None of these

54. The sum of 100 terms of the series $0.9 + 0.09 + 0.009 \ldots$ will be

(a) $1-\left(\frac{1}{10}\right)^{100}$ (b) $1+\left(\frac{1}{10}\right)^{100}$

(c) $1-\left(\frac{1}{10}\right)^{106}$ (d) $1+\left(\frac{1}{10}\right)^{10}$

55. The value of $0.2\dot{3}\dot{4}$ is

(a) $\frac{232}{990}$ (b) $\frac{232}{9990}$

(c) $\frac{232}{900}$ (d) $\frac{232}{9909}$

56. 0.14189189189 … can be expressed as a rational number

(a) $\frac{7}{3700}$ (b) $\frac{7}{50}$

(c) $\frac{525}{111}$ (d) $\frac{21}{148}$

57. The sum of the geometric progression 0.15, 0.015, 0.0015, … 20 terms is [NCERT]

(a) $\frac{1}{6}[1-(0.1)^{20}]$ (b) $[1-(0.1)^{20}]$

(c) $\frac{1}{6}[1+(0.1)^{20}]$ (d) None of these

58. If $x > 1, y > 1, z > 1$ are in GP, then $\frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in

(a) AP (b) HP

(c) GP (d) None of these

59. If a, b, c be in GP, then $\log a^n, \log b^n, \log c^n$ will be

(a) AP (b) GP

(c) HP (d) None of these

60. If $1+\lambda+\lambda^2+\ldots+\lambda^n = (1+\lambda)(1+\lambda^2)(1+\lambda^4)(1+\lambda^8)(1+\lambda^{16})$, then the value of n is (where $n \in N$)

(a) 32 (b) 16

(c) 31 (d) 15

61. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then $x, 4, z$ will be in

(a) AP (b) GP

(c) HP (d) None of these

62. In the four numbers first three are in GP and last three are in AP whose common difference is 6. If the first and last numbers are same, then first number will be

(a) 2 (b) 4

(c) 6 (d) 8

63. If $a^{1/x} = b^{1/y} = c^{1/z}$ and a, b, c are in GP, then x, y, z will be in

(a) AP (b) GP

(c) HP (d) None of these

64. In a GP the sum of three numbers is 14, if 1 is added to first two numbers and subtracted from third number the series becomes AP, then the greatest number is

(a) 8 (b) 4

(c) 24 (d) 16

65. Which of the following statement is correct ?

(a) If each term of an AP a number is added or subtracted, then the series so obtained is also an AP.

(b) The nth term of geometric series whose first term is a and common ratio r, is ar^{n-1}.

(c) If each term of a GP be raised to the same power the resulting terms are in GP.

(d) All of the above

66. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP. If $a < b < c$ and $a+b+c = \frac{3}{2}$, then the value of a is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$

(c) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

67. If the AM and GM of roots of a quadratic equations are 8 and 5 respectively, then the quadratic equation will be

(a) $x^2 - 16x - 25 = 0$

(b) $x^2 - 8x + 5 = 0$

(c) $x^2 - 16x + 25 = 0$

(d) $x^2 + 16x - 25 = 0$

68. Three numbers form a GP. If the 3rd term is decreased by 64, then the three numbers thus obtained will constitute an AP. If the second term of this AP is decreased by 8, a GP will be formed again, then the numbers will be
(a) 4, 20, 36 (b) 4, 12, 36
(c) 4, 20, 100 (d) None of these

69. The number 111...1 (91 times) is a/an
(a) even number (b) prime number
(c) not prime (d) None of these

70. The value of the sum $\sum_{r=1}^{n} \sum_{s=1}^{n} S_{rs}\, 2^r 3^s$, where $S_{rs} = 0$, if $r \neq s$ and $S_{rs} = 1$, if $r = s$, is
(a) $\frac{(5^n - 1)}{4}$ (b) $\frac{6}{5}(6^n - 1)$
(c) $\frac{5^n 6^n}{n+1}$ (d) $\frac{5}{4}(5^n - 1)$

71. Let $n(>1)$ be a positive integer, then the largest integer m such that $(n^m + 1)$ divides $(1 + n + n^2 + \ldots + n^{127})$, is
(a) 32 (b) 63 (c) 64 (d) 127

72. The sum of
$$(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + \ldots + (x+1)^{n-1}$$
is equal to
(a) $(x+2)^{n-2} - (x+1)^n$ (b) $(x+2)^{n-1} - (x+1)^{n-1}$
(c) $(x+2)^n - (x+1)^n$ (d) None of these

73. Let two numbers have arithmetic mean 9 and geometric mean 4. Then, these numbers are the roots of the quadratic equation
(a) $x^2 - 18x - 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$ (d) $x^2 + 18x + 16 = 0$

74. If sum of the series $\sum_{n=0}^{\infty} r^n = S$, for $|r| < 1$, then sum of the series $\sum_{n=0}^{\infty} r^{2n}$ is
(a) S^2 (b) $\frac{S^2}{2S+1}$
(c) $\frac{2S}{S^2 - 1}$ (d) $\frac{S^2}{2S-1}$

75. If AM of two numbers is twice of their GM, then the ratio of greatest number to smallest number is
(a) $7 - 4\sqrt{3}$ (b) $7 + 4\sqrt{3}$ (c) 21 (d) 5

76. If n geometric means between a and b be $G_1, G_2, \ldots, G_n$ and a geometric mean be G, then the true relation is
(a) $G_1 G_2 \ldots G_n = G$ (b) $G_1 G_2 \ldots G_n = G^{1/n}$
(c) $G_1 G_2 \ldots G_n = G^n$ (d) $G_1 G_2 \ldots G_n = G^{2/n}$

77. If $1 + \cos\alpha + \cos^2\alpha + \ldots \infty = 2 - \sqrt{2}$, then $\alpha, (0 < \alpha < \pi)$ is
(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

78. An infinite GP has the first term 'x' and sum 5, then x belongs to
(a) $x < -10$ (b) $-10 < x < 0$
(c) $0 < x < 10$ (d) $x > 10$

Harmonic Progression

79. If a, b, c, d are any four consecutive coefficients of any expanded binomial, then $\frac{a+b}{a}, \frac{b+c}{b}, \frac{c+d}{c}$ are in
(a) AP (b) GP
(c) HP (d) None of these

80. If the 7th term of a HP is $\frac{1}{10}$ and the 12th term is $\frac{1}{25}$, then the 20th term is
(a) $\frac{1}{41}$ (b) $\frac{1}{45}$
(c) $\frac{1}{49}$ (d) $\frac{1}{37}$

81. If $\log_a x, \log_b x, \log_c x$ be in HP, then a, b, c are in
(a) AP (b) HP
(c) GP (d) None of these

82. If the altitudes of a triangle are in AP, then the sides of the triangle are in
(a) AP
(b) HP
(c) GP
(d) arithmetico-geometric progression

Relation Among AM, GM and HM

83. The minimum value of $4^x + 4^{1-x}, x \in R$ is
[NCERT Exemplar]
(a) 2 (b) 4 (c) 1 (d) 0

84. The AM, HM and GM between two numbers are $\frac{144}{15}$, 15 and 12, but not necessarily in this order. Then, HM, GM and AM respectively are
(a) 15, 12, $\frac{144}{15}$ (b) $\frac{144}{15}$, 12, 15
(c) 12, 15, $\frac{144}{15}$ (d) $\frac{144}{15}$, 15, 12

85. If a, b and c are in AP and p, p' are the AM and GM respectively between a and b, while q, q' are the AM and GM respectively between b and c, then
(a) $p^2 + q^2 = p^2 + q'^2$ (b) $pq = p'q'$
(c) $p^2 - q^2 = p'^2 - q'^2$ (d) None of these

86. The product of n positive numbers is unity. Their sum is

(a) a positive integer (b) equal to $n + \frac{1}{n}$

(c) divisible by n (d) never less than n

87. If $2p + 3q + 4r = 15$, then the maximum value of $p^3q^5r^7$ is

(a) 2180 (b) $\frac{5^4 \cdot 3^5}{2^{15}}$ (c) $\frac{5^5 \cdot 7^7}{2^{17} \cdot 9}$ (d) 2285

88. If $a + 2b + 3c = 12$, $(a, b, c \in R^+)$, then ab^2c^3 is

(a) $\geq 2^3$ (b) $\geq 2^6$

(c) $\leq 2^6$ (d) None of these

Arithmetico-Geometric Progression

89. nth term of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots$ will be

(a) $\frac{3n+1}{5^{n-1}}$ (b) $\frac{3n-1}{5^n}$ (c) $\frac{3n-2}{5^{n-1}}$ (d) $\frac{3n+2}{5^{n-1}}$

90. If $|x| < 1$, then the sum of the series $1 + 2x + 3x^2 + 4x^3 + \ldots\infty$ will be

(a) $\frac{1}{1-x}$ (b) $\frac{1}{1+x}$ (c) $\frac{1}{(1+x^2)}$ (d) $\frac{1}{(1-x)^2}$

91. The sum of n terms of the infinite series

$$1 \cdot 3^2 + 2 \cdot 5^2 + 3 \cdot 7^2 + \ldots\infty \text{ is}$$

(a) $\frac{n}{6}(n+1)(6n^2 + 14n + 7)$ (b) $\frac{n}{6}(n+1)(2n+1)(3n+1)$

(c) $4n^3 + 4n^2 + n$ (d) None of these

92. $1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} \ldots\infty$ is equal to

(a) 3 (b) 6 (c) 9 (d) 12

93. If T_n denotes the nth term of the series $2 + 3 + 6 + 11 + 18 + \ldots$, then T_{50} is [NCERT Exemplar]

(a) $49^2 - 1$ (b) 49^2 (c) $50^2 + 1$ (d) $49^2 + 2$

94. Sum of n terms of series $12 + 16 + 24 + 40 + \ldots$ will be

(a) $2(2^n - 1) + 8n$ (b) $2(2^n - 1) + 6n$

(c) $3(2^n - 1) + 8n$ (d) $4(2^n - 1) + 8n$

Sum of n Terms of Special Series

95. The sum of the series $1 + 3x + 6x^2 + 10x^3 + \ldots\infty$ will be

(a) $\frac{1}{(1-x)^2}$ (b) $\frac{1}{1-x}$

(c) $\frac{1}{(1+x)^2}$ (d) $\frac{1}{(1-x)^3}$

96. Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers. Then, $\sum_{r=1}^{n} \frac{S_r}{s_r}$ is equal to [NCERT Exemplar]

(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{n^2 + 3n + 2}{2}$ (d) None of these

97. The sum of the series

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots \text{ to } n$$

terms is

(a) $\frac{n(n^2+1)}{n^2+n+1}$ (b) $\frac{n(n+1)}{2(n^2+n+1)}$

(c) $\frac{n(n^2-1)}{2(n^2+n+1)}$ (d) None of these

98. The sum of the series $\frac{1}{3 \times 7} + \frac{1}{7 \times 11} + \frac{1}{11 \times 15} + \ldots$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$ (d) $\frac{1}{12}$

99. The sum of the series $1 \cdot 3 \cdot 5 + 2 \cdot 5 \cdot 8 + 3 \cdot 7 \cdot 11 + \ldots$ upto n terms is

(a) $\frac{n(n+1)(9n^2 + 23n + 13)}{6}$

(b) $\frac{n(n-1)(9n^2 + 23n + 12)}{6}$

(c) $\frac{(n+1)(9n^2 + 23n + 13)}{6}$

(d) $\frac{n(9n^2 + 23n + 13)}{6}$

100. The nth term of the series

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots \text{ will be}$$

(a) $n^2 + 2n + 1$ (b) $\frac{n^2 + 2n + 1}{8}$

(c) $\frac{n^2 + 2n + 1}{4}$ (d) $\frac{n^2 - 2n + 1}{4}$

101. For any odd integer $n \geq 1$,

$$n^3 - (n-1)^3 + \ldots + (-1)^{n-1}1^3 \text{ is equal to}$$

(a) $\frac{1}{2}(n-1)^2(2n-1)$ (b) $\frac{1}{4}(n-1)^2(2n-1)$

(c) $\frac{1}{2}(n+1)^2(2n-1)$ (d) $\frac{1}{4}(n+1)^2(2n-1)$

Round II (Mixed Bag)

Only One Correct Option

1. Number of identical terms in the sequence 2, 5, 8, 11, ... upto 100 terms and 3, 5, 7, 9, 11, ... upto 100 terms, are
(a) 17 (b) 33
(c) 50 (d) 147

2. If positive numbers a^{-1}, b^{-1}, c^{-1} are in AP, then the product of roots of the equation $x^2 - kx + 2b^{101} - a^{101} - c^{101} = 0, (k \in R)$ is
(a) > 0 (b) < 0
(c) $= 0$ (d) None of these

3. If $a_1, a_2, a_3, \ldots, a_{20}$ are AM's between 13 and 67, then the maximum value of $a_1\, a_2\, a_3 \ldots a_{20}$ is equal to
(a) $(20)^{20}$ (b) $(40)^{20}$
(c) $(60)^{20}$ (d) $(80)^{20}$

4. If the set of natural numbers is partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3)$, $S_3 = \{4, 5, 6\}$ and so on. Then, the sum of the terms in S_{50} is
(a) 62525 (b) 25625
(c) 62500 (d) None of these

5. Jairam purchased a house in ₹ 15000 and paid ₹ 5000 at once. Rest money he promised to pay in annual instalment of ₹ 1000 with 10% per annum interest. How much money is to be paid by Jairam ?
(a) ₹ 21555 (b) ₹ 20475
(c) ₹ 20500 (d) ₹ 20700

6. 150 workers were engaged to finish a piece of work in a certain number of days. 4 workers dropped the second day, 4 more workers dropped the third day and so on. It takes eight more days to finish the work now. The number of days in which the work was completed is
(a) 15 (b) 20 (c) 25 (d) 30

7. If the sum of the roots of the equation $ax^2 + bx + c = 0$ be equal to the sum of the reciprocals of their squares, then bc^2, ca^2, ab^2 will be in
(a) AP (b) GP
(c) HP (d) None of these

8. If p, q, r are in AP and are positive, the roots of the quadratic equation $px^2 + qx + r = 0$ are all real for
(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| < 4\sqrt{3}$
(c) All p and r (d) No p and r

9. The sixth term of an AP is equal to 2. The value of the common difference of the AP which makes the product $T_1\, T_4\, T_5$ least, is given by
(a) 8/5 (b) 5/4
(c) 2/3 (d) None of these

10. After striking the floor, a certain ball rebounds (4/5)th of height from which it has fallen. Then, the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 m is
(a) 1260 m (b) 600 m
(c) 1080 m (d) None of these

11. Concentric circles of radii 1, 2, 3, ... , 100 cm are drawn. The interior of the smallest circle is coloured red and the angular regions are coloured alternately green and red, so that no two adjacent regions are of the same colour. Then, the total area of the green regions in sq cm is equal to
(a) 1000π (b) 5050π
(c) 4950π (d) 5151π

12. For $0 < \theta < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\theta$, $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta$, then
(a) $xyz = xz + y$ (b) $xyz = xy + z$
(c) $xyz = x + y + z$ (d) $xyz = yz + x$

13. The sum of n terms of an AP is $a\, n(n-1)$. The sum of the squares of these terms is
(a) $a^2 n^2 (n-1)^2$ (b) $\frac{a^2}{6} n(n-1)(2n-1)$
(c) $\frac{2a^2}{3} n(n-1)(2n-1)$ (d) $\frac{2a^2}{3} n(n+1)(2n+1)$

14. Let $S = \frac{8}{5} + \frac{16}{65} + \ldots + \frac{128}{2^{18}+1}$, then
(a) $S = \frac{1088}{545}$ (b) $S = \frac{545}{1088}$
(c) $S = \frac{1056}{545}$ (d) $S = \frac{545}{1056}$

15. The sum of the infinite terms of the series $\frac{5}{3^2+7^2} + \frac{9}{7^2+11^2} + \frac{13}{11^2+15^2} + \ldots$ is
(a) $\frac{1}{18}$ (b) $\frac{1}{36}$ (c) $\frac{1}{54}$ (d) $\frac{1}{72}$

16. If the ratio of AM between two positive real numbers a and b to their HM is $m : n$, then $a : b$ is equal to
(a) $\frac{\sqrt{(m-n)} + \sqrt{n}}{\sqrt{m-n} - \sqrt{n}}$ (b) $\frac{\sqrt{n} + \sqrt{m-n}}{\sqrt{n} - \sqrt{m-n}}$
(c) $\frac{\sqrt{m} + \sqrt{m-n}}{\sqrt{m} - \sqrt{m-n}}$ (d) $\frac{\sqrt{m-n} + \sqrt{m}}{\sqrt{m-n} - \sqrt{m}}$

17. The sum of

$$\frac{\frac{1}{2}\cdot\frac{2}{2}}{1^3}+\frac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3}+\frac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3}+\ldots \text{ upto } n \text{ terms is}$$

equal to

(a) $\frac{n-1}{n}$ (b) $\frac{n}{n+1}$ (c) $\frac{n+1}{n+2}$ (d) $\frac{n+1}{n}$

18. If the sum of first n terms of an AP is cn^2, then the sum of squares of these n terms is

(a) $\frac{n(4n^2-1)c^2}{6}$ (b) $\frac{n(4n^2+1)c^2}{3}$

(c) $\frac{n(4n^2-1)c^2}{3}$ (d) $\frac{n(4n^2+1)c^2}{6}$

More Than One Correct Option

19. If the non-zero numbers x, y, z are in AP and $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in AP, then

(a) $x = y = z$ (b) $y^2 = zx$

(c) $x^2 = yz$ (d) $z^2 = xy$

20. The consecutive digits of a three digit number are in GP. If the middle digit be increased by 2, then they form an AP. If 792 is subtracted from this, then we get the number constituting of same three digits but in reverse order. Then, number is divisible by

(a) 7 (b) 49

(c) 19 (d) None of these

21. For an increasing AP $a_1, a_2, \ldots, a_n$, if $a_1 + a_3 + a_5 = -12$ and $a_1a_3a_5 = 80$, then which of the following is/are true?

(a) $a_1 = -10$ (b) $a_2 = -1$

(c) $a_3 = -4$ (d) $a_5 = -2$

Comprehension Based Questions

Passage

Directions (Q. Nos. 22-24) *In a GP the sum of the first and last term is* 66, *the product of the second and the last but one is* 128 *and the sum of the terms is* 126.

22. If an increasing GP is considered, then the number of terms in GP is

(a) 9 (b) 8 (c) 12 (d) 6

23. If the decreasing GP is considered, then the sum of infinite terms is

(a) 64 (b) 128 (c) 256 (d) 729

24. In any case, the difference of the least and greatest term is

(a) 78 (b) 126

(c) 127 (d) None of these

Assertion and Reason

Directions (Q. Nos. 25-33) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

25. **Statement I** The sums of n terms of two arithmetic progressions are in the ratio $(7n+1):(4n+17)$, then the ratio of their nth terms is $7:4$.

Statement II If $S_n = ax^2 + bx + c$, then $T_n = S_n - S_{n-1}$

26. Let sum of n terms of a series $S_n = 6n^2 + 3n + 1$.

Statement I The series S_n is in AP.

Statement II Sum of n terms of an AP is always of the form $an^2 + bn$.

27. Consider first three terms of a sequence S_n is $|x-1|, 3, |x-3|$, which are in AP.

Statement I The sixth term of S_n is $7 <$ third term.

Statement II If $a, a+d, a+2d, \ldots$ are in AP $(d \neq 0)$, then sixth term is $(a+5d)$.

28. **Statement I** 3, 6, 12 are in GP, then 9, 12, 18 are in HP.

Statement II If middle term is added in three consecutive terms of a GP, resultant will be in HP.

29. **Statement I** Let $F_1(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, then $\sum_{r=1}^{n} F_1(r) = (n+1)F_1(n) - n$.

Statement II $\frac{1^{-1}+2^{-1}+3^{-1}+\ldots+n^{-1}}{n} > \left(\frac{1+2+3+\ldots+n}{n}\right)^{-1}$

or $\left(1+\frac{1}{2}+\frac{1}{3}++\ldots+\frac{1}{n}\right) > \frac{n^2}{\Sigma n}$

or $\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) > \frac{2n}{(n+1)}$

30. **Statement I** There are infinite geometric progressions for which 27, 8 and 12 are three of its terms (not necessarily consecutive).

Statement II Given terms are integers.

31. Let us define the sum of cubic number is $\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Statement I Sum of the series

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots + 11^3 = 378$$

Statement II For any odd integer

$$n \geq 1,\ n^3 - (n-1)^3 + \ldots + (-1)^{n-1}\, 1^3 = \frac{1}{4}(2n-1)(n+1)^2.$$

32. **Statement I** If the AM of two numbers is 5/2, GM of the numbers is 2, then the HM will be 8/5.

Statement II For a group of positive numbers $(\text{GM})^2 = (\text{AM}) \times (\text{HM})$.

33. **Statement I** The sum of the series

$$1 + (1+2+4) + (4+6+9) + (9+12+16) + \ldots + (361+380+400) \text{ is } 8000.$$

Statement II $\sum_{k=1}^{n} \{k^3 - (k-1)^3\} = n^3$, for any natural number n.

Previous Years' Questions

34. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ... , is [JEE Main 2013]

(a) $\frac{7}{81}(179 - 10^{20})$ (b) $\frac{7}{9}(99 - 10^{-20})$

(c) $\frac{7}{81}(179 + 10^{-20})$ (d) $\frac{7}{9}(99 + 10^{-20})$

35. If 100 times the 100th term of an AP with non-zero common difference equals the 50 times its 50th term, then the 150th term of this AP is [AIEEE 2012]

(a) –150 (b) 150 times its 50th term

(c) 150 (d) zero

36. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹ 40 more than the saving of immediately previous month. His total saving from the start of service will be ₹ 11040 after [AIEEE 2011]

(a) 19 months (b) 20 months

(c) 21 months (d) 18 months

37. A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the nth minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference –2, then the time taken by him to count all notes, is [AIEEE 2010]

(a) 24 min (b) 34 min (c) 125 min (d) 135 min

38. The sum of the infinity of the series

$$1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots \text{ is}$$

[AIEEE 2009]

(a) 3 (b) 4 (c) 6 (d) 2

39. The first two terms of a geometric progression add upto 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then first term is [AIEEE 2008]

(a) 4 (b) –4

(c) –12 (d) 12

40. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p+q)$ is [AIEEE 2007]

(a) 2 (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$

41. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then, the common ratio of this progression is equal to [AIEEE 2007]

(a) $\frac{1}{2}(1-\sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$

(c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5}-1)$

42. Let $a_1, a_2, a_3, \ldots$ be terms of an AP. If $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ is equal to [AIEEE 2006]

(a) $\frac{7}{2}$ (b) $\frac{2}{7}$

(c) $\frac{11}{41}$ (d) $\frac{41}{11}$

43. If $a_1, a_2, \ldots, a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n$ is equal to [AIEEE 2006]

(a) $(n-1)(a_1 - a_n)$

(b) na_1a_n

(c) $(n-1)a_1a_n$

(d) $n(a_1 - a_n)$

44. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, where a, b, c are in AP and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x, y, z are in [AIEEE 2005]

(a) HP (b) AGP

(c) AP (d) GP

Answers

Check Point 1

1. (b)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (a)

Check Point 2

1. (c)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (b)	**6.** (d)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (a)

Check Point 3

1. (c)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (a)	**8.** (d)

Round I

1. (c)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (a)
11. (b)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (d)	**17.** (b)	**18.** (c)	**19.** (b)	**20.** (b)
21. (b)	**22.** (b)	**23.** (c)	**24.** (d)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (c)
31. (a)	**32.** (c)	**33.** (c)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (b)	**38.** (b)	**39.** (b)	**40.** (a)
41. (c)	**42.** (b)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (a)	**49.** (b)	**50.** (a)
51. (a)	**52.** (b)	**53.** (b)	**54.** (a)	**55.** (a)	**56.** (d)	**57.** (a)	**58.** (b)	**59.** (a)	**60.** (c)
61. (c)	**62.** (d)	**63.** (a)	**64.** (a)	**65.** (d)	**66.** (d)	**67.** (c)	**68.** (c)	**69.** (c)	**70.** (b)
71. (c)	**72.** (c)	**73.** (b)	**74.** (d)	**75.** (b)	**76.** (c)	**77.** (d)	**78.** (c)	**79.** (c)	**80.** (c)
81. (c)	**82.** (b)	**83.** (b)	**84.** (b)	**85.** (c)	**86.** (d)	**87.** (c)	**88.** (c)	**89.** (c)	**90.** (d)
91. (a)	**92.** (d)	**93.** (d)	**94.** (d)	**95.** (d)	**96.** (a)	**97.** (b)	**98.** (d)	**99.** (a)	**100.** (c)
101. (d)									

Round II

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (c)
11. (b)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (a,d)
21. (a,c,d)	**22.** (d)	**23.** (b)	**24.** (d)	**25.** (d)	**26.** (d)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (b)
31. (d)	**32.** (c)	**33.** (a)	**34.** (c)	**35.** (d)	**36.** (c)	**37.** (b)	**38.** (a)	**39.** (c)	**40.** (d)
41. (d)	**42.** (c)	**43.** (c)	**44.** (a)						

the **Guidance**

Round I

1. $a_1a_2 \ldots a_n = b_n\left(\dfrac{a_1a_2 \ldots a_n}{b_n}\right) = a_nb_n\left(\dfrac{a_1a_2 \ldots a_{n-1}}{b_n}\right)$

$$= \left[(x)^{\frac{1}{2^{n-1}}} - (y)^{\frac{1}{2^{n-1}}}\right]\left(\frac{a_1a_2 \ldots a_{n-1}}{b_n}\right)$$

$$= b_{n-1} \cdot a_{n-1}\left(\frac{a_1a_2 \ldots a_{n-2}}{b_n}\right) = \ldots = \frac{x-y}{b_n}$$

2. $S_n = 3n + 2n^2$

We know, $T_n = S_n - S_{n-1}$

$\Rightarrow$ $T_n = 2n^2 + 3n - \{2(n-1)^2 + 3(n-1)\} = 4n + 1$

And $d = T_n - T_{n-1}$

$\Rightarrow$ $d = 4n + 1 - \{4(n-1) + 1\} = 4$

3. Let Ist term of AP = a and common difference = d

Given, $9 \times T_9 = 13 \times T_{13}$

$\Rightarrow$ $9(a + 8d) = 13(a + 12d)$

$\Rightarrow$ $4a + 84d = 0$

$\Rightarrow$ $a + 21d = 0$

Now, $T_{22} = a + 21d = 0$

4. Let AP series is $a, a + d, a + 2d, a + 3d, \ldots$

Now, given $a = 2$

According to the given condition,

Sum of first five terms = $\frac{1}{4}$ (Sum of next five terms)

$\Rightarrow$ $a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d)$

$$= \frac{1}{4}[a + 5d + a + 6d + a + 7d + a + 8d + a + 9d]$$

$\Rightarrow$ $$5a + 10d = \frac{1}{4}(5a + 35d)$$

$\Rightarrow$ $$4(5a + 10d) = 5a + 35d$$

$\Rightarrow$ $$20a + 40d = 5a + 35d \Rightarrow 20a - 5a = 35d - 40d$$

$\Rightarrow$ $$15a = -5d$$

$\Rightarrow$ $$15 \times 2 = -5d$$

$\Rightarrow$ $$30 = -5d \qquad (\because a = 2)$$

$\Rightarrow$ $$d = \frac{-30}{5} = -6$$

Now, $$T_n = a + (n-1)d$$

$\Rightarrow$ $$T_{20} = 2 + (20-1)(-6) = 2 + 19(-6)$$
$$= 2 - 19 \times 6 = 2 - 114 = -112$$

5. Since, a, b, c, d, e, f are in AP.

So, $b - a = c - b = d - c = e - d = f - e = k$,

where k is the common difference.

Now, $d - c = e - d \Rightarrow e + c = 2d$

$\Rightarrow$ $e - c + 2c = 2d \Rightarrow e - c = 2(d - c)$

6. Let Ist term $= a$ and common difference $= d$

Given, $$S_{2n} = 3S_n$$

$\Rightarrow$ $$\frac{2n}{2}[2a + (2n-1)d] = 3\left\{\frac{n}{2}[2a + (n-1)d]\right\}$$

$$4a + (4n-2)d = 6a + (3n-3)d$$

$\Rightarrow$ $$2a = (n+1)d \qquad \ldots(i)$$

Now, $$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)d]}{\frac{n}{2}[2a + (n-1)d]}$$

$$= \frac{\{3(n+1)d + (3n-1)d\}}{(n+1)d + (n-1)d} \qquad [\because \text{from Eq. (i)}]$$

$$= \frac{3(4nd)}{2nd} = 6$$

7. Suppose that $\angle A = x$, then $\angle B = x + 10°$,

$$\angle C = x + 20° \quad \text{and} \quad \angle D = x + 30°$$

So, we know that $\angle A + \angle B + \angle C + \angle D = 360°$

On putting these values, we get

$$(x) + (x + 10°) + (x + 20°) + (x + 30°) = 360°$$

$\Rightarrow$ $$x = 75°$$

Hence, the angles of the quadrilateral are 75°, 85°, 95°, 105°.

8. Let the number of sides of the polygon be n. Then, the sum of interior angles of the polygon

$$= (2n-4)\frac{\pi}{2} = (n-2)\pi$$

Since, the angles are in AP and $a = 120°$, $d = 5$

Therefore, $$S_n = \frac{n}{2}[2a + (n-1)d]$$

$\Rightarrow$ $$\frac{n}{2}[2 \times 120 + (n-1)5] = (n-2)180$$

$\Rightarrow$ $$n^2 - 25n + 144 = 0$$

$\Rightarrow$ $$(n-9)(n-16) = 0$$

$\Rightarrow$ $$n = 9, 16$$

Take $n = 16$

$T_{16} = a + 15d = 120° + 15(5°) = 195°$, which is impossible, an interior angle cannot be greater than 180°.

Hence, $n = 9$

9. $\because$ $$T_n = a + (n-1)d$$

Therefore, $$T_p = a + (p-1)d = \frac{1}{q} \qquad \text{(given) } \ldots(i)$$

and $$a + (q-1) = \frac{1}{p} \qquad \text{(given) } \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i),

$$d(p - 1 - q + 1) = \frac{1}{q} - \frac{1}{p}$$

$\Rightarrow$ $$d(p - q) = \frac{p - q}{pq} \Rightarrow d = \frac{1}{pq}$$

On putting the value of d in Eq. (i), we get

$$a + \frac{(p-1)}{pq} = \frac{1}{q}$$

$\Rightarrow$ $$a = \frac{1}{q} - \frac{p-1}{pq}$$

$\Rightarrow$ $$a = \frac{p - p + 1}{pq} = \frac{1}{pq}$$

Now, $$S_{pq} = \frac{pq}{2}[2a + (pq-1)d]$$

$$\left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$= \frac{pq}{2}\left[2 \times \frac{1}{pq} + (pq-1)\frac{1}{pq}\right]$$

$$= \frac{pq}{2} \times \frac{1}{pq}(2 + pq - 1) = \frac{1}{2}(pq + 1)$$

10. Given, AP is 25, 22, 19, ...

Here, $a = 25, d = -3$ and $S_n = 116$

$\because$ $$S_n = \frac{n}{2}[2a + (n-1)d]$$

$\therefore$ $$116 = \frac{n}{2}[2 \times 25 + (n-1)(-3)]$$

$\Rightarrow$ $$116 \times 2 = n(50 - 3n + 3)$$

$\Rightarrow$ $$232 = n(53 - 3n)$$

$\Rightarrow$ $$232 = 53n - 3n^2$$

$\Rightarrow$ $$3n^2 - 53n + 232 = 0$$

$\Rightarrow$ $$3n^2 - 24n - 29n + 232 = 0$$

$\Rightarrow$ $$3n(n-8) - 29(n-8) = 0$$

$\Rightarrow \quad (3n-29)(n-8)=0$

$\Rightarrow \quad n=\frac{29}{3}, n=8$

$n=\frac{29}{3}$ is useless as n cannot be fraction, only $n=8$ is valid.

Now, $\quad T_n = a+(n-1)d$

$T_8 = 25+(8-1)(-3)$

$= 25+7\times(-3) = 25-21 = 4$

11. Now, we assume $(b-c)^2, (c-a)^2, (a-b)^2$ are in AP, then we have

$$(c-a)^2-(b-c)^2=(a-b)^2-(c-a)^2$$

$\Rightarrow \quad (b-a)(2c-a-b)=(c-b)(2a-b-c) \quad \ldots(i)$

Also, if $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in AP, then

$$\frac{1}{c-a}-\frac{1}{b-c}=\frac{1}{a-b}-\frac{1}{c-a}$$

$\Rightarrow \quad \frac{b+a-2c}{(c-a)(b-c)}=\frac{c+b-2a}{(a-b)(c-a)}$

$\Rightarrow \quad (a-b)(b+a-2c)=(b-c)(c+b-2a)$

$\Rightarrow \quad (b-a)(2c-a-b)=(c-b)(2a-b-c)$

which is equal to Eq.(i), so our hypothesis is true.

12. Since, $\log 2, \log(2^n-1)$ and $\log(2^n+3)$ are in AP.

$\therefore \quad 2\log(2^n-1)=\log 2+\log(2^n+3)$

$\Rightarrow \quad (2^n-1)^2=2(2^n+3)$

$\Rightarrow \quad (2^n-5)(2^n+1)=0$

As 2^n cannot be negative hence, $2^n-5=0$

$\Rightarrow \quad 2^n=5 \Rightarrow n=\log_2 5$

13. We know that, arithmetic mean of a and $b=\frac{a+b}{2}$

But given that $\frac{a+b}{2}=\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$

$\Rightarrow \quad a^n+b^n+\frac{ab^n}{b}+\frac{ba^n}{a}=2(a^n+b^n)$

$\Rightarrow \quad \frac{a}{b}b^n+\frac{b}{a}a^n=a^n+b^n$

$\Rightarrow \quad a^n\left(\frac{a-b}{a}\right)=-b^n\left(\frac{b-a}{b}\right)$

$\Rightarrow \quad \left(\frac{a}{b}\right)^n=\left(\frac{a}{b}\right)$

$\therefore \quad n=1$

14. Let the first term is A and common difference is d.

Given, $\quad S_p=a \Rightarrow \frac{p}{2}[2A+(p-1)d]=a \quad \ldots(i)$

$S_q=b \Rightarrow \frac{q}{2}[2A+(q-1)d]=b \quad \ldots(ii)$

and $\quad S_r=c \Rightarrow \frac{r}{2}[2A+(r-1)d]=c \quad \ldots(iii)$

$\therefore \quad \frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$

$=\frac{1}{p}\times\frac{p}{2}[2A+(p-1)d](q-r)+\frac{1}{q}\times\frac{q}{2}$

$[2A+(q-1)d](r-p)+\frac{1}{r}\times\frac{r}{2}[2A+(r-1)d](p-q)$

[$\because$ from Eqs. (i), (ii), (iii)]

$=\frac{1}{2}[2A+(p-1)d](q-r)+[2A+(q-1)d](r-p)$

$+[2A+(r-1)d\}(p-q)$

$=\frac{1}{2}[2A(q-r)+(p-1)d(q-r)+2A(r-p)$

$+(q-1)d(r-p)+2A(p-q)+(r-1)d(p-q)]$

$=\frac{1}{2}2A(q-r+r-p+p-q)$

$+d[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$

$=\frac{1}{2}[2A\times(0)+d(pq-pr-q+r+qr-pq-r$

$+p+rp-rq-p+q)]$

$=\frac{1}{2}(0+d\times 0)=0$

15. Let S_n and S' be the sums of n terms of two AP's and T_{11} and T'_{11} be the respective 11th term, then

$$\frac{S_n}{S_n'}=\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']}=\frac{7n+1}{4n+27} \quad \text{(given)}$$

$\Rightarrow \quad \frac{a+\frac{(n-1)}{2}d}{a'+\frac{(n-1)}{2}d'}=\frac{7n+1}{4n+27}$

Now, put $n=21$, we get

$$\frac{a+10d}{a'+10d'}=\frac{T_{11}}{T'_{11}}=\frac{148}{111}$$

$$=\frac{4}{3}$$

16. mth mean between a and $2b$ is $a+\frac{m(2b-a)}{n+1}$.

and mth mean between $2a$ and b is $2a+\frac{m(b-2a)}{n+1}$.

According to the given condition,

$$a+\frac{m(2b-a)}{n+1}=2a+\frac{m(b-2a)}{n+1}$$

$\Rightarrow \quad m(2b-a)=a(n+1)+m(b-2a)$

$\Rightarrow \quad a(n-m+1)=bm$

$\Rightarrow \quad \frac{a}{b}=\frac{m}{n-m+1}$

17. Here, $n=20, S_n=66000, d=200$

Let first term is a.

$\because \quad S_n=\frac{n}{2}[2a+(n-1)d]$

$\therefore \quad 66000 = \frac{20}{2}[2 \times a + (20-1)200]$

$\Rightarrow \quad 6600 = 2a + 19 \times 200$

$\Rightarrow \quad 2a = 6600 - 3800$

$\Rightarrow \quad 2a = 2800$

$\Rightarrow \quad a = 1400$

Hence, he save in the first year is ₹ 1400.

18. Clearly, the series is 5, 7, 9, 11, ...

It is an AP, where $a = 5$ and $d = 2$, $S_n = 192$

$\Rightarrow \quad S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow \quad 192 = \frac{n}{2}[2 \times 5 + (n-1)2]$

$\Rightarrow \quad 192 = \frac{n}{2} \times 2(5 + n - 1)$

$\Rightarrow \quad 192 = n(n+4)$

$\Rightarrow \quad 192 = n^2 + 4n$

$\Rightarrow \quad n^2 + 4n - 192 = 0$

$\Rightarrow \quad n^2 + 16n - 12n - 192 = 0$

$\Rightarrow \quad n(n+16) - 12(n+16) = 0$

$\Rightarrow \quad (n+16)(n-12) = 0$

$\Rightarrow \quad n = -16, 12$

But $\quad n \neq -16$

$\because$ Number of days are always positive.

$\therefore$ Required number of days $= 12$

19. Let a_1, a_2, a_3 and d_1, d_2, d_3 are the first term and common difference of the three AP's, respectively.

We have, $a_1 = a_2 = a_3 = 1$ and $d_1 = 1, d_2 = 2, d_3 = 3$, therefore

$$S_1 = \frac{n}{2}(n+1) \quad \text{...(i)}$$

$$S_2 = \frac{n}{2}(2n) \quad \text{...(ii)}$$

$$S_3 = \frac{n}{2}(3n-1) \quad \text{...(iii)}$$

On adding Eqs. (i) and (iii), we get

$$S_1 + S_3 = \frac{n}{2}[(n+1) + (3n-1)] = 2\left[\frac{n}{2}(2n)\right] = 2S_2$$

Hence, correct relation is $S_1 + S_3 = 2S_2$

20. Let the common difference be d.

Sum of first p terms $= 0$

Given, $\quad S_p = 0$

$\Rightarrow \quad \frac{p}{2}[2a + (p-1)d] = 0$

$\Rightarrow \quad 2a + (p-1)d = 0$

$\Rightarrow \quad d = -\frac{2a}{p-1} \quad \text{...(i)}$

$\therefore$ Required sum of next q terms = Sum of first $(p+q)$ terms – Sum of first p terms

$= S_{p+q} - S_p$

$= \frac{p+q}{2}[2a + (p+q-1)d] - 0$

$= \frac{p+q}{2}\left[2a + (p+q-1)\left(\frac{-2a}{p-1}\right)\right]$ [from Eq. (i)]

$= \frac{p+q}{2} \times 2a\left[\frac{1}{1} - \frac{p+q-1}{p-1}\right]$

$= (p+q)\,a\left[\frac{p-1-p-q+1}{p-1}\right] = \frac{-a(p+q)q}{p-1}$

21. Here, $n = 16, S_n = 8000$

Since, we consider last place team as a first term,

$a = 275$

Let common difference be d.

$\because \quad S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore \quad 8000 = \frac{16}{2}[2 \times 275 + (16-1)d]$

$\Rightarrow \quad 1000 = (550 + 15d)$

$\Rightarrow \quad 15d = 450$

$\Rightarrow \quad d = 30$

Now, $\quad T_{16} = 275 + (16-1)30$

$= 275 + 450 = 725$

$\therefore$ The first place team will get the prize ₹ 725.

22. Since, a, b, c are in AP.

$\Rightarrow 2b = a + c$, then straight line $ax + by + c = 0$ will pass through (1,–2) because it satisfies condition $a - 2b + c = 0$ or $2b = a + c$

23. Since, $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP.

$\Rightarrow \quad \theta_2 - \theta_1 = \theta_3 - \theta_2 = \ldots = \theta_n - \theta_{n-1} = d \quad \text{...(i)}$

Now, taking only first term

$\sin d \sec\theta_1 \sec\theta_2 = \frac{\sin d}{\cos\theta_1 \cos\theta_2} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1 \cos\theta_2}$

[from Eq. (i)]

$= \frac{\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2}$

$= \frac{\sin\theta_2 \cos\theta_1}{\cos\theta_1 \cos\theta_2} - \frac{\cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2}$

$= \tan\theta_2 - \tan\theta_1$

Similarly, we can solve other terms which will be

$\tan\theta_3 - \tan\theta_2, \tan\theta_4 - \tan\theta_3, \ldots$

$\therefore \sin d(\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n)$

$= \tan\theta_2 - \tan\theta_1 + \tan\theta_3 - \tan\theta_2 + \ldots + \tan\theta_n - \tan\theta_{n-1}$

$= -\tan\theta_1 + \tan\theta_n = \tan\theta_n - \tan\theta_1$

24. Given that,

$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow \quad (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow \quad 3(a_1 + a_{24}) = 225$

$\Rightarrow \quad a_1 + a_{24} = 75 \quad \ldots(i)$

($\because$ in an AP the sum of the terms equidistant from the beginning and the end is same and is equal to the sum of the first and last term)

$\therefore \quad a_1 + a_2 + \ldots + a_{24} = \frac{n}{2}(a + l) = \frac{24}{2}(a_1 + a_{24})$

$= 12 \times 75 = 900$ [from Eq. (i)]

25. Since, $a_1, a_2, a_3, \ldots, a_n$ are in AP.

Then, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$

where d is the common difference of the given AP.

Also, $a_n = a_1 + (n-1)d$

Then, by rationalizing each term

$$\frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

$$= \frac{1}{d}(\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots+\sqrt{a_n}-\sqrt{a_{n-1}})$$

$$= \frac{1}{d}(\sqrt{a_n}-\sqrt{a_1}) \times \frac{\sqrt{a_n}+\sqrt{a_1}}{\sqrt{a_n}+\sqrt{a_1}}$$

$$= \frac{1}{d}\left(\frac{a_n - a_1}{\sqrt{a_n}+\sqrt{a_1}}\right)$$

$$= \frac{1}{d}\left[\frac{(n-1)d}{\sqrt{a_n}+\sqrt{a_1}}\right] = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

26. Since, $a_1, a_2, a_3, \ldots, a_n$ form an AP.

$\therefore \quad a_2 - a_1 = a_4 - a_3 = \ldots = a_{2n} - a_{2n-1} = d$

Let $S = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$

$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \ldots + (a_{2n-1} - a_{2n})(a_{2n-1} + a_{2n})$

$= -d(a_1 + a_2 + \ldots + a_{2n}) = -d\left[\frac{2n}{2}(a_1 + a_{2n})\right] \ldots(i)$

Also, we know $a_{2n} = a_1 + (2n-1)d$

$\Rightarrow \quad d = \frac{a_{2n} - a_1}{2n-1} \Rightarrow -d = \frac{a_1 - a_{2n}}{2n-1}$

On putting the value of d in Eq. (i), we get

$$S = \frac{n(a_1 - a_{2n})(a_1 + a_{2n})}{2n-1} = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$$

27. Now, $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{4000}a_{4001}}$

$$= \frac{1}{d}\left(\frac{a_2-a_1}{a_1a_2} + \frac{a_3-a_2}{a_2a_3} + \ldots + \frac{a_{4001}-a_{4000}}{a_{4000}a_{4001}}\right)$$

$$= \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \ldots + \frac{1}{a_{4000}} - \frac{1}{a_{4001}}\right)$$

$$= \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{4001}}\right) = \frac{4000}{a_1a_{4001}} = 10 \quad \text{(given)}$$

$\Rightarrow \quad a_1a_{4001} = 400 \quad \ldots(i)$

$a_1 + a_{4001} = a_2 + a_{4000} = 50 \quad \ldots(ii)$

$\therefore \quad (a_1 - a_{4001})^2 = (a_1 + a_{4001})^2 - 4a_1a_{4001}$

$= (50)^2 - 1600$

$\Rightarrow \quad |a_1 - a_{4001}| = 30$

28. Since, $a_1, a_2, a_3, \ldots, a_n$ are in AP.

$\Rightarrow \quad d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$

$\therefore \sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$

$$= \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} + \ldots + \frac{\sin(a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$= \frac{(\sin a_2 \cos a_1 - \cos a_2 \sin a_1)}{\sin a_1 \sin a_2} + \ldots + \frac{(\sin a_n \cos a_{n-1} + \cos a_n \sin a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \ldots + (\cot a_{n-1} - \cot a_n)$

$= \cot a_1 - \cot a_n$

29. Given, $x_n = x_{n+1}\sqrt{2}$

$\therefore \quad x_1 = x_2\sqrt{2}, x_2 = x_3\sqrt{2}, \ldots, x_n = x_{n+1}\sqrt{2}$

On multiplying $x_1 = x_{n+1}(\sqrt{2})^n$

$\Rightarrow \quad x_{n+1} = \frac{x_1}{(\sqrt{2})^n}$

Hence, $x_n = \frac{x_1}{(\sqrt{2})^{n-1}}$

Area of $S_n = x_n^2 = \frac{x_1^2}{2^{n-1}} < 1 \Rightarrow 2^{n-1} > x_1^2 \quad (\because x_1 = 10)$

$\therefore \quad 2^{n-1} > 100$

But $2^7 > 100, 2^8 > 100$ etc.

$\therefore \quad n - 1 = 7, 8, 9, \ldots \Rightarrow n = 8, 9, 10, \ldots$

30. Since a, b, c are in AP.

$\therefore \quad 2b = a + c$

$\Rightarrow \quad 2bx = (a + c)x$ for all x

$\Rightarrow \quad 2bx + 20 = (a + c)x + 20$ for all x

$\Rightarrow \quad 2(bx + 10) = (ax + 10) + (cx + 10)$

$\therefore \quad 10^{2(bx+10)} = 10^{(ax+10)+(cx+10)}$

$\Rightarrow \quad (10^{bx+10})^2 = 10^{ax+10} \cdot 10^{cx+10}$

$\Rightarrow \quad 10^{ax+10}, 10^{bx+10}, 10^{cx+10}$ are in GP for all x

31. $\because (x+1) + (x+4) + (x+7) + \ldots + (x+28) = 155$

Let n be the number of terms in the AP on LHS.

$\therefore \quad x + 28 = (x+1) + (n-1)3$

$\Rightarrow \quad n = 10$

$\therefore \quad \frac{10}{2}[(x+1) + (x+28)] = 155$

$\Rightarrow \quad x = 1$

32. **Statement I** Since, nth term of an AP is a_n.

$$\therefore \quad a_n = \frac{a_{n+k} + a_{n-k}}{2} \quad \left(\because \text{AM} = \frac{a+b}{2}\right)$$

Statement II Let a be the first term and d be the common difference of an AP.

According to the given condition,

$$S_m = S_n$$

$$\therefore \quad \frac{m}{2}[2a + (m-1)d] = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow \quad (m-n)2a + (m^2 - n^2 - m + n)d = 0$$

$$\Rightarrow \quad (m-n)[2a + (m+n-1)d] = 0 \quad \ldots\text{(i)}$$

$$\therefore \quad S_{(m+n)} = \frac{m+n}{2}[2a + (m+n-1)d] = 0$$

$$= \frac{m+n}{2}(0) \quad [\because \text{from Eq. (i)}]$$

$$= 0$$

33. The given numbers are 13, 17, ... , 97. This is an AP with the first term 13 and common difference 4. Let the number of terms be n. Then,

$$97 = 13 + (n-1)\,4$$

$$\Rightarrow \quad 4n = 88 \quad \Rightarrow \quad n = 22$$

Therefore, the sum of the numbers

$$S = \frac{n}{2}(a+l) = \frac{22}{2}(13+97) = 11(110) = 1210$$

34. The sum of integers from 1 to 100 that are divisible by 2 or 5

= Sum of series divisible by 2 + Sum of series divisible by 5 − Sum of series divisible by 2 and 5

$$= (2+4+6+\ldots+100) + (5+10+15+\ldots+100) - (10+20+30+\ldots+100)$$

$$= \frac{50}{2}\{2\times 2 + (50-1)\,2\} + \frac{20}{2}\{2\times 5 + (20-1)\,5\} - \frac{10}{2}\{10\times 2 + (10-1)\,10\}$$

$$= 25(102) + 10(105) - 5(110)$$

$$= 2550 + 1050 - 550 = 3050$$

35. Let $S = 1+2+3+\ldots+100$

$$= \frac{100}{2}(1+100) = 50(101) = 5050$$

Let $S_1 = 3+6+9+12+\ldots+99$

$$= 3(1+2+3+4+\ldots+33)$$

$$= 3\cdot\frac{33}{2}(1+33) = 99\times 17 = 1683$$

Let $S_2 = 5+10+15+\ldots+100$

$$= 5(1+2+3+\ldots+20)$$

$$= 5\cdot\frac{20}{2}(1+20) = 50\times 21 = 1050$$

Let $S_3 = 15+30+45+\ldots+90$

$$= 15(1+2+3+\ldots+6)$$

$$= 15\cdot\frac{6}{2}(1+6) = 45\times 7 = 315$$

$\therefore$ Required sum $= S - S_1 - S_2 + S_3$

$$= 5050 - 1683 - 1050 + 315 = 2632$$

36. $\because a^2, b^2, c^2$ are in AP.

$$\Rightarrow \quad \sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$$

$$\Rightarrow \quad \sin C(\sin B\cos A - \cos B\sin A)$$

$$= \sin A(\sin C\cos B - \cos C\sin B)$$

On dividing by $\sin A\sin B\sin C$, we get

$$2\cot B = \cot A + \cot C$$

$\Rightarrow$ $\cot A, \cot B, \cot C$ are in AP.

37. Let A_1, A_2, A_3 be 3 AM's between 3 and 19.

Then, $3, A_1, A_2, A_3, 19$ are in AP whose common difference is

$$d = \frac{19-3}{3+1} = 4$$

$$\therefore \quad A_1 = 3 + d = 3 + 4 = 7;\ A_2 = 3 + 2d = 3 + 2\times 4 = 11,$$

$$A_3 = 3 + 3d = 3 + 3\times 4 = 15$$

38. Since, a, b, c are in GP.

$$\Rightarrow \quad b^2 = ac$$

and $\log a - \log 2b, \log 2b - \log 3c$ and $\log 3c - \log a$ are in AP.

$$\Rightarrow 2(\log 2b - \log 3c) = \log a - \log 2b + \log 3c - \log a$$

$$\therefore \quad b^2 = ac \quad \text{and} \quad 2b = 3c$$

$$\Rightarrow \quad b = \frac{2a}{3} \quad \text{and} \quad c = \frac{4a}{9}$$

Since, $\quad a + b = \frac{5a}{3} > c, b + c = \frac{10a}{9} > a,$

$$c + a = \frac{13a}{9} > b$$

It implies that a, b, c form a triangle with a as the greatest side.

Now, let us find the greatest $\angle A$ of ΔABC by using the cosine formula.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\frac{4a^2}{9} + \frac{16a^2}{81} - a^2}{\frac{4a}{3}\cdot\frac{4a}{9}} = -\frac{29}{48} < 0$$

$\therefore$ The $\angle A$ is obtuse.

39. Given, $\quad T_4 = x \Rightarrow ar^{4-1} = x \Rightarrow ar^3 = x \quad \ldots\text{(i)}$

$$T_{10} = y \Rightarrow ar^{10-1} = y \Rightarrow ar^9 = y \quad \ldots\text{(ii)}$$

$$T_{16} = z \Rightarrow ar^{16-1} = z \Rightarrow ar^{15} = z \quad \ldots\text{(iii)}$$

Now, multiplying Eq. (i) by Eq. (iii),

$$\Rightarrow \quad ar^3 \times ar^{15} = x \times z$$

$\Rightarrow \quad a^2r^{3+15} = xz$

$\Rightarrow \quad a^2r^{18} = xz$

$\Rightarrow \quad (ar^9)^2 = xz$

$\Rightarrow \quad y^2 = xz$ [from Eq. (ii)]

Therefore, x, y, z are in GP.

40. Let $S = 8 + 88 + 888 + 8888 + \ldots + n$ terms

$\Rightarrow \quad S = 8(1 + 11 + 111 + 1111 + \ldots + n \text{ terms})$

$= \frac{8}{9}[9 + 99 + 999 + 9999 + \ldots + n \text{ terms}]$

$= \frac{8}{9}[(10-1) + (100-1) + (1000-1) + \ldots + n \text{ terms}]$

$= \frac{8}{9}[(10 + 100 + 1000 + \ldots + n \text{ terms})$

$- (1 + 1 + 1 + 1 + \ldots + n \text{ terms})]$

$= \frac{8}{9}\left[10\frac{(10^n - 1)}{10 - 1} - n\right] \quad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}, \text{ as } r > 1\right]$

$= \frac{8}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$

$= \frac{8}{9} \times \frac{10}{9} \times (10^n - 1) - \frac{8}{9} \times n = \frac{80}{81}(10^n - 1) - \frac{8n}{9}$

41. Let the GP sequence is $a, ar, ar^2, ar^3, \ldots$

Given, third term = first term + 9

$\therefore \quad T_3 = a + 9 \Rightarrow ar^2 = a + 9$

$\Rightarrow \quad ar^2 - a = 9$ ...(i)

Again, second term = fourth term + 18

$T_2 = T_4 + 18 \Rightarrow ar = ar^3 + 18$

$\Rightarrow \quad ar - ar^3 = 18$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^2 - a}{ar - ar^3} = \frac{9}{18} \Rightarrow \frac{a(r^2 - 1)}{ar(1 - r^2)} = \frac{1}{2}$$

$\Rightarrow \quad -\frac{1}{r} = \frac{1}{2} \Rightarrow r = -2$

On putting $r = -2$ in Eq. (ii), we get

$a(-2) - a(-2)^3 = 18$

$\Rightarrow \quad -2a + 8a = 18$

$\Rightarrow \quad 6a = 18 \Rightarrow a = 3$

Hence, GP series is $3, 3(-2), 3(-2)^2, 3(-3)^3, \ldots$

i.e., $3, -6, 12, -24, \ldots$

42. Suppose that the added number be x, then $x + 2, x + 14, x + 62$ are in GP.

$\therefore \quad (x + 14)^2 = (x + 2)(x + 62)$

$\Rightarrow \quad x^2 + 196 + 28x = x^2 + 64x + 124$

$\Rightarrow \quad 36x = 72 \quad \Rightarrow \quad x = 2$

43. Let r be the common ratio.

$$\therefore \quad \frac{\sum_{n=1}^{100} a_{2n}}{\sum_{n=1}^{100} a_{2n-1}} = \frac{a_2 + a_4 + a_6 + \ldots + a_{200}}{a_1 + a_3 + a_5 + \ldots + a_{199}}$$

$$= \frac{a_1(r + r^3 + r^5 + \ldots + r^{199})}{a_1(1 + r^2 + r^4 + \ldots + r^{198})} = r$$

$\Rightarrow \quad \frac{\alpha}{\beta} = r$

44. Let the first term of GP is A and common ratio is R.

Given, pth term $= T_p = a \Rightarrow AR^{p-1} = a$ ...(i)

qth term $= T_q = b \Rightarrow AR^{q-1} = b$ ...(ii)

rth term $= T_r = c \Rightarrow AR^{r-1} = c$ ...(iii)

$\therefore \quad a^{q-r}b^{r-p}c^{p-q}$

$= (AR^{p-1})^{q-r}(AR^{q-1})^{r-p}(AR^{r-1})^{p-q}$

$= A^{q-r}R^{(p-1)(q-r)}A^{r-p}R^{(q-1)(r-p)}A^{p-q}R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q}R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= A^0R^{pq-pr-q+r+qr-pq+r+p+rp-rq-p+q}$

$= A^0R^0 = 1$

45. Since, a, b, c and d are in GP.

$\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$ (say)

$\Rightarrow \quad b = ar, c = br, d = cr$

$\Rightarrow \quad b = ar, c = (ar)r, d = (br)r$

$\Rightarrow \quad b = ar, c = ar^2, d = br^2$

$\Rightarrow \quad b = ar, c = ar^2, d = (ar)r^2 = ar^3$ ...(i)

$\therefore \quad (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^2(1 + r^2 + r^4)a^2r^2(1 + r^2 + r^4)$

$= a^4r^2(1 + r^2 + r^4)^2 = [a^2r(1 + r^2 + r^4)]^2$

$= (a^2r + a^2r^3 + a^2r^5)^2$

$= (a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2 = (ab + bc + cd)^2$

[from Eq. (i)]

46. Let a be the first term and r be the common ratio.

Now, $T_3 = 4$

$\Rightarrow \quad ar^2 = 4$ ...(i)

$\therefore$ Product of first 5 terms $= a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$

$= a(ar)(ar^2)(ar^3)(ar^4)$

$= a^5r^{10}$

$= (ar^2)^5 = (4)^5$ [from Eq. (i)]

47. Since, $x, 2y$ and $3z$ are in AP.

$\therefore \quad 4y = x + 3z \quad \ldots(i)$

And $\quad x, y, z$ are in GP.

$\therefore \quad y = rx$ and $z = xr^2$

On putting the values of y and z in Eq. (i), we get

$4xr = x + 3xr^2$

$\Rightarrow \quad 3r^2 - 4r + 1 = 0$

$\Rightarrow \quad (3r - 1)(r - 1) = 0$

$\Rightarrow \quad r = \frac{1}{3}, 1$

$\therefore \quad r = \frac{1}{3} \quad (\because r \neq 1)$

48. Let the edges of rectangular block are

$a, ar, ar^2 \quad \ldots(i)$

Now, volume $= 216$ cm^3

$a(ar)(ar^2) = 216 \quad (\because$ volume of cuboid $= l \times b \times h)$

$\Rightarrow \quad (ar)^3 = 216$

$\Rightarrow \quad ar = 6$ cm $\quad \ldots(ii)$

And total surface area $= 252$ cm^2

$2[a(ar) + ar(ar^2) + a(ar^2)] = 252$

$\therefore$ From Eq. (ii),

$2(6a + 36r + 36) = 252$

$\Rightarrow \quad 12(a + 6r + 6) = 252$

$\Rightarrow \quad a + 6r = 15 \quad \ldots(iii)$

$\Rightarrow \quad a + 6 \times \left(\frac{6}{a}\right) = 15 \quad [\because$ from Eq. (ii)]

$\Rightarrow \quad a^2 - 15a + 36 = 0$

$\Rightarrow \quad (a - 12)(a - 3) = 0$

$\Rightarrow \quad a = 3, 12$

$\therefore$ From Eq. (iii),

when $a = 3, r = 2$

when $a = 12, r = \frac{1}{2}$

On putting above values in Eq. (i), we get

Edges are 3, 6, 12 or 12, 6, 3.

Hence, the length of the longest edge is 12 cm.

49. $\because a, b, c, d$ are in GP.

$\Rightarrow \quad b = ar, c = ar^2, d = ar^2 \quad \ldots(i)$

Now, $(a^n + b^n)(c^n + d^n)$

$= (a^n + a^n r^n) \times (a^n r^{2n} + a^n r^{3n}) \quad$ [from Eq. (i)]

$= a^n(1 + r^n) a^n r^{2n}(1 + r^n)$

$= a^{2n} r^{2n}(1 + r^n)^2 = (a^n r^n (1 + r^n))^2 = [a^n r^n + a^n r^{2n}]^2$

$= [(ar)^n + (ar^2)^n]^2 = (b^n + c^n)^2 \quad$ [from Eq. (i)]

Hence, they are in GP.

50. Let A be the arithmetic mean between two numbers a and b. Again, let G_1 and G_2 be the two geometric mean between two numbers a and b.

$\therefore \quad A = \frac{a + b}{2}$ and $G_1 = ar, G_2 = ar^2,$

where $\quad r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3} \quad \ldots(i)$

$\therefore \quad \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{a^2 r^2}{ar^2} + \frac{a^2 r^4}{ar}$

$= a + ar^3 = a + a\left(\frac{b}{a}\right) \quad [\because$ from Eq. (i)]

$= a + b = 2A \quad \left(\because A = \frac{a + b}{2}\right)$

51. Let first term be A and common ratio be R.

Given, pth term, $T_p = q$ and qth term, $T_q = p$.

$AR^{p-1} = q$ and $AR^{q-1} = p \quad \ldots(i)$

$\therefore \quad \frac{AR^{p-1}}{AR^{q-1}} = \frac{q}{p}$

$\Rightarrow \quad R^{p-q} = \frac{q}{p} \Rightarrow R = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$

On putting the value of R in Eq. (i), we get

$$A \cdot \left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \Rightarrow A = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

Now, $(p + q)$th term, $T_{p+q} = AR^{p+q-1}$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \times \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= \frac{q^{1 - \frac{p-1}{p-q} + \frac{p+q-1}{p-q}}}{p^{\frac{p+q-1}{p-q} - \frac{p-1}{p-q}}} = \frac{q^{\frac{p-q-p+1+p+q-1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}} = \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}} = \left(\frac{q^p}{p^q}\right)^{\frac{1}{(p-q)}}$$

52. We have,

$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0 \quad \ldots(i)$

LHS $= (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2)$
$+ (c^2p^2 - 2cdp + d^2)$

$= (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \geq 0 \quad \ldots(ii)$

Since, the sum of squares of real number is non-negative.

From Eqs. (i) and (ii), we get

$(ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

$\Rightarrow \quad ap - b = 0 = bp - c = cp - d$

$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = p$

$\therefore a, b, c$ and d are in GP.

53. $\underset{n \text{ digits}}{(666\ldots6)} = 6 + 6\times10 + 6\times10^2 + \ldots + 6\times10^{n-1}$

$$= 6(1 + 10 + 10^2 + \ldots + 10^{n-1})$$

$$= \frac{6}{9}(10^n - 1) = \frac{2}{3}(10^n - 1)$$

Similarly, $\underset{n \text{ digits}}{(888\ldots8)} = \frac{8}{9}(10^n - 1)$

Hence, required sum

$$= \frac{4}{9}(10^n - 1)^2 + \frac{8}{9}(10^n - 1)$$

$$= \frac{4}{9}(10^{2n} - 2\cdot10^n + 1 + 2\cdot10^n - 2)$$

$$= \frac{4}{9}(10^{2n} - 1)$$

54. Here, $a = 0.9 = \frac{9}{10}$ and $r = \frac{1}{10} = 0.1$

$$\therefore \quad S_{100} = a\left(\frac{1-r^{100}}{1-r}\right)$$

$$= \frac{9}{10}\left(\frac{1-\frac{1}{10^{100}}}{1-\frac{1}{10}}\right) = 1 - \frac{1}{10^{100}}$$

55. $0.2\dot{3}\dot{4} = 0.2343434\ldots$

$$= 0.2 + 0.034 + 0.00034 + \ldots$$

$$= \frac{2}{10} + 34\left(\frac{1}{10^3} + \frac{1}{10^5} + \ldots\right)$$

$$= \frac{2}{10} + 34\times\frac{1}{1000}\times\frac{100}{99}$$

$$= \frac{2}{10} + \frac{34}{990} = \frac{232}{990}$$

56. $0.14189189189\ldots$

$$= 0.14 + 0.00189 + 0.00000189 + \ldots$$

$$= \frac{14}{100} + 189\left[\frac{1}{10^5} + \frac{1}{10^8} + \ldots\infty\right]$$

$$= \frac{7}{50} + \frac{189}{999\times100}$$

$$= \frac{7}{50} + \frac{7}{3700} = \frac{21}{148}$$

57. Here, $a = 0.15, r = \frac{0.015}{0.15} = \frac{15}{1000}\times\frac{100}{15}$

$$r = \frac{1}{10} < 1 \text{ and } n = 20$$

Now, $S_n = \frac{a(1-r^n)}{1-r}$

$$\Rightarrow \quad S_{20} = \frac{0.15\left[1-\left(\frac{1}{10}\right)^{20}\right]}{1-\frac{1}{10}} = \frac{0.15\left(1-\frac{1}{10^{20}}\right)}{\frac{10-1}{10}}$$

$$= \frac{0.15\times10}{9}\left(1-\frac{1}{10^{20}}\right)$$

$$= \frac{15\times10}{900}\left[1-\left(\frac{1}{10}\right)^{20}\right] = \frac{1}{6}[1-(0.1)^{20}]$$

58. Since, x, y, z are in GP.

$$\therefore \quad y^2 = xz$$

$$\Rightarrow \quad 2\log y = \log x + \log z$$

$$\Rightarrow \quad 2(\log y + 1) = (1 + \log x) + (1 + \log z)$$

$\Rightarrow 1 + \log x, 1 + \log y, 1 + \log z$ are in AP.

$\Rightarrow \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in HP.

59. If a, b, c are in GP, then $b^2 = ac$

Taking log on both sides, we get

$$2\log_e b = \log_e a + \log_e c$$

$$\Rightarrow \quad 2n\log_e b = n\log_e a + n\log_e c$$

$$\Rightarrow \quad 2\log_e b^n = \log_e a^n + \log_e c^n$$

$\Rightarrow \log_e a^n, \log_e b^n, \log_e c^n$ be in AP.

60. LHS $= \frac{1(1-\lambda^{n+1})}{1-\lambda} = \frac{1-\lambda^{n+1}}{1-\lambda}$

and RHS $= (1+\lambda)(1+\lambda^2)(1+\lambda^4)(1+\lambda^8)(1+\lambda^{16})$

$$= \frac{(1-\lambda)(1+\lambda)(1+\lambda^2)(1+\lambda^4)(1+\lambda^8)(1+\lambda^{16})}{(1-\lambda)}$$

$$= \frac{(1-\lambda^2)(1+\lambda^2)(1+\lambda^4)(1+\lambda^8)(1+\lambda^{16})}{1-\lambda}$$

$$= \frac{(1-\lambda^{32})}{1-\lambda}$$

$$\therefore \quad \frac{1-\lambda^{n+1}}{1-\lambda} = \frac{1-\lambda^{32}}{1-\lambda}$$

$$\Rightarrow \quad 1-\lambda^{n+1} = 1-\lambda^{32}$$

$$\therefore \quad n+1 = 32 \Rightarrow n = 31$$

61. $x, 1, z$ are in AP, then $2 = x + z$ ...(i)

and $x, 2, z$ are in GP, then $4 = xz$ ...(ii)

On divide Eq. (ii) by Eq. (i), we get

$$\frac{xz}{x+z} = \frac{4}{2} \Rightarrow \frac{2xz}{x+z} = 4$$

Hence, $x, 4, z$ will be in HP.

62. Let the four numbers be $\frac{a}{r}, a, ar, 2ar - a$ ...(i)

where first three numbers are in GP and last three in AP.

Given that, the common difference of AP is 6, so

$$ar - a = 6 \quad \text{...(ii)}$$

and also given $\frac{a}{r} = 2ar - a$

$$\Rightarrow \quad \frac{a}{r} = 2(ar - a) + a$$

$\Rightarrow \quad \frac{a}{r} = 2(6) + a$ [from Eq. (ii)]

$\Rightarrow \quad \left(\frac{a}{r}\right) - a = 12$

$\Rightarrow \quad a(1-r) = 12r$

$\Rightarrow \quad -6 = 12r$ [from Eq. (ii)]

$\Rightarrow \quad r = -\frac{1}{2}$

From Eq. (ii), we get

$$a\left[\left(-\frac{1}{2}\right) - 1\right] = 6$$

$\Rightarrow \quad a = -4$

On putting the value of a and r in Eq. (i), the required numbers are $8, -4, 2, 8$.

63. Let $a^{1/x} = b^{1/y} = c^{1/z} = k$

$\Rightarrow \quad a = k^x, b = k^y, c = k^z$

Now, a, b, c are in GP.

$\Rightarrow \quad b^2 = ac \Rightarrow k^{2y} = k^x \cdot k^z = k^{x+z}$

$\Rightarrow \quad 2y = x + z$

$\therefore$ x, y, z are in AP.

64. Let three numbers in GP are $\frac{a}{r}, a, ar$.

From the given condition,

$$\frac{a}{r} + a + ar = 14$$

$\Rightarrow \quad a\left(\frac{1}{r} + 1 + r\right) = 14$ ...(i)

and $\frac{a}{r} + 1, a + 1$ and $ar - 1$ will be in AP, then

$$2(a+1) = \frac{a}{r} + 1 + ar - 1 = \frac{a}{r}(1 + r^2) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$a = 4$

and $r = 2$

So, required numbers are 2, 4, 8.

Hence, greatest number is 8.

65. (a) By the property an AP, it is correct statement.

(b) By the property a GP, it is a correct statement.

(c) By the property a GP, it is a correct statement.

66. Since, a, b, c are in AP.

$\therefore \quad b = a + d, \quad c = a + 2d,$

where d is a common difference, $d > 0$

Again, since a^2, b^2, c^2 are in GP.

$\therefore$ $a^2, (a+d)^2$ and $(a+2d)^2$ are in GP.

$\Rightarrow \quad (a+d)^4 = a^2(a+2d)^2$

or $\quad (a+d)^2 = \pm a(a+2d)$

$\Rightarrow \quad a^2 + d^2 + 2ad = \pm(a^2 + 2ad)$

On taking (+) sign, $d = 0$ (not possible as $a < b < c$)

On taking (–) sign,

$$2a^2 + 4ad + d^2 = 0$$

$\Rightarrow \quad 2a^2 + 4a\left(\frac{1}{2} - a\right) + \left(\frac{1}{2} - a\right)^2 = 0$

$$\left(\because a + b + c = \frac{3}{2} \Rightarrow a + d = \frac{1}{2}\right)$$

$\Rightarrow \quad 4a^2 - 4a - 1 = 0$

$\therefore \quad a = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$

Here, $\quad d = \frac{1}{2} - a > 0$

So, $\quad a < \frac{1}{2}$

Hence, $\quad a = \frac{1}{2} - \frac{1}{\sqrt{2}}$

67. Given that, AM = 8, GM = 5, if α, β are the roots of quadratic equation, then the required quadratic equation is

$$x^2 - x(\alpha + \beta) + \alpha\beta = 0 \quad \text{...(i)}$$

Here, $\text{AM} = \frac{\alpha + \beta}{2} = 8 \Rightarrow \alpha + \beta = 16$

and $\text{GM} = \sqrt{\alpha\beta} = 5 \Rightarrow \alpha\beta = 25$

From Eq. (i),

$$x^2 - 16x + 25 = 0$$

68. Let a, ar, ar^2 are in GP and $a, ar, ar^2 - 64$ are in AP, we get

$$a(r^2 - 2r + 1) = 64 \quad \text{...(i)}$$

Again, $a, ar - 8, ar^2 - 64$ are in GP.

$\therefore \quad (ar - 8)^2 = a(ar^2 - 64)$

$\Rightarrow \quad a(16r - 64) = 64$ ...(ii)

On solving Eqs. (i) and (ii), we get $r = 5, a = 4$

Thus, required numbers are 4, 20, 100.

69. Let $S = 1 + 10 + 10^2 + \ldots + 10^{90}$

$$= \frac{1 \cdot (10^{91} - 1)}{10 - 1} = \frac{(10^{13})^7 - 1}{10^{13} - 1} \times \frac{10^{13} - 1}{10 - 1}$$

$$= [(10^{13})^6 + (10^{13})^5 + (10^{13})^4 + \ldots + 1]$$

$$\times (10^{12} + 10^{11} + \ldots + 1)$$

$\therefore$ It is the product of two integers and hence not prime.

70. $\sum_{r=1}^{n} \sum_{s=1}^{n} S_{rs} 2^r 3^s = 2 \cdot 3 + 2^2 \cdot 3^2 + 2^3 \cdot 3^3 + \ldots + 2^n \cdot 3^n$

(as $S_{rs} = 0$, if $r \neq s$ and $S_{rs} = 1$, if $r = s$) $= \frac{6(6^n - 1)}{6 - 1} = \frac{6}{5}(6^n - 1)$

71. Since, $n^m + 1$ divides $1 + n + n^2 + \ldots + n^{127}$

Therefore, $\frac{1 + n + n^2 + \ldots + n^{127}}{n^m + 1}$ is an integer.

$\Rightarrow \dfrac{1-n^{128}}{1-n} \times \dfrac{1}{n^m+1}$ is an integer.

$\Rightarrow \dfrac{(1-n^{64})(1+n^{64})}{(1-n)(n^m+1)}$ is an integer when largest

$$m = 64$$

72. Given, sum

$$= (x+2)^{n-1}\left\{1+\left(\frac{x+1}{x+2}\right)+\left(\frac{x+1}{x+2}\right)^2+\ldots+\left(\frac{x+1}{x+2}\right)^{n-1}\right\}$$

$$= (x+2)^{n-1}\left\{\frac{1-\left(\frac{x+1}{x+2}\right)^n}{1-\left(\frac{x+1}{x+2}\right)}\right\}$$

$$= \frac{(x+2)^{n-1}\{(x+2)^n-(x+1)^n\}\cdot(x+2)}{(x+2)^n} = (x+2)^n-(x+1)^n$$

73. Let the two numbers are x_1 and x_2.

Given, $\dfrac{x_1+x_2}{2} = 9$ and $x_1 \cdot x_2 = 16$

$\Rightarrow$ $x_1 + x_2 = 18$ and $x_1x_2 = 16$

Hence, required equation is

$x^2 -$ (sum of roots) $x +$ product of roots

$\Rightarrow$ $x^2 - 18x + 16 = 0$

74. Given, $\displaystyle\sum_{n=0}^{\infty} r^n = S$

$\Rightarrow$ $1 + r + r^2 + \ldots \infty = S$

$\Rightarrow$ $\dfrac{1}{1-r} = S$

$\Rightarrow$ $1 = S - rS$

$\Rightarrow$ $r = \dfrac{S-1}{S}$

Now, $\displaystyle\sum_{n=0}^{\infty} r^{2n} = 1 + r^2 + r^4 + \ldots \infty$

$$= \frac{1}{1-r^2} = \frac{1}{1-\left(\frac{S-1}{S}\right)^2} = \frac{S^2}{S^2-(S-1)^2}$$

$$= \frac{S^2}{(S-S+1)(S+S-1)}$$

$$= \frac{S^2}{(2S-1)}$$

75. Let the numbers be x and y.

According to the question,

$$\frac{x+y}{2} = 2\sqrt{xy}$$

$\Rightarrow$ $x + y = 4\sqrt{xy}$ ...(i)

$\Rightarrow$ $(x+y)^2 = 16xy$

And we know that,

$$(x-y)^2 = (x+y)^2 - 4xy$$

$\therefore$ $(x-y)^2 = 16xy - 4xy = 12xy$

$\Rightarrow$ $x - y = 2\sqrt{3}xy$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = (2+\sqrt{3})\sqrt{xy} \text{ and } y = (2-\sqrt{3})\sqrt{xy}$$

$\therefore$ Required ratio $= \dfrac{x}{y} = \dfrac{(2+\sqrt{3})\sqrt{xy}}{(2-\sqrt{3})\sqrt{xy}}$

$$(2+\sqrt{3})^2 = 7 + 4\sqrt{3}$$

76. Since, G is the geometric mean between a and b.

$\therefore$ $G = (ab)^{1/2}$ and n geometric means $G_1, G_2, \ldots, G_n$ be inserted between a and b.

$\therefore$ $G_1 = ar^1, G_2 = ar^2, \ldots, G_n = ar^n$

Now, $G_1 \cdot G_2 \cdot G_3 \cdots G_n = a^n r^{1+2+\ldots+n}$

$$= a^n r^{n(n+1)/2}$$

But $b = ar^{n+1} \Rightarrow r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

Therefore, the required product is

$$a^n\left(\frac{b}{a}\right)^{\left(\frac{1}{n+1}\right)\frac{n(n+1)}{2}} = (ab)^{\frac{n}{2}}$$

$$= \{(ab)^{\frac{1}{2}}\}^n = G^n$$

77. We have,

$$1 + \cos\alpha + \cos^2\alpha + \ldots \infty = 2 - \sqrt{2}$$

$\Rightarrow$ $\dfrac{1}{1-\cos\alpha} = 2 - \sqrt{2}$

$\Rightarrow$ $1 - \cos\alpha = \dfrac{1}{2-\sqrt{2}} = 1 + \dfrac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\alpha = -\dfrac{1}{\sqrt{2}} \Rightarrow \alpha = \dfrac{3\pi}{4}$

78. As we know, sum of infinite terms of GP,

$$S_\infty = \begin{cases} \dfrac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \geq 1 \end{cases}$$

$\therefore$ $S_\infty = \dfrac{x}{1-r} = 5$ (thus $|r| < 1$)

$\Rightarrow$ $1 - r = \dfrac{x}{5}$

$\Rightarrow$ $r = \dfrac{5-x}{5}$ exists only when $|r| < 1$

$\Rightarrow$ $-1 < \dfrac{5-x}{5} < 1$

$\therefore$ $-10 < -x < 0 \Rightarrow 0 < x < 10$

79. Let $a = {}^nC_{r-1}, b = {}^nC_r, c = {}^nC_{r+1}$ and $d = {}^nC_{r+2}$

On substituting these values in given expression.

$$\frac{a+b}{a}, \frac{b+c}{b}, \frac{c+d}{c}$$

$$\Rightarrow \quad \frac{{}^nC_{r-1}+{}^nC_r}{{}^nC_{r-1}}, \frac{{}^nC_r+{}^nC_{r+1}}{{}^nC_r}, \frac{{}^nC_{r+1}+{}^nC_{r+2}}{{}^nC_{r+1}}$$

$$\Rightarrow \quad \frac{{}^{n+1}C_r}{{}^nC_{r-1}}, \frac{{}^{n+1}C_{r+1}}{{}^nC_r}, \frac{{}^{n+1}C_{r+2}}{{}^nC_{r+1}}$$

$\Rightarrow$ It is in HP.

80. Since, 7th term of HP is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$.

Its corresponding terms of AP is 10 and 25.

$\therefore \quad T_7 = a + (n+1)d = 10$

$\Rightarrow \quad a + (7-1)d = 10$

$\Rightarrow \quad a + 6d = 10 \quad$ ...(i)

and $\quad T_{12} = a + (12-1)d = 25$

$\Rightarrow \quad a + 11d = 25 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a = -8, d = 3$$

$\therefore \quad T_{20} = -8 + (20-1)3 = -8 + 57 = 49$

Then, 20th term of HP is $\frac{1}{49}$.

81. Since, the given series $\log_a x, \log_b x, \log_c x$ be in HP.

$\Rightarrow \quad \frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c}$ are in HP.

$\Rightarrow \quad \frac{\log a}{\log x}, \frac{\log b}{\log x}, \frac{\log c}{\log x}$ are in AP.

$\Rightarrow \quad \log_x a, \log_x b, \log_x c$ are in AP.

$\therefore$ a, b and c are in GP.

82. Let P_1, P_2, P_3 be altitudes from P, Q and R.

$\therefore \quad P_1 = c\sin Q = \lambda bc, \quad P_2 = a\sin R = \lambda ca,$

$P_3 = b\sin P = \lambda ab \quad \left(\because \frac{\sin P}{a} = \frac{\sin Q}{b} = \frac{\sin R}{c} = \lambda\right)$

Since, P_1, P_2, P_3 are in AP.

$\Rightarrow \lambda bc, \lambda ca, \lambda ab$ are in AP.

$\Rightarrow bc, ca, ab$ are in AP.

$\Rightarrow \frac{abc}{a}, \frac{abc}{b}, \frac{abc}{c}$ are in AP.

$\therefore$ a, b, c are in HP.

i.e., Sides of the triangle are in HP.

83. We know, AM $\geq$ GM

$$\therefore \quad \frac{4^x + \frac{4}{4^x}}{2} \geq \sqrt{4^x \times \frac{4}{4^x}}$$

$$\Rightarrow \quad 4^x + \frac{4}{4^x} \geq 2\sqrt{4}$$

$$\Rightarrow \quad 4^x + \frac{4}{4^x} \geq 4$$

84. Let AM = 15, GM = 12

We know, $\quad (\text{GM})^2 = \text{AM} \times \text{HM}$

$\Rightarrow \quad \text{HM} = \frac{144}{15}$

85. Since, a, b and c are in AP.

$\therefore \quad 2b = a + c \quad$ ...(i)

Also, p is AM between a and b

$\therefore \quad 2p = a + b \quad$ ...(ii)

Also, p' is GM between a and b

$\therefore \quad p' = \sqrt{ab} \quad$ ...(iii)

Similarly, $\quad 2q = b + c$ and $q' = \sqrt{bc} \quad$ ...(iv)

Now, $\quad p^2 - q^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{b+c}{2}\right)^2$

$= \frac{1}{4}[a^2 + b^2 + 2ab - (b^2 + c^2 + 2bc)]$

$= \frac{1}{4}[(a^2 - c^2) + 2b(a - c)]$

$= \frac{(a-c)}{4}(a + c + 2b)$

$= \left(\frac{a-c}{4}\right)(2b + 2b) \quad$ [$\because$ from Eq. (i)]

$= \left(\frac{a-c}{4}\right)4b = (a-c)b \quad$ ...(v)

and $\quad p'^2 - q'^2 = (\sqrt{ab})^2 - (\sqrt{bc})^2$

$= ab - bc = b(a - c) \quad$ ...(vi)

$\therefore$ From Eqs. (v) and (vi), we get

$$p^2 - q^2 = p'^2 - q'^2$$

86. Given that, $\quad x_1x_2x_3 \ldots x_n = 1 \quad$ ...(i)

We know that, $\quad$ AM $\geq$ GM

$\therefore \quad \left(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\right) \geq (x_1x_2x_3 \ldots x_n)^{1/n}$

$= (1)^{1/n} = 1 \quad$ [from Eq. (i)]

$\Rightarrow \quad x_1 + x_2 + x_3 + \ldots + x_n \geq n$

$\therefore \quad x_1 + x_2 + x_3 + \ldots + x_n$

can never be less than n.

87. Since, $\dfrac{\dfrac{2p}{3}+\dfrac{2p}{3}+\dfrac{2p}{3}+\underbrace{\dfrac{3q}{5}+\ldots+\dfrac{3q}{5}}_{5\text{ times}}+\underbrace{\dfrac{4r}{7}+\ldots+\dfrac{4r}{7}}_{7\text{times}}}{15}$

$$\geq \sqrt[15]{\left(\frac{2p}{3}\right)^3\left(\frac{3q}{5}\right)^5\left(\frac{4r}{7}\right)^7} \qquad (\because \text{AM} \geq \text{GM})$$

$$\Rightarrow \qquad p^3q^5r^7\,\frac{2^3 3^5 4^7}{3^3\,5^5\,7^7} \leq 1$$

$$\Rightarrow \qquad p^3q^5r^7 \leq \frac{5^5\,7^7}{2^3\,3^2\,4^7}$$

88. Given that, $a+2b+3c=12$

and a, b, c are positive real numbers.

Now, $\text{AM} \geq \text{GM}$

$$\Rightarrow \qquad \frac{a+b+b+c+c+c}{6} \geq \sqrt[6]{ab^2c^3}$$

$$\Rightarrow \qquad \frac{a+2b+3c}{6} \geq \sqrt[6]{ab^2c^3}$$

$$\Rightarrow \qquad ab^2c^3 \leq 2^6$$

89. The given series is clearly an AG, the corresponding AP is $1+4+7+10+\ldots$ having nth term $=3n-2$.

and corresponding GP is $1+\dfrac{1}{5}+\dfrac{1}{5^2}+\ldots$ having nth term $=\dfrac{1}{5^{n-1}}$

Hence, required nth term of the series is $\dfrac{3n-2}{5^{n-1}}$.

90. Let $S=1+2x+3x^2+4x^3+\ldots\infty$ ...(i)

$xS = \quad x+2x^2+3x^3+\ldots\infty$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)\,S=1+x+x^2+x^3+\ldots\infty$$

$$\Rightarrow \qquad S=\frac{1}{(1-x)}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}$$

91. Given series is $1\cdot 3^2+2\cdot 5^2+3\cdot 7^2+\ldots\infty$

This is an arithmetico-geometric series whose nth term is equal to

$$T_n=n(2n+1)^2=4n^3+4n^2+n$$

$$\therefore \qquad S_n=\sum_1^n T_n=\sum_1^n (4n^3+4n^2+n)$$

$$=4\sum_1^n n^3+4\sum_1^n n^2+\sum_1^n n$$

$$=4\left[\frac{n}{2}(n+1)\right]^2+\frac{4}{6}n(n+1)(2n+1)+\frac{n}{2}(n+1)$$

$$=n(n+1)\left[n^2+n+\frac{4}{6}(2n+1)+\frac{1}{2}\right]$$

$$=\frac{n}{6}(n+1)(6n^2+14n+7)$$

92. This progression is an arithmetico-geometric series.

$$\therefore \qquad S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

$$=\frac{1}{1-1/2}+\frac{2}{(1-1/2)^2}$$

$$=\frac{2}{1/2}+\frac{2}{1/4}=4+8=12$$

93. Let $S=2+3+6+11+18+\ldots+T_n$

or $S = \quad 2+3+6+11+\ldots+T_n$

On subtracting, we get

$$0=2+1+3+5+7+\ldots-T_n$$

$$\Rightarrow \qquad T_n=2+[1+3+5+\ldots(n-1)\text{ term}]$$

$$=2+\frac{n-1}{2}[2+(n-1-1)2]$$

$$T_n=2+(n-1)^2$$

$$\therefore \qquad T_{50}=2+(50-1)^2=49^2+2$$

94. Let $S_n=12+16+24+\ldots+T_n$

$S_n=12+16 \quad +\ldots+T_n$

On subtraction

$$0=12+4+8+16+\ldots-T_n$$

$$\Rightarrow \qquad T_n=12+\frac{4(2^{n-1}-1)}{2-1}$$

$$=2^{n+1}+8$$

$$S_n=\Sigma T_n=2^2+2^3+2^4+\ldots+8n$$

$$=\frac{2^2(2^n-1)}{2-1}+8n=4(2^n-1)+8n$$

95. Let $S=1+3x+6x^2+10x^3+\ldots\infty$

$xS = \quad x+3x^2+6x^3+\ldots\infty$

On subtracting, we get

$$S(1-x)=1+2x+3x^2+4x^3+\ldots\infty$$

$$x(1-x)\,S = \quad x+2x^2+3x^3+\ldots\infty$$

Again on subtracting, we get

$$S[(1-x)-x(1-x)]=(1+x+x^2+x^3+\ldots\infty)$$

$$\Rightarrow \qquad S[(1-x)(1-x)]=\frac{1}{1-x} \;\Rightarrow\; S=\frac{1}{(1-x)^3}$$

96. $$\sum_{r=1}^n \frac{S_r}{s_r}=\sum_{r=1}^n \frac{\dfrac{r^2(r+1)^2}{4}}{\dfrac{r(r+1)}{2}}$$

$$=\sum_{r=1}^n \frac{r(r+1)}{2}=\frac{1}{2}\sum_{r=1}^n (r^2+r)$$

$$=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$=\frac{1}{12}(n)(n+1)(2n+1+3)=\frac{n(n+1)(n+2)}{6}$$

97. Given series is

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots+n \text{ terms}$$

Let T_n be the nth term of the series

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots$$

Then, $T_n=\dfrac{n}{1+n^2+n^4}=\dfrac{n}{(1+n^2)^2-n^2}$

$$=\frac{n}{(n^2+n+1)(n^2-n+1)}$$

$$=\frac{1}{2}\left[\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right]$$

$$=\frac{1}{2}\left[\frac{1}{1+(n-1)n}-\frac{1}{1+n(n+1)}\right]$$

$\therefore$ $T_1=\dfrac{1}{2}\left[\dfrac{1}{1}-\dfrac{1}{1+1\cdot 2}\right]$

$$T_2=\frac{1}{2}\left[\frac{1}{1+1\cdot 2}-\frac{1}{1+2\cdot 3}\right]$$

$$T_3=\frac{1}{2}\left[\frac{1}{1+2\cdot 3}-\frac{1}{1+3\cdot 4}\right]$$

$$\ldots\quad\ldots\quad\ldots$$

$$\ldots\quad\ldots\quad\ldots$$

$$T_n=\frac{1}{2}\left[\frac{1}{1+(n-1)n}-\frac{1}{1+n(n+1)}\right]$$

On adding all these equations, we get

$$\sum_{r=1}^{n}T_r=\frac{1}{2}\left[1-\frac{1}{1+n(n+1)}\right]=\frac{n(n+1)}{2(n^2+n+1)}$$

98. Let $S=\dfrac{1}{3\times 7}+\dfrac{1}{7\times 11}+\dfrac{1}{11\times 15}+\ldots\infty$

$$=\frac{1}{4}\left[\left\{\frac{1}{3}-\frac{1}{7}\right\}+\left\{\frac{1}{7}-\frac{1}{11}\right\}+\ldots\right]$$

$$=\frac{1}{4}\left(\frac{1}{3}+0\right)=\frac{1}{12}$$

99. Let $S_n=1\cdot 3\cdot 5+2\cdot 5\cdot 8+3\cdot 7\cdot 11+\ldots+n$th term

$\therefore$ $T_n=n(2n+1)(3n+2)$

$\therefore$ $S_n=\Sigma T_n=\Sigma n(2n+1)(3n+2)$

$$=\Sigma n(6n^2+7n+2)$$

$$=\Sigma(6n^3+7n^2+2n)$$

$$=6\Sigma n^3+7\Sigma n^2+2\Sigma n$$

$$=6\left[\frac{n(n+1)}{2}\right]^2+\frac{7n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{6n(n+1)}{2}+\frac{7(2n+1)}{3}+2\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{18(n^2+n)+28n+14+12}{6}\right]$$

$$=\frac{n(n+1)}{2}\left(\frac{18n^2+46n+26}{6}\right)$$

$$=\frac{n(n+1)}{2}\times\frac{2(9n^2+23n+13)}{6}$$

$$=\frac{n(n+1)(9n^2+23n+13)}{6}$$

100. Let $S_n=\dfrac{1^3}{1}+\dfrac{1^3+2^3}{1+3}+\dfrac{1^3+2^3+3^3}{1+3+5}+\ldots$

$\therefore$ $T_n=\dfrac{1^3+2^3+3^3+\ldots+n^3}{1+3+\ldots+(2n-1)}$

$$=\frac{\Sigma n^3}{n^2}\qquad[\because 1+3+\ldots+(2n-1)=n^2]$$

$$=\frac{[n(n+1)/2]^2}{n^2}=\frac{(n+1)^2}{4}=\frac{n^2+2n+1}{4}$$

101. Let $S_n=n^3-(n-1)^3+\ldots+(-1)^{n-1}1^3$

Here, n is an odd integer.

$$S_n=1^3-2^3+3^3-4^3+\ldots+n^3$$

$$=[1^3+2^3+3^3+\ldots+(n-1)^3+n^3]$$

$$-2[2^3+4^3+6^3+\ldots+(n-1)^3]$$

$$=\Sigma n^3-2\times 2^3\left[1^3+2^3+3^3+\ldots\left(\frac{n-1}{2}\right)^3\right]$$

$$=\Sigma n^3-16\left[\Sigma\left(\frac{n-1}{2}\right)^3\right]$$

$$=\left[\frac{n(n+1)}{2}\right]^2-16\left[\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right]^2$$

$$=\frac{n^2(n+1)^2}{4}-\frac{16\left[\left(\frac{n-1}{2}\right)^2\left(\frac{n+1}{2}\right)^2\right]}{4}$$

$$=\frac{(n+1)^2}{4}[n^2-(n^2+1-2n)]$$

$$=\frac{(n+1)^2}{4}(2n-1)$$

Round II

1. $\because$ Common terms are 5, 11, 17, ...

$\therefore \quad T_n = 5 + (n-1)\,6$

$= 6n - 1$

100th term of the first sequence

$= 2 + (100-1)\,3 = 299$

and 100th term of the second sequence

$= 3 + (100-1)\,2 = 201$

Now, $\quad 201 > 6n - 1$

$\Rightarrow \quad n \le 33\frac{2}{3}$

$\Rightarrow \quad n = 33 \qquad (\because n \in N)$

2. $\because$ a^{-1}, b^{-1}, c^{-1} are in AP.

$\therefore$ a, b, c are in HP.

Now, for numbers $a^{101}, b^{101}, c^{101}$

AM > GM

$\Rightarrow \quad \frac{a^{101} + c^{101}}{2} > (\sqrt{ac})^{101} > b^{101} \qquad (\because \sqrt{ac} > b)$

$\Rightarrow \quad 2b^{101} - a^{101} - c^{101} < 0 \qquad$ ...(i)

Now, product of roots of given equation

$= \frac{2b^{101} - a^{101} - c^{101}}{1} < 0 \qquad$ [from Eq. (i)]

3. $\because 13, a_1, a_2, \ldots, a_{20}, 67$ are in AP.

$\therefore a_1 + a_2 + a_3 + \ldots + a_{20} = 20\left(\frac{13+67}{2}\right) = 800$

Also, $\quad$ AM > GM

$\Rightarrow \quad \frac{a_1 + a_2 + \ldots + a_{20}}{20} \ge (a_1 a_2 a_3 \ldots a_{20})^{1/20}$

$\Rightarrow \quad 40 \ge (a_1 \cdot a_2 \cdot a_3 \ldots a_{20})^{1/20}$

Hence, maximum value of $a_1 \cdot a_2 \cdot a_3 \ldots a_{20}$ is $(40)^{20}$.

4. From symmetry, we observe that S_{50} has 50 terms. First term of $S_1, S_2, S_3, S_4, \ldots, S_{50}$ are 1, 2, 4, 7, ... , 50.

Let T_n be the first term of nth set. Then,

$S = T_1 + T_2 + T_3 + \ldots + T_n$

$\Rightarrow \quad S = 1 + 2 + 4 + 7 + 11 + \ldots + T_{n-1} + T_n$

or $\quad S = \quad 1 + 2 + 4 + 7 + \ldots \quad + T_{n-1} + T_n$

Therefore, on subtracting

$0 = 1 + [1 + 2 + 3 + 4 + \ldots + (T_n - T_{n-1})] - T_n$

or $\quad 0 = 1 + \frac{n(n-1)}{2} - T_n$

$\Rightarrow \quad T_n = 1 + \frac{n(n-1)}{2}$

$\Rightarrow \quad T_{50} =$ First term in $S_{50} = 1226$

Therefore, sum of the terms in S_{50}

$= \frac{50}{2}[2 \times 1226 + (50-1) \times 1]$

$= 25(2452 + 49) = 25(2501) = 62525$

5. It will take 10 yr for Jairam to pay off ₹ 10000 in 10 yearly installments.

$\because$ He pays 10% annual interest on remaining amount.

$\therefore$ Money given in the first year

$= 1000 + \frac{10000 \times 10}{100} = 1000 + 1000 =$ ₹ 2000

Money given in second year

$= 1000 +$ interest of $(10000 - 1000)$

$= 1000 + \frac{9000 \times 10}{100} = 1000 + 900 =$ ₹ 1900

Similarly, money paid in third year = Rs 1800 etc.

So, money given by Jairam in 10 yr will be Rs 2000, ₹ 1900, ₹ 1800, ₹ 1700, ...

Which is in arithmetic progression, whose first term

$a = 2000 \quad$ and $\quad d = -100$

Total money given in 10 yr

$= \frac{10}{2}[2(2000) + (10-1)(-100)] =$ ₹ 15500

Therefore, total money given by Jairam

$= 5000 + 15500 =$ ₹ 20500

6. Let the number of days be n. Hence, a worker can do $\left(\frac{1}{150n}\right)$th part of the work in a day.

According to the given condition,

$[150 + 146 + 142 + \ldots + \underset{\text{terms}}{\text{upto}}\,(n+8)] \times \frac{1}{150n} = 1$

$\Rightarrow \quad \frac{n+8}{2}[300 + (n+8-1)(-4)] = 1$

$\Rightarrow \quad (n+8)(272 - 4n) = 300n$

$\Rightarrow \quad 4n^2 + 60n - 2176 = 0$

$\Rightarrow \quad n^2 + 15n - 544 = 0$

$\Rightarrow \quad n = 17, -32$

We do not take negative value

$\therefore \quad n = 17$

Therefore, number of total days

$= 17 + 8 = 25$

7. Given equation is $ax^2 + bx + c = 0$ and let the roots are α, β.

So, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Now, $\quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \frac{b^2}{a^2} - \frac{2c}{a}$

Now, $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}=\dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\dfrac{\dfrac{b^2}{a^2}-\dfrac{2c}{a}}{\dfrac{c^2}{a^2}}=\dfrac{b^2-2ac}{c^2}$

According to the given condition,

$$\alpha+\beta=\frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

$\Rightarrow$ $$-\frac{b}{a}=\frac{b^2-2ac}{c^2}$$

$\Rightarrow$ $$-bc^2=ab^2-2a^2c$$

Hence, $$2a^2c=ab^2+bc^2$$

$\Rightarrow$ ab^2, ca^2, bc^2 or bc^2, ca^2, ab^2 be in AP.

8. Since, p, q and r are positive and are in AP.

$\therefore$ $$q=\frac{p+r}{2} \quad \text{...(i)}$$

$\because$ The roots of the equation $px^2+qx+r=0$ are real.

$\Rightarrow$ $$q^2\ge 4pr \Rightarrow \left[\frac{p+r}{2}\right]^2\ge 4pr \quad \text{[from Eq. (i)]}$$

$\Rightarrow$ $$p^2+r^2-14pr\ge 0$$

$\Rightarrow$ $$\left(\frac{r}{p}\right)^2-14\left(\frac{r}{p}\right)+1\ge 0 \quad (\because p>0)$$

$\Rightarrow$ $$\left(\frac{r}{p}-7\right)^2-48\ge 0$$

$\Rightarrow$ $$\left(\frac{r}{p}-7\right)^2-(4\sqrt{3})^2\ge 0 \Rightarrow \left|\frac{r}{p}-7\right|\ge 4\sqrt{3}$$

9. $\because$ $T_6=2 \Rightarrow a+5d=2$

Now, let $P=T_1T_4T_5$

$$=a(a+3d)(a+4d)=(2-5d)(2-2d)(2-d)$$
$$=2(4-16d+17d^2-5d^3)$$

On differentiating w.r.t. d, we get

Now, $$\frac{dP}{dd}=2(-16+34d-15d^2)$$

Put $\dfrac{dP}{dd}=0$ for maxima or minima

$$-16+34d-15d^2=0\Rightarrow d=\frac{2}{3} \text{ and } \frac{8}{5}$$

Also, $$\frac{d^2P}{dd^2}=2(34-30d)$$

$$\left(\frac{d^2P}{dd^2}\right)_{d=2/3}>0$$

Thus, P is least.

Thus, the value of $d=2/3$.

10. Initially the ball falls from a height of 120 m. After striking the floor, it rebounds and goes to a height of $\frac{4}{5}\times(120)$ m. Now, it falls from a height of $\frac{4}{5}\times(120)$ m and after rebounding goes to a height of $\frac{4}{5}\left[\frac{4}{5}(120)\right]$ m. This process is continued till the ball comes to rest.

Hence, the total distance travelled is

$$120+2\left[\frac{4}{5}(120)+\left(\frac{4}{5}\right)^2(120+\ldots\infty)\right]$$

$$=120+2\left[\frac{\frac{4}{5}(120)}{1-\frac{4}{5}}\right]=1080 \text{ m}$$

11.

$$\pi[(r_2^2-r_1^2)+(r_4^2-r_3^2)+\ldots+(r_{100}^2-r_{99}^2)]$$
$$=\pi(r_1+r_2+r_3+r_4+\ldots+r_{100})$$
$$(\because r_2-r_1=r_4-r_3=\ldots=r_{100}-r_{99}=1)$$
$$=\pi(1+2+3+\ldots+r_{100})=5050\pi \text{ sq cm}$$

12. Since, $$x=\sum_{n=0}^{\infty}\cos^{2n}\theta$$

$\Rightarrow$ $$x=\cos^0\theta+\cos^2\theta+\cos^4\theta+\ldots\infty$$

which is an infinite GP with first term 1 and common ratio $\cos^2\theta$ such that $0\le\cos^2\theta\le 1$

$\therefore$ $$x=\frac{1}{1-\cos^2\theta} \quad \text{(sum of an infinite GP)}$$

$\Rightarrow$ $$x=\frac{1}{\sin^2\theta} \quad \text{...(i)}$$

Similarly, $$y=\frac{1}{\cos^2\theta} \quad \text{...(ii)}$$

and $$z=\frac{1}{1-\cos^2\theta\sin^2\theta} \quad \text{...(iii)}$$

On substituting respective values of x, y, z in the given options, we get the option (b) is correct.

13. Since, $S_n=an(n-1)$

Now, nth term of the series is

$$T_n=S_n-S_{n-1}$$

$\Rightarrow$ $$T_n=an(n-1)-a(n-1)(n-2)$$

$\Rightarrow$ $$T_n=a(n-1)[n-(n-2)]=2a(n-1)$$

Again, let the sum of squares of n terms of the series is S_1 such that

$$S = T_1^2 + T_2^2 + T_3^2 + \ldots + T_n^2 = \sum_{r=1}^{n} T_r^2$$

$$\Rightarrow \quad S = \sum_{r=1}^{n} \{2a(r-1)\}^2$$

$$\Rightarrow \quad S = \sum_{r=1}^{n} 4a^2(r-1)^2$$

$$\Rightarrow \quad S = 4a^2\left[\frac{1}{6}(n-1)n\{2(n-1)+1)\}\right]$$

$$\therefore \quad S = \frac{2a^2}{3}n(n-1)(2n-1)$$

14. $$S = \sum_{r=1}^{16n}\left(\frac{8r}{4r^4+1}\right)$$

$$= 2\sum_{r=1}^{16n}\left(\frac{1}{2r^2-2r+1} - \frac{1}{2r^2+2r+1}\right)$$

$$= 2\left(1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{13} + \frac{1}{13} - \ldots + \frac{1}{481} - \frac{1}{545}\right)$$

$$= 2\left(1 - \frac{1}{545}\right) = \frac{1088}{545}$$

15. $$\because \quad T_n = \frac{5+(n-1)4}{[3+(n-1)4]^2[7+(n-1)4]^2}$$

$$= \frac{1}{8}\left\{\frac{1}{(4n-1)^2} - \frac{1}{(4n+3)^2}\right\}$$

$$\therefore \quad S_n = T_1 + T_2 + \ldots + T_n$$

$$= \frac{1}{8}\left\{\frac{1}{3^2} - \frac{1}{7^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{11^2} - \ldots + \frac{1}{(4n-1)^2} - \frac{1}{(4n+3)^2}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{3^2} - \frac{1}{(4n+3)^2}\right\} \Rightarrow S_\infty = \frac{1}{8}\left(\frac{1}{9} - 0\right) = \frac{1}{72}$$

16. Let $\quad A = m\lambda, \quad H = n\lambda$

$$\therefore \quad G^2 = AH = mn\lambda^2$$

Also, a and b be the roots of

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 2m\lambda x + mn\lambda^2 = 0$$

$$\Rightarrow \quad x = \lambda\sqrt{m}\{\sqrt{m} \pm \sqrt{m-n}\}$$

$$\therefore \quad a : b = (\sqrt{m} + \sqrt{m-n}) : (\sqrt{m} - \sqrt{m-n})$$

or $\quad (\sqrt{m} - \sqrt{m-n}) : (\sqrt{m} + \sqrt{m-n})$

17. The general term is

$$T_n = \frac{\frac{n}{2}\cdot\frac{n+1}{2}}{1^3 + 2^3 + 3^3 + \ldots + n^3}$$

$$= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore \quad S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

18. $S_n = cn^2$

$$S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$$

$$T_n = 2cn - c$$

$$T_n^2 = (2cn-c)^2 = 4c^2n^2 + c^2 - 4c^2n$$

Sum $= \Sigma T_n^2$

$$= \frac{4c^2 + n(n+1)(2n+1)}{6} + nc^2 - 2c^2n(n+1)$$

$$= \frac{2c^2n(n+1)(2n+1) + 3nc^2 - 6c^2n(n+1)}{3}$$

$$= \frac{nc^2(4n^2+6n+2+3-6n-6)}{3} = \frac{nc^2(4n^2-1)}{3}$$

19. We have, $2y = x + z$ and $2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$.

$$\Rightarrow \quad \frac{2y}{1-y^2} = \frac{x+z}{1-xz}$$

$$\Rightarrow \quad y^2 = xz$$

So, x, y, z are in GP which is possible, if $x = y = z$.

20. Let the three digit number be xyz. According to the given condition, we have

$$y^2 = xz \quad \ldots(i)$$

$$2(y+2) = x + z \quad \ldots(ii)$$

$$\Rightarrow \quad 100x + 10y + z - 792 = 100z + 10y + x$$

$$\Rightarrow \quad x - z = 8 \quad \ldots(iii)$$

On squaring Eqs. (ii) and (iii) and subtracting, we have

$$4xz = 4(y+2)^2 - 64$$

$$\Rightarrow \quad y^2 = (y+2)^2 - 16 \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad y = 3$$

$$\Rightarrow \quad x + z = 10 \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad x = 9, z = 1$$

Hence, the number is $931 = 7^2 \times 19$.

21. $a_1 + a_3 + a_5 = -12$

$$a + a + 2d + a + 4d = -12 (d > 0)$$

$$a + 2d = -4 \quad \ldots(i)$$

$$a_1a_3a_5 = 80$$

$$a(a+2d)(a+4d) = 80$$

or $\quad a(-4)(-4-2d+4d) = 20 \quad$ [from Eq. (i)]

or $\quad (-4-2d)(-4+2d) = -20 \Rightarrow d = \pm 3$

Since, AP is increasing, so $d = +3$; $a = -10$. Hence,

$$\left.\begin{aligned} a_1 &= -10;\ a_2 = -7 \\ a_3 &= a + 2d = -10 + 6 = -4 \\ a_5 &= a + 4d = -10 + 8 = -2 \end{aligned}\right\}$$

Solutions (Q. Nos. 22-24)

Let a be the first term and r be the common ratio of the given GP.

Further, let there be n terms in the given GP. Then,

$$a_1 + a_n = 66$$

$\Rightarrow \quad a + ar^{n-1} = 66 \quad \ldots(i)$

$a_2 \times a_{n-1} = 128$

$\Rightarrow \quad ar \times ar^{n-2} = 128$

$\Rightarrow \quad a^2 r^{n-1} = 128$

$\Rightarrow \quad a \times (ar^{n-2}) = 128 \Rightarrow ar^{n-1} = \dfrac{128}{a}$

On putting this value of ar^{n-1} in Eq. (i), we get

$$a + \frac{128}{a} = 66$$

$\Rightarrow \quad a^2 - 66a + 128 = 0$

$\Rightarrow \quad (a-2)(a-64) = 0$

$\Rightarrow \quad a = 2, 64$

On putting $a = 2$ in Eq. (i), we get

$$2 + 2 \times r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

On putting $a = 64$ in Eq. (i), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

22. For an increasing GP, $r > 1$. Now,

$$S_n = 126$$

$\Rightarrow \quad 2\left(\dfrac{r^n - 1}{r - 1}\right) = 126$

$\Rightarrow \quad \dfrac{r^n - 1}{r - 1} = 63$

$\Rightarrow \quad \dfrac{r^{n-1} \times r - 1}{r - 1} = 63$

$\Rightarrow \quad \dfrac{32r - 1}{r - 1} = 63$

$\Rightarrow \quad r = 2$

$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$

23. For decreasing GP, $a = 64$ and $r = 1/2$. Hence, the sum of infinite terms is $64/[1 - 1/2] = 128$.

24. Hence, the required difference $= 128 - 64 = 64$

25. $\because \quad \dfrac{S_n}{S'_n} = \dfrac{(7n+1)}{(4n+17)} = \dfrac{n(7n+1)}{n(4n+17)}$

$\therefore \quad S_n = (7n^2 + n)\lambda, \quad S'_n = (4n^2 + 17n)\lambda$

Then, $\quad \dfrac{T_n}{T'_n} = \dfrac{S_n - S_{n-1}}{S'_n - S'_{n-1}} = \dfrac{7(2n-1)+1}{4(2n-1)+17} = \dfrac{14n-6}{8n+13}$

$\Rightarrow \quad T_n : T'_n = (14n - 6) : (8n + 13)$

26. $\because$ Sum of n terms of an AP is $S_n = \dfrac{n}{2}[2A + (n-1)D]$,

where A and D are first term and common difference.
Hence, sum always of the form $an^2 + bn$.

27. ***Case I*** $x < 1$

Then, $1 - x, 3, 3 - x$ are in AP.

$$6 = 4 - 2x \Rightarrow x = -1$$

$\therefore$ Terms are 2, 3, 4.

$\therefore$ Sixth term $= 7$

Case II $1 < x < 3$

Then, $x - 1, 3, x - 3$ are in AP.

$$6 = 2 \quad \text{(impossible)}$$

Case III $x > 3$

Then, $x - 1, 3, x - 3$ are in AP.

$\therefore \quad 6 = 2x - 4 \Rightarrow x = 5$

Then, terms are 4, 3, 2.

$\therefore$ Sixth term is -1.

28. If a, b, c are in GP, then $a + b, b + b, c + b$ are in HP.

$\Rightarrow \quad (2b) = \dfrac{2(a+b)(b+c)}{(a+b)+(c+b)}$

$\Rightarrow \quad b(a + 2b + c) = (a + b)(b + c)$

$\Rightarrow \quad b(a + c) + 2b^2 = ab + ac + b^2 + bc$

$\Rightarrow \quad b^2 = ac \quad (\because a, b, c \text{ are in GP})$

29. $\therefore \quad \displaystyle\sum_{r=1}^{n} F_1(r) = \sum_{r=1}^{n}\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right\}$

$= 1 \cdot n + \dfrac{1}{2}(n-1) + \dfrac{1}{3}(n-2) + \ldots + 1 \cdot \dfrac{1}{n}$

$= n\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right) - \left\{\dfrac{1}{2} + \dfrac{2}{3} + \ldots + \left(\dfrac{n-1}{n}\right)\right\}$

$= nF_1(n) - \left\{\left(1 - \dfrac{1}{2}\right) + \left(1 - \dfrac{1}{3}\right) + \left(1 - \dfrac{1}{4}\right) + \ldots + \left(1 - \dfrac{1}{n}\right)\right\}$

$= nF_1(n) - \{n - F_1(n)\} = (n+1)F_1(n) - n$

30. Let, if possible, 8 be the first term and 12 and 27 be mth and nth terms, respectively. Then,

$12 = ar^{m-1} = 8r^{m-1}, 27 = 8r^{n-1}$

$\Rightarrow \quad \dfrac{3}{2} = r^{m-1}, \left(\dfrac{3}{2}\right)^3 = r^{n-1} = r^{3(m-1)}$

$\Rightarrow \quad n - 1 = 3m - 3$ or $3m = n + 2$

$\Rightarrow \quad \dfrac{m}{1} = \dfrac{n+2}{3} = k$ (say)

By giving k different values, we get the integral values of m and n.

Hence, there can be infinite number of GP's whose any three terms will be 8, 12, 27 (not consecutive). Obviously, statement II is not a correct explanation of statement I.

31. For odd integer n, we have

$S_n = n^3 - (n-1)^3 + \ldots + (-1)^{n-1}1^3$

$= 1^3 - 2^3 + 3^3 - 4^3 + \ldots + n^3$

$= [1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3]$
$\quad - 2[2^3 + 4^3 + 6^3 + \ldots + (n-1)^3]$

$= \dfrac{n^2(n+1)^2}{4} - 2 \times 2^3\left[1^3 + 2^3 + \ldots + \left(\dfrac{n-1}{2}\right)^3\right]$

$= \dfrac{n^2(n+1)^2}{4} - 2^4 \dfrac{\left(\dfrac{n-1}{2}\right)^2\left(\dfrac{n-1}{2} + 1\right)^2}{4}$

$$= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2(n+1)^2}{4} = \frac{(n+1)^2}{4}[n^2 - (n-1)^2]$$

$$= \frac{1}{4}(2n-1)(n+1)^2$$

Now, putting $n = 11$ in above formula, $S_{11} = 756$. Hence, statement I is false and statement II is correct.

32. For two positive numbers $(GH)^2 = (AM) \times (HM)$. This is not true for numbers greater than 2. Hence, statement I is true but statement II is false.

33. **Statement I**

$S = (1) + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + .. + (361 + 380 + 400)$

$S = (0 + 0 + 1) + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$

Now, we can clearly observe the first elements in reach bracket.

In second bracket, the first element is $1 = 1^2$

In third bracket, the first element is $4 = 2^2$

In fourth bracket, the first elements is $9 = 3^2$

... ...

In last bracket, the first element is $361 = 19^2$

Hence, we can conclude that there are 20 brackets in all. Also, in each of the brackets there are 3 terms out of which the first and last terms are perfect squares of consecutive integers and the middle term is their product.

$\therefore$ The general term of the series is

$$T_r = (r-1)^2 + (r-1)r + (r^2)$$

The sum of the n terms of the series is

$$S_n = \sum_{r=1}^{n} [(r-1)^2 + (r-1)r + (r)^2]$$

$$S_n = \sum_{r=1}^{n} \left[\frac{r^3 - (r-1)^3}{r - (r-1)}\right]$$

$$[\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)]$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} [r^3 - (r-1)^3]$$

$$\therefore \quad S_n = (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + [n^3 - (n-1)^3]$$

$$S_n = -0^3 + (1^3 - 1^3) + (2^3 - 2^3) + (3^3 - 3^3) + \ldots + [(n-1)^3 - (n-1)^3] + n^3$$

$$S_n = n^3$$

Since, the number of terms is 20, hence substitituting $n = !20$, we get $S_{20} = 8000$

Hence, statement I is correct.

Statement II We have, already proved in the statement I that

$$S_n = \sum_{r=1}^{n} [r^3 - (r-1)^3] = n^3$$

Hence, statement II is also correct and it is a correct explanation of statement I.

34. Let $S = 0.7 + 0.77 + 0.777 + \ldots$

$$= \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \ldots + \text{upto 20 terms}$$

$$= 7\left[\frac{1}{10} + \frac{11}{10^2} + \frac{111}{10^3} + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[(1 + 1 + \ldots + \text{upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \text{upto 10 terms}\right)\right]$$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}}{1 - \frac{1}{10}}\right]$$

$$\left[\because \sum_{i=1}^{\infty} = 20 \text{ and sum of } n \text{ terms of GP, } S_n = \frac{a(1-r^n)}{1-r} \text{ where } (r < 1)\right]$$

$$= \frac{7}{9}\left[20 - \frac{1}{9}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}\right]$$

$$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + (10)^{-20}]$$

35. Let a be the first term and $d (d \neq 0)$ be the common difference of a given AP, then

$$T_{100} = a + (100 - 1)d = a + 99d$$

$$T_{50} = a + (50 - 1)d = a + 49d$$

$$T_{150} = a + (150 - 1)d = a + 149d$$

Now, according to the given condition,

$$100 \times T_{100} = 50 \times T_{50}$$

$$\Rightarrow \quad 100(a + 99d) = 50(a + 49d)$$

$$\Rightarrow \quad 2(a + 99d) = (a + 49d)$$

$$\Rightarrow \quad 2a + 198d = a + 49d$$

$$\Rightarrow \quad a + 149d = 0$$

$$T_{150} = 0$$

36. Let the time taken to save ₹ 11040 be $(n + 3)$ months.

For first 3 months he saves ₹ 200 each month.

$\therefore$ In $(n + 3)$ month,.

$$3 \times 200 + \frac{n}{2}\{2(240) + (n-1) \times 40\} = 11040$$

$$\Rightarrow \quad 600 + \frac{n}{2}\{40(12 + n - 1)\} = 11040$$

$$\Rightarrow \quad 600 + 20n(n + 11) = 11040$$

$$\Rightarrow \quad 30 + n^2 + 11n = 552$$

$$\Rightarrow \quad n^2 + 11n - 522 = 0$$

$$\Rightarrow \quad n^2 + 29n - 18n - 522 = 0$$

$\Rightarrow \quad n(n+29)-(n+29)=0$

$\Rightarrow \quad (n-18)(n+29)=0$

$\therefore \quad n=18$, neglecting $n=-29$

$\therefore$ Total time $=(n+3)=21$ months

37. Number of notes that the person counts in 10 min $=10\times 150=1500$

Since, $a_{10}, a_{11}, a_{12}, \ldots$ are in AP with common difference -2.

Let n be the time taken to count remaining 3000 notes, then

$$\frac{n}{2}[2\times 148+(n-1)\times -2]=3000$$

$\Rightarrow \quad n^2-149n+3000=0$

$\Rightarrow \quad (n-24)(n-125)=0$

$\therefore \quad n=24$ and 125

Then, the total time taken by the person to count all notes

$$=10+24=34 \text{ min}$$

38. Let $S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$

$\Rightarrow \quad S-1=\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots \quad \ldots(i)$

$\Rightarrow \quad \frac{S-1}{3}=\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\frac{14}{3^5}+\ldots \quad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}(S-1)=\frac{2}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots$$

$\Rightarrow \quad S-1=1+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots$

$\Rightarrow \quad S=2+\dfrac{\frac{2}{3}}{1-\frac{1}{3}}=2+1=3$

39. Since, $\quad a+ar=a(1+r)=12 \quad \ldots(i)$

and $\quad ar^2+ar^3=ar^2(1+r)=48 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$r^2=4 \therefore r=-2$$

(since, the series is alternately sign, so we take negative values)

On putting the value of r in Eq. (i), we get

$$a=-12$$

40. Applying AM-GM inequality in the positive real numbers p and q.

$$\frac{p^2+q^2}{2}\geq pq \qquad (\because \text{AM} \geq \text{GM})$$

$\Rightarrow \quad pq\leq \frac{1}{2} \qquad (\because p^2+q^2=1) \ldots(i)$

Now, $\quad (p+q)^2=p^2+q^2+2pq$

$\Rightarrow \quad (p+q)^2=1+2pq$

$\Rightarrow \quad (p+q)^2\leq 1+1$

$\Rightarrow \quad p+q\leq \sqrt{2}$

41. Since, each term is equal to the sum of two preceeding terms.

$\therefore \quad ar^{n-1}=ar^n+ar^{n+1}$

$\Rightarrow \quad \frac{1}{r}=1+r \Rightarrow r^2+r-1=0$

$\Rightarrow \quad r=\frac{\sqrt{5}-1}{2} \qquad \left(\because r\neq \frac{-\sqrt{5}-1}{2}\right)$

42. Given that, $\quad \frac{a_1+a_2+\ldots+a_p}{a_1+a_2+\ldots+a_q}=\frac{p^2}{q^2}$

$\therefore \quad \dfrac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\frac{p^2}{q^2},$

where d be a common difference of an AP.

$\Rightarrow \quad \frac{(2a_1-d)+pd}{(2a_1-d)+qd}=\frac{p}{q}$

$\Rightarrow \quad (2a_1-d)(p-q)=0 \Rightarrow a_1=\frac{d}{2}$

Now, $\quad \frac{a_6}{a_{21}}=\frac{a_1+5d}{a_1+20d}=\dfrac{\frac{d}{2}+5d}{\frac{d}{2}+20d}=\frac{11}{41}$

43. Since, $a_1, a_2, a_3, \ldots, a_n$ are in HP.

$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

Let d be the common difference of AP.

$\therefore \quad \frac{1}{a_2}-\frac{1}{a_1}=d \Rightarrow a_1-a_2=a_1a_2d$

Similarly, $\quad a_2-a_3=a_2a_3d$

$\ldots \ldots \ldots \ldots \ldots$

$\ldots \ldots \ldots \ldots \ldots$

$a_{n-1}-a_n=a_{n-1}a_nd$

On adding all of these, we get

$a_1-a_n=d(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n) \quad \ldots(i)$

Also, $\quad \frac{1}{a_n}=\frac{1}{a_1}+(n-1)d \Rightarrow d=\frac{a_1-a_n}{a_1a_n(n-1)}$

On putting the value of d in Eq. (i), we get

$$a_1-a_n=\frac{a_1-a_n}{a_1a_n(n-1)}(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)$$

$\Rightarrow \quad a_1a_2+a_2a_3+\ldots+a_{n-1}a_n=a_1a_n(n-1)$

44. Given that, $\quad x=\sum_{n=0}^{\infty}a^n, \quad y=\sum_{n=0}^{\infty}b^n, \quad z=\sum_{n=0}^{\infty}c^n$

$\Rightarrow \quad x=1+a+a^2+\ldots=\frac{1}{1-a} \quad \ldots(i)$

Similarly, $\quad y=\frac{1}{1-b}$ and $z=\frac{1}{1-c} \quad \ldots(ii)$

Now, a, b and c are in AP.

$\Rightarrow -a, -b$ and $-c$ are in AP.

$\Rightarrow 1-a, 1-b$ and $1-c$ are also in AP.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}$ and $\frac{1}{1-c}$ are in HP.

$\Rightarrow x, y$ and z are in HP.

4 Complex Numbers

JEE Main MILESTONE

- Complex Numbers
- Equality of Complex Numbers
- Representation of a Complex Number in a Plane
- Algebra of Complex Numbers
- Conjugate of a Complex Number
- Argument/Amplitude of Complex Numbers
- Triangle Inequality
- Square Root of a Complex Number
- Concept of Rotation
- De-Moivre's Theorem
- nth Roots of Unity
- Application of Complex Numbers in Coordinate Geometry
- Equation of a Circle

4.1 Complex Numbers

The numbers of the form $x + iy$ are known as complex numbers. Here, x and y are real numbers and $i = \sqrt{-1}$ is iota.

OR

A complex number may also be defined as an **ordered pair of real numbers** and it is denoted by the symbol (x, y). If we write $z = (x, y)$, then x is called the real part and y is called the imaginary part of z.

The Italian Mathematician Gerolamo Cardano is the first known to have introduced complex numbers. Complex numbers extend the idea of the one dimensional number line to the two dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part.

The complex number is usually denoted by z and its set is denoted by C.

$\therefore \quad C = \{x + iy : x, y \in R, i = \sqrt{-1}\}$

e.g., $7 + 2i$, $0 + i$, $1 + 0i$, etc., are complex numbers.

Integral Powers of Iota $i = \sqrt{-1}$ is called the imaginary unit.

Also, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

In general $i^{4n} = 1$, $i^{4n+1} = -i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, for any integer n.

e.g., $i^{1998} = i^{4 \times 499 + 2} = -1$

Note

- $i = -\dfrac{1}{i}$
- The sum of any four consecutive powers of i is zero. *i.e.*,
$$i^{4n+1} + i^{4n+2} + i^{4n+3} + i^{4n+4} = 0$$
- $\sqrt{-a} = i\sqrt{a}$, when a is any real number.

  Then, $\sqrt{-a}\sqrt{-b} = i\sqrt{a}\, i\sqrt{b} = -\sqrt{ab}$

  But $\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$ is wrong.

***Sample* Problem 1** *The value of* $(1+i)^4\left(1+\frac{1}{i}\right)^4$ *is*

(a) 12 (b) 2
(c) 8 (d) 16

Interpret (d) $(1+i)^4 \times \left(1+\frac{1}{i}\right)^4 = (1+i)^4 \times (1-i)^4$

$= (1-i^2)^4 = (1+1)^4 = 2^4 = 16$

Representation of Complex Numbers in the form $x + iy$

Let $z = x + iy$ is a complex number, then x is called the real part of z and is denoted by Re (z) and y is called the imaginary part of z and is denoted by Im (z).

e. g., If $z = 7 + 4i$, then Re $(z) = 7$ and Im $(z) = 4$.

A complex number z is said to be purely real, if Im$(z) = 0$ and is said to be purely imaginary, if Re $(z) = 0$. The complex number $0 = 0 + i0$ is both purely real and purely imaginary.

Every real number a can be written as $a + i0$. Therefore, every real number is considered as a complex number, whose imaginary part is zero.

4.2 Equality of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then these two numbers are equal, if

$$x_1 = x_2 \text{ and } y_1 = y_2$$

i.e., $\text{Re}(z_1) = \text{Re}(z_2)$

and $\text{Im}(z_1) = \text{Im}(z_2)$

e. g., If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, then

$$2 - iy = x + 3i$$

$\Rightarrow$ $x = 2$ and $y = -3$

Two complex numbers cannot be compared *i.e.*, no greater complex number can be find in two given complex numbers.

***Sample* Problem 2** *The real values of x and y for which the following equality hold, are respectively*

$$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$$

(a) 2, 3 *or* -2, 1/3 (b) 1, 3 *or* -1, 1/3
(c) 2, 1/3 *or* -2, 3 (d) *None of these*

Interpret (a) The given equality can be rewritten as

$$(x^4 - 3x^2) + i(2x - y) = 4 + i(2y - 5)$$

$\Rightarrow$ $x^4 - 3x^2 = 4$ and $2x - y = 2y - 5$

$\Rightarrow$ $x^4 - 3x^2 - 4 = 0$ and $2x - 3y + 5 = 0$

$\Rightarrow$ $(x^2 - 4)(x^2 + 1) = 0$

$\Rightarrow$ $x = \pm 2$ $(\because x^2 \neq -1)$

$\therefore$ At $x = 2$, $y = 3$ and at $x = -2$, $y = \frac{1}{3}$

4.3 Representation of a Complex Number in a Plane

Complex numbers can be represented as follows

(i) Geometrical Representation of a Complex Number

The complex number $z = x + iy$ may be represented graphically by the point P, whose rectangular coordinates are (x, y). Thus, each point in the plane is associated with a complex number. In the figure, P defines $z = x + iy$. It is customary to choose x-axis as real axis and y-axis as imaginary axis. Such a plane is called argand plane or argand diagram or complex plane or gaussian plane.

Distance of P from origin is $OP = \sqrt{x^2 + y^2}$. It is called the modulus of z and angle of OP with positive direction of x-axis is called argument of z.

$\therefore$ $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(ii) Trigonometrical or Polar Representation of a Complex Number

Let $z = x + iy$ is a complex number which is denoted by a point $P(x, y)$ in a complex plane, then

$OP = |z|$ and $\angle POX = \theta = \arg(z)$

In ΔPOM,

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|}$$

$\Rightarrow$ $x = |z|\cos\theta$

and $\sin\theta = \frac{PM}{OP} = \frac{y}{|z|}$

$\Rightarrow$ $y = |z|\sin\theta$

$\therefore$ $z = x + iy$

$\Rightarrow$ $z = |z|\cos\theta + i|z|\sin\theta$

$\Rightarrow \quad z = |z|(\cos\theta + i\sin\theta)$

$\Rightarrow \quad z = r(\cos\theta + i\sin\theta)$

where, $\quad r = |z|$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

This form of z is known as **polar form.**

In general, polar form is

$$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$$

where, $\quad r = |z|, \theta = \arg(z)$ and $n \in N$.

(iii) Eulerian Form of a Complex Number

We have, $\quad e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

These two are called Euler's notations.

Let z be any complex number such that $|z| = r$ and $\arg(z) = \theta$. Then, $z = x + iy = r(\cos\theta + i\sin\theta)$ can be represented in exponential or Eulerian form as

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

4.4 Algebra of Complex Numbers

(i) Addition of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

$\Rightarrow \quad \text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\quad \text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Geometrical Representation

Let the complex numbers z_1 and z_2 be represented by the points P and Q on the argand plane.

Complete the parallelogram $OPRQ$. Then, the mid-points of PQ and OR are the same. The mid-point of $PQ = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Here,
$$R = (x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$
$$= (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2$$
$$= (x_1, y_1) + (x_2, y_2)$$

Properties of Addition of Complex Numbers

(a) $z_1 + z_2 = z_2 + z_1$ (commutative law)

(b) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative law)

(c) $z + 0 = 0 + z$ (where, $0 = 0 + i0$) (additive identity law)

***Sample* Problem 3** *The value of $3(7 + i7) + i(7 + i7)$ is* [NCERT]

(a) $15 + 27i$ (b) $14 + 28i$

(c) $14 - 28i$ (d) $14 + 23i$

Interpret (b) We have, $3(7 + i7) + i(7 + i7)$

$$= 21 + 21i + 7i + 7i^2$$
$$= 21 + 28i - 7 \quad (\because i^2 = -1)$$
$$= 14 + 28i$$

(ii) Subtraction of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

$\Rightarrow \quad \text{Re}(z_1 - z_2) = \text{Re}(z_1) - \text{Re}(z_2)$

and $\quad \text{Im}(z_1 - z_2) = \text{Im}(z_1) - \text{Im}(z_2)$

Geometrical Representation

We first represent $-z_2$ by Q', so that QQ' is bisected at O. Complete the parallelogram $OPRQ'$. Then, the point R represents the difference $z_1 - z_2$.

We see that, $ORPQ$ is a parallelogram, so that

$$\mathbf{OR} = \mathbf{OP}$$

We have, in vectorial notation

$$z_1 - z_2 = \mathbf{OP} - \mathbf{OQ}$$
$$= \mathbf{OP} + \mathbf{QO}$$
$$= \mathbf{OP} + \mathbf{PR}$$
$$= \mathbf{OR} = \mathbf{QP}$$

Note The subtraction of two complex numbers is equivalent to addition of first with additive inverse of second.

***Sample* Problem 4** *The value of*

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right) \text{ is}$$

(a) $\frac{5}{3}-i\frac{17}{3}$ (b) $\frac{17}{3}-i\frac{5}{3}$

(c) $\frac{17}{3}+i\frac{5}{3}$ (d) $\frac{17}{5}-i\frac{4}{3}$

Interpret (c) $\left(\frac{1}{3}+i\frac{7}{3}+4+i\frac{1}{3}\right)+\frac{4}{3}-i$

$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)=\left(\frac{1}{3}+\frac{4}{1}+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-\frac{1}{1}\right)$$

$$=\frac{(1+12+4)}{3}+i\left(\frac{7+1-3}{3}\right)=\frac{17}{3}+i\frac{5}{3}$$

(iii) Multiplication of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$
$$\Rightarrow \quad z_1 \cdot z_2 = [\text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)] + i[\text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)]$$

Geometrical Representation

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\therefore \quad |z_1| = r_1$ and $\arg(z_1) = \theta_1$

and $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$

$\therefore \quad |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $\quad z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$\therefore \quad |z_1z_2| = r_1r_2$

and $\quad \arg(z_1z_2) = \theta_1 + \theta_2$

Let P and Q represent the complex numbers z_1 and z_2, respectively.

$\therefore \quad OP = r_1, OQ = r_2$

$\angle POX = \theta_1$ and $\angle QOX = \theta_2$

Take a point A on the real axis OX, such that $OA = 1$ unit. Complete the ΔOPA. Now, taking OQ as the base, construct a ΔOQR similar to OPA so that

$$\frac{OR}{OQ} = \frac{OP}{OA}$$

i.e., $\quad OR = OP \cdot OQ = r_1r_2 \quad (\because OA = 1)$

and $\quad \angle ROX = \angle ROQ + \angle QOX = \theta_1 + \theta_2$

Hence, R is the point representing product of complex numbers z_1 and z_2.

Properties of Multiplication of Complex Numbers

(a) $z_1 \cdot z_2 = z_2 \cdot z_1$ (commutative law)

(b) $(z_1 \cdot z_2)z_3 = z_1(z_2 \cdot z_3)$ (associative law)

(c) If $z_1 \cdot z_2 = 1 = z_2 \cdot z_1$, then z_1 and z_2 are multiplicative inverse of each other.

(d) (i) $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$ (left distribution law)

(ii) $(z_2 + z_3)z_1 = z_2 \cdot z_1 + z_3 \cdot z_1$ (right distribution law)

***Sample* Problem 5** *The real values of x and y, if*

$$\frac{(1+i)x-2i}{(3+i)}+\frac{(2-3i)y+i}{(3-i)} = i, \text{ are respectively}$$

(a) 3, −1 (b) 3, 1

(c) −3, 1 (d) −3, −1

Interpret (a) $\frac{(1+i)x-2i}{(3+i)}+\frac{(2-3i)y+i}{(3-i)} = i$

$$\Rightarrow \quad \{(1+i)x-2i\}(3-i) + \{(2-3i)y+i\}(3+i) = i(3+i)(3-i)$$

$$\Rightarrow (1+i)(3-i)x - 2i(3-i) + (2-3i)(3+i)y + i(3+i) = 10i$$

$$\Rightarrow \quad (4+2i)x - 6i - 2 + (9-7i)y + 3i - 1 = 10i$$

$$\Rightarrow \quad (4x - 2 + 9y - 1) + i(2x - 6 - 7y + 3) = 10i$$

$$\Rightarrow \quad (4x + 9y - 3) + i(2x - 7y - 3) = 10i$$

On equating real and imaginary parts on both sides, we get

$$4x + 9y = 3 \quad \text{...(i)}$$

and $\quad 2x - 7y = 13 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get $x = 3, \quad y = -1$

***Sample* Problem 6** *The multiplicative inverse of* $4 - 3i$ *is* **[NCERT]**

(a) $\frac{4}{25}-\frac{3i}{25}$ (b) $\frac{4}{25}+\frac{3i}{25}$

(c) $\frac{4}{16}+\frac{3i}{25}$ (d) *None of these*

Interpret (b) Let $z = 4 - 3i$

Then, its multiplicative inverse is

$$\frac{1}{z} = \frac{1}{4-3i} = \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{16-9i^2}$$

$$[\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{4+3i}{16+9} \quad (\because i^2 = -1)$$

$$= \frac{4+3i}{25} = \frac{4}{25} + \frac{3i}{25}$$

(iv) Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ $(\neq 0)$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \frac{1}{x_2^2 + y_2^2}[(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)]$$

Geometrical Representation

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$

$\therefore$ $|z_1| = r_1$ and $\arg(z_1) = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$

$\therefore$ $|z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \qquad (z_2 \neq 0, r_2 \neq 0)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$\therefore$ $$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}, \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

Let P and Q represent the complex numbers z_1 and z_2, respectively.

$\therefore$ $$OP = r_1, OQ = r_2$$
$$\angle POX = \theta_1 \text{ and } \angle QOX = \theta_2$$

Let OS be new position of OP. Take a point A on the real axis OX, such that $OA = 1$ unit and through A draw a line making with OA an angle equal to the $\angle OQP$ and meeting OS in R. Then, R represented by (z_1/z_2).

Now, in similar ΔOPQ and ΔOAR,

$$\frac{OR}{OA} = \frac{OP}{OQ}$$

$\Rightarrow$ $$OR = \frac{r_1}{r_2}, \text{ since } OA = 1$$

and $$\angle AOR = \angle POR - \angle POX = \theta_2 - \theta_1$$

Hence, the vecterial $\angle R$ is $-(\theta_2 - \theta_1)$.

i.e., $$\theta_1 - \theta_2$$

Note The division of two complex numbers is equivalent to multiplication of first with multiplicative inverse of second.

***Sample* Problem 7** *The value of* $\frac{z_1}{z_2}$, *where* $z_1 = 2 + 3i$ *and* $z_2 = 1 + 2i$, *is*

(a) $\frac{8}{5} + \frac{1}{5}i$ (b) $\frac{8}{5} - \frac{1}{5}i$

(c) $\frac{1}{5} - \frac{8}{5}i$ (d) *None of these*

Interpret (b) $\because$ $z_1 = 2 + 3i$ and $z_2 = 1 + 2i$

$\therefore$ $$z_2^{-1} = \frac{1}{1+2i} = \frac{1-2i}{(1+2i)(1-2i)}$$

$$= \frac{1-2i}{1+4} = \frac{1}{5} - \frac{2}{5}i$$

Thus, $$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = (2+3i)\left(\frac{1}{5} - \frac{2}{5}i\right)$$

$$= \left(\frac{2}{5} + \frac{6}{5}\right) + i\left(-\frac{4}{5} + \frac{3}{5}\right) = \frac{8}{5} - \frac{1}{5}i$$

Alternate Solution

Here, $x_1 = 2, y_1 = 3, x_2 = 1$ and $y_2 = 2$

$\therefore$ $$\frac{z_1}{z_2} = \frac{1}{x^2 + y^2}\{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)\}$$

$$= \frac{1}{1^2 + 2^2}\{(2 \times 1 + 3 \times 2) + i(3 \times 1 - 2 \times 2)\}$$

$$= \frac{1}{5}\{(6 + 2) + i(3 - 4)\} = \frac{8}{5} - \frac{1}{5}i$$

4.5 Conjugate of a Complex Number

Let $z = x + iy$ is a complex number, then conjugate of z is denoted by $\bar{z}$ or z' and is equal to $x - iy$.

Thus, $$\bar{z} = x - iy$$

Geometrically, the conjugate of z is the reflection of point image of z in the real axis.

e. g., If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$

Note From the definition it is clear that conjugate of a complex number can be obtained by replacing i by $-i$.

Properties of Conjugate of Complex Numbers

If z, z_1 and z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(iii) $z - \bar{z} = i\,[2\,\mathrm{Im}(z)]$

(iv) $z = \bar{z} \Rightarrow z$ is purely real.

(v) $z = -\bar{z} \Rightarrow z$ is purely imaginary.

(vi) $z\bar{z} = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(x) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(xi) $\bar{z}^n = (\bar{z})^n$

***Sample* Problem 8** *If $(x - iy)(3 + 5i)$ is the conjugate of $(-6 - 24i)$, then values of x and y are respectively* [NCERT]

(a) 3, –3 (b) 3, 4

(c) – 3, 4 (d) – 3, – 4

Interpret (a) $(x - iy)(3 + 5i) = 3x + 5xi - 3yi - 5yi^2$

$= 3x + (5x - 3y)i + 5y \quad (\because i^2 = -1)$

$= (3x + 5y) + (5x - 3y)i \quad \ldots(i)$

Given, $(x - iy) \cdot (3 + 5i) = (-6 - 24i)$

$\Rightarrow (3x + 5y) + i(5x - 3y) = -6 + 24i$

[using Eq. (i) and $z = (a + ib) \Rightarrow \bar{z} = (a - ib)$]

Comparing the real and imaginary parts on both sides, we get

$3x + 5y = -6$ and $5x - 3y = 24$

Solving the above equations by substitution or elimination method, we get $x = 3, y = -3$.

***Sample* Problem 9** *If $x + iy = (1 + i)(1 + 2i) \ldots (1 + ni)$, then the value of $x^2 + y^2$ is equal to*

(a) $2.5 \ldots (1 + n^2)$ (b) $2.4 \ldots (1 + n^2)$

(c) $2.5 \ldots (1 - n^2)$ (d) *None of these*

Interpret (a) Given, $x + iy = (1 + i)(1 + 2i) \ldots (1 + ni) \quad \ldots(i)$

$\therefore$ Conjugate of above complex number is

$x - iy = (1 - i)(1 - 2i) \ldots (1 - ni) \quad \ldots(ii)$

On multiplying Eqs. (i) and (ii), we get

$x^2 - (iy)^2 = (1^2 - i^2)[1 - (2i)^2] \ldots [1^2 - (ni)^2]$

$\Rightarrow x^2 + y^2 = (1 + 1)(1 + 4) \ldots (1 + n^2)$

$= 2 \cdot 5 \ldots (1 + n^2)$

Hot Spot 1 Modulus of a Complex Number

Let $z = x + iy$ is a complex number, then modulus of a complex number z is denoted by $|z|$.

$\therefore \quad |z| = \sqrt{x^2 + y^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}$

e.g., If $z = 4 + 3i$ is a complex number, then

$|z| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

In geometrical representation, the modulus of a complex number is equal to the distance between origin and point P. *i.e.*, $|OP| = \sqrt{x^2 + y^2}$

Properties of Modulus of Complex Numbers

(i) $|z| \geq 0 \Rightarrow |z| = 0$, iff $z = 0$ and $|z| > 0$, iff $z \neq 0$

(ii) $-|z| \leq \mathrm{Re}(z) \leq |z|$ and $-|z| \leq \mathrm{Im}(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n|$

$\leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

(viii) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

(ix) $|z^n| = |z_2|^n$

(x) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.

(xi) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

$= |z_1|^2 + |z_2|^2 \pm \text{Re}(z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

where, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(xiii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xv) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$

Note If z is unimodular, then $|z| = 1$. Now, if $f(z)$ is a unimodular, then it always be expressed as $f(z) = \cos\theta + i\sin\theta$, $\theta \in R$.

These types of problems are indirectly asked in the various years in AIEEE (JEE Main) 2010, 2009, 2007, 2005, 2004, 2003. *To solve such types of problems, following steps are given below*

Step I Either put $z = x + iy$ or use the step II in the given equation.

Step II Use the property of modulus.

Step III Simplify it and get the result.

***Sample* Problem 10** *Find the locus of a complex number $z = x + iy$, which satisfy the equation* $\left|\frac{z - 5i}{z + 5i}\right| = 1$.

(a) x-axis (b) y-axis

(c) origin (d) *None of these*

Interpret (a) **Step I** Put $z = x + iy$ in the given equation.

$$\therefore \quad \left|\frac{x + iy - 5i}{x + iy + 5i}\right| = 1$$

$$\Rightarrow \quad \left|\frac{x + i(y - 5)}{x + i(y + 5)}\right| = 1$$

Step II Using the property of modulus,

i.e., $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \therefore \left|\frac{x + i(y - 5)}{x + i(y + 5)}\right| = 1$

Step III Simplify it

$$\therefore \quad \sqrt{x^2 + (y - 5)^2} = \sqrt{x^2 + (y + 5)^2}$$

$$\Rightarrow \quad x^2 + (y - 5)^2 = x^2 + (y + 5)^2$$

$$\Rightarrow \quad x^2 + y^2 - 10y + 25 = x^2 + y^2 + 25 + 10y$$

$$\Rightarrow \quad 20y = 0 \Rightarrow y = 0$$

Hence, it lies on x-axis.

***Sample* Problem 11** *If $x - iy = \sqrt{\frac{a - ib}{c - id}}$, then $(x^2 + y^2)^2$ is equal to* [NCERT Exemplar]

(a) $\frac{a^2 - b^2}{c^2 - d^2}$ (b) $\frac{a^2 + b^2}{c^2 + d^2}$

(c) $\frac{a^2 + b^2}{c^2 - d^2}$ (d) *None of these*

Interpret (b) Given, $x - iy = \sqrt{\frac{a - ib}{c - id}} \Rightarrow x + i(-y) = \left(\frac{a - ib}{c - id}\right)^{1/2}$

On taking modulus both sides, we get

$$|x + i(-y)| = \left|\left(\frac{a - ib}{c - id}\right)^{\frac{1}{2}}\right|$$

$$\Rightarrow \quad \sqrt{x^2 + (-y)^2} = \left|\frac{a - ib}{c - id}\right|^{\frac{1}{2}} \quad \left(\because |x + iy| = \sqrt{x^2 + y^2} \text{ and } |z^n| = |z|^n\right)$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} = \left|\frac{a - ib}{c - id}\right|^{\frac{1}{2}}$$

On squaring both sides, we get

$$x^2 + y^2 = \left|\frac{a - ib}{c - id}\right|$$

$$\Rightarrow \quad x^2 + y^2 = \frac{|a - ib|}{|c - id|} \quad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$\Rightarrow \quad (x^2 + y^2) = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \quad (\because |x - iy| = \sqrt{x^2 + y^2})$$

On squaring both sides, we get

$$(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

4.6 Argument/Amplitude of Complex Numbers

Let $z = x + iy$ be a non-zero complex number and can be represented in the form $z = r(\cos\theta + i\sin\theta)$ where, r is the modulus and θ is the argument of z. From the figure, let z be represented by a line OP inclined at an angle θ with the positive direction of x-axis or the angle measured in the counter clockwise direction and the distance of the point from O in the direction θ is r, *i.e.*, $|z|$. Then, in right angled ΔOPM, right angled at M, we have

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

or $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Principal Value of Argument

The value of θ of the argument which satisfies the inequality $-\pi < \theta \le \pi$ is called the principal value of the argument.

Principal values of the argument are $\theta, \pi - \theta, -\pi + \theta, -\theta$ according as the complex number lies on the Ist, IInd, IIIrd or IVth quadrant.

Note Argument of 0 is not defined.

Properties of Argument of Complex Numbers

If z_1, z_2 and z_3 are three complex numbers, then

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ ($k = 0$ or 1 or – 1)

In general, $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n) + 2k\pi$ ($k = 0$ or 1 or – 1)

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$ ($k = 0$ or 1 or – 1)

(iii) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi$ ($k = 0$ or 1 or – 1)

(iv) $\arg(z^n) = n\arg(z) + 2k\pi$ ($k = 0$ or 1 or – 1)

(v) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

(vi) $\arg(\bar{z}) = -\arg(z)$

(vii) If $\arg(z) = 0 \Rightarrow z$ is real.

(viii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(ix) $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(x) $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \arg(z_1) = \arg(z_2)$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xii) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

***Sample* Problem 12** *Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then, z is equal to*

(a) w (b) $-w$ (c) $-\bar{w}$ (d) $\bar{w}$

Interpret (c) Let $|z| = |w| = r$ and $\arg(w) = \theta$

Also, $\arg(z) + \arg(w) = \pi$

$\Rightarrow \arg(z) = \pi - \theta$

Now, $z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$

$= r[-\cos\theta + i\sin\theta]$

$= -r(\cos\theta - i\sin\theta)$

$= -\bar{w}$

***Sample* Problem 13** *The modulus and argument of the complex number $\frac{1+2i}{1-3i}$ is* [NCERT]

(a) $\frac{1}{\sqrt{2}}, \frac{3\pi}{4}$ (b) $\frac{1}{\sqrt{2}}, -\frac{3\pi}{4}$

(c) $\frac{1}{2}, \frac{3\pi}{4}$ (d) *None of these*

Interpret (a) Let $z = \frac{1+2i}{1-3i}$

$\therefore \quad z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2-(3i)^2}$

$[\because (a+b)(a-b) = a^2 - b^2]$

$= \frac{1+5i+6(-1)}{1-9i^2}$ $(\because i^2 = -1)$

$= \frac{1+5i-6}{1+9} = \frac{-5+5i}{10} = \frac{-1+i}{2}$

$\Rightarrow \quad z = -\frac{1}{2} + \frac{1}{2}i$

$\therefore \quad |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$ $(\because |a+ib| = \sqrt{a^2+b^2})$

$= \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\therefore$ Now, $\tan\theta = \left|\frac{\frac{1}{2}}{-\frac{1}{2}}\right|$ $\left[\because \theta = \tan^{-1}\left|\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)\right|\right]$

$\Rightarrow \quad \tan\theta = 1 = \tan\frac{\pi}{4}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

Since, the real part of z is negative and imaginary part of z is positive, so the point lies in IInd quadrant.

$\therefore \quad \arg(z) = \pi - \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, modulus $= \frac{1}{\sqrt{2}}$ and $\arg(z) = \frac{3\pi}{4}$

4.7 Triangle Inequality

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side.

(i) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(ii) $|z_1 \pm z_2| \ge \big||z_1| - |z_2|\big|$

(iii) In general, $|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

Sample **Problem 14** *For any complex number z the minimum value of* $|z| + |z - 2|$ *is*

(a) 4 (b) 2
(c) 1 (d) 3

Interpret (b) $\therefore$ $|z| + |z - 2| = |z| + |2 - z|$

$\ge |z + (2 - z)|$ $(\because |z_1 + z_2| \le |z_1| + |z_2|)$

$\Rightarrow$ $|z| + |z - 2| \ge 2$

Check Point 1

1. If n is a positive integer, then $\left(\frac{1+i}{1-i}\right)^{4n+1}$ is equal to
(a) 1 (b) -1 (c) i (d) $-i$

2. $a + ib > c + id$ can be explained only when
(a) $b = 0, c = 0$ (b) $b = 0, d = 0$
(c) $a = 0, c = 0$ (d) $a = 0, d = 0$

3. Let z_1 and z_2 be two complex numbers such that $z_1 + z_2$ and $z_1 z_2$ both are real, then
(a) $z_1 = -z_2$ (b) $z_1 = \overline{z_2}$
(c) $z_1 = -\overline{z_2}$ (d) None of these

4. The conjugate of complex number $\frac{2-3i}{4-i}$ is
(a) $\frac{3i}{4}$ (b) $\frac{11+10i}{17}$
(c) $\frac{11-10i}{17}$ (d) $\frac{2+3i}{4i}$

5. If $\frac{z-i}{z+i}(z \ne -i)$ is purely imaginary number, then $z\bar{z}$ is equal to
(a) 0 (b) 1
(c) 2 (d) None of these

6. If $\frac{2z_1}{3z_2}$ is purely imaginary number, then $\left|\frac{z_1 - z_2}{z_1 + z_2}\right|$ is equal to
(a) $\frac{3}{2}$ (b) 1 (c) $\frac{2}{3}$ (d) $\frac{4}{9}$

7. If $z = \frac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$, then arg (z) is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$
(c) $-\frac{2\pi}{3}$ (d) None of these

8. If $(\sqrt{8} + i)^{50} = 3^{49}(a + ib)$, then $a^2 + b^2$ is equal to
(a) 3 (b) 8 (c) 9 (d) $\sqrt{8}$

9. If arg $z < 0$, then arg $(-z)$ – arg (z) is equal to
(a) π (b) $-\pi$
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

10. For any complex number z, the minimum value of $|z| + |z - 2i|$ is equal to
(a) 0 (b) 1
(c) 2 (d) None of these

4.8 Square Root of a Complex Number

Let $a + ib$ is a complex number such that $\sqrt{a + ib} = x + iy$, where x and y are real numbers.

Now, $\sqrt{a + ib} = x + iy \Rightarrow (x + iy)^2 = a + ib$

$\Rightarrow$ $(x^2 - y^2) + 2ixy = a + ib$

$\Rightarrow$ $x^2 - y^2 = a$ ...(i)

and $2xy = b$ ...(ii)

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow$ $(x^2 + y^2)^2 = a^2 + b^2$

$\Rightarrow$ $(x^2 + y^2) = \sqrt{a^2 + b^2}$ ...(iii)

$(\because x^2 + y^2 > 0)$

On solving Eqs. (i) and (iii), we get

$$x^2 = \left(\frac{1}{2}\right)(\sqrt{a^2 + b^2} + a)$$

and
$$y^2 = \left(\frac{1}{2}\right)(\sqrt{a^2 + b^2} - a)$$

$\Rightarrow$
$$x = \pm\sqrt{\left(\frac{1}{2}\right)[\sqrt{a^2 + b^2} + a]}$$

and
$$y = \pm\sqrt{\left(\frac{1}{2}\right)[\sqrt{a^2 + b^2} - a]}$$

If b is positive, then the sign of x and y from Eq. (ii) will be same *i.e.*,

$$\sqrt{a + ib} = \pm\left[\sqrt{\left(\frac{1}{2}\right)(\sqrt{a^2 + b^2} + a)} + i\sqrt{\left(\frac{1}{2}\right)(\sqrt{a^2 + b^2} - a)}\right]$$

If b is negative, then the sign of x and y will be opposite,

i.e.,

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{1}{2}\right)\left(\sqrt{a^2+b^2}+a\right)} - i\sqrt{\left(\frac{1}{2}\right)\left(\sqrt{a^2+b^2}-a\right)}\right]$$

Note If $x, y \in R$, then

- $\sqrt{x+iy} + \sqrt{x-iy} = \sqrt{2\left(\sqrt{x^2+y^2}+x\right)}$
- $\sqrt{x+iy} - \sqrt{x-iy} = i\sqrt{2\left(\sqrt{x^2+y^2}-x\right)}$

***Sample* Problem 15** *The value of*

$$(4+3\sqrt{-20})^{1/2} + (4-3\sqrt{-20})^{1/2} \text{ is}$$

(a) ± 6 (b) 0
(c) $\pm\sqrt{5}$ (d) ± 3

Interpret (a) We may write,

$$(4+3\sqrt{-20}) = (4+6i\sqrt{5})$$

Let $(4+3\sqrt{-20})^{1/2} = (x+iy)$

Then, $(4+6i\sqrt{5})^{1/2} = (x+iy)$

$\Rightarrow$ $4+6i\sqrt{5} = (x^2-y^2)+(2xy)i$

$\Rightarrow$ $x^2-y^2 = 4$ and $2xy = 6\sqrt{5}$

$\therefore$ $x^2+y^2 = \sqrt{(x^2-y^2)^2+4x^2y^2}$

$= \sqrt{(16+180)} = \sqrt{196} = 14$

On solving the equations $x^2+y^2 = 14$ and $x^2-y^2 = 4$, we get

$$x^2 = 9 \text{ and } y^2 = 5$$

$\therefore$ $x = \pm 3$ and $y = \pm\sqrt{5}$

Since, $xy > 0$, it follows that x and y are of the same sign.

$\therefore$ $x = 3, y = \sqrt{5}$ or $x = -3, y = -\sqrt{5}$

So, $(4+3\sqrt{-20})^{1/2} = (4+6i\sqrt{5})^{1/2}$

$= \pm(3+\sqrt{5}\,i)$ ...(i)

$\therefore$ $(4-3\sqrt{-20})^{1/2} = \pm(3-\sqrt{5}\,i)$ ...(ii)

Hence, $(4+3\sqrt{-20})^{1/2} + (4-3\sqrt{-20})^{1/2} = \pm 6$

[on adding Eqs. (i) and (ii)]

Alternate Solution Here, $x = 4, y = 6\sqrt{5}$

$\therefore$ $(4+3\sqrt{-20})^{1/2} + (4-3\sqrt{-20})^{1/2}$

$= \sqrt{2\{\sqrt{4^2+(6\sqrt{5})^2}+4\}} = \sqrt{2(\sqrt{16+180}+4)}$

$= \sqrt{2(14+4)} = \sqrt{36} = \pm 6$

4.9 Concept of Rotation

Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through $\angle\alpha$ in anti-clockwise sense it coincides with OQ.

Then, $\dfrac{z_3-z_1}{z_2-z_1} = \dfrac{OQ}{OP}(\cos\alpha + i\sin\alpha) = \dfrac{CA}{BA}e^{i\alpha} = \dfrac{|z_3-z_1|}{|z_2-z_1|}e^{i\alpha}$

$\left(\because \Delta OPQ \text{ and } \Delta ABC \text{ are congruent}, \therefore \dfrac{OQ}{OP} = \dfrac{CA}{BA}\right)$

or $\text{amp}\left(\dfrac{z_3-z_1}{z_2-z_1}\right) = \alpha$

***Sample* Problem 16** *The point represented by the complex number $2-i$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is*

[NCERT Exemplar]

(a) $1+2i$ (b) $-1-2i$ (c) $2+i$ (d) $-1+2i$

Interpret (b) Here, $z = 2-i$

Let z_1 be the required complex number.

$\therefore$ $|z_1| = |z| = \sqrt{2^2+1^2} = \sqrt{5}$

$\because$ $\dfrac{z_1}{z} = \dfrac{|z_1|}{|z|}e^{-i\frac{\pi}{2}}$

$\therefore$ $z_1 = \dfrac{(2-i)\sqrt{5}}{\sqrt{5}}\left[\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right]$

$= (2-i)\left(\cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}\right) = (2-i)(0-i)$

$= -(2i-i^2) = -2i-1$

4.10 De-Moivre's Theorem

A simple formula for calculating powers of complex numbers in the form of $\cos\theta$ and $\sin\theta$ is known as de-Moivre's theorem. If n is a rational number, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Applications of De-Moivre's Theorem

(i) If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

Then, $z = \cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

(ii) If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$(z)^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

where, $k = 0, 1, 2, 3, \ldots, (n-1)$

(iii) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(iv) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

(v) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

(vi) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^n$

$$= \left[\cos\left(\frac{n\pi}{2} - n\theta\right)\right] + i\left[\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$$

(vii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta \pm i\sin n\phi$

Note

- $1 = \cos 0 + i\sin 0$
- $-1 = \cos\pi + i\sin\pi$
- $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$
- $-i = \cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}$

***Sample* Problem 17** *If $a = \cos\theta + i\sin\theta$, then the value of $\dfrac{1+a}{1-a}$ is* **[NCERT]**

(a) $i\tan\dfrac{\theta}{2}$ (b) $i\cot\dfrac{\theta}{2}$

(c) $\cot\dfrac{\theta}{2}$ (d) *None of these*

Interpret (b) $a = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1+a}{1-a} = \frac{1+\cos\theta + i\sin\theta}{1-(\cos\theta + i\sin\theta)}$$

(applying componendo and dividendo)

$$= \frac{\left(2\cos^2\dfrac{\theta}{2}\right) + i\left(2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right)}{(1-\cos\theta) - i\sin\theta} \quad \left(\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right)$$

$$= \frac{2\cos^2\dfrac{\theta}{2} + 2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2} - 2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}} \quad \left(\because 1-\sin\theta = 2\sin^2\frac{\theta}{2}\right)$$

On multiplying by i in numerator and denominator,

$$= \frac{2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)}{2\sin\dfrac{\theta}{2}\left(\sin\dfrac{\theta}{2} - i\cos\dfrac{\theta}{2}\right)} = \frac{i\cot\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)}{\left(i\sin\dfrac{\theta}{2} - i^2\cos\dfrac{\theta}{2}\right)}$$

$$= \frac{i\cot\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)}{\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)} = i\cot\frac{\theta}{2}$$

***Sample* Problem 18** *If $x_r = \cos\left(\dfrac{\pi}{2^r}\right) + i\sin\left(\dfrac{\pi}{2^r}\right)$, then the value of $x_1x_2x_3\ldots\infty$ is*

(a) -1 (b) 1

(c) 0 (d) *None of these*

Interpret (a) $\because \quad x_r = \cos\left(\dfrac{\pi}{2^r}\right) + i\sin\left(\dfrac{\pi}{2^r}\right)$

$$\therefore \quad x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\therefore \; x_1 x_2 x_3 \ldots = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\ldots$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\right)$$

$$= \cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) + i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) = \cos\pi + i\sin\pi = -1$$

Hot Spot 2

Cube Roots of Unity

Let $\quad x = \sqrt[3]{1} \Rightarrow x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2 + x + 1) = 0$

Therefore, $\quad x = 1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

If second root be represented by ω, then third root will be ω^2.

$\therefore$ Cube roots of unity are $1, \omega, \omega^2$; 1 is a real root of unity and other two *i.e.*, ω and ω^2 are conjugate complex of each other.

Properties of Cube Roots of Unity

(i) $\omega^3 = 1$ or $\omega^{3r} = 1$

(ii) $\omega^{3r+1} = \omega, \; \omega^{3r+2} = \omega^2$

(iii) $1 + \omega^r + \omega^{2r} = 0$, if r is not a multiple of 3

$= 3$, if r is multiple of 3

(iv) Each complex cube root of unity is square of other and also reciprocal of each other.

(v) If a is any positive number, then $a^{1/3}$ has roots $a^{1/3}(1)$, $a^{1/3}(\omega)$, $a^{1/3}(\omega^2)$ and if a is any negative number, then $a^{1/3}$ has roots $-|a|^{1/3}, -|a|^{1/3}\omega, -|a|^{1/3}\omega^2$.

(vi) The cube roots of unity when represented on complex plane lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin. One vertex being on positive real axis.

Important Relations

(i) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(ii) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(iii) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(iv) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

These types of problems are directly or indirectly asked in various years in AIEEE (JEE Main) 2011, 2006, 2005, 2002. *The level of the question is on average. To solve such types of problems, following steps are given below*

Step I Firstly, if the roots of unity are not given, then we determine all the roots. In among them, some are real and some are complex roots.

Step II Use the property of cube roots of unity.

Step III Simplify it and get the result.

Sample **Problem 19** *If $z^2 + z + 1 = 0$, where z is a complex number, then find the value of*

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots \left(z^6 + \frac{1}{z^6}\right)^2$$

(a) 5 (b) 12 (c) –12 (d) 5

Interpret (b) **Step I** Find all roots of given equation

$\therefore \quad z^2 + z + 1 = 0$

$$z = \frac{-1 \pm \sqrt{1^2 - 4(1)}}{2(1)} = \frac{-1 \pm \sqrt{3}i}{2}$$

$\Rightarrow \quad z = \omega, \omega^2$, where $\omega = \dfrac{-1 + \sqrt{3}i}{2}$

and $\quad \omega^2 = \dfrac{-1 - \sqrt{3}i}{2}$

$$\therefore \quad \left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \left(z^4 + \frac{1}{z^4}\right)^2 + \left(z^5 + \frac{1}{z^5}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2$$

Put $z = \omega$, we get

$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega^2 + \frac{1}{\omega^2}\right)^2 + \left(\omega^3 + \frac{1}{\omega^3}\right)^2 + \left(\omega^4 + \frac{1}{\omega^4}\right)^2 + \left(\omega^5 + \frac{1}{\omega^5}\right)^2 + \left(\omega^6 + \frac{1}{\omega^6}\right)^2$$

$$= (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + (1 + 1)^2 + (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + (1 + 1)^2$$

$$= (-1)^2 + (-1)^2 + 4 + (-1)^2 + (-1)^2 + 4 \quad (\because 1 + \omega + \omega^2 = 0)$$

$$= 1 + 1 + 4 + 1 + 1 + 4 = 12$$

Similarly, for putting $z = \omega^2$, we get the result 12.

Sample **Problem 20** *If 1, ω and ω^2 be the cube roots of unity. Then, the value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ … upto $2n$ factors is*

(a) 2^{2n-1} (b) 2^{2n} (c) 2^{n-1} (d) 2^{2n+1}

Interpret (b) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ … upto $2n$ factors.

$= (1 - \omega + \omega^2)(1 - \omega^2 + \omega^3 \cdot \omega)(1 - \omega^3 \cdot \omega + \omega^6 \cdot \omega^2)$ … upto $2n$ factors

$= (1 - \omega + \omega^2)(1 - \omega^2 + \omega)(1 - \omega + \omega^2)$ … upto $2n$ factors

$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$ … upto $2n$ factors

$= [(-2\omega)(-2\omega^2)] \times [(-2\omega)(-2\omega^2)] \times \ldots$ upto $2n$ factors

$= (2^2\omega^3) \times (2^2\omega^3) \times \ldots$ to n factors

$= (2^2 \times 2^2 \times 2^2 \times \ldots$ to n factors$) = 2^{2n}$

4.11 nth Roots of Unity

Let $z = 1^{1/n}$, then $z = (\cos 0° + i \sin 0°)^{1/n}$

$\Rightarrow \quad z = (\cos 2r\pi + i \sin 2r\pi)^{1/n}, r \in I$

$\Rightarrow \quad z = \cos\dfrac{2r\pi}{n} + i \sin\dfrac{2r\pi}{n}, r = 0, 1, 2, \ldots, (n - 1)$

(using De-Moivre's theorem)

$\Rightarrow \quad z = [e^{\frac{i2\pi}{n}}]^r, r = 0, 1, 2, \ldots, (n - 1)$

$\Rightarrow \quad z = \alpha^r, \alpha = e^{\frac{i2\pi}{n}} = \cos\dfrac{2\pi}{n} + i \sin\dfrac{2\pi}{n}, r = 0, 1, 2, \ldots, (n - 1)$

Thus, nth roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where

$$\alpha = e^{\frac{i2\pi}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}.$$

Also, $\quad 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

$$\sum_{k=0}^{n-1} \alpha^k = 0 \Rightarrow \sum_{k=0}^{n-1} \cos\frac{2\pi k}{n} + i \sum_{k=0}^{n-1} \sin\frac{2\pi k}{n} = 0$$

Properties of nth Roots of Unity

(i) nth roots of unity form a GP with common ratio $e^{\frac{i2\pi}{n}}$.

(ii) Sum of nth roots of unity is always zero.

(iii) Product of nth roots of unity is $(-1)^{n-1}$.

(iv) nth roots of unity are the vertices of the regular polygon of n sides inscribed in a circle of radius unity centred at origin. One vertex being on the positive real axis.

If we put $n=3$ in $X^n=1$, $X^3=1$ Roots are $I_1\ \omega_1\ \omega_2$

Epuilitral

$n=4$
$X^4=1$ Roots are $I_1\ i_1 - i, -i$

Square

$n=5$ Root are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$

Pentegon

***Sample* Problem 21** *If* $1, a_1, a_2 \ldots, a_{n-1}$ *are the nth roots of unity, then the value of* $(1-a_1)(1-a_2)\ldots(1-a_{n-1})$ *is equal to*

(a) $\sqrt{3}$ (b) $\frac{1}{2}$

(c) n (d) 0

Interpret (c) Since, $1, a_1, a_2, \ldots, a_{n-1}$ are the nth roots of unity.

$$\therefore \quad x^n - 1 = (x-1)(x-a_1)\ldots(x-a_{n-1})$$

$$\Rightarrow \quad \frac{x^n-1}{x-1} = (x-a_1)(x-a_2)\ldots(x-a_{n-1})$$

$$\therefore \quad x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$$
$$= (x-a_1)(x-a_2)\ldots(x-a_{n-1})$$

Put $x=1$, we get

$$(1-a_1)(1-a_2)\ldots(1-a_{n-1}) = 1+1+\ldots+n \text{ times}$$
$$= n$$

Check Point **2**

1. The square root of $(-15-8i)$ is
(a) $\pm(1-4i)$ (b) $\pm(1+4i)$
(c) $\pm(-2+4i)$ (d) None of these

2. The common roots of the equations
$x^{12}-1=0$ and $x^4+x^2+1=0$ are
(a) $\pm\omega$ (b) $\pm\omega^2$
(c) $\pm\omega, \pm\omega^2$ (d) None of these

3. If α and β are imaginary cube roots of unity, then the value of $\alpha^4+\beta^{28}+\frac{1}{\alpha\beta}$ is
(a) 1 (b) -1
(c) 0 (d) None of these

4. $\dfrac{\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)^8}{\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)^8}$ is equal to
(a) -1 (b) 0
(c) 1 (d) $2i$

5. If $x+\frac{1}{x}=2\cos\frac{\pi}{10}$, then $x^5+\frac{1}{x^5}$ is equal to
(a) 0 (b) 32
(c) 10 (d) 2

6. One root of the equation $z^2-az+a-1=0$, is $(1+i)$, where a is a complex number, then other root is
(a) 1 (b) $1+i$
(c) $1-i$ (d) None of these

7. If $iz^4+1=0$, then z can take the value
(a) $\frac{1+i}{\sqrt{2}}$ (b) $\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$
(c) $\frac{1}{4i}$ (d) i

8. The value of the expression
$1(2-\omega)(2-\omega^2)+2(3-\omega)(3-\omega^2)+\ldots+(n-1)(n-\omega)(n-\omega^2)$,
where ω is an imaginary cube root of unity is
(a) $\left[\frac{n(n+1)}{2}\right]^2$ (b) $\left[\frac{n(n+1)}{2}\right]^2-n$
(c) $\left[\frac{n(n+1)}{2}\right]^2+n$ (d) None of these

9. The value of the expression
$$\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}} \text{ is}$$
(a) $\cos 49\theta + i\sin 49\theta$ (b) $\cos 49\theta - i\sin 49\theta$
(c) $-\cos 49\theta + i\sin 49\theta$ (d) None of these

4.12 Application of Complex Numbers in Coordinate Geometry

(i) Distance between Two Points

Distance between two points $P(z_1)$ and $Q(z_2)$ is

$$PQ = |z_2 - z_1|$$

(ii) Section Formula

Let $R(z)$ divides a join of $P(z_1)$ and $Q(z_2)$ in the ratio $m : n$.

(a) If $R(z)$ divides the line segment PQ internally, then

$$z = \frac{mz_2 + nz_1}{m + n}$$

(b) If $R(z)$ divides the line segment PQ externally, then

$$z = \frac{mz_2 - nz_1}{m - n}$$

Note

- Distance of a point $P(z)$ from the origin $= |z|$.
- If $R(z)$ is a mid-point of PQ, then $z = \dfrac{z_1 + z_2}{2}$.

- Three points will be collinear, if for $A(z_1)$, $B(z_2)$, $C(z_3)$,

$$AB + BC = AC$$

i.e., $\quad |z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$

(iii) Equation of Perpendicular Bisector

If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is an equidistant point from P and Q. Then,

$$|z - z_1| = |z - z_2|$$

$$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

Hence, $R(z)$ lies on perpendicular bisector of $P(z_1)$ and $Q(z_2)$.

(iv) Equation of Straight Line

(a) **Parametric form** Equation of line joining points $P(z_1)$ and $Q(z_2)$ is $z = tz_1 + (1 - t)z_2$, where $t \in R$.

(b) **Non-parametric form** Equation of line joining points $P(z_1)$ and $Q(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

(c) **General equation** General equation of straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

Note

- Three points z_1, z_2 and z_3 will be collinear, if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$
- Slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\dfrac{a}{\bar{a}}$.
- If α_1 and α_2 are slopes of two lines in a complex plane, then
  (a) lines will be parallel, if $\alpha_1 = \alpha_2$.
  (b) lines will be perpendicular, if $\alpha_1 + \alpha_2 = 0$.
- Slope of a line PQ joining two points $P(z_1)$ and $Q(z_2) = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.
- Length of perpendicular from a point $P(z_1)$ to the line $\bar{a}z + a\bar{z} + b = 0 = \dfrac{|a\bar{z}_1 + \bar{a}z_1 + b|}{|a| + |\bar{a}|} = \dfrac{|a\bar{z}_1 + \bar{a}z_1 + b|}{2|\bar{a}|}$

Some Important Results

(i) The coordinates of centroid of a ΔABC whose vertex are $A(z_1)$, $B(z_2)$ and $C(z_3)$, is $G(z) = \dfrac{z_1 + z_2 + z_3}{3}$

(ii) The triangle whose vertices are z_1, z_2 and z_3 is equilateral iff

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

or $\quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

(iii) If $\arg\left[\dfrac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right] = \pm\pi$, 0 (or purely real), then the points z_1, z_2, z_3 and z_4 are concyclic.

(iv) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0 \Rightarrow$ Locus of z is a straight line passing through z_1 and z_2.

4.13 Equation of a Circle

A circle is the locus of all those points which are equidistant from the given fixed point. This fixed point is called the centre and the fixed distance is called radius.

(i) Equation of a circle whose radius is r and centre is $C(z_0)$, is $|z - z_0| = r$.

If the centre of circle lies on the origin, then equation of circle is $|z| = r$.

(ii) The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where $a \in C$ and $b \in R$. Centre of circle is at $-a$ and radius is $\sqrt{|a|^2 - b}$.

(iii) If $P(z_1)$ and $Q(z_2)$ are the vertices of diameter of a circle, then equation of circle is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

Nature of Circle

The general equation of circle $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a

(a) real circle, if $|a|^2 > b$

(b) point circle, if $|a|^2 = b$

(c) an imaginary circle, if $|a|^2 < b$

Position of a Point with Respect to Circle

Let the general equation of circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ whose centre z_0 and radius a. There are following position with respect to circle are given below.

(a) If the point (z_0) lies inside the circle, then $|z - z_0| < a$.

(b) If the point (z_0) lies on the circle, then $|z - z_0| = a$.

(c) If the point (z_0) lies outside the circle, then $|z - z_0| > a$.

***Sample* Problem 22** *If* $\left|\frac{z-2}{z-3}\right| = 2$ *represents a circle, then centre and radius are* **[NCERT Exemplar]**

(a) $\left(\frac{10}{3}, 0\right), \frac{2}{3}$ (b) $\left(\frac{-10}{3}, 0\right), \frac{2}{3}$

(c) $\left(\frac{10}{3}, 1\right), \frac{2}{3}$ (d) *None of these*

Interpret (a) Let $z = x + iy$

$\therefore$ Given equation becomes, $\left|\frac{x+iy-2}{x+iy-3}\right| = 2$

$$\Rightarrow \quad |(x-2) + iy| = 2\,|(x-3) + iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 4[(x-3)^2 + y^2]$$

$$\Rightarrow \quad x^2 + 4 - 4x + y^2 = 4(x^2 + 9 - 6x + y^2)$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

Standard equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$\therefore \quad g = \frac{-10}{3}, f = 0, c = \frac{32}{3}$$

$\therefore$ Certre of circle $= (-g, -f) = \left(\frac{10}{3}, 0\right)$

$$\therefore \quad \text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{10}{3}\right)^2 + (0)^2 - \frac{32}{3}} = \sqrt{\frac{100}{9} - \frac{32}{3}}$$

$$= \sqrt{\frac{100 - 96}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

***Sample* Problem 23** *If z lies on the circle* $|z - 1| = 1$, *then* $\frac{z-2}{z}$ *is equal to*

(a) 0 (b) 2

(c) −1 (d) −2

Interpret (d) Since, $|z - 1| = 1$ represents a circle whose diameter is a line segment joining the points $z = 0$ and $z = 2$. If z lies on a circle, then $\frac{z-2}{z-0}$ *i.e.*, $\frac{z-2}{z}$ is purely imaginary.

Check Point 3

1. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then the area of triangle whose vertices are z_1, z_2 and z_3, is

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{4}$ (c) 1 (d) 2

2. If z_1, z_2, z_3 and z_4 are the affixes of the vertices of a parallelogram taken in order in argand plane, then

(a) $z_1 + z_3 = z_2 + z_4$ (b) $z_1 + z_2 = z_3 + z_4$

(c) $z_1 - z_3 = z_2 - z_4$ (d) None of these

3. If z be any complex number such that $|3z - 2| + |3z + 2| = 4$, then locus of z is

(a) a circle (b) an ellipse

(c) a line segment (d) None of these

4. If $z = x + iy$, then the equation $\left|\frac{2z - i}{z + 1}\right| = m$ represents a circle when m is not equal to

(a) $\frac{1}{2}$ (b) 1

(c) 2 (d) 3

5. A curve represented by $\text{Im}(z^2) = k$, where k is a non-zero real number is

(a) a pair of straight lines (b) an ellipse

(c) a parabola (d) a hyperbola

6. The region of the argand diagram defined by $|z - 1| + |z + 1| \le 4$ is

(a) interior of an ellipse

(b) exterior of an ellipse

(c) interior and boundary of an ellipse

(d) None of the above

7. If $|z + \bar{z}| + |z - \bar{z}| = 2$, then z lies on

(a) a straight line (b) a square

(c) a line (d) None of these

WORKED OUT Examples

Example 1 *For a positive integer n, the expression* $(1-i)^n\left(1-\frac{1}{i}\right)^n$ *is equal to* [NCERT]

(a) 0 (b) $2i^n$

(c) 2^n (d) 4^n

Solution (c) $(1-i)^n\left(1-\frac{1}{i}\right)^n$

$$= (1-i)^n(1+i)^n \qquad \left(\because -i = \frac{1}{i}\right)$$

$$= (1-i^2)^n = (1+1)^n = 2^n$$

Example 2 *If* $a + ib = \frac{(x+i)^2}{2x^2+1}$, *then* $a^2 + b^2$ *is equal to* [NCERT]

(a) $\frac{x^2+1}{(2x^2+1)^2}$ (b) $\frac{(x^2+1)^2}{(2x^2+1)^2}$

(c) $\frac{(x^2+1)^2}{(2x^2+1)}$ (d) *None of these*

Solution (b) Given, $a + ib = \frac{(x+i)^2}{2x^2+1}$

On taking modulus of both sides, we get

$$|a+ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$$

$$\Rightarrow \quad \sqrt{a^2+b^2} = \frac{|(x+i)^2|}{|2x^2+1|} = \frac{|x+i|^2}{2x^2+1}$$

$\because \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, ($|z^n| = |z^n|$) and $\text{Re}|(z)| = \text{Re}(z)$]

$$\Rightarrow \quad \sqrt{a^2+b^2} = \frac{(\sqrt{x^2+1})^2}{2x^2+1}$$

On squaring both sides, we get

$$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

Example 3 *A real value of x satisfies the equation* $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ $(\alpha, \beta \in R)$, *if* $\alpha^2 + \beta^2$ *is equal to* [NCERT Exemplar]

(a) 1 (b) –1 (c) 2 (d) –2

Solution (a) Given, $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$

$$\therefore \quad \frac{3-4ix}{3+4ix} \times \frac{3-4ix}{3-4ix} = \alpha - i\beta$$

$$\Rightarrow \quad \frac{9-16x^2-24ix}{9+16x^2} = \alpha - i\beta$$

On equating real and imaginary parts both sides, we get

$$\alpha = \frac{9-16x^2}{9+16x^2} \text{ and } \beta = \frac{24x}{9+16x^2}$$

$$\therefore \quad \alpha^2 + \beta^2 = \sqrt{\frac{(9-16x^2)^2}{(9+16x^2)^2} + \frac{(24x)^2}{(9+16x^2)^2}}$$

$$= \frac{1}{9+16x^2}\sqrt{81+256x^4-288x^2+576x^2}$$

$$= \frac{1}{9+16x^2}\sqrt{81+256x^4+288x^2}$$

$$= \frac{1}{9+16x^2}\sqrt{(9+16x^2)^2}$$

$$= \frac{9+16x^2}{9+16x^2} = 1$$

Example 4 *Which of the following is correct for any two complex numbers* z_1 and z_2? [NCERT Exemplar]

(a) $|z_1z_2| = |z_1||z_2|$ (b) $\arg(z_1z_2) = \arg(z_1)\cdot\arg(z_2)$

(c) $|z_1+z_2| = |z_1| + |z_2|$ (d) $|z_1+z_2| \geq |z_1| - |z_2|$

Solution (a) $|z_1z_2| = |z_1z_2|$

(b) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

(c) $|z_1+z_2| \neq |z_1| + |z_2|$

(d) $|z_1+z_2| \leq |z_1| + |z_2|$

Example 5 *The square root of* $7 + 24i$ *are*

(a) $\pm(3+4i)$ (b) $\pm(3-4i)$

(c) $\pm(4+3i)$ (d) $\pm(4-3i)$

Solution (c) Let $(x+iy)^2 = 7 + 24i$

$$\Rightarrow \quad x^2 - y^2 = 7 \quad \text{and} \quad 2xy = 24$$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2 + 4a^2b^2$

$$= 7^2 + 24^2 = 625$$

$\Rightarrow \quad x^2 + y^2 = 25$

$\Rightarrow \quad x = \pm 4$ and $y = \pm 3$

Hence, square root of $7 + 24i = \pm(4 + 3i)$.

Example 6 *If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then*

(a) $x = \frac{1}{5}$ (b) $y = \frac{3}{5}$

(c) $x + iy = \frac{1-i}{1-2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

Solution (c) Since, conjugate of $(x + iy)(1 - 2i)$ be $1 + i$.

$\therefore \quad \overline{(x + iy)(1 - 2i)} = 1 + i$

$\Rightarrow \quad x - iy = \frac{1+i}{1+2i}$

$\Rightarrow \quad x + iy = \frac{1-i}{1-2i}$

Example 7 *If $z(1 + a) = b + ic$ and $a^2 + b^2 + c^2 = 1$, then $\frac{1+iz}{1-iz}$ is equal to*

(a) $\frac{a+ib}{1+c}$ (b) $\frac{b-ic}{1+a}$

(c) $\frac{a+ic}{1+b}$ (d) *None of these*

Solution (a) $\frac{1+iz}{1-iz} = \frac{1 + i\frac{(b+ic)}{1+a}}{1 - i\frac{(b+ic)}{1+a}} = \frac{1+a-c+ib}{1+a+c-ib}$

$= \frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2 + b^2}$

$= \frac{a^2 + a + iab + ib}{1 + a + ac + c}$

$= \frac{(a+1)(a+ib)}{(a+1)(1+c)} = \frac{a+ib}{1+c}$

Example 8 *If z be a complex number. Then, the angle between z and iz is*

(a) π (b) 0

(c) $-\frac{\pi}{2}$ (d) *None of these*

Solution (c) We know that, multiplication of a complex number by $-i$ rotates through it by a right angle in clockwise direction.

Thus, required angle $= -\frac{\pi}{2}$

Example 9 *If $\frac{1}{x} + x = 2\cos\theta$, then $x^n + \frac{1}{x^n}$ is equal to*

(a) $2\cos n\theta$ (b) $2\sin n\theta$

(c) $\cos n\theta$ (d) $\sin n\theta$

Solution (a) $\because x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow \quad x = \cos\theta \pm i\sin\theta$

$\Rightarrow \quad x^n = \cos n\theta \pm i\sin n\theta$

and $\quad \frac{1}{x^n} = \cos n\theta \mp i\sin n\theta$

$\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta$

Example 10 *If $z(2 - i2\sqrt{3})^2 = i(\sqrt{3} + i)^4$, then amplitude of z is*

(a) $\frac{5\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{7\pi}{6}$

Solution (b) $z = \frac{i(\sqrt{3}+i)^4}{4(1-\sqrt{3}i)^2} = \frac{i}{4}\cdot\frac{(2+2\sqrt{3}i)^2}{-2-2\sqrt{3}i} = \frac{i(-2+2\sqrt{3}i)}{2(-1-\sqrt{3}i)}$

$= \frac{\sqrt{3}+i}{1+\sqrt{3}i} = \frac{(\sqrt{3}+i)(1-\sqrt{3}i)}{1+3}$

$= \frac{2\sqrt{3}-2i}{4} = \frac{\sqrt{3}}{2} - \frac{i}{2}$

$\therefore \quad \text{amp}(z) = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

Example 11 $(\cos 2\theta + i\sin 2\theta)^{-5}$ $(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3$ *in the form of $a + ib$ is*

(a) $\cos 25\theta + i\sin 25\theta$ (b) $i(\cos 25\theta + i\sin 25\theta)$

(c) $i(\cos 25\theta - i\sin 25\theta)$ (d) $\cos 25\theta - i\sin 25\theta$

Solution (c) $\because \sin\theta - i\cos\theta = -i(\cos\theta + i\sin\theta)$

$\therefore (\cos 2\theta + i\sin 2\theta)^{-5}(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3$

$= (-i)^3[\cos(-10\theta - 18\theta + 3\theta) + i\sin(-10\theta - 18\theta + 3\theta)]$

$= i(\cos 25\theta - i\sin 25\theta)$

Example 12 *The inequality $|z - 4| < |z - 2|$ represents the region given by*

(a) $Re(z) > 0$ (b) $Re(z) < 0$

(c) $Re(z) > 2$ (d) *None of these*

Solution (d) $\because \quad |z - 4| < |z - 2|$

$\Rightarrow \quad |z - 4|^2 < |z - 2|^2$

$\Rightarrow \quad (x-4)^2 + y^2 < (x-2)^2 + y^2$ (let $z = x + iy$)

$\Rightarrow \quad x^2 - 8x + 16 < x^2 - 4x + 4$

$\Rightarrow \quad x > 3$

$\Rightarrow \quad \text{Re}(z) > 3$

Example 13 *The complex number z, satisfying the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a* [NCERT Exemplar]

(a) *circle* (b) *line* (c) *x-axis* (d) *y-axis*

Solution (*a*) Let $z = x + iy$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 - i^2y^2}$$

$$= \frac{x^2 - 1 + iy(x+1-x+1) + y^2}{(x+1)^2 + y^2} = \frac{(x^2+y^2-1) + 2iy}{(x+1)^2+y^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + \frac{i \times 2y}{(x+1)^2+y^2}$$

Now, $$\arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left[\frac{\dfrac{2y}{(x+1)^2+y^2}}{\dfrac{x^2+y^2-1}{(x+1)^2+y^2}}\right]$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

$$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{2y}{x^2+y^2-1} \Rightarrow 1 = \frac{2y}{x^2+y^2-1}$$

$$\Rightarrow \quad x^2+y^2-1 = 2y \Rightarrow x^2+y^2-2y-1=0$$

Hence, this equation represents a circle.

Example 14 *The complex number* $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ *in polar form is* **[NCERT Exemplar]**

(*a*) $\sqrt{2}\left(\dfrac{\cos 5\pi}{12} + \dfrac{i\sin 5\pi}{12}\right)$

(*b*) $\dfrac{1}{\sqrt{2}}\left(\dfrac{\cos\pi}{12} + \dfrac{i\sin\pi}{12}\right)$

(*c*) $\left(\dfrac{\cos\pi}{12} + \dfrac{i\sin\pi}{12}\right)$

(*d*) *None of the above*

Solution (*d*) $$z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}{e^{i\pi/3}}$$

$$= \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]}{e^{i\frac{\pi}{3}}} = \frac{\sqrt{2}\left(e^{-\frac{\pi i}{4}}\right)}{e^{i\pi/3}}$$

$$= \sqrt{2}\, e^{-\frac{\pi i}{4} - \frac{i\pi}{3}} = \sqrt{2}\, e^{-\frac{7\pi}{12}}$$

$$= \sqrt{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)$$

Example 15 *If the real part of* $\dfrac{\bar{z}+2}{\bar{z}-1}$ *is 4, then the locus of the point representing z in the complex plane is a* **[NCERT Exemplar]**

(*a*) *line* (*b*) *circle* (*c*) *parabola* (*d*) *ellipse*

Solution (*b*) Let $z = x + iy$

$$\Rightarrow \quad \bar{z} = x - iy$$

Now, $$\frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1} = \frac{(x+2)-iy}{(x-1)-iy} \times \frac{(x-1)+iy}{(x-1)+iy}$$

$$= \frac{(x+2)(x-1) + iy(x+2) - iy(x-1) - i^2y^2}{(x-1)^2 - i^2y^2}$$

$$= \frac{(x^2+2x-x-2) + iy(x+2-x+1) + y^2}{(x-1)^2+y^2}$$

$$= \frac{(x^2+x-2+y^2)}{(x-1)^2+y^2} + \frac{3iy}{(x-1)^2+y^2}$$

$$\because \ \text{Re}\left(\frac{\bar{z}+2}{\bar{z}-1}\right) = 4$$

$$\therefore \quad \frac{x^2+y^2+x-2}{(x-1)^2+y^2} = 4$$

$$\Rightarrow \quad x^2+y^2+x-2 = 4(x^2+1-2x+y^2)$$

$$\Rightarrow \quad x^2+y^2+x-2 = 4x^2+4-8x+4y^2$$

$$\Rightarrow \quad 3x^2+3y^2-9x+2=0$$

$$\Rightarrow \quad x^2+y^2-3x+\frac{2}{3}=0$$

Clearly, it represents a circle.

Example 16 *For all complex numbers* z_1 and z_2 *satisfying* $|z_1| = 12$ *and* $|z_2 - 3 - 4i| = 5$, *the minimum value of* $|z_1 - z_2|$ *is*

(*a*) 0 (*b*) 2

(*c*) 7 (*d*) 17

Solution (*b*) The two circles whose centre and radius are $C_1(0,0), r_1 = 12$, $C_2(3,4), r_2 = 5$ and it passes through origin, then the centre of C_1.

Now, $$C_1C_2 = \sqrt{3^2+4^2} = 5$$

and $$r_1 - r_2 = 12 - 5 = 7$$

$$\therefore \quad C_1C_2 < r_1 - r_2$$

Hence, circle C_2 lies inside circle C_1.

From figure, the minimum distance between them is

$$AB = C_1B - C_1A = r_1 - 2r_2 = 12 - 10 = 2$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Algebraic Operations and Square Root

1. The multiplicative inverse of $\sqrt{5} + 3i$ is [NCERT]

(a) $\frac{\sqrt{5}}{14} + \frac{3i}{14}$ (b) $\frac{\sqrt{5}}{14} - \frac{3i}{14}$

(c) $\frac{\sqrt{5}}{13} - \frac{3i}{13}$ (d) $\frac{\sqrt{5}}{13} + \frac{3i}{13}$

2. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then (x, y) is equal to [NCERT Exemplar]

(a) $(0, -2)$ (b) $(-2, 0)$

(c) $(0, 2)$ (d) $(2, 0)$

3. If $\frac{(1+i)^2}{2-i} = x + iy$, then the value of $x + y$ is equal to [NCERT Exemplar]

(a) $\frac{5}{2}$ (b) $-\frac{2}{5}$ (c) $\frac{2}{5}$ (d) $-\frac{5}{2}$

4. The real value of α for which the expression $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real is, where $n \in N$ [NCERT Exemplar]

(a) $(n + 1)\frac{\pi}{2}$ (b) $(2n + 1)\frac{\pi}{2}$

(c) $n\pi$ (d) None of these

5. If $\text{Re}(z^2) = 0, |z| = 2$, then [NCERT Exemplar]

(a) $z = \mp\sqrt{2} \pm i\sqrt{2}$ (b) $z = \mp\sqrt{3} \pm i\sqrt{3}$

(c) $z = \mp\sqrt{3} \pm i\sqrt{2}$ (d) None of these

6. The value of $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$ is

(a) positive (b) negative

(c) zero (d) cannot be determined

7. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, is equal to [NCERT]

(a) i (b) $i - 1$

(c) $-i$ (d) 0

8. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then [NCERT Exemplar]

(a) $x = 2n + 1$ (b) $x = 4n$

(c) $x = 2n$ (d) $x = 4n + 1$, where $n \in N$

9. The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is [NCERT Exemplar]

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi + (-1)^n\frac{\pi}{4}$

(c) $2n\pi \pm \frac{\pi}{2}$ (d) None of these

10. Given $z = \frac{q + ir}{1 + p}$, then $\frac{p + iq}{1 + r} = \frac{1 + iz}{1 - iz}$, if

(a) $p^2 + q^2 + r^2 = 1$ (b) $p^2 + q^2 + r^2 = 2$

(c) $p^2 + q^2 - r^2 = 1$ (d) None of these

Conjugate, Modulus and Argument

11. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant, if [NCERT Exemplar]

(a) $x > y > 0$ (b) $x < y < 0$

(c) $y < x < 0$ (d) $y > x > 0$

12. The conjugate of $\frac{2-i}{(1-2i)^2}$ is [NCERT Exemplar]

(a) $\frac{2}{25} - \frac{i11}{25}$ (b) $\frac{-2}{25} - \frac{i11}{25}$

(c) $\frac{-2}{25} + \frac{i11}{25}$ (d) $\frac{2}{25} + \frac{i11}{25}$

13. $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for [NCERT Exemplar]

(a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$

(c) $x = 0$ (d) No value of x

14. The value of $(z+3)(\bar{z}+3)$ is equivalent to [NCERT Exemplar]

(a) $|z+3|^2$ (b) $|z-3|$
(c) z^2+3 (d) None of these

15. If $\frac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then the value of $|z|$ is [NCERT Exemplar]

(a) -1 (b) 1 (c) 2 (d) -2

16. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then the value of $|z_1 + z_2 + z_3 + \ldots + z_n|$ is [NCERT]

(a) 1 (b) $|z_1| + |z_2| + \ldots + |z_n|$
(c) $\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|$ (d) None of these

17. If α and β are different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to [NCERT]

(a) 0 (b) 1/2
(c) 1 (d) 2

18. If $|z+1| = z + 2(1+i)$, then the value of z is

(a) $\frac{1}{2} - 2i$ (b) $\frac{1}{2} + 2i$
(c) $\frac{1}{2} - 3i$ (d) $\frac{1}{3} - 2i$

19. If z is a complex number, then $(\bar{z}^{-1})(\bar{z})$ is equal to

(a) 1 (b) -1
(c) 0 (d) None of these

20. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right| = 1$ lies on [NCERT Exemplar]

(a) circle $x^2 + y^2 = 1$ (b) the x-axis
(c) the y-axis (d) the line $x + y = 1$

21. $|z_1 + z_2| = |z_1| + |z_2|$ is possible, if [NCERT Exemplar]

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$
(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

22. If $|z-2| = \min\{|z-1|, |z-5|\}$, where z is a complex number, then

(a) $\text{Re}(z) = \frac{3}{2}$ (b) $\text{Re}(z) = \frac{7}{2}$
(c) $\text{Re}(z) \in \left\{\frac{3}{2}, \frac{7}{2}\right\}$ (d) None of these

23. If z is a complex number, then $|3z-1| = 3|z-2|$ represents

(a) y-axis
(b) a circle
(c) x-axis
(d) a line parallel to y-axis

24. The region of the complex plane for which $\left|\frac{z-a}{z+\bar{a}}\right| = 1$ $[\text{Re}(a) \neq 0]$, is

(a) x-axis (b) y-axis
(c) the straight line $x = a$ (d) None of these

25. If $|a_k| < 1, \lambda_k \geq 0$ for $k = 1, 2, \ldots, n$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$, then the value of $|\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n|$ is

(a) equal to one (b) greater than one
(c) zero (d) less than one

26. $\arg(\bar{z})$ is equal to

(a) $\pi - \arg(z)$ (b) $2\pi - \arg(z)$
(c) $\pi + \arg(z)$ (d) $2\pi + \arg(z)$

27. If $\arg(z-1) = \arg(z+3i)$, then $x - 1 : y$ is equal to [NCERT Exemplar]

(a) 3 : 1 (b) 1 : 3
(c) 3 : 2 (d) 2 : 3

28. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then z_1 is equal to [NCERT Exemplar]

(a) $2\bar{z}_2$ (b) $\bar{z}_2$
(c) $-\bar{z}_2$ (d) None of these

29. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is [NCERT Exemplar]

(a) $\frac{|z|}{2}$ (b) $|z|$
(c) $2|z|$ (d) None of these

30. If $\tan^{-1}(\alpha + i\beta) = x + iy$, then x is equal to

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$
(b) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1+\alpha^2+\beta^2}\right)$
(c) $\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$
(d) None of the above

31. If $\bar{z}$ be the conjugate of the complex number z, then which of the following relations is false?

(a) $|z| = |\bar{z}|$ (b) $z \cdot \bar{z} = |\bar{z}|^2$
(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (d) $\arg(z) \neq \arg(\bar{z})$

32. Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then principal $\arg(z_1 z_2)$ is given by

(a) $\alpha + \beta + \pi$
(b) $\alpha + \beta - \pi$
(c) $\alpha + \beta + 2\pi$
(d) $\alpha + \beta$

33. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ is equal to [NCERT Exemplar]

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

34. The argument of $\frac{(1 - i\sqrt{3})}{(1 + i\sqrt{3})}$ is

(a) 60° (b) 120° (c) 210° (d) 240°

35. If z and w are two non-zero complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then $\bar{z}w$ is equal to [NCERT Exemplar]

(a) 1 (b) $-i$

(c) i (d) -1

Triangle Inequality

36. If $|z + 4| \le 3$, then the greatest and the least value of $|z + 1|$ are

(a) 6, – 6 (b) 6, 0

(c) 7, 2 (d) 0, – 1

37. If z is a complex number, then the minimum value of $|z| + |z - 1|$ is

(a) 1 (b) 0

(c) $\frac{1}{2}$ (d) None of these

38. If z_1, z_2 and z_3 be three complex numbers such that $|z_1 + 1| \le 1$, $|z_2 + 2| \le 2$ and $|z_3 + 4| \le 4$, then the maximum value of $|z_1| + |z_2| + |z_3|$ is

(a) 7 (b) 10

(c) 12 (d) 14

39. The maximum value of $|z|$ where z satisfies the condition $\left| z + \frac{2}{z} \right| = 2$, is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

40. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

(a) is less than 6 (b) is more than 3

(c) is less than 12 (d) lies between 6 and 12

41. Let z be a complex number satisfying $|z - 5i| \le 1$ such that amp(z) is minimum. Then, z is equal to

(a) $\frac{2\sqrt{6}}{5} + \frac{24i}{5}$ (b) $\frac{24}{5} + \frac{2\sqrt{6}\,i}{5}$

(c) $\frac{2\sqrt{6}}{5} - \frac{24i}{5}$ (d) None of these

De-Moivre's Theorem

42. The value of $\frac{4(\cos 75° + i\sin 75°)}{0.4(\cos 30° + i\sin 30°)}$ is

(a) $\frac{\sqrt{2}}{10}(1 + i)$ (b) $\frac{\sqrt{2}}{10}(1 - i)$

(c) $\frac{10}{\sqrt{2}}(1 - i)$ (d) $\frac{10}{\sqrt{2}}(1 + i)$

43. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$, then the value of θ is

(a) $4m\pi$ (b) $\frac{2m\pi}{n(n+1)}$

(c) $\frac{4m\pi}{n(n+1)}$ (d) $\frac{m\pi}{n(n+1)}$

44. $\sqrt{1 - c^2} = nc - 1$ and $z = e^{i\theta}$, then $\frac{c}{2n}(1 + nz)\left(1 + \frac{n}{z}\right)$ is equal to

(a) $1 - c\cos\theta$ (b) $1 + 2c\cos\theta$

(c) $1 + c\cos\theta$ (d) $1 - 2c\cos\theta$

45. The real part of $(1 - \cos\theta + 2i\sin\theta)^{-1}$ is

(a) $\frac{1}{3 + 5\cos\theta}$ (b) $\frac{1}{5 - 3\cos\theta}$

(c) $\frac{1}{3 - 5\cos\theta}$ (d) $\frac{1}{5 + 3\cos\theta}$

46. If $a = \cos\theta + i\sin\theta$, then $\frac{1 + a}{1 - a}$ is equal to

(a) $\cot\frac{\theta}{2}$ (b) $\cot\theta$

(c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

47. If $x + iy = \frac{3}{2 + \cos\theta + i\sin\theta}$, then $x^2 + y^2$ is equal to

(a) $3x - 4$ (b) $4x - 3$

(c) $4x + 3$ (d) None of these

48. If $z_r = \cos\frac{r\alpha}{n^2} + i\sin\frac{r\alpha}{n^2}$, where $r = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} z_1 z_2 z_3 \ldots z_n$ is equal to

(a) $\cos\alpha + i\sin\alpha$ (b) $\cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right)$

(c) $e^{i\alpha/2}$ (d) $\sqrt[3]{e^{i\alpha}}$

49. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is

(a) -1 (b) 0

(c) $-i$ (d) i

50. The imaginary part of

$(z - 1)(\cos\alpha - i\sin\alpha) + (z - 1)^{-1} \times (\cos\alpha + i\sin\alpha)$ is zero, if

(a) $|z - 1| = 2$ (b) $\arg(z - 1) = 2\alpha$

(c) $\arg(z - 1) = \alpha$ (d) $|z| = 1$

51. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $\frac{b}{c} + \frac{c}{a} + \frac{a}{b} = 1$, then $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)$ is equal to

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$
(c) 0 (d) 1

52. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$ is equal to

(a) 0 (b) $\cos(\alpha + \beta + \gamma)$
(c) $3\cos(\alpha + \beta + \gamma)$ (d) $3\sin(\alpha + \beta + \gamma)$

53. The values of $(16)^{1/4}$ are

(a) $\pm 2, \pm 2i$ (b) $\pm 4, \pm 4i$
(c) $\pm 1, \pm i$ (d) None of these

Cube Roots and *n*th Roots of Unity

54. Let $\omega_n = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, $i^2 = -1$, then $(x + y\omega_3 + z\omega_3^2)(x + y\omega_3^2 + z\omega_3)$ is equal to

(a) 0
(b) $x^2 + y^2 + z^2$
(c) $x^2 + y^2 + z^2 - yz - zx - xy$
(d) $x^2 + y^2 + z^2 + yz + zx + xy$

55. The value of $\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} + \frac{a + b\omega + c\omega^2}{c + a\omega + \omega^2}$ is

(a) 1 (b) -1
(c) 2 (d) -2

56. If α, β and γ are the cube roots of a positive number p, then for any real x, y and z, the expression $\left(\frac{\alpha x + \beta y + \gamma z}{\beta x + \gamma y + \alpha z}\right)$ is equal to

(a) $\frac{-1 - \sqrt{3}i}{2}$ (b) $\frac{-1 + \sqrt{3}i}{2}$
(c) $\frac{1 + \sqrt{3}i}{2}$ (d) $\frac{1 - \sqrt{3}i}{2}$

57. The value of $(1 + \omega^2 + 2\omega)^{3n} - (1 + \omega + 2\omega^2)^{3n}$ is equal to

(a) 0 (b) 1
(c) ω (d) ω^2

58. $\sqrt{-1 - \sqrt{1 - \sqrt{-1 - \ldots\infty}}}$ is equal to

(a) 1 (b) -1
(c) ω^2 (d) $-\omega$

59. If $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$, where α and β are complex cube roots of unity, then xyz is equal to

(a) $a^2 + b^2$ (b) $a^3 + b^3$
(c) a^3b^3 (d) $a^3 - b^3$

60. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B are respectively

(a) 0, 1 (b) 1, 1
(c) 1, 0 (d) -1, 1

61. If ω is a complex cube root of unity, then for positive integral value of n, the product of $\omega \cdot \omega^2 \cdot \omega^3 \ldots\ldots \omega^n$ will be

(a) $\frac{1 - i\sqrt{3}}{2}$ (b) $-\frac{1 - i\sqrt{3}}{2}$
(c) 1 (d) Both (b) and (c)

62. If 1, ω and ω^2 are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} \text{ is equal to}$$

(a) 0 (b) 1 (c) ω (d) ω^2

63. If $\alpha \neq 1$ is any nth root of unity, then $S = 1 + 3\alpha + 5\alpha^2 + \ldots$ upto n terms, is equal to

(a) $\frac{2n}{1 - \alpha}$ (b) $-\frac{2n}{1 - \alpha}$
(c) $\frac{n}{1 - \alpha}$ (d) $-\frac{n}{1 - \alpha}$

64. Common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(a) ω, ω^2 (b) ω, ω^3
(c) ω^2, ω^3 (d) None of these

65. The value of the expression

$$1 \cdot (2 - \omega)(2 - \omega^2) + 2 \cdot (3 - \omega)(3 - \omega^2) + \ldots + (n - 1) \cdot (n - \omega)(n - \omega^2),$$

where ω is an imaginary cube root of unity, is

(a) $\frac{1}{2}(n - 1)\, n\, (n^2 + 3n + 4)$
(b) $\frac{1}{4}(n - 1)\, n\, (n^2 + 3n + 4)$
(c) $\frac{1}{2}(n + 1)\, n\, (n^2 + 3n + 4)$
(d) $\frac{1}{4}(n + 1)\, n\, (n^2 + 3n + 4)$

66. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then the value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} \text{ is}$$

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

67. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are n roots of unity, then the value of $(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)\ldots(1 - \alpha_{n-1})$ is equal to

(a) $\sqrt{3}$ (b) 1/2
(c) n (d) 0

68. If $x^2 - x + 1 = 0$, then the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$ is

(a) 8 (b) 10
(c) 12 (d) None of these

Applications in Geometry

69. If $\text{Im}\left(\frac{z-1}{2z+1}\right) = -4$, then locus of z is

(a) an ellipse (b) a parabola
(c) a straight line (d) a circle

70. In the argand plane the complex number $z = 4 - 3i$ is turned in the clockwise sense through 180° and stretched three times. The complex number represented by the new number is

(a) $12 + 9i$ (b) $12 - 9i$ (c) $-12 - 9i$ (d) $-12 + 9i$

71. If the complex numbers z_1 and z_2 and the origin form an equilateral triangle, then $z_1^2 + z_2^2$ is equal to

(a) $z_1 z_2$ (b) $z_1 \bar{z}_2$
(c) $\bar{z}_2 z_1$ (d) $|z_1|^2 = |z_2|^2$

72. In a given parallelogram, if the points P_1 and P_2 represent two complex numbers z_1 and z_2, then the point P_3 represents the number

(a) $z_1 + z_2$
(b) $z_1 - z_2$
(c) $z_1 \times z_2$
(d) $z_1 \div z_2$

73. Let z_1 and z_2 be the non-real roots of the equation $3z^2 + 3z + b = 0$. If the origin together with the points represented by z_1 and z_2 form an equilateral triangle, then the value of b is

(a) 1 (b) 2
(c) 3 (d) None of these

74. Locus of z, if $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, is

(a) a circle (b) a semi-circle
(c) a straight line (d) None of these

75. The locus of the points z which satisfy the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, is

(a) a straight line (b) a circle
(c) a parabola (d) None of these

76. Let z be a complex number and a be a real parameter such that $z^2 + ax + a^2 = 0$, then

(a) locus of z is a pair of straight lines
(b) locus of z is a circle
(c) $\arg(z) = \pm\frac{5\pi}{3}$
(d) $|z| = -2|a|$

77. If z_1, z_2, z_3 and z_4 are the affixes of four points in the argand plane and z is the affix of a point, such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then z_1, z_2, z_3 and z_4 are

(a) concyclic (b) vertices of a parallelogram
(c) vertices of a rhombus (d) in a straight line

78. The points represented by the complex numbers $1 + i$, $-2 + 3i$ and $\frac{5}{3}i$ on the argand diagram are

(a) vertices of an equilateral triangle
(b) vertices of an isosceles triangle
(c) collinear
(d) None of the above

79. If the area of the triangle on the complex plane formed by the points z, $z + iz$ and iz is 200, then the value of $|3z|$ must be equal to

(a) 20 (b) 40 (c) 60 (d) 80

80. $\text{Re}(z^2) = 1$ is represented by

(a) the circle $x^2 + y^2 = 1$ (b) the hyperbola $x^2 - y^2 = 1$
(c) parabola or a circle (d) All of these

81. If the points z_1, z_2 and z_3 are the vertices of an equilateral triangle in the complex plane, then the value of $z_1^2 + z_2^2 + z_3^2$ is equal to

(a) $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1}$ (b) $z_1z_2 + z_2z_3 + z_3z_1$
(c) $z_1z_2 - z_2z_3 - z_3z_1$ (d) $-\frac{z_1}{z_2} - \frac{z_2}{z_3} - \frac{z_3}{z_1}$

82. If the complex numbers iz, z and $z + iz$ represent the three vertices of a triangle, then the area of the triangle is

(a) $\frac{1}{2}|z|^2$ (b) $|z|^2$ (c) $\frac{1}{2}|z - 1|$ (d) $|z - 1|^2$

83. The area of the triangle whose vertices are represented by the complex numbers 0, z, $ze^{i\alpha}$, $(0 < \alpha < \pi)$ is equal to

(a) $\frac{1}{2}|z|^2 \cos\alpha$ (b) $\frac{1}{2}|z|^2 \sin\alpha$
(c) $\frac{1}{2}|z|^2 \sin\alpha\cos\alpha$ (d) $\frac{1}{2}|z|^2$

84. If $\left|\frac{z-2}{z+2}\right| = \frac{\pi}{6}$, then the locus of z is **[NCERT]**

(a) a straight line (b) a circle
(c) a parabola (d) an ellipse

85. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then the locus of the point representing z in the complex plane is

(a) a circle (b) a parabola
(c) a hyperbola (d) an ellipse

86. The equation $z\bar{z} + (2 - 3i)z + (2 + 3i)\bar{z} + 4 = 0$ represents a circle of radius
(a) 2 (b) 3 (c) 4 (d) 6

87. When $\frac{z+i}{z+2}$ is purely imaginary, the locus described by the point z in the argand diagram is a
(a) circle of radius $\frac{\sqrt{5}}{2}$ (b) circle of radius $\frac{5}{4}$
(c) straight line (d) parabola

88. The equation of a circle whose radius and centre are r and z_0 respectively, is
(a) $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$
(b) $z\bar{z} + z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$
(c) $z\bar{z} - z\bar{z}_0 + \bar{z}z_0 - z_0\bar{z}_0 = r^2$
(d) None of the above

89. If $iz^3 + z^2 - z + i = 0$, then $|z|$ is equal to
(a) 1 (b) i (c) -1 (d) $-i$

90. The centre of the circle
$$z\bar{z} - (2 + 3i)z - (2 - 3i)\bar{z} + 9 = 0 \text{ is}$$
(a) $(2, -3)$ (b) $(2, 3)$ (c) $(-2, -3)$ (d) $(-2, 3)$

Miscellaneous

91. If $\sqrt{3} + i = (a + ib)(c + id)$, then $\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right)$ has the value
(a) $\frac{\pi}{3} + 2n\pi, n \notin I$ (b) $n\pi + \frac{\pi}{6}, n \in I$
(c) $n\pi - \frac{\pi}{3}, n \in I$ (d) $2n\pi - \frac{\pi}{3}, n \in I$

92. The number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$ is [NCERT]
(a) infinite (b) 1
(c) 2 (d) None of these

93. If in polar form $z_1 = (1, \alpha)$, $z_2 = (1, \beta)$, $z_3 = (1, \gamma)$ and $z_1 + z_2 + z_3 = 0$, then $z_1^{-1} + z_2^{-1} + z_3^{-1}$ is equal to
(a) 1 (b) 0
(c) -1 (d) None of these

94. The value of $x^4 + 9x^3 + 35x^2 - x + 4$ for $x = -5 + 2\sqrt{-4}$ is
(a) 0 (b) -160
(c) 160 (d) -164

95. If $\alpha = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$, then the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$, is
(a) $x^2 - x + 2 = 0$ (b) $x^2 + x - 2 = 0$
(c) $x^2 - x - 2 = 0$ (d) $x^2 + x + 2 = 0$

96. The maximum distance from the origin of coordinates to the point z satisfying the equation $\left|z + \frac{1}{z}\right| = a$ is
(a) $\frac{1}{2}(\sqrt{a^2 + 1} + a)$ (b) $\frac{1}{2}(\sqrt{a^2 + 2} + a)$
(c) $\frac{1}{2}(\sqrt{a^2 + 4} + a)$ (d) None of these

97. The closest distance of the origin from a curve given as $a\bar{z} + \bar{a}z + |a|^2 = 0$ is
(a) 1 (b) $\frac{\text{Re}(|a|)}{|a|}$ (c) $\frac{\text{Im}(|a|)}{|a|}$ (d) $\frac{|a|}{2}$

98. Let z_1, z_2 and z_3 be the affixes of the vertices of a triangle having the circumcentre at the origin. If z is the affix of it's orthocentre, then z is equal to
(a) $\frac{z_1 + z_2 + z_3}{3}$ (b) $\frac{z_1 + z_2 + z_3}{2}$
(c) $z_1 + z_2 + z_3$ (d) None of these

99. If $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$, then the locus of z is
(a) $|z| = 5$ (b) $|z| < 5$
(c) $|z| > 5$ (d) None of these

100. Given that the equation $z^2 + (p + iq)z + r + is = 0$, where p, q, r and s are real and non-zero roots, then
(a) $pqr = r^2 + p^2s$ (b) $prs = q^2 + r^2p$
(c) $qrs = p^2 + s^2q$ (d) $pqs = s^2 + q^2r$

Round II (Mixed Bag)

Only One Correct Option

1. If $|z - i\text{Re}(z)| = |z - \text{Im}(z)|$ (where $i = \sqrt{-1}$), then z lies on
(a) $\text{Re}(z) = 2$
(b) $\text{Im}(z) = 2$
(c) $\text{Re}(z) + \text{Im}(z) = 2$
(d) None of the above

2. If z_1 is a root of the equation $a_0^n z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \ldots, n$. Then,
(a) $|z_1| > \frac{1}{3}$ (b) $|z_1| < \frac{1}{4}$
(c) $|z_1| > \frac{1}{4}$ (d) $|z| < \frac{1}{3}$

3. The point of intersection of the curves $\arg(z - 3i) = \frac{3\pi}{4}$ and $\arg(2z + 1 - 2i) = \frac{\pi}{4}$, (where $i = \sqrt{-1}$) is

(a) $\frac{1}{4}(3 + 9i)$ (b) $\frac{1}{4}(3 - 9i)$

(c) $\frac{1}{2}(3 + 2i)$ (d) No point

4. The complex number z satisfies the condition $\left|z - \frac{25}{z}\right| = 24$. The maximum distance from the origin of coordinates to the point z is

(a) 25 (b) 30

(c) 32 (d) None of these

5. The trigonometric form of $z = (1 - i\cot 8)^3$ (where $i = \sqrt{-1}$) is

(a) $\operatorname{cosec}^3 8 \cdot e^{i\left(24 - \frac{3\pi}{2}\right)}$ (b) $\operatorname{cosec}^3 8 \cdot e^{-i\left(24 - \frac{3\pi}{2}\right)}$

(c) $\operatorname{cosec}^3 8 \cdot e^{i\left(36 - \frac{\pi}{2}\right)}$ (d) $\operatorname{cosec}^2 8 \cdot e^{-i\left(24 - \frac{\pi}{2}\right)}$

6. If ω is a complex cube root of unity, then the value of $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$ is

(a) x^3 (b) $2x^3$

(c) $3x^3$ (d) None of these

7. If $x = 9^{1/3}9^{1/9}9^{1/27}\ldots\infty$, $y = 4^{1/3}4^{-1/9}4^{1/27}\ldots\infty$ and $z = \sum_{r=1}^{\infty}(1+i)^{-r}$, then $\arg(x + yz)$ is equal to

(a) 0 (b) $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

(c) $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (d) $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

8. If z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $[(z_1 + z_2)/(z_1 - z_2)]$ may be

(a) purely imaginary

(b) real and positive

(c) real and negative

(d) None of the above

9. If $|z_1| = |z_2|$, $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to

(a) 0 (b) purely imaginary

(c) purely real (d) None of these

10. If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z\cos\alpha|$ is

(a) less than 1 (b) $\sqrt{2} + 1$

(c) $\sqrt{2} - 1$ (d) None of these

11. If $A(z_1)$, $B(z_2)$ and $C(z_3)$ are the vertices of the ΔABC such that $(z_1 - z_2)/(z_3 - z_2) = (1/\sqrt{2}) - (i/1/\sqrt{2})$, then ΔABC is

(a) equilateral (b) right angled

(c) isosceles (d) obtuse angled

12. If z_1, z_2 and z_3 are the vertices of an equilateral ΔABC such that $|z_1 - i| = |z_2 - i| = |z_3 - i|$, then $|z_1 + z_2 + z_3|$ is equal to

(a) $3\sqrt{3}$ (b) $\sqrt{3}$ (c) 3 (d) $\frac{1}{3\sqrt{3}}$

More Than One Correct Option

13. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies

(a) $|w_1| = 1$ (b) $|w_2| = 1$

(c) $\operatorname{Re}|w_1\bar{w}_2| = 0$ (d) None of these

14. The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number, is

(a) $2n\pi + \frac{\pi}{2}, n \in I$ (b) $2n\pi - \frac{\pi}{2}, n \in I$

(c) $2n\pi \pm \frac{\pi}{2}, n \in I$ (d) $2n\pi \pm \frac{\pi}{4}, n \in I$

15. If $|z_1| = |z_2| = 1$ and amp z_1 + amp $z_2 = 0$, then

(a) $z_1z_2 = 1$ (b) $z_1 + z_2 = 0$

(c) $z_1 = \bar{z}_2$ (d) None of these

16. If $|z_1| = 15$ and $|z_2 - 3 - 4i| = 5$, then

(a) $|z_1 - z_2|_{\min} = 5$ (b) $|z_1 - z_2|_{\min} = 10$

(c) $|z_1 - z_2|_{\max} = 20$ (d) $|z_1 - z_2|_{\max} = 25$

Comprehension Based Questions

Passage

Directions (Q. Nos. 17-19) *Consider the complex numbers z_1 and z_2 satisfying the relation $|z_1 + z_1|^2 = |z_1|^2 + |z_2|^2$.*

17. Complex number $z_1\bar{z}_2$ is

(a) purely real (b) purely imaginary

(c) zero (d) None of the above

18. Complex number z_1/z_2 is

(a) purely real (b) purely imaginary

(c) zero (d) None of these

19. One of the possible argument of complex number $i(z_1/z_2)$

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$

(c) 0 (d) None of these

Assertion and Reason

Directions (Q. Nos. 20-28) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*),(*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

20. Consider z_1 and z_2 are two complex numbers.

Statement I Such that $|z_1| = |z_2| + |z_1 - z_2|$, then $\text{Im}\left(\frac{z_1}{z_2}\right) = 0$.

Statement II $\arg(z) = 0 \Rightarrow z$ is purely real.

21. **Statement I** If $\left|\frac{z\,z_1 - z_2}{z\,z_1 + z_2}\right| = k$, $(z_1, z_2 \neq 0)$, then the locus of z is circle.

Statement II As $\left|\frac{z - z_1}{z - z_2}\right| = \lambda$ represents a circle, if $\lambda \notin \{0, 1\}$.

22. Consider $|z_1| = 1$, $|z_2| = 2$ and $|z_3| = 3$.

Statement I If $|z_1 + 2z_2 + 3z_3| = 6$, then the value of $|z_2z_3 + 8z_3z_1 + 27z_1z_2|$ is 36.

Statement II $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

23. **Statement I** If $\cos(1 - i) = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, then $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1$, $b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$.

Statement II $e^{i\theta} = \cos\theta + i\sin\theta$

24. **Statement I** If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, then $\frac{z_1}{z_2}$ is purely imaginary.

Statement II If z is purely imaginary, then $z + \bar{z} = 0$.

25. **Statement I** If $\arg(z_1z_2) = 2\pi$, then both z_1 and z_2 are purely real (z_1 and z_2 have principal arguments).

Statement II Principal argument of complex number and between $-\pi$ and π.

26. **Statement I** $|z_1 - a| < a$, $|z_2 - b| < b$, $|z_3 - c| < c$, where a, b and c are positive real numbers, then $|z_1 + z_2 + z_3|$ is greater than $2|a + b + c|$.

Statement II $|z_1 \pm z_2| \le |z_1| + |z_2|$

27. **Statement I** If the equation $ax^2 + bx + c = 0$, $0 < a < b < c$, has non-real complex roots z_1 and z_2, then $|z_1| > 1$, $|z_2| > 1$.

Statement II Complex roots always occur in conjugate pairs.

28. **Statement I** The product of all values of $(\cos\alpha + i\sin\alpha)^{3/5}$ is $\cos 3a + i\sin 3a$.

Statement II The product of fifth roots of unity is 1.

Previous Years' Questions

29. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to **[JEE Main 2013]**

(a) $-\theta$ (b) $\frac{\pi}{2} - \theta$
(c) θ (d) $\pi - \theta$

30. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies **[AIEEE 2012]**

(a) either on the real axis or on a circle passing through the origin
(b) on a circle with centre at the origin
(c) either on the real axis or on a circle not passing through the origin
(d) on the imaginary axis

31. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$. Then, (A, B) is equal to **[AIEEE 2011]**

(a) (1, 1) (b) (1, 0)
(c) (−1, 1) (d) (0, 1)

32. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ is equal to **[AIEEE 2010]**

(a) 0 (b) 1
(c) 2 (d) ∞

33. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to **[AIEEE 2009]**

(a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$
(c) 2 (d) $2 + \sqrt{2}$

34. The conjugate of a complex number is $\frac{1}{i-1}$. Then, that complex number is [AIEEE 2008]

(a) $\frac{1}{i-1}$ (b) $-\frac{1}{i-1}$

(c) $\frac{1}{i+1}$ (d) $-\frac{1}{i+1}$

35. If $|z+4| \le 3$, then the maximum value of $|z+1|$ is [AIEEE 2007]

(a) 4 (b) 10

(c) 6 (d) 0

36. If $z^2 + z + 1 = 0$, where z is complex number, then the value of $\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 + \ldots + \left(z^6+\frac{1}{z^6}\right)^2$ is [AIEEE 2006]

(a) 54 (b) 6

(c) 12 (d) 18

37. If the cube roots of unity are 1, ω and ω^2, then the roots of the equation $(x-1)^3 + 8 = 0$, are [AIEEE 2005]

(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$

(c) $-1, -1, -1$ (d) $-1, -1+2\omega, -1-2\omega^2$

38. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to [AIEEE 2005]

(a) $-\frac{\pi}{2}$ (b) 0

(c) $-\pi$ (d) $\frac{\pi}{2}$

39. If $w = \frac{z}{z - \frac{i}{3}}$ and $|w| = 1$, then z lies on [AIEEE 2005]

(a) a parabola (b) a straight line

(c) a circle (d) an ellipse

40. Let z and w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg(zw) = \pi$. Then, $\arg(z)$ is equal to [AIEEE 2004]

(a) $\pi/4$ (b) $\pi/2$

(c) $3\pi/4$ (d) $5\pi/2$

41. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{q}\right) \Big/ (p^2 + q^2)$ is equal to [AIEEE 2004]

(a) 1 (b) -1

(c) 2 (d) -2

42. If $|z^2 - 1| = |z|^2 + 1$, then z lies on [AIEEE 2004]

(a) the real axis

(b) the imaginary axis

(c) a circle

(d) an ellipse

43. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin z_1 and z_2 form an equilateral triangle. Then, [AIEEE 2003]

(a) $a^2 = b$ (b) $a^2 = 2b$

(c) $a^2 = 3b$ (d) $a^2 = 4b$

44. If z and w are two non-zero complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then $\bar{z}w$ is equal to [AIEEE 2003]

(a) 1 (b) -1

(c) i (d) $-i$

45. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then [AIEEE 2003]

(a) $x = 4n$, where n is any positive integer

(b) $x = 2n$, where n is any positive integer

(c) $x = 4n + 1$, where n is any positive integer

(d) $x = 2n + 1$, where n is any positive integer

46. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to [AIEEE 2002]

(a) $128\,\omega$ (b) $-128\,\omega$

(c) $128\,\omega^2$ (d) $-128\,\omega^2$

47. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then [AIEEE 2002]

(a) $x = 3, y = 1$

(b) $x = 1, y = 3$

(c) $x = 0, y = 3$

(d) $x = 0, y = 0$

Answers

Check Point 1

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (c)

Check Point 2

1. (a)	**2.** (c)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (b)

Check Point 3

1. (a)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (b)

Round I

1. (b)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (b)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (d)	**24.** (b)	**25.** (d)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (a)
31. (b)	**32.** (b)	**33.** (a)	**34.** (a)	**35.** (b)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (c)
41. (a)	**42.** (d)	**43.** (c)	**44.** (c)	**45.** (d)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (c)
51. (d)	**52.** (c)	**53.** (a)	**54.** (c)	**55.** (b)	**56.** (a)	**57.** (a)	**58.** (c)	**59.** (b)	**60.** (b)
61. (d)	**62.** (a)	**63.** (b)	**64.** (a)	**65.** (b)	**66.** (b)	**67.** (c)	**68.** (a)	**69.** (d)	**70.** (d)
71. (a)	**72.** (a)	**73.** (a)	**74.** (b)	**75.** (b)	**76.** (a)	**77.** (a)	**78.** (c)	**79.** (c)	**80.** (b)
81. (b)	**82.** (a)	**83.** (d)	**84.** (b)	**85.** (a)	**86.** (b)	**87.** (a)	**88.** (a)	**89.** (a)	**90.** (a)
91. (b)	**92.** (b)	**93.** (b)	**94.** (b)	**95.** (d)	**96.** (c)	**97.** (d)	**98.** (c)	**99.** (b)	**100.** (d)

Round II

1. (d)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (a)
11. (c)	**12.** (c)	**13.** (a,b,c)	**14.** (a,b,c)	**15.** (a,c)	**16.** (a,d)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (a)
21. (d)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (a)	**26.** (d)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (a)
31. (a)	**32.** (b)	**33.** (b)	**34.** (d)	**35.** (c)	**36.** (c)	**37.** (b)	**38.** (b)	**39.** (b)	**40.** (c)
41. (d)	**42.** (b)	**43.** (c)	**44.** (d)	**45.** (a)	**46.** (d)	**47.** (d)			

the **Guidance**

Round I

1. Let $z = \sqrt{5} + 3i$, then its multiplicative inverse is

$$\frac{1}{z} = \frac{1}{\sqrt{5}+3i} = \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{5-9i^2} \qquad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{\sqrt{5}-3i}{5+9} = \frac{\sqrt{5}-3i}{14} \qquad (\because i^2 = -1)$$

$$= \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

2. Given, $$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy \qquad \text{...(i)}$$

Now, $$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1^2-(i)^2} = \frac{1^2+(i)^2+2i}{1-(-1)}$$

$$= \frac{1-1+2i}{1+1} = \frac{2i}{2} = i$$

$$\therefore \quad \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = (i)^3 - \left(\frac{1}{i}\right)^3 = -i - \frac{1}{(-i)}$$

$$= -i + \frac{1}{i} \times \frac{i}{i} = -i + \frac{i}{-1} = -2i$$

$$\therefore \quad -2i = x + iy \qquad [\because \text{from Eq. (i)}]$$

On comparing the real and imaginary part both sides, we get

$$x = 0, y = -2$$

3. Given, $$\frac{(1+i)^2}{2-i} = x + iy$$

$$\therefore \quad \frac{1^2+(i)^2+2i}{2-i} \times \frac{2+i}{2+i} = x + iy$$

$$\Rightarrow \quad \frac{(1-1+2i)(2+i)}{2^2-(i)^2} = x + iy$$

$\Rightarrow \quad \dfrac{4i-2}{4+1} = x + iy$

$\Rightarrow \quad \dfrac{4i}{5} - \dfrac{2}{5} = x + iy$

On comparing real and imaginary parts, we get

$$x = -\frac{2}{5}, y = \frac{4}{5}$$

$\therefore \quad x + y = -\dfrac{2}{5} + \dfrac{4}{5} = \dfrac{2}{5}$

4. Let $\quad z = \dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha} \times \dfrac{(1 - 2i\sin\alpha)}{(1 - 2i\sin\alpha)}$

$$= \frac{1 - 2i\sin\alpha - i\sin\alpha + (-1)\,2\sin^2\alpha}{1^2 - (2i\sin\alpha)^2}$$

$$= \frac{1 - 2\sin^2\alpha - 3i\sin\alpha}{1 + 4\sin^2\alpha}$$

Since, z is purely real, therefore

$$\text{Im}(z) = 0$$

$$\frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0$$

$\Rightarrow \quad \sin\alpha = 0 \Rightarrow \quad \alpha = n\pi$

5. Let $\quad z = x + iy$

$\Rightarrow \quad z^2 = (x + iy)^2$

$\Rightarrow \quad z^2 = x^2 + i^2y^2 + 2ixy$

$\Rightarrow \quad z^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow \quad \text{Re}(z^2) = x^2 - y^2$

Given, $\quad \text{Re}(z^2) = 0$

$\Rightarrow \quad x^2 - y^2 = 0$

$\Rightarrow \quad x = \pm y$

$\Rightarrow \quad x = y \quad$ ...(i)

and $\quad x = -y \quad$ ...(ii)

Again, $\quad |z| = 2$

$\Rightarrow \quad |z|^2 = 4$

$\Rightarrow \quad x^2 + y^2 = 4 \quad$ ...(iii)

From Eqs. (i) and (iii), we get

$$y^2 + y^2 = 4 \Rightarrow 2y^2 = 4$$

$\Rightarrow \quad y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

Therefore, from Eq. (i), we get

$$x = \pm\sqrt{2}$$

$\therefore \quad z = \pm\sqrt{2} \pm i\sqrt{2}$

On putting the value of x from Eq. (ii) in Eq. (iii), we get

$$(-y)^2 + y^2 = 4 \Rightarrow 2y^2 = 4$$

$\Rightarrow \quad y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

From Eq. (ii), $\quad x = \mp\sqrt{2}$

$\therefore \quad z = x + iy, z = \mp\sqrt{2} \pm i\sqrt{2}$

6. Let $S = 1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$

$$= 1 - 1 + 1 - 1 + 1 - \ldots + (-1)^n$$

The value of S depends on n.

$\therefore$ The value cannot be determined.

7. As sum of any four consecutive powers of iota is zero.

$$\therefore \quad \sum_{n=1}^{13} (i^n + i^{n+1}) = (i + i^2 + \ldots + i^{13}) + (i^2 + i^3 + \ldots + i^{14})$$

$$= i + i^2 = i - 1$$

8. $\left(\dfrac{1+i}{1-i}\right)^x = \left[\dfrac{(1+i)}{(1-i)} \times \dfrac{1+i}{1+i}\right]^x$

$$= \left[\frac{(1+i)^2}{1^2 - i^2}\right]^x = \left[\frac{1^2 + i^2 + 2i}{1+1}\right]^x$$

$$= \left(\frac{1 - 1 + 2i}{2}\right)^x = i^x$$

But it is given $i^x = 1$

$\Rightarrow \quad x = 4n$, where $n \in N$

9. **Hints** $\dfrac{1 + i\cos\theta}{1 - 2i\cos\theta} = \dfrac{1 + i\cos\theta}{1 - 2i\cos\theta} \times \dfrac{1 + 2i\cos\theta}{1 + 2i\cos\theta}$

$$= \frac{1 + 3i\cos\theta + 2i^2\cos^2\theta}{1^2 - (2i\cos\theta)^2}$$

$$= \frac{1 - 2\cos^2\theta}{1 + 4\cos^2\theta} + \frac{i3\cos\theta}{1 + 4\cos^2\theta}$$

10. We have, $z = \dfrac{q + ir}{1 + p} \quad \therefore \; iz = \dfrac{-r + iq}{1 + p}$

By componendo and dividendo,

$$\frac{1 + iz}{1 - iz} = \frac{1 + p - r + iq}{1 + p + r - iq}$$

$\therefore \quad \dfrac{p + iq}{1 + r} = \dfrac{1 + iz}{1 - iz}$, if $\dfrac{p + iq}{1 + r} = \dfrac{1 + p - r + iq}{1 + p + r - iq}$

$\Rightarrow \quad p(1 + p + r) + q^2 + i\{q(1 + p + r) - pq\}$

$$= (1 + r)(1 + p - r) + iq(1 + r)$$

$\Rightarrow \quad p(1 + p + r) + q^2 = (1 + r)(1 + p - r)$

and $\quad q(1 + p + r) - pq = q(1 + r)$

(this is obviously true)

$\therefore$ The condition is

$$p(1 + p + r) + q^2 = (1 + r)(1 + p - r)$$

$\Rightarrow \quad p + p^2 + pr + q^2 = 1 + p - r + r + pr - r^2$

$\Rightarrow \quad p^2 + q^2 + r^2 = 1$

11. **Hints** $\quad \dfrac{\bar{z}}{z} = \dfrac{\overline{(x + iy)}}{x + iy} = \dfrac{x - iy}{x + iy} \times \dfrac{x - iy}{x - iy}$

$$= \frac{x^2 - y^2 - 2ixy}{x^2 - i^2y^2} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} + \frac{i(-2xy)}{x^2 + y^2}$$

Since, $\frac{\bar{z}}{z}$ lies in third quadrant, therefore its real and imaginary parts should be negative.

12. Let $z = \frac{2 - i}{(1 - 2i)^2}$

$\therefore$ Conjugate of $z = \bar{z} = \frac{2 + i}{(1 + 2i)^2}$ ($\because$ replacing i by $-i$)

$$\therefore \quad \bar{z} = \frac{2 + i}{1^2 - 4 + 4i} = \frac{2 + i}{4i - 3} \times \frac{4i + 3}{4i + 3}$$

$$= \frac{11i + 2}{(4i)^2 - (3)^2} = \frac{11i + 2}{-16 - 9} = \frac{-2}{25} - i\,\frac{11}{25}$$

13. Let $z = \sin x + i \cos 2x$

According to the given condition,

$$\bar{z} = \cos x - i \sin 2x$$

$$\therefore \quad \sin x - i \cos 2x = \cos x - i \sin 2x$$

$$\Rightarrow \quad (\sin x - \cos x) + i(\sin 2x - \cos 2x) = 0$$

On equating real and imaginary parts, we get

$$\sin x - \cos x = 0, \sin 2x - \cos 2x = 0$$

$$\Rightarrow \quad \tan x = 0 \text{ and } \tan 2x = 0$$

$$\Rightarrow \quad x = n\pi \text{ and } 2x = n\pi$$

$$\Rightarrow \quad x = n\pi \text{ and } x = \frac{n\pi}{2}$$

which is not possible.

14. $(z + 3)(\bar{z} + 3) = z\bar{z} + 3z + 3\bar{z} + (3)^2$

$$= |z|^2 + 3\left(\frac{z + \bar{z}}{2}\right) \times 2 + 3^2 \qquad (\because |z|^2 = z\bar{z})$$

$$= |z|^2 + 2 \times 3 \times |z| + 3^2 = |z + 3|^2$$

15. Let $z = x + iy$

$$\frac{z - 1}{z + 1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x - 1) + iy}{(x + 1) + iy} \times \frac{(x + 1) - iy}{(x + 1) - iy}$$

$$= \frac{(x - 1)(x + 1) - iy(x - 1) + iy(x + 1) - i^2 y^2}{(x + 1)^2 - i^2 y^2}$$

$$= \frac{x^2 - 1 + iy(x + 1 - x + 1) + y^2}{(x + 1)^2 + y^2}$$

$$\Rightarrow \quad \frac{z - 1}{z + 1} = \frac{(x^2 + y^2 - 1)}{(x + 1)^2 + y^2} + \frac{i(2y)}{(x + 1)^2 + y^2}$$

$\because \frac{z - 1}{z + 1}$ is purely imaginary.

$$\therefore \quad \text{Re}\left(\frac{z - 1}{z + 1}\right) = 0$$

$$\Rightarrow \quad \frac{x^2 + y^2 - 1}{(x + 1)^2 + y^2} = 0$$

$$\Rightarrow \quad x^2 + y^2 - 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 = 1$$

$$\Rightarrow \quad |z^2| = 1 \Rightarrow |z| = 1$$

16. We have, $|z_k| = 1, k = 1, 2, \ldots, n$

$$\Rightarrow \quad |z_k|^2 = 1 \Rightarrow z_k \bar{z}_k = 1 \Rightarrow \bar{z}_k = \frac{1}{z_k}$$

$$\therefore |z_1 + z_2 + \ldots + z_n| = |\overline{z_1 + z_2 + \ldots + z_n}| \qquad (\because |z| = |\bar{z}|)$$

$$= |\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_n|$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|$$

17. Given that, $|\beta| = 1$

$$\therefore \quad \left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{\beta\bar{\beta} - \bar{\alpha}\beta}\right|$$

$$= \left|\frac{\beta - \alpha}{\beta(\bar{\beta} - \bar{\alpha})}\right| = \frac{1}{|\beta|}\left|\frac{\beta - \alpha}{(\bar{\beta} - \bar{\alpha})}\right|$$

$$= \frac{1}{|\beta|} = 1 \qquad (\because |z| = |\bar{z}|)$$

18. Given, $|z + 1| = z + 2(1 + i)$ ...(i)

By taking option (a),

Put $z = \frac{1}{2} - 2i$ in Eq. (i), we get

$$\text{LHS} = \left|\frac{1}{2} - 2i + 1\right| = \left|\frac{3}{2} - 2i\right| = \sqrt{\frac{9}{4} + 4} = \frac{\sqrt{25}}{2} = \frac{5}{2}$$

$$\text{RHS} = \left(\frac{1}{2} - 2i\right) + 2 + 2i = \frac{5}{2}$$

$\therefore$ LHS = RHS

Hence, option (a) is correct.

19. Let $z = x + iy$

$$\Rightarrow \quad \bar{z} = x - iy$$

and $$(\bar{z}^{-1}) = \frac{1}{x - iy} = \frac{x + iy}{x^2 + y^2}$$

$$\therefore \quad (\bar{z}^{-1})\,\bar{z} = \frac{x + iy}{x^2 + y^2} \times (x - iy) = 1$$

20. Given, $\left|\frac{i + z}{i - z}\right| = 1$

Let $z = x + iy$

$$\therefore \quad \left|\frac{i + x + iy}{i - (x + iy)}\right| = 1$$

$$\Rightarrow \quad \frac{|x + i(1 + y)|}{|-x + i(1 - y)|} = 1$$

$$\Rightarrow \quad \sqrt{x^2 + (1 + y)^2} = \sqrt{(-x)^2 + (1 - y)^2}$$

$$\Rightarrow \quad x^2 + 1^2 + y^2 + 2y = x^2 + 1^2 + y^2 - 2y$$

$$\Rightarrow \quad 4y = 0 \Rightarrow y = 0$$

Hence, the given condition lies on x-axis.

21. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\because \quad |z_1 + z_2| = |z_1| + |z_2|$

$|(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$

$= |r_1\cos\theta_1 + ir_1\sin\theta_1| + |r_2\cos\theta_2 + ir_2\sin\theta_2|$

$\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)} = \sqrt{r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta_1} + \sqrt{r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2}$

$\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)} = r_1 + r_2$

$\Rightarrow r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2 = 0$

$\Rightarrow \theta_1 = \theta_2$

$\Rightarrow \arg(z_1) = \arg(z_2)$

22. $|z - 2| = \min\{|z - 1|, |z - 5|\}$

i.e., $|z - 2| = |z - 1|$, where $|z - 1| < |z - 5|$

$\Rightarrow \text{Re}(z) = \frac{3}{2}$, which satisfy $|z - 1| < |z - 5|$

Also, $|z - 2| = |z - 5|$, where $|z - 5| < |z - 1|$

$\Rightarrow \text{Re}(z) = \frac{7}{2}$, which satisfy $|z - 5| < |z - 1|$

23. $|3z - 1| = 3|z - 2|$

$\Rightarrow \left|z - \frac{1}{3}\right| = |z - 2|$

$\Rightarrow z$ is perpendicular bisector of $\left(\frac{1}{3}, 0\right)$ and $(2, 0)$.

$\Rightarrow x = \frac{7}{6}$

24. We have, $\left|\frac{z - a}{z + \bar{a}}\right| = 1$

$\Rightarrow |z - a| = |z + \bar{a}| \Rightarrow |z - a|^2 = |z + \bar{a}|^2$

$\Rightarrow (z - a)\overline{(z - a)} = (z + \bar{a})\overline{(z + \bar{a})}$

$\Rightarrow (z - a)(\bar{z} - \bar{a}) = (z + \bar{a})(\bar{z} + a) \quad [\because (\bar{\bar{a}}) = a]$

$\Rightarrow z\bar{z} - z\bar{a} - a\bar{z} + a\bar{a} = z\bar{z} + za + \bar{a}\bar{z} + \bar{a}a$

$\Rightarrow za + z\bar{a} + \bar{a}\bar{z} + a\bar{z} = 0$

$\Rightarrow (a + \bar{a})(z + \bar{z}) = 0$

$\Rightarrow z + \bar{z} = 0 \quad [\because a + \bar{a} = 2\,\text{Re}(a) \neq 0]$

$\Rightarrow 2\,\text{Re}(z) = 0 \Rightarrow 2x = 0$

$\Rightarrow x = 0 \Rightarrow y$-axis

25. We have, $|\lambda_1a_1 + \lambda_2a_2 + \ldots + \lambda_na_n|$

$\leq |\lambda_1a_1| + |\lambda_2a_2| + \ldots + |\lambda_na_n|$

$= |\lambda_1||a_1| + \ldots + |\lambda_n||a_n|$

$= \lambda_1|a_1| + \ldots + \lambda_n|a_n| < \lambda_1 + \ldots + \lambda_n \quad (\because \text{each } \lambda_k \geq 0)$

$(\because |a_k| < 1$ and so $\lambda_k|a_k| < \lambda_k$ for all $k = 1, 2, \ldots, n)$

Hence, $|\lambda_1a_1 + \lambda_2a_2 + \ldots + \lambda_na_n| < 1$

26. Let $z = x + iy$

$\therefore \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Then, $\arg(\bar{z}) = \tan^{-1}\left(-\frac{y}{x}\right) = 2\pi - \tan^{-1}\frac{y}{x}$

$= 2\pi - \arg(z)$

Since, in argument of a conjugate of a complex, the real axis is unaltered but imaginary axis be changed, hence it is given by $2\pi - \arg(z)$.

27. We have, $\arg(z - 1) = \arg(z + 3i)$

On putting $z = x + iy$,

$\Rightarrow \arg(x + iy - 1) = \arg(x + iy + 3i)$

$\Rightarrow \arg[(x - 1) + iy] = \arg[x + i(y + 3)]$

$\Rightarrow \tan^{-1}\frac{y}{x - 1} = \tan^{-1}\frac{y + 3}{x}$

$\Rightarrow \frac{y}{x - 1} = \frac{y + 3}{x} \Rightarrow xy = (x - 1)(y + 3)$

$\Rightarrow xy = xy + 3x - y - 3$

$\Rightarrow 0 = 3(x - 1) - y$

$\Rightarrow y = 3(x - 1) \Rightarrow \frac{x - 1}{y} = \frac{1}{3}$

$\Rightarrow (x - 1) : y = 1 : 3$

28. Let $z_1 = r(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r(\cos\theta_2 + i\sin\theta_2)$

Since, $|z_2| = |z_1|$

$\therefore r_2 = r_1$

Also, $\arg(z_1) + \arg(z_2) = \pi$

$\therefore \arg(z_2) = \pi - \arg(z_1)$

$\Rightarrow \arg(z_2) = \pi - \theta_1$

$\therefore z_2 = r_1\{\cos(\pi - \theta_1) + i\sin(\pi - \theta_1)\}$

$= r_1(-\cos\theta_1 + i\sin\theta_1)$

$= -r_1(\cos\theta_1 - i\sin\theta_1) = -\bar{z}_1$

$\Rightarrow z_1 = -\bar{z}_2$

29. Given, $f(z) = \frac{7 - z}{1 - z^2}$ and $z = 1 + 2i$

$\therefore f(z) = \frac{7 - (1 + 2i)}{1 - (1 + 2i)^2}$

$= \frac{6 - 2i}{1 - (1 - 4 + 4i)} = \frac{6 - 2i}{4 - 4i}$

$= \frac{6 - 2i}{4(1 - i)} \times \frac{1 + i}{(1 + i)} = \frac{6 + 4i + 2}{4(1^2 - i^2)}$

$= \frac{8 + 4i}{4(2)} = \frac{1}{2}(2 + i)$

$|f(z)| = \frac{\sqrt{4 + 1}}{2} = \frac{\sqrt{5}}{2}$

$= \frac{|z|}{2} \quad (\because z = 1 + 2i$, given $\Rightarrow |z| = \sqrt{5})$

30. We have, $\tan^{-1}(\alpha + i\beta) = x + iy$

$\Rightarrow \quad \alpha + i\beta = \tan(x + iy)$ ...(i)

Taking conjugate, we get

$(\alpha - i\beta) = \tan(x - iy)$ ...(ii)

$\therefore \quad \tan 2x = \tan[(x + iy) + (x - iy)]$

$\Rightarrow \quad \tan 2x = \dfrac{(\alpha + i\beta) + (\alpha - i\beta)}{1 - (\alpha + i\beta)(\alpha - i\beta)}$

$= \dfrac{2\alpha}{1 - (\alpha^2 + \beta^2)}$

$\therefore \quad x = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2\alpha}{1 - \alpha^2 - \beta^2}\right)$

31. Let $z = x + iy \Rightarrow \bar{z} = x - iy$

Since, $\arg(z) = \tan^{-1}\dfrac{y}{x}$

and $\arg(\bar{z}) = \tan^{-1}\left(\dfrac{-y}{x}\right)$

$\Rightarrow \quad \arg(z) \neq \arg(\bar{z})$

32. We know that principal argument of a complex number lie between $-\pi$ and π, but $\alpha + \beta > \pi$.

Therefore, principal

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \alpha + \beta$ is given by $\alpha + \beta - \pi$.

33. We have, $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$

Therefore, $z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

Now, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = \arg\left(\dfrac{z_1 z_2}{z_4 z_3}\right)$

$= \arg\left(\dfrac{|z_1|^2}{|z_3|^2}\right) = \arg\left(\left|\dfrac{z_1}{z_3}\right|^2\right) = 0$

($\because$ argument of positive real number is zero)

34. Let $z = \dfrac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$

$= \dfrac{1 - i\sqrt{3}}{1 + i\sqrt{3}} \times \dfrac{1 - i\sqrt{3}}{1 - i\sqrt{3}} = \dfrac{-1 - i\sqrt{3}}{2}$

$\therefore \quad \arg(z) = \theta = \tan^{-1}\left|\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right| = \tan^{-1}(\sqrt{3}) = 60°$

35. Since, $\arg(z) - \arg(w) = \dfrac{\pi}{2}$

$\Rightarrow \quad -\arg(z) + \arg(w) = -\dfrac{\pi}{2}$

$\Rightarrow \quad \arg(\bar{z}) + \arg(w) = -\dfrac{\pi}{2}$

$\Rightarrow \quad \arg(\bar{z}w) = -\dfrac{\pi}{2}$ ...(i)

Also given, $|zw| = i = |\bar{z}w|$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$\bar{z}w = |\bar{z}w|\, e^{i\arg(zw)}$

$= 1\, e^{-i\frac{\pi}{2}} = \cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)$

$= 0 - i\sin\dfrac{\pi}{2} = -i$

36. We have, $|z + 4| \le 3 \Rightarrow -3 \le z + 4 \le 3$

$\Rightarrow \quad -6 \le z + 1 \le 0 \Rightarrow 0 \le -(z + 1) \le 6$

$\Rightarrow \quad 0 \le |z + 1| \le 6$

Hence, greatest and least values of $|z + 1|$ are 6 and 0, respectively.

37. We know that, $|-z| = |z|$

and $|z_1 + z_2| \le |z_1| + |z_2|$

Now, $|z| + |z - 1| = |z| + |1 - z| \ge |z + (1 - z)| = |1| = 1$

Hence, minimum value of $|z| + |z - 1|$ is 1.

38. Since, $|z + a| \le a$ implies z lies on or inside a circle with centre $(-a, 0)$ and radius a, we have

$|z_1| + |z_2| + |z_3| \le 14$

39. $\left|z + \dfrac{2}{z}\right| = 2 \Rightarrow |z| - \dfrac{2}{|z|} \le 2$

$\Rightarrow \quad |z|^2 - 2|z| - 2 \le 0$

This is a quadratic equation in $|z|$.

$\therefore \quad |z| \le \dfrac{2 \pm \sqrt{4 + 8}}{2} \le 1 \pm \sqrt{3}$

Hence, maximum value of $|z|$ is $1 + \sqrt{3}$.

40. Since, $|z_1 - 1| < 4$

$\Rightarrow \quad |z_1| - |1| < 1$ ($\because$ by triangle inequality)

$\Rightarrow \quad |z_1| < 2$ ...(i)

Similarly, $|z_3 - 2| < 2$

$\Rightarrow \quad |z_2| < 4$ ...(ii)

and $|z_3 - 3| < 3$

$\Rightarrow \quad |z_3| < 6$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$|z_1| + |z_2| + |z_3| < 2 + 4 + 6$

$\Rightarrow \quad |z_1| + |z_2| + |z_3| < 12$ ...(iv)

By using triangular inequality

$|z_1 + z_2 + z_3| \le |z_1| + |z_2 + |z_3|$

$\Rightarrow \quad |z_1 + z_2 + z_3| < 12$ [$\because$ from Eq. (iv)]

41. We have, $OC = 5$ and $CA = 1$

Clearly, the point z is such that amplitude of z is minimum to the point of contact of the circle and the tangent drawn from the origin O.

Let $\theta = \angle AOC$. Then, $\angle AOC = 90° - \theta$

From figure,

$$\sin \angle AOC = \frac{1}{5}$$

$$\Rightarrow \quad \sin(90° - \theta) = \frac{1}{5}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{5}$$

and $$\sin\theta = \frac{\sqrt{24}}{5}$$

Now, $$OA = \sqrt{OC^2 - CA^2} = \sqrt{5^2 - 1^2} = \sqrt{24}$$

Now, the coordinates of A are $(\sqrt{24}\cos\theta, \sqrt{24}\sin\theta)$

i.e., $$\left(\frac{\sqrt{24}}{5}, \frac{24}{5}\right)$$

Hence, $$z = \frac{2\sqrt{6}}{5} + i\,\frac{24}{5}$$

42. $$\frac{4(\cos 75° + i\sin 75°)}{0.4(\cos 30° + i\sin 30°)}$$

$$= 10(\cos 75° + i\sin 75°)(\cos 30° - i\sin 30°)$$

$$= 10\,e^{75i}\cdot e^{-30i} = 10\,e^{45i}$$

$$= 10(\cos 45° + i\sin 45°) = \frac{10}{\sqrt{2}}(1+i)$$

43. $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$

$$\Rightarrow \quad \cos(\theta + 2\theta + 3\theta + \ldots + n\theta) + i\sin(\theta + 2\theta + 3\theta + \ldots + n\theta) = 1$$

$$\Rightarrow \quad \cos\left\{\frac{n(n+1)}{2}\theta\right\} + i\sin\left\{\frac{n(n+1)}{2}\theta\right\} = 1$$

On comparing the coefficients of real and imaginary parts on both sides, we get

$$\cos\left\{\frac{n(n+1)}{2}\theta\right\} = 1 \text{ and } \sin\left\{\frac{n(n+1)}{2}\theta\right\} = 0$$

$$\Rightarrow \quad \frac{n(n+1)}{2}\theta = 2m\pi$$

$$\Rightarrow \quad \theta = \frac{4m\pi}{n(n+1)}, \text{ where } m \in I$$

44. Here, $$\sqrt{1-c^2} = nc - 1$$

$$\Rightarrow \quad 1 - c^2 = n^2c^2 - 2nc + 1$$

$$\therefore \quad \frac{c}{2n} = \frac{1}{1+n^2} \quad \ldots(\text{i})$$

or $$\frac{c}{2n}(1+nz)\left(1+\frac{n}{z}\right) = \frac{1}{1+n^2}\left\{1 + n^2 + n\left(z + \frac{1}{z}\right)\right\}$$

$$= \frac{1}{1+n^2}\{1 + n^2 + n(2\cos\theta)\}$$

$$= \frac{(1+n^2) + 2n\cos\theta}{1+n^2}$$

$$= 1 + \left(\frac{2n}{1+n^2}\right)\cos\theta \quad \text{[using Eq. (i)]}$$

$$= 1 + c\cos\theta$$

45. We have, $\{(1-\cos\theta) + i\cdot 2\sin\theta\}^{-1}$

$$= \left(2\sin^2\frac{\theta}{2} + i\cdot 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1}\left(\sin\frac{\theta}{2} + i\cdot 2\cos\frac{\theta}{2}\right)^{-1}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-1}\cdot\frac{1}{\sin\frac{\theta}{2} + i\cdot 2\cos\frac{\theta}{2}} \times \frac{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2} - i\cdot 2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1 + 3\cos^2\frac{\theta}{2}\right)}$$

It's real part

$$= \frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1 + 3\cos^2\frac{\theta}{2}\right)} = \frac{1}{2\left(1 + 3\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{1}{2 + 3(\cos\theta + 1)} = \frac{1}{5 + 3\cos\theta}$$

46. $$\frac{1+a}{1-a} = \frac{e^{\frac{-i\theta}{2}}(1+e^{i\theta})}{e^{\frac{-i\theta}{2}}(1-e^{i\theta})} = \frac{e^{-i\left(\frac{\theta}{2}\right)} + e^{i\frac{\theta}{2}}}{e^{-i\left(\frac{\theta}{2}\right)} - e^{-i\frac{\theta}{2}}} = \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}} = i\cot\frac{\theta}{2}$$

47. $$x + iy = \frac{3}{2 + \cos\theta + i\sin\theta} = \frac{3(2 + \cos\theta - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{6 + 3\cos\theta - 3i\sin\theta}{4 + \cos^2\theta + 4\cos\theta + \sin^2\theta}$$

$$= \left[\frac{6 + 3\cos\theta}{5 + 4\cos\theta}\right] + i\left[\frac{-3\sin\theta}{5 + 4\cos\theta}\right]$$

On equating the real and imaginary parts on both sides, we get

$$x = \frac{3(2+\cos\theta)}{5 + 4\cos\theta}, \quad y = \frac{-3\sin\theta}{5 + 4\cos\theta}$$

$$\therefore \quad x^2 + y^2 = \frac{9}{(5 + 4\cos\theta)^2}(4 + \cos^2\theta + 4\cos\theta + \sin^2\theta)$$

$$= \frac{9}{5 + 4\cos\theta} = 4\left(\frac{6 + 3\cos\theta}{5 + 4\cos\theta}\right) - 3$$

$$= 4x - 3$$

48. We have, $z_r = \cos\frac{r\alpha}{n^2} + i\sin\frac{r\alpha}{n^2}$

where $r = 1, 2, 3, \ldots, n$

$$\therefore \quad z_1 = \cos\frac{\alpha}{n^2} + i\sin\frac{\alpha}{n^2};$$

$$z_2 = \cos\frac{2\alpha}{n^2} + i\sin\frac{2\alpha}{n^2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$z_n = \cos\frac{n\alpha}{n^2} + i\sin\frac{n\alpha}{n^2}$$

$$\therefore \lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n)$$

$$= \lim_{n\to\infty}\left(\cos\frac{\alpha}{n^2} + i\sin\frac{\alpha}{n^2}\right)\left(\cos\frac{2\alpha}{n^2} + i\sin\frac{2\alpha}{n^2}\right)\ldots\left(\cos\frac{n\alpha}{n^2} + i\sin\frac{n\alpha}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\cos\left\{\frac{\alpha}{n^2}(1+2+3+\ldots+n)\right\} + i\sin\left\{\frac{\alpha}{n^2}(1+2+3+\ldots+n)\right\}\right]$$

$$= \lim_{n\to\infty}\left[\cos\frac{\alpha n(n+1)}{2n^2} + i\sin\frac{\alpha n(n+1)}{2n^2}\right]$$

$$= \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} = e^{\frac{i\alpha}{2}}$$

49. Let $z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$, then by using de-Moivre's theorem

$$\therefore \quad z^k = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7} \qquad \ldots(\text{i})$$

Let
$$S = \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$$

$$= \sum_{k=1}^{6}\left[(-i)\left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right)\right]$$

$$= (-i)\sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right)$$

$$= (-i)\sum_{k=1}^{6} z^k \qquad \text{[from Eq. (i)]}$$

$$= (-i)(z + z^2 + z^3 + \ldots + z^6)$$

It is a GP of which the first term is z, number of terms is 6 and the common ratio is

$$z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} \neq 1$$

$$\therefore \quad S = (-i)\frac{z(1-z^6)}{1-z} = (-i)\frac{z - z^7}{1-z}$$

$$= (-i)\frac{z-1}{1-z} = i \qquad \left[\because z^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1\right]$$

50. Let $z - 1 = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\therefore \text{Given expression} = re^{i\theta}\cdot e^{-i\alpha} + \frac{1}{re^{i\theta}}\cdot e^{i\alpha}$$

$$= re^{i(\theta-\alpha)} + \frac{1}{r}e^{-i(\theta-\alpha)}$$

Since, imaginary part of given expression is zero, we have

$$r\sin(\theta-\alpha) - \frac{1}{r}\sin(\theta-\alpha) = 0$$

$$r^2 - 1 = 0 \quad \Rightarrow \quad r^2 = 1$$

$$\Rightarrow \quad r = 1$$

$$\Rightarrow \quad |z-1| = 1$$

or $\quad \sin(\theta-\alpha) = 0 \quad \Rightarrow \quad \theta - \alpha = 0$

$$\Rightarrow \quad \theta = \alpha$$

$$\Rightarrow \quad \arg(z-1) = \alpha$$

51. We have, $\quad a = \cos\alpha + i\sin\alpha,$

$$b = \cos\beta + i\sin\beta$$

and $\quad c = \cos\gamma + i\sin\gamma$

Now, $\quad \frac{b}{c} = \frac{\cos\beta + i\sin\beta}{\cos\gamma + i\sin\gamma} \times \frac{\cos\gamma - i\sin\gamma}{\cos\gamma - i\sin\gamma}$

$$= \cos\beta\cos\gamma + \sin\beta\sin\gamma + i[\sin\beta\cos\gamma - \sin\gamma\cos\beta]$$

$$\Rightarrow \quad \frac{b}{c} = \cos(\beta-\gamma) + i\sin(\beta-\gamma) \qquad \ldots(\text{i})$$

Similarly, $\quad \frac{c}{a} = \cos(\gamma-\alpha) + i\sin(\gamma-\alpha) \qquad \ldots(\text{ii})$

and $\quad \frac{a}{b} = \cos(\alpha-\beta) + i\sin(\alpha-\beta) \qquad \ldots(\text{iii})$

On adding Eqs. (i), (ii) and (iii), we get

$$\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) + i[\sin(\beta-\gamma) + \sin(\gamma-\alpha) + \sin(\alpha-\beta)] = 1$$

On equating real parts to on both sides, we get

$$\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = 1$$

52. We have, $\quad \cos\alpha + \cos\beta + \cos\gamma = 0 \qquad \ldots(\text{i})$

and $\quad \sin\alpha + \sin\beta + \sin\gamma = 0 \qquad \ldots(\text{ii})$

Let $\quad a = \cos\alpha + i\sin\alpha,$

$$b = \cos\beta + i\sin\beta$$

and $\quad c = \cos\gamma + i\sin\gamma$

Therefore, $\quad a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$$= 0 + i0 = 0 \qquad \text{[from Eqs. (i) and (ii)]}$$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$

$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$$

$$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

$$\Rightarrow \quad \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$

53. We have,

$$(16)^{1/4} = (2^4)^{1/4} = 2(1)^{1/4} = 2(\cos 0 + i \sin 0)^{1/4}$$

$$= 2\{\cos \frac{1}{4}(2k\pi + 0) + i \sin \frac{1}{4}(2k\pi + 0)\}$$

$$k = 0, 1, 2, 3$$

$$= 2 \times 1, 2 \times i, 2 \times -1, 2 \times -i = \pm 2, \pm 2i$$

54. We have, $\omega_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$

$$\Rightarrow \quad \omega_3 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$$

$$= -\frac{1}{2} + \frac{i\sqrt{3}}{2} = \omega \text{ and } \omega_3^2 = \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)^2$$

$$= \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \omega^2$$

$$\therefore (x + y\omega_3 + z\omega_3^2)(x + y\omega_3^2 + z\omega_3)$$

$$= (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$$

$$= x^2 + y^2 + z^2 - xy - yz - zx$$

55. Given expression

$$= \frac{\omega(a + b\omega + c\omega^2)}{b\omega + c\omega^2 + a} + \frac{\omega^2(a + b\omega + c\omega^2)}{c\omega^2 + a + b\omega}$$

$$= \omega + \omega^2 = -1 \qquad (\because 1 + \omega + \omega^2 = 0)$$

56. Since, α, β and γ be the cube roots of a positive number p.

$$\therefore \quad \alpha = p^{1/3}, \quad \beta = \omega p^{1/3}, \quad \gamma = \omega^2 p^{1/3}$$

Hence, $\frac{\alpha x + \beta y + \gamma z}{\beta x + \gamma y + \alpha z} = \frac{x + \omega y + \omega^2 z}{\omega x + \omega^2 y + z} = \frac{1}{\omega} = \omega^2$

$$= \frac{-1 - i\sqrt{3}}{2}$$

57. $(1 + \omega^2 + 2\omega)^{3n} - (1 + \omega + 2\omega^2)^{3n}$

$$= \omega^{3n} - (\omega^2)^{3n}$$

$$= (\omega^3)^n - (\omega^3)^{2n}$$

$$= 1 - 1 = 0$$

58. Let $x = \sqrt{-1 - \sqrt{-1 - \sqrt{-1 - \ldots \infty}}}$

Then, $x = \sqrt{-1 - x}$ or $x^2 = -1 - x$

or $x^2 + x + 1 = 0$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2} = \omega \text{ or } \omega^2$$

59. $x = a + b, y = a\alpha + b\beta$ and $z = a\beta + b\alpha$

Now, $xyz = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$,

where $\alpha = \omega$ and $\beta = \omega^2$

$$\therefore \quad xyz = (a + b)(a^2 + ab\omega^2 + ab\omega + b^2)$$

$$= (a + b)(a^2 - ab + b^2) = a^3 + b^3$$

60. $(1 + \omega)^7 = (1 + \omega)(1 + \omega)^6 = (1 + \omega)(-\omega^2)^6 = (1 + \omega)$

$$\Rightarrow \quad A + B\omega = 1 + \omega$$

$$\Rightarrow \quad A = 1, \ B = 1$$

61. We have, $\omega^1 \cdot \omega^2 \cdot \omega^3 \cdot \ldots \cdot \omega^n$

$$= \omega^{1+2+3+\ldots+n} = \omega^{\frac{n(n+1)}{2}} = S_n \qquad \text{(say)}$$

On putting $n = 1, 2, 3, \ldots, n$, we get

$$S_1 = \omega^1 = \omega, \quad S_2 = \omega^3 = 1,$$

$$S_3 = \omega^6 = 1, \ldots, \quad S_7 = \omega^{28} = \omega$$

$\therefore$ We always get 1 and ω.

62. $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$

$$= 1(\omega^{3n} - 1) - \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$$

$$= (1 - 1) - 0 + \omega^{2n}[\omega^n - (\omega^3)^n \omega^n] \qquad (\because \omega^{3n} = 1)$$

$$= 0 + 0 + 0 = 0$$

63. $S = 1 + 3\alpha + 5\alpha^2 + \ldots + (2n - 1)\alpha^{n-1}$ ...(i)

$\Rightarrow \quad \alpha S = \alpha + 3\alpha^2 + 5\alpha^3 + \ldots + (2n - 1)\alpha^n$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(1 - \alpha) S = 1 + 2\alpha + 2\alpha^2 + \ldots + 2\alpha^{n-1} - (2n - 1)\alpha^n$$

$$= 2(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}) - 1 - (2n - 1)\alpha^n$$

$$= \frac{2(1 - \alpha^n)}{1 - \alpha} - 2n = -2n \qquad (\because \alpha^n = 1)$$

$$\Rightarrow \quad S = \frac{-2n}{(1 - \alpha)}$$

64. The given equation $z^3 + 2z^2 + 2z + 1 = 0$ can be rewritten as $(z + 1)(z^2 + z + 1) = 0$. Its roots are -1, ω and ω^2.

Let $f(z) = z^{1985} + z^{100} + 1$

Put $z = -1$, ω and ω^2 respectively, we have

$$f(-1) = (-1)^{1985} + (-1)^{100} + 1 \neq 0$$

Therefore, -1 is not a root of the equation $f(z) = 0$.

Again, $f(\omega) = \omega^{1985} + \omega^{100} + 1$

$$= (\omega^3)^{661}\omega^2 + (\omega^3)^{33}\omega + 1 = \omega^2 + \omega + 1 = 0$$

Therefore, ω is a root of the equation $f(z) = 0$.

Similarly, $f(\omega^2) = 0$

Hence, ω and ω^2 are the common roots.

65. r th term of the given series

$$= r[(r + 1) - \omega][(r + 1) - \omega^2]$$

$$= r[(r + 1)^2 - (\omega + \omega^2)(r + 1) + \omega^3]$$

$$= r[(r + 1)^2 - (-1)(r + 1) + 1]$$

$$= r(r^2 + 3r + 3) = r^3 + 3r^2 + 3r$$

Thus, sum of the given series

$$= \sum_{r=1}^{(n-1)} (r^3 + 3r^2 + 3r)$$

$$= \frac{1}{4}(n-1)^2 n^2 + 3 \cdot \frac{1}{6}(n-1)(n)(2n-1) + 3 \cdot \frac{1}{2}(n-1)n$$

$$= \frac{1}{4}(n-1)\,n\,(n^2 + 3n + 4)$$

66. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1-\omega^2 & \omega^2 \\ 0 & \omega^2 & \omega^4 \end{vmatrix}$ $(\because C_1 \to C_1 + C_2 + C_3)$

$$= 3(\omega^2 - \omega) = 3\omega(\omega - 1)$$

67. $\because$ $1, a_1, a_2, \ldots, a_{n-1}$ are n roots of unity.

$\therefore$ $$\frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2)\ldots(x - a_{n-1})$$

$\Rightarrow$ $$x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$$
$$= (x - a_1)(x - a_2)\ldots(x - a_{n-1})$$

Put $x = 1$

$\therefore$ $(1 - a_1)(1 - a_2)\ldots(1 - a_{n-1}) = 1 + 1 + \ldots n$ times $= n$

68. Since, $x^2 - x + 1 = 0$ (given)

Solving for x, we have $x = -\omega$ and $x = -\omega^2$

Case I $x = -\omega$

$$S = \sum_{n=1}^{5} \left\{(-\omega)^n + \frac{1}{(-\omega)^n}\right\}^2$$

$\Rightarrow$ $$S = \sum_{r=1}^{5} (\omega^n + \omega^{2n})^2$$

$\Rightarrow$ $$S = (-1)^2 + (-1)^2 + 2^2 + (-1)^2 + (-1)^2$$
$$S = 8$$

Case II $x = -\omega^2$

$$S = \sum_{n=1}^{6} \left\{(-\omega^2)^n + \left(\frac{1}{-\omega^2}\right)^n\right\}^2$$

$\Rightarrow$ $$S = \sum_{n=1}^{5} (\omega^{2n} + \omega^n)^2 = 8$$

69. Let $z = x + iy$

$\therefore$ $$\frac{z-1}{2z+1} = \frac{x + iy - 1}{2x + 2iy + 1} = \frac{\{(x-1)(2x+1) + 2y^2)\} + iy(3)}{(2x+1)^2 + 4y^2}$$

According to the question,

$$\text{Im}\left(\frac{z-1}{2z+1}\right) = -4$$

$\therefore$ $$\frac{3y}{(2x+1)^2 + 4y^2} = -4$$

$\Rightarrow$ $$3y = -4\{(4x^2 + 4x + 1) + 4y^2\}$$

$\Rightarrow$ $$16x^2 + 16y^2 + 16x + 3y + 4 = 0$$

This equation represents a circle.

Hence, locus of z is a circle.

70. We have, $z = 4 - 3i$

$\therefore$ $$|z| = \sqrt{4^2 + (-3)^2} = 5$$

Let z_1 be the new complex number obtained by rotating z in the clockwise sense through $180°$, therefore

$$z_1 = -4 + 3i$$

Therefore, required complex number is

$$3(-4 + 3i) = -12 + 9i$$

71. Let OA and OB be the sides of an equilateral ΔOAB and let OA and OB represent the complex numbers z_1 and z_2, respectively.

From the equilateral ΔOAB,

$$\mathbf{AB} = z_2 - z_1$$

$\therefore$ $$\arg\left(\frac{z_2 - z_1}{z_2}\right) = \arg(z_2 - z_1) - \arg z_2 = \frac{\pi}{3}$$

and $$\arg\left(\frac{z_2}{z_1}\right) = \arg(z_2) - \arg(z_1) = \frac{\pi}{3}$$

Also, $\left|\frac{z_2 - z_1}{z_2}\right| = 1 = \left|\frac{z_2}{z_1}\right|$, since triangle is equilateral. Thus, the complex numbers $\frac{z_2 - z_1}{z_2}$ and $\frac{z_2}{z_1}$ have same modulus and same argument, which implies that the numbers are equal, that is

$$\frac{z_2 - z_1}{z_2} = \frac{z_2}{z_1} \Rightarrow z_1 z_2 - z_1^2 = z_2^2$$

$\Rightarrow$ $$z_1^2 + z_2^2 = z_1 z_2$$

72. In a parallelogram $OP_1P_2P_3$, the mid-point of P_1P_2 and OP_3 are the same. But mid-point of P_1P_2 is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

So, that the coordinates of P_3 are $(x_1 + x_2, y_1 + y_2)$.

Thus, the point P_3 corresponds to sum of the complex numbers z_1 and z_2.

$\therefore$ $$\mathbf{OP_3} = \mathbf{OP_1} + \mathbf{OP_2} = z_1 + z_2$$

73. $z_1 + z_2 = -1$ and $z_1 z_2 = \frac{b}{3}$

As z_1 and z_2 origin form an equilateral triangle, we have

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$\Rightarrow$ $$z_1^2 + z_2^2 + 0 = z_1 z_2 + 0 + 0$$

$\Rightarrow$ $$(z_1 + z_2)^2 = 3 z_1 z_2 \Rightarrow 1 = b$$

74. We have, $\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$

It is clear from the figure that it a semi-circle.

75. Let $z = x + iy$

$$\therefore \quad \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$$

$$= \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\therefore \quad \arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\frac{2y}{x^2+y^2-1}$$

$$\Rightarrow \quad \tan^{-1}\frac{2y}{x^2+y^2-1} = \frac{\pi}{3} \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow \quad x^2+y^2-1 = \frac{2}{\sqrt{3}}y$$

$$\Rightarrow \quad x^2+y^2-\frac{2}{\sqrt{3}}y-1=0$$

which is an equation of a circle.

76. $z^2 + ax + a^2 = 0 \Rightarrow z = a\omega, a\omega^2$

(where ω is a non-real root of unity)

$\Rightarrow$ Locus of z is a pair of straight lines

and $\arg(z) = \arg(a) + \arg(\omega)$

or $\arg(z) = \arg(a) + \arg(\omega^2)$

$$\Rightarrow \quad \arg(z) = \pm\frac{2\pi}{3}$$

Also, $|z| = |a||\omega|$ or $|z| = |a||\omega^2|$

$\Rightarrow \quad |z| = |a|$

77. We have, $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$

Therefore, the point having affix z is equidistant from the four points having affixes z_1, z_2, z_3 and z_4. Thus, z is the affix of either the centre of a circle or the point of intersection of diagonals of a square (or rectangle). Therefore, z_1, z_2, z_3 and z_4 are either concyclic or vertices of a square (or rectangle). Hence, z_1, z_2, z_3 and z_4 are concyclic.

78. Let $z_1 = 1 + i$, $z_2 = -2 + 3i$ and $z_3 = 0 + \frac{5}{3}i$.

$$\text{Then,} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & \frac{5}{3} & 1 \end{vmatrix}$$

$$= 1\left(3 - \frac{5}{3}\right) + 1(2) + 1\left(\frac{-10}{3}\right)$$

$$= \frac{4}{3} + 2 - \frac{10}{3}$$

$$= \frac{4+6-10}{3} = 0$$

Hence, area of triangle is zero, therefore points are collinear.

79. Let $z = x + iy$

Then, $z + iz = (x + iy) + i(x + iy) = (x - y) + i(x + y)$

and $iz = i(x + iy) = -y + ix$

If triangle be the area of the triangle formed by z, $z + iz$ and iz, then

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} x & y & 1 \\ x-y & x+y & 1 \\ -y & x & 1 \end{vmatrix}\right|$$

Applying $R_2 \to R_2 - (R_1 + R_3)$,

$$\text{then} \quad \Delta = \frac{1}{2}\left\|\begin{matrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 1 \end{matrix}\right\| = \frac{1}{2}(x^2+y^2)$$

$$= \frac{1}{2}|z|^2 = 200 \qquad \text{(given)}$$

$\Rightarrow \quad |z|^2 = 400 \Rightarrow |z| = 20$

$\therefore \quad |3z| = 3|z| = 60$

80. Let $z = x + iy$

$\Rightarrow \quad z^2 = x^2 - y^2 + 2ixy$

$\Rightarrow \quad \text{Re}(z^2) = \text{Re}(x^2 - y^2 + 2ixy)$

$\Rightarrow \quad 1 = x^2 - y^2 \qquad [\because \text{Re}(z^2) = 1 \text{ (given)}]$

81.

$$\therefore \quad \overline{AC} = \overline{AB}\, e^{i\pi/3}$$

By rotating $\frac{\pi}{3}$ in clockwise sense

$\Rightarrow \quad (z_3 - z_1) = (z_2 - z_1)\, e^{i\pi/3} \qquad \ldots\text{(i)}$

Also, $(z_1 - z_2) = (z_3 - z_2)\, e^{i\pi/3} \qquad \ldots\text{(ii)}$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

82. Given z, iz and $z + iz$ be the vertices of a triangle, then

let $z = x + iy$

$\therefore \quad iz = -y + ix, z + iz = (x - y) + i(x + y)$

Hence, required area $= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ 0 & 0 & -1 \end{vmatrix}$

$= \left[-\frac{1}{2}(x^2 + y^2)\right]$

$= \frac{1}{2}|z|^2$

83. Vertices of the triangle are $0 = 0 + i0, z = x + iy$

and $ze^{i\alpha} = (x + iy)(\cos\alpha + i\sin\alpha)$

$= (x\cos\alpha - y\sin\alpha) + i(y\cos\alpha + x\sin\alpha)$

$\therefore$ Area of triangle

$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ (x\cos\alpha - y\sin\alpha) & (y\cos\alpha + x\sin\alpha) & 1 \end{vmatrix}$

$= \frac{1}{2}[xy\cos\alpha + x^2\sin\alpha - xy\cos\alpha + y^2\sin\alpha]$

$= \frac{1}{2}(x^2 + y^2)\sin\alpha = \frac{1}{2}|z|^2\sin\alpha \qquad (\because |z| = \sqrt{x^2 + y^2})$

84. As we know, the equation of the form $\left|\frac{z-2}{z+2}\right| = n$ is a circle, if $n \neq 1$.

85. Let $z = x + iy$

$\therefore \quad \text{Re}\left(\frac{x - iy + 2}{x - iy - 1}\right) = 4 \qquad$ (given)

$\therefore \quad \text{Re}\left[\frac{(x+2) - iy}{(x-1) - iy} \times \frac{(x-1) + iy}{(x-1) + iy}\right] = 4$

$\Rightarrow \quad (x + 2)(x - 1) + y^2 = 4[(x-1)^2 + y^2]$

$\Rightarrow \quad x^2 + y^2 - 3x + 2 = 0$, which represents a circle.

86. Let $z = x + iy$, therefore given equation becomes

$(x + iy)(x - iy) + (2 - 3i)(x + iy) + (2 + 3i)(x - iy) + 4 = 0$

$\Rightarrow x^2 + y^2 + 2x + 3y - 3ix + 2iy + 2x - 2iy$

$+ 3ix + 3y + 4 = 0$

$\Rightarrow \quad x^2 + y^2 + 4x + 6y + 4 = 0$

Therefore, given equation represents a circle with radius

$= \sqrt{2^2 + 3^2 - 4} = \sqrt{4 + 9 - 4} = \sqrt{9} = 3$

87. Let $z = x + iy$

$\therefore \quad \frac{z+i}{z+2} = \frac{x + iy + i}{x + iy + 2} = \frac{x + i(y+1)}{(x+2) + iy}$

$= \frac{[x + i(y+1)] \times [(x+2) - iy]}{[(x+2) + iy] \times [(x+2) - iy]}$

$= \left[\frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2}\right] + i\left[\frac{(y+1)(x+2) - xy}{(x+2)^2 + y^2}\right]$

Since, it is purely imaginary, therefore real part must be equal to zero.

$\therefore \quad \frac{x^2 + y^2 + 2x + y}{(x+2)^2 + y^2} = 0$

$\Rightarrow \quad x^2 + y^2 + 2x + y = 0$

It represents the equation of circle and its radius

$= \sqrt{1 + \frac{1}{4} - 0} = \frac{\sqrt{5}}{2}$

Therefore, locus of z in argand diagram is a circle of radius $\frac{\sqrt{5}}{2}$.

88. Equation of circle whose centre is z_0 and radius is r, is $|z - z_0|^2 = r^2$

$\Rightarrow \quad (z - z_0)\overline{(z - z_0)} = r^2$

$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$

89. $z^3 + \frac{1}{i}z^2 - \frac{z}{i} + 1 = 0$

$\Rightarrow \quad z^3 - iz^2 + iz + 1 = 0$

$\Rightarrow \quad z^2(z - i) + i(z - i) = 0$

$\Rightarrow \quad (z - i)(z^2 + i) = 0 \quad \Rightarrow \quad |z| = 1$

90. Centre of the circle $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$ is $-a$.

Therefore, centre of the given circle

$z\bar{z} - (2 + 3i)z - (2 - 3i)\bar{z} + 9 = 0$

is $2 - 3i$ i.e., $(2, -3)$.

91. $\because \quad \sqrt{3} + i = (a + ib)(c + id)$

$= (ac - bd) + i(ad + bc)$

Here, $\quad ac - bd = \sqrt{3}$ and $ad + bc = 1$

Now, $\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right) = \tan^{-1}\left(\frac{bc + ad}{ac - bd}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= n\pi + \frac{\pi}{6}, n \in I$

92. We have, $|1 - i|^x = 2^x \quad \Rightarrow \quad (\sqrt{2})^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \quad \frac{x}{2} = x \quad \Rightarrow \quad x = 0$

Therefore, the number of non-zero integral solution is one.

93. Given, $z_1 = (1, \alpha) = \cos\alpha + i\sin\alpha = e^{i\alpha}$

$z_2 = (1, \beta) = \cos\beta + i\sin\beta = e^{i\beta}$

and $z_3 = (1, \gamma) = \cos\gamma + i\sin\gamma = e^{i\gamma}$

Also, $z_1 + z_2 + z_3 = 0$

$\therefore (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$ ...(i)

Let $E = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma}$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i0 = 0$ [$\because$ from Eq. (i)]

94. Since, $x = -5 + 4i$

$\Rightarrow x + 5 = 4i$

On squaring both sides, we get

$x^2 + 10x + 25 = -16$

$\Rightarrow x^2 + 10x + 41 = 0$ ...(i)

Let $E = x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2[(x^2 + 10x + 41) - x - 6] - x + 4$

$= x^2(0) - x^3 - 6x^2 - x + 4$ [$\because$ from Eq. (i)]

$= -x^3 - 6x^2 - x + 4$

$= -x(x^2 + 6x + 1) + 4$

$= -x(x^2 + 10x + 41 - 4x - 40) + 4$

$= 4x^2 + 40x + 4$ [$\because$ from Eq. (i)]

$= 4(x^2 + 10x + 41 - 40)$

$= 4(0) - 160$ [$\because$ from Eq. (i)]

$= -160$

95. We have, $a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$\Rightarrow a^7 = \left[\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)\right]^7$

$= \cos 2\pi + i\sin 2\pi = 1$ ...(i)

Let $S = \alpha + \beta = (a + a^2 + a^4) + (a^3 + a^5 + a^6)$

$(\because \alpha = a + a^2 + a^4, \beta = a^3 + a^5 + a^6)$

$\Rightarrow S = a + a^2 + a^3 + a^4 + a^5 + a^6 = \frac{a(1 - a^6)}{1 - a}$

$\Rightarrow S = \frac{a - a^7}{1 - a} = \frac{a - 1}{1 - a} = -1$ ...(ii)

Let $P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$

$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3$

[from Eq. (i)]

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6) = 3 + S$

$= 3 - 1 = 2$ [from Eq. (ii)]

Required equation is, $x^2 - Sx + P = 0$

$\Rightarrow x^2 + x + 2 = 0$

96. Let $z = r(\cos\theta + i\sin\theta)$

Given that, $\left|z + \frac{1}{z}\right| = a \Rightarrow \left|z + \frac{1}{z}\right|^2 = a^2$

$\Rightarrow r^2 + \frac{1}{r^2} + 2\cos 2\theta = a^2$ ...(i)

On differentiating w.r.t. θ, we get

$2r\frac{dr}{d\theta} - \frac{2}{r^3}\frac{dr}{d\theta} - 4\sin 2\theta = 0$

$\Rightarrow \frac{dr}{d\theta}\left(2r - \frac{2}{r^3}\right) = 4\sin 2\theta$

For maximum or minimum, put $\frac{dr}{d\theta} = 0$, we get $\Rightarrow \theta = 0, \frac{\pi}{2}$

$\therefore$ r is maximum for $\theta = \frac{\pi}{2}$, therefore from Eq. (i), we get

$r^2 + \frac{1}{r^2} - 2 = a^2 \Rightarrow r - \frac{1}{r} = a$

$\Rightarrow r^2 - ar - 1 = 0 \Rightarrow r = \frac{a + \sqrt{a^2 + 4}}{2}$

97. $d = \frac{a \cdot 0 + 0 \cdot \bar{a} + |a|^2}{2|a|} = \frac{|a|}{2}$

98. Let O is orthocentre, G is centroid and C is circumcentre, then

$\frac{z_1 + z_2 + z_3}{3} = \frac{2 \times 0 + 1(z)}{3} \Rightarrow z = z_1 + z_2 + z_3$

99. $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$

$\Rightarrow \frac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2$

$\Rightarrow |z|^2 - |z| + 1 < 3(2 + |z|)$

$\Rightarrow |z|^2 - 4|z| - 5 < 0$

$\Rightarrow (|z| + 1)(|z| - 5) < 0$

$\Rightarrow -1 < |z| < 5 \Rightarrow |z| < 5$ as $|z| > 0$

$\therefore$ Locus of z is $|z| < 5$.

100. Given that, $z^2 + (p + iq)z + r + is = 0$ ...(i)

Let $z = \alpha$ (where α is real) be a root of Eq. (i), then

$\alpha^2 + (p + iq)\alpha + r + is = 0$

$\Rightarrow \alpha^2 + p\alpha + r + i(q\alpha + s) = 0$

On equating real and imaginary parts, we get

$\alpha^2 + p\alpha + r = 0$ ...(ii)

and $q\alpha + s = 0 \Rightarrow \alpha = \frac{-s}{q}$

On putting the value of α in Eq. (ii), we get

$\left(\frac{-s}{q}\right)^2 + p\left(\frac{-s}{q}\right) + r = 0$

$\Rightarrow s^2 - pqs + q^2r = 0 \Rightarrow pqs = s^2 + q^2r$

Round II

1. $|z - i\text{Re}(z)| = |z - \text{Im}(z)|$

If $z = x + iy$,

then $|x + iy - ix| = |x + iy - y|$

$$\sqrt{x^2 + (y-x)^2} = \sqrt{(x-y)^2 + y^2}$$

$\Rightarrow$ $x^2 = y^2$

$\therefore$ $x = \pm y$

$\Rightarrow$ $\text{Re}(z) = \pm \text{Im}(z)$

$\Rightarrow$ $\text{Re}(z) + \text{Im}(z) = 0$

and $\text{Re}(z) - \text{Im}(z) = 0$

2. $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n = 3$

$\Rightarrow$ $|3| = |a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n|$

$\Rightarrow$ $3 \le |a_0||z|^n + |a_1||z|^{n-1} + \ldots + |a_{n-1}||z| + |a_n|$

$\Rightarrow$ $3 < 2(|z|^n + |z|^{n-1} + \ldots + |z| + 1)$

$\Rightarrow$ $1 + |z| + |z|^2 + \ldots + |z|^n > \frac{3}{2}$

If $|z| \ge 1$, the inequality is clearly satisfied. For $|z| < 1$, we must have.

$$\frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2}$$

$\Rightarrow$ $2 - 2|z|^{n+1} > 3 - 3|z| \Rightarrow 2|z|^{n+1} < 3|z| - 1$

$\Rightarrow$ $3|z| - 1 > 0 \Rightarrow |z| > \frac{1}{3}$

Since, z_1 is a root of given equation,

$\therefore$ $|z_1| > \frac{1}{3}$

3. $\because \arg(z - 3i) = \arg(x + iy - 3i) = \frac{3\pi}{4}$

$\Rightarrow$ $x < 0, y - 3 > 0$ $\left(\because \frac{3\pi}{4} \text{ is in IInd quadrant}\right)$

$$\frac{y-3}{x} = \tan\frac{3\pi}{4} = -1$$

$\Rightarrow$ $y = -x + 3$ ...(i)

$\forall x < 0$ and $y > 3$

and $\arg(2z + 1 - 2i) = \arg[(2x+1) + i(2y-2)] = \frac{\pi}{4}$

$\Rightarrow$ $2x + 1 > 0, 2y - 2 > 0$ $\left(\because \frac{\pi}{4} \text{ is in Ist quadrant}\right)$

$\therefore$ $\frac{2y-2}{2x+1} = \tan\frac{\pi}{4} = 1$

$\Rightarrow$ $2y - 2 = 2x + 1$

$\Rightarrow$ $y = x + \frac{3}{2}, \ \forall x > \frac{-1}{2}, y > 1$ ...(ii)

From Eqs. (i) and (ii), we get

It is clear from the graph that there is no point of intersection.

4. $\left|z - \frac{25}{z}\right| \ge \left||z| - \frac{25}{|z|}\right| \Rightarrow 24 \ge \left||z| - \frac{25}{|z|}\right|$

$\Rightarrow$ $-24 \le |z| - \frac{25}{|z|} \le 24$

$\Rightarrow$ $-24|z| \le |z|^2 - 25 \le 24|z|$

$\therefore$ $|z|^2 + 24|z| - 25 \ge 0$ and $|z|^2 - 24|z| - 25 \le 0$

$\Rightarrow$ $(|z| + 25)(|z| - 1) \ge 0$ and $(|z| - 25)(|z| + 1) \le 0$

$\therefore$ $|z| - 1 \ge 0$ and $|z| - 25 \le 0$

Hence, $1 \le |z| \le 25$

$\Rightarrow$ $1 \le |z - 0| \le 25$

5. $z = (1 - i\cot 8)^3$

$= \text{cosec}^3 8 (\sin 8 - i\cos 8)^3$

$= \text{cosec}^3 8 \left\{\cos\left(\frac{\pi}{2} - 8\right) - i\sin\left(\frac{\pi}{2} - 8\right)\right\}^3$

$= \text{cosec}^3 8 \left\{\cos\left(\frac{3\pi}{2} - 24\right) - i\sin\left(\frac{3\pi}{2} - 24\right)\right\}$

$= \text{cosec}^3 8 \cdot e^{-i\left(\frac{3\pi}{2} - 24\right)} = \text{cosec}^3 8 \cdot e^{i\left(24 - \frac{3\pi}{2}\right)}$

6. $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

$= \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ \omega+x+\omega^2+1 & x+\omega^2 & 1 \\ \omega^2+1+x+\omega & 1 & x+\omega \end{vmatrix}$

$(C_1 \to C_1 + C_2 + C_3)$

$= x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix}$ $(\because 1 + \omega + \omega^2 = 0)$

$= x\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & x+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & x+\omega-\omega^2 \end{vmatrix}$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$= x[(x+\omega^2-\omega)(x+\omega-\omega^2) - (1-\omega)(1-\omega^2)]$

$= x[x + 3 - 3]$

$= x^2$

7. $x = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27},+\dots} = 9^{\frac{1/3}{1-1/3}} = 9^{1/2} = 3$

$y = 4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\dots} = 4^{\frac{1/3}{1+1/3}} = 4^{1/4} = \sqrt{2}$

$$z = \sum_{r=1}^{\infty} (1+i)^{-r} = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \dots$$

$$= \frac{\frac{1}{1+i}}{1-\frac{1}{1+i}} = \frac{1}{i} = -i$$

Let $a = x + yz = 3 - \sqrt{2}i$ (fourth quadrant). Then,

$$\arg \alpha = -\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

8. Let $z_1 = a + ib$ and $z_2 = c - id$, where $a > 0$ and $d > 0$. Then,

$|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2$ ...(i)

Now, $\frac{z_1 + z_2}{z_1 - z_2} = \frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$

$$= \frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$$

$$= \frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$$

$$= \frac{-(ad+bc)i}{a^2+b^2-ac+bd} \quad \text{[using Eq. (i)]}$$

Hence, $(z_1 + z_2)/(z_1 - z_2)$ is purely imaginary. However, if $ad + bc = 0$, then $(z_1 + z_2)/(z_1 - z_2)$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$.

9. We have,

$$\arg\left(\frac{z_1}{z_2}\right) = \pi$$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \pi$

$\Rightarrow \quad \arg(z_1) = \arg(z_2) + \pi$

Let $\arg(z_2) = \theta$. Then, $\arg(z_1) = \pi + \theta$

$\therefore \quad z_1 = |z_1|[\cos(\pi+\theta) + i\sin(\pi+\theta)]$

$= |z_1|(-\cos\theta - i\sin\theta)$

and $\quad z_2 = |z_2|(\cos\theta + i\sin\theta)$

$= |z_1|(\cos\theta + i\sin\theta) \quad (\because |z_1| = |z_2|)$

$= -z_1$

$\Rightarrow \quad z_1 + z_2 = 0$

10. $|z^2 + 2z\cos\alpha| \le |z^2| + |2z\cos\alpha|$

$= |z|^2 + 2|z||\cos\alpha| \le |z|^2 + 2|z|$

$< (\sqrt{2}-1)^2 + 2(\sqrt{2}-1) = 1$

11.

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = e^{i\pi/4}$$

$$\angle CBA = \frac{\pi}{4}$$

Also, $\quad |z_1 - z_2| = |z_3 - z_2|$

Hence, ΔABC is isosceles.

12. Given that,

$$|z_1 - i| = |z_2 - i| = |z_3 - i|$$

Hence, z_1, z_2 and z_3 lie on the circle whose centre is i. Also, given that the triangle is equilateral. Hence, centroid and cirucmcentre coincides.

$\therefore \quad \frac{z_1 + z_2 + z_3}{3} = i$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 3$

13. We have, $z_1 = a + ib$ and $z_2 = c + id$

$\therefore \quad |z_1|^2 = a^2 + b^2 = 1$

and $\quad |z_2|^2 = c^2 + d^2 = 1$ ...(i)

Also, $\quad \text{Re}(z_1 \bar{z}_2) = 0$

$\Rightarrow \quad ac + bd = 0$

$\Rightarrow \quad \frac{a}{b} = -\frac{d}{c} = \lambda$ ...(ii)

From Eqs. (i) and (ii), we get

$$b^2\lambda^2 + b^2 = c^2 + \lambda^2 c^2$$

$\Rightarrow \quad b^2 = c^2$ and $a^2 = d^2$

Now, $\quad |w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

and $\quad |w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

$\text{Re}|w_1 \overline{w_2}| = (ab + cd)$

$= (b\lambda)\, b + c(-\lambda c) = \lambda(b^2 - c^2) = 0$

Hence, (a), (b) and (c) are correct answers.

14. We have, $\frac{1 + i\cos\theta}{1 - 2i\cos\theta} = \frac{(1+i\cos\theta)(1+2i\cos\theta)}{(1-2i\cos\theta)(1+2i\cos\theta)}$

$$= \frac{(1 - 2\cos^2\theta) + i\,3\cos\theta}{1 + 4\cos^2\theta}$$

Thus, $\frac{(1+i\cos\theta)}{(1-2i\cos\theta)}$ is a real number, if $\cos\theta = 0$

$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{2}$

where, n is an integer.

15. Let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad |z_2| = r_2$

Also, $\quad \arg(z_1) + \arg(z_2) = 0$

$\Rightarrow \quad \arg(z_1) = -\arg(z_2) = -\theta_2$

$\therefore \quad z_1 = r_2[\cos(-\theta_2) + i\sin(-\theta_2)]$

$= r_2[\cos\theta_2 - i\sin\theta_2]$

$\Rightarrow \quad z_1 = \bar{z}_2$

$\Rightarrow \quad z_1 = \dfrac{1}{z_2}$

$\Rightarrow \quad z_1 z_2 = 1$

16.

We have,

$$|z_1| = 15, |z_2 - 3 - 4i| = 5$$

Minimum value of $|z_1 - z_2|$ is $AB = OB - OA = 15 - 10 = 5$

Maximum value of $|z_1 - z_2|$ is

$$CA = OA + OC = 10 + 15 = 25$$

Solutions (Q. Nos. 17-19)

Given that,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = 0 \quad$ ...(i)

$\Rightarrow \quad \dfrac{z_1}{z_2} + \dfrac{\bar{z}_1}{\bar{z}_2} = 0 \quad$ (dividing by $z_2\bar{z}_2$)

$\Rightarrow \quad \dfrac{z_1}{z_2} + \overline{\left(\dfrac{z_1}{z_2}\right)} = 0 \quad$ ...(ii)

17. From Eq. (i), $z_2\bar{z}_2$ is purely imaginary.

18. From Eq. (ii), z_1/z_2 is purely imaginary.

19. Also, $i(z_1/z_2)$ is purely real. Hence, its possible arguments are 0 and π.

20. We have, $\arg(z) = 0 \Rightarrow z$ is purely real.

$\therefore$ Statement II is true.

Also, $\quad |z_1| = |z_2| + |z_1 - z_2|$

$\Rightarrow \quad |z_1 - z_2|^2 = (|z_1| - |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 0$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = 0$

$\Rightarrow \quad \arg\left(\dfrac{z_1}{z_2}\right) = 0$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely real.

$\Rightarrow \quad \operatorname{Im}\left(\dfrac{z_1}{z_2}\right) = 0$

Statement I and statement II are true and statement II is a correct explanation of statement I.

21. $\left|\dfrac{z_1 z - z_2}{z_1 z + z_2}\right| = k \Rightarrow \left|\dfrac{z - \dfrac{z_2}{z_1}}{z + \dfrac{z_2}{z_1}}\right| = k$

Clearly, if $k \neq 0, 1$, then z would lie on a circle.

If $k = 1$, z would lie on a perpendicular bisector of line segment joining $\dfrac{z_2}{z_1}$ and $-\dfrac{z_2}{z_1}$ and represents a point, if $k = 0$.

$\therefore$ Statement I is false and statement II is true.

22. $|z_2 z_3 + 8z_3 z_1 + 27 z_1 z_2| = \left|z_1 z_2 z_3\left(\dfrac{1}{z_1} + \dfrac{8}{z_2} + \dfrac{27}{z_3}\right)\right|$

$= |z_1||z_2||z_3|\left|\dfrac{1}{z_1} + \dfrac{8}{z_2} + \dfrac{27}{z_3}\right|$

$= |z_1||z_2||z_3|\left|\dfrac{\bar{z}_1}{|z_1|^2} + \dfrac{8\bar{z}_2}{|z_2|^2} + \dfrac{27\bar{z}_3}{|z_3|^2}\right|$

$= |z_1||z_2||z_3|\left|\dfrac{\bar{z}_1}{1} + \dfrac{8\bar{z}_2}{4} + \dfrac{27\bar{z}_3}{9}\right|$

$= |z_1||z_2||z_3||\overline{z_1 + 2z_2 + 3z_3}|$

$= |z_1||z_2||z_3||z_1 + 2z_2 + 3z_3| = 1 \cdot 2 \cdot 3 \cdot 6 = 36$

23. $\because \; e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \quad e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore \quad \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Now, $\cos(1 - i) = \dfrac{e^{i(1-i)} + e^{-i(1-i)}}{2} = \dfrac{e^{(i+1)} + e^{-(1+i)}}{2}$

$= \dfrac{e(e^i) + e^{-1}(e^{-i})}{2}$

$= \dfrac{e(\cos 1 + i\sin 1) + e^{-1}(\cos 1 - i\sin 1)}{2}$

$= \dfrac{1}{2}\left(e + \dfrac{1}{e}\right)\cos 1 + \dfrac{i}{2}\left(e - \dfrac{1}{e}\right)\sin 1$

$\therefore \quad a = \dfrac{1}{2}\left(e + \dfrac{1}{e}\right)\cos 1, \quad b = \dfrac{1}{2}\left(e - \dfrac{1}{e}\right)\sin 1$

24. We have, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2$

where, $\quad \theta_1 = \arg(z_1), \quad \theta_2 = \arg(z_2)$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 0 \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \text{Re}\left(\frac{z_1}{z_2}\right) = \frac{|z_1|}{|z_2|}\cos\frac{\pi}{2} = 0$

$\therefore \frac{z_1}{z_2}$ is purely imaginary.

$\therefore$ Statement I is true.

If z is purely imaginary, then $z - \bar{z} = 0$.

25. $\arg(z_1 z_2) = 2\pi \Rightarrow \arg(z_1) + \arg(z_2) = 2\pi$

$\Rightarrow \arg(z_1) = \arg(z_2) = \pi$, as principal arguments are from $-\pi$ to π.

Hence, both the complex numbers are purely real. Hence, both the statements are true and statement II is correct explanation of statement I.

26. $|z_1 + z_2 + z_3| = |z_1 - a + z_2 - b + z_3 - c + (a + b + c)|$

$\le |z_1 - a| + |z_2 - b| + |z_3 - c| + |a + b + c|$

$\le 2|a + b + c|$

Hence, $|z_1 + z_2 + z_3|$ is less than $2|a + b + c|$.

27. If roots of $ax^2 + bx + c = 0, 0 < a < b < c$, are non-real, then they will be the conjugate of each other. Hence,

$$z_2 = \bar{z}_1 \Rightarrow |z_1| = |z_2|$$

Now, $\quad z_1 z_2 = \frac{c}{a} > 1 \Rightarrow |z_1|^2 > 1$

$\Rightarrow \quad |z_1| > 1$

$\Rightarrow \quad |z_2| > 1$

28. We have,

$$(\cos\theta + i\sin\theta)^{3/5} = (\cos 3\theta + i\sin 3\theta)^{1/5}$$

$= [\cos(2r\pi + 3\theta) + i\sin(2r\pi + 3\theta)]^{1/5}$, where $r = 0, 1, 2, 3, 4$

$= e^{i\left(\frac{2r\pi + 3\theta}{5}\right)}, r = 0, 1, 2, 3, 4$

Hence, product of all values of $(\cos\theta + i\sin\theta)^{3/5}$ is

$$e^{i\frac{3\theta}{5}} e^{i\left(\frac{2\pi + 3\theta}{5}\right)} e^{i\left(\frac{4\pi + 3\theta}{2}\right)} e^{i\left(\frac{6\pi + 3\theta}{5}\right)} e^{i\left(\frac{8\pi + 3\theta}{5}\right)}$$

$$= e^{i3\theta + 4\pi i}$$

$$= e^{4i\pi} e^{i3\theta}$$

$$= \cos 3\theta + i\sin 3\theta$$

Also, product of roots of the equation $x^5 - 1 = 0$ is 1. Hence, statement II is true, but it is not a correct explanation of statement I.

29. Given, $|z| = 1, \arg z = \theta \therefore z = e^{i\theta}$

But $\quad \bar{z} = \frac{1}{z}$

$\therefore \quad \arg\left(\frac{1 + z}{1 + \frac{1}{z}}\right) = \arg(z) = \theta$

30. Since, $\frac{z^2}{z - 1}$ $(z \ne 1)$ is purely real.

Hence, $\quad \frac{z^2}{z - 1} = \frac{\bar{z}^2}{\bar{z} - 1}$

$\Rightarrow \quad z^2(\bar{z} - 1) = \bar{z}^2(z - 1)$

$\Rightarrow \quad z^2\bar{z} - z^2 = z^2 z - \bar{z}^2$

$\Rightarrow \quad zz\bar{z} - z^2 = \bar{z}^2 z z - \bar{z}^2$

$z|z|^2 - z^2 = \bar{z}|z|^2 - \bar{z}^2$

$\Rightarrow \quad z|z|^2 - \bar{z}|z|^2 = \bar{z}^2 - z^2$

$\Rightarrow \quad |z|^2(z - \bar{z}) = (\bar{z} - z)(\bar{z} + z)$

$\Rightarrow \quad |z|^2(z - \bar{z}) - (z - \bar{z})(\bar{z} + z) = 0$

$\Rightarrow \quad (z - \bar{z})\{|z|^2 - (\bar{z} + z)\} = 0$

Either $\quad (z - \bar{z}) = 0$ or $\{|z|^2 - (\bar{z} + z)\} = 0$

Now, $\quad z = \bar{z} \Rightarrow$ Locus of z is real axis

and $\quad \{|z|^2 - (\bar{z} + z)\} = 0$

$\Rightarrow \quad z\bar{z} - (\bar{z} + z) = 0$

Locus of z is a circle passing through origin.

Alternate Method

Put $z = x + iy$, then

$$\frac{z^2}{z - 1} = \frac{(x + iy)^2}{(x + iy) - 1} = \frac{(x^2 - y^2) + i(2xy)}{(x - 1) + iy}$$

$$= \frac{(x^2 - y^2) + i(2xy)}{(x - 1) + iy} \times \frac{(x - 1) - iy}{(x - 1) - iy}$$

Since, $\frac{z^2}{z - 1}$ $(z \ne 1)$ is purely real, hence its imaginary part should be equal to zero.

$\Rightarrow \quad (x^2 - y^2)(-y) + (2xy)(x - 1) = 0$

$\Rightarrow \quad y(x^2 - y^2 + 2x - 2x^2) = 0$

$\Rightarrow \quad y(x^2 + y^2 - 2x) = 0$

$\Rightarrow$ Either $\quad y = 0$

$\Rightarrow \quad x^2 + y^2 - 2x = 0$

Now, $y = 0 \Rightarrow$ Locus of z is real axis and $x^2 + y^2 - 2x = 0$.

Locus of z is a circle pasing through origin.

Locus of z is either real axis or a circle passing through origin.

31. $(1 + \omega)^7 = A + B\omega$, we know $1 + \omega + \omega^2 = 0$

$\therefore \quad 1 + \omega = -\omega^2$

$\Rightarrow \quad (-\omega^2)^7 = A + B\omega$

$\Rightarrow \quad -\omega^{14} = A + B\omega \qquad (\omega^{14} = \omega^{12} \cdot \omega^2 = \omega^2)$

$\Rightarrow \quad -\omega^2 = A + B\omega$

$\Rightarrow \quad 1 + \omega = A + B\omega$

On comparing, we get

$$A = 1, B = 1$$

32. We have, $|z-1| = |z+1| = |z-i|$

Clearly, z is the circumcentre of the triangle formed by the vertices (1, 0) and (0, 1) and (–1, 0), which is unique.

Hence, the number of complex numbers z is one.

33. $|z| = \left|\left(z - \frac{4}{z}\right) + \frac{4}{z}\right| \Rightarrow |z| < \left|z - \frac{4}{z}\right| + \frac{4}{|z|} \Rightarrow |z| \le 2 + \frac{4}{|z|}$

$$\Rightarrow \quad \frac{|z|^2 - 2|z| - 4}{|z|} \le 0$$

Since, $|z| > 0$

$$|z|^2 - 2|z| - 4 \le 0$$

$$\Rightarrow \quad [|z| - (\sqrt{5}+1)][|z| - (1-\sqrt{5})] \le 0$$

$$\Rightarrow \quad 1 - \sqrt{5} \le |z| \le \sqrt{5} + 1$$

34. Let $z = \frac{1}{i-1}$

$$\therefore \quad \bar{z} = \overline{\left(\frac{1}{i-1}\right)} = \frac{1}{-i-1} = \frac{-1}{i+1}$$

35. $|z+4| \le 3$ represents the interior and boundary of the circle with centre at $(-4, 0)$ and radius $= 3$. As -1 is an end point of a diameter of the circle, maximum possible value of $|z+1|$ is 6.

36. Given equation is

$$z^2 + z + 1 = 0$$

$$\Rightarrow \quad z = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2 \times 1}$$

$$\Rightarrow \quad z = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow \quad z = \omega, \omega^2$$

Now, $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2$

$$+ \left(z^4 + \frac{1}{z^4}\right)^2 + \left(z^5 + \frac{1}{z^5}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2$$

$$= (-1)^2 + (-1)^2 + (1+1)^2 + (-1)^2 + (-1)^2 + (1+1)^2$$

when, we put either $z = \omega$ or $z = \omega^2$, we get the same result

$$= 1 + 1 + 4 + 1 + 1 + 4 = 12$$

37. Given that, $(x-1)^3 + 8 = 0$

$$\Rightarrow \quad (x-1)^3 = (-2)^3$$

$$\Rightarrow \quad \left(\frac{x-1}{-2}\right)^3 = 1$$

$$\Rightarrow \quad \left(\frac{x-1}{-2}\right) = (1)^{1/3}$$

$\therefore$ Cube roots of $\left(\frac{x-1}{-2}\right)$ are 1, ω and ω^2.

$\Rightarrow$ Cube roots of $(x-1)$ are -2, -2ω and $-2\omega^2$

$\Rightarrow$ Cube roots of x are -1, $1-2\omega$ and $1-2\omega^2$.

38. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that,

$$|z_1 + z_2| = |z_1| + |z_2|$$

$$\therefore \quad \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

On squaring both sides, we get

$$x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2$$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$\Rightarrow \quad x_1x_2 + y_1y_2 = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

Again, squaring on both sides, we get

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2$$

$$= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2$$

$$\Rightarrow \quad (x_1y_2 - y_1x_2)^2 = 0$$

$$\Rightarrow \quad \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$$

$$\Rightarrow \quad \arg(z_1) = \arg(z_2)$$

$$\Rightarrow \quad \arg(z_1) - \arg(z_2) = 0$$

39. Given that, $w = \frac{z}{z - \frac{i}{3}}$ and $|w| = 1$

$$\Rightarrow \quad \left|\frac{z}{z - \frac{i}{3}}\right| = 1$$

$\Rightarrow \quad |z| = \left|z - \frac{i}{3}\right|$

$\Rightarrow z$ lies on perpendicular bisector of $(0, 0)$ and $\left(0, \frac{1}{3}\right)$.

Hence, z lies on a straight line.

40. Given that, $\bar{z} + i\bar{w} = 0 \Rightarrow \bar{z} = -i\bar{w}$

$\Rightarrow \quad z = iw \Rightarrow w = -iz$

and $\quad \arg(zw) = \pi$

$\Rightarrow \quad \arg(-iz^2) = \pi$

$\Rightarrow \quad \arg(-i) + 2\arg(z) = \pi$

$\Rightarrow \quad -\frac{\pi}{2} + 2\arg(z) = \pi \qquad \left[\because \arg(-i) = -\frac{\pi}{2}\right]$

$\Rightarrow \quad \arg(z) = \frac{3\pi}{4}$

41. Since, $\quad z^{1/3} = p + iq$

$\therefore \quad z = (p + iq)^3$

$= p^3 - iq^3 + 3ip^2q - 3pq^2$

Given that, $\quad z = x - iy$

$\therefore \quad x - iy = p^3 - 3pq^2 + i(3p^2q - q^3)$

$\Rightarrow \quad x = p^3 - 3pq^2$ and $-y = 3p^2q - q^3$

$\Rightarrow \quad \frac{x}{p} = p^2 - 3q^2$

and $\quad \frac{y}{q} = -3p^2 + q^2$

$\therefore \quad \frac{x}{p} + \frac{y}{q} = -2p^2 - 2q^2$

$\Rightarrow \quad \frac{1}{(p^2 + q^2)}\left(\frac{x}{p} + \frac{y}{q}\right) = -2$

42. Using the relation, if

$|z_1 + z_2| = |z_1| + |z_2|$

Then, $\quad \arg(z_1) = \arg(z_2)$

Since, $\quad |z^2 + (-1)| = |z^2| + |-1|$

Then, $\quad \arg(z^2) = \arg(-1)$

$\Rightarrow \quad 2\arg(z) = \pi \qquad [\because \arg(-1) = \pi]$

$\Rightarrow \quad \arg(z) = \frac{\pi}{2}$

$\Rightarrow z$ lies on y-axis (imaginary axis).

43. Since, origin z_1 and z_2 are the vertices of an equilateral triangle, then

$z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow \quad (z_1 + z_2)^2 = 3z_1z_2 \qquad \ldots(i)$

Again, z_1 and z_2 are the roots of the equation

$z^2 + az + b = 0$

Then, $\quad z_1 + z_2 = -a$

and $\quad z_1z_2 = b$

On putting these values in Eq. (i), we get

$(-a)^2 = 3b$

$\Rightarrow \quad a^2 = 3b$

44. Let $\quad z = r_1e^{i\theta}$ and $w = r_2e^{i\phi}$

$\Rightarrow \quad \bar{z} = r_1e^{-i\theta}$

Given, $\quad |zw| = 1$

$\Rightarrow \quad |r_1e^{i\theta} \cdot r_2e^{i\phi}| = 1$

$\Rightarrow \quad r_1r_2 = 1 \qquad \ldots(i)$

and $\quad \arg(z) - \arg(w) = \frac{\pi}{2}$

$\Rightarrow \quad \theta - \phi = \frac{\pi}{2} \qquad \ldots(ii)$

Now, $\quad \bar{z}w = r_1e^{-i\theta} \cdot r_2e^{i\phi}$

$= r_1r_2e^{-i(\theta - \phi)}$

$= 1 \cdot e^{i\pi/2} \qquad$ [from Eqs. (i) and (ii)]

$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$\Rightarrow \quad \bar{z}w = -i$

45. Now, $\left(\frac{1+i}{1-i}\right)^x = \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x$

$= \left[\frac{(1+i)^2}{1-i^2}\right]^x = \left[\frac{1-1+2i}{2}\right]^x$

$\Rightarrow \quad \left(\frac{1+i}{1-i}\right)^x = (i)^x = 1$ (given)

$\Rightarrow \quad (i)^x = (i)^{4n},$

where n is any positive integer.

$\therefore \quad x = 4n$

46. Now, $(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7 \qquad (\because 1 + \omega + \omega^2 = 0)$

$= (-2\omega^2)^7$

$= -2^7 \cdot \omega^{14} = -128(\omega^3)^4\omega^2$

$= -128\omega^2 \qquad (\because \omega^3 = 1)$

47. Given that,

$$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$$

Applying $R_1 \to R_1 + R_2$

$$\Rightarrow \quad \begin{vmatrix} 6i+4 & 0 & 0 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$$

$$\Rightarrow \quad (6i + 4)\begin{vmatrix} 3i & -1 \\ 3 & i \end{vmatrix} = x + iy$$

$\Rightarrow \quad (6i + 4)(3i^2 + 3) = x + iy$

$\Rightarrow \quad 0 + 0i = x + iy$

$\Rightarrow \quad x = 0, y = 0$

5 Quadratic Equations and Inequalities

JEE Main MILESTONE

- Real and Complex Polynomial
- Degree of a Polynomial
- Polynomial Equation
- Identity and Equation
- Roots of an Equation
- Remainder and Factor Theorem
- Quadratic Equation
- Nature of Roots
- Relation between Coefficients and Roots of an Equation
- Formation of an Equation with Given Roots
- Symmetric Function
- Condition for Common Root(s)
- Different Conditions for Roots
- Quadratic Expression
- Possible Graph of Quadratic Expression
- Position of Roots of a Quadratic Equation
- Maximum and Minimum Value of $ax^2 + bx + c$
- Inequality
- Arithmetico-Geometric Mean Inequality
- Logarithm

Quadratic Equations and Inequalities introduces students to the graphs of quadratics, teaches them to find the vertex, intercepts, discriminant, domain and range and interpret the graph in relation to these qualities. It also introduces students to the focus and directrix of parabolas, how to use a sign on number line to find the shape of a graph.

5.1 Real and Complex Polynomial

If $a_0, a_1, a_2, \ldots, a_n$ are real numbers and x is a real variable, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is known as a real polynomial with real coefficients while if $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a complex polynomial with complex coefficients.

e.g., $x^2 + 3x + 7$ is a real polynomial whereas $x^2 + (3 + 2i)x + (7 + 3i)$ is a complex polynomial.

5.2 Degree of a Polynomial

A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ which is real or complex, is a polynomial of degree n, if $a_n \neq 0$.

e.g., $x^2 + 7x + 2$ and $x^2 + (7 + 2i) + (3 + 2i)$ are two polynomials of degree 2.

Note Polynomials of degree 2, 3 and 4 are known as quadratic, cubic and biquadratic polynomials, respectively.

5.3 Polynomial Equation

If $f(x)$ is a real or complex polynomial, then $f(x) = 0$ is known as a polynomial equation.

e.g., If $x^2 + 3x + 2$ is a real polynomial, then

$x^2 + 3x + 2 = 0$ is a polynomial equation.

5.4 Identity and Equation

A statement of equality of two expressions which is satisfied for each value of variable is called an identity.

e.g., $(x - 4)^2 + 8x = x^2 + 16$ is an identity.

A statement of equality between two expressions which is satisfied for definite values of variable, is known as an equation.

e.g., $x^2 - 5x + 6 = 0$ is an equation, which is not satisfied for any value of x except 2 and 3.

5.5 Roots of an Equation

The values of a variable in an equation which satisfied the given equation are known as roots of an equation *i.e.,* if $f(x) = 0$ is a polynomial equation and $f(a) = 0$, then a is a root of $f(x) = 0$.

e.g., If $f(x) = x^2 + 3x + 2 = 0$

and $f(-1) = (-1) + 3(-1) + 2 = 1 + 2 - 3 = 0 \Rightarrow f(-1) = 0$

$\therefore$ -1 is a root of $f(x) = 0$.

Note

- There are n real or imaginary roots of an equation of order n.
- An odd degree equation has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of highest degree term is positive.
- Every equation of an even degree, whose constant term is negative and the coefficient of highest degree term is positive, has atleast two real roots, one positive and one negative.

Geometrical Interpretation of Roots of Equation

In geometrical interpretation, the real roots of the equation $f(x) = 0$ are the coordinates of the points, where the curve $y = f(x)$ crosses the x-axis.

e.g., Let $f(x) = x^4 + x^3 - 4x^2 + 2x - 12$ be a polynomial equation of 4 degree. The graph of $f(x)$ is shown below.

$\therefore$ $f(x) = (x - 2)(x + 3)(x^2 + 2)$

$\therefore$ The roots of $f(x)$ are

$x - 2 = 0, x + 3 = 0$ and $x^2 + 2 = 0$

$\Rightarrow$ $x = 2, x = -3$ and $x^2 = -2$ (complex root)

The given polynomial equation of four degree in $f(x)$ have $-3, 2$ as two real roots.

Since, we have seen in the above graph that $f(x)$ is intersecting the x-axis at the points $(-3, 0)$ and $(2, 0)$.

Sample Problem 1 *Two non-integer roots of*

$$\left(\frac{3x-1}{2x+3}\right)^4 - 5\left(\frac{3x-1}{2x+3}\right)^2 + 4 = 0 \text{ are}$$

(a) $-5/7, -2/5$ (b) $-2/5, 7/5$

(c) $5/7, 7/5$ (d) $-2/5, 3/5$

Interpret (a) Let $\left(\frac{3x-1}{2x+3}\right)^2 = t$

Then, given equation becomes

$t^2 - 5t + 4 = 0 \Rightarrow (t - 1)(t - 4) = 0 \Rightarrow t = 1$ or 4

When, $t = 1$

$$\frac{3x-1}{2x+3} = \pm 1$$

$\Rightarrow$ $x = 4, \frac{-2}{5}$

and when, $t = 4$ $\frac{3x-1}{2x+3} = \pm 2$

$\Rightarrow$ $x = -7, -5/7$

Thus, two non-integer roots of given equation are $-2/5, -5/7$.

Descartes Rule of Sign

Descartes rule of sign gives the maximum number of positive or negative real roots in the given polynomial equation.

(i) Maximum Number of Positive Real Roots

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ are the number of changes of sign from positive to negative and *vice-versa* in $f(x)$.

(ii) Maximum Number of Negative Real Roots

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ are the number of changes of sign from positive to negative and *vice-versa* in $f(x)$.

Note

(a) Descartes rule only gives the maximum number of positive or negative real roots in $f(x)$ or $f(-x)$, respectively.

(b) Descartes rule does not tell the exact number of positive or negative real roots of the polynomial equation $f(x) = 0$.

Sample Problem 2 *The maximum number of positive and negative roots in the equation*

$$x^4 - x^3 + 3x^2 - 14x + 5 = 0 \text{ are}$$

(a) 4, 0 (b) 4, 1
(c) 4, 2 (d) 2,2

Interpret (a) Let $f(x) = x^4 - x^3 + 3x^2 - 14x + 5$. Here, we see that there are four changes of sign from $\oplus \to \ominus$ or $\ominus \to \oplus$. Hence, the maximum number of positive real roots in $f(x) = 0$ are 4.

Now, $f(-x) = (-x)^4 - (-x)^3 + 3(-x)^2 - 14(-x) + 5.$

Here, we see that, there in no change of sign. Hence, no negative roots of equation exist.

5.6 Remainder and Factor Theorem

Remainder theorem states that, if a polynomial $f(x)$ is divided by a linear function $x - a$, then remainder is $f(a)$.

And factor theorem is a special case of remainder theorem, which states that, if a polynomial $f(x)$ is divisible by a linear function $x - a$, then remainder is zero, *i.e.*, if on dividing a polynomial $f(x)$ by $x - a$, the remainder is zero, then $f(x)$ is divisible by $x - a$ or $x - a$ is a factor of $f(x)$.

Note In remainder theorem, the degree of remainder is always one less than the degree of divisor.

Sample Problem 3 *On dividing the expression $2x^3 + 3px^2 - 4x + p$ by $x + 2$, the remainder is 5, then the value of p is*

(a) 0 (b) 1
(c) 2 (d) 3

Interpret (b) Let $f(x) = 2x^3 + 3px^2 - 4x + p$

$$\therefore \quad f(-2) = 5$$
$$\Rightarrow \quad 2(-2)^3 + 3p(-2)^2 - 4(-2) + p = 5$$
$$\Rightarrow \quad -16 + 12p + 8 + p = 5$$
$$\Rightarrow \quad 13p = 5 + 8 \Rightarrow p = 1$$

5.7 Quadratic Equation

If $f(x)$ is a quadratic polynomial. Then, $f(x) = 0$ is known as quadratic equation. The general form of a quadratic equation is $ax^2 + bx + c = 0, \forall\, a, b, c \in R$ or C and $a \neq 0$.

Roots of a Quadratic Equation

Roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$, where $b^2 - 4ac$ is known as discriminant and it is denoted by D.

If in a quadratic equation $ax^2 + bx + c = 0$, a, b, c are complex numbers, then we cannot assign positive or negative sign to the discriminant.

Note If one root of an equation is $a + \sqrt{b}$ or $a + ib$, then another root of an equation will be $a - \sqrt{b}$ or $a - ib$, respectively.

Sample Problem 4 *The roots of equation $27x^2 - 10x + 1 = 0$ are* [NCERT]

(a) $\dfrac{5}{23} \pm \dfrac{\sqrt{2}i}{23}$ (b) $\dfrac{5}{27} \pm \dfrac{\sqrt{2}i}{27}$
(c) $\dfrac{5}{27} \pm \dfrac{\sqrt{3}i}{27}$ (d) *None of these*

Interpret (b) Given, $27x^2 - 10x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we get

$$a = 27, b = -10, c = 1$$
$$\therefore \quad D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1$$
$$= 100 - 108 = -8 < 0$$
$$\Rightarrow \quad x = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{(10) \pm 2\sqrt{2}i}{2 \times 27}$$
$$= \frac{2(5 \pm \sqrt{2}i)}{2 \times 27} = \frac{5}{27} \pm \frac{\sqrt{2}i}{27}$$

Sample Problem 5 *The roots of the quadratic equation $x^2 - 2\sqrt{3}x - 22 = 0$ are*

(a) *imaginary* (b) *real*
(c) *real and imaginary* (d) *None of these*

Interpret (b) Given equation is $x^2 - 2\sqrt{3}x - 22 = 0$

On comparing the given equation of $ax^2 + bx + c = 0$, we get

$$a = 1, b = -2\sqrt{3}, c = -22$$
$$\therefore \quad D = b^2 - 4ac$$
$$= (-2\sqrt{3})^2 - 4 \times 1 \times (-22)$$
$$= 12 + 88 = 100 \quad = (10)^2 > 0$$

Hence, roots are real.

5.8 Nature of Roots

(i) The roots are real and distinct, iff $D > 0$.
(ii) The roots are real and equal, iff $D = 0$.
(iii) The roots are complex with non-zero imaginary part, iff $D < 0$.
(iv) The roots are rational, iff a, b, c are rational and D is perfect square.
(v) The roots are of the form $p + \sqrt{q}\,(p, q \in Q)$, iff a, b, c are rational and D is not a perfect square.

(vi) If $a = 1, b, c \in I$ and the roots are rational numbers, then these roots must be integers.

(vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

Sample Problem 6 *If a, b and c are distinct positive real numbers in AP, then the roots of the equation $ax^2 + 2bx + c = 0$ are*

(a) *imaginary* (b) *rational and equal*
(c) *rational and distinct* (d) *irrational*

Interpret (c) Since, a, b and c are in AP.

$$\therefore \quad b = \frac{a+c}{2} \quad \ldots(i)$$

Now, discriminant, $D = B^2 - AC$

$$= 4b^2 - 4ac$$

$$= 4\left(\frac{a+c}{2}\right)^2 - 4ac \quad \text{[from Eq.(i)]}$$

$$= (a+c)^2 - 4ac$$

$$= (a-c)^2 \geq 0$$

Hence, the roots of the given equation are rational and distinct.

5.9 Relation between Coefficients and Roots of an Equation

(i) **Quadratic Equation** If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

Sum of roots $\alpha + \beta = -\frac{b}{a}$ and

Product of roots $\alpha\beta = \frac{c}{a}$

Also, $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

(ii) **Cubic Equation** If α, β and γ are the roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

Sum of roots $\alpha + \beta + \gamma = -\frac{b}{a}$,

Product of two roots $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and

Product of three roots $\alpha\beta\gamma = -\frac{d}{a}$

Sample Problem 7 *If the roots of the equation $x^2 + px + q = 0$ are in the same ratio as those of the equation $x^2 + lx + m = 0$, then p^2m is equal to*

(a) lq (b) l^2q
(c) lq^2 (d) *None of these*

Interpret (b) Let the roots of each equation be in the ratio $k:1$.

Let $k\alpha$ and α be the roots of $x^2 + px + q = 0$.

Then, $k\alpha + \alpha = -p$ and $k\alpha \cdot \alpha = q$

$$\Rightarrow \quad \alpha = \frac{-p}{k+1} \quad \text{and} \quad \alpha^2 = \frac{q}{k}$$

$$\Rightarrow \quad \left(\frac{-p}{k+1}\right)^2 = \frac{q}{k}$$

$$\Rightarrow \quad \frac{k}{(k+1)^2} = \frac{q}{p^2} \quad \ldots(i)$$

Again, let $k\beta$ and β be the roots of $x^2 + lx + m = 0$.

Then, $k\beta + \beta = -l$ and $k\beta \cdot \beta = m$

$$\Rightarrow \quad \beta = -\frac{l}{k+1} \quad \text{and} \quad \beta^2 = \frac{m}{k}$$

$$\Rightarrow \quad \left(\frac{-l}{k+1}\right)^2 = \frac{m}{k}$$

$$\Rightarrow \quad \frac{k}{(k+1)^2} = \frac{m}{l^2} \quad \ldots(ii)$$

Thus, from Eqs. (i) and (ii), we get

$$\frac{q}{p^2} = \frac{m}{l^2}$$

$$\Rightarrow \quad p^2m = l^2q$$

5.10 Formation of an Equation with Given Roots

(i) **Quadratic Equation** If α and β are the roots of a quadratic equation, then the equation will be $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

e.g., 2 and –5 be the roots of a quadratic equation, then the equation will be

$$x^2 - (2 - 5)x + 2 \times (-5) = 0$$

i.e., $$x^2 + 3x - 10 = 0$$

(ii) **Cubic Equation** If α, β and γ are the roots of a cubic equation, then the equation will be

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0.$$

Sample Problem 8 *If α and β are the roots of equation $2x^2 - 5x + 7 = 0$, then the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$, is*

(a) $2x^2 - 25x + 82 = 0$ (b) $2x^2 + 25x + 82 = 0$
(c) $x^2 - 25x + 82 = 0$ (d) *None of these*

Interpret (a) Since, α and β be the roots of the equation $2x^2 - 5x + 7 = 0$, then

$$\alpha + \beta = \frac{5}{2} \quad \text{and} \quad \alpha\beta = \frac{7}{2}$$

Now, sum of roots $= (2\alpha + 3\beta) + (3\alpha + 2\beta) = 5(\alpha + \beta) = \left(\frac{25}{2}\right)$

and product of roots $= (2\alpha + 3\beta)(3\alpha + 2\beta)$

$$= 6(\alpha^2 + \beta^2) + 13\alpha\beta$$
$$= 6[(\alpha+\beta)^2 - 2\alpha\beta] + 13\alpha\beta$$
$$= \left[6 \times \left(\frac{25}{4} - 7\right) + \frac{91}{2}\right] = 41$$

The required equation is

$$x^2 - \frac{25}{2}x + 41 = 0$$
$$\Rightarrow \quad 2x^2 - 25x + 82 = 0$$

5.11 Symmetric Function

A function of α and β is said to be a symmetric function, if it remains unchanged when α and β are interchanged.

Let α and β be the roots of the equation $ax^2 + bx + c = 0$,

then $\quad \alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

The following results may be useful.

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(iii) $\alpha^4 + \beta^4 = [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

(iv) $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(v) $\alpha^2 - \beta^2 = (\alpha+\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(vi) $\alpha^3 - \beta^3 = (\alpha-\beta)[(\alpha+\beta)^2 - \alpha\beta]$

$$= [\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}]\,[(\alpha+\beta)^2 - \alpha\beta]$$

(vii) $\alpha^4 - \beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

$$= (\alpha+\beta)\,[\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}]\,[(\alpha+\beta)^2 - 2\alpha\beta]$$

Sample Problem 9 *If α and β be the roots of the equation $px^2 + qx + r = 0$, then the value of $\alpha^3\beta + \beta^3\alpha$ is*

(a) $\dfrac{r}{p^3}$ (b) $\dfrac{r(q^2 - 2pr)}{p^3}$

(c) $\dfrac{r(q^2 + 2pr)}{p^3}$ (d) $\dfrac{(q^2 - 2pr)}{p^3}$

Interpret (*b*) Since, α and β be the roots of the equation $px^2 + qx + r = 0$, therefore $\alpha + \beta = -\dfrac{q}{p}$ and $\alpha\beta = \dfrac{r}{p}$.

$$\therefore \quad \alpha^3\beta + \beta^3\alpha = \alpha\beta(\alpha^2 + \beta^2)$$
$$= (\alpha\beta)[(\alpha+\beta)^2 - 2\alpha\beta]$$
$$= \left(\frac{r}{p}\right)\left(\frac{q^2}{p^2} - \frac{2r}{p}\right)$$
$$= \frac{r(q^2 - 2pr)}{p^3}$$

5.12 Condition for Common Roots (s)

Consider two equations

$$ax^2 + bx + c = 0$$

and $\quad a'x^2 + b'x + c' = 0$

(a) **When one common root** Let a be the common roots of two equations, then

$$a\alpha^2 + b\alpha + c = 0$$

and $\quad a'\alpha^2 + b'\alpha + c' = 0$

On solving the two equations, we get

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \quad \alpha = \frac{a'c - ac'}{ab' - a'b},\ \alpha^2 = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow \quad (bc' - b'c)(ab' - a'b) = (a'c - ac')^2$$

This is the condition for roots of two quadratic equations to be common.

(b) **When two common roots** In such a case, two equations should be identical. The ratio of coefficients of x^2, x^1 and x^0 must be same

i.e., $\quad \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

Sample Problem 10 *If the equations $3x^2 - 2x + p = 0$ and $6x^2 - 17x + 12 = 0$ have a common root, then the value of p is*

(a) $\dfrac{15}{3}, \dfrac{8}{3}$ (b) $-\dfrac{15}{4}, -\dfrac{8}{3}$

(c) $\dfrac{15}{7}, \dfrac{8}{3}$ (d) $-\dfrac{15}{4}, \dfrac{8}{3}$

Interpret (*b*) Let α be a common root, then

$$3\alpha^2 - 2\alpha + p = 0 \quad \text{...(i)}$$

and $\quad 6\alpha^2 - 17\alpha + 12 = 0 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{\alpha^2}{-24 + 17p} = \frac{\alpha}{6p - 36} = \frac{-1}{39}$$

$$\Rightarrow \quad \alpha^2 = \frac{24 - 17p}{39} \text{ and } \alpha = \frac{36 - 6p}{39}$$

$$\Rightarrow \quad \left(\frac{36 - 6p}{39}\right)^2 = \frac{24 - 17p}{39}$$

$$\Rightarrow \quad 12p^2 + 77p + 120 = 0$$

$$\Rightarrow \quad (4p + 15)(3p + 8) = 0$$

$$\Rightarrow \quad p = \frac{-15}{4}, \frac{-8}{3}$$

5.13 Different Conditions for Roots

(i) If b is of opposite sign as compared to a and c, then both roots of $ax^2 + bx + c = 0$ are positive.

(ii) If a, b and c are all of same sign, then both roots of $ax^2 + bx + c = 0$ are negative.

(iii) If there is no term containing coefficient of x, then both the roots of the equation $ax^2 + bx + c = 0$ are equal in magnitude but opposite in sign.

(iv) If a and c are of opposite signs, then both the roots of the equation are of opposite sign.

(v) If the roots of $ax^2 + bx + c$ are reciprocal to each other, then $c = a$.

Sample Problem 11 *If $a \neq b$, then the roots of the equation $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are*

(a) *real and distinct*
(b) *real and equal*
(c) *imaginary*
(d) *None of the above*

Interpret (c) The given equation is

$$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$$

Now,
$$D = 4(a + b)^2 - 8(a^2 + b^2)$$
$$= -4(a^2 + b^2 - 2ab)$$
$$= -4(a - b)^2 < 0 \qquad (\because a - b \neq 0)$$

Hence, the roots of the given equation are imaginary.

Hot Spot

Roots of Quadratic Equation

In these types of problems, a quadratic equation is given and we have to find its root/solutions. These types of problems are directly or indirectly asked in every year in AIEEE (JEE Main) from 2008. *To solve such types of problem following steps are given below.*

Step I Firstly, assume the roots of a quadratic equation α and β

Step II Determine sum and product of roots.

Step III If it is required, use symmetric function.

Step IV Apply the suitable formula carefully and get the required result.

Sample Problem 12 *If the quadratic equation is $x^2 + 9x + 20 = 0$, then the roots of an equation are*

(a) $-4, -5$
(b) $+4, -5$
(c) $5, 4$
(d) *None of these*

Interpret (a)

Step I Firstly assume the roots of a quadratic equation be α and β.

Step II Sum of a roots, $\alpha + \beta = -9$ and product of a roots, $\alpha\beta = 20$.

Step III Use symmetric function

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$
$$= \sqrt{81 - 4(20)} = \sqrt{9} = 3$$

Step IV Simplify it,

On solving, we get $\alpha = -4, -5$

Check Point 1

1. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of q is
   (a) 49/4 (b) 4/49
   (c) 4 (d) None of these
2. If the roots of the equation $x^2 + 2ax + b = 0$ are real and distinct and they differ by atmost 2m, then b lies in the interval
   (a) $(a^2 - m^2, a^2)$
   (b) $[a^2 - m^2, a^2)$
   (c) $(a^2, m^2 + a^2)$
   (d) None of the above
3. The value of α, for which the equation $x^2 - (\sin\alpha - 2)x - (1 + \sin\alpha) = 0$ has roots, whose sum of square is least, is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$
4. The number of values of the triplet (a, b, c) for which, $a\cos^2 x + b\sin^2 x + c = 0$ is satisfied by all real x, is
   (a) 0 (b) 2 (c) 3 (d) infinite
5. If a and b are rational and b is not a perfect square, then the quadratic equation with rational coefficients whose one root is $\frac{1}{a + \sqrt{b}}$, is
   (a) $x^2 - 2ax + (a^2 - b) = 0$
   (b) $(a^2 - b)x^2 - 2ax + 1 = 0$
   (c) $(a^2 - b)x^2 - 2bx + 1 = 0$
   (d) None of the above
6. If tan A and tan B are the roots of the quadratic equation $x^2 - px + q = 0$, then the value of $\sin^2(A + B)$ is
   (a) $\frac{p^2}{p^2 + q^2}$ (b) $\frac{p^2}{(p + q)^2}$
   (c) $1 - \frac{p}{(1 - q)^2}$ (d) None of these
7. If the roots of $a_1x^2 + b_1x + c_1 = 0$ are α_1, β_1 and those of $a_2x^2 + b_2x + c_2 = 0$ are α_2, β_2 such that $\alpha_1\alpha_2 = \beta_1\beta_2 = 1$, then
   (a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (b) $\frac{a_1}{c_2} = \frac{b_1}{b_2} = \frac{c_1}{a_2}$
   (c) $a_1 a_2 = b_1 b_2 = c_1 c_2$ (d) None of these
8. If $(2x^2 - 3x + 1)(2x^2 + 5x + 1) = 9x^2$, then equation has
   (a) four real roots
   (b) two real and two imaginary roots
   (c) four imaginary roots
   (d) None of the above
9. If $|x^2| + |x| - 2 = 0$, then the value of x is equal to
   (a) 2
   (b) – 2
   (c) 1
   (d) None of the above

5.14 Quadratic Expression

An expression of the form $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called a quadratic expression in x. So, in general quadratic expression is represented as

$$f(x) = ax^2 + bx + c \text{ or } y = ax^2 + bx + c$$

Graph of a Quadratic Expression

We have, $y = ax^2 + bx + c = f(x)$ $(a \neq 0)$

$$\Rightarrow \quad y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \quad \text{or} \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

Now, let $y + \frac{D}{4a} = Y$ and $x + \frac{b}{2a} = X$

$$\therefore \quad Y = aX^2 \quad \Rightarrow X^2 = \frac{Y}{a}$$

(i) The shape of the curve $y = f(x)$ is parabolic.

(ii) The axis of parabola is $X = 0$ or $x + \frac{b}{2a} = 0$, *i.e.*, parallel to y-axis.

(iii) (a) If $a > 0$, then the parabola opens upwards.

(b) If $a < 0$, then the parabola opens downwards.

Sample Problem 13 *The linear factors of* $2x^2 - y^2 - x + xy + 2y - 1$ *are*

(a) $(2x + y + 1)$ *and* $(x + y - 1)$
(b) $(2x - y + 1)$ *and* $(x + y - 1)$
(c) $(2x - y + 1)$ *and* $(x + y + 1)$
(d) *None of the above*

Interpret (*b*) The corresponding equation to given expression can be rewritten as

$$2x^2 - (1 - y)x - (y^2 - 2y + 1) = 0$$

$$\therefore \quad \text{Roots } x = \frac{1 - y \pm \sqrt{(1 - y)^2 + 8(y - 1)^2}}{4}$$

$$= \frac{(1 - y) \pm 3(1 - y)}{4} = 1 - y, \frac{y - 1}{2}$$

Thus, required linear factors are $(x + y - 1)$ and $(2x - y + 1)$.

5.15 Possible Graph of Quadratic Expression

Let a quadratic expression is $f(x) = ax^2 + bx + c$.

Here, two cases arise.

***Case* I** When $a > 0$

- $D = 0$
- $f(x) = 0$ has real and equal roots.
- $f(x) \geq 0$ for all $x \in R$

(a)

- $D < 0$
- $f(x) = 0$ does not have real roots.
- $f(x) > 0$ for all $x \in R$

(b)

- $D > 0$
- Roots of $f(x) = 0$ are real and distinct.
- $f(x) > 0$, if $x < \alpha$ or $x > \beta$ and $f(x) < 0$, if $\alpha < x < \beta$

(c)

***Case* II** When $a < 0$

- $D < 0$
- $f(x) = 0$ does not have real roots.
- $f(x) < 0$ for all $x \in R$

(d)

- $D = 0$
- $f(x) = 0$ has real and equal roots.
- $f(x) \leq 0$ for all $x \in R$

(e)

- $D > 0$
- Roots of $f(x) = 0$ are real and distinct.
- $f(x) < 0$, if $x < \alpha$ or $x > \beta$ and $f(x) > 0$, if $\alpha < x < \beta$

(f)

5.16 Position of Roots of a Quadratic Equation

Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$, $a \neq 0$ and α, β be the roots of $f(x) = 0$. Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$. Then, the following hold good

(i) **Conditions for a number k**

[If both the roots of $f(x) = 0$ are less than k]

(a) $D \geq 0$ (roots may be equal) (b) $af(k) > 0$

(c) $k > -\dfrac{b}{2a}$, where $\alpha \leq \beta$

(ii) **Conditions for a number k**

[If both the roots of $f(x) = 0$ are greater than k]

(a) $D \geq 0$ (b) $f(2) > 0$ (c) $\dfrac{-b}{2a} > 2$

(iii) **Conditions for a number k**

[If k lies between the roots of $f(x) = 0$]

(a) $D > 0$ (b) $af(k) < 0$ where, $\alpha < \beta$

(iv) **Conditions for numbers k_1 and k_2**

[If exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2)]

(a) $D > 0$

(b) $f(k_1)f(k_2) < 0$, where $\alpha < \beta$

(v) **Conditions for numbers k_1 and k_2**

[If both roots of $f(x) = 0$ lies between k_1 and k_2]

(a) $D \geq 0$ (roots may be equal) (b) $af(k_1) > 0$

(c) $af(k_2) > 0$ (d) $k_1 < -\dfrac{b}{2a} < k_2$, where $\alpha \leq \beta$ and $k_1 < k_2$.

(vi) **Conditions for numbers k_1 and k_2**

[If k_1 and k_2 lie between the roots of $f(x) = 0$]

(a) $D > 0$ (b) $af(k_1) < 0$ (c) $af(k_2) < 0$, where $\alpha < \beta$.

Sample Problem 14 *For what values of $m \in R$, both roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?*

(a) $\left(\dfrac{11}{7}, \infty\right)$ (b) $\left(\dfrac{11}{9}, \infty\right)$

(c) $\left(\dfrac{11}{5}, \infty\right)$ (d) $\left(-\dfrac{11}{9}, \infty\right)$

Interpret *(b)* Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$ as both roots of $f(x) = 0$ are greater than 3, we can take $D \geq 0$, $af(3) > 0$ and $-\dfrac{b}{2a} > 3$.

(1) For $D \geq 0$: $(-6m)^2 - 4.1(9m^2 - 2m + 2) \geq 0$

$\Rightarrow \quad 8m - 8 \geq 0 \therefore m \geq 1$

or $\quad m \in [1, \infty)$ ...(i)

(2) For $af(3) > 0$: $1.(9 - 18m + 9m^2 - 2m + 2) > 0$

$\Rightarrow \quad 9m^2 - 20m + 11 > 0$

$\Rightarrow \quad (9m - 11)(m - 1) > 0$

$\Rightarrow \quad \left(m - \dfrac{11}{9}\right)(m - 1) > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup \left(\dfrac{11}{9}, \infty\right)$ ...(ii)

For $-\dfrac{b}{2a} > 3$: $\dfrac{6m}{2} > 3 \Rightarrow m > 1 \Rightarrow m \in (1, \infty)$ ...(iii)

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in \left(\dfrac{11}{9}, \infty\right)$.

Sample Problem 15 *The value of a for which the quadratic equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ possesses roots of opposite sign, lies in*

(a) $(-\infty, 1)$ (b) $(-\infty, 0)$

(c) $(1, 2)$ (d) $\left(\dfrac{3}{2}, 2\right)$

Interpret (c) The quadratic equation

$$3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$$

will have two roots of opposite sign, if it has real roots and the product of roots is negative *i.e.*, if $D > 0$ and $\alpha\beta < 0$.

$\Rightarrow \quad 4(a^2 + 1)^2 - 12(a^2 - 3a + 2) > 0$

and $\quad \dfrac{a^2 - 3a + 2}{3} < 0$

Both of these conditions are satisfied, if $a^2 - 3a + 2 < 0$ *i.e.*, if $(a - 1)(a - 2) < 0$ or $1 < a < 2$.

5.17 Maximum and Minimum Value of $ax^2 + bx + c$

$\because \quad ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2 + \left(\dfrac{4ac - b^2}{4a}\right)$

***Case* I** When $a > 0$

$\because \quad ax^2 + bx + c \geq \dfrac{4ac - b^2}{4a}$

$\therefore$ Minimum value of $ax^2 + bx + c$ is $\dfrac{4ac - b^2}{4a}$ at $x = -\dfrac{b}{2a}$.

***Case* II** When $a < 0$

$\because \quad ax^2 + bx + c \leq \dfrac{4ac - b^2}{4a}$

$\therefore$ Maximum value of $ax^2 + bx + c$ is $\dfrac{4ac - b^2}{4a}$ at $x = -\dfrac{b}{2a}$.

Check Point **2**

1. The roots of $ax^2 + bx + c = 0$, where $a \neq 0$ and coefficients are real, non-real complex and $a + c < b$, then
   (a) $4a + c > 2b$ (b) $4a + c < 2b$
   (c) $4a + c = 2b$ (d) None of the above
2. If X denotes the set of real number p for which the equation $x^2 = p(x + p)$ has its roots greater than p, then X is equal to
   (a) $\left(-2, -\frac{1}{2}\right)$ (b) $\left(\frac{-1}{2}, \frac{1}{4}\right)$
   (c) ϕ (d) $(-\infty, 0)$
3. Conditions on a and b for which $x^2 - ax - b^2$ is less than zero for atleast one positive x, are
   (a) $a > 3, b < 0$ (b) $a > 3, b > 0$
   (c) $a, b \in R$ (d) None of these
4. If $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $3a + b$ is
   (a) 2 (b) – 2 (c) 1 (d) – 1
5. If $(\lambda^2 + \lambda - 2)x^2 + (\lambda + 2)x < 1$ for all $x \in R$, then λ belongs to the interval
   (a) $(-2, 1)$ (b) $\left(-2, \frac{2}{5}\right)$ (c) $\left(\frac{2}{5}, 1\right)$ (d) $\left(-2, \frac{1}{2}\right)$
6. The value of a for which $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ may have one root less than a and other root greater than a, is
   (a) $-1 < a < 0$ (b) $a > 0$ or $a < -1$
   (c) $a \geq 0$ (d) $-\frac{1}{2} < a < 0$
7. Find all values of p, so that 6 lies between the roots of the equation $x^2 + 2(p - 3)x + 9 = 0$.
   (a) $\left(-\infty, \frac{3}{4}\right)$ (b) $\left(-\infty, -\frac{3}{4}\right)$
   (c) $\left(\frac{3}{4}, \infty\right)$ (d) None of these

5.18 Inequality

An equation having signs $>, <, \geq$ or $\leq$ at the place of sign of equality (=) is known as inequality or inequation.

OR

Let a and b are two real numbers. If $a - b$ is negative, we say that a is less than $b (a < b)$ and if $a - b$ is positive, then a is greater than $b (a > b)$.

Properties of Inequalities

(i) If $a > b$ and $b > c$, then $a > c$. Generally, if $a_1 > a_2, a_2 > a_3, \ldots, a_{n-1} > a_n$, then $a_1 > a_n$.

(ii) If $a > b$, then $a \pm c > b \pm c, \forall c \in R$

(iii) If $a > b$, then for $m > 0$, $am > bm$, $\frac{a}{m} > \frac{b}{m}$ and for $m < 0$, $bm > am$, $\frac{b}{m} < \frac{a}{m}$

(iv) If $a > b > 0$, then
   (a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} < \frac{1}{b}$
   and if $a < b < 0$, then
   (a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} > \frac{1}{b}$

(v) If $a < 0 < b$, then
   (a) $a^2 > b^2$, if $|a| > |b|$ (b) $a^2 < b^2$, if $|a| < |b|$

(vi) If $a < x < b$ and a, b are positive real numbers, then
$$a^2 < x^2 < b^2$$

(vii) If $a < x < b$ and a is negative number and b is positive number, then
   (a) $0 \leq x^2 < b^2$, if $|b| > |a|$
   (b) $0 \leq x^2 \leq a^2$, if $|a > |b|$

(viii) If $\frac{a}{b} > 0$, then
   (a) $a > 0$, if $b > 0$ (b) $a < 0$, if $b < 0$

(ix) If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then
$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n$$

(x) If $|x| < a$, then
   (a) If a is positive, then $-a < x < a$
   (b) If a is negative, then $x \in \phi$

(xi) If $a_1 > b_i$, where $i = 1, 2, 3, \ldots, n$, then
$$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$$

(xii) If $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$

(xiii) If $0 < a < 1$ and n is a positive rational number, then
   (a) $0 < a^n < 1$ (b) $a^{-n} > 1$

Algorithm of Modulus Function

The equation of the form
$$|f_1(x)| + |f_2(x)| + |f_3(x)| + \ldots + |f_n(x)| = g(x) \quad \ldots(i)$$
when $f_1(x) f_2(x) \ldots f_3(x), g(x)$ are function of x and $g(x)$ may be constant.

Equations of this form solved by the method of intervals. We first find all critical points of $f_1(x), f_2(x), \ldots f_n(x)$, if coefficient of x is +ve then graph start with +ve sign and if coefficient of x is negative the graph start with negative sign.

Then, using the definition of the absolute value, we pass from Eq. (i) to a collection of systems which do not contain the absolute value symbols.

Sample Problem 16 *The solution set of* $|3-4x| \geq 9$ *is*

[NCERT Exemplar]

(a) $\left(-\infty, \frac{3}{2}\right) \cup [3, \infty)$ (b) $\left(-\infty, -\frac{3}{2}\right] \cup [3, \infty)$

(c) $(-\infty, 2) \cup [2, \infty)$ (d) *None of these*

Interpret (b) We have, $|3-4x| \geq 9$.

$\Rightarrow$ $3-4x \leq -9$ or $3-4x \geq 9$ (since, $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$)

$\Rightarrow$ $-4x \leq -12$ or $-4x \geq 6$

$\Rightarrow$ $x \geq 3$ or $x \leq \frac{-3}{2}$ (dividing both sides by –4)

$\Rightarrow$ $x \in \left(-\infty, \frac{-3}{2}\right] \cup [3, \infty)$

Sample Problem 17 *The solution set of* $\frac{|x+3|+x}{x+2} > 1$ *is*

[NCERT Exemplar]

(a) $[-5, -2] \cup [-1, \infty)$ (b) $[-5, -2] \cup [-1, \infty)$

(c) $(-5, -2) \cup (-1, \infty)$ (d) *None of these*

Interpret (c) We have, $\frac{|x+3|+x}{x+2} > 1$

$\Rightarrow$ $\frac{|x+3|+x}{x+2} - 1 > 0 \Rightarrow \frac{|x+3|-2}{x+2} > 0$

Case I When $x+3 \geq 0$, *i.e.*, $x \geq -3$

$\therefore$ $\frac{|x+3|-2}{x+2} > 0 \Rightarrow \frac{x+3-2}{x+2} > 0 \Rightarrow \frac{x+1}{x+2} > 0$

$\Rightarrow$ $\{(x+1) > 0$ and $x+2 > 0\}$ or $\{x+1 < 0$ and $x+2 < 0\}$

$\Rightarrow$ $\{x > -1$ and $x > -2\}$ or $\{x < -1$ and $x < -2\}$

$\Rightarrow$ $x > -1$ or $x < -2$

$\Rightarrow$ $x \in (-1, \infty)$ or $x \in (-\infty, -2)$

$\Rightarrow$ $x \in (-3, -2) \cup (-1, \infty)$ (since, $x \geq -3$) ...(i)

Case II When $x+3 < 0$ *i.e.*, $x < -3$

$\therefore$ $\frac{|x+3|-2}{x+2} > 0 \Rightarrow \frac{-x-3-2}{x+2} > 0$

$\Rightarrow$ $\frac{-(x+5)}{x+2} > 0 \Rightarrow \frac{x+5}{x+2} < 0$

$\Rightarrow$ $(x+5 < 0$ and $x+2 > 0)$ or $(x+5 > 0$ and $x+2 < 0)$

$\Rightarrow$ $(x < -5$ and $x > -2)$

or $(x > -5$ and $x < -2)$ it is not possible.

$\Rightarrow$ $x \in (-5, -2)$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get $x \in (-5, -2) \cup (-1, \infty)$

5.19 Arithmetico-Geometric Mean Inequality

(i) If $a, b > 0$ and $a \neq b$, then $\frac{a+b}{2} > \sqrt{ab} > \frac{2}{\frac{1}{a}+\frac{1}{b}}$

(ii) If $a_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}$$

Sample Problem 18 *If* a, b and c *be positive number, then* $a^2 + b^2 + c^2$ *is greater than*

(a) $ab + bc + ca$ (b) $\frac{ab+bc+ca}{2}$

(c) $2(ab + bc + ca)$ (d) *None of these*

Interpret (a) $\because$ AM > GM

$\therefore$ $\frac{a^2+b^2}{2} > \sqrt{a^2b^2} \Rightarrow \frac{a^2+b^2}{2} > ab$ ...(i)

Similarly, $\frac{b^2+c^2}{2} > bc$ ...(ii)

and $\frac{c^2+a^2}{2} > ca$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$a^2 + b^2 + c^2 > ab + bc + ca$$

5.20 Logarithm

If a is a positive real number other than 1 and $a^x = m$, then x is called the logarithm of m to the base a, written as $\log_a m$. ln $\log_a m$, m should be always positive.

Note

- If $m < 0$, then $\log_a m$ will be imaginary and if $m = 0$, then $\log_a m$ will be meaningless.
- $\log_a m$ exists, if $m, a > 0$ and $a \neq 1$.

Important Results on Logarithm

(a) $a^{\log_a x} = x$; $a \neq 0, \neq 1, x > 0$ (b) $a^{\log_b x} = x^{\log_b a}$; $a, b > 0, \neq 1, x > 0$

(c) $\log_a a = 1, a > 0, \neq 1$ (d) $\log_a x = \frac{1}{\log_x a}$; $x, a > 0, \neq 1$

(e) $\log_a x = \log_a b \log_b x = \frac{\log_b x}{\log_b a}$; $a, b > 0, \neq 1, x > 0$

(f) For $m, n > 0$ and $a > 0, \neq 1$, then

(i) $\log_a (m \cdot n) = \log_a m + \log_a n$ (ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a (m^n) = n \log_a (m)$

(g) For $x > 0, a > 0, \neq 1$

(i) $\log_{a^n} (x) = \frac{1}{n} \log_a x$ (ii) $\log_{a^n} x^m = \left(\frac{m}{n}\right) \log_a x$

(h) For $x > y > 0$

(i) $\log_a x > \log_a y$, if $a > 1$ (ii) $\log_a x < \log_a y$, if $0 < a < 1$

(i) If $a > 1$ and $x > 0$, then

(i) $\log_a x > p \Rightarrow x > a^p$ (ii) $0 < \log_a x < p \Rightarrow 1 < x < a^p$

(j) If $0 < a < 1$, then

(i) $\log_a x > p \Rightarrow 0 < x < a^p$ (ii) $0 < \log_a x < p \Rightarrow a^p < x < 1$

Sample **Problem 19** *If $a^x = b^y = c^z = d^w$, then the value of $x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)$ is*

(a) $\log_a (abc)$ (b) $\log_a (bcd)$ (c) $\log_b (cda)$ (d) $\log_c (dab)$

Interpret (b) Given, $a^x = b^y = c^z = d^w$

$\Rightarrow$ $x = y \log_a b = z \log_a c = w \log_a d$

$\Rightarrow$ $y = \frac{x}{\log_a b}, z = \frac{x}{\log_a c}, w = \frac{x}{\log_a d}$

Now, $x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right) = x\left(\frac{\log_a b}{x}+\frac{\log_a c}{x}+\frac{\log_a d}{x}\right)$

$= \frac{x}{x}(\log_a bcd)$ $(\because \log_x y + \log_x z = \log_x yz)$

$= \log_a (bcd)$

Check Point **3**

1. The solution set of $\frac{1}{|x|-3} \le \frac{1}{2}$ is
   (a) $(-\infty, -5) \cup (-3, 3) \cup [5, \infty)$
   (b) $(-\infty, -5] \cup [-3, 3] \cup [5, \infty)$
   (c) $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$
   (d) None of the above
2. The solution set of $|x+1| + |x| > 3$ is
   (a) $(-\infty, -2) \cup [1, \infty)$ (b) $(-\infty, -2) \cup (1, \infty)$
   (c) $(-\infty, 2) \cup [3, \infty)$ (d) None of these
3. If $c < d$, $x^2 + (c+d)x + cd < 0$, then $x \in$
   (a) $(-d, -c]$ (b) $(-d, -c)$
   (c) R (d) S
4. If $x^2 + x + |x| + 1 \le 0$, then x lies in
   (a) $(0, \infty)$
   (b) $(-\infty, 0)$
   (c) R
   (d) ϕ
5. Solution set of the inequality $\frac{1}{2^x - 1} > \frac{1}{1 - 2^{x-1}}$, is
   (a) $(1, \infty)$ (b) $\left(0, \log_2\left(\frac{4}{3}\right)\right)$
   (c) $(-1, \infty)$ (d) $\left(0, \log_2\left(\frac{4}{3}\right)\right) \cup (1, \infty)$
6. If $a_i > 0, \forall i = 1, 2, \ldots, n$ and $a_1 a_2 \ldots a_n = 1$, then the minimum value of $(1+a_1)(1+a_2)\ldots(1+a_n)$ is
   (a) $2^{n/2}$ (b) 2^n (c) 2^{2n} (d) 1
7. If $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$, then
   (a) $-2 > x > -1$ (b) $-2 \ge x \ge -1$
   (c) $-2 < x < -1$ (d) $-2 < x \le -1$
8. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
   (a) $(2, \infty)$ (b) $(1, 2)$
   (c) $(-2, -1)$ (d) None of these

WORKED OUT
Examples

Example 1 *If α and β be the roots of equation $ax^2 + bx + c = 0$, then the value of $\left(\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}\right)$ is*

(a) $\frac{ac}{b}$ (b) $\frac{b}{ac}$

(c) $\frac{b}{c}$ (d) *None of these*

Solution (b) Since, α and β are the roots of $ax^2 + bx + c = 0$.

$$\therefore \quad \alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Now, $$\frac{1}{a\alpha + b} + \frac{1}{a\beta + b} = \frac{a\beta + b + a\alpha + b}{(a\alpha + b)(a\beta + b)}$$

$$= \frac{a(\alpha + \beta) + 2b}{a^2\alpha\beta + ab(\alpha + \beta) + b^2}$$

$$= \frac{a\left(-\frac{b}{a}\right) + 2b}{a^2\left(\frac{c}{a}\right) + ab\left(-\frac{b}{a}\right) + b^2}$$

$$= \frac{-b + 2b}{ac - b^2 + b^2} = \frac{b}{ac}$$

Example 2 *If equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have a common root, then the values of k are*

(a) -3 *or* $-\frac{27}{4}$ (b) 3 *or* $\frac{27}{4}$

(c) -3 *or* $\frac{27}{4}$ (d) 3 *or* $-\frac{27}{4}$

Solution (a) Let α be a root common to both of the given equations.

$$\therefore \quad 2\alpha^2 + k\alpha - 5 = 0 \quad \ldots(i)$$

$$\text{and} \quad \alpha^2 - 3\alpha - 4 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\frac{\alpha^2}{(-4k - 15)} = \frac{\alpha}{-5 + 8} = \frac{1}{-6 - k}$$

$$\Rightarrow \quad \alpha^2 = \frac{4k + 15}{k + 6} \quad \text{and} \quad \alpha = \frac{-3}{k + 6}$$

$$\Rightarrow \quad \left(\frac{-3}{k + 6}\right)^2 = \frac{4k + 15}{k + 6}$$

$$\Rightarrow \quad (4k + 15)(k + 6) = 9$$

$$\Rightarrow \quad 4k^2 + 39k + 81 = 0$$

$$\Rightarrow \quad k = -3 \quad \text{or} \quad k = -\frac{27}{4}$$

Example 3 *If the roots of equation $ax^2 + bx + c = 0$ are $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then the value of $(a + b + c)^2$ is*

(a) $b^2 + 4ac$ (b) $-b^2 + 4ac$

(c) $b^2 - 4ac$ (d) *None of these*

Solution (c) $\because$ Sum of roots,

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \quad \ldots(i)$$

and product of roots,

$$\frac{k+1}{k} \times \frac{k+2}{k+1} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{k+2}{k} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{2}{k} = \frac{c}{a} - 1 = \frac{c - a}{a}$$

$$\Rightarrow \quad k = \frac{2a}{c - a}$$

On putting the value of k in Eq. (i), we get

$$\frac{c + a}{2a} + \frac{2c}{c + a} = -\frac{b}{a}$$

$$\Rightarrow \quad (c + a)^2 + 4ac = -2b(a + c)$$

$$\Rightarrow \quad (a + c)^2 + 2b(a + c) = -4ac$$

$$\Rightarrow \quad (a + c)^2 + 2b(a + c) + b^2 = b^2 - 4ac$$

$$\Rightarrow \quad (a + b + c)^2 = b^2 - 4ac$$

Example 4 *Two students tried to solve a quadratic equation $x^2 + px + q = 0$. First student took a wrong value of p and then found the roots of equation as 2 and 6. Another student took a wrong value of q and then found the roots of equation as 2 and -9. The exact roots of equation are*

(a) $3, 4$ (b) $-3, -4$

(c) $-3, 4$ (d) $3, -4$

Solution (*b*) Let equation of exact roots is

$$x^2 + ax + b = 0 \quad \ldots(i)$$

According to first student,

sum of roots $= 2 + 6 = 8$

and product of roots $= 2 \times 6 = 12$

Since, first student took a wrong value of p.

$\therefore \quad q = 12$

and according to second student,

sum of roots $= 2 + (-9) = -7$

and product of roots $= 2 \times (-9) = -18$

Since, second student took a wrong value of q.

$\therefore \quad p = 7$

$\therefore$ From Eq. (i),

$$x^2 + 7x + 12 = 0$$

$$x^2 + 3x + 4x + 12 = 0$$

$$\Rightarrow \quad (x + 4)(x + 3) = 0$$

$$\Rightarrow \quad x = -3, -4$$

Example 5 *If* $x = 2 + 2^{2/3} + 2^{1/3}$, *then the value of* $x^3 - 6x^2 + 6x$ *is*

(*a*) 2 (*b*) 3 (*c*) -6 (*d*) 8

Solution (*a*) $\because \quad x = 2 + 2^{2/3} + 2^{1/3}$

$$\Rightarrow \quad (x - 2) = 2^{2/3} + 2^{1/3}$$

$$\Rightarrow \quad x^3 - 8 - 6x^2 + 12x = 2^2 + 2 + 3 \cdot 2(x - 2)$$

$$\Rightarrow \quad x^3 - 8 - 6x^2 + 12x = 4 + 2 + 6x - 12$$

$$\Rightarrow \quad x^3 - 6x^2 + 6x = 8 + 6 - 12$$

$$\Rightarrow \quad x^3 - 6x^2 + 6x = 2$$

Example 6 *If the product of roots of equation* $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ *is 7, then the value of k is*

(*a*) -2 (*b*) 2

(*c*) 4 (*d*) *None of these*

Solution (*b*) Since, product of roots of $x^2 - 3kx + 2e^{2\log_e k} - 1 = 0$ is 7.

$$\therefore \quad 2e^{2\log k} - 1 = 7$$

$$\Rightarrow \quad 2e^{2\log k} = 8 \Rightarrow e^{2\log_e k} = 4$$

$$\Rightarrow \quad k^2 = 4 \Rightarrow k = \pm 2$$

$$\Rightarrow \quad k = 2 \quad (\because k \neq -2)$$

Example 7 *The number of solutions of equation*

$$\frac{(x + 2)(x - 5)}{(x - 3)(x + 6)} = \frac{x - 2}{x + 4} \text{ is}$$

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) *infinite*

Solution (*a*) We have, $\dfrac{(x + 2)(x - 5)}{(x - 3)(x + 6)} = \dfrac{x - 2}{x + 4}$

$$\Rightarrow \quad (x + 2)(x - 5)(x + 4) = (x - 2)(x - 3)(x + 6)$$

$$\Rightarrow \quad (x^2 - 3x - 10)(x + 4) = (x^2 - 5x + 6)(x + 6)$$

$$\Rightarrow \quad x^3 - 3x^2 - 10x + 4x^2 - 12x - 40$$

$$= x^3 - 5x^2 + 6x + 6x^2 - 30x + 36$$

$$\Rightarrow \quad x^2 - 22x - 40 = x^2 - 24x + 36$$

$$\Rightarrow \quad 2x = 40 + 36$$

$$\Rightarrow \quad x = \frac{76}{2} = 38$$

Number of solutions of given equation is 1.

Example 8 *If* $x > 0$, *then solution of* $\left|x + \dfrac{1}{x}\right| < 4$ *is*

(*a*) $-2 - \sqrt{3} < x < -2 + \sqrt{3}$ (*b*) $2 - \sqrt{3} < x < 7 + \sqrt{3}$

(*c*) $2 - \sqrt{3} < x < 2 + \sqrt{3}$ (*d*) *None of these*

Solution (*a*) $\because \left|x + \dfrac{1}{x}\right| < 4 \Rightarrow \dfrac{x^2 + 1}{|x|} < 4$

$$\Rightarrow \quad |x|^2 - 4|x| + 1 < 0$$

$$\Rightarrow \quad (|x| - 2)^2 < 3$$

$$\Rightarrow \quad (|x| - 2)^2 < (\sqrt{3})^2$$

$$\Rightarrow \quad ||x| - 2| < \sqrt{3}$$

$$\Rightarrow \quad -\sqrt{3} < (|x| - 2) < \sqrt{3}$$

$$\Rightarrow \quad 2 - \sqrt{3} < |x| < 2 + \sqrt{3}$$

when $x > 0$, $\quad 2 - \sqrt{3} < x < 2 + \sqrt{3}$

and when $x < 0$, $\quad -2 - \sqrt{3} < x < -2 + \sqrt{3}$

Example 9 *For positive real numbers a, b and c, the minimum value of* $\dfrac{b + c}{a} + \dfrac{c + a}{b} + \dfrac{a + b}{c}$ *is*

(*a*) 5 (*b*) 6

(*c*) 9 (*d*) *None of these*

Solution (*b*) Since,

$$\frac{1}{6}\left(\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c}\right) \geq \left(\frac{a}{b} \cdot \frac{b}{a} \cdot \frac{b}{c} \cdot \frac{c}{b} \cdot \frac{c}{a} \cdot \frac{a}{c}\right)^{1/6}$$

$$\Rightarrow \quad \left(\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c}\right) \geq 6$$

$\therefore$ Minimum value of $\left(\dfrac{a + b}{c} + \dfrac{c + a}{b} + \dfrac{b + c}{a}\right) = 6$

Example 10 *The equation*

$$\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1 \text{ has}$$

(*a*) *no solution* (*b*) *only one solution*

(*c*) *only two solutions* (*d*) *more than two solutions*

Solution (*d*) Put $\sqrt{x - 1} = t$ or $x = t^2 + 1$, so that given equation becomes

$$\sqrt{t^2 + 4 - 4t} + \sqrt{t^2 + 9 - 6t} = 1$$

$$\Rightarrow \quad \sqrt{(t - 2)^2} + \sqrt{(t - 3)^2} = 1$$

$$\Rightarrow \quad |t - 2| + |t - 3| = 1$$

This equation is satisfied for all values of t lying between 2 and 3 *i.e.*, $2 \le t \le 3$.

Thus, the given equation is satisfied for all values of x lying between 5 and 10.

Example 11 *If the roots of equation $ax^2 + bx + c = 0$ are in the ratio $1:r$, then $\left\{\frac{(r+1)^2}{r}\right\}$ is equal to*

(a) $\frac{b^2}{ac}$ (b) $\frac{ac}{b^2}$ (c) $\frac{b^2}{a}$ (d) $\frac{b^2}{c}$

Solution (a) Let roots of equation $ax^2 + bx + c = 0$ are α and $r\alpha$, then

$$\alpha + \alpha r = -\frac{b}{a} \quad \text{...(i)}$$

and $$\alpha \cdot r\alpha = \frac{c}{a} \Rightarrow r\alpha^2 = \frac{c}{a} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left\{-\frac{b}{a}\left(\frac{1}{1+r}\right)\right\}^2 = \frac{c}{ar}$$

$$\Rightarrow \quad \frac{b^2}{a^2}\left(\frac{1}{1+r}\right)^2 = \frac{c}{ar} \Rightarrow \frac{b^2}{ac} = \frac{(1+r)^2}{r}$$

Example 12 *If one root of equation $(l - m)x^2 + lx + 1 = 0$ is twice of another and l is real, then m is*

(a) $< \frac{9}{8}$ (b) $\le \frac{9}{8}$ (c) $\ge \frac{9}{8}$ (d) $> \frac{9}{8}$

Solution (b) Let roots of equation $(l - m)x^2 + lx + 1 = 0$ are α and 2α.

Then, $$2\alpha + \alpha = -\frac{l}{l-m} \Rightarrow \alpha = -\frac{1}{3(l-m)} \quad \text{...(i)}$$

and $$(2\alpha)(\alpha) = \frac{1}{l-m} \Rightarrow \alpha^2 = \frac{1}{2(l-m)} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left(-\frac{l}{3(l-m)}\right)^2 = \frac{1}{2(l-m)}$$

$$\Rightarrow \quad \frac{2l^2}{9} = (l-m)$$

$$\Rightarrow \quad 2l^2 - 9l + 9m = 0 \quad (\because l \text{ is real})$$

$$\therefore \quad 9^2 - 72m \ge 0$$

$$\Rightarrow \quad m \le \frac{81}{72} \Rightarrow m \le \frac{9}{8}$$

Example 13 *If a, b and c are real positive numbers such that $a < b < c$, then $\frac{a^2+b^2+c^2}{a+b+c}$ lies between*

(a) $\frac{a}{c}, \frac{c}{a}$ (b) $\frac{a^2}{c}, \frac{c^2}{a}$

(c) $\frac{c}{a^2}, \frac{a}{c^2}$ (d) *None of these*

Solution (b) Since, $c > b > a$

$$\Rightarrow \quad c^2 > b^2 > a^2$$

$$\Rightarrow \quad 3a^2 < a^2 + b^2 + c^2 < 3c^2 \quad \text{...(i)}$$

and $$3a < a + b + c < 3c$$

$$\Rightarrow \quad \frac{1}{3c} < \frac{1}{a+b+c} < \frac{1}{3a} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{3a^2}{3c} < \frac{a^2+b^2+c^2}{a+b+c} < \frac{3c^2}{3a}$$

$$\Rightarrow \quad \frac{a^2}{c} < \frac{a^2+b^2+c^2}{a+b+c} < \frac{c^2}{a}$$

Example 14 *If a, b and c are distinct three positive real numbers, then $(a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is*

(a) > 1 (b) > 9

(c) < 9 (d) *None of these*

Solution (b) We know that,

$$\text{AM} > \text{GM}$$

$$\Rightarrow \quad \frac{a+b+c}{3} > (abc)^{1/3} \quad \text{...(i)}$$

and $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} > \left(\frac{1}{abc}\right)^{1/3}$$

$$\Rightarrow \quad \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) > \frac{3}{(abc)^{1/3}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left(\frac{a+b+c}{3}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) > \frac{3}{(abc)^{1/3}} \cdot (abc)^{1/3}$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) > 9$$

Example 15 *The maximum value of function $f(x) = (1-x)^3$, $(1+3x)$, $\frac{-1}{3} < x < 1$ is*

(a) 1 (b) 0

(c) 3 (d) *None of these*

Solution (a) $\because f(x) = (1-x)^3 (1 + 3x)$

$$= (1-x)(1-x)(1-x)(1+3x)$$

$$\therefore \quad (1-x) > 0 \text{ and } 1 + 3x > 0 \quad \left(\because -\frac{1}{3} < x < 1\right)$$

and $(1-x) + (1-x) + (1-x) + (1+3x) = 4$

$\therefore$ For maximum value of $f(x)$

$$1 - x = 1 + 3x \Rightarrow x = 0$$

$\therefore$ Maximum value of $f(x)$ at $x = 0$ is

$$f(0) = (1-0)^3 (1 + 0) = 1$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Remainder and Factor Theorem

1. If $(x-2)$ is a common factor of the expressions x^2+ax+b and x^2+cx+d, then $\frac{b-d}{c-a}$ is equal to

(a) -2 (b) -1 (c) 1 (d) 2

2. If x^2+ax+1 is a factor of ax^3+bx+c, then

(a) $b+a+a^2=0, a=c$ (b) $b-a+a^2=0, a=c$

(c) $b+a-a^2=0, a=0$ (d) None of these

3. If x^2-3x+2 be a factor of x^4-px^2+q, then (p, q) is equal to

(a) (3, 4) (b) (4, 5) (c) (4, 3) (d) (5, 4)

4. If x^2+px+1 is a factor of the expression ax^3+bx+c, then

(a) $a^2+c^2=-ab$ (b) $a^2-c^2=-ab$

(c) $a^2-c^2=ab$ (d) None of these

Roots of Equation

5. The roots of a equation $2x^2+x+1=0$ are [NCERT]

(a) $\frac{-1+i\sqrt{7}}{4}, \frac{-1-i\sqrt{7}}{4}$ (b) $\frac{1+i\sqrt{7}}{4}, \frac{1-i\sqrt{7}}{4}$

(c) $\frac{-1+i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}$ (d) None of these

6. If one root of the equation $x^2+(1-3i)x-2(1+i)=0$ is $-1+i$, then the other root is

(a) $-1-i$ (b) $\frac{-1-i}{2}$

(c) i (d) $2i$

7. If the difference of the roots of the equation $x^2-Px+8=0$ is 2, then the value of P is [NCERT Exemplar]

(a) ± 4 (b) ± 6

(c) ± 5 (d) None of these

8. If the sum of the squares of the roots of the equation $x^2-(a-2)x-(a+1)=0$ is least, then the value of a is [NCERT Exemplar]

(a) -1 (b) 1 (c) 2 (d) -2

9. If $x=\sqrt{1+\sqrt{1+\sqrt{1+\ldots\infty}}}$, then x is equal to

(a) $\frac{1+\sqrt{5}}{2}$ (b) $\frac{1-\sqrt{5}}{2}$

(c) $\frac{1\pm\sqrt{5}}{2}$ (d) None of these

10. If $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$, then x is equal to

(a) 2 (b) 3 (c) 6 (d) 5

11. The value of $2+\cfrac{1}{2+\cfrac{1}{2+\ldots\infty}}$ is

(a) $1-\sqrt{2}$ (b) $1+\sqrt{2}$

(c) $1\pm\sqrt{2}$ (d) None of these

12. The roots of the equation $|2x-1|^2-3|2x-1|+2=0$ are

(a) $\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$ (b) $\left\{-\frac{1}{2}, 0, \frac{3}{2}\right\}$

(c) $\left\{-\frac{3}{2}, \frac{1}{2}, 0, 1\right\}$ (d) $\left\{-\frac{1}{2}, 0, 1, \frac{3}{2}\right\}$

13. The number of real roots of the equation $e^{\sin x}-e^{-\sin x}-4=0$ are

(a) 1 (b) 2

(c) infinite (d) None of these

14. How many roots of the equation $x-\frac{2}{x-1}=1-\frac{2}{x-1}$ have ?

(a) One (b) Two

(c) Infinite (d) None of these

15. The number of real solutions of the equation $|x^2 + 4x + 3| + 2x + 5 = 0$ are
(a) 1 (b) 2
(c) 3 (d) 4

16. The roots of the equation $x^4 - 8x^2 - 9 = 0$ are
(a) $\pm 1, \pm i$ (b) $\pm 3, \pm i$
(c) $\pm 2, \pm i$ (d) None of these

17. If the roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in GP, then the roots are
(a) $1, \frac{1}{2}, \frac{1}{4}$ (b) 2, 4, 8
(c) 3, 6, 12 (d) None of these

18. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then (p, q) is equal to
(a) $(-4, 7)$ (b) $(4, -7)$
(c) $(4, 7)$ (d) $(-4, -7)$

19. The number of real roots of $3^{2x^2 - 7x + 7} = 9$ is
(a) 0 (b) 2
(c) 1 (d) 4

20. Rational roots of the equation

$$2x^4 + x^3 - 11x^2 + x + 2 = 0 \text{ are}$$

(a) $\frac{1}{2}$ and 2 (b) $\frac{1}{2}, 2, \frac{1}{4}, -2$
(c) $\frac{1}{2}, 2, 3, 4$ (d) $\frac{1}{2}, 2, \frac{3}{4}, -2$

21. The roots of the given equation

$$(p - q)x^2 + (q - r)x + (r - p) = 0 \text{ are}$$

(a) $\frac{p-q}{r-p}, 1$ (b) $\frac{q-r}{p-q}, 1$
(c) $\frac{r-p}{p-q}, 1$ (d) $1, \frac{q-r}{p-q}$

22. If $2\sin^2 \frac{\pi}{8}$ is a root of the equation $x^2 + ax + b = 0$, where a and b are rational numbers, then $a - b$ is equal to
(a) $-\frac{5}{2}$ (b) $-\frac{3}{2}$ (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

23. $\tan\alpha$ and $\tan\beta$ are the roots of the equation $x^2 + ax + b = 0$, then the value of

$\sin^2(\alpha + \beta) + a\sin(\alpha + \beta)\cos(\alpha + \beta) + b\cos^2(\alpha + \beta)$ is equal to
(a) ab (b) b
(c) $\frac{a}{b}$ (d) a

24. The coefficient of x in the equation $x^2 + px + q = 0$ was taken as 17 in place of 13 its roots were found to be -2 and -15. The roots of the original equation are
(a) 3, 10 (b) $-3, -10$
(c) $-5, -8$ (d) None of these

25. Two students while solving a quadratic equation in x, one copied the constant term incorrectly and got the roots 3 and 2. The other copied the constant term coefficient of x^2 correctly as -6 and 1 respectively the correct roots are
(a) $3, -2$ (b) $-3, 2$
(c) $-6, -1$ (d) $6, -1$

26. If α, β, γ are the roots of $x^3 + 2x^2 - 3x - 1 = 0$, then $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$ is equal to
(a) 12 (b) 13
(c) 14 (d) 15

27. If the roots of the given equation
$(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ are real, then
(a) $p \in (-\pi, 0)$ (b) $p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $p \in (0, \pi)$ (d) $p \in (0, 2\pi)$

Nature of Roots

28. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)\,Q(x) = 0$ has atleast
(a) four real roots (b) two real roots
(c) four imaginary roots (d) None of these

29. If the roots of $(a^2 + b^2)x^2 - 2(bc + ad)x + c^2 + d^2 = 0$ are equal, then
(a) $\frac{a}{b} = \frac{c}{d}$ (b) $\frac{a}{c} + \frac{b}{d} = 0$
(c) $\frac{a}{d} = \frac{b}{c}$ (d) $a + b = c + d$

30. Let $f(x) = x^2 + ax + b; a, b \in R$. If $f(1) + f(2) + f(3) = 0$, then the roots of the equation $f(x) = 0$
(a) are imaginary
(b) are real and equal
(c) are from the set $\{1, 2, 3\}$
(d) real and distinct

31. If $\sin\alpha, \sin\beta$ and $\cos\alpha$ are in GP, then roots of $x^2 + 2x\cot\beta + 1 = 0$ are always
(a) real (b) real and negative
(c) greater than one (d) non-real

32. If one root of equation $x^2 + ax + 12 = 0$ is 4 while the equation $x^2 + ax + b = 0$ has equal roots, then the value of b is
(a) $\frac{4}{49}$ (b) $\frac{49}{4}$ (c) $\frac{7}{4}$ (d) $\frac{4}{7}$

33. The solution set of the equation

$$pqx^2 - (p + q)^2 x + (p + q)^2 = 0 \text{ is}$$

(a) $\left\{\frac{p}{q}, \frac{q}{p}\right\}$ (b) $\left\{pq, \frac{p}{q}\right\}$
(c) $\left\{\frac{q}{p}, pq\right\}$ (d) $\left\{\frac{p+q}{p}, \frac{p+q}{q}\right\}$

34. If $\sin\alpha$ and $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$, then
(a) $a^2 - b^2 + 2ac = 0$ (b) $(a - c)^2 = b^2 + c^2$
(c) $a^2 + b^2 - 2ac = 0$ (d) $a^2 + b^2 + 2ac = 0$

35. If the roots of the equation $ax^2 + bx + c = 0$ of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a + b + c)^2$ is equal to
(a) $2b^2 - ac$ (b) Σa^2 (c) $b^2 - 4ac$ (d) $b^2 - 2ac$

36. If a root of the equation $ax^2 + bx + c = 0$ be reciprocal of a root of the equation $a'x^2 + b'x + c' = 0$, then
(a) $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$
(b) $(bb' - aa')^2 = (ca' - bc')(ab' - bc')$
(c) $(cc' - aa')^2 = (ba' + cb')(ab' + bc')$
(d) None of the above

37. If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is 2, then the sum of roots is
(a) 1 (b) -1 (c) 2 (d) -2

38. The system $y^{(x^2 + 7x + 12)} = 1$ and $x + y = 6, y > 0$ has
(a) no solution (b) one solution
(c) two solutions (d) more than 2 solutions

39. If $[x]^2 = [x + 2]$, where $[x]$ = the greatest integer less than or equal to x, then x must be such that
(a) $x = 2, -1$ (b) $[-1, 0) \cup [2, 3)$
(c) $x \in [-1, 0)$ (d) None of these

40. If α, β and γ are the roots of the equation $x^3 - 7x + 7 = 0$, then $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$ is
(a) 7/3 (b) 3/7
(c) 4/7 (d) 7/4

41. If the roots of the equation $\frac{\alpha}{x - \alpha} + \frac{\beta}{x - \beta} = 1$ be equal in magnitude but opposite in sign, then $\alpha + \beta$ is equal to
(a) 0 (b) 1
(c) 2 (d) None of these

42. If the roots of the equation $\frac{1}{x + p} + \frac{1}{x + q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots will be
(a) $\frac{p^2 + q^2}{2}$ (b) $-\frac{(p^2 + q^2)}{2}$
(c) $\frac{p^2 - q^2}{2}$ (d) $-\frac{(p^2 - q^2)}{2}$

43. The least value of $|a|$ for which $\tan\theta$ and $\cot\theta$ are roots of the equation $x^2 + ax + 1 = 0$, is
(a) 2 (b) 1 (c) 1/2 (d) 0

44. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is
(a) 2 (b) 4
(c) 6 (d) 8

45. If $a + b + c = 0$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are
(a) equal (b) imaginary
(c) real (d) None of these

46. If x_1, x_2 and x_3 are distinct roots of the equation $ax^2 + bx + c = 0$, then
(a) $a = b = 0, c \in R$ (b) $a = c = 0, b \in R$
(c) $b^2 - 4ac \geq 0$ (d) $a = b = c = 0$

47. If $x^2 + 2x + 2xy + my - 3 = 0$ has two rational factors, then the values of m will be
(a) $-6, -2$ (b) $-6, 2$
(c) $6, -2$ (d) $6, 2$

48. If the roots of the equation $qx^2 + px + q = 0$ are complex, where p and q are real, then the roots of the equation $x^2 - 4qx + p^2 = 0$ are
(a) real and unequal (b) real and equal
(c) imaginary (d) None of these

49. If $a < b < c < d$, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are
(a) real and distinct (b) real and equal
(c) imaginary (d) None of these

50. If the roots of the equation $x^2 + px + q = 0$ are α and β and roots of the equation $x^2 - xr + s = 0$ are α^4 and β^4, then the roots of the equation $x^2 - 4qx + 2q^2 - r = 0$ are
(a) both negative
(b) both positive
(c) both real
(d) one negative and one positive

51. If $a > 0, b > 0, c > 0$, then both the roots of the equation $ax^2 + bx + c = 0$
(a) are real and negative (b) have negative real part
(c) are rational numbers (d) None of these

52. If $(ax^2 + c)y + (a'x^2 + c') = 0$ and x is a rational function of y and ac is negative, then
(a) $ac' + a'c = 0$
(b) $\frac{a}{a'} = \frac{c}{c'}$
(c) $a^2 + c^2 = a'^2 + c'^2$
(d) $aa' + cc' = 1$

53. If roots of the equation $(a - b)x^2 + (c - a)x + (b - c) = 0$ are equal, then a, b and c are in
(a) AP (b) HP
(c) GP (d) None of these

54. If the roots of the equation $(p^2 + q^2)x^2 - 2q(p + r)x + (q^2 + r^2) = 0$ be real and equal, then p, q and r will be in

(a) AP (b) GP
(c) HP (d) None of these

55. Let α and β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then, the value of r is

(a) $\frac{2}{9}(p - q)(2q - p)$ (b) $\frac{2}{9}(q - p)(2p - q)$
(c) $\frac{2}{9}(q - 2p)(2q - p)$ (d) $\frac{2}{9}(2p - q)(2q - p)$

56. If roots of the equation $ax^2 + bx + c = 0; (a, b, c \in N)$ are rational numbers, then which of the following cannot be true ?

(a) All a, b and c are even
(b) All a, b and c are odd
(c) b is even while a and c are odd
(d) None of the above

Formation of Quadratic Equation

57. Let α, β be the roots of $x^2 - 2x\cos\phi + 1 = 0$, then the equation whose roots are α^n and β^n, is

(a) $x^2 - 2x\cos n\phi - 1 = 0$
(b) $x^2 - 2x\cos n\phi + 1 = 0$
(c) $x^2 - 2x\sin n\phi + 1 = 0$
(d) $x^2 + 2x\sin n\phi - 1 = 0$

58. If α and β be the roots of the equation $2x^2 + 2(a + b)x + a^2 + b^2 = 0$, then the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$, is

(a) $x^2 - 2ab\,x - (a^2 - b^2)^2 = 0$
(b) $x^2 - 4abx - (a^2 - b^2)^2 = 0$
(c) $x^2 - 4abx + (a^2 - b^2)^2 = 0$
(d) None of the above

59. Let α and α^2 be the roots of $x^2 + x + 1 = 0$, then the equation whose roots are α^{31} and α^{62}, is

(a) $x^2 - x + 1 = 0$ (b) $x^2 + x - 1 = 0$
(c) $x^2 + x + 1 = 0$ (d) $x^{60} + x^{30} + 1 = 0$

60. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is

(a) $ac\,x^2 + (a + c)\,bx + (a + c)^2 = 0$
(b) $ab\,x^2 + (a + c)\,bx + (a + c)^2 = 0$
(c) $ac\,x^2 + (a + b)cx + (a + c)^2 = 0$
(d) None of the above

Condition for Common Roots

61. If atleast one root of the equation $x^3 + ax^2 + bx + c = 0$ remains unchanged, when a, b and c are decreased by one, then which one of the following is always a root of the given equation ?

(a) 1
(b) -1
(c) ω, an imaginary cube root of unity
(d) i

62. If the equation $2x^2 + 3x + 5\lambda = 0$ and $x^2 + 2x + 3\lambda = 0$ have a common root, then λ is equal to

(a) 0 (b) -1
(c) 0, -1 (d) 2, -1

63. If each pair of the equation $x^2 + ax + b = 0$, $x^2 + bx + c = 0$ and $x^2 + cx + a = 0$ has a common root, then product of all common roots is

(a) $\sqrt{abc}$ (b) $2\sqrt{abc}$
(c) $\sqrt{ab + bc + ca}$ (d) $2\sqrt{ab + bc + ca}$

64. If atleast one root of $2x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in N$ is common, then the maximum value of $a + b + c$ is

(a) 10 (b) 0
(c) does not exist (d) None of these

Quadratic Expression and Position of Roots

65. If $x^2 + 2ax + b \geq c, \forall x \in R$, then

(a) $b - c \geq a^2$ (b) $c - a \geq b^2$
(c) $a - b \geq c^2$ (d) None of these

66. For all x, $x^2 + 2ax + (10 - 3a) > 0$, then the interval in which a lies, is

(a) $a < -5$ (b) $-5 < a < 2$
(c) $a > 5$ (d) $2 < a < 5$

67. The values of 'a' for which $(a^2 - 1)x^2 + 2(a - 1)x + 2$ is positive for any x, are

(a) $a \geq 1$ (b) $a \leq 1$
(c) $a > -3$ (d) $a < -3$ or $a > 1$

68. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$ (b) $2 \leq a \leq 3$
(c) $3 < a \leq 4$ (d) $a > 4$

69. If α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$ and k be a real number, then the condition, so that $\alpha < k < \beta$ is given by

(a) $ac > 0$ (b) $ak^2 + bk + c = 0$
(c) $ac < 0$ (d) $a^2k^2 + abk + ac < 0$

70. The values of a for which $2x^2 - 2(2a+1)x + a(a+1) = 0$ may have one root less than a and other root greater than a are given by
(a) $1 > a > 0$ (b) $-1 < a < 0$
(c) $a \geq 0$ (d) $a > 0$ or $a < -1$

71. One lies between the roots of the equation $-x^2 + ax + a = 0$, $a \in R$ if and only if a lies in the interval
(a) $\left(\frac{1}{2}, \infty\right)$ (b) $\left[-\frac{1}{2}, \infty\right)$
(c) $\left(-\infty, \frac{1}{2}\right)$ (d) $\left(-\infty, \frac{1}{2}\right]$

72. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta|$

Inequality

73. The solution set of $1 \leq |x-2| \leq 3$ is [NCERT Exemplar]
(a) $(-1, 1) \cup (3, 5]$ (b) $[-1, 1] \cup [3, 5]$
(c) $[-1, 1] \cup [3, 5)$ (d) None of these

74. If $|x+2| \leq 9$, then
(a) $x \in (-7, 11)$ (b) $x \in [-11, 7]$
(c) $x \in (-\infty, -7) \cup (11, \infty)$ (d) $x \in (-\infty, -7) \cup [11, \infty)$

75. If the expression $\left(mx - 1 + \frac{1}{x}\right)$ is always non-negative, then the minimum value of m must be
(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

76. The solution set of $\frac{2x-1}{3} \geq \left(\frac{3x-2}{4}\right) - \left(\frac{2-x}{5}\right)$ is [NCERT]
(a) $(4, \infty)$ (b) $[4, \infty)$
(c) $[-4, 4]$ (d) $(-\infty, 2]$

77. The solution set of $\frac{|x-2|-1}{|x-2|-2} \leq 0$ is [NCERT Exemplar]
(a) $[0, 1] \cup (3, 4)$ (b) $[0, 1] \cup [3, 4]$
(c) $[0, 1] \cup (3, 4)$ (d) None of these

78. The solution set of $1 \leq |x-2| \leq 3$ is [NCERT Exemplar]
(a) $[-1, 1] \cup (3, 5)$ (b) $(-1, 1) \cup [3, 5]$
(c) $[-1, 1] \cup [3, 5]$ (d) $[-1, 2] \cup [3, 5]$

79. $|2x-3| < |x+5|$, then x belongs to
(a) $(-3, 5)$ (b) $(5, 9)$ (c) $\left(-\frac{2}{3}, 8\right)$ (d) $\left(-8, \frac{2}{3}\right)$

80. $(x-1)(x^2 - 5x + 7) < (x-1)$, then x belongs to
(a) $(1, 2) \cup (3, \infty)$ (b) $(2, 3)$
(c) $(-\infty, 1) \cup (2, 3)$ (d) None of these

81. $x^2 - 3|x| + 2 < 0$, then x belongs to
(a) $(1, 2)$ (b) $(-2, -1)$
(c) $(-2, -1) \cup (1, 2)$ (d) $(-3, 5)$

82. $\left|x + \frac{2}{x}\right| < 3$, then x belongs to
(a) $(-2, -1) \cup (1, 2)$
(b) $(-\infty, -2) \cup (-1, 1) \cup (2, \infty)$
(c) $(-2, 2)$
(d) $(-3, 3)$

83. If $x^2 + 6x - 27 > 0$ and $x^2 - 3x - 4 < 0$, then
(a) $x > 3$ (b) $x < 4$
(c) $3 < x < 4$ (d) $x = \frac{7}{2}$

84. If $a + b = 8$, then ab is greatest when
(a) $a = 4, b = 4$ (b) $a = 3, b = 5$
(c) $a = 6, b = 2$ (d) None of these

85. If $ab = 4 (a, b \in R^+)$, then
(a) $a + b \leq 4$ (b) $a + b = 4$
(c) $a + b \geq 4$ (d) None of these

86. The minimum value of $P = bcx + cay + abz$, when $xyz = abc$, is
(a) $3abc$ (b) $6abc$
(c) abc (d) $4abc$

87. $\log_2(x^2 - 3x + 18) < 4$, then x belongs to
(a) $(1, 2)$ (b) $(2, 16)$
(c) $(1, 16)$ (d) None of these

88. If $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1 a_2 \ldots a_n = 1$, then minimum value of $(1 + a_1)(1 + a_2) \ldots (1 + a_n)$ is
(a) $2^{n/2}$ (b) 2^n
(c) 2^{2n} (d) 1

Miscellaneous

89. $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a - 2 = r$ for three distinct values of x for some $r \in R$, if $a + r$ is equal to
(a) 1 (b) 2
(c) 3 (d) does not exist

90. If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied values of x and y, then
(a) $1 \leq x \leq 3$ (b) $2 \leq x \leq 3$
(c) $-\frac{1}{3} < y < 1$ (d) $0 < y < \frac{2}{3}$

91. If x is real, then the maximum and minimum values of the expression $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ will be
(a) 2, 1 (b) 5, $\frac{1}{5}$
(c) 7, $\frac{1}{7}$ (d) None of these

92. If x is real, then expression $\frac{x+2}{2x^2+3x+6}$ takes all values in the interval

(a) $\left(\frac{1}{13}, \frac{1}{3}\right)$ (b) $\left[-\frac{1}{13}, \frac{1}{3}\right]$

(c) $\left(-\frac{1}{3}, \frac{1}{13}\right)$ (d) None of these

93. If x is real, then function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values, provided

(a) $a > b > c$ (b) $a \le b \le c$

(c) $a > c > b$ (d) $a \le c \le b$

94. For what value of λ the sum of the squares of the roots of $x^2 + (2+\lambda)x - \frac{1}{2}(1+\lambda) = 0$ is minimum ?

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{11}{4}$

95. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots, is

(a) 15 (b) 9 (c) 7 (d) 8

96. Let α and β be the roots of the equation $ax^2 + 2bx + c = 0$ and γ, δ be the roots of the equation $px^2 + 2qx + r = 0$. If α, β, γ and δ are in GP, then

(a) $q^2ac = b^2pr$ (b) $qac = bpr$

(c) $c^2pq = r^2ab$ (d) $p^2ab = a^2qr$

97. If the roots of $ax^2 + bx + c = 0$ are α, β and the roots of $Ax^2 + Bx + C = 0$ are $\alpha - k, \beta - k$, then $\frac{B^2 - 4AC}{b^2 - 4ac}$ is equal to

(a) 0 (b) 1

(c) $\left(\frac{A}{a}\right)^2$ (d) $\left(\frac{a}{A}\right)^2$

98. Let a, b and c be real numbers $a \ne 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root of γ that always satisfies

(a) $\gamma = \frac{\alpha+\beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$

(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

99. $\sin x + \cos x = y^2 - y + a$ has no value of x for any y, if a belongs to

(a) $(0, \sqrt{3})$ (b) $(-\sqrt{3}, 0)$

(c) $(-\infty, -\sqrt{3})$ (d) $(\sqrt{3}, \infty)$

100. If roots of $x^2 - ax + b = 0$ are prime numbers, then

(a) b is a prime number

(b) a is a composite number

(c) $1 + a + b$ is a prime number

(d) None of the above

Round II (Mixed Bag)

Only One Correct Option

1. The root of the equation $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$, where $i = \sqrt{-1}$, which has greater modulus, is

(a) $\frac{3-5i}{2}$ (b) $\frac{5-3i}{2}$

(c) $\frac{3+i}{2}$ (d) $\frac{3i+1}{2}$

2. If the equation $ax^2 + 2bx - 3c = 0$ has non-real roots and $(3c/4) < (a+b)$, then c is

(a) < 0 (b) > 0 (c) ≥ 0 (d) $= 0$

3. If α and β are the roots of $ax^2 + bx + c = 0$, then the equation $ax^2 - bx(x-1) + c(x-1)^2 = 0$ has roots

(a) $\frac{\alpha}{1-\alpha}, \frac{\beta}{1-\beta}$ (b) $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$

(c) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (d) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

4. If α and β are the roots of the equation $ax^2 + bx + c = 0$ such that $\beta < \alpha < 0$, then the quadratic equation whose roots are $|\alpha|, |\beta|$, is given by

(a) $|a|x^2 + |b|x + |c| = 0$ (b) $ax^2 - |b|x + c = 0$

(c) $|a|x^2 - |b|x + |c| = 0$ (d) $a|x|^2 + b|x| + c = 0$

5. The value of a for which the equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root, is

(a) -2 (b) -1

(c) 1 (d) 2

6. The number of real solutions of the equation

$$\left(\frac{9}{10}\right)^x = -3 + x - x^2 \text{ is}$$

(a) none (b) one

(c) two (d) more than two

7. Suppose the quadratic equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ are such that p, q, r and s are real and $pr = 2(q+s)$. Then,

(a) both the equations always have real roots

(b) atleast one equation always has real roots

(c) both the equations always have non-real roots

(d) atleast one equation always has real and equal roots

8. If α and β be the roots of $x^2 + px + q = 0$, then $\dfrac{(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)}{\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}}$ is equal to

(a) $-\dfrac{q}{p}$ (b) $\alpha\beta$ (c) $-\dfrac{p}{q}$ (d) ω

(ω and ω^2 are complex cube roots of unity)

9. In writing an equation of the form $ax^2 + bx + c = 0$; the coefficient of x is written incorrectly and roots are found to be equal. Again, in writing the same equation the constant term is written incorrectly and it is found that one root is equal to those of the previous wrong equation while the other is double of it. If α and β be the roots of correct equation, then $(\alpha - \beta)^2$ is equal to

(a) 5 (b) $5\alpha\beta$
(c) $-4\alpha\beta$ (d) -4

10. If the equation $ax^2 + 2bx + 3c = 0$ and $3x^2 + 8x + 15 = 0$ have a common root, where a, b and c are the lengths of the sides of a ΔABC, then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(a) 1 (b) $\dfrac{3}{2}$ (c) $\sqrt{2}$ (d) 2

11. If $x = 2 + 2^{2/3} + 2^{1/3}$, then the value of $x^3 - 6x^2 + 6x$ is

(a) 3 (b) 2 (c) 1 (d) -2

12. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2. If $p(x)$ leaves remainders a and $-a$ when divided, respectively, by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$ is

(a) $2x$ (b) $-2x$
(c) x (d) $-x$

13. If the roots of the equation $ax^2 - bx + c = 0$ are α and β, then the roots of the equation $b^2cx^2 - ab^2x + a^3 = 0$ are

(a) $\dfrac{1}{\alpha^3 + \alpha\beta}, \dfrac{1}{\beta^3 + \alpha\beta}$ (b) $\dfrac{1}{\alpha^2 + \alpha\beta}, \dfrac{1}{\beta^2 + \alpha\beta}$

(c) $\dfrac{1}{\alpha^4 + \alpha\beta}, \dfrac{1}{\beta^4 + \alpha\beta}$ (d) None of these

14. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots, then $(a + b + 1)$ is

(a) positive
(b) negative
(c) zero
(d) dependent on the sign of b

15. If roots of $x^2 - (a - 3)x + a = 0$ are such that atleast one of them is greater than 2, then

(a) $a \in [7, 9]$ (b) $a \in [7, \infty)$
(c) $a \in [9, \infty)$ (d) $a \in [7, 9)$

16. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval

(a) $-2 < m < 0$ (b) $m > 3$
(c) $-1 < m < 3$ (d) $1 < m < 4$

17. If the roots of the quadratic equation $(4p - p^2 - 5)x^2 - (2p - 1)x + 3p = 0$ lie on either side of unity, then the number of integral values of p is

(a) 1 (b) 2
(c) 3 (d) 4

18. The range of a for which the equation $x^2 + ax - 4 = 0$ has its smaller root in the interval $(-1, 2)$ is

(a) $(-\infty, -3)$ (b) $(0, 3)$
(c) $(0, \infty)$ (d) $(-\infty, -3) \cup (0, \infty)$

19. For $x^2 - (a + 3)|x| + 4 = 0$ to have real solutions, then the range of a is

(a) $(-\infty, -7] \cup [1, \infty)$ (b) $(-3, \infty)$
(c) $(-\infty, -7]$ (d) $[1, \infty)$

20. Let α and β be the roots of the equation $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If α, β, γ and δ are in GP, then integral values of p and q are respectively

(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$

More Than One Correct Option

21. The equation $x^2 + a^2x + b^2 = 0$ has two roots each of which exceeds a number c, then

(a) $a^4 > 4b^2$ (b) $c^2 + a^2c + b^2 > 0$

(c) $-\dfrac{a^2}{2} > c$ (d) None of these

22. If $c \neq 0$ and the equation $\dfrac{p}{2x} = \dfrac{a}{x + c} + \dfrac{b}{x - c}$ has two equal roots, then p can be

(a) $(\sqrt{a} - \sqrt{b})^2$ (b) $(\sqrt{a} + \sqrt{b})^2$

(c) $a + b$ (d) $a - b$

23. If α and β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$, then

(a) $h = \dfrac{1}{2}\left(\dfrac{b}{a} - \dfrac{q}{p}\right)$

(b) $\dfrac{b^2 - 4ac}{a^2} = \dfrac{q^2 - 4pr}{p^2}$

(c) $\dfrac{a}{p} = \dfrac{b}{q} = \dfrac{c}{r}$

(d) None of the above

Comprehension Based Questions

Passage

Directions (Q. Nos. 24-26) *Consider the quadratic equation* $ax^2 - bx + c = 0$, $a, b, c \in N$, *which has two distinct real roots belonging to the interval* (1, 2).

24. The least value of a is
(a) 4 (b) 6 (c) 7 (d) 5

25. The least value of b is
(a) 10 (b) 11 (c) 13 (d) 15

26. The least value of c is
(a) 4 (b) 6 (c) 7 (d) 5

Assertion and Reason

Directions (Q. Nos. 27-34) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

27. Let $a + b + c = 0$ and a, b, c are rational.

Statement I Then, the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are rational.

Statement II Discriminant of equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ is a perfect square.

28. **Statement I** If both roots of the equation $4x^2 - 2x + a = 0, a \in R$ lie in the interval $(-1, 1)$, then $-2 < a \le \frac{1}{4}$.

Statement II If $f(x) = 4x^2 - 2x + a$, then $D \ge 0$, $f(-1) > 0$ and $f(1) > 0 \Rightarrow -2 < a \le \frac{1}{4}$.

29. **Statement I** The greatest integral value of λ for which $(2\lambda - 1)x^2 - 4x + (2\lambda - 1) = 0$ has real roots, is 2.

Statement II For real roots of $ax^2 + bx + c = 0, D \ge 0$.

30. **Statement I** Let $f(x)$ be quadratic expression such that $f(0) + f(1) = 0$. If -2 is one of the root of $f(x) = 0$, then other root is 3/5.

Statement II If α and β are the zero's of $f(x) = ax^2 + bx + c$, then sum of zero's $= -b/a$, product of zero's $= c/a$.

31. **Statement I** If equations $ax^2 + bx + c = 0$, $(a, b, c \in R)$ and $2x^2 + 3x + 4 = 0$ have a common root, then $a : b : c = 2 : 3 : 4$.

Statement II Roots of $2x^2 + 3x + 4 = 0$ are imaginary.

32. **Statement I** If the equations $ax^2 + bx + c = 0$ and $x^2 - 3x + 4 = 0$ have exactly one root common, then atleast one of a, b and c is imaginary.

Statement II If a, b and c are not all real, then $ax^2 + bx + c = 0$ can have one root real and one root imaginary.

33. **Statement I** Equation $ix^2 + (i - 1)x - (1/2) - i = 0$ has imaginary roots.

Statement II If $a = i, b = i - 1$ and $c = -(1/5) - i$, then $b^2 - 4ac < 0$.

34. Let $f(x) = -x^2 + (a + 1)x + 5$.

Statement I $f(x)$ is positive for some $\alpha < x < \beta$ and for all $a \in R$.

Statement II $f(x)$ is always positive for all $x \in R$ and for same real a.

Previous Years' Questions

35. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is **[JEE Main 2013]**
(a) 1 : 2 : 3 (b) 3 : 2 : 1
(c) 1 : 3 : 2 (d) 3 : 1 : 2

36. Let α and β be real and z be a complex number. If $z^2 + az + \beta = 0$ has two distinct roots on the line $\text{Re}(z) = 1$, then it is necessary that **[AIEEE 2011]**
(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in [1, \infty)$ (d) $\beta \in (0, 1)$

37. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ is equal to **[AIEEE 2010]**
(a) -2 (b) -1 (c) 1 (d) 2

38. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ **[AIEEE 2009]**
(a) greater than $4ab$
(b) less than $4ab$
(c) greater than $-4ab$
(d) less than $-4ab$

39. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then, the common root is [AIEEE 2008]
(a) 2 (b) 1 (c) 4 (d) 3

40. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is [AIEEE 2007]
(a) $(-3, 3)$ (b) $(-3, \infty)$ (c) $(3, \infty)$ (d) $(-\infty, -3)$

41. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$, then the value of $2 + q - p$ is [AIEEE 2006]
(a) 3 (b) 0 (c) 1 (d) 2

42. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4 lie in the interval [AIEEE 2005]
(a) $m > 3$ (b) $-1 < m < 3$
(c) $1 < m < 4$ (d) $-2 < m < 0$

43. If the cube roots of unity are 1, ω and ω^2, then the roots of the equation $(x - 1)^3 + 8 = 0$, are [AIEEE 2005]
(a) $-1, 1 + 2\omega, 1 + 2\omega^2$ (b) $-1, 1 - 2\omega, 1 - 2\omega^2$
(c) $-1, -1, -1$ (d) $-1, -1 + 2\omega, -1 - 2\omega^2$

44. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assume the least value is [AIEEE 2005]
(a) 2 (b) 3 (c) 0 (d) 1

45. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals [AIEEE 2005]
(a) 1 (b) 2 (c) 3 (d) -2

46. If both the roots of the quadratic equation $x^2 - 2kx + k^2 - 5 = 0$ are less than 5, than k lies in the interval [AIEEE 2005]
(a) $[4, 5]$ (b) $(-\infty, 4)$ (c) $(6, \infty)$ (d) $(5, 6]$

47. If the equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is [AIEEE 2004]
(a) equal to α (b) greater than or equal to α
(c) smaller than α (d) greater than α

48. If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are [AIEEE 2004]
(a) 0, 1 (b) $-1, 1$
(c) $0, -1$ (d) $-1, 2$

49. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of q is [AIEEE 2004]
(a) $\frac{49}{4}$ (b) 12
(c) 3 (d) 4

50. Let two numbers have arithmetic mean 9 and geometric mean 4. Then, these numbers are the roots of the quadratic equation [AIEEE 2004]
(a) $x^2 + 18x + 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$ (d) $x^2 - 18x - 16 = 0$

51. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in [AIEEE 2003]
(a) arithmetic progression
(b) geometric progression
(c) harmonic progression
(d) arithmetico-geometric progression

52. The number of the real solutions of the equation $x^2 - 3|x| + 2 = 0$ is [AIEEE 2003]
(a) 2 (b) 4
(c) 1 (d) 3

53. The value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is [AIEEE 2003]
(a) 2/3 (b) $-2/3$
(c) 1/3 (d) $-1/3$

54. If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3, \beta^2 = 5\beta - 3$, then the equation having α/β and β/α as its roots, is [AIEEE 2002]
(a) $3x^2 + 19x + 3 = 0$ (b) $3x^2 - 19x + 3 = 0$
(c) $3x^2 - 19x - 3 = 0$ (d) $x^2 - 16x + 1 = 0$

Answers

Check Point 1

1. (a)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (a)	**9.** (c)

Check Point 2

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (b)

Check Point 3

1. (a)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (c)	**8.** (a)

Round I

1. (d)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (c)
11. (b)	**12.** (d)	**13.** (d)	**14.** (d)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (a)
21. (c)	**22.** (a)	**23.** (b)	**24.** (b)	**25.** (d)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (d)
31. (a)	**32.** (b)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (d)	**39.** (b)	**40.** (b)
41. (a)	**42.** (b)	**43.** (a)	**44.** (b)	**45.** (c)	**46.** (d)	**47.** (c)	**48.** (a)	**49.** (a)	**50.** (c)
51. (b)	**52.** (b)	**53.** (a)	**54.** (b)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (b)	**59.** (c)	**60.** (a)
61. (c)	**62.** (c)	**63.** (a)	**64.** (c)	**65.** (a)	**66.** (b)	**67.** (d)	**68.** (a)	**69.** (d)	**70.** (d)
71. (a)	**72.** (b)	**73.** (b)	**74.** (b)	**75.** (c)	**76.** (d)	**77.** (b)	**78.** (c)	**79.** (c)	**80.** (c)
81. (c)	**82.** (a)	**83.** (c)	**84.** (a)	**85.** (c)	**86.** (a)	**87.** (a)	**88.** (b)	**89.** (b)	**90.** (a)
91. (c)	**92.** (b)	**93.** (d)	**94.** (c)	**95.** (c)	**96.** (a)	**97.** (c)	**98.** (d)	**99.** (d)	**100.** (d)

Round II

1. (a)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (d)
11. (b)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (d)	**20.** (a)
21. (a,b,c)	**22.** (a,b)	**23.** (a,b)	**24.** (d)	**25.** (b)	**26.** (b)	**27.** (a)	**28.** (a)	**29.** (d)	**30.** (a)
31. (b)	**32.** (a)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (c)	**37.** (c)	**38.** (c)	**39.** (a)	**40.** (a)
41. (a)	**42.** (b)	**43.** (b)	**44.** (d)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (c)	**49.** (a)	**50.** (b)
51. (c)	**52.** (b)	**53.** (a)	**54.** (b)						

the Guidance

Round I

1. Since, $(x-2)$ is a common factor of the expressions $x^2 + ax + b$ and $x^2 + cx + d$

$\Rightarrow \quad 4 + 2a + b = 0 \quad \ldots(i)$

and $\quad 4 + 2c + d = 0 \quad \ldots(ii)$

$\Rightarrow \quad 2a + b = 2c + d$

$\Rightarrow \quad b - d = 2(c - a)$

$\Rightarrow \quad \dfrac{b-d}{c-a} = 2$

2. $x^2 + ax + 1$ must divide $ax^3 + bx + c$.

Now, $\dfrac{ax^3 + bx + c}{x^2 + ax + 1} = \dfrac{a(x-a) + (b - a + a^3)x + c + a^2}{x^2 + ax + 1}$

Remainder must be zero

$\Rightarrow \quad b - a + a^3 = 0, \quad a^2 + c = 0$

3. Given that, $x^2 - 3x + 2$ be a factor of $x^4 - px^2 + q = 0 \quad \ldots(i)$

$\Rightarrow \quad (x^2 - 3x + 2) = 0$

$\Rightarrow \quad (x-2)(x-1) = 0 \Rightarrow x = 2, 1$

On putting these values in Eq. (i), we get

$4p - q - 16 = 0 \quad \ldots(ii)$

and $\quad p - q - 1 = 0 \quad \ldots(iii)$

On solving Eqs. (ii) and (iii), we get

$p = 5$ and $q = 4$

$\Rightarrow \quad (p, q) = (5, 4)$

4. Given that, $x^2 + px + 1$ is a factor of $ax^3 + bx + c = 0$, then let $ax^3 + bx + c \equiv (x^2 + px + 1)(ax + \lambda)$, where λ is a constant. Then, equating the coefficients of like powers of x on both sides, we get

$0 = ap + \lambda, b = p\lambda + a, c = \lambda$

$\Rightarrow \quad p = -\frac{\lambda}{a} = -\frac{c}{a}$

Hence, $\quad b = \left(-\frac{c}{a}\right)c + a$

$\Rightarrow \quad ab = a^2 - c^2$

5. Given equation is $2x^2 + x + 1 = 0$.

On comparing the given equation with $ax^2 + bx + c = 0$, we get

$a = 2, b = 1, c = 1$

Now, $\quad D = b^2 - 4ac$

$= (1)^2 - 4 \times 2 \times 1$

$= 1 - 8 = -7 < 0$

$\because \quad x = \frac{-b \pm \sqrt{D}}{2a}$

$\therefore \quad x = \frac{-1 \pm \sqrt{-7}}{2(2)} = \frac{-1 \pm \sqrt{7}i}{4}$

6. Let another root of equation

$x^2 + (1 - 3i)x - 2(1 + i) = 0$. is α

$\therefore \quad \alpha + (-1 + i) = -(1 - 3i)$

$\Rightarrow \quad \alpha = 2i$

7. Let α and β be the roots of the equation $x^2 - Px + 8 = 0$

$\therefore \quad \alpha + \beta = P$ and $\alpha \cdot \beta = 8$

Now, $\quad \alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$\therefore \quad 2 = \pm\sqrt{P^2 - 32} \qquad (\because \alpha - \beta = 2$ given)

$\Rightarrow \quad P^2 - 32 = 4 \Rightarrow P^2 = 6$

$\Rightarrow \quad P = \pm 6$

8. Let α and β be the roots of the given equation

$x^2 - (a - 2)x - (a + 1) = 0$

$\therefore \quad \alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

Now, $\quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a - 2)^2 + 2(a + 1)$

$= a^2 + 4 - 4a + 2a + 2$

$= a^2 - 2a + 6$

$= (a - 1)^2 + 5$

$\therefore \alpha^2 + \beta^2$ will be minimum, if $(a - 1)^2 = 0$ *i.e.*, $a = 1$.

9. We have, $\quad x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots \infty}}}$

$\Rightarrow \quad x = \sqrt{1 + x}$

$\Rightarrow \quad x^2 = 1 + x$

$\Rightarrow \quad x^2 - x - 1 = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

As $x > 0$, we take only $x = \frac{1 + \sqrt{5}}{2}$.

10. We have,

$\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$

$\Rightarrow \quad \sqrt{3x^2 - 7x - 30} = (x + 5) - \sqrt{2x^2 - 7x - 5}$

On squaring both sides, we get

$3x^2 - 7x - 30 = x^2 + 25 + 10x + (2x^2 - 7x - 5)$

$- 2(x + 5)\sqrt{2x^2 - 7x - 5}$

$\Rightarrow \quad \sqrt{2x^2 - 7x - 5} = 5$

Again on squaring both sides, we get

$2x^2 - 7x - 30 = 0$

$\Rightarrow \quad x = 6$

11. Let $\quad x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots \infty}}$

$\Rightarrow \quad x = 2 + \frac{1}{x}$

$\Rightarrow \quad x^2 - 2x - 1 = 0$

$\Rightarrow \quad x = \frac{2 \pm \sqrt{4 + 4}}{2}$

$\Rightarrow \quad x = 1 \pm \sqrt{2}$

But the value of the given expression cannot be negative or less than 2, therefore $1 + \sqrt{2}$ is required answer.

12. Given equation is

$|2x - 1|^2 - 3|2x - 1| + 2 = 0$

Let $\quad |2x - 1| = t$

$\therefore \quad t^2 - 3t + 2 = 0$

$\Rightarrow \quad (t - 1)(t - 2) = 0 \quad \Rightarrow \quad t = 1, 2$

$\Rightarrow \quad |2x - 1| = 1$ and $|2x - 1| = 2$

$\Rightarrow \quad 2x - 1 = \pm 1$ and $2x - 1 = \pm 2$

$\Rightarrow \quad x = 1, 0$ and $x = \frac{3}{2}, -\frac{1}{2}$

13. Given equation is $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$, then given equation can be written as

$y^2 - 4y - 1 = 0 \quad \Rightarrow \quad y = 2 \pm \sqrt{5}$

But the value of $y = e^{\sin x}$ is always positive, so we take only $y = 2 + \sqrt{5}$

$\Rightarrow \quad \log_e y = \log_e(2 + \sqrt{5})$

$\Rightarrow \quad \sin x = \log_e(2 + \sqrt{5}) > 1$

Which is impossible since $\sin x$ cannot be greater than 1.

Hence, we cannot find any real value of x which satisfies the given equation.

14. We have, $x - \frac{2}{x - 1} = 1 - \frac{2}{x - 1}$. If $x \neq 1$ multiplying each term by $(x - 1)$, the given equation reduces to $x(x - 1) = (x - 1)$ or $(x - 1)^2 = 0$ or $x = 1$ which is not possible as considering $x \neq 1$.

Thus, given equation has no root.

15. We have,

$$|x^2 + 4x + 3| + 2x + 5 = 0$$

Here, two cases arise.

Case I When $x^2 + 4x + 3 > 0$

$\Rightarrow \quad x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow \quad x^2 + 6x + 8 = 0$

$\Rightarrow \quad (x + 2)(x + 4) = 0$

$\Rightarrow \quad x = -2, -4$

$x = -2$ is not satisfying the condition $x^2 + 4x + 3 > 0$. So, $x = -4$ is the only solution of the given equation.

Case II When $x^2 + 4x + 3 < 0$

$\Rightarrow \quad -(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow \quad -x^2 - 2x + 2 = 0$

$\Rightarrow \quad x^2 + 2x - 2 = 0$

$\Rightarrow \quad (x + 1 + \sqrt{3})(x + 1 - \sqrt{3}) = 0$

$\Rightarrow \quad x = -1 + \sqrt{3}, -1 - \sqrt{3}$

Hence, $x = -(1 + \sqrt{3})$ satisfy the given condition.

Since, $x^2 + 4x + 3 < 0$ while $x = -1 + \sqrt{3}$ is not satisfying the condition. Thus, number of real solutions are two.

16. $\because x^4 - 8x^2 - 9 = 0$

$\Rightarrow \quad x^4 - 9x^2 + x^2 - 9 = 0$

$\Rightarrow \quad x^2(x^2 - 9) + 1(x^2 - 9) = 0$

$\Rightarrow \quad (x^2 + 1)(x^2 - 9) = 0$

$\Rightarrow \quad x = \pm i, \pm 3$

17. Since, the roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in GP. Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta, \beta \neq 0$. Then, the product of roots is $\alpha^3 = \frac{1}{8} \Rightarrow \alpha = \frac{1}{2}$ and hence, $\beta = \frac{1}{2}$.

So, roots are $1, \frac{1}{2}, \frac{1}{4}$.

18. Since, $2 + i\sqrt{3}$ is a root of equation $x^2 + px + q = 0$. Therefore, $2 - i\sqrt{3}$ will be other root.

Now, sum of the roots $= (2 + i\sqrt{3}) + (2 - i\sqrt{3}) = -p$

$\Rightarrow \quad 4 = -p$

Product of roots $= (2 + i\sqrt{3})(2 - i\sqrt{3}) = q$

$\Rightarrow \quad 7 = q$

Hence, $\quad (p, q) = (-4, 7)$

19. Given that, $3^{2x^2 - 7x + 7} = 3^2 \Rightarrow 2x^2 - 7x + 7 = 2$

$\Rightarrow \quad 2x^2 - 7x + 5 = 0$

Now, $\quad D = b^2 - 4ac$

$= (-7)^2 - 4 \times 2 \times 5$

$= 49 - 40 = 9 > 0$

Hence, it has two real roots.

20. Given equation can be reduced to a quadratic equation.

$\therefore \quad 2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$

$\Rightarrow \quad 2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0$

Put $x + \frac{1}{x} = y$

$2(y^2 - 2) + y - 11 = 0$

$\Rightarrow \quad 2y^2 + y - 15 = 0$

$\Rightarrow \quad y = -3$ and $\frac{5}{2}$

$\Rightarrow \quad x + \frac{1}{x} = -3, \; x + \frac{1}{x} = \frac{5}{2}$

$\Rightarrow \quad x^2 + 3x + 1 = 0, 2x^2 - 5x + 2 = 0$

Only 2nd equation has rational roots as $D = 9$ and roots are $\frac{1}{2}$ and 2.

21. Given equation is

$(p - q)x^2 + (q - r)x + (r - p) = 0$

$\Rightarrow \quad x = \frac{(r - q) \pm \sqrt{(q - r)^2 - 4(r - p)(p - q)}}{2(p - q)}$

$= \frac{(r - q) \pm \sqrt{q^2 + r^2 - 2qr - 4(rp - rq - p^2 + pq)}}{2(p - q)}$

$\Rightarrow \quad x = \frac{(r - q) \pm (q + r - 2p)}{2(p - q)}$

$\Rightarrow \quad x = \frac{r - p}{p - q}, 1$

22. $\because 2\sin^2\frac{\pi}{8} = 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$ (irrational root)

So, other root is $\frac{\sqrt{2} + 1}{\sqrt{2}}$.

Sum of roots $= -a = 1 - \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 2 \Rightarrow a = -2$

Product of roots $= 1 - \frac{1}{2} = \frac{1}{2} = b$

So, $\quad a - b = -2 - \frac{1}{2} = -\frac{5}{2}$

23. Since, $\tan\alpha$ and $\tan\beta$ are the roots of the equation $x^2 + ax + b = 0$, then

$$\tan\alpha + \tan\beta = -\frac{a}{1}$$

and $\quad \tan\alpha \cdot \tan\beta = b$

$\Rightarrow \quad \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = -\frac{a}{1}$

and $$\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = b$$

$\therefore\ \sin^2(\alpha+\beta) + a\sin(\alpha+\beta)\cos(\alpha+\beta) + b\cos^2(\alpha+\beta)$

$$= \cos^2(\alpha+\beta)\,[\tan^2(\alpha+\beta) + b + a\tan(\alpha+\beta)]$$

$$= \frac{\tan^2(\alpha+\beta) + b + a\tan(\alpha+\beta)}{1+\tan^2(\alpha+\beta)}$$

$$= \frac{\frac{a}{b-1}\left(a + \frac{a}{b-1}\right) + b}{1 + \frac{a^2}{(b-1)^2}} = b$$

24. Let the equation (incorrectly written form) be

$$x^2 + 17x + q = 0$$

Since, roots are $-2, -15$.

$\therefore\quad q = 30$

So, correct equation is $x^2 + 13x + 30 = 0$

$\Rightarrow\quad x^2 + 10x + 3x + 30 = 0$

$\Rightarrow\quad (x+3)(x+10) = 0$

$\Rightarrow\quad x = -3, -10$

25. Let the correct equation be $ax^2 + bx + c = 0$ and the correct roots are α and β. Taking c wrong, the roots are 3 and 2.

$\therefore\quad \alpha + \beta = 3 + 2 = 5$ ...(i)

Also, $a = 1$ and $c = -6$

$\therefore\quad \alpha\beta = \frac{c}{a} = -6$ ...(ii)

On solving Eqs. (i) and (ii), the correct roots are 6 and -1.

26. Here, $\alpha + \beta + \gamma = -2$ ...(i)

$\alpha\beta + \beta\gamma + \gamma\alpha = -3$ ...(ii)

and $\alpha\beta\gamma = 1$ ...(iii)

On squaring Eq. (ii), we get

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma) = 9$$

$\Rightarrow\quad \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 9 - 2(1)(-2) = 13$

Now, $\alpha^{-2} + \beta^{-2} + r^{-2} = \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2}{(\alpha\beta\gamma)^2} = \frac{13}{1} = 13$

27. Given equation is $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$

Since, roots are real, its discriminant, $D \ge 0$

$\therefore\quad \cos^2 p - 4(\cos p - 1)\sin p \ge 0$

$\Rightarrow\quad \cos^2 p - 4\cos p\sin p + 4\sin p \ge 0$

$\Rightarrow\quad (\cos p - 2\sin p)^2 - 4\sin^2 p + 4\sin p \ge 0$

$\Rightarrow\quad (\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \ge 0$ ...(i)

Now, $(1 - \sin p) \ge 0$ for all real p and $\sin p > 0$ for $0 < p < \pi$.

Therefore, $4\sin p(1 - \sin p) \ge 0$ when $0 < p < \pi$ or $p \in (0, \pi)$.

28. Let all four roots are imaginary. Then, roots of both equation $P(x) = 0$ and $Q(x) = 0$ are imaginary.

Thus, $b^2 - 4ac < 0$; $d^2 - 4ac < 0$, so $b^2 + d^2 < 0$ which is impossible unless $b = 0, d = 0$.

So, if $b \ne 0$ or $d \ne 0$ atleast two roots must be real, if $b = 0$, $d = 0$, we have the equations

$$P(x) = ax^2 + c = 0$$

and $$Q(x) = -ax^2 + c = 0$$

or $x^2 = -\frac{c}{a}$; $x^2 = \frac{c}{a}$ as one of $\frac{c}{a}$ and $-\frac{c}{a}$ must be positive, so two roots must be real.

29. Since, roots are real.

$\therefore\ \{2(bc + ad)\}^2 = 4(a^2 + b^2)(c^2 + d^2)$

$\Rightarrow 4b^2c^2 + 4a^2d^2 + 8abcd = 4a^2c^2 + 4a^2d^2 + 4b^2c^2 + 4b^2d^2$

$\Rightarrow\quad 4a^2d^2 + 4b^2c^2 - 8abcd = 0$

$\Rightarrow\quad 4(ad - bc)^2 = 0$

$\Rightarrow\quad ad = bc$

$\Rightarrow\quad \frac{a}{b} = \frac{c}{d}$

30. Since, $f(1) + f(2) + f(3) = 0$

$f(1), f(2), f(3)$ all cannot be of same sign.

$\Rightarrow$ Roots are real and distinct.

31. Since, $\sin\alpha$, $\sin\beta$ and $\cos\alpha$ are in GP, then

$\sin^2\beta = \sin\alpha\cos\alpha$ ...(i)

Given equation is $x^2 + 2x\cot\beta + 1 = 0$.

$\therefore$ Discriminant, $D = b^2 - 4ac$

$= (2\cot\beta)^2 - 4 = 4(\text{cosec}^2\beta - 2)$

$= 4(\text{cosec}\,\alpha\sec\alpha - 2)$ [from Eq. (i)]

$= 4(2\,\text{cosec}\,2\alpha - 2) \ge 0$

$\therefore$ Roots are real.

32. Since, 4 is a root of $x^2 + ax + 12 = 0$

$\therefore\quad 16 + 4a + 12 = 0$

$\Rightarrow\quad a = -7$

Let the roots of the equation $x^2 + ax + b = 0$ be α and α.

$\therefore\quad 2\alpha = -a$

$\Rightarrow\quad \alpha = \frac{7}{2}$

and $\alpha\cdot\alpha = b \Rightarrow \left(\frac{7}{2}\right)^2 = b$

$\Rightarrow\quad b = \frac{49}{4}$

33. The given equation is

$$pqx^2 - (p+q)^2x + (p+q)^2 = 0.$$

$$x = \frac{(p+q)^2 \pm \sqrt{(p+q)^4 - 4pq(p+q)^2}}{2pq}$$

$$x = \frac{(p+q)^2 \pm (p^2 - q^2)}{2pq}$$

Now, taking (+ ve) sign

$$x = \frac{p+q}{q}$$

and taking (– ve) sign

$$x = \frac{p+q}{p}$$

$\therefore$ Solution set is $\left\{\frac{p+q}{p}, \frac{p+q}{q}\right\}$.

34. Since, $\sin\alpha$ and $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$, then

$$\sin\alpha + \cos\alpha = -\frac{b}{a} \quad \text{and} \quad \sin\alpha\cos\alpha = \frac{c}{a}$$

To eliminate α, we have

$$1 = \sin^2\alpha + \cos^2\alpha$$

$$\Rightarrow \quad 1 = (\sin\alpha + \cos\alpha)^2 - 2\sin\alpha\cos\alpha$$

$$\Rightarrow \quad 1 = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$\Rightarrow \quad a^2 - b^2 + 2ac = 0$$

35. $\because$

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \quad \text{...(i)}$$

and

$$\frac{k+1}{k} \cdot \frac{k+2}{k+1} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{k+2}{k} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{2}{k} = \frac{c}{a} - 1 = \frac{c-a}{a}$$

$$\Rightarrow \quad k = \frac{2a}{c-a}$$

On putting the value of k in the Eq. (i), we get

$$\frac{c+a}{2a} + \frac{2c}{c+a} = -\frac{b}{a}$$

$$\Rightarrow \quad (c+a)^2 + 4ac = -2b(a+c)$$

$$\Rightarrow \quad (a+b+c)^2 = b^2 - 4ac$$

36. Let α be a root of equation $ax^2 + bx + c = 0$, then $\frac{1}{\alpha}$ be a root of second equation, therefore

$$a\alpha^2 + b\alpha + c = 0 \quad \text{...(i)}$$

and

$$a'\frac{1}{\alpha^2} + b'\frac{1}{\alpha} + c' = 0$$

$$\Rightarrow \quad c'\alpha^2 + b'\alpha + a' = 0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\frac{\alpha^2}{ba' - b'c} = \frac{\alpha}{cc' - aa'} = \frac{1}{ab' - bc'}$$

$$\Rightarrow \quad (cc' - aa')^2 = (ba' - cb')(ab' - bc')$$

37. We have,

$$(a+1)x^2 + (2a+3)x + (3a+4) = 0$$

Let α and β be the roots of the equation.

According to the given condition,

$$\alpha\beta = 2$$

$$\Rightarrow \quad \frac{3a+4}{a+1} = 2$$

$$\Rightarrow \quad 3a + 4 = 2a + 2 \Rightarrow a = -2$$

Also,

$$\alpha + \beta = -\frac{2a+3}{a+1} = -\frac{-4+3}{-2+1} = -1$$

38. $y^{x^2+7x+12} = 1$

$$\Rightarrow \quad x^2 + 7x + 12 = 0$$

$$\Rightarrow \quad x = -3, -4$$

$$\Rightarrow \quad y = 9, 10 \quad \text{(when } y \neq 1\text{)}$$

Again when $y = 1, x = 5$.

$\therefore$ Solutions are $(-3, 9)$, $(-4, 10)$, $(5, 1)$.

39. $[x]^2 - [x] - 2 = 0$

$$\Rightarrow \quad ([x] - 2)([x] + 1) = 0$$

$$\Rightarrow \quad [x] = 2, -1$$

$$\Rightarrow \quad x \in [-1, 0) \cup [2, 3)$$

40. Here, $\Sigma\alpha = 0, \Sigma\alpha\beta = -7, \alpha\beta\gamma = -7$

$$\therefore \quad \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{\alpha^4\beta^4 + \beta^4\gamma^4 + \gamma^4\alpha^4}{\alpha^4\beta^4\gamma^4}$$

$$= \frac{\Sigma\alpha^4\beta^4}{\alpha^4\beta^4\gamma^4} \quad \text{...(i)}$$

Now, $\Sigma\alpha\beta\,\Sigma\alpha\beta\,\Sigma\alpha\beta\,\Sigma\alpha\beta = (\Sigma\alpha\beta)^2(\Sigma\alpha\beta)^2$

$$\Rightarrow \quad (-7)^4 = [\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)]$$

$$[\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)]$$

$$= (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$$

$$(\because \Sigma\alpha = \alpha + \beta + \gamma = 0)$$

$$= \alpha^4\beta^4 + \beta^4\gamma^4 + \gamma^4\alpha^4 + 2\alpha^4\beta^2\gamma^2 + 2\alpha^2\beta^4\gamma^2 + 2\alpha^2\beta^2\gamma^4$$

$$= \Sigma\alpha^4\beta^4 + 2\alpha^2\beta^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)$$

$$= \Sigma\alpha^4\beta^4 + 2\alpha^2\beta^2\gamma^2[(\Sigma\alpha)^2 - 2\Sigma\alpha\beta]$$

$$= \Sigma\alpha^4\beta^4 + 2\alpha^2\beta^2\gamma^2[0 - 2 \times (-7)]$$

$$= \Sigma\alpha^4\beta^4 + 2(-7)^2(2 \times 7)$$

$$\Rightarrow \quad \Sigma\alpha^4\beta^4 = (-7)^4 + 4(-7)^3$$

$$\Rightarrow \quad \Sigma\alpha^4\beta^4 = (-7)^3(-7 + 4) = -3(-7)^3$$

On putting this value in Eq. (i), we get

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{-3(-7)^3}{(-7)^4} = \frac{-3}{-7} = \frac{3}{7}$$

41. Given equation $\frac{\alpha}{x-\alpha} + \frac{\beta}{x-\beta} = 1$ can be rewritten as

$$x^2 - 2(\alpha + \beta)x + 3\alpha\beta = 0$$

Let its roots be α' and $-\alpha'$.

$$\Rightarrow \quad \alpha' + (-\alpha') = 2(\alpha + \beta)$$

$$\Rightarrow \quad 0 = 2(\alpha + \beta)$$

$$\Rightarrow \quad \alpha + \beta = 0$$

42. Given equation $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$ can be rewritten as

$$x^2+x(p+q-2r)+pq-pr-qr=0 \quad \text{...(i)}$$

Let roots are α and $-\alpha$, then the product of roots

$$-\alpha^2=pq-pr-qr=pq-r(p+q) \quad \text{...(ii)}$$

and sum of roots, $0=-(p+q-2r)$

$$\Rightarrow \quad r=\frac{p+q}{2} \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$-\alpha^2=pq-\frac{p+q}{2}(p+q)$$

$$=-\frac{1}{2}\{(p+q)^2-2pq\}$$

$$\Rightarrow \quad \alpha^2=-\frac{(p^2+q^2)}{2}$$

43. Given equation is $x^2+ax+1=0$.

Since, roots are $\tan\theta$ and $\cot\theta$.

$\therefore$ Product of roots, $\tan\theta\cdot\cot\theta=a\Rightarrow a=1$

Again, since roots are real.

$$\therefore \quad a^2-4\geq 0 \quad \Rightarrow \quad |a|\geq 2$$

Thus, the least value of $|a|$ is 2.

44. Given equation is $(5+\sqrt{2})x^2-(4+\sqrt{5})x+8+2\sqrt{5}=0$.

Let x_1 and x_2 are the roots of the equation.

$$\Rightarrow \quad x_1+x_2=\frac{4+\sqrt{5}}{5+\sqrt{2}} \quad \text{...(i)}$$

and $x_1x_2=\frac{8+2\sqrt{5}}{5+\sqrt{2}}=\frac{2(4+\sqrt{5})}{5+\sqrt{2}}=2(x_1+x_2)$ ...(ii)

$\therefore$ Harmonic mean $=\frac{2x_1x_2}{x_1+x_2}=\frac{4(x_1+x_2)}{(x_1+x_2)}=4$ [from Eq. (ii)]

45. Given, $a+b+c=0$ ...(i)

and $4ax^2+3bx+2c=0$

$\because \quad D=b^2-4ac$

$\therefore \quad D=(3b)^2-4(4a)(2c)$

$=9b^2-32ac$

$=9(a+c)^2-32ac$ [$\because$ from Eq. (i)]

$=9(a-c)^2+4ac>0$

Hence, roots are real.

46. Since, quadratic equation $ax^2+bx+c=0$ has three distinct roots. So, it must be an identity. So, $a=b=c=0$.

47. Given expression $x^2+2x+2xy+my-3=0$ can be rewritten as $x^2+2x(1+y)+(my-3)=0$.

But factors are rational so discriminant b^2-4ac is a perfect square.

Now, $b^2-4ac=4\{(1+y)^2-(my-3)\}\geq 0$

$\Rightarrow \quad 4\{y^2+1+2y-my+3\}\geq 0$

$\Rightarrow \quad y^2+2y-my+4\geq 0$

Hence, $2y-my=\pm 4y$ (as it is perfect square).

$\Rightarrow \quad 2y-my=4y$

$\Rightarrow \quad m=-2$

Now, taking (–)ve sign, we get $m=6$

48. The given equations are

$$qx^2+px+q=0 \quad \text{...(i)}$$

and $x^2-4qx+p^2=0$ ...(ii)

Since, root of the Eq. (i) are complex, therefore

$$p^2-4q^2<0$$

Now, discriminant of Eq. (ii) is

$$16q^2-4p^2=-4(p^2-4q^2)>0$$

Hence, roots are real and unequal.

49. Given equation can be rewritten as

$$3x^2-(a+c+2b+2d)x+ac+2bd=0$$

$\therefore$ Discriminant, D

$=(a+c+2b+2d)^2-4\cdot 3\,(ac+2bd)$

$=\{(a+2d)+(c+2b)\}^2-12(ac+2bd)$

$=\{(a+2d)-(c+2b)\}^2+4(a+2d)(c+2b)-12\,(ac+2bd)$

$=\{(a+2d)-(c+2b)\}^2-8ac+8ab+8dc-8bd$

$=\{(a+2d)-(c+2b)\}^2+8\,(c-b)(d-a)$

Which is +ve, since $a<b<c<d$.

Hence, roots are real and distinct.

50. Since, α,β are the roots of the equation $x^2+px+q=0$, then

$$\alpha+\beta=p, \quad \alpha\beta=q \quad \text{...(i)}$$

and α^4,β^4 are the roots of $x^2-xr+s=0$.

Then, $\alpha^4+\beta^4=r$ ...(ii)

and $\alpha^4\beta^4=s$

If D is discriminant of the equation $x^2-4qx+2q^2-r=0$,

then $D=16q^2-4\,(2q^2-r)=8q^2+4r$

$=8\alpha^2\beta^2+4(\alpha^4+\beta^4)$ [from Eqs. (i) and (ii)]

$=4\,(\alpha^2+\beta^2)^2\geq 0$

Hence, the equation $x^2-4qx+2q^2-r=0$ has always two real roots.

51. The roots of the equation $ax^2+bx+c=0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(i) Let $b^2-4ac>0$, $b>0$

Now, if $a>0$, $c>0$, $b^2-4ac<b^2$

$\Rightarrow$ The roots are negative.

(ii) Let $b^2 - 4ac < 0$, then the roots are given by

$$x = \frac{-b \pm i\sqrt{(4ac - b^2)}}{2a} \qquad (i = \sqrt{-1})$$

Which are imaginary and have negative part. $(\because b > 0)$

$\therefore$ In each case the root have negative real part.

52. Given, $(ax^2 + c)\,y + (a'x^2 + c') = 0$

or $x^2(ay + a') + (cy + c') = 0$

Since, x is rational, then the discriminant of the above equation must be a perfect square.

$\therefore \quad 0 - 4(ay + a')(cy + c') = 0$

$\Rightarrow -acy^2 - (ac' + a'c)\,y - a'c'$ must be a perfect square

$\Rightarrow \quad (ac' - a'c)^2 - 4aca'c' = 0$

$\Rightarrow \quad (ac' - a'c)^2 = 0$

$\Rightarrow \quad ac' = a'c$

$\Rightarrow \quad \dfrac{a}{a'} = \dfrac{c}{c'}$

53. Since, roots of the equation

$(a - b)x^2 + (c - a)x + (b - c) = 0$ are equal.

$\therefore$ Discriminant, $B^2 - 4AC = 0$

$\Rightarrow \quad (c - a)^2 - 4(a - b)(b - c) = 0$

$\Rightarrow \quad a^2 + 4b^2 + c^2 + 2ac - 4ab - 4bc = 0$

$\Rightarrow \quad (a + c - 2b)^2 = 0$

$\Rightarrow \quad a + c = 2b$

Hence, a, b, c are in AP.

54. Given equation is

$$(p^2 + q^2)x^2 - 2q(p + r)x + (q^2 + r^2) = 0.$$

Since, roots are real and equal, then

$$b^2 - 4ac = 0$$

$\Rightarrow \quad 4q^2(p + r)^2 - 4(p^2 + q^2)(q^2 + r^2) = 0$

$\Rightarrow \quad q^2(p^2 + r^2 + 2pr) - (p^2q^2 + p^2r^2 + q^4 + q^2r^2) = 0$

$\Rightarrow \quad q^2p^2 + q^2r^2 + 2pq^2r - p^2q^2 - p^2r^2 - q^4 - q^2r^2 = 0$

$\Rightarrow \quad 2pq^2r - p^2r^2 - q^4 = 0$

$\Rightarrow \quad (q^2 - pr)^2 = 0$

$\Rightarrow \quad q^2 = pr$

$\therefore$ p, q and r will be in GP.

55. The equation $x^2 - px + r = 0$ has roots α, β and the equation $x^2 - qx + r = 0$ has roots $\dfrac{\alpha}{2}, 2\beta$.

$\Rightarrow \quad r = \alpha\beta$

and $\alpha + \beta = p$

and $\dfrac{\alpha}{2} + 2\beta = q$

$\Rightarrow \quad \beta = \dfrac{2q - p}{3}$ and $\alpha = \dfrac{2(2p - q)}{3}$

$\Rightarrow \quad \alpha\beta = r = \dfrac{2}{9}(2q - p)(2p - q)$

56. For rational roots $b^2 - 4ac$ must be a perfect square of a rational number and as a, b and c are natural numbers $b^2 - 4ac$ must be a perfect square of an integer.

$$b^2 - 4ac = I^2 \Rightarrow b^2 - I^2 = 4ac$$

$\Rightarrow \quad 4ac = (b - I)(b + I)$

$\Rightarrow \quad ac = \dfrac{b - I}{2} \cdot \dfrac{b + I}{2}$

$b - I, b + I$ are both odd integers or both even integers but ac is an odd integer. So, $b - I$ and $b + I$ must be even integers. b is odd I must be odd. Now, let

$b - I = 2m$, (m is an odd integer)

$b + I = 2n$, (n is an odd integer)

$I = (n - m)$, ($n - m$ is an even integer)

So, contradiction $\Rightarrow b^2 - 4ac$ is not a perfect square. So, all a, b and c cannot be odd integers.

57. The given equation is $x^2 - 2x\cos\phi + 1 = 0$.

$$\therefore \quad x = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2} = \cos\phi \pm i\sin\phi$$

Let $\alpha = \cos\phi + i\sin\phi$, then $\beta = \cos\phi - i\sin\phi$

$\therefore \quad \alpha^n + \beta^n = (\cos\phi + i\sin\phi)^n + (\cos\phi - i\sin\phi)^n$

$= 2\cos n\phi$

and $\alpha^n\beta^n = (\cos n\phi + i\sin n\phi)(\cos n\phi - i\sin n\phi)$

$= \cos^2 n\phi + \sin^2 n\phi = 1$

$\therefore$ Required equation is $x^2 - 2x\cos n\phi + 1 = 0$

58. Since, α and β are the roots of the equation

$2x^2 + 2(a + b)x + a^2 + b^2 = 0$.

$\therefore \quad (\alpha + \beta)^2 = (a + b)^2$ and $\alpha\beta = \dfrac{a^2 + b^2}{2}$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= (a + b)^2 - 4\left(\dfrac{a^2 + b^2}{2}\right)$

$= -(a - b)^2$

Now, the required equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

$$x^2 - \{(\alpha + \beta)^2 + (\alpha - \beta)^2\}x + (\alpha + \beta)^2(\alpha - \beta)^2 = 0$$

$\Rightarrow \quad x^2 - \{(a + b)^2 - (a - b)^2\}\,x - (a + b)^2(a - b)^2 = 0$

$\Rightarrow \quad x^2 - 4abx - (a^2 - b^2)^2 = 0$

59. Since, α, α^2 be the roots of the equation $x^2 + x + 1 = 0$.

$\therefore \quad \alpha + \alpha^2 = -1 \qquad$ ...(i)

and $\alpha^3 = 1 \qquad$ ...(ii)

Now, $\alpha^{31} + \alpha^{62} = \alpha^{31}(1 + \alpha^{31})$

$\Rightarrow$ $\alpha^{31} + \alpha^{62} = \alpha^{30} \cdot \alpha(1 + \alpha^{30} \cdot \alpha)$

$\Rightarrow$ $\alpha^{31} + \alpha^{62} = (\alpha^3)^{10} \cdot \alpha\{1 + (\alpha^3)^{10} \cdot \alpha\}$

$\Rightarrow$ $\alpha^{31} + \alpha^{62} = \alpha(1 + \alpha)$ [from Eq. (ii)]

$\Rightarrow$ $\alpha^{31} + \alpha^{62} = -1$ [from Eq. (i)]

Again, $\alpha^{31} \cdot \alpha^{62} = \alpha^{93}$

$\Rightarrow$ $\alpha^{31} \cdot \alpha^{62} = [\alpha^3]^{31} = 1$

$\therefore$ Required equation is

$$x^2 - (\alpha^{31} + \alpha^{62})x + \alpha^{31} \cdot \alpha^{62} = 0$$

$\Rightarrow$ $x^2 + x + 1 = 0$

60. Since, α and β are the roots of $ax^2 + bx + c = 0$.

$\Rightarrow$ $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$ ...(i)

If roots are $\alpha + \dfrac{1}{\beta}, \beta + \dfrac{1}{\alpha}$, then

sum of roots $= \left(\alpha + \dfrac{1}{\beta}\right) + \left(\beta + \dfrac{1}{\alpha}\right) = (\alpha + \beta) + \dfrac{\alpha + \beta}{\alpha\beta}$

$= \dfrac{-b}{ac}(a + c)$ [from Eq. (i)]

and product of roots $= \left(\alpha + \dfrac{1}{\beta}\right)\left(\beta + \dfrac{1}{\alpha}\right)$

$= \alpha\beta + 1 + 1 + \dfrac{1}{\alpha\beta} = 2 + \dfrac{c}{a} + \dfrac{a}{c}$ [from Eq. (i)]

$= \dfrac{2ac + c^2 + a^2}{ac} = \dfrac{(a + c)^2}{ac}$

Hence, required equation is given by

$x^2 -$ (sum of roots) $x +$ (product of roots) $= 0$

$\Rightarrow$ $x^2 + \dfrac{b}{ac}(a + c)x + \dfrac{(a + c)^2}{ac} = 0$

$\Rightarrow$ $acx^2 + (a + c)bx + (a + c)^2 = 0$

61. Equations $x^3 + ax^2 + bx + c = 0$

and $x^3 + (a - 1)x^2 + (b - 1)x + (c - 1) = 0$ have atleast one common root, let common root be α.

$\therefore$ $\alpha^3 + a\alpha^2 + b\alpha + c = 0$

and $\alpha^3 + a\alpha^2 + b\alpha + c - \alpha^2 - \alpha - 1 = 0$

$\Rightarrow$ $\alpha^2 + \alpha + 1 = 0$

$\Rightarrow \alpha = \omega, \omega^2$ (where, ω and ω^2 are the cube roots of unity)

62. Given equations are $2x^2 + 3x + 5\lambda = 0$ and $x^2 + 2x + 3\lambda = 0$ have a common root, if

$$\frac{x^2}{(9 - 10)\lambda} = \frac{x}{(5 - 6)\lambda} = \frac{1}{(4 - 3)}$$

$\Rightarrow$ $\dfrac{x^2}{-\lambda} = \dfrac{x}{-\lambda} = \dfrac{1}{1}$

$\Rightarrow$ $x^2 = -\lambda$, $x = -\lambda$ or $\lambda = -1, 0$

63. Let the roots be $\alpha, \beta, \beta, \gamma$ and γ, α, then

$$\alpha\beta = b, \quad \beta\gamma = c \quad \text{and} \quad \gamma\alpha = a$$

$\Rightarrow$ $\alpha\beta\gamma = \sqrt{abc}$

64. Roots of the equation $2x^2 + 3x + 5 = 0$ are

$$x = \frac{-3 \pm \sqrt{9 - 40}}{6} \quad \text{(imaginary roots)}$$

Hence, both roots coincide, so on comparing

$$\frac{a}{2} = \frac{b}{3} = \frac{c}{5} = k$$

$\Rightarrow$ $a = 2k, b = 3k, c = 5k$

$\Rightarrow$ $a + b + c = 10k$

So, maximum value does not exist.

65. Let $f(x) = x^2 + 2ax + b$

$= (x + a)^2 + b - a^2$

So, minimum value of $f(x)$ is $b - a^2$.

Since, $f(x) \geq c, \forall x \in R$, hence $b - a^2 \geq c$

i.e., $b - c \geq a^2$

66. As we know, $ax^2 + bx + c > 0$ for all $x \in R$, iff $a > 0$ and $D < 0$.

$\therefore$ $x^2 + 2ax + (10 - 3a) > 0, \forall x \in R$

$\Rightarrow$ $D < 0$

$\Rightarrow$ $4a^2 - 4(10 - 3a) < 0$

$\Rightarrow$ $4(a^2 + 3a - 10) < 0$

$\Rightarrow$ $(a + 5)(a - 2) < 0$

Using number line rule

$a \in (-5, 2)$

67. We know that, the expression $ax^2 + bx + c > 0$ for all x, if $a > 0$ and $b^2 < 4ac$.

$\therefore$ $(a^2 - 1)x^2 + 2(a - 1)x + 2$ is positive for all x, if

$a^2 - 1 > 0$ and $4(a - 1)^2 - 8(a^2 - 1) < 0$

$\Rightarrow$ $a^2 - 1 > 0$ and $-4(a - 1)(a + 3) < 0$

$\Rightarrow$ $a^2 - 1 > 0$ and $(a - 1)(a + 3) > 0$

$\Rightarrow$ $a^2 > 1$ and $a < -3$ or $a > 1$

$\Rightarrow$ $a < -3$ or $a > 1$

68. Given equation is $x^2 - 2ax + a^2 + a - 3 = 0$.

If roots are real, then $D \geq 0$

$\Rightarrow$ $4a^2 - 4(a^2 + a - 3) \geq 0$

$\Rightarrow$ $-a + 3 \geq 0$

$\Rightarrow$ $a - 3 \leq 0 \Rightarrow a \leq 3$

As roots are less than 3, hence $f(3) > 0$

$$9 - 6a + a^2 + a - 3 > 0$$

$\Rightarrow \quad a^2 - 5a + 6 > 0$

$\Rightarrow \quad (a-2)(a-3) > 0$

$\Rightarrow$ Either $a < 2$ or $a > 3$.

Hence, only $a < 2$ satisfy.

69. Since, α, β are the roots of the equation $ax^2 + bx + c = 0$.

$\therefore \quad ax^2 + bx + c = a(x-\alpha)(x-\beta)$

$\Rightarrow \alpha, \beta$ be the roots of $ax^2 + bx + c = 0$. Also, $\alpha < k < \beta$

So, $\quad a(k-\alpha)(k-\beta) < 0$

Also, $\quad a^2k^2 + abk + ac = a(ak^2 + bk + c)$

$= a^2(k-\alpha)(k-\beta) < 0$

$\Rightarrow \quad a^2k^2 + abk + ac < 0$

70. The given condition suggest that a lies between the roots.

Let $\quad f(x) = 2x^2 - 2(2a+1)x + a(a+1)$

For a to lie between the roots we must have discriminant ≥ 0 and $f(a) < 0$.

Now, $\quad$ discriminant ≥ 0

$\Rightarrow \quad 4(2a+1)^2 - 8a(a+1) \geq 0$

$\Rightarrow 8\left(a^2 + a + \frac{1}{2}\right) \geq 0$, which is always true.

Also, $\quad f(a) < 0$

$\Rightarrow \quad 2a^2 - 2a(2a+1) + a(a+1) < 0$

$\Rightarrow \quad -a^2 - a < 0 \quad \Rightarrow \quad a^2 + a > 0 \quad \Rightarrow \quad a(1+a) > 0$

$\Rightarrow \quad a > 0 \quad$ or $\quad a < -1$

71. $-x^2 + ax + a = 0$

$\Rightarrow \quad x^2 - ax - a = 0$

Let $\quad f(x) = x^2 - ax - a$

$f(1) < 0$

$\Rightarrow \quad 1 - a - a < 0$

$\Rightarrow \quad 1 < 2a$

$\Rightarrow \quad a > \frac{1}{2}$

72. Since, α, β are the roots of the equation $x^2 + bx + c = 0$.

Here, $D = b^2 - 4c > 0$ because $c < 0 < b$. So, roots are real and unequal.

Now, $\quad \alpha + \beta = -b < 0 \quad$ and $\quad \alpha\beta = c < 0$

$\therefore$ One root is positive and the other is negative, then the negative root being numerically bigger. As $\alpha < \beta$, α is the negative root while β is the positive root. So, $|\alpha| > \beta$ and $\alpha < 0 < \beta$.

73. Given, $\quad 1 \leq |x-2| \leq 3$

Case I When $1 \leq |x-2|$

$\therefore \quad x - 2 \geq 1$ and $(x-2) \leq -1$

$\Rightarrow \quad x \geq 3$ and $x \leq 1$

$\Rightarrow \quad x \in (-\infty, 1] \cup [3, \infty) \quad$ ...(i)

Case II When $|x-2| \leq 3$

$\Rightarrow \quad -3 \leq x - 2 \leq 3$

$\Rightarrow \quad -1 \leq x \leq 5$

$\Rightarrow \quad x \in [-1, 5] \quad$ ...(ii)

$\therefore$ From Eqs.(i) and (ii), we get

$x \in [-1, 1] \cup [3, 5]$

74. Given, $\quad |x+2| \leq 9$

$\Rightarrow \quad -9 \leq x + 2 \leq 9$

$\Rightarrow \quad -11 \leq x \leq 7$

75. We know that, $ax^2 + bx + c \geq 0$, if $a > 0$ and $b^2 - 4ac \leq 0$

Now, $\quad mx - 1 + \frac{1}{x} \geq 0$

$\Rightarrow \quad \frac{mx^2 - x + 1}{x} \geq 0$

$\Rightarrow \quad mx^2 - x + 1 \geq 0 \quad$ and $\quad x > 0$

Now, $mx^2 - x + 1 \geq 0$, if $m > 0$ and $1 - 4m \leq 0$ or if $m > 0$ and $m \geq \frac{1}{4}$.

Thus, the minimum value of m is $\frac{1}{4}$.

76. Given, $\quad \frac{2x-1}{3} \geq \left(\frac{3x-2}{4}\right) - \left(\frac{2-x}{5}\right)$

$\therefore \quad \frac{2x-1}{3} \geq \frac{15x - 10 - 8 + 4x}{20}$

$\Rightarrow \quad \frac{2x-1}{3} \geq \frac{19x - 18}{20}$

$\Rightarrow \quad 20(2x-1) \geq 3(19x - 18)$

$\Rightarrow \quad 40x - 20 \geq 57x - 54$

$\Rightarrow \quad -17x \geq -34$

$\Rightarrow \quad 17x \leq 34 \quad (\because -a \geq -b \Rightarrow a \leq b)$

$\Rightarrow \quad x \leq 2$

$\therefore \quad x \in (-\infty, 2]$

77. Given, $\quad \frac{|x-2| - 1}{|x-2| - 2} \leq 0$

Let $\quad |x-2| = k$

$\therefore$ Given equation becomes

$\frac{k-1}{k-2} \leq 0$

$\Rightarrow \quad \frac{(k-1)(k-2)}{(k-2)^2} \leq 0$

$\Rightarrow \quad (k-1)(k-2) \leq 0$

$\Rightarrow \quad 1 \leq k \leq 2$

$\Rightarrow \quad 1 \leq |x-2| \leq 2$

Case I When $1 \leq |x-2|$

$\Rightarrow \quad |x-2| \geq 1$

$\Rightarrow \quad x - 2 \geq 1$ or $x - 2 \leq -1$

$\Rightarrow \quad x \geq 3$ and $x \leq 1 \quad$ ...(i)

Case II When $|x-2| \le 2$

$\Rightarrow \quad -2 \le x-2 \le 2$

$\Rightarrow \quad -2+2 \le x \le 2+2$

$\Rightarrow \quad 0 \le x \le 4 \quad$ ...(ii)

$\therefore$ From Eqs.(i) and (ii), we get

$$x \in [0, 1] \cup [3, 4]$$

78. We have, $1 \le |x-2| \le 3$

$\Rightarrow \quad |x-2| \ge 1$ and $|x-2| \le 3$

$\Rightarrow \quad (x-2 \le -1$ or $x-2 \ge 1)$

and $\quad (-3 \le x-2 \le 3)$

$\Rightarrow \quad (x \le 1$ or $x \ge 3)$ and $(-1 \le x \le 5)$

$\Rightarrow \quad x \in (-\infty, 1] \cup [3, \infty)$ and $x \in [-1, 5]$

Combining the solutions of two inequalities, we have

$x \in [-1, 1] \cup [3, 5]$

79. We have, $\quad |2x-3| < |x+5|$

$\Rightarrow \quad |2x-3| - |x+5| < 0$

$$\Rightarrow \quad \begin{cases} 3-2x+x+5<0, x \le -5 \\ 3-2x-x-5<0, -5<x \le \frac{3}{2} \\ 2x-3-x-5<0, x>\frac{3}{2} \end{cases}$$

$$\Rightarrow \quad \begin{cases} x>8, x \le -5 \\ x>-\frac{2}{3}, -5<x \le \frac{3}{2} \\ x<8, x>\frac{3}{2} \end{cases}$$

$$\Rightarrow \quad x \in \left(-\frac{2}{3}, \frac{3}{2}\right] \cup \left(\frac{3}{2}, 8\right)$$

$$\Rightarrow \quad x \in \left(-\frac{2}{3}, 8\right)$$

80. $(x-1)(x^2-5x+7) < (x-1)$

$\Rightarrow \quad (x-1)(x^2-5x+6) < 0$

$\Rightarrow \quad (x-1)(x-2)(x-3) < 0$

$\therefore \quad x \in (-\infty, 1) \cup (2, 3)$

81. $x^2 - 3|x| + 2 < 0$

$\Rightarrow \quad |x|^2 - 3|x| + 2 < 0$

$\Rightarrow \quad (|x|-1)(|x|-2) < 0$

$\Rightarrow \quad 1 < |x| < 2$

$\Rightarrow \quad -2 < x < -1$ or $1 < x < 2$

$\therefore \quad x \in (-2, -1) \cup (1, 2)$

82. Since, $-3 < x + \frac{2}{x} < 3$

$\Rightarrow \quad -3 < \frac{(x^2+2)x}{x^2} < 3$

$\Rightarrow \quad -3x^2 < (x^2+2)x < 3x^2 \quad (x \ne 0)$

$\Rightarrow \quad x(x^2+3x+2) > 0$

and $\quad x(x^2-3x+2) < 0 \quad (x \ne 0)$

$\Rightarrow \quad x(x+1)(x+2) > 0$ and $x(x-1)(x-2) < 0$

$\Rightarrow \quad x \in (-2, -1) \cup (0, \infty) \quad$ ...(i)

and $\quad x \in (-\infty, 0) \cup (1, 2) \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$x \in (-2, -1) \cup (1, 2)$$

83. We have, $x^2 + 6x - 27 > 0$

$\Rightarrow \quad (x+9)(x-3) > 0 \Rightarrow x < -9$ or $x > 3$

$\Rightarrow \quad x \in (-\infty, -9) \cup (3, \infty) \quad$ ...(i)

and $\quad x^2 - 3x - 4 < 0$

$\Rightarrow \quad (x-4)(x+1) < 0$

$\Rightarrow \quad -1 < x < 4 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$3 < x < 4$$

84. We know that, $\quad$ A M $\ge$ GM

$\Rightarrow \quad \frac{a+b}{2} \ge \sqrt{ab}$

$\Rightarrow \quad 4 \ge \sqrt{ab} \quad (\because a + b = 8$ given)

$\Rightarrow \quad ab \le 16$

Equality holds when numbers are equal. So, ab is equal to 16 when $a = 4$, $b = 4$.

85. Since, $\quad$ AM $\ge$ GM

$\Rightarrow \quad \frac{a+b}{2} \ge \sqrt{ab}$

$\Rightarrow \quad \frac{a+b}{2} \ge \sqrt{4} \quad (\because ab = 4$, given)

$\Rightarrow \quad a + b \ge 4$

86. Since, $\quad$ AM $\ge$ GM

$\Rightarrow \quad \frac{bcx + cay + abz}{3} \ge (a^2b^2c^2 \cdot xyz)^{1/3}$

$\Rightarrow \quad bcx + cay + abz \ge 3abc \quad (\because xyz = abc)$

87. $\log_2(x^2 - 3x + 18) < 4$

$\Rightarrow \quad x^2 - 3x + 18 < 16 \quad$ (if $\log_a b < c \Rightarrow b < a^c$)

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x-1)(x-2) < 0$

$\Rightarrow \quad x \in (1, 2)$

88. Since, $\quad \frac{(1+a_1)}{2} \ge \sqrt{1 \cdot a_1} = \sqrt{a_1}$

$$\frac{(1+a_2)}{2} \ge \sqrt{1 \cdot a_2} = \sqrt{a_2}$$

$$\vdots \qquad \vdots$$

$$\frac{(1+a_n)}{2} \ge \sqrt{1 \cdot a_n} = \sqrt{a_n}$$

$\Rightarrow \quad \frac{1}{2^n}(1+a_1)(1+a_2)\ldots(1+a_n) \geq \sqrt{a_1a_2\ldots a_n} = 1$

$\Rightarrow \quad (1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n$

89. Now, $a^2 - 3a + 2 = 0$

$\Rightarrow \quad a = 1, 2 \quad \ldots(i)$

and $a^2 - 5a + 6 = 0$

$\Rightarrow \quad a = 2, 3 \quad \ldots(ii)$

$\Rightarrow \quad a - 2 - r = 0$

At $a = 2$ [common value from Eqs. (i) and (ii)]

$r = 0$

So, $a + r = 2$

90. Given equation is $x^2 + 9y^2 - 4x + 3 = 0 \quad \ldots(i)$

or $x^2 - 4x + 9y^2 + 3 = 0$

Since, x is real.

$\therefore \quad (-4)^2 - 4(9y^2 + 3) \geq 0$

$\Rightarrow \quad 16 - 4(9y^2 + 3) \geq 0$

$\Rightarrow \quad 4 - 9y^2 - 3 \geq 0$

$\Rightarrow \quad 9y^2 - 1 \leq 0$

$\Rightarrow \quad (3y - 1)(3y + 1) \leq 0$

$\Rightarrow \quad \frac{-1}{3} \leq y \leq \frac{1}{3}$

From Eq. (i) can also be written as

$$9y^2 + 0y + x^2 - 4x + 3 = 0$$

Since, y is real.

$\therefore \quad 0^2 - 4 \cdot 9(x^2 - 4x + 3) \geq 0$

$\Rightarrow \quad x^2 - 4x + 3 \leq 0$

$\Rightarrow \quad (x - 1)(x - 3) \leq 0$

$\Rightarrow \quad 1 \leq x \leq 3$

91. Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow \quad (y - 1)x^2 + 3(y + 1)x + 4(y - 1) = 0$

$\because x$ is real.

$\therefore \quad D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad -7y^2 + 50y - 7 \geq 0$

$\Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (y - 7)(7y - 1) \leq 0 \quad \ldots(i)$

$\Rightarrow \quad y \leq 7$ and $y \geq \frac{1}{7}$

$\Rightarrow \quad \frac{1}{7} \leq y \leq 7$

Hence, maximum value is 7 and minimum value is $\frac{1}{7}$.

92. Let the given expression be y.

$\therefore \quad y = \frac{x + 2}{2x^2 + 3x + 6} \Rightarrow 2x^2y + (3y - 1)x + (6y - 2) = 0$

If $y \neq 0$, then $\Delta \geq 0$ for real x.

i.e., $b^2 - 4ac \geq 0$

$\therefore \quad (3y - 1)^2 - 8y(6y - 2) \geq 0$

$\Rightarrow \quad -39y^2 + 10y + 1 \geq 0$

$\Rightarrow \quad 39y^2 - 10y - 1 \leq 0$

$\Rightarrow \quad (13y + 1)(3y - 1) \leq 0$

$\Rightarrow \quad -\frac{1}{13} \leq y \leq \frac{1}{3}$

If $y = 0$, then $x = -2$ which is real and this value of y is included in the above range.

93. Let $y = \frac{(x - a)(x - b)}{(x - c)}$

$\Rightarrow \quad y(x - c) = x^2 - (a + b)x + ab$

$\Rightarrow \quad x^2 - (a + b + y)x + ab + cy = 0$

Now, discriminant

$$D = (a + b + y)^2 - 4(ab + cy)$$
$$= y^2 + 2y(a + b - 2c) + (a - b)^2$$

Since, x is real and y assumes all real values, we must have $D \geq 0$ for all real values of y.

$\Rightarrow \quad 4(a + b - 2c)^2 - 4(a - b)^2 \leq 0$

$\Rightarrow \quad 4(a + b - 2c + a - b)(a + b - 2c - a + b) \leq 0$

$\Rightarrow \quad 16(a - c)(b - c) \leq 0$

$\Rightarrow \quad (c - a)(c - b) \leq 0$

$\Rightarrow \quad a \leq c \leq b$ or $b \leq c \leq a$

94. Given equation is $x^2 + (2 + \lambda)x - \frac{1}{2}(1 + \lambda) = 0$.

Let α and β are the roots of the given equation.

$\Rightarrow \quad \alpha + \beta = -(2 + \lambda)$ and $\alpha\beta = -\left(\frac{1 + \lambda}{2}\right)$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\Rightarrow \quad \alpha^2 + \beta^2 = [-(2 + \lambda)]^2 + 2\frac{(1 + \lambda)}{2}$

$\Rightarrow \quad \alpha^2 + \beta^2 = \lambda^2 + 4 + 4\lambda + 1 + \lambda = \lambda^2 + 5\lambda + 5$

Now, we take the option simultaneously.

It is minimum for $\lambda = \frac{1}{2}$.

95. We have, $px^2 + qx + 1 = 0$, for real roots discriminant ≥ 0

$\Rightarrow \quad q^2 - 4p \geq 0 \Rightarrow q^2 \geq 4p$

For $p = 1, q^2 \geq 4 \Rightarrow q = 2, 3, 4$

$p = 2, q^2 \geq 8 \Rightarrow q = 3, 4$

$p = 3, q^2 \geq 12 \Rightarrow q = 4$

$p = 4, q^2 \geq 16 \Rightarrow q = 4$

Total seven solutions are possible.

96. Since, α, β and γ, δ are the roots of the equation $ax^2 + 2bx + c = 0$ and $px^2 + 2qx + r = 0$ respectively, then

$$\alpha + \beta = -\frac{2b}{a}, \quad \alpha\beta = \frac{c}{a}, \quad \gamma + \delta = -\frac{2q}{p}, \quad \gamma\delta = \frac{r}{p}$$

As given α, β, γ and δ are in GP, therefore

$$\frac{\alpha}{\gamma} = \frac{\beta}{\delta} \quad \text{...(i)}$$

But $\quad \dfrac{\alpha\beta}{\gamma\delta} = \dfrac{pc}{ar} \Rightarrow \left(\dfrac{\beta}{\delta}\right)^2 = \dfrac{pc}{ar}$ [from Eq. (i)]

Also, $\dfrac{\alpha}{\beta} = \dfrac{\gamma}{\delta} \Rightarrow \dfrac{\alpha+\beta}{\beta} = \dfrac{\gamma+\delta}{\delta} \Rightarrow \dfrac{\alpha+\beta}{\gamma+\delta} = \dfrac{\beta}{\delta}$

$$\Rightarrow \frac{bp}{aq} = \sqrt{\frac{pc}{ar}} \Rightarrow \frac{b^2p^2}{a^2q^2} = \frac{pc}{ar} \Rightarrow q^2ac = b^2pr$$

97. Since, α, β and $\alpha - k, \beta - k$ are the roots of the equations $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$, respectively.

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

and $\quad \alpha + \beta - 2k = -\dfrac{B}{A}, \quad (\alpha - k)(\beta - k) = \dfrac{C}{A}$

Now, $\quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \dfrac{(b^2 - 4ac)}{a^2} \quad$ ...(i)

Also, $\quad \{(\alpha - k) - (\beta - k)\}^2$

$$= \{(\alpha - k) + (\beta - k)\}^2 - 4(\alpha - k)(\beta - k)$$

$$= \left(-\frac{B}{A}\right)^2 - 4\left(\frac{C}{A}\right) = \frac{B^2 - 4AC}{A^2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{(b^2 - 4ac)}{a^2} = \frac{B^2 - 4AC}{A^2}$$

$$\therefore \quad \frac{B^2 - 4AC}{b^2 - 4ac} = \left(\frac{A}{a}\right)^2$$

98. Since, α and β are the roots of given equation.

Let $\quad f(x) = a^2x^2 + 2bx + 2c = 0$

Then, $\quad f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = 0$

$$= a^2\alpha^2 + 2(b\alpha + c) = a^2\alpha^2 - 2a^2\alpha^2$$

$$= -a^2\alpha^2 = -\text{ve}$$

and $\quad f(\beta) = a^2\beta^2 + 2(b\beta + c) = a^2\beta^2 + 2a^2\beta^2$

$$= 3a^2\beta^2 = +\text{ve}$$

Since, $f(\alpha)$ and $f(\beta)$ are of opposite signs, therefore by theory of equations there lies a root γ of the equation $f(x) = 0$ between α and β, *i.e.*, $\alpha < \gamma < \beta$.

99. Since, $y^2 - y + a = \left(y - \dfrac{1}{2}\right)^2 + a - \dfrac{1}{4}$

and $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$, given equation will have no real values of x for any y, if $a - \dfrac{1}{4} > \sqrt{2}$

i.e., $\quad a \in \left(\sqrt{2} + \dfrac{1}{4}, \infty\right)$

$\Rightarrow \quad a \in (\sqrt{3}, \infty) \quad$ (as $\sqrt{2} + \dfrac{1}{4} < \sqrt{3}$)

100. Let α, β are the roots of the equation $x^2 - ax + b = 0$.

$\therefore \quad \alpha + \beta = a \quad$ ...(i)

and $\quad \alpha\beta = b \quad$ ...(ii)

Roots are prime numbers, so clearly b cannot be a prime number as it is product of two prime numbers [from Eq.(ii)]. Sum of two prime numbers is always an even number except in one situation when one prime number is 2. 'a' can be a prime number and can be composite number.

Now, $\quad 1 + a + b = 1 + \alpha\beta + \alpha + \beta = (1 + \alpha)(1 + \beta)$

$(1 + \alpha), (1 + \beta)$ can be prime numbers, can be composite numbers, so $1 + a + b$ is not certain.

So, option (d) is correct.

Round II

1. The given equation is

$$2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$$

$$\Rightarrow \quad x = \frac{4(2 - i) \pm \sqrt{16(2 - i)^2 + 8(1 + i)(5 + 3i)}}{4(1 + i)}$$

$$= -\frac{i}{1 + i} \text{ or } \frac{4 - i}{1 + i} = \frac{-1 - i}{2} \text{ or } \frac{3 - 5i}{2}$$

Now, $\quad \left|\dfrac{-1 - i}{2}\right| = \sqrt{\dfrac{1}{4} + \dfrac{1}{4}} = \sqrt{\dfrac{1}{2}}$

and $\quad \left|\dfrac{3 - 5i}{2}\right| = \sqrt{\dfrac{9}{4} + \dfrac{25}{4}} = \sqrt{\dfrac{17}{2}}$

Also, $\quad \sqrt{\dfrac{17}{2}} > \sqrt{\dfrac{1}{2}}$

Hence, required root is $\dfrac{3 - 5i}{2}$.

2. Let $f(x) = ax^2 + 2bx - 3c$ and $f(x) = 0$ has non-real roots, $f(x)$ will have the same sign for all values of x.

Also, $\quad \dfrac{3c}{4} < (a + b) \Rightarrow 4a + 4b - 3c > 0$

$\Rightarrow \quad f(2) > 0 \Rightarrow f(0) > 0$

$\Rightarrow \quad c < 0$

3. We have, $\quad ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \quad$ ...(i)

$$\Rightarrow \quad a\left(\frac{x}{1 - x}\right)^2 + b\left(\frac{x}{1 - x}\right) + c = 0$$

Also, α and β be the roots of $ax^2 + bx + c = 0$.

$$\therefore \quad \alpha = \frac{x}{1 - x} \text{ and } \beta = \frac{x}{1 - x}$$

$\Rightarrow \quad x = \dfrac{\alpha}{\alpha+1}, x = \dfrac{\beta}{\beta+1}$

Hence, $\dfrac{\alpha}{\alpha+1}$ and $\dfrac{\beta}{\beta+1}$ are the required roots.

4. Since, α and β be the roots of the equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Now, sum of roots $= |\alpha| + |\beta|$

$= -\alpha - \beta \qquad (\because \beta < \alpha < 0)$

$= -\left(-\dfrac{b}{a}\right) = \left|\dfrac{b}{a}\right| \qquad (\because |\alpha| + |\beta| > 0)$

and product of roots $= |\alpha||\beta| = \left|\dfrac{c}{a}\right|$

Hence, required equation is

$$x^2 - \left|\frac{b}{a}\right|x + \left|\frac{c}{a}\right| = 0$$

$\Rightarrow \quad |a|x^2 - |b|x + |c| = 0$

5. On multiplying first equation by x, we get

$$x^4 + ax^2 + x = 0 \qquad \text{...(i)}$$

and another given equation is

$$x^4 + ax^2 + 1 = 0 \qquad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$x - 1 = 0$$

$\Rightarrow \quad x = 1$

which is a common root.

On putting this value in Eq. (ii), we get

$$1 + a + 1 = 0$$

$\Rightarrow \quad a = -2$

6. $\left(\dfrac{9}{10}\right)^x = -3 + x - x^2$

$\Rightarrow \quad \left(\dfrac{9}{10}\right)^x = -\left\{\left(x - \dfrac{1}{2}\right)^2 + \dfrac{11}{4}\right\}$

LHS is always positive while RHS is always negative.

Hence, the given equation has no solution.

7. Let the discriminant of the equation $x^2 + px + q = 0$ is D_1, then $D_1 = p^2 - 4q$

and the discriminant of the equation $x^2 + rx + s = 0$ is D_2, then $D_2 = r^2 - 4s$

$\therefore \quad D_1 + D_2 = p^2 + r^2 - 4(q + s) = p^2 + r^2 - 2pr$

(from the given relation)

$\Rightarrow \quad D_1 + D_2 = (p - r)^2 \geq 0$

Clearly, atleast one of D_1 and D_2 must be non-negative, consequently atleast one of the equation has real roots.

8. Since, α and β are the roots of the equation $x^2 + px + q = 0$, therefore

$\alpha + \beta = -p$ and $\alpha\beta = q$

Now, $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$

$= \alpha^2 + \beta^2 + (\omega^4 + \omega^2)\alpha\beta \qquad (\because \omega^3 = 1)$

$= \alpha^2 + \beta^2 - \alpha\beta \qquad (\because \omega + \omega^2 = -1)$

$= (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q$

Also, $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \dfrac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta} = \dfrac{p(3q - p^2)}{q}$

$\therefore$ The given expression $= \dfrac{(p^2 - 3q)}{\dfrac{p(3q - p^2)}{q}} = -\dfrac{q}{p}$

9. Let the correct equation is

$$ax^2 + bx + c = 0,$$

then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

When b is written incorrectly, then the roots are equal. Let these are γ and γ.

$\therefore \quad \gamma \cdot \gamma = \dfrac{c}{a} \Rightarrow \gamma^2 = \alpha\beta \qquad \text{...(i)}$

When c is written incorrectly, then the roots are γ and 2γ.

$\therefore \quad \gamma + 2\gamma = -\dfrac{b}{a} \Rightarrow 3\gamma = \alpha + \beta$

$\Rightarrow \quad 9\gamma^2 = (\alpha + \beta)^2 \Rightarrow 9\alpha\beta = (\alpha - \beta)^2 + 4\alpha\beta$

[from Eq. (i)]

$\therefore \quad (\alpha - \beta)^2 = 5\alpha\beta$

10. Discriminant of the equation $3x^2 + 8x + 15 = 0$ is given by

$$D = 64 - 180 = -116 < 0$$

So, its roots are imaginary and therefore roots are conjugate to each other. Therefore, one common root means both the roots are common.

$\therefore \quad \dfrac{a}{3} = \dfrac{2b}{8} = \dfrac{3c}{15}$

$\Rightarrow \quad \dfrac{a}{3} = \dfrac{b}{4} = \dfrac{c}{5} = k \qquad$ (say), $k \neq 0$

$\Rightarrow \quad a = 3k, \quad b = 4k, \quad c = 5k$

Now, $\quad a^2 + b^2 = c^2$

$\Rightarrow \Delta ABC$ is right angled.

$\therefore \quad \sin^2 A + \sin^2 B = \sin^2 C$

$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2\sin^2 C = 2\sin^2 90° = 2$

11. Given, $\quad x - 2 = 2^{2/3} + 2^{1/3}$

On cubing both sides, we get

$(x - 2)^3 = 2^2 + 2 + 3 \times 2^{2/3} \times 2^{1/3}(x - 2) = 6 + 6(x - 2)$

$\Rightarrow \quad x^3 - 6x^2 + 12x - 8 = -6 + 6x$

$\therefore \quad x^3 - 6x^2 + 6x = 2$

12. We are given that $p(-a) = q$ and $p(a) = -a$

[since, when a polynomial $f(x)$ is divided by $x - a$, remainder is $f(a)$].

Let the remainder, when $p(x)$ is divided by $x^2 - a^2$, be $Ax + B$. Then,

$$p(x) = Q(x)(x^2 - a^2) + Ax + B \quad \ldots(i)$$

where $Q(x)$ is the quotient.

Putting $x = a$ and $-a$ in Eq. (i), we get

$$p(a) = 0 + Aa + B \Rightarrow -a = aA + B \quad \ldots(ii)$$

and $$p(-a) = 0 - aA + B \Rightarrow a = -aA + B \quad \ldots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$B = 0 \text{ and } A = -1$$

Hence, the required remainder is $-x$.

13. On multiplying the given equation by c/a^3, we get

$$\frac{b^2c^2}{a^3}x^2 - \frac{b^2c}{a^2}x + c = 0$$

$$\Rightarrow \quad a\left(\frac{bc}{a^2}x\right)^2 - b\left(\frac{bc}{a^2}\right)x + c = 0$$

$$\Rightarrow \quad \frac{bc}{a^2}x = \alpha, \beta$$

$$\Rightarrow \quad (\alpha + \beta)\alpha\beta x = a, \beta$$

$$\Rightarrow \quad x = \frac{1}{(\alpha + \beta)\alpha}, \frac{1}{(\alpha + \beta)\beta}$$

14. $D = b^2 - 4a < 0 \Rightarrow a > 0$

Therefore, the graph is concave upwards.

$$f(x) > 0, \forall x \in R$$

$$\Rightarrow \quad f(-1) > 0 \Rightarrow a + b + 1 > 0$$

15. $x^2 - (a - 3)x + a = 0$

Now, $$D = (a - 3)^2 - 4a$$
$$= a^2 - 10a + 9 = (a - 1)(a - 9)$$

Case I Both the roots are greater than 2.

$$D \geq 0, f(2) > 0, -\frac{B}{2A} > 2$$

$$\Rightarrow \quad (a - 1)(a - 9) \geq 0; 4 - (a - 3)2 + a > 0; \frac{a - 3}{2} > 2$$

$$\Rightarrow \quad a \in (-\infty, 1] \cup [9, \infty); a < 10, a > 7$$

$$\Rightarrow \quad a \geq [9, 10] \quad \ldots(i)$$

Case II One root is greater than 2 and the other root is less than or equal to 2. Hence,

$$f(2) \leq 0$$

$$\Rightarrow \quad 4 - (a - 3)2 + a \leq 0$$

$$\Rightarrow \quad a \geq 10 \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$a \in [9, 10) \cup [10, \infty)$$

$$\Rightarrow \quad a \in [9, \infty)$$

16. The given equation is

$$x^2 - 2mx + m^2 - 1 = 0$$

$$\Rightarrow \quad (x - m)^2 - 1 = 0$$

$$\Rightarrow \quad (x - m + 1)(x - m - 1) = 0$$

$$\Rightarrow \quad x = m - 1, m + 1$$

From given condition,

$m - 1 > -2$ and $m + 1 < 4$

$$\Rightarrow \quad m > -1 \text{ and } m > 3$$

Hence, $$-1 < m < 3$$

17. Note that coefficient of x^2 is $(4p - p^2 - 5) < 0$. Therefore, the graph is concave downward. According to the question, 1 must lie between the roots. Hence,

$f(1) > 0$

$$\Rightarrow \quad 4p - p^2 - 5 - 2p + 1 + 3p > 0$$

$$\Rightarrow \quad -p^2 + 5p - 4 > 0$$

$$\Rightarrow \quad p^2 - 5p + 4 < 0$$

$$\Rightarrow \quad (p - 4)(p - 1) < 0$$

$$\Rightarrow \quad 1 < p < 4$$

$$\Rightarrow \quad p \in \{2, 3\}$$

18. Clearly, $f(-1) > 0$, $f(2) < 0$. Now,

$$f(0) = -4 < 0$$

$$\Rightarrow \quad f(-1) = 1 - a - 4 > 0$$

and $$f(2) = 4 + 2a - 4 < 0$$

$$\Rightarrow \quad a < -3 \text{ and } a < 0$$

$$\Rightarrow \quad a \in (-\infty, -3)$$

19. $$a = \frac{x^2 + 4}{|x|} - 3 = |x| + \frac{4}{|x|} - 3$$

$$\geq 2\sqrt{|x|\frac{4}{|x|}} - 3 \quad (\because \text{AM} \geq \text{GM})$$

$$= 4 - 3 = 1$$

$$\Rightarrow \quad a \geq 1$$

20. Let r be the common ratio of the GP. Since, $\alpha, \beta, \gamma, \delta$ are in GP, then $\beta = \alpha r, \gamma = \alpha r^2$ and $\delta = \alpha r^3$.

For equation, $$x^2 - x + p = 0$$

$$\therefore \quad \alpha + \beta = 1$$

$$\Rightarrow \quad \alpha + \alpha r = 1$$

$$\Rightarrow \quad \alpha(1 + r) = 1 \quad \ldots(i)$$

and $$\alpha\beta = p \Rightarrow \alpha(\alpha r) = p$$

$$\Rightarrow \quad \alpha^2 r = p \quad \ldots(ii)$$

For equation, $$x^2 - 4x + q = 0$$

$$\gamma + \delta = 4$$

$$\Rightarrow \quad \alpha r^2 + \alpha r^3 = 4$$

$$\Rightarrow \quad \alpha r^2(1 + r) = 4 \quad \ldots(iii)$$

and $\gamma\delta = q \Rightarrow \alpha r^3 \cdot \alpha r^2 = q$

$\Rightarrow \alpha^2 r^5 = q$ ...(iv)

On dividing Eq. (iii) by Eq. (i), we get

$$r^2 = 4 \Rightarrow r = \pm 2$$

If we take $r = 2$, then α is not integral, so we take $r = -2$. On substituting $r = -2$ in Eq. (i), we get

$$\alpha = -1$$

Now, from Eq. (ii), we have

$$p = \alpha^2 r = (-1)^2(-2) = -2$$

and from Eq. (iv), we have

$$q = \alpha^2 r^5 = (-1)^2(-2)^5 = -32$$

$$(p, q) = (-2, -32)$$

21. Since, the roots of equation are real.

$\therefore$ $B^2 - 4AC > 0$

$\Rightarrow$ $a^4 > 4b^2$

Hence, option (a) is correct.

If $f(x) = x^2 + a^2x + b^2$ ($\because$ c lies outside the roots)

$\therefore f(c) > 0$, then $c^2 + a^2c + b^2 > 0$

Hence, option (b) is correct.

Also, (x-coordinate of vertex) $> c$

$\Rightarrow$ $-\dfrac{a^2}{2} > c$

Hence, option (c) is correct.

22. The given equation can be written as

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

$\Rightarrow$ $p(x^2 - c^2) = 2(a+b)x^2 - 2c(a-b)x$

$\Rightarrow$ $(2a + 2b - p)x^2 - 2c(a-b)x + pc^2 = 0$

Now, $c^2(a-b)^2 - pc^2(2a + 2b - p) = 0$ ($\because$ equal roots)

$\Rightarrow$ $(a-b)^2 - 2p(a+b) + p^2 = 0$ ($\because c^2 \neq 0$)

$\Rightarrow$ $[p - (a+b)]^2 = (a+b)^2 - (a-b)^2$

$\Rightarrow$ $p = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

23. $\because$ $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

$$\alpha + h + \beta + h = \frac{-q}{p}$$

and $(\alpha + h)(\beta + h) = \dfrac{r}{p}$

$\therefore$ $-\dfrac{q}{p} = \alpha + \beta + 2h = -\dfrac{b}{a} + 2h$

$\Rightarrow$ $h = \dfrac{1}{2}\left(\dfrac{b}{a} - \dfrac{q}{p}\right)$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$$

Also, $(\alpha - \beta)^2 = [(\alpha + h) - (\beta + h)]^2$

$$= \frac{q^2 - 4pr}{p^2} \Rightarrow \frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$$

Solutions (Q. Nos. 24-26)

$$ax^2 - bx + c = 0$$

Let $f(x) = ax^2 - bx + c$ be the corresponding quadratic expression and α, β be the roots of $f(x) = 0$. Then,

$$f(x) = a(x - \alpha)(x - \beta)$$

Now, $af(1) > 0$, $af(2) > 0$, $1 < -\dfrac{b}{2a} < 2$, $b^2 - 4ac > 0$

$\Rightarrow a(1-\alpha)(1-\beta) > 0$, $a(2-\alpha)(2-\beta) > 0$, $2a < b < 4a$, $b^2 - 4ac > 0$

$\Rightarrow$ $a^2(1-\alpha)(1-\beta)(2-\alpha)(2-\beta) > 0$

$\Rightarrow$ $a^2(\alpha - 1)(2-\alpha)(\beta - 1)(2-\beta) > 0$

As $f(1)$ and $f(2)$ both are integers and $f(1) > 0$ and $f(2) > 0$, so $f(1)f(2) > 0$

$\Rightarrow$ $f(1)f(2) \geq 1$

$\Rightarrow$ $1 \leq a^2(\alpha - 1)(2-\alpha)(\beta - 1)(2-\beta)$

Now, $\dfrac{(\alpha - 1) + (2 - \alpha)}{2} \geq \{(\alpha - 1)(2 - \alpha)\}^{1/2}$

$$(\alpha - 1)(2 - \alpha) \leq \frac{1}{4}$$

Similarly,

$$(\beta - 1)(2 - \beta) \leq \frac{1}{4}$$

$\Rightarrow$ $(\alpha - 1)(2-\alpha)(\beta - 1)(2-\beta) < \dfrac{1}{16}$

24. As $\alpha \neq \beta$, so

$$a^2 > 16 \Rightarrow a \geq 5$$

25. $b^2 > 20c$ and $b > 10$

$\Rightarrow$ $b \geq 11$

Also, $b^2 > 100$

26. $c > 5 \Rightarrow c \geq 6$

27. Let $A = b + c - a$, $B = c + a - b$, $C = a + b - c$

$\therefore$ $A + B + C = 0$

And given equation becomes

$$Ax^2 + Bx + C = 0$$

$\therefore$ $D = B^2 - 4AC$

$= (A + C)^2 - 4AC = (A - C)^2$

$= 4(c - a)^2 =$ perfect square

Hence, roots of

$$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$$

are rationals.

Hence option (a) is correct.

28. Let $f(x) = 4x^2 - 2x + a$

Also, both roots of $f(x) = 0$ lie in the interval $(-1, 1)$.

$\therefore D \geq 0$, $f(-1) > 0$ and $f(1) > 0$.

Now, $D \geq 0$

$\Rightarrow (-2)^2 - 4 \cdot 4 \cdot a \geq 0$

$\Rightarrow a \leq \frac{1}{4}$ ...(i)

and $f(-1) > 0$

$\Rightarrow 4(-1)^2 - 2(-1) + a > 0$

$\Rightarrow a > -6$ ...(ii)

Also, $f(1) > 0 \Rightarrow 4(1)^2 - 2(1) + a > 0$

$\Rightarrow a > -2$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$-2 < a \leq \frac{1}{4}$$

29. For real roots, $D \geq 0$

$\Rightarrow (-4)^2 - 4(2\lambda - 1)(2\lambda - 1) \geq 0$

$\Rightarrow (2\lambda - 1)^2 \leq 4$

$\Rightarrow -2 \leq 2\lambda - 1 \leq 2$

$\Rightarrow -\frac{1}{2} \leq \lambda \leq \frac{3}{2}$

$\therefore$ Integral values of λ are 0 and 1.

Hence, greatest integral value of $\lambda = 1$

30. Since, $x = -2$ is a root of $f(x)$.

$\therefore f(x) = (x + 2)(ax + b)$

But $f(0) + f(1) = 0$

$\therefore 2b + 3a + 3b = 0 \Rightarrow -\frac{b}{a} = \frac{3}{5}$

Hence, option (a) is correct.

31. $D = (3)^2 - 4 \cdot 2 \cdot 4 = -23 < 0$

$\therefore$ Roots of $2x^2 + 3x + 4 = 0$ are imaginary.

Now, $\because 2, 3, 4 \in R$.

$\therefore$ Roots are conjugate to each other.

$\because$ One root is common in $ax^2 + bx + c = 0$ and $2x^2 + 3x + 4 = 0$ (given), then other roots is also common.

$\because$ Roots are conjugate $(a, b, c \in R)$.

Hence, both equation are identical

$\therefore a : b : c = 2 : 3 : 4$

32. $a^2 + bx + c = 0$ has two complex conjugate roots only, if all the coefficients are real. If all the coefficients are not real, then it is not necessary that both the roots are imaginary. Hence, statement I is true.

Now, equation $x^2 - 3x + 4 = 0$ has two complex conjugate roots. If $ax^2 + bx + c = 0$ has all coefficients real, then there will be two common roots. But if there is only one root common, then atleast one of a, b, c must be non-real.

Thus, both the statements are true and statement II is correct explanation of statement I.

33. $ix^2 + (i - 1)x - \frac{1}{2} - i = 0$

$$\Rightarrow x = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4(i)\left(-\frac{i}{2} - i\right)}}{2i}$$

$$= \frac{-(i-1) \pm \sqrt{-4}}{2i}$$

Thus, roots are imaginary. Also, we have $b^2 - 4ac = -4 < 0$ but this is not the correct reason for which roots are imaginary as coefficients of the equation are imaginary.

Hence, both the statements are correct but statement II is not correct explanation of statement I.

34. Here, $f(x)$ is a downward parabola.

$$D = (a + 1)^2 + 20 > 0$$

From the graph, clearly statement I is true but statement II is false.

35. Given equation are

$x^2 + 2x + 3 = 0$ ...(i)

and $ax^2 + bx + c = 0$ ...(ii)

Since, Eq. (i) has imaginary roots.

So, Eq. (ii) will also have both roots same as Eq.(i).

Thus, $\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

Hence, $a : b : c$ is $1 : 2 : 3$.

36. Let $z = x + iy$, given $\text{Re}(z) = 1$

$\therefore x = 1 \Rightarrow z = 1 + iy$

Since, the complex roots are conjugate to each other.

$\therefore z = 1 + iy$ and $1 - iy$ are two roots of $z^2 + \alpha z + \beta = 0$.

$\because$ Product of roots $= \beta$

$\Rightarrow (1 + iy)(1 - iy) = \beta$

$\therefore \beta = 1 + y^2 \geq 1 \Rightarrow \beta \in [1, \infty)$

37. Since, α and β are roots of the equation $x^2 - x + 1 = 0$.

$\Rightarrow \alpha + \beta = 1, \alpha\beta = 1 \Rightarrow x = \frac{1 \pm \sqrt{3}i}{2}$

$\Rightarrow x = \frac{1 + \sqrt{3}i}{2}$ or $\frac{1 - \sqrt{3}i}{2}$

$\Rightarrow x = -\omega$ or $-\omega^2$

Thus, $\alpha = -\omega^2$, then $\beta = -\omega$

$\Rightarrow$ $\alpha = -\omega$, then $\beta = -\omega^2$ (where, $\omega^3 = 1$)

Hence, $\alpha^{2009} + \beta^{2009} = (-\omega)^{2009} + (-\omega^2)^{2009}$

$= -[(\omega^3)^{669} \cdot \omega^2 + (\omega^3)^{1339} \cdot \omega]$

$= -(\omega^2 + \omega) = -(-1) = 1$

38. Given, $bx^2 + cx + a = 0$ has imaginary roots.

$\therefore$ $c^2 - 4ab < 0$

$\Rightarrow$ $c^2 < 4ab$

$\Rightarrow$ $-c^2 > -4ab$ ...(i)

Let $f(x) = 3b^2x^2 + 6bcx + 2c^2$

Here, $3b^2 > 0$

So, the given expression has a minimum value.

$\therefore$ Minimum value $= \dfrac{-D}{4a}$

$= \dfrac{4AC - B^2}{4A}$

$= \dfrac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)}$

$= -\dfrac{12b^2c^2}{12b^2} = -c^2 > -4ab$ [from Eq. (i)]

39. Let the roots of $x^2 - 6x + a = 0$ be $\alpha, 4\beta$ and that of $x^2 - cx + 6 = 0$ be α and 3β.

$\therefore \alpha + 4\beta = 6$ and $4\alpha\beta = a$ and $\alpha + 3\beta = c$ and $3\alpha\beta = 6$

$\Rightarrow$ $\dfrac{a}{6} = \dfrac{4}{3} \Rightarrow a = 8$

$\therefore$ $x^2 - 6x + 8 = 0$

$\Rightarrow$ $(x - 4)(x - 2) = 0 \Rightarrow x = 2, 4$

and $x^2 - cx + 6 = 0$

$\Rightarrow$ $2^2 - 2c + 6 = 0$

$\Rightarrow$ $c = 5$

$\therefore$ $x^2 - 5x + 6 = 0$

$\Rightarrow$ $x = 2, 3$

Hence, common root is 2.

40. Since, $\alpha + \beta = -a$ and $\alpha\beta = 1$

Now, $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{a^2 - 4}$

Thus, $\sqrt{a^2 - 4} < \sqrt{5}$

$\Rightarrow$ $a^2 < 9$

$\Rightarrow$ $|a| < 3$

41. Here, $\tan 30° + \tan 15° = -p$

and $\tan 30° \tan 15° = q$

Now, $2 + q - p = 2 + \tan 30° \tan 15° + (\tan 30° + \tan 15°)$

$= 2 + \tan 30° \tan 15° + 1 - \tan 30° \tan 15° = 3$

42. Since, both roots of equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4.

$\therefore$ $D \geq 0$, $-2 < -\dfrac{b}{2a} < 4$, $f(4) > 0$ and $f(-2) > 0$

Now, $D \geq 0$

$\Rightarrow$ $4m^2 - 4(m^2 - 1) \geq 0$

$\Rightarrow$ $4 > 0, \forall m \in R$ ...(i)

$-2 < -\dfrac{b}{2a} < 4$

$\Rightarrow$ $-2 < \left(\dfrac{2m}{2 \cdot 1}\right) < 4$

$\Rightarrow$ $-2 < m < 4$ ...(ii)

$f(4) > 0$

$\Rightarrow$ $m^2 - 8m + 15 > 0$

$\Rightarrow$ $(m - 3)(m - 5) > 0$

$\Rightarrow$ $-\infty < m < 3$ and $5 < m < \infty$ ...(iii)

and $f(-2) > 0$

$\Rightarrow$ $m^2 + 4m + 3 > 0$

$\Rightarrow$ $(m + 3)(m + 1) > 0$

$\Rightarrow$ $-\infty < m < -3$ and $-1 < m < \infty$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), we get m lies between -1 and 3.

43. Given that, $(x - 1)^3 + 8 = 0 \Rightarrow (x - 1)^3 = (-2)^3$

$\Rightarrow$ $\left(\dfrac{x - 1}{-2}\right)^3 = 1$

$\Rightarrow$ $\left(\dfrac{x - 1}{-2}\right) = (1)^{1/3}$

$\therefore$ Cube roots of $\left(\dfrac{x - 1}{-2}\right)$ are 1, ω and ω^2.

$\Rightarrow$ Cube roots of $(x - 1)$ are -2, -2ω and $-2\omega^2$.

$\Rightarrow$ Cube roots of x are $-1, 1 - 2\omega$ and $1 - 2\omega^2$.

44. Let α and β be the roots of equation

$x^2 - (a - 2)x - a - 1 = 0$, then

$\alpha + \beta = a - 2$ and $\alpha\beta = -a - 1$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\Rightarrow$ $\alpha^2 + \beta^2 = (a - 2)^2 + 2(a + 1)$

$\Rightarrow$ $\alpha^2 + \beta^2 = a^2 - 2a + 6$

$\Rightarrow$ $\alpha^2 + \beta^2 = (a - 1)^2 + 5$

The value of $\alpha^2 + \beta^2$ will be least, if $a - 1 = 0$.

$\Rightarrow$ $a = 1$

45. Let n and $(n + 1)$ be the two consecutive roots of $x^2 - bx + c = 0$. Then, $n + (n + 1) = b$ and $n(n + 1) = c$

$\therefore$ $b^2 - 4c = (2n + 1)^2 - 4n(n + 1)$

$= 4n^2 + 4n + 1 - 4n^2 - 4n = 1$

46. Let $f(x) = x^2 - 2kx + k^2 + k - 5$

Since, both roots are less than 5. Then, $D \geq 0$,

$$-\frac{b}{2a} < 5 \quad \text{and} \quad f(5) > 0$$

Now, $D = 4k^2 - 4(k^2 + k - 5)$

$= -4k + 20 \geq 0$

$\Rightarrow \quad k \leq 5 \quad$ ...(i)

$\Rightarrow \quad -\frac{b}{2a} < 5 \Rightarrow k < 5 \quad$ ...(ii)

and $\quad f(5) > 0$

$\Rightarrow \quad 25 - 10k + k^2 + k - 5 > 0$

$\Rightarrow \quad k^2 - 9k + 20 > 0$

$\Rightarrow \quad (k-5)(k-4) > 0$

$\Rightarrow \quad k < 4$ and $k > 5 \quad$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$k < 4$$

47. Let $\quad f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0 \quad$ ...(i)

$f(0) = 0$ and $f(\alpha) = 0$

$\therefore$ According to the Rolle's theorem

$$f'(x) = 0$$

has atleast one root between $(0, \alpha)$.

$\Rightarrow f'(x) = 0$ has a positive root less than α.

48. Since, $(1-p)$ is a root of quadratic equation

$$x^2 + px + (1-p) = 0 \quad \text{...(i)}$$

So, $(1-p)$ satisfied the above equation.

$\therefore \quad (1-p)^2 + p(1-p) + (1-p) = 0$

$\Rightarrow \quad (1-p)[1 - p + p + 1] = 0$

$\Rightarrow \quad (1-p)2 = 0$

$\Rightarrow \quad p = 1$

Then, from Eq. (i), we get

$$x^2 + x = 0$$

$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0, -1$

49. Since, one of the roots of equation $x^2 + px + 12 = 0$ is 4.

$\therefore \quad 16 + 4p + 12 = 0$

$\Rightarrow \quad 4p = -28$

$\Rightarrow \quad p = -7$

So, the other equation is $x^2 - 7x + q = 0$ whose roots are equal. Let the roots are α and α.

$\therefore \quad$ Sum of roots $= \alpha + \alpha = \frac{7}{1} \Rightarrow \alpha = \frac{7}{2}$

and product of roots $= \alpha \cdot \alpha = q$

$\Rightarrow \quad \left(\frac{7}{2}\right)^2 = q \Rightarrow q = \frac{49}{4}$

50. Let α and β be two numbers whose arithmetic mean is 9 and geometric mean is 4.

$\therefore \quad \alpha + \beta = 18$

and $\quad \alpha\beta = 16$

$\therefore$ Required equation is

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

$\Rightarrow \quad x^2 - 18x + 16 = 0$

51. **Hint** $\quad \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Also given, $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$\Rightarrow \quad \alpha + \beta = \left(\frac{\alpha + \beta}{\alpha\beta}\right)^2 - \frac{2}{\alpha\beta}$

$\Rightarrow \quad \left(-\frac{b}{a}\right) = \left(\frac{-b/a}{c/a}\right)^2 - \frac{2}{c/a}$

52. Given equation is $x^2 - 3|x| + 2 = 0$

***Case* I** When $x > 0$, then $|x| = x$

$\therefore \quad x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0$

$\Rightarrow \quad x = 1, 2$

***Case* II** When $x < 0$, then $|x| = -x$

$\therefore \quad x^2 + 3x + 2 = 0 \Rightarrow (x+1)(x+2) = 0$

$\Rightarrow \quad x = -1, -2$

Hence, four solutions are possible.

53. **Hint** Since, one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, then let their roots be α and 2α.

$\therefore \quad \alpha + 2\alpha = -\frac{(3a-1)}{(a^2 - 5a + 3)} \Rightarrow 3\alpha = -\frac{(3a-1)}{(a^2 - 5a + 3)}$

and $\quad \alpha \cdot 2\alpha = \frac{2}{(a^2 - 5a + 3)} \Rightarrow 2\alpha^2 = \frac{2}{(a^2 - 5a + 3)}$

54. Since, $\quad \alpha^2 = 5\alpha - 3$

$\Rightarrow \quad \alpha^2 - 5\alpha + 3 = 0$

and $\quad \beta^2 = 5\beta - 3$

$\Rightarrow \quad \beta^2 - 5\beta + 3 = 0$

These two equations shows that α and β are the roots of the equation

$$x^2 - 5x + 3 = 0$$

$\therefore \quad \alpha + \beta = 5$ and $\alpha\beta = 3$

Now, $\quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{25 - 6}{3} = \frac{19}{3}$ and $\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

$\therefore$ Required equation is

$$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 0$$

$\Rightarrow \quad x^2 - \frac{19}{3}x + 1 = 0$

$\Rightarrow \quad 3x^2 - 19x + 3 = 0$

6 Permutations and Combinations

JEE Main MILESTONE

- Fundamental Principles of Counting
- Factorial Notation
- Exponent of Prime p in $n!$
- Permutation
- Circular Permutation
- Combination
- Division of Objects into Groups
- Applications of Permutation and Combination
- Number of Divisors
- Dearrangements
- Number of Integral Solution of Linear Equations

6.1 Fundamental Principles of Counting [FPC]

The basis of all the techniques given in this chapter are based on two fundamental principles of counting namely addition and multiplication principles. Using these principles formulae for number of ways of arrangements and selections of different objects are derived.

(i) Fundamental Principle of Multiplication

If an operation can be performed in m different ways, following which a second operation can be performed in n different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

e.g., A hall has 12 gates. After entering into the hall the man come out through a different gate in 11 ways.

Hence, by the fundamental principle of multiplication, the total number of ways of man come out through different gates $= 12 \times 11 = 132$.

(ii) Fundamental Principle of Addition

If an operation can be performed in m different ways and another operation, which is independent of the first operation can be performed in n different ways. Then, either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exclusive operations.

e.g., There are 25 students in a class in which 15 boys and 10 girls. The class teacher select either a boy or a girl for monitor of the class. Since, there are 15 ways to select a boy and there are 10 ways to select a girl. Hence, by the fundamental principle of addition, the number of ways in which either a boy or a girl can be chosen as a monitor $= 10 + 15 = 25$ ways.

***Sample* Problem 1** *How many 3-digits even numbers can be formed from the digits 1, 2, 3, 4, 5, 6, if the digits can be repeated?* **[NCERT]**

(a) 108 (b) 107
(c) 110 (d) *None of these*

Interpret (a) Even digits = 2, 4, 6

$\therefore$ Number of ways to fill the unit place = 3,
Number of ways to fill the ten place = 6,
Number of ways to fill the hundred place = 6

H	T	U

Hence, by FPC total number of ways $= 3 \times 6 \times 6 = 108$

6.2 Factorial Notation

The product of first n natural numbers is denoted by $n!$ and read as 'factorial n'.

Thus, $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$

e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

and $4! = 4 \times 3! = 4 \times 3 \times 2!$
$= 4 \times 3 \times 2 \times 1 = 24$

Properties of Factorial Notation

(i) $0! = 1! = 1$

(ii) Factorials of negative integers and fractions are not defined.

(iii) $n! = n(n-1)! = n(n-1)(n-2)!$

(iv) $\frac{n!}{r!} = n(n-1)(n-2)\ldots(r+1)$

***Sample* Problem 2** *The units and ten's place digits in* $1! + 2! + 3! + \ldots + 49!$ *are respectively*

(a) 1, 3 (b) 3, 1
(c) 1, 1 (d) *None of these*

Interpret (b) We know that, $n!$ is divisible by 100 for $n \geq 10$ and $1! + 2! + 3! + \ldots + 9! = 409113$

$\therefore$ $1! + 2! + 3! + \ldots + 49! = 409113 +$ a multiple of 100

Hence, one's and ten's digit in $1! + 2! + 3! + \ldots + 49!$ are 3 and 1, respectively.

6.3 Exponent of Prime p in n!

Let n be a positive integer and p be a prime number. Then, last integer amongst $1, 2, 3, \ldots, (n-1), n$ which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}$.

e.g., $\left[\frac{12}{5}\right] = 2, \left[\frac{15}{5}\right] = 3$ etc.

Let $E_p(n!)$ denotes the exponent of prime p in $n!$, then

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^a}\right]$$

where a is a greatest positive integer such that $p^a \leq n \leq p^{a+1}$.

***Sample* Problem 3** *Let*

$E = \left[\frac{1}{3} + \frac{1}{50}\right] + \left[\frac{1}{3} + \frac{2}{50}\right] + \left[\frac{1}{3} + \frac{3}{50}\right] + \ldots +$ *upto* 50 *terms, then the exponent of 2 in* $E!$ *is*

(a) 17 (b) 25
(c) 15 (d) *None of these*

Interpret (c) Let $E = \left[\frac{1}{3} + \frac{x}{50}\right]$

where, $x = 1, 2, \ldots, 50$

For $1 \leq x \leq 33, \frac{1}{3} < \frac{1}{3} + \frac{x}{50} < 1$

$\therefore$ $\left[\frac{1}{3} + \frac{x}{50}\right] = 0$, for $1 \leq x \leq 33$

For $34 \leq x \leq 50$, $1 < \frac{1}{3} + \frac{x}{50} < \frac{4}{3}$

$\Rightarrow$ $\left[\frac{1}{3} + \frac{x}{50}\right] = 1$, for $34 \leq x \leq 50$

Thus, $E = 17$

and $p = 2$

$\because$ $2^4 < 17 < 2^5$

$\therefore$ $a = 4$

Exponent of 2 in $(17)!$

$$= \left[\frac{17}{2}\right] + \left[\frac{17}{4}\right] + \left[\frac{17}{8}\right] + \left[\frac{17}{16}\right] = 8 + 4 + 2 + 1 = 15$$

6.4 Permutation

Each of different arrangements which can be made by taking some or all of a number of things is called a permutation.

e.g., Arrangements of objects taking 2 at a time from given 3 objects (a, b, c) are ab, bc, ca, cb, ac, ba, then total number of arrangements is 6 each of which is known as permutation.

Meaning of ^nP_r

Number of permutations of n distinct objects taking r at a time is denoted by ^nP_r.

$$^nP_r = \frac{n!}{(n-r)!}, \forall 0 \leq r \leq n$$
$$= n(n-1)(n-2)\ldots(n-r+1), \forall n \in N \text{ and } r \in W.$$

Properties of ^nP_r

(i) The number of permutations of n distinct object taken all at a time is $^nP_n = n!$.

(ii) $^nP_0 = 1$, $^nP_1 = n$ and $^nP_{n-1} = n!$

(iii) $^nP_r = n \cdot {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1} + {}^{n-1}P_r$

(iv) $^{n-1}P_r = (n-r)\,{}^{n-1}P_{r-1}$

Important Results on Permutations

(i) The number of permutations of n things taken all at a time p are alike of one kind, q are alike of second kind, r are alike of third kind and remaining are distinct, is $\frac{n!}{p!\,q!\,r!}$.

(ii) The number of permutations of n different things, taken r at a time when each thing may be repeated any number of times is n^r.

(iii) **Number of permutations under certain conditions**

(a) Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement is $r \cdot {}^{n-1}P_{r-1}$.

(b) Number of permutations of n different things taken r at a time, when a particular thing is never taken in each arrangement is $^{n-1}P_r$.

(c) Number of permutations of n different things taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$

(d) Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m! \times (n-m+1)!$

(e) Number of permutations of n different things, taken r at a time when p particular things are to be always included in each arrangement is $p!\{r-(p-1)\}\,{}^{n-p}P_{r-p}$.

Hot Spot 1 Arrangement Types Problems

Arrangement types of problems usually have alphabate letters or digits, you are asked to find the words or numbers that can be formed under the conditions given in the question. These types of problems are directly or indirectly asked in various years of AIEEE (JEE Main) 2008, 2005, 2004, 2003. To solve such types of problems, following steps are given below.

Step I Firstly, separate the identical and distinct objects.

Step II Use the properties or results, if require.

Step III Use the formula $^nP_r = \frac{n!}{(n-r)!}$ and simplify it to get the result.

***Sample* Problem 4** *How many numbers of 4-digits can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, if atleast one digit is repeated?*

(a) 1562 (b) 1560
(c) 1561 (d) *None of these*

Interpret (c) **Step I** Separate the identical and different objects. Here, all objects are distinct.

Step II Use the properties or result.

When all digits are repeated, then number of ways $= 7^4$.

Step III Use the formula.

$\therefore$ Required number of ways $= 7^4 - {}^7P_4 = 2401 - 840 = 1561$

***Sample* Problem 5** *Find n, if* $^{n-1}P_3 : {}^nP_4 = 1:9$. [NCERT]

(a) 9 (b) 8 (c) 7 (d) 5

Interpret (a) Given, $^{n-1}P_3 : {}^nP_4 = 1:9$

$$\therefore \quad \frac{(n-1)!}{(n-1-3)!} : \frac{n!}{(n-4)!} = 1:9$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-4)!} : \frac{n(n-1)!}{(n-4)!} = 1:9$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n(n-1)!} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{9}$$

$$\Rightarrow \quad n = 9$$

***Sample* Problem 6** *In how many of the distinct permutations of the letters in the 'MISSISSIPPI' do the four I's not come together ?* [NCERT]

(a) 35810 (b) 33810
(c) 33820 (d) *None of these*

Interpret (b) The word MISSISSIPPI has 11 letters in which

M $\rightarrow$ 1 time
I $\rightarrow$ 4 times
S $\rightarrow$ 4 times
P $\rightarrow$ 2 times

The number of permutations of the word 'MISSISSIPPI' in which 4 I's and 4 S's are alike

$$= \frac{11!}{4!\,4!\,2!} \quad \ldots(i)$$

If all the I's are together, then it will be considered as one letter and remaining 7 letters and 1 I's letter (including 4 I's) will be considered as 8 letters. So, the number of permutations is $\frac{8!}{4!\,2!}$.

Hence, total number of arrangements $= \frac{11!}{4!\,4!\,2!} - \frac{8!}{4!\,2!}$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} - \frac{8 \times 7 \times 6 \times 5 \times 4!}{4! \times 2 \times 1}$$

$$= 34650 - 840 = 33810$$

***Sample* Problem 7** *If the different permutations of all the letters of the word 'EXAMINATION' are listed as in a dictionary, how many words are there in this list before the first word starting with E?* [**NCERT**]

(a) 908300 *ways* (b) 907200 *ways*
(c) 907300 *ways* (d) *None of these*

Interpret (*b*) The letters of given word are A, A, E, I, I, M, N, N, O, T, X *i.e.*, word starting with A are formed with the letters 2I's, 2N's, A, E, X, M, T, O (total 10 letters).

Hence, number of words formed by these letters

$$= \frac{10!}{2!\,2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4}$$

$$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 2 \times 1 = 907200 \text{ ways}$$

6.5 Circular Permutation

If objects are arranged along a closed curve, then permutation is known as circular permutation.

In other words, the permutation in a row has a beginnings and end but there is no beginning and end in circular permutation. So, we need to consider one object is fixed and the remaining objects are arranged in $(n-1)!$ ways.

e.g., Consider five persons A, B, C, D and E to be seated on the circumference of a circular table in order (which has no head). Now, shifting A, B, C, D and E one position in anti-clockwise direction, we will get arrangements as follows

we see that arrangements in all figures are same.

$\therefore$ The number of circular permutations of n different things taken all at a time is $\frac{{}^nP_n}{n} = (n-1)!$, if clockwise and anti-clockwise orders are taken as different.

Note In a circular permutation, if the position are given by number, then it is treated as a linear arrangement.

Important Results on Circular Permutation

(i) Number of circular permutations of n different things taken all at a time $= (n-1)!$. If clockwise and anti-clockwise orders are taken as different.

(ii) The number of circular permutations of n different things taken all at a time $= \frac{1}{2}(n-1)!$. If clockwise and anti-clockwise orders are taken as not different.

***Sample* Problem 8** *Three boys and three girls are to be seated around a round table, in a circle. Among them the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour, then the number of such arrangements is*

(a) 2 (b) 4
(c) 2^3 (d) *None of these*

Interpret (*b*) As shown in figure 1, 2 and *X* are the three boys and 3, 4 and *Y* are three girls. Boy *X* will have neighbours as boys 1 and 2 and the girl *Y* will have neighbours as girls 3 and 4.

Number of ways in which 1 and 2 can be arranged $= {}^2P_2 = 2! = 2$

Also, number of ways in which 3 and 4 can be arranged $= {}^2P_2$
$= 2! = 2$

Hence, required number of permutations $= 2 \times 2 = 4$.

Check Point 1

1. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, then the value of n is equal to
   (a) 4 (b) 3 (c) 2 (d) 1
2. Seven different lecturers are to deliver lectures in seven periods of a class on a particular day. A, B and C are three of the lecturers. The number of ways in which a routine for the day can be made such that A delivers his lecture before B and B before C, is
   (a) 420 (b) 120 (c) 210 (d) 840
3. A letter lock contains 5 rings each marked with for different letters. The number of all possible unsuccessful attempts to open the lock is
   (a) 625 (b) 1024 (c) 624 (d) 1023
4. The number of ways in which four particular persons A, B, C, D and six more persons can stand in a queue so that A always stands before B, B before C and C before D, is
   (a) 7! 4! (b) 10! – 7! 4!
   (c) $\frac{10!}{4!}$ (d) None of these
5. The number of ways in which 7 pictures can be hung from 5 picture nails on the wall is
   (a) 7^5 (b) 5^7
   (c) 2520 (d) None of these
6. The letters of the word RACHIT are written in all possible manner and words are written as in dictionary. The rank of word RACHIT is
   (a) 365 (b) 702 (c) 481 (d) 480
7. The number of ways in which 10 different diamonds can be arranged to form a necklace, is
   (a) 181440 (b) 161400
   (c) 261960 (d) None of these
8. How many numbers of 4-digits can be formed by using the digits 1, 2, 3, 4, 5, 6, 7 if atleast one digit is repeated ?
   (a) ${}^7P_4$ (b) 7^4
   (c) $7^4 - {}^7P_4$ (d) None of these
9. If a represents the number of permutations of (x + 2) things taken together, b represents the number of permutation of 11 things taken together out of x things, and c represents the number of permutation of (x – 11) things taken together so that a = 182bc, then x is equal to
   (a) 15 (b) 12 (c) 10 (d) 18
10. There are 4 parcels and 5 post offices. In how many ways can 4 parcels be got registered?
   (a) 20 (b) 4^5 (c) 5^4 (d) $5^4 - 4^5$

6.6 Combination

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

e.g., The groups made by taking 2 objects at a time from three objects (a, b, c) are ab, bc, ca. Then, the number of groups is 3 each of which is known as combination.

Note In permutation order of objects is important whereas in combination order of objects is not important.

Meaning of ${}^nC_r$

The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.

Then, $${}^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!} \qquad (0 \le r \le n)$$

$$= \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{r\,(r-1)\,(r-2)\ldots 2\cdot 1}$$

$$n \in N \text{ and } r \in W$$

If $r > n$, then ${}^nC_r = 0$

Properties of ${}^nC_r$

(a) ${}^nC_r$ is a natural number.

(b) ${}^nC_0 = {}^nC_n = 1$, ${}^nC_1 = n$

(c) ${}^nC_r = {}^nC_{n-r}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) ${}^nC_x = {}^nC_y$

$\Leftrightarrow \quad x = y$

or $\quad x + y = n$

(f) $n \cdot {}^{n-1}C_{r-1} = (n - r + 1)\,{}^nC_{r-1}$

(g) If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

(h) If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$

or $\quad {}^nC_{\frac{n-1}{2}}$.

(i) ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

(j) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(k) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

(l) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$

(m) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

(n) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Important Results on Combinations

(i) The number of combinations of n different things, taken r at a time, where p particular things occur is ${}^{n-p}C_{r-p}$.

(ii) The number of combinations of n different things, taken r at a time, where p particular things never occur is ${}^{n-p}C_r$.

(iii) The total number of combinations of n different things taken one or more at a time or the number of ways of n different things selecting atleast one of them is

$$ {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1 $$

(iv) The number of combinations of n identical things taking $r\,(r \le n)$ at a time is 1.

(v) The number of ways of selecting r things out of n alike things is $(n+1)$, (where $r = 0, 1, 2, 3, \ldots, n$).

(vi) If out of $(p+q+r)$ things, p are alike of one kind, q are alike of second kind and rest are alike of third kind, then the total number of combinations is

$$ [(p+1)(q+1)(r+1)] - 1 $$

(vii) If out of $(p+q+r+t)$ things, p are alike of one kind, q are alike of second kind, r are alike of third kind and t are different, then the total number of combinations is $(p+1)(q+1)(r+1)2^t - 1$

***Sample* Problem 9** *If ${}^{2n}C_3 : {}^nC_3 = 11:1$, then the value of n is.* **[NCERT]**

(a) 5
(b) 16
(c) 6
(d) 4

Interpret (c) ${}^{2n}C_3 : {}^nC_3 = 11:1$

$$ \therefore \quad \frac{2n(2n-1)(2n-2)}{6} : \frac{n(n-1)(n-2)}{6} = 11:1 $$

$$ \Rightarrow \quad 2n(2n-1)2(n-1) : n(n-1)(n-2) = 11:1 $$

$$ \Rightarrow \quad 2n(2n-1)2(n-1) \times \frac{1}{n(n-1)(n-2)} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{4(2n-1)}{n-2} = \frac{11}{1} $$

$$ \Rightarrow \quad 8n - 4 = 11n - 22 $$

$$ \Rightarrow \quad 3n = 18 $$

$$ \Rightarrow \quad n = 6 $$

Hot Spot 2

Selection Types Problems

Selection types of problems usually have a group of different persons or objects out of these objects or persons some are related following the conditions given in the questions. These types of problems are directly or indirectly asked in various years of AIEEE (JEE Main) 2013, 2011, 2010, 2008, 2006, 2004, 2003.

To solve such types of problems, following steps are given below.

Step I Use the properties or result, if require.

Step II Use the formula ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ and simplify it to get the result.

***Sample* Problem 10** *Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls, if each selection consists of 3 balls of each colour.* **[NCERT]**

(a) 2200 (b) 2000 (c) 2050 (d) 2400

Interpret (b) **Step I** Use the properties or results. Since, 3 red balls out of 6 red balls can be selected in ${}^6C_3$ ways, 3 white balls out of 5 white balls can be selected in ${}^5C_3$ ways and 3 blue balls out of 5 blue balls can be selected in ${}^5C_3$ ways.

Total number of ways selecting 9 balls (3 balls of each colour) is

$$ = {}^6C_3 \times {}^5C_3 \times {}^5C_3 $$

$$ = {}^6C_3 \times {}^5C_2 \times {}^5C_2 \qquad (\because {}^nC_r = {}^nC_{n-r}) $$

Step II Use the formulae,

$$ = \frac{6 \times 5 \times 4}{6} \times \frac{5 \times 4}{2} \times \frac{5 \times 4}{2} $$

$$ \left[\because {}^nC_3 = \frac{n(n-1)(n-2)}{6} \text{ and } {}^nC_2 = \frac{n(n-1)}{2} \right] $$

$$ = 20 \times 10 \times 10 = 2000 $$

6.7 Division of Objects into Groups

Objects are Different

(i) The number of ways of dividing n different objects into 3 groups of p, q and r things

$$(p + q + r = n) \text{ is}$$

(a) $\dfrac{n!}{p!\,q!\,r!}$; p, q and r are unequal.

(b) $\dfrac{n!}{p!\,2!\,(q!)^2}$; $q = r$

(c) $\dfrac{n!}{3!\,(p!)^3}$; $p = q = r$

(ii) The number of ways of dividing n different objects into r groups is

$$\frac{1}{r!}\left[r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \binom{r}{3}(r-3)^n + \ldots\right]$$

(iii) The number of ways of dividing n different objects into r groups taking into account the order of the groups and also the order of objects in each group is

$${}^{(n+r-1)}P_n = r\,(r+1)\,(r+2)\ldots(r+n-1).$$

Objects are Identical

(i) The number of ways of dividing n identical objects among r persons such that each gets 1,2,3,... or k objects is the coefficient of x^{n-r} in the expansion of $(1 + x + x^2 + \ldots + x^{k-1})^r$.

(ii) The number of ways of dividing n identical objects among r persons such that each one may get atmost n objects is $\dbinom{n+r-1}{r-1}$,

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(iii) The total number of ways of dividing n identical objects among r persons, each one of whom, receives atleast one item is ${}^{n-1}C_{r-1}$.

OR

The number of ways in which n identical things can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

Sample **Problem 11** *The number of ways in which 9 identical balls can be placed in three identical boxes is* **[NCERT]**

(a) 55 (b) $\dfrac{9!}{(3!)^4}$

(c) $\dfrac{9!}{(3!)^3}$ (d) 12

Interpret (*d*) $\therefore$ Required number of ways

$$= ({}^3C_1 \times {}^2C_1 \times {}^1C_1) \times 2$$

$$= 3 \times 2 \times 1 \times 2 = 12$$

Arrangement in Groups

(a) The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}P_n$ or $n!\,{}^{n-1}C_{r-1}$.

(b) The number of ways in which n different things can be distributed into r different groups is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1}\cdot{}^rC_{r-1}$$

or coefficient of x^n in $n!\,(e^x - 1)^r$.

Here, blank groups are not allowed.

(c) The number of ways in which n identical things can be distributed into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$, according as blank groups are or are not admissible.

6.8 Applications of Permutation and Combination

Functional Applications

(i) The number of all permutations (arrangements) of n different objects taken r at a time,

(a) when a particular object is to be always included in each arrangement is ${}^{n-1}C_{r-1} \times r!$

(b) when a particular object is never taken in each arrangement is ${}^{n-1}C_r \times r!$

(ii) If the sets A has m elements and B has n elements, then

(a) the number of functions from A to B is n^m.

(b) the number of one-one functions from A to B is ${}^nP_m$, $m \le n$.

(c) the number of onto functions from A to B is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \ldots,\ m \le n.$$

(d) the number of increasing (decreasing) functions from A to B is $\dbinom{n}{m}$, $m \le n$.

(e) the number of non-decreasing (non-increasing) functions from A to B is

$$\binom{m+n-1}{m}, \quad m \le n.$$

(f) the number of bijections from A to B is $n!$, if $m = n$.

(g) the number of bijections from A to A such that $f(x) \ne x, \forall\ x \in A$, is

$$m!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{(-1)^m}{m!}\right].$$

Sample **Problem 12** *Let* $A = \{x_1, x_2, x_3, x_4, x_5\}$, $B = \{y_1, y_2, y_3, y_4, y_5\}$. *Then, the number of one one mappings, from A to B such that* $f(x_i) \neq y_i$, $i = 1, 2, \ldots, 5$ *is* [**NCERT**]

(a) 40 (b) 44 (c) 6 (d) 24

Interpret (b) Here, we use the formula, the number of bijections from A to A such that $f(x) \neq x, \forall\, x \in A$ is

$$m!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{(-1)^m}{m!}\right]$$

Here, $m = 5$

$\therefore$ Required number of ways $= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

$$= 120\left(\frac{60 - 20 + 5 - 1}{120}\right) = 44$$

Geometrical Applications

(i) Out of n non-concurrent and non-parallel straight lines the number of point of intersection are ${}^nC_2$.

(ii) The number of straight lines passing through n points $= {}^nC_2$.

(iii) The number of straight lines passing through n points out of which m are collinear $= {}^nC_2 - {}^mC_2 + 1$.

(iv) In a polygon, the total number of diagonals out of n points (no three points are collinear) $= {}^nC_2 - n = \frac{n(n-3)}{2}$.

(v) Number of triangles formed by joining n points is ${}^nC_3$.

(vi) Number of triangles formed by joining n points out of which m are collinear are ${}^nC_3 - {}^mC_3$.

(vii) The number of parallelogram in two systems of parallel lines (when Ist set contains m parallel lines and 2nd set contains n parallel lines) $= {}^nC_2 \times {}^mC_2$

(viii) The number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} n^2$.

Sample **Problem 13** *The total number of selections of atmost n things with* 10 *points as vertices, n of them being collinear, is* 110. *Then, n is* [**NCERT**]

(a) 13 (b) 4 (c) 5 (d) 6

Interpret (c) Out of 10 points, let n points be collinear. Then, number of triangles are ${}^{10}C_3 - {}^nC_3 = 110$

$$\therefore \quad \frac{10 \times 9 \times 8}{3 \times 2} - \frac{n(n-1)(n-2)}{3 \times 2} = 110$$

$$\Rightarrow \quad 720 - 660 = n(n-1)(n-2)$$

$$\Rightarrow \quad n(n-1)(n-2) = 60$$

$$\Rightarrow \quad n(n-1)(n-2) = 5 \times 4 \times 3$$

$$\Rightarrow \quad n = 5$$

Sample **Problem 14** *In a polygon the number of diagonals is* 54. *The number of sides of the polygon is*

(a) 10 (b) 12

(c) 9 (d) *None of these*

Interpret (b) Let number of sides of polygon is n. Since, number of sides of polygon is equal to number of vertices of polygon.

$\because$ Number of diagonals of polygon $= 54$

$$\therefore \quad {}^nC_2 - n = 54$$

$$\Rightarrow \quad \frac{n(n-1)}{2} - n = 54$$

$$\Rightarrow \quad n^2 - 3n = 108$$

$$\Rightarrow \quad n^2 - 3n - 108 = 0$$

$$\Rightarrow \quad (n+9)(n-12) = 0$$

$$\Rightarrow \quad n = 12 \qquad (\because n \text{ cannot be negative})$$

6.9 Number of Divisors

Let $N = p_1^{\alpha_1} . p_2^{\alpha_2} . p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3 \ldots, p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_k$ are natural numbers, then

(i) the total number of divisors of N including 1 and N

$$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$$

(ii) the total number of divisors of N excluding 1 and N

$$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 2$$

(iii) the total number of divisors of N excluding 1 or N

$$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 1$$

(iv) the sum of these divisors

$$= (p_1^0 + p_1^1 + p_1^2 + \ldots + p_1^{\alpha_1})$$

$$(p_2^0 + p_2^1 + p_2^2 + \ldots + p_2^{\alpha_2}) \ldots (p_k^0 + p_k^1 + p_k^2 + \ldots + p_k^{\alpha_k})$$

(use sum of GP in each bracket)

(v) the number of ways in which N can be resolved as a product of two factors is $\frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$, if N is not a perfect square

and $\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) + 1]$, if N is a perfect square.

(vi) the number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal 2^{n-1}, where n is the number of different factors in N.

Sample **Problem 15** *The total number of divisors of 5880 is* [**NCERT**]

(a) 48 (b) 24 (c) 96 (d) 16

Interpret (a) $5880 = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^2$

$\therefore$ Total number of divisors $= (3+1)(1+1)(1+1)(2+1)$

$= 4 \times 2 \times 2 \times 3 = 48$

6.10 Dearrangements

If n distinct objects are arranged in a row, then the number of ways in which they can be dearranged so that none of them occupies its original place is

$$n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right\}$$

and it is denoted by $D(n)$.

If r $(0 \le r \le n)$ objects occupy the places assigned to them *i.e.*, their original places and none of the remaining $(n-r)$ objects occupies its original places, then the number of such ways is

$$D(n-r) = {}^nC_r \cdot D(n-r)$$

$$= {}^nC_r \cdot (n-r)!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right\}$$

Sample **Problem 16** *Ajay writes letters to his five friends and addresses the corresponding. The number of ways can the letters be placed in the envelops so that atleast two of them are in the wrong envelopes are*

(a) 120 (b) 125 (c) 119 (d) 124

Interpret (c) Required number of ways $= \sum_{r=2}^{5} {}^5C_{5-r}\, D(r)$

$$= \sum_{r=2}^{5} \frac{5!}{r!(5-r)!} \cdot r!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^r}{r!}\right\}$$

$$= \sum_{r=2}^{5} \frac{5!}{(5-r)!}\left\{\frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^r}{r!}\right\}$$

$$= \frac{5!}{3!}\left\{\frac{1}{2!}\right\} + \frac{5!}{2!}\left\{\frac{1}{2!} - \frac{1}{3!}\right\} + \frac{5!}{1!}\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\} + \frac{5!}{0!}\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$= 10 + 20 + (60 - 20 + 5) + (60 - 20 + 5 - 1)$

$= 10 + 20 + 45 + 44 = 119$

6.11 Number of Integral Solutions of Linear Equations

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n \quad \ldots(i)$$

where $x_1, x_2, \ldots, x_r$ and n are non-negative integers objects are to be divided into r groups where a group may contain any number of objects. Therefore, total number of solutions of Eq. (i),

$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + \ldots + x^n)^r$

$= {}^{n+r-1}C_r$ or ${}^{n+r-1}C_{n-1}$

Sample **Problem 17** *Total number of non-negative integral solutions of $x_1 + x_2 + x_3 = 10$ is equal to*

(a) ${}^{12}C_3$ (b) ${}^{10}C_3$ (c) ${}^{12}C_2$ (d) ${}^{10}C_2$

Interpret (c) Number of non-negative integral solutions

$= {}^{12}C_{10} = {}^{12}C_2$

Check Point 2

1. The value of $({}^7C_0 + {}^7C_1) + ({}^7C_1 + {}^7C_2) + \ldots + ({}^7C_6 + {}^7C_7)$ is
   (a) $2^7 - 1$ (b) $2^8 - 2$ (c) $2^8 - 1$ (d) 2^8
2. The number of products that can be formed with 10 prime numbers taken two or more at a time is
   (a) 2^{10} (b) $2^{10} - 1$ (c) $2^{10} - 11$ (d) $2^{10} - 10$
3. How many ways can 6 coins be chosen from 20, one rupees coins, 10 fifty paise coins, 7 twenty paise coins ?
   (a) 28 (b) 56 (c) ${}^{37}C_6$ (d) 38
4. A person wishes to make up as many different parties as he can out of 20 friends. Each party consists of the same number of friends. How many should be invited at a time ?
   (a) 8 (b) 9 (c) 10 (d) 11
5. Total number of divisors of 5880 is equal to
   (a) 48 (b) 24 (c) 96 (d) 16
6. There are 6 letters and 6 directed envelopes. Find the number of ways in which all letters are put in the wrong envelopes.
   (a) 260 (b) 265
   (c) 270 (d) 275
7. The number of non-negative integral solutions of $x + y + z \le n$, where $n \in N$ is
   (a) ${}^{n+3}C_3$ (b) ${}^{n+4}C_4$ (c) ${}^{n+5}C_5$ (d) ${}^{n+2}C_2$
8. The number of ways in which a committee of 6 members can be formed from 8 gentlemen and 4 ladies so that the committee contains atleast 3 ladies is
   (a) 252 (b) 672 (c) 444 (d) 420
9. Eight straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. The number of parts into which these lines divide the plane, is
   (a) 29 (b) 32 (c) 36 (d) 37
10. The total number of ways in which 9 different boys can be distributed among three different children, so that the youngest gets 4, the middle gets 3 and the oldest gets 2, is
   (a) 137 (b) 236 (c) 1240 (d) 1260

WORKED OUT
Examples

Example 1 *The value of* ${}^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \ldots + n \cdot {}^nP_n$ *is*

(*a*) ${}^{n+1}P_{n+1} - 1$ (*b*) ${}^{n+1}P_{n+1}$

(*c*) ${}^{n+1}C_{n+1} - 1$ (*d*) *None of these*

Solution (*a*) ${}^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \ldots + n \cdot {}^nP_n$

$= (1)! + 2(2)! + 3(3)! + \ldots + n(n)!$ $(\because x! = {}^xP_x, x \in N)$

$= (2-1)1! + (3-1)2! + (4-1)3! + \ldots + [(n+1)-1]n!$

$= (2!-1!) + (3!-2!) + (4!-3!) + \ldots + [(n+1)! - n!]$

$= (n+1)! - 1$ $[\because x(x-1)! = x!, x \in N]$

$= {}^{n+1}P_{n+1} - 1$

Example 2 *How many 3-digits even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?* **[NCERT]**

(*a*) 60 (*b*) 61

(*c*) 59 (*d*) 62

Solution (*a*) Even digits = 2, 4, 6

$\therefore$ Number of ways to fill the unit place = 3,

Number of ways to fill the tenth place (in rest of 5-digits) = 5,

H T U

Number of ways of fill the hundred place (in rest of 4 digits) = 4

Hence, by FPC total number of ways $= 3 \times 5 \times 4 = 60$

Example 3 *Find the number of 5 card combinations out of a deck of 52 cards, if there is exactly one ace in each combination.* **[NCERT]**

(*a*) 778315 (*b*) 778325

(*c*) 778320 (*d*) 778328

Solution (*c*) Number of ways selecting one ace from 4 aces $= {}^4C_1$

Number of ways selecting 4 cards out of 48 cards $= {}^{48}C_4$

Hence, by FPC, 5 cards out of 52 cards with exactly one ace can be selected in

${}^4C_1 \times {}^{48}C_4$ ways

$$= 4 \times \frac{48 \times 47 \times 46 \times 45}{24}$$

$$\left[\because {}^nC_1 = n \text{ and } {}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{24}\right]$$

$$= \frac{48 \times 47 \times 46 \times 45}{6}$$

$= 8 \times 47 \times 46 \times 45$

$= 778320$ ways

Example 4 *In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl, if each cricket team of 11 must include exactly 4 bowlers?* **[NCERT]**

(*a*) 3960 (*b*) 3961

(*c*) 3962 (*d*) *None of these*

Solution (*a*)

We are to select 11 players including exactly 4 bowlers.

Hence, 4 bowlers will be selected from 5 bowlers and remaining 7 players will be selected from 12 batsmen.

Now, 4 bowlers out of 5 bowlers can be selected in ${}^5C_4$ ways.

7 players out of 12 players can be selected in ${}^{12}C_7$ ways.

Hence, by FPC, total number of ways selecting the eleven players

$= {}^5C_4 \times {}^{12}C_7 = {}^5C_1 \times {}^{12}C_5$ $(\because {}^nC_r = {}^nC_{n-r})$

$$= \frac{5 \times 12 \times 11 \times 10 \times 9 \times 8}{120}$$

$$\left[\because {}^nC_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}\right]$$

$= 5 \times 11 \times 9 \times 8 = 55 \times 72 = 3960$ ways

Example 5 *In how many different ways can five boys and five girls form a circle such that the boys and girls set alternate ?*

(*a*) 4!

(*b*) 2880

(*c*) 5!

(*d*) *None of the above*

Solution (*b*) After fixing up one boy on the table the remaining can be arranged in 4! ways but boys and girls are to sit alternate.

There will be 5 places, between two boys, these five places can be filled by 5 girls in 5! ways.

The required number of ways $= 4! \times 5! = 2880$.

Example 6 *In a certain test, there are n questions. In the test 2^{n-1} students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \ldots, n$. If the total number of wrong answers given is 2047, then the value of n is*

(a) 11 (b) 2047
(c) 12 (d) *None of these*

Solution (a) The number of students answering exactly $i\ (1 \le i \le n-1)$ questions wrongly is $2^{n-i} - 2^{n-i-1}$. The number of students answering all n questions wrongly is 2047. Thus, the total number of wrong answers is

$$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + \ldots + (n-1)(2^1 - 2^0) + n(2^0)$$
$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^0$$
$$= 2^n - 1$$

Thus, $2^n - 1 = 2047$

$\Rightarrow$ $2^n = 2048 = 2^{11}$

$\Rightarrow$ $n = 11$

Example 7 *Four boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes be identical ?*

(a) ${}^{33}C_4$ (b) ${}^{33}C_2$
(c) 5456 (d) *None of these*

Solution (c) Clearly, 30 mangoes can be distributed among 4 boys such that each boy can receive any number of mangoes.

Hence, total number of ways $= {}^{30+4-1}C_{4-1} = {}^{33}C_3$

$$= \frac{33 \cdot 32 \cdot 31}{1 \cdot 2 \cdot 3} = 5456$$

Example 8 *The sum of all five digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 (repetition of digits is not allowed), is*

(a) 11111 (b) 3999960
(c) 399960 (d) 396990

Solution (b) Required sum

$$= (5-1)!(1+2+3+4+5)\left(\frac{10^5 - 1}{9}\right)$$
$$= 24 \cdot 15 \cdot 11111 = 3999960$$

Example 9 *If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^nC_2$.* **[NCERT]**

(a) 8 (b) 7
(c) 9 (d) 6

Solution (c) Given,

$${}^nC_{r-1} = 36 \quad \ldots(i)$$
$${}^nC_r = 84 \quad \ldots(ii)$$
$${}^nC_{r+1} = 126 \quad \ldots(iii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{36}{84} \qquad \left[\because {}^nC_r = \frac{n!}{(n-r)!\,r!} \text{ and } n! = n(n-1)!\right]$$

$$\Rightarrow \frac{n!}{(r-1)!\{n-(r-1)\}!} \times \frac{r!(n-r)!}{n!} = \frac{3}{7}$$

$$\Rightarrow \frac{1}{(r-1)!(n-r+1)!} \times \frac{r(r-1)!(n-r)!}{1} = \frac{3}{7}$$

$$\Rightarrow \frac{1 \cdot r}{(n-r+1)(n-r)!} \times (n-r)! = \frac{3}{7}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow 7r = 3n - 3r + 3$$

$$\Rightarrow 10r - 3n = 3 \quad \ldots(iv)$$

Now, dividing Eq. (ii) by Eq. (iii),

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \frac{n!}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{n!} = \frac{14}{21}$$

$$\Rightarrow \frac{1}{r!(n-r)(n-r-1)!} \times \frac{(r+1)r!(n-r-1)!}{1} = \frac{2}{3}$$

$$\Rightarrow \frac{r+1}{n-r} = \frac{2}{3}$$

$$\Rightarrow 3r + 3 = 2n - 2r$$

$$\Rightarrow 2n - 5r = 3 \quad \ldots(v)$$

Now, multiplying Eq. (iv) by 2 and Eq. (v) by 3 and adding both the equations, we get

$$20r - 6n = 6 \quad \ldots(vi)$$

and $$6n - 15r = 9 \quad \ldots(vii)$$

Now, adding Eqs. (vi) and (vii), we get

$$5r = 15$$

$$\Rightarrow r = 3$$

Now, from Eq. (v),

$$2n = 3 + 15$$

$$\Rightarrow 2n = 18$$

$$\Rightarrow n = 9$$

Example 10 *Eighteen guests are to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three on other side of the table. Find the number of ways in which the seating arrangement can be made.* **[NCERT Exemplar]**

(a) $\frac{11!}{5!6!}(9!)(8!)$ (b) $\frac{11!}{5!6!}(9!)(9!)$

(c) $\frac{11!}{6!6!}(9!)(9!)$ (d) *None of these*

Solution (b) After seating 4 on one side and 3 on the other side, we have to select out of 11; 5 on one side and 6 on the other side.

Now, there are 9 on each side of the long table and each cannot be arranged in 9! ways.

$\therefore$ The number of ways in which the seating arrangements can be made $= \frac{11!}{5!6!}(9!)(9!)$.

Example 11 *A seven digit number divisible by 9 is to be formed by using 7-digit out of numbers* $\{1, 2, 3, \ldots, 9\}$. *The number of ways in which this can be done, is*

(a) 7! (b) 2 (7!) (c) 3 (7!) (d) 4 (7!)

Solution (d) Sum of 7-digit = A multiple of 9

We know that, sum of numbers 1, 2 , 3, ..., 9 is 45. So, two left numbers should also have sum as 9. The pairs to the left are $(1, 8), (2, 7), (3, 6), (4, 5)$ with each part left number of 7-digits numbers is 7!.

So, with all 4 pairs $= 4 \times 7!$

Example 12 *The number of ways of dividing 15 men and 15 women into 15 couples, each consisting of a man and a woman, is*

(a) 1240 (b) 1840 (c) 1820 (d) 2005

Solution (a) The number of ways of choosing first couple

$= (15) \times (15) = 15^2$

The number of ways of choosing second couple $= 14 \times 14 = (14)^2$

and so on

Thus, required number of ways

$$= 15^2 + 14^2 + \ldots + 1^2 = \frac{15 \times (15+1) \times (30+1)}{6} = 1240$$

$$\left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

Example 13 *The letters of the word MODESTY are written in all possible orders and these words are written out as in a dictionary, then the rank of the word MODESTY is*

(a) 5040 (b) 720 (c) 1681 (d) 2520

Solution (c) The number of words start with $D = 6! = 720$

The number of words start with $E = 6! = 720$

The number of words start with $MD = 5! = 120$

The number of words start with $ME = 5! = 120$

Now, the first word start with MO is MODESTY.

Hence, rank of MODESTY $= 720 + 720 + 120 + 120 = 1681$

Example 14 *A bag contains four one rupee coins, two twenty five-paisa coins and five ten-paisa coins. In how many ways can amount, not les than ₹ 1 be taken out from the bag? (Consider coins of the same denominations to be identical)*

(a) 71 (b) 72 (c) 73 (d) 80

Solution (c) When atleast one one rupee coin is selected we can select any number of twenty-five paisa coins and ten paisa coins. Then, number of ways of such selection is $4(2+1)(5+1) = 72$ as we can select zero or more twenty five paisa and ten paisa coins to ensure that amount selected is ₹ 1 or more.

But when none of the one rupee coins is selected, we have to select all twenty five paisa coins and ten paisa coins to ensure sum of ₹ 1.which can be done only in one way. Then, the total number of ways is 73.

Example 15 *Rajdhani Express going from Bombay to Delhi stops at five intermediate stations, 10 passenger enter the train during the journey with 10 different ticket of two classes. The number of different sets of tickets they may have is*

(a) ${}^{15}C_{10}$ (b) ${}^{20}C_{10}$

(c) ${}^{30}C_{10}$ (d) *None of these*

Solution (c) For a particular class the total number of different tickets from first intermediate station is 5. Similarly, number of different tickets from second intermediate station is 4. So, the total number of different tickets is $5 + 4 + 3 + 2 + 1 = 15$. And same number of tickets for another class is equal to total number of different tickets, which is equal to 30 and number of selection is ${}^{30}C_{10}$.

Example 16 *Five distinct letters are to be transmitted through a communication channel. A total number of 15 blanks is to be inserted between the two letters with atleast three between every two. The number of ways in which this can be done, is*

(a) 20 (b) 2

(c) 6 (d) 2400

Solution (a) For $1 \le (i) \le 4$, let $x_i (\ge 3)$ be the number of blanks between ith and $(i+1)$th letters. Then,

$$x_1 + x_2 + x_3 + x_4 = 15 \quad \ldots(i)$$

The number of solutions of Eq. (i)

$= \text{Coefficient of } t^{15} \text{ in } (t^3 + t^4 + \ldots)^4$

$= \text{Coefficient of } t^3 \text{ in } (1-t)^{-4}$

$= \text{Coefficient of } t^3 \text{ in } [1 + {}^4C_1 + {}^5C_2 t^2 + {}^6C_3 t^3 + \ldots]$

$= {}^6C_3 = 20$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Fundamental Principles of Counting

1. How many 5-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once? [NCERT]

(a) 336 (b) 337
(c) 335 (d) None of these

2. Given 5 flags of different colours, how many different signals can be generated, if each signal requires the use of 2 flags, one below the other. [NCERT]

(a) 18 (b) 20
(c) 19 (d) 23

3. From a committee of 8 persons, in how many ways can we choose a chairman and a vice-chairman assuming one person cannot hold more than one position? [NCERT]

(a) 54 (b) 55
(c) 52 (d) 56

Factorial Notation

4. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, then the value of x is [NCERT]

(a) 63 (b) 64
(c) 66 (d) 65

5. The value of $2^n\,[1\cdot 3\cdot 5\ldots(2n-3)(2n-1)]$ is

(a) $\frac{(2n)!}{n!}$ (b) $\frac{(2n)!}{2^n}$
(c) $\frac{n!}{(2n)!}$ (d) None of these

6. The products of any r consecutive natural numbers is always divisible by

(a) $r!$ (b) r^2
(c) r^n (d) None of these

7. The exponent of 3 in 100! is

(a) 47 (b) 48 (c) 49 (d) 50

Permutation

8. If ${}^5P_r = 2\,{}^6P_{r-1}$, then the value of r is [NCERT]

(a) 10 (b) 3
(c) 0 (d) None of these

9. In how many ways can the letters of the word PERMUTATIONS be arranged, if the words start with P and end with S? [NCERT]

(a) 1814400 (b) 1814405
(c) 1824050 (d) None of these

10. How many different non-digit numbers can be formed from the digits of the number 223355888 by rearrangement of the digits so that the odd digits occupy even places?

(a) 16 (b) 36 (c) 60 (d) 180

11. How many even numbers of 3 different digits can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (repetition of digits is not allowed)?

(a) 224 (b) 280
(c) 324 (d) None of these

12. The total number of 9-digit numbers which have all different digits is [NCERT Exemplar]

(a) 10! (b) 9! (c) $9\times 9!$ (d) $10\times 10!$

13. The number of words which can be formed out of the letters of the word ARTICLE, so that vowels occupy the even place is [NCERT Exemplar]

(a) 1440 (b) 144 (c) 7! (d) ${}^4C_4\times{}^3C_3$

14. Eight chairs are numbered 1 to 8. Two women and 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs 1 to 4 and then men select from the remaining chairs. Find the total number of possible arrangements. [NCERT Exemplar]

(a) 1440 (b) 1450
(c) 1460 (d) None of these

15. If the letters of the word RACHIT are arranged in all possible ways as listed in dictionary. Then, what is the rank of the word RACHIT? [NCERT Exemplar]
(a) 479 (b) 480 (c) 481 (d) 482

16. Find the number of different words that can be formed from the letters of the word TRIANGLE, so that no vowels are together. [NCERT Exemplar]
(a) 14000 (b) 14500
(c) 14400 (d) 14402

17. The number of different four digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit only once is [NCERT Exemplar]
(a) 120 (b) 96 (c) 24 (d) 100

18. The sum of the digits in unit place of all the numbers formed with the help of 3, 4, 5 and 6 taken all at a time is [NCERT Exemplar]
(a) 432 (b) 108 (c) 36 (d) 18

19. The number of 5-digits telephone numbers having atleast one of their digits repeated, is [NCERT Exemplar]
(a) 90000 (b) 100000
(c) 30240 (d) 69760

20. We are to form different words with the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which begin with I and end with R, then m_1/m_2 is equal to
(a) 30 (b) 60 (c) 90 (d) 180

21. If the letters of the word KRISNA are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word KRISNA is
(a) 324 (b) 341
(c) 359 (d) None of these

22. If the letters of the word MOTHER are written in all possible orders and these words are written out as in a dictionary, then the rank of the word MOTHER is
(a) 240 (b) 261
(c) 308 (d) 309

23. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together. [NCERT Exemplar]
(a) ${}^{n-3}C_{r-3}(r-2)!\,3!$ (b) ${}^{n-3}C_{r-3}(r-3)!\,3!$
(c) ${}^{n-3}C_{r-3}(r-2)!$ (d) None of these

24. There are 10 persons named $P_1, P_2, P_3, \ldots, P_{10}$. Out of 10 persons, 5 persons are to be arranged in a line such that in each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements. [NCERT Exemplar]
(a) 4210 (b) 4200 (c) 4203 (d) 4205

25. If a denotes the number of permutations of $x+2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x-11$ things taken all at a time such that $a = 182\,bc$, then the value of x is
(a) 15 (b) 12 (c) 10 (d) 18

26. In a circus there are ten cages for accommodating ten animals. Out of these four cages are so small that five out of 10 animals cannot enter into them. In how many ways will it be possible to accommodate ten animals in these ten cages?
(a) 66400 (b) 86400
(c) 96400 (d) None of these

27. The total number of permutations of $n\ (>1)$ different things taken not more than r at a time, when each thing may be repeated any number of times is
(a) $\dfrac{n(n^n-1)}{n-1}$ (b) $\dfrac{n^r-1}{n-1}$
(c) $\dfrac{n(n^r-1)}{n-1}$ (d) None of these

28. The number of ways in which 10 candidates $A_1, A_2, \ldots, A_{10}$ can be ranked such that A_1 is always above A_{10} is
(a) 5! (b) 2(5!)
(c) 10! (d) $\frac{1}{2}(10!)$

Circular Permutation

29. If eleven members of a committee sit at a round table so that the President and Secretary always sit together, then the number of arrangements is
(a) $10! \times 2$ (b) 10!
(c) $9! \times 2$ (d) None of these

30. The number of ways in which seven persons can be arranged at a round table, if two particular persons may not sit together is
(a) 480 (b) 120
(c) 80 (d) None of these

31. In how many ways can 15 members of a council sit along a circular table, when the Secretary is to sit on one side of the Chairman and the Deputy Secretary on the other side?
(a) $2 \times 12!$ (b) 24
(c) $2 \times 15!$ (d) None of these

32. 20 persons are invited for a party. In how many different ways can they and the host be seated at circular table, if the two particular persons are to be seated on either side of the host?
(a) 20 ! (b) 2·18 !
(c) 18 ! (d) None of these

33. In how many ways can 5 boys and 5 girls sit in a circle so that no two boys sit together?
(a) $5! \times 5!$ (b) $4! \times 5!$
(c) $\frac{5! \times 5!}{2}$ (d) None of these

Combination

34. How many numbers lying between 10 and 1000 can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (repetition of digits is allowed) ?
(a) 1024 (b) 810
(c) 2346 (d) None of these

35. Let A be a set containing 10 distinct elements. Then, the total number of distinct functions from A to A is
(a) 10! (b) 10^{10}
(c) 2^{10} (d) $2^{10} - 1$

36. How many 10-digit numbers can be written by using the digits 1 and 2 ?
(a) ${}^{10}C_1 + {}^{9}C_2$ (b) 2^{10}
(c) ${}^{10}C_2$ (d) 10!

37. The number of times the digits 3 will be written when listing the integers from 1 to 1000 is
(a) 269 (b) 300 (c) 271 (d) 302

38. The figures 4, 5, 6, 7, 8 are written in every possible order. The number of numbers greater than 56000 is
(a) 72 (b) 96
(c) 90 (d) 98

39. If ${}^{n}C_{12} = {}^{n}C_8$, then n is equal to [NCERT Exemplar]
(a) 20 (b) 12 (c) 6 (d) 30

40. The value of ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to
(a) ${}^{47}C_6$ (b) ${}^{52}C_5$
(c) ${}^{52}C_4$ (d) None of these

41. ${}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2}$ is equal to [NCERT Exemplar]
(a) ${}^{n+1}C_r$ (b) ${}^{n+1}C_{r+1}$
(c) ${}^{n+2}C_r$ (d) ${}^{n+2}C_{r+1}$

42. If ${}^{n}C_3 + {}^{n}C_4 > {}^{n+1}C_3$, then
(a) $n > 6$ (b) $n > 7$
(c) $n < 6$ (d) None of these

43. In how many ways can a student choose a program of 5 courses, if 9 courses are available and 2 specific courses are compulsory for every student? [NCERT]
(a) 34 (b) 36
(c) 35 (d) 37

44. Every body in a room shakes hands with everybody else. The total number of hand shakes is 66. The total number of persons in the room is [NCERT Exemplar]
(a) 11 (b) 12
(c) 13 (d) 14

45. The number of ways in which a team of eleven players can be selected from 22 players always including 2 of them and excluding 4 of them is [NCERT Exemplar]
(a) ${}^{16}C_{11}$ (b) ${}^{16}C_5$
(c) ${}^{16}C_9$ (d) ${}^{20}C_9$

46. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if atleast one black ball is to be included in the draw?
(a) 64 (b) 45
(c) 46 (d) None of these

47. The number of ways in which we can choose a committee from four men and six women so that the committee includes atleast two men and exactly twice as many women as men is [NCERT Exemplar]
(a) 94 (b) 126
(c) 128 (d) None of these

48. A five digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done, is [NCERT Exemplar]
(a) 216 (b) 240
(c) 600 (d) 3125

49. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which hall can be illuminated. [NCERT Exemplar]
(a) $2^{10} - 2$ (b) $2^{10} - 1$
(c) $2^{10} + 1$ (d) None of these

50. Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated. [NCERT Exemplar]
(a) 111 (b) 112
(c) 113 (d) None of these

51. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Find the number of ways in which the student can make the choice. [NCERT Exemplar]
(a) 3 (b) 2 (c) 4 (d) 5

52. A sports team of 11 students is to be constituted, choosing atleast 5 from Class XI and atleast 5 from Class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted? [NCERT Exemplar]

(a) ${}^{20}C_5 \times {}^{20}C_6$ (b) $2\,({}^{20}C_5 \times {}^{20}C_6)$
(c) $2\,({}^{20}C_5)^2$ (d) None of these

53. A pack of 52 cards are shuffled together. The number of ways in which a man can be dealt 26 cards so that he does not get two cards of the same suit and same denomination, is

(a) ${}^{52}C_{26} \cdot 2^{26}$ (b) ${}^{104}C_{26}$
(c) $2 \cdot {}^{52}C_{26}$ (d) None of these

54. In how many ways can 21 English and 19 Hindi books be placed in a row so that no two Hindi books are together?

(a) 1540 (b) 1450 (c) 1504 (d) 1405

55. 4 buses runs between Bhopal and Gwalior. If a man goes from Gwalior to Bhopal by a bus and comes back to Gwalior by another bus, then the total possible ways are

(a) 12 (b) 16 (c) 4 (d) 8

56. In a football championship, there were played 153 matches. Every team played one match with each other. The number of teams participating in the championship is

(a) 17 (b) 18 (c) 9 (d) 13

57. A father with 8 children takes them 3 at a time to the zoological gardens, as often as he can without taking the same 3 children together more than once. The number of times he will go the garden, is

(a) 336 (b) 112
(c) 56 (d) None of these

58. A car will hold 2 in the front seat and 1 in the rear seat. If among 6 persons 2 can drive, then number of ways in which the car can be filled, is

(a) 10 (b) 20
(c) 30 (d) None of these

59. Six X's have to be placed in the square of the figure such that each row contains atleast one 'X'. In how many different ways can this be done?

(a) 28 (b) 27
(c) 26 (d) None of these

60. A question paper is divided into two parts A and B and each part contains 5 questions. The number of ways in which a candidate can answer 6 questions selecting atleast two questions from each part is

(a) 80 (b) 100
(c) 200 (d) None of these

61. In an examination there are three multiple choice questions and each question has 4 choices. Number of ways in which a student can fail to get all answers correct, is

(a) 11 (b) 12 (c) 27 (d) 63

62. There were two women participating in a chess tournament. Every participant played two games with the other participants. The number of games that the men played between themselves proved to exceed by 66 the number of games that the men played with the women. The number of participants is

(a) 6 (b) 11
(c) 13 (d) None of these

63. A lady gives a dinner party for six guests. The number of ways in which they may be selected from among ten friends, if two of the friends will not attend the party together, is

(a) 112 (b) 140
(c) 164 (d) None of these

64. A person is permitted to select atleast one and atmost n coins from a collection of $2n + 1$ (distinct) coins. If the total number of ways in which he can select coins is 255, then n is equal to

(a) 4 (b) 8 (c) 16 (d) 32

65. In an steamer, there are stalls for 12 animals and there are horses, cows and calves (not less than 12 each) ready to be shipped in how many ways can the ship load be made?

(a) $3^{12} - 1$ (b) 3^{12}
(c) $(12)^3 - 1$ (d) $(12)^3$

66. In an examination of 9 papers a candidate has to pass in more papers, then the number of papers in which he fails in order to be successful. The number of ways in which he can be unsuccessful, is

(a) 255 (b) 256 (c) 193 (d) 319

Division of Objects into Groups

67. Given 5 different green dyes, four different blue dyes and three different red dyes, the number of combinations of dyes which can be chosen taking atleast one green and one blue dye is [NCERT Exemplar]

(a) 3600 (b) 3720 (c) 3800 (d) 3600

68. There are four balls of different colours and four boxes of colours same as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour, is
(a) 8 (b) 7
(c) 9 (d) None of these

69. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many ways can we place the balls so that no box remains empty?
(a) 50 (b) 100
(c) 150 (d) 200

70. The number of ways of dividing 52 cards amongst four players so that three players have 17 cards each and the fourth players just one card, is
(a) $\frac{52!}{(17!)^3}$ (b) 52!
(c) $\frac{52!}{17!}$ (d) None of these

71. 18 mice were placed in two experimental groups and one control group with all group equally large. In how many ways can the mice be placed into three groups? [NCERT Exemplar]
(a) $\frac{181}{(6!)^2}$ (b) $\frac{181}{(6!)^3}$
(c) $\frac{180}{(6!)^3}$ (d) None of these

72. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions. [NCERT Exemplar]
(a) 779 (b) 781
(c) 780 (d) 782

73. In how many ways can ₹16 be divided into 4 persons when none of them get less than ₹ 3?
(a) 70 (b) 35 (c) 64 (d) 192

74. A library has a copies of one book, b copies of each of two books, c copies of each of three books and single copies of d books. The total number of ways in which these books can be distributed, is
(a) $\frac{(a+b+c+d)!}{a!\,b!\,c!}$ (b) $\frac{(a+2b+3c+d)!}{a!\,(b!)^2(c!)^3}$
(c) $\frac{(a+2b+3c+d)}{a!\,b!\,c!}$ (d) None of these

75. Eleven books consisting of 5 Mathematics, 4 Physics and 2 Chemistry are placed on a shelf. The number of possible ways of arranging them on the assumption that the books of the same subject are all together, is
(a) 4! 2! (b) 11!
(c) 5! 4! 3! 2! (d) None of these

76. Three boys of class X, four boys of class XI and five boys of class XII sit in a row. The total number of ways in which these boys can sit so that all the boys of same class sit together is equal to
(a) $(3!)^2\,(4!)\,(5!)$ (b) $(3!)\,(4!)^2\,(5!)$
(c) $(3!)\,(4!)\,(5!)$ (d) $(3!)\,(4!)\,(5!)^2$

Applications of Permutation and Combination

77. The number of mappings (functions) from the set $A = \{1, 2, 3\}$ into the set $B = \{1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \le f(j)$, whenever $i < j$, is
(a) 84 (b) 90
(c) 88 (d) None of these

78. Let $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 4, 5\}$ that are onto and $f(x) \ne i$ is equal to
(a) 9 (b) 44
(c) 16 (d) None of these

79. The number of triangles that are formed by choosing the vertices from a set of 12 points, seven of which lie on the same line is [NCERT Exemplar]
(a) 105 (b) 15
(c) 175 (d) 185

80. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is [NCERT Exemplar]
(a) 6 (b) 18 (c) 12 (d) 9

81. The maximum number of points of intersection of 6 circles is
(a) 25 (b) 24
(c) 50 (d) 30

82. If a polygon has 44 diagonals, then the number of its sides are [NCERT Exemplar]
(a) 11 (b) 7
(c) 8 (d) None of these

83. The number of diagonals in a polygon of m sides is
(a) $\frac{1}{2!}\,m\,(m-5)$ (b) $\frac{1}{2!}\,m\,(m-1)$
(c) $\frac{1}{2!}\,m\,(m-3)$ (d) $\frac{1}{2!}\,m\,(m-2)$

84. The number of triangles that can be formed by 5 points in a line and 3 points on a parallel line is
(a) ${}^8C_3$ (b) ${}^8C_3 - {}^5C_3$
(c) ${}^8C_3 - {}^5C_3 - 1$ (d) None of these

85. The straight lines I_1, I_2, I_3 are parallel and lie in the same plane. A total numbers of m points are taken on I_1, n points on I_2, k points on I_3. The maximum number of triangles formed with vertices at these points is
(a) ${}^{m+n+k}C_3$
(b) ${}^{m+n+k}C_3 - {}^{m}C_3 - {}^{n}C_3 - {}^{k}C_3$
(c) ${}^{m}C_3 + {}^{n}C_3 + {}^{k}C_3$
(d) None of the above

86. Six points in a plane be joined in all possible ways by indefinite straight lines and if no two of them be coincident or parallel and no three pass through the same point (with the exception of the original 6 points). The number of distinct points or intersection is equal to
(a) 105 (b) 45
(c) 51 (d) None of these

87. The greatest possible number of points of intersection of 8 straight lines and 4 circles is
(a) 32 (b) 64 (c) 76 (d) 104

88. There are n distinct points on the circumference of a circle. The number of pentagons that can be formed with these points as vertices is equal to the number of possible triangles. Then, the value of n is
(a) 7 (b) 8
(c) 15 (d) 30

Number of Divisors

89. The number of divisors of 9600 including 1 and 9600 are
(a) 60 (b) 58 (c) 48 (d) 46

90. Number of divisors of the form $(4n+2)$, $n \geq 0$ of the integer 240 is
(a) 4 (b) 8 (c) 10 (d) 3

91. The number of divisors of the number of 38808 (excluding 1 and the number itself) is
(a) 70 (b) 72
(c) 71 (d) None of these

92. If a, b, c, d, e are prime integers, then the number of divisors of ab^2c^2de excluding 1 as a factor is
(a) 94 (b) 72 (c) 36 (d) 71

Miscellaneous

93. The number of ways can 10 letters be placed in 10 marked envelopes, so that no letter is in the right envelope are
(a) $10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!}\right)$
(b) $10!\left(1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots - \frac{1}{10!}\right)$
(c) $\left\{1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots - \frac{1}{10!}\right\}$
(d) $9!\left\{1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots - \frac{1}{10!}\right\}$

94. The number of ordered triplets of positives integers which are solutions of the equation $x + y + z = 100$, is
(a) 6005 (b) 4851
(c) 5081 (d) None of these

95. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to [NCERT Exemplar]
(a) 60 (b) 120 (c) 7200 (d) 720

96. There are 5 historical monuments, 6 gardens and 7 shopping malls in the city. In how many ways a tourist can visit the city, if he visits atleast one shopping mall?
(a) $2^5 \cdot 2^6 \cdot (2^7 - 1)$ (b) $2^4 \cdot 2^6(2^7 - 1)$
(c) $2^5 \cdot 2^6(2^6 - 1)$ (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. In a city no two persons have identical set of teeth and there is no person without a tooth. Also, no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, the maximum population of the city is
(a) 2^{32} (b) $(32)^2 - 1$ (c) $2^{32} - 1$ (d) 2^{32-1}

2. A rectangle with sides $2m - 1$ and $2n - 1$ divided into squares of unit length. The number of rectangle which can be formed with sides of odd length is
(a) m^2n^2 (b) $mn\,(m+1)\,(n+1)$
(c) 4^{m+n-1} (d) None of these

3. The lock of a safe consists of five discs each of which features the digits $0, 1, 2, \ldots, 9$. The safe can be opened by dialing a special combination of the digits. The number of days sufficient enough to open the safe. If the work day lasts 13 h and 5 s are needed to dial one combination of digits is
(a) 9 (b) 10
(c) 11 (d) 12

4. The interior angles of a regular polygon measure 160° each. The number of diagonals of the polygon are
(a) 97 (b) 105 (c) 135 (d) 146

5. Let A be the set of 4-digit numbers $a_1\ a_2\ a_3\ a_4$, where $a_1 < a_2 < a_3 < a_4$, then $n(A)$ is equal to
(a) 84 (b) 126
(c) 210 (d) None of these

6. If the total number of m elements subsets of the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$ is λ times the number of 3 elements subsets containing a_4, then n is
(a) $(m-1)\lambda$ (b) $m\lambda$ (c) $(m+1)\lambda$ (d) 0

7. Sixteen men compete with one another in running, swimming and riding. How many prize lists could be made, if there were altogether 6 prizes of different values, one for running, 2 for swimming and 3 for riding?
(a) $16 \times 15 \times 14$ (b) $16^3 \times 15^2 \times 14$
(c) $16^3 \times 15 \times 14^2$ (d) $16^2 \times 15 \times 14$

8. The number of ways in which we can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers is
(a) 27378 (b) 27405
(c) 27399 (d) None of these

9. The number of different seven digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number is
(a) ${}^7P_2 2$ (b) ${}^7C_2 2^5$
(c) ${}^7C_2 5^2$ (d) None of these

10. If the difference of the number of arrangements of three things from a certain number of dissimilar things and the number of selections of the same number of things from them exceeds 100, then the least number of dissimilar things is
(a) 8 (b) 6 (c) 5 (d) 7

11. A person always prefers to eat 'parantha' and 'vegetable dish' in his meal. How many ways can he make his platter in a marriage party, if there are three types of paranthas, four types of 'vegetable dish', three types of 'salads' and two types of 'sauces'?
(a) 3360 (b) 4096
(c) 3000 (d) None of these

12. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices on these points is
(a) $3p^2(p-1)+1$ (b) $3p^2(p-1)$
(c) $p^2(4p-3)$ (d) None of these

13. In how many different ways can the first 12 natural numbers be divided into three different groups such that numbers in each group are in AP?
(a) 1 (b) 5
(c) 6 (d) 4

14. Two packs of 52 cards are shuffled together. The number of ways in which a man can be dealt 20 cards, so that he does not get two cards of the same suit and same denomination is
(a) ${}^{56}C_{20} \times 2^{20}$ (b) ${}^{104}C_{20}$
(c) $2 \times {}^{52}C_{20}$ (d) None of these

More Than One Correct Option

15. The number of ways of arranging seven persons (having A, B, C and D among them) in a row, so that A, B, C and D are always in order A-B-C-D (not necessarily together) is
(a) 210 (b) 5040
(c) $6 \times {}^7C_3$ (d) ${}^7P_3$

16. Two players P_1 and P_2 plays a series of $2n$ games. Each game can result in either a win or loss for P_1. Total number of ways in which P_1 can win the series of these games, is equal to
(a) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$
(c) $\frac{1}{2}\left(2^n - \frac{2n!}{n!\,n!}\right)$ (d) None of these

17. Total number of ways of giving atleast one coin out of three 25 paise and two 50 paise coins to a beggar is
(a) 32 (b) 12
(c) 11 (d) ${}^{12}P_1 - 1$

Comprehension Based Questions

Passage

Directions (Q. Nos. 18-20) *Different words are being formed by arranging the letters of the word 'SUCCESS'. All the words obtained by written in the form of a dictionary.*

18. The number of words in which the two C are together but no two S are together, is
(a) 120 (b) 96 (c) 24 (d) 420

19. The number of words in which the consonants appear in alphabetic order is
(a) 42 (b) 40 (c) 420 (d) 280

20. The rank of the word 'SUCCESS' in the dictionary is
(a) 328 (b) 329
(c) 330 (d) 331

Assertion and Reason

Directions (Q. Nos. 21-32) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

21. **Statement I** The number of different car licence plates can be constructed, if the licences contain three letters of the English alphabet followed by a three digit number is $(26)^3 \times (900)$ (if repetitions are allowed).

Statement II The number of permutations of n different things taken r at a time when each things may be repeated any number of times is n^r.

22. **Statement I** A number of four different digit is formed with the help of the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. Then, number of ways which are exactly divisible by 4 is 200.

Statement II A number divisible by 4, if unit place digit divisible by 4.

23. **Statement I** The number of non-negative integral solutions of $x_1 + x_2 + x_3 + \ldots + x_n = r$ is ${}^{r+n-1}C_r$.

Statement II The number of ways in which n identical things can be distributed into r different groups is ${}^{n+r-1}C_n$.

24. **Statement I** From a group of 8 men and 4 women at team of 5, including atleast one woman can be formed in 736 ways.

Statement II Number of ways of selecting atleast one woman from m men and n women is ${}^{m+n}C_n - {}^{m}C_n$.

25. **Statement I** Product of five consecutive natural numbers is divisible by 4!.

Statement II Product of n consecutive natural numbers is divisible by $(n+1)!$.

26. **Statement I** Number of rectangle on a chess board is ${}^8C_2 \times {}^8C_2$.

Statement II To form a rectangle we have to select any two of the horizontal line and any two of the vertical line.

27. **Statement I** The sum of the digits in the tens place of all numbers formed with the help of 2, 3, 4, 5 taken all at a time is 84.

Statement II The sum of the digits in the units place of all numbers formed with the help of $a_1, a_2, \ldots, a_n$ taken all at a time is
$(n-1)!(a_1 + a_2 + \ldots + a_n)$ (repetition of digits is not allowed).

28. **Statement I** Number of ways in which 10 identical toys can be distributed among 3 students, if each receives atleast one toys is ${}^9C_2$.

Statement II Number of positive integral solutions of $x + y + z + w = 7$ is ${}^6C_2$.

29. **Statement I** Let $E = \{1, 2, 3, 4\}$ and $F = \{a, b\}$. Then, the number of onto functions from E to F is 14.

Statement II Number of ways in which four distinct objects can be distributed into two different boxes is 14, if no box remains empty.

30. **Statement I** Number of ways in which Indian team (11 players) can bat, if Yuvraj wants to bat before Dhoni and Pathan wants to bat after Dhoni is 11!/3!.

Statement II Yuvraj, Dhoni and Pathan can be arranged in batting order in 3! ways.

31. **Statement I** The number of ways distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.

Statement II The number of ways of choosing any 3 places, from 9 different places is ${}^9C_3$.

32. In a shop there are five types of ice-creams available. A child buys six ice-creams.

Statement I The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.

Statement II The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

Previous Years' Questions

33. Let A and B two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is [JEE Main 2013]
(a) 256 (b) 220
(c) 219 (d) 211

34. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is [JEE Main 2013]
(a) 7 (b) 5
(c) 10 (d) 8

35. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is [AIEEE 2012]
(a) 880 (b) 629
(c) 630 (d) 879

36. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can formed such that $Y \subseteq X, Z \subseteq X$ and $Y \cap Z$ is empty, is [AIEEE 2012]
(a) 5^2 (b) 3^5 (c) 2^5 (d) 5^3

37. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is [AIEEE 2010]
(a) 3 (b) 36 (c) 66 (d) 108

38. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the shelf so that the dictionary is always in the middle. Then, the number of such arrangements is [AIEEE 2009]
(a) atleast 500 but less than 750
(b) atleast 750 but less than 1000
(c) atleast 1000
(d) less than 500

39. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent ? [AIEEE 2008]
(a) $7 \cdot {}^6C_4 \cdot {}^8C_4$ (b) $8 \cdot {}^6C_4 \cdot {}^7C_4$
(c) $6 \cdot 7 \cdot {}^8C_4$ (d) $6 \cdot 8 \cdot {}^7C_4$

40. The set $S = \{1, 2, 3, \ldots, 12\}$ is to be partitioned into three sets A, B and C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is [AIEEE 2007]
(a) $12!/3!(4!)^3$
(b) $12!/3!(3!)^4$
(c) $12!/(4!)^3$
(d) $12!/(3!)^4$

41. At an election, a voter may vote for any number of candidates not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for atleast one candidate, then the number of ways in which he can vote, is [AIEEE 2006]
(a) 6210 (b) 385
(c) 1110 (d) 5040

42. If the letters of the word SACHIN are arranged in all possible ways and these words are written out as in dictionary, then the word SACHIN appears at serial number [AIEEE 2005]
(a) 602 (b) 603
(c) 600 (d) 601

43. How many ways are there to arrange the letters in the word GARDEN with the vowels in alphabetical order? [AIEEE 2004]
(a) 120 (b) 240 (c) 360 (d) 480

44. The number of ways of distributing 8 identical balls in 3 distinct boxes, so that none of the boxes is empty, is [AIEEE 2004]
(a) 5 (b) 21
(c) 3^8 (d) ${}^8C_3$

45. A student is to answer 10 out of 13 questions in an examination such that he must choose atleast 4 from the first five questions. The number of choices available to him is [AIEEE 2003]
(a) 140 (b) 196
(c) 280 (d) 346

46. The number of ways in which 6 men and 5 women can dine at a round table, if no two women are to sit together, is given by [AIEEE 2003]
(a) $6! \times 5!$ (b) 30
(c) $5! \times 4!$ (d) $7! \times 5!$

47. If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ equals [AIEEE 2003]
(a) ${}^{n+2}C_r$ (b) ${}^{n+2}C_{r+1}$
(c) ${}^{n+1}C_r$ (d) ${}^{n+1}C_{r+1}$

Answers

Check Point 1

1. (a)	**2.** (d)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (c)

Check Point 2

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (a)	**9.** (d)	**10.** (d)

Round I

1. (a)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (c)	**17.** (c)	**18.** (b)	**19.** (d)	**20.** (a)
21. (a)	**22.** (d)	**23.** (a)	**24.** (b)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (a)
31. (a)	**32.** (b)	**33.** (b)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (c)
41. (c)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (c)	**46.** (a)	**47.** (a)	**48.** (a)	**49.** (b)	**50.** (b)
51. (a)	**52.** (b)	**53.** (a)	**54.** (a)	**55.** (a)	**56.** (b)	**57.** (c)	**58.** (b)	**59.** (c)	**60.** (c)
61. (d)	**62.** (c)	**63.** (b)	**64.** (a)	**65.** (b)	**66.** (b)	**67.** (b)	**68.** (c)	**69.** (c)	**70.** (a)
71. (b)	**72.** (c)	**73.** (b)	**74.** (b)	**75.** (c)	**76.** (a)	**77.** (a)	**78.** (b)	**79.** (d)	**80.** (d)
81. (d)	**82.** (a)	**83.** (c)	**84.** (c)	**85.** (b)	**86.** (c)	**87.** (d)	**88.** (b)	**89.** (c)	**90.** (a)
91. (a)	**92.** (d)	**93.** (a)	**94.** (b)	**95.** (c)	**96.** (a)				

Round II

1. (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (d)
11. (a)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (a,b,d)	**16.** (a,c)	**17.** (c,d)	**18.** (c)	**19.** (a)	**20.** (d)
21. (d)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (a)	**28.** (d)	**29.** (a)	**30.** (a)
31. (a)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (d)	**36.** (b)	**37.** (d)	**38.** (c)	**39.** (a)	**40.** (c)
41. (b)	**42.** (d)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (a)	**47.** (b)			

the **Guidance**

Round I

1. Number of ways to fill the IIIrd place = 8

Number of ways to fill the IVth place = 7

6	7			
I	II	III	IV	V

Number of ways to fill the Vth place = 6

Hence, by FPC total number of ways $= 8 \times 7 \times 6 = 56 \times 6$

$= 336$

2. Number of ways choose the first flag = 5

Number of ways to choose the second flag from the rest of four flags = 4

Hence, by FPC total number of ways $= 5 \times 4 = 20$

3. Out of 8 persons, one person can be chosen for chairman in 8 ways. Now, a person cannot hold more than one position *i.e.*, out of 7 persons, one person can be chosen for vice-chairman in 7 ways.

Hence, by FPC total number of ways $= 8 \times 7 = 56$

4. Given, $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!} \Rightarrow \frac{1}{6!} + \frac{1}{7 \times 6!} = \frac{x}{8 \times 7 \times 6!}$

$$\frac{1}{6!}\left[1 + \frac{1}{7}\right] = \frac{x}{8 \times 7 \times 6!} \Rightarrow \frac{1}{1} + \frac{1}{7} = \frac{x}{8 \times 7}$$

$$\Rightarrow \quad \frac{7+1}{7} = \frac{x}{8 \times 7}$$

$$\Rightarrow \quad x = 8 \times 8 = 64$$

5. Now, $[1 \cdot 3 \cdot 5 \ldots (2n-3)(2n-1)]2^n$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2n-1)\,(2n)2^n}{2 \cdot 4 \cdot 6 \ldots 2n}$$

$$= \frac{(2n)!\,2^n}{2^n(1 \cdot 2 \cdot 3 \ldots n)} = \frac{(2n)!}{n!}$$

6. The product of r consecutive natural numbers

$$= 1 \cdot 2 \cdot 3 \cdot 4 \ldots r = r!$$

The natural number will divided by $r!$.

7. Now, $100! = 1 \cdot 2 \cdot 3 \cdot \ldots 98 \cdot 99 \cdot 100$

$$= (1 \cdot 2 \cdot 4 \cdot 5 \ldots 98 \cdot 100)(3 \cdot 6 \cdot 9 \cdot \ldots 96 \cdot 99)$$

$$= K \cdot 3^{33}(1 \cdot 2 \cdot 3 \ldots 32 \cdot 33)$$

$(\because \text{let } K = 1 \cdot 2 \cdot 4 \cdot 5 \ldots 98 \cdot 100)$

$= [K(1\cdot 2\cdot 4\ldots 31\cdot 32)]\, 3^{33}\cdot(3\cdot 9\cdot 12\ldots 30\cdot 33)$

$= K_1\cdot 3^{33}\cdot 3^{11}(1\cdot 2\cdot 3\ldots 10\cdot 11)$

$[\because \text{let } K(1\cdot 2\cdot 4\ldots 31\cdot 32) = K_1]$

$= K_1(1\cdot 2\cdot 4\ldots 10\cdot 11)3^{33}\cdot 3^{11}(3\cdot 6\cdot 9\cdot 12)$

$= K_2 3^{33}\cdot 3^{11}\cdot 3^4(1\cdot 2\cdot 3\cdot 4)$

$= K_3\cdot 3^{33}\cdot 3^{11}\cdot 3^4\cdot 3 \qquad [\because \text{let } K_2(1\cdot 2\cdot 3\cdot 4) = K_3]$

$= K_3\cdot 3^{49}$

Hence, exponent of 3 is 49.

8. Given, ${}^5P_r = 2\ {}^6P_{r-1}$

$$\frac{5!}{(5-r)!} = 2\times\frac{6!}{(6-r+1)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2\times\frac{6\times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{1}{(5-r)!} = \frac{12}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad 42 - 7r - 6r + r^2 = 12$$

$$\Rightarrow \quad r^2 - 13r + 30 = 0$$

$$\Rightarrow \quad r^2 - 10r - 3r + 30 = 0$$

$$\Rightarrow \quad r(r-10) - 3(r-10) = 0$$

$$\Rightarrow \quad (r-10)(r-3) = 0$$

$$\therefore \quad r = 10, 3$$

But we will take only $r = 3$ as $0 \le r \le 5$.

9. The word 'PERMUTATIONS' has

P	—	1 time
E	—	1 time
R	—	1 time
M	—	1 time
U	—	1 time
T	—	2 times
A	—	1 time
I	—	1 time
O	—	1 time
N	—	1 time
S	—	1 time

(i) Word start with P and end with S *i.e.*,

P											S

First and last position are already filled by P and S, respectively.

Remaining 10 positions can be filled $= \dfrac{10!}{2!}$ ways

$$= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2!}{2!}$$

$= 720\times 7\times 6\times 5\times 4\times 3$

$= 720\times 2520 = 1814400$

10. In a nine digits number, there are four even places for the four odd digits 3, 3, 5, 5.

$\therefore$ Required number of ways $= \dfrac{4!}{2!\,2!}\cdot\dfrac{5!}{2!\,3!} = 60$

11. The number will be even, if last digit is either 2, 4, 6 or 8 *i.e.*, the last digit can be filled in 4 ways and remaining two digits can be filled in ${}^8P_2$ ways. Hence, required number of numbers of three different digits $= {}^8P_2\times 4 = 224$.

12. The total number of 9-digit numbers, having all digits are different $= 9\times {}^9P_8$

$$= \frac{9\times 9!}{1!} = 9\times 9!$$

($\because$ in first place we select any one of nine numbers except zero. In rest of the eight places, we select any eight numbers from remaining 9 numbers.)

13. In a word ARTICLE, vowels are A, E, I and consonants are C, L, R, T.

In a seven letter word, there are three even places in which three vowels are placed in 3! way. In rest of the four places, four consonants are placed in 4! ways.

$\therefore$ Required number of ways $= 3!\times 4!$

$= 6\times 24 = 144$

14. Two women occupy the chair from 1 to 4 in ${}^4P_2$ ways and 3 men occupy the remaining chairs in ${}^6P_3$ ways.

$\therefore$ Required number of ways $= {}^4P_2\times {}^6P_3$

$= 12\times 120 = 1440$

15. In a word RACHIT, the number of words beginning with A, C, H, I is 5! and the next word we get RACHIT.

$\therefore$ Required number of words $= 4\times 5! + 1$

$= 4\times 120 + 1 = 481$

16. In a word TRIANGLE, vowels are (A, E, I) and consonants are (G, L, N, R, T).

First, we fix the 5 consonants in alternate position in 5! ways.

_G_L_N_R_T_

In rest of the six blank position, three vowels can be arranged in ${}^6P_3$ ways.

$\therefore$ Total number of different words $= 5!\times {}^6P_3$

$= 120\times 6\times 5\times 4 = 14400$

17. Given, total number of digits are 4.

$\therefore$ Total four digit numbers can be formed

$= {}^4P_4 = 4! = 24$

18. The sum of the digit in unit place of all the numbers formed

$= 3!(3 + 4 + 5 + 6)$

$= 6\times 18 = 108$

19. Using the digits 0, 1, 2, ... , 9 the number of five digits telephone numbers which can be formed is 10^5 (since, repetition is allowed).

The number of five digits telephone numbers, which have none of the digits repeated

$$= {}^{10}P_5 = 30240$$

$\therefore$ The required number of telephone number

$$= 10^5 - 30240 = 69760$$

20. In the word INTEGER, we have 5 letters other than 'I' and 'N' of which two are identical (E's). We can arrange these letters in $\frac{5!}{2!}$ ways. In any such arrangements, 'I' and 'N' can be placed in 6 available gaps in ${}^6P_2$ ways.

So, required number of ways $= \frac{5!}{2!} \cdot {}^6P_2 = m_1$. Now, if word start with 'I' and end with 'R', then the remaining letters are 5.

So, total number of ways $= \frac{5!}{2!} = m_2$

$$\therefore \quad \frac{m_1}{m_2} = \frac{5!}{2!} \cdot \frac{6!}{4!} \cdot \frac{2!}{5!} = 30$$

21. The number of words starting from A are $= 5! = 120$

The number of words starting from I are $= 5! = 120$

The number of words starting from KA are $= 4! = 24$

The number of words starting from KI are $= 4! = 24$

The number of words starting from KN are $= 4! = 24$

The number of words starting from KRA are $= 3! = 6$

The number of words starting from KRIA are $= 2! = 2$

The number of words starting from KRIN are $= 2! = 2$

The number of words starting from KRISA are $= 1! = 1$

The number of words starting from KRISNA are $= 1! = 1$

Hence, rank of the word KRISNA

$$= 2(120) + 3(24) + 6 + 2(2) + 2(1) = 324$$

22. The number of words starting from E are $= 5! = 120$

The number of words starting from H are $= 5! = 120$

The number of words starting from ME are $= 4! = 24$

The number of words starting from MH are $= 4! = 24$

The number of words starting from MOE are $= 3! = 6$

The number of words starting from MOH are $= 3! = 6$

The number of words starting from MOR are $= 3! = 6$

The number of words starting from MOTE are $= 2! = 2$

The number of words starting from MOTHER are $= 1! = 1$

Hence, rank of the word MOTHER

$$= 2(120) + 2(24) + 3(6) + 2 + 1$$
$$= 309$$

23. Since, three particular things must occur together. Therefore, we have to select $r - 3$ person from $n - 3$. And there are $r - 3$ person can be arranged in $(r - 2)!\,3!$.

$\therefore$ Required number of permutation

$$= {}^{n-3}C_{r-3}(r-2)!\,3!$$

24. In out of 10 persons, P_1 is always consider and P_4 and P_5 is not consider.

i.e., We have to select 4 persons out of 7 person and after that they arrange it.

$\therefore$ Required number of ways $= {}^7C_4 \times 5! = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 120$

$$= 35 \times 120 = 4200$$

25. **Hint** We have, $\quad a = {}^{x+2}P_{x+2} = (x+2)!$

and $\quad b = {}^xP_{11} = \frac{x!}{(x-11)!}$ and $c = {}^{x-11}P_{x-11} = (x-11)!$

26. At first we have to a accommodate those 5 animals in cages which cannot enter in 4 small cages, therefore number of ways are ${}^6P_5$ and rest of the five animals arrange in $5!$ ways.

Total number of ways $= 5! \times {}^6P_5$

$$= 120 \times 720 = 86400$$

27. When we arrange one things at a time, the number of possible permutations is n. When we arrange them two at a time the number of possible permutations are $n \times n = n^2$ and so on. Thus, the total number of permutations are

$$n + n^2 + \ldots + n^r = \frac{n(n^r - 1)}{n - 1} \qquad (\because n > 1)$$

28. Two positions for A_1 and A_{10} can be selected in ${}^{10}C_2$ ways. Rest eight students can be ranked in 8! ways. Hence, total number of ways is ${}^{10}C_2 \times 8! = (1/2)(10!)$.

29. Since, out of eleven members two members sit together, then the number of arrangements $= 9! \times 2$

($\because$ two numbers can be sit in two ways.)

30. $\because$ Remaining 5 can be seated in 4! ways.

Now, on cross marked five places 2 person can sit in ${}^5P_2$ ways.

So, number of arrangements

$$= 4! \times \frac{5!}{3!} = 24 \times 20 = 480 \text{ ways}$$

31. Since, total members are 15 but three special members constitute one member.

Therefore, required number of arrangements are $12! \times 2$, because, chairman remains between the two specified persons and the person can sit in two ways.

32. There are total $20+1=21$ persons. The two particular persons and the host be taken as one unit so that these remain $21-3+1=19$ persons be arranged in round table in 18! ways. But the two persons on either sides of the host can themselves be arranged in 2! ways.

$\therefore$ Required number of ways $=2!\times 18!=2\cdot 18!$

33. First we fix the alternate position of girls and they arrange in 4! ways and in the five places five boys can be arranged in ${}^5P_5$ ways.

$\therefore$ Total number of ways $=4!\times {}^5P_5=4!\times 5!$

34. ***Case* I** When number in two digits.

Total number of ways $={}^9C_1\times {}^9C_1=9\times 9=81$

***Case* II** When number in three digits

Total number of ways $={}^9C_1\times {}^9C_1\times {}^9C_1=9\times 9\times 9=729$

$\therefore$ Total number of ways $=81+729=810$

35. Since, each element of a domain can image any of the elements of codomain.

$\therefore$ Required number of ways $=10\times 10\ldots\times 10=10^{10}$

36. Each digit can be placed in 2 ways.

$\therefore$ Required number of ways $=2^{10}$

37. Any number between 1 to 999 is a 3-digit number xyz where the digits x, y, z are any digits from 0 to 9.

Now, we first count the numbers in which 3 occurs once only. Since, 3 can occur at one place in ${}^3C_1$ ways, there are ${}^3C_1\cdot(9\times 9)=3\cdot 9^2$ such numbers.

Again, 3 can occur in exactly two places in ${}^3C_2\cdot(9)$ such numbers. Lastly ,3 can occur in all the three digits in one such number only 333.

$\therefore$ The number of times 3 occurs

$$=1\times(3\times 9^2)+2\times(3\times 9)+3\times 1=300$$

38. The number forms by the figure 4, 5, 6, 7, 8 which is greater than 56000 is in two cases.

***Case* I** Let the ten thousand digit place number be greater than 5. The number of numbers

$$={}^3C_1\times {}^4C_1\times {}^3C_1\times {}^2C_1\times {}^1C_1=3\times 4\times 3\times 2\times 1=72$$

***Case* II** Let the ten thousand digit number be 5 and thousand digit number be either 6 or greater than 6.

Then, the number of numbers

$$={}^3C_1\times {}^3C_1\times {}^2C_1\times {}^1C_1=3\times 3\times 2\times 1=18$$

$\therefore$ Required number of ways $=72+18=90$

39. Given, $\quad {}^nC_{12}={}^nC_8$

$\therefore \quad {}^nC_{n-12}={}^nC_8 \qquad (\because {}^nC_r={}^nC_{n-r})$

$\Rightarrow \quad n-12=8\Rightarrow n=20$

40. $${}^{47}C_4+\sum_{r=1}^{5}{}^{52-r}C_3={}^{47}C_4+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{47}C_3$$

$$={}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+({}^{47}C_3+{}^{47}C_4)={}^{52}C_4$$

41. ${}^nC_r+2\,{}^nC_{r-1}+{}^nC_{r-2}={}^nC_r+{}^nC_{r-1}+{}^nC_{r-1}+{}^nC_{r-2}$

$$={}^{n+1}C_r+{}^{n+1}C_{r-1}={}^{n+2}C_r$$

42. ${}^nC_3+{}^nC_4>{}^{n+1}C_3$

$\Rightarrow \quad {}^{n+1}C_4>{}^{n+1}C_3 \quad ({}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1})$

$\Rightarrow \quad \dfrac{{}^{n+1}C_4}{{}^{n+1}C_3}>1\Rightarrow \dfrac{n-2}{4}>1\Rightarrow n>6$

43. Total number of available courses $=9$

Out of these 5 courses have to be chosen. But it is given that 2 courses are compulsory for every student *i. e.*, you have to choose only 3 courses instead of 5, out of 7 instead of 9.

It can be done in ${}^7C_3$ ways $=\dfrac{7\times 6\times 5}{6}=35$ ways

44. Let total number of persons be n.

Since, total number of hand shakes $=66$

$\therefore \quad {}^nC_2=66\Rightarrow \dfrac{n(n-1)}{2}=66$

$\Rightarrow \quad n^2-n-132=0\Rightarrow (n-12)(n+11)=0$

$\Rightarrow \quad n=12 \quad (\because n$ cannot be negative)

45. $\therefore$ Required number of ways $={}^{22-4-2}C_{11-2}={}^{16}C_9$

46. A selection of 3 balls so as to include atleast one black ball, can be made in the following 3 mutually exclusive ways

(i) The number of ways in which 1 black balls and 2 others are selected

$$={}^3C_1\times {}^6C_2=3\times 15=45$$

(ii) The number of ways in which 2 black balls and 1 other are selected

$$={}^3C_2\times {}^6C_1=3\times 6=18$$

(iii) The number of ways in which 3 black balls and no other are selected $={}^3C_3=1$

$\therefore$ Total numbers of ways $=45+18+1=64$

47. The number of ways in which we can choose a committee = Choose two men and four women + Choose three men and six women

$$={}^4C_2\times {}^6C_4+{}^4C_3\times {}^6C_6$$

$$=6\times 15+4\times 1=90+4=94$$

48. Since, a five digit number is formed using digits { 0, 1, 2, 3, 4 and 5} divisible by 3 *i.e.*, only possible when sum of digits is multiple of 3 which gives two cases.

***Case* I** {using digits 0, 1, 2, 4, 5}

Number of numbers

$$={}^4C_1\times {}^4C_1\times {}^3C_1\times {}^2C_1\times {}^1C_1=4\times 4\times 3\times 2\times 1=96$$

***Case* II** {using digits 1, 2, 3, 4, 5}

Number of numbers

$$={}^5C_1\times {}^4C_1\times {}^3C_1\times {}^2C_1\times {}^1C_1=5\times 4\times 3\times 2\times 1=120$$

$\therefore$ Total numbers formed $=120+96=216$

49. Total number of ways =

$$^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + \ldots + {}^{10}C_{10}$$

$$= 2^{10} - 1 \qquad (\because {}^nC_0 + {}^nC_1 + {}^nc_2 + \ldots = 2^n)$$

50. Any number divisible by 5, if either 0 or 5 in unit place

6			0 or 5
Th	H	T	U

In unit place, the number of ways = $^2C_1 = 2$

In thousand place, number 6 is fixed. In ten and hundred place the number of ways of selection $= 8 \times 7$.

$\therefore$ Required number of ways $= 2 \times 8 \times 7 = 112$

51. Since, questions 1 and 2 are compulsory, so students has to select two question in out of four questions.

$\therefore$ Required number of ways = $^3C_2 = 3$

52. There are two cases arises

***Case* I** Choose six from class XI and five from class XII

$\therefore$ Number of ways = $^{20}C_6 \times {}^{20}C_5$

***Case* II** Choose five from class XI and six from class XII

$\therefore$ Number of ways = $^{20}C_5 \times {}^{20}C_6$

$\therefore$ Required number of ways $= 2(^{20}C_5 \times {}^{20}C_6)$

53. $\because$ 26 cards can be chosen out of 52 cards in $^{52}C_{26}$ ways. There are two ways in which each card can be dealt because a card can be either from the first pack or from the second.

$\therefore$ Total number of ways = $^{52}C_{26} \cdot 2^{26}$

54. First we fix the alternate position of English books. Then, there are 22 vacant places for Hindi books.

Hence, total number of ways = $^{22}C_{19} = \dfrac{22!}{3!9!} = 1540$

55. Man goes from Gwalior to Bhopal in 4 ways and they come back in 3 ways.

$\therefore$ Total number of ways = $^4C_1 \times {}^3C_1 = 4 \times 3 = 12$ ways

56. Let there are n teams.

Each team play to every other team in $^nC_2$ ways

$\therefore \quad {}^nC_2 = 153$ (given)

$\Rightarrow \quad \dfrac{n!}{(n-2)!2!} = 153$

$\Rightarrow \quad n(n-1) = 306$

$\Rightarrow \quad n^2 - n - 306 = 0$

$\Rightarrow \quad (n-18)(n+17) = 0$

$\Rightarrow \quad n = 18 \quad (\because n$ is never negative)

57. The number of times he will go to the garden is same as the number of selecting 3 children from 8.

Therefore, the required number of ways = $^8C_3 = 56$

58. Since, 2 persons can drive the car, therefore we have to select 1 from these two. This can be done in $^2C_1$ ways. Now, from the remaining 5 persons we have to select 2 which can be done in $^5C_2$ ways.

Therefore, the required number of ways in which the car can be filled = $^5C_2 \times {}^2C_1 = 10 \times 2 = 20$

59. In all, we have 8 squares in which 6 'X' have to be placed and it can be done in $^8C_6 = 28$ ways.

But this includes the possibility that either the top or horizontal row does not have any 'X'. Since, we want each row must have atleast one 'X', these two possibilities are to be excluded.

Hence, required number of ways $= 28 - 2 = 26$

60. The number of ways that the candidate may select

(i) if 2 questions from A and 4 questions from B

$$= {}^5C_2 \times {}^5C_4 = 50$$

(ii) 3 questions from A and 3 questions from B

$$= {}^5C_3 \times {}^5C_3 = 100$$

and (iii) 4 questions from A and 2 questions from B

$$= {}^5C_4 \times {}^5C_2 = 50$$

Hence, total number of ways $= 50 + 100 + 50 = 200$

61. Each question can be answered in 4 ways and all question can be answered correctly in only one way.

So, required number of ways $= (^4C_1)^3 - 1 = 4^3 - 1 = 63$

62. Let there be 'n' men participants. Then, the number of games that the men play between themselves is $2 \cdot {}^nC_2$ and the number of games that the men played with the women is $2 \cdot (2n)$.

$\therefore \quad 2 \cdot {}^nC_2 - 2 \cdot 2n = 66$ (given)

$\Rightarrow \quad n(n-1) - 4n - 66 = 0 \Rightarrow n^2 - 5n - 66 = 0$

$\Rightarrow \quad (n+5)(n-11) = 0 \Rightarrow n = 11$

$\therefore$ Number of participants = 11 men + 2 women = 13

63. There are two cases arise

***Case* I** They do not invite the particular friend

$$= {}^8C_6 = 28$$

***Case* II** They invite one particular friend

$$= {}^8C_5 \times {}^2C_1 = 112$$

$\therefore$ Required number of ways $= 28 + 112 = 140$

64. Since, the person is allowed to select atmost n coins out of $(2n+1)$ coins, therefore in order to select one, two, three, ..., n coins. Thus, if T is the total number of ways of selecting atleast one coin, then

$$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 255 \qquad \ldots(i)$$

Using the binomial theorem

$$^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n+1}$$

$$= (1+1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow \quad {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow \quad 1 + 2(T) + 1 = 2^{2n+1}$$

$$\Rightarrow \quad 1 + T = \frac{2^{2n+1}}{2} = 2^{2n} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 1 + 255 = 2^{2n}$$

$$\Rightarrow \quad 2^{2n} = 2^8 \Rightarrow n = 4$$

65. First stall can be filled in 3 ways, second stall can be filled in 3 ways and so on.

$\therefore$ Number of ways of loading steamer

$$= {}^3C_1 \times {}^3C_1 \times \ldots \times {}^3C_1 \text{ (12 times)}$$

$$= 3 \times 3 \times \ldots \times 3 \text{ (12 times)} = 3^{12}$$

66. $\because$ The candidate is unsuccessful, if he fails in 9 or 8 or 7 or 6 or 5 papers.

$\therefore$ Numbers of ways to be unsuccessful

$$= {}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5$$

$$= {}^9C_0 + {}^9C_1 + {}^9C_2 + {}^9C_3 + {}^9C_4$$

$$= \frac{1}{2}({}^9C_0 + {}^9C_1 + \ldots + {}^9C_9)$$

$$= \frac{1}{2}(2^9) = 2^8 = 256$$

67. In each dye of chosen, there are two possibility either chose or reject it.

$\therefore$ The total number of ways in which atleast one green and one blue dye is chosen

$$= (2^5 - 1)(2^4 - 1)2^3 = 31 \times 15 \times 8 = 3720$$

68. The number of ways in which four different balls can be placed in four different boxes

$$= {}^4C_1 + {}^3C_1 + {}^2C_1 + {}^1C_1$$

$$= 4 + 3 + 2 + 1 = 10$$

$\therefore$ Required number of ways $= 10 - 1 = 9$

(since, only one way in which the same ball have a same box).

69. Let the boxes be marked as A, B, C. We have to ensure that no box remains empty and all five balls have to put in. There will be two possibilities.

(i) Any two box containing one ball each and 3rd box containing 3 balls. Number of ways

$$= A(1)\, B(1)\, C(3)$$

$$= {}^5C_1 \cdot {}^4C_1 \cdot {}^3C_3 = 5 \cdot 4 \cdot 1 = 20$$

Since, the box containing 3 balls could be any of the three boxes A, B, C. Hence, the required number of ways $20 \times 3 = 60$.

(ii) Any two box containing 2 balls each and 3rd containing 1 ball, the number of ways

$$= A(2)\, B(2)\, C(1) = {}^5C_2 \cdot {}^3C_2 \cdot {}^1C_1$$

$$= 10 \times 3 \times 1 = 30$$

Since, the box containing 1 ball could be any of the three boxes A, B, C. Hence, the required number of ways $= 30 \times 3 = 90$.

Hence, total number of ways $= 60 + 90 = 150$

70. For the first player, distribute the cards in ${}^{52}C_{17}$ ways. Now, out of 35 cards left 17 cards can be put for second player in ${}^{35}C_{17}$ ways. Similarly, for third player put them in ${}^{18}C_{17}$ ways. One card for the last player can be put in ${}^1C_1$ way. Therefore, the required number of ways for the proper distribution

$$= {}^{52}C_{17} \times {}^{35}C_{17} \times {}^{18}C_{17} \times {}^1C_1$$

$$= \frac{52!}{35!17!} \times \frac{35!}{18!17!} \times \frac{18!}{17!1!} \times 1! = \frac{52!}{(17!)^3}$$

71. $\dfrac{\text{Total arrangement}}{\text{Equally likely arrangement}} = \dfrac{18!}{6!6!6!} = \dfrac{18!}{(6!)^3}$

72. Total number of ways

= (Attempt 3 from group I and 4 from group II)

+ (Attempt 4 from group I and 3 from group II)

+ (Attempt 5 from group I and 2 from group II)

+ (Attempt 2 from group I and 5 from group II)

$$= {}^6C_3 \times {}^6C_4 + {}^6C_4 \times {}^6C_3 + {}^6C_5 \times {}^6C_2 + {}^6C_2 \times {}^6C_5$$

$$= 2({}^6C_3 \times {}^6C_4) + 2({}^6C_5 \times {}^6C_2) = 2(20 \times 15) + 2(6 \times 15)$$

$$= 600 + 180 = 780$$

73. Required number of ways

$$= \text{Coefficient of } x^{16} \text{ in } (x^3 + x^4 + x^5 + \ldots + x^{16})^4$$

$$= \text{Coefficient of } x^{16} \text{ in } x^{12}(1 + x + x^2 + \ldots + x^{12})^4$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - x^{13})^4 (1 - x)^{-4}$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - 13x^5 + \ldots) \times \left[1 + 4x + \ldots + \frac{(r+1)(r+2)(r+3)}{3!} x^r\right]$$

$$= \frac{(4+1)(4+2)(4+3)}{3!} = 35$$

74. Total number of books $= a + 2b + 3c + d$

$\therefore$ The total number of arrangements

$$= \frac{(a + 2b + 3c + d)!}{a!(b!)^2(c!)^3}$$

75. Since, the books consisting of 5 Mathematics, 4 Physics and 2 Chemistry can be put together of the same subject in $5!\,4!\,2!$ ways.

But these subject books can be arranged itself in 3! ways

$\therefore$ Required number of ways $= 5!\,4!\,3!\,2!$.

76. We can think of three packets. One consisting of three boys of class X, other consisting of four boys of class XI and last one consisting of five boys of class XII. These packets can be arranged in 3! ways and contents of these packets can be further arranged in 3!4! and 5! ways, respectively. Hence, the total number of ways is $3! \times 3! \times 4! \times 5!$.

77. If the function is one one, then select any three from the set B in ${}^7C_3$ ways *i.e.*, 35 ways.

If the function is many one, then there are two possibilities. All three corresponds to same element number of such functions $= {}^7C_1 = 7$ ways. Two corresponds to same element. Select any two from the set B. The larger one corresponds to the larger and the smaller one corresponds to the smaller the third may corresponds to any two.

Number of such functions $= {}^7C_2 \times 2 = 42$

So, the required number of mappings $= 35 + 7 + 42 = 84$

78. Total number of functions

$$= \text{Number of dearrangement of 5 objects}$$
$$= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

79. We know, a triangle will be formed by taking three points at a time.

$\therefore$ Required number of triangles $= {}^{12}C_3 - {}^7C_3$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} - \frac{7 \times 6 \times 5}{3 \times 2} = 220 - 35 = 185$$

80. Total number of parallelogram formed

$$= {}^4C_2 \times {}^3C_2 = 6 \times 3 = 18$$

81. Two circles intersect maximum at two distinct points.

Now, two circles can be selected in ${}^6C_2$ ways.

$\therefore$ Total number of points of intersection are

$${}^6C_2 \times 2 = 30$$

82. Let n be the number of diagonals of a polygon.

Then, $\quad {}^nC_2 - n = 44$

$\Rightarrow \quad \dfrac{n(n-1)}{2} - n = 44 \Rightarrow n^2 - 3n - 88 = 0$

$\Rightarrow \quad n = -8$ or $11 \quad \therefore n = 11$

83. Required number of diagonals $= {}^mC_2 - m$

$$= \frac{m(m-1)}{2!} - m = \frac{m}{2!}(m-3)$$

84. The triangle will be formed by joining any three non-collinear points.

$\therefore$ Required number of ways $= {}^8C_3 - {}^5C_3 - {}^3C_3$

$$= {}^8C_3 - {}^5C_3 - 1$$

85. Total number of points are $m + n + k$, the triangles formed by these points $= {}^{m+n+k}C_3$

Joining of three points on the same line gives no triangle, the number of such triangles is ${}^mC_3 + {}^nC_3 + {}^kC_3$

$\therefore$ Required number of triangles

$$= {}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$$

86. Number of lines from 6 points $= {}^6C_2 = 15$

Points of intersection obtained from these lines $= {}^{15}C_2 = 105$

Now, we find the number of times, the original 6 points come.

Consider one point say A_1. Joining A_1 to remaining 5 points, we get 5 lines and any two lines from these 5 lines gives A_1 as the point of intersection.

$\therefore A_1$ is common in ${}^5C_2 = 10$ times out of 105 points of intersections.

Similar is the case with other five points.

$\therefore$ 6 original points come $6 \times 10 = 60$ times in points of intersection.

Hence, the number of distinct points of intersection

$$= 105 - 60 + 6 = 51$$

87. The required number of points

$$= {}^8C_2 \times 1 + {}^4C_2 \times 2 + ({}^8C_1 \times {}^4C_1) \times 2$$
$$= 28 + 12 + 32 \times 2 = 104$$

88. Since, there are n distinct points on a circle.

For making a pentagon it requires a five points.

According to given condition,

$${}^nC_5 = {}^nC_3 \quad \Rightarrow \quad n = 8$$

89. $\because$ The factors of $9600 = 2^7 \times 3^1 \times 5^2$

$\therefore$ The number of divisors $= (7+1)(1+1)(2+1)$

$$= 8 \times 2 \times 3 = 48$$

90. Since, $\quad 240 = 2^4 \cdot 3 \cdot 5$

$\therefore$ Total number of divisors $= (4+1)(1+1)(1+1) = 20$

Out of these 2, 6, 10 and 30 are of the form $4n + 2$.

91. Since, $\quad 38808 = 2^3 \times 3^2 \times 7^2 \times 11^1$

$\therefore$ Number of divisors $= 4 \times 3 \times 3 \times 2 - 2 = 72 - 2 = 70$

92. The number of divisors of ab^2c^2de

$$= (1+1)(2+1)(2+1)(1+1)(1+1) - 1$$
$$= 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 - 1 = 71$$

93. The required number of ways is equal to the number of dearrangements of 10 objects.

$\therefore$ Required number of ways

$$= 10!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!}\right\}$$

94. The number of ordered triples of positive integers which are solution of $x + y + z = 100$

$=$ coefficient of x^{100} in $(x + x^2 + x^3 + \ldots)^3$

$=$ coefficient of x^{100} in $x^3(1-x)^{-3}$

$=$ coefficient of x^{97} in

$$\left\{1 + 3x + 6x^2 + \ldots + \frac{(n+1)(n+2)}{2}x^n + \ldots\right\}$$

$$= \frac{(97+1)(97+2)}{2} = 49 \times 99 = 4851$$

95. Given, 4 vowels and 5 consonants

$\therefore$ Total number of words $= {}^4C_2 \times {}^5C_3 \times 5! = 6 \times 10 \times 120$

$$= 7200$$

96. For each historical monument, there are two possibilities either he visit or does not visit.

Total number of ways $= 2^5 \cdot 2^6 (2^7 - 1)$

Round II

1. We have, 32 places for teeth. For each place, we have two choices either there is a tooth or there is no tooth. Therefore, the number of ways to fill up these places is 2^{32}. As there is no person without a tooth, the maximum population is $2^{32}-1$.

2. For length, number of choices is

$$(2m-1)+(2m-3)+\ldots+3+1=m^2$$

Similarly, for breadth number of choices is

$$(2n-1)+(2n-3)+\ldots+3+1=n^2$$

Hence, required number of choices is m^2n^2.

3. Total time required = (total number of dials required to sure open the lock) $\times 5\,s$

$$=10^5\times 5\,s=\frac{500000}{60\times 60\times 13}\text{ days}$$

$$=10.7\text{ days}$$

Hence, 11 days are enough to open the safe.

4. Let n be the number of sides of the polygon.

$$n\cdot 160^\circ=(n-2)\cdot 180^\circ\Rightarrow 20^\circ\cdot n=360^\circ$$

$$\therefore\quad n=18$$

Then number of diagonals $={}^{18}C_2-18=153-18=135$

5. Required number of ways $={}^9C_4=126$

6. Total number of m elements subsets of $A={}^nC_m$ ...(i)

and number of m elements subsets of A each containing the element $a_4={}^{n-1}C_{m-1}$

According to the question, ${}^nC_m=\lambda\cdot{}^{n-1}C_{m-1}$

$$\Rightarrow\quad \frac{n}{m}\cdot{}^{n-1}C_{m-1}=\lambda\cdot{}^{n-1}C_{m-1}$$

$$\Rightarrow\quad \lambda=\frac{n}{m}\quad\Rightarrow\quad n=m\lambda$$

7. Number of ways of giving one prize for running $=16$

Number of ways of giving two prizes for swimming

$$=16\times 15$$

Number of ways of giving three prizes for riding

$$=16\times 15\times 14$$

$\therefore$ Required ways of giving prizes

$$=16\times 16\times 15\times 16\times 15\times 14$$

$$=16^3\times 15^2\times 14$$

8. The number of ways of selecting four numbers from 1 to 30 without any restriction is ${}^{30}C_4$. The number of ways of selecting four consecutive [*i.e.*, (1, 2, 3, 4), (2, 3, 4, 5), ..., (27, 28, 29, 30)] number is 27. Hence, the number of ways of selecting four integers which excludes consecutive four selections is

$${}^{30}C_4-27=\frac{30\times 29\times 28\times 27}{24}-27=27378$$

9. Other than 2, remaining five places can be filled by 1 and 3 for each place. The number of ways for five places is $2\times 2\times 2\times 2\times 2=2^5$. For 2, selecting 2 places out of 7 is ${}^7C_2$. Hence, for required number of ways is ${}^7C_2\times 2^5$.

10. **Hint** ${}^nP_3-{}^nC_3>100$

$$\Rightarrow\quad \frac{n!}{(n-3)!}-\frac{n!}{3!(n-3)!}>100$$

11. The number of ways he can select atleast one parantha is $2^3-1=7$. The number of ways he can select atleast one vegetable dish is $2^4-1=15$. The number of ways be can select zero or more items from salads and sauces is 2^5. Hence, the total number of ways is $7\times 15\times 32=3360$.

12. Select any three points from total $3p$ points, which can be done in ${}^{3p}C_3$ ways. But this also includes selection of three collinear points. Now, three collinear points from each straight line can be selected in ${}^pC_3$ ways. Then, the number of triangles is ${}^{3p}C_3-3\cdot{}^pC_3=p^2(4p-3)$.

13. No group of four numbers from the first 12 natural numbers can have the common difference 4.

If one group including 1 is selected with the common difference 1, then the other two group can have the common difference 1 or 2.

If one group including 1 is selected with the common difference 2, then one of the other two group can have the common difference 2 and the remaining group will have common difference 1.

If one group including 1 is selected with the common difference 3, then the other two groups can have the common difference 3.

Therefore, the required number of ways is $2+1+1=4$.

14. 20 cards can be chosen out of 52 cards in ${}^{52}C_{20}$ ways. There are two ways in which each card can be dealt, because a card can be either from the first pack or from the second. Hence, the total number of ways is ${}^{52}C_{20}\times 2^{20}$.

15. Total number of arrangements $=7!$

Number of arrangements of A, B, C, D among themselves $=4!$

$\therefore$ Number of arrangements when A, B, C, D occur in a particular order

$$=\frac{7!}{4!}=210\;={}^7P_3=3!\times{}^7C_3$$

16. Total number of ways $=\sum_{r=1}^{n}{}^{2n}C_{n+r}$

$$={}^{2n}C_{n+1}\ldots+{}^{2n}C_{2n}=\frac{1}{2}(2^{2n}-{}^{2n}C_n)$$

$$=\frac{1}{2}\left(2^{2n}-\frac{2n!}{n!n!}\right)$$

17. Required number = Number of selections of one or more out of three 25 paise coins and two 50 paise coins

$= 4 \times 3 - 1 = 11 = {}^{12}P_1 - 1$

18. Considering CC as single object, U, CC, E can be arranged in 3! ways.

$\times U \times CC \times E \times$

Now, the three S are to be placed in four available places.

Hence, required number of ways $= {}^4P_3 = 24$

19. Total number of ways $= \dfrac{7!}{2!\,3!} = 420$

Consonants in SUCCESS are S, C, C, S, S.

Number of ways arranging consonants $= \dfrac{5!}{2!\,3!} = 10$

Hence, number of words in which the consonants appear in alphabetic order $= \dfrac{420}{10} = 42$

20. The alphabetic order is C, E, S, U. The number of words beginning with C is $\dfrac{6!}{3!} = 120$ and number of words beginning with E is $\dfrac{6!}{2!3!} = 60$

Number of words beginning with SC, $\dfrac{5!}{2!} = 60$,

beginning with SE, $\dfrac{5!}{2!2!} = 30$ and number of words

beginning SS, $\dfrac{5!}{2!} = 60$

After which the next word is SUCCESS.

Thus, the rank of SUCCESS is

$120 + 60 + 60 + 30 + 60 + 1 = 331$

21. Total letters $= 26$ (*i.e.*, $A, B, C, \ldots, Y, Z$) and total digit number $= 10$ (*i.e.*, 0, 1, 2, 3, ..., 9)

$\because$ Repetition of letters is allowed.

$\therefore$ The three letters can be filled by $26 \times 26 \times 26 = (26)^3$ ways and three digit numbers on plate by 999 ways.

(*i.e.*, 001, 002, ..., 999)

$\therefore$ Required number of ways $= (26)^3 \times 999$

22. For the number exactly divisible by 4, then last two digit must be divisible by 4, the last two digits are *viz*.

12, 16, 24, 32, 36, 52, 56, 64, 72, 76

Total 10 ways. Now, the remaining two first places on the left of 4-digit numbers are to be filled from the remaining 5-digits and this can be done in ${}^5P_2 = 20$ ways.

$\therefore$ Required number of ways $= 20 \times 10 = 200$

23. The number of non-negative integral solutions

$=$ Coefficient of x^r in $(1 + x + x^2 + \ldots)^n$

$=$ Coefficient of x^r in $(1 - x)^{-n}$

$= {}^{n+r-1}C_r$ or ${}^{n+r-1}C_{r-1}$

24. Number of required ways

= (1 woman, 4 men) or (2 women, 3 men) or (3 Women, 2 men) or (4 women, 1 man)

$= {}^4C_1 \times {}^8C_4 + {}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_4 \times {}^8C_1$

$= 736$

25. Product of n consecutive natural number

$= (m+1)(m+2)(m+3)\ \ldots (m+n), m \in$ whole number

$= \dfrac{(m+n)!}{m!} = n! \times \dfrac{(m+n)!}{m!\,n!}$

$= n! \times {}^{m+n}C_m$

$\Rightarrow$ Product is divisible by $n!$, then it is always divisible by $(n-1)!$ but not by $(n+1)!$.

26. In a chess board there are 9 horizontal and 9 vertical lines. Number of rectangles of any size are ${}^9C_2 \times {}^9C_2$.

Hence, option (d) is correct.

27. Sum of the digits in the tenth place

= Sum of the digits in the unit's place

$= (4-1)!\,(2 + 3 + 4 + 5) = 3! \times 14 = 84$

28. Since, each student receive atleast one toy. Then, firstly we give each student one toy and the remaining 7 toys can be distributed in three students in ${}^{7-1}C_{3-1} = {}^6C_2$ ways.

Hence, statement I is false and statement II is true.

29. In onto function each image must be assigned atleast one pre-image. Now, if we consider the images a and b as two different boxes, then four distinct objects 1, 2, 3 and 4 (pre-images) can be distributed in $2^4 - {}^2C_1(2-1)^4 = 14$ ways.

Hence, both statements are correct and statement II is correct explanation of statement I.

30. The batting order of 11 players can be decided in 11! ways. Now, Yuvraj, Dhoni and Pathan can be arranged in 3! ways. But the order of these three players is fixed, *i.e.*, Yuvraj-Dhoni-Pathan. Now, 11! answer is 3! times more, hence the required number of ways is 11!/3!.

31. Let the number of ways of distributing n identical objects, among r persons such that each person gets atleast one object is same as the number of ways of selecting $(r-1)$ places out of $(n-1)$ different places, *i.e.*, ${}^{n-1}C_{r-1}$.

32. Since, the number of ways that child can buy the six ice-creams is equal to the number of different ways of arranging 6A's and 4B's in a row.

$\therefore$ Number of ways to arrange 6A's and 4B's in a row.

$= \dfrac{10!}{6!\,4!} = {}^{10}C_4$

and number of integral solution of the equation

$x_1 + x_2 + x_3 + x_4 + x_5 = 6 = {}^{6+5-1}C_{5-1} = {}^{10}C_4 \neq {}^{10}C_5$

Statement I is false and statement II is true.

33. Given, $n(A) = 2, n(B) = 4 \therefore n(A \times B) = 8$

The number of subsets of $A \times B$ having 3 or more elements

$$= {}^8C_3 + {}^8C_4 + \ldots + {}^8C_8$$
$$= ({}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$$

$\therefore \quad 2^n = {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n$

$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 = 256 - 1 - 8 - 28 = 219$$

34. **Hint** $\quad T_n = {}^nC_3$

$$T_{n+1} = {}^{n+1}C_3$$

$\therefore \quad T_{n+1} - T_n = {}^{n+1}C_3 - {}^nC_3 = 10$ (given)

35. The number of ways to choose zero or more white balls

$= (10 + 1)$

($\because$ all white balls are mutually identical)

Number of ways to choose zero or more green balls

$= (9 + 1)$

($\because$ all green balls are mutually identical)

Number of ways to choose zero or more black balls

$= (7 + 1)$

($\because$ all black balls are mutually identical)

Hence, number of ways to choose zero or more balls of any colour $= (10 + 1)(9 + 1)(7 + 1)$

Also, number of ways to choose a total of zero balls $= 1$

Hence, the number of ways to choose atleast one ball (irrespective of any colour)

$$= (10 + 1)(9 + 1)(7 + 1) - 1 = 880 - 1 = 879$$

36. The number of different ordered pairs (Y, Z) such that $Y \subseteq X, Z \subseteq X$ and $Y \cap Z = \phi$. Since, $Y \subseteq X, Z \subseteq X$, hence we can only use the elements of X to construct sets Y and Z.

$n(Y)$	Number of ways to make Y	Number of ways to make Z such that $Y \cap Z = \phi$
0	${}^5C_0$	2^5
1	${}^5C_1$	2^4
2	${}^5C_2$	2^3
3	${}^5C_3$	2^2
4	${}^5C_4$	2^1
5	${}^5C_5$	2^0

Hence, total number of ways to construct sets Y and Z such that $Y \cap Z = \phi$ is

$${}^5C_0 \times 2^5 + {}^5C_1 \times 2^{5-1} + \ldots + {}^5C_5 \times 2^{5-5} = (2 + 1)^5 = 3^5$$

37.

The number of ways in which two balls from urn A and two balls from urn B can be selected $= {}^3C_2 \times {}^9C_2 = 3 \times 36 = 108$

38. The number of ways in which 4 novels can be selected

$$= {}^6C_4 = 15$$

The number of ways in which 1 dictionary can be selected

$$= {}^3C_1 = 3$$

4 novels can be arranged in 4! ways.

$\therefore$ The total number of ways

$$= 15 \times 4! \times 3 = 15 \times 24 \times 3 = 1080$$

39. Given word is MISSISSIPPI.

Here, I = 4 times, S = 4 times,

P = 2 times, M = 1 time

_M_I_I_I_I_P_P_

Required number of words $= {}^8C_4 \times \dfrac{7!}{4!2!} = {}^8C_4 \times \dfrac{7 \times 6!}{4!2!}$

$$= 7 \cdot {}^8C_4 \cdot {}^6C_4$$

40. $\therefore$ Required number of ways $= {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$$= \frac{12!}{8!4!} \times \frac{8!}{4!4!} \times 1 = \frac{12!}{(4!)^3}$$

41. Total number of ways $= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

$$= 10 + 45 + 120 + 210 = 385$$

42. In the word SACHIN order of alphabets is A, C, H, I, N, S. Number of words start with $A = 5!$, so with C, H, I, N, now words start with S and after that ACHIN are in ascending orders of position, so $5 \cdot 5! = 600$ words are in dictionary before words start with S and position of this words is 601.

43. Total number of ways in which all letters can be arranged in alphabetical order $= 6!$

There are two vowels in the word GARDEN.

Total number of ways in which these two vowels can be arranged $= 2!$

$\therefore$ Total number of required ways $= \dfrac{6!}{2!} = 360$

44. Required number of ways $= {}^{8-1}C_{3-1} = {}^7C_2 = \dfrac{7!}{2!5!} = \dfrac{7 \cdot 6}{2 \cdot 1} = 21$

45. The number of choices available to him

$$= {}^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5 = 5 \times \frac{8 \times 7}{2} + 1 \times \frac{8 \times 7 \times 6}{3 \times 2}$$
$$= 5 \times 4 \times 7 + 8 \times 7 = 140 + 56 = 196$$

46. The number of ways to sit men $= 5!$

and the number of ways to sit women $= {}^6P_5$

Total number of ways $= {}^6P_5 \times 5! = 6! \times 5!$

47. ${}^nC_{r+1} + {}^nC_{r-1} + 2\,{}^nC_r = {}^nC_{r+1} + {}^nC_r + {}^nC_{r-1} + {}^nC_r$

$$= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1}$$

7 Mathematical Induction

JEE Main MILESTONE

- Peano's Theorem
- Principle of Mathematical Induction
- Simple Applications of Mathematical Induction

7.1 Peano's Theorem

The statement or proposition $P(n)$ is assumed to be true for all values of $n \in N$,

(a) $P(1)$ is true

(b) $P(m)$ is true

$\Rightarrow$ $P(m + 1)$ is true

7.2 Principle of Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n such that

(i) The statement is true for $n = 1$, *i.e.*, $P(1)$ is true and

(ii) If the statement is true for $n = k$ (where, k is some positive integer), then the statement is also true for $n = k + 1$, *i.e.*, truth of $P(k)$ implies the truth of $P(k + 1)$.

Then, $P(n)$ is true for all natural numbers n.

There are following steps given below

Step I Verification step Actual verification of the proposition for the starting value 'i'.

Step II Induction step Assuming the proposition to be true for 'k', $k \geq i$ and then providing that it is true for the value $(k + 1)$ which is the next higher integer.

Step III Generalization step Combine the above two steps.

The word induction means the generalization from particular cases or facts. In algebra or in other disciplines of mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements the well suited principle that is used based on specific technique, is the principle of mathematical induction.

Hot Spot

Solution by Mathematical Induction

In these types of problems, generally a series is given and we have to find its name.
These types of problems are directly asked in various years in AIEEE (JEE Main) 2008, 2005, 2004.

Step I Firstly, we consider the given expression is equal to $P(n)$.

Step II Put $n = 1$ and verify it.

Step III Assume given expression is true for $P(k)$.

Step IV Verify for $P(k + 1)$. Hence, we can say that it is true for all natural numbers.

Sample Problem 1 *The series*

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2) \text{ is equal to}$$

(a) $\dfrac{n(n+1)(n+2)(n+3)}{8}$ (b) $\dfrac{n(n+1)(n+2)(n+3)}{4}$

(c) $\dfrac{n(n+1)(n+2)(n+3)}{3}$ (d) *None of these*

Interpret (b) **Step I** Consider the given expression is equal to $P(n)$.

Let $P(n) : 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2)$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Step II Put $n = 1$ and verify it

$$\therefore \quad P(1) : \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1\times 2\times 3\times 4}{4} = 1\cdot 2\cdot 3$$

which is true.

Step III Assume given expression is true for $P(k)$.

$$\therefore \quad P(k) : 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)k+3)}{4}$$

Step IV Verify for $P(k + 1)$.

$$\therefore \quad P(k+1) : 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + k(k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$\therefore P(k + 1)$ is true, when $P(k)$ is true.

Hence, the principle of mathematical induction, the statement is true for all natural numbers n.

Sample Problem 2 *For all $n \in N$,*

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha + (n-1)\beta\}$$

is equal to [NCERT Exemplar]

(a) $\dfrac{\cos\left[\alpha + \frac{(n+1)}{2}\beta\right]\sin\frac{n\beta}{2}}{\sin\beta}$ (b) $\dfrac{\cos\left[\alpha + \frac{(n-1)}{2}\beta\right]\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

(c) $\dfrac{\cos\left[\alpha + \frac{(n-1)}{2}\beta\right]\sin\frac{n\beta}{2}}{\sin\beta}$ (d) *None of these*

Interpret (b) Let the given statement be $P(n)$.

i.e.,

$$P(n) : \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Step I For $n = 1$,

$$P(1) : \frac{\cos\left[\alpha + \frac{1-1}{2}\beta\right]\sin\left(\frac{1\times\beta}{2}\right)}{\sin\frac{\beta}{2}} = \frac{\cos(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \cos\alpha$$

which is true.

Step II Let it is true for $n = k$, *i.e.,*

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (k-1)\beta]$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} \quad \ldots\text{(i)}$$

Step III For $n = k + 1$,

$$\cos\alpha + \cos(\alpha+\beta) + \ldots + \cos[\alpha + (k-1)\beta] + \cos(\alpha + k\beta)$$

$$= \frac{\cos\left[\alpha + \frac{(k-1)}{2}\beta\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} + \cos(\alpha + k\beta) \quad \text{[using Eq. (i)]}$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2} + \cos(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2} + 2\cos(\alpha+k\beta)\sin\frac{\beta}{2}}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left[\alpha + \left(\frac{k-1}{2}\right)\beta + \frac{k\beta}{2}\right] - \sin\left[\left(\alpha + \frac{k-1}{2}\beta\right) - \frac{k\beta}{2}\right] + \sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha + k\beta - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$[\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$

$$= \frac{\sin\left(\alpha + k\beta - \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right) + \sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha + k\beta - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\frac{\alpha + k\beta + \frac{\beta}{2} + \alpha - \frac{\beta}{2}}{2}\sin\frac{\alpha + k\beta + \frac{\beta}{2} - \alpha + \frac{\beta}{2}}{2}}{2\sin\frac{\beta}{2}}$$

$$\left(\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right)$$

$$= \frac{2\cos\left(\frac{2\alpha + k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left[\alpha + \frac{\{(k+1)-1\}\beta}{2}\right]\sin\left[\frac{\beta(k+1)}{2}\right]}{\sin\frac{\beta}{2}}$$

Therefore, $P(k+1)$ is true when $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

7.3 Simple Applications of Mathematical Induction

Mathematical induction can be used in various types of problems.

Type I **Use identity type problems.**

This can be explained by given example.

***Sample* Problem 3** *The identity* $1^3 + 2^3 + 3^3 + \ldots + n^3$ *is equal to* **[NCERT]**

(a) $\left[\frac{n(n-1)}{2}\right]^2$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{\{n(n+1)\}^2}{2}$ (d) $\left[\frac{n(n+1)}{2}\right]^2$

Interpret (*d*) Let the given statement be $P(n)$.

i.e., $P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Step I For $n = 1$,

$$P(1): \left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{1\times 2}{2}\right]^2 = 1^2 = 1 = 1^3$$

which is true.

Step II Let it is true for $n = k$,

i.e., $1^3 + 2^3 + 3^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ ...(i)

Step III For $n = k+1$, $(1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3) + (k+1)^3$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \quad \text{[using Eq. (i)]}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{(k+1)^3}{1} = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

On taking $(k+1)^2$ common in numerator part,

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2[(k+1)+1]^2}{4}$$

$$= \left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2$$

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

Type II **Use divisibility type problems.**

This can be explained by given example.

***Sample* Problem 4** *For all* $n \in N$, $a^n - b^n$ *is divisible by*

(a) $(a+b)$ (b) $(a-b)$

(c) $2(a-b)$ (d) *None of these*

Interpret (*b*) Let $P(n) = a^n - b^n$

Step I For $n = 1$,

$P(1) = a - b$, which is divisible by $a - b$. Therefore, the result is true for $n = 1$.

Step II Assume that the result is true for $n = k$, *i.e.*, $P(k) = a^k - b^k$ is divisible by $a - b$.

$\Rightarrow$ $P(k) = (a-b)r$, where r is an integer.

Step III For $n = k+1$,

$\therefore$ $P(k+1) = a^{k+1} - b^{k+1}$

Now, $a^k - b^k)a^{k+1} - b^{k+1}(a$

$$\underline{-\ a^{k+1} \mp ab^k}$$

$$ab^k - b^{k+1} = b^k(a-b)$$

$\therefore$ $a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a-b)$

i.e., $P(k+1) = aP(k) + b^k(a-b)$

But we know that $P(k)$ is divisible by $a - b$. Also, $b^k(a - b)$ is clearly divisible by $a - b$.

Hence, $P(k + 1)$ is divisible by $a - b$. This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Type III Use inequality type problems.

This can be explained by given example.

Sample Problem 5 *The sum of series is less than*

(a) $\frac{1}{8}(2n+1)^2$ (b) $\frac{1}{12}(2n+1)^2$

(c) $\frac{1}{8}(2n+1)$ (d) *None of these*

Interpret (a) Let the given statement be $P(n)$.

i.e., $P(n): 1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n+1)^2$

Step I For $n = 1$,

$$1 < \frac{1}{8}(2 \cdot 1 + 1)^2 \Rightarrow 1 < \frac{1}{8} \times 3^2 \Rightarrow 1 < \frac{9}{8}$$

which is true.

Step II Let it is true for $n = k$,

$$1 + 2 + 3 + \ldots + k < \frac{1}{8}(2k+1)^2 \quad \ldots(i)$$

Step III For $n = k + 1$,

$$(1 + 2 + 3 + \ldots + k) + (k + 1) < \frac{1}{8}(2k+1)^2 + (k+1) \text{ [using Eq. (i)]}$$

$$= \frac{(2k+1)^2}{8} + \frac{k+1}{1} = \frac{(2k+1)^2 + 8k + 8}{8}$$

$$= \frac{4k^2 + 1 + 4k + 8k + 8}{8} = \frac{4k^2 + 12k + 9}{8}$$

$$= \frac{(2k+3)^2}{8} = \frac{(2k+2+1)^2}{8} = \frac{[2(k+1)+1]^2}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + k + (k+1) < \frac{[2(k+1)+1]^2}{8}$$

Therefore, $P(k + 1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

Type IV Use recursion type of problems.

This can be explained by given example.

Sample Problem 6 *If* $u_{n+1} = 3u_n - 2u_{n-1}$ *and* $u_0 = 2$, $u_1 = 3$, *then* u_n *is equal to*

(a) $1 - 2^n$ (b) $2^n + 1$ (c) $2^n - 1$ (d) $2^n + 2$

Interpret (b) $\because$ $u_{n+1} = 3u_n - 2u_{n-1}$ ...(i)

Step I Given, $u_1 = 3 = 2 + 1 = 2^1 + 1$

which is true for $n = 1$,

Put $n = 1$ in Eq. (i),

then $u_{1+1} = 3u_1 - 2u_{1-1}$

$\Rightarrow$ $u_2 = 3u_1 - 2u_0$

$= 3 \cdot 3 - 2 \cdot 2 = 5 = 2^2 + 1$

which is true for $n = 2$.

Therefore, the result are true for $n = 1$ and $n = 2$.

Step II Assume it is true for $n = k$, then it is also true for $n = k - 1$.

Then, $u_k = 2^k + 1$ ...(ii)

and $u_{k-1} = 2^{k-1} + 1$ ...(iii)

Step III On putting $n = k$ in Eq. (i), we get

$$u_{k+1} = 3u_k - 2u_{k-1}$$

$$= 3(2^k + 1) - 2(2^{k-1} + 1) \quad \text{[from Eqs. (ii) and (iii)]}$$

$$= 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2$$

$$= 3 \cdot 2^k + 3 - 2^k - 2 = (3 - 1)2^k + 1$$

$$= 2 \cdot 2^k + 1 = 2^{k+1} + 1$$

This shows that the result is true for $n = k + 1$, hence by the principle of mathematical induction the result is true for all $n \in N$.

Check Point 1

1. For positive integer n, $10^{n-2} > 81n$, if
   (a) $n > 5$ (b) $n \geq 5$
   (c) $n < 5$ (d) $n > 6$
2. If $n \in N$, $7^{2n} - 48n - 1$ is divisible by
   (a) 25 (b) 26
   (c) 1234 (d) 2304
3. For all $n \geq 2$, $n^2(n^4 - 1)$ is divisible by
   (a) 60 (b) 50
   (c) 40 (d) 70
4. For each $n \in N$, the correct statement is
   (a) $2^n < n$ (b) $n^2 > 2n$
   (c) $n^4 < 10^n$ (d) $2^{3n} > 7n + 1$
5. If $n \in N$, then the highest positive integer which divides $n(n-1)(n-2)$ is
   (a) 3 (b) 6 (c) 9 (d) 12
6. For every positive integer n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is
   (a) an integer (b) a rational number
   (c) a negative real number (d) an odd integer

WORKED OUT Examples

Example 1 *The sum of series* $1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + n\cdot 3^n$ *is*

(a) $\dfrac{(2n+1)3^{n+1}+3}{4}$ (b) $\dfrac{(2n-1)3^{n+1}+3}{4}$

(c) $\dfrac{(2n-1)3^{n-1}-3}{4}$ (d) *None of these*

Solution (b) Let the given statement be $P(n)$.

i.e., $P(n): 1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + n\cdot 3^n = \dfrac{(2n-1)3^{n+1}+3}{4}$

Step I For $n = 1$,

$$P(1): \frac{(2\cdot 1-1)3^{1+1}+3}{4} = \frac{3^2+3}{4} = \frac{9+3}{4} = \frac{12}{4}$$

$\Rightarrow$ $3 = 1\cdot 3$

which is true.

Step II Let it is true for $n = k$,

i.e., $1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + k\cdot 3^k = \dfrac{(2k-1)3^{k+1}+3}{4}$ ...(i)

Step III For $n = k+1$,

$$(1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + k\cdot 3^k) + (k+1)3^{k+1}$$

$$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1} \quad \text{[using Eq. (i)]}$$

$$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

On taking 3^{k+1} common in first and last term of numerator part,

$$= \frac{3^{k+1}(2k-1+4k+4)+3}{4}$$

$$= \frac{3^{k+1}(6k+3)+3}{4}$$

On taking 3 common in first term of numerator part,

$$= \frac{3^{k+1}\cdot 3(2k+1)+3}{4}$$

$$= \frac{3^{(k+1)+1}(2k+2-1)+3}{4}$$

$$= \frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

Example 2 *By mathematical induction*

$\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \ldots + \dfrac{1}{n(n+1)(n+2)}$ *is equal to* **[NCERT]**

(a) $\dfrac{n(n+1)}{4(n+2)(n+3)}$ (b) $\dfrac{n(n+3)}{4(n+1)(n+2)}$

(c) $\dfrac{n(n+2)}{4(n+1)(n+3)}$ (d) *None of these*

Solution (b) Let $P(n): \dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \ldots + \dfrac{1}{n(n+1)(n+2)}$

$$= \frac{n(n+3)}{4(n+1)(n+2)} \quad \ldots\text{(i)}$$

Step I For $n = 1$,

$$\text{LHS} = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6} \quad \text{and RHS} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$$

$\therefore$ $P(1)$ is true.

Step II Let $P(k)$ is true, then

$$P(k): \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

Step III For $n = k+1$, then

$$P(k+1): \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\text{LHS} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \text{[from Eq. (i)]}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)} = \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \text{RHS}$$

Hence, $P(k+1)$ is true.

Hence, by principle of mathematical induction for all $n \in N$, $P(n)$ is true.

Example 3 *By principle of mathematical induction* $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta)$, $\forall\, n \in N$ *is equal to*

[**NCERT Exemplar**]

(a) $\dfrac{\sin 2^n \theta}{2^n \sin\theta}$ (b) $\dfrac{\cos 2^n \theta}{2^n \sin\theta}$

(c) $\dfrac{\sin 2^n \theta}{2^{n-1} \sin\theta}$ (d) *None of these*

Solution (a) Let

$$P(n) : \cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin 2^n \theta}{2^n \sin\theta} \quad \ldots(\text{i})$$

Step I For $n = 1$,

$$\text{LHS} = \cos\theta$$

and $$\text{RHS} = \frac{\sin 2\theta}{2\sin\theta} = \cos\theta$$

$\therefore$ $P(1)$ is true.

Step II Let $P(k)$ is true, then

$$P(k) : \cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{k-1}\theta) = \frac{\sin 2^k\theta}{2^k \sin\theta}$$

Step III For $n = k + 1$

$$P(k+1) : \cos\theta \cos 2\theta \ldots \cos(2^k\theta) = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta}$$

$$\text{LHS} = \cos\theta \cos 2\theta \ldots \cos 2^{(k-1)}\theta \cos 2^k\theta$$

$$= \frac{\sin(2^k\theta)}{2^k \sin\theta} \cdot \cos(2^k\theta)$$

$$= \frac{2\sin(2^k\theta)\cdot\cos(2^k\theta)}{2^{k+1}\sin\theta} = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta} = \text{RHS}$$

$\therefore$ For $n = k + 1$, $P(n)$ is true.

Hence, by principle for mathematical induction for all $n \in N$, $P(n)$ is true.

Example 4 *If* $\dfrac{4^n}{n+1} < \dfrac{(2n)!}{(n!)^2}$, *then* $P(n)$ *is true for*

(a) $n \geq 1$ (b) $n > 0$

(c) $n < 0$ (d) $n \geq 2$

Solution (d) Let $P(n) : \dfrac{4^n}{n+1} < \dfrac{(2n)!}{(n!)^2}$

For $n = 2$,

$$P(2) : \frac{4^2}{2+1} < \frac{4!}{(2)^2}$$

$$\Rightarrow \quad \frac{16}{3} < \frac{24}{4}$$

which is true.

Let for $n = m \geq 2$, $P(m)$ is true.

i.e., $$\frac{4^m}{m+1} < \frac{(2m)!}{(m!)^2} \quad \ldots(\text{i})$$

Now, $$\frac{4^{n+1}}{m+2} = \frac{4^m}{m+1} \cdot \frac{4(m+1)}{m+2}$$

$$< \frac{(2m)!}{(m!)^2} \cdot \frac{4(m+1)}{(m+2)}$$

$$= \frac{(2m)!(2m+1)(2m+2)\,4(m+1)(m+1)^2}{(2m+1)(2m+2)(m!)^2(m+1)^2(m+2)}$$

$$= \frac{[2(m+1)]!}{[(m+1)!]^2} \cdot \frac{2(m+1)^2}{(2m+1)(m+2)} < \frac{[2(m+1)]!}{[(m+1)!]^2}$$

Hence, for $n \geq 2$, $P(n)$ is true.

Example 5 *If* $a_n = 2^{2^n} + 1$, *then for* $n > 1$, *last digit of* a_n *is*

(a) 3 (b) 5

(c) 7 (d) 8

Solution (c) For $n = 2$, $a_n = 2^{2^2} + 1 = 16 + 1 = 17$

Assume that, $a_k = 2^{2^k} + 1 = 10\,m + 7$, where $k > 1$ and m is some positive integer.

Now, $$a_{k+1} = 2^{2^{k+1}} + 1 = (2^{2^k})^2 + 1 = (10\,m + 6)^2 + 1$$

$$= 10\,(10\,m^2 + 12\,m + 3) + 7$$

Hence, last digit of a_n is 7, $\forall\, n > 1$.

Example 6 *By principle of mathematical induction* $3^{2n+2} - 8n - 9$ *is divisible by*

(a) 9 (b) 8 (c) 7 (d) 6

Solution (b) Let the given statement be $P(n)$.

i.e., $P(n) : 3^{2n+2} - 8n - 9$ is divisible by 8.

Step I For $n = 1$,

i.e., $$P(1) : 3^{2\times 1+2} - 8\times 1 - 9 = 3^4 - 8 - 9 = 81 - 17$$

$$= 64 = 8 \times 8$$

which is divisible by 8.

Step II Let it is true for $n = k$,

i.e., $$3^{2k+2} - 8k - 9 = 8\lambda \quad \ldots(\text{i})$$

Step III For $n = k + 1$,

$$3^{2(k+1)+2} - 8(k+1) - 9$$

$$= 3^{2k+2+2} - 8k - 8 - 9 = 3^{2k+2}3^2 - 8k - 17$$

$$= (8\lambda + 8k + 9)3^2 - 8k - 17 \quad \text{[using Eq. (i)]}$$

$$= (8\lambda + 8k + 9)9 - 8k - 17$$

$$= 72\lambda + 72k + 81 - 8k - 17$$

$$= 72\,\lambda + 64k + 64 = 8(9\lambda + 8k + 8)$$

which is divisible by 8.

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Principle of Mathematical Induction

1. For $n \in N$, $\left(\frac{1}{5}\right)n^5 + \left(\frac{1}{3}\right)n^3 + \left(\frac{7}{15}\right)n$ is [NCERT Exemplar]

(a) an integer (b) a natural number
(c) a positive fraction (d) None of these

2. The value of $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right)$ is [NCERT]

(a) $(n+1)$ (b) $(n+1)^2$
(c) $2(n+1)^2$ (d) None of these

3. For all $n \in N$, $n(n+1)(n+5)$ is a multiple of [NCERT Exemplar]

(a) 4 (b) 3 (c) 5 (d) 7

4. If $P(n)$ is a statement $(n \in N)$ such that, if $P(k)$ is true, $P(k+1)$ is true for $k \in N$, then $P(n)$ is true

(a) for all n (b) for all $n > 1$
(c) for all $n > 2$ (d) Nothing can be said

5. If $P(n)$ is a statement such that $P(3)$ is true. Assuming $P(k)$ is true $\Rightarrow P(k+1)$ is true for all $k \geq 3$, then $P(n)$ is true

(a) for all n (b) for $n \geq 3$
(c) for $n > 4$ (d) None of these

6. Let $P(n): n^2 + n + 1$ is an even integer. If $P(k)$ is assumed true $\Rightarrow P(k+1)$ is true. Therefore, $P(n)$ is true

(a) for $n > 1$ (b) for all $n \in N$
(c) for $n > 2$ (d) None of these

7. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then for $n \in N$, A^n is equal to

(a) $\begin{bmatrix} \cos^n\theta & \sin^n\theta \\ -\sin^n\theta & \cos^n\theta \end{bmatrix}$ (b) $\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

(c) $\begin{bmatrix} n\cos\theta & n\sin\theta \\ -n\sin\theta & n\cos\theta \end{bmatrix}$ (d) None of these

8. If $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then for $x \in N$, A^{10} is equal to

(a) $\begin{pmatrix} 1 & 10x \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & x^{10} \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 10 & x \\ 0 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} 10x & 0 \\ 0 & 10x \end{pmatrix}$

Applications of Mathematical Induction

9. If $1 + 5 + 12 + 22 + 35 + \ldots$ to n terms $= \frac{n^2(n+1)}{2}$, nth term of series is

(a) $\frac{n(4n-1)}{3}$ (b) $\frac{n(3n-1)}{2}$

(c) $\frac{n(3n+1)}{2}$ (d) $\frac{n(4n+1)}{3}$

10. The smallest positive integer n for which $n! < \left(\frac{n+1}{2}\right)^n$ holds, is

(a) 1 (b) 2 (c) 3 (d) 4

11. The inequality $n! > 2^{n-1}$ is true for

(a) $n > 2$ (b) $n \in N$
(c) $n > 3$ (d) None of these

12. The greatest positive integer, which divides $(n+2)(n+3)(n+4)(n+5)(n+6)$ for all $n \in N$, is

(a) 4 (b) 120 (c) 240 (d) 24

13. If m, n are any two odd positive integer with $n < m$, then the largest positive integers which divides all the numbers of the type $m^2 - n^2$ is

(a) 4 (b) 6 (c) 8 (d) 9

14. For all $n \in N$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by [NCERT Exemplar]

(a) 19 (b) 17
(c) 23 (d) 25

15. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is [NCERT Exemplar]
(a) 1 (b) 2
(c) 3 (d) 4

16. For all $n \in N$, $41^n - 14^n$ is a multiple of
(a) 26 (b) 27
(c) 25 (d) None of these

17. For all $n \in N$, $2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by
(a) 2 (b) 9
(c) 3 (d) 11

18. For each $n \in N$, $10^{2n-1} + 1$ is divisible by [NCERT]
(a) 11 (b) 13
(c) 9 (d) None of these

19. $2^{3n} - 7n - 1$ is divisible by
(a) 64 (b) 36
(c) 49 (d) 25

20. For each $n \in N$, $3^{2n} - 1$ is divisible by [NCERT Exemplar]
(a) 8 (b) 16
(c) 32 (d) None of these

21. $x(x^{n-1} - n\alpha^{n-1}) + \alpha^n(n-1)$ is divisible by $(x - \alpha)^2$ for
(a) $n > 1$
(b) $n > 2$
(c) all $n \in N$
(d) None of the above

22. If $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, if n is
(a) a positive integer (b) an even positive integer
(c) an odd positive integer (d) None of these

23. If $49^n + 16^n + k$ is divisible by 64 for $n \in N$, then the least negative integral value of k is [NCERT Exemplar]
(a) −1 (b) −2
(c) −3 (d) −4

Round II (Mixed Bag)

Only One Correct Option

1. For all positive integral values of n, $3^{2n} - 2n + 1$ is divisible by
(a) 2 (b) 4 (c) 8 (d) 12

2. If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by
(a) 113 (b) 123
(c) 133 (d) None of these

3. For natural number n, $2^n(n-1)! < n^n$, if
(a) $n < 2$ (b) $n > 2$
(c) $n \geq 2$ (d) never

4. For a positive integer n,
let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{(2^n) - 1}$. Then,
(a) $a(100) \leq 100$ (b) $a(100) > 100$
(c) $a(200) \leq 100$ (d) None of these

5. $10^n + 3(4^{n+2}) + 5$ is divisible by $(n \in N)$ [NCERT Exemplar]
(a) 7 (b) 5 (c) 9 (d) 17

6. Let $P(n)$ denotes the statement that $n^2 + n$ is odd. It is seen that $P(n) \Rightarrow P(n+1)$, $P(n)$ is true for all
(a) $n > 1$ (b) n
(c) $n > 2$ (d) None of these

7. If $P(n) = 2 + 4 + 6 + \ldots + 2n, n \in N$, then $P(k) = k(k+1) + 2 \Rightarrow P(k+1) = (k+1)(k+2) + 2$ for all $k \in N$. So, we can conclude that $P(n) = n(n+1) + 2$ for
(a) all $n \in N$ (b) $n > 1$
(c) $n > 2$ (d) nothing can be said

8. The remainder when 5^{99} is divided by 13, is
(a) 6 (b) 8
(c) 9 (d) 10

9. For all positive integers $n > 1$, $\{x(x^{n-1} - na^{n-1}) + a^n(n-1)\}$ is divisible by
(a) $(x - a)^2$ (b) $x - a$
(c) $2(x - a)$ (d) $x + a$

10. For all $n \in N$, $\int_0^{\pi} \frac{\sin(2nx)}{\sin x} dx$ is equal to
(a) 0 (b) π
(c) $\frac{\pi}{2}$ (d) $-\pi$

More Than One Correct Option

11. The product of three consecutive natural numbers is divisible by
(a) 2 (b) 3 (c) 6 (d) 4

Comprehension Based Questions

Directions (Q. Nos. 12 and 13) *Let us consider the series* $S_n = 2 \cdot 7^n + 3 \cdot 5^n - 5$.

12. S_n is divisible by the multiple of
(a) 5 (b) 7
(c) 24 (d) None of these

13. If S_n is divisible for every n, then S_n is
(a) > 0 (b) > 1
(c) > 5 (d) None of these

Assertion and Reason

Directions (Q. Nos. 14-17) *Each of these questions contains two statements : Statement* I *(Assertion) and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

14. **Statement I** $\frac{(n+2)!}{(n-1)!}$ is divisible by 6.

Statement II Product of 3 consecutive integers is divisible by 3!.

15. **Statement I** For all $n \in N$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

Statement II For all $n \in N$, $10^{2n-1} + 1$ is divisible by 11.

16. **Statement I** If $P(n)$ is defined as $2^n < n!$, then $P(n)$ is true for all $n > 3$.

Statement II In $P(n)$, n may be negative.

17. **Statement I** For every natural number $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Statement II For every natural number $n \geq 2$

$$\sqrt{n(n+1)} < n + 1$$ [AIEEE 2008]

Previous Years' Questions

18. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction ? [AIEEE 2005]

(a) $A^n = 2^{n-1}A + (n-1)I$
(b) $A^n = nA + (n-1)I$
(c) $A^n = 2^{n-1}A - (n-1)I$
(d) $A^n = nA - (n-1)I$

19. Let $S(k) = 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$. Then, which of the following is true? [AIEEE 2004]

(a) $S(1)$ is correct
(b) $S(k) \Rightarrow S(k+1)$
(c) $S(k) \not\Rightarrow S(k+1)$
(d) Principle of mathematical induction can be used to prove the formula

Answers

Check Point 1

1. (b) **2.** (d) **3.** (a) **4.** (c) **5.** (b) **6.** (a)

Round I

1. (b) **2.** (b) **3.** (b) **4.** (d) **5.** (b) **6.** (d) **7.** (b) **8.** (a) **9.** (b) **10.** (b)
11. (a) **12.** (b) **13.** (c) **14.** (b) **15.** (a) **16.** (b) **17.** (d) **18.** (a) **19.** (c) **20.** (a)
21. (c) **22.** (a) **23.** (a)

Round II

1. (a) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (d) **7.** (d) **8.** (b) **9.** (b) **10.** (a)
11. (a,b,c) **12.** (c) **13.** (a) **14.** (a) **15.** (b) **16.** (c) **17.** (b) **18.** (d) **19.** (b)

the Guidance

Round I

1. Let $P(n): \left(\frac{1}{5}\right)n^5 + \left(\frac{1}{3}\right)n^3 + \left(\frac{7}{15}\right)n$

$$\therefore \quad P(1) = \left(\frac{1}{5}\right)(1)^5 + \left(\frac{1}{3}\right)(1)^3 + \left(\frac{7}{15}\right)1$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1 \in N$$

$$P(2) = \left(\frac{1}{5}\right)(2)^5 + \left(\frac{1}{3}\right)(2)^3 + \left(\frac{7}{15}\right)2$$

$$= \frac{32}{5} + \frac{8}{3} + \frac{14}{15}$$

$$= \frac{96+40+14}{15} = \frac{150}{15} = 10 \in N$$

$\therefore$ By mathematical induction $P(n)$ is a natural number.

2. Let the given statement be $P(n)$.

i.e., $P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

Step I For $n = 1$, *i.e.,* $P(1) = (1+1)^2 = 2^2 = 4 = \left(1+\frac{3}{1}\right)$

which is true.

Step II Let it is true for $n = k$,

i.e., $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right) = (k+1)^2 \quad \ldots(i)$

Step III For $n = k+1$,

$$\left\{\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right)\right\}\left\{1+\frac{2k+2+1}{(k+1)^2}\right\}$$

$$= (k+1)^2\left\{1+\frac{2k+3}{(k+1)^2}\right\} \quad \text{[using Eq. (i)]}$$

$$= (k+1)^2\left[\frac{(k+1)^2+2k+3}{(k+1)^2}\right] = k^2+2k+1+2k+3$$

$$= (k+2)^2 = [(k+1)+1]^2$$

$$[\because (a+b)^2 = a^2+2ab+b^2]$$

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

3. Let the given statement be $P(n)$.

i.e., $P(n): n(n+1)(n+5)$ is a multiple of 3.

Step I For $n = 1$,

$1(1+1)(1+5) = 1 \times 2 \times 6 = 12 = 3 \times 4$

which is a multiple of 3, that is true.

Step II Let it is true for $n = k$,

i.e., $k(k+1)(k+5) = 3\lambda$

$\Rightarrow \quad k(k^2+5k+k+5) = 3\lambda$

$\Rightarrow \quad k^3+6k^2+5k = 3\lambda \quad \ldots(i)$

Step III For $n = k+1$,

$(k+1)(k+1+1)(k+1+5)$

$= (k+1)(k+2)(k+6) = (k^2+2k+k+2)(k+6)$

$= (k^2+3k+2)(k+6) = k^3+6k^2+3k^2+18k+2k+12$

$= (k^3)+9k^2+20k+12$

$= (3\lambda - 6k^2 - 5k)+9k^2+20k+12 \quad$ [using Eq. (i)]

$= 3\lambda + 3k^2 + 15k + 12 = 3(\lambda + k^2 + 5k + 4)$

which is a multiple of 3.

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

4. Unless we prove $P(1)$ is true, nothing can be said.

5. Since, $P(3)$ is true.

Assume $P(k)$ is true $\Rightarrow P(k+1)$ is true means, if $P(3)$ is true $\Rightarrow P(4)$ is true $\Rightarrow P(5)$ is true and so on. So, statement is true for all $n \geq 3$.

6. Given, $P(n): n^2+n+1$

At $n = 1$, $P(1): 3$, which is not an even integer.

$\therefore$ $P(1)$ is not true (principle of induction is not applicable).

Also, $n(n+1)+1$ is always an odd integer.

7. Given, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\therefore$ By induction, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

8. Since, $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad A^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}$

$\therefore$ By induction, $A^{10} = \begin{pmatrix} 1 & 10x \\ 0 & 1 \end{pmatrix}$

9. Let $P(n) : 1 + 5 + 12 + 22 + 35 + \ldots + (n \text{ terms}) = \dfrac{n^2(n+1)}{2}$

nth term of LHS $= P(n) - P(n-1)$

$\Rightarrow \quad P(n) - P(n-1) = \dfrac{n^2(n+1)}{2} - \dfrac{(n-1)^2 n}{2}$

$\Rightarrow \quad P(n) - P(n-1) = \dfrac{n}{2}\{n^2 + n - n^2 + 2n - 1\}$

$\therefore \quad T_n = P(n) - P(n-1)$

$= \dfrac{n}{2}(3n-1)$

10. Given, $\quad n! < \left(\dfrac{n+1}{2}\right)^n$

At $n = 1$, $\quad 1! \not< 1$

At $n = 2$, $\quad 2! < \left(\dfrac{3}{2}\right)^2$

$\Rightarrow \quad 2 < 2.25$ which is true.

11. Let $P(n) \equiv n! > 2^{n-1}$

$P(3) \equiv 6 > 4$

Let $P(k) \equiv k! > 2^{k-1}$ is true.

$\therefore \quad P(k+1) = (k+1)! = (k+1)k!$

$> (k+1)\, 2^{k-1}$

$> 2^k \quad$ (as $k + 1 > 2$)

12. Let $P(n) : (n+2)(n+3)(n+4)(n+5)(n+6)$

$\therefore$ Put $n = 1$

$P(1) = (1+2)(1+3)(1+4)(1+5)(1+6)$

$= 3 \times 4 \times 5 \times 6 \times 7 = 120 \times 21$

Put $n = 2$,

$P(2) = (2+2)(2+3)(2+4)(2+5)(2+6)$

$= 4 \times 5 \times 6 \times 7 \times 8 = 120 \times 56$

Hence, $P(n)$ is always divisible by 120.

13. Let $\quad m = 2k + 1, n = 2k - 1 \ (k \in N)$

$\therefore \quad m^2 - n^2 = 4k^2 + 1 + 4k - 4k^2 + 4k - 1 = 8k$

Hence, all the numbers of the form $m^2 - n^2$ are always divisible by 8.

14. $3 \cdot 5^{2n+1} + 2^{3n+1}$

Put $\quad n = 1$

$3 \times 5^3 + 2^4 = 391$

which is divisible by 17

15. $x - 1$ is always a factor of $x^n - 1$.

$\Rightarrow \quad k = 1$

16. Let the given statement be $P(n)$.

i.e., $\quad P(n) : 41^n - 14^n$ is a multiple of 27.

Step I For $n = 1$,

i.e., $\quad P(1) = 41^1 - 14^1 = 27 = 1 \times 27$

which is a multiple of 27 that is true.

Step II Let it is true for $n = k$,

i.e., $\quad 41^k - 14^k = 27\lambda \quad$ ...(i)

Step III For $n = k + 1$, $41^{k+1} - 14^{k+1}$

$= 41^k 41 - 14^k 14$

$= (27\lambda + 14^k)41 - 14^k 14 \quad$ [using Eq. (i)]

$= 27\lambda \times 41 + 14^k \times 41 - 14^k \times 14$

$= 27\lambda \times 41 + 14^k(41 - 14)$

$= 27\lambda \times 41 + 14^k \times 27 = 27(41\lambda + 14^k)$

which is multiple of 27.

Therefore, $P(k+1)$ is true when $P(k)$ is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

17. Let $P(n) = 2 \cdot 4^{2n+1} + 3^{3n+1}$

$P(1) \equiv 128 + 81 = 209$ (divisible by 11 only)

18. Let $\quad P(n) = 10^{2n-1} + 1$

$P(1) = 10 + 1 = 11$

Let $\quad P(k) \equiv 10^{2k-1} + 1 = 11\, l$ is true

Now, $\quad P(k+1) \equiv 10^{2k+1} + 1$

$= (11l - 1)\, 100 + 1$

$= 1100l - 99 = 11 l_1$

So, $P(k+1)$ is true.

19. Let $\quad P(n) = 2^{3n} - 7n - 1$

$\therefore \quad P(1) = 0$

$P(2) = 49$

$P(1)$ and $P(2)$ are divisible by 49.

Let $\quad P(k) \equiv 2^{3k} - 7k - 1 = 49l$

$\therefore \quad P(k+1) \equiv 2^{3k+3} - 7k - 8$

$= 8\,(49l + 7k + 1) - 7k - 8$

$= 49(8l) + 49k = 49 l_1$

Hence, by mathematical induction $2^{3n} - 7n - 1$ is divisible by 49.

20. Let $P(n) = 3^{2n} - 1$

At $n = 1$, $P(1) = 8$ which is divisible by 8.

$\therefore$ $P(1)$ is true.

Let $P(k)$ is true, then

$P(k) \equiv 3^{2k} - 1 = 8l$

$\therefore \quad P(k+1) \equiv 3^{2k+2} - 1 = (8l + 1)\, 9 - 1 = 72\, l + 8 = 8 l_1$

$\therefore$ $P(n)$ is divisible by 8, $\forall\, n \in N$.

21. Let $P(n) = x(x^{n-1} - n\alpha^{n-1}) + \alpha^n (n-1) = (x-\alpha)^2 g(x)$

$P(1) \equiv 0$ is true.

Let $P(k)$ is true.

i.e., $x(x^{k-1} - k\alpha^{k-1}) + \alpha^k (k-1) = (x-\alpha)^2 g(x)$

Now, $P(k+1) \equiv x[x^k - (k+1)\alpha^k] + \alpha^{k+1}(k)$

$\equiv (x-\alpha)^2 [xg(x) + k\alpha^{k-1}]$ (true)

So, holds for all $n \in N$.

22. Let $P(n) \equiv x^{2n-1} + y^{2n-1} = \lambda(x+y)$

$P(1) \equiv x + y = \lambda_1 (x+y)$

$P(2) \equiv x^3 + y^3 = \lambda_2 (x+y)$

Hence, for $\forall\, n \in N$, $P(n)$ is true.

23. Let $P(n) = 49^n + 16^n + k$

For $n = 1$, we get

$$P(1) = 49^1 + 16^1 + k = 65 + k$$

As $P(1)$ is divisible by 64, we take

$$k = -1$$

$\therefore$ $P(1) = 65 - 1 = 64$, which is divisible by 64.

Round II

1. On putting $n = 2$ in $3^{2n} - 2n + 1$, we get

$$3^{2\times 2} - 2\times 2 + 1 = 81 - 4 + 1 = 78$$

which is divisible by 2.

2. On putting $n = 1$ in $11^{n+2} + 12^{2n+1}$, we get

$$11^{1+2} + 12^{2\times 1+1} = 11^3 + 12^3 = 3059$$

which is divisible by 133.

3. The condition $2^n (n-1)! < n^n$ is satisfied for $n > 2$.

4. It can be proved with the help of mathematical induction that $\frac{n}{2} < a(n) \le n$.

$\therefore$ $\frac{200}{2} < a(200)$

$\Rightarrow$ $a(200) > 100$

and $a(100) \le 100$

5. On putting $n = 2$ in $10^n + 3(4^{n+2}) + 5$, we get

$$10^2 + 3\times 4^4 + 5 = 100 + 768 + 5 = 873$$

which is divisible by 9.

6. $P(n) = n^2 + n$

It is always odd but square of any odd number is always odd and also sum of two odd number is always even. So, for no any n for which this statement is true.

7. It is obvious, nothing can be said.

8. $5^{99} = 5(5^2)^{49} = 5(25)^{49} = 5(26-1)^{49}$

$= 5 \times 26 \times$ (positive term) $- 5$

So, when it is divided by 13, it gives the remainder -5 or 8.

9. Let $P(n) : x(x^{n-1} - na^{n-1}) + a^n(n-1)$, where $n > 1$

$P(2) : x(x-2a) + a^2 = (x-a)^2$

$P(3) : x(x^2 - 3a^2) + 2a^3 = x^3 - 3a^2x + 2a^3$

$\Rightarrow$ $P(3) = (x-a)(x^2 + a - 2a^2)$

which are divisible by $(x-a)$

10. Let $P(n) = \int_0^\pi \frac{\sin 2nx}{\sin x} dx$

Step I For $n = 1$,

$\Rightarrow$ $P(1) = \int_0^\pi \frac{\sin 2x}{\sin x} dx = 2\int_0^\pi \cos x dx = 0$

Step II For $n = m$,

let us assume that the proposition is true for $n = m$

$\therefore$ $P(m) = \int_0^\pi \frac{\sin 2kx}{\sin x} dx = 0$ ...(i)

Step III Let us prove the proposition for $n = m + 1$,

if the proposition is true for $n = m + 1$, then it was also true for $n = m$. Thus, the proposition will be true for all $n \in N$

$$P(m+1) - P(m) = \int_0^\pi \frac{\sin[2(k+1)x] - \sin 2kx}{\sin x} dx$$

$$P(m+1) - P(m) = \int_0^\pi \frac{2\cos(2k+1)x \sin x}{\sin x} dx$$

$\Rightarrow$ $P(m+1) - P(m) = 2\left[\frac{\sin(2k+1)x}{(2k+1)}\right]_0^\pi$

$\Rightarrow$ $P(m+1) - P(m) = \frac{2}{2k+1}(0-0) = 0$

$\therefore$ $P(m+1) = P(m)$

Since, the proposition is true for $n = m + 1$, thus it was also true for $n = m$. Hence, the proposition is true, $\forall\, n \in N$.

11. Let $P(n) : n(n+1)(n+2)$, where $n \in N$

$\Rightarrow$ $P(1) : (1)(2)(3) = 6$

Also, $P(2) : (2)(3)(4) = 24$

which is divisible by 2, 3 and 6.

12. Let $P(n) : 2\cdot 7^n + 3\cdot 5^n - 5$

Then, $P(1) : 2\cdot 7 + 3\cdot 5 - 5 = 24$

Let $P(m)$ be true.

$\therefore$ $P(m) : 2\cdot 7^m + 3\cdot 5^m - 5$

$= 24k, k \in N$ ...(i)

Now, $P(m+1) - P(m)$:

$$2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m)$$
$$= 2 \cdot 7^m (7-1) + 3 \cdot 5^m (5-1)$$
$$= 12(7^m + 5^m)$$

Since, 7^m and 5^m are odd integers and for all $m \in N$, their sum must be even say $7^m + 5^m = 2\lambda, \lambda \in N$.

$\therefore \quad P(m+1) - P(m) : 24\lambda$

$\Rightarrow \quad P(m+1) : 24\lambda + P(m)$

$\Rightarrow \quad P(m+1) : 24(\lambda + k)$ [from Eq. (i)]

Here, $P(m+1)$ is divisible by 24.

13. S_n is divisible for every n. So, S_n is greater than zero.

14. $\dfrac{(n+2)!}{(n-1)!} = \dfrac{(n+2)(n+1)n(n-1)!}{(n-1)!}$

$$= (n+2)(n+1)n$$

It is a product of 3 consecutive integrals and it is divisible by 3!.

15. Let $P_1(n) : 3 \cdot 5^{2n+1} + 2^{3n+1}$

$$P_1(1) : 3 \cdot 5^3 + 2^4 = 3(125) + 16 = 391$$
$$P_1(2) : 3 \cdot 5^5 + 2^7 = 3 \cdot 3125 + 128 = 9503$$

16. Since, $P(n) : 2^n < n!$

Put $n = 1, 2, 3$ respectively, we get $2 \not< 1!$, $4 \not< 2$ and $8 \not< 6$

At $n = 4$

$16 < 24$ which is true.

Hence, it is true for $n > 3$.

17. Let $\quad P(n) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{n}}$

Now, $\quad P(2) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} > \sqrt{2}$

Let us assume that $P(k) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{k}} > \sqrt{k}$ is true

$\therefore P(k+1) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{k}} + \dfrac{1}{\sqrt{k+1}} > \sqrt{k+1}$ has to be true

$$\text{LHS} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}}$$

Since, $\quad \sqrt{k(k+1)} > k (\forall k \le 0)$

$\therefore \quad \dfrac{\sqrt{k(h+1)} + 1}{\sqrt{k+1}} > \dfrac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$

Let $\quad P(n) = \sqrt{n(n+1)} < n + 1$

Statement I is correct.

Now, $\quad P(2) = \sqrt{2 \times 3} < 3$

If $P(k) = \sqrt{k(k+1)} < (k+1)$ is true.

Now, $\quad P(k+1) = \sqrt{(k+1)(k+2)} < (k+2)$ has to be true

Since, $\quad (k+1) < k + 2$

$\therefore \quad \sqrt{(k+1)(k+2)} < (k+2)$

Hence, statement II is not a correct explanation of statement I.

18. Given that, $\quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$
$$\vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots$$

$A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ can be verified by induction. Now, taking options

(b) $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} + \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \neq \begin{bmatrix} 2n-1 & 0 \\ n & 2n-1 \end{bmatrix}$

(d) $nA - (n-1)I = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$$

Hence, option (d) is the required answer.

19. $S(k) = 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$

Put $k = 1$ in both sides, we get

$$\text{LHS} = 1$$

and $\quad \text{RHS} = 3 + 1 = 4$

$\Rightarrow \quad \text{LHS} \neq \text{RHS}$

Put $(k+1)$ in place of k

$$\text{LHS} = 1 + 3 + 5 + \ldots + (2k-1) + (2k+1)$$
$$\text{RHS} = 3 + (k+1)^2 = 3 + k^2 + 2k + 1$$

Let $\quad \text{LHS} = \text{RHS}$

$\Rightarrow \quad 1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = 3 + k^2 + 2k + 1$

$\Rightarrow \quad 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$

If $S(k)$ is true, then $S(k+1)$ is also true.

Hence, $\quad S(k) \Rightarrow S(k+1)$

8 Binomial Theorem and its Applications

JEE Main MILESTONE

- Binomial Theorem
- Binomial Theorem For Positive Integer Index
- General Term is the Expansion of $(x + a)^n$
- Middle Term in Binomial Expansion
- Greatest Term in the Expansion of $(x + a)^n$
- Binomial Theorem for Negative/Rational Index
- Important Expansions
- Multinomial Theorem

8.1 Binomial Theorem

An algebraic expression consisting of two terms with +ve or –ve sign between, then is called binomial expression.

e.g., $(a+2b), \left(\frac{p}{x^2}-\frac{q}{x^4}\right)$

Binomial theorem is the formula obtained by which the binomial raised to the power of n, where n can be positive or negative or fraction.

There are two types of Binomial Theorem.

(i) Binomial theorem for positive Integral Index.

(ii) Binomial theorem for negative/Rational Index.

In elementary algebra, the ***binomial theorem*** *describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the power $(x+y)^n$ into a sum involving terms of the form ax^by^c, where the exponents b and c are non-negative integers with $b + c = n$ and the coefficient a of each term is a specific positive integer depending on n and b.*

8.2 Binomial Theorem for Positive Integer Index

If n is a positive integer and $x, a \in R$, then

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_na^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r \quad \text{...(i)}$$

and $$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - \ldots + (-1)^na^n$$

$$= \sum_{r=0}^{n} (-1)^r {}^nC_r x^{n-r} a^r \quad \text{...(ii)}$$

Here, ${}^nC_0, {}^nC_1, {}^nC_2, .., {}^nC_n$ are known as binomial coefficients or it can be rewritten as $C_0, C_1, C_2, \ldots, C_n$.

Also, $$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

On adding and subtracting Eqs. (i) and (ii) respectively, we get

$$(x+a)^n + (x-a)^n = 2\,[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \ldots]$$

= 2 (sum of odd places terms)

($\because$ the last term may be ${}^nC_n a^n$ or ${}^nC_{n-1}\, x a^{n-1}$ according as n is even or odd, respectively)

and $$(x+a)^n - (x-a)^n = 2\,[{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots]$$

= 2 (sum of even places terms)

($\because$ the last term may be ${}^nC_{n-1} x\, a^{n-1}$ or ${}^nC_n a^n$ as n is even or odd respectively)

Note

- The coefficient of x^r in $(a+x)^n$ is ${}^nC_r$.
- The coefficient of x^r in $(a-x)^n$ is $(-1)^n \cdot {}^nC_r$.

Pascal's Triangle

Let us consider the following expansion given below.

(i) $(x+a)^1 = {}^1C_0 x + {}^1C_1 a = x + a$

(ii) $(x+a)^2 = {}^2C_0 x^2 + {}^2C_1 xa + {}^2C_2 xa^2 = x^2 + 2xa + a^2$

(iii) $(x+a)^3 = {}^4C_0 x^4 + {}^4C_1 x^3 a + {}^4C_2 x^2 a^2 + {}^4C_3 x a^3 + {}^4C_4 a^4$

$= x^4 + 4x^3 a + 6x^2 a^2 + 4xa^3 + a^4$

and so on

The given below, the Pascal's Triangle showing the coefficients of various terms in a binomial expansion.

In the above triangle, each row is bounded on 1 on both sides. Any entry, except the first and the last, in the row is the sum of two entries above it in the preceding row.

Properties of Binomial Theorem

(i) The sum of indices of x and a in each term is n.

(ii) The number of terms in $(x+a)^n$ is $(n+1)$.

(iii) The coefficient of terms equidistant from the beginning and the end are equal.

(iv) (i) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \ldots]$

(ii) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots]$

(v) If n is odd, then $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$, whereas if n is even, then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms and $\{(x+a)^n - (x-a)^n\}$ has $\frac{n}{2}$ terms.

(vi) In the binomial expansion of $(x+a)^n$, the rth term from the end is $(n-r+2)$th term from the beginning.

Hot Spot

Coefficient of x^n or General Term of $(x+a)^n$

In these type of equation a either a binomial expression or an expression convertable to binomial expression is given. We have to find the coefficient of a particular power of variable a general term in the expansion of given binomial expression.

These types of problems are directly or indirectly asked in various years in AIEEE (JEE Main) 2005, 2004, 2002.

To solve such types of problems following steps are given below.

Step I Firstly, if the given expression is not in binomial expression, convert it into binomial expression.

Step II Take a coefficient of x^n in the expansion of $(x+a)^n$ is ${}^nC_n$.

Step III Simplify it and get the result.

Sample **Problem 1** *Find the coefficient of x^9 in $(1+3x+3x^2+x^3)^{15}$.*

(a) $\frac{45!}{46!\,9!}$ (b) $\frac{45!}{46!\,8!}$

(c) $-\frac{45!}{46!\,9!}$ (d) $\frac{45!}{46!\,7!}$

Interpret (*d*) **Step I** Firstly convert the expression into standard form.

i.e., $\{(1+x)^3\}^{15} = (1+x)^{45}$

Step II Taking the coefficient of x^9 *i.e.,* ${}^{45}C_9$.

Step III Simplify, ${}^{45}C_9 = \dfrac{45!}{36!9!}$

***Sample* Problem 2** *The coefficient of x^5 in the product $(1+2x)^6(1-x)^7$ is* [NCERT]

(*a*) −171 (*b*) 171 (*c*) 172 (*d*) −172

Interpret (*b*)

$$(1+2x)^6(1-x)^7 = \{{}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6\}$$
$$\times\{{}^7C_0 - {}^7C_1x + {}^7C_2x^2 - {}^7C_3x^3 + {}^7C_4x^4 - {}^7C_5x^5 + {}^7C_6x^6 - {}^7C_7x^7\}$$
$$= ({}^6C_0 + {}^6C_1\times 2\times x + {}^6C_2\times 4\times x^2 + {}^6C_3\times 8\times x^3 + {}^6C_4\times 2^4\times x^4 + {}^6C_5\times 2^5\times x^5 + {}^6C_6\, 2^6\times x^6)$$
$$\times({}^7C_0 - {}^7C_1x + {}^7C_2x^2 - {}^7C_3x^3 + {}^7C_4x^4 - {}^7C_5x^5 + {}^7C_6x^6 - {}^7C_7x^7)$$

For coefficient of x^5, in $(1+2x)^6(1-x)^7$

$$= {}^6C_0(-{}^7C_5) + {}^6C_1\times 2\times {}^7C_4 + {}^6C_2\times 4\times(-{}^7C_3) + {}^6C_3\times 8\times {}^7C_2 + {}^6C_4\times 2^4\times(-{}^7C_1) + {}^6C_5\times 2^5({}^7C_0)$$
$$= -{}^6C_0\times {}^7C_2 + {}^6C_1\times 2\times {}^7C_3 - {}^6C_2\times 4\times {}^7C_3 + {}^6C_3\times 8\times {}^7C_2 - {}^6C_2\times 16\times {}^7C_1 + {}^6C_1\times 2^5$$
$$(\because {}^nC_r = {}^nC_{n-r})$$
$$= -\frac{1\times 7\times 6}{2} + \frac{6\times 2\times 7\times 6\times 5}{6} - \frac{6\times 5}{2}\times 4\times\frac{7\times 6\times 5}{6} + \frac{6\times 5\times 4}{6}\times\frac{8\times 7\times 6}{2} - \frac{6\times 5}{2}\times 16\times 7 + 6\times 32$$
$$= -21 + 12\times 35 - 15\times 28\times 5 + 20\times 24\times 7 - 15\times 16\times 7 + 192$$
$$= -21 + 420 - 420\times 5 + 3360 - 1680 + 192$$
$$= 3972 - 2100 - 21 - 1680$$
$$= 3972 - 3801 = 171$$

8.3 General Term in the Expansion of $(x+a)^n$

Let us consider the binomial expansion

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_r\,x^{n-r}\,a^r$$

In this expansion first term is ${}^nC_0\,x^n$, second term is ${}^nC_2\,x^{n-2}\,a^2$ and so on. On the basis of that we can say that rth term is $T_{r-1} = {}^nC_r\,x^{n-r}\,a^r$.

Note General term in the expansion of $(x-a)^n$ is

$$T_{r+1} = (-1)^r\cdot {}^nC_r\,x^{n-r}a^r$$

General term in the expansion of $(1+x)^n$ is

$$T_{r+1} = {}^nC_rx^r$$

General term in the expansion of $(1-x)^n$ is

$$T_{r+1} = (-1)^r\,{}^nC_rx^r$$

***Sample* Problem 3** *The term independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$ is*

(*a*) $\dfrac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}$ (*b*) $\dfrac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}2^n$

(*c*) $\dfrac{1\cdot 3\cdot 5\ldots(2n+1)}{(n+1)!}2^n$ (*d*) *None of these*

Interpret (*b*) Let $(r+1)$th term is independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$.

$\therefore$ $T_{r+1} = {}^{2n}C_r\,x^{2n-2r}$

This term is independent of x.

$\therefore$ $2n-2r=0 \Rightarrow r=n$

Thus, $T_{r+1} = {}^{2n}C_n$

$$= \frac{(2n)!}{n!n!} = \frac{1\cdot 2\cdot 3\cdot 4\ldots(2n-1)(2n)}{n!n!}$$
$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{1\cdot 2\ldots n\}2^n}{n!n!} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}2^n$$

***Sample* Problem 4** *The coefficient of x^p in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{2n}$ is* [NCERT]

(*a*) $\dfrac{(2n)!}{\left(\dfrac{4n-p}{3}\right)!\left(\dfrac{2n+p}{3}\right)!}$ (*b*) $\dfrac{(2n)!}{\left(\dfrac{4n-p}{3}\right)!\left(\dfrac{4n-p}{3}\right)!}$

(*c*) $\dfrac{(n)!}{\left(\dfrac{4n-p}{3}\right)!\left(\dfrac{2n+p}{3}\right)!}$ (*d*) *None of these*

Interpret (a) The general term in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{2n}$ is

$$T_{r+1} = {}^{2n}C_r\,(x^2)^{2n-r}\left(\frac{1}{x}\right)^r = {}^{2n}C_r\,x^{4n-2r}\,\frac{1}{x^r} \Rightarrow T_{r+1} = {}^{2n}C_r\,x^{4n-3r}$$

On putting $4n-3r=p$

$\Rightarrow$ $3r = 4n-p \Rightarrow r = \dfrac{4n-p}{3}$

$\Rightarrow$ Coefficient of $x^p = {}^{2n}C_{\frac{4n-p}{3}} = \dfrac{(2n)!}{\left(2n-\dfrac{4n-p}{3}\right)!\left(\dfrac{4n-p}{3}\right)!}$

$$= \frac{2n!}{\left(\dfrac{2n+p}{3}\right)!\left(\dfrac{4n-p}{3}\right)!}$$

8.4 Middle Term in Binomial Expansion

Consider binomial expansion

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_na^n$$

or $$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + (-1)^n\,{}^nC_na^n$$

(i) If n is an even number, then $\left(\frac{n}{2}+1\right)$th term is middle term in the expansion of $(x+a)^n$ or $(x-a)^n$

$$\therefore \quad T_{\frac{n}{2}+1} = {}^nC_{n/2}\,x^{n/2}a^{n/2}$$

or $$(-1)^{n/2}x^{n/2}a^{n/2}$$

(ii) If n is an odd number, then $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are middle terms in the expansion of $(x+a)^n$ or $(x-a)^n$

i.e., $$\frac{T_{n+1}}{2} = {}^2C_{\frac{n-1}{2}}x^{\frac{n+1}{2}}a^{\frac{n-1}{2}}$$

and $$T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}a^{\frac{n+1}{2}}$$

or $$T_{\frac{n+1}{2}} = (-1)^{\frac{n+1}{2}}\,{}^nC_{\frac{n-1}{2}}(x)^{\frac{n+1}{2}}a^{\frac{n-1}{2}}$$

and $$T_{\frac{n+3}{2}} = (-1)^{\frac{n+3}{2}}\,{}^nC_{\frac{n+1}{2}}(x)^{\frac{n-1}{2}}a^{\frac{n+1}{2}}$$

Note

- When there are two middle terms in the expansion, then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.

***Sample* Problem 5** *The middle term in the expansion of* $\left(x^2+\frac{1}{x^2}+2\right)^n$ *is*

(a) $\frac{2n!}{n!n!}$ (b) $\frac{2n!}{n!}$

(c) $\frac{n!}{(n-1)}$ (d) *None of these*

Interpret (a) $\because \left(x^2+\frac{1}{x^2}+2\right)^n = \left(x+\frac{1}{x}\right)^{2n}$

$\therefore$ Number of terms $= 2n+1$

Here, $2n$ is even, then middle term will be $(n+1)$th term. Thus, middle term,

$$T_{n+1} = {}^{2n}C_nx^{2n-n}\left(\frac{1}{x}\right)^n = \frac{2n!}{n!n!}$$

8.5 Greatest Term in the Expansion of $(x+a)^n$

If T_r and T_{r+1} be the rth and $(r+1)$th terms in the expansion of $(x+a)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_rx^{n-r}a^r}{{}^nC_{r-1}x^{n-r+1}a^{r-1}}$$

$$= \left(\frac{n-r+1}{r}\right)\left|\frac{a}{x}\right|$$

$\because$ $$\frac{T_{r+1}}{T_r} \geq 1$$

$\therefore$ $$\frac{n-r+1}{r}\left|\frac{a}{x}\right| \geq 1$$

$\Rightarrow$ $$(n-r+1)|a| \geq r|x|$$

$\Rightarrow$ $$r \leq \frac{(n+1)|a|}{|x|+|a|}$$

Let $$\frac{(n+1)|a|}{|x|+|a|} = I + f$$

where, I is an integer and $0 \leq f < 1$.

Greatest Binomial Coefficient

The greatest coefficient depends upon the value of n.

n	Number of greatest Coefficient (s)	Greatest coefficient
Even	1	${}^nC_{\frac{n+2}{2}}$
odd	2	${}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$

It is clearly that, greatest binomial coefficient compounds to the coefficient of middle term.

***Sample* Problem 6** *Numerically the greatest term in the expansion of* $(2+3x)^9$ *when* $x = \frac{3}{2}$, *is*

(a) $\frac{7\times 3^3}{2}$ (b) $\frac{7\times 3^{13}}{2}$

(c) $\frac{3^{13}}{2}$ (d) *None of these*

Interpret (b) Since, $(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now, in the expansion of $\left(1+\frac{3}{2}x\right)^9$, we have

$$T_{\frac{r+1}{T_r}} = \left(\frac{9-r+1}{r}\right)\left|\frac{3}{2}x\right|$$

$$=\left(\frac{10-r}{r}\right)\left(\frac{9}{4}\right)=\frac{90-9r}{4r} \qquad \left(\because x=\frac{3}{2}\right)$$

$$\because \quad \frac{T_{r+1}}{T_r}\geq 1 \Rightarrow \frac{90-9r}{4r}\geq 1 \Rightarrow 90\geq 13r$$

$$\Rightarrow \quad r\leq \frac{90}{13} \Rightarrow r\leq 6\frac{12}{13}$$

$\therefore$ Maximum value of r is 6.

So, greatest term $=2^9\cdot T_{6+1}$

$$=2^9\cdot {}^9C_6\left(\frac{3}{2}x\right)^6$$

$$=2^9\cdot {}^9C_3\left(\frac{3}{2}\times\frac{3}{2}\right)^6$$

$$=2^9\cdot\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{3^{12}}{2^{12}}=\frac{7\times 3^{13}}{2}$$

Properties of Binomial Coefficients

(i) If ${}^nC_r={}^nC_s$, then either $r=s$ or $r+s=n$

(ii) ${}^nC_0+{}^nC_1+\ldots+{}^nC_n=2^n$

(iii) ${}^nC_0+{}^nC_2+{}^nC_4+\ldots={}^nC_1+{}^nC_3+\ldots=2^{n-1}$

(iv) ${}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\ldots(-1)^n\,{}^nC_n=0$

(v) $\dfrac{{}^nC_r}{{}^nC_{r-1}}=\dfrac{n-r+1}{r}$

(vi) ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$

(vii) ${}^{n+1}C_{r+1}=\dfrac{n+1}{r+1}\cdot{}^nC_r$

(viii) ${}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\ldots+n\,{}^nC_n=n\cdot 2^{n-1}$

(ix) ${}^nC_1-2\,{}^nC_2+3\,{}^nC_3-\ldots=0$

(x) ${}^nC_0+2\,{}^nC_1+3\,{}^nC_2+\ldots+(n+1)\,{}^nC_n=(n+2)2^{n-1}$

(xi) $C_0C_r+C_1C_{r+1}+\ldots+C_{n-r}C_n=\dfrac{(2n)!}{(n-r)!(n+r)!}$

(xii) $C_0^2+C_1^2+C_2^2+\ldots+C_n^2=\dfrac{(2n)!}{(n!)^2}$

(xiii) $C_0^2-C_1^2+C_2^2-C_3^2+\ldots=\begin{cases}0, & \text{if } n \text{ is odd}\\ (-1)^{n/2}, & \text{if } n \text{ is even}\end{cases}$

(xiv) $C_1^2-2C_2^2+3C_3^2-\ldots+(-1)^n n\cdot C_n^2$

$$=(-1)^{\frac{n}{2}-1}\cdot\frac{n}{2}\cdot\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \text{ when } n \text{ is even.}$$

***Sample* Problem 7** $\sum_{r=0}^{n} 3^r\,{}^nC_r$ *is equal to* [NCERT]

(a) 4^{n-1} (b) 4^n (c) $\dfrac{4^n}{3}$ (d) 3^{n-1}

Interpret (b) $\because \sum_{r=0}^{n}{}^nC_r\times 3^r$

$$={}^nC_0 3^0+{}^nC_1 3+{}^nC_2 3^2+{}^nC_3 3^3+\ldots+{}^nC_n 3^n$$

$$={}^nC_0+{}^nC_1 3+{}^nC_2 3^2+{}^nC_3 3^3+\ldots+{}^nC_n 3^n=(1+3)^n=4^n$$

***Sample* Problem 8** *The value of*

$${}^{47}C_4+\sum_{j=0}^{3}{}^{50-j}C_3+\sum_{k=0}^{5}{}^{56-k}C_{53-k} \text{ is}$$

(a) ${}^{57}C_4$ (b) ${}^{57}C_3$ (c) ${}^{57}C_5$ (d) ${}^{57}C_6$

Interpret (a) We have, ${}^{47}C_4+\sum_{j=0}^{3}{}^{50-j}C_3+\sum_{k=0}^{5}{}^{56-k}C_{53-k}$

$$={}^{47}C_4+({}^{50}C_3+{}^{49}C_3++{}^{48}C_3+{}^{47}C_3)+({}^{56}C_{53}+{}^{55}C_{52}+{}^{54}C_{51}+{}^{53}C_{50}+{}^{52}C_{49}+{}^{51}C_{48})$$

$$={}^{47}C_4+({}^{47}C_3+{}^{48}C_3+{}^{49}C_3+{}^{50}C_3)+({}^{56}C_3+{}^{55}C_3+{}^{54}C_3+{}^{53}C_3+{}^{52}C_3+{}^{51}C_3)$$

$$(\because {}^nC_r={}^nC_{n-r})$$

$$=({}^{47}C_4+{}^{47}C_3)+({}^{48}C_3+{}^{49}C_3+{}^{50}C_3)+({}^{51}C_3+{}^{52}C_3+\ldots+{}^{56}C_3)$$

$$(\because {}^nC_r+{}^nC_{r-1}={}^{n+1}C_r)$$

$$=({}^{48}C_4+{}^{48}C_3)+{}^{49}C_3+{}^{50}C_3+\ldots+{}^{56}C_3={}^{57}C_4$$

Applications of Binomial Theorem

(i) *R-f* Factor Relation

Here, we are going to discuss problems involving

$$(\sqrt{A}+B)^n=I+f,$$

where I and n are positive integers.

$$0\leq f\leq 1, \left|A-B^2\right|=k \text{ and } \left|\sqrt{A}-B\right|<1$$

Approach for these type of problems can be learnt from following examples.

***Sample* Problem 9** *Integral part of* $(7+4\sqrt{3})^n$ *is* $(n\in N)$

(a) *an even number*

(b) *an odd number*

(c) *an even or an odd number depending upon the value of n*

(d) *None of the above*

Interpret (b) Here, $\forall\, n\in N, (7+4\sqrt{3})^n\notin N$

$\therefore$ Denote $(7+4\sqrt{3})^n$ by $I+f$,

where, I is an integer and $f \in R$ such that $0 < f < 1$

$\because \quad 0 < 7 - 4\sqrt{3} < 1$

$\therefore$ We can denote $(7 - 4\sqrt{3})^n$ by G.

where, $G \in R$ such that $0 < G < 1$

Now, $I + f = (7 + 4\sqrt{3})^n = 7^n + {}^nC_1 7^{n-1}(4\sqrt{3})$

$$+ {}^nC_2 7^{n-2}(4\sqrt{3})^2 + \ldots \quad \ldots(i)$$

$$G = (7 - 4\sqrt{3})^n = 7^n - {}^nC_1 7^{n-1}(4\sqrt{3}) + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \ldots \quad \ldots(ii)$$

To cancel irrational terms we add Eqs. (i) and (ii), we get

$$I + f + G = 2(7^n + {}^nC_2 7^{n-2}(48) + {}^nC_4 7^{n-4}(48)^2 + \ldots)$$

$$= 2k, \text{ where } k \text{ is an integer}$$

$\because$ I is an integer.

$\therefore$ $f + G = 2k - I$ is an integer. ...(iii)

Now, $0 < f < 1$

and $0 < G < 1$

$\Rightarrow$ $0 < f + G < 2$ ...(iv)

Form Eqs. (iii) and (iv), $f + G = 1$

Now, form Eq. (iii) $I = 2k - 1$

$\Rightarrow$ Integral part of $(7 + 4\sqrt{3})^n$

i.e., I is an odd integer.

(ii) Divisibility Problem

In the expansion,

$$(1 + \alpha)^n = 1 + {}^nC_1\alpha + {}^nC_2\alpha^2 + \ldots + {}^nC_n\alpha^n.$$

We can conclude that,

(i) $(1 + \alpha)^n - 1 = {}^nC_1\alpha + {}^nC_2\alpha^2 + \ldots + {}^nC_n\alpha^n$ is divisible by α *i.e.*, it is a multiple of α

***Sample* Problem 10** *For all $n \in R$, $9^{n+1} - 8n - 9$ is divisible by*

(a) 64 (b) 63 (c) 62 (d) 65

Interpret (a) $9^{n+1} - 8n - 9 = 9^n \times 9 - 8n - 9$

$= (1 + 8)^n \times 9 - 8n - 9$

$= ({}^nC_0 + {}^nC_1 8 + {}^nC_2 8^2 + {}^nC_3 8^3 + \ldots + {}^nC_n 8^n)\,9 - 8n - 9$

$= (1 + 8n + {}^nC_2 8^2 + {}^nC_3 8^3 + \ldots + {}^nC_n 8^n)\,9 - 8n - 9$

$= 9 + 72n + ({}^nC_2 8^2 + {}^nC_3 8^3 + \ldots + {}^nC_n 8^n)\,9 - 8n - 9$

$= (72n - 8n) + 8^2({}^nC_2 + {}^nC_3 8 + \ldots + {}^nC_n 8^{n-2})\,9$

$= 64n + 64\,({}^nC_2 + {}^nC_3 8 + \ldots + {}^nC_n 8^{n-2})\,9$

$= 64\,[n + ({}^nC_2 + {}^nC_3 8 + \ldots + {}^nC_n 8^{n-2})\,9]$

$= 64 \times$ Some constant numbers = Divisible by 64

(iii) Differentiability Problem

Sometimes to generalise the result we use the differentiation.

***Sample* Problem 11** *Series*

$${}^nC_1 + 1 \cdot {}^nC_2 + 2 \cdot {}^nC_3 + \ldots + n \cdot {}^nC_n \text{ is equal to}$$

(a) $n2^{n-1}$ (b) 2^{n-1} (c) $n2^n$ (d) 2^{3-1}

Interpret (a) Since,

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$$

On differentiating w.r.t., x, we get

$$n(1 + x)^{n-1} = 0 + {}^nC_1 + 2 \cdot x \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n x^{n-1}$$

Put $x = 1$, we get

$$n(1 + 1)^{n-1} = {}^nC_1 + 2 \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n$$

$$\Rightarrow \quad n\,2^{n-1} = {}^nC_1 + 2 \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n$$

Check Point 1

1. The term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is
   (a) 17 / 54 (b) 15 / 54
   (c) 13 / 54 (d) None of these
2. The 14th term from the end in the expansion of $(\sqrt{x} - \sqrt{y})^{17}$ is
   (a) ${}^{17}C_5(-x^6)(-\sqrt{y})^5$ (b) ${}^{17}C_6(\sqrt{x})^{11}y^3$
   (c) ${}^{17}C_4 x^{13/2}y^2$ (d) None of these
3. The ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ and the term independent of x in $\left(x - \frac{2}{x}\right)^{10}$ is
   (a) 1 : 16 (b) 1 : 8 (c) 64 : 1 (d) 1: 32
4. The value of $(1.0025)^{10}$ correct to five decimal places is
   (a) 1.025281 (b) 1.034285
   (c) 1.005293 (d) None of these
5. The number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$ is
   (a) 5 (b) 6 (c) 7 (d) 4
6. $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$ is equal to
   (a) 101 (b) $70\sqrt{2}$ (c) $140\sqrt{2}$ (d) $120\sqrt{2}$
7. If the coefficient of rth, (r + 1)th and (r + 2)th term in the expansion of $(1 + x)^{14}$ are in AP, then the value of r is
   (a) 5 or 8 (b) 4 or 9 (c) 5 or 9 (d) 6 or 7
8. ${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9$ is equal to
   (a) 2^9 (b) 2^{10} (c) $2^{10} - 1$ (d) 2^8
9. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n$ is equal to
   (a) $(n - 1)2^{n-1}$ (b) $n2^{n-1}$ (c) $n2^n$ (d) $n2^{n-2}$
10. For all $n \in N$, $2^{4n} - 15n - 1$ is divisible by
   (a) 225 (b) 125
   (c) 325 (d) None of these

8.6 Binomial Theorem for Negative/Rational Index

Let n is a rational number and x is a real number such that $|x|<1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

e.g., (i) The expansion of $(x+a)^n$

$$= a^n\left(1+\frac{x}{a}\right)^n$$

$$= a^n\left\{1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \ldots\right\}$$

The above expansion is valid only, when $\left|\frac{x}{a}\right| < 1$.

(ii) The expansion $(2+3x)^{-5}$ upto four terms in decreasing power of x, is as follows

$$(2+3x)^{-5} = \left[3x\left(1+\frac{2}{3x}\right)\right]^{-5}$$

$$= \frac{1}{243x^5}\left[1 + (-5)\left(\frac{2}{3x}\right) + \frac{(-5)(-6)}{2!}\left(\frac{2}{3x}\right)^2 + \frac{(-5)(-6)(-7)}{3!}\left(\frac{2}{3x}\right)^3 + \ldots\right]$$

$$= \frac{1}{243x^5}\left[1 - \frac{10}{3x} + \frac{20}{3x^2} - \frac{280}{27x^3} + \ldots\right]$$

$$= \frac{1}{243}\left[\frac{1}{x^5} - \frac{10}{3}\cdot\frac{1}{x^6} + \frac{20}{3}\cdot\frac{1}{x^7} - \frac{280}{27}\cdot\frac{1}{x^8} + \ldots\right]$$

Note

- If n is a whole number, then there is no need of condition $|x|<1$.
- In the expansion of $(1+x)^n$,
  - (i) if n is natural number, then there are finite number terms exist.
  - (ii) if n is negative/rational number, then there are infinite number of terms exist.

General Term in the Expansion of $(1+x)^n$

General term in the expansion of $(1+x)^n$ is as follows

$$T_{r+1} = \frac{[n(n-1)(n-2)\ldots\{n-(r-1)\}]}{r!}x^r$$

Note

- General term in the expansion of $(1-x)^n$ is
$$T_{r+1} = (-1)^r\frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}x^r$$
- General term in the expansion of $(1+x)^{-n}$ is
$$T_{r+1} = (-1)^r\frac{n(n+1)(n+2)+\ldots+(n+(r-1))}{r!}x^r$$
- General term in the expansion of $(1-x)^{-n}$ is
$$T_{r+1} = \frac{n(n+1)(n+2)+\ldots+(n+(r-1))}{r!}x^r$$

8.7 Important Expansions

(i) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{(-1)^r n(n+1)(n+2)\ldots(n+r-1)x^r}{r!} + \ldots$

(ii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(iii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots + (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)x^r}{r!} + \ldots$

(iv) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$

(v) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(vi) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r(r+1)x^r + \ldots$

(vii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots$

(viii) $(1+x)^{-3} = 1 - 3x + 6x^2 - \ldots$

(ix) $(1-x)^{-3} = 1 + 3x + 6x^2 + \ldots$

Note

If the value of x is so small, then on leaving square and higher powers of x, we get $(1+x)^n = 1 + nx$

***Sample* Problem 12** *If $(r+1)$th term is the first negative term in the expansion of $(x+1)^{7/2}$, then the value of r is*

(a) 5 (b) 6
(c) 4 (d) 7

Interpret (a) The general term of $(x+1)^{7/2}$ is

$$T_{r+1} = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\ldots\left(\frac{7}{2}-r+1\right)x^r}{r!}$$

This will be the first negative term, when $\frac{7}{2} - r + 1 < 0 \Rightarrow r > \frac{9}{2}$

$\therefore \quad r = 5$

***Sample* Problem 13** *Assume that x is so small that its, second and higher power may be neglected, then the simplified value of $\frac{(1-2x)^{2/3}(4+5x)^{3/2}}{\sqrt{1-x}}$ is*

(a) $8 - \frac{25}{3}x$ (b) $8 + \frac{25}{3}x$ (c) $8 + \frac{25}{4}x$ (d) $4 + \frac{25}{3}x$

Interpret (b) Using binomial expansion for rational index

$$\frac{(1-2x)^{2/3}(4+5x)^{3/2}}{\sqrt{1-x}}$$

$$= (1-2x)^{2/3}\cdot 4^{3/2}\left(1+\frac{5}{4}x\right)^{3/2}(1-x)^{-1/2}$$

$$= 8(1-2x)^{2/3}\cdot\left(1+\frac{5}{4}x\right)^{3/2}(1-x)^{-1/2}$$

$$=8\left[\left\{1+\frac{2}{3}\times(-2x)+\ldots\right\}\times\left\{1+\frac{3}{2}\times\frac{5}{4}x+\ldots\right\}\times\left\{1+\left(-\frac{1}{2}\right)(-x)+\ldots\right\}\right]$$

$$=8\left[\left(1-\frac{4}{3}x\right)\left(1+\frac{15}{8}x\right)\left(1+\frac{1}{2}x\right)\right]$$

($\because$ neglecting x^2 and higher powers of x)

$$=8\left[\left(1-\frac{4}{3}x+\frac{15}{8}x\right)\left(1+\frac{x}{2}\right)\right] \quad (\because \text{ neglecting } x^2)$$

$$=8\left[\left(1+\frac{13}{24}x\right)\left(1+\frac{x}{2}\right)\right] \quad (\because \text{ neglecting } x^2)$$

$$=8\left[1+\frac{1}{2}x+\frac{13}{24}x\right] \quad (\because \text{ neglecting } x^2)$$

$$=8\left[1+\frac{25}{24}x\right]$$

$$=8+\frac{25}{3}x$$

8.8 Multinomial Theorem

If n is a positive integer and $a_1, a_2, \ldots, a_k \in R$, then

$$(a_1+a_2+\ldots+a_k)^n=\sum_{r_1+r_2+\ldots+r_k=n}\frac{n!\,a_1^{r_1}\cdot a_2^{r_2}\ldots a_k^{r_k}}{r_1!\,r_2!\ldots r_k!}$$

Note

The number of terms in $(x+y+z)^n$ is ${}^{n+2}C_2$.

The number of terms in the expansion of $(a_1+a_2+\ldots+a_k)^n$ is ${}^{n+k-1}C_{k-1}$.

Coefficient of $x^{n_1}\cdot y^{n_2}\cdot z^{n_3}$ in the expansion of $(x+y+z)^n$ is $\frac{n!}{n_1!n_2!n_3!}$ where, $n=n_1+n_2+n_3$.

Sum of coefficients in the expansion of $(a+bx+cx^2)^n$ is $(a+b+c)^n$.

***Sample* Problem 14** *The coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ is*

(a) 6 (b) 60
(c) 61 (d) *None of these*

Interpret (*b*) In this case, we write

$$a^3b^4c^5=(ab)^x(bc)^y(ca)^z$$

$$\Rightarrow \quad a^3b^4c^5=a^{z+x}\cdot b^{x+y}\cdot c^{y+z}$$

$$\Rightarrow \quad z+x=3, x+y=4, y+z=5 \quad \ldots(i)$$

On adding these equations, we get

$$2(x+y+z)=12$$

$$\therefore \quad x+y+z=6 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x=1, y=3, z=2$$

Therefore, the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6=$ The coefficient of $(ab)^1(bc)^3(ca)^2$ in the expansion of $(bc+ca+ab)^6=\frac{6!}{1!3!2!}=60$

Check Point 2

1. The 4th term in the expansion of $(1-2x)^{3/2}$ will be
(a) $-\frac{3}{4}x^4$ (b) $\frac{x^2}{2}$
(c) $-\frac{x^3}{2}$ (d) $\frac{3}{4}x^4$

2. If $(r+1)$th term is the first negative term in the expansion of $(1+x)^{7/2}$, then the value of r is
(a) 5 (b) 6
(c) 4 (d) 7

3. If x is so small that its square and higher powers may be neglected, then the value of $\frac{(8+3x)^{2/3}}{(2+3x)(4-5x)^{1/2}}$ is
(a) $1-\frac{3}{2}x$ (b) $1+\frac{5}{8}x$
(c) $1-\frac{5}{8}x$ (d) None of these

4. Sum of infinite series $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{2^2}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{1}{2^3}+\ldots\infty$ is
(a) $2^{1/3}$ (b) $4^{1/3}$ (c) $8^{1/3}$ (d) None of these

5. If is very small in magnitude compared with a such that $\left(\frac{a}{a+x}\right)^{1/2}+\left(\frac{a}{a-x}\right)^{1/2}=2+k\cdot\frac{x^2}{a^2}$, then the value of k is
(a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) $\frac{3}{4}$ (d) 1

6. The approximate value of $(7.995)^{1/3}$ correct to four decimal places is
(a) 1.9995 (b) 1.9996 (c) 1.9990 (d) 1.9991

7. The number of dissimilar terms in the expansion of $(a+b-c)^{2n-1}-(a+b-c)^{2n+1}$ is
(a) $(n+1)^2$ (b) $(n-1)^2$
(c) $4n^2-1$ (d) None of these

WORKED OUT Examples

Example 1 *The coefficient of* $\frac{1}{x^{17}}$ *in the expansion of* $\left(x^4 - \frac{1}{x^3}\right)^{15}$ *is*

[**NCERT Exemplar**]

(*a*) 1365 (*b*) −1365

(*c*) −1450 (*d*) *None of these*

Solution (*b*) General term in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is $T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$$= {}^{15}C_r\,(-1)^r\,x^{60-7r}$$

For coefficient of $\frac{1}{x^{17}}$ *i.e.*, x^{-7}

Put $60 - 7r = -17 \Rightarrow r = 11$

$\therefore$ Coefficient of $\frac{1}{x^{17}} = {}^{15}C_{11}(-1)^{11} = -1365$

Example 2 *Using binomial theorem, which number is larger* $(1\cdot 1)^{1000}$ *or* 1000*?*

[**NCERT**]

(*a*) $(1\cdot 1)^{1000}$ (*b*) 1000

(*c*) *Cannot to determined* (*d*) *None of these*

Solution (*b*) $(1\cdot 1)^{10000} = (1 + 0\cdot 1)^{10000}$

$= {}^{10000}C_0\,(1)^{10000} + {}^{10000}C_1\,(1)^{9999}\,(0\cdot 1) + \ldots$ other terms

$= 1\times 1 + 10000 \times 0.1 + \ldots$ other terms $\quad (\because {}^nC_0 = 1, {}^nC_1 = n)$

$= 1 + 1000 +$ other terms

$= (1001 +$ other terms$) > 1000$

$\Rightarrow \quad (1\cdot 1)^{10000} > 1000$

Example 3 *The middle term in the expansion of* $\left(x - \frac{1}{x}\right)^{2n}$ *is*

[**NCERT**]

(*a*) $\frac{1\times 3\times 5\times\ldots\times(2n+1)}{n!}(-2)^n$ (*b*) $\frac{1\times 3\times 5\times\ldots\times(2n-1)}{n!}(-2)^n$

(*c*) $\frac{1\times 3\times 5\times\ldots\times(2n-1)}{n!}2^n$ (*d*) *None of these*

Solution (*b*) $N = 2n$ (even), total terms $= 2n + 1$

$\therefore$ Middle term $= \frac{(2n+1+1)\text{th}}{2}$ term

$= (n+1)$ th the middle term

$$T_{r+1} = {}^{2n}C_r(x)^{2n-r}\left(-\frac{1}{x}\right)^r$$

Putting $r = n$,

$$T_{n+1} = {}^{2n}C_n(x)^{2n-1}\left(-\frac{1}{x}\right) = \frac{2n!}{n!\,n!}(x)^n(1)^n\,\frac{1}{(x^n)}$$

$$= \frac{2n\,(2n-1)(2n-2)(2n-3)\ldots 3\cdot 2\cdot 1}{n!\,n!}(-1)^n$$

$$= \frac{2^n\,[n(n-1)(n-2)\ldots 3\cdot 2\cdot 1][(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1]}{n!\,n!}\times(-1)^n$$

$$= \frac{2^n \times n![1\times 3\times 5\ldots\times(2n-1)]}{n!\,n!}(-1)^n$$

$$= \frac{2^n \times [1\times 3\times 5\ldots\times(2n-1)]}{n!}(-1)^n$$

$$= \frac{1\cdot 3\cdot 5\ \ldots\ (2n-1)}{n!}(-2)^n$$

Example 4 *If the Coefficient of second, third and fourth terms in the expansion of* $(1+x)^{2n}$ *are in* AP*, then* $2n^2 - 9n$ *is equal to*

[**NCERT Exemplar**]

(*a*) −7 (*b*) 7

(*c*) 6 (*d*) −6

Solution (*a*) The general term of $(1+x)^{2n}$ is

$$T_{r+1} = {}^{2n}C_r\,x^r$$

$$T_2 = {}^{2n}C_1x^2,\ T_3 = {}^{2n}C_2x^3,\ T_4 = {}^{2n}C_3x^4$$

Since , coefficients are in AP.

$\Rightarrow \quad {}^{2n}C_1,\ {}^{2n}C_2,\ {}^{2n}C_3$, are in AP

$\Rightarrow \quad 2\times {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$

$\Rightarrow \quad 2 = \frac{{}^{2n}C_1}{{}^{2n}C_2} + \frac{{}^{2n}C_3}{{}^{2n}C_2}$

$\Rightarrow \quad 2 = \frac{2}{(2n-2+1)} + \frac{2n-3+1}{3}$

$\Rightarrow \quad 2 = \frac{2}{2n-1} + \frac{2n-2}{3}$

$\Rightarrow \quad 2n^2 - 9n + 7 = 0$

$\therefore \quad 2n^2 - 9n = -7$

Example 5 *The coefficient of x^5 in the expansion of $(1+2x)^6(1-x)^7$ is*

(a) 192 (b) 171
(c) 21 (d) *None of these*

Solution (b) $(1+2x)^6(1-x)^7 = [1 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6]$
$\times [1 - {}^7C_1x + {}^7C_2x^2 - {}^7C_3x^3 + {}^7C_4x^4 \ldots]$

$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + \ldots)$
$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \ldots)$

$\therefore$ Coefficient of x^5 in the expansion of $(1+2x)^6(1-x)^7$

$= 1 \times (-21) + 12 \times 35 + 60 \times (-35) + 160 \times 21 + 240 \times (-7) + 192$

$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$

Example 6 *The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ is*

(a) 900 (b) 909
(c) 990 (d) 999

Solution (c) Let $E = (1 + x + x^2 + x^3)^{11}$

$= [(1+x)(1+x^2)]^{11}$

$= (1+x)^{11}(1+x^2)^{11}$

$\therefore$ General term of E is,

$T(r,k) = [{}^{11}C_r x^r)({}^{11}C_k(x^2)^k]$

$= {}^{11}C_r \cdot {}^{11}C_k\, x^{r+2k}$

For taking coefficient of x^4, we take

$r + 2k = 4$

r	0	1	2	3	4	5
$k = \frac{4-r}{2}$	2	x	1	x	0	x

$\therefore$ Coefficient of $x^4 = T_{(0,2)} + T_{(2,1)} + T_{(4,0)}$

$= {}^{11}C_0\,{}^{11}C_2 + {}^{11}C_2\,{}^{11}C_1 + {}^{11}C_4\,{}^{11}C_0$

$= 55 + 55 \times 11 + 330$

$= 55 + 605 + 330 = 990$

Example 7 *In the expansion of $(1+x)^n$,*

$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots + n\frac{C_n}{C_{n-1}}$ *is equal to*

(a) $\frac{(n+1)}{2}$ (b) $\frac{n}{2}$
(c) $\frac{n(n+1)}{2}$ (d) *None of these*

Solution (c) Since, we know that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

$\therefore \quad \frac{C_1}{C_0} = n,\ \frac{C_2}{C_1} = \frac{n-1}{2},\ \frac{C_3}{C_2} = \frac{n-2}{3}, \ldots$

Thus, $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots = n + 2\cdot\frac{n-1}{2} + 3\cdot\frac{n-2}{3} + \ldots + n\cdot\frac{1}{n}$

$= [n + (n-1) + (n-2) + \ldots + 1]$

$= \frac{n(n+1)}{2}$

Example 8 $(C_0 + C_1)(C_1 + C_2) \ldots (C_{n-1} + C_n)$ *is equal to*

(a) $(C_0C_1C_2 \ldots C_{n-1})(n+1)$
(b) $(C_0C_1C_2 \ldots C_{n-1})(n+1)^n$
(c) $\frac{(C_0C_1C_2 \ldots C_{n-1})(n+1)^n}{n!}$
(d) *None of the above*

Solution (c) $(C_0 + C_1)(C_1 + C_2) \ldots (C_{n-1} + C_n)$

$= C_0\left(1 + \frac{C_1}{C_0}\right) C_1\left(1 + \frac{C_2}{C_1}\right) \ldots C_{n-1}\left(1 + \frac{C_n}{C_{n-1}}\right)$

$= \{C_0C_1C_2 \ldots C_{n-1}\}\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right) \ldots \left(1 + \frac{C_n}{C_{n-1}}\right)$

$= \{C_0C_1C_2 \ldots C_{n-1}\}(n+1)\left(1 + \frac{n-1}{2}\right) \ldots \left(1 + \frac{1}{n}\right)$

$= \{C_0C_1C_2 \ldots C_{n-1}\}(n+1)\left(\frac{n+1}{2}\right)\left(\frac{n+1}{3}\right) \ldots \left(\frac{n+1}{n}\right)$

$= \frac{\{C_0C_1C_2 \ldots C_{n-1}\}(n+1)^n}{n!}$

Example 9 $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n$ *is equal to*

(a) $\frac{(2n)!}{(n-r)!(n+r)!}$ (b) $\frac{n!}{r!(n+r)!}$
(c) $\frac{n!}{(n-r)!}$ (d) None of these

Solution (a) We know,

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + \ldots$ ...(i)

and $\left(1 + \frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + \ldots + C_r\frac{1}{x^r}$
$+ C_{r+1}\frac{1}{x^{r+1}} + C_{r+2}\frac{1}{x^{r+2}} \ldots C_n\frac{1}{x^n}$ ...(ii)

On multiplying Eqs. (i) and (ii), equating coefficient of x^r in $\frac{1}{x^n}(1+x)^{2n}$ or the coefficient of x^{n+r} in $(1+x)^{2n}$, we get the value of required expression which is

$${}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$$

Example 10 $(100)^{50} + (99)^{50}$

(*a*) $< (101)^{50}$ (*b*) (101) (*c*) $> (101)^{50}$ (*d*) $> (101)$

Solution (*a*) Since, $(101)^{50} = (101+1)^{50} = 100^{50} + {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} + \ldots + 1$ ... (i)

and $(99)^{50} = (100-1)^{100} = 100^{50} - {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} - \ldots + 1$ ... (ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(101)^{50} - (99)^{50} = 2\{{}^{50}C_1 100^{49} + {}^{50}C_3 100^{47} + \ldots\}$$
$$= 2 \times {}^{50}C_1 100^{49} + (2 \times {}^{50}C_3 \times 100^{47} + \ldots\}$$
$$= 100 \times 100^{49} + \text{a positive number}$$
$$> 100^{50}$$

$\Rightarrow$ $(101)^{50} - (99)^{50} > (100)^{50}$

$\Rightarrow$ $(101)^{50} > (100)^{50} + (99)^{50}$

Example 11 *The sum of the series*

$$1 + \frac{1}{3^2} + \frac{1\cdot 4}{1\cdot 2}\cdot\frac{1}{3^4} + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{1}{3^6} + \ldots \text{ is}$$

(*a*) $\sqrt{\frac{3}{2}}$ (*b*) $\left(\frac{3}{2}\right)^{1/3}$

(*c*) $\sqrt{\frac{1}{3}}$ (*d*) *None of these*

Solution (*b*) We know, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots + x^n$

Here, $nx = \frac{1}{3^2}$ ...(i)

and $\frac{n(n-1)}{2}x^2 = \frac{1\cdot 4}{1\cdot 2}\cdot\frac{1}{3^4}$ ...(ii)

From Eqs. (i) and (ii), we get

$$n = -\frac{1}{3} \text{ and } x = -\frac{1}{3}$$

$\therefore$ Required sum $= \left(1 - \frac{1}{3}\right)^{-1/3} = \left(\frac{3}{2}\right)^{1/3}$

Example 12 *If* $(1 + x - 2x^2)^6 = 1 + a_1 x + a_2 x^2 + \ldots + a_{12}x^{12}$, *then the expression* $a_2 + a_4 + a_6 + \ldots + a_{12}$ *has the value*

(*a*) 32 (*b*) 63 (*c*) 64 (*d*) 31

Solution (*d*) $(1 + x - 2x^2)^6 = 1 + a_1 x + a_2 x^2 + \ldots + a_{12}x^{12}$

On putting $x = 1$ and $x = -1$ and adding the results.

$$64 = 2(1 + a_2 + a_4 + \ldots)$$

$\Rightarrow$ $a_2 + a_4 + a_6 + \ldots + a_{12} = 31$

Example 13 *For* $\forall\, n \in N$, $2^{3n+3} - 7n - 8$ *is divisible by*

(*a*) 49 (*b*) 7

(*c*) 343 (*d*) *None of these*

Solution (*d*) Now, $2^{3n+3} - 7n - 8$

$$= (2^3)^{n+1} - 7(n+1) - 1$$
$$= (1+7)^{n+1} - 7(n+1) - 1$$
$$= {}^{n+1}C_0 + {}^{n+1}C_1 7 + {}^{n+1}C_2 7^2 + {}^{n+1}C_3 7^3 \ldots + {}^{n+1}C_{n+1}\, 7^{n+1} - 7(n+1) - 1$$
$$= 7^2\,({}^{n+1}C_2 + {}^{n+1}C_3 7^1 \ldots + {}^{n+1}C_{n+1}\, 7^{n-1},$$

which is divisible by 49.

Example 14 *If* $(2+\sqrt{3})^n = I + f$ *where I and n are positive integers and* $0 < f < 1$, *then* $(1-f)(I+f)$

(*a*) -1 (*b*) 1

(*c*) 2 (*d*) *None of these*

Solution (*b*) $I + f = (2+\sqrt{3})^n = 2^n + {}^nC_1 2^{n-1}\sqrt{3} + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_3 2^{n-3}(\sqrt{3})^3 + \ldots$ ... (i)

Now, $0 < 2 - \sqrt{3} < 1$

$\Rightarrow$ $0 < (2-\sqrt{3})^n < 1$

Let $(2-\sqrt{3})^n = f'$ where $0 < f' < 1$

$\Rightarrow$ $f' = 2^n - {}^nC_1 2^{n-1}\sqrt{3} + {}^nC_2 2^{n-2}(\sqrt{3})^2 - {}^nC_3 2^{n-3}(\sqrt{3})^3 + \ldots$ ... (ii)

On adding Eqs. (i) and (ii), we get

$$I + f + f' = 2[2^n + {}^nC_2 2^{n-2}\cdot 3 + \ldots] \quad \text{...(iii)}$$

$\Rightarrow$ $I + f + f' =$ even integer

Now, $0 < f < 1$

and $0 < f' < 1$

$\Rightarrow$ $0 < f + f' < 2$

Hence, from Eq. (iii) we conclude that $f + f'$ is an integer between 0 and 2.

$\Rightarrow$ $f + f' = 1$

$\Rightarrow$ $f' = 1 - f$ ...(iv)

$\therefore$ $I + f = (2+\sqrt{3})^n$, $f' = 1 - f = (2-\sqrt{3})^n$

$\Rightarrow$ $(I+f)(1-f) = [(2+\sqrt{3})(2-\sqrt{3})]^n = (4-3)^n = 1$

$\Rightarrow$ $(I+f)(1-f) = 1$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Binomial Theorem for Positive Integer Index

1. The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification will be [NCERT Exemplar]

(a) 202 (b) 51
(c) 50 (d) None of these

2. If $(1+ax)^n = 1+8x+24x^2+\ldots$, then the values of a and n are

(a) 2, 4 (b) 2, 3 (c) 3, 6 (d) 1, 2

3. By using the first three terms of its expansion , the approximate value of $(0.99)^5$ is [NCERT]

(a) 0.949 (b) 0.951
(c) 0.954 (d) None of these

General, Middle and Greatest Term in the Binomial Expansion

4. The coefficient of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ are in the ratio [NCERT Exemplar]

(a) 1:2 (b) 1:3 (c) 3:1 (d) 2:1

5. The value of $(1.002)^{12}$ upto fourth place of decimal is

(a) 1.0242 (b) 1.0245
(c) 1.0004 (d) 1.0254

6. The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^n$ is

(a) ${}^nC_4$ (b) ${}^nC_4 + {}^nC_2$
(c) ${}^nC_4 + {}^nC_2 + {}^nC_2$ (d) ${}^nC_4 + {}^nC_2 + {}^nC_1 \cdot {}^nC_2$

7. In the expansion of the following expression $1+(1+x)+(1+x)^2+\ldots+(1+x)^n$, the coefficient of x^4 $(0 \le k \le n)$ is

(a) ${}^{n+1}C_{k+1}$ (b) ${}^nC_k$
(c) ${}^nC_{n-k-1}$ (d) None of these

8. The coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$ is

(a) ${}^{51}C_5$ (b) ${}^9C_5$
(c) ${}^{31}C_6 - {}^{21}C_6$ (d) ${}^{30}C_5 + {}^{20}C_5$

9. The coefficient of t^{24} in the expansion of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(a) ${}^{12}C_6 + 2$ (b) ${}^{12}C_5$
(c) ${}^{12}C_6$ (d) ${}^{12}C_7$

10. The coefficient of x^{53} in the following expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m$ is

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$
(c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

11. In the polynomial $(x-1)(x-2)(x-3)\ldots(x-100)$, the coefficient of x^{99} is

(a) 5050 (b) – 5050 (c) 100 (d) 99

12. If in the expansion of $(1+x)^m(1-x)^n$, the coefficient of x and x^2 are 3 and – 6 respectively, then m is

(a) 6 (b) 9 (c) 12 (d) 24

13. If for positive integers $r>1$, $n>2$, the coefficient of the $(3r)$th and $(r+2)$th powers of x in the expansion of $(1+x)^{2n}$ are equal, then

(a) $n=2r$ (b) $n=3r$
(c) $n=2r+1$ (d) None of these

14. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio $1:7:42$, then the value of n is

(a) 60 (b) 70
(c) 55 (d) None of these

15. The digit at the unit place in the number $19^{2005}+11^{2005}-9^{2005}$ is

(a) 2 (b) 1 (c) 0 (d) 8

16. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$ is equal to

(a) $\frac{a_2}{a_2+a_3}$ (b) $\frac{1}{2}\frac{a_2}{(a_2+a_3)}$

(c) $\frac{2a_2}{a_2+a_3}$ (d) $\frac{2a_3}{a_2+a_3}$

17. If the coefficients of p th, $(p+1)$th and $(p+2)$th terms in the expansion of $(1+x)^n$ are in AP, then

(a) $n^2-2np+4p^2=0$

(b) $n^2-n(4p+1)+4p^2-2=0$

(c) $n^2-n(4p+1)+4p^2=0$

(d) None of the above

18. If $(1+x-2x^2)^6=1+a_1x+a_2x^2+\ldots+a_{12}x^{12}$, then the expression $a_2+a_4+a_6+\ldots+a_{12}$ has the value

(a) 32 (b) 63

(c) 64 (d) None of these

19. If $(1-x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, then $a_0+a_2+a_4+\ldots+a_{2n}$ is equal to

(a) $\frac{3^n+1}{2}$ (b) $\frac{3^n-1}{2}$

(c) $\frac{1-3^n}{2}$ (d) $3^n+\frac{1}{2}$

20. The 4th term in the expansion of $(x-2y)^{12}$ is [NCERT]

(a) $1760x^3y^9$ (b) $-1760x^9y^3$

(c) $1760x^9y^3$ (d) None of these

21. The rth term in the expansion of $(a+2x)^n$ is

(a) $\frac{n(n+1)\ \ldots(n-r+1)}{r!}a^{n-r+1}(2x)^r$

(b) $\frac{n(n-1)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}(2x)^{r-1}$

(c) $\frac{n(n+1)\ldots(n-r)}{(r+1)}a^{n-r}(x)^r$

(d) None of the above

22. The $(r+1)$th term in the expansion of $(1-x)^{-4}$ will be

(a) $\frac{x^r}{r!}$ (b) $\frac{(r+1)(r+2)(r+3)}{6}x^r$

(c) $\frac{(r+2)(r+3)}{2}x^r$ (d) None of these

23. In the expansion of $(1+a)^{m+n}$, the coefficients of a^m and a^n are [NCERT]

(a) equal (b) not equal

(c) do not say anything (d) None of these

24. Given the integers, $r>1, n>2$ and coefficients of $(3r)$th and $(r+2)$nd terms in the binomial expansion of $(1+x)^{2n}$ are equal, then [NCERT Exemplar]

(a) $n=2r$ (b) $n=3r$

(c) $n=2r+1$ (d) None of these

25. If the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal, then the value of a is [NCERT]

(a) $\frac{5}{7}$ (b) $\frac{9}{7}$ (c) $\frac{7}{9}$ (d) $\frac{-9}{7}$

26. The two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 1:4 are [NCERT Exemplar]

(a) 3rd and 4th (b) 4th and 5th

(c) 5th and 6th (d) 6th and 7th

27. If the coefficients of 2nd, 3rd and the 4th terms in the expansion of $(1+x)^n$ are in AP, then value of n is [NCERT Exemplar]

(a) 2 (b) 7 (c) 11 (d) 14

28. If in the expansion of $(a-2b)^n$, the sum of 4th and 5th term is zero, then the value of $\frac{a}{b}$ is

(a) $\frac{n-4}{5}$ (b) $\frac{n-3}{2}$

(c) $\frac{5}{n-4}$ (d) $\frac{5}{2(n-4)}$

29. The value of x in the expression $(x+x^{\log_{10}x})^5$, if the third term in the expansion is 1000000, is

(a) 10 (b) 11

(c) 12 (d) None of these

30. The first 3 terms in the expansion of $(1+ax)^n$ $(n\neq 0)$ are $1, 6x$ and $16x^2$. Then, the values of a and n are respectively

(a) 2 and 9 (b) 3 and 2

(c) $\frac{2}{3}$ and 9 (d) $\frac{3}{2}$ and 6

31. If the last term in the binomial expansion of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is

(a) 210 (b) 420

(c) 105 (d) None of these

32. If x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, then

(a) $n-2k$ is a multiple of 2

(b) $n-2k$ is a multiple of 3

(c) $k=0$

(d) None of the above

33. The coefficient of the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is

(a) $\frac{1}{3}$ (b) $\frac{19}{54}$

(c) $\frac{17}{54}$ (d) $\frac{1}{4}$

34. If the term free from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, then the value of k is [NCERT Exemplar]

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

35. The coefficient of x in the expansion of $(1-3x+7x^2)(1-x)^{16}$ is [NCERT Exemplar]

(a) 19 (b) -19
(c) 18 (d) -18

36. The term independent of x in the expansion of $\left(3x-\frac{2}{x^2}\right)^{15}$ is [NCERT Exemplar]

(a) $-3003\,(3^{10})\,(2^5)$ (b) $-3003\,(3^{10})\,(2^4)$
(c) $3003\,(3^{10})\,(2^5)$ (d) None of these

37. The middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$ is

(a) $\frac{1\cdot 3\cdot 5\ldots(2n-3)}{n!}$ (b) $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}$
(c) $\frac{1\cdot 3\cdot 5\ldots(2n+1)}{n!}$ (d) None of these

38. If the middle term of $\left(\frac{1}{x}+x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then value of x is [NCERT Exemplar]

(a) $2n\pi+\frac{\pi}{6}$ (b) $n\pi+\frac{\pi}{6}$
(c) $n\pi+(-1)^n\frac{\pi}{6}$ (d) $n\pi+(-1)^n\frac{\pi}{3}$

39. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then the value of p is [NCERT Exemplar]

(a) ± 3 (b) ± 1
(c) ± 2 (d) None of these

40. If n is even, then the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924\,x^6$, then n is equal to

(a) 10 (b) 12
(c) 14 (d) None of these

41. The largest term in the expansion of $(3+2x)^{50}$, where $x=\frac{1}{5}$ is

(a) 5th (b) 3th
(c) 7th (d) 6th

42. If the sum of the coefficients in the expansion of $(x-2y+3z)^n$ is 128, then the greatest coefficient in the expansion of $(1+x)^n$ is

(a) 35
(b) 20
(c) 10
(d) None of the above

43. The greatest value of the term independent of x, as α varies over R, in the expansion of $\left(x\cos\alpha+\frac{\sin\alpha}{x}\right)^{20}$ is

(a) ${}^{20}C_{10}$ (b) ${}^{20}C_{15}$
(c) ${}^{20}C_{19}$ (d) None of these

44. The interval in which x must lie, so that the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient, is

(a) $\left(\frac{n-1}{n},\frac{n}{n-1}\right)$ (b) $\left(\frac{n}{n+1},\frac{n+1}{n}\right)$
(c) $\left(\frac{n}{n+2},\frac{n+2}{n}\right)$ (d) None of these

Properties of Binomial Coefficients

45. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ is equal to

(a) $(n-1)\,a_n$ (b) na_n
(c) $\frac{1}{2}na_n$ (d) None of these

46. $\sum_{r=0}^{n}(-1)^r\,({}^nC_r)\,\frac{1+rx}{1+nx}$ is equal to

(a) 1 (b) -1 (c) n (d) 0

47. $\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\ldots+\binom{30}{20}\binom{30}{30}$ is equal to

(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$
(c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

48. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, then $C_0^2+C_1^2+C_2^2+C_3^2+\ldots+C_n^2$ is equal to

(a) $\frac{n!}{n!\,n!}$ (b) $\frac{(2n)!}{n!\,n!}$
(c) $\frac{(2n)!}{n!}$ (d) None of these

49. $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\ldots+15\frac{C_{15}}{C_{14}}$ is equal to

(a) 100 (b) 120
(c) -120 (d) None of these

50. If $n\in N, n>1$, then value of $E=a-{}^nC_1(a-1)+{}^nC_2(a-2)+\ldots+(-1)^n(a-n)\,{}^nC_n$ is

(a) a (b) 0 (c) a^2 (d) 2^n

51. ${}^nC_0-\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2-\ldots+(-1)^n\frac{{}^nC_n}{n+1}$ is equal to

(a) n (b) $\frac{1}{n}$ (c) $\frac{1}{n+1}$ (d) $\frac{1}{n-1}$

52. If $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, then

$\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$ is equal to

(a) $\frac{n^{n-1}}{(n-1)}!$ (b) $\frac{(n+1)^{n-1}}{(n-1)!}$

(c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n+1)^{n+1}}{n!}$

53. If C_r stands for ${}^nC_r$, the sum of the given series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\cdot[C_0^2 - 2C_1^2 + 3C_2^2 - \ldots + (-1)^n (n+1) C_n^2],$$

where n is an even positive integer, is

(a) 0 (b) $(-1)^{n/2}(n+1)$

(c) $(-1)^n(n+2)$ (d) $(-1)^{n/2}(n+2)$

54. If n is odd, then $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2$ is equal to

(a) 0 (b) 1

(c) ∞ (d) $\frac{n!}{\left(\frac{n}{2}\right)^2!}$

Applications of Binomial Theorem

55. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n$, then the value of $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$ will be

(a) $(n+2)2^{n-1}$ (b) $(n+1)2^n$

(c) $(n+1)2^{n-1}$ (d) $(n+2)2^n$

56. If $n > (8+3\sqrt{7})^{10}$, $n \in N$, then the least value of n is

(a) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10}$

(b) $(8+3\sqrt{7})^{10} + (8-3\sqrt{7})^{10}$

(c) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10} + 1$

(d) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10} - 1$

57. Let $R = (2+\sqrt{3})^{2n}$ and $f = R - [R]$, where $[\cdot]$ denotes the greatest integer function, then $R(1-f)$ is equal to

(a) 1 (b) 2^{2n}

(c) $2^{2n} - 1$ (d) ${}^{2n}C_n$

58. If a_r is the coefficient of x^r, in the expansion of $(1+x+x^2)^n$, then $a_1 - 2a_2 + 3a_3 - \ldots - 2n\,a_{2n}$ is equal to

(a) 0 (b) n (c) $-n$ (d) $2n$

59. If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then

$a_1 - 2a_2 + 3a_3 \ldots - 2na_{2n}$ is equal to

(a) n (b) $-n$ (c) 0 (d) $2n$

60. If $(1+x)^{15} = C_0 + C_1x + C_2x^2 + \ldots + C_{15}x^{15}$, then $C_2 + 2C_3 + 3C_4 + \ldots + 14C_{15}$ is equal to

(a) $14 \cdot 2^{14}$ (b) $13 \cdot 2^{14} + 1$

(c) $13 \cdot 2^{14} - 1$ (d) None of these

61. If a and d are two complex numbers, then the sum to $(n+1)$ terms of the following series

$aC_0 - (a+d)C_1 + (a+2d)C_2 - \ldots + \ldots$ is

(a) $\frac{a}{2^n}$ (b) na

(c) 0 (d) None of these

62. $49^n + 16n - 1$ is divisible by

(a) 3 (b) 19 (c) 64 (d) 29

63. If $A = 1000^{1000}$ and $B = (1001)^{999}$, then

(a) $A > B$ (b) $A = B$

(c) $A < B$ (d) None of these

Binomial Theorem for Negative/Rational Index

64. Cube root of 217 is

(a) 6.01 (b) 6.04

(c) 6.02 (d) None of these

65. For $|x| < 1$, the constant term in the expansion of $\frac{1}{(x-1)^2(x-2)}$ is

(a) 2 (b) 1 (c) 0 (d) $-\frac{1}{2}$

66. The expansion of $\frac{1}{(4-3x)^{1/2}}$ by binomial theorem will be valid, if

(a) $x < 1$

(b) $|x| < 1$

(c) $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

(d) None of the above

67. The expression $\frac{1}{\sqrt[3]{6-3x}}$ is equal to

(a) $6^{1/3}\left[1+\frac{x}{6}+\frac{2x^2}{6^2}+\ldots\right]$ (b) $6^{-1/3}\left[1+\frac{x}{6}+\frac{2x^2}{6^2}+\ldots\right]$

(c) $6^{1/3}\left[1-\frac{x}{6}+\frac{2x^2}{6^2}-\ldots\right]$ (d) $6^{-1/3}\left[1-\frac{x}{6}+\frac{2x^2}{6^2}-\ldots\right]$

68. If the value of x is so small that x^2 and greater powers can be neglected, then $\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$ is equal to

(a) $1+\frac{5}{6}x$ (b) $1-\frac{5}{6}x$

(c) $1+\frac{2}{3}x$ (d) $1-\frac{2}{3}x$

69. If $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to $a + bx$ for small values of x, then (a, b) is equal to

(a) $\left(1, \dfrac{35}{24}\right)$ (b) $\left(1, -\dfrac{35}{24}\right)$

(c) $\left(2, \dfrac{35}{12}\right)$ (d) $\left(2, -\dfrac{35}{12}\right)$

Miscellaneous

70. The number of terms in the expansion of $(a + b + c)^n$ will be

(a) $n + 1$

(b) $n + 3$

(c) $\dfrac{(n+1)(n+2)}{2}$

(d) None of the above

71. Let $[x]$ denote the greatest integer less than or equal to x. If $x = (\sqrt{3}+1)^5$, then $[x]$ is equal to

(a) 75 (b) 50 (c) 76 (d) 152

72. $1+\dfrac{1}{3}x+\dfrac{1\cdot 4}{3\cdot 6}x^2+\dfrac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3+\ldots$ is equal to

(a) x (b) $(1+x)^{1/3}$

(c) $(1-x)^{1/3}$ (d) $(1-x)^{-1/3}$

73. $1+\dfrac{2\cdot 1}{3\cdot 2}+\dfrac{2\cdot 5}{3\cdot 6}\left(\dfrac{1}{2}\right)^2+\dfrac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}\cdot\left(\dfrac{1}{2}\right)^3+\ldots$ is equal to

(a) $2^{1/3}$ (b) $3^{1/4}$

(c) $4^{1/3}$ (d) $3^{1/3}$

74. $\dfrac{1}{1!(n-1)!}+\dfrac{1}{3!(n-3)!}+\dfrac{1}{5!(n-5)!}+\ldots$ is equal to

(a) $\dfrac{2^n}{n!}$ (b) $\dfrac{2^{n-1}}{n!}$

(c) 0 (d) None of these

75. The coefficient of x^n in the polynomial $(x + {}^nC_0)(x+3\,{}^nC_1)(x+5\,{}^nC_2)\ldots[x+(2n+1)\,{}^nC_n]$ is

(a) $n\cdot 2^n$ (b) $n\cdot 2^{n+1}$

(c) $(n+1)\,2^n$ (d) $n\cdot 2^n+1$

76. Let T_n denotes the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1}-T_n = 21$, then n is equal to

(a) 5 (b) 7

(c) 6 (d) 4

77. The coefficient of x^7 in $(1+3x-2x^3)^{10}$ is equal to

(a) 62640 (b) 26240

(c) 64620 (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. The coefficient of $x^{2^{m+1}}$ in the expansion of

$$E=\frac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^m})},\ |x|<1$$

is

(a) 3 (b) 2

(c) 1 (d) 0

2. The coefficient of $x^r[0\le r\le(n-1)]$ in the expansion of $(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+\ldots+(x+2)^{n-1}$ is

(a) ${}^nC_r(3^r-2^n)$ (b) ${}^nC_r(3^{n-r}-2^{n-r})$

(c) ${}^nC_r(3^r+2^{n-1})$ (d) None of these

3. The sum of the series

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots+m\text{ terms}\right)\text{ is}$$

(a) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$ (b) $\dfrac{2^{mn}-1}{2^n-1}$

(c) $\dfrac{2^{mn}+1}{2^n+1}$ (d) None of these

4. The value of x, for which the 6th term in the expansion of

$$\left\{2^{\log_2\sqrt{(9^{x-1}+7)}}+\frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right\}^7 \text{ is 84, is equal to}$$

(a) 4 (b) 3 (c) 2 (d) 5

5. The sum $\displaystyle\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, $\left[\text{where, }\dbinom{p}{q}=0,\text{ if } p<q\right]$ is maximum, when m is

(a) 5 (b) 10 (c) 15 (d) 20

6. If ${}^{n-1}C_r=(k^2-3)\cdot{}^nC_{r+1}$, then k is belongs to

(a) $(-\infty,-2]$ (b) $[2,\infty)$

(c) $[-\sqrt{3},\sqrt{3}]$ (d) $(\sqrt{3},2]$

7. If $(5+2\sqrt{6})^n = I+f$; $n, I\in N$ and $0\le f<1$, then I is equal to

(a) $\dfrac{1}{f}-f$ (b) $\dfrac{1}{1+f}-f$

(c) $\dfrac{1}{1+f}+f$ (d) $\dfrac{1}{1-f}-f$

8. The coefficients of $x^2 y^2$, yzt^2 and $xyzt$ in the expansion of $(x + y + z + t)^4$ are in the ratio
(a) 4 : 2 : 1 (b) 1 : 2 : 4
(c) 2 : 4 : 1 (d) 1 : 4 : 2

9. If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2 + \frac{1}{x}\right)^m$ is 46, then the coefficient of the term that does not contain x is
(a) 84 (b) 92 (c) 98 (d) 106

More Than One Correct Option

10. The last digit of $3^{3^{4n}} + 1$, $n \in N$, is
(a) ${}^4C_3$ (b) ${}^8C_7$
(c) 8 (d) 4

11. The number $101^{100} - 1$ is divisible by
(a) 100 (b) 1000
(c) 10000 (d) 100000

12. Which of the following is/are correct ?
(a) $101^{50} - 99^{50} > 100^{50}$
(b) $101^{50} - 100^{50} > 99^{50}$
(c) $(1000)^{1000} > (1001)^{999}$
(d) $(1001)^{999} > (1000)^{1000}$

Comprehension Based Questions

Passage

Directions (Q. Nos. 13-15) *The 2nd, 3rd and 4th terms in the expansion of* $(x + a)^n$ *are* 240, 720 *and* 1080, *respectively.*

13. The value of $(x - a)^n$ can be
(a) 64 (b) –1
(c) –32 (d) None of these

14. The value of least term in the expansion is
(a) 16 (b) 160
(c) 32 (d) 81

15. The sum of odd numbered terms is
(a) 1664 (b) 2376
(c) 1562 (d) 1486

Assertion and Reason

Directions (Q. Nos. 16-25) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

16. **Statement I** The number of terms in the expansion of $\left(x + \frac{1}{x} + 1\right)^n$ is $2n + 1$.

Statement II $(a_1 + a_2 + a_3 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

17. **Statement I** If $\sum_{r=1}^{n} r^3 \left(\frac{{}^nC_r}{{}^nC_{r-1}}\right) = 196$, then the sum of the coefficients of power x in the expansion of the polynomial $(x - 3x^2 + x^3)^n$ is – 1.

Statement II $\frac{{}^nC_r}{{}^nC_{r-1}} = \left(\frac{n - r + 1}{r}\right)$, $\forall\ n \in N$ and $r \in W$.

18. Three consecutive binomial coefficients are given.

Statement I They cannot be in GP and HP.

Statement II They always are in AP.

19. **Statement I** $(\sqrt{2} - 1)^n$ can be expressed as of $\sqrt{N} - \sqrt{(N - 1)}$ for all $N > 1$ and n is positive integer.

Statement II $(\sqrt{2} - 1)^n$ can be expressed as $A + B\sqrt{2}$, where A and B are integers and n is positive integer.

20. **Statement I** If n is an odd prime, then the integral part of $(\sqrt{5} + 2)^n - 2^{n+1}$ is divisible by $2n$

Statement II If n is prime, then ${}^nC_1, {}^nC_2, \ldots, {}^nC_{n-1}$ must be divisible by n.

21. **Statement I** If n is even, then
$${}^{2n}C_1 + {}^{2n}C_3 + \ldots + {}^{2n}C_{n-1} = 2^{2n-1}.$$
Statement II ${}^{2n}C_1 + {}^{2n}C_3 + \ldots + {}^{2n}C_{2n-1} = 2^{2n-1}$

22. **Statement I** The term independent of x in the expansion of $\left(x + \frac{1}{x} + 2\right)^m$ is $\frac{(4m)!}{(2m!)^2}$.

Statement II The coefficient of x^6 in the expansion of $(1 + x)^n$ is ${}^nC_6$.

23. **Statement I** Greatest term in the expansion of $(1+x)^{12}$, when $x = 11/10$ is 7th.

Statement II Seventh term in the expansion of $(1+x)^{12}$ has the factor ${}^{12}C_6$ which is greatest value of ${}^{12}C_r$.

24. **Statement I** $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\ n \in N$.

Statement II $(1+x)^n - nx - 1$ is divisible by x^2, $\forall\ n \in N$.

25. **Statement I** $\sum_{r=0}^{n} (r + 1) \cdot {}^nC_r = (n + 2)2^{n-1}$

Statement II $\sum_{r=0}^{n} (r + 1)\ {}^nC_r \cdot x^r = (1 + x)^n + nx(1 + x)^{n-1}$

Previous Years' Questions

26. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is [JEE Main 2013]

(a) 4 (b) 120 (c) 210 (d) 310

27. If n is a positive integer, then $(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}$ is [AIEEE 2012]

(a) an irrational number
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

28. The coefficient of x^7 in the expansion of $(1-x-x^2+x^3)^6$ is [AIEEE 2011]

(a) -132 (b) -144 (c) 132 (d) 144

29. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is [AIEEE 2009]

(a) 0 (b) 2 (c) 7 (d) 8

30. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ is equal to [AIEEE 2007]

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$ (c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$

31. The sum of the series
${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10}$ is [AIEEE 2007]

(a) $-{}^{20}C_{10}$ (b) $\frac{1}{2}\,{}^{20}C_{10}$ (c) 0 (d) ${}^{20}C_{10}$

32. If the expansion in powers of x of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, then a_n is [AIEEE 2006]

(a) $\frac{a^n-b^n}{b-a}$ (b) $\frac{a^{n+1}-b^{n+1}}{b-a}$
(c) $\frac{b^{n+1}-a^{n+1}}{b-a}$ (d) $\frac{b^n-a^n}{b-a}$

33. For natural numbers m, n, if

$$(1-y)^m(1+y)^n = 1 + a_1y + a_2y^2 + \ldots$$

and $a_1 = a_2 = 10$, then (m, n) is [AIEEE 2006]

(a) 35, 20 (b) 45, 35 (c) 35, 45 (d) 20, 65

34. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is [AIEEE 2005]

(a) ${}^{56}C_4$ (b) ${}^{56}C_3$
(c) ${}^{55}C_3$ (d) ${}^{55}C_4$

35. If the coefficients of rth, $(r+1)$ th and $(r+2)$ th terms in the binomial expansion of $(1+y)^m$ are in AP, then m and r satisfy the equation [AIEEE 2005]

(a) $m^2 - m(4r-1) + 4r^2 + 2 = 0$
(b) $m^2 - m(4r+1) + 4r^2 - 2 = 0$
(c) $m^2 - m(4r+1) + 4r^2 + 2 = 0$
(d) $m^2 - m(4r-1) + 4r^2 - 2 = 0$

36. If the coefficient of x^7 in $\left[ax^2 + \frac{1}{bx}\right]^{11}$ equals to the coefficient of x^{-7} in $\left[ax - \frac{1}{bx^2}\right]^{11}$, then a and b satisfy the relation [AIEEE 2005]

(a) $ab = 1$ (b) $\frac{a}{b} = 1$ (c) $a + b = 1$ (d) $a - b = 1$

37. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{3/2} - \left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$ may be approximated as [AIEEE 2005]

(a) $\frac{x}{2} - \frac{3}{8}x^2$ (b) $-\frac{3}{8}x^2$ (c) $3x + \frac{3}{8}x^2$ (d) $1 - \frac{3}{8}x^2$

38. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same, if α is equal to [AIEEE 2004]

(a) $-\frac{5}{3}$ (b) $\frac{10}{3}$ (c) $-\frac{3}{10}$ (d) $\frac{3}{5}$

39. If $s_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$, then $\frac{t_n}{s_n}$ is equal to [AIEEE 2004]

(a) $\frac{n}{2}$ (b) $\frac{n}{2} - 1$ (c) $n - 1$ (d) $\frac{2n-1}{2}$

40. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is [AIEEE 2004]

(a) $(n-1)$ (b) $(-1)^n(1-n)$
(c) $(-1)^{n-1}(n-1)^2$ (d) $(-1)^{n-1}n$

41. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is [AIEEE 2003]

(a) 32 (b) 33 (c) 34 (d) 35

42. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is [AIEEE 2003]

(a) 7th term (b) 5th term (c) 8th term (d) 6th term

43. The coefficient of x^5 in $(1 + 2x + 3x^2 + \ldots)^{-3/2}$ is [AIEEE 2002]

(a) 21 (b) 25
(c) 26 (d) None of these

44. If $|x| < 1$, then the coefficient of x^n in expansion of $(1 + x + x^2 + x^3 + \ldots)^2$ is [AIEEE 2002]

(a) n (b) $n - 1$ (c) $n + 2$ (d) $n + 1$

Answers

Check Point 1

1. (a)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (a)
11. (d)									

Check Point 2

1. (b)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (a)

Round I

1. (b)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (c)
11. (b)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (a)	**20.** (b)
21. (b)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (a)	**30.** (c)
31. (a)	**32.** (b)	**33.** (c)	**34.** (c)	**35.** (b)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (c)	**40.** (b)
41. (c)	**42.** (a)	**43.** (d)	**44.** (b)	**45.** (c)	**46.** (d)	**47.** (c)	**48.** (b)	**49.** (b)	**50.** (b)
51. (c)	**52.** (c)	**53.** (d)	**54.** (a)	**55.** (a)	**56.** (b)	**57.** (a)	**58.** (a)	**59.** (b)	**60.** (b)
61. (c)	**62.** (c)	**63.** (a)	**64.** (a)	**65.** (d)	**66.** (d)	**67.** (b)	**68.** (b)	**69.** (b)	**70.** (c)
71. (d)	**72.** (d)	**73.** (c)	**74.** (b)	**75.** (c)	**76.** (b)	**77.** (a)			

Round II

1. (c)	**2.** (b)	**3.** (a)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (a,d)
11. (a,b,c)	**12.** (a,b,c)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (d)
31. (b)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (b)
41. (b)	**42.** (c)	**43.** (d)	**44.** (d)						

the **Guidance**

Round I

1. In $(x+a)^{100} + (x-a)^{100}$ n is even.

$\therefore$ Total number of terms $= \frac{n}{2} + 1 = \frac{100}{2} + 1 = 51$

2. Given that, $(1+ax)^n = 1 + 8x + 24x^2 + \ldots$

$\Rightarrow \; 1 + \frac{n}{1}ax + \frac{n(n-1)}{1\cdot 2}a^2x^2 + \ldots = 1 + 8x + 24x^2 + \ldots$

On comparing the coefficients of x, x^2, we get

$$na = 8, \frac{n(n-1)}{1\cdot 2}a^2 = 24$$

$\Rightarrow \quad na(n-1)a = 48$

$\Rightarrow \quad 8(8-a) = 48$

$\Rightarrow \quad 8 - a = 6$

$\Rightarrow \quad a = 2$

$\Rightarrow \quad n = 4$

3. $(0.99)^5 = (1-0.01)^5$

$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2$

(ignore the other terms)

$= 1 - 5\times 1\times 0.01 + \frac{5\times 4}{2}\times 1\times 0.01\times 0.01$

$= 1 - 0.05 + 10\times 0.0001 = 1 - 0.05 + 0.001$

$= 1.001 - 0.05 = 0.951$

4. Coefficient of x^n in $(1+x)^{2n} = {}^{2n}C_n$

and coefficient of x^n in $(1+x)^{2n-1} = {}^{2n-1}C_n$

$\therefore$ Required ratio $= \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}} = 2:1$

5. We have, $(1.002)^{12}$ or it can be rewritten as $(1+0.002)^{12}$

$\Rightarrow \quad (1.002)^{12} = 1 + {}^{12}C_1(0.002) + {}^{12}C_2(0.002)^2 + {}^{12}C_3(0.002)^3 + \ldots$

We want the answer upto 4 decimal places and as such we have left further expansion.

$\therefore \quad (1.002)^{12} = 1 + 12(0.002) + \frac{12\cdot 11}{1\cdot 2}(0.002)^2 + \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}(0.002)^3 + \ldots$

$= 1 + 0.024 + 2.64\times 10^{-4} + 1.76\times 10^{-6} + \ldots$

$= 1.0242$

6. $(1+x+x^2+x^3)^n = \{(1+x)^n (1+x^2)^n\}$

$$= (1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n)$$
$$(1 + {}^nC_1 x^2 + {}^nC_2 x^4 + \ldots + {}^nC_n x^{2n})$$

Therefore, the coefficient of $x^4 = {}^nC_2 + {}^nC_2 {}^nC_1 + {}^nC_4$

$$= {}^nC_4 + {}^nC_2 + {}^nC_1 {}^nC_2$$

7. The given expression is

$1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$ being in GP.

Let $S = 1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$

$$= \frac{(1+x)^{n+1} - 1}{(1+x) - 1} = x^{-1}[(1+x)^{n+1} - 1]$$

$\therefore$ The coefficient of x^k in S.

= The coefficient of x^{k+1} in $[(1+x)^{n+1} - 1]$

$= {}^{n+1}C_{k+1}$

8. $(1+x)^{21} + (1+x)^{22} + \ldots + (1+x)^{30}$

$$= (1+x)^{21} [1 + (1+x)^1 + \ldots + (1+x)^9]$$
$$= (1+x)^{21} \left[\frac{(1+x)^{10} - 1}{(1+x) - 1}\right]$$
$$= \frac{1}{x}[(1+x)^{31} - (1+x)^{21}]$$

$\therefore$ Coefficient of x^5 in the given expression

= Coefficient of x^5 in $\frac{1}{x}[(1+x)^{31} - (1+x)^{21}]$

= Coefficient of x^6 in $[(1+x)^{31} - (1+x)^{21}]$

$= {}^{31}C_6 - {}^{21}C_6$

9. We have, $(1+t^2)^{12} (1+t^{12}) (1+t^{24})$

$$= (1 + {}^{12}C_1 t^2 + {}^{12}C_2 t^4 + \ldots + {}^{12}C_6 t^{12}$$
$$+ \ldots + {}^{12}C_{12} t^{24} + \ldots)(1 + t^{12} + t^{24} + t^{36})$$

$\therefore$ Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$$= {}^{12}C_6 + {}^{12}C_{12} + 1 = {}^{12}C_6 + 2$$

10. The given sigma expansion

$\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m$ can be written as

$$[(x-3) + 2]^{100} = (x-1)^{100} = (1-x)^{100}$$

$\therefore$ Coefficient of x^{53} in $(1-x)^{100} = (-1)^{53}\ {}^{100}C_{53}$

$$= -{}^{100}C_{53}$$

11. We have, $(x-1)(x-2)(x-3)\ldots(x-100)$

Number of terms $= 100$

$\therefore$ Coefficient of x^{99} in $(x-1)(x-2)(x-3)\ldots(x-100)$

$$= (-1-2-3-\ldots-100)$$
$$= -(1+2+\ldots+100) = -\frac{100 \times 101}{2} = -5050$$

12. $(1+x)^m (1-x)^n = \left\{1 + mx + \frac{m(m-1)x^2}{2!} + \ldots\right\}$

$$\left[1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots\right]$$

$$= 1 + (m-n)x + \left[\frac{n^2-n}{2} - mn + \frac{(m^2-m)}{2}\right]x^2 + \ldots$$

Given, $m - n = 3 \Rightarrow n = m - 3$

and $\frac{n^2-n}{2} - mn + \frac{m^2-m}{2} = -6$

$\Rightarrow \frac{(m-3)(m-4)}{2} - m(m-3) + \frac{m^2-m}{2} = -6$

$\Rightarrow m^2 - 7m + 12 - 2m^2 + 6m + m^2 - m + 12 = 0$

$\Rightarrow -2m + 24 = 0 \Rightarrow m = 12$

13. In the expansion of $(1+x)^{2n}$, the general term

$$= {}^{2n}C_k x^k, 0 \le k \le 2n$$

As given for $r > 1, n > 2$, ${}^{2n}C_{3r} = {}^{2n}C_{r+2}$

$\Rightarrow$ Either $3r = r + 2$ or $3r = 2n - (r+2)$ $(\because {}^nC_r = {}^nC_{n-r})$

$\Rightarrow r = 1$ or $n = 2r + 1$

We take the relation only

$n = 2r + 1$ $(\because r > 1)$

14. Let $(r+1)$th, $(r+2)$th and $(r+3)$th be three consecutive terms.

Then, ${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1 : 7 : 42$

Now, $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{1}{7} \Rightarrow \frac{r+1}{n-r} = \frac{1}{7} \Rightarrow n - 8r = 7$ ...(i)

and $\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = \frac{7}{42}$

$\Rightarrow \frac{r+2}{n-r-1} = \frac{1}{6}$

$\Rightarrow n - 7r = 13$ ...(ii)

On solving Eqs. (i) and (ii), we get $n = 55$

15. $(19)^{2005} + (11)^{2005} - (9)^{2005}$

$$= (10+9)^{2005} + (10+1)^{2005} - (9)^{2005}$$
$$= \{9^{2005} + {}^{2005}C_1 (9)^{2004} \times 10 + \ldots\}$$
$$+ ({}^{2005}C_0 + {}^{2005}C_1 10 + \ldots) - (9)^{2005}$$

$= ({}^{2005}C_1 9^{2004} \times 10 + \text{multiple of } 10) + (1 + \text{multiple of } 10)$

$\therefore$ Unit digit $= 1$

16. Let a_1, a_2, a_3, a_4 be respectively the coefficients of $(r+1)$th, $(r+2)$th, $(r+3)$th and $(r+4)$th terms in the expansion of $(1+x)^n$. Then,

$a_1 = {}^nC_r, a_2 = {}^nC_{r+1}, a_3 = {}^nC_{r+2}, a_4 = {}^nC_{r+3}$

Now, $\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} + \frac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}}$

$$= \frac{{}^nC_r}{{}^{n+1}C_{r+1}} + \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} \qquad (\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$$

$$= \frac{{}^nC_r}{\frac{n+1}{r+1}\,{}^nC_r} + \frac{{}^nC_{r+2}}{\frac{n+1}{r+3}\,{}^nC_{r+2}} \qquad \left(\because {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}\right)$$

$$= \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1}$$

$$= 2\frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}} = 2\frac{{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{2a_2}{a_2 + a_3}$$

17. Coefficients of pth, $(p+1)$th and $(p+2)$th terms in the expansion $(1+x)^n$ are ${}^nC_{p-1}$, ${}^nC_p$, ${}^nC_{p+1}$, respectively.

Since, these are in AP.

$$\therefore \qquad 2\,{}^nC_p = {}^nC_{p-1} + {}^nC_{p+1}$$

$$\Rightarrow 2\frac{n!}{(n-p)!\,p!} = \frac{n!}{(n-p+1)!\,(p-1)!} + \frac{n!}{(n-p-1)!\,(p+1)!}$$

$$\Rightarrow \frac{2}{(n-p)!\,p!} = \frac{p}{(n-p+1)\,(n-p)!\,p!} + \frac{n-p}{(n-p)!\,(p+1)\,p!}$$

$$\Rightarrow \qquad \frac{2}{1} = \frac{p}{(n-p+1)} + \frac{n-p}{p+1}$$

$$\Rightarrow \qquad n^2 - n(4p+1) + 4p^2 - 2 = 0$$

18. Given that, $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \ldots + a_{12}x^{12}$

On putting $x = 1$ and $x = -1$ and adding the results, we get

$$64 = 2(1 + a_2 + a_4 + \ldots + a_{12})$$

$$\therefore \qquad a_2 + a_4 + a_6 + \ldots + a_{12} = 31$$

19. $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$

On putting $x = 1$, we get

$$(1 - 1 + 1)^n = a_0 + a_1 + a_2 + \ldots + a_{2n}$$

$$\Rightarrow \qquad 1 = a_0 + a_1 + a_2 + \ldots + a_{2n} \qquad \ldots(\text{i})$$

Again, putting $x = -1$, we get

$$3^n = a_0 - a_1 + a_2 - \ldots + a_{2n} \qquad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get

$$\frac{3^n + 1}{2} = a_0 + a_2 + a_4 + \ldots + a_{2n}$$

20. General term in the expansion of $(x - 2y)^{12}$ is

$$T_{r+1} = {}^{12}C_r(x)^{12-r}(-2y)^r$$

On putting $r = 3$, we get

$$T_{3+1} = {}^{12}C_3x^{12-3}(-2y)^3$$

$$T_4 = \frac{12 \times 11 \times 10}{6}x^9(-2)^3y^3$$

$$= 2 \times 11 \times 10x^9(-8)y^3$$

$$= 220 \times (-8)x^9y^3 = -1760x^9y^3$$

21. The rth term of $(a + 2x)^n$ is ${}^nC_{r-1}(a)^{n-r+1}(2x)^{r-1}$

$$= \frac{n!}{(n-r+1)!\,(r-1)!}a^{n-r+1}(2x)^{r-1}$$

$$= \frac{n(n-1)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}(2x)^{r-1}$$

22. $(1-x)^{-4} = 1 \cdot x^0 + 4x^1 + \frac{4 \cdot 5}{2}x^2 + \ldots$

$$= \left[\frac{1 \cdot 2 \cdot 3}{6}x^0 + \frac{2 \cdot 3 \cdot 4}{6}x + \frac{3 \cdot 4 \cdot 5}{6}x^2 + \frac{4 \cdot 5 \cdot 6}{6}x^3 + \ldots + \frac{(r+1)(r+2)(r+3)}{6}x^r + \ldots\right]$$

Therefore, $\quad T_{r+1} = \frac{(r+1)(r+2)(r+3)}{6}x^r$

23. In the expansion of $(1 + a)^{m+n}$, the general term is

$$T_{r+1} = {}^{m+n}C_r(a)^r$$

For coefficient of a^m, putting $r = m$

Then, $\quad T_{m+1} = {}^{m+n}C_m(a)^m$

$\Rightarrow$ Coefficient of $a^m = {}^{m+n}C_m \qquad \ldots(\text{i})$

For coefficient of a^n, putting $r = n$, then

$$T_{n+1} = {}^{m+n}C_na^n$$

$\Rightarrow$ Coefficient of $a^n = {}^{m+n}C_n$

$$= {}^{m+n}C_{m+n-n} \qquad (\because {}^nC_r = {}^nC_{n-r})$$

$$= {}^{m+n}C_m \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), the coefficient of a^m is equal to coefficient of a^n.

24. The general term in the expansion of $(1 + x)^{2n}$ is

$$T_{r+1} = {}^{2n}C_rx^r$$

$$T_{3r} = {}^{2n}C_{3r-1}x^{3r-1}$$

and $\quad T_{r+2} = {}^{2n}C_{r+1}x^{r+1}$

Now, $\quad {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$$\Rightarrow \qquad 3r - 1 + r + 1 = 2n \Rightarrow n = 2r$$

25. The general term in the expansion of $(3 + ax)^9$ is

$$T_{r+1} = {}^9C_r3^{9-r}a^rx^r$$

For coefficient of x^2, put $r = 2$

$$T_{2+1} = {}^9C_23^{9-2}a^2x^2$$

$\Rightarrow$ Coefficient of $x^2 = {}^9C_23^7a^2 \qquad \ldots(\text{i})$

For coefficient of x^3, put $r = 3$

$$T_{3+1} = {}^9C_33^{9-3}a^3x^3 = {}^9C_33^6a^3x^3$$

$\Rightarrow$ Coefficient of $x^3 = {}^9C_33^6a^3 \qquad \ldots(\text{ii})$

Given, coefficient of x^2 = coefficient of x^3

$$\Rightarrow \qquad {}^9C_23^7a^2 = {}^9C_33^6a^3 \quad [\text{from Eqs. (i) and (ii)}]$$

$\Rightarrow \quad \frac{9\times 8}{2}\times 3\times 1=\frac{9\times 8\times 7}{6}\times 1\times a$

$\Rightarrow \quad \frac{3}{2}=\frac{7}{6}\times a \Rightarrow a=\frac{9}{7}$

26. Given expression is $(1+x)^{24}$.

Let two successive terms are $(r+1)$th and $(r+2)$th terms

$\Rightarrow \quad T_{r+1}={}^{24}C_r x^r$ and $T_{r+2}={}^{24}C_{r+1}x^{r+1}$

Now, ratio of coefficients $=\frac{1}{4}$

$\Rightarrow \quad \frac{{}^{24}C_r}{{}^{24}C_{r+1}}=\frac{1}{4}$

$\Rightarrow \quad \frac{r+1}{24-r}=\frac{1}{4}\Rightarrow 4r+4=24-r$

$\Rightarrow \quad r=4$

$\therefore$ Required terms are 5th and 6th.

27. Given expression is $(1+x)^n$. Coefficients of 2nd, 3rd and 4th terms are ${}^nC_1$, ${}^nC_2$ and ${}^nC_3$, respectively.

Since, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$ are in AP.

$\Rightarrow \quad 2{}^nC_2={}^nC_1+{}^nC_3$

$\Rightarrow \quad 2=\frac{{}^nC_1}{{}^nC_2}+\frac{{}^nC_3}{{}^nC_2}$

$\Rightarrow \quad 2=\frac{2}{n-1}+\frac{n-2}{3}$

$\Rightarrow \quad n^2-9n+14=0\Rightarrow n=2,7$

But $\quad n\neq 2$

$\therefore \quad n=7$

28. Here, $T_4={}^nC_3(a)^{n-3}(-2b)^3$

and $\quad T_5={}^nC_4(a)^{n-4}(-2b)^4$

Given, $\quad T_4+T_5=0$

$\Rightarrow \quad {}^nC_3(a)^{n-3}(-2b)^3+{}^nC_4(a)^{n-4}(-2b)^4=0$

$\Rightarrow \quad (a)^{n-4}(-2b)^3[a\,{}^nC_3+{}^nC_4(-2b)]=0$

$\Rightarrow \quad \frac{a}{b}=\frac{2\,{}^nC_4}{{}^nC_3}$

$=\frac{2\cdot n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1}\times\frac{3\cdot 2\cdot 1}{n(n-1)(n-2)}=\frac{n-3}{2}$

29. Given expression is $(x+x^{\log_{10}x})^5$.

$\therefore \quad T_3={}^5C_2\cdot x^3(x^{\log_{10}x})^2=10^6$ (given)

Put $x=10$, then $10^4\cdot 10^2=10^6$ is satisfied. Hence, $x=10$

30. Given that,

$T_1={}^nC_0=1$ ...(i)

$T_2={}^nC_1ax=6x$

$\Rightarrow \quad \frac{n!}{(n-1)!}a=6$

$\Rightarrow \quad na=6$ ...(ii)

and $\quad T_3={}^nC_2(ax)^2=16x^2$

$\Rightarrow \quad \frac{n(n-1)}{2}a^2=16$ ...(iii)

Only option (c) is satisfying Eqs. (ii) and (iii).

31. Last term of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is

$T_{n+1}={}^nC_n(2^{1/3})^{n-n}\left(-\frac{1}{\sqrt{2}}\right)^n$

$={}^nC_n(-1)^n\frac{1}{2^{n/2}}=\frac{(-1)^n}{2^{n/2}}$

Also, we have

$\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}=3^{-(5/3)\log_3 2^3}=2^{-5}$

Thus, $\quad \frac{(-1)^n}{2^{n/2}}=2^{-5}\Rightarrow\frac{(-1)^n}{2^{n/2}}=\frac{(-1)^{10}}{2^5}$

$\Rightarrow \quad \frac{n}{2}=5 \quad\Rightarrow\quad n=10$

Now, $\quad T_5=T_{4+1}={}^{10}C_4(2^{1/3})^{10-4}\left(-\frac{1}{\sqrt{2}}\right)^4$

$=\frac{10!}{4!6!}(2^{1/3})^6(-1)^4(2^{-1/2})^4$

$=210(2)^2(1)(2^{-2})=210$

32. The general term in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$ is given by

$T_{r+1}={}^{n-3}C_r(x)^{n-3-r}\left(\frac{1}{x^2}\right)^r={}^{n-3}C_r\,x^{n-3-3r}$

As x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, we must have $n-3-3r=2k$ for some non-negative integer r.

$\Rightarrow \quad 3(1+r)=n-2k$

$\Rightarrow n-2k$ is a multiple of 3.

33. The general term in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is

$T_{r+1}={}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

$={}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$ ...(i)

Now, the coefficients of the terms x^0, x^{-1} and x^{-3} in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is

For x^0, $\quad 18-3r=0 \quad\Rightarrow\quad r=6$

For x^{-1}, there exists no integer value of r

For x^{-3}, $\quad 18-3r=-3 \quad\Rightarrow\quad r=7$

Now, the coefficient of the term independent of x in the expansion of

$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$=1\cdot {}^9C_6(-1)^6\left(\frac{3}{2}\right)^{9-6}\left(\frac{1}{3}\right)^6+0+2\cdot {}^9C_7(-1)^7\left(\frac{3}{2}\right)^{9-7}\left(\frac{1}{3}\right)^7$

$=\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{3^3}{2^3}\cdot\frac{1}{3^6}+2\cdot\frac{9\cdot 8}{1\cdot 2}(-1)\frac{3^2}{2^2}\cdot\frac{1}{3^7}$

$=\frac{7}{18}-\frac{2}{27}=\frac{17}{54}$

34. General term of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is

$$T_{r+1}={}^{10}C_r(\sqrt{x})^{10-r}\left(-\frac{k}{x^2}\right)^r$$

$$={}^{10}C_r\,x^{\frac{10-r}{2}-2r}(-k)^r={}^{10}C_r(-k)^r x^{\frac{10-5r}{2}-}$$

The term free from x,

Put $\frac{10-5r}{2}=0\Rightarrow r=2$

$\Rightarrow$ ${}^{10}C_2(-k)^2=405$

$\Rightarrow$ $\frac{10\times 9}{2}\times k^2=405$

$\therefore$ $k=\pm 3$

35. $(1-3x+7x^2)(1-x)^{16}$

$=(1-3x+7x^2)({}^{16}C_0-{}^{16}C_1x+{}^{16}C_2x^2+\ldots)$

After multiplying, the term containing x is $-{}^{16}C_1x-3x^{16}C_0x$

$\therefore$ Coefficient of $x=-16-3=-19$

36. The general term of $\left(3x-\frac{2}{x^2}\right)^{15}$ is

$$T_{r+1}={}^{15}C_r(3x)^{15-r}\left(\frac{-2}{x^2}\right)^r$$

$$={}^{15}C_r(3)^{15-r}(-2)^r x^{15-3r}$$

For independent term of x,

Put $15-3r=0\Rightarrow r=5$

$\therefore$ Coefficient $={}^{15}C_5(3)^{15-5}(-2)^5=-3003(3^{10})(2^5)$

37. Given expression is $\left(x+\frac{1}{2x}\right)^{2n}$.

$\therefore$ Middle term $={}^{2n}C_n(x)^n\left(\frac{1}{2x}\right)^n$

$$=\frac{2n!}{n!n!2^n}=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}$$

38. $\left(\frac{1}{x}+x\sin x\right)^{10}$

Here, $n=10$ (even)

$\Rightarrow$ Middle term $=\left(\frac{10}{2}+1\right)$th $=6$th

$$T_6={}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5$$

$\Rightarrow$ $252(\sin x)^5=\frac{63}{8}$

$\Rightarrow$ $(\sin x)^5=\frac{1}{32}$

$\Rightarrow$ $\sin x=\frac{1}{2}\Rightarrow \sin x=\sin\pi/6$

$\therefore$ $x=n\pi+(-1)^n\frac{\pi}{6}$

39. Given expression is $\left(\frac{p}{2}+2\right)^8$

Here, $n=8$ (even)

$\Rightarrow$ Middle term $=\left(\frac{8}{2}+1\right)$th $=5$th

$$T_5={}^8C_4(p/2)^{8-4}(2)^4$$

$\Rightarrow$ $\frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}\times\frac{p^4}{2^4}\times 2^4=1120$

$\Rightarrow$ $P^4=16\therefore p=\pm 2$

40. Since, n is even, therefore $\left(\frac{n}{2}+1\right)$th term is the middle term.

$\therefore$ $T_{\frac{n}{2}+1}={}^nC_{n/2}(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2}=924x^6$ (given)

$\Rightarrow$ $x^{n/2}=x^6\Rightarrow n=12$

41. $\because$ $(3+2x)^{50}=3^{50}\left(1+\frac{2x}{3}\right)^{50}$

Here, $T_{r+1}=3^{50}\,{}^{50}C_r\left(\frac{2x}{3}\right)^r$

and $T_r=3^{50}\,{}^{50}C_{r-1}\left(\frac{2x}{3}\right)^{r-1}$

But $x=\frac{1}{5}$ (given)

$\therefore$ $\frac{T_{r+1}}{T_r}\geq 1\Rightarrow\frac{{}^{50}C_r}{{}^{50}C_{r-1}}\frac{2}{3}\cdot\frac{1}{5}\geq 1$

$\Rightarrow$ $102-2r\geq 15r\Rightarrow r\leq 6$

42. Sum of the coefficients in the expansion of $(x-2y+3z)^n$ is $(1-2+3)^n=2^n$ (put $x=y=z=1$)

$\therefore$ $2^n=128\Rightarrow n=7$

Therefore, the greatest coefficient in the expansion of $(1+x)^7$ is ${}^7C_3$ or ${}^7C_4$ because both are equal to 35.

43. The general term in the expansion of $\left(x\cos\alpha+\frac{\sin\alpha}{x}\right)^{20}$ is

${}^{20}C_r(x\cos\alpha)^{20-r}\left(\frac{\sin\alpha}{x}\right)^r={}^{20}C_rx^{20-2r}(\cos\alpha)^{20-r}(\sin\alpha)^r$

For this term to be independent of x, we get

$20-2r=0\Rightarrow r=10$

Let $\beta =$ Term independent of x

$= {}^{20}C_{10}(\cos\alpha)^{10}(\sin\alpha)^{10}$

$= {}^{20}C_{10}(\cos\alpha\sin\alpha)^{10} = {}^{20}C_{10}\left(\frac{\sin 2\alpha}{2}\right)^{10}$

Thus, the greatest possible value of β is ${}^{20}C_{10}\left(\frac{1}{2}\right)^{10}$.

44. Here, the greatest coefficient is ${}^{2n}C_n$.

$\therefore \quad {}^{2n}C_n x^n > {}^{2n}C_{n+1}x^{n-1} \Rightarrow x > \frac{n}{n+1}$

and $\quad {}^{2n}C_n x^n > {}^{2n}C_{n-1}x^{n+1} \Rightarrow x < \frac{n+1}{n}$

$\therefore$ x must lie in the interval $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

45. Let $b = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-(n-r)}{{}^nC_r}$

$= n\sum_{r=0}^{n}\frac{1}{{}^nC_r} - \sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$

$= na_n - \sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}} \qquad (\because {}^nC_r = {}^nC_{n-1})$

$= n\,a_n - b$

$\Rightarrow \quad 2b = na_n \Rightarrow b = \frac{n}{2}a_n$

46. Let $\quad E = \sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1+rx}{1+nx}\right)$

$= \left(\frac{1}{1+nx}\right)\sum_{r=0}^{n}(-1)^r\,{}^nC_r(1+rx)$

$= \left(\frac{1}{1+nx}\right)\left\{\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r + x\sum_{r=0}^{n}r(-1)^r\,{}^nC_r\right\}$

$= \left(\frac{1}{1+nx}\right)(0+0) = 0$

$[\because {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots(-1)^n\,{}^nC_n = 0]$

47. Let

$A = \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$

or $A = {}^{30}C_0\cdot{}^{30}C_{10} - {}^{30}C_1\cdot{}^{30}C_{11}$

$+ {}^{30}C_2\cdot{}^{30}C_{12} - \ldots + {}^{30}C_{20}\cdot{}^{30}C_{30}$

$=$ Coefficient of x^{20} in $(1+x)^{30}(1-x)^{30}$

$=$ Coefficient of x^{20} in $(1-x^2)^{30}$

$=$ Coefficient of x^{20} in $\sum_{r=0}^{30}(-1)^r\,{}^{30}C_r(x^2)^r$

$= (-1)^{10}\,{}^{30}C_{10} \qquad$ (for coefficient of x^{20}, let $r = 10$)

$= {}^{30}C_{10}$

48. We have,

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n \qquad \ldots$(i)

and $\quad \left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\left(\frac{1}{x}\right)^2 + \ldots + C_n\left(\frac{1}{x}\right)^n \qquad \ldots$(ii)

On multiplying Eqs. (i) and (ii) and taking the coefficient of constant terms in right hand side

$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

In right hand side $(1+x)^n\left(1+\frac{1}{x}\right)^n$ or in $\frac{1}{x^n}(1+x)^{2n}$ or term containing x^n in $(1+x)^{2n}$. Clearly, the coefficient of x^n in $(1+x)^{2n}$ is equal to ${}^{2n}C_n = \frac{(2n)!}{n!n!}$.

49. We know that,

$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

On putting $n = 15$, then $\frac{15\times(15+1)}{2} = 15\times 8 = 120$

50. Given equation can be rewritten as

$E = a[{}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\,{}^nC_n]$

$+ [{}^nC_1 - (2)({}^nC_2) + (3)({}^nC_3) - \ldots + (-1)^n(n)({}^nC_n)]$

$\Rightarrow \qquad E = 0 + 0 = 0 \qquad$ (by properties)

51. Given, $\quad {}^nC_0 - \frac{1}{2}\,{}^nC_1 + \frac{1}{3}\,{}^nC_2 - \ldots + (-1)^n\frac{{}^nC_n}{n+1}$

At $n = 1$, ${}^1C_0 - \frac{1}{2}\,{}^1C_1 = 1 - \frac{1}{2} = \frac{1}{2}$

At $n = 2$, ${}^2C_0 - \frac{1}{2}\,{}^2C_1 + \frac{1}{3}\,{}^2C_2 = 1 - 1 + \frac{1}{3} = \frac{1}{3}$

Which is satisfied only in option (c).

52. We have, $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$

$= \left(1+\frac{n}{1}\right)\left[1+\frac{\frac{n(n-1)}{2!}}{n}\right]\ldots\left(1+\frac{1}{n}\right)$

$= \frac{(1+n)}{1}\cdot\frac{(1+n)}{2}\cdot\frac{(1+n)}{3}\ldots\frac{(1+n)}{n} = \frac{(n+1)^n}{n!}$

53. We have, $C_0^2 - 2C_1^2 + 3C_2^2 - \ldots + (-1)^n(n+1)C_n^2$

$= [C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^nC_n^2]$

$- [C_1^2 - 2C_2^2 + 3C_3^2 - \ldots + (-1)^n n C_n^2]$

$= (-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\cdot -(-1)^{n/2-1}\cdot\frac{1}{2}n\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$

$= (-1)^{n/2}\cdot\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\cdot\left(1+\frac{n}{2}\right)$

Therefore, the value of the given expression is

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\times(-1)^{n/2}\cdot\frac{(n)!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\left(1+\frac{n}{2}\right)=(-1)^{n/2}(2+n)$$

54. We know that,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$$

$$(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - \ldots + (-1)^nC_n$$

On multiplying both equations and equating the coefficient of x^n, we get

$$C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2 = {}^nC_{n/2}(-1)^{n/2}(x^2)^{n/2}$$

Above is possible only when $\frac{n}{2}$ is an integer *i.e.*, n is even and in case n is odd, then term x^n will not occur.

55. Since, $x(1+x)^n = xC_0 + C_1x^2 + C_2x^3 + \ldots + C_nx^{n+1}$

On differentiating w.r.t. x, we get

$$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + \ldots + (n+1)C_nx^n$$

Put $x = 1$, we get

$$C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = 2^n + n2^{n-1} = 2^{n-1}(n+2)$$

56. Let $f = (8-3\sqrt{7})^{10}$, here $0 < f < 1$

$\therefore$ $(8+3\sqrt{7})^{10} + (8-3\sqrt{7})^{10}$ is an integer, hence this is the value of n.

57. Given that, $R = (2+\sqrt{3})^{2n}$ and $f = R - [R]$

As $0 < 2-\sqrt{3} < 1$, we get $0 < F = (2-\sqrt{3})^{2n} < 1$

We have, $R + F = (2+\sqrt{3})^{2n} + (2-\sqrt{3})^{2n}$

$$= 2[{}^{2n}C_0 2^{2n} + {}^{2n}C_2 2^{2n-2}(\sqrt{3})^2 + {}^{2n}C_4(2^{2n-4})(\sqrt{3})^4 + \ldots + {}^{2n}C_{2n}(\sqrt{3})^{2n}]$$

$\Rightarrow$ $R + F$ is an even integer.

$\Rightarrow$ $[R] + f + F$ is an even integer.

$\Rightarrow$ $f + F$ is an integer.

But $0 \le f < 1$ and $0 < F < 1$

$\Rightarrow$ $0 < f + F < 2$

But the only integer between 0 and 2 is 1. Thus,

$f + F = 1 \Rightarrow 1 - f = F$

Now, $R(1-f) = RF = (2+\sqrt{3})^{2n}(2-\sqrt{3})^{2n}$

$$= (4-3)^{2n} = 1^{2n} = 1$$

58. Let us take $a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n} = (1+x+x^2)^n$

On differentiating both sides w.r.t. x, we get

$$a_1 + 2a_2x + \ldots + 2na_{2n}x^{2n-1} = n(1+x+x^2)^{n-1}(2x+1)$$

Put $x = -1$

$\Rightarrow$ $a_1 - 2a_2 + 3a_3 - \ldots - 2na_{2n} = -n$

59. We have,

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{2n}x^{2n}$$

On differentiating both sides, we get

$$n(1+x+x^2)^{n-1}(1+2x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + 2na_{2n}x^{2n-1}$$

On putting $x = -1$, we get

$$n(1-1+1)^{n-1}(1-2) = a_1 - 2a_2 + 3a_3 - \ldots - 2na_{2n}$$

$\Rightarrow$ $a_1 - 2a_2 + 3a_3 - \ldots - 2na_{2n} = -n$

60. We have, $(1+x)^{15} = C_0 + C_1x + C_2x^2 + \ldots + C_{15}x^{15}$

$$\Rightarrow \frac{(1+x)^{15}-1}{x} = C_1 + C_2x + \ldots + C_{15}x^{14}$$

On differentiating both sides w.r.t. x, we get

$$\frac{x\cdot 15(1+x)^{14} - (1+x)^{15} + 1}{x^2} = C_2 + 2C_3x + \ldots + 14C_{15}x^{13}$$

On putting $x = 1$, we get

$$C_2 + 2C_3 + \ldots + 14C_{15} = 15\cdot 2^{14} - 2^{15} + 1 = 13\cdot 2^{14} + 1$$

61. We can write,

$aC_0 - (a+d)C_1 + (a+2d)C_2 - \ldots$ upto $(n+1)$ terms

$= a(C_0 - C_1 + C_2 - \ldots) + d(-C_1 + 2C_2 - 3C_3 + \ldots)$ ...(i)

We know,

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^nC_nx^n \quad \text{...(ii)}$$

On differentiating Eq. (ii) w.r.t. x, we get

$$-n(1-x)^{n-1} = -C_1 + 2C_2x - \ldots + (-1)^nC_n nx^{n-1} \quad \text{...(iii)}$$

On putting $x = 1$ in Eqs. (ii) and (iii), we get

$$C_0 - C_1 + C_2 - \ldots + (-1)^nC_n = 0 \quad \text{...(iv)}$$

and $$-C_1 + 2C_2 - \ldots + (-1)^n nC_n = 0 \quad \text{...(v)}$$

From Eq. (i),

$aC_0 - (a+d)C_1 + (a+2d)C_2 - \ldots$ upto $(n+1)$ terms

$= a\cdot 0 + d\cdot 0 = 0$ [from Eqs. (iv) and (v)]

62. We have, $49^n + 16n - 1 = (1+48)^n + 16n - 1$

$$= 1 + {}^nC_1(48) + {}^nC_2(48)^2 + \ldots + {}^nC_n(48)^n + 16n - 1$$

$$= (48n + 16n) + {}^nC_2(48)^2 + {}^nC_3(48)^3 + \ldots + {}^nC_n(48)^n$$

$$= 64n + 8^2[{}^nC_2\cdot 6^2 + {}^nC_3\cdot 6^3\cdot 8 + {}^nC_4\cdot 6^4 8^2 + \ldots + {}^nC_n\cdot 6^n\cdot 8^{n-2}]$$

Hence, $49^n + 16n - 1$ is divisible by 64.

63. Since, $\left(1+\frac{1}{n}\right)^n < 3$ for $\forall\, n \in N$

Now, $$\frac{(1001)^{999}}{(1000)^{1000}} = \frac{1}{1001}\cdot\left(\frac{1001}{1000}\right)^{1000}$$

$$= \frac{1}{1001}\left(1+\frac{1}{1000}\right)^{1000} < \frac{1}{1001}\cdot 3 < 1$$

$$(1001)^{999} < (1000)^{1000}$$

$\therefore$ $B < A$

64. Cube root of $217 = (217)^{1/3} = (6^3 + 1)^{1/3} = 6\left(1+\frac{1}{6^3}\right)^{1/3}$

$$\therefore\ 6\left(1+\frac{1}{6^3}\right)^{1/3} = 6\left[1+\frac{1}{3\times 216}-\frac{1\times 2}{3\times 3\times 2}\left(\frac{1}{216}\right)^2+\ldots\right]$$

$$= 6\left(1+\frac{1}{648}\right) \qquad \text{(neglect the other terms)}$$

$$= 6.01$$

65. $\because \frac{1}{(x-1)^2(x-2)} = \frac{1}{-2(1-x)^2\left(1-\frac{x}{2}\right)}$

$$= -\frac{1}{2}\left[(1-x)^{-2}\left(1-\frac{x}{2}\right)^{-1}\right]$$

$$= -\frac{1}{2}\left[(1+2x+\ldots)\left(1+\frac{x}{2}+\ldots\right)\right]$$

$\therefore$ Coefficient of constant term is $-\frac{1}{2}$.

66. The given expression of $\frac{1}{(4-3x)^{1/2}}$ can be rewritten as

$4^{-1/2}\left(1-\frac{3}{4}x\right)^{-1/2}$ and it is valid only when $\left|\frac{3}{4}x\right| < 1$

$$\Rightarrow \qquad -\frac{4}{3} < x < \frac{4}{3}$$

67. $\frac{1}{(6-3x)^{1/3}} = (6-3x)^{-1/3} = 6^{-1/3}\left[1-\frac{x}{2}\right]^{-1/3}$

$$= 6^{-1/3}\left[1+\left(-\frac{1}{3}\right)\left(-\frac{x}{2}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2\cdot 1}\left(-\frac{x}{2}\right)^2+\ldots\right]$$

$$= 6^{-1/3}\left[1+\frac{x}{6}+\frac{2x^2}{6^2}+\ldots\right]$$

68. Given expression $\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$ can be rewritten as

$$\frac{(1+x)^{1/2}+(1-x)^{2/3}}{1+x+(1+x)^{1/2}}$$

$$= \frac{\left[1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right]+\left[1-\frac{2}{3}x-\frac{1}{9}x^2-\ldots\right]}{1+x+\left[1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right]}$$

$$= \frac{2-\frac{1}{6}x-\frac{17}{72}x^2+\ldots}{2+\frac{3}{2}x-\frac{1}{8}x^2+\ldots} = \frac{\left[1-\frac{1}{12}x-\frac{17}{144}x^2+\ldots\right]}{\left[1+\frac{3}{4}x-\frac{1}{16}x^2+\ldots\right]}$$

$$= \left[1-\frac{1}{12}x-\frac{17}{144}x^2+\ldots\right]\left[1+\frac{3}{4}x-\frac{1}{16}x^2+\ldots\right]^{-1}$$

$$= 1-\frac{5}{6}x+\ldots$$

$$= 1-\frac{5}{6}x \qquad \text{(on neglecting the higher powers of } x\text{)}$$

69. $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{2\left(1-\frac{x}{4}\right)^{1/2}}$

$$= \frac{\left[\left\{1+\frac{1}{2}(-3x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2+\ldots\right\} + \left\{1+\frac{5}{3}(-x)+\frac{5}{3}\cdot\frac{2}{3}\cdot\frac{1}{2}(-x)^2+\ldots\right\}\right]}{2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2+\ldots\right]}$$

$$= \frac{2\left(1-\frac{19}{12}x-\frac{41}{144}x^2-\ldots\right)}{2\left(1-\frac{x}{8}-\frac{1}{128}x^2-\ldots\right)}$$

$$= \left(1-\frac{19}{12}x-\frac{41}{144}x^2-\ldots\right)\left(1-\frac{x}{8}-\frac{1}{128}x^2\ldots\right)^{-1}$$

$$= 1-\frac{35}{24}x+\ldots$$

On neglecting higher powers of x, we get

$$a+bx = 1-\frac{35}{24}x \Rightarrow a = 1, \quad b = -\frac{35}{24}$$

70. The number of terms in the expansion of $(a+b+c)^n$

$$= \frac{(n+1)(n+2)}{2}$$

71. $x = (\sqrt{3}+1)^5 = (\sqrt{3})^5 + {}^5C_1(\sqrt{3})^4 + {}^5C_2(\sqrt{3})^3 + {}^5C_3(\sqrt{3})^2 + {}^5C_4(\sqrt{3}) + {}^5C_5$

$$= 9\sqrt{3}+45+30\sqrt{3}+30+5\sqrt{3}+1$$

$$= 76+44\sqrt{3}$$

$$\therefore \quad [x] = [(\sqrt{3}+1)^5] = (76+44\sqrt{3}) = (76)+(44\times 1.732)$$

$$= 76+(76.2) = 76+76 = 152$$

72. Let $1+\frac{1}{3}x+\frac{1\cdot 4}{3\cdot 6}x^2+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3+\ldots = (1+y)^n$

$$= 1+ny+\frac{n(n-1)}{2!}y^2+\ldots$$

On comparing the terms, we get

$$ny = \frac{1}{3}x, \quad \frac{n(n-1)}{2!}y^2 = \frac{1\cdot 4}{3\cdot 6}x^2$$

On solving, we get

$$n = -\frac{1}{3}, \quad y = -x$$

$\therefore$ Required expansion is $(1-x)^{-1/3}$.

73. Let $S = 1+\frac{2\cdot 1}{3\cdot 2}+\frac{2\cdot 5}{3\cdot 6}\left(\frac{1}{2}\right)^2+\frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}\left(\frac{1}{2}\right)^3+\ldots$

$$= 1+\frac{\frac{2}{3}}{1}\left(\frac{1}{2}\right)+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{2!}\left(\frac{1}{2}\right)^2+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)}{3!}\left(\frac{1}{2}\right)^3+\ldots$$

$$= \left(1-\frac{1}{2}\right)^{-2/3} = \left(\frac{1}{2}\right)^{-2/3} = 2^{2/3} = 4^{1/3}$$

$$\left[\because (1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\ldots\right]$$

74. Given that, $\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\ldots$

$$=\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}+\frac{n!}{3!(n-3)!}+\frac{n!}{5!(n-5)!}+\ldots\right]$$

$$=\frac{1}{n!}({}^nC_1+{}^nC_3+{}^nC_5+\ldots)=\frac{2^{n-1}}{n!}$$

75. There are total $(n+1)$ factors, let $P(x)=0$

Let $(x+{}^nC_0)(x+3\,{}^nC_1)(x+5\,{}^nC_2)\ldots[x+(2n+1)\,{}^nC_n]$

$$=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$$

Clearly, $a_n=1$ and roots of the equation $P(x)=0$ are $-{}^nC_0, -3\,{}^nC_1, \ldots$

Sum of roots $=-a_{n-1}/a_n=-{}^nC_0-3\,{}^nC_1-5\,{}^nC_2\ldots$

$\Rightarrow \quad a_{n-1}=(n+1)\,2^n$

76. According to given condition, $T_n={}^nC_3$

and $\quad T_{n+1}-T_n=21 \Rightarrow {}^{n+1}C_3-{}^nC_3=21$

$$\Rightarrow \frac{1}{6}(n+1)(n)(n-1)-\frac{1}{6}n(n-1)(n-2)=21$$

$$\Rightarrow \frac{n(n-1)}{6}[(n+1)-(n-2)]=21$$

$$\Rightarrow \frac{n(n-1)\cdot 3}{6}=21 \Rightarrow n(n-1)=42 \Rightarrow n=7$$

77. The coefficient of x^7 in $(1+3x-2x^3)^{10}$

$$=\frac{\sum 10!}{n_1!n_2!n_3!}(1)^{n_1}(3)^{n_2}(-2)^{n_3},$$

where $n_1+n_2+n_3=10$, $n_2+3n_3=7$

Different possibilities are as follow

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

$$\therefore \text{Coefficient of } x^7=\frac{10!}{3!7!}(1)^3(3)^7(-2)^0+\frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1+\frac{10!}{7!1!2!}(1)^7(3)^1(-2)^2$$

$$=62640$$

Round II

1. Multiplying the numerator and denominator by $1-x$, we have

$$E=\frac{1-x}{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})}$$

$$=\frac{1-x}{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^m})}$$

$$=\frac{1-x}{(1-x^4)(1+x^4)\ldots(1+x^{2^m})}=\frac{1-x}{(1-x^{2^{m+1}})}$$

$$=(1-x)(1-x^{2^{m+1}})^{-1}$$

$$=(1-x)(1+x^{2^{m+1}}+x^{2^{m+2}}+\ldots)$$

$\therefore$ Coefficient of $x^{2^{m+1}}$ is 1.

2. We have, $(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+\ldots+(x+2)^{n-1}$

$$=\frac{(x+3)^n-(x+2)^n}{(x+3)-(x+2)}$$

$$=(x+3)^n-(x+2)^n$$

$$\left(\because \frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a^1+x^{n-3}a^2+\ldots+a^{n-1}\right)$$

Therefore, the coefficient of x^r in the given expression

$=$ coefficient of x^r in $[(x+3)^n-(x+2)^n]$

$$={}^nC_r\,3^{n-r}-{}^nC_r\,2^{n-r}$$

$$={}^nC_r\,(3^{n-r}-2^{n-r})$$

3. $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots \text{ upto } m \text{ terms}\right\}$

$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\frac{1}{2^r}+\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\,\frac{3^r}{2^{2r}}+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{7^r}{2^{3r}}+\ldots$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\ldots \text{ upto } m \text{ terms}$$

$$=\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}\ldots \text{ upto } m \text{ terms}$$

$$=\frac{\frac{1}{2^n}\left\{1-\left(\frac{1}{2^n}\right)^m\right\}}{\left(1-\frac{1}{2^n}\right)}=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

4. We have, $\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right]^7$

$$=\left[\sqrt{9^{x-1}+7}+\frac{1}{(3^{x-1}+1)^{1/5}}\right]^7$$

$$\therefore \quad T_6={}^7C_5\,(\sqrt{9^{x-1}+7})^{7-5}\left[\frac{1}{(3^{x-1}+1)^{1/5}}\right]^5$$

$$={}^7C_5\,(9^{x-1}+7)\frac{1}{(3^{x-1}+1)}$$

$$\Rightarrow \quad 84={}^7C_5\,\frac{(9^{x-1}+7)}{(3^{x-1}+1)}$$

$\Rightarrow \quad 9^{x-1} + 7 = 4(3^{x-1} + 1)$

$\Rightarrow \quad \frac{3^{2x}}{9} + 7 = 4\left(\frac{3^x}{3} + 1\right)$

$\Rightarrow \quad 3^{2x} - 12(3^x) + 27 = 0$

$\Rightarrow \quad y^2 - 12y + 27 = 0 \quad$ (put $y = 3^x$)

$\Rightarrow \quad (y-3)(y-9) = 0$

$\Rightarrow \quad y = 3, 9 \Rightarrow 3^x = 3, 9 \Rightarrow x = 1, 2$

5. $\because \sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i} = \sum_{i=0}^{m} {}^{10}C_i \cdot {}^{20}C_{m-1}$

$= $ Coefficient of x^m in the expansion of

$(1+x)^{10}(1+x)^{20} = {}^{30}C_m$

It is maximum, when

$$m = \frac{30}{2} = 15$$

6. $\because \quad (k^2 - 3) = \frac{{}^{n-1}C_r}{{}^{n}C_{r+1}} = \frac{{}^{n-1}C_r}{\left(\frac{n}{r+1}\right){}^{n-1}C_r} = \left(\frac{r+1}{n}\right) \quad$ ...(i)

$\because \quad 0 \le r \le n-1$

$\Rightarrow \quad 1 \le r+1 \le n$

$\Rightarrow \quad \frac{1}{n} \le \frac{r+1}{n} \le 1$

$\Rightarrow \quad \frac{1}{n} \le k^2 - 3 \le 1 \quad$ [from Eq. (i)]

$\Rightarrow \quad 3 + \frac{1}{n} \le k^2 \le 4$

When $n \to \infty, 3 \le k^2 \le 4$ or $k \in [-2, -\sqrt{3}] \cup (\sqrt{3}, 2]$

7. $\because \quad I + f + f' = (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n = 2k$ (even integer)

$\therefore \quad f + f' = 1$

Now, $\quad (I + f)f' = (5 + 2\sqrt{6})^n(5 - 2\sqrt{6})^n = (1)^n = 1$

$\Rightarrow \quad (I + f)(1 - f) = 1$ or $I = \frac{1}{(1-f)} - f$

8. Coefficient of x^2y^2 in $(x + y + z + t)^4 = \frac{4!}{2!2!} = 6$ and coefficient of yzt^2 in $(x + y + z + t)^4$

$$= \frac{4!}{1!1!1!2!} = 12$$

Also, coefficient of $xyzt$ in

$$(x + y + z + t)^4 = \frac{4!}{1!1!1!1!} = 24$$

$\therefore$ Required ratio is $6 : 12 : 24 = 1 : 2 : 4$

9. Given, $\quad {}^mC_0 + {}^mC_1 + {}^mC_2 = 46$

$\Rightarrow \quad 2m + m(m-1) = 90$

$\Rightarrow \quad m^2 + m - 90 = 0 \Rightarrow m = 9$ as $m > 0$

Now, $(r+1)$th term of $\left(x^2 + \frac{1}{x}\right)^m$ is

$${}^mC_r\,(x^2)^{m-r}\left(\frac{1}{x}\right)^r = {}^mC_r\,x^{2m-3r}$$

For this to be independent of x put

$$2m - 3r = 0 \Rightarrow r = 6$$

$\therefore$ Coefficient of the term independent of x is ${}^9C_6 = 84$.

10. $\because 3^{4n} = 81^n = (1 + 80)^n = 1 + 80\lambda, \lambda \in N$

$\therefore \quad 3^{3^{4n}} = 3^{1+80\lambda} = 3 \cdot 3^{80\lambda} = 3 \cdot (9)^{40\lambda} = 3(10-1)^{40\lambda}$

$= 3(1 + 10\mu) = 3 + 30\ \mu$

$\therefore$ Last digit of $3^{3^{4n}} + 1$ is 4.

11. $(101)^{100} - 1 = (1 + 100)^{100} - 1$

$= 1 + {}^{100}C_1(100) + {}^{100}C_2(100)^2 + \ldots$

$+ {}^{100}C_{100}(100)^{100} - 1$

$= 10^4\lambda, \forall \lambda \in N$

12. $\because (101)^{50} - (99)^{50} = (100 + 1)^{50} - (100 - 1)^{50}$

$= 2\{{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47} + {}^{50}C_5(100)^{45} + \ldots\}$

$= (100)^{50} + 2\{{}^{50}C_3(100)^{47} + {}^{50}C_5(100)^{45} + \ldots\}$

$> (100)^{50}$

$\Rightarrow \quad (101)^{50} - (99)^{50} > (100)^{50}$

$\Rightarrow \quad (101)^{50} - (100)^{50} > (99)^{50}$

Also, $\left(\frac{1001}{1000}\right)^{999} = \left(1 + \frac{1}{1000}\right)^{999}$

$= 1 + {}^{999}C_1\left(\frac{1}{1000}\right) + {}^{999}C_2\left(\frac{1}{1000}\right)^2 + \ldots$

$< 1 + 1 + 1 + 1 + \ldots + 1$

$= 1000$

$\therefore \quad \left(\frac{1001}{1000}\right)^{999} < 1000$

$\Rightarrow \quad (1001)^{999} < (1000)^{1000}$

Solutions (Q. Nos. 13-15)

2nd term is ${}^nC_1x^{n-1}a = 240$ ...(i)

3rd term is ${}^nC_2x^{n-2}a^2 = 720$ ...(ii)

4th term is ${}^nC_3x^{n-3}a^3 = 1080$ ...(iii)

On multiplying Eqs. (i) and (iii) and dividing by the square of Eq. (ii), we get

$$\frac{{}^nC_1 \times {}^nC_3}{({}^nC_2)^2} = \frac{240 \times 1080}{(720)^2}$$

$\Rightarrow \quad \frac{n \times n(n-1)(n-2)(2!)^2}{n^2(n-1)^2 \times 3!} = \frac{1}{2}$

$\Rightarrow \quad 4(n-2) = 3(n-1) \quad (\because n \ne 1)$

$\Rightarrow \quad n = 5$

On putting $n=5$, from Eqs. (i) and (ii), we get

$$5x^4a=240$$

and $$10x^3a^2=270$$

$\Rightarrow$ $$\frac{(5x^4a)^2}{10x^3a^2}=\frac{(240)^2}{720} \text{ or } x^5=32$$

$\therefore$ $$x=2$$

$\therefore$ $$a=\frac{240}{5x^4}=\frac{48}{2^4}=3$$

13. Hence, $x=2, a=3$ and $n=5$.

$$(x-a)^n=(2-3)^5=-1$$

14. Also,

$$(2+3)^5=2^5+{}^5C_1 2^4\times 3+{}^5C_2 2^3\times 3^2+{}^5C_3 2^2\times 3^3+{}^5C_4 2\times 3^4+{}^5C_5 3^5$$

$$=32+240+720+1080+810+243$$

Hence, least value of the term is 32.

15. Sum of odd numbered terms is $32+720+810=1562$

16. Given expression $=\left\{1+\left(x+\frac{1}{x}\right)\right\}^n$

$$=1+{}^nC_1\left(x+\frac{1}{x}\right)+{}^nC_2\left(x+\frac{1}{x}\right)^2+{}^nC_3\left(x+\frac{1}{x}\right)^3+\ldots+\left(x+\frac{1}{x}\right)^n$$

This will be of the form

$$=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\frac{b_1}{x}+\frac{b_2}{x^2}+\frac{b_3}{x^3}+\ldots+\frac{b_n}{x^n}$$

$\therefore$ Number of terms $=1+n+n=2n+1$

17. $$\sum_{r=1}^{n} r^3\left(\frac{n-r+1}{r}\right)^2=\sum_{r=1}^{n} r(n-r+1)^2$$

$$=\sum_{r=1}^{n} r\{(n+1)^2-2(n+1)r+r^2\}$$

$$=(n+1)^2\sum r-2(n+1)\sum r^2+\sum r^3$$

$$=(n+1)^2\times\frac{n(n+1)}{2}-\frac{2(n+1)\times n(n+1)(2n+1)}{6}+\left[\frac{n(n+1)}{2}\right]^2$$

$$=\frac{(n+1)^2\cdot n\cdot(n+2)}{12}=14^2 \quad \text{(given)}$$

$$=7^2\times 2^2=\frac{7^2\cdot 6\cdot 8}{12} \quad \therefore n=6$$

Sum of coefficients $=(1-3+1)^6=(-1)^6=1$

18. The coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$

Statement I If ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in AP, then

$$2\cdot{}^nC_r={}^nC_{r-1}+{}^nC_{r+1}$$

$\Rightarrow$ $$2=\frac{{}^nC_{r-1}}{{}^nC_r}+\frac{{}^nC_{r+1}}{{}^nC_r}$$

$$=\frac{r}{n-r+1}+\frac{n-r}{r+1}$$

Then, we get $(n-2r)^2=n+2$, which is true.

Statement II If ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in GP, then

$$\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{{}^nC_{r+1}}{{}^nC_r}$$

$\Rightarrow$ $$\frac{n-r+1}{r}=\frac{n-r}{r+1}$$

$\Rightarrow$ $$(n-r+1)(r+1)=(n-r)r$$

We get, $$n+1=0$$

$\therefore$ $n=-1$, which is not possible, since n is a positive integer.

Statement III If ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in HP, then

$$\frac{2}{{}^nC_r}=\frac{1}{{}^nC_{r-1}}+\frac{1}{{}^nC_{r+1}}$$

$\Rightarrow$ $$2=\frac{{}^nC_r}{{}^nC_{r-1}}+\frac{{}^nC_r}{{}^nC_{r+1}}$$

$\Rightarrow$ $$2=\frac{n-r+1}{r}+\frac{r+1}{n-r}$$

$\Rightarrow$ $$2r(n-r)=(n-r+1)(n-r)+r(r+1)$$

$\Rightarrow$ $$n^2-4nr+4r^2+n=0$$

$\Rightarrow$ $$(n-2r)^2+n=0$$

This is not possible, as $(n-2r)^2\geq 0$

and n is a positive integer.

19. For $n=1$, $\sqrt{2}-1=\sqrt{2}-\sqrt{1}$

For $n=2$, $(\sqrt{2}-1)^2=3-2\sqrt{2}=\sqrt{9}-\sqrt{8}$

For $n=3$, $(\sqrt{2}-1)^3=5\sqrt{2}-7=\sqrt{50}-\sqrt{49}$

$\therefore$ Statement I is true for all natural numbers by mathematical induction.

Statement II is also true.

$\because$ $(\sqrt{2}-1)^n$ = Rational part + Irrational part

The irrational part will have only $\sqrt{2}$.

Hence, statement II is not a correct explanation of statement I.

20. Let $(\sqrt{5}+2)^n=N+f$, where N is an integer and $0<f<1$.

Let $(\sqrt{5}-2)^n=f'$, then $0<f'<1$

Let $(\sqrt{5}+2)^n-(\sqrt{5}-2)^n$ = integer $\quad(\because n$ is odd)

$\therefore$ $$N=(\sqrt{5}+2)^n-(\sqrt{5}-2)^n$$

$$=2[{}^nC_1\cdot 2\cdot 5^{(n-1)/2}+{}^nC_3\cdot 2^3\cdot 5^{(n-3)/2}+\ldots]$$

$\Rightarrow$ N is divisible by $2n$ on using statement II

(if n is prime and $r<n$, then there is no factor which will cut n $\Rightarrow$ ${}^nC_r$ will be divisible by n)

21. Since, n is even, put $n = 2$

$$\text{LHS} = {}^4C_1 = 4 \quad \text{and} \quad \text{RHS} = 2^3 = 8$$

Hence, statement I is false but statement II is true.

22. $\left(x + \frac{1}{x} + 2\right)^m = \left(\frac{x^2 + 2x + 1}{x}\right)^m = \frac{(1+x)^{2m}}{x^m}$

Term independent of x is coefficient of x^m in the expansion of $(1+x)^{2m} = {}^{2m}C_m = \frac{(2m)!}{(m!)^2}$

Hence, statement I is false and statement II is true.

23. $\frac{T_{r+1}}{T_r} = \frac{12 - r + 1}{r}\frac{11}{10}$

Let $\quad T_{r+1} \geq T_r \Rightarrow 13 - r \geq 1.1r$

$\Rightarrow \quad 13 \geq 2.1r \Rightarrow r \leq 6.19$

Hence, the greatest term occurs for $r = 6$.

Hence, 7th term is the greatest term. Also, the binomial coefficient of 7th term is ${}^{12}C_6$ which is the greatest binomial coefficient.

But this is not the reason for which T_7 is the greatest.

Here, it is coincident that the greatest term has the greatest binomial coefficient.

Hence, statement I is true, statement II is true but statement II is not a correct explanation of statement I.

24. $(1+x)^n - nx - 1 = (1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n) - nx - 1 \quad \ldots(i)$

$= {}^nC_2x^2 + \ldots + {}^nC_nx^n = x^2({}^nC_2 + {}^nC_3x + \ldots + {}^nC_nx^{n-2})$

Hence, $(1+x)^n - nx - 1$ is divisible by x^2.

Now, in Eq. (i), replace x by 8 and n by $n+1$.

Then, we have

$(1+8)^{n+1} - (n+1)8 - 1 = 8^2({}^nC_2 + {}^nC_38 + \ldots + {}^nC_n8^{n-2})$

$\Rightarrow \quad 9^{2n+2} - 8n - 9 = 8^2({}^nC_2 + {}^nC_38 + \ldots + {}^nC_n8^{n-2})$,

which is divisible by 64.

Hence, both the statements are correct and statement II is a correct explanation of statement I.

25. Since, $\quad \sum_{r=0}^{n} {}^nC_r \cdot x^r = (1+x)^n$

On multiplying by x, we get

$$\sum_{r=0}^{n} {}^nC_r \cdot x^{r+1} = x(1+x)^n$$

On differentiating w.r.t. x, we get

$$\sum_{r=0}^{n} (r+1) \cdot {}^nC_r \cdot x^r = (1+x)^n + nx(1+x)^{n-1}$$

Statement II is true.

If $x = 1$, then

$$\sum_{r=0}^{n} (r+1) \cdot {}^nC_r = 2^n + n(2)^{n-1} = (n+2)2^{n-1}$$

$\therefore$ Statement I is true and statement II is a correct explanation of statement I.

26. $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$

$$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[\frac{(x^{1/3} + 1)(x^{2/3} + 1 - x^{1/3})}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}}\right]^{10} = (x^{1/3} - x^{-1/2})^{10}$$

$\therefore$ The general term is

$$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}(-x^{-1/2})^r$$

$$= {}^{10}C_r(-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$$

For independent of x, put

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$\Rightarrow \quad 20 - 2r - 3r = 0$

$\Rightarrow \quad 20 = 5r \Rightarrow r = 4$

$\therefore \quad T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$

27. $(\sqrt{3} + 1)^{2n} = {}^{2n}C_0(\sqrt{3})^{2n} + {}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_2(\sqrt{3})^{2n-2} + \ldots + {}^{2n}C_{2n}(\sqrt{3})^{2n-2n}$

and $(\sqrt{3} - 1)^{2n} = {}^{2n}C_0(\sqrt{3})^{2n}(-1)^0 + {}^{2n}C_1(\sqrt{3})^{2n-1}(-1)^1 + {}^{2n}C_2(\sqrt{3})^{2n-2}(-1)^2 + \ldots + {}^{2n}C_{2n}(\sqrt{3})^{2n-2n}(-1)^{2n}$

$\therefore \quad (\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n} = 2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + \ldots + {}^{2n}C_{2n-1}(\sqrt{3})^{2n-(2n-1)}]$

which is most certainly an irrational number because of odd powers of $\sqrt{3}$ in each of the terms.

28. Here, $(1 - x - x^2 + x^3)^6 = \{(1-x) - x^2(1-x)\}^6$

$$= \{(1-x)(1-x^2)\}^6 = (1-x)^6 \cdot (1-x^2)^6$$

$$= \left\{\sum_{r=0}^{6} (-1)^r \, {}^6C_r \cdot x^r\right\}\left\{\sum_{s=0}^{6} (-1)^s \, {}^6C_s \cdot x^{2s}\right\}$$

$$= \sum_{r=0}^{6} \sum_{s=0}^{6} (-1)^{r+s} \cdot {}^6C_r \cdot {}^6C_s \cdot x^{r+2s}$$

For coefficient of x^7, $+ r + 2s = 7$

i.e., $\quad (s = 1, r = 5)$ or $(s = 2, r = 3)$ or $(s = 3, r = 1)$

$\therefore$ Coefficient of x^7 is

$\{(-1)^{5+1} \cdot {}^6C_5 \cdot {}^6C_1\} + \{(-1)^{3+2} \cdot {}^6C_3 \cdot {}^6C_2\} + \{(-1)^{1+3} \cdot {}^6C_1 \cdot {}^6C_3\}$

$= (36) - (20)(15) + 6(20)$

$= 36 - 300 + 120 = -144$

29. $8^{2n} - (62)^{2n+1} = (1+63)^n - (63-1)^{2n+1}$

$= (1+63)^n + (1-63)^{2n+1}$

$= [1 + {}^nC_1 63 + {}^nC_2(63)^2 + \ldots + (63)^n]$

$+ [1 - {}^{(2n+1)}C_1 63 + {}^{(2n+1)}C_2(63)^2$

$+ \ldots + (-1)(63)^{(2n+1)}]$

$= 2 + 63\,[{}^nC_1 + {}^nC_2(63) + \ldots + (63)^{n-1}$

$- {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2(63) - \ldots - (63)^{(2n)}]$

$\therefore$ Remainder is 2.

30. Since, in a binomial expansion of $(a-b)^n, n \geq 5$, the sum of 5th and 6th terms is equal to zero.

$\therefore \quad {}^nC_4 a^{n-4}(-b)^4 + {}^nC_5 a^{n-5}(-b)^5 = 0$

$\Rightarrow \quad \dfrac{n!}{(n-4)!\,4!} a^{n-4} \cdot b^4 - \dfrac{n!}{(n-5)!\,5!} a^{n-5} b^5 = 0$

$\Rightarrow \quad \dfrac{n!}{(n-5)!\,4!} a^{n-5} \cdot b^4 \left(\dfrac{a}{n-4} - \dfrac{b}{5}\right) = 0$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{n-4}{5}$

31. We know that,

$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + \ldots + {}^{20}C_{10}x^{10} + \ldots + {}^{20}C_{20}x^{20}$

On putting $x = -1$, we get

$0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_{11} + \ldots + {}^{20}C_{20}$

$\Rightarrow 0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_9 + \ldots + {}^{20}C_0$

$[\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow \quad 0 = 2({}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9) + {}^{20}C_{10}$

$\Rightarrow \quad {}^{20}C_{10} = 2({}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10})$

$\Rightarrow \quad {}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10} = \dfrac{1}{2}\,{}^{20}C_{10}$

32. $\because \quad (1-ax)^{-1}(1-bx)^{-1}$

$= (1 + ax + a^2x^2 + \ldots)(1 + bx + b^2x^2 + \ldots)$

$\therefore \quad a_n$ = coefficient of x^n in $(1-ax)^{-1}(1-bx)^{-1}$

$= a^0b^n + ab^{n-1} + \ldots + a^nb^0$

$= a^0b^n \left[1 + \dfrac{a}{b} + \left(\dfrac{a}{b}\right)^2 + \ldots\right] = a^0b^n \left[\dfrac{\left(\dfrac{a}{b}\right)^{n+1} - 1}{\dfrac{a}{b} - 1}\right]$

$= \dfrac{b^n(a^{n+1} - b^{n+1})}{a-b} \cdot \dfrac{b}{b^{n+1}}$

$= \dfrac{a^{n+1} - b^{n+1}}{a-b}$

33. $(1-y)^m(1+y)^n = 1 + a_1y + a_2y^2 + a_3y^3 + \ldots$

On differentiating w.r.t. y, we get

$-m(1-y)^{m-1}(1+y)^n + (1-y)^m n(1+y)^{n-1}$

$= a_1 + 2a_2y + 3a_3y^2 + \ldots \quad \ldots(\text{i})$

On putting $y = 0$ in Eq. (i), we get

$-m + n = a_1 = 10 \quad \ldots(\text{ii})$

$(\because a_1 = 10$, given)

On again differentiating Eq. (i), we get

$-m[-(m-1)(1-y)^{m-2}(1+y)^n + (1-y)^{m-1}n(1+y)^{n-1}]$

$+ n[-m(1-y)^{m-1}(1+y)^{n-1} + (1-y)^m(n-1)(1+y)^{n-2}]$

$= 2a_2 + 6a_3y + \ldots \quad \ldots(\text{iii})$

On putting $y = 0$ in Eq. (iii), we get

$-m[-(m-1) + n] + n[-m + (n-1)] = 2a_2 = 20$

$\Rightarrow \quad m(m-1) - mn - mn + n(n-1) = 20$

$\Rightarrow \quad m^2 + n^2 - m - n - 2mn = 20$

$\Rightarrow \quad (m-n)^2 - (m+n) = 20$

$\Rightarrow \quad 100 - (m+n) = 20$

$\Rightarrow \quad m + n = 80 \quad \ldots(\text{iv})$

On solving Eqs. (ii) and (iv), we get

$m = 35$ and $n = 45$

34. ${}^{50}C_4 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3 + {}^{50}C_3$

$= {}^{50}C_3 + {}^{50}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{51}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= {}^{52}C_4 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{53}C_4 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{54}C_4 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{55}C_4 + {}^{55}C_3 = {}^{56}C_4$

35. Since, the coefficient of given terms are ${}^mC_{r-1}, {}^mC_r, {}^mC_{r+1}$ respectively and they are in AP.

$\therefore \quad {}^mC_{r-1} + {}^mC_{r+1} = 2\,{}^mC_r$

$\Rightarrow \quad \dfrac{m!}{(r-1)!(m-r+1)!} + \dfrac{m!}{(r+1)!(m-r-1)!} = 2\,\dfrac{m!}{r!(m-r)!}$

$\Rightarrow \quad \dfrac{1}{(m-r+1)(m-r)} + \dfrac{1}{(r+1)r} = \dfrac{2}{r(m-r)}$

$\Rightarrow \quad \dfrac{r(r+1) + (m-r+1)(m-r)}{r(r+1)(m-r+1)(m-r)} = \dfrac{2}{r(m-r)}$

$\Rightarrow \quad r^2 + r + m^2 + r^2 - 2mr + m - r$

$= 2(mr - r^2 + r + m - r + 1)$

$\Rightarrow \quad 4r^2 - 4mr - m - 2 + m^2 = 0$

$\Rightarrow \quad m^2 - m(4r+1) + 4r^2 - 2 = 0$

36. Let x^7 is contained in $(r+1)$th term in the expansion of $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$.

$\therefore \quad T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\dfrac{1}{bx}\right)^r = {}^{11}C_r \dfrac{a^{11-r}}{b^r} \cdot x^{22-3r}$

For coefficient of x^7, put

$$22 - 3r = 7$$

$\Rightarrow$ $$r = 5 \therefore T_6 = {}^{11}C_5 \frac{a^6}{b^5} \cdot x^7$$

$\therefore$ Coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is

${}^{11}C_5 \frac{a^6}{b^5}$.

Similarly, coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

is ${}^{11}C_6 \frac{a^5}{b^6}$.

According to the given condition,

$$^{11}C_5 \frac{a^6}{b^5} = {}^{11}C_6 \frac{a^5}{b^6} \Rightarrow \frac{a^6}{b^5} = \frac{a^5}{b^6}$$

$\Rightarrow$ $$ab = 1$$

37. $$\frac{(1+x)^{3/2} - \left(1 + \frac{1}{2}x\right)^3}{(1-x)^{1/2}}$$

$$= \frac{\left(1 + \frac{3}{2}x + \frac{\frac{3}{2}\cdot\frac{1}{2}}{2}x^2\right) - \left(1 + \frac{3x}{2} + \frac{3\cdot 2}{2}\cdot\frac{x^2}{4}\right)}{(1-x)^{1/2}}$$

(neglecting higher powers of x^2)

$$= -\frac{3x^2}{8}(1-x)^{-1/2} = -\frac{3x^2}{8}\left(1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2}\cdot x^2\right)$$

$$= -\frac{3x^2}{8}$$ ($\because$ higher powers of x^2 can be neglected)

38. The coefficient of the middle term in powers of x of $(1 + \alpha x)^4 = {}^4C_2\,\alpha^2$

The coefficient of the middle term in powers of x of $(1 - \alpha x)^6 = {}^6C_3(-\alpha)^3$

According to the given condition,

$$^4C_2\alpha^2 = {}^6C_3(-\alpha)^3$$

$\Rightarrow$ $$\frac{4!}{2!2!}\alpha^2 = -\frac{6!}{3!3!}\alpha^3 \Rightarrow 6\alpha^2 = -20\alpha^3$$

$\Rightarrow$ $$\alpha = -\frac{6}{20} \Rightarrow \alpha = -\frac{3}{10}$$

39. Given that, $$s_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$$

$$s_n = \sum_{r=0}^{n} \frac{1}{{}^nC_{n-r}} \qquad (\because {}^nC_r = {}^nC_{n-r})$$

$\Rightarrow$ $$ns_n = \sum_{r=0}^{n} \frac{n}{{}^nC_{n-r}}$$

$\Rightarrow$ $$ns_n = \sum_{r=0}^{n}\left(\frac{n-r}{{}^nC_{n-r}} + \frac{r}{{}^nC_{n-r}}\right)$$

$\Rightarrow$ $$ns_n = \sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}} + \sum_{r=0}^{n}\frac{r}{{}^nC_r}$$

$\Rightarrow$ $$ns_n = \left(\frac{n}{{}^nC_n} + \frac{n-1}{{}^nC_{n-1}} + \ldots + \frac{1}{{}^nC_1}\right) + \sum_{r=0}^{n}\frac{r}{{}^nC_r}$$

$\Rightarrow$ $$ns_n = t_n + t_n \qquad \left(\because t_n = \sum_{r=0}^{n}\frac{r}{{}^nC_r}, \text{ given}\right)$$

$\Rightarrow$ $$ns_n = 2t_n \Rightarrow \frac{t_n}{s_n} = \frac{n}{2}$$

40. The coefficient of x^n in expansion of $(1+x)(1-x)^n$.

= Coefficient of x^n in $(1-x)^n$ + Coefficient of x^{n-1} in $(1-x)^n$

$$= (-1)^n \frac{n!}{n!0!} + (-1)^{n-1}\frac{n!}{1!(n-1)!}$$

$$= (-1)^n\left[\frac{n!}{n!0!} - \frac{n!}{1!(n-1)!}\right] = (-1)^n(1-n)$$

41. The general term of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is

$$T_{r+1} = {}^{256}C_r(3)^{(256-r)/2}(5)^{r/8}$$

For integral terms, $\frac{256-r}{2}$ and $\frac{r}{8}$ are both positive integers.

i.e., $$r = 0, 8, 16, 24, 32, \ldots, 256$$

Hence, total number of terms are 33.

42. Since, $(r+1)$th term in the expansion of $(1+x)^{27/5}$

$$= \frac{\frac{27}{5}\left(\frac{27}{5} - 1\right)\ldots\left(\frac{27}{5} - r + 1\right)}{r!}x^r$$

Now, this term will be negative, if the last factor in numerator is the only one negative factor.

$\Rightarrow$ $$\frac{27}{5} - r + 1 < 0 \Rightarrow \frac{32}{5} < r$$

$\Rightarrow 6.4 < r \Rightarrow$ least value of r is 7.

Thus, first negative term will be 8th.

43. $(1 + 2x + 3x^2 + \ldots)^{-3/2} = [(1-x)^{-2}]^{-3/2} = (1-x)^3$

So, coefficient of x^5 in $(1 + 2x + 3x^2 + \ldots)^{-3/2}$

= Coefficient of x^5 in $(1-x)^3 = 0$

44. $(1 + x + x^2 + x^3 + \ldots)^2 = [(1-x)^{-1}]^2 = (1-x)^{-2}$

Coefficient of x^n in $(1 + x + x^2 + \ldots)^2$

= Coefficient of x^n in $(1-x)^{-2}$

$= {}^{n+2-1}C_{2-1} = {}^{n+1}C_1$

$= n + 1$

9 Trigonometric Function and Equations

JEE Main MILESTONE

- Measure of Angles
- Trigonometric Ratios
- Trigonometric Functions
- Graph of Trigonometric Functions
- Trigonometric Identities
- Complementary and Supplementary Angles
- Variation of Values of Trigonometrical Ratios in Different Quadrants
- Trigonometrical Ratios of Some Useful Angles between 0° and 90°
- Trigonometric Ratios of Compound Angles
- Maximum and Minimum Greatest and Least Value of $a \sin \theta + b \cos \theta$
- Trigonometric Equations
- Solutions or Roots of a Trigonometric Equation
- Solution of Trigonometric Equation of the Form $a \cos \theta + b \sin \theta = c$
- Solution of Trigonometric Inequalities by Graphs

9.1 Measure of Angles

The measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving ray are respectively called the initial and terminal sides (arms). Also, the revolving line is called the generating side. *e.g.*, If initial and final positions of the revolving ray are OP and OQ, then the angle formed will be $\angle POQ$.

If the rotation is in clockwise sense, the angle measured is negative and if the rotation is in anti-clockwise sense, the angle measured is positive.

Trigonometry is a branch of Mathematics that studies triangles and the relationships between the lengths of their sides and the angles between those sides. Trigonometry defines the trigonometric functions, which describe those relationships and have applicability to physical phenomena, such as waves.

System of Measurement of Angles

There are three systems of measurement

(i) Sexagesimal system

(ii) Centesimal system

(iii) Circular system

(i) **Sexagesimal system** In this system each angle is divided into 90 equal parts and each part is known as a degree. Thus, a right angle is equal to 90 degrees. One degree is denoted as 1°.

Each degree is divided into 60 equal parts each of which is known as one minute. One minute is denoted as $1'$. Each minute is consist of 60 parts, each part is known as a second. One second is denoted by $1''$.

Hence, 1 right angle = 90° (90 degree)

1 degree = $1^\circ = 60'$ (60 minute)

1 minute = $1' = 60''$ (60 second)

(ii) **Centesimal system** In this system each angle is divided into 100 equal parts and one part is known as a grade. Thus, one right angle is equal to 100 grade. One grade is denoted as 1^g.

One grade is divided into 100 equal parts, one part is known as a minute and is denoted as $1'$.

One minute is also divided into 100 equal parts, one part is known as a second which is denoted by $1''$.

Hence, 1 right angle = 100^g (100 grade)

1 grade = $1^g = 100'$ (100 minute)

1 minute = $1' = 100''$ (100 second)

(iii) **Circular system** If the angle subtended by an arc of length l to the centre of circle of radius r is θ, then $\theta = \frac{l}{r}$.

If the length of arc is equal to the radius of the circle, then the angle subtended at the centre of the circle will be one radian. One radian is denoted by 1^c.

The ratio of the circumference of the circle to the diameter of the circle is denoted by a greek letter π and it is a constant quantity.

$$\therefore \quad \frac{\text{Circumference of circle}}{\text{Diameter of circle}} = \pi$$

This constant quantity π is a irrational quantity and generally its approximate value is $\frac{22}{7}$ and its general value upto six places of decimal is 3.142857.

Relation Among Degree, Radian and Grade

The number of radians of an angle subtended by an arc of a circle at the centre is equal to the ratios of arc and radius.

$$180^\circ = \pi^c = 100^g$$

and $$1 \text{ radian} = 57^\circ 17' 44.8''$$

Note If measure of an angle is given in degree, then to convert it into radians, multiply the measure of an angle by $\frac{\pi}{180^\circ}$ and if the measure of an angle is given in radians, then to convert it into degree, write 180° at the place of π.

***Sample* Problem 1** *Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm.* $\left(\text{use } \pi = \frac{22}{7}\right)$ **[NCERT]**

(a) $11^\circ 36'$ (b) $12^\circ 36'$
(c) $12^\circ 30'$ (d) $12^\circ 34'$

Interpret (b) Here, $l = 22$ cm and $r = 100$ cm

Using the formula $\theta = \frac{l}{r}$, we have

$$\Rightarrow \quad \theta = \frac{22}{100} = \frac{11}{50} \text{ rad} \qquad \left(\because \theta = \frac{l}{r}\right)$$

$$= \left(\frac{11}{50} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11 \times 18 \times 7}{5 \times 22}\right)^\circ = \left(\frac{63}{5}\right)^\circ$$

$$= 12^\circ + \left(\frac{3}{5}\right)^\circ$$

$$= 12^\circ + \frac{3}{5} \times 60' \qquad (\because 1^\circ = 60')$$

$$= 12^\circ 36'$$

***Sample* Problem 2** *If the three angles of a quadrilateral are* 60°, 60^g *and* $\frac{5\pi}{6}$ *radians. Then, the fourth angle is*

(a) 60° (b) 96°
(c) 96^g (d) *None of these*

Interpret (b) First angle = 60°

$$\text{Second angle} = 60^g = 60 \times \frac{90}{100} = 54^\circ$$

$$\text{Third angle} = \frac{5\pi}{6} \text{ radians}$$

$$= \frac{5 \times 180^\circ}{6} = 150^\circ$$

$$\text{Fourth angle} = 360^\circ - (60^\circ + 54^\circ + 150^\circ)$$

$$= 360^\circ - 264^\circ$$

$$= 96^\circ$$

9.2 Trigonometric Ratios

Let us take a right angled ΔABC right angled at B.

Let $\angle CAB = \theta$, then

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{Base}}{\text{Perpendicular}} = \frac{AB}{BC}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{AC}{AB}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{AC}{BC}$$

Except these six ratios, there are two other terms which are known as 'versed sine' and 'coversed sine', respectively.

i.e., $\text{ver}\sin\theta = 1 - \cos\theta$

and $\text{cover}\sin\theta = 1 - \sin\theta$

Note Trigonometric ratios of any angle is always constant .

***Sample* Problem 3** *In a right angle ΔABC, whose base is 6 and perpendicular height is 8. Find the trigonometric ratios* sin *A*, cos *A* and tan *A, respectively.* **[NCERT]**

(a) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$ (b) $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}$

(c) $\frac{4}{5}, \frac{1}{5}, \frac{2}{3}$ (d) *None of these*

Interpret (a) By pythagoras theorem,

$$AC^2 = AB^2 + BC^2$$

$$= 6^2 + 8^2 = 36 + 64 = 100$$

$$\Rightarrow \quad AC = 10$$

$$\therefore \quad \sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{8}{10} = \frac{4}{5}$$

$$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{6}{10} = \frac{3}{5}$$

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB} = \frac{8}{6} = \frac{4}{3}$$

9.3 Trigonometric Functions

Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B', as shown in the figure.

Let $\angle AOP = \theta$

$$\left(\because \angle AOP = \frac{\text{arc } AP}{\text{radius } OP} = \frac{\theta}{1} = \theta^c, \text{ using } \theta = \frac{l}{r}\right)$$

Now, the six trigonometric functions may be defined as under

(i) $\cos\theta = \frac{OM}{OP} = x$ (ii) $\sin\theta = \frac{PM}{OP} = y$

(iii) $\sec\theta = \frac{OP}{OM} = \frac{1}{x}, x \neq 0$ (iv) $\text{cosec}\,\theta = \frac{OP}{PM} = \frac{1}{y}, y \neq 0$

(v) $\tan\theta = \frac{PM}{OM} = \frac{y}{x}, x \neq 0$ (vi) $\cot\theta = \frac{OM}{PM} = \frac{x}{y}, y \neq 0$

9.4 Graph of Trigonometric Functions

(i) Graph of sin x

Facts Related to sin *x*

(a) Domain = R

(b) Range = $[-1, 1]$

(c) Period = 2π

(d) Graph of $\sin x$ is continuous for all real values of x.

(ii) Graph of cos x

Facts Related to cos *x*

(a) Domain = R

(b) Range = $[-1, 1]$

(c) Period = 2π

(d) Graph of $\cos x$ is continuous for all real values of x

(iii) Graph of tan x

Facts Related to tan x

(a) Domain = $R \sim (2n+1)\frac{\pi}{2}$, $n \in I$

(b) Range = $(-\infty, \infty)$

(c) Period = π

(d) Graph of $\tan x$ is discontinuous at $x = \frac{m\pi}{2}$, where m is an odd integer.

(iv) Graph of cot x

Facts Related to cot x

(a) Domain = $R \sim n\pi$, $n \in I$

(b) Range = $(-\infty, \infty)$

(c) Period = π

(d) Graph of $\cot x$ is discontinuous at $x = m\pi$, where m is an integer.

(v) Graph of sec x

Facts Related to sec x

(a) Domain = $R \sim (2n+1)\frac{\pi}{2}$, $n \in I$

(b) Range = $(-\infty, -1] \cup [1, \infty)$

(c) Period = 2π

(d) Graph of $\sec x$ is discontinuous at $x = \frac{m\pi}{2}$, where m is an odd integer.

(vi) Graph of cosec x

Facts Related to cosec x

(a) Domain = $R \sim n\pi$, $n \in I$

(b) Range = $(-\infty, -1] \cup [1, \infty)$

(c) Period = 2π

(d) Graph of $\operatorname{cosec} x$ is discontinuous at $x = m\pi$, where m is an integer.

9.5 Trigonometric Identities

A trigonometric equation is an identity, if it is true for all values of the angle or angles involved.

Some Important Identities are Given Below

(i) $\cos^2\theta + \sin^2\theta = 1$ or $\cos^2\theta = 1 - \sin^2\theta$

or $\sin^2\theta = 1 - \cos^2\theta$

(ii) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$

or $\tan^2\theta = \sec^2\theta - 1$

(iii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

Transformation of One Trigonometric Ratio to Another Trigonometric Ratios

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{(1-\cos^2\theta)}$	$\frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$	$\frac{1}{\sqrt{(1+\cot^2\theta)}}$	$\frac{\sqrt{(\sec^2\theta-1)}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{(1-\sin^2\theta)}$	$\cos\theta$	$\frac{1}{\sqrt{(1+\tan^2\theta)}}$	$\frac{\cot\theta}{\sqrt{(1+\cot^2\theta)}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$	$\frac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{(\sec^2\theta-1)}$	$\frac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\cot\theta$	$\frac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{(1-\cos^2\theta)}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{(\sec^2\theta-1)}}$	$\sqrt{(\operatorname{cosec}^2\theta-1)}$
$\sec\theta$	$\frac{1}{\sqrt{(1-\sin^2\theta)}}$	$\frac{1}{\cos\theta}$	$\sqrt{(1+\tan^2\theta)}$	$\frac{\sqrt{(1+\cot^2\theta)}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{(1-\cos^2\theta)}}$	$\frac{\sqrt{(1+\tan^2\theta)}}{\tan\theta}$	$\sqrt{(1+\cot^2\theta)}$	$\frac{\sec\theta}{\sqrt{(\sec^2\theta-1)}}$	$\operatorname{cosec}\theta$

Note Above table is applicable only when $\theta \in (0°, 90°)$.

***Sample* Problem 4** $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$ *is*

(a) ≥ 9 (b) ≤ 9
(c) $= 9$ (d) *None of these*

Interpret (a) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$$= \sin^2\theta + \operatorname{cosec}^2\theta + 2 + \cos^2\theta + \sec^2\theta + 2$$

$$= 7 + \tan^2\theta + \cot^2\theta \quad \ldots(i)$$

And we know that, AM $\geq$ GM

$$\therefore \quad \frac{\tan^2\theta + \cot^2\theta}{2} \geq \sqrt{\tan^2\theta\cot^2\theta}$$

$$\Rightarrow \quad \tan^2\theta + \cot^2\theta \geq 2 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 \geq 7 + 2$$

$$(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 \geq 9$$

Hot Spot

Trigonometric Expression

Trigonometric ratios and function types of problems are directly or indirectly asked in various years in AIEEE (JEE Main) 2013, 2010, 2009, 2002. *To solve such types of problem following steps are given below.*

Step I Firstly use the trigonometric ratios or identities.

Step II If it is require, use transformation of trigonometric ratios or identities.

Step III After that, simplify it.

***Sample* Problem 5** *The expression*

$$\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right]$$

is equal to [NCERT]

(a) 1 (b) 0
(c) 2 (d) -1

Interpret (a) **Step I** Use the trigonometric ratios.

$$\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right]$$

$$= (\sin x)(\cos x)[\tan x + \cot x]$$

$$= \sin x\cos x\left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right] \quad \left[\begin{array}{l}\because \cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta \\ \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta\end{array}\right]$$

Step II Use trigonometric identities.

$$= \sin x\cos x\left[\frac{\sin^2 x + \cos^2 x}{\cos x\sin x}\right] = \frac{\sin x\cos x}{\sin x\cos x}$$

Step III Simplifies it

$$= 1$$

9.6 Complementary and Supplementary Angles

If the sum of two angles is equal to a right angle, then these angles are known as complementary angles of each other. Thus, θ and $90° - \theta$ are complementary angles of each other.

Now, if the sum of two angles is equal to two right angles, then these angles are known as supplementary angles of each other.

Thus, θ and $180° - \theta$ are supplementary angles of each other.

e.g., 23° and 67° are complementary angles of each other while 167° and 13° are supplementary angles of each other.

Trigonometric Ratios of Complementary and Supplementary Angles

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\text{cosec}\,\alpha$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\text{cosec}\,\theta$
$90° - \theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\text{cosec}\,\theta$	$\sec\theta$
$90° + \theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$	$-\tan\theta$	$-\text{cosec}\,\theta$	$\sec\theta$
$180° - \theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$	$-\cot\theta$	$-\sec\theta$	$\text{cosec}\,\theta$
$180° + \theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$\cot\theta$	$-\sec\theta$	$-\text{cosec}\,\theta$

Note

- $\sin\{n\pi + (-1)^n\theta\} = \sin\theta, n \in I$
- $\cos(2n\pi \pm \theta) = \cos\theta, n \in I$
- $\tan(n\pi + \theta) = \tan\theta, n \in I$

***Sample* Problem 6** *The value of* $\text{cosec}(-1410°)$ *is* [NCERT]

(*a*) 3 (*b*) 2 (*c*) −3 (*d*) −1

Interpret (*b*) $\text{cosec}(-1410°) = -\text{cosec}(1410°)$

$[\because \text{cosec}(-\theta) = -\text{cosec}\,\theta]$

$= -\text{cosec}(360 \times 4 - 30)°$

$= -(-\text{cosec}\,30°) = \text{cosec}\,30° = 2$ $[\because \text{cosec}(2n\pi - \theta) = -\text{cosec}\,\theta]$

Check Point 1

1. If $\text{cosec}\,\theta - \cot\theta = p$, then the value of $\text{cosec}\,\theta$ is
   (a) $\frac{1}{2}\left(p + \frac{1}{p}\right)$ (b) $\frac{1}{2}\left(1 - \frac{1}{p}\right)$
   (c) $\left(1 + \frac{1}{p}\right)$ (d) $\left(1 - \frac{1}{p}\right)$

2. If $\tan\alpha = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$, then $\sin\alpha$ is equal to
   (a) $\pm\frac{\sin\theta + \cos\theta}{\sqrt{2}}$ (b) $\pm\frac{\sin\theta - \cos\theta}{\sqrt{2}}$
   (c) $\pm(\sin\theta - \cos\theta)$ (d) None of these

3. If $\text{cosec}\,\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then $a^2b^2(a^2 + b^2)$ is equal to
   (a) 2 (b) $\sin^2\theta$
   (c) $\cos^2\theta$ (d) 1

4. If $2\cos^2 B - 1 = \tan^2 A$, then $\cos A \cos B$ is equal to
   (a) $\pm\frac{1}{2}$ (b) $\pm\frac{1}{3}$
   (c) $\pm\frac{1}{\sqrt{2}}$ (d) None of these

5. The value of the expression $\frac{\tan\theta}{1 - \cot\theta} + \frac{\cot\theta}{1 - \tan\theta} - \sec\theta\,\text{cosec}\,\theta$ is equal to
   (a) −1 (b) 0
   (c) 1 (d) None of these

6. The angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm is
   (a) 25° 27′ (b) 23° 12′
   (c) 25° 12′ (d) None of these

7. If $\text{cosec}\,A + \cot A = \frac{11}{2}$, then $\tan A$ is equal to
   (a) $\frac{21}{22}$ (b) $\frac{15}{16}$
   (c) $\frac{117}{43}$ (d) $\frac{44}{117}$

8. If $3\sin\theta + 5\cos\theta = 5$, then the value of $5\sin\theta - 3\cos\theta$ is
   (a) ± 9 (b) ± 3
   (c) ± 4 (d) ± 5

9. If $\sec A = a + \left(\frac{1}{4a}\right)$, then $\sec A + \tan A$ is equal to
   (a) $2a$ or $\frac{1}{2a}$ (b) a or $\frac{1}{a}$
   (c) $2a$ or $\frac{1}{a}$ (d) a or $\frac{1}{2a}$

10. If $x\sin^3\theta + y\cos^3\theta = \cos\theta\sin\theta$ and $x\sin\theta = y\cos\theta$, then
    (a) $x^2 + y^2 = 2$
    (b) $x^2 + y^2 = 4$
    (c) $x^2 + y^2 = 3$
    (d) $x^2 + y^2 = 1$

9.7 Variation of Values of Trigonometrical Ratios in Different Quadrants

(i)

	II Except sin and cosec, all ratios are negative.	**I** All trigonometric ratios are positive.
	III Except tan and cot, all ratios are negative.	**IV** Except cos and sec, all ratios are negative.

(ii)

	II	**I**
sine	decreases from 1 to 0	increases from 0 to 1
cosine	decreases from 0 to –1	decreases from 1 to 0
tangent	increases from –∞ to 0	increases from 0 to ∞
cotangent	decreases from 0 to –∞	decreases from ∞ to 0
secant	increases from –∞ to –1	increases from 1 to ∞
cosecant	increases from 1 to ∞	decreases from ∞ to 1
	III	**IV**
sine	decreases from 0 to –1	increases from –1 to 0
cosine	increases from –1 to 0	increases from 0 to 1
tangent	increases from 0 to ∞	increases from –∞ to 0
cotangent	decreases from ∞ to 0	decreases from 0 to –∞
secant	decreases from –1 to –∞	decreases from ∞ to 1
cosecant	increases from –∞ to –1	decreases from –1 to –∞

9.8 Trigonometrical Ratios of Some Useful Angles between 0° and 90°

Angle	0°/0	15°/$\frac{\pi}{12}$	18°/$\frac{\pi}{10}$	22.5°/$\frac{\pi}{8}$	30°/$\frac{\pi}{6}$	36°/$\frac{\pi}{5}$	45°/$\frac{\pi}{4}$	54°/$\frac{3\pi}{10}$	60°/$\frac{\pi}{3}$	67.5°/$\frac{3\pi}{8}$	72°/$\frac{2\pi}{5}$	75°/$\frac{5\pi}{12}$	90°/$\frac{\pi}{2}$
sin θ	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1
cos θ	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0
tan θ	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	1	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{3}$	$\sqrt{2}+1$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞
cot θ	∞	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\sqrt{2}+1$	$\sqrt{3}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	1	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	0
sec θ	1	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}-1$	$\sqrt{2}$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	2	$\sqrt{4+2\sqrt{2}}$	$\sqrt{5}+1$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	∞
cosec θ	∞	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	$\sqrt{5}+1$	$\sqrt{4+2\sqrt{2}}$	2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{2}$	$\sqrt{5}-1$	$\frac{2}{\sqrt{3}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	1

***Sample* Problem 7** *The value of*

$$\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ \text{ is}$$ **[NCERT Exemplar]**

(a) 0 (b) 1
(c) 2 (d) 3

Interpret (b) $\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ \ldots \tan 89^\circ$

$= (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ) \ldots (\tan 44^\circ \tan 46^\circ) \tan 45^\circ$

$= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ) \ldots \tan 45^\circ = 1 \cdot 1 \ldots = 1$

***Sample* Problem 8** *The value of*

$$2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} \text{ is}$$ **[NCERT Exemplar]**

(a) 10 (b) –10 (c) 12 (d) 14

Interpret (a) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3}$

$$= 2\sin^2\left(\pi - \frac{\pi}{4}\right) + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3}$$

$$= 2\sin^2\frac{\pi}{4} + 2\times\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2 \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= 2\times\left(\frac{1}{\sqrt{2}}\right)^2 + 2\times\frac{1}{2} + 2\times 4 = 1 + 1 + 8 = 10$$

9.9 Trigonometric Ratios of Compound Angles

An angle made up of the algebraic sum of two or more angles is called compound angles.

Sum and Difference Formulae

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$

(viii) $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

(ix) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(x) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(xi) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

Also,

$\sin(A+B+C) = \sin A\cos B\cos C(\tan A + \tan B + \tan C) - \tan A\tan B\tan C$

(xii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

Also, $\cos(A+B+C)$

$= \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

***Sample* Problem 9** *If* $\cos\theta + \sin\phi = m$ *and* $\sin\theta + \cos\phi = n$, *then* $\frac{1}{2}(m^2 + n^2 - 2)$ *is equal to* **[NCERT Exemplar]**

(a) $\sin(\theta - \phi)$ (b) $\sin(\theta + \phi)$
(c) $\cos(\theta - \phi)$ (d) $\cos(\theta + \phi)$

Interpret (b) Given $m = \cos\theta + \sin\phi$ ...(i)

and $n = \sin\theta + \cos\phi$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$m^2 + n^2 = \cos^2\theta + \sin^2\phi + 2\sin\phi\cos\theta + \sin^2\theta + \cos^2\phi + 2\sin\theta\cos\phi$$

$$\Rightarrow \quad m^2 + n^2 = 2 + 2(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

$$\Rightarrow \quad \sin(\theta + \phi) = \frac{m^2 + n^2 - 2}{2}$$

***Sample* Problem 10** *The expression* $\dfrac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)}$ *is equal to* **[NCERT Exemplar]**

(a) $\left(\dfrac{1 - \tan x}{1 + \tan x}\right)^2$ (b) $\left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2$

(c) $\dfrac{1 + \tan x}{1 - \tan x}$ (d) *None of these*

Interpret (b) $\dfrac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \dfrac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} \times \dfrac{1 + \tan\frac{\pi}{4}\tan x}{\tan\frac{\pi}{4} - \tan x}$

$$\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \text{ and } \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}\right]$$

$$= \frac{1 + \tan x}{1 - \tan x} \times \frac{1 + \tan x}{1 - \tan x} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$$

Product into Sum and Difference Formulae

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ } $A > B$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

***Sample* Problem 11** $\dfrac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta}$ *is equal to*

(*a*) $\dfrac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta}$

(*b*) $\dfrac{\sin 7\theta + \sin 3\theta}{\cos 7\theta + \cos 3\theta}$

(*c*) $\dfrac{\sin 7\theta + \sin 3\theta}{\cos 7\theta - \cos 3\theta}$

(*d*) *None of the above*

Interpret (*a*) $\dfrac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta}$

$$= \frac{2\sin 8\theta \cos\theta - 2\sin 6\theta \cos 3\theta}{2\cos 2\theta \cos\theta - 2\sin 3\theta \sin 4\theta}$$

$$= \frac{(\sin 9\theta + \sin 7\theta) - (\sin 9\theta + \sin 3\theta)}{(\cos 3\theta + \cos\theta) + (\cos 7\theta - \cos\theta)}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta}$$

Sum and Difference into Product Formulae

(i) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

***Sample* Problem 12** *The expression* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$ *is equal to* [**NCERT**]

(*a*) $-\cot 3x$

(*b*) $\cot 3x$

(*c*) $\cot 2x$

(*d*) $-\cot 2x$

Interpret (*b*)

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2} + \cos 3x}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2} + \sin 3x} \quad \text{(by formulae)}$$

$$= \frac{2\cos 3x\cos x + \cos 3x}{2\sin 3x\cos x + \sin 3x}$$

$$= \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)} = \cot 3x$$

Trigonometric Ratios of Multiples of an Angle

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$= 2\cos^2\theta - 1 = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

(iii) $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$

(iv) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(v) $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

(vi) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(vii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(viii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(ix) $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1} = \dfrac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$

***Sample* Problem 13** $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$ *is equal to*

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

Interpret (*d*) $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cdot\cos 20°}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20°\cdot\cos 20°}$$

$$= \frac{4(\sin 60°\cos 20° - \cos 60°\sin 20°)}{\sin 40°}$$

$$= \frac{4\sin 40°}{\sin 40°} = 4$$

Trigonometric Ratios of Submultiple Angles

(i) $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = \dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$

(ii) $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta = 1 - 2\sin^2\frac{1}{2}\theta$

$$= 2\cos^2\frac{1}{2}\theta - 1 = \frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

(iii) $\tan\theta = \dfrac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$ (iv) $\cot\theta = \dfrac{\cot^2\frac{1}{2}\theta - 1}{2\cot\frac{1}{2}\theta}$

Note

- $1 - \cos A = 2\sin^2\frac{A}{2}$
- $1 + \cos A = 2\cos^2\frac{A}{2}$
- $\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha + (n-1)\beta]$
$$= \frac{\sin\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$
- $\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$
$$= \frac{\cos\left\{\alpha + (n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

***Sample* Problem 14** $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ *upto n terms is equal to*

(a) 1 (b) 2 (c) 3 (d) 0

Interpret (d) $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ upto n terms

$$= \frac{\sin\frac{n\cdot 2\pi}{2n}}{\sin\frac{2\pi}{2n}}\cdot\sin\left(\frac{2\cdot\frac{\pi}{n} + (n-1)\frac{2\pi}{n}}{2}\right)$$

$$= \frac{\sin\pi}{\sin\frac{\pi}{n}}\cdot\sin\left(\frac{2\pi + 2n\pi - 2\pi}{2n}\right) = \frac{\sin^2\pi}{\sin\frac{\pi}{n}} = 0$$

$\therefore \sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ upto n terms $= 0$

9.10 Maximum and Minimum (Greatest and Least) Value of $a\sin\theta + b\cos\theta$

Let $a = r\cos\alpha$ and $b = r\sin\alpha$

$\therefore \quad a\sin\theta + b\cos\theta = r(\cos\alpha\sin\theta + \sin\alpha\cos\theta)$

$= r\sin(\theta + \alpha)$

where, $r = \sqrt{a^2 + b^2}$

and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$\because$ Maximum and minimum values of $\sin(\theta + \alpha)$ are 1 and respectively.

Therefore,

(i) Maximum value of $a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}$

(ii) Minimum value of $a\sin\theta + b\cos\theta = -\sqrt{a^2 + b^2}$

***Sample* Problem 15** *The maximum and minimum value of* $6\sin x\cos x + 4\cos 2x$ *are respectively*

(a) 5, 5 (b) −5, 5
(c) 5, −5 (d) −4, 4

Interpret (c) $\because$ $6\sin x\cos x + 4\cos 2x = 3\sin 2x + 4\cos 2x$

$\therefore \quad -\sqrt{3^2 + 4^2} \le 3\sin 2x + 4\cos 2x \le \sqrt{3^2 + 4^2}$

$\Rightarrow \quad -5 \le 3\sin 2x + 4\cos 2x \le 5$

Thus, maximum and minimum value of

$6\sin x\cos x + 4\cos 2x$ are 5 and −5 respectively.

Check Point **2**

1. The expression $3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right\}$ is equal to

(a) 0 (b) 1
(c) 3 (d) $\sin 4\alpha + \cos 6\alpha$

2. $\cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20}$ is equal to

(a) 0 (b) −1
(c) 1 (d) None of these

3. If $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\alpha = -\frac{3}{5}$, where θ does not lie in the third quadrant and α lies in the third quadrant, then $\frac{2\tan\alpha + \sqrt{3}\tan\theta}{\cot^2\theta + \cos\alpha}$ is equal to

(a) $\frac{5}{22}$ (b) $\frac{7}{22}$
(c) $\frac{9}{22}$ (d) $\frac{22}{5}$

4. The value of $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ is equal to

(a) 2 (b) 1/2
(c) 3/2 (d) None of these

5. If $\frac{\sqrt{1 - \sin\alpha}}{\sqrt{1 + \sin\alpha}} = \sec\alpha - \tan\alpha$, then the quadrants in which α lies are

(a) II and III (b) I and IV
(c) I and II (d) III or IV

6. The expression $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$ is equal to

(a) 0 (b) 1
(c) −1 (d) $\frac{1}{2}$

7. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ is equal to

(a) 3 (b) 2
(c) 1 (d) 0

8. If $\sin B = \frac{1}{5}\sin(2A + B)$, then $\frac{\tan(A+B)}{\tan A}$ is equal to
(a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) $\frac{3}{4}$ (d) None of these

9. The minimum value of $9\tan^2\theta + 4\cot^2\theta$ is
(a) 13 (b) 9
(c) 6 (d) 12

10. If $(1+\tan\alpha)(1+\tan 4\alpha) = 2$, $\alpha \in (0, \pi/16)$, then α is equal to
(a) $\frac{\pi}{20}$ (b) $\frac{\pi}{30}$ (c) $\frac{\pi}{40}$ (d) $\frac{\pi}{60}$

11. $\frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$ is equal to
(a) $\tan 3\theta$ (b) $\cot 3\theta$
(c) $\tan 6\theta$ (d) $\cot 6\theta$

12. The greatest value of $4\sin\theta + 3\cos\theta + 10$ is
(a) 13 (b) 15 (c) −15 (d) −13

9.11 Trigonometric Equations

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

e.g., $\cos^2\theta - \sin\theta = \frac{1}{2}$, $\tan m\theta = \cot n\theta$

etc., are trigonometric equations.

9.12 Solutions or Roots of a Trigonometric Equation

A value of the unknown angle which satisfies the given equation, is called a solution or root of the equation.

The trigonometric equation may have infinite number of solutions and can be classified as

(i) Principal solution
(ii) General solution

(i) Principal Solution

The least value of unknown angle which satisfies the given equation, is called a principal solution of trigonometric equation.

(ii) General Solution

We know that trigonometric functions are periodic and solutions of trigonometric equations can be generalized with the help of the periodicity of the trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Note A function $f(x)$ is said to be a periodic function, if a least positive real number T is such that $f(x+T) = f(x)$, then T is known as period of function $f(x)$.

General Solutions of Trigonometric Equations

(i)	$\sin\theta = 0$	$\theta = n\pi, n \in I$
(ii)	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in I$
(iii)	$\tan\theta = 0$	$\theta = n\pi, n \in I$
(iv)	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], n \in I$
(v)	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, \alpha \in [0, \pi], n \in I$
(vi)	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), n \in I$
(vii)	$\left.\begin{array}{l}\sin^2\theta = \sin^2\alpha \\ \cos^2\theta = \cos^2\alpha \\ \tan^2\theta = \tan^2\alpha\end{array}\right\}$	$\theta = n\pi \pm \alpha, n \in I$
(viii)	$\sin\theta = 1$	$\theta = (4n+1)\frac{\pi}{2}, n \in I$
(ix)	$\cos\theta = 1$	$\theta = 2n\pi, n \in I$
(x)	$\cos\theta = -1$	$\theta = (2n+1)\pi, n \in I$
(xi)	$\sin x = \sin\alpha$ and $\cos x = \cos\alpha$	$x = 2n\pi + \alpha$

Note

- If θ is an odd multiple of $\frac{\pi}{2}$ *i.e.*, when $\theta = (2n+1)\frac{\pi}{2}, n \in I$, then $\sec\theta$ and $\tan\theta$ are not defined.
- If θ is an even multiple of $\frac{\pi}{2}$, *i.e.*, when $\theta = n\pi, n \in I$, then $\operatorname{cosec}\theta$ and $\cot\theta$ are not defined.
- The general solution of equations containing $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ is equivalent to that of the equation involving $\sin\theta$, $\cos\theta$, $\tan\theta$.
- If value of θ which satisfies the two trigonometric equations, is least positive value α, then $\theta = 2n\pi + \alpha, n \in I$.

  e.g., $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$,
  then $\theta = 2n\pi + \alpha, n \in I$

  When $\cos\theta = 0$, then $\sin\theta = 1$ or -1

  $\Rightarrow \quad \theta = \left(n + \frac{1}{2}\right)\pi$

  If $\sin\theta = 1$, then n is even and if $\sin\theta = -1$, then n is odd.
  Similarly, when $\sin\theta = 0$, then $\cos\theta = 1$ or -1

  $\Rightarrow \quad \theta = n\pi$

  If $\cos\theta = 1$, then n is even and if $\cos\theta = -1$, then n is odd.

***Sample* Problem 17** *The solution of the equation* $\sin x + \sin 3x + \sin 5x = 0$ *is* [**NCERT**]

(a) $\frac{n\pi}{3}, n\pi \pm \frac{\pi}{3}$ (b) $\frac{n\pi}{2}, n\pi \pm \frac{\pi}{3}$

(c) $\frac{n\pi}{3}, n\pi \pm \frac{\pi}{4}$ (d) *None of these*

Interpret (a) $\sin x + \sin 3x + \sin 5x = 0$

$\Rightarrow \quad (\sin 5x + \sin x) + \sin 3x = 0$

$\Rightarrow \quad 2 \sin \frac{5x + x}{2} \cos \frac{5x - x}{2} + \sin 3x = 0$

$$\left[\because \sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)\right]$$

$\Rightarrow \quad 2 \sin 3x \cos 2x + \sin 3x = 0$

$\Rightarrow \quad \sin 3x (2 \cos 2x + 1) = 0$

Either $\sin 3x = 0$ or $2 \cos 2x + 1 = 0$

When $\sin 3x = 0$,

Then, $3x = n\pi \Rightarrow x = \frac{n\pi}{3}$

When $2 \cos 2x + 1 = 0$,

Then, $\cos 2x = -\frac{1}{2}$

$\Rightarrow \quad \cos 2x = -\cos \frac{\pi}{3}$

$\Rightarrow \quad \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right) \quad [\because \cos(\pi - \theta) = -\cos\theta]$

$\Rightarrow \quad \cos 2x = \cos \frac{2\pi}{3}$

$\Rightarrow \quad 2x = 2n\pi \pm \frac{2\pi}{3}$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3}$

***Sample* Problem 18** *The solution of* $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ *is*

(a) $x = n\pi, n \in I$ (b) $x = \frac{n\pi}{2}, n \in I$

(c) $x = \frac{n\pi}{3}, n \in I$ (d) *None of these*

Interpret (c) We know that,

$$\tan 3x = \tan(x + 2x) = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$$

$\Rightarrow \quad \tan 3x - \tan x \tan 2x \tan 3x = \tan x + \tan 2x$

$\Rightarrow \quad \tan x \tan 2x \tan 3x = \tan 3x - \tan x - \tan 2x \quad \ldots(i)$

Now, $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$

$\Rightarrow \quad \tan x + \tan 2x + \tan 3x = \tan 3x - \tan x - \tan 2x$

[from Eq. (i)]

$\Rightarrow \quad 2 \tan x = -2 \tan 2x$

$\Rightarrow \quad \tan 2x = \tan(-x)$

$2x = n\pi + (-x) \Rightarrow \quad = n\pi$

$\Rightarrow \quad x = \frac{n\pi}{3}$, where $n \in I$

9.13 Solution of Trigonometric Equation of the Form $a \cos\theta + b \sin\theta = c$

Let the equation is

$$a \cos\theta + b \sin\theta = c$$

On dividing by $\sqrt{a^2 + b^2}$ both sides, we get

$$\frac{a}{\sqrt{a^2+b^2}} \cos\theta + \frac{b}{\sqrt{a^2+b^2}} \sin\theta = \frac{c}{\sqrt{a^2+b^2}} \quad \ldots(i)$$

Let $\tan\alpha = \frac{b}{a}$

$\therefore \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

and $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

From Eq. (i), we get

$$\cos\theta \cos\alpha + \sin\theta \sin\alpha = \frac{c}{\sqrt{a^2+b^2}}$$

$\Rightarrow \quad \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}}$

If $|c| > \sqrt{a^2+b^2}$,

then equation $a \cos\theta + b \sin\theta = c$ has no solution and if,

$$|c| \le \sqrt{a^2+b^2},$$

then let $\cos\phi = \frac{|c|}{\sqrt{a^2+b^2}}$

$\Rightarrow \quad \cos(\theta - \alpha) = \cos\phi$

$\Rightarrow \quad \theta - \alpha = 2n\pi \pm \phi$

$\Rightarrow \quad \theta = 2n\pi \pm \phi + \alpha, n \in I$

Moment to Think!

(i) While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.

(ii) Never cancel terms, containing unknown terms on the two sides, which are in product. It may cause loss of genuine solution.

(iii) The answer should not contain such values of angles which make any of the terms undefined or infinite.

(iv) Domain should not change, if it changes, necessary corrections must be made.

(v) Check that denominator is not zero at any stage while solving equations.

***Sample* Problem 19** *The solution of the equation* $k \cos x - 3 \sin x = k + 1$ *is possible only if*

(a) $k \in (-\infty, 4]$ (b) $k \in (-\infty, \infty)$

(c) $k \in [4, \infty)$ (d) *None of these*

Interpret (a) Here, $k \cos x - 3 \sin x = k + 1$ could be rewritten as

$$\frac{k}{\sqrt{k^2+9}} \cos x - \frac{3}{\sqrt{k^2+9}} \sin x = \frac{k+1}{\sqrt{k^2+9}}$$

$$\Rightarrow \quad \cos(x + \phi) = \frac{k+1}{\sqrt{k^2+9}}$$

where, $\cos \phi = \dfrac{k}{\sqrt{k^2+9}}$ and $\sin \phi = \dfrac{3}{\sqrt{k^2+9}}$

which posses solution only, if

$$-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1 \Rightarrow \left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$$

$$\Rightarrow \quad (k+1)^2 \le k^2 + 9$$

$$\Rightarrow \quad k^2 + 2k + 1 \le k^2 + 9$$

$$\Rightarrow \quad k \le 4$$

Hence, the interval in which $k \cos x - 3 \sin x = k + 1$ admits solution for k is $(-\infty, 4]$.

9.14 Solution of Trigonometric Inequalities by Graphs

To solve trigonometric inequality, we use graph of the trigonometric functions.

***Sample* Problem 20** *The solution set of inequality* $\sin x > \frac{1}{2}$ *is*

(a) $\bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$

(b) $\bigcup_{n \in I} \left(2n\pi - \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

(c) $\bigcup_{n \in I} \left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$

(d) $\bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

Interpret (d) When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. From the graph of $y = \sin x$, it is obvious that, between 0 and 2π,

$$\sin x < \frac{1}{2} \text{ for } \frac{\pi}{6} < x < \frac{5\pi}{6}$$

Hence, $\sin x > \frac{1}{2}$

$$\Rightarrow \quad 2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$$

The required solution set is $\bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$.

***Sample* Problem 21** *The solution set of inequality* $\cos x \ge -\frac{1}{2}$ *is*

(a) $\bigcup_{n \in I} \left(2\pi n - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$

(b) $\bigcup_{n \in I} \left(2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$

(c) $\bigcup_{n \in I} \left(n\pi - \frac{2\pi}{3}, n\pi + \frac{2\pi}{3}\right)$

(d) $\bigcup_{n \in I} \left(2n\pi - \frac{2\pi}{3}, n\pi - \frac{\pi}{3}\right)$

Interpret (a) From the graph of $y = \cos x$, it is obvious that $\cos x \ge -\frac{1}{2}$

if $-\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$

Hence, $\cos x \ge -\frac{1}{2}$

$$\Rightarrow \quad 2n\pi - \frac{2\pi}{3} \le x \le 2n\pi + \frac{2\pi}{3}$$

The required solution set is

$$\bigcup_{n \in I} \left(2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$$

Check Point **3**

1. If $4\sin^2\theta + 2(\sqrt{3}+1)\cos\theta = 4+\sqrt{3}$, the general value of θ is

(a) $2n\pi + \frac{\pi}{3}$ (b) $2n\pi + \frac{\pi}{4}$

(c) $n\pi - \frac{\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{6}, 2n\pi \pm \frac{\pi}{3}$

2. The solution set of $\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1$ and $\frac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x}$ is (k is an integer)

(a) $k\pi + (-1)^k \frac{\pi}{4}$ (b) $\frac{\pi(8k+3)}{4}$

(c) $\frac{\pi(8k+1)}{4}$ (d) None of these

3. The minimum value of $2^{\sin x} + 2^{\cos x}$ is

(a) 1 (b) 2

(c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $2^{1-\frac{1}{\sqrt{2}}}$

4. If $\frac{1-\tan x}{1+\tan x} = \tan y$ and $x - y = \frac{\pi}{6}$, then x and y are respectively

(a) $\frac{5\pi}{24}, \frac{\pi}{24}$ (b) $-\frac{7\pi}{24}, -\frac{11\pi}{24}$

(c) $-\frac{115\pi}{24}, -\frac{119\pi}{24}$ (d) All of these

5. The equation $\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$ possesses a solution, if

(a) $a > -3$

(b) $a < -2$

(c) $-3 \le a \le -2$

(d) a is any positive integer

6. The general solution of the equation $\tan^2\theta + 2\sqrt{3}\tan\theta = 1$ is given by

(a) $\frac{\pi}{2}$ (b) $\left(n + \frac{1}{2}\right)\pi$

(c) $(6n+1)\frac{\pi}{12}$ (d) $\frac{n\pi}{12}$

7. If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$, then θ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \cos^{-1}\left(\frac{3}{5}\right)$

(c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) $\frac{\pi}{3}, \pi - \cos^{-1}\left(\frac{3}{5}\right)$

8. If $1 + \sin\theta + \sin^2\theta + \ldots\infty = 4 + 2\sqrt{3}, 0 < \theta < \pi, \theta \ne \frac{\pi}{2}$, then

(a) $\theta = \frac{\pi}{6}$ (b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{3}$ or $\frac{\pi}{6}$ (d) $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

9. If $0 < \theta < 2\pi$, then the interval of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$

(c) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (d) $\left(\frac{41\pi}{48}, \pi\right)$

10. If $2\sin x + 1 \ge 0$ and $x \in [0, 2\pi]$, then the solution set for x is

(a) $\left[0, \frac{7\pi}{6}\right]$ (b) $\left[0, \frac{7\pi}{6}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$

(c) $\left[\frac{11\pi}{6}, 2\pi\right]$ (d) None of these

WORKED OUT
Examples

Example 1 *If* $\cos(\alpha+\beta)=\frac{4}{5}$ *and* $\sin(\alpha-\beta)=\frac{5}{13}$, *where* α *lies between* 0 *and* $\frac{\pi}{4}$, *find the value of* $\tan 2\alpha$. **[NCERT Exemplar]**

(*a*) $\frac{56}{33}$ (*b*) $\frac{54}{33}$

(*c*) $\frac{56}{29}$ (*d*) *None of these*

Solution (*a*) We have, $\cos(\alpha+\beta)=\frac{4}{5}$ and $\sin(\alpha-\beta)=\frac{5}{13}$

In $\Delta\, ABC$, by using Pythagoras theorem,

$$BC=\sqrt{(AC)^2+(AB)^2}=\sqrt{9+16}=5$$

$$\therefore \quad \tan(\alpha+\beta)=\frac{3}{4}$$

Similarly, in ΔDEF, by using Pythagoras theorem

$$EF=\sqrt{DF^2+DE^2}=\sqrt{5^2+12^2}=13$$

$$\therefore \quad \tan(\alpha-\beta)=\frac{5}{12}$$

$$\therefore \quad \tan 2\alpha=\tan[(\alpha+\beta)+(\alpha-\beta)]$$

$$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$\tan 2\alpha=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}=\frac{(36+20)}{12\times 4}\times\frac{12\times 4}{(48-15)}=\frac{56}{33}$$

Example 2 *If* $\sec x\cos 5x+1=0$, *where* $0<x\le\frac{\pi}{2}$, *then find the value of x.* **[NCERT Exemplar]**

(*a*) $\frac{\pi}{6},\frac{\pi}{4}$ (*b*) $\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{2}$

(*c*) $\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{2}$ (*d*) *None of these*

Solution (*b*) Given, $\sec x\cos 5x+1=0$

$$\Rightarrow \quad \frac{1}{\cos x}\times\cos 5x+\frac{1}{1}=0$$

$$\Rightarrow \quad \frac{\cos 5x+\cos x}{\cos x}=0$$

$$\Rightarrow \quad \cos 5x+\cos x=0$$

$$\Rightarrow \quad 2\cos\frac{5x+x}{2}\cos\frac{5x-x}{2}=0$$

$$\left(\because \cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right)$$

$$\Rightarrow \quad 2\cos 3x\cos 2x=0$$

$$\Rightarrow \quad \cos 3x=0 \text{ or } \cos 2x=0$$

$$\Rightarrow \quad 3x=(2n+1)\frac{\pi}{2} \quad \text{or} \quad 2x=(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x=(2n+1)\frac{\pi}{6} \text{ or } x=(2n+1)\frac{\pi}{4}$$

If $n=0$, then $x=\frac{\pi}{6}$ or $x=\frac{\pi}{4}$

If $n=1$, then $x=3\times\frac{\pi}{6}=\frac{\pi}{2}$ or $x=\frac{3\pi}{4}>\frac{\pi}{2}$

Therefore, the values of x are $\frac{\pi}{6},\frac{\pi}{4}$ and $\frac{\pi}{2}$.

Example 3 *If* $\sin(\theta+\alpha)=a$ *and* $\sin(\theta+\beta)=b$, *then* $\cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$ *is equal to* **[NCERT Exemplar]**

(*a*) $1-2a^2+2b^2$ (*b*) $1+2a^2+2b^2$

(*c*) $1-2a^2-2b^2$ (*d*) *None of these*

Solution (*c*) We can write $\cos(\alpha-\beta)=\cos[(\theta+\alpha)-(\theta+\beta)]$

$$=\cos(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\alpha)\sin(\theta+\beta)$$

We have to find the value of $(\alpha-\beta)$ and we are given $(\theta+\alpha)$, so we write $(\alpha-\beta)$ in terms of $(\theta+\alpha)$ and $(\theta+\beta)$

$$[\because \cos(A-B)=\cos A\cos B+\sin A\sin B)]$$

$$=\sqrt{1-\sin^2(\theta+\alpha)}\sqrt{1-\sin^2(\theta+\beta)}+ab$$

$$[\because \sin(\theta+\alpha)=a \text{ and } \sin(\theta+\beta)=b]$$

$$\Rightarrow \quad \cos(\alpha-\beta)=\sqrt{1-a^2}\sqrt{1-b^2}+ab$$

Now, $\cos 2(\alpha-\beta)=2\cos^2(\alpha-\beta)-1$

$$=2(\sqrt{1-a^2}\sqrt{1-b^2}+ab)^2-1$$

$$\therefore \quad \cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$$

$$= 2(\sqrt{1-a^2}\sqrt{1-b^2} + ab)^2 - 1 - 4ab(\sqrt{1-a^2}\sqrt{1-b^2} + ab)$$
$$= 2[(1-a^2)(1-b^2) + a^2b^2 + 2ab\sqrt{(1-a^2)(1-b^2)}]$$
$$-1 - 4ab[\sqrt{(1-a^2)(1-b^2)} + ab]$$
$$= 2(1 - a^2 - b^2 + a^2b^2) + 2a^2b^2 + 4ab\sqrt{(1-a^2)(1-b^2)}$$
$$-1 - 4ab\sqrt{(1-a^2)(1-b^2)} - 4a^2b^2$$
$$= 2 - 2a^2 - 2b^2 + 2a^2b^2 + 2a^2b^2 - 1 - 4a^2b^2$$
$$= 2 - 1 - 2a^2 - 2b^2 + 4a^2b^2 - 4a^2b^2 = 1 - 2a^2 - 2b^2$$

Example 4 *Find the value of the expression*

$$\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$$ [**NCERT Exemplar**]

(a) $\frac{2}{3}$ (b) $-\frac{3}{2}$

(c) $\frac{3}{2}$ (d) $-\frac{2}{3}$

Solution (c) $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\left(\frac{\pi}{2} + \frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{2} + \frac{3\pi}{8}\right)$$
$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}$$
$$\left[\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right]$$
$$= \left(\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}\right) + \left(\cos^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8}\right)$$
$$= \left(\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8} + 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$$
$$+ \left(\cos^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + 2\sin^2\frac{3\pi}{8}\cos^2\frac{3\pi}{8} - 2\sin^2\frac{3\pi}{8}\cos^2\frac{3\pi}{8}\right)$$
$$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right)^2 - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$$
$$+ \left(\cos^2\frac{3\pi}{8} + \sin^2\frac{3\pi}{8}\right)^2 - 2\sin^2\frac{3\pi}{8}\cos^2\frac{3\pi}{8}$$
$$= 1 - \frac{1}{2}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 + 1 - \frac{1}{2}\left(2\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}\right)^2$$
$$= 2 - \frac{1}{2}\left(\sin 2\times\frac{\pi}{8}\right)^2 - \frac{1}{2}\left(\sin 2\times\frac{3\pi}{8}\right)^2$$
$$= 2 - \frac{1}{2}\sin^2\frac{\pi}{4} - \frac{1}{2}\sin^2\frac{3\pi}{4} = 2 - \frac{1}{2}\times\left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}\times\left(\frac{1}{\sqrt{2}}\right)^2$$
$$\left[\because \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$
$$= 2 - \frac{1}{2}\times\frac{1}{2} - \frac{1}{2}\times\frac{1}{2} = 2 - \frac{1}{4} - \frac{1}{4} = 2 - \frac{1}{2} = \frac{3}{2}$$

Example 5 *The value of expression*

$$2\cos\frac{\pi}{3}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$ *is equal to* [**NCERT**]

(a) -1 (b) 0

(c) 1 (d) 2

Solution (b) $2\cos\frac{\pi}{3}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$$= \cos\left(\frac{9\pi}{13} + \frac{\pi}{13}\right) + \cos\left(\frac{9\pi}{13} - \frac{\pi}{13}\right) + \cos\frac{5\pi}{13} + \cos\frac{3\pi}{13}$$
(by formulae)
$$= \cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{5\pi}{13} + \cos\frac{3\pi}{13}$$
$$= \left(\cos\frac{10\pi}{13} + \cos\frac{3\pi}{13}\right) + \left(\cos\frac{8\pi}{13} + \cos\frac{5\pi}{13}\right)$$
$$= 2\cos\frac{\frac{10\pi}{13} + \frac{3\pi}{13}}{2}\cos\frac{\frac{10\pi}{13} - \frac{3\pi}{13}}{2}$$
$$+ 2\cos\frac{\frac{8\pi}{13} + \frac{5\pi}{13}}{2}\cos\frac{\frac{8\pi}{13} - \frac{5\pi}{13}}{2}$$ (by formulae)
$$= 2\cos\frac{\pi}{2}\cos\frac{7\pi}{26} + 2\cos\frac{\pi}{2}\cos\frac{3\pi}{26} = 0 + 0 = 0 \quad \left(\because \cos\frac{\pi}{2} = 0\right)$$

Example 6 *If* $\cos 5\theta = a_0 + a_1\cos\theta + a_2\cos^2\theta + a_3\cos^3\theta + a_4\cos^4\theta + a_5\cos^5\theta$, *then the value of* $a_0 + \sum_{i=1}^{5} a_i$ *is equal to*

(a) 0 (b) 1 (c) 2 (d) 3

Solution (b) $\cos 5\theta = \cos(3\theta + 2\theta)$

$$= \cos 3\theta\cos 2\theta - \sin 3\theta\sin 2\theta$$
$$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1)$$
$$-(3\sin\theta - 4\sin^3\theta)(2\sin\theta\cos\theta)$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2\cos\theta\sin^2\theta(3 - 4\sin^2\theta)$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta$$
$$-2\cos\theta(1 - \cos^2\theta)\{3 - 4(1 - \cos^2\theta)\}$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2\cos\theta(1 - \cos^2\theta)(4\cos^2\theta - 1)$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2\cos\theta(5\cos^2\theta - 4\cos^4\theta - 1)$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

But $\cos 5\theta = a_0 + a_1\cos\theta + a_2\cos^2\theta + a_3\cos^3\theta + a_4\cos^4\theta + a_5\cos^5\theta$

Hence, $a_0 = 0$, $a_1 = 5$, $a_2 = 0$, $a_3 = -20$, $a_4 = 0$ and $a_5 = 16$

Thus, $a_0 + \sum_{i=1}^{5} a_i = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$

$$= 0 + 5 + 0 - 20 + 0 + 16 = 21 - 20 = 1$$

Example 7 *If* $\cos^3\theta + \cos^3(120° + \theta) + \cos^3(240° + \theta) = a\cos 3\theta$, *then the value of a is*

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{7}{4}$

Solution (c) LHS $= \frac{1}{4}[4\cos^3\theta + 4\cos^3(120° + \theta)$
$+ 4\cos^3(240° + \theta)]$

$$= \frac{1}{4}[3\cos\theta + \cos 3\theta + 3\cos(120° + \theta)$$
$$+ \cos(360° + 3\theta) + \cos(240° + \theta) + \cos(720° + 3\theta)]$$
$$(\because 4\cos^3\theta = 3\cos\theta + \cos 3\theta)$$
$$= \frac{1}{4}[3\cos\theta + \cos 3\theta + \cos 3\theta + 3\{\cos(120° + \theta)$$
$$+ \cos(240° + \theta) + \cos 3\theta\}]$$
$$= \frac{1}{4}[3\cos\theta + 3\cos 3\theta + 3 \times 2\cos(180° + \theta)\cos 60°]$$
$$= \frac{1}{4}\left[3\cos\theta + 3\cos 3\theta + 3 \times 2\,(-\cos\theta)\frac{1}{2}\right]$$
$$= \frac{1}{4}[3\cos\theta + 3\cos 3\theta - 3\cos\theta]$$
$$= \frac{1}{4}[3\cos 3\theta] = \frac{3}{4}\cos 3\theta$$

and $\quad$ RHS $= a\cos 3\theta$

$\therefore \quad a = \frac{3}{4}$

Example 8 *If in any ΔABC, the value of $\angle A$ is obtained from the equation $3\cos A + 2 = 0$, then the equation whose roots are $\sin A$ and $\tan A$, is*

(a) $6x^2 + \sqrt{5}x - 5 = 0$ $\quad$ (b) $6x^2 + 5x - 5 = 0$

(c) $x^2 + \sqrt{5}x - 5 = 0$ $\quad$ (d) *None of these*

Solution (a) The given equation is

$$3\cos A + 2 = 0$$

$\therefore \quad \cos A = -\frac{2}{3}$

$\because$ A is angle of a triangle.

$\therefore \quad A < \pi \Rightarrow \sin A$ is positive.

$\therefore \quad \sin A = \sqrt{(1 - \cos^2 A)} = \frac{\sqrt{5}}{3}$

$\Rightarrow \quad \tan A = \frac{\sin A}{\cos A} = -\frac{\sqrt{5}}{2}$

$\therefore \quad \sin A + \tan A = -\frac{\sqrt{5}}{6}$

and $\quad \sin A \tan A = -\frac{5}{6}$

Hence, the equation whose roots are $\sin A$ and $\tan A$, is

$$x^2 - (\sin A + \tan A)x + \sin A \tan A = 0$$

$\Rightarrow \quad x^2 + \frac{\sqrt{5}}{6}x + \left(-\frac{5}{6}\right) = 0$

$\Rightarrow \quad 6x^2 + \sqrt{5}x - 5 = 0$

Example 9 *If $x + y + z = xyz$, then the value of $x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2)$ is*

(a) xyz $\quad$ (b) $2xyz$

(c) $4xyz$ $\quad$ (d) *None of these*

Solution (c) Let $x = \tan A$, $y = \tan B$ and $z = \tan C$

Given, $\quad x + y + z = xyz$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$$= -\tan C(1 - \tan A \tan B)$$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow \quad \tan(A + B) = \tan(-C)$

$\Rightarrow \quad A + B = n\pi - C$

$\Rightarrow \quad A + B + C = n\pi$

$\Rightarrow \quad 2A + 2B + 2C = 2n\pi$

$\Rightarrow \quad \tan(2A + 2B + 2C) = \tan 2n\pi = 0$

$$\Rightarrow \quad \frac{\left[\begin{array}{l}\tan 2A + \tan 2B + \tan 2C \\ \quad - \tan 2A \tan 2B \tan 2C\end{array}\right]}{\left[\begin{array}{l}1 - \tan 2A \tan 2B \\ \quad - \tan 2B \tan 2C - \tan 2C \tan 2A\end{array}\right]} = 0$$

$\Rightarrow \quad \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

$$\Rightarrow \quad \frac{2\tan A}{1 - \tan^2 A} + \frac{2\tan B}{1 - \tan^2 B} + \frac{2\tan C}{1 - \tan^2 C}$$
$$= \frac{2\tan A}{1 - \tan^2 A} \cdot \frac{2\tan B}{1 - \tan^2 B} \cdot \frac{2\tan C}{1 - \tan^2 C}$$

Hence, on putting $\tan A = x$, $\tan B = y$, $\tan C = z$, we get

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}$$

On multiplying both sides by $\frac{1}{2}(1 - x^2)(1 - y^2)(1 - z^2)$, we get

$$x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2) = 4xyz$$

Example 10 *If* $\operatorname{cosec}\theta = \frac{x^2 - y^2}{x^2 + y^2}$, *where x and y are two unequal non-zero real numbers, then the number of real values of θ is*

(a) 0 $\quad$ (b) 1

(c) 2 $\quad$ (d) *infinite*

Solution (a) $\because$ $x \neq y$ and they are real.

So, $\quad x^2 + y^2 > x^2 - y^2$

$\therefore \quad \operatorname{cosec}\theta < 1$, if $x > y$

and $\quad \operatorname{cosec}\theta > -1$, if $x < y$

$\therefore \quad -1 < \operatorname{cosec}\theta < 1$

which is not possible because we know that $\operatorname{cosec}\theta \geq 1$ or $\operatorname{cosec}\theta \leq -1$.

Hence, there is no real value of θ exist.

Example 11 *The maximum and minimum values of* $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right)$ *are respectively*

(a) 10, 4 $\quad$ (b) 10, – 4

(c) –10, 4 $\quad$ (d) *None of these*

Solution (*b*) $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right)$

$$= 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3}$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$$

Here, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

$$\le \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2}$$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 + 3 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 7 + 3, \forall\ \theta$

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10, \forall\ \theta$

Example 12 *If* $2^{\sin x} + 2^{\cos x} \ge 2^{1-y}$, *then the value of y is*

(*a*) 1 (*b*) $\sqrt{2}$ (*c*) 1/2 (*d*) $1/\sqrt{2}$

Solution (*d*) We know that,

$$\text{AM} \ge \text{GM}$$

$\therefore \quad \frac{1}{2}(2^{\sin x} + 2^{\cos x}) \ge \sqrt{(2^{\sin x}\cdot 2^{\cos x})}$

$\Rightarrow \quad \frac{1}{2}(2^{\sin x} + 2^{\cos x}) \ge 2^{(\sin x + \cos x)/2}$

$\Rightarrow \quad 2^{\sin x} + 2^{\cos x} \ge 2^{1 + (\sin x + \cos x)/2}$

But $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \ge -\sqrt{2}$

$\therefore \quad 2^{\sin x} + 2^{\cos x} \ge 2^{1-\sqrt{2}/2}$

$\Rightarrow \quad 2^{\sin x} + 2^{\cos x} \ge 2^{1-(1/\sqrt{2})}, \forall\ x \in R$

$\therefore \quad y = \frac{1}{\sqrt{2}}$

Example 13 *The solution set of* $\tan 2x \tan 3x = 1$ *is*

(*a*) $\left\{(2n+1)\frac{\pi}{10} : n \in I, n \ne 5k + 2, k \in I\right\}$

(*b*) $\left\{(2n+1)\frac{\pi}{10}, n \in I\right\}$

(*c*) $\{(2n+1)\pi, n \in I, n \ne 5k + 2, k \in I\}$

(*d*) *None of the above*

Solution (*a*) $\tan 2x \tan 3x = 1$

$\Rightarrow \quad \sin 2x \sin 3x = \cos 2x \cos 3x$

$\Rightarrow \quad \cos 3x \cos 2x - \sin 3x \sin 2x = 0$

$\Rightarrow \quad \cos(3x + 2x) = 0$

$\Rightarrow \quad 5x = (2n+1)\frac{\pi}{2}, n \in I$

For this x, $\cos 2x \ne 0$, $\cos 3x \ne 0$

$\therefore$ x is not odd multiple of $\frac{\pi}{2}$.

$\Rightarrow \quad (2n+1)\frac{\pi}{10} \ne (2k+1)\frac{\pi}{2}, k \in I$

$\Rightarrow \quad 2n + 1 \ne 10k + 5 \Rightarrow n \ne 5k + 2, k \in I$

Hence, required set of solution

$$= \left\{(2n+1)\frac{\pi}{10} : n \in I, n \ne 5k + 2, k \in I\right\}$$

Example 14 *If* $3\sin^2\theta + 2\sin^2\phi = 1$ *and* $3\sin 2\theta = 2\sin 2\phi, 0 < \theta < \frac{\pi}{2}$ *and* $0 < \phi < \frac{\pi}{2}$, *then the value of* $\theta + 2\phi$ *is*

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{2}$

(*c*) π (*d*) *None of these*

Solution (*b*) Given that,

$$3\sin^2\theta + 2\sin^2\phi = 1$$

$\Rightarrow \quad 3\sin^2\theta = \cos 2\phi$ ...(i)

and $\quad 3\sin\theta\cos\theta = \sin 2\phi$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$9\sin^2\theta(\sin^2\theta + \cos^2\theta) = 1$$

$\Rightarrow \quad \sin\theta = \frac{1}{3}$ and $\cos\theta = \frac{2\sqrt{2}}{3}$

$\therefore \quad \cos 2\phi = 3 \times \frac{1}{9} = \frac{1}{3}$ and $\sin 2\phi = \frac{2\sqrt{2}}{3}$

$\therefore \quad \cos(\theta + 2\phi) = \cos\theta\cos 2\phi - \sin\theta\sin 2\phi$

$$= \frac{2\sqrt{2}}{3}\cdot\frac{1}{3} - \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = 0$$

and $\quad \theta + 2\phi < \frac{3\pi}{2} \quad \therefore\ \theta + 2\phi = \frac{\pi}{2}$

Example 15 *The number of solutions of*

$$|\cot x| = \cot x + \frac{1}{\sin x}, x \in [0, 3\pi] \text{ is}$$

(*a*) 1 (*b*) 0

(*c*) 2 (*d*) *None of these*

Solution (*c*) $|\cot x| = \cot x + \frac{1}{\sin x}, \quad x \in [0, 3\pi]$

Let $\quad \cot x > 0$

$\Rightarrow \cot x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$ (which is not possible)

Let $\quad \cot x < 0$

$\Rightarrow \quad -\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow \quad -2\cot x = \frac{1}{\sin x}$

$\Rightarrow \quad \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{8\pi}{3}$

$\therefore$ Number of solutions is 2.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Trigonometric Ratio, Function and Identities

1. Find the radian measure corresponding to the degree – 47°30′. [NCERT]

(a) $\frac{-19\pi}{72}$ rad (b) $\frac{19\pi}{72}$ rad
(c) $\frac{13\pi}{72}$ rad (d) None of these

2. Find the degree measure corresponding to the radian $\frac{11}{16}$. [NCERT]

(a) 39°22′32″ (b) 39°22′30″
(c) 39°23′33″ (d) None of these

3. If in two circles, arcs of the same length subtend angles of 60° and 75° at the centre, find the ratio of their radii. [NCERT]

(a) 4 : 5 (b) 5 : 4 (c) 6 : 5 (d) 5 : 6

4. Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.

(a) $\frac{5\pi}{13}$ cm (b) $\frac{5\pi}{12}$ cm
(c) $\frac{7\pi}{12}$ cm (d) None of these

5. If $\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = y$, then $\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is equal to [NCERT Exemplar]

(a) y (b) $2y$
(c) y^2 (d) $\frac{y}{2}$

6. The value of the expression

$$1 - \frac{\sin^2 y}{1+\cos y} + \frac{1+\cos y}{\sin y} - \frac{\sin y}{1-\cos y} \text{ is equal to}$$

(a) 0 (b) 1
(c) $\sin y$ (d) $\cos y$

7. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to [NCERT Exemplar]

(a) 1 (b) 4
(c) 2 (d) None of these

8. If $f(x) = \cos^2 x + \sec^2 x$, then [NCERT Exemplar]

(a) $f(x) < 1$ (b) $f(x) = 1$
(c) $2 < f(x) < 1$ (d) $f(x) \geq 2$

9. Which of the following number is rational ?

(a) $\sin 15°$ (b) $\cos 15°$
(c) $\sin 15° \cos 15°$ (d) $\sin 15° \cos 75°$

10. If θ lies in the second quadrant, then the value of $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ is equal to

(a) $2\sec\theta$ (b) $-2\sec\theta$
(c) $2\operatorname{cosec}\theta$ (d) None of these

11. If $\sin\theta = -\frac{4}{5}$ and θ lies in the third quadrant, then $\cos\frac{\theta}{2}$ is equal to [NCERT Exemplar]

(a) $\frac{1}{\sqrt{5}}$ (b) $-\frac{1}{\sqrt{5}}$
(c) $\sqrt{\frac{2}{5}}$ (d) $-\sqrt{\frac{2}{5}}$

12. Which of the following is not correct? [NCERT Exemplar]

(a) $\sin\theta = -\frac{1}{5}$ (b) $\cos\theta = 1$
(c) $\sec\theta = \frac{1}{2}$ (d) $\tan\theta = 20$

13. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is [NCERT Exemplar]

(a) $\frac{1}{\sqrt{2}}$ (b) 0
(c) 1 (d) – 1

14. If $\tan\theta = 3$ and θ lies in third quadrant, then the value of $\sin\theta$ is [NCERT Exemplar]

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$

(c) $\frac{-3}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$

15. The value of $\tan 75° - \cot 75°$ is equal to [NCERT Exemplar]

(a) $2\sqrt{3}$ (b) $2 + \sqrt{3}$

(c) $2 - \sqrt{3}$ (d) 1

16. Which of the following is correct ? [NCERT Exemplar]

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$

(c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{18°} \sin 1$

17. If $\tan\alpha = \frac{m}{m+1}$, $\tan\beta = \frac{1}{2m+1}$, then $\alpha + \beta$ is equal to [NCERT Exemplar]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

18. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{\operatorname{cosec}^2\alpha + 2\cot\alpha}$ is equal to

(a) $1 + \cot\alpha$ (b) $1 - \cot\alpha$

(c) $-1 - \cot\alpha$ (d) $-1 + \cot\alpha$

19. If $\tan A + \sin A = m$ and $\tan A - \sin A = n$, then $\frac{(m^2 - n^2)^2}{mn}$ is equal to

(a) 4 (b) 3

(c) 16 (d) 9

20. The value of $6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$ is

(a) -3 (b) 0

(c) 1 (d) 3

21. If $\frac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = x$, then $\frac{1 - \cos\alpha - \sin\alpha}{\cos\alpha}$ is equal to

(a) $\frac{1}{x}$ (b) x

(c) $1 - x$ (d) None of these

22. If $\sin\theta + \cos\theta = m$ and $\sec\theta + \operatorname{cosec}\theta = n$, then $n(m+1)(m-1)$ is equal to

(a) m (b) n

(c) $2m$ (d) $2n$

23. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x + y + z$ is equal to

(a) 1 (b) 0

(c) -1 (d) None of these

24. If x lies in IInd quadrant, then $\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$ is equal to

(a) $\sin\frac{x}{2}$ (b) $\tan\frac{x}{2}$

(c) $\sec\frac{x}{2}$ (d) $\operatorname{cosec}\frac{x}{2}$

25. If $\tan\theta, 2\tan\theta + 2$ and $3\tan\theta + 3$ are in GP, then the value of $\frac{7 - 5\cot\theta}{9 - 4\sqrt{\sec^2\theta - 1}}$ is

(a) $\frac{12}{5}$ (b) $-\frac{33}{28}$

(c) $\frac{33}{100}$ (d) $\frac{12}{13}$

26. If A, B, C and D are the angles of a cyclic quadrilateral, then $\cos A + \cos B + \cos C + \cos D$ is equal to

(a) $2(\cos A + \cos C)$ (b) $2(\cos A + \cos B)$

(c) $2(\cos A + \cos D)$ (d) 0

27. If $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ and $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}, 0 < A, B < \frac{\pi}{2}$, then

(a) $\tan A = \frac{\sqrt{3}}{\sqrt{5}}$ (b) $\tan A = \frac{\sqrt{5}}{\sqrt{3}}$

(c) $\tan A = 2$ (d) $\tan B = 2$

28. If $a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$, then $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

(a) $2a^2$ (b) $2a^{1/3}$

(c) $2a^{2/3}$ (d) $2a^3$

29. If $\cos\theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$, then $\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$ is equal to

(a) $\sin 2\theta$ (b) $\cos 2\theta$

(c) $\tan 2\theta$ (d) None of these

30. The value of $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is equal to [NCERT Exemplar]

(a) 0 (b) 1

(c) 3 (d) $\sin 4\alpha + \sin 6\alpha$

31. If $\tan x + \cot x = 2$, then $\sin^{2n} x + \cos^{2n} x$ is equal to

(a) 2^n (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) None of these

32. If $-\pi \le x \le \pi$, $-\pi \le y \le \pi$ and $\cos x + \cos y = 2$, then the value of $\cos(x - y)$ is

(a) -1 (b) 0

(c) 1 (d) None of these

33. Let n be a positive integer such that $\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}$. Then,

(a) $n = 6$ (b) $n = 1, 2, 3, \ldots, 8$
(c) $n = 5$ (d) None of these

Trigonometric Ratios of Sum and Difference of Two and Three Angles

34. If $m\sin\theta = n\sin(\theta+2\alpha)$, then $\tan(\theta+\alpha)\cot\alpha$ is equal to [NCERT Exemplar]

(a) $(m+n)(m-n)$ (b) $\frac{m-n}{m+n}$
(c) $\frac{m+n}{m-n}$ (d) $2(m+n)(m-n)$

35. The value of $\frac{\sin(B+A)+\cos(B-A)}{\sin(B-A)+\cos(B+A)}$ is equal to

(a) $\frac{\cos B+\sin B}{\cos B-\sin B}$ (b) $\frac{\cos A+\sin A}{\cos A-\sin A}$
(c) $\frac{\cos A-\sin A}{\cos A+\sin A}$ (d) None of these

36. The expression

$\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B$ is

(a) dependent on B (b) dependent on A and B
(c) dependent on A (d) independent of A and B

37. If $\frac{\sin A-\sin C}{\cos C-\cos A}=\cot B$, then A, B and C are in

(a) AP (b) GP
(c) HP (d) None of these

38. If $\tan\theta=\frac{1}{2}$ and $\tan\phi=\frac{1}{3}$, then the value of $\theta+\phi$ is [NCERT Exemplar]

(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

39. The value of $\frac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$ is [NCERT Exemplar]

(a) 1 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2

40. If $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ and α,β lies between 0 and $\frac{\pi}{4}$, then $\tan 2\alpha$ is equal to

(a) $\frac{16}{63}$ (b) $\frac{56}{33}$
(c) $\frac{28}{33}$ (d) None of these

41. The value of $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n$ (where, n is an even) is

(a) $2\tan^n\left(\frac{A-B}{2}\right)$ (b) $2\cot^n\left(\frac{A-B}{2}\right)$
(c) 0 (d) None of these

42. If $\frac{\pi}{2}<\alpha<\pi$, $\pi<\beta<\frac{3\pi}{2}$; $\sin\alpha=\frac{15}{17}$ and $\tan\beta=\frac{12}{5}$, then the value of $\sin(\beta-\alpha)$ is

(a) $-\frac{171}{221}$ (b) $-\frac{21}{221}$ (c) $\frac{21}{221}$ (d) $\frac{171}{221}$

43. If $\cos x+\cos y+\cos\alpha=0$ and $\sin x+\sin y+\sin\alpha=0$, then $\cot\left(\frac{x+y}{2}\right)$ is equal to

(a) $\sin\alpha$ (b) $\cos\alpha$
(c) $\cot\alpha$ (d) $\sin\left(\frac{x+y}{2}\right)$

44. If $\tan\alpha=(1+2^{-x})^{-1}$, $\tan\beta=(1+2^{x+1})^{-1}$, then $\alpha+\beta$ equals

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

45. If $\cos(\theta-\alpha)=a$, $\cos(\theta-\beta)=b$, then $\sin^2(\alpha-\beta)+2ab\cos(\alpha-\beta)$ is equal to

(a) a^2+b^2 (b) a^2-b^2
(c) b^2-a^2 (d) $-a^2-b^2$

46. If $\cos A=m\cos B$ and $\cot\frac{A+B}{2}=\lambda\tan\frac{B-A}{2}$, then λ is

(a) $\frac{m}{m-1}$ (b) $\frac{m+1}{m}$
(c) $\frac{m+1}{m-1}$ (d) None of these

47. The value of $\sin(45^\circ+\theta)-\cos(45^\circ-\theta)$ is [NCERT Exemplar]

(a) $2\cos\theta$ (b) $2\sin\theta$
(c) 1 (d) 0

48. $\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)$ is equal to [NCERT Exemplar]

(a) $\sin 2(\theta+\phi)$ (b) $\cos 2(\theta+\phi)$
(c) $\sin 2(\theta-\phi)$ (d) $\cos 2(\theta-\phi)$

49. The value of $\cos 12^\circ+\cos 84^\circ+\cos 156^\circ+\cos 132^\circ$ is [NCERT Exemplar]

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) $\frac{1}{8}$

50. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

51. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta$ is equal to [NCERT Exemplar]
(a) $2\cos(\alpha+\beta)$ (b) $-2\cos(\alpha+\beta)$
(c) $3\cos(\alpha+\beta)$ (d) None of these

52. If $\cos(\theta-\alpha) = a$, $\sin(\theta-\beta) = b$, then $\cos^2(\alpha-\beta) + 2ab\sin(\alpha-\beta)$ is equal to
(a) $4a^2b^2$ (b) $a^2 - b^2$
(c) $a^2 + b^2$ (d) $-a^2b^2$

Trigonometric Ratios of Multiple and Submultiple Angles

53. The value of $\sin 20° \sin 40° \sin 60° \sin 80°$ is equal to
(a) $-\frac{3}{16}$ (b) $\frac{5}{16}$
(c) $\frac{3}{16}$ (d) $-\frac{5}{16}$

54. The value of $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$ is equal to
(a) $\frac{1}{16}$ (b) 0
(c) $-\frac{1}{8}$ (d) $-\frac{1}{16}$

55. The value of $\cos^2 48° - \sin^2 12°$ is [NCERT Exemplar]
(a) $\frac{\sqrt{5}+1}{8}$ (b) $\frac{\sqrt{5}-1}{8}$
(c) $\frac{\sqrt{5}+1}{5}$ (d) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

56. If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$, then the value of $\frac{\sin(\alpha+\beta+\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}$ is
(a) < 1 (b) > 1
(c) 1 (d) None of these

57. If angle θ be divided into two parts such that the tangent of one part is k times the tangent of the other and ϕ is their difference, then $\sin\theta$ is equal to
(a) $\frac{k+1}{k-1}\sin\phi$ (b) $\frac{k-1}{k+1}\sin\phi$
(c) $\frac{2k-1}{2k+1}\sin\phi$ (d) None of these

58. If $A + B + C = \pi$ and $\cos A = \cos B \cos C$, then $\tan B \tan C$ is equal to
(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) $-\frac{1}{2}$

59. If $\cos(\theta+\phi) = m\cos(\theta-\phi)$, then $\frac{1-m}{1+m}\cot\phi$ is equal to [NCERT Exemplar]
(a) $\tan\theta$ (b) $-\tan\theta$
(c) $2\tan\theta$ (d) None of these

60. The value of $\sin 12° \sin 48° \sin 54°$ is equal to
(a) $\frac{1}{16}$ (b) $\frac{1}{32}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

61. The value of $1 + \cos 56° + \cos 58° - \cos 66°$ is equal to
(a) $2\cos 28° \cos 29° \cos 33°$
(b) $4\cos 28° \cos 29° \sin 33°$
(c) $4\cos 28° \cos 29° \cos 33°$
(d) $2\cos 28° \cos 29° \sin 33°$

62. $\tan\frac{2\pi}{5} - \tan\frac{\pi}{15} - \sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}$ is equal to
(a) $-\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) 1 (d) $\sqrt{3}$

63. The sum $S = \sin\theta + \sin 2\theta + \ldots + \sin n\theta$ equals
(a) $\sin\frac{1}{2}(n+1)\theta \sin\frac{n\theta}{2} / \sin\frac{\theta}{2}$
(b) $\cos\frac{1}{2}(n+1)\theta \sin\frac{n\theta}{2} / \sin\frac{\theta}{2}$
(c) $\sin\frac{1}{2}(n+1)\theta \cos\frac{n\theta}{2} / \sin\frac{\theta}{2}$
(d) $\cos\frac{1}{2}(n+1)\theta \cos\frac{n\theta}{2} / \sin\frac{\theta}{2}$

64. If $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m \cos mx$, where $c_0, c_1, c_2, \ldots, c_n$ are constants and $c_n \neq 0$, then the value of n is
(a) 15 (b) 6 (c) 1 (d) 0

65. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, then the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is equal to
(a) 1 (b) 2
(c) 0 (d) $3\cos\theta$

66. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to
(a) $1 - 4\cos A \cos B \cos C$ (b) $4\sin A \sin B \sin C$
(c) $1 + 2\cos A \cos B \cos C$ (d) $1 - 4\sin A \sin B \sin C$

67. If $\alpha + \beta - \gamma = \pi$, then $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to
(a) $2\sin\alpha \sin\beta \cos\gamma$ (b) $2\cos\alpha \cos\beta \cos\gamma$
(c) $2\sin\alpha \sin\beta \sin\gamma$ (d) None of these

68. If $\alpha + \beta + \gamma = 2\pi$, then
(a) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
(b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$
(c) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
(d) None of the above

69. The value of $\frac{\cos 12° - \sin 12°}{\cos 12° + \sin 12°} + \frac{\sin 147°}{\cos 147°}$ is equal to

(a) 1 (b) −1
(c) 0 (d) None of these

70. If $\frac{\sin x}{\sin y} = \frac{1}{2}, \frac{\cos x}{\cos y} = \frac{3}{2}$, where $x, y \in \left(0, \frac{\pi}{2}\right)$, then the value of $\tan(x + y)$ is equal to

(a) $\sqrt{13}$ (b) $\sqrt{14}$
(c) $\sqrt{17}$ (d) $\sqrt{15}$

71. If $\sin\theta + \cos\theta = x$, then $\sin^6\theta + \cos^6\theta = \frac{1}{4}[4 - 3(x^2 - 1)^2]$ for

(a) all real x (b) $x^2 \le 2$
(c) $x^2 > 2$ (d) None of these

72. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is equal to

(a) 11 (b) 12 (c) 13 (d) 14

73. The value of $\tan 3A - \tan 2A - \tan A$ is equal to [NCERT Exemplar]

(a) $\tan 3A \tan 2A \tan A$
(b) $-\tan 3A \tan 2A \tan A$
(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$
(d) None of the above

74. The value of $\tan 20° \tan 40° \tan 60° \tan 80°$ is equal to

(a) 1 (b) 2
(c) 3 (d) $\sqrt{\frac{3}{2}}$

75. The value of $\sin 36° \sin 72° \sin 108° \sin 144°$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) $\frac{3}{4}$ (d) $\frac{5}{16}$

76. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT Exemplar]

(a) $\frac{2\cos x}{\sqrt{\cos 2x}}$ (b) $\frac{\cos x}{\sqrt{\cos 2x}}$
(c) $\frac{2\cos 2x}{\sqrt{\cos x}}$ (d) None of these

77. If $(a - b)\sin(\theta + \phi) = (a + b)\sin(\theta - \phi)$ and $a\tan\frac{\theta}{2} - b\tan\frac{\phi}{2} = c$, then the value of $\sin\phi$ is equal to

(a) $\frac{2ab}{a^2 - b^2 - c^2}$ (b) $\frac{2bc}{a^2 - b^2 - c^2}$
(c) $\frac{2bc}{a^2 - b^2 + c^2}$ (d) $\frac{2ab}{a^2 - b^2 + c^2}$

78. The value of $2\cos x - \cos 3x - \cos 5x$ is equal to

(a) $16\cos^3 x \sin^2 x$ (b) $16\sin^3 x \cos^2 x$
(c) $4\cos^3 x \sin^2 x$ (d) $4\sin^3 x \cos^2 x$

79. Consider the following statements :

I. If $\sin A = \sin B$, then we have $\sin 2A = \sin 2B$ always.

II. The value of $\cos\frac{\pi}{7}\cos\frac{4\pi}{7}\cos\frac{5\pi}{7}$ is $\frac{1}{4}$.

Which of the statements given above is/are correct ?

(a) Only (I) (b) Only (II)
(c) Both (I) and (II) (d) Neither (I) nor (II)

80. The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\ldots\cos\frac{32\pi}{65}$ is

(a) $\frac{1}{32}$ (b) $\frac{1}{64}$ (c) $-\frac{1}{32}$ (d) $-\frac{1}{64}$

81. $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$ is equal to

(a) 1 (b) 3/2 (c) 2 (d) 1/4

82. If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its roots, then $\tan\alpha + \tan\beta$ is equal to [NCERT Exemplar]

(a) $-\frac{2b}{a+c}$ (b) $\frac{2b}{a+c}$
(c) $\frac{3b}{a+c}$ (d) None of these

83. The value of

$$\cos^2 A(3 - 4\cos^2 A)^2 + \sin^2 A(3 - 4\sin^2 A)^2$$

is equal to

(a) $\cos 4A$ (b) $\sin 4A$
(c) 1 (d) None of these

84. If $4n\alpha = \pi$, then the value of

$$\tan\alpha \cdot \tan 2\alpha \cdot \tan 3\alpha \cdot \tan 4\alpha \ldots \tan(2n-2)\alpha \tan(2n-1)\alpha$$

is

(a) 0 (b) 1
(c) −1 (d) None of these

85. If $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = 3$, then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $n\pi \pm \frac{\pi}{6}$
(c) $2n\pi \pm \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

Maximum and Minimum Value of Trigonometric Equation

86. If in a ΔABC, $\angle C = 90°$, then the maximum value of $\sin A \sin B$ is

(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these

87. If $A = \sin^2\theta + \cos^4\theta$, then for all real values of θ

(a) $1 \le A \le 2$ (b) $\frac{3}{4} \le A \le 1$

(c) $\frac{13}{16} \le A \le 1$ (d) $\frac{3}{4} \le A \le \frac{13}{16}$

88. If $y = 1 + 4\sin^2 x \cos^2 x$, then

(a) $1 \le y \le 2$ (b) $-1 \le y \le 1$

(c) $-3 \le y \le 3$ (d) None of these

89. The maximum value of $\cos^2\left(\frac{\pi}{3} - x\right) - \cos^2\left(\frac{\pi}{3} + x\right)$ is

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{2}$

90. The maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ is attained at

(a) $x = \frac{\pi}{12}$ (b) $x = \frac{\pi}{6}$

(c) $x = \frac{\pi}{3}$ (d) $x = \frac{\pi}{2}$

91. The greatest and least value of $\sin x \cos x$ are respectively

(a) $1, -1$ (b) $\frac{1}{2}, -\frac{1}{2}$

(c) $\frac{1}{4}, -\frac{1}{4}$ (d) $2, -2$

92. The maximum value of

$$4\sin^2 x + 3\cos^2 x + \sin\frac{x}{2} + \cos\frac{x}{2} \text{ is}$$

(a) $4 + \sqrt{2}$ (b) $3 + \sqrt{2}$

(c) 9 (d) 4

93. $\cos 2\theta + 2\cos\theta$ is always

(a) greater than $-\frac{3}{2}$

(b) less than or equal to $\frac{3}{2}$

(c) greater than or equal to $-\frac{3}{2}$ and less than or equal to 3

(d) None of the above

94. The maximum value of $(\cos\alpha_1)(\cos\alpha_2)\ldots(\cos\alpha_n)$ under the restriction $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \frac{\pi}{2}$ and $(\cot\alpha_1)(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$, is

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

95. The minimum value of $3\cos x + 4\sin x + 8$ is [NCERT Exemplar]

(a) 5 (b) 9 (c) 7 (d) 3

Solution of Trigonometric Equations

96. If $\sin\theta = \sqrt{3}\cos\theta$, $-\pi < \theta < 0$, then θ is equal to

(a) $-\frac{5\pi}{6}$ (b) $-\frac{4\pi}{6}$ (c) $\frac{4\pi}{6}$ (d) $\frac{5\pi}{6}$

97. Total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$ is equal to

(a) 2 (b) 3

(c) 5 (d) None of these

98. Total number of solutions of the equation $3x + 2\tan x = \frac{5\pi}{2}$ in $x \in [0, 2\pi]$, is equal to

(a) 1 (b) 2 (c) 3 (d) 4

99. Find the general solution of the equation $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ [NCERT Exemplar]

(a) $2n\pi \pm \frac{\pi}{4} - \frac{5\pi}{12}$ (b) $2n\pi \pm \frac{\pi}{4} + \frac{5\pi}{12}$

(c) $2n\pi \pm \pi - \frac{3\pi}{11}$ (d) None of these

100. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

101. The number of solutions of the equation $1 + \sin x \sin^2\frac{x}{2} = 0$ in $[-\pi, \pi]$ is

(a) zero (b) one (c) two (d) three

102. The number of values of x lying in the interval $(-\pi, \pi)$ which satisfy the equation

$$8^{(1 + |\cos x| + \cos^2 x + |\cos^3 x| + \ldots\infty)} = 4^3, \text{ is}$$

(a) 3 (b) 4 (c) 5 (d) 6

103. If $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$, then find the general value of θ. [NCERT Exemplar]

(a) $2n\pi \pm \frac{1}{4}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $2n\pi \pm \frac{\pi}{3}$ (d) None of these

104. If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ. [NCERT Exemplar]

(a) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{5}{3}, \frac{4\pi}{3}$ (d) None of these

105. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has

(a) no solution (b) two solutions

(c) three solutions (d) None of these

106. The number of solutions of the equation $|\cos x| = 2[x]$, where $[\cdot]$ is the greatest integer, is

(a) one (b) two

(c) infinite (d) nil

107. The number of ordered pairs (x, y) satisfying $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is
(a) 0 (b) 1
(c) 2 (d) ∞

108. If $\tan 2x = \tan \frac{2}{x}$, then the value of x is
(a) $\frac{n\pi \pm \sqrt{n^2\pi^2 + 16}}{4}$ (b) $\frac{n\pi}{4}$
(c) $\frac{n\pi \pm \sqrt{n^2\pi^2 - 16}}{4}$ (d) None of these

109. The most general solutions of the equation $\sec x - 1 = (\sqrt{2} - 1)\tan x$ are given by
(a) $n\pi + \frac{\pi}{8}$ (b) $2n\pi, 2n\pi + \frac{\pi}{4}$
(c) $2n\pi$ (d) None of these

110. Find the general solution of the equation $5\cos^2\theta + 7\sin^2 - 6 = 0$. [NCERT Exemplar]
(a) $(2n + 1)\frac{\pi}{4}$ (b) $(2n - 1)\frac{\pi}{4}$
(c) $(2n + 3)\frac{\pi}{4}$ (d) None of these

111. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is
(a) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}, \pi\right\}$
(c) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (d) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

112. If $4\cos\theta - 3\sec\theta = 2\tan\theta$, then θ is equal to
(a) $n\pi + (-1)^n \frac{\pi}{10}$ (b) $n\pi + (-1)^n \frac{\pi}{6}$
(c) $n\pi + (-1)^n \frac{3\pi}{10}$ (d) $n\pi$

113. If the equation $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$, $0 \le x \le 2\pi$, then the number of values of x is
(a) 6 (b) 7
(c) 4 (d) 5

114. If $\sec x \cos 5x + 1 = 0$, where $0 < x < 2\pi$, then x is equal to
(a) $\frac{\pi}{5}, \frac{\pi}{4}$ (b) $\frac{\pi}{5}$
(c) $\frac{\pi}{4}$ (d) None of these

115. The most general values of θ satisfying $\tan\theta + \tan\left(\frac{3\pi}{4} + \theta\right) = 2$ is/are
(a) $n\pi \pm \frac{\pi}{3}, n \in I$ (b) $2n\pi + \frac{\pi}{3}, n \in I$
(c) $2n\pi \pm \frac{\pi}{3}, n \in I$ (d) $2n\pi + (-1)^n \frac{\pi}{3}, n \in I$

116. The general value of θ satisfying $\sin^2\theta + \sin\theta = 2$ is
(a) $n\pi + (-1)^n \frac{\pi}{6}$ (b) $2n\pi + \frac{\pi}{4}$
(c) $n\pi + (-1)^n \frac{\pi}{2}$ (d) $n\pi + (-1)^n \frac{\pi}{3}$

117. The general solution of $\sin^2\theta \sec\theta + \sqrt{3}\tan\theta = 0$ is
(a) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, \ \theta = n\pi, n \in I$
(b) $\theta = n\pi, n \in I$
(c) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, n \in I$
(d) $\theta = \frac{n\pi}{2}, n \in I$

118. The general solution of
$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$
is [NCERT Exemplar]
(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$
(c) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ (d) $2n\pi + \cos^{-1}\frac{3}{2}$

119. The equation $3\sin^2 x + 10\cos x - 6 = 0$ is satisfied, if
(a) $x = n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (b) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$
(c) $x = n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$ (d) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$

120. If the equation $\sin^2\theta - \cos\theta = \frac{1}{4}$, then the value of θ lying in the interval $0 \le \theta \le 2\pi$ is
(a) $\frac{\pi}{3}, \frac{5\pi}{3}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}$ (c) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (d) $\frac{3\pi}{5}, \frac{\pi}{5}$

121. The smallest positive root of the equation $\tan x - x = 0$ is in
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\pi, \frac{3\pi}{2}\right)$
(c) $\left(\frac{\pi}{2}, \pi\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$

122. If $2\cos^2 x + 3\sin x - 3 = 0$, $0 \le x \le 180°$, then the value of x is
(a) 30°, 90°, 150° (b) 60°, 120°, 180°
(c) 0°, 30°, 150° (d) 45°, 90°, 135°

123. The most general value of θ satisfying the equation $(1 + 2\sin\theta)^2 + (\sqrt{3}\tan\theta - 1)^2 = 0$ are given by
(a) $n\pi \pm \frac{\pi}{6}$ (b) $n\pi + (-1)^n \frac{7\pi}{6}$
(c) $2n\pi + \frac{7\pi}{6}$ (d) $2n\pi + \frac{11\pi}{6}$

124. If the equation $\cos 3x \cos^3 x + \sin 3x \sin^3 x = 0$, then x is equal to
(a) $(2n + 1)\frac{\pi}{4}$ (b) $(2n - 1)\frac{\pi}{4}$
(c) $\frac{n\pi}{4}$ (d) None of these

125. If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then
(a) $\theta = \frac{(6n+1)\pi}{18}, \forall n \in I$
(b) $\theta = \frac{(6n+1)\pi}{9}, \forall n \in I$
(c) $\theta = \frac{(3n+1)\pi}{9}, \forall n \in I$
(d) None of the above

126. If $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$, then the general value of θ is
(a) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (b) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$
(c) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (d) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

127. Total number of solutions of
$\sin^4 x + \cos^4 x = \sin x \cdot \cos x$ in $[0, 2\pi]$ is equal to
(a) 2 (b) 4
(c) 6 (d) 8

128. The general value of x for which $\cos 2x, \frac{1}{2}$ and $\sin 2x$ are in AP, are given by
(a) $n\pi, n\pi + \frac{\pi}{2}$ (b) $n\pi, n\pi + \frac{\pi}{4}$
(c) $n\pi + \frac{\pi}{4}, \frac{3n\pi}{4}$ (d) None of these

129. Set of values of x lying in $[0, 2\pi]$ satisfying the inequality $|\sin x| > 2\sin^2 x$ contains
(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$ (b) $\left(0, \frac{7\pi}{6}\right)$
(c) $\frac{\pi}{6}$ (d) None of these

130. The solution of the equation $\sin^{10} 2x = 1 + \cos^{10} x$ is
(a) $x = (2n+1)\frac{\pi}{2}$ (b) $x = n\pi$
(c) $x = (2n+1)\frac{\pi}{4}$ (d) None of these

131. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi$, $0 \le y \le 2\pi, 0 \le z \le 2\pi$ has
(a) one solution (b) two sets of solutions
(c) four sets of solutions (d) no solution

132. The number of solutions of the equation $\sin^5 x - \cos^5 x = \frac{1}{\cos x} - \frac{1}{\sin x}$ $(\sin x \ne \cos x)$ is
(a) 0 (b) 1
(c) infinite (d) None of these

133. The number of ordered pairs (x, y), where $x, y \in [0, 10]$ satisfying $\left(\sqrt{\sin^2 x - \sin x + \frac{1}{2}}\right) \cdot 2^{\sec^2 y} \le 1$ is
(a) 0 (b) 16 (c) infinite (d) 12

134. The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}, x < \frac{\pi}{9}$ has
(a) no real solution
(b) one real solution
(c) more than one real solutions
(d) None of the above

135. The general solution of $e^{-1/\sqrt{2}}(e^{\sin x} + e^{\cos x}) = 2$ is
(a) $x = m\pi$ (b) $x = \frac{(4m+1)\pi}{4}$
(c) $x = \frac{(4m+1)\pi}{2}$ (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. If $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are the roots of the equation $x^3 - px^2 - r = 0$, then the value of $(1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)$ is equal to
(a) $(p-r)^2$ (b) $1 + (p-r)^2$
(c) $1 - (p-r)^2$ (d) None of these

2. If α, β, γ and δ are four solutions of the equation $\tan\left(\theta + \frac{\pi}{4}\right) = 3\tan 3\theta$, then $\tan\alpha\tan\beta\tan\gamma\tan\delta$ equals
(a) 3 (b) 1/3
(c) $-\frac{1}{3}$ (d) None of these

3. The least value of $\operatorname{cosec}^2 x + 25\sec^2 x$ is
(a) 0 (b) 26 (c) 28 (d) 36

4. If $x\sin a + y\sin 2a + z\sin 3a = \sin 4a$
$x\sin b + y\sin 2b + z\sin 3b = \sin 4b$
$x\sin c + y\sin 2c + z\sin 3c = \sin 4c$
Then, the roots of the equation
$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0, a, b, c \ne n\pi,$$
are
(a) $\sin a, \sin b, \sin c$
(b) $\cos a, \cos b, \cos c$
(c) $\sin 2a, \sin 2b, \sin 2c$
(d) $\cos 2a, \cos 2b, \cos 2c$

5. If $0° < \theta < 180°$, then $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2(1+\cos\theta)}}}}$, then being n number of 2's, is equal to

(a) $2\cos\left(\frac{\theta}{2^n}\right)$ (b) $2\cos\left(\frac{\theta}{2^{n-1}}\right)$
(c) $2\cos\left(\frac{\theta}{2^{n+1}}\right)$ (d) None of these

6. If $\tan\alpha/2$ and $\tan\beta/2$ are the roots of the equation $8x^2 - 26x + 15 = 0$, then $\cos(\alpha + \beta)$ is equal to
(a) $-\frac{627}{725}$ (b) $\frac{627}{725}$
(c) -1 (d) None of these

7. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$ for all real θ, then
(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$
(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n - 3$

8. If θ is an acute angle and $\tan\theta = \frac{1}{\sqrt{7}}$, then the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$ is
(a) 3/4 (b) 1/2 (c) 2 (d) 5/4

9. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2\theta} = \tan^4\theta + 2\tan^2\theta$ is
(a) 1 (b) 2
(c) 3 (d) None of these

10. If $\sin^4 x + \cos^4 y + 2 = 4\sin x\cos y$ and $0 \le x, y \le \frac{\pi}{2}$, then $\sin x + \cos y$ is equal to
(a) -2 (b) 0 (c) 2 (d) $\frac{3}{2}$

11. The equation $\cos^4 x - (\lambda + 2)\cos^2 x - (\lambda + 3) = 0$ possesses a solution, if
(a) $\lambda > -3$ (b) $\lambda < -2$
(c) $-3 \le \lambda \le -2$ (d) λ is any positive integer

12. The arithmetic mean of the roots of the equation
$$4\cos^3 x - 4\cos^2 x - \cos(315\pi + x) = 1$$
in the interval (0, 315) is equal to
(a) 50π (b) 51π (c) 100π (d) 315π

13. If $\frac{\sin\theta}{6}$, $\cos\theta$ and $\tan\theta$ are in GP, then the general value of θ is
(a) $2n\pi \pm \frac{\pi}{3}, n \in I$ (b) $2n\pi \pm \frac{\pi}{6}, n \in I$
(c) $2n\pi + (-1)^n\frac{\pi}{3}, n \in I$ (d) $n\pi + \frac{\pi}{3}, n \in I$

14. Set $a, b \in [-\pi, \pi]$ be such that $\cos(a - b) = 1$ and $\cos(a + b) = \frac{1}{e}$. The number of pairs of a and b satisfying the above system of equations is
(a) 0 (b) 1 (c) 2 (d) 4

15. The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$ is
(a) zero (b) one (c) two (d) four

16. Let α and β be any two positive values of x for which $2\cos x, |\cos x|$ and $1 - 3\cos^2 x$ are in GP. The minimum value of $|\alpha - \beta|$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) None of these

More Than One Correct Option

17. If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then
(a) $\sin\alpha - \cos\alpha = \pm\sqrt{2}\sin\theta$
(b) $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$
(c) $\cos 2\theta = \sin 2\alpha$
(d) $\sin 2\theta + \cos 2\alpha = 0$

18. Let $0 \le \theta \le \pi/2$ and $x = X\cos\theta + Y\sin\theta$, $y = X\sin\theta - Y\cos\theta$ such that $x^2 + 4xy + y^2 = aX^2 + bY^2$, where a and b are constants. Then,
(a) $a = -1, b = 3$ (b) $\theta = \frac{\pi}{4}$
(c) $a = 3, b = -1$ (d) $\theta = \frac{\pi}{3}$

19. If $\cos x + \cos y - \cos(x + y) = \frac{3}{2}$, then
(a) $x + y = 0$ (b) $x = 2y$
(c) $x = y$ (d) $\cos^2\left(\frac{x - y}{2}\right) \ge 1$

20. $\tan|x| = |\tan x|$, if
(a) $x \in \{-\pi(2k + 1)/2, -\pi k\}, k \in I$
(b) $x \in [\pi k, \pi(2k + 1)/2], k \in I$
(c) $x \in \{-\pi k, -\pi(2k - 1)/2\}, k \in I$
(d) $x \in \{\pi(2k - 1)/2, \pi k\}, k \in I$

Comprehension Based Questions

Passage

Directions (Q. Nos. 21-23) *Whenever the terms on the two sides of the equation are of different nature, then equations are known as non-standard form, some of them are in the form of an ordinary equation but cannot be solved by standard procedures.*
Non-standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, inequalities.

21. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ is
(a) 1 (b) 2 (c) 3 (d) 4

22. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}, 0 < x \le \frac{x}{2}$ has
(a) one real solution
(b) more than one real solutions
(c) no real solution
(d) None of the above

23. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is
(a) 0 (b) 1 (c) 2 (d) 3

Assertion and Reason

Directions (Q. Nos. 24-35) *Each of these question contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these question also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

24. **Statement I** $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha = \cot\alpha$
Statement II $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

25. **Statement I** $\cos^3\alpha + \cos^3\left(\alpha + \frac{2\pi}{3}\right) + \cos^3\left(\alpha + \frac{4\pi}{3}\right)$
$$= 3\cos\alpha\cos\left(\alpha + \frac{2\pi}{3}\right)\cos\left(\alpha + \frac{4\pi}{3}\right).$$
Statement II If $a + b + c = 0 \Leftrightarrow a^3 + b^3 + c^3 = 3abc$

26. **Statement I** If $xy + yz + zx = 1$, then
$$\Sigma\frac{x}{(1+x^2)} = \frac{2}{\sqrt{\prod(1+x^2)}}.$$
Statement II In a ΔABC
$\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$

27. **Statement I** If α and β are two distinct solutions of the equatio $a\cos x + b\sin x = c$, then $\tan\left(\frac{\alpha+\beta}{2}\right)$ is independent of c.
Statement II Solution of $a\cos x + b\sin x = c$ is possible, if
$$-\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$$

28. **Statement I** The number of real solutions of the equation $\cos(x) = 7^x + 7^{-x}$ is zero.
Statement II Since, $|\cos x| \le 1$

29. **Statement I** The equation $\sin x = x^2 + x + 1$ has no solution.
Statement II The curve $y = \sin x$ and $y = x^2 + x + 1$ do not intersect each other when graph is observed.

30. **Statement I** $\cos 36° > \sin 36°$
Statement II $\cos 36° > \tan 36°$

31. **Statement I** $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = -\frac{1}{2}$
Statement II $\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex 7th root of unity.

32. **Statement I** If $\sin x + \cos x = \sqrt{\left(y + \frac{1}{y}\right)}, x \in [0, \pi]$, then $x = \frac{\pi}{4}, y = 1$
Statement II AM $\ge$ GM

33. **Statement I** If $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$, then $\sin\theta + \cos\theta = \pm\sqrt{2}$
Statement II $-\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$

34. **Statement I** The equation $\sin(\cos x) = \cos(\sin x)$ does not possess real roots.
Statement II If $\sin x > 0$, then $2n\pi < x < (2n+1), n \in I$

35. **Statement I** If $\sin^2 A = \sin^2 B$ and $\cos^2 A = \cos^2 B$, then $A = n\pi + B, n \in I$
Statement II If $\sin A = \sin B$ and $\cos A = \cos B$, then $A = n\pi + B, n \in I$

Previous Years' Questions

36. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta, BC = p$ and $CD = q$, then AB is equal to **[JEE Main 2013]**
(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta + q\sin\theta}$
(c) $\frac{p^2+q^2}{p^2\cos\theta + q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

37. The expression $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$ can be written as **[JEE Main 2013]**
(a) $\sin A\cos A + 1$
(b) $\sec A\,\text{cosec}\,A + 1$
(c) $\tan A + \cot A$
(d) $\sec A + \text{cosec}\,A$

38. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the $\angle R$ is equal to [AIEEE 2012]

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

39. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has [AIEEE 2012]
(a) infinite number of real roots
(b) no real roots
(c) exactly one real root
(d) exactly four real roots

40. If $A = \sin^2 x + \cos^4 x$, then for all real x [AIEEE 2011]
(a) $\frac{13}{16} \le A \le 1$ (b) $1 \le A \le 2$
(c) $\frac{3}{4} \le A \le \frac{13}{16}$ (d) $\frac{3}{4} \le A \le 1$

41. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and let $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to [AIEEE 2010]

(a) $\frac{25}{16}$ (b) $\frac{56}{33}$
(c) $\frac{19}{12}$ (d) $\frac{20}{7}$

42. Let A and B denote the statements

$$A : \cos\alpha + \cos\beta + \cos\gamma = 0$$
$$B : \sin\alpha + \sin\beta + \sin\gamma = 0$$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then [AIEEE 2009]

(a) A is true and B is false (b) A is false and B is true
(c) both A and B are true (d) both A and B are false

43. The number of values of x in $[0, 3\pi]$ such that $2\sin^2 x + 5\sin x - 3 = 0$ is [AIEEE 2006]
(a) 1 (b) 2
(c) 4 (d) 6

44. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is equal to [AIEEE 2006]

(a) $\frac{-(4+\sqrt{7})}{3}$ (b) $\frac{1+\sqrt{7}}{4}$
(c) $\frac{1-\sqrt{7}}{4}$ (d) $\frac{4-\sqrt{7}}{3}$

45. In a ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0, a \ne 0$, then [AIEEE 2005]

(a) $b = a + c$ (b) $b = c$
(c) $c = a + b$ (d) $a = b + c$

46. Let α and β such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\frac{21}{65}$, $\cos\alpha + \cos\beta = -\frac{27}{65}$, then $\cos\left(\frac{\alpha - \beta}{2}\right)$ is [AIEEE 2004]

(a) $-\frac{3}{\sqrt{130}}$ (b) $\frac{3}{\sqrt{130}}$ (c) $\frac{6}{65}$ (d) $-\frac{6}{65}$

47. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by [AIEEE 2004]

(a) $2(a^2 + b^2)$ (b) $2\sqrt{a^2 + b^2}$
(c) $(a + b)^2$ (d) $(a - b)^2$

48. In a ΔABC, medians AD and BE are drawn. If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of the ΔABC is [AIEEE 2003]

(a) 8 / 3 sq units (b) 16 / 3 sq units
(c) $32/3\sqrt{3}$ sq units (d) 64 / 3 sq units

49. $\sin^2\theta = \frac{4xy}{(x+y)^2}$ is true, if and only if [AIEEE 2002]

(a) $x - y \ne 0$ (b) $x = -y$
(c) $x = y$ (d) $x \ne 0, y \ne 0$

50. The value of $\frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ}$ is [AIEEE 2002]

(a) 1 (b) $\sqrt{3}$ (c) $\sqrt{3}/2$ (d) 2

51. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is [AIEEE 2002]

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$
(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these

52. If $\sin(\alpha + \beta) = 1$, $\sin(\alpha - \beta) = \frac{1}{2}$, then $\tan(\alpha + 2\beta)\tan(2\alpha + \beta)$ is equal to [AIEEE 2002]
(a) 1 (b) -1
(c) zero (d) None of these

53. If $y = \sin^2\theta + \operatorname{cosec}^2\theta, \theta \ne 0$, then [AIEEE 2002]
(a) $y = 0$ (b) $y \le 2$
(c) $y \ge -2$ (d) $y > 2$

54. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to [AIEEE 2002]

(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$
(c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

Answers

Check Point 1

1. (a)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (d)

Check Point 2

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (a)	**7.** (d)	**8.** (b)	**9.** (d)	**10.** (a)
11. (c)	**12.** (b)								

Check Point 3

1. (d)	**2.** (b)	**3.** (d)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (d)	**8.** (d)	**9.** (c)	**10.** (b)

Round I

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (d)	**9.** (c)	**10.** (b)
11. (b)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (c)
21. (d)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (d)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (b)
31. (d)	**32.** (c)	**33.** (a)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (a)	**38.** (d)	**39.** (c)	**40.** (b)
41. (b)	**42.** (d)	**43.** (c)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (c)
51. (b)	**52.** (c)	**53.** (c)	**54.** (d)	**55.** (a)	**56.** (a)	**57.** (a)	**58.** (b)	**59.** (a)	**60.** (c)
61. (b)	**62.** (d)	**63.** (a)	**64.** (b)	**65.** (c)	**66.** (d)	**67.** (a)	**68.** (a)	**69.** (c)	**70.** (d)
71. (b)	**72.** (c)	**73.** (a)	**74.** (c)	**75.** (d)	**76.** (a)	**77.** (b)	**78.** (a)	**79.** (d)	**80.** (b)
81. (b)	**82.** (b)	**83.** (c)	**84.** (b)	**85.** (d)	**86.** (a)	**87.** (b)	**88.** (a)	**89.** (c)	**90.** (a)
91. (b)	**92.** (a)	**93.** (c)	**94.** (a)	**95.** (d)	**96.** (b)	**97.** (a)	**98.** (c)	**99.** (a)	**100.** (c)
101. (a)	**102.** (b)	**103.** (c)	**104.** (b)	**105.** (a)	**106.** (d)	**107.** (a)	**108.** (a)	**109.** (b)	**110.** (a)
111. (c)	**112.** (a)	**113.** (d)	**114.** (c)	**115.** (a)	**116.** (c)	**117.** (b)	**118.** (b)	**119.** (b)	**120.** (a)
121. (b)	**122.** (a)	**123.** (c)	**124.** (a)	**125.** (c)	**126.** (a)	**127.** (a)	**128.** (b)	**129.** (a)	**130.** (d)
131. (a)	**132.** (a)	**133.** (b)	**134.** (a)	**135.** (b)					

Round II

1. (b)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (c)
11. (c)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (b)	**16.** (d)	**17.** (a,b,c,d)	**18.** (b,c)	**19.** (c,d)	**20.** (a,b)
21. (a)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (b)
31. (d)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (b)	**39.** (b)	**40.** (d)
41. (b)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (c)	**46.** (a)	**47.** (d)	**48.** (c)	**49.** (c)	**50.** (c)
51. (b)	**52.** (a)	**53.** (d)	**54.** (b)						

the **Guidance**

Round I

1. $-47^\circ\, 30' = -\left(47 + \frac{30}{60}\right)^\circ = -\left(47 + \frac{1}{2}\right)^\circ \qquad \left[\because 1' = \left(\frac{1}{60}\right)^\circ\right]$

$= -\left(\frac{94+1}{2}\right)^\circ = -\frac{95^\circ}{2}$

$= -\frac{95}{2} \times \frac{\pi}{180} = -\frac{19\pi}{72}$ rad

2. $\left(\frac{11}{16}\right)^c = \left(\frac{11}{16} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{8} \times \frac{90}{22} \times 7\right)^\circ = \left(\frac{45 \times 7}{8}\right)^\circ$

$\left[\because 1^c = \left(\frac{180}{\pi}\right)^\circ\right]$

$= \left(\frac{315}{8}\right)^\circ = 39^\circ + \left(\frac{3}{8}\right)^\circ$

$= 39^\circ + \frac{3}{8} \times 60'$

$= 39^\circ + \left(\frac{45}{2}\right)' \qquad (\because 1^\circ = 60')$

$= 39^\circ + 22' + \frac{1''}{2}$

$= 39^\circ\, 22' + \frac{1 \times 60''}{2}$

$= 39^\circ\, 22'\, 30'' \qquad (\because 1' = 60'')$

3.

Using the formula, $\theta = \frac{l}{r}$

For first circle, $60° \times \frac{\pi}{180°} = \frac{l}{r_1}$ ...(i)

And for second circle, $75° \times \frac{\pi}{180°} = \frac{l}{r_2}$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{60 \times \frac{\pi}{180}}{75 \times \frac{\pi}{180}} = \frac{\frac{l}{r_1}}{\frac{l}{r_2}} \Rightarrow \frac{r_2}{r_1} = \frac{4}{5}$$

$\Rightarrow$ $r_1 : r_2 = 5 : 4$

4. Let s be the length of the arc subtending an angle θ radian at the centre of a circle of radius r. Then, $\theta = \frac{s}{r}$

Here, $r = 5$ cm and $\theta = 15° = 15° \times \frac{\pi}{180°} = \frac{\pi}{12}$ rad

$\because$ $\theta = \frac{s}{r}$

$\Rightarrow$ $\frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12}$ cm ($\because r = 5$ given)

5. $\therefore \frac{1 - \cos\alpha + \sin\alpha}{1 + \sin\alpha} \times \frac{1 + \cos\alpha + \sin\alpha}{1 + \cos\alpha + \sin\alpha}$

$$= \frac{(1 + \sin\alpha)^2 - \cos^2\alpha}{(1 + \sin\alpha)(1 + \cos\alpha + \sin\alpha)}$$

$$= \frac{1^2 + \sin^2\alpha + 2\sin\alpha - \cos^2\alpha}{(1 + \sin\alpha)(1 + \cos\alpha + \sin\alpha)}$$

$$= \frac{\sin^2\alpha + 2\sin\alpha + \sin^2\alpha}{(1 + \sin\alpha)(1 + \cos\alpha + \sin\alpha)}$$

$$= \frac{2\sin\alpha(1 + \sin\alpha)}{(1 + \sin\alpha)(1 + \cos\alpha + \sin\alpha)}$$

$$= \frac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = y$$

6. The given expression can be written as

$$\frac{1 + \cos y - \sin^2 y}{1 + \cos y} + \frac{(1 - \cos^2 y) - \sin^2 y}{\sin y(1 - \cos y)}$$

$$= \frac{\cos y(1 + \cos y)}{1 + \cos y} + 0 = \cos y$$

7. Given, $\sin\theta + \operatorname{cosec}\theta = 2$

On squaring both sides, we get

$$\sin^2\theta + \operatorname{cosec}^2\theta + 2 = 4$$

$\Rightarrow$ $\sin^2\theta + \operatorname{cosec}^2\theta = 2$

8. Given, $f(x) = \cos^2 x + \sec^2 x$

Using AM $\geq$ GM

$\therefore$ $\frac{\cos^2 x + \sec^2 x}{2} \geq \sqrt{\cos^2 x \times \sec^2 x}$

$\Rightarrow$ $\cos^2 x + \sec^2 x \geq 2$ $\therefore$ $f(x) \geq 2$

9. Now, on taking option one by one, we get

(a) $\sin 15° = \sin(45° - 30°) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$ = irrational

(b) $\cos 15° = \cos(45° - 30°) = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ = irrational

(c) $\sin 15° \cos 15° = \frac{1}{2}(2\sin 15° \cos 15°)$

$= \frac{1}{2}\sin 30° = \frac{1}{4}$ = rational

(d) $\sin 15° \cos 75° = \sin 15° \sin 15° = \sin^2 15°$

$= \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2 = \frac{4 - 2\sqrt{3}}{8}$ = irrational

10. $\sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}} + \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \frac{1 - \sin\theta + 1 + \sin\theta}{\sqrt{1 - \sin^2\theta}}$

$$= \frac{2}{\sqrt{\cos^2\theta}} = \frac{2}{|\cos\theta|}$$

$$= \frac{2}{-\cos\theta} = -2\sec\theta \quad \left(\because \frac{\pi}{2} < \theta < \pi\right)$$

11. Given that, $\sin\theta = -\frac{4}{5}$ and θ lies in the IIIrd quadrant.

$\Rightarrow$ $\cos\theta = -\sqrt{1 - \frac{16}{25}} = -\frac{3}{5}$

Now, $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}} = \pm\sqrt{\frac{1 - \frac{3}{5}}{2}} = \pm\sqrt{\frac{1}{5}}$

But we take $\cos\frac{\theta}{2} = -\frac{1}{\sqrt{5}}$. Since, if θ lies in IIIrd quadrant, then $\frac{\theta}{2}$ will be in IInd quadrant.

Hence, $\cos\frac{\theta}{2} = -\frac{1}{\sqrt{5}}$

12. Since, $\sec\theta \notin (-1, 1)$

$\therefore$ $\sec\theta \neq \frac{1}{2}$

13. $\cos 1° \cos 2° \cos 3° \ldots \cos 90° \ldots \cos 179°$

$= \cos 1° \cos 2° \cos 3° \ldots 0 \ldots \cos 179° = 0$

14. Given, $\tan\theta = \frac{3}{1}$ $\therefore$ $AC^2 = AB^2 + BC^2$

$= 1^2 + 3^2 = 1 + 9 = 10$

$AC = \sqrt{10}$

$\sin\theta = -\frac{3}{\sqrt{10}}$

($\because \theta$ lies in IIIrd quadrant, so $\sin\theta$ will be negative)

15. $\therefore \tan 75° - \cot 75°$

$$= \tan(45° + 30°) - \cot(45° + 30°)$$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} - \frac{\cot 45° \cot 30° - 1}{\cot 45° + \cot 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} - \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{(\sqrt{3} + 1)}{(\sqrt{3} - 1)} - \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{(3 + 1 + 2\sqrt{3})}{3 - 1} - (3 + 1 - 2\sqrt{3})$$

$$= \frac{4\sqrt{3}}{2} = 2\sqrt{3}$$

16. $\because 1 \text{ radian} = \frac{180°}{\pi} = 57°30' \text{ app}$

Since, $\sin\theta$ is increasing from 0 to 90°

$\therefore \quad \sin 1 > \sin 1°$

17. Given $\tan\alpha = \frac{m}{m+1}$, $\tan\beta = \frac{1}{2m+1}$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \times \frac{1}{2m+1}} = \frac{2m^2 + m + m + 1}{(m+1)(2m+1) - m}$$

$$= \frac{2m^2 + 2m + 1}{2m^2 + 3m + 1 - m} = \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1$$

$\Rightarrow \quad \tan(\alpha + \beta) = 1$

$\Rightarrow \quad \alpha + \beta = \frac{\pi}{4}$

18. We have, $\sqrt{\operatorname{cosec}^2\alpha + 2\cot\alpha}$

$$= \sqrt{1 + \cot^2\alpha + 2\cot\alpha} = |1 + \cot\alpha|$$

But $\quad \frac{3\pi}{4} < \alpha < \pi$

$\Rightarrow \quad \cot\alpha < -1 \Rightarrow 1 + \cot\alpha < 0$

Hence, $\quad |1 + \cot\alpha| = -(1 + \cot\alpha)$

19. Since, $\quad \tan A + \sin A = m$

and $\quad \tan A - \sin A = n$

$\therefore \quad m + n = 2\tan A$

and $\quad m - n = 2\sin A$

Also, $\quad mn = (\tan A + \sin A)(\tan A - \sin A)$

$$= \tan^2 A - \sin^2 A$$

Now, $\quad \frac{(m^2 - n^2)^2}{mn} = \frac{(m+n)^2(m-n)^2}{mn} = \frac{(2\tan A)^2(2\sin A)^2}{\tan^2 A - \sin^2 A}$

$$= \frac{16\tan^2 A \sin^2 A}{\sin^2 A \tan^2 A} = 16$$

20. $6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$

$$= 6[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)]$$
$$- 9[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] + 4$$

$$= 6[1 - 3\sin^2\theta\cos^2\theta] - 9(1 - 2\sin^2\theta\cos^2\theta) + 4$$

$$= 6 - 9 + 4 = 1$$

21. Given, $\quad \frac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = x$

$\Rightarrow \quad \frac{2\sin\alpha(1 - \cos\alpha - \sin\alpha)}{(1 + \cos\alpha + \sin\alpha)(1 - \cos\alpha - \sin\alpha)} = x$

$\Rightarrow \quad \frac{2\sin\alpha(1 - \cos\alpha - \sin\alpha)}{1 - \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha} = x$

$\Rightarrow \quad \frac{1 - \cos\alpha - \sin\alpha}{\cos\alpha} = -x$

22. Given that, $\quad \sin\theta + \cos\theta = m \quad$ ...(i)

and $\quad \sec\theta + \operatorname{cosec}\theta = n \quad$ ...(ii)

Now, $\quad n(m+1)(m-1) = n(m^2 - 1)$

$$= (\sec\theta + \operatorname{cosec}\theta)\, 2\sin\theta\cos\theta$$
$$(\because m^2 = 1 + 2\sin\theta\cos\theta)$$

$$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} 2\sin\theta\cos\theta$$

$$= 2m \quad \text{[from Eq. (i)]}$$

23. We have, $\quad \frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$

Therefore, each ratio is equal to

$$\frac{x + y + z}{\cos\theta + \cos\left(\theta - \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{2\pi}{3}\right)}$$

$$= \frac{x + y + z}{\cos\theta + 2\cos\theta\cos\frac{2\pi}{3}} = \frac{x + y + z}{0}$$

$\Rightarrow \quad x + y + z = 0$

24. We have, $\quad \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}$

$$= \frac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \sin\frac{x}{2} - \cos\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2} - \sin\frac{x}{2} + \cos\frac{x}{2}} = \tan\frac{x}{2}$$

25. Since, $\quad (2\tan\theta + 2)^2 = \tan\theta(3\tan\theta + 3)$

$\Rightarrow \quad 4\tan^2\theta + 8\tan\theta + 4 = 3\tan^2\theta + 3\tan\theta$

$\Rightarrow \quad \tan^2\theta + 5\tan\theta + 4 = 0$

$\Rightarrow \quad (\tan\theta + 4)(\tan\theta + 1) = 0$

$\Rightarrow \quad \tan\theta = -4 \quad (\because \tan\theta \neq -1)$

$$\therefore \quad \frac{7 - 5\cot\theta}{9 - 4\sqrt{\tan^2\theta}} = \frac{7 + \frac{5}{4}}{9 - 4(-4)} = \frac{33}{100}$$

26. Given that, $ABCD$ is a cyclic quadrilateral.

So, $A + C = 180° \Rightarrow A = 180° - C$

$\Rightarrow \cos A = \cos(180° - C) = -\cos C$

$\Rightarrow \cos A + \cos C = 0$ ...(i)

Similarly, $\cos B + \cos D = 0$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$\cos A + \cos B + \cos C + \cos D = 0$$

27. Given that, $2 \sin A = \sqrt{3} \sin B$

$\Rightarrow 2\sqrt{5} \sin A = \sqrt{15} \sin B$ ...(i)

and $2 \cos A = \sqrt{5} \cos B$

$\Rightarrow 2\sqrt{3} \cos A = \sqrt{15} \cos B$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$20 \sin^2 A + 12 \cos^2 A = 15$$

$\Rightarrow 8 \sin^2 A = 3 \Rightarrow \sin^2 A = \frac{3}{8}$

$\Rightarrow \cos^2 A = \frac{5}{8}$

$\therefore \frac{\sin^2 A}{\cos^2 A} = \frac{3}{5} \Rightarrow \tan A = \sqrt{\frac{3}{5}}$

28. Given, $a \cos^3 \alpha + 3a \cos \alpha \sin^2 \alpha = m$

and $a \sin^3 \alpha + 3a \cos^2 \alpha \sin \alpha = n$

$\therefore (m + n) = a \cos^3 \alpha + 3a \cos \alpha \sin^2 \alpha + 3a \cos^2 \alpha \sin \alpha + a \sin^3 \alpha$

$= a (\cos \alpha + \sin \alpha)^3$

and similarly, $(m - n) = a(\cos \alpha - \sin \alpha)^3$

$\therefore (m + n)^{2/3} + (m - n)^{2/3}$

$= a^{2/3} \{(\cos \alpha + \sin \alpha)^2 + (\cos \alpha - \sin \alpha)^2\}$

$= a^{2/3}\{2 (\cos^2 \alpha + \sin^2 \alpha)\} = 2 a^{2/3}$

29. Given that, $\cos \theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$

$\Rightarrow x + \frac{1}{x} = 2 \cos \theta$ ...(i)

We know that,

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

$= (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2$

$= 2 \cos 2\theta$ [from Eq. (i)]

$\therefore \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) = \frac{1}{2} \times 2 \cos 2\theta = \cos 2\theta$

30. We have, $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$

$= 3[(-\cos \alpha)^4 + (-\sin \alpha)^4] - 2 [\cos^6 \alpha + \sin^6 \alpha]$

$= 3[(\cos^2 \alpha + \sin^2 \alpha)^2 - 2 \sin^2 \alpha \cos^2 \alpha]$

$- 2[(\cos^2 \alpha + \sin^2 \alpha)^3 - 3 \cos^2 \alpha \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha)]$

$= 3 - 6 \sin^2 \alpha \cos^2 \alpha - 2 + 6 \sin^2 \alpha \cos^2 \alpha = 3 - 2 = 1$

31. Since, $\tan x + \frac{1}{\tan x} = 2$

$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

$\therefore \sin x = \frac{1}{\sqrt{2}}$

and $\cos x = \frac{1}{\sqrt{2}}$

Hence, $\sin^{2n} x + \cos^{2n} x = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$

32. Maximum value of $\cos \theta = 1$

So, the equation can have solution only when

$\cos x = 1, \quad \cos y = 1$

$\Rightarrow x = 0, \quad y = 0$

$\Rightarrow \cos (x - y) = \cos 0 = 1$

33. $\because \sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$

On squaring both sides, we get

$$\sin^2\left(\frac{\pi}{2n}\right) + \cos^2\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{n}\right) = \frac{n}{4}$$

$\Rightarrow \sin\left(\frac{\pi}{n}\right) = \frac{n}{4} - 1$

$\Rightarrow \sin\left(\frac{\pi}{n}\right) = \frac{n - 4}{4}$

$\Rightarrow n = 6$ only

34. Given, $m \sin \theta = n \sin (\theta + 2\alpha)$

$\Rightarrow \frac{m}{n} = \frac{\sin(\theta + 2\alpha)}{\sin \theta}$

Applying componendo and dividendo rule, we get

$$\frac{m + n}{m - n} = \frac{\sin(\theta + 2\alpha) + \sin \theta}{\sin (\theta + 2\alpha) - \sin \theta}$$

$= \frac{2 \sin(\theta + \alpha) \cos \alpha}{2 \cos (\theta + \alpha) \sin \alpha}$

$= \tan (\theta + \alpha) \cot \alpha$

35. $\frac{\sin (B + A) + \cos (B - A)}{\sin (B - A) + \cos (B + A)}$

$= \frac{\sin (B + A) + \sin (90° - \overline{B - A})}{\sin (B - A) + \sin (90° - \overline{A + B})}$

$= \frac{2 \sin (A + 45°) \cos (45° - B)}{2 \sin (45° - A) \cos (45° - B)}$

$= \frac{\sin (A + 45°)}{\sin (45° - A)} = \frac{\frac{1}{\sqrt{2}} \sin A + \frac{1}{\sqrt{2}} \cos A}{\frac{1}{\sqrt{2}} \cos A - \frac{1}{\sqrt{2}} \sin A}$

$= \frac{\cos A + \sin A}{\cos A - \sin A}$

36. We have,

$$\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B$$
$$=\cos^2(A-B)+\cos^2 B-\cos(A-B)\times[\cos(A-B)+\cos(A+B)]$$
$$=\cos^2 B-\cos(A-B)\cos(A+B)$$
$$=\cos^2 B-(\cos^2 A-\sin^2 B)$$
$$=1-\cos^2 A=\sin^2 A$$

Hence, it depends on A.

37. Given, $\dfrac{\sin A-\sin C}{\cos C-\cos A}=\cot B$

$$\Rightarrow \quad \frac{2\cos\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)}{2\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)}=\cot B$$

$$\Rightarrow \quad \cot\left(\frac{A+C}{2}\right)=\cot B$$

$$\Rightarrow \quad B=\frac{A+C}{2}$$

Hence, A, B and C will be in AP.

38. Given, $\tan\theta=\dfrac{1}{2}$ and $\tan\phi=\dfrac{1}{3}$

$$\therefore \quad \tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$
$$=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\frac{\frac{5}{6}}{\frac{5}{6}}=1\Rightarrow \theta+\phi=\frac{\pi}{4}$$

39. $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}=\dfrac{\cos^2 15^\circ-\sin^2 15^\circ}{\cos^2 15^\circ+\sin^2 15^\circ}$

$$=\frac{\cos 2\times 15^\circ}{1}=\cos 30^\circ=\frac{\sqrt{3}}{2}$$

40. We have, $\cos(\alpha+\beta)=\dfrac{4}{5}$ and $\sin(\alpha-\beta)=\dfrac{5}{13}$

$$\Rightarrow \quad \sin(\alpha+\beta)=\frac{3}{5} \quad \text{and} \quad \cos(\alpha-\beta)=\frac{12}{13}$$

$$\Rightarrow \quad (\alpha+\beta)=\sin^{-1}\frac{3}{5} \quad \text{and} \quad (\alpha-\beta)=\sin^{-1}\frac{5}{13}$$

$$\therefore \quad 2\alpha=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}$$
$$=\sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{9}{25}}\right]$$
$$=\sin^{-1}\left(\frac{3}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{4}{5}\right)=\sin^{-1}\left(\frac{36}{65}+\frac{20}{65}\right)$$

$$\Rightarrow \quad \sin 2\alpha=\frac{56}{65}$$

$$\therefore \quad \tan 2\alpha=\frac{56}{33}$$

41. $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n$

$$=\left[\frac{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}\right]^n+\left[\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}\right]^n$$

$$=\cot^n\left(\frac{A-B}{2}\right)+\cot^n\left(\frac{B-A}{2}\right)$$
$$=\cot^n\left(\frac{A-B}{2}\right)+(-1)^n\cot\left(\frac{A-B}{2}\right)$$
$$=2\cot^n\left(\frac{A-B}{2}\right) \qquad (\because n \text{ is even})$$

42. Given, $\sin\alpha=\dfrac{15}{17}$, $\tan\beta=\dfrac{12}{5}$

Since, $\dfrac{\pi}{2}<\alpha<\pi, \quad \pi<\beta<\dfrac{3\pi}{2}$

$$\therefore \quad \cos\alpha=-\frac{8}{17}, \quad \sin\beta=-\frac{12}{13}$$

and $\cos\beta=-\dfrac{5}{13}$

Now, $\sin(\beta-\alpha)=\sin\beta\cos\alpha-\cos\beta\sin\alpha$

$$=-\frac{12}{13}\left(\frac{-8}{17}\right)-\left(\frac{-5}{13}\right)\left(\frac{15}{17}\right)$$
$$=\frac{96}{221}+\frac{75}{221}=\frac{171}{221}$$

43. Given equations may be written as

$\cos x+\cos y=-\cos\alpha$

and $\sin x+\sin y=-\sin\alpha$

$$\Rightarrow \quad 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=-\cos\alpha \qquad \text{...(i)}$$

$$\text{and} \quad 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=-\sin\alpha \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}=\frac{\cos\alpha}{\sin\alpha}$$

$$\Rightarrow \quad \cot\left(\frac{x+y}{2}\right)=\cot\alpha$$

44. We have, $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\because \quad \tan\alpha=\frac{1}{1+2^{-x}} \text{ and } \tan\beta=\frac{1}{1+2^{x+1}}$$

$$\therefore \quad \tan(\alpha+\beta) = \frac{\dfrac{1}{1+\dfrac{1}{2^x}} + \dfrac{1}{1+2^{x+1}}}{1 - \dfrac{1}{1+\dfrac{1}{2^x}} \cdot \dfrac{1}{1+2^{x+1}}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{2^x + 2\cdot 2^{2x} + 2^x + 1}{1 + 2^x + 2\cdot 2^x + 2\cdot 2^{2x} - 2^x}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = 1$$

$$\Rightarrow \quad \alpha+\beta = \frac{\pi}{4}$$

45. We know that,

$$\alpha - \beta = (\theta-\beta) - (\theta-\alpha)$$

$$\therefore \quad \cos(\alpha-\beta) = \cos(\theta-\beta)\cos(\theta-\alpha) + \sin(\theta-\beta)\sin(\theta-\alpha)$$

$$= ab + \sqrt{1-a^2}\sqrt{1-b^2}$$

and $\quad \sin(\alpha-\beta) = \pm\,(a\sqrt{1-b^2} - b\sqrt{1-a^2})$

$$\Rightarrow \quad \sin^2(\alpha-\beta) = a^2 + b^2 - 2a^2b^2 - 2\,ab\sqrt{1-a^2}\sqrt{1-b^2}$$

$$\Rightarrow \quad \sin^2(\alpha-\beta) = a^2 + b^2 - 2\,a^2b^2 - 2\,ab[\cos(\alpha-\beta) - ab]$$

$$\therefore \quad \sin^2(\alpha-\beta) = a^2 + b^2 - 2\,ab\cos(\alpha-\beta)$$

$$\Rightarrow \quad \sin^2(\alpha-\beta) + 2\,ab\cos(\alpha-\beta) = a^2 + b^2$$

46. We have, $\quad \cos A = m\cos B$

$$\Rightarrow \quad \frac{\cos A}{\cos B} = \frac{m}{1}$$

$$\Rightarrow \quad \frac{\cos A + \cos B}{\cos A - \cos B} = \frac{m+1}{m-1}$$

$$\Rightarrow \quad \frac{2\cos\dfrac{A+B}{2}\cos\dfrac{B-A}{2}}{2\sin\dfrac{A+B}{2}\sin\dfrac{B-A}{2}} = \frac{m+1}{m-1}$$

$$\Rightarrow \quad \cot\frac{A+B}{2} = \left(\frac{m+1}{m-1}\right)\tan\frac{B-A}{2}$$

But $\quad \cot\dfrac{A+B}{2} = \lambda\tan\dfrac{B-A}{2}$

$$\therefore \quad \lambda = \frac{m+1}{m-1}$$

47. $\sin(45^\circ + \theta) - \cos(45^\circ - \theta)$

$$= \sin 45^\circ\cos\theta + \cos 45^\circ\sin\theta - [\cos 45^\circ\cos\theta + \sin 45^\circ\sin\theta]$$

$$= \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta - \left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$$

$$= 0$$

48. $\cos 2\theta\cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi)$

$$= \cos 2\theta\cos 2\phi + \sin(\theta-\phi+\theta+\phi)\sin(\theta-\phi-(\theta+\phi))$$

$$(\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B))$$

$$= \cos 2\theta\cos 2\phi + \sin 2\theta\sin(-2\phi)$$

$$= \cos 2\theta\cos 2\phi - \sin 2\theta\sin 2\phi$$

$$= \cos(2\theta + 2\phi) = \cos 2(\theta+\phi)$$

49. $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$

$$= \cos 156^\circ + \cos 84^\circ + \cos 132^\circ + \cos 12^\circ$$

$$= 2\cos\left(\frac{156^\circ + 84^\circ}{2}\right)\cos\left(\frac{156^\circ - 84^\circ}{2}\right) + 2\cos\left(\frac{132^\circ + 12^\circ}{2}\right)\cos\left(\frac{132^\circ - 12^\circ}{2}\right)$$

$$= 2\cos 120^\circ\cos 36^\circ + 2\cos 72^\circ\cos 60^\circ$$

$$= 2\left(-\frac{1}{2}\right)\cos 36^\circ + 2\cos 72^\circ \times \frac{1}{2}$$

$$= -\cos 36^\circ + \cos 72^\circ$$

$$= -\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4} = -\frac{1}{2}$$

50. Given equation can be written as

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)$$

$$= \left(1+\cos\frac{\pi}{8} + \cos\frac{7\pi}{8} + \cos\frac{\pi}{8}\cos\frac{7\pi}{8}\right)\left(1+\cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\right)$$

$$= \left(1+\cos\frac{\pi}{8} - \cos\frac{\pi}{8} + \cos\frac{\pi}{8}\cos\frac{7\pi}{8}\right)\left(1-\cos\frac{5\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\right)$$

$$= \left(1+\cos\frac{\pi}{8}\cos\frac{7\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\right)$$

$$= \frac{1}{4}\left(2 + 2\cos\frac{\pi}{8}\cos\frac{7\pi}{8}\right)\left(2 + 2\cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\right)$$

$$= \frac{1}{4}\left(2 + \cos\frac{3\pi}{4} + \cos\pi\right)\left(2 + \cos\frac{\pi}{4} + \cos\pi\right)$$

$$= \frac{1}{4}\left(1+\cos\frac{3\pi}{4}\right)\left(1+\cos\frac{\pi}{4}\right) = \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1+\cos\frac{\pi}{4}\right)$$

$$= \frac{1}{4}\left(1-\cos^2\frac{\pi}{4}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$$

51. Given, $\cos\alpha + \cos\beta = 0$ and $\sin\alpha + \sin\beta = 0$

On squaring and subtracting both the equations, we get

$$(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0$$

$$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta) - (\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta) = 0$$

$$\Rightarrow \quad (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + 2[\cos\alpha\cos\beta - \sin\alpha\sin\beta] = 0$$

$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta) = 0$$

$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta = -2\cos(\alpha+\beta)$$

52. Now, $\sin(\alpha-\beta) = \sin(\theta-\beta-(\theta-\alpha))$

$$= \sin(\theta-\beta)\cos(\theta-\alpha) - \cos(\theta-\beta)\sin(\theta-\alpha)$$

$$= ba - \sqrt{1-b^2}\sqrt{1-a^2}$$

and $\quad \cos(\alpha-\beta) = \cos(\theta-\beta-(\theta-\alpha))$

$= \cos(\theta-\beta)\cos(\theta-\alpha) + \sin(\theta-\beta)\sin(\theta-\alpha)$

$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$

$\therefore\ \cos^2(\alpha-\beta) + 2\,ab\sin(\alpha-\beta)$

$= (a\sqrt{1-b^2} + b\sqrt{1-a^2})^2 + 2\,ab\,(ab - \sqrt{1-a^2}\,\sqrt{1-b^2})$

$= a^2 + b^2$

53. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \frac{1}{2}\sin 20^\circ \sin 60^\circ (2\sin 40^\circ \sin 80^\circ)$

$= \frac{1}{2}\sin 20^\circ \sin 60^\circ (\cos 40^\circ - \cos 120^\circ)$

$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\sin 20^\circ\left(1 - 2\sin^2 20^\circ + \frac{1}{2}\right)$

$= \frac{\sqrt{3}}{4}\sin 20^\circ\left(\frac{3}{2} - 2\sin^2 20^\circ\right)$

$= \frac{\sqrt{3}}{8}(3\sin 20^\circ - 4\sin^3 20^\circ)$

$= \frac{\sqrt{3}}{8}\sin 60^\circ = \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2} = \frac{3}{16}$

54. $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5} = \frac{\sin 2^4\frac{\pi}{5}}{2^4\sin\frac{\pi}{5}}$

$\left(\because \cos x\cos 2x\cos 4x\ldots\cos 2nx = \frac{\sin 2^n x}{2^n\sin x}\right)$

$= \frac{\sin\frac{16\pi}{5}}{16\sin\frac{\pi}{5}} = \frac{\sin\left(3\pi + \frac{\pi}{5}\right)}{16\sin\frac{\pi}{5}}$

$= \frac{-\sin\frac{\pi}{5}}{16\sin\frac{\pi}{5}} = -\frac{1}{16}$

55. $\cos^2 48^\circ - \sin^2 12^\circ = \cos(48^\circ + 12^\circ)\cos(48^\circ - 12^\circ)$

$[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$

$= \cos 60^\circ \cos 36^\circ$

$= \frac{1}{2}\times\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8}$

56. $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma$

$+ \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$

$\Rightarrow \sin(\alpha+\beta+\gamma) - \sin\alpha - \sin\beta - \sin\gamma$

$= \sin\alpha(\cos\beta\cos\gamma - 1) + \sin\beta(\cos\alpha\cos\gamma - 1)$

$+ \sin\gamma(\cos\alpha\cos\beta - 1) - \sin\alpha\sin\beta\sin\gamma$

$\Rightarrow \sin(\alpha+\beta+\gamma) - \sin\alpha - \sin\beta - \sin\gamma < 0$

$\Rightarrow \sin(\alpha+\beta+\gamma) < \sin\alpha + \sin\beta + \sin\gamma$

$\Rightarrow \frac{\sin(\alpha+\beta+\gamma)}{\sin\alpha + \sin\beta + \sin\gamma} < 1$

57. Let $A + B = \theta$ and $A - B = \phi$

Then, $\tan A = k\tan B$

$\Rightarrow \frac{\tan A}{\tan B} = \frac{k}{1}$

$\Rightarrow \frac{k}{1} = \frac{\sin A\cos B}{\cos A\sin B}$

Applying componendo and dividendo rule, we get

$\frac{k+1}{k-1} = \frac{\sin A\cos B + \cos A\sin B}{\sin A\cos B - \cos A\sin B}$

$= \frac{\sin(A+B)}{\sin(A-B)} = \frac{\sin\theta}{\sin\phi}$

$\Rightarrow \sin\theta = \frac{k+1}{k-1}\sin\phi$

58. Since, $A + B + C = \pi$

$\Rightarrow A = \pi - (B + C)$

We have, $\cos A = \cos B\cos C$

$\Rightarrow \cos[\pi - (B+C)] = \cos B\cos C$

$\Rightarrow -\cos(B+C) = \cos B\cos C$

$\Rightarrow -[\cos B\cos C - \sin B\sin C] = \cos B\cos C$

$\Rightarrow \sin B\sin C = 2\cos B\cos C$

$\Rightarrow \tan B\tan C = 2$

59. $\frac{1}{m} = \frac{\cos(\theta-\phi)}{\cos(\theta+\phi)}$

Applying componendo and dividendo rule,

$\frac{1-m}{1+m} = \frac{\cos(\theta-\phi) - \cos(\theta+\phi)}{\cos(\theta-\phi) + \cos(\theta+\phi)}$

$= \frac{2\sin(\theta)\sin(\phi)}{2\cos(\theta)\cos(-\phi)} = \tan\theta\tan\phi$

$\therefore \left(\frac{1-m}{1+m}\cot\phi\right) = \tan\theta$

60. Now, $\sin 12^\circ \sin 48^\circ \sin 54^\circ$

$= \frac{1}{2}(\cos 36^\circ - \cos 60^\circ)\cos 36^\circ$

$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right]\left[\frac{\sqrt{5}+1}{4}\right]$

$= \frac{1}{2}\left[\frac{\sqrt{5}-1}{4}\right]\left[\frac{\sqrt{5}+1}{4}\right]$

$= \frac{5-1}{32} = \frac{4}{32} = \frac{1}{8}$

61. $1 + \cos 56^\circ + \cos 58^\circ - \cos 66^\circ$

$= 2\cos^2 28^\circ + 2\sin 62^\circ \sin 4^\circ$

$= 2\cos^2 28^\circ + 2\cos 28^\circ\cos 86^\circ$

$= 2\cos 28^\circ(\cos 28^\circ + \cos 86^\circ)$

$= 2\cos 28^\circ(2\cos 57^\circ\cos 29^\circ)$

$= 4\cos 28^\circ\cos 29^\circ\sin 33^\circ$

62. Now, $\tan\frac{\pi}{3} = \tan\left(\frac{6\pi}{15} - \frac{\pi}{15}\right) = \dfrac{\tan\frac{6\pi}{15} - \tan\frac{\pi}{15}}{1 + \tan\frac{6\pi}{15}\tan\frac{\pi}{15}}$

$\Rightarrow \tan\frac{6\pi}{15} - \tan\frac{\pi}{15} = \sqrt{3} + \sqrt{3}\tan\frac{6\pi}{15}\tan\frac{\pi}{15}$

$\Rightarrow \tan\frac{6\pi}{15} - \tan\frac{\pi}{15} - \sqrt{3}\tan\frac{6\pi}{15}\tan\frac{\pi}{15} = \sqrt{3}$

$\Rightarrow \tan\frac{2\pi}{5} - \tan\frac{\pi}{15} - \sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15} = \sqrt{3}$

63. We have,

$$S = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$$

We know that,

$\sin\theta + \sin(\theta + \beta) + \sin(\theta + 2\beta) + \ldots n$ terms

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\frac{\theta + \theta + (n-1)\beta}{2}\right]$$

Put $\beta = \theta$,

$$\therefore \quad S = \frac{\sin\frac{n\theta}{2}\cdot\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

64. We have, $\sin^3 x\sin 3x = \sum_{m=0}^{n} c_m \cos mx$

Now, $\sin^3 x\sin 3x = \frac{1}{4}(3\sin x - \sin 3x)\sin 3x$

$= \frac{3}{8}\cdot 2\sin x\sin 3x - \frac{1}{8}\cdot 2\sin^2 3x$

$= \frac{3}{8}(\cos 2x - \cos 4x) - \frac{1}{8}(1 - \cos 6x)$

$= -\frac{1}{8} + \frac{3}{8}\cos 2x - \frac{3}{8}\cos 4x + \frac{1}{8}\cos 6x \quad \ldots(i)$

$$\text{RHS} = \sum_{m=0}^{n} c_m \cos mx = c_0 + c_1\cos x + c_2\cos 2x + c_3\cos 3x + \ldots + c_n\cos nx \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$n = 6$$

65. We have, $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right)$

$= z\cos\left(\theta + \frac{4\pi}{3}\right) = k$ (say)

$\Rightarrow \cos\theta = \frac{k}{x}, \quad \cos\left(\theta + \frac{2\pi}{3}\right) = \frac{k}{y}$ and $\cos\left(\theta + \frac{4\pi}{3}\right) = \frac{k}{z}$

$\therefore \quad \frac{k}{x} + \frac{k}{y} + \frac{k}{z} = \cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)$

$= \cos\theta - \cos\left(\frac{\pi}{3} - \theta\right) - \cos\left(\frac{\pi}{3} + \theta\right)$

$= \cos\theta - 2\cos\frac{\pi}{3}\cos\theta = 0$

$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

66. Since, $A + B + C = \frac{3\pi}{2}$

$\therefore \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + \cos 2C$

$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A - B) + 1 - 2\sin^2 C$

$= 1 - 2\sin C\left[\cos(A - B) + \sin\left(\frac{3\pi}{2} - (A + B)\right)\right]$

$= 1 - 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 1 - 4\sin A\sin B\sin C$

67. We have, $\alpha + \beta - \gamma = \pi$

Now, $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \sin^2\alpha + \sin(\beta - \gamma)\sin(\beta + \gamma)$

$= \sin^2\alpha + \sin(\pi - \alpha)\sin(\beta + \gamma) \quad (\because \alpha + \beta - \gamma = \pi)$

$= \sin^2\alpha + \sin\alpha\sin(\beta + \gamma)$

$= \sin\alpha[\sin\alpha + \sin(\beta + \gamma)]$

$= \sin\alpha[\sin(\pi - (\beta - \gamma)) + \sin(\beta + \gamma)]$

$= \sin\alpha[\sin(\beta - \gamma) + \sin(\beta + \gamma)]$

$= \sin\alpha[2\sin\beta\cos\gamma]$

$= 2\sin\alpha\sin\beta\cos\gamma$

68. We have, $\alpha + \beta + \gamma = 2\pi$

$\Rightarrow \quad \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$

$\Rightarrow \quad \frac{\alpha}{2} + \frac{\beta}{2} = \pi - \frac{\gamma}{2}$

$\Rightarrow \quad \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(\pi - \frac{\gamma}{2}\right)$

$\Rightarrow \quad \dfrac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = -\tan\frac{\gamma}{2}$

$\Rightarrow \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

69. $\dfrac{\cos 12^\circ - \sin 12^\circ}{\cos 12^\circ + \sin 12^\circ} + \dfrac{\sin 147^\circ}{\cos 147^\circ} = \dfrac{1 - \tan 12^\circ}{1 + \tan 12^\circ} + \tan 147^\circ$

$= \tan(45^\circ - 12^\circ) + \tan(180^\circ - 33^\circ)$

$= \tan 33^\circ - \tan 33^\circ = 0$

70. Given, $\frac{\sin x}{\sin y} = \frac{1}{2}, \quad \frac{\cos x}{\cos y} = \frac{3}{2}$

$\Rightarrow \quad \frac{\tan x}{\tan y} = \frac{1}{3}$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y} = \dfrac{4\tan x}{1 - 3\tan^2 x}$

Also, $\sin y = 2\sin x, \quad \cos y = \frac{2}{3}\cos x$

$\Rightarrow \sin^2 y + \cos^2 y = 4\sin^2 x + \frac{4\cos^2 x}{9} = 1$

$\Rightarrow \quad 36\tan^2 x + 4 = 9\sec^2 x = 9(1 + \tan^2 x)$

$\Rightarrow \quad 27 \tan^2 x = 5$

$\Rightarrow \quad \tan x = \dfrac{\sqrt{5}}{3\sqrt{3}}$

$\Rightarrow \quad \tan(x+y) = \dfrac{\dfrac{4\sqrt{5}}{3\sqrt{3}}}{1-\dfrac{15}{27}} = \sqrt{15}$

71. Since, $\sin\theta + \cos\theta = x$ ...(i)

and $\sin^6\theta + \cos^6\theta = \dfrac{1}{4}[4 - 3(x^2-1)^2]$

On squaring Eq. (i), we get

$\sin 2\theta = x^2 - 1 \le 1 \quad (\because \sin 2\theta \le 1)$

$\Rightarrow \quad x^2 \le 2 \Rightarrow -\sqrt{2} \le x \le \sqrt{2}$

Now, $\sin^6\theta + \cos^6\theta = (\sin^2\theta + \cos^2\theta)^3$
$- 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta = 1 - \dfrac{3}{4}\sin^2 2\theta$

$= 1 - \dfrac{3}{4}(x^2-1)^2 = \dfrac{1}{4}[4 - 3(x^2-1)^2]$

Thus, the given result will hold true only when $x^2 \le 2$ and not for all real values of x.

72. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(1 - \sin 2x)^2 + 6(1 + \sin 2x) + 4\{(\sin^2 x + \cos^2 x)^3$
$- 3\sin^2 x\cos^2 x \cdot (\sin^2 x + \cos^2 x)\}$

$= 3(1 - 2\sin 2x + \sin^2 2x) + 6 + 6\sin 2x$
$+ 4\{1 - 3\sin^2 x\cos^2 x\}$

$= 3\{1 - 2\sin 2x + \sin^2 2x + 2 + 2\sin 2x\} + 4\left\{1 - \dfrac{3}{4}\sin^2 2x\right\}$

$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$

73. Now, $\tan 3A = \tan(2A + A)$

$\tan 3A = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$\Rightarrow \quad \tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow \quad \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$

74. We have, $\tan 20° \tan 40° \tan 60° \tan 80°$

$= \dfrac{\sin 20° \sin 40° \sin 80°}{\cos 20° \cos 40° \cos 80°} \tan 60°$

Here, numerator $= (\sin 20° \sin 40° \sin 80°)$

$= \dfrac{\sin 20°}{2}(2\sin 40° \sin 80°)$

$= \dfrac{\sin 20°}{2}(\cos 40° - \cos 120°)$

$= \dfrac{1}{2}\sin 20°\left(1 - 2\sin^2 20° + \dfrac{1}{2}\right)$

$= \dfrac{1}{2}\sin 20°\left(\dfrac{3}{2} - 2\sin^2 20°\right)$

$= \dfrac{1}{4}[3\sin 20° - 4\sin^3 20°] = \dfrac{\sin 60°}{4} = \dfrac{\sqrt{3}}{8}$

Now, denominator $= \cos 20° \cos 40° \cos 80°$

$= \dfrac{\sin 2^3 20°}{2^3 \sin 20°} = \dfrac{\sin 160°}{8\sin 20°}$

$= \dfrac{\sin 20°}{8\sin 20°} = \dfrac{1}{8}$

Hence, $\tan 20° \tan 40° \tan 80° = \dfrac{\dfrac{\sqrt{3}}{8}}{\dfrac{1}{8}} = \sqrt{3}$

$\Rightarrow \quad \tan 20° \tan 40° \tan 60° \tan 80° = \sqrt{3}\cdot\sqrt{3} = 3$

75. $\sin 36° \sin 72° \sin 108° \sin 144°$

$= \sin^2 36° \sin^2 72°$

$= \dfrac{1}{4}[(2\sin^2 36°)(2\sin^2 72°)]$

$= \dfrac{1}{4}[(1 - \cos 72°)(1 - \cos 144°)]$

$= \dfrac{1}{4}[(1 - \sin 18°)(1 + \cos 36°)]$

$= \dfrac{1}{4}\left[\left(1 - \dfrac{\sqrt{5}-1}{4}\right)\left(1 + \dfrac{\sqrt{5}+1}{4}\right)\right]$

$= \dfrac{1}{4}\left[1 + \left(\dfrac{\sqrt{5}+1}{4}\right) - \left(\dfrac{\sqrt{5}-1}{4}\right) - \left(\dfrac{4}{16}\right)\right]$

$= \dfrac{1}{4}\left[1 + \dfrac{1}{2} - \dfrac{1}{4}\right] = \dfrac{5}{16}$

76. $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$

$= \sqrt{\dfrac{1+\dfrac{b}{a}}{1-\dfrac{b}{a}}} + \sqrt{\dfrac{1-\dfrac{b}{a}}{1+\dfrac{b}{a}}} = \dfrac{1+\dfrac{b}{a}+1-\dfrac{b}{a}}{\sqrt{\left(1-\dfrac{b}{a}\right)\left(1+\dfrac{b}{a}\right)}}$

(dividing by a in Nr and Dr of both the terms)

$= \dfrac{2}{\sqrt{1-\dfrac{b^2}{a^2}}} = \dfrac{2}{\sqrt{1-\tan^2 x}} = \dfrac{2}{\sqrt{1-\dfrac{\sin^2 x}{\cos^2 x}}} \quad \left(\because \tan x = \dfrac{b}{a}\right)$

$= \dfrac{2\cos x}{\sqrt{\cos^2 x - \sin^2 x}} = \dfrac{2\cos x}{\sqrt{\cos 2x}} \quad (\because \cos^2 x - \sin^2 x = \cos 2x)$

77. Since, $(a-b)\sin(\theta+\phi) = (a+b)\sin(\theta-\phi)$

$\Rightarrow \quad a\{\sin(\theta+\phi) - \sin(\theta-\phi)\}$
$= b\{\sin(\theta-\phi) + \sin(\theta+\phi)\}$

$\Rightarrow \quad 2a\sin\phi\cos\theta = 2b\sin\theta\cos\phi$

$\Rightarrow \quad a\tan\phi = b\tan\theta$

$\Rightarrow \quad \dfrac{2a\tan\dfrac{\phi}{2}}{1-\tan^2\dfrac{\phi}{2}} = \dfrac{2b\tan\dfrac{\theta}{2}}{1-\tan^2\dfrac{\theta}{2}}$ ...(i)

Since, $a\tan\dfrac{\theta}{2} - b\tan\dfrac{\phi}{2} = c$ (given)

$$\Rightarrow \quad \tan\frac{\theta}{2} = \frac{b\tan\frac{\phi}{2} + c}{a} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{a\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} = \frac{b\dfrac{\left(b\tan\frac{\phi}{2}+c\right)}{a}}{1-\dfrac{\left(b\tan\frac{\phi}{2}+c\right)^2}{a^2}}$$

$$\Rightarrow \quad \tan\frac{\phi}{2}(a^2-b^2-c^2) = bc\left(1+\tan^2\frac{\phi}{2}\right)$$

Now, $$\sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{2bc}{a^2-b^2-c^2}$$

78. $2\cos x - \cos 3x - \cos 5x = 2\cos x - 2\cos x\cos 4x$

$$= 2\cos x(1-\cos 4x)$$
$$= 2\cos x\, 2\sin^2 2x$$
$$= 4\cos x(2\sin x\cos x)^2$$
$$= 16\sin^2 x\cos^3 x$$

79. **Statement I** We have, $\sin A = \sin B$

$\Rightarrow \quad A = B$ or $A = \pi - B$

Now, $\sin 2A = \sin 2B$ is satisfied by $A = B$ but it is not satisfied by $A = \pi - B$.

Statement II

$$\cos\frac{\pi}{7}\cos\frac{4\pi}{7}\cos\frac{5\pi}{7} = \cos\frac{\pi}{7}\cos\frac{4\pi}{7}\cos\left(\pi-\frac{2\pi}{7}\right)$$
$$= -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$$
$$= -\frac{\sin\left(2^3\frac{\pi}{7}\right)}{2^3\sin\frac{\pi}{7}} = -\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} = \frac{1}{8}$$

80. $$\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\ldots\cos\frac{32\pi}{65} = \cos\frac{\pi}{65}\cdot\cos\frac{2\pi}{65}\ldots\cos\frac{2^5\pi}{65}$$
$$= \frac{\sin\frac{2^6\pi}{65}}{2^6\sin\frac{\pi}{65}} = \frac{\sin\frac{64\pi}{65}}{64\sin\frac{\pi}{65}}$$
$$= \frac{\sin\left(\pi-\frac{\pi}{65}\right)}{64\sin\frac{\pi}{65}} = \frac{1}{64}$$

81. $$\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$$
$$= \frac{1}{4}\left[\left(2\sin^2\frac{\pi}{8}\right)^2 + \left(2\sin^2\frac{3\pi}{8}\right)^2\right] + \frac{1}{4}\left[\left(2\sin^2\frac{\pi}{8}\right)^2 + \left(2\sin^2\frac{3\pi}{8}\right)^2\right]$$
$$= \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right] + \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right]$$
$$= \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right] + \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right]$$
$$= \frac{1}{4}(3) + \frac{1}{4}(3) = \frac{3}{2}$$

82. Given, $a\cos 2\theta + b\sin 2\theta = c$

$$\therefore \quad a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = c$$
$$\Rightarrow \quad a - a\tan^2\theta + 2b\tan\theta = c(1+\tan^2\theta)$$
$$\Rightarrow \quad \tan^2\theta(a+c) - 2b\tan\theta + c - a = 0$$

Since, α and β are the roots of the above equation.

$$\therefore \quad \tan\alpha + \tan\beta = \frac{2b}{a+c}$$

83. $\cos^2 A(3-4\cos^2 A)^2 + \sin^2 A(3-4\sin^2 A)^2$

$$= (3\cos A - 4\cos^3 A)^2 + (3\sin A - 4\sin^3 A)^2$$
$$= (-\cos 3A)^2 + (\sin 3A)^2 = 1$$

84. $\tan\alpha\tan 2\alpha\ldots\tan(2n-2)\alpha\tan(2n-1)\alpha$

$$= \{\tan\alpha\tan(2n-1)\alpha\}\{\tan 2\alpha\tan(2n-2)\alpha\}\ldots\{\tan(n-1)\alpha\tan(n+1)\alpha\}\tan n\alpha$$
$$= \left\{\tan\alpha\tan\left(\frac{\pi}{2}-\alpha\right)\right\}\left\{\tan 2\alpha\tan\left(\frac{\pi}{2}-2\alpha\right)\right\}\ldots\tan\frac{\pi}{4} \quad \left(\because n\alpha = \frac{\pi}{4}\right)$$
$$= 1$$

85. Given that, $$\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$$
$$\Rightarrow \quad \frac{2\sin^2\theta}{2\cos^2\theta} = 3$$
$$\Rightarrow \quad \tan^2\theta = (\sqrt{3})^2$$
$$\Rightarrow \quad \tan^2\theta = \tan^2\frac{\pi}{3}$$
$$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}$$

86. $\because \quad \sin A\sin B = \frac{1}{2}\times 2\sin A\sin B$

$$= \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$= \frac{1}{2}[\cos(A-B) - \cos 90^\circ]$$
$$(\because A+B+C = 180^\circ \text{ and } \angle C = 90^\circ, \text{ given})$$
$$= \frac{1}{2}\cos(A-B) \le \frac{1}{2}$$

$\therefore$ Maximum value of $\sin A\sin B = \frac{1}{2}$

87. We have, $A = \sin^2\theta + \cos^4\theta$

$= \sin^2\theta + \cos^2\theta\cos^2\theta$

$\le \sin^2\theta + \cos^2\theta$ (since, $\cos^2\theta \le 1$)

$\Rightarrow \sin^2\theta + \cos^4\theta \le 1 \Rightarrow A \le 1$

Again, $\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta$

$= \cos^4\theta - \cos^2\theta + 1$

$= \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$

Hence, $\frac{3}{4} \le A \le 1$

88. Since, $y = 1 + 4\sin^2 x\cos^2 x$

$\Rightarrow y = 1 + \sin^2 2x$

We know that,

$0 \le \sin^2 2x \le 1$

$\Rightarrow 1 \le 1 + \sin^2 2x \le 2$

$\Rightarrow 1 \le y \le 2$

89. $\cos^2\left(\frac{\pi}{3} - x\right) - \cos^2\left(\frac{\pi}{3} + x\right)$

$= \left[\cos\left(\frac{\pi}{3} - x\right) + \cos\left(\frac{\pi}{3} + x\right)\right]\left[\cos\left(\frac{\pi}{3} - x\right) - \cos\left(\frac{\pi}{3} + x\right)\right]$

$= \left(2\cos\frac{\pi}{3}\cos x\right)\left(2\sin\frac{\pi}{3}\sin x\right)$

$= \sin\frac{2\pi}{3}\sin 2x = \frac{\sqrt{3}}{2}\sin 2x$

Hence, maximum value of given expression is $\frac{\sqrt{3}}{2}$.

90. We have, $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{6}\right) + \frac{1}{\sqrt{2}}\cos\left(x + \frac{\pi}{6}\right)\right]$

$= \sqrt{2}\cos\left[x + \frac{\pi}{6} - \frac{\pi}{4}\right]$

$= \sqrt{2}\cos\left(x - \frac{\pi}{12}\right)$

Hence, maximum value will be at $x = \frac{\pi}{12}$.

91. Let $f(x) = \sin x\cos x = \frac{1}{2}\sin 2x$

We know that, $-1 \le \sin 2x \le 1$

$\Rightarrow -\frac{1}{2} \le \frac{1}{2}\sin 2x \le \frac{1}{2}$

Thus, the greatest and least value of $f(x)$ are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

92. Maximum value of $4\sin^2 x + 3\cos^2 x$ *i.e.*, $\sin^2 x + 3$ is 4 and that of $\sin\frac{x}{2} + \cos\frac{x}{2}$ is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$, both attained at $x = \frac{\pi}{2}$.

Hence, the given function has maximum value $4 + \sqrt{2}$.

93. $\cos 2\theta + 2\cos\theta = 2\cos^2\theta - 1 + 2\cos\theta$

$= 2\left(\cos\theta + \frac{1}{2}\right)^2 - \frac{3}{2}$

$\ge -\frac{3}{2}$ $\left[\because 2\left(\cos\theta + \frac{1}{2}\right)^2 \ge 0, \forall\theta\right]$

Then, maximum value of $\cos 2\theta + 2\cos\theta$ is 3.

94. Since, $(\cot\alpha_1)(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$

$\Rightarrow (\cos\alpha_1)(\cos\alpha_2)\ldots(\cos\alpha_n) = (\sin\alpha_1)(\sin\alpha_2)\ldots(\sin\alpha_n)$

$\Rightarrow \cos^2\alpha_1\cos^2\alpha_2\ldots\cos^2\alpha_n$

$= \frac{\sin 2\alpha_1\sin 2\alpha_2\ldots\sin 2\alpha_n}{2^n}$

$\Rightarrow \cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n$

$= \left(\frac{\sin 2\alpha_1\sin 2\alpha_2\ldots\sin 2\alpha_n}{2^n}\right)^{1/2}$

Since, maximum value of $\sin\alpha = 1$

$\therefore$ Maximum value of $\cos\alpha_1\ldots\cos\alpha_n = \frac{1}{2^{n/2}}$

95. We know that, $-\sqrt{3^2 + 4^2} \le 3\cos x + 4\sin x \le \sqrt{3^2 + 4^2}$

$\Rightarrow -\sqrt{25} \le 3\cos x + 4\sin x \le \sqrt{25}$

$\Rightarrow -5 \le 3\cos x + 4\sin x \le 5$

$\Rightarrow -5 + 8 \le 3\cos x + 4\sin x + 8 \le 5 + 3$

$\Rightarrow 3 \le 3\cos x + 4\sin x + 8 \le 8$

Hence, minimum value of given expression is 3.

96. Given that, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = n\pi + \frac{\pi}{3}$

For $-\pi < \theta < 0$ put $n = -1$, we get

$\theta = -\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$ or $\frac{-4\pi}{6}$

97. $\because \cos x = \sqrt{1 - \sin 2x}$

$\Rightarrow \cos x = |\sin x - \cos x|$

There are two cases arise.

***Case* I** $\sin x \le \cos x$

$\Rightarrow \cos x = \cos x - \sin x$

$\Rightarrow \sin x = 0$

where, $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

$\Rightarrow x = 2\pi$, neglecting $x = \pi$

***Case* II** $\sin x > \cos x$

$\Rightarrow \tan x = 2$

where, $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\because \tan x = 2$

$\Rightarrow x = \tan^{-1}(2)$

Thus, the given equation has two solutions.

98. We have, $3x + 2\tan x = \frac{5\pi}{2}$

$\Rightarrow \quad \tan x = \frac{5\pi}{4} - \frac{3x}{2}$

Let $\quad \frac{5\pi}{4} - \frac{3x}{2} = y$

and $\quad y = \tan x$

$\therefore$ Graph of $y = \frac{5\pi}{4} - \frac{3x}{2}$ and $y = \tan x$ meet exactly three times in $[0, 2\pi]$. Thus, the number of solutions of given equation is 3.

99. Given $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ...(i)

Let $\quad (\sqrt{3} - 1) = r\cos\alpha$ and $(\sqrt{3} + 1) = r\sin\alpha$

$\therefore \quad r^2(\cos^2 + \sin^2\alpha) = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2$

$\Rightarrow \quad r^2 = 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$

$\Rightarrow \quad r^2 = 8 \Rightarrow r = 2\sqrt{2}$

And $\quad \frac{r\sin\alpha}{r\cos\alpha} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = -\frac{\sqrt{3}+1}{(1-\sqrt{3})}$

$= -\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$\Rightarrow \quad \tan\alpha = -\tan\left(\frac{7\pi}{12}\right)$

$\Rightarrow \quad \alpha = -\frac{5\pi}{12}$

$\therefore$ From Eq. (i) becomes,

$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = 2$

$\Rightarrow \quad 2\sqrt{2}\cos(\alpha - \theta = 2)$

$\Rightarrow \quad \cos(\theta - \alpha) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta - \alpha = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{4} - \frac{5\pi}{12}$

100. $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad 3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$\Rightarrow \quad 3\sin x(\sin x - 2) - 1(\sin x - 2) = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}$ or 2

$\Rightarrow \quad \sin x = \frac{1}{3} \quad (\because \sin x \neq 2)$

Let $\quad \sin^{-1}\frac{1}{3} = \alpha, 0 < \alpha < \frac{\pi}{2}$

Then, $\alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha, 5\pi - \alpha$ are the solutions in $[0, 5\pi]$.

$\therefore$ Required number of solutions = 6

101. Since, $\quad 1 + \sin x\sin^2\frac{x}{2} = 0$

$\therefore \quad 1 + \sin x\left(\frac{1 - \cos x}{2}\right) = 0$

$\Rightarrow \quad 2 + \sin x - \sin x\cos x = 0$

$\Rightarrow \quad \sin 2x - 2\sin x = 4$

which is not possible for any x in $[-\pi, \pi]$.

102. Now, $1 + |\cos x| + \cos^2 x + |\cos^3 x| + \ldots\infty = \frac{1}{1 - |\cos x|}$

$\therefore \quad 8^{\frac{1}{1-|\cos x|}} = 4^3$

$\Rightarrow \quad 2^{\frac{3}{1-|\cos x|}} = 2^6 \Rightarrow 1 = 2 - 2|\cos x|$

$\Rightarrow \quad |\cos x| = \frac{1}{2}$

$\Rightarrow \quad \cos x = \pm\frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}$

$\therefore$ Number of solutions = 4

103. Given, $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$

$\therefore \quad \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{2}{\sin\theta}$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 2\cos\theta, \sin\theta \neq 0$

$\Rightarrow \quad 1 = 2\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}$

104. Given, $\quad 2\sin^2\theta = 3\cos\theta$

$\therefore \quad 2(1 - \cos^2\theta) = 3\cos\theta$

$\Rightarrow \quad 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow \quad (\cos\theta + 2)(2\cos\theta - 1) = 0$

$\Rightarrow \quad \cos\theta = -2, \cos\theta = \frac{1}{2}$

But $\cos\theta \in [-1, 1]$

$\therefore \quad \cos\theta \neq -2$

Now, $\quad \cos\theta = \frac{1}{2}$

As $\quad \theta \in [0, 2\pi]$

$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

105. Given, $e^{\sin x} - e^{-\sin x} - 4 = 0$

$\Rightarrow \quad e^{2\sin x} - 4e^{\sin x} - 1 = 0$

$\Rightarrow \quad e^{\sin x} = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 + \sqrt{5}$

$\Rightarrow \quad \sin x = \log(2+\sqrt{5}) \quad [\because \log(2-\sqrt{5})$ is not defined$]$

Since, $2+\sqrt{5} > e \Rightarrow \log(2+\sqrt{5}) > 1$

$\Rightarrow \sin x > 1$, which is not possible.

Hence, no solution exist.

106. From graph $|\cos x|$ and $2[x]$ does not cut each other for any real value of x. Hence, number of solution is nil.

107. $2\sin x = 5x^2 + 2x + 3$

$\Rightarrow \quad 2\sin x = 4x^2 + (x+1)^2 + 2$

But $\quad 2\sin x \le 2$

and $4x^2 + (x+1)^2 + 2 > 2$, so it has no solution.

108. Since, $\tan 2x = \tan\dfrac{2}{x}$

$\Rightarrow \quad 2x = n\pi + \dfrac{2}{x} \Rightarrow 2x^2 - n\pi x - 2 = 0$

$\Rightarrow \quad x = \dfrac{n\pi \pm \sqrt{n^2\pi^2 + 16}}{4}$

109. $\because \sec x - 1 = (\sqrt{2}-1)\tan x$

$\Rightarrow \quad 1 - \cos x = (\sqrt{2}-1)\sin x$

$\Rightarrow \quad \sin\dfrac{x}{2}\left\{\sin\dfrac{x}{2} - (\sqrt{2}-1)\cos\dfrac{x}{2}\right\} = 0$

$\Rightarrow \quad \sin\dfrac{x}{2} = 0$ or $\tan\dfrac{x}{2} = \sqrt{2}-1 = \tan\dfrac{\pi}{8}$

$\Rightarrow \quad \dfrac{x}{2} = n\pi$ or $\dfrac{x}{2} = n\pi + \dfrac{\pi}{8}$

$\therefore \quad x = 2n\pi, 2n\pi + \dfrac{\pi}{4}$

110. $5\cos^2\theta + 7\sin^2\theta - 6 = 0$

$\Rightarrow \quad 5\cos^2\theta + 7(1-\cos^2\theta) - 6 = 0$

$\Rightarrow \quad 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0$

$\Rightarrow \quad -2\cos^2\theta + 1 = 0$

$\Rightarrow \quad -(2\cos^2\theta - 1) = 0$

$\Rightarrow \quad \cos 2\theta = 0 \quad (\because \cos 2\theta = 2\cos^2\theta - 1)$

$\Rightarrow \quad 2\theta = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{4}$

111. We have, $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = -\dfrac{5}{4}$ which is not possible.

$\therefore \quad 2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$\therefore$ Solution set is $\left\{\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right\} \in [0, 2\pi]$.

112. Since, $\quad 4\cos\theta - 3\sec\theta = 2\tan\theta$

$\Rightarrow \quad 4\cos\theta - \dfrac{3}{\cos\theta} = 2\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \quad 4\cos^2\theta - 3 = 2\sin\theta$

$\Rightarrow \quad 4 - 4\sin^2\theta - 3 = 2\sin\theta$

$\Rightarrow \quad 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm \sqrt{5}}{4}$

Either $\sin\theta = \dfrac{-1+\sqrt{5}}{4}$ or $\sin\theta = \dfrac{-1-\sqrt{5}}{4}$

$\Rightarrow \quad \sin\theta = \sin\dfrac{\pi}{10}$ or $\sin\theta = \sin\left(-\dfrac{3\pi}{10}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\dfrac{\pi}{10}$ or $\theta = n\pi + (-1)^n\left(-\dfrac{3\pi}{10}\right)$

113. We have, $\cos 3x + \cos 2x = \sin\dfrac{3x}{2} + \sin\dfrac{x}{2}$

$\Rightarrow \quad 2\cos\dfrac{5x}{2}\cos\dfrac{x}{2} = 2\sin x\cos\dfrac{x}{2}$

Either $\quad \cos\dfrac{x}{2} = 0 \Rightarrow \dfrac{x}{2} = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x = (2n+1)\pi$ or $\cos\dfrac{5x}{2} = \sin x$

$\Rightarrow \quad \cos\dfrac{5x}{2} = \cos\left(\dfrac{\pi}{2} - x\right)$

$\Rightarrow \quad \dfrac{5x}{2} = 2n\pi \pm \left(\dfrac{\pi}{2} - x\right)$

Taking the +ve sign $\dfrac{7x}{2} = 2n\pi + \dfrac{\pi}{2} \Rightarrow x = \dfrac{4n\pi}{7} + \dfrac{\pi}{7}$

Taking –ve sign

$\dfrac{3x}{2} = 2n\pi - \dfrac{\pi}{2} \Rightarrow x = \dfrac{4n\pi}{3} - \dfrac{\pi}{3}$

For $0 \le x \le 2\pi$

$x = \dfrac{\pi}{7}, \dfrac{5\pi}{7}, \dfrac{9\pi}{7}, \dfrac{13\pi}{7}, \pi$

Thus, number of solutions = 5

114. $\sec x \cos 5x = -1$

$\Rightarrow \quad \cos 5x = -\cos x$

$\Rightarrow \quad 5x = 2n\pi \pm (\pi - x)$

$\Rightarrow \quad x = \dfrac{(2n+1)\pi}{6}$ or $\dfrac{(2n-1)\pi}{4}$

The possible values of x which lies in the interval $(0, 2\pi)$ are $\dfrac{\pi}{4}, \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \dfrac{5\pi}{6}, \dfrac{5\pi}{4}, \dfrac{7\pi}{6}, \dfrac{7\pi}{4}, \dfrac{9\pi}{6}$ and $\dfrac{11\pi}{6}$.

115. Since, $\tan\theta + \tan\left(\dfrac{3\pi}{4} + \theta\right) = 2$

$\therefore \quad \tan\theta + \dfrac{-1 + \tan\theta}{1 + \tan\theta} = 2$

$\Rightarrow \quad \tan\theta + \tan^2\theta - 1 + \tan\theta = 2 + 2\tan\theta$

$\Rightarrow \quad \tan^2\theta = 3$

$\Rightarrow \quad \tan^2\theta = (\sqrt{3})^2 = \tan^2\dfrac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi \pm \dfrac{\pi}{3}, n \in I$

116. We have, $\sin^2\theta + \sin\theta - 2 = 0$

$\Rightarrow \quad (\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow \quad \sin\theta = 1, \quad \sin\theta = -2$

But $\quad \sin\theta \neq -2$

$\therefore \quad \sin\theta = 1 = \sin\dfrac{\pi}{2}$

$\Rightarrow \quad \theta = n\pi + (-1)^n \dfrac{\pi}{2}$

117. The given equation can be rewritten as

$\tan\theta(\sin\theta + \sqrt{3}) = 0$

$\Rightarrow \quad \tan\theta = 0$ but $\sin\theta + \sqrt{3} \neq 0$

$\Rightarrow \quad \tan\theta = 0$

$\Rightarrow \quad \theta = n\pi, n \in I$

118. We have,

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\Rightarrow \sin x + \sin 3x - 3\sin 2x = \cos x + \cos 3x - 3\cos 2x$

$\Rightarrow 2\sin 2x\cos x - 3\sin 2x - 2\cos 2x\cos x + 3\cos 2x = 0$

$\Rightarrow \quad \sin 2x(2\cos x - 3) - \cos 2x(2\cos x - 3) = 0$

$\Rightarrow \quad (\sin 2x - \cos 2x)(2\cos x - 3) = 0$

$\Rightarrow \quad \sin 2x = \cos 2x \qquad \left(\because \cos x \neq \dfrac{3}{2}\right)$

$\Rightarrow \quad 2x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right)$

Taking +ve sign

$$x = \frac{n\pi}{2} + \frac{\pi}{8}$$

119. We have,

$3\sin^2 x + 10\cos x - 6 = 0$

$\Rightarrow \quad (\cos x - 3)(3\cos x - 1) = 0$

$\Rightarrow \quad \cos x \neq 3$ or $\cos x = \dfrac{1}{3}$

$\Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\dfrac{1}{3}\right)$

120. The given equation can be rewritten as

$1 - \cos^2\theta - \cos\theta = \dfrac{1}{4}$

$\Rightarrow \quad \cos^2\theta + \cos\theta - \dfrac{3}{4} = 0$

$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0$

$\Rightarrow \quad \cos\theta = \dfrac{-4 \pm \sqrt{16 + 48}}{8} = \dfrac{1}{2}, -\dfrac{3}{2}$

Since, $\cos\theta = -\dfrac{3}{2}$ is not possible, so we take

$\cos\theta = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow \quad \theta = 2n\pi \pm \dfrac{\pi}{3} \qquad \ldots(i)$

For the given interval, put $n = 0$ and $n = 1$ in Eq. (i) we get

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

121.

It is clear from the figure that, the smallest positive root of a given equation is in $\left(\pi, \dfrac{3\pi}{2}\right)$.

122. We have, $2\cos^2 x + 3\sin x - 3 = 0$

$\Rightarrow \quad 2 - 2\sin^2 x + 3\sin x - 3 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) = 0$

$\Rightarrow \quad \sin x = \dfrac{1}{2}$ or $\sin x = 1$

$\Rightarrow \quad x = 30°, 150°, 90°$

123. $(1 + 2\sin\theta)^2 + (\sqrt{3}\tan\theta - 1)^2 = 0$

$\Rightarrow \quad 1 + 2\sin\theta = 0$

and $\quad \sqrt{3}\tan\theta - 1 = 0$

$\therefore \quad \sin\theta = -\frac{1}{2} \Rightarrow \theta = m\pi + (-1)^m\left(-\frac{\pi}{6}\right)$

and $\quad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = m\pi + \frac{\pi}{6}$

For common values, m must be odd.

i.e., $\quad m = 2n+1 \Rightarrow \theta = 2n\pi + \frac{7\pi}{6}$

124. $\because \quad \sin 3x = 3\sin x - 4\sin^3 x$

$\therefore \quad \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

and $\quad \cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow \quad \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$\therefore \quad \cos 3x\cos^3 x + \sin 3x \sin^3 x$

$= \frac{1}{4}[\cos^2 3x + 3\cos x\cos 3x + 3\sin x\sin 3x - \sin^2 3x]$

$= \frac{1}{4}[3\cos 2x + \cos 6x] = \cos^3 2x$

$\Rightarrow \quad \cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2}$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{4}$

125. $\because \quad \tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3} \Rightarrow \tan 3\theta = \tan\frac{\pi}{3}$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{3} \Rightarrow \theta = \frac{(3n+1)\pi}{9}, n \in I$

126. Given that,

$\tan^2\theta - \tan\theta - \sqrt{3}\tan\theta + \sqrt{3} = 0$

$\Rightarrow \quad \tan\theta(\tan\theta - 1) - \sqrt{3}(\tan\theta - 1) = 0$

$\Rightarrow \quad (\tan\theta - \sqrt{3})(\tan\theta - 1) = 0$

$\Rightarrow \quad \theta = n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{4}$

127. Given, $\quad \sin^4 x + \cos^4 x = \sin x \cdot \cos x$

$\Rightarrow \quad (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x = \sin x \cdot \cos x$

$\Rightarrow \quad 1 - \frac{\sin^2 2x}{2} = \frac{\sin 2x}{2}$

$\Rightarrow \quad \sin^2 2x + \sin 2x - 2 = 0$

$\Rightarrow \quad (\sin 2x + 2)(\sin 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 1 \qquad (\because \sin 2x \geq -1)$

$\therefore \quad 2x = (4n+1)\frac{\pi}{2}$

$\Rightarrow \quad x = (4n+1)\frac{\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$

Hence, two solutions exist.

128. Since, $\cos 2x, \frac{1}{2}, \sin 2x$ are in AP.

$\Rightarrow \quad \cos 2x + \sin 2x = 1$

$\Rightarrow \quad \sin 2x = 1 - \cos 2x = 2\sin^2 x$

$\Rightarrow \quad 2\sin x(\cos x - \sin x) = 0$

$\Rightarrow \quad \sin x = 0$ or $\cos x - \sin x = 0$

$\Rightarrow \quad x = n\pi$ or $\tan x = 1$

$\Rightarrow \quad x = n\pi$ or $x = n\pi + \frac{\pi}{4}$

Thus, required values of x are $n\pi$ and $n\pi + \frac{\pi}{4}$.

129. $|\sin x| > 2\sin^2 x$

$\Rightarrow \quad |\sin x|(2|\sin x| - 1) < 0$

$\Rightarrow \quad 0 < |\sin x| < \frac{1}{2}$

$\Rightarrow \quad x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\pi, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

130. We have, $\sin^{10} 2x = 1 + \cos^{10} x$

Minimum value of RHS $= 1$ and maximum value of LHS $= 1$. Therefore, solution is possible only when $\sin^{10} 2x = 1$ and $\cos^{10} x = 0$. But this is not possible. Therefore, it has no solution.

131. Given equation, $\sin x + \sin y + \sin z = -3$ is satisfied only when $x = y = z = \frac{3\pi}{2}$, for $x, y, z \in [0, 2\pi]$.

132. $\because \quad \sin^5 x - \cos^5 x = \frac{\sin x - \cos x}{\sin x\cos x}$

$\Rightarrow \quad \sin x\cos x\left[\frac{\sin^5 x - \cos^5 x}{\sin x - \cos x}\right] = 1$

$\Rightarrow \quad \frac{1}{2}\sin 2x[\sin^4 x + \sin^3 x\cos x + \sin^2 x\cos^2 x$

$+ \sin x\cos^2 x + \cos^4 x] = 1$

$\Rightarrow \quad \sin 2x[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$+ \sin x\cos x(\sin^2 x + \cos^2 x) + \sin^2 x\cos^2 x] = 2$

$\Rightarrow \quad \sin 2x[1 - \sin^2 x\cos^2 x + \sin x\cos x] = 2$

$\Rightarrow \quad \sin^3 2x - 2\sin^2 2x - 4\sin 2x + 8 = 0$

$\Rightarrow \quad (\sin 2x - 2)^2(\sin 2x + 2) = 0$

$\Rightarrow \quad \sin 2x = \pm 2$, which is not possible for any x.

133. $\sqrt{\sin^2 x - \sin x + \frac{1}{2}} = \sqrt{\left(\sin x - \frac{1}{2}\right)^2 + \frac{1}{4}} \geq \frac{1}{2}, \forall x$

and $\sec^2 y \geq 1, \forall y$, so $2^{\sec^2 y} \geq 2$. Hence, the above inequality holds only for those values of x and y for which $\sin x = \frac{1}{2}$ and $\sec^2 y = 1$. Hence, $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ and $y = 0, \pi, 2\pi, 3\pi$. Hence, required number of ordered pairs are 16.

134. Since, $2\cos^2\frac{x}{2}\sin^2 x < 2$

But $x^2 + \frac{1}{x^2} \geq 2$

Thus, the equation has no solution.

135. $e^{\sin x} + e^{\cos x} = 2e^{1/\sqrt{2}}$ ...(i)

$\Rightarrow$ $e^{\sin x} > 0$ and $e^{\cos x} > 0$

$\therefore$ $e^{\sin x} + e^{\cos x} \geq 2\sqrt{e^{\sin x + \cos x}}$ ($\because$ AM $\geq$ GM)

$\Rightarrow$ $e^{\sin x} + e^{\cos x} \geq 2\, e^{1/\sqrt{2}}$ ...(ii)

Since, equality holds

$\Rightarrow$ $e^{\sin x} = e^{\cos x}$

$\Rightarrow$ $\sin x = \cos x$

$\Rightarrow$ $\tan x = 1 \Rightarrow x = m\pi + \frac{\pi}{4}$

$\Rightarrow$ $x = (4m+1)\frac{\pi}{4}$

Round II

1. From the given equations, we have $\Sigma \tan\alpha = p$

$$\Sigma \tan\alpha \tan\beta = 0$$

and $\tan\alpha\tan\beta\tan\gamma = r$

$\therefore$ $(1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)$

$= 1 + \Sigma\tan^2\alpha + \Sigma\tan^2\alpha\tan^2\beta + \tan^2\alpha\tan^2\beta\tan^2\gamma$

$= 1 + (\Sigma\tan\alpha)^2 - 2\Sigma\tan\alpha\tan\beta + (\Sigma\tan\alpha\tan\beta)^2$

$- 2\tan\alpha\tan\beta\tan\gamma\,\Sigma\tan\alpha + \tan^2\alpha\tan^2\beta\tan^2\gamma$

$= 1 + p^2 - 2pr + r^2 = 1 + (p-r)^2$

2. $\frac{1+\tan\theta}{1-\tan\theta} = 3\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

On simplification, we get

$$3\tan^4\theta - 6\tan^2\theta + 8\tan\theta - 1 = 0$$

$\therefore$ Product of roots

$$= \tan\alpha\cdot\tan\beta\cdot\tan\gamma\cdot\tan\delta = -\frac{1}{3}$$

3. $\operatorname{cosec}^2 x + 25\sec^2 x = 26 + \cot^2 x + 25\tan^2 x$

$$= 26 + 10 + (\cot x - 5\tan x)^2 \geq 36$$

4. Equation first can be written as

$x\sin a + y \times 2\sin a\cos a + z \times \sin a(3 - 4\sin^2 a)$

$= 2 \times 2\sin a\cos a\cos 2a$

$\Rightarrow$ $x + 2y\cos a + z(3 + 4\cos^2 a - 4)$

$= 4\cos a(2\cos^2 a - 1)$ as $\sin a \neq 0$

$\Rightarrow$ $8\cos^3 a - 4z\cos^2 a - (2y+4)\cos a + (z - x) = 0$

$\Rightarrow$ $\cos^3 a - \left(\frac{z}{2}\right)\cos^2 a - \left(\frac{y+2}{4}\right)\cos a + \left(\frac{z-x}{8}\right) = 0$

which shows that $\cos a$ is a root of the equation

$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$$

Similarly, from second and third equation we can verify that $\cos b$ and $\cos c$ are the roots of the given equation.

5. $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2(1+\cos\theta)}}}}$ (n numbers of 2's)

$= \sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{\left(2+2\cos\frac{\theta}{2}\right)}}}}$

[$(n-1)$ numbers of 2's]

... ...

... ...

$= \sqrt{2 + 2\cos(\theta/2^{n-1})}$

$= \sqrt{2\{1 + 2\cos^2(\theta/2^n) - 1\}} = 2\cos(\theta/2^n)$

6. $\because$ $\tan\alpha/2 + \tan\beta/2 = \frac{26}{8} = \frac{13}{4}$

and $\tan\alpha/2\tan\beta/2 = \frac{15}{8}$

$\therefore$ $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\alpha/2 + \tan\beta/2}{1 - \tan\alpha/2\tan\beta/2}$

$$= \frac{\frac{13}{4}}{1 - \frac{15}{8}} = -\frac{26}{7}$$

Now, $\cos(\alpha+\beta) = \frac{1 - \tan^2\left(\frac{\alpha+\beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha+\beta}{2}\right)}$

$$= \frac{1 - \left(-\frac{26}{7}\right)^2}{1 + \left(-\frac{26}{7}\right)^2} = \frac{49 - 676}{49 + 676} = -\frac{627}{725}$$

7. Given, $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta = b_0 + b_1\sin\theta + b_2\sin^2\theta + \ldots + b_n\sin^n\theta$ ...(i)

Putting $\theta = 0$ in Eq. (i), we get $0 = b_0$

Again, Eq. (i) can be written as $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$

$$\frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{n} b_r \sin^{r-1}\theta$$

On taking limit as $\theta \to 0$, we get

$$\lim_{\theta \to 0} \frac{\sin n\theta}{\sin \theta} = b_1$$

$$\Rightarrow \quad \lim_{\theta \to 0} n\left(\frac{\sin n\theta}{n\theta}\right)\left(\frac{\theta}{\sin \theta}\right) = b_1$$

$$\Rightarrow \quad n = b_1$$

Hence, $\quad b_0 = 0;\ b_1 = n$

8. Given, $\quad \tan\theta = \frac{1}{\sqrt{7}}$

$$\therefore \quad \frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta} = \frac{(1+\cot^2\theta) - (1+\tan^2\theta)}{(1+\cot^2\theta) + (1+\tan^2\theta)}$$

$$= \frac{(\cot^2\theta - \tan^2\theta)}{2 + \tan^2\theta + \cot^2\theta}$$

$$= \frac{7 - \frac{1}{7}}{2 + \frac{1}{7} + 7} = \frac{48}{64} = \frac{3}{4}$$

9. $(\sqrt{3})^{\sec^2\theta} = (\tan^2\theta + 1)^2 - 1$

$$= (\sec^2\theta)^2 - 1$$

Put $\quad \sec^2\theta = x \quad (\because x \geq 1)$

Then, $\quad (\sqrt{3})^x = x^2 - 1$

Let $\quad y = (\sqrt{3})^x = (x^2 - 1) \quad (x > 1)$

Now, graphs of $y = (\sqrt{3})^x$ and $y = x^2 - 1$ intersect at one point

i.e., $x = 2$, then $y = 3$

Then, $\quad \sec^2\theta = 2$

$$\Rightarrow \quad \sec\theta = \pm\sqrt{2}$$

Therefore, there are two values of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

10. The given equation is

$\sin^4 x + \cos^4 y + 2 = 4 \sin x \cos y$

$$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2\sin^2 x + 2\cos^2 y - 4\sin x \cos y = 0$$

$$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2(\sin x - \cos y)^2 = 0$$

which is possible only when

$$\sin^2 x - 1 = 0, \quad \cos^2 y - 1 = 0, \quad \sin x - \cos y = 0$$

$$\Rightarrow \quad \sin^2 x = 1, \quad \cos^2 y = 1, \quad \sin x = \cos y$$

As $0 \leq x, y \leq \frac{\pi}{2}$

We get $\quad \sin x = \cos y = 1$

$$\therefore \quad \sin x + \cos y = 1 + 1 = 2$$

11. $\cos^4 x - (\lambda + 2)\cos^2 x - (\lambda + 3) = 0$

$$\Rightarrow \quad (\cos^2 x)^2 - (\lambda + 2)\cos^2 x - (\lambda + 3) = 0$$

$$\therefore \quad \cos^2 x = \frac{(\lambda + 2) \pm \sqrt{(\lambda + 2)^2 + 4(\lambda + 3)}}{2}$$

$$= \frac{(\lambda + 2) \pm (\lambda + 4)}{2} = \lambda + 3, -1$$

$$\Rightarrow \quad \cos^2 x = \lambda + 3 \quad (\because \cos^2 x \neq -1)$$

But $\quad 0 \leq \cos^2 x \leq 1$

$$\Rightarrow \quad 0 \leq \lambda + 3 \leq 1$$

$$\Rightarrow \quad -3 \leq \lambda \leq -2$$

12. $\because \cos(315\pi + x) = (-1)^{315}\cos x = -\cos x$

$$\therefore \quad 4\cos^3 x - 4\cos^2 x - \cos(315\pi + x) = 1$$

$$\Rightarrow \quad 4\cos^3 x - 4\cos^2 x + \cos x - 1 = 0$$

$$\Rightarrow \quad (4\cos^2 x + 1)(\cos x - 1) = 0$$

$$\Rightarrow \quad \cos x = 1,\ 4\cos^2 x + 1 \neq 0$$

$$\Rightarrow \quad \cos x = \cos 0$$

$$\Rightarrow x = 2n\pi, n \in I$$

$$\therefore x = 2\pi, 4\pi, 6\pi, 8\pi, \ldots, 100\pi \quad (\because 0 < x < 315)$$

(*i.e.*, $100\pi < 315 < 101\pi$)

Required arithmetic mean

$$= \frac{2\pi + 4\pi + 6\pi + 8\pi + \ldots + 100\pi}{50}$$

$$= \frac{2\pi(1 + 2 + 3 + 4 + \ldots + 50)}{50} = \frac{2\pi \cdot \frac{50}{2} \cdot 51}{50} = 51\pi$$

13. Since, $\cos^2\theta = \frac{1}{6}\sin\theta \cdot \tan\theta$

$$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$$

As $\cos\theta = \frac{1}{2}$ satisfied the equation.

$$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \text{ (other values of } \cos\theta \text{ are imaginary)}$$

$$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$$

14. Given, $\quad \cos(a - b) = \cos 0$

$$\therefore \quad a - b = 2n\pi, n \in I$$

$$\Rightarrow \quad a - b = -2\pi, 0, 2\pi$$

$$\therefore \quad a + b = 2\pi + 2a, 2a, 2a - 2\pi$$

$$\Rightarrow \quad \cos(a + b) = \cos 2a, \cos 2a, \cos 2a = \frac{1}{e}$$

$$y = \cos 2a = \frac{1}{e}$$

Hence, number of solutions is 4.

15. $\because \quad x^3 + x^2 + 4x + 2\sin x = 0$

$\Rightarrow \quad x^3 + (x+2)^2 + 2\sin x = 4$

$x = 0$, satisfies this equation.

Now, in $0 < x \le \pi$, $x^3 + (x+2)^2 + 2\sin x > 4$

and in $\pi < x \le 2\pi$, $x^3 + (x+2)^2 + 2\sin x > 27 + 25 - 2 = 50$

Hence, $x = 0$ is the only solution.

16. $\because$ $2\cos x, |\cos x|, 1 - 3\cos^2 x$ are in GP.

$\therefore \quad \cos^2 x = 2\cos x \cdot (1 - 3\cos^2 x)$

$\Rightarrow \quad 6\cos^3 x + \cos^2 x - 2\cos x = 0$

$\therefore \quad \cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$\therefore \quad x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right) \qquad (\because \alpha, \beta$ are positive)

If $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}$

Then, $\quad |\alpha - \beta| = \frac{\pi}{6}$

17. $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} = \frac{\tan\alpha - 1}{\tan\alpha + 1}$

$\Rightarrow \quad \tan\theta = \tan\left(\alpha - \frac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi + \alpha - \left(\frac{\pi}{4}\right), n \in I$

$\Rightarrow \quad 2\theta = 2n\pi + 2\alpha - \frac{\pi}{2}$

$\therefore \quad \sin 2\theta = \sin\left(2\alpha - \frac{\pi}{2}\right) = -\cos 2\alpha$

and $\quad \cos 2\theta = \cos\left(2\alpha - \frac{\pi}{2}\right) = \sin 2\alpha$

and $\quad \sin\alpha - \cos\alpha = \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\{\theta - n\pi\}$

$= \pm\sqrt{2}\sin\theta$

and $\quad \sin\alpha + \cos\alpha = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left\{\frac{\pi}{2} + \theta - n\pi\right\}$

$= \sqrt{2}\cos(\theta - n\pi)$

$= \pm\sqrt{2}\cos\theta$

18. $x^2 + 4xy + y^2 = aX^2 + bY^2$

$\Rightarrow \quad (X\cos\theta + Y\sin\theta)^2$

$+ 4(X\cos\theta + Y\sin\theta)(X\sin\theta - Y\cos\theta)$

$+ (X\sin\theta - Y\cos\theta)^2 = aX^2 + bY^2$

$\Rightarrow \quad X^2(1 + 4\sin\theta\cos\theta) + Y^2(1 - 4\sin\theta\cos\theta)$

$+ 4XY(\sin^2\theta - \cos^2\theta)$

$= aX^2 + bY^2$

On comparing both sides, we get

$1 + 4\sin\theta\cos\theta = a, \quad 1 - 4\sin\theta\cos\theta = b$

and $\quad \sin^2\theta - \cos^2\theta = 0$

$\therefore \quad \theta = \frac{\pi}{4}$, then $a = 3, b = -1$

19. Given, $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$

$\Rightarrow \quad 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - 2\cos^2\left(\frac{x+y}{2}\right) + 1 = \frac{3}{2}$

$\Rightarrow \quad 2\cos^2\left(\frac{x+y}{2}\right) - 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \frac{1}{2} = 0$

Now, $\cos\left(\frac{x+y}{2}\right)$ is always real, then discriminant ≥ 0

$\Rightarrow \quad 4\cos^2\left(\frac{x-y}{2}\right) - 4 \ge 0$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) \ge 1$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) = 1$

$\Rightarrow \quad \frac{x-y}{2} = 0$

$\Rightarrow \quad x = y$

20. $\tan|x| = |\tan x|$

Let $\quad y = \tan|x|$ and $y = |\tan x|$

It is clear that

$-\frac{3\pi}{2} < x \le -\pi, \; -\frac{\pi}{2} < x < \frac{\pi}{2}, \pi \le x < \frac{3\pi}{2}$

$\therefore \quad x \in \left(-\frac{\pi(2k+1)}{2}, -\pi k\right]$

and $\quad x \in \left[\pi k, \frac{\pi(2k+1)}{2}\right), k \in I$

21. $\because$ AM $\geq$ GM

$\therefore \quad 3^x + 3^x \geq 2$

$\Rightarrow \quad 2\cos\left(\frac{x}{2}\right) \geq 2$

$\Rightarrow \quad \cos\left(\frac{x}{2}\right) \geq 1$

$\therefore \quad \cos\left(\frac{x}{2}\right) = 1 \quad \left(\because \cos\frac{x}{2} \text{ is never} > 1\right)$

$\Rightarrow \quad \frac{x}{2} = 2n\pi, n \in I$

$\therefore \quad x = 4n\pi$

At $n = 0$ it satisfies the given equation but other values of n do not satisfy the equation.

22. Let $\quad y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$

$\therefore \quad y = x^2 + x^{-2} \geq 2 \quad (\because \text{AM} \geq \text{GM})$

$\Rightarrow \quad y \geq 2 \quad \ldots(\text{i})$

and $\quad y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x$

$= (1 + \cos x) \cdot \sin^2 x$

$= (\text{ a number} < 2) \cdot (\text{a number} \leq 1) < 2$

i.e., $\quad y < 2 \quad \ldots(\text{ii})$

No value of y can be obtained by the relations (i) and (ii).

$\Rightarrow$ No real solution of the equation exists.

23. $\because \quad$ AM $\geq$ GM

$\therefore \quad 5^x + 5^{-x} \geq 2$

$\Rightarrow \quad \sin e^x \geq 2 \quad (\because \sin e^x = 5^x + 5^{-x})$

But the value of $\sin(e^x)$ can never be > 1.

Hence, the given equation has no solution.

24. LHS

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot\alpha - \tan\alpha = 2\cot 2\alpha)$

$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot 2\alpha - \tan 2\alpha = 2\cot 4\alpha)$

$= \cot\alpha - 8(\cot 8\alpha - \tan 8\alpha) + 16\cot 16\alpha$

$= \cot\alpha - 16\cot 16\alpha + 16\cot 16\alpha$

$(\because \cot 8\alpha - \tan 8\alpha = 2\cot 16\alpha)$

$= \cot\alpha = \text{RHS}$

25. $\because \quad \cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right)$

$= \cos\alpha + 2\cos(\alpha + \pi)\cos\frac{\pi}{3}$

$= \cos\alpha + (-2\cos\alpha)\left(\frac{1}{2}\right) = 0$

$\therefore \quad \cos^3\alpha + \cos^3\left(\alpha + \frac{2\pi}{3}\right) + \cos^3\left(\alpha + \frac{4\pi}{3}\right)$

$= 3\cos\alpha\cos\left(\alpha + \frac{2\pi}{3}\right)\cos\left(\alpha + \frac{4\pi}{3}\right)$

26. Let $\quad x = \cot A, \quad y = \cot B, \quad z = \cot C$

$\therefore \quad \cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

$\because \quad A + B + C = 180°$

$\therefore \quad \Sigma\frac{x}{(1 + x^2)} = \Sigma\frac{\cot A}{(1 + \cot^2 A)}$

$= \frac{1}{2}\Sigma\frac{2\tan A}{(1 + \tan^2 A)} = \frac{1}{2}\Sigma\sin 2A$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}(4\sin A\sin B\sin C) = 2\sin A\sin B\sin C$

$= \frac{2}{\sqrt{(1 + \cot^2 A)(1 + \cot^2 B)(1 + \cot^2 C)}}$

$= \frac{2}{\sqrt{(1 + x^2)(1 + y^2)(1 + z^2)}} = \frac{2}{\sqrt{\prod(1 + x^2)}}$

and $\sin 2A + \sin 2B - \sin 2C$

$= 2\sin(A + B)\cos(A - B) - 2\sin C\cos C$

$= 2\sin C\{\cos(A - B) - \cos C\}$

$= 2\sin C\{\cos(A - B) + \cos(A + B)\}$

$= 2\sin C(2\cos A\cos B)$

$= 4\cos A\cos B\sin C$

27. $\because \quad a\cos x + b\sin x = c$

$\Rightarrow \quad a\left\{\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}\right\} + b\left\{\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right\} = c$

$\Rightarrow \quad (a + c)\tan^2\left(\frac{x}{2}\right) - 2b\tan\left(\frac{x}{2}\right) + (c - a) = 0$

$\therefore \quad \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = \frac{2b}{(a + c)}$

and $\quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{c - a}{a + c}$

Now, $\quad \tan\left(\frac{\alpha + \beta}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1 - \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$

$= \frac{\frac{2b}{a + c}}{1 - \left(\frac{c - a}{a + c}\right)} = \frac{b}{a} = \text{independent of } c$

Also, $\quad -\sqrt{(a^2 + b^2)} \leq a\cos x + b\sin x \leq \sqrt{(a^2 + b^2)}$

$\therefore \quad -\sqrt{(a^2 + b^2)} \leq c \leq \sqrt{(a^2 + b^2)}$

28. Since, $7^x + 7^{-x} \geq 2$

But $\cos x \leq 1$

Hence, no real solution exist.

Hence, option (a) is correct.

29. Let $y = \sin x$ and $y = x^2 + x + 1$

Since, $-1 \leq \sin x \leq 1$ and $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

It is clear from the graph that no two curves intersect.

30. Since, $\cos\theta > \sin\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$

So, statement I is true.

Now, $\cos 36° > \sin 36°$

$\Rightarrow \cos^2 36° > \sin 36°$

$\Rightarrow 1 + \cos 72° > 2 \sin 36° = 2 \sin (30° + 6°)$

$\Rightarrow 1 + \sin 18° > 2(\sin 30° \cos 6° + \cos 30° \sin 6°)$

$\Rightarrow 1 + 2 \sin 9° \cos 9° > \cos 6° + 2 \cos 30° \sin 6°$

which is true.

31. Let $S = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

and $C = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right)$

$\therefore \quad C + iS = \alpha + \alpha^2 + \alpha^4$ ...(i)

where, $\alpha = \cos\left(\frac{2\pi}{7}\right) + i \sin\left(\frac{2\pi}{7}\right)$

is complex 7th root of unity.

Then, $C - iS = \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^4 = \alpha^6 + \alpha^5 + \alpha^3$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2C = \alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^5 + \alpha^3 = -1$$

$\Rightarrow C = -\frac{1}{2}$ and $C^2 + S^2 = 2$

$\Rightarrow S^2 = 2 - \left(\frac{1}{2}\right)^2 = \frac{7}{4}$

$\Rightarrow S = \frac{\sqrt{7}}{2}$

32. $\because \quad y + \frac{1}{y} \geq 2$ (AM $\geq$ GM)

$\Rightarrow \sqrt{\left(y + \frac{1}{y}\right)} \geq \sqrt{2}$

or $\sin x + \cos x \geq \sqrt{2}$

and $|\sin x + \cos x| \leq \sqrt{2}$

Hence, $y + \frac{1}{y} = 2$ and $\sin x + \cos x = \sqrt{2}$

which is possible for $y = 1, x = \frac{\pi}{4}$.

33. $\because \quad \tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

$= \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$

$\therefore \quad \frac{\pi}{2}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$

$\Rightarrow \sin\theta + \cos\theta = 2n + 1, n \in I$

$\because \quad -\sqrt{2} \leq \sin\theta + \cos\theta \leq \sqrt{2}$

$\therefore n = 0; -1$

Then, $\sin\theta + \cos\theta = 1, -1$

34. $\sin(\cos x) = \cos(\sin x)$

$\Rightarrow \cos(\sin x) = \cos\left(\frac{\pi}{2} - \cos x\right)$

$\Rightarrow \sin x = 2n\pi \pm \left(\frac{\pi}{2} - \cos x\right), n \in I$

$\Rightarrow \sin x \pm \cos x = \left(2n \pm \frac{1}{2}\right)\pi$

On squaring, we get

$$1 \pm \sin 2x = \left(2n \pm \frac{1}{2}\right)^2 \pi^2$$

$\Rightarrow |\sin 2x| = \left(2n \pm \frac{1}{2}\right)^2 \pi^2 - 1$

But $\left(2n \pm \frac{1}{2}\right)^2 \pi^2 > 2$ for all $n \in I$

$\therefore |\sin 2x| > 1$, which is imposible.

Hence, the given equation does not possess real roots.

$\because \quad \sin x > 0$

(x lies in I and II quadrant)

$\therefore \quad 2n\pi < x < (2n + 1)\pi, n \in I$

35. $\because \quad \sin^2 A = \sin^2 B$ and $\cos^2 A = \cos^2 B$

$\therefore \quad \cos 2A = \cos 2B$

$\Rightarrow 2A = 2n\pi \pm B, n \in I$

$\Rightarrow A = n\pi \pm B$

$\Rightarrow A = n\pi + B, n \in I$

($\because$ given equation in square form)

Now, $\sin A = \sin B$

$\Rightarrow A = n\pi + (-1)^n B, n \in I$

If n is even, then

$$A = n\pi + B$$

and $\cos A = \cos B$

$\Rightarrow$ $A = 2n\pi \pm B, n \in I$

Hence, statement I is true but statement II is false.

36. Let $AB = x$

In ΔDAM, $\tan(\pi - \theta - \alpha) = \dfrac{p}{x-q}$

$\Rightarrow$ $\tan(\theta + \alpha) = \dfrac{p}{q-x}$

$\Rightarrow$ $q - x = p \cot(\theta + \alpha)$

$\Rightarrow$ $x = q - p\cot(\theta + \alpha)$

$$= q - p\left(\frac{\cot\theta\cot\alpha - 1}{\cot\alpha + \cot\theta}\right) \qquad \left(\because \cot\alpha = \frac{q}{p}\right)$$

$$= q - p\left(\frac{\frac{q}{p}\cot\theta - 1}{\frac{q}{p} + \cot\theta}\right)$$

$$= q - p\left(\frac{q\cot\theta - p}{q + p\cot\theta}\right)$$

$$= q - p\left(\frac{q\cos\theta - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$$

$\Rightarrow$ $x = \dfrac{q^2\sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta + q\sin\theta}$

$\Rightarrow$ $AB = \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

37. Given expression is

$$\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin A\cos A}$$

$$= \frac{1 + \sin A\cos A}{\sin A\cos A} = 1 + \sec A \operatorname{cosec} A$$

38. Given, $3\sin P + 4\cos Q = 6$ ...(i)

and $4\sin Q + 3\cos P = 1$ ...(ii)

On squaring and adding the Eqs. (i) and (ii), we get

and $9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q)$

$+ 2 \times 3 \times 4(\sin P\cos Q + \sin Q\cos P) = 37$

$\Rightarrow$ $24\{\sin(P+Q)\} = 37 - 25$

$\therefore$ $\sin(P+Q) = \dfrac{1}{2}$

Since, P and Q are angles of ΔPQR, hence $0° < P, Q < 180°$

$\therefore$ $P + Q = 30°$ or $150°$

$\Rightarrow$ $R = 150°$ or $30°$ (respectively)

Hence, two cases arise here.

***Case* I** When $R = 150°$

$P + Q = 30° \Rightarrow 0 < P, Q < 30°$

$\Rightarrow$ $\sin P < \dfrac{1}{2}, \cos Q < 1$

$\Rightarrow$ $3\sin P + 4\cos Q < \dfrac{3}{2} + 4$

$\Rightarrow$ $3\sin P + 4\cos Q < \dfrac{11}{2} < 6$

$\therefore 3\sin P + 4\cos Q = 6$ is not possible.

***Case* II** When $R = 30°$

Hence, $R = 30°$ is the only possibility.

39. Given equation is

$$e^{\sin x} - e^{-\sin x} = 4 \Rightarrow e^{\sin x} - \frac{1}{e^{\sin x}} = 4$$

Now, let $y = e^{\sin x}$

Hence, we get

$$y - \frac{1}{y} = 4$$

$\Rightarrow$ $y^2 - 4y - 1 = 0$

$\therefore$ $y = \dfrac{4 \pm \sqrt{16+4}}{2}$

$\Rightarrow$ $y = 2 \pm \sqrt{5}$

$\therefore$ $e^{\sin x} = 2 \pm \sqrt{5}$

Since, sine is a bounded function *i.e.*, $-1 \le \sin x \le 1$. Hence, we get

$$e^{-1} \le e^{\sin x} \le e \Rightarrow e^{\sin x} \in \left[\frac{1}{e}, e\right]$$

Also, it is obvious that $2 + \sqrt{5} > e$

and $2 - \sqrt{5} < \dfrac{1}{e} \Rightarrow 2 \pm \sqrt{5} \in \left[\dfrac{1}{e}, e\right]$

$\therefore e^{\sin x} = 2 + \sqrt{5}$ is not possible for any $x \in R$ and

$e^{\sin x} = 2 - \sqrt{5}$ is also not possible for any $x \in R$

Hence, we can say that the given equation has no solution.

40. $A = \sin^2 x + \cos^4 x = \cos^4 x - \cos^2 x + \dfrac{1}{4} + \dfrac{3}{4}$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

where, $0 \le \left(\cos^2 x - \frac{1}{2}\right)^2 \le \frac{1}{4}$

$\therefore \quad \frac{3}{4} \le A \le 1$

41. $\cos(\alpha+\beta) = \frac{4}{5} \Rightarrow \alpha+\beta \in$ 1st quadrant

$\sin(\alpha-\beta) = \frac{5}{13} \Rightarrow (\alpha-\beta) \in$ 1st quadrant

$\Rightarrow \quad 2\alpha = (\alpha+\beta)+(\alpha-\beta)$

$\therefore \quad \tan 2\alpha = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$

$$= \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}} = \frac{56}{33}$$

42. $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta) = -\frac{3}{2}$

$\Rightarrow \quad 2[\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)]+3=0$

$\Rightarrow \quad 2[\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)]$
$+\sin^2\alpha+\cos^2\alpha+\sin^2\beta+\cos^2\beta+\sin^2\gamma$
$+\cos^2\gamma = 0$

$\therefore \quad (\sin\alpha+\sin\beta+\sin\gamma)^2+(\cos\alpha+\cos\beta+\cos\gamma)^2 = 0$

43. $(2\sin x-1)(\sin x+3)=0$

$\Rightarrow \quad \sin x = \frac{1}{2} \qquad (\because \sin x \ne -3)$

$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

44. Since, $\cos\left(x-\frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}} \Rightarrow \tan\left(x-\frac{\pi}{4}\right) = \sqrt{7}$

$\Rightarrow \quad \frac{\tan x-1}{\tan x+1} = \sqrt{7}$

$\therefore \quad \tan x = \frac{\sqrt{7}+1}{1-\sqrt{7}} = \frac{-(4+\sqrt{7})}{3}$

45. Since, $\tan\frac{P}{2}+\tan\frac{Q}{2} = -\frac{b}{a}$ and $\tan\frac{P}{2}\tan\frac{Q}{2} = \frac{c}{a}$

Also, $\frac{P}{2}+\frac{Q}{2}+\frac{R}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \frac{P+Q}{2} = \frac{\pi}{4} \qquad \left(\because \angle R = \frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{\tan\frac{P}{2}+\tan\frac{Q}{2}}{1-\tan\frac{P}{2}\tan\frac{Q}{2}} = 1$

$\Rightarrow \quad \frac{-\frac{b}{a}}{1-\frac{c}{a}} = 1$

$\therefore \quad c = a+b$

46. $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = -\frac{21}{65}$

$2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) = -\frac{27}{65}$

On squaring and adding, we get

$4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1170}{(65)^2}$

$\Rightarrow \quad \cos\left(\frac{\alpha-\beta}{2}\right) = \frac{-3\sqrt{130}}{130}$

$\therefore \quad \cos\left(\frac{\alpha-\beta}{2}\right) = \frac{-3}{\sqrt{130}} \quad \left(\because \frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2}\right)$

47. $u = \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$

$u^2 = a^2\cos^2\theta+b^2\sin^2\theta+a^2\sin^2\theta+b^2\cos^2\theta$
$+2\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}\times\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$

$\Rightarrow \quad u^2 = (a^2+b^2)+2\sqrt{\{a^2+(b^2-a^2)\sin^2\theta\}\times\{a^2+(b^2-a^2)\cos^2\theta\}}$

$\Rightarrow \quad u^2 = (a^2+b^2)+2\sqrt{a^4+a^2(b^2-a^2)+(b^2-a^2)^2\sin^2\theta\cos^2\theta}$

$\Rightarrow \quad u^2 = (a^2+b^2)+2\sqrt{a^2b^2+\left(\frac{b^2-a^2}{2}\right)^2\sin^2 2\theta}$

$\therefore \quad \min(u^2) = a^2+b^2+2ab = (a+b)^2$

and $\quad \max(u^2) = a^2+b^2+(a^2+b^2) = 2(a^2+b^2)$

Now, $\quad \max(u^2)-\min(u^2) = (a-b)^2$

48. Since, the centroid G divides the line AD in the ratio 2 : 1.

$\therefore \quad AG = \frac{8}{3}$ and $DG = \frac{4}{3}$

In ΔABG, $\quad \tan\frac{\pi}{3} = \frac{AG}{BG}$

$\Rightarrow \quad BG = AG\cot\frac{\pi}{3}$

$\therefore \quad BG = \frac{8}{3}\times\frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}$

Area of $\Delta ADB = \frac{1}{2}\times AD\times BG = \frac{1}{2}\times 4\times\frac{8}{3\sqrt{3}} = \frac{16}{3\sqrt{3}}$

$\therefore$ Area of $\Delta ABC = 2\times$ Area of ΔADB

$= 2\times\frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}}$ sq units

49. Since, $\sin^2\theta \le 1$

$\Rightarrow \quad \dfrac{4xy}{(x+y)^2} \le 1$

$\Rightarrow \quad (x-y)^2 \ge 0$

which is true for all real values of x and y provided $x + y \ne 0$, otherwise $\dfrac{4xy}{(x+y)^2}$ will be meaningless.

50. $\because \quad \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\therefore \quad \dfrac{1-\tan^2 15°}{1+\tan^2 15°} = \cos 30° = \dfrac{\sqrt{3}}{2}$

51. Since, $\tan\theta = -\dfrac{4}{3}$

$\therefore \quad h = \sqrt{16+9} = \sqrt{25} \Rightarrow h = 5$

$\therefore \quad \sin\theta = \dfrac{4}{5}$

But $\tan\theta$ is negative which is possible only if θ lie in second and fourth quadrants.

$\therefore \quad \sin\theta$ may be $\dfrac{4}{5}$ or $-\dfrac{4}{5}$.

52. Since, $\sin(\alpha+\beta) = 1$

$\Rightarrow \quad \sin(\alpha+\beta) = \sin\dfrac{\pi}{2}$

$\Rightarrow \quad \alpha + \beta = \dfrac{\pi}{2} \quad \ldots(i)$

and $\quad \sin(\alpha-\beta) = \dfrac{1}{2}$

$\Rightarrow \quad \alpha - \beta = \dfrac{\pi}{6} \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$\alpha = \frac{\pi}{3}, \quad \beta = \frac{\pi}{6}$$

$\therefore \quad \tan(\alpha+2\beta)\tan(2\alpha+\beta) = \tan\left(\dfrac{2\pi}{3}\right)\tan\left(\dfrac{5\pi}{6}\right)$

$= \left(-\cot\dfrac{\pi}{6}\right)\left(-\cot\dfrac{\pi}{3}\right)$

$= \sqrt{3} \times \dfrac{1}{\sqrt{3}} = 1$

53. Since, $\quad y = \sin^2\theta + \operatorname{cosec}^2\theta = (\sin\theta - \operatorname{cosec}\theta)^2 + 2$

$\Rightarrow \quad y \ge 2$

But $\theta \ne 0$, then

$$y > 2$$

54. Since, α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$

$\therefore \quad 25\cos^2\alpha + 5\cos\alpha - 12 = 0$

$\Rightarrow \quad (5\cos\alpha + 4)(5\cos\alpha - 3) = 0$

$\Rightarrow \quad \cos\alpha = -\dfrac{4}{5}, \dfrac{3}{5}$

But $\dfrac{\pi}{2} < \alpha < \pi$

$\therefore \quad$ We take $\cos\alpha = -\dfrac{4}{5} \quad (\because \cos\alpha < 0)$

$\Rightarrow \quad \sin\alpha = \dfrac{3}{5}$

$\therefore \quad \sin 2\alpha = 2\sin\alpha\cos\alpha = -2 \times \dfrac{3}{5} \times \dfrac{4}{5} = -\dfrac{24}{25}$

10 Properties of Triangles, Heights and Distances

JEE Main MILESTONE

- Relation between the Sides and Angles of Triangle
- Trigonometric Ratios of Half Angles of a Triangle
- Area of Triangles
- Different Types of Circle Connected with Triangle
- Regular Polygon
- Angle of Elevation
- Angle of Depression
- Bearing

10.1 Relation between the Sides and Angles of Triangle

In a ΔABC, angles are denoted by A, B and C and the lengths of corresponding sides opposite to these angles are denoted by a, b and c, respectively. Area and perimeter of a triangle are denoted by Δ and $2s$, respectively. Also,

Semi-perimeter of the triangle is $s = \dfrac{a+b+c}{2}$.

The sum of all angles of a triangle is $180°$ *i.e.*, $\angle A + \angle B + \angle C = 180°$.

Note

- The longest side of a triangle have corresponding largest angle and *vice-versa*.
- The smallest side of a triangle have corresponding smallest angle and *vice-versa*.

Properties of triangle is a part of trigonometry which is a branch of Mathematics. In which we study about triangles and relationship between their sides and angles. Height and distance is an application of trigonometry. Here with the help of trigonometry, we find the height of any object or distance of an object from any point/object which cannot measured directly.

Sine Rule

In any ΔABC, the sines of the angles are proportional to the lengths of the opposite sides, *i.e.*,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

It can also be written as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

Then, $a = k \sin A, b = k \sin B, c = k \sin C$

Cosine Rule

In any ΔABC, cosine of an angle can express in terms of sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection Rule

In any ΔABC,

$$a = b\cos C + c\cos B$$
$$b = a\cos C + c\cos A$$
$$c = a\cos B + b\cos A$$

Napier's Analogy

In any ΔABC,

$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}, \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

and $$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

***Sample* Problem 1** *In a ΔABC, if $\angle A = 60°$, $a = 5$, $b = 4$, then c is a root of the equation*

(a) $c^2 - 5c - 9 = 0$ (b) $c^2 - 4c - 9 = 0$
(c) $c^2 - 10c + 25 = 0$ (d) $c^2 - 5c - 41 = 0$

Interpret (b) Given, $\angle A = 60°$, $a = 5, b = 4$

$$\because \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \quad \cos 60° = \frac{(4)^2 + (c)^2 - (5)^2}{2 \times 4 \times c}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{16 + c^2 - 25}{8c} \Rightarrow 4c = c^2 - 9$$

$$\Rightarrow \quad c^2 - 4c - 9 = 0$$

***Sample* Problem 2** *In a ΔABC,*

$$\Sigma (b + c)\tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right)$$

is equal to

(a) a (b) b (c) c (d) 0

Interpret (d) Now, $(b + c)\tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right)$

$$= (b + c)\tan\frac{A}{2} \times \frac{b-c}{b+c}\cot\frac{A}{2} = b - c$$

$$\therefore \quad \Sigma (b + c)\tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2} = b - c + c - a + a - b$$

$$= 0$$

10.2 Trigonometric Ratios of Half Angles of a Triangle

In a ΔABC, if the sides of the triangle are a, b, c and corresponding angles are A, B and C respectively and s is the semi-perimeter, then

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (iv) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(v) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (vi) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(vii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (viii) $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

(ix) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

***Sample* Problem 3** *If in ΔABC, $\sin\frac{A}{2}, \sin\frac{C}{2} = \sin\frac{B}{2}$ and $2s$ is the perimeter of the triangle, then s is*

(a) $2b$ (b) b
(c) $3b$ (d) $4b$

Interpret (a) Given, $\sin\frac{A}{2}, \sin\frac{C}{2} = \sin\frac{B}{2}$

$$\therefore \quad \sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$\Rightarrow \quad \frac{s-b}{b} = 1$$

$$\Rightarrow \quad s = 2b$$

10.3 Area of Triangles

In a ΔABC, if the sides of the triangle are a, b, c and corresponding angles are A, B and C respectively, then area of triangle

(i) **When two sides and angle between them is known**

$$\Delta = \frac{1}{2}ab\sin C, \Delta = \frac{1}{2}bc\sin A,$$

$$\Delta = \frac{1}{2}ca\sin B$$

(ii) **When one side and corresponding angles are known**

$$\Delta = \frac{c^2\sin A\sin B}{2\sin C}$$

$$\Delta = \frac{a^2\sin B\sin C}{2\sin A}$$

$$\Delta = \frac{b^2\sin C\sin A}{2\sin B}$$

(iii) **When all the three sides are known**

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

It is known as **Hero's formula.**

***Sample* Problem 4** *The area of the ΔABC, in which $a = 1$, $b = 2$, $\angle c = 60°$, is*

(a) 4 *sq units* (b) $\frac{1}{2}$ *sq unit* (c) $\frac{\sqrt{3}}{2}$ *sq unit* (d) $\sqrt{3}$ *sq units*

Interpret (c) Given, $a = 1, b = 2, \angle C = 60°$

$$\therefore \text{Area of triangle} = \frac{1}{2}ab\sin C = \frac{1}{2}\times 1\times 2\times \sin 60°$$

$$= \sin 60° = \frac{\sqrt{3}}{2} \text{ sq unit}$$

***Sample* Problem 5** *If the angles of a triangle are 30° and 45° and the inclined side is $(\sqrt{3}+1)$ cm. Then, area of triangle is*

(a) $\frac{1}{2}\sqrt{3}$ *cm*^{2} (b) $\frac{1}{2}(\sqrt{3}+1)$ *cm*^{2}

(c) $(\sqrt{3}+1)$ *cm*^{2} (d) *None of these*

Interpret (b) We know that,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \quad \frac{\sin 105°}{\sqrt{3}+1} = \frac{\sin 30°}{b} = \frac{\sin 45°}{c}$$

$$\Rightarrow \quad b = \frac{\sqrt{3}+1}{2\sin 105°}, \quad c = \frac{\sqrt{3}+1}{\sqrt{2}\sin 105°}$$

$$\therefore \text{ Area of } \Delta ABC = \frac{1}{2}bc\sin A$$

$$= \frac{1}{2}\cdot\frac{(\sqrt{3}+1)^2}{2\sqrt{2}\sin 105°}$$

$$= \frac{1}{2}\cdot\frac{(\sqrt{3}+1)^2}{(\sqrt{3}+1)}$$

$$= \frac{1}{2}(\sqrt{3}+1) \text{ cm}^2$$

10.4 Different Types of Circle Connected with Triangle

(i) Circumcircle of a Triangle

The circle passing through the vertices of a ΔABC is called circumcircle (circumscribed). Its radius R is called the circumradius and its centre is known as circumcentre. Circumcentre is the point of intersection of perpendicular bisectors of the sides.

(a) $R = \frac{a}{2\sin A} = \frac{c}{2\sin C}$ (b) $R = \frac{abc}{4\Delta}$

***Sample* Problem 6** *In any ΔABC,*

$a\cos A + b\cos B + c\cos C$ is equal to

(a) $4R$

(b) $\sin A \sin B \sin C$

(c) $R\sin A\sin B\sin C$

(d) $4R\sin A\sin B\sin C$

Interpret (d) $a\cos A + b\cos B + c\cos C$

$$= 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C$$

$$(\because a = 2R\sin A, b = 2R\sin B \text{ and } c = 2R\sin C)$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= R[2\sin(A+B)\cos(A-B) + 2\sin C\cos C]$$

$$= 2R[\sin(\pi - C)\cos(A-B) + \sin C\cos\{\pi - (A+B)\}]$$

$$= 2R[\sin C\{\cos(A-B) - \cos(A+B)\}]$$

$$= 4R\sin A\sin B\sin C$$

(ii) Incircle of a Triangle

The circle touching all the sides of a triangle internally is known as an incircle (inscribed) of the triangle. Its centre is called incentre and its radius r is called the inradius of the circle. Incentre is the point of intersection of bisectors of the angles of the triangle.

(a) $r = \frac{\Delta}{s}$

(b) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

(d) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

***Sample* Problem 7** *In a ΔABC, $\cos A + \cos B + \cos C$ is equal to*

(a) $\frac{r}{R}$ (b) $1 - \frac{r}{R}$

(c) $\frac{R}{r}$ (d) $1 + \frac{r}{R}$

Interpret (d) $\cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \left(1 + \frac{r}{R}\right) \qquad \left(\because r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

(iii) Escribed Circles of a Triangle

The circle touching BC and the two sides AB and AC produced of ΔABC externally is called the escribed circle opposite to A. Its radius is denoted by r_1. The centre of this circle is known as excentre and denoted by I. It is the point, where the external bisectors of the $\angle B$ and $\angle C$ and the interval bisector of the $\angle A$ meet.

Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively and excentres are denoted by I_2 and I_3. r_1, r_2 and r_3 are called the exradii of ΔABC. Here,

(a) $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$$= \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

(b) $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

$$= \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

(c) $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2}$

$$= \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

Note

- $r_1 + r_2 + r_3 = 4R + r$
- $r_1r_2 + r_2r_3 + r_3r_1 = s^2 = \dfrac{r_1r_2r_3}{r}$.
- $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{s}{\Delta} = \dfrac{1}{r}$

and $\quad r_1r_2r_3 = r^2\left(\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}\right)^2$

***Sample* Problem 8** *If $r_1 = r_2 + r_3 + r$, then triangle is*

(a) *right angled* (b) *obtuse angled*
(c) *equilateral* (d) *None of these*

Interpret (a) $\because\ r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \quad \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$$

$$\Rightarrow \quad \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$$

$$\Rightarrow \quad s^2 - (b+c)s + bc = s^2 - as$$

$$\Rightarrow \quad s(-a+b+c) = bc$$

$$\Rightarrow \quad \frac{(b+c+a)(b+c-a)}{2} = bc$$

$$\Rightarrow \quad (b+c)^2 - a^2 = 2bc$$

$$\Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc \Rightarrow b^2 + c^2 = a^2$$

$$\therefore \quad \angle A = 90°$$

Thus, the triangle is right angled triangle.

(iv) Orthocentre of a Triangle

In a ΔABC, AD, BE and CF are perpendiculars from the vertices A, B and C respectively to the opposite sides. These three perpendiculars are concurrent at the point P which is called the orthocentre of the ΔABC. The ΔDEF is called the pedal triangle of ΔABC.

(a) $PA = 2R\cos A$, $PB = 2R\cos B$, $PC = 2R\cos C$

(b) $PD = 2R\cos B\cos C$, $PE = 2R\cos C\cos A$,
$PF = 2R\cos A\cos B$

Note If p_1, p_2 and p_3 are perpendicular's drawn from the vertices to opposite side of the triangle, then

- $\Delta = \dfrac{1}{2}ap_1, \Delta = \dfrac{1}{2}bp_2, \Delta = \dfrac{1}{2}cp_3$.
- $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{r}$
- $p_1p_2p_3 = \dfrac{(abc)^2}{8R^2}$
- $\dfrac{\cos A}{p_1} + \dfrac{\cos B}{p_2} + \dfrac{\cos C}{p_3} = \dfrac{1}{R}$

***Sample* Problem 9** *If p_1, p_2 and p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, then p_1, p_2 and p_3 is equal to*

(a) $a^2b^2c^2$ (b) $2a^2b^2c^2$

(c) $\frac{4a^2b^2c^2}{R^2}$ (d) $\frac{a^2b^2c^2}{8R^2}$

Interpret (*d*) Since, $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$$\therefore \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

$$\therefore \quad p_1p_2p_3 = \frac{8\Delta^3}{abc} = \frac{8}{abc}\left(\frac{abc}{4R}\right)^3 = \frac{a^2b^2c^2}{8R^2}$$

10.5 Regular Polygon

A regular polygon is a polygon which has all its sides as well as all its angles are equal. The circle passing through all the vertices of a regular polygon is called its circumscribed circle and the circle which touches all the sides of a regular polygon is called its inscribed circle.

Radius of circumcircle, $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

and **radius of incircle**, $r = \frac{a}{2}\cot\frac{\pi}{n}$

where, a is the length of the side of a regular polygon.

$$\text{Area of a regular polygon} = \frac{1}{4}na^2\cot\frac{\pi}{n}$$

$$= nr^2\tan\frac{\pi}{n} = \frac{n}{2}R^2\sin\frac{2\pi}{n}$$

Note If the polygon has n sides, then sum of the internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$.

***Sample* Problem 10** *If the area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2. Then, $\frac{A_1}{A_2}$ is equal to*

(a) $\frac{2\pi}{5}$ (b) $\frac{2\pi}{5}\sec\frac{\pi}{10}$

(c) $\sec\frac{\pi}{10}$ (d) *None of these*

Interpret (*b*) In $\Delta OAB, OA = OB = r$

and $\angle AOB = 72°$

$$\therefore \text{ Area of } \Delta AOB = \frac{1}{2}r\cdot r\sin 72°$$

$$= \frac{1}{2}r^2\cos 18°$$

Area of pentagon = 5 (Area of ΔAOB)

$$\Rightarrow \quad A_2 = \frac{5}{2}r^2\cos 18° \quad \text{...(i)}$$

Also we know that,

area of circle $= \pi r^2$

$$\Rightarrow \quad A_1 = \pi r^2 \quad \text{...(ii)}$$

$$\text{Thus,} \quad \frac{A_1}{A_2} = \frac{\pi r^2}{\frac{5}{2}r^2\cos 18°} = \frac{2\pi}{5}\sec\frac{\pi}{10}$$

Check Point 1

1. If in a ΔABC, $3a = b + c$, then $\cot\frac{B}{2}\cot\frac{C}{2}$ is equal to

(a) 1 (b) $\sqrt{3}$

(c) 2 (d) None of these

2. If in a ΔABC, $\sin A : \sin B : \sin C = 1 : 2 : 3$, if $b = 4$ cm, then perimeter of the triangle is

(a) 6 cm (b) 24 cm (c) 12 cm (d) 8 cm

3. If in a ΔABC, $a\tan A + b\tan B = (a+b)\tan\left(\frac{A+B}{2}\right)$, then

(a) $A = B$ (b) $A = -B$ (c) $A = 2B$ (d) $B = 2A$

4. If in a ΔABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then $\cos A$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{5}{7}$

(c) $\frac{19}{35}$ (d) None of these

5. In a ΔABC, $2ca\sin\frac{A-B+C}{2}$ is equal to

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

6. If the angles of a triangle be in the ratio 1 : 2 : 7, then the ratio of its greatest side to the least side is

(a) 1 : 2 (b) 2 : 1

(c) $(\sqrt{5}+1) : (\sqrt{5}-1)$ (d) $(\sqrt{5}-1) : (\sqrt{5}+1)$

7. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of

(a) $\tan\frac{\pi}{n} : \frac{\pi}{n}$ (b) $\cos\frac{\pi}{n} : \frac{\pi}{n}$

(c) $\sin\frac{\pi}{n} : \frac{\pi}{n}$ (d) $\cot\frac{\pi}{n} : \frac{\pi}{n}$

8. The circumradius of an equilateral triangle of side $2\sqrt{3}$ cm, is
(a) 1 cm (b) $\sqrt{3}$ cm (c) 2 cm (d) $2\sqrt{3}$ cm

9. If the sides of a triangle are 13, 14, 15, then inradius of this triangle is
(a) $\frac{67}{4}$ (b) $\frac{65}{4}$ (c) 4 (d) 24

10. $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to
(a) 1/r (b) r/R
(c) R/r (d) 1/R

11. If in a ΔABC, $8R^2 = a^2 + b^2 + c^2$, then the triangle is
(a) right angled (b) isosceles
(c) equilateral (d) None of these

12. If r, r_1, r_2, r_3 have their usual meanings, then value of $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ is
(a) 1
(b) 0
(c) $\frac{1}{r}$
(d) None of the above\

10.6 Angle of Elevation

If '*O*' be the observer's eye and *OX* be the horizontal line through *O*. If object *P* is at a higher level than eye, then $\angle POX$ is called the angle of elevation.

10.7 Angle of Depression

If '*O*' be the observer's eye and *OX* is a horizontal line. If object *P* is at a lower level than *O*, then the $\angle POX$ is called the angle of depression.

Note
- Angle of elevation and depression are always acute angle.
- Angle of elevation of an object from an observer is same as angle of depression of an observer from the object.

Hot Spot

Height and Distance

In these types of problems, certain information like angles, distances, heights etc. are given and on the basis of these, figure is drawn and required information is calculated.

These types of problems are directly asked in various years in AIEEE (JEE Main) 2008, 2007, 2004, 2003. *To solve such types of problem following steps are given.*

Step I Draw a possible figure, showing all distances and angles.

Step II In the problem, there is a right angled triangle in which one of active angle and one side is given. First find the other side using a proper trigonometric ratio.

Step III In the problem, if there is not a right angled triangle, then sine rule should be applied.

Step IV Find the length of a particular side from two different triangles containing that side common and then equate the two values thus obtained.

Sample Problem 11 *The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45°. If the angle of elevation of the top of the complete pillar at the same point is to be 60°. Then, find the height of the incomplete pillar is to be increased.*
(a) $100(\sqrt{3}+1)$ *m* (b) $100(\sqrt{3}-1)$ *m*
(c) $50(\sqrt{3}-1)$ *m* (d) *None of these*

Interpret (*b*) **Step I** Draw a figure.

Let *OP* be the incomplete and *OQ* be the complete pillar. Let *A* be a point on the horizontal plane, such that $OA = 100$ m. Then, $\angle OAP = 45°$ and $\angle OAQ = 60°$.

Step II Find the other sides

In ΔOAP, $\tan 45° = \frac{OP}{OA}$

$\therefore$ $OP = OA = 100$ m ...(i)

In ΔOAQ, $\tan 60° = \frac{OQ}{OA}$

$\Rightarrow$ $\sqrt{3} = \frac{OQ}{OA} = \frac{OP + x}{OA}$ ...(ii)

Step III Since, it is a right angled, so there is no need to use a sine rule.

Step IV Equating two triangles.

On equating Eqs. (i) and (ii), we get

$$\sqrt{3} = \frac{100 + x}{100} \Rightarrow 100\sqrt{3} = 100 + x$$

$$\Rightarrow \quad x = 100(\sqrt{3} - 1) \text{ m}$$

***Sample* Problem 12** *The angle of elevation of the top of a tower at a point on the ground is 30°. If on walking 20 m toward the tower, the angle of elevation becomes 60°, then the height of the tower is*

(a) 10 *m* (b) $\frac{10}{\sqrt{3}}$ *m*

(c) $10\sqrt{3}$ *m* (d) *None of these*

Interpret (c) Let height of the tower *AB* is *h* m.

In ΔABD,

$$\tan 60° = \frac{h}{x} \Rightarrow \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad x = \frac{h}{\sqrt{3}} \quad \text{...(i)}$$

In ΔABC, $\tan 30° = \frac{h}{20 + x} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{20 + x}$

$$\Rightarrow \quad \sqrt{3}h = 20 + \frac{h}{\sqrt{3}} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)h = 20 \Rightarrow h = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \text{ m}$$

10.8 Bearing

If the observer and the object are *O* and *P* be on the same level respectively, then bearing is defined. To measure the bearing the four standard directions East, West, North and South are taken as the cardinal directions. Angle between the line of observations. *i.e.*, *OP* and any one standard direction is measured.

Thus, $\angle POE$ is called the bearing of the point *P* with respect to *O* measured from East to North.

In other words the bearing of *P* as seen from *O* is the direction in which *P* is seen from *O*.

Note North-East means equally inclined to North and East. ENE means equally inclined to East and North-East.

***Sample* Problem 13** *A man standing 30 m South of a tower of height h walks 60 m to the East and finds the angle of elevation of the top of the tower to be 30°. Then, the height of tower is equal to*

(a) $15\sqrt{10}$ *m* (b) $15\sqrt{3}$ *m*

(c) $10\sqrt{15}$ *m* (d) *None of these*

Interpret (c) Let the lines East-West and North-South meet at the foot of the tower *T*.

Let the man be standing at *P*, 30 m South of the tower. The man walks 60 m to *Q* parallel to the line *WE*. The angle of elevation of the tower at *Q* is 30°.

Hence, $TQ = h \cot 30° = \sqrt{3}h$

From the ΔPQT,

$$QT^2 = PT^2 + PQ^2$$

$$\Rightarrow \quad 3h^2 = (30)^2 + (60)^2 = 4500$$

$$\Rightarrow \quad h = 10\sqrt{15} \text{ m}$$

Some Important Facts

(i) In an isosceles triangle, the median is perpendicular to the base. ΔABC is isosceles and $AD \perp BC$.

(ii) In similar triangle, the corresponding sides are proportional.

(iii) The exterior angle of a triangle is equal to the sum of interior opposite angles.

In ΔABC, $\theta = \theta_1 + \theta_2$

(iv) In a triangle, the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.

In ΔABC, *AD* is bisector of $\angle A$.

$$\therefore \quad \frac{BD}{DC} = \frac{c}{b}$$

(v) In a ΔABC, if $DE \| AB$, then $\frac{AB}{DE} = \frac{BC}{DC} = \frac{AC}{EC}$

(vi) In a right angled ΔABC, the mid-point *D* of hypotenuse *AC* is equidistant from its vertices *A*, *B* and *C*.

i.e., $AD = BD = CD$

(vii) ***m-n* Theorem**

In a ΔABC, *D* divides *BC* in the ratio *m*:*n*, then

(a) $(m + n)\cot\theta = n\cot B - m\cot C$

(b) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(viii) **Appolonius Theorem**

If in ΔABC, *AD* is median, then

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

(ix) Angles in the same segment of a circle are equal.

(x) The angle subtended at the centre by an arc is double of the angle subtended at any point of the remaining arc by the same arc.

(xi) **Ptolemy's Theorem**

Incyclic quadrilateral *ABCD*,

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

***Sample* Problem 14** *A train travelling on one of two intersecting railway lines, subtends at a certain station on the other line, an angle α when the front of the carriage reaches the junction and an angle β when the end of the carriage reaches it. Then, the two lines are inclined to each other at an angle θ determined by $2\cot\theta$ is equal to*

(a) $\tan\alpha - \tan\beta$
(b) $\cot\beta - \cot\alpha$
(c) $\cot\alpha - \cot\beta$
(d) $\cot\alpha + \cot\beta$

Interpret (c) l_1 and l_2 are two intersecting lines.

In ΔABC, by m-n theorem, we have

$$(x + x)\cot\theta = x\cot\alpha - x\cot\beta \Rightarrow 2\cot\theta = \cot\alpha - \cot\beta$$

Check Point **2**

1. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45°. If the angle of elevation of the top of the complete pillar at the same point is 60°, then the difference in the height of complete and incomplete pillar is
(a) $50\sqrt{2}$ m (b) 100 m
(c) $100(\sqrt{3}-1)$ m (d) $100(\sqrt{3}+1)$ m

2. A balloon is coming down at the rate of 4 m/min and its angle of elevation is 45° from a point on the ground which has been reduced to 30° after 10 min. Balloon will be on the ground at a distance of how many metres from the observer?
(a) $20\sqrt{3}$ (b) $20(3+\sqrt{3})$
(c) $10(3+\sqrt{3})$ (d) None of these

3. From an aeroplane vertically over a straight horizontal road, the angles of depression of two consecutive milestones on opposite sides of the aeroplane are observed to be α and β. The height in miles of the aeroplane above the road is
(a) $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha\ \tan\beta}$ (b) $\dfrac{\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$
(c) $\dfrac{\cot\alpha\cot\beta}{\cot\alpha + \cot\beta}$ (d) None of these

4. A tree is broken by wind, its upper part touches the ground at a point 10 m from the foot of the tree and makes an angle 45° with the ground. The entire length of tree is
(a) 15 m (b) 20 m
(c) $10(1+\sqrt{2})$ m (d) $10\left(1+\dfrac{\sqrt{3}}{2}\right)$ m

5. Each side of a square subtends an angle of 60° at the top of a tower h metres high standing in the centre of the square. If a is the length of each side of the square, then
(a) $2a^2 = h^2$ (b) $2h^2 = a^2$
(c) $3a^2 = 2h^2$ (d) $2h^2 = 3a^2$

6. From a station A due West of a tower the angle of elevation of the top of the tower is seen to be 45°. From a station B, 10 metre from A and in the direction 45° South of East the angle of elevation is 30°, the height of the tower is
(a) $5\sqrt{2}(\sqrt{5}+1)$ m (b) $\dfrac{5\sqrt{2}(\sqrt{5}+1)}{2}$ m
(c) $\dfrac{5(\sqrt{5}+1)}{2}$ m (d) None of these

7. A vertical lamp-post 6 m high, stands at a distance of 2 m from a wall, 4 m high. A 1.5 m tall man starts to walk away from the wall on the other side of the wall, in line with the lamp-post. The maximum distance to which the man can walk remaining in the shadow is
(a) 5/2 m (b) 3/2 m
(c) 4 m (d) None of these

WORKED OUT

Examples

Example 1 *In* ΔABC, $\dfrac{\cos^2\frac{A}{2}}{a}+\dfrac{\cos^2\frac{B}{2}}{b}+\dfrac{\cos^2\frac{C}{2}}{c}$ *is equal to*

(a) $\dfrac{s^2}{2abc}$ (b) $\dfrac{s^2}{abc}$

(c) $\dfrac{s^2}{3abc}$ (d) *None of these*

Solution (b) We have, $\dfrac{\cos^2\frac{A}{2}}{a}+\dfrac{\cos^2\frac{B}{2}}{b}+\dfrac{\cos^2\frac{C}{2}}{c}$

$$=\left\{\frac{\frac{s(s-a)}{bc}}{a}\right\}+\left\{\frac{\frac{s(s-b)}{ca}}{b}\right\}+\left\{\frac{\frac{s(s-c)}{ab}}{c}\right\}$$

$$=\frac{s(s-a+s-b+s-c)}{abc}$$

$$=\frac{s[3s-(a+b+c)]}{abc}=\frac{s(3s-2s)}{abc}=\frac{s^2}{abc}$$

Example 2 *If in a triangle* $\dfrac{b+c}{11}=\dfrac{c+a}{12}=\dfrac{a+b}{13}$, *then* $\cos A:\cos B:\cos C$ *is equal to*

(a) $7:12:95$ (b) $7:19:25$

(c) $7:91:52$ (d) $71:92:5$

Solution (b) Let $\dfrac{b+c}{11}=\dfrac{c+a}{12}=\dfrac{a+b}{13}=k$

$\Rightarrow$ $b+c=11k,\quad c+a=12k,\quad a+b=13k$ ...(i)

and $2(a+b+c)=36k$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a=7k,\quad b=6k,\quad c=5k$$

Now, $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

$$=\frac{36k^2+25k^2-49k^2}{60k^2}=\frac{1}{5}=\frac{7}{35}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$=\frac{25k^2+49k^2-36k^2}{70k^2}=\frac{38}{70}=\frac{19}{35}$$

and $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

$$=\frac{49k^2+36k^2-25k^2}{84k^2}=\frac{60}{84}=\frac{25}{35}$$

$\Rightarrow$ $\cos A:\cos B:\cos C=7:19:25$

Example 3 *In* ΔABC, $a=2b$ *and* $|A-B|=\dfrac{\pi}{3}$, *then* $\angle C$ *is equal to*

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$

(c) $\dfrac{\pi}{4}$ (d) *None of these*

Solution (b) Since, $a=2b$ ...(i)

$\Rightarrow$ $\angle A>\angle B$

and $|A-B|=\dfrac{\pi}{3}\Rightarrow A-B=\dfrac{\pi}{3}$ ...(ii)

By Napier's analogy ($\because \angle A>\angle B$)

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$\Rightarrow$ $\tan\dfrac{\pi}{6}=\dfrac{2b-b}{2b+b}\cot\dfrac{C}{2}\Rightarrow\dfrac{1}{\sqrt{3}}=\dfrac{1}{3}\cot\dfrac{C}{2}$

$\Rightarrow$ $\cot\dfrac{C}{2}=\sqrt{3}$

$\Rightarrow$ $\dfrac{\angle C}{2}=\dfrac{\pi}{6}\Rightarrow\angle C=\dfrac{\pi}{3}$

Example 4 *In a* ΔABC, $a=1$ *and the sum of perimeter is six times the AM of the sines of the angles, then* $\angle A$ *is equal to*

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi}{6}$ (d) *None of these*

Solution (c) Since, $1+b+c=6\cdot\dfrac{\sin A+\sin B+\sin C}{3}$

$$=2\left(\frac{1}{2R}+\frac{b}{2R}+\frac{c}{2R}\right)=\frac{1}{R}(1+b+c)$$

$\therefore$ $R=1$

$\Rightarrow$ $\dfrac{1}{\sin A}=2R=2$

$\Rightarrow$ $\angle A=\dfrac{\pi}{6}$

Example 5 *If in a ΔABC,*

$$\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C,$$

then the value of determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ *is equal to*

(a) 0 (b) 1 (c) 2 (d) 3

Solution (a) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a+b+c)(bc + ca + ab - a^2 - b^2 - c^2)$

$= -(a^3 + b^3 + c^3 - 3abc)$

$= -8R^3(\sin^3 A + \sin^3 B + \sin^3 C - 3 \sin A \sin B \sin C)$

$= -8R^3(3 \sin A \sin B \sin C - 3 \sin A \sin B \sin C) = 0$

Example 6 *If in a ΔABC, $a^2 + b^2 + c^2 = 8R^2$, where* R = circumradius, then the triangle is

(a) *equilateral* (b) *isosceles*
(c) *right angled* (d) *None of these*

Solution (c) $a^2 + b^2 + c^2 = 8R^2$

$\Rightarrow \quad \sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow \quad \cos 2A + \cos 2B + \cos 2C + 1 = 0$

$\therefore \quad -4 \cos A \cdot \cos B \cdot \cos C = 0 \Rightarrow A$ or B or $C = \frac{\pi}{2}$

Example 7 *In any ΔABC, $16R^2 rr_1 r_2 r_3$ is equal to*

(a) $2a^2b^2c^2$ (b) $a^2b^2c^2$
(c) $3a^2b^2c^2$ (d) *None of these*

Solution (b) We have, $16R^2 rr_1 r_2 r_3$

$$= 16 \cdot \frac{a^2b^2c^2}{16\Delta^2} \cdot \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}$$

$$= \frac{a^2b^2c^2\Delta^2}{s(s-a)(s-b)(s-c)}$$

$$= \frac{a^2b^2c^2\Delta^2}{\Delta^2} = a^2b^2c^2$$

Example 8 *The area of a circle is A_1 and the area of a regular pentagon inscribed in the circle is A_2. Then, $A_1 : A_2$ is*

(a) $\frac{\pi}{5} \cos \frac{\pi}{10}$ (b) $\frac{2\pi}{5} \sec \frac{\pi}{10}$
(c) $\frac{2\pi}{5} \operatorname{cosec} \frac{\pi}{10}$ (d) *None of these*

Solution (b) In the ΔOAB, $OA = OB = r$ and

$\angle AOB = \frac{360°}{5} = 72°$

$\therefore \quad \text{ar}(\Delta AOB) = \frac{1}{2} \cdot 6r \cdot r \cdot \sin 72° = \frac{1}{2} r^2 \cos 18°$

$\Rightarrow \quad \text{ar (pentagon)} = \frac{5}{2} r^2 \cos 18°$

$\therefore \quad A_1 : A_2 = \frac{2\pi r^2}{5r^2 \cos 18°} = \frac{2\pi}{5} \sec \frac{\pi}{10}$

Example 9 *The height of the chimney when it is found that on walking towards it 50 m in the horizontal line through its base, the angle of elevation of its top changes from 30° to 60°, is*

(a) 25 *m* (b) $25\sqrt{2}$ *m*
(c) $25\sqrt{3}$ *m* (d) *None of these*

Solution (c) Let PQ be the chimney whose height is h metres.

In ΔBPQ, $\tan 60° = \frac{PQ}{BP}$

$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$

$\Rightarrow \quad h = \sqrt{3}\, x$ ...(i)

and in ΔAPQ, $\tan 30° = \frac{PQ}{AP}$

$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{50 + x}$

$\Rightarrow \quad 50 + x = h\sqrt{3}$

$\Rightarrow \quad 50 + x = 3x$ [using Eq. (i)]

$\Rightarrow \quad x = 25$

$\therefore$ Height of the chimney $= 25\sqrt{3}$ m

Start Practice for JEE Main

Round I (Topically Divided Problems)

Relation between the Sides and Angles of a Triangle

1. In a ΔABC, $\operatorname{cosec} A(\sin B \cos C + \cos B \sin C)$ is equal to

(a) $\frac{c}{a}$ (b) $\frac{a}{c}$

(c) 1 (d) $\frac{c}{ab}$

2. If in a ΔABC, $2b^2 = a^2 + c^2$, then $\frac{\sin 3B}{\sin B}$ is equal to

(a) $\frac{c^2 - a^2}{2ca}$ (b) $\frac{c^2 - a^2}{ca}$

(c) $\left(\frac{c^2 - a^2}{ca}\right)^2$ (d) $\left(\frac{c^2 - a^2}{2ca}\right)^2$

3. $\sin A : \sin C = \sin(A - B) : \sin(B - C)$, then a^2, b^2 and c^2 are in

(a) AP (b) GP

(c) HP (d) None of these

4. If a, b and c are sides of a triangle, then

(a) $\sqrt{a} + \sqrt{b} > \sqrt{c}$

(b) $\left|\sqrt{a} - \sqrt{b}\right| > \sqrt{c}$ (if c is smallest)

(c) $\sqrt{a} + \sqrt{b} < \sqrt{c}$

(d) None of the above

5. In a ΔABC, if $2s = a + b + c$ and $(s - b)(s - c) = x \sin^2 \frac{A}{2}$, then the value of x is

(a) bc (b) ca (c) ab (d) abc

6. If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side to the perimeter is

(a) $\sqrt{3} : (2 + \sqrt{3})$ (b) 1 : 6

(c) $1 : (2 + \sqrt{3})$ (d) 2 : 3

7. In a ΔABC, $\angle C = 60°$, then $\frac{1}{a + c} + \frac{1}{b + c}$ is equal to

(a) $\frac{1}{a + b + c}$ (b) $\frac{2}{a + b + c}$

(c) $\frac{3}{a + b + c}$ (d) None of these

8. In a ΔABC, if a, b and c are in AP, then the value of $\frac{\sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{B}{2}}$ is

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) -1

9. In a triangle, the lengths of the two larger sides are 10 cm and 9 cm, respectively. If the angles of the triangle are in AP, then the length of the third side in cm can be

(a) $5 - \sqrt{6}$ only (b) $5 + \sqrt{6}$ only

(c) $5 - \sqrt{6}$ or $5 + \sqrt{6}$ (d) neither $5 - \sqrt{6}$ nor $5 + \sqrt{6}$

10. Points D, E are taken on the side BC of a ΔABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x + y)\sin(y + z)}{\sin x \sin z}$ is equal to

(a) 1 (b) 2

(c) 4 (d) None of these

11. In a ΔABC, if $a = 2x$, $b = 2y$ and $\angle C = 120°$, then the area of the triangle is

(a) xy sq unit (b) $xy\sqrt{3}$ sq unit

(c) $3xy$ sq unit (d) $2xy$ sq unit

12. In ΔABC, if $\tan \frac{A}{2} \tan \frac{C}{2} = \frac{1}{2}$, then a, b and c are in

(a) AP (b) GP

(c) HP (d) None of these

13. In a ΔABC, if $2s = a + b + c$, then the value of the $\frac{s(s-a)}{bc} - \frac{(s-b)(s-c)}{bc}$ is equal to

(a) $\sin A$ (b) $\cos A$
(c) $\tan A$ (d) None of these

14. In a ΔABC, $(b + c - a) \tan \frac{A}{2}$ is equal to

(a) $\frac{2\Delta}{s}$ (b) $\frac{\Delta}{s}$
(c) $\frac{\Delta s}{bc}$ (d) $\frac{s}{a}R$

15. In a ΔPQR as shown in figure given that $x : y : z :: 2 : 3 : 6$, then the value of $\angle QPR$ is

(a) $\frac{\pi}{6}$
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$
(d) None of the above

16. In ΔABC, AD is median and $\angle A = 60°$, then $4AD^2$ is equal to

(a) $b^2 + c^2 - bc$ (b) $2b^2 + c^2 - 2bc$
(c) $b^2 + c^2 + 2bc$ (d) None of these

17. If $\cos^2 A + \cos^2 C = \sin^2 B$, then ΔABC is

(a) equilateral (b) right angled
(c) isosceles (d) None of these

18. In a ΔABC, $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, then the number of possible triangles are

(a) 1 (b) 0
(c) 2 (d) infinite

19. If p_1, p_2 and p_3 are altitudes of a ΔABC drawn from the vertices A, B and C and Δ the area of the triangle, then $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is equal to

(a) $\frac{a+b+c}{\Delta}$ (b) $\frac{a^2+b^2+c^2}{4\Delta^2}$
(c) $\frac{a^2+b^2+c^2}{\Delta^2}$ (d) None of these

20. If x, y and z are perpendicular drawn from the vertices of triangle having sides a, b and c, then the value of $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b}$ will be

(a) $\frac{a^2+b^2+c^2}{2R}$ (b) $\frac{a^2+b^2+c^2}{R}$
(c) $\frac{a^2+b^2+c^2}{4R}$ (d) $\frac{2(a^2+b^2+c^2)}{R}$

21. If in a ΔABC, CD is the angular bisector of the $\angle ACB$, then CD is equal to

(a) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (b) $\frac{a+b}{ab}\cos\frac{C}{2}$
(c) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (d) None of these

22. In ΔABC, $(a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$ is equal to

(a) a^2 (b) b^2
(c) c^2 (d) None of these

23. In a ΔABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is

(a) $\frac{16}{9}$ (b) $\frac{16}{7}$ (c) $\frac{11}{7}$ (d) $\frac{7}{16}$

24. If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, then the remaining fourth side is

(a) 2 (b) 3 (c) 4 (d) 5

Circle Connected with Triangle

25. The inradius of the triangle whose sides are 3, 5, 6, is

(a) $\sqrt{\frac{8}{7}}$ (b) $\sqrt{8}$
(c) $\sqrt{7}$ (d) $\sqrt{\frac{7}{8}}$

26. In a ΔABC, $\cos A + \cos B + \cos C$ is equal to

(a) $1 + \frac{r}{R}$ (b) $1 - \frac{r}{R}$
(c) $1 - \frac{R}{r}$ (d) $1 + \frac{R}{r}$

27. In a triangle $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, then the triangle is

(a) right angled (b) equilateral
(c) isosceles (d) None of these

28. If the sides of a triangle are in the ratio 3 : 7 : 8, then $R : r$ is equal to

(a) 2 : 7 (b) 7 : 2
(c) 3 : 7 (d) 7 : 3

29. In a triangle $r_1 > r_2 > r_3$, then

(a) $a > b > c$ (b) $a < b < c$
(c) $a > b$ and $b < c$ (d) $a < b$ and $b > c$

30. Consider the following statements

I. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin (A - B)}{\sin (B - C)}$, then a^2, b^2 and c^2 are in AP.

II. If exradius r_1, r_2 and r_3 of a ΔABC are in HP, then the sides a, b and c are in AP.

Which of these is/are correct ?

(a) Only (I) (b) Only (II)
(c) Both (I) and (II) (d) None of these

31. In a ΔABC, $a \cot A + b \cot B + c \cot C$ is equal to

(a) $r + R$ (b) $r - R$
(c) $2(r + R)$ (d) $2(r - R)$

32. The value of $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$ is

(a) 0 (b) $\frac{a^2 + b^2 + c^2}{\Delta^2}$
(c) $\frac{\Delta^2}{a^2 + b^2 + c^2}$ (d) $\frac{a^2 + b^2 + c^2}{\Delta}$

33. In an equilateral triangle, the ratio of the inradius, circumradius and exradius are

(a) 1 : 3 : 2 (b) 1 : 1 : 1
(c) 2 : 3 : 4 (d) 1 : 2 : 3

34. If r_1, r_2 and r_3 are exradii of any triangle, then $r_1r_2 + r_2r_3 + r_3r_1$ is equal to

(a) $\frac{\Delta}{r}$ (b) $\frac{\Delta^2}{r^2}$
(c) $\frac{r}{\Delta}$ (d) $\frac{r^2}{\Delta^2}$

35. If a flag staff of 6 m high placed on the top of a tower throws a shadow of $2\sqrt{3}$ m along the ground, then the angle (in degrees) that the sun makes with the ground is

(a) 60° (b) 80°
(c) 75° (d) None of these

Heights and Distances

36. When the elevation of sun changes from 45° to 30°, the shadow of a tower increases by 60 m, then the height of the tower is

(a) $30\sqrt{3}$ m (b) $30(\sqrt{2} + 1)$ m
(c) $30(\sqrt{3} - 1)$ m (d) $30(\sqrt{3} + 1)$ m

37. A flag staff 20 m long standing on a wall 10 m high subtends an angle whose tangent is 0.5 at a point on the ground. If θ is the angle subtended by the wall at this point, then

(a) $\tan \theta = 1$ (b) $\tan \theta = 3$
(c) $\tan \theta = \frac{1}{2}$ (d) None of these

38. A ladder rests against a wall making an angle α with the horizontal. The foot of the ladder is pulled away from the wall through a distance x, so that it slides a distance y down the wall making an angle β with the horizontal. The correct relation is

(a) $x = y \tan\left(\frac{\alpha + \beta}{2}\right)$ (b) $y = x \tan\left(\frac{\alpha + \beta}{2}\right)$
(c) $x = y \tan(\alpha + \beta)$ (d) $y = x \tan(\alpha + \beta)$

39. At a distance $2h$ m from the foot of a tower of height h m the top of the tower and a pole at the top of the tower subtend equal angles. Height of the pole should be

(a) $\frac{5h}{3}$ m (b) $\frac{4h}{3}$ m
(c) $\frac{7h}{5}$ m (d) $\frac{3h}{2}$ m

40. A house of height 100 m subtends a right angle at the window of an opposite house. If the height of the window be 64 m, then the distance between the two houses is

(a) 48 m (b) 36 m
(c) 54 m (d) 72 m

41. An observer standing on a 300 m high tower observes two boats in the same direction their angles of depression are 60° and 30°, respectively. The distance between boats is

(a) 173.2 m (b) 346.4 m
(c) 25 m (d) 72 m

42. An observer on the top of tree, finds the angle of depression of a car moving towards the tree to be 30°. After 3 min this angle becomes 60°. After how much more time, the car will reach the tree ?

(a) 4 min (b) 4.5 min
(c) 1.5 min (d) 2 min

43. The horizontal distance between two towers is 60 m and the angle of depression of the top of the first tower as seen from the top of the second is 30°. If the height of the second tower be 150 m, then the height of the first tower is

(a) $(150 - 60\sqrt{3})$ m (b) 90 m
(c) $(150 - 20\sqrt{3})$ m (d) None of these

44. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60° and after 10 s the elevation is observed to be 30°. The uniform speed of the aeroplane (in km/h) is

(a) 240
(b) $240\sqrt{3}$
(c) $60\sqrt{3}$
(d) None of the above

45. A flag staff is upon the top of a building. If at a distance of 40 m from the base of building the angles of elevation of the topes of the flag staff and building are 60° and 30° respectively, then the height of the flag staff is

(a) 46.19 m (b) 50 m
(c) 25 m (d) None of these

46. At a point on the ground the angle of elevation of a tower is such that its cotangent is $\frac{3}{5}$. On walking 32 m towards the tower the cotangent of the angle of elevation is $\frac{2}{5}$. The height of the tower is

(a) 160 m (b) 120 m
(c) 64 m (d) None of these

47. From the bottom of a pole of height h the angle of elevation of the top of a tower is α and the pole subtends an angle β at the top of the tower. The height of the tower is

(a) $\frac{h\tan(\alpha-\beta)}{\tan(\alpha-\beta)-\tan\alpha}$ (b) $\frac{h\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$
(c) $\frac{\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$ (d) None of these

48. The top of a hill observed from the top and bottom of a building h is at angles of elevation p and q, respectively. The height of hill is

(a) $\frac{h\cot q}{\cot q-\cot p}$ (b) $\frac{h\cot p}{\cot p-\cot q}$
(c) $\frac{h\tan p}{\tan p-\tan q}$ (d) None of these

49. From the top of a cliff of height a, the angle of depression of the foot of a certain tower is found to be double the angle of elevation of the top of the tower of height h. If θ be the angle of elevation, then its value is

(a) $\cos^{-1}\sqrt{\frac{2h}{a}}$ (b) $\sin^{-1}\sqrt{\frac{2h}{a}}$
(c) $\sin^{-1}\sqrt{\frac{a}{2-h}}$ (d) $\tan^{-1}\sqrt{3-\frac{2h}{a}}$

50. From a point a metres above a lake the angle of elevation of a cloud is α and the angle of depression of its reflection is β. The height of the cloud is

(a) $\frac{a\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ m (b) $\frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$ m
(c) $\frac{a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}$ (d) None of these

51. A flag staff of 5 m high stands on a building of 25 m high. At an observer at a height of 30 m, the flag staff and the building subtend equal angles. The distance of the observer from the top of the flag staff is

(a) $\frac{5\sqrt{3}}{2}$ m (b) $5\sqrt{\frac{3}{2}}$ m
(c) $5\sqrt{\frac{2}{3}}$ m (d) None of these

52. The angles of depression of the top and the foot of a chimney as seen from the top of a second chimney, which is 150 m high and standing on the same level as the first are θ and ϕ respectively, then the distance between their tops when $\tan\theta = \frac{4}{3}$ and $\tan\phi = \frac{5}{2}$, is

(a) $\frac{150}{\sqrt{3}}$ m (b) $100\sqrt{3}$ m
(c) 150 m (d) 100 m

53. From the tower 60 m high angles of depression of the top and bottom of a house are α and β, respectively. If the height of the house is $\frac{60\sin(\beta-\alpha)}{x}$, then x is equal to

(a) $\sin\alpha\sin\beta$ (b) $\cos\alpha\cos\beta$
(c) $\sin\alpha\cos\beta$ (d) $\cos\alpha\sin\beta$

54. The angle of elevation of the top of the tower observed from each of the three points A, B, C on the ground forming a triangle is the same $\angle\alpha$. If R is the circumradius of the ΔABC, then the height of the tower is

(a) $R\sin\alpha$ (b) $R\cos\alpha$
(c) $R\cot\alpha$ (d) $R\tan\alpha$

55. A spherical balloon of radius r subtends an $\angle\alpha$ at the eye of an observer. If the angle of elevation of the centre of the balloon be β, then height of the centre of the balloon is

(a) $r\,\text{cosec}\left(\frac{\alpha}{2}\right)\sin\beta$ (b) $r\,\text{cosec}\,\alpha\sin\left(\frac{\beta}{2}\right)$
(c) $r\sin\left(\frac{\alpha}{2}\right)\text{cosec}\,\beta$ (d) $r\sin\alpha\,\text{cosec}\left(\frac{\beta}{2}\right)$

56. A ladder leaves again a wall at an $\angle\alpha$ to the horizontal. Its foot is pulled away through a distance a_1, so that it slides a distance b_1 down the wall and rests inclined at $\angle\beta$ with the horizontal. It foot is further pulled aways through a_2, so that it slides a further distance b_2 down the wall and is now, inclined at an $\angle\gamma$. If $a_1 a_2 = b_1 b_2$, then

(a) $\alpha+\beta+\gamma$ is greater than π
(b) $\alpha+\beta+\gamma$ is equal to π
(c) $\alpha+\beta+\gamma$ is less than π
(d) nothing can be said about $\alpha+\beta+\gamma$

57. ABC is a triangular park with $AB = AC = 100$ m. A clock tower is situated at the mid-point of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1} 3.2$ and $\operatorname{cosec}^{-1} 2.6$, respectively. The height of the tower is

(a) 50 m (b) 25 m
(c) 40 m (d) None of these

58. The angle of elevation of the top of a tower from a point A due South of the tower is α and from a point B due East of the tower is β. If $AB = d$, then the height of the tower is

(a) $\dfrac{d}{\sqrt{\tan^2\alpha - \tan^2\beta}}$ (b) $\dfrac{d}{\sqrt{\tan^2\alpha + \tan^2\beta}}$
(c) $\dfrac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$ (d) $\dfrac{d}{\sqrt{\cot^2\alpha - \cot^2\beta}}$

59. $ABCD$ is a square plot. The angle of elevation of the top of a pole standing at D from A or C is 30° and that from B is θ, then $\tan\theta$ is equal to

(a) $\sqrt{6}$ (b) $1/\sqrt{6}$ (c) $\sqrt{3}/2$ (d) $\sqrt{2/3}$

60. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 ft and then, finds that the tower subtends an angle of 30°. The height of tower is

(a) $20(\sqrt{6} - \sqrt{2})$ ft (b) $40(\sqrt{6} - \sqrt{2})$ ft
(c) $40(\sqrt{6} + \sqrt{2})$ ft (d) None of these

61. If A and B are two points on one bank of a straight river and C, D are two other points on the other bank of river. If direction from A to B is same as that from C to D and $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, then CD is equal to

(a) $\dfrac{a\sin\beta\sin\gamma}{\sin\alpha\sin(\alpha+\beta+\gamma)}$ (b) $\dfrac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$
(c) $\dfrac{a\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta+\gamma)}$ (d) None of these

62. The angle of elevation of the top of a hill from a point is α. After walking b m towards the top up a slope inclined at an $\angle\beta$ to the horizon, the angle of elevation of the top becomes γ. Then, the height of the hill is

(a) $\dfrac{b\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$ (b) $\dfrac{b\sin\alpha\sin(\gamma-\alpha)}{\sin(\gamma-\beta)}$
(c) $\dfrac{b\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$ (d) $\dfrac{\sin(\gamma-\beta)}{b\sin\alpha\sin(\gamma-\alpha)}$

63. The angle of elevation of a cloud at a point 2500 m high above a lake is 15° and the angle of depression of its image in the lake is 45°, the height of the cloud above the surface of the lake is

(a) $2500\sqrt{3}$ m (b) 2500 m
(c) $500\sqrt{3}$ m (d) 500 m

Round II (Mixed Bag)

Only One Correct Option

1. In an ambiguous case of solving a triangle when $a = \sqrt{5}$, $b = 2$, $\angle A = \dfrac{\pi}{6}$, the two possible values of third side are c_1 and c_2, then

(a) $|c_1 - c_2| = 2\sqrt{6}$ (b) $|c_1 - c_2| = 4\sqrt{6}$
(c) $|c_1 - c_2| = 4$ (d) $|c_1 - c_2| = 6$

2. In an isosceles ΔABC, $AB = AC$. If vertical $\angle A$ is 20°, then $a^3 + b^3$ is equal to

(a) $3a^2b$ (b) $3b^2c$
(c) $3c^2a$ (d) abc

3. If A, B, C, D are the angles of a quadrilateral, then $\dfrac{\Sigma\tan A}{\Sigma\cot A}$ is equal to

(a) $\Pi\tan A$ (b) $\Pi\cot A$
(c) $\Sigma\tan^2 A$ (d) $\Sigma\cot^2 A$

4. In a ΔABC, $(a+b+c)(b+c-a) = kbc$, if

(a) $k < 0$ (b) $k > 6$
(c) $0 < k < 4$ (d) $k > 4$

5. In an acute angled ΔABC, $r + r_1 = r_2 + r_3$ and $\angle B > \dfrac{\pi}{3}$, then

(a) $b + 2c < 2a < 2b + 2c$ (b) $b + 4c < 4a < 2b + 4c$
(c) $b + 4c < 4a < 4b + 4c$ (d) $b + 3c < 3a < 3b + 3c$

6. For a ΔABC, $R = \dfrac{5}{2}$ and $r = 1$. Let I be the incentre of the triangle and D, E and F be the feet of the perpendiculars from I to BC, CA and AB, respectively. The value of $\dfrac{ID \times IE \times IF}{IA \times IB \times IC}$ is equal to

(a) $\dfrac{5}{2}$ (b) $\dfrac{5}{4}$ (c) $\dfrac{1}{10}$ (d) $\dfrac{1}{5}$

7. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of

(a) $\tan\left(\dfrac{\pi}{n}\right) : \dfrac{\pi}{n}$ (b) $\cos\left(\dfrac{\pi}{n}\right) : \dfrac{\pi}{n}$
(c) $\sin\left(\dfrac{\pi}{n}\right) : \dfrac{\pi}{n}$ (d) $\cot\left(\dfrac{\pi}{n}\right) : \dfrac{\pi}{n}$

8. The radii r_1, r_2, r_3 of the described circles of the ΔABC are in HP. If the area of the triangle is 24 cm^2 and its perimeter is 24 cm, then the length of its largest side is

(a) 10 (b) 9 (c) 8 (d) 7

9. If PQ be a vertical tower subtending angles α, β and γ at the points A, B and C respectively on the line in the horizontal plane through the foot D of tower and on the same side of it, then $BC \cot\alpha - CA \cot\beta + AB \cot\gamma$ is equal to

(a) 0 (b) 1
(c) 2 (d) None of these

10. A flag staff stands in the centre of a rectangular field whose diagonal is 1200 m and subtends angles 15° and 45° at the mid-points of the sides of the field. The height of the flag staff is

(a) 200 m (b) $300\sqrt{2+\sqrt{3}}$ m
(c) $300\sqrt{2-\sqrt{3}}$ m (d) 400 m

11. At each end of a horizontal line of length $2a$, the angular elevation of the peak of a vertical tower is θ and that at its middle point it is ϕ. The height of the peak is

(a) $a \sin\theta \sin\phi$
(b) $\dfrac{a \sin\theta \sin\phi}{\sqrt{\sin(\theta+\phi)\sin(\phi-\theta)}}$
(c) $\dfrac{a \cos\theta \cos\phi}{\sqrt{\cos(\phi+\theta)\cos(\phi-\theta)}}$
(d) None of the above

12. In a cubical hall $ABCDPQRS$ with each side 10 m, G is the centre of the wall $BCRQ$ and T is the mid-point of the side AB. The angle of elevation of G at the point T is

(a) $\sin^{-1}(1/\sqrt{3})$ (b) $\cos^{-1}(1/\sqrt{3})$
(c) $\cot^{-1}(1/\sqrt{3})$ (d) None of these

13. Two vertical poles 20 m and 80 m stands apart on a horizontal plane. The height of the point of intersection of the lines joining the top of each pole to the foot of the other is

(a) 15 m (b) 16 m (c) 18 m (d) 50 m

More Than One Correct Option

14. Let ABC be an isosceles triangle with base BC. If r is the radius of the circle inscribed in the ΔABC and r_1 be the radius of the circle escribed opposite to the angle A, then the product $r_1 r$ can be equal to

(a) $R^2 \sin^2 A$ (b) $R^2 \sin^2 2B$
(c) $\frac{1}{2}a^2$ (d) $\frac{a^2}{4}$

where R is the radius of the circumcircle of the ΔABC.

15. The area of a regular polygon of n sides is (where, r is inradius, R is circumradius and a is side of the triangle)

(a) $\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ (b) $nr^2 \tan\left(\frac{\pi}{n}\right)$
(c) $\frac{na^2}{4}\cot\frac{\pi}{n}$ (d) $nR^2 \tan\left(\frac{\pi}{n}\right)$

Comprehension Based Questions

Passage

Directions (Q. Nos. 16-18) *Given that*

$$\Delta = 6,\ r_1 = 2,\ r_2 = 3,\ r_3 = 6.$$

16. Circumradius R is equal to

(a) 2.5 (b) 3.5 (c) 1.5 (d) 4.2

17. Inradius is equal to

(a) 2 (b) 1 (c) 1.5 (d) 2.5

18. Difference between the greatest and the least angle is

(a) $\cos^{-1}\frac{4}{5}$ (b) $\tan^{-1}\frac{3}{4}$
(c) $\cos^{-1}\frac{3}{5}$ (d) None of these

Assertion and Reason

Directions (Q. Nos. 19-21) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

19. **Statement I** In any right angled triangle $\dfrac{a^2+b^2+c^2}{R^2}$ is always equal to 8.

Statement II $a^2 = b^2 + c^2$

20. In acute angled ΔABC, $a > b > c$

Statement I $r_1 > r_2 > r_3$

Statement II $\cos A < \cos B < \cos C$

21. **Statement I** If I is incentre of ΔABC and I_1 excentre opposite to A and P is the intersection of II_1 and BC, then $IP \cdot I_1P = BP \cdot PC$.

Statement II In a ΔABC, I is incentre and I_1 is excentre opposite to A, then IBI_1C must be square.

Previous Years' Questions

22. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is **[AIEEE 2010]**

(a) there is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

(b) there is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$

(c) there is a regular polygon with $\frac{r}{R} = \frac{2}{3}$

(d) there is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

23. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that $CD = 7$m. From D the angle of elevation of the point A is 45°. Then, the height of the pole is **[AIEEE 2008]**

(a) $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}+1}\right)$ m (b) $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}-1}\right)$ m

(c) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$ m (d) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$ m

24. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that $AB (= a)$ subtends an angle of 60° at the foot of the tower and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is **[AIEEE 2007]**

(a) $\frac{2a}{\sqrt{3}}$ (b) $2a\sqrt{3}$

(c) $\frac{a}{\sqrt{3}}$ (d) $\sqrt{3}$

Answers

Check Point 1

1. (c)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (b)
11. (a)	**12.** (c)								

Check Point 2

1. (c)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (a)

Round I

1. (c)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (c)
11. (b)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (d)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (a)
21. (c)	**22.** (c)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (c)
31. (c)	**32.** (b)	**33.** (d)	**34.** (b)	**35.** (a)	**36.** (d)	**37.** (a)	**38.** (a)	**39.** (a)	**40.** (a)
41. (b)	**42.** (c)	**43.** (d)	**44.** (b)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (b)	**49.** (d)	**50.** (b)
51. (b)	**52.** (d)	**53.** (d)	**54.** (d)	**55.** (a)	**56.** (c)	**57.** (b)	**58.** (c)	**59.** (b)	**60.** (b)
61. (b)	**62.** (a)	**63.** (a)							

Round II

1. (c)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (c)
11. (b)	**12.** (a)	**13.** (b)	**14.** (a,b,d)	**15.** (a,b,c)	**16.** (a)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (b)
21. (c)	**22.** (d)	**23.** (c)	**24.** (c)						

the Guidance

Round I

1. We have, cosec A (sin B cos C + cos B sin C)

$$= \left(\frac{\sin B}{\sin A}\cos C + \frac{\sin C}{\sin A}\cos B\right)$$

$$= \left(\frac{b}{a}\cos C + \frac{c}{a}\cos B\right) = 1$$

$$(\because\ a = b\cos C + c\cos B)$$

2. $\dfrac{\sin 3B}{\sin B} = \dfrac{3\sin B - 4\sin^3 B}{\sin B} = 3 - 4\sin^2 B$

$$= 3 - 4(1 - \cos^2 B)$$

$$= -1 + \frac{4(a^2 + c^2 - b^2)^2}{4(ac)^2} = -1 + \frac{\left(\dfrac{a^2 + c^2}{2}\right)^2}{(ac)^2}$$

$$= -1 + \frac{(a^2 + c^2)^2}{4(ac)^2} \qquad [\because 2b^2 = a^2 + c^2 \text{ (given)}]$$

$$= \frac{(a^2 + c^2)^2 - 4a^2c^2}{4(ac)^2} = \left(\frac{c^2 - a^2}{2ac}\right)^2$$

3. $\because\ \dfrac{\sin A}{\sin C} = \dfrac{\sin(A - B)}{\sin(B - C)}$

$$\Rightarrow \frac{\sin A}{\sin C} = \frac{\sin A\cos B - \cos A\sin B}{\sin B\cos C - \cos B\sin C}$$

$$\Rightarrow \frac{a}{c} = \frac{a\cos B - b\cos A}{b\cos C - c\cos B}$$

$$\Rightarrow ab\cos C + bc\cos A = 2ac\cos B$$

$$\Rightarrow \frac{a^2 + b^2 - c^2}{2} + \frac{b^2 + c^2 - a^2}{2} = 2\,\frac{c^2 + a^2 - b^2}{2}$$

$$\Rightarrow a^2 + c^2 = 2b^2$$

$\Rightarrow$ a^2, b^2 and c^2 are in AP.

4. $\because\ (\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})$

$$= (\sqrt{a} + \sqrt{b})^2 - c$$

$$= a + b - c + 2\sqrt{ab} > 0$$

$$\therefore \quad \sqrt{a} + \sqrt{b} > \sqrt{c}$$

5. We have, $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s - b)(s - c)}{bc}}$

$$\Rightarrow bc\sin^2\frac{A}{2} = (s - b)(s - c)$$

On comparing with $x\sin^2\dfrac{A}{2} = (s - b)(s - c)$,

we get $x = bc$

6. Here, ratio of angles are 4 : 1 : 1.

$$\Rightarrow 4x + x + x = 180°$$

$$\Rightarrow x = 30°$$

$$\therefore \angle A = 120°,\ \angle B = \angle C = 30°$$

Thus, the ratio of longest side to the perimeter $= \dfrac{a}{a + b + c}$

Let $b = c = x$

$$\therefore a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow a^2 = 2x^2 - 2x^2\cos A = 2x^2(1 - \cos A)$$

$$\Rightarrow a^2 = 4x^2\sin^2\frac{A}{2}$$

$$\Rightarrow a = 2x\sin\frac{A}{2}$$

$$\Rightarrow a = 2x\sin 60° = \sqrt{3}x$$

Thus, required ratio is

$$\frac{a}{a + b + c} = \frac{\sqrt{3}x}{x + x + \sqrt{3}x} = \frac{\sqrt{3}}{2 + \sqrt{3}}$$

7. We have, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow \cos 60° = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow a^2 + b^2 - c^2 = ab$$

$$\Rightarrow b^2 + bc + a^2 + ac = ab + ac + bc + c^2$$

$$\Rightarrow b(b + c) + a(a + c) = (a + c)(b + c)$$

On dividing by $(a + c)(b + c)$ and add 2 on both sides, we get

$$1 + \frac{b}{a + c} + 1 + \frac{a}{b + c} = 3$$

$$\Rightarrow \frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}$$

8. Since a, b and c are in AP.

$$\Rightarrow 2b = a + c$$

$$\Rightarrow 3b = 2s \Rightarrow s = \frac{3b}{2}$$

Now, $\dfrac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}} = \sqrt{\dfrac{ac\,(s-b)(s-c)(s-b)(s-a)}{(s-a)(s-c)\times bc\times ab}}$

$$= \frac{s-b}{b} = \frac{\frac{3b}{2}-b}{b} = \frac{1}{2}$$

9. We know that, in triangle larger side has an larger angle opposite to it. Since, $\angle A$, $\angle B$ and $\angle C$ are in AP.

$\Rightarrow \quad 2B = A + C$

$\because \quad A + B + C = \pi$

$\Rightarrow \quad B = 60°$

$\therefore \quad \cos B = \dfrac{a^2 + c^2 - b^2}{2\,ac}$

$\Rightarrow \quad \cos 60° = \dfrac{1}{2} = \dfrac{100 + a^2 - 81}{20a}$

$\Rightarrow \quad a^2 + 19 = 10a$

$\Rightarrow \quad a^2 - 10a + 19 = 0$

$\therefore \quad a = \dfrac{10 \pm \sqrt{100-76}}{2} = 5 \pm \sqrt{6}$

10. From ΔABC, $\dfrac{\sin(y+z)}{DC} = \dfrac{\sin C}{AD}$

From ΔABD, $\dfrac{\sin x}{BD} = \dfrac{\sin B}{AD}$

From ΔAEC, $\dfrac{\sin z}{EC} = \dfrac{\sin C}{AE}$

From ΔABE, $\dfrac{\sin(x+y)}{BE} = \dfrac{\sin B}{AE}$

$$\therefore \quad \frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} = \frac{\dfrac{BE\sin B}{AE}\times\dfrac{DC\sin C}{AD}}{\dfrac{BD\sin B}{AD}\times\dfrac{EC\sin C}{AE}}$$

$$= \frac{BE}{AE}\times\frac{DC}{AD}\times\frac{AD}{BD}\times\frac{AE}{EC}$$

$$= \frac{2BD\times 2EC}{BD\times EC} = 4$$

11. We have, $a = 2x$, $b = 2y$ and $\angle C = 120°$

Area of triangle, $\Delta = \dfrac{1}{2}ab\sin C$

$$= \frac{1}{2}\times 2x\times 2y\sin 120°$$

$$= xy\sqrt{3} \text{ sq units}$$

12. We have, $\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{2}$

$\Rightarrow \quad \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{s-b}{s} = \dfrac{1}{2} \Rightarrow 2s - 2b - s = 0$

$\Rightarrow \quad a + c - 3b = 0$

13. We have, $\dfrac{s(s-a)}{bc} - \dfrac{(s-b)(s-c)}{bc}$

$$= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \frac{\cos 2A}{2}$$

$$= \cos A$$

14. $(b + c - a)\tan\frac{A}{2} = (2s - 2a)\tan\frac{A}{2}$

$$= 2(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= 2\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{2\Delta}{s}$$

15. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{\frac{y}{z}+\frac{x}{z}}{1-\frac{y}{z}\times\frac{x}{z}}$$

$$= \frac{\frac{3}{6}+\frac{2}{6}}{1-\frac{3}{6}\times\frac{2}{6}} = 1$$

$\Rightarrow \quad \alpha + \beta = \angle QPR = \dfrac{\pi}{4}$

16. $\because \angle A = 60°$, $\cos 60° = \dfrac{b^2 + c^2 - a^2}{2\,bc}$

$\Rightarrow \quad b^2 + c^2 - a^2 = bc$

Now, $AB^2 + AC^2 = 2(AD^2 + BD^2)$

$\Rightarrow \quad c^2 + b^2 = 2AD^2 + 2\left(\dfrac{a}{2}\right)^2$

$\Rightarrow \quad 2b^2 + 2c^2 - a^2 = 4AD^2$

$\Rightarrow \quad b^2 + c^2 + bc = 4AD^2 \quad (\because b^2 + c^2 - a^2 = bc)$

17. Given that, $\cos^2 A + \cos^2 C = \sin^2 B$

Obviously it is not an equilateral triangle because $A = B = C = 60°$ does not satisfy the given condition. But $B = 90°$, then $\sin^2 B = 1$ and

$$\cos^2 A + \cos^2 C = \cos^2 A + \cos^2\left(\frac{\pi}{2} - A\right)$$

$$= \cos^2 A + \sin^2 A = 1$$

Hence, this satisfies the condition, so it is a right angled triangle but not necessarily isosceles triangle.

18. Given that, $a = 5, b = 7$ and $\sin A = \dfrac{3}{4}$

As we know, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\Rightarrow \quad \dfrac{3}{4\times 5} = \dfrac{\sin B}{7} \Rightarrow \sin B = \dfrac{21}{20}$

which is not possible because its value is greater than one.

19. We have, $\frac{1}{2}ap_1 = \Delta,\ \frac{1}{2}bp_2 = \Delta,\ \frac{1}{2}cp_3 = \Delta$

where a, b and c are the sides of a triangle.

$\Rightarrow \quad p_1 = \frac{2\Delta}{a},\ p_2 = \frac{2\Delta}{b},\ p_3 = \frac{2\Delta}{c}$

$\therefore \quad \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{a^2}{4\Delta^2} + \frac{b^2}{4\Delta^2} + \frac{c^2}{4\Delta^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$

20. Let area of triangle be Δ, then according to the question

$$\Delta = \frac{1}{2}ax = \frac{1}{2}by = \frac{1}{2}cz$$

$$\therefore \quad \frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{b}{c}\left(\frac{2\Delta}{a}\right) + \frac{c}{a}\left(\frac{2\Delta}{b}\right) + \frac{a}{b}\left(\frac{2\Delta}{c}\right)$$

$$= \frac{2\Delta(b^2+c^2+a^2)}{abc}$$

$$= \frac{2(a^2+b^2+c^2)}{abc}\cdot\frac{abc}{4R} \qquad \left(\because \Delta = \frac{abc}{4R}\right)$$

$$= \frac{a^2+b^2+c^2}{2R}$$

21. Area of ΔCAB = area of ΔCAD + area of ΔCDB

$\Rightarrow \quad \frac{1}{2}ba\sin C = \frac{1}{2}b\cdot CD\sin\frac{C}{2} + \frac{1}{2}a\cdot CD\sin\frac{C}{2}$

$\Rightarrow \quad CD = \frac{2ab}{a+b}\cos\frac{C}{2}$

22. $(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}$

$$= (a^2+b^2-2ab)\cos^2\frac{C}{2} + (a^2+b^2+2ab)\sin^2\frac{C}{2}$$

$$= a^2+b^2+2ab\left(\sin^2\frac{C}{2} - \cos^2\frac{C}{2}\right)$$

$$= a^2+b^2-2ab\cos C$$

$$= a^2+b^2-(a^2+b^2-c^2)$$

$$= c^2 \qquad \left(\because \cos C = \frac{a^2+b^2-c^2}{2ab}\right)$$

23. We have, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$

$\therefore \quad \frac{R}{r} = \frac{abc}{4\Delta}\cdot\frac{s}{\Delta} = \frac{abc}{4(s-a)(s-b)(s-c)}$

Since, $a:b:c = 4:5:6$

$\Rightarrow \quad \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say)

Thus, $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{15k}{2}-4k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-6k\right)}$

$$= \frac{120k^3\cdot 2}{k^3\cdot 7\cdot 5\cdot 3} = \frac{16}{7}$$

24. In ΔABD,

$$\cos 60° = \frac{2^2+5^2-BD^2}{2(2)(5)}$$

$\Rightarrow \quad BD^2 = 19$

Now, in ΔBCD,

$$\cos 120° = \frac{CD^2+9-19}{(2)(3)(CD)}$$

$\Rightarrow \quad CD^2 + 3CD - 10 = 0$

$\Rightarrow \quad CD = -5, 2$

$\Rightarrow \quad CD = 2 \qquad (\because CD \neq -5)$

25. Given that, $a = 3, b = 5, c = 6$

Now, $s = \frac{a+b+c}{2} = 7$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{7(7-3)(7-5)(7-6)}$$

$$= \sqrt{7\cdot 4\cdot 2\cdot 1} = 2\sqrt{14}$$

$\therefore \quad r = \frac{\Delta}{s} = \frac{2\sqrt{14}}{7} = \sqrt{\frac{8}{7}}$

26. We know that, in any ΔABC,

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore \quad \cos A + \cos B + \cos C = 1 + \frac{r}{R}$

27. Since, $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$

$\therefore \quad \left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right) = 2$

$\Rightarrow \quad \frac{(b-a)(c-a)}{(s-a)^2} = 2$

$\Rightarrow 2bc - 2ab - 2ac + 2a^2 = b^2 + c^2 + a^2 + 2bc - 2ab - 2ac$

$\Rightarrow \quad a^2 = b^2 + c^2$

So, triangle is right angled.

28. Let the sides be $a = 3x$, $b = 7x$, $c = -8x$. Then,

$2s = a + b + c \Rightarrow s = 9x$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{9x\times 6x\times 2x\times x} = 6\sqrt{3}x^2$$

Now, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$

$\therefore \quad \frac{R}{r} = \frac{sabc}{4\Delta^2} = \frac{9x\times 3x\times 7x\times 8x}{4\times 108\times x^4} = \frac{7}{2}$

29. We know that, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

Given that, $r_1 > r_2 > r_3$

$\Rightarrow \frac{\Delta}{s-a} > \frac{\Delta}{s-b} > \frac{\Delta}{s-c}$

$\Rightarrow \frac{1}{s-a} > \frac{1}{s-b} > \frac{1}{s-c}$

$\Rightarrow s-a < s-b < s-c$

($s-a$, $s-b$, $s-c$ are positive)

$\Rightarrow -a < -b < -c$

$\Rightarrow a > b > c$

30. **Statement I** $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

$\Rightarrow \sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$

$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\Rightarrow b^2 - c^2 = a^2 - b^2$

$\Rightarrow 2b^2 = a^2 + c^2$

$\Rightarrow a^2, b^2$ and c^2 are in AP.

Statement II r_1, r_2 and r_3 are in HP.

$\Rightarrow \frac{1}{r_1}, \frac{1}{r_2}$ and $\frac{1}{r_3}$ are in AP.

$\Rightarrow \frac{2}{r_2} = \frac{1}{r_1} + \frac{1}{r_3}$

$\Rightarrow 2(s-b) = s-a+s-c$

$\Rightarrow 2b = (a+c)$

$\Rightarrow a, b$ and c are in AP.

Hence, both of these statements are correct.

31. $a\cot A + b\cot B + c\cot C$

$= \frac{a}{\sin A}\cos A + \frac{b}{\sin B}\cos B + \frac{c}{\sin C}\cos C$

$= 2R(\cos A + \cos B + \cos C)$

$= 2R\left(1 + \frac{r}{R}\right) = 2(r+R)$

32. Since, $\frac{1}{r_1^2} = \frac{(s-a)^2}{\Delta^2}, \frac{1}{r_2^2} = \frac{(s-b)^2}{\Delta^2}, \frac{1}{r_3^2} = \frac{(s-c)^2}{\Delta^2}$

and $\frac{1}{r^2} = \frac{s^2}{\Delta^2}$

$\therefore \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$

$= \frac{s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2}{\Delta^2}$

$= \frac{4s^2 + a^2 + b^2 + c^2 - 2s(a+b+c)}{\Delta^2}$

$= \frac{a^2 + b^2 + c^2}{\Delta^2}$

33. The area of an equilateral triangle $= \frac{\sqrt{3}}{4}a^2$, where a is side.

Also, $s = \frac{a+a+a}{2} = \frac{3a}{2}$

$\therefore$ Inradius, $r = \frac{\Delta}{s} = \frac{\sqrt{3}a^2 \times 2}{4 \times 3a} = \frac{a}{2\sqrt{3}}$ ...(i)

Circumradius, $R = \frac{abc}{4\Delta} = \frac{a^3}{\sqrt{3}a^2} = \frac{a}{\sqrt{3}}$ ...(ii)

and exradius, $r_1 = \frac{\Delta}{s-a} = \frac{(\sqrt{3}/4)a^2}{a/2} = \frac{\sqrt{3}}{2}a$

Hence, $r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2}a = 1:2:3$

34. $\therefore r_1r_2 + r_2r_3 + r_3r_1$

$= \frac{\Delta}{(s-a)} \times \frac{\Delta}{(s-b)} + \frac{\Delta}{s-b} \times \frac{\Delta}{s-c} + \frac{\Delta}{s-c} \times \frac{\Delta}{s-a}$

$= \frac{\Delta^2}{(s-b)(s-a)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-a)(s-c)}$

$= \frac{\Delta^2[(s-c)+(s-a)+(s-b)]}{(s-a)(s-b)(s-c)}$

$= \frac{\Delta^2[3s-(a+b+c)]}{(s-a)(s-b)(s-c)} \times \frac{s}{s} = \frac{\Delta^2 s^2}{\Delta^2} = s^2 = \frac{\Delta^2}{r^2}$

35. In ΔDCB, $\tan\theta = \frac{h}{x}$ ...(i)

and in ΔECA, $\tan\theta = \frac{h+6}{2\sqrt{3}+x}$

$\Rightarrow \frac{h}{x} = \frac{h+6}{2\sqrt{3}+x}$ [from Eq. (i)]

$\Rightarrow 2\sqrt{3}h + hx = hx + 6x$

$\Rightarrow 2\sqrt{3}h = 6x$

$\Rightarrow h = \frac{6x}{2\sqrt{3}}$

From Eq. (i), we get

$\tan\theta = \frac{6x}{2\sqrt{3}x} = \sqrt{3} \Rightarrow \theta = 60°$

36. Let the height of the tower be h metres.

In ΔBCD, $\tan 45° = \frac{h}{BC}$

$\Rightarrow BC = h$ ...(i)

In ΔACD, $\tan 30° = \dfrac{h}{AC}$

$\Rightarrow$ $AC = \dfrac{h}{\tan 30°}$

$\Rightarrow$ $AB + BC = \sqrt{3}\,h$

$\Rightarrow$ $60 + h = \sqrt{3}\,h$ [from Eq. (i)]

$\Rightarrow$ $h = \dfrac{60}{\sqrt{3}-1}$

$\Rightarrow$ $h = \dfrac{60(\sqrt{3}+1)}{2}$

$\Rightarrow$ $h = 30(\sqrt{3}+1)$ m

37. Given, $\tan\phi = 0.5 = \dfrac{1}{2}$

In ΔABC, $\tan\theta = \dfrac{10}{AB}$

$\Rightarrow$ $AB = \dfrac{10}{\tan\theta}$

In ΔABD,

$\tan(\theta+\phi) = \dfrac{30}{AB}$

$\Rightarrow$ $\dfrac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \dfrac{30\tan\theta}{10}$

$\Rightarrow$ $\tan\theta + \dfrac{1}{2} = 3\tan\theta - \dfrac{3}{2}\tan^2\theta$

$\Rightarrow$ $3\tan^2\theta - 4\tan\theta + 1 = 0$

$\Rightarrow$ $(3\tan\theta - 1)(\tan\theta - 1) = 0$

$\Rightarrow$ $\tan\theta = \dfrac{1}{3}, 1$

$\therefore$ $\tan\theta = 1$

38. Let l be the length of the ladder

i.e., $BP = CQ = 1$

In ΔPAB, $\cos\alpha = \dfrac{PA}{PB}$ and $\sin\alpha = \dfrac{AB}{PB}$

$\Rightarrow$ $PA = l\cos\alpha$ and $AB = l\sin\alpha$ ...(i)

In ΔQAC, $\cos\beta = \dfrac{AQ}{QC}$

$\Rightarrow$ $AQ = l\cos\beta$

and $AC = l\sin\beta$ ...(ii)

Now, $CB = AB - CA$

$= l\sin\alpha - l\sin\beta$ [from Eqs. (i) and (ii)]

$= l(\sin\alpha - \sin\beta)$

and $QP = AQ - PA$

$= l\cos\beta - l\cos\alpha$

$= l(\cos\beta - \cos\alpha)$

$\therefore$ $\dfrac{CB}{QP} = \dfrac{l(\sin\alpha - \sin\beta)}{l(\cos\beta - \cos\alpha)}$

$\Rightarrow$ $\dfrac{y}{x} = \dfrac{2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)}{2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)}$

$\Rightarrow$ $\dfrac{y}{x} = \cot\left(\dfrac{\alpha+\beta}{2}\right)$

$\Rightarrow$ $x = y\tan\left(\dfrac{\alpha+\beta}{2}\right)$

39. In ΔABD, $\tan\alpha = \dfrac{h}{2h}$

$\Rightarrow$ $\tan\alpha = \dfrac{1}{2}$ ...(i)

In ΔABC, $\tan 2\alpha = \dfrac{h+p}{2h}$

$\Rightarrow$ $\dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{h+p}{2h}$

$\Rightarrow$ $\dfrac{2\left(\dfrac{1}{2}\right)}{1-\left(\dfrac{1}{2}\right)^2} = \dfrac{h+p}{2h}$

$\Rightarrow$ $\dfrac{4}{3} = \dfrac{h+p}{2h}$

$\Rightarrow$ $8h = 3h + 3p$

$\Rightarrow$ $5h = 3p \Rightarrow p = \dfrac{5h}{3}$ m

40. In ΔDAB, $\tan\theta = \frac{64}{d}$

$\Rightarrow \quad d = 64\cot\theta$ ...(i)

In ΔCDE, $\tan(90° - \theta) = \frac{(100-64)}{d}$

$\Rightarrow \quad d = 36\tan\theta$ ...(ii)

From Eqs. (i) and (ii), we get

$d^2 = 36 \times 64 \Rightarrow d = 48$ m

41. In ΔBCD, $\cot 60° = \frac{BC}{300}$

$\Rightarrow \quad BC = 300 \times \frac{1}{\sqrt{3}}$ ...(i)

In ΔACD, $\cot 30° = \frac{AC}{300}$

$\Rightarrow \quad AC = 300\sqrt{3}$ ...(ii)

$\therefore$ Distance between two boats = AB

$= AC - BC = 300\sqrt{3} - \frac{300}{\sqrt{3}}$ [using Eqs. (i) and (ii)]

$= 300\frac{(3-1)}{\sqrt{3}} = \frac{600 \times \sqrt{3}}{3} = 346.4$ m

42. In ΔABC, $\tan 60° = \frac{h}{BC}$

$\Rightarrow \quad BC = h\cot 60°$ ...(i)

In ΔABD, $\tan 30° = \frac{h}{BD}$

$\Rightarrow \quad BD = h\cot 30°$

$\Rightarrow \quad BC + CD = h\cot 30°$

$\Rightarrow \quad CD = h\cot 30° - BC$

$\Rightarrow \quad d = h\cot 30° - h\cot 60°$ [from Eq. (i)]

$\therefore$ Speed of car = $\frac{\text{Distance from } D \text{ to } C}{\text{Time taken}}$

$= \frac{d}{3} = \frac{h\cot 30° - h\cot 60°}{3}$

$\therefore$ Time taken from C to B = $\frac{\text{distance}}{\text{speed}} = \frac{h\cot 60°}{\frac{h(\cot 30° - \cot 60°)}{3}}$

$= \frac{\frac{1}{\sqrt{3}} \times 3}{\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)} = \frac{3}{2} = 1.5$ min

43. In ΔABC, $\tan 30° = \frac{BC}{AC}$

$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h-150}{60}$

$\Rightarrow \quad h - 150 = \frac{60}{\sqrt{3}}$

$\Rightarrow \quad h = (150 + 20\sqrt{3})$ m

44. In ΔADF, $\tan 60° = \frac{1}{AF}$

$\Rightarrow \quad AF = \cot 60° = \frac{1}{\sqrt{3}}$ ...(i)

In ΔABC, $\quad \tan 30° = \frac{1}{AB}$

$\Rightarrow \quad AB = \cot 30°$

$\Rightarrow \quad AF + FB = \sqrt{3}$

$\Rightarrow \quad d = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ [using Eq. (i)]

$\therefore$ Speed of aeroplane $= \frac{\text{Distance from } D \text{ to } C}{\text{Time taken}}$

$$= \frac{\frac{2}{\sqrt{3}}}{10} \times 60 \times 60 = 240\sqrt{3} \text{ km/h}$$

45. In ΔABC,

$$\tan 30° = \frac{BC}{40}$$

$\Rightarrow \quad BC = \frac{40}{\sqrt{3}}$ ...(i)

In ΔDAB, $\quad \tan 60° = \frac{BD}{40}$

$\Rightarrow \quad BD = 40\sqrt{3}$

$\Rightarrow \quad DC + BC = 40\sqrt{3}$

$\Rightarrow \quad DC + \frac{40}{\sqrt{3}} = 40\sqrt{3}$ [from Eq. (i)]

$\Rightarrow \quad DC = 40\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{80}{\sqrt{3}}$

$\Rightarrow \quad DC = 46.19$ m

46. Given that, $\cot\alpha = \frac{3}{5}$ and $\cot\beta = \frac{2}{5}$

In ΔBCD, $\quad \tan\beta = \frac{h}{BC}$

$\Rightarrow \quad BC = h\cot\beta$

$\Rightarrow \quad BC = \frac{2h}{5}$ ...(i)

and in ΔACD, $\quad \tan\alpha = \frac{h}{32 + BC}$

$\Rightarrow \quad h = \left(32 + \frac{2h}{5}\right)\frac{5}{3}$ [using Eq. (i)]

$\Rightarrow \quad 3h = 160 + 2h$

$\Rightarrow \quad h = 160$ m

47. In ΔABD, $\quad \tan\alpha = \frac{H}{d}$

$\Rightarrow \quad d = H\cot\alpha$ ...(i)

In ΔECD, $\quad \tan(\alpha - \beta) = \frac{H - h}{d}$

$\Rightarrow \quad \tan(\alpha - \beta) = \frac{H - h}{H\cot\alpha}$ [from Eq. (i)]

$\Rightarrow \quad H[1 - \cot\alpha\tan(\alpha - \beta)] = h$

$\Rightarrow \quad H = \frac{h\cot(\alpha - \beta)}{\cot(\alpha - \beta) - \cot\alpha}$

48. Let AD be the building of height h and BP be the hill. Then,

$\tan q = \frac{h + x}{y}$ ...(i)

and $\quad \tan p = \frac{x}{y}$

$\Rightarrow \quad y = x\cot p$ ...(ii)

From Eqs. (i) and (ii), we get

$$\tan q = \frac{h + x}{x\cot p}$$

$\Rightarrow \quad x\cot p = (h + x)\cot q$

$\Rightarrow \quad x = \frac{h\cot q}{\cot p - \cot q}$

$\Rightarrow \quad h + x = \dfrac{h \cot q}{\cot p - \cot q} + h$

$\therefore \quad$ Height of hill $= \dfrac{h \cot p}{\cot p - \cot q}$

49. In $\Delta\, DEC$, $\quad \tan\theta = \dfrac{DE}{CE}$

$\Rightarrow \quad \tan\theta = \dfrac{h-a}{CE}$

$\Rightarrow \quad CE = \dfrac{h-a}{\tan\theta}$ ...(i)

In ΔABC,

$\tan 2\theta = \dfrac{a}{AB} = \dfrac{a}{CE}$ $\quad (\because CE = AB)$

$\Rightarrow \quad \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{a}{\dfrac{h-a}{\tan\theta}}$ [from Eq. (i)]

$\Rightarrow \quad a\tan\theta = (h-a) \times \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \quad a\tan\theta(1-\tan^2\theta) - 2(h-a)\tan\theta = 0$

$\Rightarrow \quad \tan\theta(a - a\tan^2\theta - 2h + 2a) = 0$

$\because \quad \tan\theta \neq 0$

$\therefore \quad \tan^2\theta = \dfrac{-2h+3a}{a}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{\dfrac{-2h+3a}{a}}$

$= \tan^{-1}\sqrt{3 - \dfrac{2h}{a}}$

50. In ΔPMC, $\quad \tan\alpha = \dfrac{h-a}{PM}$

$\Rightarrow \quad PM = (h-a)\cot\alpha$ ...(i)

In $\Delta PMC'$, $\quad \tan\beta = \dfrac{h+a}{PM}$

$\Rightarrow \quad h + a = PM\tan\beta$

$\Rightarrow \quad h = (h-a)\cot\alpha\tan\beta - a$

$\Rightarrow \quad h(1-\cot\alpha\tan\beta) = -a(1+\cot\alpha\tan\beta)$

$\Rightarrow \quad h = \dfrac{a(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}{\sin\beta\cos\alpha - \sin\alpha\cos\beta}$

$= \dfrac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$ m

51. In ΔDBC, $\quad \tan\alpha = \dfrac{5}{x}$ ...(i)

and in ΔDAC,

$\tan 2\alpha = \dfrac{30}{x}$

$\Rightarrow \quad \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{30}{x}$

$\Rightarrow \quad \dfrac{2\left(\dfrac{5}{x}\right)}{1-\dfrac{25}{x^2}} = \dfrac{30}{x}$

$\Rightarrow \quad \dfrac{10x}{x^2-25} = \dfrac{30}{x}$

$\Rightarrow \quad 10x^2 = 30x^2 - 750$

$\Rightarrow \quad 20x^2 = 750$

$\Rightarrow \quad x^2 = \dfrac{75}{2}$

$\Rightarrow \quad x = 5\sqrt{\dfrac{3}{2}}$ m

52. Given that,

$\tan\theta = \dfrac{4}{3}$ and $\tan\phi = \dfrac{5}{2}$ ...(i)

In ΔABE, $\quad \tan\phi = \dfrac{150}{d}$

$\Rightarrow \quad d = 150 \cot \phi$

$= 150 \times \frac{2}{5} = 60$ m ...(ii)

In ΔDCE, $\tan\theta = \frac{h}{d}$

$\Rightarrow \quad \frac{4}{3} = \frac{h}{d}$ [from Eq. (i)]

$\Rightarrow \quad h = \frac{4}{3}(60)$ [from Eq. (ii)]

$\Rightarrow \quad h = 80$ m

Now, in ΔDCE, $DE^2 = DC^2 + CE^2$

$\Rightarrow \quad x^2 = 60^2 + 80^2 = 10000$

$\Rightarrow \quad x = 100$ m

53. In ΔABD, $\tan\beta = \frac{60}{d}$

$\Rightarrow \quad d = 60 \cot\beta$ ...(i)

In ΔDEC, $\tan\alpha = \frac{DC}{EC}$

$\Rightarrow \quad DC = d \tan\alpha$

$\Rightarrow \quad 60 - h = d \tan\alpha$ $(\because BC = EA = h)$

$\Rightarrow \quad 60 - h = 60 \cot\beta \tan\alpha$ [from Eq. (i)]

$\Rightarrow \quad h = 60\left(1 - \frac{\cos\beta}{\sin\beta} \cdot \frac{\sin\alpha}{\cos\alpha}\right)$

$\Rightarrow \quad h = \frac{60 \sin(\beta - \alpha)}{\cos\alpha \sin\beta}$

$\Rightarrow \quad \frac{60 \sin(\beta - \alpha)}{x} = \frac{60 \sin(\beta - \alpha)}{\cos\alpha \sin\beta}$ (given)

$\Rightarrow \quad x = \cos\alpha \sin\beta$

54. Let OP be the tower. Since, the tower make equal angles at the vertices of the triangle, therefore foot of tower is at the circumcentre.

In ΔOAP, $\tan\alpha = \frac{OP}{OA}$

$\Rightarrow \quad OP = OA \tan\alpha$

$\Rightarrow \quad OP = R \tan\alpha$ $(\because OA = R$, given)

55. Since, $\angle QPC = \alpha$

$\therefore \quad \angle QPB = \angle BPC = \frac{\alpha}{2}$

In ΔPQB, $\sin\frac{\alpha}{2} = \frac{r}{l}$

$\Rightarrow \quad l = r \sec\frac{\alpha}{2}$ ...(i)

and in ΔPOB, $\sin\beta = \frac{h}{l}$

$\Rightarrow \quad h = l \sin\beta$

$\Rightarrow \quad h = r \operatorname{cosec}\frac{\alpha}{2} \sin\beta$ [from Eq. (i)]

56. $\because \quad \frac{a_1}{b_1} = \tan\left(\frac{\alpha + \beta}{2}\right)$ ...(i)

and $\frac{a_2}{b_2} = \tan\left(\frac{\beta + \gamma}{2}\right)$ ...(ii)

Since, $a_1 a_2 = b_1 b_2$

$\Rightarrow \quad \frac{a_1}{b_1} = \frac{b_2}{a_2}$

$\Rightarrow \quad \tan\left(\frac{\alpha + \beta}{2}\right) = \frac{1}{\tan\left(\frac{\beta + \gamma}{2}\right)}$

$\Rightarrow \quad \tan\left(\frac{\alpha + \beta}{2}\right) \tan\left(\frac{\beta + \gamma}{2}\right) = 1$

$\therefore \quad \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \alpha + \beta + \gamma = \pi - \beta < \pi$

57. Let DP is clock tower standing at the middle point D of BC.

Let $\angle PAD = \alpha = \cot^{-1} 3.2 \Rightarrow \cot\alpha = 3.2$

and $\angle PBD = \beta = \text{cosec}^{-1} 2.6 \Rightarrow \text{cosec}\,\beta = 2.6$

$\therefore \quad \cot\beta = \sqrt{(\text{cosec}^2\beta - 1)} = \sqrt{5.76} = 2.4$

In ΔPAD and ΔPBD,

$AD = h\cot\alpha = 3.2\,h$ and $BD = h\cot\beta = 2.4\,h$

In ΔABD, $\quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad 100^2 = [(3.2)^2 + (2.4)^2]h^2 = 16h^2$

$\Rightarrow \quad h = \dfrac{100}{4} \Rightarrow h = 25$ m

58. Let OP be the tower whose height is h metres.

In ΔOAP, $\quad \tan\alpha = \dfrac{OP}{OA}$

$\Rightarrow \quad OA = h\cot\alpha \quad$ ...(i)

In ΔOBP, $\quad \tan\beta = \dfrac{OP}{OB}$

$\Rightarrow \quad OB = h\cot\beta \quad$ ...(ii)

Now, in ΔOAB, $\quad AB^2 = OA^2 + OB^2$

$\Rightarrow \quad d^2 = h^2(\cot^2\alpha + \cot^2\beta)$

[from Eqs. (i) and (ii)]

$\Rightarrow \quad h = \dfrac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$

59. Let PD be a pole.

In ΔDAP, $\quad \tan 30° = \dfrac{DP}{AD}$

$\Rightarrow \quad DP = \dfrac{a}{\sqrt{3}}$

In ΔPDB, $\quad \tan\theta = \dfrac{DP}{BD}$

$\Rightarrow \quad \tan\theta = \dfrac{a/\sqrt{3}}{\sqrt{2}a} = \dfrac{1}{\sqrt{6}}$

60. Let BC be the declivity and BA be the tower.

$\therefore$ In ΔABC, on applying sine rule

$\dfrac{BC}{\sin 75°} = \dfrac{AB}{\sin 30°}$

$\Rightarrow \quad AB = \dfrac{80\sin 30°}{\sin 75°}$

$= \dfrac{40 \times 2\sqrt{2}}{\sqrt{3}+1}$

$= 40(\sqrt{6} - \sqrt{2})$ ft

61. $\because \quad \angle CDA = \beta, \quad \angle DCB = \gamma,$

and $\quad \angle ACB = \pi - (\alpha + \beta + \gamma)$

In ΔABC, on applying sine rule, we get

$\dfrac{AB}{\sin[\pi - (\alpha + \beta + \gamma)]} = \dfrac{CA}{\sin\gamma}$

$\Rightarrow \quad CA = \dfrac{a\sin\gamma}{\sin(\alpha + \beta + \gamma)}$

Now, in ΔCAD, on applying sine rule

$\dfrac{CA}{\sin\beta} = \dfrac{CD}{\sin\alpha}$

$\Rightarrow \quad CD = \dfrac{CA \cdot \sin\alpha}{\sin\beta}$

$= \dfrac{a\sin\gamma\sin\alpha}{\sin\beta\sin(\alpha + \beta + \gamma)}$

62. In ΔABE,

$\sin\beta = \dfrac{BE}{b}$

$\Rightarrow \quad BE = h_1 = b\sin\beta$

Using sine rule, in $\Delta\, AED$

$\dfrac{\sin(\alpha - \beta)}{ED} = \dfrac{\sin(\gamma - \alpha)}{b}$

$\Rightarrow \quad ED = \dfrac{b\sin(\alpha - \beta)}{\sin(\gamma - \alpha)}$

Now, in ΔFED,

$$\sin\gamma = \frac{h_2}{ED} \Rightarrow \eta_2 = \frac{\beta\sin(\alpha-\beta)\sin\gamma}{\sin(\gamma-\alpha)}$$

$\therefore$ Total height of the hill $= CD = h_1 + h_2$

$$= b\sin\beta + \frac{b\sin(\alpha-\beta)\sin\gamma}{\sin(\gamma-\alpha)}$$

$$= \frac{b[\sin\beta\sin(\gamma-\alpha) + \sin(\alpha-\beta)\sin\gamma]}{\sin(\gamma-\alpha)} = \frac{b\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$$

63. Let h be the height of the cloud above the surface of the lake. The arrangement(s) given in the problem is/are shown in the figure.

Now, in $\Delta AMC'$, we have

$$\tan 45° = \frac{AM}{MC'}$$

$\Rightarrow$ $AM = MC'$

$\Rightarrow$ $AM = (h + 2500)$ ...(i)

Also, in ΔAMC, we have

$$\tan 15° = \frac{MC}{AM}$$

$\Rightarrow$ $MC = AM\tan 15°$

On substituting respective values, we get

$$(h - 2500) = (h + 2500)\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$$

$$\Rightarrow \frac{h-2500}{h+2500} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Using componendo and dividendo, we have

$$\frac{h}{-2500} = \frac{\sqrt{3}}{-1}$$

$\Rightarrow$ $h = 2500\sqrt{3}$

Round II

1. We have, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or $c^2 - 2bc\cos A + b^2 - a^2 = 0$

$\therefore$ $c_1 + c_2 = 2b\cos A = 2 \times 2 \times \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$

and $c_1c_2 = b^2 - a^2 = 4 - 5 = -1$

$\therefore$ $|c_1 - c_2| = \sqrt{(c_1 + c_2)^2 - 4c_1c_2}$

$= \sqrt{12 + 4} = \sqrt{16} = 4$

2. Given, $\angle A = 20°$

$\therefore$ $\angle B = \angle C = 80°$

Then, $b = c$

$\therefore$ $\dfrac{a}{\sin 20°} = \dfrac{b}{\sin 80°} = \dfrac{c}{\sin 80°}$

$\Rightarrow$ $\dfrac{a}{\sin 20°} = \dfrac{b}{\cos 10°}$

$\Rightarrow$ $a = 2b\sin 10°$ ...(i)

$\therefore$ $a^3 + b^3 = 8b^3\sin^3 10° + b^3$

$= b^3\{2(4\sin^3 10°) + 1\}$

$= b^3\{2(3\sin 10° - \sin 30°) + 1\}$

$= b^3\{6\sin 10°\} = 3b^2(2b\sin 10°)$

$= 3b^2a$ [from Eq.(i)]

$= 3ac^2$ ($\because b = c$)

3. $\because$ $A + B + C + D = 2\pi$

$\Rightarrow$ $\tan(A + B + C + D) = 0$

$\Rightarrow$ $\dfrac{\Sigma\tan A - \Sigma\tan A\tan B\tan C}{1 - \Sigma\tan A\tan B + \tan A\tan B\tan C\tan D} = 0$

$\Rightarrow$ $\Sigma\tan A - \Sigma\tan A\tan B\tan C = 0$

$\Rightarrow$ $\Sigma\tan A = \tan A\tan B\tan C\tan D\,\Sigma\cot A$

$\Rightarrow$ $\dfrac{\Sigma\tan A}{\Sigma\cot A} = \prod\tan A$

4. $(a + b + c)(b + c - a) = kbc$

$\Rightarrow$ $2s(2s - 2a) = kbc \Rightarrow \dfrac{s(s-a)}{bc} = \dfrac{k}{4}$

$\Rightarrow$ $\cos^2\left(\dfrac{A}{2}\right) = \dfrac{k}{4}$

$\because$ $0 < \cos^2\left(\dfrac{A}{2}\right) < 1$

$\therefore$ $0 < \dfrac{k}{4} < 1 \Rightarrow 0 < k < 4$

5. $r - r_2 = r_3 - r_1$

$\Rightarrow$ $\dfrac{\Delta}{s} - \dfrac{\Delta}{s-b} = \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s-a}$

$\Rightarrow$ $\dfrac{-b}{s(s-b)} = \dfrac{c-a}{(s-a)(s-c)}$

$\dfrac{(s-a)(s-c)}{s(s-b)} = \dfrac{a-c}{b} \Rightarrow \tan^2\dfrac{B}{2} = \dfrac{a-c}{b}$

But $\frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

$\Rightarrow \quad \tan^2 \frac{B}{2} \in \left(\frac{1}{3}, 1\right)$

$\Rightarrow \quad \frac{1}{3} < \frac{a-c}{b} < 1$

$\Rightarrow \quad b < 3a - 3c < 3b$

$\Rightarrow \quad b + 3c < 3a < 3b + 3c$

6. In ΔAIF and AIE,

$$\frac{IF}{\sin A/2} = AI = \frac{IE}{\sin A/2}$$

$\Rightarrow \quad AI^2 = \frac{IE \cdot IF}{\sin^2 A/2}$

$\Rightarrow \quad \frac{ID \times IE \times IF}{IA \times IB \times IC} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R} = \frac{1}{10}$

7. Let r be the radius of the circle and A_1 be its area.

Then, $A_1 = \pi r^2$

Since, the perimeter of the circle is same as the perimeter of a regular polygon of n sides, therefore $2\pi r = na$, where a is the length of one side of the regular polygon.

$\Rightarrow \quad a = \frac{2\pi r}{n}$

Let A_2 be the area of the polygon.

Then, $A_2 = \frac{1}{4}\pi a^2 \cot\left(\frac{\pi}{n}\right) = \frac{\pi^2 r^2}{n}\cot\left(\frac{\pi}{n}\right)$

$\therefore \quad A_1 : A_2 = \pi r^2 : \frac{\pi^2 r^2}{n}\cot\left(\frac{\pi}{n}\right) = \tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

8. Given, r_1, r_2 and r_3 are in HP.

$\because$ a, b and c are in AP.

$\therefore \quad a + b + c = 24 = 2s$

$\Rightarrow \quad 2b + b = 24 \Rightarrow b = 8 \Rightarrow c = 16 - a$

Also, given $\sqrt{s(s-a)(s-b)(s-c)} = 24$

$\Rightarrow \quad \sqrt{12(12-a)(12-8)(a-4)} = 24$

$\Rightarrow \quad (12-a)(a-4) = 12$

$\Rightarrow \quad a^2 - 16 + 60 = 0$

$\Rightarrow \quad a = 6, 10$

9. Let h be the height of the tower, then

$h = AQ \tan\alpha = BQ \tan\beta = CQ \tan\gamma$

$\Rightarrow \quad BC = BQ - CQ = h(\cot\beta - \cot\gamma)$

$\Rightarrow \quad CA = h(\cot\alpha - \cot\gamma)$

and $AB = h(\cot\alpha - \cot\beta)$

Now, $BC \cot\alpha - CA \cot\beta + AB \cot\gamma$

$= h[\cot\alpha(\cot\beta - \cot\gamma) - \cot\beta(\cot\alpha - \cot\gamma) + \cot\gamma(\cot\alpha - \cot\beta)]$

$= 0$

10. Let OP be the flag staff of height h standing at the centre O of the field.

In ΔOEP, $OE = h\cot 15° = h(2 + \sqrt{3})$

and in ΔOFP, $OF = h\cot 45° = h$

$\therefore \quad EF = h\sqrt{1 + (2+\sqrt{3})^2}$

$= 2h\sqrt{2+\sqrt{3}}$

Since, $AC = 1200$

$\Rightarrow \quad 2EF = 4h\sqrt{(2+\sqrt{3})}$

$\Rightarrow \quad h = \frac{300}{\sqrt{2+\sqrt{3}}}$

$= 300\sqrt{2-\sqrt{3}}$ m

11. Let the height of the vertical tower $PQ = h$, C is the middle point of line segment AB. Since, PQ is perpendicular to the plane QAB. $\therefore \angle PQA = \angle PQC = \angle PQB = 90°$, we get

$\frac{PQ}{QA} = \tan\theta \Rightarrow QA = h\cot\theta$

Similarly, $QB = h\cot\theta$ and $QC = h\cot\phi$

Since, $QA = QB$, the ΔQAB is an isosceles triangle.

Here, QCA is a right angled triangle in which $\angle QCA = 90°$.

$\therefore \quad QC^2 + AC^2 = QA^2$

$\Rightarrow \quad h^2 \cot^2 \phi + a^2 = h^2 \cot^2 \theta$

$\Rightarrow \quad h^2 (\cot^2 \theta - \cot^2 \phi) = a^2$

or $\quad h^2 = \dfrac{a^2}{\cot^2 \theta - \cot^2 \phi}$

$$\Rightarrow \quad h^2 = \frac{a^2}{(\text{cosec}^2 \theta - 1) - (\text{cosec}^2 \phi - 1)}$$

$$= \frac{a^2}{\text{cosec}^2 \theta - \text{cosec}^2 \phi}$$

$$= \frac{a^2}{\dfrac{1}{\sin^2 \theta} - \dfrac{1}{\sin^2 \phi}}$$

$$\Rightarrow \quad h^2 = \frac{a^2 \sin^2 \theta \sin^2 \phi}{\sin^2 \phi - \sin^2 \theta} = \frac{a^2 \sin^2 \theta \sin^2 \phi}{\sin(\phi - \theta) \sin(\phi - \theta)}$$

Hence, the required height h of the peak

$$= \frac{a \sin \theta \sin \phi}{\sqrt{\sin(\phi + \theta) \sin(\phi - \theta)}}$$

12. Let H be the mid-point of BC. Since, $\angle TBH = 90°$, then $TH^2 = BT^2 + BH^2 = 5^2 + 5^2 = 50$

Also since,

$\angle THG = 90°$, $TG^2 = TH^2 + GH^2$

$= 50 + 25 = 75$

Let θ be the required angle of elevation of G at T.

Then, $\quad \sin \theta = \dfrac{GH}{TG} = \dfrac{5}{5\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \sin^{-1}(1/\sqrt{3})$

13. Let PQ and RS be the poles of height 20 m and 80 m subtending angles α and β at R and P, respectively. Let h be the height of the point T, the intersection of QR and PS.

Then, $\quad PR = h \cot \alpha + h \cot \beta$

$= 20 \cot \alpha = 80 \cot \beta$

$\Rightarrow \quad \cot \alpha = 4 \cot \beta$

$\Rightarrow \quad \dfrac{\cot \alpha}{\cot \beta} = 4$

Again, $\quad h \cot \alpha + h \cot \beta = 20 \cot \alpha$

$\Rightarrow \quad (h - 20) \cot \alpha = -h \cot \beta$

$\Rightarrow \quad \dfrac{\cot \alpha}{\cot \beta} = \dfrac{h}{20 - h} = 4$

$\Rightarrow \quad h = 80 - 4h \Rightarrow h = 16$ m

14. $\quad r = \dfrac{\Delta}{s}, r_1 = \dfrac{\Delta}{s - a}$

$$r_1 r = \frac{\Delta^2}{s(s-a)} = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$$

$$= (s-b)(s-c) = (s-b)^2 \qquad (\because b = c)$$

$$= \frac{(2s - 2b)^2}{4} = \frac{(a + b + c - 2b)^2}{4} \qquad (\because b = c)$$

$$= \frac{a^2}{4} = \frac{4R^2 \sin^2 A}{4} = R^2 \sin^2 A$$

Also, if $\quad \angle B = \theta \Rightarrow \angle A = \pi - 2\theta$

$\therefore \quad r_1 r = R^2 \sin^2(\pi - 2\theta) = R^2 \sin^2 2\theta = R^2 \sin^2 2B$

15. Area of polygon,

$\Delta = n$ times the area of the ΔOBC

$$= \frac{na^2}{4} \cot \frac{\pi}{n} \qquad \text{...(i)}$$

Now, $\quad a = 2r \tan \dfrac{\pi}{n}$

$\Rightarrow \quad \Delta = nr^2 \tan\left(\dfrac{\pi}{n}\right)$ [from Eq. (i)]

Also, $\quad a = 2R \sin \dfrac{\pi}{n}$

$\Rightarrow \quad \Delta = \dfrac{nR^2}{2} \sin\left(\dfrac{2\pi}{n}\right)$ [from Eq.(i)]

Solutions (Q. Nos. 16-18)

We have,

$r_1 = \dfrac{\Delta}{s - a} = 2, r_2 = \dfrac{\Delta}{s - b} = 3, r_3 = \dfrac{\Delta}{s - c} = 6$

Given $\Delta = 6$,

$\therefore \quad s - a = 3 \qquad$ ...(i)

$s - b = 2 \qquad$ ...(ii)

$s - c = 1 \qquad$ ...(iii)

On adding Eqs. (i) and (ii), $2s - a - b = 5$

$\Rightarrow \quad a + b + c - a - b = 5 \Rightarrow c = 5$

On adding Eqs. (i) and (iii), $2s - a - c = 4 \Rightarrow b = 4$ and adding Eqs (ii) and (iii), $2s - b - c = 3$

$\therefore \quad a = 3$

Hence, the sides of the triangle are $a = 3$, $b = 4$, $c = 5$.

16. Also, $R = \frac{abc}{4\Delta} = \frac{60}{24} = 2.5$

17. $r = \frac{\Delta}{5} = \frac{6}{6} = 1$

18. Since, the triangle is right angled, greatest angle is 90°. Also, the least angle is opposite to side a, which is $\sin^{-1}\frac{3}{5}$.

$$\Rightarrow \quad 90° - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{3}{5}$$

19. $\because \; a^2 = b^2 + c^2$, then $\angle A = \frac{\pi}{2}$

$$\therefore \quad a = 2R \sin A = 2R$$

$$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{R^2} = \frac{a^2 + a^2}{R^2} = \frac{2a^2}{R^2} = \frac{2(2R)^2}{R^2} = 8$$

20. $a > b > c$

$$\Rightarrow \quad s - a > s - b > s - c$$

$$\Rightarrow \quad \frac{\Delta}{s-a} > \frac{\Delta}{s-b} > \frac{\Delta}{s-c}$$

$$\Rightarrow \quad r_1 > r_2 > r_3$$

Also, cosine is a decreasing function for $x \in [0, (\pi/2)]$

For $a > b > c \Rightarrow A > B > C \Rightarrow \cos A < \cos B < \cos C$

Hence, both statements are true, but statements II does not explain statement I.

21. $\angle ICI_1 = \frac{\pi}{2}$, $\angle IBI_1 = \frac{\pi}{2}$

$\therefore BICI_1$ is a cyclic quadrilateral.

$$\Rightarrow \quad BP \cdot PC = IP \cdot I_1P$$

22. $\therefore \quad \frac{a}{2R} = \sin\frac{\pi}{n}$ and $\frac{a}{2r} = \tan\frac{\pi}{n}$

$$\therefore \quad \frac{r}{R} = \cos\frac{\pi}{n}$$

$n = 3$ gives, $\quad \frac{r}{R} = \frac{1}{2}$

$n = 4$ gives, $\quad \frac{r}{R} = \frac{1}{\sqrt{2}}$

$n = 6$ gives, $\quad \frac{r}{R} = \frac{\sqrt{3}}{2}$

23. In ΔABC, $\quad BC = h \cot 60°$

and in ΔABD, $\quad BD = h \cot 45°$

Since, $\quad BD - BC = DC$

$$\Rightarrow \quad h \cot 45° - h \cot 60° = 7$$

$$h = \frac{7}{\cot 45° - \cot 60°} = \frac{7}{\left(1 - \frac{1}{\sqrt{3}}\right)}$$

$$= \frac{7\sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{7\sqrt{3}}{2}(\sqrt{3} + 1) \text{ m}$$

24. Let h be the height of a tower

Since, $\quad \angle AOB = 60°$

$\therefore \Delta OAB$ is an equilateral.

$$\therefore \quad OA = OB = AB = a$$

In ΔOAC, $\quad \tan 30° = \frac{h}{a}$

$$\Rightarrow \quad h = \frac{a}{\sqrt{3}}$$

11 Cartesian Coordinate System

JEE Main MILESTONE

- Cartesian System of Rectangular Axes
- Cartesian Coordinates in a Plane Point
- Distance Formula
- Area of Some Geometrical Figures
- Different Points Related to a Triangle
- Translation of Axes
- Slope of a Line
- Parallel and Perpendicular Line on the Coordinate Axes
- Intercepts of a Line on the Coordinate Axes

11.1 Cartesian System of Rectangular Axes

The position of a point in a plane is determined with reference to two intersecting straight lines called the **coordinate axes** and their point of intersection is called the **origin**. If these two axes of reference (generally, we call them x and y-axes) cut each other at right angle, then they are called rectangular axes otherwise they are called oblique axes.

The French Mathematician Rene Descartes invented the cartesian system. In a cartesian coordinates form we have a point, which shows the particular place in a segment for representation. Cartesian coordinates system consist of many coordinates which are polar, rectangular etc. In a cartesian coordinates we have collinear point coordinates, mid-point coordinates and the equidistant point coordinates.

The horizontal axes, OX and OX' are called positive and negative directions respectively of x-axis and OY and OY' are called positive and negative directions respectively of y-axis.

Quadrants

Let $X'OX$ and $Y'OY$ be the coordinate axes. We observe that the two axes divide the Euclidean plane into four regions called quadrants. The regions XOY, $X'OY$, $X'OY'$ and XOY' are respectively known as first, second, third and fourth quadrants. The ray OX is taken as positive x-axis, OX' as negative x-axis. Also, the ray OY is called positive y-axis and OY' as negative y-axis.

Sign of Coordinates in a Quadrant

Quadrant		x	y
I.	XOY	>0	>0
II.	$X'OY$	<0	>0
III.	$X'OY'$	<0	<0
IV.	XOY'	>0	<0

11.2 Cartesian Coordinates in a Plane Point

Let $X'OX$ and $Y'OY$ be two perpendicular axes in the plane paper intersecting at O and P be any point in the plane.

Draw PM perpendicular to OX. Let $OM = x$, $PM = y$. Then, the position of the point P in the plane with respect to coordinate axes is represented by the ordered pair (x, y). The order pair (x, y) is called the coordinates of point P, *i.e.*,

$OM = x = x$-coordinate or abscissa of the point P.

and $PM = y = y$-coordinate or ordinate of the point P.

Note

- The ordinate of every point on x-axis is 0.
- The abscissa of every point on y-axis is 0.
- The abscissa and ordinate of the origin O are both zero.

11.3 Distance Formula

The distance between any two points in the plane is the length of the line segment joining them.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or
$$D = \sqrt{\begin{matrix}(\text{difference of abscissa})^2 \\ + (\text{difference of ordinates})^2\end{matrix}}$$

If O is the origin and $P(x_1, y_1)$ is any point, then from the distance formula, we have

$$OP = \sqrt{(x - 0)^2 + (y - 0)^2}$$
$$= \sqrt{x^2 + y^2}$$

Note

- If the line PQ is parallel to y-axis, then $PQ = |y_2 - y_1|$.
- If the line PQ is parallel to x-axis, then $PQ = |x_2 - x_1|$.

Properties of Quadrilaterals

Rectangle

(a) Opposite sides are equal
(b) Diagonals are unequal
(c) Diagonals bisect each other

Parallelogram

(a) Opposite sides are equal
(b) Diagonals are equal
(c) Diagonals bisect each other

Rhombus

(a) All four sides are equal
(b) Diagonals are unequal
(c) Diagonals bisect each other at right angle

Square

(a) All four sides are equal
(b) Diagonals are equal
(c) Diagonals bisect each other at right angles

Inferences

(a) Every parallelogram is not a rectangle but every rectangle is a parallelogram.
(b) Even rhombus is not a square but every square is a rhombus.
(c) Every parallelogram is not a rhombus but every rhombus is a parallelogram.
(d) Every rectangle is not a square but every square is a rectangle.

***Sample* Problem 1** *Find a point on the X-axis which is equidistant from the points* (7, 6) *and* (3, 4). **[NCERT]**

(a) $\left(\frac{15}{2}, 0\right)$ (b) $\left(\frac{7}{2}, 0\right)$

(c) $\left(-\frac{15}{2}, 0\right)$ (d) *None of these*

Interpret (a) Let any point P on the X-axis is $(x, 0)$ as for a point on X-axis, y-coordinate is zero and the given points are $A(7, 6)$ and $B(3, 4)$.

Given, $PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2$

where, $x_1 = x, x_2 = 7, y_1 = 0, y_2 = 6, x_3 = 3, y_3 = 4$

$\Rightarrow \quad (x - 7)^2 + (0 - 6)^2 = (x - 3)^2 + (0 - 4)^2$

[$\because$ distance between two points $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$]

$\Rightarrow \quad x^2 + 49 - 14x + 36 = x^2 - 9 - 6x + 16$

$\Rightarrow \quad -14x + 6x = 25 - 36 - 49$

$\Rightarrow \quad -8x = 25 - 85 \Rightarrow -8x = -60$

$\Rightarrow \quad x = \frac{60}{8} \Rightarrow x = \frac{15}{2}$

$\therefore$ Point P on X-axis $= \left(\frac{15}{2}, 0\right)$

***Sample* Problem 2** *The figure formed by four points* (1, – 2), (3, 6), (5, 10) *and* (3, 2) *is a*

(a) *square* (b) *rectangle*

(c) *rhombus* (d) *parallelogram*

Interpret (d) Let $A \equiv (1, -2), B \equiv (3, 6), C \equiv (5, 10)$ and $D \equiv (3, 2)$

Then, $AB = \sqrt{(1-3)^2 + (-2-6)^2} = 2\sqrt{17}$

$BC = \sqrt{(3-5)^2 + (6-10)^2} = 2\sqrt{5}$

$CD = \sqrt{(5-3)^2 + (10-2)^2} = 2\sqrt{17}$

$AD = \sqrt{(1-3)^2 + (-2-2)^2} = 2\sqrt{5}$

$AC = \sqrt{(1-5)^2 + (-2-10)^2} = 4\sqrt{10}$

and $BD = \sqrt{(3-3)^2 + (6-2)^2} = 4$

Clearly, $AB = CD, BC = AD$ and $AC \neq BD$.

Hence, $ABCD$ is a parallelogram.

Section Formula

If a point $P(x, y)$ divides the line segment joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$, then coordinates of point P are

(i) when it divides AB internally

$$x = \frac{mx_2 + nx_1}{m + n}, y = \frac{my_2 + ny_1}{m + n}$$

(ii) when it divides AB externally

$$x = \frac{mx_2 - nx_1}{m - n}, y = \frac{my_2 - ny_1}{m - n}$$

Note

- The coordinates of the points of trisection P and Q of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$ and $\left(\frac{x_1 + 2x_2}{3}, \frac{y_1 + 2y_2}{3}\right)$, respectively.
- x-axis divides the line joining $(x_1, y_1), (x_2, y_2)$ in the ratio $y_1 : y_2$ and y-axis divides the same line segment in the ratio $x_1 : x_2$.
- The mid-point of the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Straight line $ax + by + c = 0$ divides the join of points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $-\frac{ax_1 + by_1 + c_1}{ax_2 + by_2 + c_2}$.

  If ratio is negative, then line divides externally and if ratio is positive, then lines divides internally.

***Sample* Problem 3** *The ratio in which* $y - x + 2 = 0$ *divides the line joining* (3, – 1) *and* (8, 9) *is*

(a) 2 : 3 *(externally)* (b) 2 : 3 *(internally)*

(c) 3 : 2 *(externally)* (d) 3 : 2 *(internally)*

Interpret (b) Let the required ratio be $\lambda : 1$, then point of division is $\left(\frac{8\lambda + 3}{\lambda + 1}, \frac{9\lambda - 1}{\lambda + 1}\right)$. Now, this point must satisfy the equation of the line $y - x + 2 = 0$.

$$\Rightarrow \quad \left(\frac{9\lambda - 1}{\lambda + 1}\right) - \left(\frac{8\lambda + 3}{\lambda + 1}\right) + 2 = 0 \Rightarrow \lambda = \frac{2}{3}$$

So, the required ratio is $\frac{2}{3} : 1$ *i.e.*, 2 : 3 (internally) since, λ is positive.

Alternative

$\therefore$ Required ratio $= -\left(\frac{-1 - 3 + 2}{9 - 8 + 2}\right) = \frac{2}{3} \quad \left(\because \frac{-ax_1 + by_1 + c_1}{ax_2 + by_2 + c_2}\right)$

***Sample* Problem 4** *Suppose* $A(1, 1)$ *and* $B(2, -3)$ *are two points and D is a point on AB produced such that* $AD = 3AB$. *Then, coordinates of D are*

(a) (3, –12) (b) (4, –11) (c) (11, 4) (d) (–11, 7)

Interpret (b) Given, $AD = 3AB$

$\therefore \quad BD = 2AB$

Thus, D divides AB externally in the ratio $AD : BD = 3 : 2$.

$\therefore$ The coordinates of D are $\left[\frac{3(2) - 2(1)}{3 - 2}, \frac{3(-3) - 2(1)}{3 - 2}\right]$

$$= \left(\frac{6 - 2}{1}, \frac{-9 - 2}{1}\right) = (4, -11)$$

Check Point 1

1. The ratio in which the line joining the points (2, 3) and (4, 1) divides the segment joining the points (1, 2) and (4, 3) is
   (a) 1 : 2 (b) (3 : 1)
   (c) (1 : 4) (d) 1 : 1
2. The coordinates of three vertices of a rectangle are (2, –2), (8, 4) and (5, 7), the 4th vertex is
   (a) (1,1) (b) (1, –1)
   (c) (–1, 1) (d) (–1, –1)
3. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30) are vertices of
   (a) an isosceles triangle (b) equilateral triangle
   (c) a right angled triangle (d) None of these
4. x-axis divides line segment joining points (2, – 3) and (5, 6) in the ratio
   (a) 1: 2 (b) 2 : 1 (c) 2 : – 1 (d) –1: 2
5. Let P and Q be points in the line joining A(–2, 5) and B(3, 1) such that AP = PQ = QB. Then, the mid-point of PQ is
   (a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(-\frac{1}{2}, 4\right)$
   (c) (2, 3) (d) (–1, 4)
6. If the points (1, 1), (–1, – 1) and $(-\sqrt{3}, k)$ are vertices of an equilateral triangle, then the value of k will be
   (a) 1 (b) –1 (c) $\sqrt{3}$ (d) $-\sqrt{3}$
7. The points which trisect the line segment joining the points (0, 0) and (9, 12) are
   (a) (3, 4), (6, 8) (b) (4, 3), (6, 8)
   (c) (4, 3), (8, 6) (d) (3, 4), (8, 6)
8. The points (–2, – 1), (1, 0), (4, 3) and (1, 2) are the vertices of a
   (a) parallelogram (b) rectangle
   (c) square (d) None of these

11.4 Area of Some Geometrical Figures

Area of Triangle

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are the coordinates of the vertices of ΔABC.

$\therefore$ Area of ΔABC

$$= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

By Stair Method

$$\text{Area of triangle} = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|[(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)]\right|$$

Condition of Collinearity

Three points are said to be collinear, if area of triangle is zero.

i.e., $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Area of Triangle, when Three Lines are Given

If three sides of a triangle are

$$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0 \text{ and } a_3x + b_3y + c_3 = 0,$$

then the area of triangle is given by

$$\Delta = \left|\frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}\right|$$

Area of Trapezium

If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$ are the vertices of a trapezium, then

Area of trapezium

$$= \frac{1}{2}\text{(Sum of parallel sides)} \times \text{(Distance between parallel sides)}$$

$$= \frac{1}{2}\left|[(y_2 + y_1)(x_1 - x_2) + (y_1 + y_3)(x_3 - x_1) - (y_2 + y_3)(x_3 - x_2)]\right|$$

Area of Quadrilateral

If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$ are the vertices of a quadrilateral $ABCD$, then

$$\text{Area of quadrilateral} = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1)\}\right|$$

Note

The area of polygon whose vertices are (x_1, y_1), $(x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ is given by

$$\frac{1}{2}\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{matrix}\right| = \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)\}\right|$$

***Sample* Problem 5** *The coordinates of A, B and C are $(0, 3), (-3, 5)$ and $(4, -2)$ respectively and P is any point (x, y). Then, the ratio of the area of ΔPBC and ΔABC is*

(a) $\left|\frac{x + y + 2}{7}\right|$ (b) $\left|\frac{x + y - 2}{49}\right|$

(c) $\left|\frac{x + y - 2}{7}\right|$ (d) *None of these*

Interpret (c) We have,

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\left|[x(5 + 2) - 3(-2 - y) + 4(y - 5)]\right|}{\frac{1}{2}\left|[6(5 + 2) - 3(-2 - 3) + 4(3 - 5)]\right|}$$

$$= \frac{|7x + 7y - 14|}{|42 + 15 - 8|}$$

$$= \frac{7|x + y - 2|}{|49|} = \left|\frac{x + y - 2}{7}\right|$$

***Sample* Problem 6** *The area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7, -3)$, $D(12, 2)$ and $E(0, -3)$, is*

(a) $\frac{137}{2}$ *sq units* (b) $\frac{137}{3}$ *sq units*

(c) $\frac{137}{4}$ *sq units* (d) *None of these*

Interpret (a) The required area $= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_1 & y_1 \end{matrix}\right\| = \frac{1}{2}\left\|\begin{matrix} 1 & 1 \\ 7 & 21 \\ 7 & -3 \\ 12 & 2 \\ 0 & -3 \\ 1 & 1 \end{matrix}\right\|$

$$= \frac{1}{2}\left|[(21 - 21 + 14 - 36 + 0) - (7 + 147 - 36 + 0 - 3)]\right|$$

$$= \frac{137}{2} \text{ sq units}$$

***Sample* Problem 7** *If points $(k, 2 - 2k), (-k + 1, 2k), (-4 - k, 6 - 2k)$ are collinear, then k is equal to*

(a) 2, 3 (b) 1, 0

(c) $\frac{1}{2}, -1$ (d) 1, 2

Interpret (c) Since, points are collinear

$$\therefore \quad \Delta = \frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right| = 0$$

$$\therefore \quad \frac{1}{2}[k\{2k - (6 - 2k)\} + (1 - k)\{6 - 2k - (2 - 2k)\} + (-4 - k)\{2 - 2k - 2k\}] = 0$$

$$\Rightarrow \quad k(4k - 6) + (1 - k)4 + (k + 4)(4k - 2) = 0$$

$$\Rightarrow \quad 8k^2 + 4k - 4 = 0$$

$$\Rightarrow \quad 2k^2 + k - 1 = 0$$

$$\Rightarrow \quad (k + 1)(2k - 1) = 0$$

$$\Rightarrow \quad k = -1, \frac{1}{2}$$

11.5 Different Points Related to a Triangle

Centroid of a Triangle

The point of intersection of the medians of a triangle is called the centroid of the triangle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC, then coordinates of centroid are given by $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Incentre of a Triangle

The point of intersection of internal angle bisectors of triangle is called the incentre of the triangle. If we draw a perpendicular on each side of a triangle through the incentre I, then they all will be of equal lengths, these are called in radius of inscribed circle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of ΔABC, then coordinates of incentre of triangle are given by $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$

where a, b and c be the sides of the triangle.

Circumcentre of a Triangle

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. The distance between the circumcentre and any vertex of triangle is called the **circumradius** of the circumscribed circle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of ΔABC, then the coordinates of circumcentre of triangle is given by using $OB = OA = OC$

or $C\left(\frac{ax_1 \cos A + bx_2 \cos B + cx_3 \cos C}{a \cos A + b \cos B + c \cos C}, \frac{ay_1 \cos A + by_2 \cos B + cy_3 \cos C}{a \cos A + b \cos B + c \cos C}\right)$

or $C\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

where a, b and c be the sides of a triangle.

Orthocentre of a Triangle

The orthocentre of a triangle is the point of intersection of altitudes (*i.e.,* the lines through the vertices and perpendicular to opposite sides. The coordinates of orthocentre of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given as

$O\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

If equation of lines are given, then firstly determine the two altitudes of a triangle

Equation for an AP

$$\frac{a_1x + b_1y + c_1}{a_1a_3 + b_1b_3} = \frac{a_2x + b_2y + c_2}{a_2a_3 + b_2b_3}$$

and for a BM

$$\frac{a_1x + b_1y + c_1}{a_1a_2 + b_1b_2} = \frac{a_3x + b_3y + c_3}{a_2a_3 + b_2b_3}$$

After solving AP and BM, we get the orthocentre.

Note

- The centroid of a triangle divides median in the ratio 2 : 1.
- For the right angled triangle, the coordinates of vertex containing the right angle are actually the coordinates of orthocentre.
- The orthocentre (O), centroid (G) and circumcentre (C), of any triangle lie in a same straight line and G divides the join of O and C in the ratio of 2 : 1.
- The circumcentre of a right angled triangle is the mid-point of hypotenuse.
- In an equilateral triangle orthocentre, centroid, circumcentre, incentre coincide.

Excentre of a Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. Let a, b, c be the lengths of the sides BC, CA and AB or a, b and c be the lengths of the sides opposite to $\angle A$, $\angle B$ and $\angle C$ respectively, then the circle which touches the side BC and two sides AB and AC produced is called the escribed circle opposite to the $\angle A$. The bisector of the external angles B and C meet at E_1 (which is the centre of the escribed circle) such that

$$E_1\left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

Similarly, $E_2\left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$

and $E_3\left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$

Sample **Problem 8** *The coordinates of the circumcentre of triangle whose vertices are $A(5, -1)$, $B(-1, 5)$ and $C(6, 6)$ are*

(a) $\left(\frac{23}{8}, -\frac{23}{8}\right)$ (b) $\left(\frac{23}{8}, \frac{23}{8}\right)$

(c) $(5, 6)$ (d) *None of these*

Interpret (b) Let circumcentre be $O(x, y)$, then

$$(OA)^2 = (OB)^2 = (OC)^2 = (\text{Radius})^2 = R^2 \quad \ldots(i)$$

$$\Rightarrow \quad (x-5)^2 + (y+1)^2 = (x+1)^2 + (y-5)^2 = (x-6)^2 + (y-6)^2$$

Taking first two relations

$$(x-5)^2 + (y+1)^2 = (x+1)^2 + (y-5)^2$$

$$\Rightarrow \quad (x-5)^2 - (x+1)^2 = (y-5)^2 - (y+1)^2$$

$$\Rightarrow \quad (x-2)(-12) = (y-2)(-12)$$

$$\Rightarrow \quad x - 2 = y - 2$$

$$\therefore \quad x = y \quad \ldots(ii)$$

Taking last two relations

$$(x+1)^2 + (y-5)^2 = (x-6)^2 + (y-6)^2$$

$$\Rightarrow \quad (x+1)^2 + (x-5)^2 = (x-6)^2 + (x-6)^2 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad 2x^2 - 8x + 26 = 2x^2 - 24x + 72$$

$$\Rightarrow \quad 16x = 46$$

$$\Rightarrow \quad x = y = \frac{23}{8}$$

$$\therefore \quad \text{Circumcentre} = \left(\frac{23}{8}, \frac{23}{8}\right)$$

Sample **Problem 9** *If G be the centroid and I be the incentre of the triangle with vertices $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$ and $GI = \frac{25}{3}\sqrt{205}\,\lambda$, then the value of λ is*

(a) 1/25
(b) 1/5
(c) 1/125
(d) *None of the above*

Interpret (a) Coordinates of centroid are

$$G \equiv \left(\frac{-36 + 20 + 0}{3}, \frac{7 + 7 - 8}{3}\right)$$

i.e., $G \equiv \left(-\frac{16}{3}, 2\right)$

and $a = BC = \sqrt{(0-20)^2 + (-8-7)^2} = 25$

$b = AC = \sqrt{(0+36)^2 + (-8-7)^2} = 39$

and $c = BA = \sqrt{(-36-20)^2 + (7-7)^2} = 56$

Therefore, the coordinates of I are

$$\left[\frac{25 \times (-36) + 39 \times 20 + 56 \times 0}{25 + 39 + 56}, \frac{25 \times 7 + 39 \times 7 + 56 \times (-8)}{25 + 39 + 56}\right]$$

$$\therefore \quad I \equiv (-1, 0)$$

$$GI = \sqrt{\left(-1 + \frac{16}{3}\right)^2 + (0-2)^2} = \frac{1}{3}\sqrt{205}$$

But given, $GI = \frac{25}{3}\sqrt{205}\,\lambda$

$$\therefore \quad \frac{1}{3}\sqrt{205} = \frac{25}{3}\sqrt{205}\,\lambda$$

$$\Rightarrow \quad \lambda = \frac{1}{25}$$

Hot Spot

Locus

The curve described by any point which moves under given conditions is called locus of the point.

Equation of Locus

A relation $f(x, y) = 0$ between x and y which is satisfied by each point on the locus and such that each point satisfying the equation is on the locus is called the equation of the locus.

Locus of a point types of problem are directly or indirectly asked in various years in AIEEE (JEE Main) 2011, 2004, 2003.

To solve such types of problem following steps are given below.

Step I Assume (h, k) be the coordinates of any moving point say P.

Step II Mathematically or geometrically apply the given conditions on $P(h, k)$.

Step III Eliminate the variable term, if any in the expression obtained in step II with the help of given conditions.

Step IV Replace h by x and k by y. After replacement, a general equation of the locus is obtained.

Sample **Problem 10** *The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation of its locus.*

Interpret **Step I** Firstly consider the moving point.

Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given point.

Step II Applying mathematically condition,

$$\therefore \quad PA^2 + PB^2 = 2c^2 \quad \text{(given)}$$

$$\Rightarrow \quad (h-a)^2 + (k-0)^2 + (h+a)^2 + (k-0)^2 = 2c^2$$

$$\Rightarrow \quad h^2 - 2ah + a^2 + k^2 + h^2 + 2ah + a^2 + k^2 = 2c^2$$

$$\Rightarrow \quad h^2 + k^2 = c^2 - a^2$$

Step III Eliminate the variable terms. Here, there is no variable terms left.

Step IV Replace h by x and k by y.

$$\therefore \quad x^2 + y^2 = c^2 - a^2$$

Sample **Problem 11** *The equation of the locus of a point which moves so that the difference of its distances from the points $(3, 0)$ and $(-3, 0)$ is 4 units, is*

(a) *an ellipse* (b) *a parabola*
(c) *a hyperbola* (d) *a circle*

Interpret (c) Let $P(x_1, y_1)$ be the moving point whose locus is required. Here,

$PB > PA$

By hypothesis,

$$PB - PA = 4$$

$$\Rightarrow \sqrt{(x_1+3)^2 + y_1^2} - \sqrt{(x_1-3)^2 + y_1^2} = 4$$

$$\Rightarrow \sqrt{(x_1+3)^2 + y_1^2} = 4 + \sqrt{(x_1-3)^2 + y_1^2}$$

On squaring both sides, we get

$$(x_1+3)^2 + y_1^2 = 16 + (x_1-3)^2 + y_1^2 + 8\sqrt{(x_1-3)^2 + y_1^2}$$

$$\Rightarrow \quad (x_1+3)^2 - (x_1-3)^2 = 16 + 8\sqrt{(x_1-3)^2 + y_1^2}$$

$$\Rightarrow \quad 12x_1 = 16 + 8\sqrt{x_1^2 + y_1^2 - 6x_1 + 9}$$

$$\Rightarrow \quad 3x_1 = 4 + 2\sqrt{x_1^2 + y_1^2 - 6x_1 + 9}$$

$$\Rightarrow \quad (3x_1 - 4) = 2\sqrt{x_1^2 + y_1^2 - 6x_1 + 9}$$

Again, on squaring both sides, we get

$$9x_1^2 - 24x_1 + 16 = 4x_1^2 + 4y_1^2 - 24x_1 + 36$$

$$\Rightarrow \quad 5x_1^2 - 4y_1^2 = 20 \Rightarrow \frac{x_1^2}{4} - \frac{y_1^2}{5} = 1$$

Changing (x_1, y_1) by (x, y), then $\frac{x^2}{4} - \frac{y^2}{5} = 1$

which is the required locus of P.

This equation represents a hyperbola.

11.6 Translation of Axes

Shifting of Origin

Let $P(x, y)$ is the point with respect to the origin O. Now, the origin $O(0, 0)$ is shifted to a new point say $O'(h, k)$, then the coordinates of same point P with respect to the new origin O' be $P'(x', y')$ in such a way that

$$x = x' + h, \; y = y' + k$$

Hence, if the origin is shifted to (h, k) without rotation of axes, then the new equation of the curve can be obtained by putting $(x + h)$ and $(y + k)$ in place of x and y, respectively.

Rotation of Axes without Changing Origin

Let OX and OY be the original system of coordinate axes. Let OX' and OY' be the new axes are obtained by rotating the original axes through an angle θ.

Let P be a point in the plane whose coordinates are (x, y) and (X, Y) referred to old and new axes, respectively.

Now, $\quad x = OL - ML = X\cos\theta - Y\sin\theta$

and $\quad y = PQ + NL = Y\cos\theta + X\sin\theta$

If it is required to shift the new coordinates axes back, then replace X by $x\cos\theta + y\sin\theta$ and Y by $-x\sin\theta + y\cos\theta$.

Note

- These relations can also obtained from the table

	$x\downarrow$	$y\downarrow$
$X\rightarrow$	$\cos\theta$	$\sin\theta$
$Y\rightarrow$	$-\sin\theta$	$\cos\theta$

Sample Problem 12 *If the axes be turned through an angle* $\tan^{-1}2$. *What does the equation* $4xy - 3x^2 = a^2$ *become ?*

(a) $X^2 - 4Y^2 = a^2$

(b) $X^2 + 4Y^2 = a^2$

(c) $X^2 + 4Y^2 = -a^2$

(d) *None of the above*

Interpret (a) $\theta = \tan^{-1}2 \Rightarrow \tan\theta = 2$

So, $\quad \cos\theta = \frac{1}{\sqrt{5}}, \sin\theta = \frac{2}{\sqrt{5}}$

$$x = X\cos\theta - Y\sin\theta = \frac{X-2Y}{\sqrt{5}}$$

and $\quad y = X\sin\theta + Y\cos\theta = \frac{2X+Y}{\sqrt{5}}$

The equation $4xy - 3x^2 = a^2$ reduces to

$$\frac{4(X-2Y)}{\sqrt{5}}\cdot\frac{(2X+Y)}{\sqrt{5}} - 3\left(\frac{X-2Y}{\sqrt{5}}\right)^2 = a^2$$

$\Rightarrow \quad 4(2X^2 - 2Y^2 - 3XY) - 3(X^2 - 4XY + 4Y^2) = 5a^2$

$\Rightarrow \quad 5X^2 - 20Y^2 = 5a^2$

$\Rightarrow \quad X^2 - 4Y^2 = a^2$

Double Translation

(Origin Shifted and Axes Rotated)

If origin is shifted to the point $O'(h, k)$ and at the same time the directions of axes are rotated through an angle θ in the anti-clockwise sense such that new coordinates of $P(x, y)$ become (X, Y). Then,

$$x = h + X\cos\theta - Y\sin\theta$$

and $\quad y = k + X\sin\theta + Y\cos\theta$

or above can be written in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

If it is required to shift to coordinate axes back to their original positions, then replace X by

$$(x-h)\cos\theta + (y-k)\sin\theta \text{ and } Y \text{ by}$$
$$-(x-h)\sin\theta + (y-k)\cos\theta.$$

Sample Problem 13 *If the axes are shifted to the point* $(-2, -3)$ *and then they are rotated through an angle of* $45°$ *in anti-clockwise sense, then the equation* $2x^2 + 4xy - 5y^2 + 20x - 22y - 24 = 0$ *becomes*

(a) $X^2 - 14XY - 7Y^2 - 12 = 0$

(b) $X^2 - 10XY - 2Y^2 - 12 = 0$

(c) $X^2 + 14XY + 7Y^2 - 12 = 0$

(d) *None of the above*

Interpret (a) Let (X, Y) be the coordinates of a point with respect to the new axes. Then,

$$x = -2 + X\cos\frac{\pi}{4} - Y\sin\frac{\pi}{4} = -2 + \frac{X-Y}{\sqrt{2}}$$

and $\quad y = -3 + X\sin\frac{\pi}{4} + Y\cos\frac{\pi}{4} = -3 + \frac{X+Y}{\sqrt{2}}$

On substituting these value of x and y in the equation, we get

$$2x^2 + 4xy - 5y^2 + 20x - 22y - 24 = 0$$

$$2\left\{-2 + \frac{X-Y}{\sqrt{2}}\right\}^2 + 4\left\{\left(-2 + \frac{X-Y}{\sqrt{2}}\right)\left(-3 + \frac{X+Y}{\sqrt{2}}\right)\right\}$$
$$-5\left\{-3 + \frac{X+Y}{\sqrt{2}}\right\}^2 + 20\left\{-2 + \frac{X-Y}{\sqrt{2}}\right\}$$
$$-22\left\{-3 + \frac{X+Y}{\sqrt{2}}\right\} - 24 = 0$$

$\Rightarrow \quad X^2 - 14XY - 7Y^2 - 12 = 0$

Hence, the given equation reduces to

$$X^2 - 14XY - 7Y^2 - 12 = 0$$

11.7 Slope of a Line

The trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anti-clockwise sense is called the slope or gradient of the line. The slope of a line is generally denoted by m. Thus, $m = \tan\theta$.

Note The angle of inclination of the line lies between 0° and 180°, *i.e.*, $0 < \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$.

Slope of a Line in Terms of Coordinates

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line making an angle θ with the positive direction of x-axis. Then, its slope m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissa}}$$

Sample **Problem 14** *Consider the following population and year graph (in figure), find the slope of the line AB and using it, find what will be the population in the year 2010?* **[NCERT]**

(a) 104.8 *crore* (b) 104.5 *crore*
(c) 105 *crore* (d) *None of these*

Interpret (b) Let the population in the year 2010 will be k crore. Since, A, B and C are collinear.

$\therefore$ Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\Rightarrow \quad \frac{97 - 92}{1995 - 1985} = \frac{k - 97}{2010 - 1995}$$

$$(\because x_1 = 1985, x_2 = 1995, x_3 = 2010, y_1 = 92, y_2 = 97, y_3 = k)$$

$$\Rightarrow \quad \frac{5}{10} = \frac{k - 97}{15}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{k - 97}{15} \Rightarrow k - 97 = \frac{15}{2}$$

$$\Rightarrow \quad k = 97 + \frac{15}{2} = 97 + 7.5 = 104.5$$

Hence, slope of $AB = \frac{1}{2}$ and population in the year 2010 = 104.5 crore.

11.8 Parallel and Perpendicular Line on the Coordinate Axes

A line parallel to x-axis makes an angle of 0° with x-axis. Therefore, its slope is tan 0° = 0. A line parallel to y-axis *i.e.*, perpendicular to x-axis makes an angle of 90° with x-axis, so its slope is $\tan\frac{\pi}{2} = \infty$. Also, the slope of a line equally inclined with axes is 1 or −1 as it makes an angle of 45° or 135° with x-axis.

Line Parallel to x-axis

Let AB be a straight line parallel to x-axis at a distance b from it. Then, clearly the ordinate of each point on AB is b. Thus, AB can be considered at the locus of a point at a distance b from x-axis. Thus, if $P(x, y)$ is any point on AB, then $y = b$.

Line Parallel to y-axis

Let AB be a line parallel to y-axis and at a distance a from it. Then, the abscissa of every point on AB is a. So, it can be treated as the locus of a point at a distance a from y-axis. Thus, if $P(x, y)$ is any point on AB, then $x = a$.

11.9 Intercepts of a Line on the Coordinate Axes

The equation of a line which cuts off intercepts a and b respectively, from the x and y axes is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The intercept made by a line on x-axis can also be obtained by putting $y = 0$ in its equation. Similarly, y-intercept is the value of y obtained from the line when x is replaced by zero.

Sample Problem 15 *If three points* $(h, 0)$, (a, b) *and* $(0, k)$ *lie on a line, then* $\frac{a}{h} + \frac{b}{k}$ *is equal to* **[NCERT]**

(a) –1 (b) 1
(c) 2 (d) –2

Interpret (b) Given points are $A(h, 0)$, $B(a, b)$ and $C(0, k)$. If A, B and C are collinear, then

slope of AB = slope of BC = slope of CA

Let $(x_1, y_1) \to A(h, 0), (x_2, y_2) \to B(a, b), (x_3, y_3) \to C(0, k)$

Now, slope of AB = slope of BC = slope of CA

$$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_1 - y_3}{x_1 - x_3}$$

$$= \frac{b-0}{a-h} = \frac{k-b}{0-a} = \frac{0-k}{h-0}$$

Taking first two terms,

$$\Rightarrow \quad \frac{b}{a-h} = \frac{k-b}{-a}$$

$$\Rightarrow \quad -ab = (a-h)(k-b)$$

$$\Rightarrow \quad -ab = ak - ab - hk + bh$$

$$\Rightarrow \quad ak + bh = hk$$

Dividing each term by (hk),

$$\Rightarrow \quad \frac{ak}{hk} + \frac{bh}{hk} = \frac{hk}{hk} \Rightarrow \frac{a}{h} + \frac{b}{k} = 1$$

Check Point 2

1. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then the locus of centroid of triangle is
   (a) $x^2 + y^2 - 2x - 4y + 1 = 0$
   (b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
   (c) $x^2 + y^2 - 2x - 4y + 3 = 0$
   (d) None of the above
2. If the vertices of a triangle are $(am_1^2, 2am_1)$, $(am_2^2, 2am_2)$, $(am_3^2, 2am_3)$, then its area is
   (a) $a(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)$
   (b) $(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)$
   (c) $a^2(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)$
   (d) $a(m_2 + m_3)(m_3 + m_1)(m_1 + m_2)$
3. The area of the triangle with vertices at the point $(a, b + c)$, $(b, c + a)$, $(c, a + b)$ is
   (a) 0 (b) $a + b + c$
   (c) $ab + bc + ca$ (d) None of these
4. The points $(-a, -b)$, $(0, 0)$, (a, b) and (a^2, ab) are
   (a) collinear
   (b) vertices of a parallelogram
   (c) vertices of a rectangle
   (d) None of the above
5. If the vertices of a triangle be $(a, b - c)$, $(b, c - a)$ and $(c, a - b)$, then the centroid of the triangle lies
   (a) at origin (b) on x-axis
   (c) on y-axis (d) None of these
6. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is
   (a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
   (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$
7. If $A(-a, 0)$ and $B(a, 0)$ are two fixed points, then the locus of the point at which AB subtends a right angle is
   (a) $x^2 + y^2 = 2a^2$ (b) $x^2 - y^2 = a^2$
   (c) $x^2 + y^2 + a^2 = 0$ (d) $x^2 + y^2 = a^2$
8. The equation of the locus of all points equidistant from the point $(4, 2)$ and x-axis is
   (a) $x^2 + 8x + 4y - 20 = 0$
   (b) $x^2 - 8x - 4y + 20 = 0$
   (c) $y^2 - 4y - 8x + 20 = 0$
   (d) None of the above
9. By rotating the axes through 180°, the equation $x - 2y + 3 = 0$ changes to
   (a) $x + 2y - 3 = 0$ (b) $x - 2y + 3 = 0$
   (c) $x - 2y - 3 = 0$ (d) None of these
10. If the axes are rotated through an angle of 45° in the clockwise direction, the coordinates of a point in the new system are $(0, -2)$, then its original coordinates are
    (a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, \sqrt{2})$
    (c) $(\sqrt{2}, -\sqrt{2})$ (d) $(-\sqrt{2}, -\sqrt{2})$

WORKED OUT
Examples

Example 1 *The base of an equilateral triangle with side 2a lies along the Y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.* **[NCERT]**

(a) $(-\sqrt{3}a, 0), (0, a), (0, -a)$ (b) $(\sqrt{3}a, 0), (0, -a), (0, a)$

(c) $(-\sqrt{3}a, 0), (0, a), (a, 0)$ (d) *None of these*

Solution (a) Let BC be the base of a triangle which lies on Y-axis and third vertex may be $A(h, 0)$ or A'.

Since, ΔABC is an equilateral, then $AB = BC$

$$AB^2 = BC^2$$

$$\Rightarrow \quad (h-0)^2 + (0-a)^2 = (2a)^2$$

[$\because$ distance between two points (x_1, y_1) and (x_2, y_2) $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$]

For distance AB, $(x_1, y_1) = (a, 0)$, $(x_2, y_2) = (0, a)$

$\Rightarrow h^2 + a^2 = 4a^2 \Rightarrow h^2 = 3a^2 \Rightarrow h = \pm\sqrt{3}a$ (taking square root)

Hence, the vertices of triangle are $(\sqrt{3}a, 0), (0, a), (0, -a)$

or $(-\sqrt{3}a, 0), (0, a), (0, -a)$

Example 2 *The coordinates of point on the line joining the points $P(3, -4)$ and $Q(-2, 5)$ that is twice as far from P as from Q, are*

(a) (7, 14) (b) (–7, 14) (c) (14, 7) (d) (7, 15)

Solution (b) Let $A(x, y)$ be the required point,

$\therefore \quad PA = 2PQ$ (given)

Thus, A divides PQ internally or externally in the ratio 2 : 1.

If A divides PQ internally in the ratio 2 : 1, then coordinates of A are

$$x = \frac{2\times(-2) + 1\times 3}{2+1} \text{ and } y = \frac{2\times 5 + 1\times(-4)}{2+1}$$

$$\Rightarrow \quad x = -\frac{1}{3} \text{ and } y = 2$$

$\Rightarrow$ Coordinates of A are $\left(-\frac{1}{3}, 2\right)$.

Again, if A divides PQ externally in the ratio 2 : 1, then coordinates of A are,

$$x = \frac{2\times(-2) - 1\times 3}{2-1} \text{ and } y = \frac{2\times 5 - 1\times(-4)}{2-1}$$

$$\Rightarrow \quad x = -7 \text{ and } y = 14$$

$\therefore$ Coordinates of A are $(-7, 14)$.

Example 3 *Find the area of the quadrilateral, whose vertices are (–4, 5), (0, 7), (5, – 5) and (–4, – 2).* **[NCERT]**

(a) $\frac{121}{2}$ *sq units* (b) $\frac{111}{2}$ *sq units*

(c) $\frac{105}{2}$ *sq units* (d) *None of these*

Solution (a) Given vertices $A(-4, 5)$, $B(0, 7)$, $C(5, -5)$ and $D(-4, -2)$ are shown below in the X-Y plane. Joining all the vertices a quadrilateral $ABCD$ is formed.

Area of quadrilateral $ABCD$ = Area of ΔADC + Area of ΔABC ...(i)

Now, area of

$$\Delta ADC = \frac{1}{2}\left|[-4(-2+5) - 4(-5-5) + 5(5+2)]\right|$$

$$\left[\because \text{area} = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right]$$

Here, $(x_1, y_1) = (-4, 5), (x_2, y_2) = (-4, -2), (x_3, y_3) = (5, -5)$

$$= \frac{1}{2}[-4\times 3 - 4(-10) + 5\times 7]$$

$$= \frac{1}{2}[-12 - 40 + 35] = \frac{1}{2}\left|[75 - 12]\right| = \frac{63}{2}$$

Area of $\Delta ABC = \frac{1}{2}[-4(7-5) + 0(-5-5) + 5(5-7)]$

[here $(x_1, y_1) = (-4, 5), (x_2, y_2) = (0, 7)$ and $(x_3, y_3) = (5, -5)$]

$$= \frac{1}{2}\left|[-4\times 12 + 0\times(-10) + 5(-2)]\right| = \frac{1}{2}|-48 - 10| = \frac{|-58|}{2} = 29$$

$\therefore$ From Eq. (i), area of quadrilateral $ABCD$

$$= \frac{63}{2} + 29 = \frac{63 + 58}{2} = \frac{121}{2} \text{ sq units}$$

Example 4 *The points (3, 3), (h, 0) and (0, k) are collinear, if $\frac{1}{h} + \frac{1}{k}$ is equal to*

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 1 (d) 2

Solution (b) Given that, $P \equiv (3, 3)$, $Q \equiv (h, 0)$ and $R \equiv (0, k)$. Now, these points are collinear, if area of $\Delta PQR = 0$

$$\Rightarrow \quad \frac{1}{2}[3(0-k)+h(k-3)+0(3-0)]=0$$

$$\Rightarrow \quad \frac{1}{2}(-3k+hk-3h)=0$$

$$\Rightarrow \quad 3k+3h=hk$$

$$\Rightarrow \quad \frac{1}{h}+\frac{1}{k}=\frac{1}{3}$$

Example 5 *The coordinates of circumcentre of the triangle whose vertices are* $(-2,-3)$, $(-1,0)$, $(7,-6)$, *are*

(a) $(3, 3)$ (b) $(3,-3)$
(c) $(-3,3)$ (d) *None of these*

Solution (b) Let the vertices of triangle are $P\equiv(-2,-3)$, $Q\equiv(-1,0)$ and $R\equiv(7,-6)$. Let $A(x,y)$ be the circumcentre of ΔPQR.

$$\therefore \quad AP^2=AQ^2$$

$$\Rightarrow \quad (x+2)^2+(y+3)^2=(x+1)^2+y^2$$

$$\Rightarrow \quad 4x+6y+13=2x+1$$

$$\Rightarrow \quad 2x+6y=-12$$

$$\Rightarrow \quad x+3y=-6 \qquad \text{...(i)}$$

Similarly, $AP^2=AR^2$

$$\Rightarrow \quad (x+2)^2+(y+3)^2=(x-7)^2+(y+6)^2$$

$$\Rightarrow \quad 4x+6y+13=-14x+12y+85$$

$$\Rightarrow \quad 18x-6y=72 \Rightarrow 3x-y=12 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(x,y)\equiv(3,-3)$$

Hence, circumcentre is $A(3,-3)$.

Example 6 *If coordinates of vertices of triangle are* $(3,-5)$ *and* $(-7,4)$ *and its centroid is* $(2,-1)$. *The coordinates of third vertex are*

(a) $(10,2)$ (b) $(-10,2)$ (c) $(-10,-2)$ (d) $(10,-2)$

Solution (d) Let (x,y) be the coordinates of third vertex, then

$$\frac{x+3-7}{3}=2 \quad \text{and} \quad \frac{y-5+4}{3}=-1$$

$$\Rightarrow \quad x-4=6 \quad \text{and} \quad y-1=-3$$

$$\Rightarrow \quad x=10 \quad \text{and} \quad y=-2$$

Thus, the coordinates of the third vertex are $(10,-2)$.

Example 7 *The equation of the locus of a point equidistant from the points* $A(1,3)$ *and* $B(-2,1)$ *is*

(a) $6x+4y=0$ (b) $6x+y=5$
(c) $x+4y=5$ (d) $6x+4y=5$

Solution (d) Let $P(h,k)$ be any point on the locus, then given

$$PA^2=PB^2$$

$$\Rightarrow \quad (h-1)^2+(k-3)^2=(h+2)^2+(k-1)^2$$

$$\Rightarrow \quad 6h+4k=5$$

Hence, locus of (h,k) is $6x+4y=5$.

Example 8 *The origin is shifted to* $(1,2)$. *The equation* $y^2-8x-4y+12=0$ *changes to* $y^2=4ax$, *then* a *is equal to*

(a) 1 (b) 2 (c) -2 (d) -1

Solution (b) Let $P(x,y)$ be the original position of the point w.r.t. the original axes. Let us move the origin at new position to (h,k).

Hence, the position of the same point P in the new system is

$$x'=x-h$$

$$y'=y-k$$

Here, $(h,k)=(1,2)$

$$\therefore \quad x'=(x-1), \quad y'=(y-2)$$

According to question,

$$y^2-8x-4y+12\equiv(y-2)^2-4a(x-1)$$

$$\Rightarrow \quad y^2-8x-4y+12\equiv y^2-4y+4-4ax+4a$$

On comparing respective coefficients, we get

$$4a=8\Rightarrow a=2$$

Example 9 *If* $A(-a,0)$ *and* $B(a,0)$ *are two fixed points, then the locus of the point at which* AB *subtends a right angle is*

(a) $x^2+y^2=2a^2$ (b) $x^2-y^2=a^2$
(c) $x^2+y^2+a^2=0$ (d) $x^2+y^2=a^2$

Solution (d) Let $P(h,k)$ represents all those points subtending a right angle at A and B.

$$\therefore \quad m_{AP}\cdot m_{PB}=-1$$

$$\Rightarrow \quad \left(\frac{k-0}{h+a}\right)\left(\frac{k-0}{h-a}\right)=-1$$

$$\Rightarrow \quad k^2=-(h^2-a^2)$$

$$\Rightarrow \quad k^2+h^2=a^2$$

Replace k by y and h by x, we get

$$x^2+y^2=a^2$$

Example 10 *The centroid of the triangle is* $(3,3)$ *and the orthocentre is* $(-3,5)$, *then its circumcentre is*

(a) $(0,4)$ (b) $(0,8)$
(c) $(6,2)$ (d) $(6,-2)$

Solution (c) Since, the centroid divides the join of orthocentre and circumcentre in the ratio 2 : 1.

Let the circumcentre of triangle is (α,β).

$$\therefore \quad G(3,3)=G\left(\frac{2\alpha-3}{2+1},\frac{2\beta+5}{2+1}\right)$$

$$\Rightarrow \quad G(3,3)=G\left(\frac{2\alpha-3}{3},\frac{2\beta+5}{3}\right)$$

$$\Rightarrow \quad \frac{2\alpha-3}{3}=3 \text{ and } \frac{2\beta+5}{3}=3$$

$$\Rightarrow \quad 2\alpha-3=9 \text{ and } 2\beta+5=9$$

$$\Rightarrow \quad 2\alpha=12 \quad \text{and} \quad 2\beta=4$$

$$\Rightarrow \quad \alpha=6 \quad \text{and} \quad \beta=2$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

System of Coordinates and Distance between Points

1. The distance of the point $(a\cos\alpha, a\sin\alpha)$ from the origin is independent of
(a) α (b) a
(c) a, α (d) None of these

2. If the point (x, y) be equidistant from the points $(6, -1)$ and $(2, 3)$, then $x - y$ is equal to
(a) 3 (b) -2
(c) 4 (d) 8

3. The distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$, where $a > 0$ is
(a) $2a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ (b) $a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$
(c) $2\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ (d) None of these

4. The points P is equidistant from A (1, 3), $B(-3, 5)$ and $C(5, -1)$, then PA is equal to
(a) 5 (b) $5\sqrt{5}$ (c) 25 (d) $5\sqrt{10}$

5. The line joining $A(b\cos\alpha, b\sin\alpha)$ and $B(a\cos\beta, a\sin\beta)$ is produced to the point $M(x, y)$, so that $AM : MB = b : a$, then

$$x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) \text{ is}$$

(a) -1 (b) 0
(c) 1 (d) $a^2 + b^2$

6. P, Q, R and S are the points on the line joining the points $P(a, x)$ and $T(b, y)$ such that $PQ = QR = RS = ST$, then $\left(\frac{5a+3b}{8}, \frac{5x+3y}{8}\right)$ is the mid-point of
(a) PQ (b) QR
(c) RS (d) ST

7. Three vertices of a parallelogram taken in order are $(-1, -6)$, $(2, -5)$ and $(7, 2)$. The fourth vertex is
(a) (1, 4) (b) (4, 1)
(c) (1, 1) (d) (4, 4)

8. The coordinates of points A, B and C are (x_1, y_1), (x_2, y_2) and (x_3, y_3) and point D divides AB in the ratio $l : k$. If P divides line DC in the ratio $m : (k + l)$, coordinates of P are
(a) $\left(\frac{kx_1 + lx_2 + mx_3}{k+l+m}, \frac{ky_1 + ly_2 + my_3}{k+l+m}\right)$
(b) $\left(\frac{lx_1 + mx_2 + kx_3}{l+m+k}, \frac{ly_1 + my_2 + ky_3}{l+m+k}\right)$
(c) $\left(\frac{mx_1 + kx_2 + lx_3}{m+k+l}, \frac{my_1 + ky_2 + ly_3}{m+k+l}\right)$
(d) None of the above

9. y-axis divides the segment joining points $(-3, -4)$ and $(1, -2)$ in the ratio
(a) 1 : 3 (b) 2 : 3 (c) 3 : 1 (d) 3 : 2

10. Let AB be divided internally and externally at P and Q in the same ratio. Then, AP, AB and AQ are is
(a) AP (b) GP
(c) HP (d) None of these

11. The points whose coordinates are $x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$ divides the join of (x_1, y_1) and (x_2, y_2) in the ratio
(a) $\frac{t}{(1+t)}$ (b) $\frac{(1+t)}{t}$ (c) $\frac{t}{(1-t)}$ (d) $\frac{(1-t)}{t}$

Area of Some Geometrical Figures

12. If $A(6, -3)$, $B(-3, 5)$, $C(4, -2)$, $P(\alpha, \beta)$, then the ratio of the areas of the ΔPBC, ΔABC is
(a) $|\alpha + \beta|$ (b) $|\alpha - \beta|$
(c) $|\alpha + \beta + 2|$ (d) $|\alpha + \beta - 2|$

13. The straight lines $x = y$, $x - 2y = 3$ and $x + 2y = -3$ form a triangle, which is
(a) isosceles (b) equilateral
(c) right angled (d) None of these

14. Length of the median from B on AC, where $A(-1, 3)$, $B(1, -1)$, $C(5, 1)$ is
(a) $\sqrt{18}$ (b) $\sqrt{10}$
(c) $2\sqrt{3}$ (d) 4

15. The centre of a circle which passes through points (1, 1), (2, 3) and (–2, 2) is
(a) (3, 4) (b) $\left(-\frac{1}{14}, -\frac{39}{14}\right)$
(c) $\left(-\frac{1}{14}, \frac{39}{14}\right)$ (d) $\left(-\frac{39}{14}, \frac{1}{14}\right)$

16. The points $A(2a, 4a)$, $B(2a, 6a)$ and $C(2a + \sqrt{3}a, 5a)$, $a > 0$ are the vertices of
(a) an isosceles triangle
(b) a right angled triangle
(c) an acute angled triangle
(d) None of the above

17. Two opposite vertices of a rectangle are (1, 3) and (5, 1). If the rest two vertices lie on the line $y - x + \lambda = 0$, then λ is equal to
(a) 1 (b) –1
(c) 2 (d) 3

18. Find the value of x for which the points $(x, -1)$, (2, 1) and (4, 5) are collinear. **[NCERT]**
(a) –1 (b) 1
(c) 2 (d) –2

19. Area of quadrilateral whose vertices are (2, 3), (3, 4), (4, 5) and (5, 6), is equal to
(a) 0 (b) 4
(c) 6 (d) None of these

20. If A and B are two points having coordinates (3, 4) and $(5, -2)$ respectively and P is a point such that $PA = PB$ and area of $\Delta PAB = 10$ sq units, then the coordinates of P are
(a) (7, 4) and (13, 2)
(b) (7, 2) and (1, 0)
(c) (2, 7) and (4, 13)
(d) None of the above

21. If area of triangle with vertices (0, 0), (0, 6) and (α, β) is 15 sq units, then
(a) $\alpha = \pm 5, \beta = 5$
(b) $\alpha = \pm 10, \beta = 5$
(c) $\alpha = \pm 5, \beta = 2$
(d) $\alpha = \pm 5$, β can take any real value

22. If the area of the triangle with vertices $(x, 0)$, (1, 1) and (0, 2) is 4 sq units, then the value of x is
(a) –2 (b) –4
(c) –6 (d) 8

23. If an equilateral triangle has one vertex at the point (0, 0) and another at $(3, \sqrt{3})$, then the coordinates of the third vertex is
(a) $(0, 2\sqrt{3})$ (b) $(0, -2\sqrt{3})$
(c) $(-1, 2\sqrt{3})$ (d) None of these

24. Let the opposite angular points of a square be (3, 4) and (1, –1). Then, the coordinates of the remaining angular points are
(a) $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$ (b) $\left(\frac{9}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$
(c) $\left(-\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$ (d) None of these

25. If the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear, then lines $a_i x + b_i y + 1 = 0$ for $i = 1, 2, 3$, are
(a) concurrent (b) identical
(c) parallel (d) None of these

26. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are
(a) vertices of an equilateral triangle
(b) vertices of a right angled triangle
(c) vertices of an isosceles triangle
(d) None of the above

27. The length of altitude through A of the ΔABC, where $A \equiv (-3, 0)$, $B \equiv (4, -1)$, $C \equiv (5, 2)$, is
(a) $\frac{2}{\sqrt{10}}$ (b) $\frac{4}{\sqrt{10}}$
(c) $\frac{11}{\sqrt{10}}$ (d) $\frac{22}{\sqrt{10}}$

28. An equilateral triangle has each side equal to a. If the coordinates of its vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the square of the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ equals to
(a) $3a^4$ (b) $\frac{3}{4}a^4$
(c) $4a^4$ (d) None of these

29. If the vertices of a triangle have integral coordinates, the triangle cannot be
(a) an equilateral triangle (b) a right angled triangle
(c) an isosceles triangle (d) None of these

Different Points Related to a Triangle

30. The x-coordinate of the incentre of the triangle where the mid-point of the sides are (0, 1), (1, 1) and (1, 0), is
(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $2 - \sqrt{2}$ (d) $1 - \sqrt{2}$

31. The orthocentre of the triangle formed by (0, 0), (8, 0) and (4, 6) is
(a) $\left(4, \frac{8}{3}\right)$ (b) (3, 4) (c) (4, 3) (d) (−3, 4)

32. If orthocentre and circumcentre of triangle are respectively (1, 1) and (3, 2), then the coordinates of its centroid are
(a) $\left(\frac{7}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{3}, \frac{7}{3}\right)$
(c) (7, 5) (d) None of these

33. The incentre of the triangle formed by lines $x = 0$, $y = 0$ and $3x + 4y = 12$, is at
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) (1, 1) (c) $\left(1, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 1\right)$

34. If $t_1 + t_2 + t_3 = -t_1t_2t_3$, then orthocentre of the triangle formed by the points $[at_1t_2, a(t_1 + t_2)], [at_2t_3, a(t_2 + t_3)]$ and $[at_3t_1, a(t_3 + t_1)]$, lies on
(a) $(a, 0)$ (b) $(-a, 0)$ (c) $(0, a)$ (d) $(0, -a)$

35. The coordinates of the circumcentre of the triangle with vertices (8, 6), (8, −2) and (2, −2) are
(a) $\left(6, \frac{2}{3}\right)$ (b) (8, 2) (c) (5, −2) (d) (5, 2)

36. If two vertices of a triangle are (−2, 3) and (5, −1). Orthocentre lies at the origin and centroid on the line $x + y = 7$, then the third vertex lies at
(a) (7, 4) (b) (8, 14)
(c) (12, 21) (d) None of these

37. If two vertices of an equilateral triangle are (0, 0) and $(3, 3\sqrt{3})$, then the third vertex is
(a) (3, −3) (b) (−3, 3)
(c) $(-3, 3\sqrt{3})$ (d) None of these

38. If the vertices P, Q, R of a ΔPQR are rational points, which of the following points of the ΔPQR is (are) always rational points ?
(a) Centroid (b) Incentre
(c) Circumcentre (d) Orthocentre

(A rational point is a point both of whose coordinates are rational numbers)

39. ABC is a triangle with vertices $A(-1, 4)$, $B(6, -2)$ and $C(-2, 4)$. D, E and F are the points which divide each AB, BC and CA respectively in the ratio 3 : 1 internally. Then, the centroid of ΔDEF is
(a) (3, 6) (b) (1, 2)
(c) (4, 8) (d) (−3, 6)

40. If (0, 1) is the orthocentre and (2, 3) is the centroid of a triangle. Then, its circumcentre is
(a) (3, 2) (b) (1, 0)
(c) (4, 3) (d) (3, 4)

41. The centroid of a triangle is (2, 7) and two of its vertices are (4, 8) and (−2, 6). The third vertex is
(a) (0, 0) (b) (4, 7) (c) (7, 4) (d) (7, 7)

Locus of a Point

42. The locus of a point P which moves such that $2PA = 3PB$, where coordinates of points A and B are (0, 0) and (4, −3), is
(a) $5x^2 - 5y^2 - 72x + 54y + 225 = 0$
(b) $5x^2 + 5y^2 - 72x + 54y + 225 = 0$
(c) $5x^2 + 5y^2 + 72x - 54y + 225 = 0$
(d) $5x^2 + 5y^2 - 72x - 54y - 225 = 0$

43. The locus of a point whose difference of distance from points (3, 0) and (−3, 0) is 4, is
(a) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (b) $\frac{x^2}{5} - \frac{y^2}{4} = 1$
(c) $\frac{x^2}{2} - \frac{y^2}{3} = 1$ (d) $\frac{x^2}{3} - \frac{y^2}{2} = 1$

44. What is the equation of the locus of a point which moves such that 4 times its distance from the x-axis is the square of its distance from the origin ?
(a) $x^2 + y^2 - 4y = 0$ (b) $x^2 + y^2 - 4|y| = 0$
(c) $x^2 + y^2 - 4x = 0$ (d) $x^2 + y^2 - 4|x| = 0$

45. A rod of length l slides with its ends on two perpendicular lines. The locus of a point which divides it in the ratio 1 : 2, is
(a) $36x^2 + 9y^2 = 4l^2$ (b) $36x^2 + 9y^2 = l^2$
(c) $9x^2 + 36y^2 = 4l^2$ (d) $9x^2 - 36y^2 = 4l^2$

46. A point moves in such a way that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is $2a$. Then, the locus of the points is
(a) $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$ (b) $\frac{x^2}{a^2} - \frac{y^2}{a^2(1 - e^2)} = 1$
(c) $\frac{x^2}{a^2(1 - e^2)} + \frac{y^2}{a^2} = 1$ (d) None of these

47. Coordinates of two points are $A(1, 0)$ and $B(-1, 0)$ and Q is a point which satisfies $AQ - BQ = \pm 1$. The locus of the point Q is
(a) $12x^2 + 4y^2 = 3$ (b) $12x^2 - 4y^2 = 3$
(c) $12x^2 - 4y^2 + 3 = 0$ (d) $12x^2 + 4y^2 + 3 = 0$

48. A point moves is such a way that the sum of squares of its distances from $A(2, 0)$ and $B(-2, 0)$ is always eaual to the square of the distance between A and B, then the locus of point P is
(a) $x^2 + y^2 - 2 = 0$ (b) $x^2 + y^2 + 2 = 0$
(c) $x^2 + y^2 + 4 = 0$ (d) $x^2 + y^2 - 4 = 0$

49. If two points $A(a, 0)$ and $B(-a, 0)$ are stationary and if $\angle A - \angle B = \theta$ in ΔABC, the locus of C is
(a) $x^2 + y^2 + 2xy \tan\theta = a^2$
(b) $x^2 - y^2 + 2xy \tan\theta = a^2$
(c) $x^2 + y^2 + 2xy \cot\theta = a^2$
(d) $x^2 - y^2 + 2xy \cot\theta = a^2$

50. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then the locus of centroid of triangle is
(a) $x^2 + y^2 - 2x - 4y + 1 = 0$
(b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 - 2x - 4y + 3 = 0$
(d) None of the above

Translation of Axes

51. The coordinate axes rotated through an angle 135°. If the coordinates of a point P in the new system are known to be $(4, -3)$, then the coordinates of P in the original system are
(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$
(c) $\left(-\frac{1}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$ (d) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

52. If a point $P(4, 3)$ is rotated through an angle 45° in anti-clockwise direction about origin, then coordinates of P in new position are
(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $\left(-\frac{7}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $\left(\frac{1}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$

53. If origin is shifted to $(7, -4)$, then point $(4, 5)$ shifted to
(a) $(-3, 9)$ (b) $(3, 9)$
(c) $(11, 1)$ (d) None of these

54. If the axes are rotated through an angle 60°, the coordinates of a point in the new system are $(2, -\sqrt{3})$, then its original coordinates are
(a) $\left(\frac{5}{3}, -\frac{\sqrt{2}}{3}\right)$ (b) $\left(-\frac{5}{3}, \frac{\sqrt{2}}{3}\right)$
(c) $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(-\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$

55. Without change of axes the origin is shifted to (h, k), then from the equation $x^2 + y^2 - 4x + 6y - 7 = 0$ the terms containing linear powers are missing. Then, point (h, k) is
(a) $(3, 2)$ (b) $(-3, 2)$
(c) $(2, -3)$ (d) $(-2, -3)$

56. A line L has intercepts a and b on the coordinate axes. Keeping the origin fixed, the axes are rotated through a fixed angle. Now, the same line has intercepts p and q on the new axes too many
(a) $a^2 + p^2 = b^2 + q^2$ (b) $a^2 + b^2 = p^2 + q^2$
(c) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} - \frac{1}{q^2}$ (d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$

Miscellaneous

57. If points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are such that x_1, x_2, x_3 and y_1, y_2, y_3 are in AP, then
(a) A, B and C are concyclic points
(b) A, B and C are collinear points
(c) A, B and C are vertices of an equilateral triangle
(d) None of the above

58. The median BE and AD of a triangle with vertices $A(0, b)$, $B(0,0)$ and $C(a, 0)$ are perpendicular to each other, if
(a) $a = \frac{b}{2}$ (b) $b = \frac{a}{2}$
(c) $ab = 1$ (d) $a = \pm\sqrt{2}b$

Round II (Mixed Bag)

Only One Correct Option

1. A straight line with negative slope passing through the point $(1, 4)$ meets the coordinate axes at A and B. The minimum value of $OA + OB$ is equal to
(a) 5 (b) 6 (c) 9 (d) 8

2. The equation
$$\sqrt{(x-2)^2 + (y-1)^2} + \sqrt{(x+2)^2 + (y-4)^2} = 5$$
represents
(a) circle (b) ellipse
(c) line segment (d) None of these

3. Let equation of the side BC of a ΔABC be $x + y + 2 = 0$. If coordinates of its orthocentre and circumcentre are $(1, 1)$ and $(2, 0)$ respectively, then radius of the circumcircle of ΔABC is
(a) 3 (b) $\sqrt{10}$
(c) $2\sqrt{2}$ (d) None of these

4. If the points $(1, 2)$ and $(3, 4)$ were to be on the same side of the line $3x - 5y + a = 0$, then
(a) $1 < a < 6$ (b) $7 < a < 11$
(c) $a > 11$ (d) $a < 7$ or $a > 11$

5. Without change of axes the origin is shifted to (h, k), then from the equation

$$x^2 + y^2 - 4x + 6y - 7 = 0$$

the terms containing linear powers are missing. The point (h, k) is

(a) (3, 2) (b) (–3, 2)
(c) (2, – 3) (d) (–2, – 3)

6. If $P = (1, 0)$, $Q = (-1, 0)$ and $R = (2, 0)$ are three given points, then the locus of the point $S(x, y)$ satisfying the relation $SQ^2 + SR^2 = 2\,SP^2$ is

(a) a straight line parallel to x-axis
(b) a circle through the origin
(c) a circle with centre at the origin
(d) a straight line parallel to y-axis

7. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$, are

(a) (2, 0) and (4, 4) (b) (2, 0) and (–4, – 4)
(c) (2, 0) and (–4, 4) (d) (–2, 0) and (4, 4)

8. Given points are $A(0, 4)$ and $B(0, -4)$, the locus of $P(x, y)$ such that $|AP - BP| = 6$, is

(a) $9x^2 - 7y^2 + 63 = 0$ (b) $9x^2 + 7y^2 - 63 = 0$
(c) $9x^2 + 7y^2 + 63 = 0$ (d) None of these

9. The orthocentre of the triangle formed by the points (0, 0), (4, 0) and (3, 4) is

(a) (2, 0) (b) $\left(\frac{3}{2}, 2\right)$
(c) $\left(\frac{3}{4}, 3\right)$ (d) $\left(3, \frac{3}{4}\right)$

10. The area of the region bounded by the lines $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is

(a) 1 (b) 2
(c) 3 (d) 4

11. The middle point of the line segment joining (3, – 1) and (1, 1) is shifted by two units (in the sense of increasing y) perpendicular to the line segment. Then, the coordinates of the point in the new position are

(a) $(2 - \sqrt{2}, 2)$ (b) $(2, 2 - \sqrt{3})$
(c) $(2 + \sqrt{2}, \sqrt{3})$ (d) None of these

12. ABC is an isosceles triangle, if the coordinates of the base are $B(1, 3)$ and $C(-2, 7)$, the coordinates of vertex A can be

(a) (1, 6) (b) $\left(-\frac{1}{2}, 5\right)$
(c) $\left(\frac{5}{6}, 6\right)$ (d) $\left(-8, \frac{1}{8}\right)$

13. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$. Then, Q is obtained from P by

(a) clockwise rotation around the origin through angle α
(b) anti-clockwise rotation around origin through angle α
(c) reflection in the line through the origin with slope $\tan\alpha$
(d) reflection in the line through the origin with slope $\tan\frac{\alpha}{2}$

14. A line joining $A(2, 0)$ and $B(3, 1)$ is rotated about A in anti-clockwise direction through 15°. Find the equation of the line in the new position. If B goes to C in the new position, then coordinates of C are

(a) $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ (b) $\left(2 - \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$
(c) $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)$ (d) None of these

15. Two points $P(a, 0)$ and $Q(-a, 0)$ are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is 2α, then

(a) locus of R is $x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$
(b) locus of R is $x^2 + y^2 + 2xy\cot\alpha - a^2 = 0$
(c) locus of R is a hyperbola, if $\alpha = \frac{\pi}{4}$
(d) locus of R is a circle, if $\alpha = \frac{\pi}{4}$

More Than One Correct Option

16. The area of a triangle is 5 sq units. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. The coordinates of the third vertex can be

(a) (–3/2, 3/2) (b) (3/4, –3/2)
(c) (7/2, 13/2) (d) (–1/4, 11/4)

17. The point $(p + 1, 1), (2p + 1, 3)$ and $(2p + 2, 2p)$ collinear, if

(a) $p = -1$ (b) $p = 1/2$
(c) $p = 2$ (d) $p = -\frac{1}{2}$

18. The coordinates of the points A and B are respectively (–3, 2) and (2, 3). P and Q are points on the line joining A and B such that $AP = PQ = QB$. A square $PQRS$ is constructed on PQ as one side, the coordinates of R can be

(a) $\left(-\frac{4}{3}, \frac{7}{3}\right)$ (b) $\left(0, \frac{13}{3}\right)$
(c) $\left(\frac{1}{3}, \frac{8}{3}\right)$ (d) $\left(\frac{2}{3}, 1\right)$

Comprehension Based Questions

Passage

Directions (Q. Nos.19-21) *Consider a ΔPQR with coordinates of its vertices as $P(-8, 5)$; $Q(-15, -19)$ and $R(1, -7)$. The bisector of the interior angle of P has the equation which can be written in the form $ax + 2y + c = 0$.*

19. The distance between the orthocentre and the circumcentre of the ΔPQR is
(a) 25/2 (b) 29/2 (c) 37/2 (d) 51/2

20. Radius of the incircle of the ΔPQR is
(a) 4 (b) 5 (c) 6 (d) 8

21. The sum of $a + c$ is
(a) 129 (b) 78
(c) 89 (d) None of these

Assertion and Reason

Directions (Q. Nos. 22-25) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

22. **Statement I** If the circumcentre of a triangle lies at the origin and centroid is the middle point of the line joining the points (2, 3) and (4, 7), then its orthocentre lies on the line $5x - 3y = 0$.

Statement II The circumcentre, centroid and the orthocentre of a triangle lies on the same line.

23. **Statement I** If the origin is shifted to the centroid of the triangle with vertices (0, 0), (3, 3) and (3, 6) without rotation of axes, then the vertices of the triangle in the new system of coordinates are $(-2, 0), (1, 3)$ and $(1, -3)$.

Statement II If the origin is shifted to the point (2, 3) without rotation of the axes, then the coordinates of the point $P(\alpha - 1, \alpha + 1)$ in the new system of coordinates are $(\alpha - 3, \alpha - 2)$.

24. **Statement I** The maximum area of triangle formed by the point (0, 0), $(a \cos\theta, b \sin\theta), (a \cos\theta, -b \sin\theta)$ is $\frac{1}{2}|ab|$.

Statement II Maximum value of $\sin\theta$ is 2.

25. **Statement I** If sum of algebraic distances from points $A(1, 1), B(2, 3), C(0, 2)$ is zero on the line $ax + by + c = 0$, then $a + 3b + c = 0$.

Statement II The centroid of triangle is (1, 2).

Previous Years' Questions

26. The x-coordinate of the incentre of the triangle that has the coordinates of mid-points of its sides as (0, 0), (1, 1) and (1, 0) is **[AIEEE 2011]**
(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$ (c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$

27. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k is equal to **[AIEEE 2012]**
(a) $\frac{29}{5}$ (b) 5 (c) 6 (d) $\frac{11}{5}$

28. If $A(2, -3)$ and $B(-2,1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is **[AIEEE 2011]**
(a) $2x - 3y = 1$ (b) $x - y = 1$
(c) $2x + 3y = 1$ (d) $2x + 3y = 3$

29. Three distinct points A, B and C given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then, the circumcentre of the ΔABC is at the point **[AIEEE 2009]**
(a) $\left(\frac{5}{4}, 0\right)$ (b) $\left(\frac{5}{2}, 0\right)$
(c) $\left(\frac{5}{3}, 0\right)$ (d) (0, 0)

30. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which k can take is given by **[AIEEE 2007]**
(a) $\{1, 3\}$ (b) $\{0, 2\}$ (c) $\{-1, 3\}$ (d) $\{-3, -2\}$

31. If a vertex of a triangle is (1, 1) and the mid-point of two sides of a triangle through this vertex are (–1, 2) and (3, 2), then the centroid of the triangle is **[AIEEE 2005]**
(a) $\left(-\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(-1, \frac{7}{3}\right)$ (c) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (d) $\left(1, \frac{7}{3}\right)$

32. Let $A(2, -3)$ and $B(-2, 1)$ be the vertices of a ΔABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line **[AIEEE 2004]**
(a) $2x + 3y = 9$ (b) $2x - 3y = 7$
(c) $3x + 2y = 5$ (d) $3x - 2y = 3$

Answers

Check Point 1

1. (d) **2.** (c) **3.** (d) **4.** (a) **5.** (a) **6.** (c) **7.** (a) **8.** (a)

Check Point 2

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (b) **6.** (d) **7.** (d) **8.** (b) **9.** (c) **10.** (d)

Round I

1. (a)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (d)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (b)
21. (d)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (d)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (c)
31. (a)	**32.** (a)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (d)
41. (b)	**42.** (b)	**43.** (a)	**44.** (b)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (d)	**49.** (d)	**50.** (b)
51. (d)	**52.** (c)	**53.** (a)	**54.** (c)	**55.** (c)	**56.** (b)	**57.** (b)	**58.** (d)		

Round II

1. (c)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (a)	**8.** (a)	**9.** (d)	**10.** (a)
11. (d)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (a)	**16.** (a,c)	**17.** (c,d)	**18.** (b,d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (a)	**23.** (d)	**24.** (c)	**25.** (d)	**26.** (b)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (c)
31. (d)	**32.** (a)								

the Guidance

Round I

1. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $O \equiv (0, 0)$,

then $|OP| = \sqrt{(a\cos\alpha - 0)^2 + (a\sin\alpha - 0)^2}$

$= \sqrt{(a^2\cos^2\alpha + a^2\sin^2\alpha)}$

$= \sqrt{a^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{a^2}$

$= |a|$, which is independent of α.

2. Let $P \equiv (x, y)$, $A \equiv (6, -1)$ and $B \equiv (2, 3)$

By the given condition, $PA = PB$

$\Rightarrow \quad \sqrt{(x-6)^2 + (y+1)^2} = \sqrt{(x-2)^2 + (y-3)^2}$

$\Rightarrow \quad (x-6)^2 + (y+1)^2 = (x-2)^2 + (y-3)^2$

$\Rightarrow \quad x^2 - 12x + 36 + y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$

$\Rightarrow \quad 8x - 8y = 24$

$\Rightarrow \quad x - y = 3$

3. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $Q \equiv (a\cos\beta, a\sin\beta)$

Then, $|PQ| = \sqrt{(a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2}$

$= \sqrt{a^2\{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\}}$

$= \sqrt{a^2\{(\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + (\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta)\}}$

$= \sqrt{a^2\{1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\}}$

$= \sqrt{a^2[2 - 2\cos(\alpha - \beta)]}$

$= \sqrt{2a^2[1 - \cos(\alpha - \beta)]} = \sqrt{\left[2a^2 \cdot 2\sin^2\left(\frac{\alpha - \beta}{2}\right)\right]}$

$= \sqrt{4a^2\sin^2\left(\frac{\alpha - \beta}{2}\right)} = \left|2a\sin\left(\frac{\alpha - \beta}{2}\right)\right|$

$= 2a\left|\sin\left(\frac{\alpha - \beta}{2}\right)\right| \quad (\because a > 0)$

4. For finding the distance AP, first we find out the perpendicular bisector of AB and AC.

Perpendicular bisector of $A(1, 3)$ and $B(-3, 5)$ is

$2x(x_1 - x_2) + 2y(y_1 - y_2) = (x_1^2 + y_1^2) - (x_2^2 + y_2^2)$

$\Rightarrow \quad 2x(1+3) + 2y(3-5) = (1+9) - (9+25)$

$\Rightarrow \quad 2x - y + 6 = 0 \quad \ldots(i)$

Similarly, perpendicular bisector of $A(1, 3)$ and $C(5, -1)$ is

$$2x(1-5)+2y(3+1)=(1+9)-(25+1)$$

$\Rightarrow$ $$-8x+8y=-16$$

$\Rightarrow$ $$x-y-2=0 \quad \text{...(ii)}$$

Point of intersection of Eqs. (i) and (ii) is

$P=(-8,-10)$

Then, the distance between P and A is

$$PA=\sqrt{(1+8)^2+(3+10)^2}=\sqrt{81+169}$$
$$=\sqrt{250}=\sqrt{25\times 10}=5\sqrt{10}$$

5. Given, M divides AB in the ratio $b : a$ (externally).

$\therefore$ $$x=\frac{ba\cos\beta-ab\cos\alpha}{b-a}$$

and $$y=\frac{ab\sin\beta-ab\sin\alpha}{b-a}$$

$\Rightarrow$ $$\frac{x}{y}=\frac{\cos\beta-\cos\alpha}{\sin\beta-\sin\alpha}$$

$\Rightarrow$ $$\frac{x}{y}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)}$$

$\Rightarrow$ $$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=0$$

6. R is mid-point of PT.

Point $\left(\frac{5a+3b}{8}, \frac{5x+3y}{8}\right)$ divides PT in ratio 3 : 5 and that is mid-point of QR.

7. Let the fourth vertex be $D(x, y)$.

We know that, two diagonals of a parallelogram are bisect each other.

$\therefore$ $$\frac{-1+7}{2}=\frac{2+x}{2} \Rightarrow x=4$$

and $$\frac{-6+2}{2}=\frac{-5+y}{2} \Rightarrow y=1$$

$\therefore$ Fourth vertex of D is (4, 1).

8. Coordinates of point D are

$$D\left(\frac{lx_2+kx_1}{k+l}, \frac{ly_2+ky_1}{k+l}\right)$$

Coordinates of point P are

$$P\left(\frac{lx_2+kx_1+mx_3}{k+l+m}, \frac{ly_2+ky_1+my_3}{k+l+m}\right)$$

9. Let y-axis cuts in the ratio $\lambda : 1$, where $x=0$

$$x=\frac{\lambda\cdot 1+1(-3)}{\lambda+1}=0$$

$\Rightarrow$ $$\lambda-3=0 \Rightarrow \lambda=3$$

y-axis cuts in the ratio 3 : 1

10. Let P and Q divides the join of AB in the ratio of $\lambda : 1$ both internally and externally respectively as shown below.

Let coordinates of points A and B are (x_1, y_1) and (x_2, y_2)

$\therefore$ $$P\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right)$$

and $$Q\left(\frac{\lambda x_2-x_1}{\lambda-1}, \frac{\lambda y_2-y_1}{\lambda-1}\right)$$

Now, $$|AP|=\sqrt{\left(x_1-\frac{\lambda x_2+x_1}{\lambda+1}\right)^2+\left(y_1-\frac{\lambda y_2+y_1}{\lambda+1}\right)^2}$$

$\Rightarrow$ $$|AP|=\sqrt{\left(\frac{\lambda x_1-\lambda x_2}{\lambda+1}\right)^2+\left(\frac{\lambda y_1-\lambda y_2}{\lambda+1}\right)^2}$$

$\Rightarrow$ $$|AP|=\left(\frac{\lambda}{\lambda+1}\right)\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Also, $$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

and $$|AQ|=\sqrt{\left(x_1-\frac{\lambda x_2-x_1}{\lambda-1}\right)^2+\left(y_1-\frac{\lambda y_2-y_1}{\lambda-1}\right)^2}$$

$\Rightarrow$ $$|AQ|=\sqrt{\left(\frac{\lambda x_1-x_1-\lambda x_2+x_1}{\lambda-1}\right)^2+\left(\frac{\lambda y_1-y_1-\lambda y_2+y_1}{\lambda-1}\right)^2}$$

$\Rightarrow$ $$|AQ|=\left(\frac{\lambda}{\lambda-1}\right)\sqrt{(x_2-x)_1^2+(y_2-y_1)^2}$$

Clearly, $|AP|, |AB|$ and $|AQ|$ are in AP.

11. Since, $x=x_1+t(x_2-x_1)$

$\Rightarrow$ $$x=x_1+t\,x_2-t\,x_1 \Rightarrow x=(1-t)\,x_1+tx_2$$

$\Rightarrow$ $$x=\frac{(1-t)x_1+tx_2}{(1-t)+t}$$

Similarly, $$y=\frac{(1-t)y_1+ty_2}{(1-t)+t}$$

$\Rightarrow$ The point (x, y) divides the join of (x_1, y_1) and (x_2, y_2) in the ratio of $t : (1-t)$

12. $\because$ Area of $\Delta PBC = \frac{1}{2}\left\| \begin{matrix} \alpha & \beta & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{matrix} \right\| = \frac{1}{2}|7\alpha + 7\beta - 14|$

Also, Area of $\Delta ABC = \frac{1}{2}\left\| \begin{matrix} 6 & -3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{matrix} \right\|$

$$= \frac{1}{2}|42 - 21 - 14| = \frac{7}{2}$$

$$\therefore \quad \frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{7}{2}|\alpha + \beta - 2|}{\frac{7}{2}}$$

$$= |\alpha + \beta - 2|$$

13. The lines of a triangle are $x = y$, $x - 2y = 3$ and $x + 2y = -3$ intersection points at sides are $A(-3, -3)$, $B(-1, -1)$ and $C\left(0, -\frac{3}{2}\right)$

$$\therefore \quad AB = \sqrt{4+4} = 2\sqrt{2}$$

$$AC = \sqrt{9 + \frac{9}{4}} = \frac{3\sqrt{5}}{2}$$

and $$BC = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

Now, $$\left(\frac{\sqrt{5}}{2}\right)^2 + (2\sqrt{2})^2 = \frac{5}{4} + 8 = \frac{37}{4}$$

$\therefore$ Hence, option (d) is correct.

14. Let BE be the median from B on AC, where E is the mid-point of AC.

According to mid-point formula, coordinates of E are $\left(\frac{5-1}{2}, \frac{1+3}{2}\right)$ *i.e.*, (2, 2).

$\therefore$ Length of median from $B = |BE|$.

$$\Rightarrow \quad |BE| = \sqrt{(2-1)^2 + (2+1)^2} = \sqrt{10}$$

15. Let $C(x, y)$ be centre of the circle passing through the points $P(1, 1)$, $Q(2, 3)$, $R(-2, 2)$

$$\therefore \quad |PC| = |QC| = |RC|$$

Case I $|PC| = |QC|$

$$(x-1)^2 + (y-1)^2 = (x-2)^2 + (y-3)^2$$

$$\Rightarrow \quad 2x + 4y = 11 \quad \text{...(i)}$$

Case II $|QC| = |RC|$

$$(x-2)^2 + (y-3)^2 = (x+2)^2 + (y-2)^2$$

$$\Rightarrow \quad 8x + 2y = 5 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii) for x and y, we have the coordinates of centre of circle are

$$C(x, y) = \left(-\frac{1}{14}, \frac{39}{14}\right)$$

16. Given vertices are $A(2a, 4a)$, $B(2a, 6a)$ and $C(2a + \sqrt{3}a, 5a)$, $a > 0$

Now, $$AB = \sqrt{(2a - 2a)^2 + (4a - 6a)^2} = 2a$$

$$BC = \sqrt{(\sqrt{3}a)^2 + a^2} = 2a$$

and $$CA = \sqrt{(\sqrt{3}a)^2 + (-a)^2} = 2a$$

$$\therefore \quad AB = BC = CA$$

Hence, triangle is an equilateral triangle, therefore it is an acute angled triangle.

17. The mid-point of (1, 3) and (5, 1) satisfies

$$y - x + \lambda = 0 \quad \text{...(i)}$$

Here, mid-point of given points, is (3, 2), which is satisfied the given Eq. (i)

$$\therefore \quad 2 - (3) + \lambda = 0 \Rightarrow \lambda = 1$$

18. Since, points are $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear.

Here, $x_1 = x, y_1 = -1, x_2 = 2, y_2 = 1, x_3 = 4$ and $y_3 = 5$

Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\Rightarrow \quad \frac{1+1}{2-x} = \frac{5-1}{4-2} \qquad \left(\because \text{slope of line} = \frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$\Rightarrow \quad \frac{2}{2-x} = \frac{4}{2}$$

$$\Rightarrow \quad 2 - x = 1 \Rightarrow x = 2 - 1 = 1$$

19. The vertices of quadrilateral $ABCD$ is A (2, 3), B (3, 4), C(4, 5) and D (5, 6).

$$\text{Area of quadrilateral} = \frac{1}{2}|[x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1] - [y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1]|$$

$$= \frac{1}{2}|[2 \times 4 + 3 \times 5 + 4 \times 6 + 5 \times 3] - [3 \times 3 + 4 \times 4 + 5 \times 5 + 6 \times 2]|$$

$$= \frac{1}{2}|[8 + 15 + 24 + 15] - [9 + 16 + 25 + 12]|$$

$$= \frac{1}{2}|[62 - (62)]| = 0$$

20. Let P is a point on the perpendicular bisector of AB, its equation is

$$(y - 1) = \frac{1}{3}(x - 4) \quad \Rightarrow \quad x - 3y - 1 = 0$$

So, general point is $P(3h + 1, h)$.

$$\text{Area of } \Delta PAB = \frac{1}{2}\left| \begin{matrix} 3h+1 & h & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{matrix} \right| = \pm 10$$

$$\Rightarrow \quad h = 2, 0$$

$\therefore$ Position of points are (7, 2) and (1, 0).

21. Area of triangle $= \frac{1}{2} \times 6 \times |\alpha| = 15$

$$\Rightarrow \quad |\alpha| = 5 \quad \Rightarrow \quad \alpha = \pm 5$$

and β can take any real value.

22. Given that, $x_1 = x, \quad x_2 = 1, \quad x_3 = 0$

and $\quad y_1 = 0, \quad y_2 = 1, \quad y_3 = 2$

$\therefore$ Area of triangle

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[x(1-2) + 1(2-0) + 0(0-1)]$$

$$= \frac{1}{2}[-x + 2 + 0] = \frac{1}{2}(2-x)$$

But area of triangle is 4 sq units

$$\therefore \quad \frac{1}{2}(2-x) = 4$$

$$\Rightarrow \quad 2 - x = 8 \quad \Rightarrow \quad x = -6$$

23. Let $O \equiv (0, 0)$ and $A \equiv (3, \sqrt{3})$ be the given points and let $B \equiv (x, y)$ be the required point. Then,

$OA = OB = AB$

$$\Rightarrow \quad (OA)^2 = (OB)^2 = (AB)^2$$

$$\Rightarrow \quad (3-0)^2 + (\sqrt{3} - 0)^2 = (x-0)^2 + (y-0)^2$$

$$= (x-3)^2 + (y - \sqrt{3})^2$$

$$\Rightarrow \quad 12 = x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$$

Taking first two members, then

$$x^2 + y^2 = 12 \quad \text{...(i)}$$

and taking last two members, then

$$6x + 2\sqrt{3}y = 12 \text{ or } y = \sqrt{3}(2 - x) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x^2 + 3(2-x)^2 = 12 \Rightarrow 4x^2 - 12x = 0$$

$$\Rightarrow \quad x = 0, 3$$

On putting $x = 0$, 3 respectively in Eq. (ii), we get $y = 2\sqrt{3}, -\sqrt{3}$

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$.

Short Cut Method

$$\left(\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

i.e., $$\left(\frac{0 + 3 \mp \sqrt{3}(\sqrt{3} - 0)}{2}, \frac{0 + \sqrt{3} \pm \sqrt{3}(3-0)}{2}\right)$$

$$\Rightarrow \quad = \left(\frac{3 \mp 3}{2}, \frac{\sqrt{3} \pm 3\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad = (0, 2\sqrt{2}) \text{ or } (3, -\sqrt{3})$$

24. Let $A(3, 4)$ and $C(1, -1)$ be the given angular points of a square $ABCD$ and let $B(x, y)$ be the unknown vertex.

Then,

$$AB = BC$$

$$\Rightarrow \quad (AB)^2 = (BC)^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$$

$$\Rightarrow \quad 4x + 10y - 23 = 0$$

$$\Rightarrow \quad x = \left(\frac{23 - 10y}{4}\right) \quad \text{...(i)}$$

Also in ΔABC,

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$\Rightarrow (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0 \quad \text{...(ii)}$$

On substituting the value of x from Eq. (i) into Eq. (ii), we get

$$\left(\frac{23 - 10y}{4}\right)^2 + y^2 - 4\left(\frac{23 - 10y}{4}\right) - 3y - 1 = 0$$

$$\Rightarrow \quad 4y^2 - 12y + 5 = 0 \text{ or } (2y - 1)(2y - 5) = 0$$

$$\therefore \quad y = \frac{1}{2} \text{ or } \frac{5}{2}$$

On putting $y = \frac{1}{2}$ in Eq. (i), we get $x = \frac{9}{2}$,

and putting $y = \frac{5}{2}$, in Eq. (i) we get $x = -\frac{1}{2}$

Hence, the required vertices of the square are $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$

25. For collinearity, $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$

For concurrency to lines $a_i x + b_i y + 1 = 0$, $i = 1, 2, 3$, we have $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$, so lines are concurrent.

26. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, then the points are collinear.

27. In ΔABC the vertices are $A(-3, 0)$, $B(4, -1)$ and $C(5, 2)$.

$\therefore \quad BC = \sqrt{(5-4)^2 + (2+1)^2}$

$= \sqrt{1+9} = \sqrt{10}$

Area of ΔABC

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}[-3(-1-2) + 4(2-0) + 5(0+1)]$

$= \frac{1}{2}[9 + 8 + 5] = 11$

As we know that, area of triangle $= \frac{1}{2} \times BC \times AL$

$\Rightarrow \quad 11 = \frac{1}{2} \times \sqrt{10} \times AL$

$\Rightarrow \quad AL = \frac{2 \times 11}{\sqrt{10}} = \frac{22}{\sqrt{10}}$

28. Since, the triangle is an equilateral triangle, therefore its area is given by

$$\Delta = \frac{\sqrt{3}}{4} a^2 \quad \text{...(i)}$$

Also, if $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of same triangle, then

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{2} a^2$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{4} a^4$$

29. Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

= A rational number if vertices have integral coordinates only.

If triangle is equilateral, then area

$= \frac{\sqrt{3}}{4}[(x_1 - x_2)^2 + (y_1 - y_2)^2] =$ irrational quantity

So, triangle cannot be equilateral.

30. Since, (0, 1), (1, 1) and (1, 0) are mid-points of sides AB, BC and CA, respectively.

$\therefore$ Coordinates of A, B and C are (0, 0), (0, 2) and (2, 0), respectively.

Now, $\quad AB = 2, \quad BC = 2\sqrt{2}, \quad CA = 2$

$\therefore$ x-coordinate of incentre

$= \frac{0 + 0 + 2 \cdot 2}{2 + 2\sqrt{2} + 2} \qquad \left(\because x = \frac{ax_1 + bx_2 + cx_3}{a + b + c}\right)$

$= \frac{2}{2 + \sqrt{2}} = 2 - \sqrt{2}$

31. Line perpendicular to OA passing through B is $x = 4$.

Slope of $AB = -\frac{3}{2}$

Line perpendicular to AB through origin is $y = \frac{2}{3}x$.

$\therefore$ The point of intersection of a line $x = 4$ and $y = \frac{2}{3}x$ is $\left(4, \frac{8}{3}\right)$.

32. As we know that orthocentre, centroid and circumcentre are collinear and the centroid divides the line segment joining orthocentre and circumcentre in the ratio 2 : 1. If the coordinates of orthocentre and circumcentre are (1, 1) and (3, 2) respectively, the coordinate of centroid is

$$\left(\frac{2 \cdot 3 + 1 \cdot 1}{2+1}, \frac{2 \cdot 2 + 1 \cdot 1}{2+1}\right) = \left(\frac{7}{3}, \frac{5}{3}\right)$$

33. Given equation of lines are

$x = 0, \quad y = 0$ and $3x + 4y = 12$

Incentre is on the line $y = x$ (Angle bisector of OA and OB)

Angle bisector of $y = 0$ and $3x + 4y = 12$ is

$\pm 5y = 3x + 4y - 12$

$\Rightarrow \quad 3x + 9y = 12$ and $3x - y = 12$

Here, $3x + 9y = 12$ internal bisector.

So, intersection point of $y = x$ and $3x + 9y = 12$ is (1, 1).

$\therefore$ The required point of the incentre of triangle is (1, 1).

34. Let $A[at_1t_2, a(t_1 + t_2)]$, $B[at_2t_3, a(t_2 + t_3)]$, $C[at_3t_1, a(t_3 + t_1)]$

Slope of $AB(m_{AB}) = \frac{a(t_3 - t_1)}{at_2(t_3 - t_1)} = \frac{1}{t_2}$

Equation of line through C perpendicular to AB is

$$y - a(t_3 + t_1) = -t_2(x - at_3t_1)$$

$$\Rightarrow \quad y - a(t_3 + t_1) = -t_2x + at_1t_2t_3 \quad \text{...(i)}$$

Similarly, equation of line through B perpendicular to CA is

$$y - a(t_2 + t_3) = -t_1(x - at_2t_3)$$

$$\Rightarrow \quad y - a(t_2 + t_3) = -t_1x + at_1t_2t_3 \quad \text{...(ii)}$$

Using $t_1t_2t_3 = -(t_1 + t_2 + t_3)$ in Eqs. (i) and (ii), we get

$$y = -t_2x - at_2 \text{ and } y = -t_1x - at_1$$

$$\Rightarrow \quad t_2(x + a) = t_1(x + a) \Rightarrow x = -a, \quad y = 0$$

35. Triangle is right angled triangle. In right angled triangle mid-point of hypotenuse is circumcentre.

So, coordinates of the circumcentre are (5, 2).

36. Let $O(0, 0)$ be the orthocentre, $A(h, k)$ be the third vertex and $B(-2, 3)$ and $C(5, -1)$ the other two vertices. Then, the slope of the line through A and O is $\frac{k}{h}$, while the line through B and C has the slope $\frac{(-1-3)}{(5+2)} = -\frac{4}{7}$. By the poperty of the orthocentre, these two lines must be perpendicular, so we have

$$\left(\frac{k}{h}\right)\left(-\frac{4}{7}\right) = -1$$

$$\Rightarrow \quad \frac{k}{h} = \frac{7}{4} \quad \text{...(i)}$$

Also, $$\frac{5-2+h}{3} + \frac{-1+3+k}{3} = 7$$

$$\Rightarrow \quad h + k = 16 \quad \text{...(ii)}$$

which is not satisfied by the points given in the options (a), (b) or (c).

37.

$\angle AOM$ is 30°. Hence, required point of B is $(-3, 3\sqrt{3})$.

38. Since, coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, the centroid is always a rational point.

39. Let $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are coordinates of the points D, E and F, which divide each AB, BC and CA respectively in the ratio 3 : 1 (internally).

$$\therefore \quad x_1 = \frac{3 \times 6 - 1 \times 1}{4} = \frac{17}{4}$$

$$y_1 = \frac{-2 \times 3 + 4 \times 1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

Similarly, $x_2 = 0$, $y_2 = \frac{5}{2}$ and $x_3 = -\frac{5}{4}$, $y_3 = 4$

Let (x, y) be the coordinates of centroid of ΔDEF

$$\therefore \quad x = \frac{1}{3}\left(\frac{17}{4} + 0 - \frac{5}{4}\right) = 1$$

and $$y = \frac{1}{3}\left(-\frac{1}{2} + \frac{5}{2} + 4\right) = 2$$

$\therefore$ Coordinates of centroid are (1, 2).

40. Given orthocentre $O(0, 1)$, centroid $G(2, 3)$. Let the circumcentre be (x, y).

Since, we know that centroid divide orthocentre and circumcentre in the ratio 2 : 1.

$$\therefore \quad G(2, 3) = G\left[\frac{2(x) + 1(0)}{2+1}, \frac{2(y) + 1(1)}{2+1}\right]$$

$$\Rightarrow \quad G(2, 3) = G\left(\frac{2x}{3}, \frac{2y+1}{3}\right)$$

$$\therefore \quad 2 = \frac{2x}{3} \text{ and } 3 = \frac{2y+1}{3}$$

$$\Rightarrow \quad x = 3 \text{ and } y = 4$$

41. Let the third vertex of the triangle be (x, y), then

$$\frac{x + 4 + (-2)}{3} = 2 \Rightarrow x = 4$$

and $$\frac{y + 8 + 6}{3} = 7 \Rightarrow y = 7$$

$\therefore$ The coordinates of the third vertex are (4, 7).

42. Let $P(h, k)$ be the required point, then

$$4PA^2 = 9PB^2$$

$$\Rightarrow \quad 4(h^2 + k^2) = 9(h - 4)^2 + 9(k + 3)^2$$

$$\Rightarrow \quad 4h^2 + 4k^2 = 9(h^2 + 16 - 8h) + 9(k^2 + 9 + 6k)$$

$$\Rightarrow \quad 5h^2 + 5k^2 - 72h + 54k + 225 = 0$$

$\therefore$ Required locus of $P(h, k)$ is

$$5x^2 + 5y^2 - 72x + 54y + 225 = 0$$

43. Let the point be $P(h, k)$.

It is given that difference of the distance from points $A(3, 0)$ and $B(-3, 0)$ is 4 *i.e.*, $PA - PB = 4$

$$\Rightarrow \quad \sqrt{(h-3)^2+k^2} - \sqrt{(h+3)^2+k^2} = 4$$

$$\Rightarrow \quad \sqrt{(h-3)^2+k^2} = 4 + \sqrt{(h+3)^2+k^2}$$

On squaring both sides, we get

$$(h-3)^2+k^2 = 16 + (h+3)^2 + k^2 + 8\sqrt{(h+3)^2+k^2}$$

$$\Rightarrow \quad h^2+9-6h+k^2 = 16+h^2+9+6h+k^2+8\sqrt{(h+3)^2+k^2}$$

$$\Rightarrow \quad -6h = 16+6h+8\sqrt{(h+3)^2+k^2}$$

$$\Rightarrow \quad -8\sqrt{(h+3)^2+k^2} = 12h+16$$

Again, squaring both sides, we get

$$64\{(h+3)^2+k^2\} = (12h+16)^2$$

$$\Rightarrow \quad 64(h^2+9+6h+k^2) = 144h^2+256+2\cdot 16\cdot 12h$$

$$\Rightarrow \quad 64(h^2+9+6h+k^2) = 16(9h^2+16+24h)$$

$$\Rightarrow \quad 4(h^2+9+6h+k^2) = 9h^2+16+24h$$

$$\Rightarrow \quad 4h^2+36+24h+4k^2 = 9h^2+16+24h$$

$$\Rightarrow \quad 5h^2-4k^2 = 20$$

$$\Rightarrow \quad \frac{h^2}{4} - \frac{k^2}{5} = 1$$

Hence, the locus of point P is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

44. Let (h, k) be the point.

According to the question,

$$4\sqrt{(h-h)^2+k^2} = h^2+k^2$$

$$\Rightarrow \quad 4|k| = h^2+k^2$$

Locus of the point is

$$4|y| = x^2+y^2 \quad \Rightarrow \quad x^2+y^2-4|y| = 0$$

45. $\therefore \quad h = \frac{1(a)+2\cdot 0}{2+1}$

and $\quad k = \frac{1\cdot 0+2\cdot b}{2+1}$

$$\Rightarrow \quad h = \frac{a}{3} \text{ and } k = \frac{2b}{3}$$

$$\Rightarrow \quad a = 3h \text{ and } b = \frac{3k}{2}$$

$$\therefore \quad 9h^2 + \frac{9k^2}{4} = l^2 \Rightarrow 36h^2+9k^2 = 4l^2$$

Hence, locus of a point is $36x^2 + 9y^2 = 4l^2$

46. Since, $A(ae, 0)$ and $B(-ae, 0)$ be the given points and let $P(h, k)$ be any point whose distance from A and B is constant *i.e.*, $2a$.

i.e., $\quad |PA| + |PB| = 2a$

$$\Rightarrow \quad \sqrt{(h-ae)^2+k^2} + \sqrt{(h+ae)^2+k^2} = 2a \quad \text{...(i)}$$

Let us assume

$$\{(h-ae)^2+k^2\} - \{(h+ae)^2+k^2\} = -4aeh \quad \text{...(ii)}$$

On dividing Eq. (ii) by Eq. (i), we have

$$\frac{\{(h-ae)^2+k^2\} - \{(h+ae)^2+k^2\}}{\sqrt{(h-ae)^2+k^2} + \sqrt{(h+ae)^2+k^2}} = \frac{4aeh}{2a}$$

$$\Rightarrow \quad \sqrt{(h-ae)^2+k^2} - \sqrt{(h+ae)^2+k^2} = -2eh \quad \text{...(iii)}$$

$$\{\because a-b = (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})\}$$

On adding Eqs. (i) and (iii), we have

$$2\sqrt{(b-ae)^2+k^2} = 2a-2eh$$

$$\Rightarrow \quad 2\sqrt{(h-ae)^2+k^2} = 2(a-eh)$$

On squaring both sides, we have

$$(h-ae)^2+k^2 = (a-eh)^2$$

$$\Rightarrow \quad h^2+a^2e^2-2aeh+k^2 = a^2+e^2h^2-2aeh$$

$$h^2-e^2h^2+k^2 = a^2-a^2e^2$$

$$\Rightarrow \quad h^2(1-e^2)+k^2 = a^2(1-e^2)$$

Replacing h by x and k by y, we get the locus of point $P(h, k)$ which is the locus of an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

47. Let $Q(x, y)$ be a point.

Such that, $\quad AQ - BQ = \pm 1$

$$\Rightarrow \quad \sqrt{(x-1)^2+y^2} - \sqrt{(x+1)^2+y^2} = \pm 1$$

Assuming $[(x-1)^2+y^2] + [(x+1)^2+y^2]$

$$-2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2} = 1$$

$$\Rightarrow \quad 2x^2+2y^2+1 = 2\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}$$

On squaring, we have

$$12x^2-4y^2 = 3$$

48. Let $P(h, k)$ be the point such that

$$|PA|^2 + |PB|^2 = |AB|^2$$

$$\Rightarrow \quad (h-2)^2+k^2+(h+2)^2+k^2 = 4^2+0$$

$$\Rightarrow \quad 2h^2+8+2k^2 = 16$$

$$\Rightarrow \quad h^2+k^2 = 4$$

$\therefore$ Locus of P is $x^2+y^2 = 4$.

49. Slope of line $AC = m_{AC} = \dfrac{k-0}{h-a} = \tan\alpha$

Slope of line BC $= m_{BC} = \dfrac{k-0}{h+a} = \tan\beta$

Since, $\angle A = \pi - \alpha$ (from figure)

$\therefore$ $\tan A = \tan(\pi - \alpha) = -\tan\alpha = -\dfrac{k}{h-a}$ ...(i)

Similarly, $\angle B = \beta$

$\therefore$ $\tan B = \tan\beta = \dfrac{k}{h+a}$ ...(ii)

Also, $A - B = \theta$ (given)

$\therefore$ $\tan(A - B) = \tan\theta$

$\Rightarrow$ $\dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \tan\theta$

$\Rightarrow$ $\dfrac{\left(\dfrac{-k}{h-a}\right) - \left(\dfrac{k}{h+a}\right)}{1 + \left(\dfrac{-k}{h-a}\right)\left(\dfrac{k}{h+a}\right)} = \tan\theta$ [using Eqs. (i) and (ii)]

$\Rightarrow$ $\tan\theta = \dfrac{-k(h+a) - k(h-a)}{(h^2 - a^2) + (-k^2)}$

$\Rightarrow$ $\tan\theta = \dfrac{-2hk}{h^2 - k^2 - a^2}$

$\Rightarrow$ $h^2 - k^2 + 2hk\cot\theta = a^2$

Replacing h by x and k by y, we have the locus of point C is

$$x^2 - y^2 + 2xy\cot\theta = a^2.$$

50. Let $G(h, k)$ be the centroid of the triangle having vertices $A(\cos\alpha, \sin\alpha), B(\sin\alpha, -\cos\alpha), C(1, 2)$

Then, $h = \dfrac{\cos\alpha + \sin\alpha + 1}{3}$

and $k = \dfrac{\sin\alpha - \cos\alpha + 2}{3}$ (by definition)

$\Rightarrow$ $3h - 1 = \cos\alpha + \sin\alpha$

and $3k - 2 = \sin\alpha - \cos\alpha$

$\Rightarrow$ $(3h-1)^2 + (3k-2)^2 = 2$ (squaring and adding)

$\Rightarrow$ $9(h^2 + k^2) - 6h - 12k + 3 = 0$

$\Rightarrow$ $3(h^2 + k^2) - 2h - 4k + 1 = 0$

$\therefore$ Locus of centroid $G(h, k)$ is

$$3(x^2 + y^2) - 2x - 4y + 1 = 0.$$

which is a circle.

51. We know, if coordinate axes are rotated, then

$$p = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

It is rotated at an angle 135° *i.e.*, $\theta = 135°$ and the new point be

$$p = [4\cos(90° + 45°) + 3\sin(90° + 45°),$$
$$4\sin(90° + 45°) - 3\cos(90° + 45°)]$$
$$= [-4\sin 45° + 3\cos 45°, 4\cos 45° + 3\sin 45°]$$
$$= \left[4\cdot\left(\frac{-1}{\sqrt{2}}\right) + 3\cdot\frac{1}{\sqrt{2}}, 4\cdot\frac{1}{\sqrt{2}} + 3\cdot\frac{1}{\sqrt{2}}\right] = \left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

52. Slope of line $OP = \dfrac{3}{4}$, let new position is $Q(x, y)$.

Slope of $OQ = \dfrac{y}{x}$ also $x^2 + y^2 = OQ^2 = 25 = (OP^2)$

$$\tan 45° = \left|\frac{\dfrac{y}{x} - \dfrac{3}{4}}{1 + \dfrac{3y}{4x}}\right| \Rightarrow \pm 1 = \frac{4y - 3x}{4x + 3y}$$

Straight lines and pair of straight lines

$$4x + 3y = 4y - 3x \text{ or } -4x - 3y = 4y - 3x$$

$$x = \frac{1}{7}y \quad \text{...(i)}$$

$$-x = 7y \quad \text{...(ii)}$$

Correct relation is $x = \dfrac{1}{7}y$ as new point must lies in Ist quadrant.

$$x^2 + 49x^2 = 25$$

$$\Rightarrow \quad x = -\frac{1}{\sqrt{2}}, \quad y = \frac{7}{\sqrt{2}}$$

53. Let $X = x - h, \quad Y = y - k$

$\Rightarrow$ $0 = 7 - h, \quad 0 = -4 - k$

$\Rightarrow$ $h = 7, \quad k = -4$

Hence, $X = x - 7$ and $Y = y + 4$, then the point (4, 5) shifted to $(-3, 9)$.

54. Let $P(x', y')$ be the coordinates of the point obtained by rotating the axes through an angle of 60°.

$\therefore$ The transformation matrix can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} \dfrac{x}{2} + \dfrac{\sqrt{3}y}{2} \\ -\dfrac{\sqrt{3}x}{2} + \dfrac{y}{2} \end{bmatrix}$$

$\Rightarrow$ $x + \sqrt{3}y = 4$ and $\sqrt{3}x - y = 2\sqrt{3}$

On solving the above equations, we have

$$(x, y) \equiv \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$$

55. Let the new coordinates of point $P(x, y)$ after shifting the origin to (h, k) is $P(x', y')$.

$\Rightarrow$ $x' = x - h$ and $y' = y - k$

$\Rightarrow$ $x = x' + h$ and $y = y' + k$

Since, (x, y) satisfy the equation, we have

$$(x' + h)^2 + (y' + k)^2 - 4(x' + h) + 6(y' + k) - 7 = 0$$

$$\Rightarrow \quad (x')^2 + (y')^2 + 2(h - 2)x' + 2(k + 3)y' + (h^2 + k^2 - 4h + 6k - 7) = 0$$

Since, the terms containing linear powers are missing

$\therefore$ $h - 2 = 0$ and $k + 3 = 0$

$\Rightarrow$ $(h, k) \equiv (2, -3)$

56. Since, the line L makes intercepts a and b along axes.

$\Rightarrow$ The line L passes through $(a, 0)$ and $(0, b)$.

Similarly, the same line passes through $(p, 0)$ and $(0, q)$ after rotation

$$\therefore \quad \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} a\cos\theta \\ -a\sin\theta \end{bmatrix}$$

$\Rightarrow$ $p = a\cos\theta$ or $-a\sin\theta = 0$ ...(i)

Also, $$\begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} -b\sin\theta \\ b\cos\theta \end{bmatrix}$$

$\Rightarrow$ $-b\sin\theta = 0$ or $b\cos\theta = q$ ...(ii)

From Eqs. (i) and (ii), we get it is clear that

$\sin\theta = 0$

$\Rightarrow$ $\theta = n\pi$

$\Rightarrow$ $p = a(\pm 1)$ or $q = b(\pm 1)$ [using Eqs. (i) and (ii)]

On squaring and adding, we have

$$a^2 + b^2 = p^2 + q^2$$

57. Since, y_1, y_2, y_3 and x_1, x_2, x_3 are in AP.

$\therefore$ $y_2 - y_1 = y_3 - y_2$ and $x_2 - x_1 = x_3 - x_2$

So, $$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

So, points are collinear.

58. The vertices of a ΔABC are $A(0, b)$, $B(0, 0)$ and $C(a, 0)$.

Mid-point of E and D are $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $\left(\frac{a}{2}, 0\right)$.

The slope of median BE, $m_1 = \frac{b}{a}$

and slope of AD, $m_2 = -\frac{2b}{a}$

Since, the medians are perpendicular to each other.

$\therefore$ $m_1 m_2 = -1$

$\Rightarrow$ $\frac{b}{a} \times \left(\frac{-2b}{a}\right) = -1$

$\Rightarrow$ $-2b^2 = -a^2 \Rightarrow a = \pm\sqrt{2}b$

Round II

1. Let $\angle OAB = \theta$

Then, $OA + AB = 1 + 4\cot\theta + 4 + \tan\theta$

$= 5 + 4\cot\theta + \tan\theta \geq 5 + 4 = 9$

(using AM $\geq$ GM)

2. Let $A(2, 1)$, $B(-2, 4)$ $\therefore AB = 5$

Hence, the locus is the line segment AB.

3. Since, reflection of the orthocentre of ΔABC in base BC will always lie on the circumcircle of the ΔABC, therefore coordinate of a point lying on the circumcircle is $\left(1 - \frac{1 \times 4}{2}, 1 - \frac{1 \times 4}{2}\right)$ *i.e.*, $(-1, -1)$ and coordinates of the circumcentre is $(2, 0)$.

$\therefore$ Radius of the circumcircle of $\Delta ABC = \sqrt{(2+1)^2 + (1)^2}$

$= \sqrt{10}$

4. $(3 - 10 + a)(9 - 20 + a) > 0$

or $(a - 7)(a - 11) > 0$

$\therefore$ $a \in (-\infty, 7) \cup (11, \infty)$

5. Let the new coordinates be $P(x', y')$ after shifting origin to $P(x', y')$ *i.e.*, $x = x' + h$ and $y = y' + k$

$\therefore$ $(x' + h)^2 + (y' + k)^2 - 4(x' + h) + 6(y' + k) - 7 = 0$

$\Rightarrow \quad (x')^2 + (y')^2 + 2(h-2)x' + 2(k+3)y'$
$+ (h^2 + k^2 - 4h + 6k - 7) = 0$

According to the question,

$h - 2 = 0$ and $k + 3 = 0$

$\Rightarrow \quad (h, k) = (2, -3)$

6. $(x+1)^2 + y^2 + (x-2)^2 + y^2 = 2[(x-1)^2 + y^2]$

On simplification, we get $2x + 3 = 0$

7. The diagonals meet at the mid-point of AC, *i.e.*, at $(3, 2)$ which lies on $y = 2x + c$

$\therefore \quad c = -4$

Let $\quad B = (\alpha, 2\alpha - 4)$

$\because \quad AB \perp BC$

$\Rightarrow \quad \left(\frac{2\alpha - 7}{\alpha - 1}\right)\left(\frac{2\alpha - 5}{\alpha - 5}\right) = -1$

$\therefore \quad \alpha^2 - 6\alpha + 8 = 0 \Rightarrow \alpha = 2, 4$

The other two vertices are (2, 0) and (4, 4).

8. $BP - AP = \pm 6$ or $BP = AP \pm 6$

$\Rightarrow \quad \sqrt{x^2 + (y+4)^2} = \sqrt{x^2 + (y-4)^2} \pm 6$

On squaring and simplifying, we get

$4y - 9 = \pm 3\sqrt{x^2 + (y-4)^2}$

Again squaring, we get

$9x^2 - 7y^2 + 63 = 0$

9. Let $H(3, \alpha)$ is the orthocentre.

$\therefore$ Slope of $BH \times$ Slope of $AC = -1$

$\Rightarrow \quad -\alpha \cdot \frac{4}{3} = -1$

$\Rightarrow \quad \alpha = \frac{3}{4}$

Hence, orthocentre of a triangle is $\left(3, \frac{3}{4}\right)$.

10. $y = |x - 2| \Rightarrow y = x - 2$ and $y = 2 - x$

Area of shaded region $= 2\,ABC = 2 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 1$

11. Let P be the middle point of the line segment joining $A(3, -1)$ and $B(1, 1)$ is $(2, 0)$.

Let P be shifted to Q by 2 unit and y-coordinate of Q is greater than that of P.

Now, slope of $AB = \frac{1-(-1)}{1-3} = 1$,

then slope of $PQ = -1$

The coordinate of Q becomes $(2 \pm 2\cos\theta, 0 \pm 2\sin\theta)$, where $\tan\theta = 1$

i.e., $\quad (2 \pm \sqrt{2}, \pm\sqrt{2})$

As, y-coordinate of Q is greater than that of P.

$\therefore$ We take $\quad Q = (2 + \sqrt{2}, \sqrt{2})$

12. Let the vertex $A(x, y)$ is equidistant from B and C.

$\therefore \quad (x-1)^2 + (y-3)^2 = (x+2)^2 + (y-7)^2$

$\Rightarrow \quad 6x - 8y + 43 = 0$

Only the option (c) satisfy it.

13. OP is inclined at angle θ with x-axis and OQ is inclined at angle $\alpha - \theta$ with x-axis.

The bisector of $\angle POQ$ is inclined at angle

$\frac{\theta + \alpha - \theta}{2} = \frac{\alpha}{2}$ with x-axis

14. Since, $AB = AC = \sqrt{2}$

The slope AB is 1. Hence, AB is inclined at 45° with the x-axis and AC is inclined at 60° with the x-axis. Equation of AC is

$$y = \sqrt{3}\,(x - 2).$$

The coordinates of C is $(2 + \sqrt{2}\cos 60°, 0 + \sqrt{2}\sin 60°)$

or $$\left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$$

15. Let $\angle RPQ = \theta$ and $\angle RQP = \phi$

$\therefore \quad \theta - \phi = 2\alpha$

Let $RM \perp PQ$, so that $RM = k$,

$$MP = a - h \quad \text{and} \quad MQ = a + h$$

Then, $$\tan\theta = \frac{RM}{MP} = \frac{k}{a-h}$$

and $$\tan\phi = \frac{RM}{MQ} = \frac{k}{a+h}$$

Again now, $2\alpha = \theta - \phi$

$$\therefore \quad \tan 2\alpha = \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{k(a+h) - k(a-h)}{a^2 - h^2 + k^2}$$

$$\Rightarrow \quad a^2 - h^2 + k^2 = 2hk\cot 2\alpha$$

Hence, the locus is $x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$

16. As the third vertex lies on the line $y = x + 3$, its coordinates are of the form $(x, x + 3)$. The area of the triangle with vertices $(2, 1), (3, -2)$ and $(x, x + 3)$ is given by

$$\frac{1}{2}\left\|\begin{matrix} x & x+3 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{matrix}\right\|$$

$$\Rightarrow \quad |2x - 2| = 5 \quad \text{(given)}$$

$$\therefore \quad 2x - 2 = \pm 5$$

$$\Rightarrow \quad x = 3/2,\ 7/2$$

Thus, the coordinates of the third vertex are (7/2, 13/2) or (–3/2, 3/2)

17. Let $A \equiv (p + 1, 1)$, $B = (2p + 1, 3)$ and $C \equiv (2p + 2, 2p)$

$\therefore$ Slope of AB = slope of AC

$$\Rightarrow \quad \frac{3-1}{2p+1-p-1} = \frac{2p-1}{2p+2-p-1}$$

$$\Rightarrow \quad \frac{2}{p} = \frac{2p-1}{p+1}$$

$$\Rightarrow \quad 2p + 2 = 2p^2 - p$$

$$\Rightarrow \quad 2p^2 - 3p - 2 = 0$$

$$(2p + 1)(p - 2) = 0$$

$$\Rightarrow \quad p = 2, -\frac{1}{2}$$

18. P, Q divides AB in the ratio of 1 : 2 and 2 : 1 respectively and hence the coordinates of P and Q are

$$\left[\frac{1\times 2 + 2(-3)}{3}, \frac{1\times 3 + 2\times 2}{3}\right]$$

and $$\left[\frac{2\times 2 + 1(-3)}{3}, \frac{2\times 3 + 1\times 2}{3}\right]$$

$$\Rightarrow \quad P\left(-\frac{4}{3}, \frac{7}{3}\right) \text{ and } Q\left(\frac{1}{3}, \frac{8}{3}\right), \text{ then}$$

$$PQ = \sqrt{\left(\frac{1}{3} + \frac{4}{3}\right)^2 + \left(\frac{8}{3} - \frac{7}{3}\right)^2} = \frac{\sqrt{26}}{3}$$

Let the coordinates of R be (x, y).

Then, $$QR = PQ \Rightarrow \left(x - \frac{1}{3}\right)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{26}{9} \quad \text{...(i)}$$

and QR is perpendicular to PQ.

$$\Rightarrow \quad \frac{y - \frac{8}{3}}{x - \frac{1}{3}} \times \frac{\frac{8}{3} - \frac{7}{3}}{\frac{1}{3} + \frac{4}{3}} = -1$$

$$\Rightarrow \quad \left(y - \frac{8}{3}\right) = -5\left(x - \frac{1}{3}\right) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$26\left(x - \frac{1}{3}\right)^2 = \frac{26}{9}, x - \frac{1}{3} = \pm\frac{1}{3}$$

$$\Rightarrow \quad x = 0 \text{ or } x = \frac{2}{3}$$

From Eq. (ii), when $x = 0, y = \frac{13}{3}$ and when $x = \frac{2}{3}, y = 1$.

So, the required coordinates of R are $\left(0, \frac{13}{3}\right)$ or $\left(\frac{2}{3}, 1\right)$.

19. Since, triangle is right angled, circumcentre is the mid-point of PQ and orthocentre is $R(1, -7)$.Hence,

$$RM = \left|\sqrt{\left(\frac{23}{2}+1\right)^2}\right| = 12\frac{1}{2} = \frac{25}{2}$$

20. The coordinates of the incentre are given as follows

$$\bar{x} = \frac{20(-8)+15(-15)+25(1)}{20+15+25}$$

$$= \frac{-360}{60} = -6$$

$$\bar{y} = \frac{20(5)+15(-19)+25(-7)}{60} = -6$$

Hence, incentre I is $(-6, -6)$.

Now, equation of side PR is

$$y + 7 = \frac{12}{-9}(x-1)$$

$\Rightarrow$ $4x - 4 = -3y - 21$

$\Rightarrow$ $4x + 3y + 17 = 0$

Inradius is given as the distance of I from side PR,

i.e., $\frac{|-24-18+17|}{5} = 5$

21. Coordinates of D using section formula are $(-5. -23/2)$ and $m_{PD} = -\frac{11}{2}$

$\therefore$ Equation of PD is $11x + 2y + 78 = 0$.

$\Rightarrow$ $a + c = 89$

22. The orthocentre lies on the line joining the points (0, 0) and (3, 5) *i.e.*, $5x - 3y = 0$.

Also, statement II is true.

23. Statement II is true as the coordinates of the point P in new system are $(\alpha - 1 - 2, \alpha + 1 - 3)$.

In statement I, the centroid is (2, 3), so the coordinates of the vertices in the new system of coordinates are $(-2, -3)$, (1, 0), (1, 3).

24. Area of triangle

$$A = \frac{1}{2}|-ab\sin\theta\cos\theta - ab\sin\theta\cos\theta|$$

$$= |ab\sin\theta\cos\theta| = \left|\frac{ab}{2}\sin 2\theta\right|$$

$\therefore$ $A_{max} = \frac{|ab|}{2}$

25. We know that, if sum of algebraic distances from three points on the variable line is zero, then the line always passes through the mean of the given point, which is centroid of triangle formed by given three points. But centroid of triangle is (1, 2). Hence, the line must pass through it, for which $a + 2b + c = 0$. Therefore, statement I is false and statement II is true.

26. Given mid-points of a triangle are (0, 1), (1, 1) and (1, 0). Plotting these points on a graph paper and make a triangle.

So, the sides of a triangle will be 2, 2 and $\sqrt{2^2+2^2}$ *i.e.*, $2\sqrt{2}$

x-coordinate of incentre $= \frac{2\times 0 + 2\sqrt{2}\cdot 0 + 2\cdot 2}{2+2+2\sqrt{2}}$

$$= \frac{2}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= 2 - \sqrt{2}$$

27. Using section formula, the coordinates of the point P, which divides AB internally in the ratio 3 : 2 are

$$P\left(\frac{3\times 2 + 2\times 1}{3+2}, \frac{3\times 4 + 2\times 1}{3+2}\right) \equiv P\left(\frac{8}{5}, \frac{14}{5}\right)$$

Also, since the line L passes through P, hence substituting the coordinates of $P\left(\frac{8}{5}, \frac{14}{5}\right)$ in the equation of $L: 2x + y = k$, we get

$\Rightarrow$ $2\left(\frac{8}{5}\right) + \left(\frac{14}{5}\right) = k$

$\Rightarrow$ $k = 6$

28. The third vertex lies on $2x + 3y = 9$

i.e., $\left(x, \frac{9-2x}{3}\right)$

$\therefore$ Locus of centroid is

$$\left(\frac{2-2+x}{3}, \frac{-3+\frac{9-2x}{3}+1}{3}\right) = (h, k)$$

$\therefore \quad h = \frac{x}{3}$

$k = \frac{3-2x}{9}$

$\Rightarrow \quad 9k = 3 - 2(3h)$

$\Rightarrow \quad 9k = 3 - 6h$

$\Rightarrow \quad 2h + 3k = 1$

Hence, locus of a point is

$\Rightarrow \quad 2x + 3y = 1$

29. Let (x, y) denotes the coordinates in A, B and C plane.

Then, $\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} = \frac{1}{9}$

$\Rightarrow \quad 9x^2 + 9y^2 - 18x + 9 = x^2 + y^2 + 2x + 1$

$\Rightarrow \quad 8x^2 + 8y^2 - 20x + 8 = 0$

$\Rightarrow \quad x^2 + y^2 - \frac{5}{2}x + 1 = 0$

$\therefore A, B$ and C lie on a circle with $C(5/4, 0)$.

30. Now, $AB = \sqrt{(1-h)^2 + (1-k)^2}$

$BC = \sqrt{(2-1)^2 + (1-1)^2} = 1$

and $CA = \sqrt{(h-2)^2 + (k-1)^2}$

Since, $AC^2 = AB^2 + BC^2$

$\Rightarrow \quad h^2 + 4 - 4h + k^2 + 1 - 2k$

$= 1 + h^2 - 2h + k^2 + 1 - 2k$

$\Rightarrow \quad h = 1 \quad \ldots(i)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times AB \times BC$

$\Rightarrow \quad 1 = \frac{1}{2} \times \sqrt{(1-h)^2 + (1-k)^2} \times 1$

$\Rightarrow \quad 2 = \sqrt{(k-1)^2}$ [from Eq. (i)]

$\Rightarrow \quad 4 = k^2 + 1 - 2k$

$\Rightarrow \quad (k-3)(k+1) = 0$

$\Rightarrow \quad k = -1, 3$

31. Let ABC be the triangle with $A(1, 1)$. The mid-point of AB is $(-1, 2)$.

$\Rightarrow \quad B(-3, 3)$

The mid-point of AC is $(3, 2)$

$\Rightarrow \quad C(5, 3)$

Hence, centroid is $\left(1, \frac{7}{3}\right)$.

32. Let the third vertex be $C(x_1, y_1)$.

The centroid is $\left(\frac{2-2+x_1}{3}, \frac{-3+1+y_1}{3}\right)$.

i.e., $\left(\frac{x_1}{3}, \frac{y_1-2}{3}\right)$

It lies on the line $2x + 3y = 1$,

$\therefore \quad \frac{2}{3}(x_1) + 3\left(\frac{y_1-2}{3}\right) = 1$

Hence, the locus of C is $2x + 3y = 9$.

12 Straight Lines

JEE Main MILESTONE

- Straight Line
- Slope or Gradient of a Line
- Equations of Lines Parallel to Axes
- Various Forms of Equations of a Line
- Angle between Two Lines
- Condition of Concurrency of Three Lines
- Equation of a Line Parallel to a Given Line
- Equation of a Line Perpendicular to a Given Line
- Area of Parallelogram
- Distance of a Point from a Line
- Position of Two Points (x_1, y_1) and (x_2, y_2) Relative to a Given Line
- Distance between Two Parallel Lines
- Equation of Internal and External Bisectors of Angles between Two Lines
- Foot of Perpendicular
- Image of a Point with Respect to a Line
- Family of Lines

12.1 Straight Line

A straight line is the locus of all those points which are collinear with two given points. Since, we know that one and only one line can be drawn from any two given points. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two such given fixed points. Let $P(x, y)$ be a point such that the points A, B and P always lie in the same straight line, then the locus of all such points will be the straight line through AB.

Straight line equations have simple variable expressions with no exponents on them.

The various 'standard' forms are often holdovers from a few centuries ago, when Mathematicians couldn't handle very complicated equations, so they tended to obsess about the simple cases. Now-a-days, you likely needn't worry too much about the 'standard' forms, this lesson will only cover the more helpful forms.

Note

- We can have one and only one line through a fixed point in a given direction.
- We can have infinitely many lines through a given point .

12.2 Slope or Gradient of a Line

If a line AB makes an angle θ with the positive x-axis, then m or $\tan\theta$ is called the slope or gradient of the line.

i.e., $m = \tan\theta$

If $0° < \theta < 90°$, then θ is an acute angle.

If $\theta = 0°$, either the line is x-axis or it is parallel to x-axis.

If $90° < \theta < 180°$, then θ is an obtuse angle.

Slope of a Line when Coordinates of Any Two Points are Given

If a line passes through $A(x_1, y_1)$ and $B(x_2, y_2)$, then the slope of a line is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Note

- If a angle θ is measured from x-axis in anti-clockwise or clockwise direction, then sign will be taken as positive or negative.
- Slope of any line parallel to x-axis is zero.
- Slope of any line parallel to y-axis *i.e.*, perpendicular to x-aixs is infinite.
- If three points A, B and C are collinear, then
  slope of AB = slope of BC = slope of CA

***Sample* Problem 1** *Find the angle between the x-axis and the line joining the points* (3, – 1) *and* (4, – 2). **[NCERT]**

(*a*) 130° (*b*) 135°
(*c*) 150° (*d*) *None of these*

Interpret (*b*) Given,

$(x_1, y_1) = (3, -1)$

and $(x_2, y_2) = (4, -2)$

Slope of line $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2+1}{4-3}$

$\tan\theta = \dfrac{-1}{1}$

$\Rightarrow \quad \tan\theta = -1$

$\Rightarrow \quad \tan\theta = -\tan 45°$ $\quad (\because 1 = \tan 45°)$

$\Rightarrow \quad \tan\theta = \tan(180° - 45°)$ $\quad [\because \tan(180° - \theta) = -\tan\theta]$

$\Rightarrow \quad \tan\theta = \tan 135°$

$\Rightarrow \quad \theta = 135°$

12.3 Equations of Lines Parallel to Axes

Equation of a Line Parallel to x-axis

Let AB be a straight line parallel to x-axis at a distance b from it. Then, the ordinate of every point on AB is b. So, it can be treated as the locus of a point at a distance b from x-axis, then $y = b$ is an equation of line AB.

If a line parallel to x-axis is at a distance b and below x-axis, then equation is $y = -b$

Equation of x-axis is $y = 0$.

Equation of a Line Parallel to y-axis

Let AB be a straight line parallel to y-axis and at a distance a from it. Then, the abscissa of every point on AB is a. So, it can be treated as the locus of a point at a distance a from y-axis, then $x = a$ is an equation of line AB. If a line parallel to y-axis is at a distance a and to the left of y-axis, then its equation is $x = -a$.

Equation of y-axis is $x = 0$

***Sample* Problem 2** *Find the equation of line, which is parallel to y-axis and at a 3 distance from left of the origin is*

(*a*) $x = 3$ (*b*) $y = -3$ (*c*) $x = -3$ (*d*) $y = 3$

Interpret (*c*) We know, the equation of line which is parallel to y-axis is of the form $y = k$.

Here, we take $k = -3$, because it is left side of the origin.

$\therefore \quad y = -3$

12.4 Various Forms of Equations of a Line

(i) Slope intercept Form

The equation of a line with slope m and making an intercept c on y-axis is

$$y = mx + c$$

Note If the line passes through the origin, then $c = 0$, then equation of the line will become $y = mx$.

(ii) Point slope Form

The equation of a line which passes through a point $P(x_1, y_1)$ and has the slope m, is $y - y_1 = m(x - x_1)$.

(iii) Two point Form

The equation of a line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

Note If a line AB is passes through two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the equation of line is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$

(iv) Intercept Form

If a line makes intercept a and b on x-axis and y-axis respectively, then the equation of line is $\frac{x}{a}+\frac{y}{b}=1$.

intercept may be –ve but length of intercept is always +ve.

(v) Normal (Perpendicular) Form

If perpendicular distance of the origin from the line is p and perpendicular drawn from origin to the line is making an angle α from the positive direction of x-axis, then equation of line is $x\cos\alpha + y\sin\alpha = p,\ 0 \le \alpha \le \pi$

(vi) Parametric Form or the Distance Form or Symmetric Form

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

where, r is the directed distance between the points (x, y) (any point on the line) and (x_1, y_1).

Note

- $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ are called parametric equations of straight line AB.
- If point P above Q, then r is positive and if P below Q, then r is negative.

(vii) General Form

General equation of first degree in x and y is $ax + by + c = 0$, which represents an equation of line. Slope of this line is given by

$$-\frac{(\text{coefficient of } x)}{(\text{coefficient of } y)}=-\frac{a}{b}$$

Given equation of a line can be written as $y=\frac{-ax}{b}+\left(\frac{-c}{b}\right)$

The intercept form of a line is $\frac{x}{c/a}+\frac{y}{c/b}=-1$

***Sample* Problem 3** *The normal form of the equation* $x-\sqrt{3}y+8=0$ *is* **[NCERT]**

(*a*) $x\cos 120° - y\sin 120° = 4$
(*b*) $x\cos 120° + y\sin 120° = 4$
(*c*) $x\cos 120° + y\sin 120° = 8$
(*d*) *None of the above*

Interpret (*b*) Given, equation of line is

$$x-\sqrt{3}y+8=0$$

$$\Rightarrow \quad x-\sqrt{3}y=-8$$

$$\Rightarrow \quad -x+\sqrt{3}y=8 \quad \ldots(i)$$

Now, divide by $\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$

$$\Rightarrow \quad \sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{1+3}=2, \text{we get}$$

$$-\frac{1}{2}x+\frac{\sqrt{3}}{2}y=\frac{8}{2}$$

$$\Rightarrow \quad -\cos 60° x + \sin 60° y = 4$$

($\because$ convert in form of $x\cos\alpha + y\sin\alpha = p$)

($\because$ $\cos x$ is negative and $\sin x$ is positive, it is possible in second quadrant.)

$$\Rightarrow \quad x\cos(180° - 60°) + y\sin(180° - 60°) = 4$$

$$\Rightarrow \quad x\cos 120° + y\sin 120° = 4$$

***Sample* Problem 4** *Equation of a line passing through* $(2, 2\sqrt{3})$ *and inclined with x-axis at an angle of* 75°. **[NCERT]**

(*a*) $(2+\sqrt{3})x - y - 4 = 0$
(*b*) $(2-\sqrt{3})x - y - 4 = 0$
(*c*) $(2-\sqrt{3})x + y - 4 = 0$
(*d*) *None of the above*

Interpret (*a*) Here, $(x_1, y_1) = (2, 2\sqrt{3})$,

$$\Rightarrow \quad m = \tan 75°$$

$$= \tan(45° + 30°)$$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$$\Rightarrow \quad m = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow m = \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2}{3-1}$$

(on multiplying numerator and denominator by $\sqrt{3}+1$)

$$\Rightarrow \quad m = \frac{3+1+2\sqrt{3}}{2} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

Equation of line in one point form is

$$y - y_1 = m(x - x_1) \Rightarrow y - 2\sqrt{3} = (2+\sqrt{3})(x-2)$$

$$\Rightarrow \quad y - 2\sqrt{3} = (2+\sqrt{3})x - 2(2+\sqrt{3})$$

$$\Rightarrow \quad y - 2\sqrt{3} = (2+\sqrt{3})x - 4 - 2\sqrt{3}$$

$$\Rightarrow \quad (2+\sqrt{3})x - y - 4 = 0$$

12.5 Angle between Two Lines

Let the equations of two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$, then angle between these lines is given by

$$\theta = \tan^{-1}\left|\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)\right|$$

If θ is the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then

$$\theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

If two lines are parallel to each other, then

$$m_1 = m_2, \quad a_2b_1 - a_1b_2 = 0 \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

If two lines are perpendicular to each other, then $m_1m_2 = -1$ or $a_1a_2 + b_1b_2 = 0$

Note Coordinates of point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\left\{\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right), \left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)\right\}$$

Method to Determine Acute/obtuse Angle of a Triangle

If three vertices of a triangle are given, then firstly determine the sides of a triangle, then using the cosine formula of the corresponding to the largest side of a triangle.

i.e., $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

(i) If $\cos A > 0$, then it is acute angle.

(ii) If $\cos A < 0$, then it is obtuse angle.

(iii) $\cos A$, then it is right angle.

Remark In three types of problem please avoid to use the formula $\tan\theta = \frac{|m_1 - m_2|}{1 + m_1m_2}$ (*think yourself*)

***Sample* Problem 5** *One of the equation of the straight lines which passes through the point (3, 2) and are inclined to $x - 2y = 3$ at an angle of 45°, is*

(a) $3x - y - 9 = 0$ (b) $3x - y - 7 = 0$

(c) $x + 3y - 7 = 0$ (d) *None of these*

Interpret (d) Equation of straight line passing through (3, 2) and having slope m is

$$y - 2 = m(x - 3) \quad \ldots(i)$$

and slope of line $x - 2y = 3$ is $\frac{1}{2}$.

$$\therefore \quad \tan 45° = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$$

$$\Rightarrow \quad \pm 1 = \frac{m - \frac{1}{2}}{1 + \frac{m}{2}} \Rightarrow m = 3, -\frac{1}{3}$$

$\therefore$ From Eq. (i), the equation of straight lines are

$$y - 2 = 3(x - 3) \Rightarrow 3x - y - 7 = 0$$

and $$y - 2 = -\frac{1}{3}(x - 3) \Rightarrow x + 3y - 9 = 0$$

12.6 Condition of Concurrency of Three Lines

Let the equation of three lines are $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv a_2x + b_2y + c_2 = 0$ and $L_3 \equiv a_3x + b_3y + c_3 = 0$, then three lines will be concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Another condition Three lines L_1, L_2 and L_3 are concurrent iff there exists constants λ_1, λ_2 and λ_3 not all zero such that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$

i.e., $$\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$$

***Sample* Problem 6** *The lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x = 0$ are concurrent, if $b_1c_2 - b_2c_1$ is equal to*

(a) 10 (b) 1

(c) 0 (d) *None of these*

Interpret (c) Let the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x = 0$ are concurrent.

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad a_3 | b_1c_2 - b_2c_1 | = 0$$

$$\Rightarrow \quad b_1c_2 - b_2c_1 = 0 \quad (\because a_3 \neq 0)$$

12.7 Equation of a Line Parallel to a Given Line

Let $ax + by + c = 0$ be any straight line, then the equation of a line parallel to given line is $ax + by + \lambda = 0$, where λ is an arbitrary constant, which can be determined by some given conditions.

Note Parallel lines differ by a constant.

***Sample* Problem 7** *Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point $(-2, 3)$.* [NCERT]

(a) $3x - 4y + 18 = 0$ (b) $3x + 4y + 18 = 0$
(c) $3x - 4y - 18 = 0$ (d) *None of these*

Interpret (a) Given equation of a line is

$$3x - 4y + 2 = 0 \quad \ldots(i)$$

Equation of line parallel to Eq. (i) is

$$3x - 4y + k = 0 \quad \ldots(ii)$$

$\because$ Line (ii) passes through the point $(-2, 3)$ *i.e.,* put $x = -2, y = 3$ in Eq. (ii).

$\therefore$ $3(-2) - 4 \times 3 + k = 0$

$\Rightarrow$ $-6 - 12 + k = 0 \Rightarrow k = 18$

Put $k = 18$ in Eq. (ii), we get

$$3x - 4y + 18 = 0$$

12.8 Equation of a Line Perpendicular to a Given Line

Let $ax + by + c = 0$ is an equation of straight line, then equation of straight line perpendicular to given line is $bx - ay + \lambda = 0$, where λ is an arbitrary constant.

***Sample* Problem 8** *Find the equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x-intercept 3.* [NCERT]

(a) $7x + y + 21 = 0$ (b) $7x + y - 21 = 0$
(c) $7x - y - 21 = 0$ (d) *None of these*

Interpret (b) Given equation of line is

$$x - 7y + 5 = 0 \quad \ldots(i)$$

Equation of line perpendicular to Eq. (i) is

$$7x + y + k = 0 \quad \ldots(ii)$$

$\because$ Line (ii) passes through the point $(3, 0)$ *i.e.,* put $x = 3, y = 0$ in Eq. (i)

$$7 \times 3 + 0 + k = 0$$

$\Rightarrow$ $k = -21$

Put $k = -21$ in Eq. (ii), we get

$$7x + y - 21 = 0$$

12.9 Area of Parallelogram

Area of Parallelogram *ABCD* whose side *AB, BC, CD* and *DA* are represented by

$$lx + my + n_1 = 0, ax + by + c_1 = 0,$$
$$lx + my + n_2 = 0 \text{ and } ax + by + c_2 = 0,$$

then area of parallelogram is given by

$$\frac{|n_2 - n_1||c_2 - c_1|}{\begin{vmatrix} a & b \\ l & m \end{vmatrix}_{2\times 2}} \text{ or } \frac{p_1 p_2}{\sin\theta}$$

12.10 Distance of a Point from a Line

Let $ax + by + c = 0$ be any equation of line and $P(x_1, y_1)$ be any point in space, then the perpendicular distance d of point P from the line is given by

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

The length of the perpendicular from the origin to the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$

Note

- If the length of perpendicular drawn from the origin to the line $ax + by + c = 0$ is p, then
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$
- The equation of a line which makes a triangle of centroid (x_1, y_1) with coordinates axes, is $\frac{x}{3x_1} + \frac{y}{3y_1} = 1$

***Sample* Problem 9** *If p is the length of perpendicular from origin to the line whose intercepts on the axes are a and b, then $\frac{1}{a^2} + \frac{1}{b^2}$ is equal to* [NCERT]

(a) p^2 (b) $\frac{1}{p^2}$
(c) $2p^2$ (d) $\frac{1}{2p^2}$

Interpret (*b*) Equation of line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}-1=0$$

Its distance from origin is

$$\left|\frac{0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|=p \qquad \left(\because d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|\right)$$

$$\Rightarrow \qquad p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \Rightarrow \frac{1}{p}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

On squaring both sides, we get

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

12.11 Position of Two Points (x_1, y_1) and (x_2, y_2) Relative to a Given Line

Let the equation of line be $ax+by+c=0$ and $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two given points.

(i) If ax_1+by_1+c and ax_2+by_2+c both are of the same sign and hence, $\frac{ax_1+by_1+c}{ax_2+by_2+c}>0$, then the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on same side of line $ax+by+c=0$

(ii) If ax_1+by_1+c and ax_2+by_2+c both are of opposite signs to each other and hence, $\frac{ax_1+by_1+c}{ax_2+by_2+c}<0$, then the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on opposite side of the line $ax+by+c=0$

(iii) If origin lie on line, then the line is known as origin side.

Note

- A point (α, β) will lie on origin side of the line $ax+by+c=0$, if $a\alpha+b\beta+c$ and c have same sign.
- A point (α, β) will lie on non-origin side of the line $ax+by+c=0$, if $a\alpha+b\beta+c$ and c have opposite sign.

***Sample* Problem 10** *The points* (2, 1) *and* (– 3, 5) *are lies along the line* $3x-2y+1=0$

(a) *same side* (b) *opposite side*
(c) *can't say* (d) *None of these*

Interpret (*b*) Let $\lambda=3x-2y+1$

Then, the value of λ at (2, 1) is

$$3\times 2-2\times 1+1=\lambda_1 \quad \text{(say)}$$

$$\Rightarrow \qquad \lambda_1=5$$

and the value of λ at $(-3, 5)$ is

$$3\times(-3)-2\times 5+1=\lambda_2 \quad \text{(say)}$$

$$\Rightarrow \qquad \lambda_2=-18$$

Since, λ_1 and λ_2 are of opposite signs, therefore, the two points are on the opposite sides of the given line.

12.12 Distance between Two Parallel Lines

Distance between two parallel lines $a_1x+b_1y+c_1=0$ and $a_1x+b_1y+c_2=0$ is given by

$$d=\left|\frac{c_1 \sim c_2}{\sqrt{a_1^2+b_1^2}}\right|$$

***Sample* Problem 11** *The distance between the lines* $5x-12y+65=0$ *and* $10x-24y-39=0$ *is*

(a) $\frac{\sqrt{169}}{5}$ (b) $\frac{\sqrt{169}}{2}$
(c) $\sqrt{169}$ (d) *None of these*

Interpret (*b*) Given lines are $5x-12y+65=0$

and $\quad 10x-24y-39=0 \Rightarrow 5x-12y-\frac{39}{2}=0$

Since, given lines are parallel.

$\therefore$ Distance between two lines

$$=\frac{|c_2-c_1|}{\sqrt{a^2+b^2}}=\frac{\left|-\frac{39}{2}-65\right|}{\sqrt{5^2+(-12)^2}}=\frac{\left|-\frac{169}{2}\right|}{\sqrt{25+144}}=\frac{\frac{169}{2}}{\sqrt{169}}=\frac{\sqrt{169}}{2}$$

12.13 Equation of Internal and External Bisectors of Angles between Two Lines

The equation of bisectors of the angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(i) If $c_1 > 0, c_2 > 0$

(ii) If $a_1a_2 + b_1b_2 > 0$, then the bisector for positive sign gives the obtuse angle bisector and for negative sign gives the acute angle bisector.

(iii) If $a_1a_2 + b_1b_2 < 0$, then bisector corresponding to positive and negative give the acute and obtuse angle bisectors respectively.

***Sample* Problem 12** *The equation of line bisecting the obtuse angle between $y - x = 2$ and $2y + x = 5$ is $\frac{y-x-2}{\sqrt{2}} = \frac{x+2y-5}{n}$, where n is*

(a) 5 (b) $\sqrt{2}$

(c) $\sqrt{5}$ (d) *None of these*

Interpret (c) Equations of given lines are

$$-x + y = 2 \quad \ldots(i)$$

and

$$x + 2y = 5 \quad \ldots(ii)$$

Since, $a_1a_2 + b_1b_2 = -1 + 2 = 1 > 0$

$\therefore$ Equation of bisector of the obtuse angle is

$$\frac{x-y+2}{\sqrt{2}} = \frac{-x-2y+5}{\sqrt{5}}$$

$$\Rightarrow \quad \frac{y-x-2}{\sqrt{2}} = \frac{x+2y-5}{\sqrt{5}}$$

Hence, $n = \sqrt{5}$

12.14 Foot of Perpendicular

The foot M of perpendicular drawn from a point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

$$\frac{x - x_1}{a} = -y\frac{y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

12.15 Image of a Point with Respect to a Line

The image of a point (x_1, y_1) with respect to the line mirror $ax + by + c = 0$ be $Q(x, y)$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

***Sample* Problem 13** *Find the image of the point (3, 8) with respect to the line x + 3y = 7 assuming the line to be plane mirror.* **[NCERT]**

(a) (1, 4) (b) (–1, –4)

(c) (–1, 4) (d) (1, –4)

Interpret (b) Given point $(x_1, y_1) = (3, 8)$ and line $x + 3y - 7 = 0$

Using the relation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\therefore \quad \frac{x-3}{1} = \frac{y-8}{3} = \frac{-2(1\times 3 + 3\times 8 - 7)}{1^2 + 3^2}$$

$$\Rightarrow \quad \frac{x-3}{1} = \frac{y-8}{3} = \frac{-2(20)}{10} \Rightarrow \frac{x-3}{1} = \frac{y-8}{3} = -4$$

$$\Rightarrow \quad x = -4 + 3, y = -12 + 8$$

$$\Rightarrow \quad x = -1, y = -4$$

12.16 Family of Lines

If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two non-parallel lines, then

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$$

represents the family of all those lines, which passes through the point of intersection of given lines.

Note

- If a straight line is such that the algebraic sum of the perpendiculars drawn upon it from any numbers of fixed points is zero, then the line always passes through a fixed point.
- If there are three points are given such that algebric sum of perpendicular drawn from it zero, then fix point is centoid of Δ.

***Sample* Problem 14** *The straight lines $x(a + 2b) + y(a + 3b) = a + b$, for different values of a and b pass through a fixed point.*

(a) (2, 1) (b) (–2, 1)

(c) (2, –1) (d) (–2, –1)

Interpret (c) Given equation can be rewritten as

$$a(x + y - 1) + b(2x + 3y - 1) = 0 \quad \ldots(i)$$

Both a and b cannot be simultaneously zero, therefore atleast one of a and b is non-zero.

Let $a \neq 0$. Now, from Eq. (i) can be rewritten as

$$x + y - 1 + \frac{b}{a}(2x + 3y - 1) = 0$$

$$\Rightarrow \quad x + y - 1 + k(2x + 3y - 1) = 0$$

where $k = \frac{b}{a}$ ...(ii)

From Eq.(ii), it is clear that lines (ii) pass through the point of intersection of lines

$$x + y - 1 = 0 \quad \ldots(iii)$$

and

$$2x + 3y - 1 = 0 \quad \ldots(iv)$$

On solving Eqs. (iii) and (iv), we get

$$x = 2, y = -1$$

WORKED OUT
Examples

Example 1 *The point (4, 1) undergoes the following two successive transformations*

(i) *Reflection about the line $y = x$*

(ii) *Translation through a distance 2 units along the positive x-axis*

Then, the final coordinates of the point are **[NCERT Exemplar]**

(a) (4, 3) (b) (3, 4)

(c) (1, 4) (d) $\left(\frac{7}{2}, \frac{7}{2}\right)$

Solution (b) The image of the point (4, 1) after reflection about the line $y = x$ is (1, 4).

And then translation through a distance of 2 units along x-axis.

$\therefore$ Final position of a point is $(1 + 2, 4)$ *i.e.*, (3, 4).

Example 2 *The vertices of ΔPQR are $P(2, 1)$, $Q(-2, 3)$ and $R(4, 5)$. Find equation of the median through the vertex R.* **[NCERT]**

(a) $3x - 4y - 8 = 0$ (b) $4x - 3y + 8 = 0$

(c) $3x - 4y + 8 = 0$ (d) $3x + 4y - 8 = 0$

Solution (c) Since, median bisects the opposite side *i.e.*, S is the mid-point of PQ.

$$S = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{2-2}{2}, \frac{1+3}{2}\right) = \left(0, \frac{4}{2}\right) = (0, 2)$$

$$(\because x_1 = 2, y_1 = 1, x_2 = -2, y_1 = 3)$$

$\therefore$ Equation of line RS by using

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 5 = \frac{2-5}{0-4}(x - 4) \quad (\because x_1 = 4, y_1 = 5, x_2 = 0, y_2 = 2)$$

$$\Rightarrow \quad y - 5 = \frac{-3}{-4}(x - 4)$$

$$\Rightarrow \quad 4y - 20 = 3x - 12$$

$$\Rightarrow \quad 3x - 4y + 8 = 0$$

Example 3 *The owner of a milk store finds that he can sell 980 L of milk each week at ₹ 14 per litre and 1220 L of milk each week at ₹ 16 per litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at ₹ 17 per litre?* **[NCERT]**

(a) 1240 L (b) 1340 L

(c) 1350 L (d) *None of these*

Solution (b) Let price and litre be denoted in ordered pair (x, y), where x denotes the ₹ per litre and y denotes the quantity of milk in litre.

Given, (14, 980) and (16, 1220) are two points.

Let linear relations *i.e.*, linear equation is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 980 = \frac{1220 - 980}{16 - 14}(x - 14)$$

$$\Rightarrow \quad y - 980 = \frac{240}{2}(x - 14)$$

$$(\because x_1 = 14, y_1 = 980, x_2 = 16, y_2 = 1220)$$

$$\Rightarrow \quad y - 980 = 120(x - 14)$$

$$\Rightarrow \quad y - 980 = 120x - 120 \times 14$$

$$\Rightarrow \quad 120x - y = 1680 - 980$$

$$\Rightarrow \quad 120x - y = 700$$

When price $x = 17$,

$$\Rightarrow \quad 120 \times 17 - y = 700$$

$$\Rightarrow \quad y = 2040 - 700$$

$$y = 1340$$

He will sell weekly 1340 L milk at the rate of ₹ 17 L.

Example 4 *Find the equation of line parallel to Y-axis and drawn through the point of intersection of the line $x - 7y + 5 = 0$ and $3x + y = 0$.* **[NCERT]**

(a) $x + 22 = 0$ (b) $22x + 5 = 0$

(c) $x + 44 = 0$ (d) $22x - 5 = 0$

Solution (*b*) Given, equation of lines are

$$x - 7y + 5 = 0 \quad \ldots(i)$$

and $$3x + y = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), to find the point of intersection, from Eq. (i),

$$x = 7y - 5$$

On putting the value of x in Eq. (ii), we get

$$3(7y - 5) + y = 0 \Rightarrow 21y - 15 + y = 0$$

$$\Rightarrow \quad 22y = 15$$

$$\Rightarrow \quad y = \frac{15}{22}$$

$\therefore$ From Eq. (i), $$x = \frac{7 \times 15}{22} - 5 = \frac{105 - 110}{22}$$

$$\Rightarrow \quad x = -\frac{5}{22}$$

Hence, the point is $\left(-\frac{5}{22}, \frac{15}{22}\right)$

Hence, equation of required line by using

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1) = \left(\frac{-5}{22}, \frac{15}{22}\right)$ and $m = \frac{1}{0}$ as line parallel to Y-axis, is

$$y - \frac{15}{22} = \frac{1}{0}\left(x + \frac{5}{22}\right) \Rightarrow 0 = x + \frac{5}{22}$$

$$\Rightarrow \quad 22x + 5 = 0$$

Example 5 *The angle between the line $x + y = 3$ and the line joining the points (1, 1) and (– 3, 4) is*

(*a*) $\tan^{-1}(7)$ (*b*) $\tan^{-1}(-1/7)$

(*c*) $\tan^{-1}(1/7)$ (*d*) *None of these*

Solution (*c*) Let $P = (1, 1)$ and $Q = (-3, 4)$

$\therefore$ Slope of line PQ, $m_1 = \frac{1-4}{1+3} = \frac{-3}{4}$

Given line is $x + y = 3$ or $x + y - 3 = 0$ ...(i)

Its slope, $m_2 = \frac{-1}{1} = -1$

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{-3}{4} + 1}{1 + \frac{3}{4}}\right| = \frac{1}{7}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{7}\right)$$

Example 6 *Point $R(h, k)$ divides a line segment between the axes in the ratio 1 : 2. Find equation of the line.* **[NCERT]**

(*a*) $2kx + hy = 3hk$ (*b*) $2kx + hy = 2hk$

(*c*) $2kx - hy = 3hk$ (*d*) *None of these*

Solution (*a*) Let equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(i)$$

Let a point $R(h, k)$ divide line AB in the ratio 1 : 2.

By using internally ratio

$$R(h, k) = \left(\frac{1x_2 + 2x_1}{1+2}, \frac{1y_2 + 2y_1}{1+2}\right)$$

$$\Rightarrow \quad h = \frac{1 \times 0 + 2 \times a}{1+2}, \; k = \frac{1 \times b + 2 \times 0}{1+2}$$

[$\because P(x, y)$ divide the line $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ internally

$$\therefore \quad P(x, y) = \left(\frac{nx_2 + mx_1}{n+m}, \frac{ny_2 + my_1}{n+m}\right)\Bigg]$$

$$\Rightarrow \quad h = \frac{2a}{3}, \; k = \frac{b}{3} \Rightarrow a = \frac{3h}{2}, \; b = 3k$$

On putting the values of a and b in Eq. (i), we get

$$\frac{x}{\frac{3h}{2}} + \frac{y}{3k} = 1$$

$$\Rightarrow \quad \frac{2x}{3h} + \frac{y}{3k} = 1$$

$$\Rightarrow \quad \frac{2kx + hy}{3hk} = 1$$

$$\Rightarrow \quad 2kx + hy = 3hk$$

Example 7 *If p and p' be the perpendicular from the origin upon the straight lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$. Then, $4p^2 + p'^2$ is equal to* **[NCERT]**

(*a*) a^2 (*b*) a (*c*) a^3 (*d*) a^4

Solution (*a*) Here, p = Length of perpendicular from (0, 0) to

$$x \sec\theta + y \operatorname{cosec}\theta = a$$

$$= \left|\frac{0 \sec\theta + 0 \operatorname{cosec}\theta - a}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}\right|$$

$$= \frac{a \cos\theta \sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = a \sin\theta \cos\theta$$

Similarly, $p' = a \cos 2\theta$

$$\therefore \quad 4p^2 + p'^2 = 4a^2 \sin^2\theta \cos^2\theta + a^2 \cos^2 2\theta$$

$$= a^2(2 \sin\theta \cos\theta)^2 + a^2 \cos^2 2\theta$$

$$= a^2(\sin^2 2\theta + \cos^2 2\theta) = a^2$$

Example 8 *A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.* **[NCERT]**

(*a*) $102x + 119y - 125 = 0$ (*b*) $119x + 102y - 125 = 0$

(*c*) $119x + 102y - 60 = 0$ (*d*) $119x + 102y + 125 = 0$

Solution (*b*) Given equation of straight path is

$$2x - 3y + 4 = 0 \quad \ldots(i)$$

and $$3x + 4y - 5 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\therefore \qquad x = -\frac{1}{17}, \quad y = \frac{22}{17}$$

Suppose, the person is standing at point P, then he can reach the path.

AB in least time if he goes straight to it *i.e.,* perpendicular to it. It means we have to find the equation of PQ.

$\because$ Line $PQ \perp$ Line AB

$\therefore$ Slope of $PQ \times$ Slope of $AB = -1$

$$\Rightarrow \qquad m \times -\left(\frac{6}{-7}\right) = -1$$

$$\Rightarrow \qquad m \times \frac{6}{7} = -1 \;\Rightarrow\; m = -\frac{7}{6}$$

Equation of PQ by using

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1) = \left(-\frac{1}{17}, \frac{22}{17}\right)$ and $m = -\frac{7}{6}$ is

$$y - \frac{22}{17} = -\frac{7}{6}\left(x + \frac{1}{17}\right)$$

$$\Rightarrow \qquad 6 \times 17y - 22 \times 6 = -7 \times 17x - 7$$

$$\Rightarrow \qquad 102y - 132 = -119x - 7$$

$$\Rightarrow \qquad 119x + 102y - 132 + 7 = 0$$

$$\Rightarrow \qquad 119x + 102y - 125 = 0$$

Example 9 *The equation of straight line cutting off an intercept – 2 from y-axis and being equally inclined to the axes are*

(a) $y = x + 2, y = x - 2$ (b) $y = -x - 2, y = x - 2$
(c) $y = x - 2, y = x - 2$ (d) *None of these*

Solution (*b*)Since, the required line is equally inclined with the coordinate axes, therefore it makes an angle of 45° or 135° with the x-axis.

So, its slope is either $m = \tan 45°$ or $m = \tan 135°$

i.e., $\qquad m = 1$ or -1

It is given that $c = -2$

Hence, the equation of the lines are

$$y = x - 2 \quad \text{or} \quad y = -x - 2$$

Example 10 *If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2)$ is equal to* [NCERT]

(a) –1 (b) 0 (c) 1 (d) 2

Solution (*b*) Given equation of lines are

$$y = m_1x + c_1 \qquad \ldots(i)$$

$$y = m_2x + c_2 \qquad \ldots(ii)$$

$$y = m_3x + c_3 \qquad \ldots(iii)$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{c_2 - c_1}{m_1 - m_2} \text{ and } y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

On putting the values of x and y in Eq. (iii), we get

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + \frac{c_3}{1}$$

$$\Rightarrow \qquad \frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + m_1c_3 - m_2c_3}{m_1 - m_2}$$

$$\Rightarrow \qquad m_1c_2 - m_2c_1 = m_3c_2 - m_3c_1 + m_1c_3 - m_2c_3$$

$$\Rightarrow \qquad m_1c_2 - m_2c_1 - m_3c_2 + m_3c_1 - m_1c_3 + m_2c_3 = 0$$

$$\Rightarrow \qquad m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

Example 11 *The equation of line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$, are*

(a) $y = 2x \pm 5$ (b) $x = 2y \pm 5$
(c) $y = x \pm 5$ (d) *None of these*

Solution (*a*) Let c be the intercept on y-axis, then the equation of the line is

$$y = 2x + c \qquad \ldots(i)$$

$$\Rightarrow \qquad -2x + y = c$$

$$\Rightarrow \qquad -\frac{2x}{\sqrt{(-2)^2 + 1^2}} + \frac{y}{\sqrt{(-2)^2 + 1^2}} = \frac{c}{\sqrt{(-2)^2 + 1^2}}$$

$$\Rightarrow \qquad -\frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{c}{\sqrt{5}} \qquad \ldots(ii)$$

This is the normal form, therefore RHS denotes the length of the perpendicular from the origin. But the length of the perpendicular from the origin is $\sqrt{5}$.

$$\therefore \qquad \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5} \;\Rightarrow\; |c| = 5$$

On putting $c = \pm 5$ in Eq. (i), we get the equation of line as $y = 2x \pm 5$

Example 12 *The equation of the bisector of the obtuse angle between the lines $3x - 4y + 7 = 0$ and $-12x - 5y + 2 = 0$ is*

(a) $21x + 77y + 101 = 0$ (b) $21x + 77y - 101 = 0$
(c) $21x + 77y = 0$ (d) *None of these*

Solution (*b*) $a_1a_2 + b_1b_2 = (3) \times (-12) + (-4) \times (-5)$

$$= -36 + 20 = -16$$

$$\Rightarrow \qquad a_1a_2 + b_1b_2 \le 0 \qquad \text{(obtuse bisector)}$$

$\therefore$ The equation of obtuse bisector is

$$\frac{(3x-4y+7)}{\sqrt{3^2+(-4)^2}} = -\frac{-12x-5y+2}{\sqrt{(-12)^2+(-5)^2}}$$

$$\Rightarrow \quad 13(3x-4y+7) = -5(-12x-5y+2)$$

$\Rightarrow$ $21x+77y-101=0$ is the obtuse angle bisector.

Example 13 *The locus of a point which moves such that its distance from the point* (0, 0) *is twice its distance from the y-axis, is*

(a) $x^2-y^2=0$ (b) $x^2-3y^2=0$

(c) $3x^2-y^2=0$ (d) *None of these*

Solution (c) By hypothesis

$$|OP| = 2|PM|$$

$$\Rightarrow \quad \sqrt{x_1^2+y_1^2} = 2|x_1|$$

On squaring both sides, we get

$$x_1^2+y_1^2 = 4x_1^2 \Rightarrow 3x_1^2-y_1^2=0$$

$\therefore$ Locus of the point is

$$3x^2-y^2=0$$

Example 14 *Find the distance of the line* $4x+7y+5=0$ *from the point* (1, 2) *along the line* $2x-y=0$. **[NCERT]**

(a) $\frac{23}{7}\sqrt{5}$ *sq units* (b) $\frac{23}{18}\sqrt{5}$ *sq units*

(c) $\frac{23}{8}\sqrt{5}$ *sq units* (d) *None of these*

Solution (b) The equation of line AB is

$$4x+7y+5=0 \quad \text{...(i)}$$

and equation of PQ is $2x-y=0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = -\frac{5}{18} \text{ and } y = -\frac{5}{9}$$

$\therefore$ Coordinate of $Q = \left(-\frac{5}{18}, -\frac{5}{9}\right)$

Length of $QP = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$= \sqrt{\left(1+\frac{5}{18}\right)^2+\left(2+\frac{5}{9}\right)^2}$$

$$= \sqrt{\left(\frac{23}{18}\right)^2+\left(\frac{23}{9}\right)^2} = \frac{23}{9}\sqrt{\frac{1}{4}+1} = \frac{23}{9}\times\frac{\sqrt{5}}{2}$$

$$= \frac{23}{18}\sqrt{5} \text{ sq units}$$

Example 15 *If the lines* $y=3x+1$ *and* $2y=x+3$ *are equally inclined to the line* $y=mx-4$, *find the value of m.* **[NCERT]**

(a) $\frac{2\pm5\sqrt{2}}{7}$ (b) $\frac{1\pm3\sqrt{2}}{7}$

(c) $\frac{1\pm5\sqrt{2}}{7}$ (d) *None of these*

Solution (c) Let the angle between the lines $y=3x+1$ and $y=mx+4$ is θ.

Here, slopes of the lines are $m_1=3, m_2=m$

$$\tan\theta = \left|\frac{3-m}{1+3m}\right| \quad \text{...(i)}$$

$$\left(\because \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|\right)$$

Again, it is given that angle between the lines $2y=x+3$ and $y=mx+4$ is also θ.

Here, slopes are $m_1=\frac{1}{2}$ and $m_2=m$.

$$\therefore \quad \tan\theta = \left|\frac{\frac{1}{2}-m}{1+\frac{1}{2}\times m}\right| \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left|\frac{3-m}{1+3m}\right| = \left|\frac{\frac{1}{2}-m}{1+\frac{m}{2}}\right|$$

$$\Rightarrow \quad \frac{3-m}{1+3m} = \pm\left(\frac{1-2m}{2+m}\right)$$

Taking positive sign,

$$(3-m)(2+m) = (1-2m)(1+3m)$$

$$\Rightarrow \quad 6+3m-2m-m^2 = 1+3m-2m-6m^2$$

$$\Rightarrow \quad 6 = 1-5m^2$$

$$\Rightarrow \quad m^2 = \frac{-5}{5} = -1$$

which is not possible as $m^2 \geq 0$.

Again, taking negative sign, we get

$$\frac{3-m}{1+3m} = -\frac{1-2m}{2+m}$$

$$\Rightarrow \quad (3-m)(2+m) = (2m-1)(1+3m)$$

$$\Rightarrow \quad 6+3m-2m-m^2 = 2m+6m^2-1-3m$$

$$\Rightarrow \quad 6+m-m^2 = 6m^2-m-1$$

$$\Rightarrow \quad 6m^2-m-1-6-m+m^2=0$$

$$\Rightarrow \quad 7m^2-2m-7=0$$

$$\Rightarrow \quad m = \frac{2\pm\sqrt{4+4\times7\times7}}{2\times7} = \frac{2\pm2\sqrt{1+49}}{2\times7}$$

$$m = \frac{1\pm\sqrt{50}}{7} \Rightarrow m = \frac{1\pm5\sqrt{2}}{7}$$

Note When we remove the mod sign, then we use $a\pm$ sign.

$$|a|=|b| \Rightarrow a=\pm b$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Equation of Straight Line in Standard Form

1. For specifying a straight line, how many geometrical parameters should be known? [NCERT Exemplar]
(a) 1 (b) 2 (c) 4 (d) 3

2. Slope of a line which cuts of intercepts of equal lengths on the axes is [NCERT Exemplar]
(a) –1 (b) –0 (c) 2 (d) $\sqrt{3}$

3. If the line $\frac{x}{a}+\frac{y}{b}=1$ passes through the points (2, –3) and (4, –5), then (a, b) is [NCERT Exemplar]
(a) (1, 1) (b) (–1, 1)
(c) (1, –1) (d) (–1, –1)

4. The equations of the lines which pass through the point (3, –2) and are inclined at 60° to the line $\sqrt{3}x+y=1$ is [NCERT Exemplar]
(a) $y+2=0, \sqrt{3}x-y-2-3\sqrt{3}=0$
(b) $x-2=0, \sqrt{3}x-y+2+3\sqrt{3}=0$
(c) $\sqrt{3}x-y-2-3\sqrt{3}=0$
(d) None of the above

5. Equation to the straight line cutting off an intercept 2 from the negative direction of the axis of y and inclined at 30° to the positive direction of x-axis, is
(a) $y+x-\sqrt{3}=0$ (b) $y-x+2=0$
(c) $y-\sqrt{3}x-2=0$ (d) $\sqrt{3}y-x+2\sqrt{3}=0$

6. In the adjacent figure, equation of refracted ray is

(a) $y=\sqrt{3}x+1$ (b) $y+\sqrt{3}x-3=0$
(c) $\sqrt{3}x+y-\sqrt{3}=0$ (d) None of these

7. The determinant $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$ represents
(a) parabola (b) a straight line
(c) a circle (d) None of these

8. If the intercept of a line between the coordinate axes is divided by the point (–5, 4) in the ratio 1 : 2, then find the equation of the line. [NCERT Exemplar]
(a) $8x-5y+60=0$ (b) $8x-6y+60=0$
(c) $8x+5y+60=0$ (d) None of these

9. The straight line whose sum of the intercepts on the axes is equal to half of the product of the intercepts, passes through the point whose coordinates are
(a) (1, 1) (b) (2, 2) (c) (3, 3) (d) (4, 4)

10. If non-zero numbers a, b, c are in HP, then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point. That point is
(a) $\left(1,-\frac{1}{2}\right)$ (b) $(1, -2)$
(c) $(-1,-2)$ (d) $(-2,2)$

11. If the coordinates of the middle point of the portion of a line intercepted between the coordinate axes is (3, 2), then the equation of the line will be [NCERT Exemplar]
(a) $2x+3y=12$ (b) $3x+2y=12$
(c) $4x-3y=6$ (d) $5x-2y=10$

12. If the straight line $ax+by+c=0$ always passes through $(1,-2)$, then a, b, c are in
(a) AP (b) HP
(c) GP (d) None of these

13. The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is [NCERT Exemplar]

(a) 1 : 2 (b) 3 : 7 (c) 2 : 3 (d) 2 : 5

14. A straight line through $P(1, 2)$ is such that its intercept between the axes is bisected at P. Its equation is

(a) $x + y = -1$ (b) $x + y = 3$
(c) $x + 2y = 5$ (d) $2x + y = 4$

Equation of Straight Line Through Intersection Point

15. Equation of a line, which is intersecting the X-axis at a distance of 3 units to the left of origin with slope –2, is [NCERT]

(a) $2x + y + 6 = 0$ (b) $2x - y + 6 = 0$
(c) $2x - y - 6 = 0$ (d) None of these

16. Find the equations of the lines, which cut off intercepts on the axes whose sum and product are 1 and – 6, respectively. [NCERT]

(a) $2x + 3y + 6 = 0$ or $3x - 2y + 6 = 0$
(b) $2x - 3y - 6 = 0$ or $3x - 2y + 6 = 0$
(c) $2x - 3y - 6 = 0$ or $3x + 2y + 6 = 0$
(d) None of the above

17. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through (1, – 1) and parallel to PS is

(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$
(c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$

18. Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2). [NCERT Exemplar]

(a) $x - 7y + 12 = 0,\ 7x + y - 16 = 0$
(b) $x - 7y + 6 = 0, 7x + y - 16 = 0$
(c) $x - 7y + 12 = 0, 7x + y + 16 = 0$
(d) None of the above

19. The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are [NCERT Exemplar]

(a) $\sqrt{3}x + y - \sqrt{3} = 0, \sqrt{3}x - y - \sqrt{3} = 0$
(b) $\sqrt{3}x + y + \sqrt{3} = 0, \sqrt{3}x - y + \sqrt{3} = 0$
(c) $x + \sqrt{3}y - \sqrt{3} = 0, x - \sqrt{3}y - \sqrt{3} = 0$
(d) None of the above

20. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$. [NCERT]

(a) $3x + y + 7 = 0$ (b) $3x - y - 7 = 0$
(c) $3x + 2y - 7 = 0$ (d) None of these

21. A ray of light passing through the point (1, 2) reflects on the X-axis at point A and the reflected ray passes through the point (5, 3). Find coordinates of A. [NCERT]

(a) $\left(\frac{13}{5}, 0\right)$ (b) $\left(-\frac{13}{5}, 0\right)$
(c) $\left(0, \frac{13}{5}\right)$ (d) None of these

22. Find the equation of the line passing through the point (5, 2) and perpendicular to the line joining the points (2, 3) and (3, –1). [NCERT Exemplar]

(a) $x - 4y + 3 = 0$ (b) $x + 4y - 3 = 0$
(c) $x - 4y - 3 = 0$ (d) None of these

23. Find the equation of the line passing through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$. [NCERT Exemplar]

(a) $4x - 3y + 3 = 0$ (b) $3x + 4y + 3 = 0$
(c) $4x + 3y - 3 = 0$ (d) $3x - 4y - 3 = 0$

24. Find the equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3, 2) is $\frac{7}{5}$. [NCERT Exemplar]

(a) $3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$
(b) $3x + 4y + 6 = 0$ and $4x + 3y + 1 = 0$
(c) $3x - 4 - 6 = 0$ and $4x + 3y + 1 = 0$
(d) None of the above

25. The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y = x$ is [NCERT Exemplar]

(a) $x - y = 5$ (b) $x + y = 5$
(c) $x + y = 1$ (d) $x - y = 1$

26. Find equation of the line passing through the point (2, 2) and cutting off intercepts the axes whose sum is 9. [NCERT]

(a) $x + 2y = 6$ (b) $2x + y = 8$
(c) $2x + y = 5$ (d) $2x - y = 6$

27. The equation of the straight line joining the origin to the point of intersection of $y - x + 7 = 0$ and $y + 2x - 2 = 0$, is

(a) $3x + 4y = 0$ (b) $3x - 4y = 0$
(c) $4x - 3y = 0$ (d) $4x + 3y = 0$

28. The line passing through $\left(-1, \frac{\pi}{2}\right)$ and perpendicular to $\sqrt{3}\sin\theta + 2\cos\theta = \frac{4}{r}$, is

(a) $2 = \sqrt{3}r\cos\theta - 2r\sin\theta$ (b) $5 = -2\sqrt{3}r\sin\theta + 4r\cos\theta$
(c) $2 = \sqrt{3}r\cos\theta + 2r\sin\theta$ (d) $5 = 2\sqrt{3}r\sin\theta - 4r\cos\theta$

29. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio $1 : n$. Find the equation of the line. [NCERT]

(a) $x(n + 1) + 3(n + 1)y = n + 11$
(b) $x(n + 1) - 3(n + 1)y = n + 11$
(c) $x(n + 1) + 3(n + 1)y = n - 11$
(d) None of the above

30. A line AB makes zero intercepts on x-axis and y-axis and it is perpendicular to another line CD which is $3x + 4y + 6 = 0$. The equation of line AB is

(a) $y = 4$ (b) $4x - 3y + 8 = 0$
(c) $4x - 3y = 0$ (d) $4x - 3y + 6 = 0$

31. Equation of a line passing through the line of intersection of lines $2x - 3y + 4 = 0$, $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + 3 = 0$, is

(a) $119x + 102y + 125 = 0$ (b) $119x + 102y = 125$
(c) $119x - 102y = 125$ (d) None of these

32. If the lines $x = a + m$, $y = -2$ and $y = mx$ are concurrent, then least value of $|a|$ is

(a) 0 (b) $\sqrt{2}$
(c) $2\sqrt{2}$ (d) None of these

33. The equation of straight line through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$, is

(a) $3x + 4y + 5 = 0$ (b) $3x + 4y - 10 = 0$
(c) $3x + 4y - 5 = 0$ (d) $3x + 4y + 6 = 0$

34. The lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0\,(a \neq b \neq c)$ are concurrent, if

(a) $a^3 + b^3 + c^3 + 3abc = 0$
(b) $a^2 + b^2 + c^2 - 3abc = 0$
(c) $a + b + c = 0$
(d) None of the above

35. The line $\frac{x}{a} - \frac{y}{b} = 1$ cuts the x-axis at P. The equation of the line through P perpendicular to the given line is

(a) $x + y = ab$ (b) $x + y = a + b$
(c) $ax + by = a^2$ (d) $bx + ay = b^2$

36. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is [NCERT Exemplar]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) 1 (d) $\frac{4}{3}$

37. The value of λ for which the lines $3x + 4y = 5$, $5x + 4y = 4$ and $\lambda x + 4y = 6$ meet at a point is

(a) 2 (b) 1
(c) 4 (d) 3

38. Find the values of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$. [NCERT]

(a) 210°, 1 (b) 210°, 2
(c) 220°, 3 (d) None of these

39. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h. [NCERT]

(a) $-\frac{22}{9}$ (b) $\frac{22}{9}$
(c) $\frac{22}{7}$ (d) $\frac{22}{13}$

Angle between Two Lines

40. Find the angle between the lines $y = (2 - \sqrt{3})(x + 5)$ and $y = (2 + \sqrt{3})(x - 7)$. [NCERT Exemplar]

(a) $-\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$
(c) $\frac{\pi}{3}$ or $-\frac{2\pi}{3}$ (d) None of these

41. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$. [NCERT]

(a) 35° (b) 30° (c) 45° (d) 60°

42. In what direction should a line be drawn through the point (1, 2) so that its point of intersection with the line $x + y = 4$ is at a distance of $\frac{\sqrt{6}}{3}$ from the given point? [NCERT Exemplar]

(a) 75° (b) 60° (c) 90° (d) 45°

43. The equation of the sides of a triangle are $x - 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x - 4y = 0$ passes through the

(a) incentre (b) centroid
(c) orthocentre (d) circumcentre

44. The triangle formed by the lines $x + y = 0, 3x + y = 4$, $x + 3y = 4$ is

(a) isosceles (b) equilateral
(c) right angled (d) None of these

45. Two lines are drawn through (3, 4) each of which makes angle of 45° with line $x - y = 2$, then area of the triangle formed by these lines is

(a) 9 sq units (b) 9/2 sq units
(c) 2 sq units (d) 2/9 sq unit

46. The inclination of the straight line passing through the point $(-3, 6)$ and the mid-point of the line joining the points $(4, -5)$ and $(-2, 9)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

Distance between Two Lines and Distance of a Point from a Line

47. Distance between the lines $5x + 3y - 7 = 0$ and $15x + 9y + 14 = 0$ is

(a) $\frac{35}{\sqrt{34}}$ (b) $\frac{1}{3\sqrt{34}}$ (c) $\frac{35}{3\sqrt{34}}$ (d) $\frac{35}{2\sqrt{34}}$

48. The distance of the point of intersection of lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is [NCERT Exemplar]

(a) $\frac{130}{17\sqrt{29}}$ (b) $\frac{13}{7\sqrt{29}}$ (c) $\frac{130}{7}$ (d) $\frac{13}{7}$

49. The coordinates of the foot of perpendicular from the point (2, 3) on the line $y = 3x + 4$ is given by [NCERT Exemplar]

(a) $\left(\frac{37}{10}, \frac{-1}{10}\right)$ (b) $\left(\frac{-1}{10}, \frac{37}{10}\right)$ (c) $\left(\frac{10}{37}, -10\right)$ (d) $\left(\frac{2}{3}, -\frac{1}{3}\right)$

50. A point equidistant from the lines $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is [NCERT Exemplar]

(a) (1, –1) (b) (1, 1) (c) (0, 0) (d) (0, 1)

51. The length of perpendicular from the point $(a\cos\alpha, a\sin\alpha)$ upon the straight line $y = x\tan\alpha + c$, $c > 0$, is

(a) c (b) $c\sin^2\alpha$ (c) $c\cos\alpha$ (d) $c\sec^2\alpha$

52. The distance of the point (3, 5) from the line $2x + 3y - 14 = 0$ measured parallel to line $x - 2y = 1$, is

(a) $\frac{7}{\sqrt{5}}$ (b) $\frac{7}{\sqrt{13}}$ (c) $\sqrt{5}$ (d) $\sqrt{13}$

53. Equation of the line passing through (1, 2) and parallel to the line $y = 3x - 1$ is [NCERT Exemplar]

(a) $y + 2 = x + 1$ (b) $y + 2 = 3(x + 1)$
(c) $y - 2 = 3(x - 1)$ (d) $y - 2 = x - 1$

Bisector of Lines

54. P is a point on either of the two lines $y - \sqrt{3}|x| = 2$ at a distance 5 units from their point of intersection. The coordinates of the foot of the perpendicular from P on the bisector of the angle between them are

(a) $\left(0, \frac{4 + 5\sqrt{3}}{2}\right)$ or $\left(0, \frac{4 - 5\sqrt{3}}{2}\right)$ depending on which the point P is taken

(b) $\left(0, \frac{4 + 5\sqrt{3}}{2}\right)$

(c) $\left(0, \frac{4 - 5\sqrt{3}}{2}\right)$

(d) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

55. The equations of perpendicular bisectors of sides AB and AC of a ΔABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the coordinates of vertex A are $(1, -2)$, then equation of BC is

(a) $23x + 14y - 40 = 0$
(b) $14x - 23y + 40 = 0$
(c) $23x - 14y + 40 = 0$
(d) $14x + 23y - 40 = 0$

56. The equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is

(a) $99x - 27y - 81 = 0$ (b) $11x - 3y + 9 = 0$
(c) $21x + 77y - 101 = 0$ (d) $21x + 77y + 101 = 0$

57. The equation of bisectors of the angles between the lines $|x| = |y|$ are

(a) $y = \pm x$ and $x = 0$ (b) $x = \frac{1}{2}$ and $y = \frac{1}{2}$
(c) $y = 0$ and $x = 0$ (d) None of these

Family of Lines

58. Equation of straight line belonging to families of straight lines $(x + 2y) + \lambda(3x + 2y + 1) = 0$ and $(x - 2y) + \mu(x - y + 1) = 0$ is

(a) $6x + 5y = 2$ (b) $5x - 6y + 4 = 0$
(c) $5x + 6y = 4$ (d) None of these

59. Equations of diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are [NCERT Exemplar]

(a) $y = x$, $y + x = 1$
(b) $y = x$, $x + y = 2$
(c) $2y = x$, $y + x = \frac{1}{3}$
(d) $y = 2x$, $y + 2x = 1$

60. Lines $2x + y = 1$ and $2x + y = 7$ are

(a) on the same side of a point $\left(0, \frac{1}{2}\right)$

(b) on the opposite side of a point $\left(0, \frac{1}{2}\right)$

(c) same lines
(d) perpendicular lines

61. Locus of the image of the point (2, 3) in the line $(x - 2y + 3) + \lambda(2x - 3y + 4) = 0$ is $(\lambda \in R)$

(a) $x^2 + y^2 - 3x - 4y - 4 = 0$

(b) $2x^2 + 3y^2 + 2x + 4y - 7 = 0$

(c) $x^2 + y^2 - 2x - 4y + 4 = 0$

(d) None of the above

Applications of Straight Line in Geometry

62. A square of area 25 sq units is formed by taking two sides as $3x + 4y = k_1$ and $3x + 4y = k_2$, then $|k_1 - k_2|$ is
(a) 5 units (b) 1 unit
(c) 25 units (d) None of these

63. Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$. If the orthocentre of the triangle is the origin, then coordinates of third vertex are
(a) (4, 7) (b) $(-4, -7)$
(c) $(-4, 7)$ (d) None of these

64. Orthocentre of triangle with vertices (0, 0), (3, 4) and (4, 0) is
(a) (3, 5/4) (b) (3, 12)
(c) (3, 3/4) (d) (3, 9)

65. Orthocentre of the triangle formed by the lines $x + y = 1$ and $xy = 0$ is
(a) (0, 0) (b) (0, 1)
(c) (1, 0) (d) $(-1, 1)$

66. If t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$, where λ is an arbitrary constant. Then, the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point whose coordinates are
(a) $(a, 0)$ (b) $(-a, 0)$ (c) $(0, a)$ (d) $(0, -a)$

67. The point moves such that the area of the triangle formed by it with the points (1, 5) and (3, – 7) is 21 sq units. The locus of the point is
(a) $6x + y - 32 = 0$ (b) $6x - y + 32 = 0$
(c) $x + 6y - 32 = 0$ (d) $6x - y - 32 = 0$

Round II (Mixed Bag)

Only One Correct Option

1. A system of lines is given as $y = m_i x + c_i$, where m_i can take any value out of 0, 1, – 1 and when m_i is positive, then c_i can be 1 or – 1, when m_i equal 0, c_i can be 0 or 1 and when m_i equals to – 1, c_i can take 0 or 2. Then, the area enclosed by all these straight line is
(a) $\frac{3}{\sqrt{2}}(\sqrt{2} - 1)$ sq units (b) $\frac{3}{\sqrt{2}}$ sq units
(c) $\frac{3}{2}$ sq units (d) None of these

2. The number of points on the line $3x + 4y = 5$, which are at a distance of $\sec^2\theta + 2\,\text{cosec}^2\theta$, $\theta \in R$, from the point (1, 3) is
(a) 1 (b) 2 (c) 3 (d) infinite

3. The base BC of ΔABC is bisected at (p, q) and equation of sides AB and AC are $px + qy = 1$ and $qx + py = 1$ respectively. Then, the equation of the median through A is
(a) $(2pq - 1)(px + qy - 1) = (p^2 + q^2 - 1)(qx + py - 1)$
(b) $(px + qy - 1)(qx + py - 1) = 0$
(c) $(px + qy - 1)(qx - py - 1) = 0$
(d) None of the above

4. A beam of light is sent along the line $x - y = 1$. Which after refracting from the x-axis enters the opposite side by turning through 30° towards the normal at the point of incidence on the x-axis. Then, the equation of the refracted ray is
(a) $(2 - \sqrt{3})x - y = 2 + \sqrt{3}$ (b) $(2 + \sqrt{3})x - y = 2 + \sqrt{3}$
(c) $(2 - \sqrt{3})x + y = 2 + \sqrt{3}$ (d) None of these

5. If $5a + 4b + 20c = t$, then the value of t for which the line $ax + by + c - 1 = 0$ always passes through a fixed point is
(a) 0 (b) 20
(c) 30 (d) None of these

6. Consider the family of lines

$$(x + y - 1) + \lambda(2x + 3y - 5) = 0$$

and $$(3x + 2y - 4) + \mu(x + 2y - 6) = 0,$$

equation of a straight line that belongs to both the families is
(a) $x - 2y - 8 = 0$ (b) $x - 2y + 8 = 0$
(c) $2x + y - 8 = 0$ (d) $2x - y - 8 = 0$

7. If the distance of any point (x, y) from the origin is defined as $d(x, y) = \max\{|x|, |y|\}$, $d(x, y) = a$, non-zero constant, then the locus is a
(a) circle (b) straight line
(c) square (d) triangle

8. The straight line $ax + by + c = 0$ where $abc \neq 0$ will pass through the first quadrant if
(a) $ac > 0$, $bc > 0$
(b) $c > 0$ and $bc < 0$
(c) $bc > 0$ and/or $ac > 0$
(d) $ac < 0$ and/or $bc < 0$

9. The lines $y = m_1x$, $y = m_2x$ and $y = m_3x$ make equal intercepts on the line $x + y = 1$, then
(a) $2(1 + m_1)(1 + m_3) = (1 + m_2)(2 + m_1 + m_3)$
(b) $(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$
(c) $(1 + m_1)(1 + m_2) = (1 + m_3)(2 + m_1 + m_3)$
(d) None of the above

10. One diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is (1, 2). Then, the equations of the sides of the square passing through this vertex are
(a) $23x + 7y = 9, 7x + 23y = 53$
(b) $23x - 7y + 9 = 0, 7x + 23y + 53 = 0$
(c) $23x - 7y - 9 = 0, 7x + 23y - 53 = 0$
(d) None of the above

11. A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then,
(a) $a = 64/115, b = 112/15$ (b) $a = 14/15, b = -8/115$
(c) $a = 64/115, b = -8/115$ (d) $a = 64/15, b = 14/15$

12. The straight lines $4ax + 3by + c = 0$, where $a + b + c = 0$, are concurrent at the point
(a) (4, 3) (b) (1/4, 1/3)
(c) (1/2, 1/3) (d) None of these

More Than One Correct Option

13. Two sides of a triangle are the lines $(a + b)x + (a - b)y - 2ab = 0$ and $(a - b)x + (a + b)y - 2ab = 0$. If the triangle is isosceles and the third side passes through point $(b - a, a - b)$, then the equation of third side can be
(a) $x + y = 0$ (b) $x = y + 2(b - a)$
(c) $x - b + a = 0$ (d) $y - a + b = 0$

14. If $bx + cy = a$, where a, b, c are of the same sign, be a line such that the area enclosed by the line and the axes of reference is $\frac{1}{8}$ sq unit, then
(a) b, a, c are in GP (b) $b, 2a, c$ are in GP
(c) $b, \frac{a}{2}, c$ are in AP (d) $b, -2a, c$ are in GP

15. Consider the straight lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$. The line $6x + 6y + 7 = 0$ is
(a) bisector of the angle including origin
(b) bisector of acute angle
(c) bisector of obtuse angle
(d) None of the above

Comprehension Based Questions

Passage

Directions (Q. Nos. 16-18) *Let L be the line belonging to the family of the straight lines* $(a + 2b)x + (a - 3b)y + a - 8b = 0$, $a, b \in R$, *which is farthest from the point* (2, 2).

16. The equation of line L is
(a) $x + 4y + 7 = 0$ (b) $2x + 3y + 4 = 0$
(c) $4x - y - 6 = 0$ (d) None of these

17. Area formed by the line L with coordinate axis is
(a) 4/3 sq units (b) 9/2 sq units
(c) 49/8 sq units (d) None of these

18. If L is concurrent with the lines $x - 2y + 1 = 0$ and $3x - 4y + \lambda = 0$, then the value of λ is
(a) 2 (b) 1 (c) –4 (d) 5

Assertion and Reason

Directions (Q. Nos. 19-21) *Each of these questions contains two statements : Statement* I *(Assertion) and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

19. Consider a, b and c are variables.

Statement I Such that $3a + 2b + 4c = 0$, then the family of lines given by $ax + by + c = 0$ pass through a fixed point (3, 2).

Statement II The equation $ax + by + c = 0$ will represent a family of straight line passing through a fixed point iff there exists a linear relation between a, b and c.

20. **Statement I** Each point on the line $y - x + 12 = 0$ is equidistant from the lines $4y + 3x - 12 = 0$, $3y + 4x - 24 = 0$.

Statement II The locus of a point which is equidistant from two given lines is the angular bisector of the two lines.

21. **Statement I** If the points (1, 2) and (3, 4) be on the same side of the line $3x - 5y + \lambda = 0$, then $\lambda < 7$ or $\lambda > 11$

Statement II If the points (x_1, y_1) and (x_2, y_2) be on the same side of the line $f(x, y) \equiv ax + by + c = 0$, then $\frac{f(x_1, y_1)}{f(x_2, y_2)} < 0$

Previous Years' Questions

22. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then, the set of all possible values of a in the interval **[AIEEE 2011]**

(a) $(-1, 1]$ (b) $(0, \infty)$ (c) $[1, \infty)$ (d) $(-1, \infty)$

23. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then, the distance between L and K is **[AIEEE 2010]**

(a) $\frac{23}{\sqrt{15}}$ (b) $\sqrt{17}$ (c) $\frac{17}{\sqrt{15}}$ (d) $\frac{23}{\sqrt{17}}$

24. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for **[AIEEE 2009]**

(a) exactly one value of p
(b) exactly two values of p
(c) more than two values of p
(d) no values of p

25. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept -4. Then, a possible value of k is **[AIEEE 2008]**

(a) 4 (b) 1 (c) 2 (d) -2

26. Let $P = (-1, 0), Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. The equation of the bisector of the $\angle PQR$ is **[AIEEE 2007]**

(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$

(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

27. A straight line through the point $A(3, 4)$ is such that is intercept between the axis is bisected at A. Its equation is **[AIEEE 2006]**

(a) $4x + 3y = 24$ (b) $3x + 4y = 25$
(c) $x + y = 7$ (d) $3x - 4y + 7 = 0$

28. If (a, a^2) falls inside the angle made by the lines, $y = \frac{x}{2}, x > 0$ and $y = 3x, x > 0$, then $a \in$ **[AIEEE 2006]**

(a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(-3, -\frac{1}{2}\right)$ (c) $\left(0, \frac{1}{2}\right)$ (d) $(3, \infty)$

29. If non-zero numbers a, b and c are in HP, then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point. That point is **[AIEEE 2005]**

(a) $\left(1, -\frac{1}{2}\right)$ (b) $(1, -2$

(c) $(-1, -2)$ (d) $(-1, 2)$

30. The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$, is **[AIEEE 2005]**

(a) above the x-axis at a distance of (2/3) from it
(b) above the x-axis at a distance of (3/2) from it
(c) below the x-axis at a distance of (2/3) from it
(d) below the x-axis at a distance of (3/2) from it

31. The equation of the bisector of the actue angles between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is **[AIEEE 2004]**

(a) $99x - 27y - 81 = 0$
(b) $11x - 3y + 9 = 0$
(c) $21x + 77y - 101 = 0$
(d) $21x + 77y + 101 = 0$

32. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a ΔABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line **[AIEEE 2004]**

(a) $2x + 3y = 9$ (b) $2x - 3y = 7$
(c) $3x + 2y = 5$ (d) $3x - 2y = 3$

33. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is -1, is **[AIEEE 2004]**

(a) $\frac{x}{2} + \frac{y}{3} = -1, \frac{x}{-2} + \frac{y}{1} = -1$

(b) $\frac{x}{2} - \frac{y}{3} = -1, \frac{x}{-2} + \frac{y}{1} = -1$

(c) $\frac{x}{2} + \frac{y}{3} = 1, \frac{x}{2} + \frac{y}{1} = 1$

(d) $\frac{x}{2} - \frac{y}{3} = 1, \frac{x}{-2} + \frac{y}{1} = 1$

Answers

Round I

1. (b)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (b)
11. (a)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (b)
21. (a)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (c)
31. (b)	**32.** (c)	**33.** (c)	**34.** (c)	**35.** (c)	**36.** (d)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (b)
41. (b)	**42.** (a)	**43.** (c)	**44.** (a)	**45.** (b)	**46.** (d)	**47.** (c)	**48.** (a)	**49.** (b)	**50.** (c)
51. (c)	**52.** (c)	**53.** (c)	**54.** (b)	**55.** (d)	**56.** (c)	**57.** (c)	**58.** (b)	**59.** (a)	**60.** (a)
61. (c)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (a)	**66.** (b)	**67.** (a)			

Round II

1. (c)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (c)
11. (c)	**12.** (b)	**13.** (a,b)	**14.** (b,d)	**15.** (a,b)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (d)	**20.** (a)
21. (c)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (d)
31. (c)	**32.** (a)	**33.** (d)							

the **Guidance**

Round I

1. Let the equation of a line be $y = mx + c$, where m and c are parameters.

$\therefore$ Number of parameters $= 2$

2. Slope of a line $= \tan\theta$

$$= \left|\frac{a}{a}\right| = \pm 1$$

($\because$ line can be equally intercept in any of the four quadrants)

$\therefore$ slope $= -1$

3. $\frac{x}{a} + \frac{y}{b} = 1$

It passes $(2, -3)$ and $(4, -5)$, then $\frac{2}{a} - \frac{3}{b} = 1$ and $\frac{4}{a} - \frac{5}{b} = 1$

On solving, we get

$$a = -1, b = -1$$

4. Equation of a line is $y - y_1 = \left(\frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}\right)(x - x_1)$

$$\Rightarrow \quad y + 2 = \left(\frac{-\sqrt{3} \pm \tan 60°}{1 \mp (-\sqrt{3})\tan 60°}\right)(x - 3)$$

$$\Rightarrow \quad y + 2 = \frac{-\sqrt{3} \pm \sqrt{3}}{1 \mp (-3)}(x - 3)$$

$$\Rightarrow \quad y + 2 = \frac{-\sqrt{3} + \sqrt{3}}{1 - (-3)}(x - 3) \text{ and } y + 2 = \frac{-\sqrt{3} - \sqrt{3}}{1 + (-3)}(x - 3)$$

$$\Rightarrow \quad y + 2 = 0$$

and $\quad \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

5. Let the equation of line is

$$y = mx + c$$

$\because \quad m = \tan 30° = \frac{1}{\sqrt{3}}$

and $\quad c = -2$

($\because$ it is intercepted in negative axis of y with an angle of 30°)

$\therefore$ The equation of required line is

$$y = \frac{x}{\sqrt{3}} - 2$$

$$\Rightarrow \quad \sqrt{3}y - x + 2\sqrt{3} = 0$$

6. Slope of refracted ray is

$$-\tan 60° = -\sqrt{3}$$

It passes through $(1, 0)$

$\therefore \quad y = -\sqrt{3}(x - 1)$

$$\Rightarrow \quad \sqrt{3}x + y - \sqrt{3} = 0$$

7. The given determinant represents a straight line.

8. $-5 = \frac{2x_1 + 0}{3} \Rightarrow x_1 = \frac{-15}{2}$

and $\quad 4 = \frac{0 + y_1}{3}$

$$\Rightarrow \quad y_1 = 12$$

$\therefore$ Equation of line is $\frac{x}{-15/2} + \frac{y}{12} = 1$

$$\Rightarrow \quad 8x - 5y + 60 = 0$$

9. Let a and b the intercepts made by the straight line on the axes.

Given that, $a + b = \frac{ab}{2}$

$\Rightarrow \quad \frac{2a + 2b}{ab} = 1 \Rightarrow \frac{2}{a} + \frac{2}{b} = 1$

On comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we get $x = 2$, $y = 2$

$\therefore$ Required point is $(2, 2)$.

So, the straight line passes through the point $(2, 2)$.

10. Since, a, b and c are in HP.

$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{1}{\alpha} - \frac{2}{\beta} + \frac{1}{\chi} = 0$

So, straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point $(1, -2)$.

11. P is the mid-point of A and B.

$\Rightarrow \quad 3 = \frac{a+0}{2} \Rightarrow a = 6$

and $\quad 2 = \frac{b+0}{2} \Rightarrow b = 4$

$\therefore$ Equation of a line AB is

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{6} + \frac{y}{4} = 1$

$\Rightarrow \quad 2x + 3y = 12$

12. Since, $ax + by + c = 0$ is always passes through $(1, -2)$.

$\therefore \quad a - 2b + c = 0$

$\Rightarrow \quad 2b = a + c$

Therefore, a, b and c are in AP.

13. Lines are, $L_1 : 3x + 4y + 2 = 0$, $L_2 : 3x + 4y + 5 = 0$,

$L_3 : 3x + 4y - 5 = 0$

Ratio of distance $= \frac{\text{Distance between } L_1 \text{ and } L_2}{\text{Distance between } L_1 \text{ and } L_3}$

$= \frac{\frac{|2-5|}{\sqrt{3^2+5^2}}}{\frac{|2+5|}{\sqrt{3^2+5^2}}} = \frac{3}{7}$

$\left(\because \text{ distance between lines} = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}\right)$

14. Since, $P(1, 2)$ is mid-point of AB. Therefore, coordinate of A and B are $(2, 0)$ and $(0, 4)$ respectively.

$\therefore$ Equation of line AB is

$y - 0 = \frac{4}{-2}(x - 2)$

$\Rightarrow \quad 2x + y = 4$

15. Since, the line intersect the X-axis to the left of origin. It means it intersect the negative X-axis. Clearly, line AB passes through the point $(-3, 0)$ and $m = -2$.

Equation of line in one point form is

$y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - 0 = -2(x + 3)$

$\Rightarrow \quad y = -2x - 6$

$\Rightarrow \quad 2x + y + 6 = 0$

16. Let the intercept form of a line is $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

Given, $\quad a + b = 1$ ...(ii)

and $\quad ab = -6$ ...(iii)

For solving Eqs. (ii) and (iii), to find the value of a and b.

Put $b = \frac{-6}{a}$ from Eq. (i) in Eq. (ii),

$a - \frac{6}{a} = 1$

$\Rightarrow \quad a^2 - 6 = a \Rightarrow a^2 - a - 6 = 0$

$\Rightarrow \quad (a + 2)(a - 3) = 0$

$\Rightarrow \quad a = 3, -2$

and $\quad b = -2, 3$

Put $a = 3, -2$ respectively in Eq. (i), we get the required equations of lines

$\frac{x}{3} + \frac{y}{-2} = 1$ or $\frac{x}{-2} + \frac{y}{3} = 1$

$\Rightarrow \quad 2x - 3y = 6$ or $-3x + 2y = 6$

$\Rightarrow \quad 2x - 3y - 6 = 0$ or $3x - 2y + 6 = 0$

17. Since, S is mid-point of QR.

$\therefore$ Coordinate of S are

$\left(\frac{6+7}{2}, \frac{-1+3}{2}\right) = \left(\frac{13}{2}, 1\right)$

$\therefore$ Slope of $PS = \frac{2-1}{2 - \frac{13}{2}} = -\frac{2}{9}$

The required equation which is passing through $(1, -1)$ and slope $-\frac{2}{9}$, is

$y + 1 = -\frac{2}{9}(x - 1)$

$\Rightarrow \quad 9y + 9 = -2x + 2$

$\Rightarrow \quad 2x + 9y + 7 = 0$

18. Let ABC be an isosceles right angled triangle at A with $AB = AC$.

$\Rightarrow \quad \angle ABC = \angle ACB = 45°$

Now, we have to determine the angle between AC and BC.

The slope of line BC *i.e.*, $3x + 4y = 4$ is

$$m_2 = -\frac{3}{4}$$

Let the slope of the line AC is $m_1 = m$.

$\because$ Angle between two lines is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \tan 45° = \left|\frac{m - \left(\frac{-3}{4}\right)}{1 + m\left(\frac{-3}{4}\right)}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}\right| \Rightarrow \pm 1 = \frac{4m + 3}{4 - 3m}$$

Taking positive sign, $1 = \dfrac{4m + 3}{4 - 3m}$

$\Rightarrow \quad 4 - 3m = 4m + 3$

$\Rightarrow \quad 4 - 3 = 4m + 3m$

$\Rightarrow \quad 7m = 1$

$\Rightarrow \quad m = \dfrac{1}{7}$

Taking negative sign, $-1 = \dfrac{4m + 3}{4 - 3m}$

$\Rightarrow \quad -4 + 3m = 4m + 3$

$\Rightarrow \quad -4 - 3 = m \Rightarrow m = -7$

Hence, equation of line AB and line AC are

$$y - 2 = \frac{1}{7}(x - 2) \text{ and } y - 2 = -7(x - 2)$$

$\Rightarrow \quad 7y - 14 = x - 2$ and $y - 2 = -7x + 14$

$\Rightarrow \quad x - 7y + 12 = 0$ and $7x + y - 16 = 0$

19. Let slope of a line be m.

Now, the equation of a line passing through (1, 0) is

$$y - 0 = m(x - 1) \Rightarrow mx - y - m = 0$$

Distance from origin $= \dfrac{\sqrt{3}}{2}$

$$\Rightarrow \quad \frac{|-m|}{\sqrt{1 + m^2}} = \frac{\sqrt{3}}{2} \Rightarrow 4m^2 = 3(1 + m^2)$$

$\Rightarrow \quad m^2 = 3$

$\Rightarrow \quad m = \pm\sqrt{3}$

$\therefore$ Equations are

$$\sqrt{3}x - y - \sqrt{3} = 0 \text{ and } -\sqrt{3}x - y + \sqrt{3} = 0$$

$\Rightarrow \quad \sqrt{3}x - y - \sqrt{3} = 0$ and $\sqrt{3}x + y - \sqrt{3} = 0$

20. Equation of line l_1 is

$$x - 2y = 3 \quad \ldots(i)$$

Slope of Eq. (i),

$$m_1 = -\left(\frac{1}{-2}\right) \Rightarrow m_1 = \frac{1}{2}$$

Angle between two lines is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Given, $\quad \theta = 45°, m_1 = \dfrac{1}{2}, m_2 = m$ (let)

$$\therefore \quad \tan 45° = \left|\frac{\frac{1}{2} - m}{1 + \frac{1}{2}m}\right| \Rightarrow 1 = \left|\frac{1 - 2m}{2 + m}\right| \Rightarrow \frac{1 - 2m}{2 + m} = \pm 1$$

Taking positive sign,

$$\frac{1 - 2m}{2 + m} = 1 \Rightarrow 1 - 2m = 2 + m$$

$$\Rightarrow \quad 1 - 2 = 3m \Rightarrow m = -\frac{1}{3}$$

Taking negative sign,

$$\frac{1 - 2m}{2 + m} = -1$$

$\Rightarrow \quad 1 - 2m = -2 - m \Rightarrow 1 + 2 = 2m - m$

$\Rightarrow \quad m = 3$

Now, equation of line l_2 by using

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2 = -\frac{1}{3}(x - 3) \quad \left(\text{when } m = -\frac{1}{3}, x_1 = 3, y_1 = 2\right)$$

$\Rightarrow \quad 3y - 6 = -x + 3$

$\Rightarrow \quad x + 3y - 9 = 0$

Again, $\quad y - 2 = 3(x - 3) \quad \left(\text{when } m = 3, x_1 = 3, y_1 = 2\right)$

$\Rightarrow \quad y - 2 = 3x - 9 \Rightarrow 3x - y - 9 + 2 = 0$

$\Rightarrow \quad 3x - y - 7 = 0$

21. In a figure, PA is the incident ray and AR is the reflected ray, which makes an angle θ from the X-axis.

It is clear from the figure that

$$AS \perp OX$$

It means AS bisect the angle PAR.

Then, $\angle PAS = \angle RAS$

$\Rightarrow$ $\angle RAX = \angle PAO = \theta$ (let)

$\Rightarrow$ $\angle XAP = 180 - \theta$

Slope of $AR = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-0}{5-k}$ ...(i)

[where point A is $(k, 0)$]

Slope of $AP = \tan(180° - \theta)$

$$= -\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-0}{1-k} \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii),

$$\frac{3}{5-k} = -\frac{2}{1-k} \Rightarrow 3 - 3k = -10 + 2k$$

$$\Rightarrow \quad 5k = 13 \Rightarrow k = \frac{13}{5}$$

Hence, the coordinate of A is $\left(\dfrac{13}{5}, 0\right)$.

22. Slope of line joining $(2, 3)$ and $(3, -1)$ is

$$m_1 = \frac{-1-3}{3-2} = -4$$

Slope of line perpendicular to it is, $m_2 = \dfrac{-1}{-4} = \dfrac{1}{4}$

$\therefore$ Equation of a line passing through $(5, 2)$ is

$$y - 2 = \frac{1}{4}(x - 5)$$

$$\Rightarrow \quad x - 4y + 3 = 0$$

23. Equation of a line passing through the intersection of given lines is

$$(2x + y - 5) + \lambda(x + 3y + 8) = 0$$

$$\Rightarrow \quad x(2 + \lambda) + y(1 + 3\lambda) - 5 + 8\lambda = 0 \quad \text{...(i)}$$

It is parallel to $3x + 4y = 7$

$$\Rightarrow \quad \frac{-(2+\lambda)}{1+3\lambda} = \frac{-3}{4}$$

$$\Rightarrow \quad \lambda = 1$$

Put in Eq. (i),

$$3x + 4y + 3 = 0$$

24. Equation of a line passing through the point of intersection of lines is

$$x - y + 1 + \lambda(2x - 3y + 5) = 0$$

$$\Rightarrow \quad x(1 + 2\lambda) + y(-1 - 3\lambda) + 1 + 5\lambda = 0 \quad \text{...(i)}$$

Its distance from point $(3, 2) = \dfrac{7}{5}$

$$\Rightarrow \quad \frac{|3(1+2\lambda) + 2(-1-3\lambda) + 1 + 5\lambda|}{\sqrt{(1+2\lambda)^2 + (-1-3\lambda)^2}} = \frac{7}{5}$$

$$\Rightarrow \quad \frac{|2 + 5\lambda|}{\sqrt{13\lambda^2 + 10\lambda + 2}} = \frac{7}{5}$$

On squaring, we get

$$25(4 + 25\lambda^2 + 20\lambda) = 49(13\lambda^2 + 10\lambda + 2)$$

$$\Rightarrow \quad 6\lambda^2 - 5\lambda - 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{6}$$

On putting $\lambda = 1, -\dfrac{1}{6}$ in Eq. (i) respectively, we get

$$3x - 4y + 6 = 0 \text{ and } 4x - 3y + 1 = 0$$

25. Equation of a line perpendicular to $y = x$ is $x + y + \lambda = 0$ and passes through $(3, 2)$.

$\therefore$ $3 + 2 + \lambda = 0 \Rightarrow \lambda = -5$

$\Rightarrow$ Equation is $x + y - 5 = 0$

26. Equation of line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{...(i)}$$

Given, $a + b = 9$ ...(ii)

And line (i) passes through the point $(2, 2)$ *i.e.*, it will satisfy the line (i) *i.e.*, put $x = 2, y = 2$ in Eq. (i),

$$\Rightarrow \quad \frac{2}{a} + \frac{2}{b} = 1 \quad \text{...(iii)}$$

On solving Eqs. (i) and (ii), we get

$\Rightarrow$ $a = 6$ or $3 \Rightarrow b = 3$ or 6

Hence, Eq. (i) becomes

$$\frac{x}{6} + \frac{y}{3} = 1 \quad \text{or} \quad \frac{x}{3} + \frac{y}{6} = 1$$

$\Rightarrow$ $3x + 6y = 18$ or $6x + 3y = 18$

$\Rightarrow$ $x + 2y = 6$ or $2x + y = 6$

27. The intersection point of $y - x + 7 = 0$ and $y + 2x - 2 = 0$ is $(3, -4)$.

$\therefore$ Equation of straight line joining from origin to the point $(3, -4)$ is

$$y - 0 = \frac{-4}{3}(x - 0) \Rightarrow 3y = -4x$$

$$\Rightarrow \quad 4x + 3y = 0$$

28. Any line which is perpendicular to $\sqrt{3}\sin\theta + 2\cos\theta = \dfrac{4}{r}$ is

$$\sqrt{3}\sin\left(\frac{\pi}{2} + \theta\right) + 2\cos\left(\frac{\pi}{2} + \theta\right) = \frac{k}{r} \quad \text{...(i)}$$

Since, it is passing through $\left(-1, \dfrac{\pi}{2}\right)$.

$$\therefore \quad \sqrt{3}\sin\pi + 2\cos\pi = \frac{k}{-1} \Rightarrow k = 2$$

On putting $k = 2$ in Eq. (i), we get

$$\sqrt{3}\cos\theta - 2\sin\theta = \frac{2}{r}$$

$$\Rightarrow \quad 2 = \sqrt{3}r\cos\theta - 2r\sin\theta$$

29. Let the given points are $A(1, 0)$ and $B(2, 3)$. Let the line PQ divide AB in the ratio $1 : n$ at R.

Using internally ratio

$$R=\left(\frac{1\times x_2+n\times x_1}{1+n},\ \frac{1\times y_2+n\times y_1}{1+n}\right)$$
$$=\left(\frac{1\times 2+n\times 1}{1+n},\ \frac{1\times 3+n\times 0}{1+n}\right)$$
$$=\left(\frac{n+2}{n+1},\ \frac{3}{1+n}\right)$$

Also, $PQ \perp AB$

Let slope of line PQ is m.

$\therefore$ Slope of line $PQ \times$ Slope of line $AB=-1$ $\quad(\because m_1 m_2=-1)$

$$\Rightarrow \quad m\times\frac{y_2-y_1}{x_2-x_1}=-1 \Rightarrow m\times\frac{3-0}{2-1}=-1$$

$$\Rightarrow \quad m\times 3=-1 \ \Rightarrow\ m=-\frac{1}{3}$$

Now, equation of line PQ by using

$$y-y_1=m(x-x_1)$$

$$\Rightarrow \quad y-\frac{3}{1+n}=\frac{-1}{3}\left(x-\frac{n+2}{n+1}\right)$$

$$\Rightarrow \quad \frac{3(n+1)y-9}{1+n}=\frac{-x(n+1)+(n+2)}{n+1}$$

$$\Rightarrow \quad 3(n+1)y-9=-x(n+1)+(n+2)$$

$$\Rightarrow \quad x(n+1)+3(n+1)y=n+2+9$$

$$\Rightarrow \quad x(n+1)+3(n+1)y=n+11$$

30. Given line AB makes 0 intercepts on x-axis and y-axis or $(x_1, y_1)\equiv(0,0)$ and the line is perpendicular to line CD, $3x+4y+6=0$.

$\therefore$ Slope of required line which is perpendicular to $3x+4y+6=0$ is 4/3.

$\therefore$ Required line which is passing through origin and having slope 4/3, is

$$y-0=\frac{4}{3}(x-0)\Rightarrow 4x-3y=0$$

31. The point of intersection of lines $2x-3y+4=0$ and $3x+4y-5=0$ is $\left(-\frac{2}{34}, \frac{22}{17}\right)$.

The slope of required line which is perpendicular to $6x-7y+3=0$ is $-\frac{7}{6}$.

$\therefore$ Equation of required line

$$y-\frac{22}{17}=-\frac{7}{6}\left(x+\frac{2}{34}\right)$$

$$\Rightarrow \quad \frac{6(17y-22)}{17}=-\frac{7(34x+2)}{34}$$

$$\Rightarrow \quad 119x+102y=125$$

32. From the given equations, we get

$$m^2+am+2=0$$

Since, m is real, $a^2\geq 8\Rightarrow |a|\geq 2\sqrt{2}$

So, least value of $|a|$ is $2\sqrt{2}$.

33. The intersection point of lines $x-2y=1$ and $x+3y=2$ is $\left(\frac{7}{5}, \frac{1}{5}\right)$.

$\because$ Required line is parallel to $3x+4y=0$.

$\therefore$ The slope of required line $=-\frac{3}{4}$

$\therefore$ Equation of required line which passes through $\left(\frac{7}{5}, \frac{1}{5}\right)$ and having slope $-\frac{3}{4}$, is

$$y-\frac{1}{5}=\frac{-3}{4}\left(x-\frac{7}{5}\right)$$

$$\Rightarrow \quad \frac{3x}{4}+y=\frac{21}{20}+\frac{1}{5}$$

$$\Rightarrow \quad \frac{3x+4y}{4}=\frac{21+4}{20}$$

$$\Rightarrow \quad 3x+4y=5\Rightarrow 3x+4y-5=0$$

34. Since, the given lines are concurrent.

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0 \ \Rightarrow\ a^3+b^3+c^3-3abc=0$$

$$\Rightarrow \quad (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$$

$$\Rightarrow \quad \frac{(a+b+c)}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}=0$$

$$\Rightarrow \quad a+b+c=0 \qquad (\text{as } a\neq b\neq c)$$

35. Line perpendicular to the given line $\frac{x}{a}-\frac{y}{b}=1$ is

$$\frac{1}{b}x+\frac{1}{a}y+\lambda=0 \qquad \ldots(\text{i})$$

According to the question, line (i) is passing through the point $P(a, 0)$.

$$\therefore \quad \frac{a}{b}+0+\lambda=0\Rightarrow\lambda=-\frac{a}{b}$$

On putting the value of λ in Eq. (i), we get

$$\frac{x}{b}+\frac{y}{a}-\frac{a}{b}=0\Rightarrow ax+by=a^2$$

36. Equation of a line perpendicular to $3x+y=3$ is

$$x-3y+\lambda=0$$

It passes (2, 3).

$$\Rightarrow \quad 2-6+\lambda=0\Rightarrow\lambda=4$$

$\therefore$ Equation is

$$x-3y+4=0$$

i.e., $\quad y=\frac{x}{3}+\frac{4}{3}$

So, y-intercept is $\frac{4}{3}$.

37. Given lines are $3x + 4y = 5$, $5x + 4y = 4$ and $\lambda x + 4y = 6$ These three lines meet at a point, if the point of intersection of first two lines lies on the third line.

Now, point of intersection of line $3x + 4y = 5$ and $5x + 4y = 4$ is $\left(-\frac{1}{2}, \frac{13}{8}\right)$.

The line $\lambda x + 4y = 6$ passes through the point $\left(-\frac{1}{2}, \frac{13}{8}\right)$.

$$\therefore \quad \lambda\left(-\frac{1}{2}\right) + 4\left(\frac{13}{8}\right) = 6 \Rightarrow -\lambda + 13 = 12$$

$$\Rightarrow \quad \lambda = 1$$

38. Given, equation of line is $\sqrt{3}x + y + 2 = 0$

$$\Rightarrow \quad \sqrt{3}\,x + y = -2 \Rightarrow -\sqrt{3}x - y = 2$$

On dividing above equation by

$$\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$$

i.e., $\sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

$$\Rightarrow \quad \frac{-\sqrt{3}}{2}x - \frac{1}{2}y = \frac{2}{2}$$

$$\Rightarrow \quad -\cos 30^\circ x - \sin 30^\circ y = 1$$

($\because$ for convert in the form of $x\cos\alpha + y\sin\alpha = p$)

$$\Rightarrow \quad \cos(180^\circ + 30^\circ)x + \sin(180^\circ + 30^\circ)y = 1$$

($\because \cos\theta$ and $\sin\theta$ both are negative in third quadrant)

$$\Rightarrow \quad (\cos 210^\circ)x + (\sin 210^\circ)y = 1$$

On comparing with $x\cos\theta + y\sin\theta = p$, we get

$$\theta = 210^\circ \quad \text{and} \quad p = 1$$

39. Equation of line through $(h, 3)$ and $(4, 1)$ by using

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 3 = \frac{1-3}{4-h}(x - h)$$

$$\Rightarrow \quad y - 3 = \frac{-2}{4-h}(x - h)$$

$$\Rightarrow \quad y - 3 = -\frac{2x}{4-h} + \frac{2h}{4-h}$$

$$\Rightarrow \quad y = -\frac{2x}{4-h} + \left(3 + \frac{2h}{4-h}\right) = 0 \qquad \text{...(i)}$$

Given equation of line is

$$7x - 9y - 19 = 0$$

$$\Rightarrow \quad y = \frac{7}{9}x - \frac{19}{9} \qquad \text{...(ii)}$$

Compare lines (i) and (ii) with $y = mx + c$

$$\therefore \quad m_1 = -\frac{2}{4-h} \text{ and } m_2 = \frac{7}{9}$$

$\because$ Lines (i) and (ii) are perpendicular to each other, *i.e.*,

Slope of line (i) $\times$ Slope of line (ii) $= -1$

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad -\frac{2}{4-h} \times \frac{7}{9} = -1 \Rightarrow 14 = 9(4 - h)$$

$$\Rightarrow \quad 14 = 36 - 9h \Rightarrow 9h = 22 \Rightarrow h = \frac{22}{9}$$

40. $y = (2 - \sqrt{3})(x + 5)$ and $y = (2 + \sqrt{3})(x - 7)$

Slopes are,

$$m_1 = 2 - \sqrt{3}, m_2 = 2 + \sqrt{3}$$

Angle between two lines,

$$\tan\theta = \left|\frac{2 + \sqrt{3} - (2 - \sqrt{3})}{1 + (2 + \sqrt{3})(2 - \sqrt{3})}\right|$$

$$\Rightarrow \quad \tan\theta = |\sqrt{3}|$$

i.e., $\tan\theta = \sqrt{3}$ and $\tan\theta = -\sqrt{3}$

$$\therefore \quad \theta = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

41. Given, equation of lines are

$$\sqrt{3}x + y = 1 \Rightarrow y = -\sqrt{3}x + 1$$

and $x + \sqrt{3}y = 1 \Rightarrow \sqrt{3}y = -x + 1$

$$\Rightarrow \quad y = \frac{-1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$$

On comparing with $y = m_1x + c_1$ and $y = m_2x + c_2$, we get

$$m_1 = -\sqrt{3} \text{ and } m_2 = -\frac{1}{\sqrt{3}}$$

$$\therefore \quad \tan\theta = \left|\frac{-\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)}{1 + (-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right)}\right| \qquad \left(\because \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|\right)$$

$$\Rightarrow \quad \tan\theta = \left|\frac{-\sqrt{3} + \frac{1}{\sqrt{3}}}{1+1}\right| = \left|\frac{\frac{-3+1}{\sqrt{3}}}{2}\right|$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \tan 30^\circ \Rightarrow \theta = 30^\circ$$

42. Let equation of line is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$\therefore \quad \frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \frac{\sqrt{6}}{3}$$

$$\Rightarrow \quad \frac{x-1}{\cos\theta} = \frac{\sqrt{6}}{3}, \frac{y-2}{\sin\theta} = \frac{\sqrt{6}}{3}$$

$$x = \frac{\sqrt{6}}{3}\cos\theta + 1, \; y = \frac{\sqrt{6}}{3}\sin\theta + 2$$

It satisfies $x + y = 4$

$$\therefore \quad \frac{\sqrt{6}}{3}\cos\theta + 1 + \frac{\sqrt{6}}{3}\sin\theta + 2 = 4$$

$$\Rightarrow \quad \frac{\sqrt{6}}{3}\cos\theta + \frac{\sqrt{6}}{3}\sin\theta = 1$$

$\Rightarrow \quad \sqrt{6}\cos\theta + \sqrt{6}\sin\theta = 3$

$\Rightarrow \quad 6\cos^2\theta + 6\sin^2\theta + 2\sqrt{6}\sqrt{6}\cos\theta\sin\theta = 9$

$\Rightarrow \quad 6(\cos^2\theta + \sin^2\theta) + 2\times 6\cos\theta\sin\theta = 9$

$\Rightarrow \quad 6 + 6\sin 2\theta = 9$

$\Rightarrow \quad 6\sin 2\theta = 9 - 6 \Rightarrow \sin 2\theta = \frac{3}{6}$

$\Rightarrow \quad \sin 2\theta = \frac{1}{2} = \sin 30°$

$\Rightarrow \quad 2\theta = 30° \Rightarrow \theta = 15°$

or $\quad \sin 2\theta = \sin(180° - 30°)$

$\Rightarrow \quad 2\theta = 150° \Rightarrow \theta = 75°$

43. Two sides $x - 3y = 0$ and $3x + y = 0$ are perpendicular to each other. Therefore, its orthocentre is the point of intersection of

$$x - 3y = 0$$

and $\quad 3x + y = 0$ *i.e.*, $(0, 0)$

So, the line $3x - 4y = 0$ passes through the orthocentre of triangle.

44. The vertices of triangle are the intersection points of the lines

$$x + y = 0 \quad \ldots(i)$$

$$3x + y = 4 \quad \ldots(ii)$$

and $\quad x + 3y = 4 \quad \ldots(iii)$

On solving Eqs. (i) and (ii), (ii) and (iii), (iii) and (i), we get the vertices of triangles are $A(-2, 2)$, $B(1, 1)$ and $C(2, -2)$.

Now, $\quad AB = \sqrt{(1+2)^2 + (1-2)^2} = \sqrt{9+1} = \sqrt{10}$

$BC = \sqrt{(2-1)^2 + (-2-1)^2} = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$AC = \sqrt{(2+2)^2 + (-2-2)^2} = \sqrt{16+16} = 4\sqrt{2}$

$\because \quad AB = BC$

So, triangle is isosceles.

45. The equation of lines are

$$y - y_1 = \frac{m_1 \pm m_2}{1 \mp m_1 m_2}(x - x_1)$$

Since, $\quad m_1 = 1, \quad m_2 = 1$

$\therefore \quad y - 4 = \frac{1 \pm 1}{1 \mp 1}(x - 3)$

$\Rightarrow \quad y = 4 \quad \text{or} \quad x = 3$

Hence, the lines which make the triangle are $x - y = 2$, $x = 3$ and $y = 4$.

The intersection points of these lines are (6, 4), (3, 1) and (3, 4).

$\therefore$ Area of triangle

$= \frac{1}{2}|6(1-4) + 3(4-4) + 3(4-1)| = \frac{1}{2}|6(-3) + 3(0) + 3(3)|$

$= \frac{1}{2}|-18 + 0 + 9| = \frac{9}{2}$ sq units

46. Mid-point of the line joining the points $(4, -5)$ and $(-2, 9)$ is $\left(\frac{4-2}{2}, \frac{-5+9}{2}\right)$ *i.e.*, $(1, 2)$.

$\therefore$ Inclination of straight line passing through point $(-3, 6)$ and mid-point $(1, 2)$ is

$$m = \frac{2-6}{1+3} = \frac{-4}{4} = -1$$

$\therefore \quad \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$

47. Given equation of lines are

$$5x + 3y - 7 = 0 \quad \ldots(i)$$

and $\quad 15x + 9y + 14 = 0 \quad \text{or} \quad 5x + 3y + \frac{14}{3} = 0 \quad \ldots(ii)$

$\because$ Lines (i) and (ii) are parallel and c_1 and c_2 are of opposite signs, therefore these lines are on opposite sides of the origin.

So, the distance between them is

$$\left|\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right| + \left|\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right| = \left|-\frac{7}{\sqrt{5^2 + 3^2}}\right| + \left|\frac{14}{3\sqrt{5^2 + 3^2}}\right|$$

$$= \left|-\frac{7}{\sqrt{34}}\right| + \left|\frac{14}{3\sqrt{34}}\right| = \frac{35}{3\sqrt{34}}$$

48. On solving the equations $2x - 3y + 5 = 0$ and $3x + 4y = 0$, we get

$$x = \frac{-60}{51}, y = \frac{15}{17}$$

Point of intersection is $\left(\frac{-60}{51}, \frac{15}{17}\right)$

$\therefore \quad$ Distance $= \dfrac{\left|5\left(\frac{-60}{51}\right) - 2\left(\frac{15}{17}\right)\right|}{\sqrt{25+4}} = \frac{130}{17\sqrt{29}}$

49. Let the foot be $A(\alpha, \beta)$.

$$\beta = 3\alpha + 4 \quad \ldots(i)$$

and $\quad m_1 \times m_2 = -1$

$$\frac{\beta - 3}{\alpha - 2} \times 3 = -1$$

$\Rightarrow \quad 3\beta = -\alpha + 11 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$\alpha = -\frac{1}{10}, \beta = \frac{37}{10}$$

50. Lines are

$$L_1 : 4x + 3y + 10 = 0, L_2 : 5x - 12y + 26 = 0$$

$$L_3 : 7x + 24y - 50 = 0$$

By option method,

Distance of lines from (0, 0) are equal.

i.e., $\left|\frac{0+0+10}{5}\right| = \left|\frac{0+0+26}{13}\right| = \left|\frac{0+0-50}{25}\right|$

$\Rightarrow \quad 2 = 2 = 2$

51. The given line is $x \tan\alpha - y + c = 0$

or $\quad x \sin\alpha - y \cos\alpha + c \cos\alpha = 0$

$\therefore$ Length of perpendicular from $(a \cos\alpha, a \sin\alpha)$

$$= \frac{a \cos\alpha \sin\alpha - a \sin\alpha \cos\alpha + c \cos\alpha}{\sqrt{\sin^2\alpha + \cos^2\alpha}}$$

$$= \frac{c \cos\alpha}{1} = c \cos\alpha$$

52. Let the equation of line parallel to $x - 2y = 1$ is

$$x - 2y + \lambda = 0$$

Since, it is passes through (3, 5).

$\therefore \quad 3 - 10 + \lambda = 0$

$\Rightarrow \quad \lambda = 7$

$\therefore$ Line is $x - 2y + 7 = 0$

The point of intersection $2x + 3y - 14 = 0$ and $x - 2y + 7 = 0$ is (1, 4).

$\therefore$ The distance between (3, 5) and (1, 4)

$$= \sqrt{(3-1)^2 + (5-4)^2}$$

$$= \sqrt{4+1} = \sqrt{5}$$

53. Slope of a line $y = 3x - 1$ is $m = 3 \Rightarrow$ slope of a line parallel to it is, $m_1 = 3$

$\therefore$ Equation of a line is

$$y - 2 = 3(x - 1)$$

54. Given that, $\quad y = \sqrt{3}\,|x| + 2$

and $\quad PQ = 5$, so $QR = \frac{5\sqrt{3}}{2}$

So, coordinates of R are $\left(0, 2 + \frac{5\sqrt{3}}{2}\right)$ or $\left(0, \frac{4 + 5\sqrt{3}}{2}\right)$.

55. Let $B(x_1, y_1)$ and $C(x_2, y_2)$ are the vertices of a triangle.

$P\left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$ lies on the line $x - y + 5 = 0$.

$\therefore \quad \frac{x_1+1}{2} - \frac{y_1-2}{2} = -5$

$\Rightarrow \quad x_1 - y_1 = -13 \quad \ldots(i)$

Also, $\quad PN \perp AB$

$\therefore \quad \frac{y_1+2}{x_1-1} = -1$

$\Rightarrow \quad y_1 + 2 = -x_1 + 1$

$\Rightarrow \quad x_1 + y_1 = -1 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$x_1 = -7 \quad \text{and} \quad y_1 = 6$$

$\therefore$ Coordinates of B are $(-7, 6)$.

Similarly, the coordinates of C are $\left(\frac{11}{5}, \frac{2}{5}\right)$.

$\therefore$ Equation of BC is

$$(y - 6) = \frac{\frac{2}{5} - 6}{\frac{11}{5} + 7}(x + 7) \Rightarrow 14x + 23y - 40 = 0$$

56. Equation of the straight lines are

$3x - 4y + 7 = 0 \quad \ldots(i)$

and $\quad 12x + 5y - 2 = 0 \quad \ldots(ii)$

The equation of bisectors of the acute angle between these lines are

$$\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}} = \frac{12x + 5y - 2}{\sqrt{12^2 + 5^2}}$$

$\Rightarrow \quad \frac{3x - 4y + 7}{5} = \frac{12x + 5y - 2}{13}$

$\Rightarrow \quad 39x - 52y + 91 = 60x + 25y - 10$

$\Rightarrow \quad 21x + 77y - 101 = 0$

57. The equation of lines are $x + y = 0$ and $x - y = 0$.

$\therefore$ The equation of bisectors of the angles between these lines are

$$\frac{x+y}{\sqrt{1+1}} = \pm \frac{x-y}{\sqrt{1+1}}$$

$\Rightarrow \quad x + y = \pm (x - y)$

Taking positive sign, $x + y = (x - y)$

$\Rightarrow \quad y = 0$

Taking negative sign, $x + y = -(x - y)$

$\Rightarrow \quad x = 0$

Hence, the equation of bisectors are $x = 0$ and $y = 0$.

58. Equation belonging to both families will pass through two fixed points. First intersection point of lines $x + 2y = 0$ and $3x + 2y + 1 = 0$, is $\left(-\frac{1}{2}, \frac{1}{4}\right)$ and second intersection point of lines $x - 2y = 0$ and $x - y + 1 = 0$, is $(-2, -1)$.

Line passing through $\left(-\frac{1}{2}, \frac{1}{4}\right)$ and $(-2, -1)$ is

$$y - \frac{1}{4} = \frac{-1 - \frac{1}{4}}{-2 + \frac{1}{2}}\left(x + \frac{1}{2}\right) \Rightarrow 5x - 6y + 4 = 0$$

59. Equation of diagonal OB is $y = x$

And equation of diagonal AC is

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$x + y = 1$$

60. Since, $2x + y = 1$ and $2x + y = 7$ are parallel lines. $(2x + y - 1)(2x + y - 7)$ is positive at point $\left(0, \frac{1}{2}\right)$. So, lines are in the same side of a point.

61. The family of lines $(x - 2y + 3) + \lambda(2x - 3y + 4) = 0$ are concurrent at point $P(1, 2)$.

If image of point $A(2, 3)$ in the above variable line is $B(h, k)$, then $AP = BP$

$\Rightarrow \quad (h-1)^2 + (k-2)^2 = (2-1)^2 + (3-2)^2$

Hence, locus of point P is $x^2 + y^2 - 2x - 4y + 4 = 0$

62. Each side of square is 5 units, distance between given lines is 5 units,

i.e., $\quad \left|\frac{k_1 - k_2}{5}\right| = 5$

$\Rightarrow \quad |k_1 - k_2| = 25$

63. Let the coordinates of the third vertex A be (h, k).

Also, $\quad AD \perp BC$

$\therefore \quad \frac{k-0}{h-0} \times \left(\frac{4}{-7}\right) = -1$

$\Rightarrow \quad 7h = 4k \quad$ ...(i)

and $\quad OB \perp AC$

$\Rightarrow \quad \frac{k-3}{h+2} \times \left(-\frac{1}{5}\right) = -1$

$\Rightarrow \quad 5h - k + 13 = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$h = -4, \quad k = -7$$

Hence, the coordinates of third vertex are $(-4, -7)$.

64. Now, we take $BD \perp AC$ and $AE \perp BC$.

Slope of $BD = -\frac{3}{4}$

Equation of BD, $y - 0 = \frac{-3}{4}(x - 4)$

$\Rightarrow \quad 4y = -3x + 12$

$\Rightarrow \quad 3x + 4y - 12 = 0 \quad$ ...(i)

and $\quad$ slope of $AE = \frac{1}{4}$

$\therefore$ Equation of AE, $y - 0 = \frac{1}{4}(x - 0)$

$\Rightarrow \quad x - 4y = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = 3, \quad y = \frac{3}{4}$$

$\therefore$ Orthocentre of the triangle is $\left(3, \frac{3}{4}\right)$.

65. We have, $\quad x + y = 1 \quad$ ...(i)

and $\quad xy = 0 \quad$ ...(ii)

On putting $x = 1 - y$ from Eq. (i) into Eq. (ii), we get

$$(1 - y)\, y = 0$$

$\Rightarrow \quad y = 0, 1$

At $y = 0 \Rightarrow x = 1$

and at $y = 1 \Rightarrow x = 0$

So, coordinates of the vertices of a triangle are (0, 0), (1, 0) and (0, 1).

Since, the triangle is right angled at $O(0, 0)$.

So, point (0, 0) is its orthocentre.

66. Since, t_1, t_2 are the roots of the equation $t^2 + \lambda t + 1 = 0$.

$\therefore \quad t_1 + t_2 = -\lambda, \quad t_1 t_2 = 1$

The equation of a line passing through $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_2 = \frac{2}{t_1 + t_2}(x - at_2^2) \Rightarrow y - 2at_2 = -\frac{2}{\lambda}(x - at_2^2)$$

$$\Rightarrow \lambda y - 2a\lambda t_2 = -2x + 2at_2^2 \Rightarrow \lambda y + 2x = 2a(\lambda t_2 + t_2^2)$$

$$\Rightarrow \quad \lambda y + 2x = 2a(-1) \Rightarrow 2(x + a) + \lambda y = 0$$

So, fixed point is $(-a, 0)$.

67. Let the locus of point be (x, y).

Area of triangle with points $(x, y), (1, 5)$ and $(3, -7)$ is 21 sq units.

$$\therefore \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 5 & 1 \\ 3 & -7 & 1 \end{vmatrix} = 21$$

$$\Rightarrow \quad \frac{1}{2}[x(5+7) - y(1-3) + 1(-7-15)] = 21$$

$$\Rightarrow \quad \frac{1}{2}[12x + 2y - 22] = 21 \Rightarrow 6x + y - 32 = 0$$

Round II

1. Lines are $y = 1, y = 0$

$$y = -x, \quad y = -x + 2$$
$$y = x + 1, \quad y = x - 1$$

Area of $OABCDE$ = Area of $OBGF$

$$= \frac{3}{2} \times 1 = \frac{3}{2} \text{ sq units}$$

2. The perpendicular distance of $(1, 3)$ from the line $3x + 4y = 5$ is 2 units while

$$\sec^2\theta + 2\,\text{cosec}^2\theta \geq 3 \quad (\text{as } \sec^2\theta, \text{cosec}^2\theta \geq 1)$$

So, there will be two such points on the line.

3. Any line through A is given by

$$(px + qy - 1) + \lambda(qx + py - 1) = 0$$

which is passing through (p, q).

Hence, $$\lambda = -\frac{(p^2 + q^2 - 1)}{2pq - 1}$$

Thus, the required line is

$$(px + qy - 1) - \frac{(p^2 + q^2 - 1)}{(2pq - 1)} \cdot (qx + py - 1) = 0$$

$$\Rightarrow (2pq - 1)(px + qy - 1) - (p^2 + q^2 - 1)(qx + py - 1) = 0$$

4. From figure refracted ray makes an angle of 75° with positive direction of x-axis and passes through the point $(1, 0)$.

$\therefore$ Its equation is

$$(y - 0) = \tan(45° - 30°)(x - 1)$$

or $$y = (2 - \sqrt{3})(x - 1)$$

5. Equation of line $\frac{ax}{c-1} + \frac{by}{c-1} + 1 = 0$ has two independent parameters. It can pass through a fixed point, if it contains only one independent parameter. Now, there must be one relation between $\frac{a}{c-1}$ and $\frac{b}{c-1}$ independent of a, b and c so that $\frac{a}{c-1}$ can be expressed in terms of $\frac{b}{c-1}$ and straight line contains only one independent parameter. Now, that given relation can be expressed as $\frac{5a}{c-1} + \frac{4b}{c-1} = \frac{t - 20c}{c-1}$

RHS in independent of c, if $t = 20$

6. The family of lines

$$(x + y - 1) + \lambda(2x + 3y - 5) = 0$$

passes through a point such that

$$x + y - 1 = 0$$
$$2x + 3y - 5 = 0$$

i.e., $(-2, 3)$ and family of lines

$$(3x + 2y - 4) + \mu(x + 2y - 6) = 0$$

passes through a point such that

$$3x + 2y - 4 = 0$$

and $x + 2y - 6 = 0$ *i.e.*, $(-1, 7/2)$

$\therefore$ Equation of the straight line that belongs to both the families passes through $(-2, 3)$ and $(-1, 7/2)$ is

$$y-3=\frac{\frac{7}{2}-3}{-1+2}(x+2)$$

$\Rightarrow \quad y-3=\frac{x+2}{2} \quad \Rightarrow \quad x-2y+8=0$

7. $d(x,y)=\max\{|x|,|y|\}$ ...(i)

but $d(x,y)=a$ ...(ii)

From Eqs. (i) and (ii),

$$a=\max\{|x|,|y|\}$$

if $|x|>|y|$, then $a=|x|$

$\therefore \quad x=\pm a$

and if $|y|>|x|$, then $a=|y|$

$\therefore \quad y=\pm a$

Therefore, locus represents a straight line.

8. If the line meets the x and y-axes at A and B, then $A\equiv(-c/a,0), B\equiv(0,-c/b)$. The line will pass through the first quadrant, if

$$-c/a>0 \text{ and /or } -c/b>0$$

$\Rightarrow \quad ac<0$ and /or $bc<0$

9.

On solving the given lines, we get

$$A=\left(\frac{1}{1+m_1},\frac{m_1}{1+m_1}\right), C=\left(\frac{1}{1+m_3},\frac{m_3}{1+m_3}\right)$$

If $AB=BC$, then mid-point of AC lies on

$$y=m_2x$$

$$\Rightarrow \quad \frac{\frac{m}{1+m_1}+\frac{m_3}{1+m_3}}{2}=m_2\left[\frac{\frac{1}{1+m_1}+\frac{1}{1+m_3}}{2}\right]$$

$\Rightarrow m_1(1+m_3)+m_3(1+m_1)=m_2[1+m_3+1+m_1]$

$\Rightarrow (1+m_1)(m_3-m_2)=(1+m_2)(m_2-m_1)$

10. Slope of BD is 8/15 and angle made by BD with DC and BC is 45°. So, let slope of DC be m. Then,

$$\tan 45°=\pm\frac{m-\frac{8}{15}}{1+\frac{8}{15}m}$$

$\Rightarrow \quad (15+8m)=\pm(15m-8)$

$\Rightarrow \quad m=\frac{23}{7}$ and $-\frac{7}{23}$

Hence, the equations of DC and BC are

$$y-2=\frac{23}{7}(x-1)\Rightarrow 23x-7y-9=0$$

and $\quad y-2=-\frac{7}{23}(x-1)\Rightarrow 7x+23y-53=0$

11. $ax+by=1$ will be one of the bisectors of the given line. Equation of bisectors of the given lines are

$$\frac{3x+4y-5}{5}=\pm\left(\frac{5x-12y-10}{13}\right)$$

$\Rightarrow \quad 64x-8y=115$ or $14x+112y=15$

$\Rightarrow \quad a=\frac{64}{115}, b=-\frac{8}{115}$ or $a=\frac{14}{15}, b=\frac{12}{115}$

12. The set of lines is $4ax+3by+c=0$, where $a+b+c=0$. Eliminating c, we get

$$4ax+3by-(a+b)=0$$

$\Rightarrow \quad a(4x-1)+b(3y-1)=0$

This passes through the intersection of the lines $4x-1=0$ and $3y-1=0$, *i.e.*, $x=1/4$, $y=1/3$, *i.e.*, (1/4, 1/3).

13. As $\quad (a+b)x+(a-b)y-2ab=0$

and $\quad (a-b)x+(a+b)y-2ab=0$

The equations of the angle bisectors are

$$(a+b)x+(a-b)y-2ab=\pm\{(a-b)x+(a+b)y-2ab\}$$

$\Rightarrow \quad 2bx-2by=0$ *i.e.*, $x=y$ ...(i)

and $\quad 2ax+2ay-4ab=0$ *i.e.*, $x+y=2b$ ...(ii)

$\therefore$ Equation of third side as given by Eq. (i) is

$x-y=k$ satisfying the point $(b-a, a-b)$

$\therefore \quad k=2b-2a$

$\therefore$ The line is $x-y=2(b-a)$

From Eq. (ii), $x+y-2b=k$ is passing through the point $(b-a, a-b)$

$\therefore \quad k=-2b$

$\therefore$ The line is $x+y=0$.

14. $bx+cy=a \quad \Rightarrow \quad \frac{x}{(a/b)}+\frac{y}{(a/c)}=1$

Area of $\Delta OAB=\frac{1}{8}$ (given)

$\Rightarrow \quad \frac{1}{2}\cdot\frac{a}{b}\cdot\frac{a}{c}=\pm\frac{1}{8}$

$\Rightarrow \quad \dfrac{a^2}{bc} = \pm \dfrac{1}{4}$

$\Rightarrow \quad (2a)^2 = \pm bc$

So, $b, \pm 2a, c$ are in GP.

15. Given lines are $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$

$\Rightarrow \quad x + 2y + 4 = 0$ and $-4x - 2y + 1 = 0$

Here, $(1)(-4) + (2)(-2) = -8 < 0$

So, bisector of the angle is acute angle bisector including origin.

Its equation is

$$\frac{x + 2y + 4}{\sqrt{5}} = \frac{(-4x - 2y + 1)}{2\sqrt{5}} \Rightarrow 6x + 6y + 7 = 0$$

16. Given lines $(x + y + 1) + b(2x - 3y - 8) = 0$ are concurrent at point of intersection of the lines $x + y + 1 = 0$ and $2x - 3y - 8 = 0$, which is $(1, -2)$. Now, line through $A(1, -2)$ which is farthest from the point $B(2, 2)$, is perpendicular to AB. Now, slope of AB is 4. Then, the required line is $y + 2 = -(1/4)(x - 1)$ or $x + 4y + 7 = 0$

17. Also, this line $x + 4y + 7 = 0$ meets axis at $C(-7, 0)$ and $D(0, -7/4)$. Then, area of ΔOCD is $1/2\,|(-7)(-7/4)|$ or 49/8.

18. Lines $x + 4y + 7 = 0$ and $x - 2y + 1 = 0$ intersect at $(-3, -1)$ which must satisfy the line $3x - 4y + \lambda = 0$. Then, $-9 + 4 + \lambda = 0 \Rightarrow \lambda = 5$.

19. First, let the equation $ax + by + c = 0$ represent a family of straight lines passing through (a, b) for different values of a, b and c.

Then, we have to show that there is a linear relation between a, b and c and have to prove that the equation $ax + by + c = 0$ represents a family of lines passing through a fixed point. Let the linear relation be

$$la + mb + nc = 0$$

$$\Rightarrow \quad \left(\frac{l}{n}\right)a + \left(\frac{m}{n}\right)b + c = 0$$

$\Rightarrow ax + by + c = 0$ always passes through a fixed point $\left(\dfrac{l}{n}, \dfrac{m}{n}\right)$.

$\therefore \quad 3a + 2b + 4c = 0$

$$\Rightarrow \quad \left(\frac{3}{4}\right)a + \left(\frac{2}{4}\right)b + c = 0$$

and $ax + by + c = 0$ represents a system of concurrent lines passing through $\left(\dfrac{3}{4}, \dfrac{1}{2}\right)$.

Thus, statement I is false and II is true.

20. Equation of bisector of $4y + 3x - 12 = 0$ and $3y + 4x - 24 = 0$ is

$$\frac{4y + 3x - 12}{\sqrt{16 + 9}} = \pm \frac{3y + 4x - 24}{\sqrt{9 + 16}}$$

$\Rightarrow \quad y - x + 12 = 0$ and $7y + 7x - 36 = 0$

So, the line $y - x + 12 = 0$ is the angular bisector.

21. Let $f(x, y) \equiv 3x - 5y + \lambda$

$\because$ Points (1, 2) and (3, 4) be on the same side of the line $3x - 5y + \lambda = 0$, then

$$\frac{f(1,2)}{f(3,4)} > 0 \Rightarrow \frac{3 - 10 + \lambda}{9 - 20 + \lambda} > 0$$

$$\Rightarrow \quad \frac{\lambda - 7}{\lambda - 11} > 0$$

$\therefore \quad \lambda < 7$ or $\lambda > 11$

22. As $x + y = |a|$ and $ax - y = 1$, intersect in first quadrant.

$\therefore x$ and y- intercepts are positive.

$$\therefore \quad x = \frac{1 + |a|}{1 + a} \geq 0$$

and $\quad y = \dfrac{a|a| - 1}{a + 1} \geq 0$

$\Rightarrow \quad 1 + a \geq 0$ and $a|a| - 1 \geq 0$

$\Rightarrow \quad a \geq -1$ and $a|a| \geq 1 \quad$ ...(i)

If $\quad -1 \leq a < 0$

$\Rightarrow \quad -a^2 > 1 \quad$ (not possible)

If $\quad a \geq 0 \Rightarrow a^2 \geq 1$

$\Rightarrow \quad a \geq 1$

$\therefore \quad a > 1$ or $a \in [1, \infty) \quad$ ...(ii)

23. Since, the line L is passing through the point (13, 32).

Therefore,

$$\frac{13}{5} + \frac{32}{b} - 1 \Rightarrow \frac{32}{b} = \frac{8}{5} \Rightarrow b = -20$$

The line K is parallel to the line L, its equation must be

$$\frac{x}{5} - \frac{y}{20} = a \text{ or } \frac{x}{5a} - \frac{y}{20a} = 1$$

On comparing with $\dfrac{x}{c} + \dfrac{y}{3} = 1$, we get

$$20a = -3,\ c = 5a = -\frac{3}{4}$$

Hence, the distance between lines

$$= \frac{|a - 1|}{\sqrt{\dfrac{1}{25} + \dfrac{1}{400}}} = \left|\frac{-3}{20} - 1\right| \Big/ \sqrt{\frac{17}{400}} = \frac{23}{\sqrt{17}}$$

24. Line perpendicular to same line are parallel to each other.

$\therefore \quad -p(p^2 + 1) = p^2 + 1 \Rightarrow p = -1$

So, there is exactly one value of p.

25. Since, slope of $PQ = \dfrac{4 - 3}{1 - k} = \dfrac{1}{1 - k}$

Slope of $AM = (k-1)$

$\therefore$ Equation of AM is

$$y - \frac{7}{2} = (k-1)\left[x - \left(\frac{k+1}{2}\right)\right]$$

For y-intercept, $x = 0, y = -4$

$$-4 - \frac{7}{2} = -(k-1)\left(\frac{k+1}{2}\right)$$

$$\Rightarrow \quad \frac{15}{2} = \frac{k^2-1}{2}$$

$$\Rightarrow \quad k^2 - 1 = 15$$

$$\Rightarrow \quad k^2 = 16$$

$$\Rightarrow \quad k = \pm 4$$

26. Now, slope of $QR = \dfrac{3\sqrt{3}-0}{3-0} = \sqrt{3} = \tan\theta \Rightarrow \theta = \dfrac{\pi}{3}$

$\therefore$ The angle between $\angle PQR$ is $\dfrac{2\pi}{3}$, so the line QM makes an angle $\dfrac{2\pi}{3}$ from positive direction of x-axis. Slope of the line $QM = \tan\dfrac{2\pi}{3} = -\sqrt{3}$

Hence, equation of line QM is $y = -\sqrt{3}x$

$$\Rightarrow \quad \sqrt{3}x + y = 0$$

27. Since, A is the mid-point of line PQ.

$$\therefore \quad 3 = \frac{a+0}{2} \Rightarrow a = 6$$

and $$4 = \frac{0+b}{2} \Rightarrow b = 8$$

Hence, the equation of line is

$$\frac{x}{6} + \frac{y}{8} = 1 \Rightarrow 4x + 3y = 24$$

28. The point (a, a^2) lies in the sector bounded by the lines $x - 2y = 0$ and $3x - y = 0$

$$\therefore \quad (a - 2a^2)(3a - a^2) < 0$$

$$\Rightarrow \quad \left[a - \frac{1}{2}\right](a-3) < 0 \therefore a \in \left(\frac{1}{2}, 3\right)$$

29. Since, a, b and c are in HP. Then, $\dfrac{1}{a}, \dfrac{1}{b}$ and $\dfrac{1}{c}$ are in AP.

$$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

Hence, straight line $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{1}{c} = 0$ is always passes through a fixed point $(1, -2)$.

30. Equation of a line passing through the intersection of lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is

$$(ax + 2by + 3b) + \lambda(bx - 2ay - 3a) = 0 \quad \ldots(i)$$

Now, this line is parallel to x-axis, so coefficient of $x = 0$

$$\Rightarrow \quad a + \lambda b = 0 \Rightarrow \lambda = -\frac{a}{b}$$

On putting this value in Eq. (i), we get

$$b(ax + 2by + 3b) - a(bx - 2ay - 3a) = 0$$

$$\Rightarrow \quad 2b^2y + 3b^2 + 2a^2y + 3a^2 = 0$$

$$\Rightarrow \quad 2(b^2 + a^2)y + 3(b^2 + a^2) = 0 \Rightarrow y = -\frac{3}{2}$$

Therefore, the required line is below x-axis, at a distance $\dfrac{3}{2}$ from it.

31. Given lines are $3x - 4y + 7 = 0$

$$\Rightarrow \quad y = \frac{3}{4}x + \frac{7}{4} \quad \ldots(i)$$

and $$12x + 5y - 2 = 0$$

$$\Rightarrow \quad y = -\frac{12}{5}x + \frac{2}{5} \quad \ldots(ii)$$

The equation of the bisectors of the angles between these lines are

$$\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}} = \pm\frac{12x + 5y - 2}{\sqrt{12^2 + 5^2}}$$

$\therefore$ Required equation of the bisector of the acute angle between these lines is

$$\frac{3x - 4y + 7}{5} = \frac{12x + 5y - 2}{13}$$

$$\Rightarrow \quad 39x - 52y + 91 = 60x + 25y - 10$$

$$\therefore \quad 21x + 77y - 101 = 0$$

32. Let the coordinates of vertex C is (x, y) and the coordinates of centroid of the triangle is (x_1, y_1).

$$\therefore \quad x_1 = \frac{x+2-2}{3} \text{ and } y_1 = \frac{y-3+1}{3}$$

$$\Rightarrow \quad x_1 = \frac{x}{3} \text{ and } y_1 = \frac{y-2}{3}$$

Since, the centroid lies on the line $2x + 3y = 1$.

$$\therefore \quad 2x_1 + 3y_1 = 1 \Rightarrow 2\frac{x}{3} + 3\frac{(y-2)}{3} = 1$$

$$\Rightarrow \quad 2x + 3y - 6 = 3 \Rightarrow 2x + 3y = 9$$

33. Let the line intercept the x-axis at a and y-axis at b distance.

Since, $$a + b = -1 \Rightarrow b = -(a+1)$$

$\therefore$ Equation of line is $\dfrac{x}{a} + \dfrac{y}{a+1} = 1$

$$\therefore \quad \frac{4}{a} + \frac{3}{a+1} = 1 \Rightarrow \frac{4a + 4 - 3a}{a(a+1)} = 1$$

$$\Rightarrow \quad a + 4 = a^2 + a \Rightarrow a = \pm 2$$

Hence, equation of lines are

$$\frac{x}{2} + \frac{y}{3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1$$

13 Circles

JEE Main MILESTONE

- Definition
- Standard Form of Equation of circle
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13.1 Definition

A circle is defined as the locus of all such points in a plane, which remains at constant distance from a fixed point. Here, the fixed point (C) is called the **centre** of the circle and the constant distance is called its **radius**.

*A **circle** is a simple shape of Euclidean geometry that is the set of all points in a plane which are at fixed distance (radius) from a point (centre) but it also a simple closed curve which divides the plane into two regions : an interior and an exterior.*
Also, it may be defined as a special ellipse in which the two foci are coincident and the eccentricity is 0.

13.2 Standard Form of Equation of Circle

(i) Equation of Circle in General Form

The general equation of second degree may represents a circle. If the coefficient of x^2 and coefficient of y^2 are identical and the coefficient of xy becomes zero.

i.e., $$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

represents a circle, if

(a) $a = b$, *i.e.*, coefficient of x^2 = coefficient of y^2

(b) $h = 0$, *i.e.*, coefficient of $xy = 0$, then Eq. (i) reduces as,

$x^2 + y^2 + 2gx + 2fy + c = 0$, whose centre and radius are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

Nature of the Circle

If $g^2 + f^2 - c > 0$, then the radius of circle will be real and real circle is possible.

If $g^2 + f^2 - c = 0$, then the radius of circle is zero and the circle is called a point circle.

If $g^2 + f^2 - c < 0$, then the radius of circle will be imaginary, so in this case, no real circle is possible.

(ii) Equation of Circle in Standard Form

Let $C(h, k)$ be the centre of circle and $CP(= r)$ be the radius of circle, then equation of circle is

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{...(i)}$$

Which is the equation of the circle in central form.

Now, if origin (0, 0) be the centre of circle, then Eq. (i) becomes,

$$x^2 + y^2 = r^2 \quad \text{...(ii)}$$

(iii) Equation of Circle in Diameter Form

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle and let $P(x, y)$ be any point on the circle.

$\therefore \quad \angle APB = 90°$

Slope of AP, $\quad m_1 = \left(\dfrac{y - y_1}{x - x_1}\right)$

and slope of BP, $\quad m_2 = \left(\dfrac{y - y_2}{x - x_2}\right)$

For perpendicular, $m_1 \cdot m_2 = -1 \therefore AP \cdot BP = -1$

$$\Rightarrow \quad \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Which is the required equation of circle in diameter form.

(iv) Equation of Circle in Parametric Form

A general point $P(x, y)$ on the circle in terms of parameters r and θ can be expressed as

$$x = h + r\cos\theta \text{ and } y = k + r\sin\theta$$

$$\Rightarrow \quad x - h = r\cos\theta \text{ and } y - k = r\sin\theta$$

On squaring and then adding, we get

$$(x - h)^2 + (y - k)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

which is the central equation of the circle satisfied by each fixed value of r (*i.e.*, radius) and varying θ.

If the circle is centered at origin, then the coordinates of centre are $C(0, 0)$, then the parametric coordinates are $x = r\cos\theta$ and $y = r\sin\theta$.

(v) Equation of Circle through Three Non-collinear Points

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three non-collinear points, then the equation of circle through these three non-collinear points is given by

$$\begin{vmatrix} x^2 + y^2 & x^2 & y^2 & 1 \\ x_1^2 + y_1^2 & x_1^2 & y_1^2 & 1 \\ x_2^2 + y_2^2 & x_2^2 & y_2^2 & 1 \\ x_3^2 + y_3^2 & x_3^2 & y_3^2 & 1 \end{vmatrix} = 0$$

***Sample* Problem 1** *Find the centre and radius of the circle.* **[NCERT]**

(a) $(2, -4), \sqrt{65}$ (b) $(2, 4), \sqrt{65}$
(c) $(-2, -4), \sqrt{65}$ (d) *None of these*

Interpret (*b*) Given equation of circle is

$$x^2 + y^2 - 4x - 8y - 45 = 0$$

Comparing it with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We get $\quad 2g = -4, 2f = -8, c = -45$

$\Rightarrow \quad g = -2, f = -4, c = -45$

$\because$ As centre and radius of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$C(-g, -f) \text{ and } r = \sqrt{g^2 + f^2 - c}$$

$\therefore$ Centre $= (-g, -f) = (2, 4)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (4)^2 - (-45)}$

$= \sqrt{4 + 16 + 45} = \sqrt{65}$

***Sample* Problem 2** *Find the equation of circle having centre* $\left(\frac{1}{2}, \frac{1}{4}\right)$ *and radius* $\frac{1}{12}$. **[NCERT]**

(a) $36x^2 + 36y^2 - 36x - 18y - 11 = 0$

(b) $36x^2 + 36y^2 + 36x + 18y + 11 = 0$

(c) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$

(d) *None of the above*

Interpret (c) Given centre $= (h, k) = \left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $r = \frac{1}{12}$

$\therefore$ Equation of circle by using $(x-h)^2 + (y-k)^2 = r^2$ is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{4}\right)^2 = \left(\frac{1}{12}\right)^2$$

$$\Rightarrow \quad x^2 + \frac{1}{4} - 2 \cdot \frac{1}{2} x + y^2 + \frac{1}{16} - 2 \cdot \frac{y}{4} = \frac{1}{144}$$

$[\because (a-b)^2 = a^2 + b^2 - 2ab]$

$$\Rightarrow \quad x^2 + y^2 - x - \frac{y}{2} + \frac{1}{4} + \frac{1}{16} - \frac{1}{144} = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - \frac{y}{2} + \frac{36 + 9 - 1}{144} = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - \frac{y}{2} + \frac{44}{144} = 0 \Rightarrow x^2 + y^2 - x - \frac{y}{2} + \frac{11}{36} = 0$$

$$\Rightarrow \quad 36x^2 + 36y^2 - 36x - 18y + 11 = 0$$

Some Particular Cases of the Central Form

***Case* I** When the circle passes through the origin (0, 0) Let the equation of circle be

$$(x-h)^2 + (y-k)^2 = r^2 \quad \text{...(i)}$$

$\because$ It passes through origin (0, 0).

$$\therefore \quad h^2 + k^2 = r^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

$$\Rightarrow \quad x^2 + h^2 - 2hx + y^2 - 2ky + k^2 = h^2 + k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky = 0$$

***Case* II** When the circle touches abscissa (x-axis) Let the centre of circle be $C(h, k)$ and it touches x-axis at point P, then the radius of circle is $CP = k$.

$\therefore$ Equation of circle is

$$(x-h)^2 + (y-k)^2 = (CP)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 = 0$$

***Case* III** When the circle touches ordinate (y-axis) Let the centre of circle be $C(h, k)$ and it touches y-axis at point P with radius $CP = h$

$\therefore$ Equation of circle is

$$(x-h)^2 + (y-k)^2 = (CP)^2 = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + k^2 = 0$$

***Case* IV** When the circle touches both abscissa and ordinate (x-axis and y-axis) In this case $h = k = \alpha$.

$\therefore$ Equation of circle is

$$(x-\alpha)^2 + (y-\alpha)^2 = \alpha^2$$

$$\Rightarrow \quad x^2 + y^2 - 2\alpha x - 2\alpha y + \alpha^2 = 0$$

***Case* V** When the circle passes through O (0, 0) and centre lies on abscissa (x-axis) In this case $k = 0$, then equation of circle is

$$(x-h)^2 + (y-0)^2 = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx = 0$$

***Case* VI** When the circle passes through O (0, 0) *and centre lies on ordinate* (y-axis)

In this case $h = 0$, then equation of circle is

$$(x-0)^2 + (y-k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ky = 0$$

***Case* VII** When the circle touches x-axis and cut off an intercepts on y-axis of length 2l.

In ΔCMQ, $\quad k^2 = h^2 + l^2$

$$\Rightarrow \quad h = \sqrt{k^2 - l^2}$$

$\therefore$ Equation of circle is

$$[x - \sqrt{(k^2 - l^2)}]^2 + (y-k)^2 = k^2$$

Case VIII When the circle touches y-axis and cut off an intercept on x-axis of length 2m.

In ΔCPQ, $\quad h^2 = k^2 + m^2$

$$\Rightarrow \quad k = \sqrt{h^2 - m^2}$$

$\therefore$ Equation of circle is

$$(x-h)^2 + [y - \sqrt{(h^2 - m^2)}]^2 = h^2$$

Note The length of intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on x-axis is $2\sqrt{g^2 - c}$ and on y-axis is $2\sqrt{f^2 - c}$.

***Sample* Problem 3** *The equation of the circle which passes through the origin and makes intercepts of length a and b on the x and y axes respectively, are* [NCERT]

(a) $x^2 + y^2 \pm ax \pm by = 0$ (b) $x^2 \pm y^2 = 0$

(c) $x^2 \pm ax \pm by = 0$ (d) *None of these*

Interpret (a) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

Since, the circle passes through the origin, we get $c = 0$ and given the intercepts on x and y axes are a and b.

Then, $2\sqrt{g^2 - c} = a$

$\Rightarrow$ $2\sqrt{g^2 - 0} = a$

$\therefore$ $g = \pm \frac{a}{2}$

and $2\sqrt{f^2 - c} = b$

$\Rightarrow$ $2\sqrt{f^2 - 0} = b$

$\therefore$ $f = \pm \frac{b}{2}$

Hence, the equation of circle from Eq.(i) is

$$x^2 + y^2 \pm ax \pm by = 0$$

Hot Spot 1 Central Form of Circle

In this types of problem some central form of conditions are given and we have to find the equation of circle. These types of problems are directly asked in various years of JEE (Main) 2013, 2011, 2005, 2004. To solve such types of problem following steps are given below.

Step I Firstly write the given condition.

Step II Using the appropriate case which is defined in heading 13.3 *i.e.*, some particular cases of the central form.

Step III Simplify it and get the result.

***Sample* Problem 4** *Find the equation of the circle which touches x-axis and whose centre is* (1, 2). [NCERT Exemplar]

(a) $x^2 + y^2 - 2x + 4y + 1 = 0$ (b) $x^2 + y^2 - 2x - 4y + 1 = 0$

(c) $x^2 + y^2 - 2x - 4y - 1 = 0$ (d) *None of these*

Interpret (b) **Step I** Write the given condition.

Given centre of circle is (1, 2).

Since, the required circle touches x-axis, so radius should be $r = 2$

Step II Using appropriate case

$\therefore$ Required equation of circle is

$$(x - 1) + (y - 2)^2 = 2^2$$

Step III Simplify it.

$\Rightarrow$ $x^2 - 2x + 1 - y^2 - 4y + 4 = 4$

$\Rightarrow$ $x^2 + y^2 - 2x - 2y + 1 = 0$

13.3 Cyclic Quadrilateral

Quadrilateral, whose all four vertices lie on a circle is called a cyclic quadrilateral as shown in figure. The four vertices are said to be concyclic.

13.4 Position of a Point with Respect to a Circle

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is the equation of the circle and $P(x_1, y_1)$ is any point in the plane of the circle, then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

***Case* I** If $S_1 > 0$, then the point lies outside the circle.

***Case* II** If $S_1 = 0$, then the point lies on the circle.

***Case* III** If $S_1 < 0$, then the point lies inside the circle.

Maximum and Minimum Distance of a Point from a Circle

Let any point $P(x_1, y_1)$ and circle $x^2 + y^2 + 2gx + 2fy + c = 0$

The centre and radius of the circle are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

The maximum and minimum distance from $P(x_1, y_1)$ to the given circle are

$$PB = CB + PC = r + PC$$

and $PA = |CP - CA| = |PC - r|$ (P inside or outside)

where $r = \sqrt{(g^2 + f^2 - c)}$

Note If $PC < r$, then P inside, $PC > r$, then P outside.

***Sample* Problem 5** *Find the position of a point* (–2.5, 3.5) *with respect to the circle* $x^2 + y^2 = 25$. **[NCERT]**

(a) *inside* (b) *outside*
(c) *on* (d) *None of these*

Interpret (a) Given equation of circle is

$$x^2 + y^2 = 25$$

Let $S = x^2 + y^2 - 25$ ...(i)

Put $(x, y) = (-2.5, 3.5)$ in Eq. (i), we get

$$\Rightarrow \quad S = (-2.5)^2 + (3.5)^2 - 25$$
$$= 6.25 + 12.25 - 25$$
$$= 18.50 - 25 = -6.50 < 0$$

$\therefore$ Point lies inside the circle.

Hot Spot 2

Intersection of a Line and a Circle

Let $y = mx + c$ and $x^2 + y^2 = r^2$ be the equation of a line and a circle, respectively.

$$\therefore \quad x^2 + (mx + c)^2 = r^2 \quad (\because y = mx + c)$$
$$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + (c^2 - r^2) = 0 \quad ...(i)$$

Which is quadratic in x, then **three cases arises.**

***Case* I** Roots of Eq. (i) are real and distinct, if

$$D = r^2(1 + m^2) - c^2 > 0$$

i.e., if $r > \dfrac{c}{\sqrt{1 + m^2}}$

Hence, the line meets the circle at two distinct points.

***Case* II** Roots of Eq. (i) are coincident, if

$$D = r^2(1 + m^2) - c^2 = 0,$$

i.e., if $r = \dfrac{c}{\sqrt{1 + m^2}}$

Hence, the line touches the circle.

***Case* III** Roots of Eq. (i) are imaginary, if

$$D = r^2(1 + m^2) - c^2 < 0,$$

i.e., if $r < \dfrac{c}{\sqrt{1 + m^2}}$

Hence, the line will not intersect the circle at all.

Note Length of perpendicular from (0, 0) on $y = mx + c$ is $\dfrac{c}{\sqrt{1 + m^2}}$

Condition of Tangency

(i) The line $y = mx + c$ touches the circle $x^2 + y^2 = r^2$, iff $c = \pm r\sqrt{1 + m^2}$

(ii) The line $lx + my + n = 0$ will touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, iff

$$(l^2 + m^2)(g^2 + f^2 - c) = (lg + mf - n)^2$$

Note

- If $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$, then the coordinates of point of contact are $\left(-\dfrac{mr^2}{c}, \dfrac{r^2}{c}\right)$.
- If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then the coordinates of point of contact are $\left(-\dfrac{ar^2}{c}, -\dfrac{br^2}{c}\right)$.
- The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = r^2$ is $2\sqrt{\dfrac{a^2(1 + m^2) - c^2}{1 + m^2}}$

These types of problems are directly or indirectly asked in various years of AIEEE (JEE Main) 2010, 2005, 2004

To solve such types of problem following steps are given below.

Step I Firstly write the given equation of line and circle.

Step II Find the intersection point of line and circle.

Step III According to the given question we apply the other condition, if given.

Step IV Simplify it and get the result.

***Sample* Problem 6** *The range of values of m for which the line* $y = mx + 2$ *cuts the circle* $x^2 + y^2 = 1$ *at distance or coincident points is* **[NCERT]**

(a) $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$ (b) $[-\sqrt{3}, \sqrt{3}]$
(c) $[\sqrt{3}, \infty)$ (d) *None of these*

Interpret (a) **Step I** Write the given equations. Given equation of line is $y = mx + c$ and equation of circle is $x^2 + y^2 = 1$.

Step II We need to find the intersection point.

Step III Apply the other condition.

The length of the perpendicular from the centre (0, 0) to the line

$$= \frac{m(0) + (0) + 2}{\sqrt{1 + m^2}} = \frac{2}{\sqrt{1 + m^2}}$$

$\therefore$ The radius of the circle $= 1$

Step IV Simplify it.

For the line to cut the circle at distinct or coincident points,

$$\frac{2}{\sqrt{1 + m^2}} < 1$$
$$\Rightarrow \quad 1 + m^2 \geq 4 \Rightarrow m^2 \geq 3$$

***Sample* Problem 7** *The line* $(x-2)\cos\theta + (y-2)\sin\theta = 1$ *touches a circle for all values of* θ*, then the equation of circle is*

(a) $x^2 + y^2 - 4x - 4y + 7 = 0$ (b) $x^2 + y^2 + 4x + 4y + 7 = 0$

(c) $x^2 + y^2 - 4x - 4y - 7 = 0$ (d) *None of the above*

Interpret (a) Given line is

$$(x-2)\cos\theta + (y-2)\sin\theta = 1 = \cos^2\theta + \sin^2\theta$$

On comparing, we get

$$x - 2 = \cos\theta \quad \ldots(i)$$

and

$$y - 2 = \sin\theta \quad \ldots(ii)$$

On squaring and then adding Eqs. (i) and (ii), we get

$$(x-2)^2 + (y-2)^2 = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \quad (x-2)^2 + (y-2)^2 = 1$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 4y + 7 = 0$$

13.5 Tangent to a Circle

A straight line is a tangent to a circle, if the distance of the centre from the line equals the radius.

Different Forms of the Equation of Tangents

Point Form

The equation of tangent at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is

$$T \equiv xx_1 + yy_1 = r^2$$

Also, the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $P(x_1, y_1)$ is

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note When we convert a carterian equation into a tangent equation, we replace. x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$

Slope Form

The equation of a tangent of slope m to the circle $x^2 + y^2 = r^2$ is $T \equiv y = mx \pm r\sqrt{(1 + m^2)}$ and the coordinates of the point of contact are $\left(\pm \frac{rm}{\sqrt{1+m^2}}, \mp \frac{r}{\sqrt{1+m^2}}\right)$.

Parametric Form

The equation of tangent to the circle $x^2 + y^2 = r^2$ at the point $(r\cos\theta, r\sin\theta)$ is $T \equiv x\cos\theta + y\sin\theta = r$

T=0 is Possible Only

(i) if points is on the curve $\rightarrow$ Tangent

(ii) if point is outside $\rightarrow$ Chord of contact

(iii) if point is outside/inside $\rightarrow$ Pole and poler

***Sample* Problem 8** *The equation of tangent to the circle* $x^2 + y^2 - 2ax = 0$ *at the point* $[a(1+\cos\alpha), a\sin\alpha]$ *is*

(a) $x\cos\alpha + y\sin\alpha = a$

(b) $x\cos\alpha + y\sin\alpha = 1 + \cos\alpha$

(c) $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$

(d) *None of the above*

Interpret (c) The equation of tangent of $x^2 + y^2 - 2ax = 0$ at $[a(1+\cos\alpha), a\sin\alpha]$ is

$$xa(1+\cos\alpha) + ya\sin\alpha - a[x + a(1+\cos\alpha)] = 0$$

$$\Rightarrow \quad ax\cos\alpha + ay\sin\alpha - a^2(1+\cos\alpha) = 0$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha = a(1+\cos\alpha)$$

13.6 Normal to a Circle

The normal to a circle at any point is a straight line which is perpendicular to the tangent at that point and always passes through the centre of the circle.

Different Forms of the Equation of Tangents

Point Form

The equation of tangent at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is

$$\frac{x}{x_1} = \frac{y}{y_1}$$

OR

The equation of tangent at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

Slope Form

The equation of a normal of slope m to the circle $x^2 + y^2 = r^2$ is

$$my = -x \pm r\sqrt{1 + m^2}$$

Parametric Form

The equation of normal at the point $P(r\cos\theta, r\sin\theta)$ to the circle $x^2 + y^2 = r^2$ is

$$\frac{x}{r\cos\theta} = \frac{y}{r\sin\theta}$$

$$\Rightarrow \quad y = x\tan\theta$$

Note Normal always passes through the centre of the circle.

***Sample* Problem 9** *The equation of normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6) is*

(a) $14x - 5y + 40 = 0$ (b) $14x + 5y + 40 = 0$
(c) $14x - 5y - 40 = 0$ (d) $14x + 5y - 40 = 0$

Interpret (c) Equation of the normal to the given circle at (5, 6) is

$$\frac{x-5}{5-\frac{5}{2}} = \frac{y-6}{6+1}$$

$$\Rightarrow \quad \frac{x-5}{\frac{5}{2}} = \frac{y-6}{7}$$

$$\Rightarrow \quad \frac{2x-10}{5} = \frac{y-6}{7}$$

$$\Rightarrow \quad 14x - 70 = 5y - 30$$

$$\Rightarrow \quad 14x - 5y - 40 = 0$$

13.7 Equation of Pair of Tangents

Let $P(x_1, y_1)$ be any point lying outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of pair of tangents represented by PA and PB is $SS_1 = T^2$

where, $S = x^2 + y^2 + 2gx + 2fy + c$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

and $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Similarly, equation of pair of tangents at the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is

$$SS_1 = T^2$$

where, $S = x^2 + y^2 - r^2$

$$S_1 = x_1^2 + y_1^2 - r^2$$

and $T = xx_1 + yy_1 - r^2$

Length of the Tangent

The length of the tangent drawn from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}\,; \sqrt{S_1}$$

***Sample* Problem 10** *The equation of pair of tangents to the circle $x^2 + y^2 = 16$ drawn from the point (1, 4) is*

(a) $x^2 + y^2 - x - 4y = 0$ (b) $x^2 + y^2 + x - 4y = 0$
(c) $x^2 + y^2 - 2x - 8y = 0$ (d) *None of these*

Interpret (d) Equation of pair of tangents to the circle $x^2 + y^2 = 16$ at the point (1, 4) is

$$(x^2 + y^2 - 16)(1 + 16 - 16) = (x + 4y - 16)^2$$

$$\Rightarrow \quad x^2 + y^2 - 16 = (x + 4y - 16)^2$$

$$\Rightarrow \quad x^2 + y^2 - 16 = x^2 + 16y^2 + 256 + 8xy - 128 - 32x$$

$$\Rightarrow \quad 15y^2 + 8xy - 32x + 112 = 0$$

13.8 Director Circle

The locus of the point of intersection of two perpendicular tangents to a given circle is known as director circle.

If the equation of circle is $x^2 + y^2 = a^2$, then the equation of the director circle to this circle is $x^2 + y^2 = 2a^2$

***Sample* Problem 11** *The locus of the point of intersection of two perpendicular tangents to the circle $x^2 + y^2 = 16$ is*

(a) $x^2 + y^2 = 20$ (b) $x^2 + y^2 = 32$
(c) $x^2 + y^2 = 16$ (d) *None of these*

Interpret (b) The required equation of circle is

$$x^2 + y^2 = 2(16) = 32$$

13.9 Chord of Contact

From any external point, two tangents can be drawn to a given circle. The chord joining the points of contact of the two tangents is called the chord of contact of tangents.

The equation of the chord of contact of tangents drawn from the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ and to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

***Sample* Problem 12** *The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Then, a, b and c are in*

(a) *AP* (b) *HP*
(c) *GP* (d) *None of these*

Interpret (c) Let $P(a\cos\theta, a\sin\theta)$ be any point on the circle $x^2 + y^2 = a^2$

Then, equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = b^2$ is

$$ax\cos\theta + ay\sin\theta = b^2 \quad \text{...(i)}$$

This touches the circle $x^2 + y^2 = c^2$ ...(ii)

$\therefore$ Length of perpendicular from (0, 0) to Eq. (i) = radius of Eq. (ii)

$$\Rightarrow \quad \frac{|0 + 0 - b^2|}{\sqrt{(a^2 \cos^2\theta + a^2 \sin^2\theta)}} = c \quad \Rightarrow \quad b^2 = ac$$

$\Rightarrow$ a, b and c are in GP.

13.10 Equation of the Chord Bisected at a Given Point

Let $P(x_1, y_1)$ is the mid-point of chord AB of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of circle bisected at the point $P(x_1, y_1)$ is

$$T = S_1$$

$$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$$

Similarly, for the circle $x^2 + y^2 = r^2$, equation of chord is

$$T = S_1 \Rightarrow xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2$$

***Sample* Problem 13** *The middle point of the chord intercepted on the line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$ is*

(a) $\left(-\frac{nl}{l^2 + m^2}, -\frac{nm}{l^2 + m^2}\right)$ (b) $\left(\frac{nl}{l^2 + m^2}, \frac{nm}{l^2 + m^2}\right)$

(c) $\left(-\frac{l}{l^2 + m^2}, \frac{m}{l^2 + m^2}\right)$ (d) *None of these*

Interpret (a) Let (x_1, y_1) be the middle point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $lx + my + n = 0$. Then, equation of the chord of the circle $x^2 + y^2 = a^2$, whose middle point is (x_1, y_1), is

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

$$\Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2 \quad \text{...(i)}$$

Clearly, $lx + my + n = 0$ and Eq. (i) represent the same line.

$$\therefore \quad \frac{x_1}{l} = \frac{y_1}{m} = -\frac{x_1^2 + y_1^2}{n} = \lambda \quad \text{(say)}$$

$$\Rightarrow \quad \left.\begin{aligned} x_1 &= l\lambda \\ y_1 &= m\lambda \end{aligned}\right\} \quad \text{...(ii)}$$

and $$x_1^2 + y_1^2 = -n\lambda$$

$$\Rightarrow \quad l^2\lambda^2 + m^2\lambda^2 = -n\lambda \quad \text{[from Eq. (ii)]}$$

$$\therefore \quad \lambda = -\frac{n}{l^2 + m^2}$$

$\therefore$ From Eq. (ii), we get

$$x_1 = -\frac{nl}{l^2 + m^2}, \quad y_1 = -\frac{mn}{l^2 + m^2}$$

Hence, the required point is $\left(-\frac{nl}{l^2 + m^2}, -\frac{nm}{l^2 + m^2}\right)$

13.11 Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

The equation of the diameter bisecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is

$$x + my = 0$$

Note The diameter corresponding to a system of parallel chords always passes through the centre of the circle and perpendicular to the parallel chords.

***Sample* Problem 14** *The equation of the diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which corresponds to the chord $ax + by + d = 0$ is*

(a) $bx - ay + bg - af = 0$ (b) $ax - by - bg + af = 0$

(c) $bx + ay - bg + af = 0$ (d) *None of these*

Interpret (a) The diameter of circle perpendicular to the chord $ax + by + d = 0$ is

$$bx - ay + \lambda = 0 \quad \text{...(i)}$$

Which passes through centre $(-g, -f)$ of the circle, then

$$-bg + af + \lambda = 0$$

$$\Rightarrow \quad \lambda = bg - af$$

From Eq. (i), the equation of the diameter is

$$bx - ay + bg - af = 0$$

13.12 Pole and Polar

If through a point $P(x_1, y_1)$ (within or without a circle) there can be drawn any straight line to meet the given circle at Q and R, then locus of the point of intersection of the tangents at Q and R is called the polar of P and P is called the **pole** of the **polar**.

The polar of point $P(x_1, y_1)$ with respect to the circle

$$x^2 + y^2 = r^2 \text{ is } xx_1 + yy_1 = r^2$$

and to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Conjugate Points

Two points A and B are conjugate points with respect to a given circle if each lies on the polar of the other with respect to the circle.

Conjugate Lines

If two lines be such that the pole of one lies on the other, then they are called conjugate lines with respect to the given circle.

***Sample* Problem 15** *The polar drawn from (–1, 2) to the circles $S_1 = x^2 + y^2 + 6y + 7 = 0$ and $S_2 = x^2 + y^2 + 6x + 1 = 0$ are*

(a) *parallel* (b) *equal*
(c) *perpendicular* (d) *None of these*

Interpret (*d*) The equation of polar to circle S_1 is $x - 5y - 13 = 0$ and circle S_2 is $x + y - 1 = 0$
It is clear that they do not parallel, it means they are intersecting.

Check Point 1

1. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ will represent a real circle, if
(a) $g^2 + f^2 - c < 0$ (b) $g^2 + f^2 - c \geq 0$
(c) always (d) None of these

2. The tangents to $x^2 + y^2 = a^2$ having inclinations α and β intersect at P. If $\cot\alpha + \cot\beta = 0$, then the locus of P is
(a) $x + y = 0$ (b) $x - y = 0$
(c) $xy = 0$ (d) None of these

3. If the points (2, 0), (0, 1), (4, 5) and (0, c) are concyclic, then the value of c is
(a) 1 (b) $\frac{14}{3}$
(c) 5 (d) None of these

4. The equation of circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is
(a) 1 (b) 2
(c) 3 (d) None of these

5. The coordinates of the middle point of the chord cut off $2x - 5y + 18 = 0$ by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$, are
(a) (1, 4) (b) (2, 4)
(c) (4, 1) (d) (1, 1)

6. The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular, if
(a) $h = r$ (b) $h = -r$
(c) $r^2 = h^2$ (d) All of these

7. The tangent to $x^2 + y^2 = 9$ which is parallel to y-axis and does not lie in the third quadrant touches the circle at the point
(a) (3, 0) (b) (–3, 0)
(c) (0, 3) (d) (0, –3)

8. Equation of the circle with centre on the y-axis and passing through the origin and (2, 3) is
(a) $x^2 + y^2 + 13y = 0$ (b) $3x^2 + 3y^2 - 13y = 0$
(c) $x^2 + y^2 + 13x + 3 = 0$ (d) $6x^2 + 6y^2 - 13x = 0$

9. Tangents drawn from the point (4, 3) to the circle $x^2 + y^2 - 2x - 4y = 0$ are inclined at an angle
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

10. Polar of origin (0, 0) with respect to the circle $x^2 + y^2 + 2\lambda x + 2\mu y + c = 0$ touches the circle $x^2 + y^2 = r^2$, if
(a) $c = r(\lambda^2 + \mu^2)$ (b) $r = c(\lambda^2 + \mu^2)$
(c) $c^2 = r^2(\lambda^2 + \mu^2)$ (d) $r^2 = c^2(\lambda^2 + \mu^2)$

13.13 Family of Circles

(i) Family of circles passing through the point of intersection of line $P \equiv lx + my + n = 0$ and circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $S + \lambda P = 0$

(ii) Family of circles passing through the points of intersection of two given circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is $S_1 + \lambda S_2 = 0$

(iii) Equation of family of circles passing through two points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) The equation of the family of circles touching the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is

$$x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$$

or $$S + \lambda T = 0$$

where, $T = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.

***Sample* Problem 16** *The equation of the circle through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$, is*

(a) $x^2 + y^2 + x + 2y = 0$ (b) $x^2 + y^2 - x - 2y = 0$
(c) $x^2 + y^2 - x + 2y = 0$ (d) *None of these*

Interpret (*b*) Equation of any circle through points of intersection of the given circle and the line is

$$(x^2 + y^2 - 2x - 4y + 4) + \lambda(x + 2y - 4) = 0$$

$$\Rightarrow \quad x^2 + y^2 + (\lambda - 2)x + (2\lambda - 4)y + 4(1 - \lambda) = 0 \quad \ldots(i)$$

It will touch the line $x + 2y = 0$, if the solution of Eq. (i) and $x = -2y$ be unique.

Hence, the roots of the equation

$$(-2y)^2 + y^2 + (\lambda - 2)(-2y) + (2\lambda - 4)y + 4(1 - \lambda) = 0$$

or $5y^2 + 4(1 - \lambda) = 0$ must be equal.

Then, $$0 = 4 \cdot 5 \cdot 4 \cdot (1 - \lambda) \Rightarrow \lambda = 1$$

$\therefore$ From Eq. (i), the required circle is

$$x^2 + y^2 - x - 2y = 0$$

13.14 Concentric Circles

Two circles having the same centre but different radii r_1 and r_2 respectively, are called concentric circles. Thus, the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2gx + 2fy + \mu = 0$ are concentric circles. Therefore, the equations of concentric circles differ only in constant term.

***Sample* Problem 17** *If (–3, 2) lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is concentric with the circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then c is equal to*

(a) 11 (b) –11 (c) 24 (d) 100

Interpret (*b*) Equation of family of concentric circles to the circle $x^2 + y^2 + 6x + 8y - 5 = 0$

$$x^2 + y^2 + 6x + 8y + \lambda = 0$$

which is similar to $x^2 + y^2 + 2gx + 2fy + c = 0$

Thus, the point (– 3, 2) lies on the circle

$$x^2 + y^2 + 6x + 8y + c = 0$$

$$\therefore \quad (-3)^2 + (2)^2 + 6(-3) + 8(2) + c = 0$$

$$\Rightarrow \quad 9 + 4 - 18 + 16 + c = 0 \quad \therefore \quad c = -11$$

13.15 Common Tangents to Two Circles

(i) Circles Neither Touching Nor Intersecting

If the distance between the centres is more than the sum of their radii, then geometrically the two circles neither touch nor intersect each other.

Mathematically, $|C_1C_2| > r_1 + r_2$

In such a case there are two direct common tangents and two transverse common tangents.

(ii) Circles Touching Each Other Externally

If the distance between the centres equals the sum of their radii, then geometrically the two circles touch each other externally. The point P where the two circles touch each other is called the point of contact. P divides the line joining centres internally in the ratio of their radii *i.e.*, $\frac{C_1P}{C_2P} = \frac{r_1}{r_2}$

Mathematically, $|C_1C_2| = r_1 + r_2$

In such a case, there are 3 common tangents to two touching circles.

(iii) Circles Intersecting Each Other

If the distance between centres is less than the sum of their radii, then geometrically, the two circles intersect each other.

Mathematically,

$$|C_1C_2| < r_1 + r_2$$

In such a case, there are two direct common tangents.

(iv) Circles Touching Each Other Internally

If the distance between centres equals the difference of the radii, then geometrically, the two circles touch each other internally. The point of contact P divides the line joining the centres externally in the ratio of their radii

i.e., $$\frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

Mathematically, $|C_1C_2| = |r_1 - r_2|$

In such a case, there is one common tangent to two circles.

(V) One Circle Lies Inside Another

If distance between centres is less than the difference of the radii, then geometrically, one circle lies inside the another.

Mathematically, $|C_1C_2| < |r_1 - r_2|$

In such a case, there is no common tangent.

Note Length of an external common tangent and internal common tangent to two circles is given by length of external common tangent

$$L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$$

and length of internal common tangent.

$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$, where d is the common distance between centres of two circles Q_1 and r_1 and r_2 are the radii of two circles, where $|C_1C_2| = d$.

***Sample* Problem 18** *The circles* $x^2 + y^2 - 10x + 16 = 0$ *and* $x^2 + y^2 = r^2$ *intersect each other at two distinct points, if*

(a) $r < 2$ (b) $r > 8$ (c) $2 < r < 8$ (d) $2 \le r \le 8$

Interpret (c) The centre and radii of two circles are $C_1(1, -3)$, $C_2(0, 0)$

and $r_1 = \sqrt{25 + 0 - 16} = 3,\ r_2 = r$

Now, $C_1C_2 = 5$

For intersection of two circles

$$r_2 - r_1, < C_1C_2 < r_1 + r_2 \Rightarrow r - 3 < 5 < 3 + r$$

$$\Rightarrow \quad r < 8 \text{ and } r > 2 \Rightarrow 2 < r < 8$$

13.16 Angle of Intersection of Two Circles

The angle of intersection of two intersecting circles is the angle between their tangents at the point of intersection,

$$\cos\alpha = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$$

where, d is distance between centres of the circles.

Condition of Orthogonality

If the angle between the circles is 90°, then the circles are said to be orthogonal circle. Let $S_1 = 0$ and $S_2 = 0$ be any two circles, then condition of orthogonality is $2(g_1g_2 + f_1f_2) = c_1 + c_2$

***Sample* Problem 19** *The angle between the circles*

$$S : x^2 + y^2 - 4x + 6y + 11 = 0$$

and $S' : x^2 + y^2 - 2x + 8y + 13 = 0$ *is*

(a) 45° (b) 90°
(c) 60° (d) *None of these*

Interpret (a) Centres and radii of circles S and S' are $C_1(2, -3)$, $r_1 = \sqrt{2}$; $C_2(1, -4), r_2 = 2$.

Distance between centres,

$$d = |C_1C_2| = \sqrt{(2-1)^2 + (-3+4)^2} = \sqrt{2}$$

If angle between the circles is θ, then

$$\cos\theta = \left|\frac{2 + 4 - 2}{2\sqrt{2}\cdot 2}\right| = \frac{1}{\sqrt{2}} \therefore \theta = 45°$$

13.17 Common Chord of Two Circles

The common chord joining the point of intersection of two given circles is called their common chord.

The equation of the common chord of two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is $$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

i.e., $$S_1 - S_2 = 0$$

Length of Common Chord

Length of common chord $= PQ = 2(PM)$

$$= 2\sqrt{[(C_1P)^2 - (C_1M)^2]}$$

where, C_1P = radius of circle $S_1 = 0$

and C_1M = length of perpendicular from C_1 on common chord PQ.

***Sample* Problem 20** *The equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 2 = 0$ and $x^2 + y^2 + 2x - 3y - 4 = 0$ is*

(a) $x^2 + y^2 + 2x + 2y + 2 = 0$ (b) $x^2 + y^2 + 2x + 2y - 1 = 0$

(c) $x^2 + y^2 + 2x + 2y + 1 = 0$ (d) $x^2 + y^2 + 2x + 2y + 3 = 0$

Interpret (c) Equation of common chord is

$$S_1 - S_2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x + 3y + 2 - (x^2 + y^2 + 2x - 3y - 4) = 0$$

$$\Rightarrow \quad 6y + 6 = 0 \Rightarrow y = -1$$

On putting $y = -1$ in first circle, we get

$$x^2 + 1 + 2x - 3 + 2 = 0$$

$$\Rightarrow \quad x^2 + 2x = 0 \Rightarrow x = 0, -2$$

$\therefore$ End points of diameter are (0, – 1) and (– 2, – 1)

$\therefore$ Equation of circle is

$$(x - 0)(x + 2) + (y + 1)(y + 1) = 0$$

$$\Rightarrow \quad x^2 + 2x + y^2 + 2y + 1 = 0$$

13.18 Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the length of the tangents drawn from it to the two circles are equal. The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$.

Properties of Radical Axis

(i) The equation of radical axis and the common chord of two circles are identical.

(ii) Radical axis is perpendicular to the straight line which joins the centres of the circles.

(iii) The radical axis bisects common tangents of two circles.

(iv) The radical axis of three circles taken in pairs are concurrents.

(v) **Radical centre** The point of intersection of radical axes of three circles whose centres are non-collinear, taken in pairs, is called their radical centre. The circle with centre at the radical centre and radius equal to the length of the tangent drawn from it to one of the circles, intersects all the three circles orthogonally.

Note

- Coordinates of radical centre can be found by solving the equations
$$S_1 = S_2 = S_3 = 0$$
- A system of circles (or family of circles) is said to be coaxial system of circles if every pair of the circles in the system has the same radical axis.

***Sample* Problem 21** *The equation of the three circles are given $x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$, $x^2 + y^2 + 10y + 24 = 0$. Find the coordinates of the point P such that the tangents drawn from it to the circles are equal in length.* **[NCERT]**

(a) $(6, -1)$ *and* $\left(2, -\frac{5}{2}\right)$ (b) $(6, 1)$ *and* $\left(2, -\frac{5}{2}\right)$

(c) $(6, -1)$ *and* $\left(2, \frac{5}{2}\right)$ (d) *None of these*

Interpret (a) We know that, point from which lengths of tangents are equal in length is radical centre of the given three circles. Now, radical axis of the first two circles is

$$(x^2 + y^2 - 1) - (x^2 + y^2 - 8x + 15) = 0 \text{ i.e., } x - 2 = 0 \quad \ldots(i)$$

and radical axis of the second and third circles is

$$(x^2 + y^2 - 8x + 15) - (x^2 + y^2 + 10y + 24) = 0$$

i.e., $$8x + 10y + 9 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), the coordinates of the radical centre *i.e.*, of point P are $P(2, -5/2)$.

Now, centre of the circle touching the line $x + y = 5$ at $(6, -1)$ lie on the line $\perp$ to $x + y = 5$ at $(6, -1)$ *i.e.*, on $x - y = 7$. Let centre be (h, k) Then, $h - k = 7$ *i.e.*, $h = k + 7$. Also, circles passes through the points $(6, -1)$ and $P(2, -5/2)$.

13.19 Coaxial System of Circles

A system (or a family) of circles, every pair of which have the same radical axis, are called coaxial circles.

(i) The equation of a system of coaxial circles, when the equation of the radical axis and of one circle of the system are

$$P \equiv lx + my + n = 0$$

and $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is

$S + \lambda P = 0$ (λ is an arbitrary constant)

(ii) The equation of a coaxial system of circles, where the equation of any two circles of the system are

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

respectively, is $S_1 + \lambda(S_1 - S_2) = 0$ $(\lambda \neq -1)$

or $S_2 + \lambda_1(S_1 - S_2) = 0$ $(\lambda_1 \neq -1)$

Other form $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$

(iii) The equation of a system of coaxial circles in the simplest form is

$$x^2 + y^2 + 2gx + c = 0$$

where, g is variable and c is a constant.

The common radical axis is the y-axis (since, centre on x-axis) and the equation of a system of other coaxial circles in the simplest form is

$$x^2 + y^2 + 2fy + c = 0$$

where, f is variable and c, a constant (since centre on y-axis). The common radical axis is the x-axis.

***Sample* Problem 22** *The equation of the system of circle coaxial with the circles*

$$x^2 + y^2 + 4x + 2y + 1 = 0 \text{ and } x^2 + y^2 - 2x + 6y - 6 = 0$$

Find the equation of that particular circle whose centre lies on the radical axis.

(a) $13(x^2 + y^2) - 98x + 56y + 9 = 0$

(b) $26(x^2 + y^2) + 98x + 56y + 9 = 0$

(c) $26(x^2 + y^2) - 98x - 56y + 9 = 0$

(d) *None of the above*

Interpret (b) Given circles are

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 1 = 0$$

and $$S_2 \equiv x^2 + y^2 - 2x + 6y - 6 = 0$$

$\therefore$ Radical axis is $S_1 - S_2 = 0$

i.e., $$6x - 4y + 7 = 0 \quad \ldots(i)$$

Now, system of coaxial circle is

$$S_1 + \lambda(S_1 - S_2) = 0$$

$$\Rightarrow \quad (x^2 + y^2 + 4x + 2y + 1) + \lambda(6x - 4y + 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x(2 + 3\lambda) + 2y(1 - 2\lambda) + 1 = 7\lambda = 0 \quad \ldots(ii)$$

Its centre $[-(2 + 3\lambda), -(1 - 2\lambda)]$ lies on Eq. (i), we get

$$\therefore \quad 6 \times -(2 + 3\lambda) - 4 \times -(1 - 2\lambda) + 7 = 0$$

$$\Rightarrow \quad -12 - 18\lambda + 4 - 8\lambda + 7 = 0$$

$$\Rightarrow \quad -26\lambda - 1 = 0$$

$$\therefore \quad \lambda = -\frac{1}{26}$$

Substituting the value of λ in Eq. (ii), the equation of circle is

$$x^2 + y^2 + 2x\left(2 - \frac{3}{26}\right) + 2y\left(1 + \frac{2}{26}\right) + 1 - \frac{7}{26} = 0$$

$$\Rightarrow \quad 26(x^2 + y^2) + 98x + 56y + 9 = 0$$

13.20 Image of the Circle by the Line Mirror

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and line mirror is $lx + my + n = 0$ in this condition, radius of circle remains unchanged but centres changes. Let the centre of imaged circle be (x_1, y_1).

Then,

$$\frac{x_1 - (-g)}{l} = \frac{y_1 - (-f)}{m}$$

$$= \frac{-2(-lg - mf + c)}{(l^2 + m^2)}$$

we get $$x_1 = \frac{(l^2g - m^2g + 2mlf - 2nl)}{(l^2 + m^2)}$$

and $$y_1 = \frac{(m^2f - l^2f + 2mlg - 2mn)}{(l^2 + m^2)}$$

$\therefore$ Required imaged circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

where, $$r = \sqrt{(g^2 + f^2 - c)}$$

***Sample* Problem 23** *Find the equation of the image of the circle* $x^2 + y^2 + 16x - 24y + 183 = 0$ *by the line mirror* $4x + 7y + 13 = 0$.

(a) $x^2 + y^2 + 32x + 4y + 235 = 0$

(b) $x^2 + y^2 - 32x - 4y + 235 = 0$

(c) $x^2 + y^2 + 32x + 4y - 235 = 0$

(d) *None of the above*

Interpret (a) The given circle and line are

$$x^2 + y^2 + 16x - 24y + 183 = 0 \quad \ldots(i)$$

and $$4x + 7y + 13 = 0 \quad \ldots(ii)$$

Centre and radius of cirde (i) are (– 8, 12) and 5, respectively. Let the centre of the imaged circle be (x_1, y_1).

Hence, (x_1, y_1) be the image of the point (– 8, 12) with respect to the line $4x + 7y + 13 = 0$, then

$$\frac{x_1 - (-8)}{4} = \frac{y_1 - 12}{7} = \frac{-2[4(-8) + 7(12) + 13]}{(4^2 + 7^2)}$$

$$\Rightarrow \quad \frac{x_1 + 8}{4} = \frac{y_1 - 12}{7} = -2$$

$$\therefore \quad x_1 = -16, y_1 = -2$$

$\therefore$ Equation of the imaged circle is

$$(x + 16)^2 + (y + 2)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 + 32x + 4y + 235 = 0$$

Check Point **2**

1. The number of common tangents to the circles
$$x^2 + y^2 + 2x + 8y - 23 = 0$$
and $x^2 + y^2 - 4x - 10y + 19 = 0$ are
(a) 4 (b) 2 (c) 3 (d) 1

2. The equation of the unit circle concentric with $x^2 + y^2 - 8x + 4y - 8 = 0$ is
(a) $x^2 + y^2 - 8x + 4y - 8 = 0$ (b) $x^2 + y^2 - 8x + 4y + 8 = 0$
(c) $x^2 + y^2 - 8x + 4y - 28 = 0$ (d) $x^2 + y^2 - 8x + 4y + 19 = 0$

3. If a circle passes through the point (1, 2) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the equation of the locus of its centre is
(a) $x^2 + y^2 - 3x - 8y + 1 = 0$ (b) $x^2 + y^2 - 2x - 6y - 7 = 0$
(c) $2x + 4y - 9 = 0$ (d) $2x + 4y - 1 = 0$

4. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation
(a) $x^2 - 6x - 10y + 14 = 0$ (b) $x^2 - 10x - 6y + 14 = 0$
(c) $y^2 - 6x - 10y + 14 = 0$ (d) $y^2 - 10x - 6y + 14 = 0$

5. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then the length of the common chord of these two circles is
(a) $2\sqrt{g^2 + f^2 - c}$ (b) $2\sqrt{g'^2 + f'^2 - c'}$
(c) $2\sqrt{g^2 + f^2 + c}$ (d) $2\sqrt{g'^2 + f'^2 + c'}$

6. If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, then
(a) $g = \frac{3}{4}$ and $f \neq 2$ (b) $g \neq \frac{3}{4}$ and $f = 2$
(c) $g = \frac{3}{4}$ or $f = 2$ (d) None of these

7. The common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin an angle equal to
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

8. The circles whose equations are $x^2 + y^2 + c^2 = 2ax$ and $x^2 + y^2 + c^2 = 2by$ will touch one another externally, if
(a) $\frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a^2}$ (b) $\frac{1}{c^2} + \frac{1}{a^2} = \frac{1}{b^2}$
(c) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (d) None of these

9. The radical axis of the co axial system of circles with limiting point (1, 2) and (–2, 1) is
(a) $x + 3y = 0$
(b) $3x + y = 0$
(c) $2x + 3y = 0$
(d) $3x + 2y = 0$

WORKED OUT Examples

Example 1 *The equation of circle whose centre and radius are* (1, –1) *and* 4 *respectively, is*

(*a*) $x^2 + y^2 - 2x - 2y - 14 = 0$

(*b*) $x^2 + y^2 - 2x + 2y - 14 = 0$

(*c*) $x^2 + y^2 - 2x + 2y + 14 = 0$

(*d*) *None of the above*

Solution (*b*) We know that, equation of circle is

$$(x-h)^2 + (y-k)^2 = r^2 \quad \ldots(\text{i})$$

Here, centre is (1, –1) and radius is 4.

∴Eq. (i) becomes

$$(x-1)^2 + (y+1)^2 = 4^2$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 + 2y + 1 = 16$$

$$\Rightarrow \quad x^2 + y^2 - 2x + 2y - 14 = 0$$

Example 2 *Find the equation of circle which passes through the three points* (0, 1), (2, 0) *and* (5, 6) *is*

(*a*) $x^2 + y^2 - 5x - 7y + 6 = 0$ (*b*) $x^2 + y^2 + 5x - 7y + 6 = 0$

(*c*) $x^2 + y^2 - 5x + 7y - 6 = 0$ (*d*) *None of these*

Solution (*a*) Let the general equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(\text{i})$$

Since, the given points satisfy this equation, we have

$$1 + 2f + c = 0 \quad \ldots(\text{ii})$$

$$4 + 4g + c = 0 \quad \ldots(\text{iii})$$

and $$61 + 10g + 12f + c = 0 \quad \ldots(\text{iv})$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$3 + 4g - 2f = 0 \quad \ldots(\text{v})$$

On subtracting Eq. (iii) from Eq. (iv), we get

$$57 + 6g + 12f = 0 \quad \ldots(\text{vi})$$

On multiplying Eq. (v) by 6 and adding Eqs. (v) and (vi), we get

$$\begin{array}{r} 57 + 6g + 12f = 0 \\ 18 + 24g - 12f = 0 \\ \hline 75 + 30g \quad = 0 \end{array}$$

$$\Rightarrow \quad g = -\frac{75}{30} = -\frac{5}{2}$$

and $$f = -\frac{7}{2}$$

∴ From Eq. (iv), we get

$$61 + 10\left(-\frac{5}{2}\right) + 12\left(-\frac{7}{2}\right) + c = 0$$

$$\Rightarrow \quad 61 - 25 - 42 + c = 0$$

$$\therefore \quad c = 6$$

On putting all these values in Eq. (i), we get required equation of circle which is

$$x^2 + y^2 - 5x - 7y + 6 = 0$$

Example 3 *One of the equation of a circle of radius* 13 *whose centre lies on x-axis and passes through the point* (4, 5), *is*

(*a*) $x^2 + y^2 - 32x + 37 = 0$ (*b*) $x^2 + y^2 + 16x + 87 = 0$

(*c*) $x^2 + y^2 - 32x - 105 = 0$ (*d*) $x^2 + y^2 + 16x - 105 = 0$

Solution (*d*) ∵ Centre lies on *x*-axis, therefore centre of circle is $C(r, 0)$ which passes through $P(4, 5)$.

∴ Equation of circle is $(r-4)^2 + (0-5)^2 = 13^2$

$$\Rightarrow \quad r^2 - 8r + 16 + 25 = 169$$

$$\Rightarrow \quad r^2 - 8r + 16 = 144$$

$$\Rightarrow \quad r - 4 = \pm 12 \quad \Rightarrow \quad r = 16, -8$$

Thus, the coordinates of centre are (16, 0), (–8, 0).

Hence, the equation of the required circle are

$$(x-16)^2 + (y-0)^2 = 13^2$$

and $$(x+8)^2 + (y-0)^2 = 13^2$$

$$\Rightarrow \quad x^2 + y^2 - 32x + 87 = 0$$

and $$x^2 + y^2 + 16x - 105 = 0$$

Example 4 *The equation of the circle, which touches both the axes and whose radius is* 4, *is*

(*a*) $x^2 + y^2 \pm 8x \pm 8y + 16 = 0$

(*b*) $x^2 + y^2 \pm x \pm y + 16 = 0$

(*c*) $x^2 + y^2 \pm x \pm y - 16 = 0$

(*d*) *None of the above*

Solution (*a*) Equation of required circle is

$$(x-4)^2 + (y-4)^2 = 4^2$$

$$\Rightarrow \quad x^2 + y^2 - 8x - 8y + 16 = 0$$

$\because$ The circle may lie in any one of the four quadrants. So, there are four circles and equation of these circles are given by

$$x^2 + y^2 \pm 8x \pm 8y + 16 = 0$$

Example 5 *The equation of circle which passes through the origin and cuts off intercepts 5 and 6 from the positive parts of the axes respectively is* $\left(x - \frac{5}{2}\right)^2 + (y-3)^2 = \lambda$, *where* λ *is*

(*a*) $\frac{61}{4}$ (*b*) $\frac{6}{4}$

(*c*) $\frac{1}{4}$ (*d*) 0

Solution (*a*) From figure, we have

$$OP = 5, \quad OQ = 6$$

and $$OM = \frac{5}{2}, \quad CM = 3$$

$\therefore$ In ΔOMC, $OC^2 = OM^2 + MC^2$

$$\Rightarrow \quad OC^2 = \left(\frac{5}{2}\right)^2 + (3)^2$$

$$\Rightarrow \quad OC = \frac{\sqrt{61}}{2}$$

Thus, the required circle has its centre $\left(\frac{5}{2}, 3\right)$ and radius $\frac{\sqrt{61}}{2}$.

Hence, its equation is $\left(x - \frac{5}{2}\right)^2 + (y-3)^2 = \left(\frac{61}{4}\right)$.

Hence, $$\lambda = \frac{61}{4}$$

Example 6 *If* $y = 5x$ *is a chord of the circle* $x^2 + y^2 + 20x = 0$, *the equation of a circle with this chord as diameter is*

(*a*) $3x^2 + 3y^2 + 10x + 50y = 0$

(*b*) $13x^2 + 13y^2 + 10x + 50y = 0$

(*c*) $13x^2 + 13y^2 + x + y = 0$

(*d*) *None of the above*

Solution (*b*) We have, $y = 5x$ and $x^2 + y^2 + 20x = 0$

$\therefore$ The point of intersection of the given chord and the given circle are obtained by solving

$$y = 5x$$

and $$x^2 + y^2 + 20x = 0$$

$$\therefore \quad x^2 + 25x^2 + 20x = 0$$

$$\Rightarrow \quad x = 0, -\frac{10}{13}$$

$$\therefore \quad y = 0, -\frac{50}{13}$$

Required equation of the circle is

$$(x-0)\left(x + \frac{10}{13}\right) + (y-0)\left(y + \frac{50}{13}\right) = 0$$

$$\Rightarrow \quad 13x^2 + 10x + 13y^2 + 50y = 0$$

$$\Rightarrow \quad 13x^2 + 13y^2 + 10x + 50y = 0$$

Example 7 *The parametric equations of the circle* $x^2 + y^2 + mx + my = 0$ *are*

(*a*) $x = -\frac{m}{2} + \frac{m}{\sqrt{2}}\cos\theta, \quad y = \frac{m}{2} + \frac{m}{\sqrt{2}}\sin\theta$

(*b*) $x = -\frac{m}{2} + \frac{m}{\sqrt{2}}\cos\theta, \quad y = -\frac{m}{2} + \frac{m}{\sqrt{2}}\sin\theta$

(*c*) $x = 0, y = 0$

(*d*) *None of the above*

Solution (*b*) Here, $$x^2 + y^2 + mx + my = 0$$

$$\Rightarrow \quad (x^2 + mx) + (y^2 + my) = 0$$

$$\Rightarrow \quad \left(x^2 + mx + \frac{m^2}{4}\right) + \left(y^2 + my + \frac{m^2}{4}\right) = \frac{m^2}{2}$$

$$\Rightarrow \quad \left[x - \left(-\frac{m}{2}\right)\right]^2 + \left[y - \left(-\frac{m}{2}\right)\right]^2 = \left(\frac{m}{\sqrt{2}}\right)^2$$

So, the parametric equations of the circles are

$$x = -\frac{m}{2} + \frac{m}{\sqrt{2}}\cos\theta$$

and $$y = -\frac{m}{2} + \frac{m}{\sqrt{2}}\sin\theta$$

Example 8 *The equation of a circle which cuts the three circles* $x^2 + y^2 - 3x - 6y + 14 = 0$, $x^2 + y^2 - x - 4y + 8 = 0$, $x^2 + y^2 + 2x - 6y + 9 = 0$ *orthogonally is*

(*a*) $x^2 + y^2 - 2x - 4y + 1 = 0$

(*b*) $x^2 + y^2 + 2x + 4y + 1 = 0$

(*c*) $x^2 + y^2 - 2x + 4y + 1 = 0$

(*d*) $x^2 + y^2 - 2x - 4y - 1 = 0$

Solution (*a*) The circle having centre at the radical centre of three given circles and radius equal to the length of the tangent from it to any one of three circles cuts all the three circles orthogonally. The given circles are

$$x^2 + y^2 - 3x - 6y + 14 = 0 \quad \text{...(i)}$$

$$x^2 + y^2 - x - 4y + 8 = 0 \quad \text{...(ii)}$$

$$x^2 + y^2 + 2x - 6y + 9 = 0 \quad \text{...(iii)}$$

The radical axis of Eqs. (i), (ii) and (iii) are respectively.

$$x + y - 3 = 0 \quad \ldots(iv)$$

and $$3x - 2y + 1 = 0 \quad \ldots(v)$$

On solving Eqs. (iv) and (v), we get $x = 1, y = 2$

Thus, the coordinates of the radical centre are (1, 2).

The length of the tangent from (1, 2) to Eq. (i) is

$$r = \sqrt{1 + 4 - 3 - 12 + 14} = 2$$

Hence, the required circle is $(x-1)^2 + (y-2)^2 = 2^2$

$$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 1 = 0$$

Example 9 *Centre of the circle whose radius is 3 and which touches internally the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ at the point $(-1, -1)$, is*

(a) $\left(\frac{7}{5}, \frac{-4}{5}\right)$ (b) $\left(\frac{4}{5}, \frac{7}{5}\right)$ (c) $\left(\frac{3}{5}, \frac{4}{5}\right)$ (d) $\left(\frac{7}{5}, \frac{3}{5}\right)$

Solution (b) The given circle is

$$x^2 + y^2 - 4x - 6y - 12 = 0 \quad \ldots(i)$$

whose centre is $C_1(2, 3)$ and radius $r_1 = C_1A = 5$

If $C_2(h, k)$ is the centre of the circle of radius 3 which touches the circle (i) internally at the point $A(-1, -1)$, then $C_2A = 3$

and $$C_1C_2 = C_1A - C_2A = 5 - 3 = 2$$

Thus, $C_2(h, k)$ divide C_1A in the ratio 2 : 3 internally.

$$\therefore \quad h = \frac{2(-1) + 3 \cdot 2}{2 + 3} = \frac{4}{5} \text{ and } k = \frac{2(-1) + 3 \cdot 3}{2 + 3} = \frac{7}{5}$$

Hence, required centre is $\left(\frac{4}{5}, \frac{7}{5}\right)$.

Example 10 *Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.*

(a) (1, 2) (b) (1, – 2)

(c) (– 1, 2) (d) *None of these*

Solution (a) Since, the radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore$ Radical centre = orthocentre

Given sides are $$4x - 7y + 10 = 0 \quad \ldots(i)$$

$$x + y - 5 = 0 \quad \ldots(ii)$$

and $$7x + 4y - 15 = 0 \quad \ldots(iii)$$

Since, lines (i) and (iii) are perpendiculars the point of intersection of (i) and (iii) is (1, 2) the orthocentre of the triangle. Hence, radical centre is (1, 2).

Example 11 *A ΔPQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is equal to*

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/6$ (d) $\pi/4$

Solution (d) We know that in circle, $\angle QPR = \frac{1}{2}\angle QOR$; $O(0, 0)$ being the centre of the given circle $x^2 + y^2 = 25$

Let m_1 = slope of $OQ = 4/3$ and m_2 = slope of $OR = -3/4$

As, $$m_1m_2 = -1, \angle QOR = \pi/2$$

$$\Rightarrow \quad \angle QPR = \frac{1}{2}(\angle QOR) = \pi/4$$

Example 12 *Which of the following is not the circles which pass through the origin and cut off chords of length a from each of the lines y = x and y = – x.*

(a) $x^2 + y^2 \pm \sqrt{2}ax = 0$ (b) $x^2 + y^2 \pm \sqrt{2}ax \pm \sqrt{2}ay = 0$

(c) $x^2 + y^2 \pm \sqrt{2}ay = 0$ (d) *None of these*

Solution (c) Coordinates of point A are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

and coordinates of point B are $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$

Now, from the geometry A and B are end points of diameter of the circle.

Then, the equation of circle is

$$\left(x - \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0$$

$$\Rightarrow \quad x^2 + y^2 - \sqrt{2}ax = 0$$

Similarly, circle with C and D as end points of diameter is

$$x^2 + y^2 + \sqrt{2}ax = 0$$

With similar arguments circles with A and C and B and D as end points of diameter are given by

$$x^2 + y^2 \pm \sqrt{2}ay = 0$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Standard Form of Equation of Circle

1. If $2(x^2 + y^2) + 4\lambda x + \lambda^2 = 0$ represents a circle of meaningful radius, then the range of real values of λ is

(a) R (b) $(0, +\infty)$
(c) $(-\infty, 0)$ (d) None of these

2. Find the equation of circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$. [NCERT]

(a) $x^2 + y^2 - 2ax - 2by - 2b^2 = 0$
(b) $x^2 + y^2 + 2ax + by + 2b^2 = 0$
(c) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
(d) None of the above

3. Find the equation of the circle which touches the both axes in first quadrant and whose radius is a. [NCERT Exemplar]

(a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
(b) $x^2 + y^2 - 2ax + 2ay - a^2 = 0$
(c) $x^2 + y^2 + 2ax - 2ay + a^2 = 0$
(d) None of the above

4. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the coordinate of the other end of the diameter.

(a) (2, 1) (b) (1, 2)
(c) (1, 1) (d) None of these

5. The centre of a circle is (2, – 3) and the circumference is 10π. Then, the equation of the circle is

(a) $x^2 + y^2 + 4x + 6y + 12 = 0$
(b) $x^2 + y^2 - 4x + 6y + 12 = 0$
(c) $x^2 + y^2 - 4x + 6y - 12 = 0$
(d) $x^2 + y^2 - 4x - 6y - 12 = 0$

6. The circle $x^2 + y^2 - 8x + 4y + 4 = 0$ touches

(a) x-axis (b) y-axis
(c) both axis (d) neither x-axis nor y-axis

7. Equation of a circle which passes through (3, 6) and touches the axes is [NCERT Exemplar]

(a) $x^2 + y^2 + 6x + 6y + 3 = 0$
(b) $x^2 + y^2 - 6x - 6y - 9 = 0$
(c) $x^2 + y^2 - 6x - 6y + 9 = 0$
(d) None of the above

8. Equation of the circle with centre on the y-axis and passing through the origin and the point (2, 3) is [NCERT Exemplar]

(a) $x^2 + y^2 + 13y = 0$
(b) $3x^2 + 3y^2 + 13x + 3 = 0$
(c) $6x^2 + 6y^2 - 26y = 0$
(d) $x^2 + y^2 + 13x + 3 = 0$

9. If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 sq units, then find the equation of the circle. [NCERT Exemplar]

(a) $x^2 + y^2 - 2x + 2y = 40$
(b) $x^2 + y^2 - 2x - 2y = 47$
(c) $x^2 + y^2 - 2x + 2y = 47$
(d) None of the above

10. Circle $x^2 + y^2 - 2x - \lambda x - 1 = 0$ passes through two fixed points, coordinates of the points are

(a) $(0, \pm 1)$ (b) $(\pm 1, 0)$
(c) (0, 1) and (0, 2) (d) (0, – 1) and (0, – 2)

11. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$. [NCERT Exemplar]

(a) $x^2 + y^2 - 4x - 10y + 25 = 0$
(b) $x^2 + y^2 - 4x - 10y - 25 = 0$
(c) $x^2 + y^2 - 4x + 10y - 25 = 0$
(d) None of the above

12. The circle $x^2 + y^2 - 10x - 14y + 24 = 0$ cuts an intercepts on y-axis of length
(a) 5 (b) 10
(c) 1 (d) None of these

Intersection of Line and Circle

13. If the line $3x - 4y - k = 0, (k > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $k + a + b$ is equal to
(a) 20 (b) 22
(c) -30 (d) -28

14. The line $3x - 2y = k$ meets the circle $x^2 + y^2 = 4r^2$ at only one point, if k^2 is
(a) $20r^2$ (b) $52r^2$
(c) $\frac{52}{9}r^2$ (d) $\frac{20}{9}r^2$

15. A line through (0, 0) cuts the circle $x^2 + y^2 - 2ax = 0$ at A and B, then locus of the centre of the circle drawn AB as diameter is
(a) $x^2 + y^2 - 2ay = 0$ (b) $x^2 + y^2 + ay = 0$
(c) $x^2 + y^2 + ax = 0$ (d) $x^2 + y^2 - ax = 0$

16. If the line $\frac{x}{a} + \frac{y}{b} = 1$ moves such that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$, where c is a constant, then the locus of the foot of the perpendicular from the origin to the line is
(a) straight line (b) circle
(c) parabola (d) ellipse

17. A line through $P(1, 4)$ intersect a circle $x^2 + y^2 = 16$ at A and B, then $PA \cdot PB$ is equal to
(a) 1 (b) 2 (c) 3 (d) 4

18. If a circle passes through the point of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is
(a) 2 (b) 4
(c) 6 (d) 3

Tangent and Normal to the Circle

19. The equation of the tangents to the circle $x^2 + y^2 = 4$, which are parallel to $x + 2y + 3 = 0$, are
(a) $x - 2y = 2$ (b) $x + 2y = \pm 2\sqrt{3}$
(c) $x + 2y = \pm 2\sqrt{5}$ (d) $x - 2y = \pm 2\sqrt{5}$

20. If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{1 + m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$, is
(a) $\frac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\frac{\sqrt{a^2 - 4b^2}}{2b}$
(c) $\frac{2b}{a - 2b}$ (d) $\frac{b}{a - 2b}$

21. The tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at
(a) (6, 7) (b) (– 6, 7)
(c) (6, – 7) (d) (– 6, – 7)

22. The square of the length of the tangent from (3, – 4) to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$, is
(a) 20 (b) 30 (c) 40 (d) 50

23. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is
(a) $c_1 - c$ (b) $c - c_1$ (c) $\sqrt{c - c_1}$ (d) $\sqrt{c_1 - c}$

24. Centre of circle whose normals are
$$x^2 - 2xy - 3x + 6y = 0, \text{ is}$$
(a) $\left(3, \frac{3}{2}\right)$ (b) $\left(3, -\frac{3}{2}\right)$
(c) $\left(\frac{3}{2}, 3\right)$ (d) None of these

25. The equation of normal to the circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$ at (1, 1) is
(a) $2x + y = 3$ (b) $x - 2y = 3$
(c) $x + 2y = 3$ (d) None of these

26. The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then, the equation of the circle is
(a) $x^2 + y^2 + 2x - 2y - 13 = 0$
(b) $x^2 + y^2 - 2x - 2y - 11 = 0$
(c) $x^2 + y^2 - 2x + 2y + 12 = 0$
(d) $x^2 + y^2 - 2x - 2y + 14 = 0$

27. If line $ax + by = 0$ touches $x^2 + y^2 + 2x + 4y = 0$ and is a normal to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$, then value of (a, b) will be
(a) (2,1) (b) (1, –2) (c) (1, 2) (d) (–1, 2)

28. The area of the triangle formed by the positive x-axis and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is
(a) $2\sqrt{3}$ sq units (b) $3\sqrt{2}$ sq units
(c) $\sqrt{6}$ sq unit (d) None of these

29. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle. [NCERT Exemplar]
(a) 3/4 (b) 4/3 (c) 1/4 (d) 7/4

30. Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant. [NCERT Exemplar]
(a) $x^2 + y^2 + 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 4x - 4y + 4 = 0$
(c) $x^2 + y^2 + 4x + 4y + 4 = 0$
(d) None of the above

31. The value of c, for which the line $y = 2x + c$ is a tangent to the circle $x^2 + y^2 = 16$, is
(a) $-16\sqrt{5}$ (b) $4\sqrt{5}$
(c) $16\sqrt{5}$ (d) 20

32. Two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P. Then, the locus of P has the equation
(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = 3a^2$
(c) $x^2 + y^2 = 4a^2$ (d) None of these

33. From a point on the circle $x^2 + y^2 = a^2$, two tangents are drawn to the circle $x^2 + y^2 = a^2 \sin^2 \alpha$. The angle between them is
(a) α (b) $\frac{\alpha}{2}$
(c) 2α (d) None of these

34. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the ΔPAB as P moves on the circle is
(a) a parabola (b) a circle
(c) an ellipse (d) None of these

35. The locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 2 : 3, is
(a) $5x^2 + 5y^2 + 60x - 7 = 0$
(b) $5x^2 + 5y^2 - 60x - 7 = 0$
(c) $5x^2 + 5y^2 + 60x + 7 = 0$
(d) $5x^2 + 5y^2 + 60x + 12 = 0$

Chord of Contact and Diameter

36 Radius of circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre, is
(a) 1 (b) $\sqrt{3}$
(c) $\frac{\sqrt{3}}{2}$ (d) None of these

37. The condition that the chord $x \cos \alpha + y \sin \alpha - p = 0$ of $x^2 + y^2 - a^2 = 0$ may subtend a right angle at the centre of circle, is
(a) $a^2 = 2p^2$ (b) $p^2 = 2a^2$
(c) $a = 2p$ (d) $p = 2a$

38. Which of the following is a point on the common chord of the circles $x^2 + y^2 + 2x - 3y + 6 = 0$ and $x^2 + y^2 + x - 8y - 13 = 0$?
(a) $(1, -2)$ (b) $(1, 4)$ (c) $(1, 2)$ (d) $(1, -4)$

39. The locus of the middle point of the chords of the circle $x^2 + y^2 = a^2$ such that the chords pass through a given point (x_1, y_1), is
(a) $x^2 + y^2 - xx_1 - yy_1 = 0$
(b) $x^2 + y^2 = x_1^2 + y_1^2$
(c) $x + y = x_1 + y_1$
(d) $x + y = x_1^2 + y_1^2$

40. If two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where, $pq \neq 0$) are bisected by the x-axis, then
(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

41. If the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then a, b and c are in
(a) AP (b) GP
(c) HP (d) None of these

42. The equation of the circle on the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ as diameter, is
(a) $x^2 + y^2 = 2ab(bx + ay)$
(b) $x^2 + y^2 = bx + ay$
(c) $(a^2 + b^2)(x^2 + y^2) = 2ab(bx - ay)$
(d) $(a^2 + b^2)(x^2 + y^2) = 2(bx + ay)$

43. The equation of the circle and its chord are respectively $x^2 + y^2 = a^2$ and $x \cos \alpha + y \sin \alpha = p$. The equation of the circle of which this chord is a diameter is
(a) $x^2 + y^2 - 2px \cos \alpha - 2py \sin \alpha + 2p^2 - a^2 = 0$
(b) $x^2 + y^2 - 2px \cos \alpha - 2py \sin \alpha + p^2 - a^2 = 0$
(c) $x^2 + y^2 + 2px \cos \alpha + 2py \sin \alpha + 2p^2 - a^2 = 0$
(d) None of the above

44. Consider the circle $x^2 + (y - 1)^2 = 9, (x - 1)^2 + y^2 = 25$. They are such that
(a) these circles touch each other
(b) one of these circles lies entirely inside the other
(c) each of these circles lies outside the other
(d) they intersect in two points

Family of Circles

45. If the circle $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$, then $c + d$ is equal to
(a) 60 (b) 50 (c) 40 (d) 30

46. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
(a) $4x^2 + 4y^2 - 30x - 10y = 25$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $4x^2 + 4y^2 - 17x - 10y + 25 = 0$
(d) None of the above

47. The equation of the circle which passes through the intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and whose centre lies on $13x + 30y = 0$, is
(a) $x^2 + y^2 + 30x - 13y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $2x^2 + 2y^2 + 30x - 13y - 25 = 0$
(d) $x^2 + y^2 + 30x - 13y + 25 = 0$

48. If $(-3, 2)$ lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is concentric with the circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then c is equal to
(a) 11 (b) -11 (c) 24 (d) 100

49. Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area. [NCERT Exemplar]
(a) $x^2 + y^2 - 6x + 12y - 15 = 0$
(b) $x^2 + y^2 - 6x - 12y + 15 = 0$
(c) $x^2 + y^2 - 6x + 12y + 15 = 0$
(d) None of the above

50. Two circles with centres (2, 3) and (5, 6) cut orthogonally. If radius of both circles are equal, then radius is equal to
(a) 1 (b) 2 (c) 3 (d) 4

51. If P is a point such that the ratio of the squares of the lengths of the tangents from P to the circles $x^2 + y^2 + 2x - 4y - 20 = 0$ and $x^2 + y^2 - 4x + 2y - 44 = 0$ is 2 : 3, then the locus of P is a circle with centre
(a) $(7, -8)$ (b) $(-7, 8)$
(c) $(7, 8)$ (d) $(-7, -8)$

52. Equation of the circle passing through the point (3, 4) and concentric with the circle $x^2 + y^2 - 2x - 4y + 1 = 0$ is
(a) $x^2 + y^2 - 2x - 4y = 0$
(b) $x^2 + y^2 - 2x - 4y + 3 = 0$
(c) $x^2 + y^2 - 2x - 4y - 3 = 0$
(d) None of the above

53. The length of the transversal common tangent to the circle $x^2 + y^2 = 1$ and $(x - t)^2 + y^2 = 1$ is $\sqrt{21}$, then t is equal to
(a) ± 2 (b) ± 5
(c) ± 3 (d) None of these

54. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the ΔPAB is
(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

55. If two circles, each of radius 5 units, touch each other at (1, 2) and the equation of their common tangent is $4x + 3y = 10$, then equation of the circle a portion of which lies in all the quadrants, is
(a) $x^2 + y^2 - 10x - 10y + 25 = 0$
(b) $x^2 + y^2 + 6x + 2y - 15 = 0$
(c) $x^2 + y^2 + 2x + 6y - 15 = 0$
(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

56. C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_2 which does not pass through C_2, is
(a) $x - \sqrt{3}y + 2 = 0$ (b) $\sqrt{3}x - y + 2 = 0$
(c) $\sqrt{3}x - y - 2 = 0$ (d) $x + \sqrt{3}y + 2 = 0$

57. The locus of centre of a circle $x^2 + y^2 - 2x - 2y + 1 = 0$, which rolls outside the circle $x^2 + y^2 - 6x + 8y = 0$, is
(a) $x^2 + y^2 - 2x - 2y - 34 = 0$
(b) $x^2 + y^2 - 6x - 8y + 11 = 0$
(c) $x^2 + y^2 - 6x + 8y - 11 = 0$
(d) None of the above

58. The locus of the centre of a circle which cuts orthogonally the circle $x^2 + y^2 - 20x + 4 = 0$ and which touches $x = 2$, is
(a) $y^2 = 16x + 4$ (b) $x^2 = 16y$
(c) $x^2 = 16y + 4$ (d) $y^2 = 16x$

59. The centre of the circle, which cuts orthogonally each of the three circles $x^2 + y^2 + 2x + 17y + 4 = 0$ and $x^2 + y^2 + 7x + 6y + 11 = 0$, $x^2 + y^2 - x + 22y + 3 = 0$, is
(a) (3, 2) (b) (1, 2)
(c) (2, 3) (d) (0, 2)

60. The locus of centres of family of circle passing through the origin and cutting the circle $x^2 + y^2 + 4x - 6y - 13 = 0$ orthogonally, is
(a) $4x + 6y + 13 = 0$ (b) $4x - 6y + 13 = 0$
(c) $4x + 6y - 13 = 0$ (d) $4x - 6y - 13 = 0$

61. The locus of the centre of circle which cuts the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 4x + 6y + 4 = 0$ orthogonally, is
(a) $12x + 8y + 5 = 0$ (b) $8x + 12y + 5 = 0$
(c) $8x - 12y + 5 = 0$ (d) None of these

62. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
(a) 2 or $-\frac{3}{2}$ (b) -2 or $\frac{3}{2}$
(c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$

63. The circles $x^2 + y^2 + 2g_1x - a^2 = 0$ and $x^2 + y^2 + 2g_2x - a^2 = 0$ cut each other orthogonally. If p_1 and p_2 are perpendiculars from $(0, a)$ and $(0, -a)$ on a common tangent of these circles, then p_1p_2 is equal to

(a) $\frac{a^2}{2}$ (b) a^2
(c) $2a^2$ (d) $a^2 + 2$

64. If the circle $S_1 : x^2 + y^2 = 16$ intersects another circle S_1 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to $\frac{3}{4}$, the coordinates of the centre of S_2 are

(a) $\left(-\frac{9}{5}, \frac{12}{5}\right), \left(\frac{9}{5}, -\frac{12}{5}\right)$ (b) $\left(-\frac{9}{5}, -\frac{12}{5}\right), \left(\frac{9}{5}, \frac{12}{5}\right)$
(c) $\left(\frac{12}{5}, \frac{9}{5}\right), \left(-\frac{12}{5}, \frac{9}{5}\right)$ (d) None of these

Miscellaneous

65. The locus of the mid-point of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$, which makes an angle of $120°$ at the centre, is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 + x + y - 1 = 0$
(c) $x^2 + y^2 - 2x - 2y - 1 = 0$
(d) None of the above

66. The lengths of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$, $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are in the ratio

(a) 1 : 2 (b) 2 : 3
(c) 3 : 4 (d) None of these

67. If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, then

(a) $g = \frac{3}{4}$ and $f \neq 2$ (b) $g \neq \frac{3}{4}$ and $f = 2$
(c) $g = \frac{3}{4}$ or $f = 2$ (d) None of these

68. $P(a, b)$ be any point such that the length of tangents from P to both the circles $x^2 + y^2 - 6x - 8y = 0$ and $x^2 + y^2 - 12x - 16y + 12 = 0$ are equal, then

(a) $3a + 4b - 6 = 0$ (b) $3a - 4b + 6 = 0$
(c) $6a - 8b + 12 = 0$ (d) $4a - 3b + 7 = 0$

69. If the area of the circle $4x^2 + 4y^2 - 8x + 16y + k = 0$ is 9π sq units, then the value of k is

(a) 4 (b) 16
(c) -16 (d) ± 16

70. The area of the circle centred at (1, 2) and passing through (4, 6) is [NCERT Exemplar]

(a) 5π (b) 10π
(c) 25π (d) None of these

71. If a circle passes through the point (0, 0), $(a, 0)$ and $(0, b)$, then find the coordinates of its centre. [NCERT Exemplar]

(a) $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ (b) $\left(\frac{a}{2}, -\frac{b}{2}\right)$
(c) $\left(-\frac{a}{2}, \frac{b}{2}\right)$ (d) None of these

72. The area of square inscribed in a circle $x^2 + y^2 - 6x - 8y = 0$ is

(a) 100 sq units (b) 50 sq units
(c) 25 sq units (d) None of these

73. The distinct points $A(0, 0), B(0, 1), C(1, 0)$ and $D(2a, 3a)$ are concyclic, then

(a) 'a' can attain only rational values
(b) a is irrational
(c) cannot be concyclic for any a
(d) None of the above

74. The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is [NCERT Exemplar]

(a) $x^2 + y^2 = 9a^2$
(b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = 4a^2$
(d) $x^2 + y^2 = a^2$

75. If $\left(m_i, \frac{1}{m_i}\right), i = 1, 2, 3, 4$ are concyclic points, then the value of $m_1 m_2 m_3 m_4$ is

(a) 1 (b) -1
(c) 0 (d) None of these

76. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is

(a) 4 (b) $2\sqrt{5}$
(c) 5 (d) $3\sqrt{5}$

77. If the abscissae and ordinates of two points P and Q are roots of the equations $x^2 + 2ax - b^2 = 0$ and $y^2 + 2py - q^2 = 0$ respectively, then the equation of the circle with PQ as diameter, is

(a) $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
(b) $x^2 + y^2 - 2ax - 2py + b^2 + q^2 = 0$
(c) $x^2 + y^2 - 2ax - 2py - b^2 - q^2 = 0$
(d) $x^2 + y^2 + 2ax + 2py + b^2 + q^2 = 0$

78. Let PQ and PS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point x on the circumference of the circle, then $2r$ equals

(a) $\sqrt{PQ \cdot RS}$ (b) $\frac{PQ + RS}{2}$

(c) $\frac{2PQ \cdot RS}{PQ + RS}$ (d) $\sqrt{\frac{PQ^2 + RS^2}{2}}$

79. If $(a \cos \theta_i, a \sin \theta_i)$, $i = 1, 2, 3$ represent the vertices of an equilateral triangle inscribed in a circle, then

(a) $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$
(b) $\sec \theta_1 + \sec \theta_2 + \sec \theta_3 = 0$
(c) $\tan \theta_1 + \tan \theta_2 + \tan \theta_3 = 0$
(d) $\cot \theta_1 + \cot \theta_2 + \cot \theta_3 = 0$

80. The range of α, for which the point (α, α) lies inside the region bounded by the curves $y = \sqrt{1 - x^2}$ and $x + y = 1$ is

(a) $\frac{1}{2} < \alpha < \frac{1}{\sqrt{2}}$ (b) $\frac{1}{2} < \alpha < \frac{1}{3}$

(c) $\frac{1}{3} < \alpha < \frac{1}{\sqrt{3}}$ (d) $\frac{1}{4} < \alpha < \frac{1}{2}$

81. A line meets the coordinate axes in A and B. A circle is circumscribed about the ΔOAB. The distances from the points A and B of the side AB to the tangent at O are equal to m and n respectively. Then, the diameter of the circle is

(a) $m(m + n)$ (b) $n(m + n)$
(c) $m - n$ (d) None of these

82. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 can be represented by

(a) $x + y = 0$ (b) $x - y = 0$
(c) $7x + y = 0$ (d) $x - 7y = 0$

83. If the area of the quadrilateral formed by the tangent from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the pair of radii at the points of contact of these tangents to the circle is 8 sq units, then c is a root of the equation

(a) $c^2 - 32c + 64 = 0$ (b) $c^2 - 34c + 64 = 0$
(c) $c^2 + 2c - 64 = 0$ (d) $c^2 + 34c - 64 = 0$

84. Consider the following statements

I. Circle $x^2 + y^2 - x - y - 1 = 0$ is completely inside the circle $x^2 + y^2 - 2x + 2y - 7 = 0$.

II. Number of common tangents of the circles

$$x^2 + y^2 + 14x + 12y + 21 = 0$$

and $$x^2 + y^2 + 2x - 4y - 4 = 0 \text{ is } 4.$$

Which of these is/are correct?

(a) Only I (b) Only II
(c) Both I and II (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. The set of values of c so that the equations $y = |x| + c$ and $x^2 + y^2 - 8|x| - 9 = 0$ have no solution, is

(a) $(-\infty, -3) \cup (3, \infty)$ (b) $(-3, 3)$
(c) $(-\infty, 5\sqrt{2}) \cup (5\sqrt{2}, \infty)$ (d) $(5\sqrt{2} - 4, \infty)$

2. Points $(-6, 0)$, $(0, 6)$ and $(-7, 7)$ are the vertices of ΔABC. The incircle of the triangle has the equation

(a) $x^2 + y^2 - 9x - 9y + 36 = 0$
(b) $x^2 + y^2 + 9x - 9y + 36 = 0$
(c) $x^2 + y^2 + 9x + 9y - 36 = 0$
(d) $x^2 + y^2 + 18x - 18y + 36 = 0$

3. Two rods of lengths a and b slide along the x-axis and y-axis respectively in such a manner that their ends are concyclic. The locus of the centre of the circle passing through the end points is

(a) $4(x^2 + y^2) = a^2 + b^2$
(b) $x^2 + y^2 = a^2 + b^2$
(c) $4(x^2 - y^2) = a^2 - b^2$
(d) $x^2 - y^2 = a^2 - b^2$

4. The range of values of a such that the angle θ between the pair of tangents drawn from $(a, 0)$ to the circle $x^2 + y^2 = 1$ satisfies $\frac{\pi}{2} < \theta < \pi$, is

(a) $(1, 2)$ (b) $(1, \sqrt{2})$
(c) $(-\sqrt{2}, -1)$ (d) $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

5. Two points P and Q are taken on the line joining the points $A(0, 0)$ and $B(3a, 0)$ such that $AP = PQ = QB$. Circles are drawn on AP, PQ and QB as diameters.

The locus of the points, the sum of the squares of the tangents from which to the three circles is equal to b^2, is

(a) $x^2 + y^2 - 3ax + 2a^2 - b^2 = 0$
(b) $3(x^2 + y^2) - 9ax + 8a^2 - b^2 = 0$
(c) $x^2 + y^2 - 5ax + 6a^2 - b^2 = 0$
(d) $x^2 + y^2 - ax - b^2 = 0$

6. The equations of the sides AB, BC and CA of a ΔABC are $x + y = 1$, $4x - y + 4 = 0$ and $2x + 3y = 6$. Circles are drawn on AB, BC and CA as diameters. The point of concurrence of the common chord is

(a) centroid of the triangle (b) orthocentre
(c) circumcentre (d) incentre

7. An isosceles right angled triangle is inscribed in the circle $x^2 + y^2 = r^2$. If the coordinates of an end of the hypotenuse are (a, b), the coordinates of the vertex are
(a) $(-a, -b)$ (b) $(b, -a)$
(c) (b, a) (d) $(-b, -a)$

8. If OA and OB are equal perpendicular chord of the circles $x^2 + y^2 - 2x + 4y = 0$, then equations of OA and OB are (where, O is origin)
(a) $3x + y = 0$ and $3x - y = 0$
(b) $3x + y = 0$ and $3y - x = 0$
(c) $x + 3y = 0$ and $y - 3x = 0$
(d) $x + y = 0$ or $x - y = 0$

9. Equation of chord of the circle $x^2 + y^2 - 3x - 4y - 4 = 0$, which passes through the origin such that the origin divides it in the ratio 4 : 1, is
(a) $x = 0$ (b) $24x + y = 0$
(c) $7x + 24y = 0$ (d) $7x - 24y = 0$

10. In a ΔABC, right angled at A, on the leg AC as diameter, a semi-circle is described. If a chord joins A with the point of intersection D of the hypotenuse and the semi-circle, then the length of AC equal to
(a) $\dfrac{AB \times AD}{\sqrt{AB^2 + AD^2}}$ (b) $\dfrac{AB \times AD}{AB + AD}$
(c) $\sqrt{AB \times AD}$ (d) $\dfrac{AB \times AD}{\sqrt{AB^2 - AD^2}}$

11. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is
(a) $8\sqrt{3}$ sq units (b) $4\sqrt{3}$ sq units
(c) $6\sqrt{3}$ sq units (d) None of these

12. A pair of tangents are drawn to a unit circle with centre at the origin and these tangents intersect at A enclosing an angle of 60°. The area enclosed by these tangents and the arc of the circle is
(a) $\dfrac{2}{\sqrt{3}} - \dfrac{\pi}{6}$ (b) $\sqrt{3} - \dfrac{\pi}{3}$
(c) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{6}$ (d) $\sqrt{3}\left(1 - \dfrac{\pi}{6}\right)$

13. A straight line with slope 2 and y-intercept 5 touches the circle, $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then, the coordinates of Q are
(a) $(-6, 11)$ (b) $(-9, -13)$
(c) $(-10, -15)$ (d) $(-6, -7)$

14. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents $(a > b \geq 2)$, then
(a) $\theta = 2\cos^{-1}\left(\dfrac{a-b}{a+b}\right)$ (b) $\theta = 2\tan^{-1}\left(\dfrac{a+b}{a-b}\right)$
(c) $\theta = 2\sin^{-1}\left(\dfrac{a+b}{a-b}\right)$ (d) $\theta = 2\sin^{-1}\left(\dfrac{a-b}{a+b}\right)$

15. The equation of a line inclined at an angle $\pi/4$ to the x-axis, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is
(a) $2x - 2y - 3 = 0$ (b) $2x - 2y + 3 = 0$
(c) $x - y + 6 = 0$ (d) $x - y - 6 = 0$

More Than One Correct Option

16. The equation of a circle C_1 is $x^2 + y^2 = 4$. The locus of the intersection of orthogonal tangents to the circle is the curve C_2 and the locus of the intersection of perpendicular tangents to the curve C_2 is the curve C_3. Then,
(a) C_3 is a circle
(b) the area enclosed by the curve C_3 is 8π
(c) C_2 and C_3 are circles with the same centre
(d) None of the above

17. The equation of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are
(a) $x = 0$ (b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$

18. $A\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ is a point on the circle $x^2 + y^2 = 1$ and B is another point on the circle such that arc length $AB = \dfrac{\pi}{2}$ units. Then, coordinates of B can be
(a) $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$ (b) $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$
(c) $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$ (d) None of these

Comprehension Based Questions

Passage

Directions (Q. Nos. 19 and 20) *P is a variable point on the line $L = 0$. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch it at Q and R. The parallelogram PQSR is completed.*

19. If $L \equiv 2x + y = 6$, then the locus of circumcentre of ΔPQR is
(a) $2x - y = 4$ (b) $2x + y = 3$
(c) $x - 2y = 4$ (d) $x + 2y = 3$

20. If $P \equiv (2, 3)$, then the centre of circumcircle of ΔQRS is

(a) $\left(\frac{2}{13}, \frac{7}{26}\right)$ (b) $\left(\frac{2}{13}, \frac{3}{26}\right)$

(c) $\left(\frac{3}{13}, \frac{9}{26}\right)$ (d) $\left(\frac{3}{13}, \frac{2}{13}\right)$

Assertion and Reason

Directions (Q. Nos. 21- 27) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) and (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

21. **Statement I** Circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.

Statement II Centre of first circle lie on the second circle.

22. Let C_1 and C_2 be two circles with C_2 lying inside C_1.

Statement I A circle C lying inside C_1 touches C_1 internally and C_2 externally. Then, the locus of the centre of C is an ellipse.

Statement II If A and B are foci and P be any point on the ellipse, then $AP + BP =$ constant.

23. **Statement I** A ray of light incident at the point $(-3, -1)$ gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. If the reflected ray touches the circle, then equation of the reflected ray is $4y - 3x = 5$.

Statement II The angle of incidence = angle of reflection *i.e.*, $\angle i = \angle r$

24. Consider two circles

$$S \equiv x^2 + y^2 + 2gx + 2fy = 0$$

and $$S' \equiv x^2 + y^2 + 2g'x + 2f'y = 0$$

Statement I If two circles S and S' touch each other, then $f'g = fg'$.

Statement II Two circles touch each other, if line joining their centres is perpendicular to all possible common tangents.

25. **Statement I** Number of circles passing through $(-2, 1), (-1, 0), (-4, 3)$ is 1.

Statement II Through three non-collinear points in a plane only one circle can be drawn.

26. **Statement I** Circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 7 = 0$ intersect each other at two distinct points.

Statement II Circles with centres C_1 and C_2 and radii r_1 and r_2 intersect at two distinct points, if $|C_1C_2| < r_1 + r_2$.

27. Consider $L_1 : 2x + 3y + p - 3 = 0$ and $L_2 : 2x + 3y + p + 3 = 0$, where p is a real number and $C : x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement I If line L_1 is a chord of circle C, then L_2 is not always a diameter of circle C.

Statement II If line L_1 is a diameter of circle C, then L_2 is not a chord of circle C.

Previous Years' Questions

28. The circle passing through (1, –2) and touching the axis to x at (3, 0) also passes through the point [JEE Main 2013]

(a) $(-5, 2)$ (b) $(2, -5)$ (c) $(5, -2)$ (d) $(-2, 5)$

29. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is [AIEEE 2012]

(a) $\frac{10}{3}$ (b) $\frac{3}{5}$ (c) $\frac{6}{5}$ (d) $\frac{5}{3}$

30. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$, $(c > 0)$ touch each other if [AIEEE 2011]

(a) $|a| = c$ (b) $a = 2c$ (c) $|a| = 2c$ (d) $2|a| = c$

31. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is [AIEEE 2011]

(a) $x^2 + y^2 + x + y - 2 = 0$

(b) $x^2 + y^2 - 2x - 2y + 1 = 0$

(c) $x^2 + y^2 - x - y = 0$

(d) $x^2 + y^2 + 2x + 2y - 7 = 0$

32. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points, if [AIEEE 2010]

(a) $-85 < m < -35$ (b) $-35 < m < 15$

(c) $15 < m < 65$ (d) $35 < m < 85$

33. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) and [AIEEE 2009]

(a) all values of p

(b) all except one value of p

(c) all except two values of p

(d) exactly one value of p

34. The point diameterically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is [AIEEE 2008]
(a) (3, 4) (b) (3, – 4)
(c) (– 3, 4) (d) (– 3, – 4)

35. Consider a family of circles which are passing through the point (– 1, 1) and are tangent to x-axis. If (h, k) is the centre of circle, then [AIEEE 2007]
(a) $k \geq 1/2$ (b) $-1/2 \leq k \leq 1/2$
(c) $k \leq 1/2$ (d) $0 < k < 1/2$

36. Any chord of the circle $x^2 + y^2 = 25$ subtends a right angle at the centre. Then, the locus of the centroid of the triangle made by the chord and a moving point P on the circle is [AIEEE 2007]
(a) parabola
(b) circle
(c) rectangular hyperbola
(d) ellipse

37. Let C be the circle with centre (0, 0) and radius 3. The equation of the locus of the mid-points of the chords of the circle C that subtend an angle $2\pi/3$ at its centre is [AIEEE 2006]
(a) $x^2 + y^2 = \frac{27}{4}$ (b) $x^2 + y^2 = \frac{9}{4}$
(c) $x^2 + y^2 = \frac{3}{2}$ (d) $x^2 + y^2 = 1$

38. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π sq units, the equation of the circle is [AIEEE 2006]
(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 - 2x + 2y - 62 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 + 2x - 2y - 47 = 0$

39. If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector, then [AIEEE 2005]
(a) $3a^2 + 2ab + 3b^2 = 0$
(b) $3a^2 + 10ab + 3b^2 = 0$
(c) $3a^2 - 2ab + 3b^2 = 0$
(d) $3a^2 - 10ab + 3b^2 = 0$

40. A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is [AIEEE 2005]
(a) a parabola (b) a hyperbola
(c) a circle (d) an ellipse

41. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q, then the line $5x + by - a = 0$ passes through P and Q for [AIEEE 2005]
(a) no value of a
(b) exactly one value of a
(c) exactly two values of a
(d) infinitely many values of a

42. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is [AIEEE 2004]
(a) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(c) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(d) $2ax - 2by - (a^2 + b^2 + 4) = 0$

43. The smallest circle with centre on y-axis and passing through the point (7, 3) has radius [AIEEE 2004]
(a) $\sqrt{58}$ (b) 7
(c) 3 (d) 4

44. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is [AIEEE 2004]
(a) $x^2 + y^2 - 2x + 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 + 2x - 2y - 23 = 0$

45. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is [AIEEE 2004]
(a) $x^2 + y^2 - x - y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 + x + y = 0$
(d) $x^2 + y^2 + x - y = 0$

46. How many common tangents can be drawn to the following circles $x^2 + y^2 = 6x$ and

$$x^2 + y^2 + 6x + 2y + 1 = 0\,?$$ [AIEEE 2004]

(a) 4 (b) 3
(c) 2 (d) 1

47. A variable circle passes through the fixed point $A(p, q)$ and touches x-axis. The locus of the other end of the diameter through A is [AIEEE 2004]
(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(y - p)^2 = 4qx$ (d) $(y - q)^2 = 4px$

Answers

Check Point 1

1. (b)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (d)	**10.** (c)

Check Point 2

1. (c)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (d)	**8.** (c)	**9.** (b)

Round I

1. (a)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (c)	**9.** (c)	**10.** (a)
11. (a)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (a)
21. (d)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (c)
31. (b)	**32.** (a)	**33.** (c)	**34.** (b)	**35.** (c)	**36.** (a)	**37.** (a)	**38.** (d)	**39.** (a)	**40.** (d)
41. (b)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (b)	**46.** (b)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (c)
51. (b)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (b)	**56.** (b)	**57.** (c)	**58.** (d)	**59.** (a)	**60.** (d)
61. (c)	**62.** (a)	**63.** (b)	**64.** (a)	**65.** (a)	**66.** (a)	**67.** (c)	**68.** (a)	**69.** (c)	**70.** (c)
71. (d)	**72.** (b)	**73.** (a)	**74.** (c)	**75.** (a)	**76.** (c)	**77.** (a)	**78.** (a)	**79.** (a)	**80.** (a)
81. (d)	**82.** (b)	**83.** (b)	**84.** (a)						

Round II

1. (d)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (d)
11. (a)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (a)	**16.** (a,c)	**17.** (a,c)	**18.** (a,b)	**19.** (b)	**20.** (c)
21. (b)	**22.** (a)	**23.** (b)	**24.** (b)	**25.** (d)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (a)
31. (c)	**32.** (b)	**33.** (c)	**34.** (d)	**35.** (a)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (a)
41. (a)	**42.** (b)	**43.** (b)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (a)			

the Guidance

Round I

1. Since, $x^2 + y^2 + 2\lambda x + \frac{\lambda^2}{2} = 0$ is a circle representing meaningful radius.

$$\Rightarrow \quad \lambda^2 - \frac{\lambda^2}{2} \geq 0 \qquad \left(\because r = \sqrt{\lambda^2 - \frac{\lambda^2}{2}}\right)$$

$$\Rightarrow \quad \lambda^2 \geq 0$$

which is always true, $\forall\ \lambda \in R$.

2. Given centre $= (h, k) = (-a, -b)$ and radius $r = \sqrt{a^2 - b^2}$

$\therefore$ Equation of circle by using $(x-h)^2 + (y-k)^2 = r^2$ is

$$(x + a)^2 + (y + b)^2 = (\sqrt{a^2 - b^2})^2$$

$$\Rightarrow \quad x^2 + a^2 + 2ax + y^2 + b^2 + 2by = a^2 - b^2$$

$$\Rightarrow \quad x^2 + y^2 + 2ax + 2by + 2b^2 = 0$$

3. Let the centre of circle in first quadrant $be(a, a)$.

$\therefore$ Equation of circle is

$$(x - a)^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 + a^2 - 2ay = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

4. The centre of the given circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (2, 3).

Let the other end of diameter is (x_1, y_1).

Since, $\quad$ centre $= \left(\frac{x_1 + 3}{2}, \frac{y_1 + 4}{2}\right)$

$$\therefore \quad (2, 3) = \left(\frac{x_1 + 3}{2}, \frac{y_1 + 4}{2}\right)$$

$$\Rightarrow \quad 2 = \frac{x_1 + 3}{2}$$

$$3 = \frac{y_1 + 4}{2} \Rightarrow x_1 = 1, y_1 = 2$$

$\therefore$ Coordinates of other end of the diameter is (1, 2).

5. It is given, centre is (2, –3) and circumference of circle $= 10\pi$

$$\Rightarrow \quad 2\pi r = 10\pi$$

$$\Rightarrow \quad r = 5$$

The equation of circle, if centre is (2, –3) and radius is 5, is

$$(x - 2)^2 + (y + 3)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 6y + 13 = 25$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 6y - 12 = 0$$

6. We have equation of circle is

$$x^2 + y^2 - 8x + 4y + 4 = 0$$

On comparing with standard equation of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ we get}$$

$$g = -4, \quad f = 2 \quad \text{and} \quad c = 4$$

$\therefore$ Coordinates of the centre $= (-g, -f) = (4, -2)$

$\therefore$ Radius of the circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(4)^2 + (-2)^2 - 4}$$

$$= \sqrt{16 + 4 - 4} = 4$$

Here, radius of circle is equal to x-coordinate of the centre.

$\therefore$ Circle touches y-axis.

7. Let the equation of circle, which touches both the axes is

$$(x - a)^2 + (y - a)^2 = a^2$$

$\Rightarrow$ $x^2 + y^2 - 2ax - 2ay + 2a^2 = a^2$

$\Rightarrow$ $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(i)

Since, it is passes through (3, 6).

$\therefore$ $(3)^2 + (6)^2 - 2a(3) - 2a(6) + a^2 = 0$

$\Rightarrow$ $9 + 36 - 18a + a^2 = 0$

$\Rightarrow$ $a^2 - 18a + 45 = 0$

$\Rightarrow$ $a^2 - 15a - 3a + 45 = 0$

$\Rightarrow$ $a(a - 15) - 3(a - 15) = 0$

$\Rightarrow$ $(a - 3)(a - 15) = 0$

$\Rightarrow$ $a = 3, 15$

On putting $a = 3$ in Eq. (i), we get

$$x^2 + y^2 - 2(3)x - 2(3)y + 3^2 = 0$$

$\Rightarrow$ $x^2 + y^2 - 6x - 6y + 9 = 0$

8. Let the centre of circle be $(0, k)$.

Let equation of circle be

$$x^2 + y^2 - 2(0)x - 2ky + c = 0$$

$\Rightarrow$ $x^2 + y^2 - 2ky + c = 0$ ...(i)

Since, it is passes through (0, 0) and (2, 3).

$\therefore$ $0^2 + 0^2 - 2k(0) + c = 0$

$\Rightarrow$ $c = 0$

and $(2)^2 + (3)^2 - 2k(3) + c = 0$

$\Rightarrow$ $13 - 6k + c = 0$

$\Rightarrow$ $13 - 6k + 0 = 0$

$\Rightarrow$ $k = \frac{13}{6}$

On putting the value of k and c in Eq. (i), we get

$$x^2 + y^2 - 2\left(\frac{13}{6}\right)y + 0 = 0$$

$\Rightarrow$ $6x^2 + 6y^2 - 26y = 0$

9. The intersection point of the lines $2x - 3y = 5$ and $3x - 4y = 5$ is $(1, -1)$, which is equal to the centre of the circle.

Since, area of circle $= \pi r^2$

$\therefore$ $154 = \frac{22}{7} r^2$

$\Rightarrow$ $r^2 = 7^2 \Rightarrow r = 7$

$\therefore$ Equation of circle is

$$(x - 1)^2 + (y + 1)^2 = 7^2$$

$\Rightarrow$ $x^2 - 2x + 1 + y^2 + 2y + 1 = 49$

$\Rightarrow$ $x^2 + y^2 - 2x + 2y = 47$

10. $(x^2 + y^2 - 2x - 1) + \lambda x = 0$, they pass through intersection points of line $x = 0$ and circle $x^2 + y^2 - 2x - 1 = 0$.

$\Rightarrow$ $y = \pm 1$

$\therefore$ Required points are $(0, \pm 1)$.

11. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fg + c = 0 \quad \text{...(i)}$$

Since, it is passing through the points (2, 3) and (4, 5).

$\therefore$ $2^2 + 3^2 + 2g(2) + 2f(3) + c = 0$

$\Rightarrow$ $4g + 6f + c + 13 = 0$ ...(ii)

and $4^2 + 5^2 + 2g(4) + 2f(5) + c = 0$

$\Rightarrow$ $8g + 10f + c + 41 = 0$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$$4g + 4f + 28 = 0$$

$\Rightarrow$ $g + f + 7 = 0$ ...(iv)

Also, centre $(-g, -f)$ lies on line $y - 4x + 3 = 0$

$\therefore$ $-f + 4g + 3 = 0$ ...(v)

On solving Eqs. (iv) and (v), we get

$$g = -2, f = -5$$

On putting $g = -2$, $f = -5$ in Eq. (ii), we get

$$4(-2) + 6(-5) + c + 13 = 0 \Rightarrow c = 25$$

$\therefore$ From Eq. (i),

$$x^2 + y^2 - 4x - 10y + 25 = 0$$

12. If circle $x^2 + y^2 - 10x - 14y + 24 = 0$ cuts an intercept on y-axis, then

length of intercept $= 2\sqrt{f^2 - c} = 2\sqrt{49 - 24} = 10$

13. Since, the given line touches the given circle, the length of the perpendicular from the centre (2, 4) of the circle to the line $3x - 4y - k = 0$ is equal to the radius $\sqrt{4 + 16 + 5} = 5$ of the circle.

$\therefore$ $\frac{3 \times 2 - 4 \times 4 - k}{\sqrt{9 + 16}} = \pm 5$

$\Rightarrow$ $k = 15$ $(\because k > 0)$

Now, equation of the tangent at (a, b) to the given circle is

$$xa + yb - 2(x + a) - 4(y + b) - 5 = 0$$

$\Rightarrow \quad (a-2)x + (b-4)y - (2a + 4b + 5) = 0$

If it represents the given line $3x - 4y - k = 0$

Then, $\dfrac{a-2}{3} = \dfrac{b-4}{-4} = \dfrac{2a+4b+5}{k} = l$ (say)

$\Rightarrow \quad a = 3l + 2, \quad b = 4 - 4l$

and $\quad 2a + 4b + 5 = kl$

$\Rightarrow \quad 2(3l + 2) + 4(4 - 4l) + 5 = 15l \quad (\because k = 15)$

$\Rightarrow \quad l = 1 \Rightarrow a = 5, \quad b = 0$

$\therefore \quad k + a + b = 15 + 5 + 0 = 20$

14. Given equation of line is

$$3x - 2y = k \quad \ldots(i)$$

and equation of circle is

$$x^2 + y^2 = 4r^2 \quad \ldots(ii)$$

Eq. (i) can be rewritten as $y = \dfrac{3}{2}x - \dfrac{k}{2}$

$\Rightarrow \quad m = \dfrac{3}{2}, \quad c = -\dfrac{k}{2}$

The line will meet the circle in one point, if

$$c = a\sqrt{1 + m^2}$$

$\Rightarrow \quad -\dfrac{k}{2} = (2r)\sqrt{1 + \left(\dfrac{3}{2}\right)^2}$

On squaring, we get

$\Rightarrow \quad \dfrac{k^2}{4} = 4r^2 \times \dfrac{13}{4} \Rightarrow k^2 = 52r^2$

15. Let AB is a chord and its equation is $y = mx$. ...(i)

Equation of CM which is perpendicular to AB, is

$$x + my = \lambda$$

It passes through the centre $(a, 0)$.

$\Rightarrow \quad x + my = a \quad \ldots(ii)$

On eliminating m from Eqs. (i) and (ii), we get

$$x^2 + y^2 = ax$$

$\Rightarrow x^2 + y^2 - ax = 0$ is the locus of the centre of the required circle.

16. Equation of line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$ ...(i)

Let P be the foot of the perpendicular drawn from the origin to the line whose coordinates are (x_1, y_1).

Since, $\quad OP \perp AB$.

$\therefore$ Slope of $OP \times$ Slope of $AB = -1$

$\Rightarrow \quad \left(\dfrac{y_1}{x_1}\right)\left(\dfrac{b}{-a}\right) = -1$

$\Rightarrow \quad by_1 = ax_1 \quad \ldots(ii)$

Since, P lies on the line AB, then

$$\frac{x_1}{a} + \frac{y_1}{b} = 1$$

$\Rightarrow \quad bx_1 + ay_1 = ab \quad \ldots(iii)$

From Eqs. (ii) and (iii), we get

$$x_1 = \frac{ab^2}{a^2 + b^2}$$

and $\quad y_1 = \dfrac{a^2b}{a^2 + b^2}$

Now, $\quad x_1^2 + y_1^2 = \left(\dfrac{ab^2}{a^2 + b^2}\right)^2 + \left(\dfrac{a^2b}{a^2 + b^2}\right)^2$

$\Rightarrow \quad x_1^2 + y_1^2 = \dfrac{a^2b^4}{(a^2 + b^2)^2} + \dfrac{a^4b^2}{(a^2 + b^2)^2}$

$\Rightarrow \quad x_1^2 + y_1^2 = \dfrac{a^2b^2(a^2 + b^2)}{(a^2 + b^2)^2}$

$\Rightarrow \quad x_1^2 + y_1^2 = \dfrac{a^2b^2}{(a^2 + b^2)}$

$\Rightarrow \quad x_1^2 + y_1^2 = \dfrac{1}{\dfrac{1}{a^2} + \dfrac{1}{b^2}}$

But $\quad \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$ (given)

$\therefore \quad x_1^2 + y_1^2 = c^2$

Thus, the locus of $P(x_1, y_1)$ is $x^2 + y^2 = c^2$ which is the equation of circle.

17. Since, $PA \cdot PB = PT^2$, where PT is length of tangent.

Here, $\quad PT = \sqrt{S_1} = \sqrt{1^2 + 4^2 - 16} = 1$

$\therefore \quad PA \cdot PB = 1$

18. Using the result $a_1a_2 = b_1b_2$, we get

$$\lambda \cdot 1 = (-1) \cdot (-2) \Rightarrow \lambda = 2$$

19. Centre of circle is (0, 0).

Equation of tangent which is parallel to $x + 2y + 3 = 0$ is

$x + 2y + \lambda = 0 \quad \ldots(i)$

As we know perpendicular distance from centre (0, 0) to $x + 2y + \lambda = 0$ should be equal to radius.

$\therefore \quad \dfrac{0 + 2 \times 0 + \lambda}{\sqrt{1^2 + 2^2}} = \pm 2$

$\Rightarrow \quad \lambda = \pm 2\sqrt{5}$

On putting the value of λ in Eq. (i), we get $x + 2y = \pm 2\sqrt{5}$ which represents the required equation of tangents.

20. Given that, any tangent to the circle $x^2 + y^2 = b^2$ is $y = mx - b\sqrt{1 + m^2}$. It touches the circle $(x - a)^2 + y^2 = b^2$, then

$$\frac{ma - b\sqrt{1 + m^2}}{\sqrt{m^2 + 1}} = b$$

$\Rightarrow \quad ma = 2b\sqrt{1 + m^2}$

$\Rightarrow \quad m^2a^2 = 4b^2 + 4b^2m^2$

$\therefore \quad m = \pm \dfrac{2b}{\sqrt{a^2 - 4b^2}}$

21. The tangent at (1, 7) to the parabola $x^2 = y - 6$ is

$$x = \frac{1}{2}(y + 7) - 6$$

$\Rightarrow \quad 2x = y + 7 - 12$

$\Rightarrow \quad y = 2x + 5$

which is also tangent to the circle

$$x^2 + y^2 + 16x + 12y + c = 0$$

$\therefore \quad x^2 + (2x + 5)^2 + 16x + 12(2x + 5) + c = 0$

$\Rightarrow \quad 5x^2 + 60x + 85 + c = 0$

must have equal roots.

Let α and β are the roots of the equation.

$\Rightarrow \quad \alpha + \beta = -12 \Rightarrow \alpha = -6 \qquad (\because \alpha = \beta)$

$\therefore \quad x = -6$

and $\quad y = 2x + 5 = -7$

$\Rightarrow$ Point of contact is (– 6, – 7).

22. Required length of tangent from the point (3, – 4) to the circle

$$x^2 + y^2 - 4x - 6y + 3 = 0$$

$$= \sqrt{3^2 + 4^2 - 4(3) - 6(-4) + 3}$$

$$= \sqrt{40}$$

$\therefore$ Square of length of tangent = 40

23. Let $P(\alpha, \beta)$ be a point on

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then, the length of the tangent from $P(\alpha, \beta)$ to the circle $\quad x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is

$$|T| = \sqrt{\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c_1} \qquad \ldots(i)$$

Since, $P(\alpha, \beta)$ lies on

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\therefore \quad \alpha^2 + \beta^2 + 2g\alpha + 2f\beta = -c$

On substituting respective value, we have

$$|T| = \sqrt{-c + c_1}$$

$\therefore$ Length of the tangent $= \sqrt{c_1 - c}$

24. Given equation can be rewritten as

$x(x - 2y) - 3(x - 2y) = 0 \Rightarrow x = 3$ and $x = 2y$ are two normals. Their intersection point is the centre $\left(3, \dfrac{3}{2}\right)$.

25. The equation of tangent to the given circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$ at point (1, 1) is

$$2x + 2y - (x + 1) - \frac{5}{2}(y + 1) + 3 = 0$$

$\Rightarrow \quad x - \dfrac{1}{2}y - \dfrac{1}{2} = 0$

$\Rightarrow \quad 2x - y - 1 = 0$

$\Rightarrow \quad y = 2x - 1$

Slope of tangent = 2 , therefore slope of normal $= -\dfrac{1}{2}$.

Hence, equation of normal at point (1, 1) and having slope $\left(-\dfrac{1}{2}\right)$ is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$\Rightarrow \quad 2y - 2 = -x + 1 \Rightarrow x + 2y = 3$

26. Since, normal passes through the centre of the circle.

$\therefore$ The required circle is the circle with ends of diameter as (3, 4) and (– 1, – 2).

$\therefore$ It's equation is

$$(x - 3)(x + 1) + (y - 4)(y + 2) = 0$$

$\Rightarrow \quad x^2 - 2x - 3 + y^2 - 2y - 8 = 0$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y - 11 = 0$

27. Since, the line $ax + by = 0$ touches the circle $x^2 + y^2 + 2x + 4y = 0$, distance of the centre (– 1, – 2) from the line = radius

$\therefore \quad \left| \dfrac{-a - 2b}{\sqrt{a^2 + b^2}} \right| = \sqrt{(-1)^2 + (-2^2)}$

$\Rightarrow \quad (a + 2b)^2 = 5(a^2 + b^2)$

$\Rightarrow \quad a^2 + 4b^2 + 4ab = 5a^2 + 5b^2$

$\Rightarrow \quad (2a - b)^2 = 0 \Rightarrow b = 2a$

Also, $ax + by = 0$ is normal to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$, the centre (2, – 1) should lie on $ax + by = 0$

$\therefore \quad 2a - b = 0 \Rightarrow b = 2a$

Hence, $\quad a = 1, b = 2$

28.

Equation of tangent to $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ on the circle is

$$x + \sqrt{3}y = 4 \quad \ldots(i)$$

Now, any line perpendicular to Eq. (i) and passing through $(1, \sqrt{3})$ or centre (0, 0) is normal to circle at $(1, \sqrt{3})$.

$\therefore$ Equation of normal is

$$\sqrt{3}x - y + \lambda = 0$$

$$\lambda = 0$$

Since, this line passes through origin.

$$\therefore \quad \sqrt{3}(0) - (0) + \lambda = 0$$

$$\Rightarrow \quad \lambda = 0$$

$\Rightarrow$ Equation of normal is

$$\sqrt{3}x - y = 0$$

Intercept of tangent on x-axis $A(4, 0)$

Now, $\quad \text{ar}(\Delta OAP) = \frac{1}{2}(AP)(OP)$

$$\therefore \quad \text{ar}(\Delta OAP) = \frac{1}{2} \times 2\sqrt{12} = 2\sqrt{3}$$

29. Given lines $\quad 3x - 4y + 4 = 0$

and $\quad 6x - 8y - 7 = 0$ or $3x - 4y - \frac{7}{2} = 0$

are tangent to the circle.

Also, given lines are parallel.

$$\therefore \quad \text{Radius} = \frac{1}{2}(\text{Distance between two parallel lines})$$

$$= \frac{1}{2} \frac{\left|4 + \frac{7}{2}\right|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{1}{2} \frac{\left|\frac{15}{2}\right|}{\sqrt{9 + 16}} = \frac{7}{4\sqrt{25}} = \frac{15}{4 \times 5} = \frac{3}{4}$$

30. Let a be the radius of the circle and centre of circle will be $(-a, -a)$.

Since, the perpendicular distance from centre to the tangent is equal to the radius of the circle.

$$\therefore \quad a = \frac{3(-a) - 4(-a) + 8}{\sqrt{3^2 + (-4)^2}}$$

$$\Rightarrow \quad a = \frac{a + 8}{5}$$

$$\Rightarrow \quad 4a = 8$$

$$\Rightarrow \quad a = 2$$

$\therefore$ Equation of circle is

$$(x + 2)^2 + (y + 2)^2 = 2^2$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 + 4y + 4 = 4$$

$$\Rightarrow \quad x^2 + y^2 + 4x + 4y + 4 = 0$$

31. Given that, $\quad y = 2x + c \quad \ldots(i)$

and $\quad x^2 + y^2 = 16 \quad \ldots(ii)$

We know that, if $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1 + m^2}$, here $m = 2$, $a = 4$

$$\therefore \quad c = \pm 4\sqrt{1 + 2^2} = \pm 4\sqrt{5}$$

32. We know that, if two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P, then the point P lies on a director circle.

Thus, the equation of director circle to the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

which is the required locus of point P.

33. Let PQ and RP be the two tangents and P be the point on the circle $x^2 + y^2 = a^2$ whose coordinates are $(a \cos t, a \sin t)$ and $\angle OPQ = \theta$

Now, PQ = Length of tangent from P on the circle

$$x^2 + y^2 = a^2 \sin^2 \alpha$$

$$\therefore \quad PQ = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t - a^2 \sin^2 \alpha}$$

$$= \sqrt{a^2(\cos^2 t + \sin^2 t) - a^2 \sin^2 \alpha}$$

$$= a \cos \alpha \quad (\because \cos^2 t + \sin^2 t = 1)$$

and $\quad OQ$ = Radius of the circle $(x^2 + y^2 = a^2 \sin^2 \alpha)$

$$\Rightarrow \quad OQ = a \sin \alpha$$

$$\therefore \quad \tan \theta = \frac{OQ}{PQ}$$

$$= \frac{a \sin \alpha}{a \cos \alpha} = \tan \alpha$$

$$\Rightarrow \quad \theta = \alpha$$

$\therefore$ Angle between tangents $= \angle QPR = 2\theta = 2\alpha$

34. Given equation of circle is $x^2 + y^2 = r^2$. Let any point on the circle is $P(r \cos \theta, r \sin \theta)$ and let the coordinates of centroid of the triangle be (α, β).

Then, $\quad \alpha = \frac{r + r \cos \theta}{3}$

$\Rightarrow \quad \frac{r}{3}\cos\theta = \alpha - \frac{r}{3}$

and $\quad \beta = \frac{r + r\sin\theta}{3}$

$\Rightarrow \quad \frac{r}{3}\sin\theta = \beta - \frac{r}{3}$

Now, $\quad \left(\alpha - \frac{r}{3}\right)^2 + \left(\beta - \frac{r}{3}\right)^2 = \frac{r^2}{9}$

$\therefore$ The locus is $\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$ which is a circle.

35. Let $P(x_1, y_1)$ be any point outside the circle. Length of tangent to the circle $x^2 + y^2 + 4x + 3 = 0$ is

$$\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}$$

and length of tangent of the circle $x^2 + y^2 - 6x + 5 = 0$ is

$$\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}$$

$\therefore$ According to question,

$$\frac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \frac{2}{3}$$

$\Rightarrow \quad 9x_1^2 + 9y_1^2 + 36x_1 + 27 - 4x_1^2 - 4y_1^2 + 24x_1 - 20 = 0$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$

$\therefore$ Locus of point is $5x^2 + 5y^2 + 60x + 7 = 0$.

36. In $\Delta\, ADB$, $\quad AD = \frac{1}{\sqrt{2}}\operatorname{cosec} 45° = 1$

37. The combined equation of the lines joining the origin to the points of intersection of $x\cos\alpha + y\sin\alpha = p$ and $x^2 + y^2 - a^2 = 0$ is a homogeneous equation of second degree given by

$$x^2 + y^2 - a^2\left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$$

$\Rightarrow \quad [x^2(p^2 - a^2\cos^2\alpha) + y^2(p^2 - a^2\sin^2\alpha)$
$\qquad - 2xya^2\sin\alpha\cos\alpha = 0]$

The lines given by this equation are at right angle, if

$$(p^2 - a^2\cos^2\alpha) + (p^2 - a^2\sin^2\alpha) = 0$$

$\Rightarrow \quad 2p^2 = a^2(\sin^2\alpha + \cos^2\alpha)$

$\Rightarrow \quad a^2 = 2p^2$

38. Let the equation of circles are

$$S_1 \equiv x^2 + y^2 + 2x - 3y + 6 = 0 \quad \ldots(i)$$

$$S_2 \equiv x^2 + y^2 + x - 8y - 13 = 0 \quad \ldots(ii)$$

$\therefore$ Equation of common chord is

$$S_1 - S_2 = 0$$

$\Rightarrow \quad (x^2 + y^2 + 2x - 3y + 6) - (x^2 + y^2 + x - 8y - 13) = 0$

$\Rightarrow \quad x + 5y + 19 = 0 \quad \ldots(iii)$

In the given option only the point (1, – 4) satisfied the Eq. (iii).

39. Let $P(x_1, y_1)$ be the point, then the chord of contact of tangents drawn from P to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

$\therefore \quad x^2 + y^2 = a^2\left(\frac{xx_1 + yy_1}{a^2}\right)$

$\Rightarrow \quad x^2 + y^2 - xx_1 - yy_1 = 0$

40. Suppose AB is a chord of the circle through $A(p, q)$ having $M(h, 0)$ as its mid-point. The coordinates of B are $(-p + 2h, -q)$.

As, B lies on the circle

$x^2 + y^2 = px + qy$, we have

$$(-p + 2h)^2 + (-q)^2$$

$$= p(-p + 2h) + q(-q)$$

$\Rightarrow \quad 2p^2 + 2q^2 - 6ph + 4h^2 = 0$

$\Rightarrow \quad 2h^2 - 3ph + p^2 + q^2 = 0 \quad \ldots(i)$

As, there are two distinct chords from $A(p, q)$ which are bisected on x-axis, there must be two distinct values of h satisfying Eq. (i).

$$D = 9p^2 - (4)(2)(p^2 + q^2) > 0 \Rightarrow p^2 > 8q^2$$

41. Let the point on $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$

Equation of chord of contact is

$$ax\cos\theta + ay\sin\theta = b^2$$

It touches circle $x^2 + y^2 = c^2$

$\therefore \quad \left|\frac{-b^2}{\sqrt{a^2\cos^2\theta + a^2\sin^2\theta}}\right| = c$

$\Rightarrow \quad b^2 = ac$

$\therefore a$, b and c are in GP.

42. The equation of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ is

$$\mathrm{I} \equiv S_1 - S_2 = 0$$

$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 - a^2 - x^2 - y^2 - b^2 - 2by + b^2 = 0$

$\Rightarrow \quad ax + by = 0 \quad \ldots(i)$

Now, the equation of required circle is

$$S_1 + \lambda L = 0$$

$\therefore \quad \{(x - a)^2 + y^2 - a^2\} + \lambda\{ax + by\} = 0$

$\Rightarrow \quad x^2 + y^2 + x(a\lambda - 2a) + \lambda by = 0 \quad \ldots(ii)$

Since, Eq. (i) is a diameter of Eq. (ii), then

$$a\left(-\frac{a\lambda - 2a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) = 0$$

$\Rightarrow \quad \lambda = \frac{2a^2}{a^2 + b^2}$

On putting the value of λ in Eq. (ii), we get

$$(a^2 + b^2)(x^2 + y^2) = 2\,ab(bx - ay)$$

which is the required equation of circle.

43. The equation of the required circle passing through $x^2 + y^2 = a^2$

and $$x \cos\alpha + y \sin\alpha - p = 0$$

$$x^2 + y^2 - a^2 + \lambda(x \cos\alpha + y \sin\alpha - p) = 0 \quad \ldots(i)$$

The centre of this circle is $\left(-\frac{\lambda}{2}\cos\alpha, -\frac{\lambda}{2}\sin\alpha\right)$. Since, the centre lies on $x \cos\alpha + y \sin\alpha = p$

$\therefore$ $$\lambda = -2p$$

On putting the value of λ in Eq. (i), equation of the required circle is

$$x^2 + y^2 - a^2 - 2px\cos\alpha - 2py\sin\alpha + 2p^2 = 0$$

44. Centres and radii of the given circles are

Centres : $C_1(0, 1), C_2(1, 0)$

Radii : $r_1 = 3, r_2 = 5$

Clearly, $C_1C_2 = \sqrt{2} < r_2 - r_1$. Therefore, one circle lies entirely inside the other.

45. Given that, circle $S_1 \equiv x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle

$$S_2 \equiv x^2 + y^2 - 2x + 8y - d = 0.$$

The common chord of the given circles is

$$S_1 - S_2 = 0$$

$\Rightarrow$ $x^2 + y^2 + 4x + 22y + c - x^2 - y^2 + 2x - 8y + d = 0$

$\Rightarrow$ $$6x + 14y + c + d = 0 \quad \ldots(i)$$

So, Eq. (i) passes through the centre of the second circle, *i.e.*, (1, – 4).

$\therefore$ $$6 - 56 + c + d = 0$$

$\Rightarrow$ $$c + d = 50$$

46. The required equation of circle is

$$(x^2 + y^2 + 13x - 3y) + \lambda\left(11x + \frac{1}{2}y + \frac{25}{2}\right) = 0 \quad \ldots(i)$$

It passes through (1, 1).

$\therefore$ $$12 + \lambda(24) = 0$$

$\Rightarrow$ $$\lambda = -\frac{1}{2}$$

On putting in Eq. (i), we get

$$x^2 + y^2 + 13x - 3y - \frac{11}{2}x - \frac{1}{4}y - \frac{25}{4} = 0$$

$\Rightarrow$ $4x^2 + 4y^2 + 52x - 12y - 22x - y - 25 = 0$

$\Rightarrow$ $$4x^2 + 4y^2 + 30x - 13y - 25 = 0$$

47. Let the equation of circles be

$$S_1 \equiv x^2 + y^2 + 13x - 3y = 0 \quad \ldots(i)$$

and $$S_2 \equiv 2x^2 + 2y^2 + 4x - 7y - 25 = 0 \quad \ldots(ii)$$

The equation of intersecting circle is $\lambda S_1 + S_2 = 0$.

$\Rightarrow$ $\lambda(x^2 + y^2 + 13x - 3y) + \left(x^2 + y^2 + 2x - \frac{7y}{2} - \frac{25}{2}\right) = 0$

$\Rightarrow$ $\left[x^2(1 + \lambda) + y^2(1 + \lambda) + x(2 + 13\lambda) - y\left(\frac{7}{2} + 3\lambda\right) - \frac{25}{2}\right]$

$= 0 \quad \ldots(iii)$

$\therefore$ $$\text{Centre} = \left[-\frac{(2 + 13\lambda)}{2(1 + \lambda)}, \frac{(7/2) + 3\lambda}{2(1 + \lambda)}\right]$$

$\because$ Centre lies on $13x + 30y = 0$.

$\Rightarrow$ $$-13\left(\frac{2 + 13\lambda}{2}\right) + 30\left[\frac{(7/2) + 3\lambda}{2}\right] = 0$$

$\Rightarrow$ $$-26 - 169\lambda + 105 + 90\lambda = 0 \Rightarrow \lambda = 1$$

Hence, putting the value of x in Eq. (iii), then required equation of circle is

$$4x^2 + 4y^2 + 30x - 13y - 25 = 0$$

48. Equation of family of concentric circles to the circle $x^2 + y^2 + 6x + 8y - 5 = 0$ is $x^2 + y^2 + 6x + 8y + \lambda = 0$.

which is similar to $x^2 + y^2 + 2gx + 2fy + c = 0$. Since, it is equation of concentric circle to the circle $x^2 + y^2 + 6x + 8y - 5 = 0$. Thus, the point (–3, 2) lies on the circle $x^2 + y^2 + 6x + 8y + c = 0$.

$\Rightarrow$ $$(-3)^2 + (2)^2 + 6(-3) + 8(2) + c = 0$$

$\Rightarrow$ $$9 + 4 - 18 + 16 + c = 0$$

$\Rightarrow$ $$c = -11$$

49. Centre of given circle $x^2 + y^2 - 6x + 12y + 15 = 0$ is (3, –6).

$\therefore$ $$\text{Radius} = \sqrt{(3)^2 + (-6)^2 - 15} = \sqrt{30}$$

$$\text{Area of circle} = \pi r^2 = \pi(\sqrt{30})^2 = 30\pi$$

Area of require circle = 2 (Area of given circle)

$\therefore$ $$\pi R^2 = 2 \times 30\pi = 60\pi$$

$\Rightarrow$ $$R^2 = 60 \Rightarrow R = 2\sqrt{15}$$

$\therefore$ Equation of required circle is

$$(x - 3)^2 + (y + 6)^2 = (2\sqrt{15})^2$$

$\Rightarrow$ $$x^2 + 9 - 6x + y^2 + 36 + 12y = 60$$

$\Rightarrow$ $$x^2 + y^2 - 6x + 12y - 15 = 0$$

50. Using the result $(C_1C_2)^2 = r_1^2 + r_2^2$, we get

$$(2 - 5)^2 + (3 - 6)^2 = r^2 + r^2$$

$\Rightarrow$ $$2r^2 = 18 \Rightarrow r^2 = 9$$

$\Rightarrow$ $$r = 3$$

51. Let $P(x_1, y_1)$ be the point outside the circle. From the given condition

$$\frac{x_1^2 + y_1^2 + 2x_1 - 4y_1 - 20}{x_1^2 + y_1^2 - 4x_1 + 2y_1 - 44} = \frac{2}{3}$$

$\Rightarrow$ $$3x_1^2 + 3y_1^2 + 6x_1 - 12y_1 - 60$$

$$= 2x_1^2 + 2y_1^2 - 8x_1 + 4y_1 - 88$$

$$\Rightarrow \quad x_1^2 + y_1^2 + 14x_1 - 16y_1 + 28 = 0$$

Thus, the locus of point is

$$x^2 + y^2 + 14x - 16y + 28 = 0$$

$\therefore$ Coordinates of centre of circle are (–7, 8).

52. Let the equation of the concentric circles be $x^2 + y^2 - 2x - 4y + \lambda = 0$, it passes through (3, 4).

$$\therefore \quad 3^2 + 4^2 - 2(3) - 4(4) + \lambda = 0$$

$$\Rightarrow \quad \lambda = -3$$

Thus, the equation of concentric circle is

$$x^2 + y^2 - 2x - 4y - 3 = 0.$$

53. Given two circles are

$$C_1 : x^2 + y^2 = 1 \quad \text{and} \quad C_2 : (x - t)^2 + y^2 = 1$$

$\Rightarrow$ Centre of $C_1(0, 0)$, centre of $C_2(t, 0)$.

Radius, $r_1 = 1$, radius, $r_2 = 1$

$\therefore$ Length, $\sqrt{(t)^2 - (1+1)^2} = \sqrt{21}$

$$\Rightarrow \quad t^2 - 4 = 21$$

$$\Rightarrow \quad t^2 = 25 \Rightarrow t = \pm 5$$

54. For required circle, $P(1, 8)$ and $O(3, 2)$ will be the end points of its diameter.

$$\therefore \quad (x-1)(x-3) + (y-8)(y-2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 10y + 19 = 0$$

55. The centres of the two circles will lie on the line through $P(1, 2)$ perpendicular to the common tangent $4x + 3y = 10$. If C_1 and C_2 are the centres of these circles, then $PC_1 = 5 = r_1$, $PC_2 = -5 = r_2$. Also, C_1, C_2 lie on the line $\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$, where $\tan\theta = \frac{3}{4}$. When $r = r_1$ the coordinates of C_1 are $(5\cos\theta + 1, 5\sin\theta + 2)$ or (5, 5) as $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$.

When $r = r_2$, the coordinates of C_2 are (–3, –1).

The circle with centre $C_1(5, 5)$ and radius 5 touches both the coordinate axes and hence lies completely in the first quadrant.

Therefore, the required circle is with centre (–3, –1) and radius 5, so its equation is

$$(x+3)^2 + (y+1)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 + 6x + 2y - 15 = 0$$

Since, the origin lies inside the circle, a portion of the circle lies in all the quadrants.

56. Equation of any circle through (0, 0) and (1, 0) is

$$(x-0)(x-1) + (y-0)(y-0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$$

If it represents C_3, its radius = 1

$$\Rightarrow \quad 1 = \left(\frac{1}{4}\right) + \left(\frac{\lambda^2}{4}\right) \Rightarrow \lambda = \pm\sqrt{3}$$

As, the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$.

Since, C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

So, let the equation of a common tangent be

$$\sqrt{3}x - y + k = 0$$

It will touch C_1, if $\left|\frac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}x - y + 2 = 0$.

which is the equation of required locus.

57. Centre of required circle = (3, – 4)

Radius of required circle = 5 + 1 = 6

$\therefore$ Locus of circle is

$$(x-3)^2 + (y+4)^2 = 36$$

$$\Rightarrow \quad x^2 - 6x + 9 + y^2 + 16 + 8y = 36$$

$$\Rightarrow \quad x^2 + y^2 - 6x + 8y - 11 = 0$$

58. Let the general equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It cuts the circle $x^2 + y^2 - 20x + 4 = 0$ orthogonally, then by the condition, $2(g_1g_2 + f_1f_2) = c_1 + c_2$

$$2(-10g + 0 \times f) = c + 4 \Rightarrow -20g = c + 4 \quad \text{...(ii)}$$

$\because$ Circle (i) touches the line $x = 2$ or $x + 0y - 2 = 0$

$\therefore$ Perpendicular distance from centre to the tangent = radius.

$$\Rightarrow \quad \left|\frac{-g + 0 - 2}{\sqrt{1^2 + 0^2}}\right| = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \quad (g+2)^2 = g^2 + f^2 - c$$

$$\Rightarrow \quad g^2 + 4 + 4g = g^2 + f^2 - c$$

$$\Rightarrow \quad 4g + 4 = f^2 - c \quad \text{...(iii)}$$

On eliminating c from Eqs. (ii) and (iii), we get

$$-16g + 4 = f^2 + 4 \Rightarrow f^2 + 16g = 0$$

Hence, the locus of $(-g, -f)$ is $y^2 - 16x = 0$.

(replacing $-f$ and $-g$ by x and y)

59. Let general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

If the circle (i) cuts orthogonally each of the given three circles.

Then, condition is

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

Applying the condition one by one, we get

$$2g + 17f = c + 4 \quad \text{...(ii)}$$

$$7g + 6f = c + 11 \quad \text{...(iii)}$$

and $$-g + 22f = c + 3 \quad \text{...(iv)}$$

On solving Eqs. (ii), (iii) and (iv), we get

$$g = -3, \quad f = -2$$

Therefore, the centre of the circle is (3, 2).

60. Let the family of circles passing through origin be

$$x^2 + y^2 + 2gx + 2fy = 0$$

They intersect circle $x^2 + y^2 + 4x - 6y - 13 = 0$ orthogonally.

So, $2g(2) - 2f(3) = -13$

Hence, locus of $(-g, -f)$ is

$$-4x + 6y + 13 = 0$$

$$\Rightarrow \quad 4x - 6y - 13 = 0$$

61. Let the equation of circle be

$$x^2 + y^2 + 2hx + 2ky + c = 0 \quad \text{...(i)}$$

The locus of whose centre is to be obtained, since the circle cuts

$$x^2 + y^2 + 4x - 6y + 9 = 0 \quad \text{...(ii)}$$

and $$x^2 + y^2 - 4x + 6y + 4 = 0 \quad \text{...(iii)}$$

Orthogonally, then

$$2h(2) + 2k(-3) = c + 9$$

$$\Rightarrow \quad 4h - 6k = c + 9 \quad \text{...(iv)}$$

and $$2h(-2) + 2k(3) = c + 4$$

$$\Rightarrow \quad -4h + 6k = c + 4 \quad \text{...(v)}$$

On solving Eqs. (iv) and (v), we get

$$c + 9 = -c - 4$$

$$\Rightarrow \quad 2c = -13 \quad \text{...(vi)}$$

On putting the value of c in Eq. (iv), we get

$$\Rightarrow \quad 8h - 12k = 5 \quad \text{...(vii)}$$

Centre of the given circle is $(-h, -k)$.

$\therefore$ Locus of $(-h, -k)$ from Eq. (vii) is

$$8(-x) - 12(-y) = 5$$

$$\Rightarrow \quad 8x - 12y + 5 = 0$$

62. Given equations of circles are $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$. They intersect each other orthogonally.

$$\therefore \quad 2 \cdot 1 \cdot 0 + 2 \cdot k \cdot k = 6 + k$$

$$\Rightarrow \quad 2k^2 - k - 6 = 0$$

$$\Rightarrow \quad (2k + 3)(k - 2) = 0 \Rightarrow k = 2, -\frac{3}{2}$$

63. Since, the given circles cut each other orthogonally.

$$\therefore \quad g_1g_2 + a^2 = 0 \quad \text{...(i)}$$

If $lx + my = 1$ is a common tangent of these circles, then

$$\frac{-lg_1 - 1}{\sqrt{l^2 + m^2}} = \pm\sqrt{g_1^2 + a^2}$$

$$\Rightarrow \quad (lg_1 + 1)^2 = (l^2 + m^2)(g_1^2 + a^2)$$

$$\Rightarrow \quad m^2g_1^2 - 2lg_1 + a^2(l^2 + m^2) - 1 = 0$$

Similarly, $$m^2g_2^2 - 2lg_2 + a^2(l^2 + m^2) - 1 = 0$$

So that g_1 and g_2 are the roots of the equation

$$m^2g^2 - 2lg + a^2(l^2 + m^2) - 1 = 0$$

$$\Rightarrow \quad g_1g_2 = \frac{a^2(l^2 + m^2) - 1}{m^2} = -a^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad a^2(l^2 + m^2) = 1 - a^2m^2 \quad \text{...(ii)}$$

Now, $$p_1p_2 = \frac{|ma - 1|}{\sqrt{l^2 + m^2}} \cdot \frac{|-ma - 1|}{\sqrt{l^2 + m^2}}$$

$$= \frac{|1 - m^2a^2|}{l^2 + m^2} = a^2 \quad \text{[from Eq. (ii)]}$$

64. Let $P(h, k)$ be the coordinates of the centre of circle S_2. Then, its equation is

$$(x - h)^2 + (y - k)^2 = 5^2$$

The equation of S_1 is $x^2 + y^2 = 4^2$ and so the equation of the common chord of S_1 and S_2 is

$$S_1 - S_2 = 0$$

$$\Rightarrow \quad 2hx + 2ky = h^2 + k^2 - 9 \quad \text{...(i)}$$

Let p be the length of the perpendicular from the centre (0, 0) of S_1 to common chord

$$\therefore \quad p = \left|\frac{h^2 + k^2 - 9}{\sqrt{4h^2 + 4k^2}}\right|$$

Now, the length of the comon chord $= 2\sqrt{4^2 - p^2}$

It will be of maximum length if $p = 0$

$$\Rightarrow \quad h^2 + k^2 - 9 = 0 \quad \text{...(ii)}$$

The slope of Eq. (i) is $\frac{3}{4}$ (given)

$$\therefore \quad -\frac{h}{k} = \frac{3}{4} \Rightarrow k = -\frac{4h}{3} \quad \text{...(iii)}$$

On substituting the respective value of k in Eq. (ii), we have

$$h = \pm\frac{9}{5} \text{ and } k = \pm\frac{12}{5} \quad \text{[using Eq. (iii)]}$$

$\therefore$ The centres of circle C_2 are

$$C\left(\frac{9}{5}, -\frac{12}{5}\right) \text{ or } C\left(-\frac{9}{5}, \frac{12}{5}\right)$$

65. Now, the coordinates of the centre and radius of the given circle are (1, 1) and 2 respectively.

Let AB be the chord subtending an angle of $\left(\frac{2\pi}{3}\right)$ at the centre.

Also, $M(h, k)$ be the mid-point of AB.

$\therefore$ In ΔOAM,

$$AM = OA \cdot \sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$\therefore$ $OM^2 = OA^2 - AM^2 = 4 - (\sqrt{3})^2 = 1$

But $OM^2 = (h-1)^2 + (k-1)^2$

$\Rightarrow$ $(h-1)^2 + (k-1)^2 = 1$

$\therefore$ The locus of (h, k) is $(x-1)^2 + (y-1)^2 = 1$

$\Rightarrow$ $x^2 + y^2 - 2x - 2y + 1 = 0$

66. Let $P(h, k)$ be a point on the circle

$$15x^2 + 15y^2 - 48x + 64y = 0$$

Then, the lengths PT_1 and PT_2 of the tangents from $P(h, k)$ to

$$5x^2 + 5y^2 - 24x + 32y + 75 = 0$$

and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$

respectively are

$$PT_1 = \sqrt{h^2 + k^2 - \frac{24}{5}h + \frac{32}{5}k + 15}$$

and $$PT_2 = \sqrt{h^2 + k^2 - \frac{48}{5}h + \frac{64}{5}k + 60}$$

Since, (h, k) lies on

$$15x^2 + 15y^2 - 48x + 64y = 0$$

$\therefore$ $h^2 + k^2 = \frac{48}{15}h - \frac{64}{15}k$

$\Rightarrow$ $PT_1 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{24}{5}h + \frac{32}{5}k + 15}$

$\therefore$ $PT_1 = \sqrt{\frac{32}{15}k - \frac{24}{15}h + 15}$

Similarly,

$$PT_2 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{48}{5}h + \frac{64}{5}k + 60}$$

$\Rightarrow$ $PT_2 = \sqrt{-\frac{96}{15}h + \frac{128}{15}k + 60}$

$\therefore$ $PT_2 = 2\sqrt{-\frac{24}{15}h + \frac{32}{15}k + 15} = 2(PT_1)$

Hence, $PT_1 : PT_2 = 1 : 2$

67. The radical axis of the two given circles

$$S_1 : x^2 + y^2 + 2gx + 2fy + c = 0$$

and $$S_2 : x^2 + y^2 + \frac{3}{2}x + 4y + c = 0$$

is $$2x\left(g - \frac{3}{4}\right) + 2y(f-2) = 0$$

$\Rightarrow$ $$x\left(g - \frac{3}{4}\right) + y(f-2) = 0$$

It touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$

$\therefore$ $r = d$ (condition to touch)

$\Rightarrow$ $$\left|\frac{-\left(g - \frac{3}{4}\right) - (f-2)}{\sqrt{\left(g - \frac{3}{4}\right)^2 + (f-2)^2}}\right| = 1$$

$\Rightarrow$ $$\left(g - \frac{3}{4}\right)^2 + (f-2)^2 + 2\left(g - \frac{3}{4}\right)(f-2) = \left(g - \frac{3}{4}\right)^2 + (f-2)^2$$

$\Rightarrow$ $\left(g - \frac{3}{4}\right)(f-2) = 0$

$\therefore$ $g = \frac{3}{4}$ and $f = 2$

68. Since, $S_1 : x^2 + y^2 - 6x - 8y = 0$

$$S_2 : x^2 + y^2 - 12x - 16y + 12 = 0$$

Now, according to given condition, we have

$$|PT_1| = |PT_2|$$

$\Rightarrow$ $\sqrt{a^2 + b^2 - 6a - 8b} = \sqrt{a^2 + b^2 - 12a - 16b + 12}$

$\Rightarrow$ $a^2 + b^2 - 6a - 8b = a^2 + b^2 - 12a - 16b + 12$

$\therefore$ $6a + 8b = 12$

$\Rightarrow$ $3a + 4b - 6 = 0$

69. Given equation of circle can be rewritten as

$$x^2 + y^2 - 2x + 4y + \frac{k}{4} = 0$$

$\therefore$ Radius of circle $= \sqrt{1 + 4 - \frac{k}{4}} = \sqrt{5 - \frac{k}{4}}$

Area of circle $= 9\pi$ (given)

$\Rightarrow$ $\pi\left(5 - \frac{k}{4}\right) = 9\pi$

$\Rightarrow$ $5 - 9 = \frac{k}{4} \Rightarrow k = -16$

70. Let equation of circle having centre (1, 2) is

$$(x-1)^2 + (y-2)^2 = r^2$$

Since, it passes through (4, 6).

$\therefore$ $(4-1)^2 + (6-2)^2 = r^2$

$\Rightarrow$ $r^2 = 9 + 16 = 25$

$\Rightarrow$ $r = 5$

$\therefore$ Area of circle $= \pi r^2$

$= \pi(5)^2 = 25\pi$

71. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since, it is passes through (0, 0), $(a, 0)$ and $(0, b)$.

$\therefore \quad 0^2 + 0^2 + 2g(0) + 2f(0) + c = 0 \Rightarrow c = 0$

and $\quad a^2 + 0^2 + 2g(a) + 2f(0) + c = 0$

$\Rightarrow \quad a^2 + 2g(a) + 0 = 0$

$\Rightarrow \quad g = -\dfrac{a}{2}$

and $\quad 0^2 + b^2 + 2g(0) + 2f(b) + 0 = 0$

$\Rightarrow \quad b^2 + 2f(b) = 0$

$\Rightarrow \quad f = -\dfrac{b}{2}$

$\therefore$ Equation of centre is $= (-g, -f) = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$.

72. Diameter of circle is diagonal of square.

Radius of the circle = 5

or diameter of the circle = 10

$\therefore \quad$ Area of square $= \dfrac{(10)^2}{2} = 50$ sq units

73. Since, B and C are the ends of diameter as $\angle BAC$ is $90°$.

$\therefore$ Equation of circle is

$$x(x-1) + y(y-1) = 0$$

$\Rightarrow \quad x^2 + y^2 - x - y = 0$

Now, point D satisfies this equation.

$\Rightarrow \quad 4a^2 + 9a^2 - 5a = 0$

$\Rightarrow \quad a(13a - 5) = 0$

$\Rightarrow \quad a = 0, \quad a = \dfrac{5}{13}$

74. Let equation of circle be

$$x^2 + y^2 = r^2$$

Since, in an equilateral triangle, the centroid coincides with the centre of the circle.

$\therefore$ Radius of circle, $\quad r = \dfrac{2}{3}(3a) = 2a$

On putting $r = 2a$ in Eq. (i), we get

$$x^2 + y^2 = (2a)^2$$

$\Rightarrow \quad x^2 + y^2 = 4a^2$

75. Let the equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since, $\left(m, \dfrac{1}{m}\right)$ lies on this circle.

$\therefore \quad m^2 + \dfrac{1}{m^2} + 2gm + \dfrac{2f}{m} + c = 0$

$\Rightarrow \quad m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

$\Rightarrow \quad m_1 m_2 m_3 m_4 = 1$

76. Let any point $P(x_1, y_1)$ outside the circle. Then, equation of tangent to the circle $x^2 + y^2 + 6x + 6y = 2$ at the point P is

$$xx_1 + yy_1 + 3(x + x_1) + 3(y + y_1) - 2 = 0 \quad \ldots(i)$$

The Eq. (i) and the line $5x - 2y + 6 = 0$ intersect at a point Q on y-axis *i.e.*, $x = 0$.

$\Rightarrow \quad 5(0) - 2y + 6 = 0 \Rightarrow y = 3$

$\therefore$ Coordinates of Q are (0, 3).

Point Q satisfies Eq. (i).

$\therefore \quad 3x_1 + 6y_1 + 7 = 0 \quad \ldots(ii)$

Distance between P and Q is given by

$$PQ^2 = x_1^2 + (y_1 - 3)^2$$

$$= x_1^2 + y_1^2 - 6y_1 + 9 = 11 - 6x_1 - 12y_1$$

$$(\because x_1^2 + y_1^2 + 6x_1 + 6y_1 - 2 = 0)$$

$$= 11 - 2(3x_1 + 6y_1)$$

$$= 11 - 2(-7) = 25 \quad \text{[from Eq. (ii)]}$$

$\therefore \quad PQ = 5$

77. Let x_1 and x_2 are the roots of the equation

$$x^2 + 2ax - b^2 = 0.$$

$\therefore \quad x_1 + x_2 = -2a$ and $x_1 x_2 = -b^2$

Also, y_1 and y_2 are roots of the equation

$$y^2 + 2py - q^2 = 0.$$

$\therefore \quad y_1 + y_2 = -2p$ and $y_1 y_2 = -q^2$

The equation of the circle with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as the end points of diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$\Rightarrow \quad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0$

$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

78. Let RS and PQ are the tangents at the extremities of diameter of circle.

In ΔRQP, $\tan\theta = \dfrac{PQ}{PR} = \dfrac{PQ}{2r} \quad \ldots(i)$

Also, in ΔSRP

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{RS}{RP} = \frac{RS}{2r}$$

$\Rightarrow \quad \cot\theta = \dfrac{RS}{2r} \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$\tan\theta \cdot \cot\theta = \frac{PQ \cdot RS}{4r^2}$$

$\Rightarrow \quad 4r^2 = PQ \cdot RS \Rightarrow 2r = \sqrt{(PQ)(RS)}$

79. Given points lie on a circle $x^2 + y^2 = a^2$ and in case of an equilateral triangle centroid is same as circumcentre. Circumcentre of given triangle is at origin or centroid is at origin.

$$\frac{a\cos\theta_1 + a\cos\theta_2 + a\cos\theta_3}{3} = 0$$

and $$\frac{a\sin\theta_1 + a\sin\theta_2 + a\sin\theta_3}{3} = 0$$

$$\Sigma\cos\theta_1 = 0, \quad \Sigma\sin\theta_1 = 0$$

80. The point should lies on the opposite side of the origin of the line $x + y - 1 = 0$

Then, $\alpha + \alpha - 1 > 0$

$\Rightarrow$ $2\alpha > 1 \Rightarrow \alpha > \frac{1}{2}$ ...(i)

Also, $(\alpha^2 + \alpha^2) < 1$

$\Rightarrow$ $\left(-\frac{1}{\sqrt{2}}\right) < \alpha < \left(\frac{1}{\sqrt{2}}\right)$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{2} < \alpha < \frac{1}{\sqrt{2}}$$

81. Let the equation of circle be

$x^2 + y^2 + 2gx + 2fgy = 0$ (passing through origin)

$$\text{Radius} = \sqrt{g^2 + f^2}$$

Now, equation of tangent at $O(0, 0)$ is

$$x(0) + y(0) + g(x) + f(y) = 0$$

$\Rightarrow$ $gx + fy = 0$

Distance from

$$A(2g, 0) = \frac{2g^2}{\sqrt{g^2 + f^2}} = m$$

and distance from B

$$(0, 2f) = \frac{2f^2}{\sqrt{g^2 + f^2}} = n$$

$\Rightarrow$ $\frac{2r^2}{r} = m + n \Rightarrow 2r = m + n$

82. Let the equation of L_1 be $y = mx$. Since, the intercepts made by the circle on L_1 and L_2 are equal, their distances from the centre of the circle are also equal.

The centre of the given circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

$\therefore$ $$\left|\frac{\frac{1}{2} - \frac{3}{2} - 1}{\sqrt{1+1}}\right| = \left|\frac{m \times \frac{1}{2} + \frac{3}{2}}{\sqrt{m^2 + 1}}\right|$$

$\Rightarrow$ $$\frac{2}{\sqrt{2}} = \frac{|m + 3|}{2\sqrt{m^2 + 1}}$$

$\Rightarrow$ $8(m^2 + 1) = (m + 3)^2$

$\Rightarrow$ $7m^2 - 6m - 1 = 0$

$\Rightarrow$ $(m - 1)(7m + 1) = 0$

$\Rightarrow$ $m = 1$ or $m = -\frac{1}{7}$

So, the equations representing L_1 are

$$y = x \text{ or } y = \left(-\frac{1}{7}\right)x$$

$\Rightarrow$ $x - y = 0$ or $x + 7y = 0$

83. Let OA and OB be the tangents from the origin to the given circle with centre $C(-3, 5)$ and radius

$$\sqrt{9 + 25 - c} = \sqrt{34 - c}.$$

Then, area of the quadrilateral,

$$OACB = 2 \times \text{Area of the } \Delta OAC = 2 \times \left(\frac{1}{2}\right) \times OA \times AC$$

Now, OA = Length of the tangent from the origin to the given circle $= \sqrt{c}$

and AC = Radius of the circle $= \sqrt{34 - c}$

So, that $\sqrt{c}\sqrt{34 - c} = 8$ (given)

$\Rightarrow$ $c(34 - c) = 64$

$\Rightarrow$ $c^2 - 34c + 64 = 0$

84. **Statement I** The centre and radius of circle

$$x^2 + y^2 - x - y - 1 = 0$$

are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\sqrt{\frac{3}{2}}$ respectively and the centre and radius of circle

$$x^2 + y^2 - 2x + 2y - 7 = 0$$

are (1, –1) and 3 respectively.

Distance between the centres is $\sqrt{\frac{5}{2}} < 3 - \sqrt{\frac{3}{2}}$

$(\because C_1C_2 < r_1 - r_2)$

$\therefore$ First circle is completely inside the second circle.

Statement II The centre and radius of circle

$$x^2 + y^2 + 14x + 12y + 21 = 0$$

are (–7, 6) and 8 respectively and the centre and radius of circle

$$x^2 + y^2 + 2x - 4y - 4 = 0$$

are (–1, 2) and 1 respectively.

Distance between the centres is $2\sqrt{13} > 8 + 1$

$(\because C_1C_2 > r_1 + r_2)$

These two circles intersect each other, therefore the number of common tangents is 2.

Hence, only first statements is correct.

Round II

1. Since, $y = |x| + c$ and $x^2 + y^2 - 8|x| - 9 = 0$ both are symmetrical about y-axis for $x > 0, y = x + c$. Equation of tangent to circle $x^2 + y^2 - 8x - 9 = 0$ which is parallel to

$$y = x + c \text{ is } y = (x - 4) + 5\sqrt{1+1}$$

$$\Rightarrow \quad y = x + (5\sqrt{2} - 4)$$

For no solution $c > 5\sqrt{2} - 4$,

$$\therefore \quad c \in (5\sqrt{2} - 4, \infty)$$

2. The triangle is isosceles and therefore the median through C is the bisector of $\angle C$. The equation of the angle bisector can be taken as $y = -x$ and $l = (-a, a)$, where a is positive.

Equation of AC is $y - 0 = -7(x + 6)$ or $7x + y + 42 = 0$ and equation of AB is $x - y + 6 = 0$.

The length of the perpendicular from l to AB and AC are equal.

$$\therefore \quad \left|\frac{-7a + a + 42}{\sqrt{50}}\right| = \left|\frac{-a - a + 6}{\sqrt{2}}\right|$$

Giving the positive value $a = \frac{9}{2}$

$\therefore$ Centre is $\left(-\frac{9}{2}, \frac{9}{2}\right)$ and radius $= \frac{3}{\sqrt{2}}$

The equation of the circle is

$$\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{9}{2}$$

$$\Rightarrow \quad x^2 + y^2 + 9x - 9y + 36 = 0$$

3. Let $C(h, k)$ be the centre of the circle passing through the end points of the rod AB and PQ of lengths a and b respectively. CL and CM be perpendiculars drawn from C on AB and PQ respectively. Then, $CA = CP$ (radii of the same circle)

$$\Rightarrow \quad k + \frac{a^2}{4} = h^2 + \frac{b^2}{4} \qquad \left(\because AL = \frac{a}{2} \text{ and } MP = \frac{b}{2}\right)$$

$$\Rightarrow \quad 4(h^2 - k^2) = a^2 - b^2$$

Hence, locus of (h, k) is

$$4(x^2 - y^2) = a^2 - b^2.$$

4. Equation of pair of tangents is

$$(a^2 - 1)y^2 - x^2 + 2ax - a^2 = 0$$

If θ be the angle between the tangents, then

$$\tan\theta = \frac{2\sqrt{(h^2 - ab)}}{a + b}$$

$$= \frac{2\sqrt{-(a^2 - 1)(-1)}}{a^2 - 2} = \frac{2\sqrt{a^2 - 1}}{a^2 - 2}$$

$\because$ θ lies in II quadrant, then $\tan\theta < 0$

$$\therefore \quad \frac{2\sqrt{a^2 - 1}}{a^2 - 2} < 0$$

$$\Rightarrow \quad a^2 - 1 > 0 \quad \text{and} \quad a^2 - 2 < 0$$

$$\Rightarrow \quad 1 < a^2 < 2$$

$$\Rightarrow \quad a \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

5. Since, $AP = PQ = QB$. The coordinates of P are $(a, 0)$ and of Q are $(2a, 0)$ the centre of the circles on AP, PQ and QB as diameters are respectively

$C_1\left(\frac{a}{2}, 0\right)$, $C_2\left(\frac{3a}{2}, 0\right)$ and $C_3\left(\frac{5a}{2}, 0\right)$ and the radius of each one of them is $\left(\frac{a}{2}\right)$.

Hence, the equations of the circles with centre C_1, C_2 and C_3 are respectively

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}; \quad \left(x - \frac{3a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

and $$\left(x - \frac{5a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$$

So that, if $S(h, k)$ be any point on the locus, then

$$\left(h - \frac{a}{2}\right)^2 + \left(h - \frac{3a}{2}\right)^2 + \left(h - \frac{5a}{2}\right)^2 + 3\left(k^2 - \frac{a^2}{4}\right) = b^2$$

$$\Rightarrow \quad 3(h^2 + k^2) - 9ah + 8a^2 = b^2$$

Hence, the locus of $S(h, k)$ is

$$3(x^2 + y^2) - 9ax + 8a^2 - b^2 = 0$$

6. Since, $\angle ADB = \angle ADC = 90°$, circles on AB and AC as diameters pass through D and therefore the altitude AD is the common chord. Similarly, the other two common chords are the other two altitudes and hence they concur at the orthocentre.

7. Since, the hypotenuse of a right angled triangle inscribed in a circle is a diameter of the circle. If the coordinates of the end C of the hypotenuse BC are (a, b), the coordinates of B

are $(-a, -b)$. Equation of BC is $\frac{y}{x} = \frac{b}{a}$. If A is the vertex of the isosceles triangle, then OA is perpendicular to BC and the equation of AO is $\frac{y}{x} = -\frac{a}{b}$ which meets the circle $x^2 + y^2 = r^2$ at points for which

$$\left(\frac{a^2}{b^2} + 1\right) x^2 = r^2 = a^2 + b^2$$

$$[\because (a, b) \text{ lies on } x^2 + y^2 = r^2]$$

$\Rightarrow \quad x^2 = b^2 \Rightarrow x = \pm b$

$\Rightarrow \quad y = \mp a$

$\therefore$ Coordinates of A are $(-b, a)$ or $(b, -a)$.

8. Let the equation of the chord OA of the circle

$$x^2 + y^2 - 2x + 4y = 0 \quad \text{...(i)}$$

by $\quad y = mx \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\Rightarrow \quad x^2 + m^2x^2 - 2x + 4mx = 0$

$\Rightarrow \quad (1 + m^2)x^2 - (2 - 4m)x = 0$

$\Rightarrow \quad x = 0$ and $x = \frac{2 - 4m}{1 + m^2}$

Hence, the points of intersection are

$$(0, 0) \text{ and } A\left(\frac{2 - 4m}{1 + m^2}, \frac{m(2 - 4m)}{1 + m^2}\right).$$

$$\Rightarrow \quad OA^2 = \left(\frac{2 - 4m^2}{1 + m^2}\right)^2 (1 + m^2)$$

$$= \frac{(2 - 4m)^2}{1 + m^2}$$

Since, OAB is an isosceles right-angled triangle

$$OA^2 = \frac{1}{2} AB^2,$$

where AB is a diameter of the given circle $OA^2 = 10$

$$\Rightarrow \quad \frac{(2 - 4m)^2}{1 + m^2} = 10$$

$\Rightarrow \quad 4 - 16m + 16m^2 = 10(1 + m)^2$

$\Rightarrow \quad 3m^2 - 8m - 3 = 0$

$\Rightarrow \quad m = 3$ or $-\frac{1}{3}$

Hence, the required equation are $y = 3x$ or $x + 3y = 0$.

9. $y = mx$ be the chord.

Then, point of intersection are given by

$$x^2(1 + m^2) - x(3 + 4m) - 4 = 0.$$

Therefore, $x_1 + x_2 = \frac{3 + 4m}{1 + m^2}$ and $x_1x_2 = \frac{-4}{1 + m^2}$

Since, (0, 0) divides the chord in the ratio 1 : 4.

Therefore, $\quad x_2 = -4x_1$

$\therefore \quad -3x_1 = \frac{3 + 4m}{1 + m^2}$ and $4x_1^2 = -\frac{-4}{1 + m^2}$

$\therefore \quad 9 + 9m^2 = 9 + 16m^2 + 24m$

i.e., $\quad m = 0, -\frac{24}{7}$

Therefore, the lines are $y = 0$ and $y + 24x = 0$.

10.

ΔABC and ΔDBA are similar

$\Rightarrow \quad lx = y\sqrt{l^2 + x^2}$

$\Rightarrow \quad l^2x^2 = y^2(l^2 + x^2)$

$\Rightarrow \quad l^2(x^2 - y^2) = x^2y^2$

$\Rightarrow \quad l = \frac{xy}{\sqrt{x^2 - y^2}} = \frac{AB \times AD}{\sqrt{AB^2 - AD^2}}$

11.

Circles with centres (2, 0) and (–2, 0) each with radius 4.

$\Rightarrow y$-axis is their common chord.

ΔABC is equilateral.

Hence, area of $ADBC$ is $\frac{2\sqrt{3}}{4}(4)^2 = 8\sqrt{3}$.

12. $r = 1; L = \sqrt{3}$

Area of quadrilateral $= \sqrt{3}$

Sector $= \frac{1}{2} \cdot 1 \cdot \frac{2\pi}{3} = \frac{\pi}{3}$

Shaded region $= \sqrt{3} - \frac{\pi}{3}$

13. From the figure, we have

$$y_1 = 2x_1 + 5 \text{ and } \frac{y_1 + 6}{x_1 + 8} \times 2 = -1$$

$$\Rightarrow \quad x_1 = -6 \text{ and } y_1 = -7$$

14.

From ΔMLN,

$$\sin\alpha = \frac{a-b}{a+b}$$

$$\therefore \quad \alpha = \sin^{-1}\left(\frac{a-b}{a+b}\right)$$

$\therefore$ Angle between AB and AD

$$= 2\alpha = \sin^{-1}\left(\frac{a-b}{a+b}\right).$$

15.

Let equation of line be $y = x - c$ or $y - x = c$ ...(i)

Perpendicular from (0, 0) on line (i) is $\left|\frac{-c}{\sqrt{2}}\right| = \frac{c}{\sqrt{2}}$

In ΔAON, $\sqrt{2^2 - \left(\frac{c}{\sqrt{2}}\right)^2} = AN$

and in ΔCPM, $\sqrt{3^2 - \left(\frac{2-c}{\sqrt{2}}\right)^2} = CM$

Given, $AN = CM$

$$\Rightarrow \quad 4 - \frac{c^2}{2} = 9 - \frac{(2-c)^2}{2}$$

$$\Rightarrow \quad c = -\frac{1}{2}$$

Therefore, equation of the line is $y = x - \frac{3}{2}$ or

$$2x - 2y - 3 = 0.$$

16. $\because C_2$ is the director circle of C_1.

$\therefore$ Equation of C_2 is $x^2 + y^2 = 2(2)^2 = 8$

Again, C_3 is the director circle of C_2. Hence, the equation of C_3 is

$$x^2 + y^2 = 2(8) = 16$$

17. The given equation is

$$(x-r)^2 + (y-h)^2 = r^2$$

Tangents are $x = 0$

and $\quad y = x\tan\left(\frac{\pi}{2} - 2\alpha\right)$

$$= x\cot 2\alpha$$

$$= \frac{x(1-\tan^2\alpha)}{2\tan\alpha}$$

$$\Rightarrow \quad y = \frac{x\left(1 - \frac{r^2}{h^2}\right)}{2\left(\frac{r}{h}\right)} \qquad \left(\because \text{in } \Delta ODC, \tan\alpha = \frac{r}{h}\right)$$

$$\Rightarrow \quad (h^2 - r^2)x - 2rhy = 0$$

18. Let $O \equiv (0, 0)$ be the centre of the circle.

$\because$ Arc length $AB = \frac{\pi}{2} = \frac{1}{4}$ (circumference of the circle)

$$\therefore \quad \angle AOB = \frac{\pi}{2}$$

$$\text{Slope of } OB = -\frac{1}{\text{Slope of } OA}$$

$$\Rightarrow \quad \text{Slope of } OB = -\frac{1}{1} = -1 \qquad \text{...(i)}$$

Let $B \equiv (a, \pm\sqrt{1-a^2})$

$$\therefore \quad \pm\frac{\sqrt{1-a^2}}{a} = -1 \qquad \text{[from Eq. (i)]}$$

$\therefore$ B can be $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ but possible points are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

19. $\because \quad L \equiv 2x + y = 6$

Let $\quad P \equiv (\lambda, 6 - 2\lambda)$

$$\because \quad \angle PQO = \angle PRO = \frac{\pi}{2}$$

$\therefore$ OP is diameter of circumcircle PQR, then centre is $\left(\frac{\lambda}{2}, 3 - \lambda\right)$.

$$\therefore \quad x = \frac{\lambda}{2} \Rightarrow \lambda = 2x \text{ and } y = 3 - \lambda,$$

then $\quad 2x + y = 3$

which is the required locus.

20. $\because P \equiv (2, 3)$

$\therefore$ Equation of QR is $2x + 3y = 4$

Let $S \equiv (\lambda, \beta)$

$$\Rightarrow \quad \frac{\alpha - 2}{2} = \frac{\beta - 3}{3} = \frac{-2(4 + 9 - 4)}{(4 + 9)}$$

$$= -\frac{18}{13}$$

$$\therefore \quad \alpha = \frac{10}{-13}, \quad \beta = \frac{-15}{13}$$

Now, equation of circumcircle of ΔQRS is

$$(x^2 + y^2 - 4) + \lambda(2x + 3y - 4) = 0$$

$\because$ It passes through $S = \left(\frac{-10}{13}, \frac{-15}{13}\right)$, then

$$\left(\frac{100}{169} + \frac{225}{169} - 4\right) + \lambda\left(-\frac{20}{13} - \frac{45}{13} - 4\right) = 0$$

$$\Rightarrow \quad \left(-\frac{351}{169}\right) + \lambda(-9) = 0$$

$$\therefore \quad \lambda = -\frac{39}{169} = -\frac{3}{13}$$

$\therefore$ Circumcentre is $\left(-\lambda, -\frac{3\lambda}{2}\right) \equiv \left(\frac{3}{13}, \frac{9}{26}\right)$.

21. Equation of common chord of these circles is

$$(x^2 + y^2 - 6x - 4y + 9) - (x^2 + y^2 - 8x - 6y + 23) = 0$$

$$\Rightarrow \quad 2x + 2y - 14 = 0$$

$$\Rightarrow r \quad x + y - 7 = 0$$

Since, centre of the second circle *i.e.*, (4, 3) lie on it. Hence, $x + y - 7 = 0$ is a diameter of second circle and hence first circle bisects the circumference of the second circle.

22. Let the given circles C_1 and C_2 have centres O_1 and O_2 with radii r_1 and r_2 respectively. Let centre of circle C is at O radius is r.

$\because \quad OO_2 = r + r_2$

$OO_1 = r_1 - r$

$\Rightarrow \quad OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$.

$\therefore$ Locus of O is an ellipse with foci O_1 and O_2.

23. Equation of tangent of the given circle is

$$y = mx \pm \sqrt{(1 + m^2)}$$

If it passes through (–3, –1), then we have

$$-1 = -3m \pm \sqrt{(1 + m^2)}$$

$$\Rightarrow \quad 1 + m^2 = (3m - 1)^2 \Rightarrow 8m^2 - 6m = 0$$

$$\therefore \quad m = 0, \quad m = \frac{3}{4}$$

Thus, reflected ray is $y + 1 = \frac{3}{4}(x + 3) \Rightarrow 4y - 3x = 5$

24. Let $C_1 \equiv (-g, -f)$, $r_1 = \sqrt{(g^2 + f^2)}$

and $C_2 \equiv (-g', -f')$, $r_2 = \sqrt{(g'^2 + f'^2)}$

$\therefore \quad |C_1C_2| = r_1 \pm r_2$

$$\Rightarrow \sqrt{(g - g')^2 + (f - f')^2} = \sqrt{(g^2 + f^2)} \pm \sqrt{(g'^2 + f'^2)}$$

$$\Rightarrow \quad (g - g')^2 + (f - f')^2 = g^2 + f^2 + g'^2 + f'^2 \pm 2\sqrt{(g^2 + f^2)(g'^2 + f'^2)}$$

$$(-gg' - ff')^2 = (g^2 + f^2)(g'^2 + f'^2)$$

$$\Rightarrow \quad g^2f'^2 + g'^2f^2 - 2gg'ff' = 0 \quad (gf' - g'f)^2 = 0$$

$$\Rightarrow \quad gf' = g'f$$

and line joining centres may not be parallel to common tangents.

25. Statement II is true.

Now, $\begin{vmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -4 & 3 & 1 \end{vmatrix} = -2(0 - 3) - 1(-1 + 4) + 1(-3 - 0)$

$$= 6 - 3 - 3 = 0$$

$\therefore$ Given points are collinear.

$\Rightarrow$ No circle will pass through the given points.

26. The two circles having centres at C_1 and C_2 and radii r_1 and r_2 respectively intersect at two distinct points, if

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$\Rightarrow$ Statement II is not true.

Here, $C_1 = (0, 0), C_2 = (4, 0)$ and $r_1 = 2, \quad r_2 = 3$

$\therefore \quad C_1C_2 = 4, \quad |r_1 - r_2| = 1, \quad r_1 + r_2 = 5$

that satisfies above condition.

27. Distance between two parallel lines L_1 and L_2 is

$$d = \left|\frac{p - 3 - p - 3}{\sqrt{4 + 9}}\right| = \frac{6}{\sqrt{13}}$$

And radius of given circle $= q$ clearly, $d < 2$

$\Rightarrow$ Statement II is false and statement I is true.

28. Let the equation of circle be

$$(x - 3)^2 + (y - 0)^2 + \lambda y = 0$$

As it passes through (1, –2)

$$\therefore \quad (1 - 3)^2 + (-2)^2 + \lambda(-2) = 0$$

$$\Rightarrow \quad 4 + 4 - 2\lambda = 0$$

$$\Rightarrow \quad \lambda = 4$$

$\therefore$ Equation of circle is

$$(x - 3)^2 + y^2 + 4y = 0$$

By hit and trial method, we see that point (5, –2) satisfies equation of circle.

29. Let us assume that the coordinates of the centre of the circle are $C(h, k)$ and its radius is r.

Since, the circle touches x-axis at (1, 0), hence its radius should be equal to ordinate of centre.

$$\Rightarrow \quad r = k$$

Hence, the equation of the circle is

$$(x-h)^2+(y-k)^2=k^2$$

Also, given that the circle passes through the points (1, 0) and (2, 3). Hence, substituting them in the equation of the circle, we get

$$(1-h)^2+(0-k)^2=k^2 \quad \ldots(i)$$

$$(2-h)^2+(3-k)^2=k^2 \quad \ldots(ii)$$

From Eq. (i), we get

$$h=1$$

On substituting in Eq. (ii), we get

$$(2-1)^2+(3-k)^2=k^2$$

$$\Rightarrow \quad k=\frac{5}{3}$$

$\therefore$ The diameter of the circle is $2k=\frac{10}{3}$.

30. $x^2+y^2-ax=0$ and $x^2+y^2=c^2$ touch each other.

(i) If circles touch internally,

$$\left|c-\frac{a}{2}\right|=\frac{a}{2}$$

$$\Rightarrow \quad c-\frac{a}{2}=\frac{a}{2}$$

$$\Rightarrow \quad c=a, c>0$$

$$\therefore \quad |a|=c$$

(ii) If circles touches externally,

$$\left|c+\frac{a}{2}\right|=\frac{a}{2}$$

$$\Rightarrow \quad c+\frac{a}{2}=\frac{a}{2}$$

$\therefore$ $c=0$, *i.e.*, not possible as $c>0$

$\therefore$ The circles should touch internally and $|a|=c$.

31. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.

$\Rightarrow (x-1)(x-0)+(y-0)(y-1)=0 \Rightarrow x^2+y^2-x-y=0$

32. Since, the coordinates of the centre of the circle are (2, 4).

Also, $\quad r^2=4+16+5=25$

The line will intersect the circle at two distinct points, if the distance of (2, 4) from $3x-4y=m$ is less than radius of the circle.

$$\therefore \quad \frac{|6-16-m|}{5}<5 \Rightarrow -25<10+m<25$$

$$\therefore \quad -35<m<15$$

33. Let $\quad S\equiv x^2+y^2+3x+7y+2p-5=0$

and $\quad S'\equiv x^2+y^2+2x+2y-p^2=0$

Equation of the required circle is $S+\lambda S'=0$

As, it passes through (1, 1), the value of

$$\lambda=-(7+2p)/(6-p^2)$$

Here, λ is not defined at $p=\pm\sqrt{6}$

Hence, it is true for all except two values of p.

34. Given equation can be rewritten as

$$(x+1)^2+(y+2)^2=(2\sqrt{2})^2$$

Let required point be $Q(\alpha,\beta)$.

Then, mid-point of $P(1,0)$ and $Q(\alpha,\beta)$ is the centre of the circle.

i.e., $\quad \frac{\alpha+1}{2}=-1$

and $\quad \frac{\beta+0}{2}=-2$

$\Rightarrow \quad \alpha=-3$ and $\beta=-4$

$\therefore$ Required point is $(-3,-4)$.

35. Circle with centre (h,k) and touching x-axis is

$$x^2+y^2-2hx-2ky+h^2=0$$

Since, $(-1,1)$ lies on it.

$$\Rightarrow \quad 2+2h-2k+h^2=0 \Rightarrow k=\frac{(h+1)^2+1}{2}\geq\frac{1}{2}$$

36. Let the centroid of the triangle be (h,k). Then, $h=\frac{0+5\cos\theta+5}{3}$ and $k=\frac{5+5\sin\theta+0}{3}$

$$\Rightarrow \quad 3h=5(1+\cos\theta) \text{ and } 3k=5(1+\sin\theta)$$

$$\Rightarrow \quad \frac{(3h-5)}{5}=\cos\theta \text{ and } \frac{(3k-5)}{5}=\sin\theta$$

$$\Rightarrow \quad \frac{9h^2+25-30h}{25}+\frac{9k^2+25-30k}{25}=1$$

$$\Rightarrow \quad 9h^2+9k^2+50-30h-30k-25=0$$

$$\Rightarrow \quad 9(x^2+y^2)-30x-30y+25=0$$

Hence, it represents a circle.

37. The locus is the circle with centre $(0, 0)$ and radius $3\cos\frac{\pi}{3} = \frac{3}{2}$

Its equation is $x^2 + y^2 = \frac{9}{4}$.

38. The given equations of diameters are

$$3x - 4y - 7 = 0 \quad \ldots(i)$$

and $$2x - 3y - 5 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 1 \text{ and } y = -1$$

$\therefore$ Centre of circle is $(1, -1)$.

Let r be the radius of circle, then

$$\pi r^2 = 49\pi$$

$\Rightarrow$ $r = 7$ unit

$\therefore$ Equation of required circle is

$$(x-1)^2 + (y+1)^2 = 49$$

$\Rightarrow$ $x^2 + y^2 - 2x + 2y + 1 + 1 = 49$

$\Rightarrow$ $x^2 + y^2 - 2x + 2y - 47 = 0$

39. Equation of pair of lines is $ax^2 + 2(a+b)xy + by^2 = 0$

Since, $4\theta = \pi \Rightarrow \theta = \frac{\pi}{4}$ ...(i)

Angle between lines is given by

$$\tan\theta = 2\frac{\sqrt{H^2 - AB}}{|A+B|}$$

$\Rightarrow$ $1 = \frac{2\sqrt{(a+b)^2 - ab}}{|a+b|}$ [from Eq. (i)]

$\Rightarrow$ $a^2 + b^2 + 2ab = 4(a^2 + b^2 + ab)$

$\Rightarrow$ $3a^2 + 3b^2 + 2ab = 0$

40. Since, circle touches the x-axis and also touches circle with the centre at (0, 3) and radius 2, then

$$C_1C_2 = r_1 + r_2$$

$\Rightarrow$ $h^2 + (k-3)^2 = (|k| + 2)^2$

$\Rightarrow$ $h^2 + k^2 + 9 - 6k = k^2 + 4 + 4|k|$

$\therefore$ Locus of centre of circle is

$$x^2 = -5 + 6y + 4|y|$$

$\Rightarrow$ $x^2 = 10y - 5$ $(\because y > 0)$

This equation represents a parabola. Thus, locus of the centre of the circle is a parabola.

41. The common chord of the circles is $S - S' = 0$

$\Rightarrow$ $5ax + (c-d)y + a + 1 = 0$

But it is same as $5x + by - a = 0$

$\therefore$ $\frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}$

$\Rightarrow a^2 + a + 1 = 0$ has no real value of a.

42. Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. The orthogonality gives $c - 4 = 0, c = 4$.

The circle passes through (a, b).

$\Rightarrow$ $a^2 + b^2 + 2ga + 2fb + 4 = 0$.

Replacing g by $-x$ and f by $-y$, we get the locus of the centre $2ax + 2by - (a^2 + b^2 + 4) = 0$

43. Let the centre of circle on y-axis be $(0, k)$.

Let $d = \sqrt{(7-0)^2 + (3-k)^2}$

$\Rightarrow$ $d^2 = 7^2 + (3-k)^2 = D$ (say)

On differentiating w.r.t. k, we get

$$\frac{dD}{dk} = 0 + 2(3-k)(-1)$$

Put $\frac{dD}{dk} = 0 \Rightarrow k = 3$

Now, $\frac{d^2D}{dk^2} = 2 < 0$ minima,

$\therefore$ Minimum value at $k = 3$ is $d = 7$

Hence, the radius of smallest circle is 7.

44. The lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ are diameters of circle.

On solving these equations, we get

$$x = 1 \text{ and } y = -1$$

Therefore, the centre of circle is $(1, -1)$ and circumference of circle $= 10\pi$

$\Rightarrow$ $2\pi r = 10\pi \Rightarrow r = 5$

$\therefore$ Equation of circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

$\Rightarrow$ $(x-1)^2 + (y+1)^2 = 5^2$

$\Rightarrow$ $x^2 + 1 - 2x + y^2 + 2y + 1 = 25$

$\Rightarrow$ $x^2 + y^2 - 2x + 2y - 23 = 0$

45. The equation of line is $y = x$ ...(i)

and equation of circle is

$$x^2 + y^2 - 2x = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x^2 + x^2 - 2x = 0$$

$$\Rightarrow \quad 2x^2 - 2x = 0$$

$$\Rightarrow \quad 2x(x-1) = 0$$

$$\Rightarrow \quad x = 0, \ x = 1$$

When, $x = 0, y = 0$ and when $x = 1, y = 1$

Let coordinates of A are (0, 0) and coordinates of B are (1, 1).

$\therefore$ Equation of circle (AB as a diameter) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad (x-0)(x-1) + (y-0)(y-1) = 0$$

$$\Rightarrow \quad x(x-1) + y(y-1) = 0$$

$$\Rightarrow \quad x^2 - x + y^2 - y = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - y = 0$$

46. Given circles are $x^2 + y^2 = 6x$ ...(i)

and $x^2 + y^2 + 6x + 2y + 1 = 0$ ...(ii)

From Eq. (i), $x^2 - 6x + y^2 = 0$

$$\Rightarrow \quad (x-3)^2 + y^2 = 3^2$$

$\therefore$ Centre (3, 0), $r = 3$

From Eq. (ii), $x^2 + 6x + y^2 + 2y + 1 + 3^2 = 3^2$

$$\Rightarrow \quad (x+3)^2 + (y+1)^2 = 3^2$$

$\therefore$ Centre (–3, 1), radius = 3.

Now, distance between centres

$$= \sqrt{(3+3)^2 + 1} = \sqrt{37} > r_1 + r_2 = 6$$

$\therefore$ Circles do not cut each other.

$\Rightarrow$ 4 tangents (two direct and two transversal) are possible.

47. In a circle AB is as a diameter where the coordinates of A are (p, q) and let the coordinates of B are (x_1, y_1).

Equation of circle in diameter form is

$$(x - p)(x - x_1) + (y - q)(y - y_1) = 0$$

$$\Rightarrow \quad x^2 - (p + x_1)x + px_1 + y^2 - (y_1 + q)y + qy_1 = 0$$

$$\Rightarrow \quad x^2 - (p + x_1)x + y^2 - (y_1 + q)y + px_1 + qy_1 = 0$$

Since, the circle touches x-axis. $\therefore \ y = 0$

$$\Rightarrow \quad x^2 - (p + x_1)x + px_1 + qy_1 = 0$$

Also, the discriminant of above equation will be equal to zero because circle touches x-axis.

$$\therefore \quad (p + x_1)^2 = 4(px_1 + qy_1)$$

$$\Rightarrow \quad p^2 + x_1^2 + 2px_1 = 4px_1 + 4qy_1$$

$$\Rightarrow \quad x_1^2 - 2px_1 + p^2 = 4qy_1$$

$$\Rightarrow \quad (x_1 - p)^2 = 4qy_1$$

Therefore, the locus of point B is

$$(x - p)^2 = 4qy$$

14 Parabola

JEE Main MILESTONE

- Conic Section
- Equation of Conic Section
- Recognisation of Conics
- Centre of Conics
- Parabola
- Terms Related to all Parabolas (in Standard Form)
- Position of a Point with Respect to the Parabola
- Equation of Chord
- Intersection of Line and Parabola
- Condition of Tangency
- Tangent to the Parabola
- Normal to the Parabola
- Conormal Points
- Pair of Tangents
- Chord of Contact
- Diameter

14.1 Conic Section

The locus of a point P which moves in a plane such that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line, is known as conic section.

The fixed point is called the **focus** of the conic and this fixed line is called the directrix of the conic. Also, this constant ratio is called the eccentricity of the conic and is denoted by e. Thus,

$$\frac{PS}{PM} = \text{Constant} = e \Rightarrow PS = ePM$$

Eccentricity	Shape
$e = 0$	Circle
$0 < e < 1$	Ellipse
$e = 1$	Parabola
$e > 1$	Hyperbola
$e = \infty$	Pair of straight lines

A parabola is a two dimensional, minor symmetrical curve, which is approximately U-shaped.

A parabola is as a conic section, created from the intersection of a right circular conical surface and a plane, which is parallel to a straight line on the conical surface and perpendicular to another plane which includes both the axis of the cone and also the same straight line on its surface.

14.2 Equation of Conic Section

If the focus is (α, β) and the directrix is $ax + by + c = 0$, then the equation of the conic section whose eccentricity is e, is

$$\sqrt{(x-\alpha)^2+(y-\beta)^2} = e\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \qquad (x-\alpha)^2+(y-\beta)^2 = e^2\frac{(ax+by+c)^2}{a^2+b^2}$$

Some Important Definitions

(i) Centre

The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

(ii) Axis

The straight line passing through the focus and perpendicular to the directrix, is called axis of the conic section.

(iii) Vertex

The points of intersection of the conic section and the axis, is called the vertex of conic section.

(iv) Latusrectum

The chord passing through the focus and perpendicular to the axis is called latusrectum of conic section.

(v) Focal Chord

A chord of a conic passing through the focus is called a focal chord.

(vi) Double Ordinate

A straight line drawn perpendicular to the axis and terminated at both ends of the curve is a double ordinate of the conic section.

14.3 Recognisation of Conics

The equation of conics represented by general equation of second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \text{...(i)}$$

can be recognised easily by the condition given below.

***Case* I** When $\Delta = abc+2fgh-af^2-bg^2-ch^2=0$

In this case Eq. (i) represents the degenerate conic.

Condition	Nature of Conic
$\Delta = 0$ and $ab-h^2=0$	A pair of straight lines or empty set
$\Delta = 0$ and $ab-h^2 \neq 0$	A pair of intersecting straight lines
$\Delta = 0$ and $ab-h^2 < 0$	Real or imaginary pair of straight lines
$\Delta = 0$ and $ab-h^2 > 0$	Point

***Case* II** $\Delta = abc+2fgh-af^2-bg^2-ch^2 \neq 0$

In this case Eq. (i) represents the non-degenerate conic.

Condition	Nature of Conic
$\Delta \neq 0, h=0, a=b$	A circle
$\Delta \neq 0, ab-h^2=0$	A parabola
$\Delta \neq 0, ab-h^2>0$	An ellipse or empty set
$\Delta \neq 0, ab-h^2<0$	A hyperbola
$\Delta \neq 0, ab-h^2<0$ and $a+b=0$	A rectangular hyperbola

(a) **Circle :** plane perpendicular to cone axis

(b) **Ellipse :** plane not perpendicular to axis not parallel to side of cone

(c) **Parabola :** plane parallel to side of cone

(d) **Hyperbola :** plane parallel to cone axis

14.4 Centre of Conics

Centre of the conic is the point which bisect every chord of the conic passing through it. If the equation of conic is $ax^2+2hxy+by^2+2gx+2fy+c=0$, then the coordinates of centre of conic are $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$.

***Sample* Problem 1** *The centre of conic*

$$14x^2-4xy+11y^2-44x-58y+71=0 \text{ is}$$

(a) (2, 3) (b) (3, 2)
(c) (−2 , −3) (d) *None of these*

Interpret (a) The equation of conic is

$$14x^2-4xy+11y^2-44x-58y+71=0$$

On comparing with

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

we get

$a=14, \quad b=11, \quad c=71, \quad h=-2, \quad g=-22, \quad f=-29$

$\therefore$ Centre of conic is

$$\left[\frac{(-2)(-29)-(11)(-22)}{(14)(11)-(-2)^2}, \frac{(-22)(-2)-(14)(-29)}{(14)(11)-(-2)^2}\right]$$

i.e., (2, 3).

14.5 Parabola

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (*i.e.*, focus) is always equal to its distance from a fixed straight line (*i.e.*, directrix).

Mathematically, $\frac{|PS|}{|PM|} = e = 1$

where, e is called eccentricity.

Note If the vertex of the parabola is (h, k), then generalised form of the parabola is $(y-k)^2 = 4a(x-h)^2$. Its focus is at $(a+h, k)$ equation of directrix is $x + a - h = 0$ and parametric equations are $x = h + at^2$, $y = k + 2at$.

***Sample* Problem 2** *Find the equation of parabola having focus at (−1, − 2) and directrix $x - 2y + 3 = 0$.* **[NCERT]**

(*a*) $4x^2 + y^2 + 4xy + 2x + 32y - 16 = 0$

(*b*) $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

(*c*) $x^2 + 4y^2 + 4xy + 4x + 32y + 16 = 0$

(*d*) $4x^2 + y^2 + 4xy - 4x + 32y + 16 = 0$

Interpret (*b*) By definition of parabola, if P is any point on the parabola, S be the focus and M is a point on directrix. Use $SP = PM$.

Given, $S = (-1, -2)$ and $P = (x, y)$

i.e., $\sqrt{(x+1)^2 + (y+2)^2} = \frac{x - 2y + 3}{\sqrt{1+4}}$

($\because$ distance between two points $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\Rightarrow$ $(x+1)^2 + (y+2)^2 = \frac{(x-2y+3)^2}{5}$ (squaring both sides)

$\Rightarrow$ $5x^2 + 5 + 10x + 5y^2 + 20 + 20y$

$= x^2 + 4y^2 + 9 - 4xy - 12y + 6x$

$\Rightarrow$ $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

14.6 Terms Related to all Parabolas (in Standard Form)

(a)	Equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
(b)	Graphs				
(c)	Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
(d)	Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
(e)	Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
(f)	Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
(g)	Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
(h)	Extremities of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
(i)	Length of latusrectum	$4a$	$4a$	$4a$	$4a$
(j)	Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
(k)	Parametric equations	$\begin{cases} x = at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = -at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}$	$\begin{cases} x = 2at \\ y = -at^2 \end{cases}$
(l)	Parametric coordinates of any point on parabola	$P(at^2, 2at)$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$
(m)	Focal distance of any point $P(h, k)$ on the parabola	$h + a$	$h - a$	$k + a$	$k - a$
(n)	Equation of latusrectum	$x = a$	$x + a = 0$	$y = a$	$y + a = 0$

***Sample* Problem 3** *If the equation of parabola is $x^2 = 6y$, then the coordinates of focus and axis of the parabola are* **[NCERT]**

(a) $\left(0, \frac{3}{2}\right)$, *y-axis* (b) $\left(0, -\frac{3}{2}\right)$, *y-axis*

(c) $\left(0, \frac{3}{2}\right)$, *x-axis* (d) *None of these*

Interpret (a) Given parabola is $x^2 = 6y$, which is of the form $x^2 = 4ay$ *i.e.*, focus lies on the positive direction of y-axis.

Here, $4a = 6 \Rightarrow a = \frac{6}{4} = \frac{3}{2}$

$\therefore$ Focus $= (0, a) = \left(0, \frac{3}{2}\right)$

Axis $= y$-axis.

***Sample* Problem 4** *If focus $(0, -3)$ and directrix $y = 3$, then equation of parabola is* **[NCERT]**

(a) $y^2 = 12x$ (b) $x^2 = 12y$

(c) $y^2 = -12x$ (d) $x^2 = -12y$

Interpret (d) Given focus $= (0, -3)$ and directrix $y = 3$. Since, focus lies on the negative direction of y-axis *i.e.*, equation of parabola will be of the form $x^2 = -4ay$ with $a = 3$.

$\therefore$ Required equation is

$$x^2 = -4(3y) \Rightarrow x^2 = -12y$$

14.7 Position of a Point with Respect to the Parabola

Let $S \equiv y^2 = 4ax$ be the equation of the parabola and $P(x_1, y_1)$, be any point in the region of the parabola, then

$$S_1 \equiv y_1^2 - 4ax_1$$

If $S_1 > 0$, then P lies outside the parabola.
If $S_1 = 0$, then P lies on the parabola.
If $S_1 < 0$, then P lies inside the parabola.

***Sample* Problem 5** *The position of point $(1, 4)$ with respect to the parabola $y^2 + 9 - 6y = 5x$ is*

(a) *on the curve* (b) *inside the curve*

(c) *outside the curve* (d) *None of these*

Interpret (b) Let $S = y^2 + 9 - 6y - 5x$

At point $(1, 4)$

$$S_1 = (4)^2 + 9 - 6(4) - 5(1)$$

$$\Rightarrow \quad S_1 = 16 + 9 - 24 - 5$$

$$= -4 < 0$$

Hence, given point lies inside the curve.

14.8 Equation of Chord

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are two points on the parabola $y^2 = 4ax$ such that PQ is a focal chord. Then, the equation of chord is

$$y(t_1 + t_2) = 2x + 2at_1t_2$$

If the chord PQ is a focal chord of the parabola, then $(a, 0)$ must satisfy this equation

$$0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$$

Thus, if t is the parameter for one end of a focal chord, then parameter for other end is $\frac{-1}{t}$ and the coordinates of the end points of a focal chord PQ of the parabola $y^2 = 4ax$ can be taken as $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

Length of Focal Chord

Let $P(at^2, 2at)$ be the one end of a focal chord PQ of the parabola $y^2 = 4ax$. Then, the lengths of focal chord is $a\left(t + \frac{1}{t}\right)^2$, where t is the parameter for one end of the chord.

If l_1 and l_2 are two lengths of focal segments.

Then, $$4a = \frac{l_1 l_2}{l_1 + l_2}$$

i.e., Length of latusrectum

$= 2$ (Harmonic mean of the focal segments)

Note The length of focal chord having parameters t_1 and t_2 for its end point is $a(t_2 - t_1)^2$.

***Sample* Problem 6** *The length of focal chord of parabola $y^2 = 4ax$ makes an angle α with the x-axis, is*

(a) $4a \operatorname{cosec} \alpha$ (b) $4 \operatorname{cosec}^2 \alpha$

(c) $4a \operatorname{cosec}^2 \alpha$ (d) *None of these*

Interpret (c) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord PQ which makes an angle α with the axis of the parabola. Then,

$$PQ = a(t_2 - t_1)^2 = a[(t_2 + t_1)^2 - 4t_1t_2]$$

$$= a[(t_2 + t_1)^2 + 4] \quad (\because t_1t_2 = -1) \quad \ldots(i)$$

$$\because \quad \tan\alpha = \text{Slope of } PQ = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$$

$$\Rightarrow \quad \tan\alpha = \frac{2}{t_2 + t_1}$$

$$\Rightarrow \quad t_2 + t_1 = 2\cot\alpha \quad \ldots(ii)$$

On substituting the value of $t_2 + t_1$ from Eq. (ii) in Eq. (i), we get

$$PQ = a(4\cot^2\alpha + 4) = 4a \operatorname{cosec}^2\alpha$$

Check Point **1**

1. The equation $\lambda x^2 + 4xy + y^2 + \lambda x + 3y + 2 = 0$ represents a parabola, if λ is
   (a) -4 (b) 4
   (c) 0 (d) None of these
2. If the vertex of a parabola is the point $(-3, 0)$ and the directrix is the line $x + 5 = 0$, then equation of parabola is
   (a) $y^2 = 8(x + 3)$ (b) $x^2 = 8(y + 3)$
   (c) $y^2 = -8(x + 3)$ (d) $y^2 = 8(x + 5)$
3. The focus of the parabola $y^2 = 4y - 4x$ is
   (a) (0, 2) (b) (1, 2) (c) (2, 0) (d) (2, 1)
4. If the point $P(4, -2)$ is one end of the focal chord PQ of the parabola $y^2 = x$, then the slope of the tangent at Q is
   (a) $-\frac{1}{4}$ (b) $\frac{1}{4}$ (c) 4 (d) -4
5. The length of latusrectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x - 12y + 17)^2$ is
   (a) 14/13 (b) 28/13
   (c) 12/13 (d) None of these
6. $x - 2 = t^2$, $y = 2t$ are the parametric equations of the parabola
   (a) $y^2 = 4x$ (b) $y^2 = -4x$
   (c) $x^2 = -4y$ (d) $y^2 = 4(x - 2)$
7. The latusrectum of the parabola $y^2 = 5x + 4y + 1$ is
   (a) 5/4 (b) 10 (c) 5 (d) 5/2
8. The equation of the parabola having vertex at the origin axis on the y-axis and passing through the point $(6, -3)$ is
   (a) $y^2 = 12x + 6$ (b) $x^2 = 12y$
   (c) $x^2 = -12y$ (d) $y^2 = -12x + 6$
9. The equation of parabola having vertex (0, 0), passing through (5, 2) and symmetric with respect to y-axis is
   (a) $3x^2 = 25y$ (b) $2x^2 = 25y$
   (c) $2y^2 = 25x$ (d) None of these

14.9 Intersection of Line and Parabola

Let the parabola be $y^2 = 4ax$ ...(i)

and the given line be $y = mx + c$ ...(ii)

On eliminating x from Eqs. (i) and (ii), then

$$y^2 = 4a\left(\frac{y - c}{m}\right)$$

$$\Rightarrow \quad my^2 - 4ay + 4ac = 0 \quad \text{...(iii)}$$

It is a quadratic equation in y.

Discriminant, $D = (-4a)^2 - 4(4ac)m = 16a(a - cm)$

(i) If $D > 0$ *i.e.*, $a > cm$, then line intersect the parabola at two distinct points.

(ii) If $D > 0$ *i.e.*, $a = cm$, then line intersect the parabola at two coincident points *i.e.*, at one point.

(iii) If $D < 0$ *i.e.*, $a < cm$, then line neither touch nor intersect the parabola.

Sample **Problem 7** *If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then $PA \cdot PB$ is equal to [where, $P = (\sqrt{3}, 0)$]*

(a) $\frac{4(\sqrt{3} + 2)}{3}$ (b) $\frac{4(2 - \sqrt{3})}{3}$

(c) $\frac{4\sqrt{3}}{3}$ (d) $\frac{2(\sqrt{3} + 2)}{3}$

Interpret (a) Given, $P = (\sqrt{3}, 0)$

$\therefore$ Equation of line AB is

$$\frac{x - \sqrt{3}}{\cos 60°} = \frac{y - 0}{\sin 60°} = r \text{ (say)}$$

$$\Rightarrow \quad x = \sqrt{3} + \frac{r}{2}$$

and $$y = \frac{r\sqrt{3}}{3}$$

$\therefore$ Point $\left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ lie on $y^2 = x + 2$.

$$\therefore \quad \frac{3r^2}{4} = \sqrt{3} + \frac{r}{2} + 2$$

$$\Rightarrow \quad \frac{3r^2}{4} - \frac{r}{2} - (2 + \sqrt{3}) = 0$$

Let the roots be r_1 and r_2.

$\therefore$ Product of roots $= PA \cdot PB = r_1 r_2 = \left|\frac{-(2 + \sqrt{3})}{3/4}\right|$

$$= \frac{4(2 + \sqrt{3})}{3}$$

14.10 Condition of Tangency

Let the line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$.

$\therefore \quad (mx + c)^2 = 4ax$

$\Rightarrow \quad m^2x^2 + c^2 + 2mcx = 4ax$

$\Rightarrow \quad m^2x^2 + 2(mc - 2a)x + c^2 = 0$

Since, the line is a tangent to the parabola.

$\therefore \quad \Delta = 0$

$4[(mc - 2a)^2 - 4m^2c^2 = 0 \quad (\because \Delta = B^2 - 4AC)$

$\Rightarrow \quad 4[m^2c^2 + 4a^2 - 4amc] - 4m^2c^2 = 0$

$\Rightarrow \quad 4[4a^2 - 4amc] = 0$

$\Rightarrow \quad 4a^2 = 4amc$

$\Rightarrow \quad 4a(a - mc) = 0$

$\Rightarrow \quad mc = a$

$\Rightarrow \quad c = \dfrac{a}{m}$

***Sample* Problem 8** *If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$, then the value of m is* **[NCERT Exemplar]**

(a) 1 (b) −1
(c) 2 (d) −2

Interpret (a) Given equation of line is

$y = mx + 1 \quad$ ...(i)

and equation of parabola is $y^2 = 4x \quad$ ...(ii)

From Eq. (i), $y = mx + 1$ put in Eq. (ii), we get

$(mx + 1)^2 = 4x$

$\Rightarrow \quad m^2x^2 + 1 + 2mx - 4x = 0$

$\Rightarrow \quad x^2(m^2) + x(2m - 4) + 1 = 0$

$\because$ For tangent, discriminant is zero *i.e.*, $D = 0$

$\therefore \quad (2m - 4)^2 - 4m^2 \times 1 = 0 \quad (\because D = b^2 - 4ac = 0)$

$\Rightarrow \quad 4m^2 + 16 - 16m - 4m^2 = 0$

$\Rightarrow \quad m = 1$

14.11 Tangent to the Parabola

A line which intersect the parabola at only one point is called the tangent to the parabola.

Equation of Tangent in Different Form

(i) Point Form

The equation of tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is $yy_1 = 2a(x + x_1)$. The equation of tangents to all standard forms of parabola at point (x_1, y_1) are given below.

Equation of parabola	Equation of tangent
$y^2 = 4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

Note The equation of tangent at (x_1, y_1) to any second degree curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\dfrac{x + x_1}{2}$, y by $\dfrac{y + y_1}{2}$ and xy by $\dfrac{xy_1 + x_1y}{2}$ and without changing the constant (if any) in the equation of the curve.

(ii) Slope Form

The equation of tangent to the parabola $y^2 = 4ax$ in slope form is $y = mx + \dfrac{a}{m}$, where m is the slope of the tangent.

The equation of tangent of slope m to the parabola $(y - k)^2 = 4a(x - h)$ is given by $y - k = m(x - h) + \dfrac{a}{m}$ and coordinates of point of contact are $\left(h + \dfrac{a}{m^2}, k + \dfrac{2a}{m}\right)$.

(iii) Parametric Form

The equation of tangent at the point $(at^2, 2at)$ or t to the parabola $y^2 = 4ax$ is $ty = x + at^2$. The parametric equation of tangents to all standard forms of parabola are given below.

Equation of parabola	Point of contact	Equation of tangent
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

Point of Intersection of Tangents at Any Two Points on the Parabola

Let the parabola be $y^2 = 4ax$. Let two points on the parabola are $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$.

Equation of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$t_1y = x + at_1^2$$

and

$$t_2y = x + at_2^2$$

On solving these equations, we get

$$x = at_1t_2, \quad y = a(t_1 + t_2)$$

Thus, the coordinates of the point of intersection of tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are $[at_1t_2, a(t_1+t_2)]$.

Note

- x-coordinate, at_1t_2 = GM of at_1^2, at_2^2
- y-coordinate, $a(t_1+t_2)$ = AM of $2at_1, 2at_2$

Important Results on Tangents

(i) The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.

(ii) The tangents at the extremities of a focal chord of a parabola intersect at right angle on the directrix.

(iii) The portion of the tangent to a parabola cut-off between the directrix and the curve subtends a right angle at the focus.

(iv) The perpendicular drawn from the focus on any tangent to a parabola intersect it at the point, where it cuts the tangent at the vertex.

(v) The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

(vi) The tangent at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.

Sample **Problem 9** *If the line* $\frac{x}{l}+\frac{y}{m}=1$ *touches the parabola* $y^2 = 4a(x+b)$, *then* $m^2(l+b)$ *is equal to*

(a) al^2 (b) $-al^2$ (c) l^2 (d) $-a$

Interpret (b) The given parabola is

$$y^2 = 4a(x+b) \quad \text{...(i)}$$

Vertex of this parabola is $(-b, 0)$.

Now, shifting (0, 0) at $(-b, 0)$, then

$$x = X + (-b)$$

and $$y = Y + 0$$

$\Rightarrow$ $$x + b = X$$

and $$y = Y \quad \text{...(ii)}$$

From Eq. (i), $$Y^2 = 4aX \quad \text{...(iii)}$$

and the line $$\frac{x}{l}+\frac{y}{m}=1$$

reduces to $$\frac{X-b}{l}+\frac{Y}{m}=1$$

$$\Rightarrow \quad Y = m\left(1-\frac{X-b}{l}\right)$$

$$\Rightarrow \quad Y = \left(-\frac{m}{l}\right)X + m\left(1+\frac{b}{l}\right) \quad \text{...(iv)}$$

The line (iv) will touch the parabola (iii), if

$$m\left(1+\frac{b}{l}\right) = \frac{a}{\left(-\frac{m}{l}\right)}$$

$$\Rightarrow \quad \frac{m^2}{l}\left(1+\frac{b}{l}\right) = -a$$

$$\Rightarrow \quad m^2(l+b) + al^2 = 0$$

14.12 Normal to the Parabola

A line which is perpendicular to the tangent of the parabola is called the normal to the parabola.

Equation of Normal in Different Forms

(i) Point Form

The equation of normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$(y - y_1) = -\frac{y_1}{2a}(x - x_1)$$

(ii) Slope Form

The equation of normal to the parabola $y^2 = 4ax$ in the slope form is

$$y = mx - 2am - am^3$$

The equation of normals to various standard forms of the parabola in terms of the slope of the normal are as given below.

Equation of parabola	Equation of normal	Slope of normal	Point of contact
$y^2 = -4ax$	$y = mx + 2am + am^3$	m	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$1/m$	$(-2am, am^2)$
$x^2 = -4ay$	$x = my + 2am + am^3$	$1/m$	$(2am, -am^2)$

Note The line $y = mx + c$ will be a normal to the parabola $y^2 = 4ax$, if $c = -2am - am^3$ and point of contact are $(am^2, -2am)$.

(iii) Parametric Form

The equation of normal at the point $(at^2, 2at)$ or t to the parabola $y^2 = 4ax$ is $y + tx = 2at + at^3$.

Point of Intersection of Normals Drawn Any Two Points on the Parabola

If two normals are drawn to the parabola $y^2 = 4ax$ at the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at a point R, then coordinates of the point R are $[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$.

Important Results on Normals

(i) The normal drawn at a point $P(at_1^2, 2at_1)$ to the parabola $y^2 = 4ax$ meets again the parabola at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

(ii) The tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.

(iii) If the normals at point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola, then $t_1t_2 = 2$.

(iv) If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

(v) The normal chord of a parabola at a point whose ordinate is equal to the abscissa, subtends a right angle at the focus.

(vi) The normal at any point of a parabola is equally inclined to the focal distance to the point and the axis of the parabola.

***Sample* Problem 10** *The normal meet the parabola $y^2 = 4ax$ at that point where the abscissae of the point is equal to the ordinate of the point is*

(a) $(6a, -9a)$ (b) $(-9a, 6a)$
(c) $(-6a, 9a)$ (d) $(9a, -6a)$

Interpret (d) If a normal drawn from the point $(at_1^2, 2at_1)$ to the parabola $y^2 = 4ax$ meets at point $(at_2^2, 2at_2)$ of same parabola, then $t_2 = -t_1 - \frac{2}{t_1}$.

Given condition is $x = y$ ...(i)

$\because$ $y^2 = 4ax$ (given)

$\therefore$ $y^2 = 4ay$ [$\because$ from Eq. (i)]

$\Rightarrow$ $y(y - 4a) = 0$

$\Rightarrow$ $y = 0$ or $y = 4a$

$\therefore$ Point $(0, 0)$ and $(4a, 4a)$

$\Rightarrow$ $2at_1 = 4a \Rightarrow t_1 = \frac{4a}{2a} = 2$

$t_2 = -2 - \frac{2}{2} = -3$

$\therefore$ Point $(at_2^2, 2at_2) = [a \times (-3)^2, 2a(-3)] = (9a, -6a)$

14.13 Conormal Points

The points on the parabola through which normals drawn are concurrent *i.e.*, pass through the same point are called conormal points. The conormal points are also called the feet of the normals. Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are called conormal points.

Results on Conormal Points

(i) The algebraic sum of the slopes of the normals at conormal points is zero.

(ii) The sum of the ordinates of the conormal points is zero.

(iii) The centroid of the triangle formed by the conormal points on a parabola lies on its axis.

***Sample* Problem 11** *Find the locus of the point through which pass three normals to the parabola $y^2 = 4ax$ such that two of them make angles α and β respectively with the axis such that $\tan\alpha \tan\beta = 2$.*

(a) $x^2 - 4ay = 0$ (b) $y^2 - 4ax = 0$
(c) $x^2 + 4ay = 0$ (d) $y^2 + 4ax = 0$

Interpret (b) Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$. The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k), then $k = mh - 2am - am^3$

$\Rightarrow$ $am^3 + m(2a - h) + k = 0$ ...(i)

Let roots of Eq. (i) be m_1, m_2, m_3, then from Eq. (i),

$$m_1m_2m_3 = -\frac{k}{a} \quad ...(ii)$$

Also, $m_1 = \tan\alpha, m_2 = \tan\beta$ and $\tan\alpha \tan\beta = 2$ ...(iii)

$\therefore$ $m_1m_2 = 2$

From Eqs. (ii) and (iii), $2m_3 = -\frac{k}{a}$

$\Rightarrow \quad m_3 = -\frac{k}{2a}$

Which being a root of Eq. (i) must satisfy it

i.e., $\quad am_3^3 + m_3(2a - h) + k = 0$

$\Rightarrow \quad a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a - h) + k = 0$

$\Rightarrow \quad -\frac{k^3}{8a^2} - k + \frac{kh}{2a} + k = 0$

$\Rightarrow \quad k^2 - 4ah = 0$

$\therefore$ Required locus of (h, k) is $y^2 - 4ax = 0$

14.14 Pair of Tangents

Let $P(x_1, y_1)$ be any point outside the parabola $S = y^2 - 4ax$, then the equation of the pair of tangents drawn from the point P to the parabola is given by

$$SS_1 = T^2$$

where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

Hot Spot

Equation of Pair of Tangents

In this types of problems, sometimes we make a equation of pair of tangent from a given point and a curve. These types of problems are directly or indirectly asked in various years of AIEEE (JEE Main) 2010, 2008 following steps are given below.

Step I Use the formula $T^2 = SS_1$.

Step II Use the result, if and only if a pair of tangents are drawn from a directrix point to the parabola, then they are always perpendicular.

Step III Apply the suitable formula and simplify it.

***Sample* Problem 12** *If a point $P(1, 4)$ is outside the parabola $y^2 + 4y = 3x$, then equation of pair of tangent is*

(a) $76y^2 + 9x^2 - 72xy + 126x - 272y + 312 = 0$

(b) $76y^2 - 9x^2 + 72xy + 126x + 272y + 312 = 0$

(c) $76y^2 + 9x^2 + 72xy - 126x - 30y + 310 = 0$

(d) *None of the above*

Interpret (a) **Step I** Use the result $T^2 = SS_1$.

Let $\quad S = y^2 + 4y - 3x$

At point $P(1, 4)$,

$$S_1 = (4)^2 + 4(1) - 3(1) = 16 + 4 - 3 = 17$$

$$T = yy_1 + 2(y + y_1) - \frac{3}{2}(x + x_1)$$

$$T = y(4) + 2(y + 4) - \frac{3}{2}(x + 1)$$

$$= 4y + 2y + 8 - \frac{3}{2}x - \frac{3}{2}$$

$$= 6y + \frac{13}{2} - \frac{3}{2}x = \frac{1}{2}(12y - 3x + 13)$$

$\therefore$ Equation of pair of tangent is

$$SS_1 = T^2$$

$$(y^2 + 4y - 3x)(17) = \frac{1}{4}[(12y - 3x + 13)^2]$$

Step II Here, no need to use the result, that tangents are drawn from a directrix to the parabola are perpendicular.

Step III Simplify it,

$$68y^2 + 272y - 204x = 144y^2 + 9^2 + 169 - 72xy - 78x + 312$$

$$\Rightarrow \quad 76y^2 + 9x^2 - 72xy + 126x - 272y + 312 = 0$$

14.15 Chord of Contact

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(h, k)$, then QR is called chord of contact of the parabola $y^2 = 4ax$.

Let $\quad Q \equiv (x_1, y_1)$ and $R \equiv (x_2, y_2)$

Equation of the tangent PQ is

$$yy_1 = 2a(x + x_1) \quad \text{...(i)}$$

and equation of the tangent PR is

$$yy_2 = 2a(x + x_2) \quad \text{...(ii)}$$

Since, lines (i) and (ii) pass through (h, k), then

$$ky_1 = 2a(h + x_1) \quad \text{...(iii)}$$

and $$ky_2 = 2a(h + x_2) \quad \text{...(iv)}$$

Hence, it is clear $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on $yk = 2a(x + h)$ which is chord of contact QR.

Equation of the Chord Bisected at a Given Point

The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$\Rightarrow$ $$T = S_1$$

where, $$T = yy_1 - 2a(x + x_1)$$

and $$S_1 = y_1^2 - 4ax_1$$

***Sample* Problem 13** *The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{n}$, where n is*

(a) 2a (b) a
(c) a/2 (d) None of these

Interpret (a) Equation of QR (chord of contact) is

$$yy_1 = 2a(x + x_1)$$

$\Rightarrow$ $$yy_1 - 2a(x + x_1) = 0$$

$\because$ PM = length of perpendicular from $P(x_1, y_1)$ on QR

$$= \frac{|y_1y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}} = \frac{(y_1^2 - 4ax_1)}{\sqrt{y_1^2 + 4a^2}}$$

$$\left[\begin{array}{l}\text{since, } P(x_1, y_1) \text{ lies outside the parabola} \\ \therefore \quad y_1^2 - 4ax_1 > 0\end{array}\right]$$

Now, area of $\Delta PQR = \frac{1}{2} \cdot QR \cdot PM$

$$= \frac{1}{2} \cdot \frac{1}{|a|} \cdot \sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)} \cdot \frac{(y_1^2 - 4ax_1)}{\sqrt{(y_1^2 + 4a^2)}}$$

$$= \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}, \text{ if } a > 0$$

14.16 Diameter

The locus of the mid-points of a system of parallel chords of a conic is known as its diameter.

Equation of Diameter of a Parabola

Let $y = mx + c$ be a system of parallel chords of the parabola $y^2 = 4ax$

Here, m is a constant and c is a variable.

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in the points say, P and Q whose ordinates (say y_1 and y_2) are the roots of the equation

$\therefore$ $$y^2 = 4a\left(\frac{y - c}{m}\right)$$

$\Rightarrow$ $$my^2 - 4ay + 4ac = 0$$

$\therefore$ $$y_1 + y_2 = \frac{4a}{m}$$

Let $M(h, k)$ be the mid-point of PQ. Then,

$$k = \frac{y_1 + y_2}{2}$$

$\Rightarrow$ $$k = \frac{2a}{m}$$

Hence, the locus of (h, k) is $y = \frac{2a}{m}$

Check Point **2**

1. A tangent to a parabola $y^2 = 4ax$ is inclined at $\frac{\pi}{3}$ with the axis of the parabola. The point of contact is
   (a) $\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right)$ (b) $(3a, -2\sqrt{3}a)$
   (c) $(3a, 2\sqrt{3}a)$ (d) None of these
2. The tangents and normals at the ends of the latusrectum of a parabola form a
   (a) cyclic quadrilateral (b) rectangle
   (c) square (d) None of these
3. If the normal to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ cuts the parabola again at $Q(aT^2, 2aT)$, then
   (a) $-2 \le T \le 2$ (b) $T \in (-\infty, -8) \cup (8, \infty)$
   (c) $T^2 < 8$ (d) $T^2 \ge 8$
4. If tangents at A and B on the parabola $y^2 = 4ax$ intersect at point C, then ordinates of A, C and B are
   (a) always in AP (b) always in GP
   (c) always in HP (d) None of these
5. The locus of the middle points of the focal chords of parabola $y^2 = 4ax$ is
   (a) $y^2 = a(x - a)$ (b) $y^2 = 2a(x - a)$
   (c) $y^2 = 4a(x - a)$ (d) None of these
6. The equation of tangent to the parabola $y^2 = 9x$ which goes through the point (4, 10), is
   (a) $x + 4y + 1 = 0$ (b) $9x + 4y + 4 = 0$
   (c) $x - 4y + 36 = 0$ (d) $9x + 4y + 9 = 0$
7. If the tangent at the point P(2, 4) to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then the mid-point of the QR is
   (a) (2, 4) (b) (4,2)
   (c) (7, 9) (d) None of these
8. The tangent to the parabola $y^2 = 4ax$ at the point (a, 2a) makes with x-axis an angle equal to
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$
9. The normals at three points P, Q, R of the parabola $y^2 = 4ax$ meet in (h, k). The centroid of ΔPQR lies on
   (a) $x = 0$ (b) $y = 0$ (c) $x = -a$ (d) $y = a$
10. At what point on the parabola $y^2 = 4x$, the normal makes equal angles with the coordinate axes?
    (a) (4, 4) (b) (9, 6) (c) $(4, -4)$ (d) $(1, -2)$
11. The circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q. Then, the length PQ is equal to
    (a) 2 (b) $2\sqrt{2}$
    (c) 4 (d) None of these

WORKED OUT Examples

Example 1 *The equation of parabola whose focus and vertex are the points* (0, 2) *and* (0, 4) *respectively is*

(*a*) $x^2 + y = 32$ (*b*) $x^2 + 8y = 2$

(*c*) $x^2 + 8y = 32$ (*d*) *None of these*

Solution (*c*) Since, the focus and vertex of the parabola are on y-axis, therefore its directrix is parallel to x-axis and axis of parabola is y-axis. Let the equation of the directrix be $y = k$. The directrix meets the axis of the parabola at $(0, k)$. But vertex is the mid-point of the line segment joining the focus to the point where directrix meets axis of the parabola.

$$\therefore \quad \frac{k+2}{2} = 4 \Rightarrow k = 6$$

Thus, the equation of directrix is $y = 6$.

Let (x, y) be a point on the parabola, then

$$(x-0)^2 + (y-2)^2 = \left(\frac{|y-6|}{\sqrt{1^2}}\right)^2$$

$$\Rightarrow \quad x^2 + 8y = 32$$

Example 2 *The focus of the parabola* $y^2 - x - 2y + 2 = 0$ *is*

(*a*) $\left(\frac{1}{4}, 1\right)$ (*b*) $\left(\frac{5}{4}, 1\right)$ (*c*) $\left(1, \frac{5}{4}\right)$ (*d*) $\left(\frac{5}{4}, 0\right)$

Solution (*b*) We have, $y^2 - 2y = x - 2$

$$\Rightarrow \quad (y-1)^2 = 1(x-1)$$

$$\Rightarrow \quad Y^2 = X$$

where, $Y = y - 1$ and $X = x - 1$

Here, $a = \frac{1}{4}$

$\therefore$ Focus is $X = a, Y = 0$

$$\Rightarrow \quad x - 1 = \frac{1}{4}, y - 1 = 0$$

$\Rightarrow$ Focus of parabola is $\left(\frac{5}{4}, 1\right)$.

Example 3 *The equation of the parabola, the extremities of whose latusrectum are* (1, 2) *and* (1, – 4), *is*

(*a*) $(x+1)^2 = 3(2y-5)$ (*b*) $(y+1)^2 = 3(2x-5)$

(*c*) $(y+1)^2 = -3(2x+5)$ (*d*) *None of these*

Solution (*d*) Let the extremities be $L = (1, 2)$ and $L' = (1, -4)$

$$\therefore \quad \text{Slope of } L'L = \frac{2+4}{1-1} = \frac{6}{0} = \infty$$

Hence, latusrectum $L'L$ is perpendicular to x-axis. Therefore, axis of the parabola will be parallel to x-axis and tangent at the vertex will be parallel to y-axis.

Let the equation of the parabola be

$$(y-\beta)^2 = 4a(x-\alpha)$$

Now, $L'L = |4a|$

$$\therefore \quad 6 = |4a| \Rightarrow 4a = \pm 6 \quad \text{...(i)}$$

Hence, from Eq. (i), equation of parabola becomes

$$(y-\beta)^2 = \pm 6(x-\alpha) \quad \text{...(ii)}$$

Since, L and L' lie on Eq. (ii), therefore

$$(2-\beta)^2 = \pm 6(1-\alpha) \quad \text{...(iii)}$$

and $$(4+\beta)^2 = \pm 6(1-\alpha) \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$(2-\beta)^2 = (4+\beta)^2$$

$$\Rightarrow \quad 12\beta = -12 \Rightarrow \beta = -1$$

$\therefore$ From Eq. (iii),

$$9 = \pm 6(1-\alpha) \Rightarrow \alpha = \frac{-1}{2}, \frac{5}{2}$$

On putting the values of α and β in Eq. (ii), we get

$$(y+1)^2 = 6\left(x + \frac{1}{2}\right) \Rightarrow (y+1)^2 = 3(2x+1)$$

and $$(y+1)^2 = -6\left(x - \frac{5}{2}\right) \Rightarrow (y+1)^2 = -3(2x-5)$$

Example 4 *If* $y = mx + c$ *touches the parabola* $y^2 = 4a(x + a)$, *then* $am + \frac{a}{m}$ *is equal to*

(*a*) y (*b*) x

(*c*) c (*d*) *None of these*

Solution (*c*) If the line $y = mx + c$ touches $y^2 = 4a(x+a)$, then the equation $(mx + c)^2 = 4a(x + a)$ has equal roots. By which, we get $c = am + \frac{a}{m}$.

Example 5 *The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is*

(a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

Solution (b) Any line through origin is $y = mx$. Since, it is a tangent to $y^2 = 4a(x-a)$, it will cut it in two coincident points.

$\therefore$ Roots of $m^2x^2 - 4ax + 4a^2$ are equal.

i.e., $b^2 - 4ac = 0$

$\Rightarrow$ $16a^2 - 16a^2m^2 = 0$

$\Rightarrow$ $m^2 = 1$ or $m = 1, -1$

$\therefore$ Product of slopes $= -1$

Hence, required angle is right angle.

Example 6 *Any tangent to a parabola $y^2 = 4ax$ and perpendicular to it from the focus meet on the line $x = 0$, is*

(a) $x = 0$ (b) $y = 0$ (c) $x = a$ (d) $y = b$

Solution (a) Equation of tangent is

$$ty = x + at^2 \quad \text{...(i)}$$

Its slope $= \frac{1}{t}$

Any line perpendicular to tangent and passing through the focus, is $y - 0 = -t(x - a)$.

$$\Rightarrow \quad y = -tx + at \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we find the coordinates of the point of intersection as $(0, at)$ and clearly it lies on the line $x = 0$.

Example 7 *The common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is*

(a) $x + y + a = 0$ (b) $x + y = a$

(c) $x + y = 0$ (d) *None of these*

Solution (a) The equation of any tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$. If it touches $x^2 = 4ay$, the equation $x^2 = 4a\left(mx + \frac{a}{m}\right)$ has equal roots.

$\Rightarrow$ $mx^2 - 4am^2x - 4a^2 = 0$ has equal roots.

$\Rightarrow$ $16a^2m^4 = -16a^2m \Rightarrow m = -1$

On putting $m = -1$ in $y = mx + \frac{a}{m}$, we get

$$y = -x - a \Rightarrow x + y + a = 0$$

Example 8 *If a chord which is normal to the parabola at one end subtend a right angle at the vertex, then angle to the axis is*

(a) $\tan^{-1}\sqrt{2}$ (b) 0

(c) π (d) $\pi/2$

Solution (a) The equation to any chord which is normal, is

$$y = mx - 2am - am^3$$

$$\Rightarrow \quad mx - y = 2am + am^3$$

Let the equation of parabola is $y^2 = 4ax$.

The straight lines joining the origin to the intersection of these two, given by the equation

$$y^2(2am + am^3) - 4ax(mx - y) = 0$$

If these be at right angles, then

$$2am + am^3 - 4am = 0$$

$$(\because \text{coefficient of } y^2 + \text{coefficient of } x^2 = 0)$$

$\Rightarrow$ $m = \pm\sqrt{2}$

$\Rightarrow$ $\tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}\sqrt{2}$

Example 9 *On the parabola $y^2 = 4ax$, three points E, F, G are taken, so that their ordinates are in geometrical progression. Then, the tangents at E and G intersect on the ordinate of*

(a) E (b) G

(c) F (d) *None of these*

Solution (c) Given parabola is

$$y^2 = 4ax \quad \text{...(i)}$$

Let the coordinates of E, F and G be respectively $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.

Since, ordinates of E, F and G are in GP.

$\therefore$ $(2at_2)^2 = (2at_1)(2at_3) \Rightarrow t_2^2 = t_1t_3$ ...(ii)

The tangents at E and G are

$$t_1y = x + at_1^2 \quad \text{...(iii)}$$

and $$t_3y = x + at_3^2 \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $x = at_1t_3 = at_2^2$. Since, the coordinate of the point of intersection is at_2^2, the point lies on line $x = at_2^2$ *i.e.,* on the ordinate of $F(at_2^2, 2at_2)$.

Example 10 *A variable chord PQ of the parabola $y^2 = 4ax$, subtends a right angle at the vertex, then the locus of the points of intersection of the normal at P and Q is*

(a) *a parabola* (b) *a hyperbola*

(c) *a circle* (d) *None of these*

Solution (a) Let P be $(at_1^2, 2at_1)$ and Q be $(at_2^2, 2at_2)$. Since, PQ subtends a right angle at the vertex (0, 0).

Hence, $$t_1t_2 = -4 \quad \text{...(i)}$$

If (h, k) be the point of intersection of normals at P and Q, then

$$h = 2a + a(t_1^2 + t_2^2 + t_1t_2) \quad \text{...(ii)}$$

and $$k = -at_1t_2(t_1 + t_2) \quad \text{...(iii)}$$

In order to find the locus of (h, k), we have to eliminate t_1 and t_2 between Eqs. (i), (ii) and (iii)

$$k = 4a(t_1 + t_2) \quad \text{...(iv)}$$

[from Eqs. (i) and (iii)]

and $$h - 2a = a[(t_1 + t_2)^2 - t_1t_2]$$

$$\Rightarrow \quad h - 2a = a\left[\frac{k^2}{16a^2} + 4\right] \quad \text{[from Eq. (iv)]}$$

$$\Rightarrow \quad h - 6a = \frac{k^2}{16a}$$

Hence, the required locus is $y^2 = 16a(x - 6a)$.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Basic Terms of Parabola

1. Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ $(abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0)$ represents a parabola, if

(a) $h^2 = ab$ (b) $h^2 > ab$
(c) $h^2 < ab$ (d) None of these

2. The equation $(x-2)^2 + (y-3)^2 = \left(\frac{3x+4y-2}{5}\right)^2$ represents

(a) a parabola (b) a pair of straight lines
(c) an ellipse (d) a hyperbola

3. The equation of latusrectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$, then length of the latusrectum is

(a) $4\sqrt{2}$ (b) $2\sqrt{2}$
(c) 8 (d) $8\sqrt{2}$

4. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with the directrix

(a) $x = -a$ (b) $x = -\frac{a}{2}$
(c) $x = 0$ (d) $x = \frac{a}{2}$

5. The coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4, is [NCERT Exemplar]

(a) (– 2, – 4), (2, – 4) (b) (2, 4), (2, – 4)
(c) (1, 4), (2, – 4) (d) (2, – 4), (– 2, 4)

6. If the parabola $y^2 = 4ax$ passes through the point (3, 2), then the length of its latusrectum is [NCERT Exemplar]

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$
(c) $\frac{1}{3}$ (d) 4

7. If the equation of parabola is $x^2 = -9y$, then equation of directrix and length of latusrectum are [NCERT]

(a) $y = -\frac{9}{4}, 8$ (b) $x = \frac{9}{4}, 9$
(c) $y = \frac{9}{4}, 9$ (d) None of these

8. If $(9a, 6a)$ is a point bounded in region formed by parabola $y^2 = 16x$ and $x = 9$, then

(a) $a \in (0, 1)$ (b) $a < \frac{1}{4}$
(c) $a < 1$ (d) $0 < a < 4$

9. The length of the latusrectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x - 12y + 17)^2$ is

(a) 14/13 (b) 12/13
(c) 28/13 (d) None of these

10. The equation $y^2 - 8y - x + 19 = 0$ represents

(a) a parabola whose focus is $\left(\frac{1}{4}, 0\right)$ and directrix is $x = \frac{-1}{4}$
(b) a parabola whose vertex is (3, 4) and directrix is $x = \frac{11}{4}$
(c) a parabola whose focus is $\left(\frac{13}{4}, 4\right)$ and vertex is (0, 0)
(d) a curve which is not a parabola

11. The parametric coordinates of any point on the parabola $y^2 = 4ax$ can be

(a) $(a - at^2, -2at)$ (b) $(a - at^2, 2at)$
(c) $(a\sin^2 t, -2a\sin t)$ (d) $(a\sin t, -2a\cos t)$

12. The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. The cartesian equation of its directrix is

(a) $x = 0$
(b) $x + 1 = 0$
(c) $y = 0$
(d) None of the above

13. The latusrectum of the parabola $y^2 = 4ax$ whose focal chord is PSQ such that $SP = 3$ and $SQ = 2$, is given by
(a) $\frac{24}{5}$ (b) $\frac{12}{5}$ (c) $\frac{6}{5}$ (d) $\frac{1}{5}$

14. If a focal chord of the parabola $y^2 = ax$ is $2x - y - 8 = 0$, then the equation of the directrix is
(a) $x + 4 = 0$ (b) $x - 4 = 0$
(c) $y - 4 = 0$ (d) $y + 4 = 0$

15. The position of the point $(-2, 2)$ with respect to the parabola $y^2 - 4y + 9x + 13 = 0$ is
(a) inside (b) outside
(c) on (d) None of these

16. The length of the chord of the parabola $x^2 = 4y$ passing through the vertex and having slope $\cot\alpha$ is
(a) $4\cos\alpha \operatorname{cosec}^2\alpha$ (b) $4\tan\alpha \sec\alpha$
(c) $4\sin\alpha \sec^2\alpha$ (d) None of these

17. The line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$. Then, one of the value of k is
(a) $\frac{1}{8}$ (b) 8 (c) 4 (d) $\frac{1}{4}$

Equation of Parabola and Intersection of Two Curves

18. If the focus of a parabola is $(0, -3)$ and its directrix is $y = 3$ and its equation is [NCERT Exemplar]
(a) $x^2 = -12y$ (b) $x^2 = 12y$
(c) $y^2 = -12x$ (d) $y^2 = 12x$

19. If the vertex of the parabola is the point $(-3, 0)$ and the directrix is the line $x + 5 = 0$, then its equation is [NCERT Exemplar]
(a) $y^2 = 8(x + 3)$ (b) $x^2 = 8(y + 3)$
(c) $y^2 = -8(x + 3)$ (d) $y^2 = 8(x + 5)$

20. If the point $(0, 4)$ and $(0, 2)$ are respectively the vertex and focus of a parabola, then the equation of the parabola is [NCERT Exemplar]
(a) $x^2 + 8y = 16$ (b) $x^2 + 8y = 32$
(c) $x^2 + 16y = 30$ (d) $x^2 - 8y = 32$

21. The equation of parabola having vertex $(0, 0)$ passing through $(2, 3)$ and axis is along x-axis is [NCERT]
(a) $x^2 = \frac{9}{2}y$ (b) $y^2 = \frac{9}{2}x$
(c) $y^2 = -\frac{9}{2}x$ (d) $x^2 = -\frac{9}{2}y$

22. The angle made by a double ordinate of length $8a$ at the vertex of the parabola $y^2 = 4ax$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

23. The angle of intersection between the curves $x^2 = 4(y + 1)$ and $x^2 = -4(y + 1)$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) 0 (d) $\frac{\pi}{2}$

Tangent and Normal to the Parabola

24. If the line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$, then point of contact is
(a) $(a\tan^2\alpha, -2a\tan\alpha)$ (b) $(a\tan^2\alpha, 2a\tan\alpha)$
(c) $(a\tan\alpha, 2a\tan\alpha)$ (d) None of these

25. Two perpendicular tangents to $y^2 = 4ax$ always intersect on the line, if
(a) $x = a$ (b) $x + a = 0$
(c) $x + 2a = 0$ (d) $x + 4a = 0$

26. The equation of the tangent to the parabola $y^2 = 9x$ which goes through the point $(4, 10)$, is
(a) $x + 4y + 1 = 0$ (b) $9x + 4y + 4 = 0$
(c) $x + 4y + 36 = 0$ (d) $9x - 4y + 4 = 0$

27. The tangents drawn from the ends of latusrectum of $y^2 = 12x$ meets at
(a) directrix (b) vertex
(c) focus (d) None of these

28. If tangents at A and B on the parabola $y^2 = 4ax$ intersect at point C, then ordinates of A, C and B are
(a) always in AP (b) always in GP
(c) always in HP (d) None of these

29. The number of real tangents that can be drawn to the curve $y^2 + 2xy + x^2 + 2x + 3y + 1 = 0$ from the point $(1, -2)$ is
(a) one (b) two
(c) zero (d) None of these

30. Number of common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is
(a) 4 (b) 3 (c) 2 (d) 1

31. If the tangent to the parabola $y^2 = ax$ makes an angle of 45° with x-axis, then the point of contact is
(a) $\left(\frac{a}{2}, \frac{a}{2}\right)$ (b) $\left(\frac{a}{4}, \frac{a}{4}\right)$
(c) $\left(\frac{a}{2}, \frac{a}{4}\right)$ (d) $\left(\frac{a}{4}, \frac{a}{2}\right)$

32. Tangent at the vertex divides the distance between directrix and latusrectum in the ratio
(a) 1 : 1
(b) 1 : 2
(c) depends on directrix and focus
(d) None of the above

33. The tangent drawn at any point P to the parabola $y^2 = 4ax$ meets the directrix at the point K, then the angle which KP subtends at its focus is
(a) 30° (b) 45° (c) 60° (d) 90°

34. If the tangents at P and Q on the parabola meet in T, then SP, ST and SQ are in
(a) AP (b) GP
(c) HP (d) None of these

35. AB, AC are tangents to a parabola $y^2 = 4ax$; p_1, p_2, p_3 are the lengths of the perpendiculars from A, B, C on any tangent to the curve, then p_2, p_1, p_3 are in
(a) AP (b) GP
(c) HP (d) None of these

36. Consider the following statements
I. The equation of the parabola whose focus is at the origin is $y^2 = 4a(x + a)$.
II. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if $ln = am^2$.
Which of these is/are correct ?
(a) Only (I) (b) Only (II)
(c) Both of these (d) None of these

37. The number of distinct normals that can be drawn to parabola $y^2 = 16x$ from the point (2, 0), is
(a) 1 (b) 2 (c) 3 (d) 0

38. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola again at the point
(a) $(-18, -12)$ (b) $(-18, 12)$
(c) $(18, 12)$ (d) $(18, -12)$

39. Three distinct normals to the parabola $y^2 = x$ are drawn through a point $(c, 0)$, then
(a) $c = \frac{1}{4}$ (b) $c = \frac{1}{2}$
(c) $c > \frac{1}{2}$ (d) None of these

40. The arithmetic mean of the ordinates of the feet of the normals from (3, 5) to the parabola $y^2 = 8x$ is
(a) 4 (b) 0
(c) 8 (d) None of these

41. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of ordinates of P and Q is
(a) $4a^2$ (b) $2a^2$
(c) $-4a^2$ (d) $8a^2$

42. If the tangent and normal at any point P of a parabola meet the axes in T and G respectively, then
(a) $ST \neq SG = SP$ (b) $ST - SG \neq SP$
(c) $ST = SG = SP$ (d) $ST = SG \cdot SP$

43. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$, then
(a) $|b| > \frac{1}{2\sqrt{2}}$ (b) $|b| < \frac{1}{2\sqrt{2}}$
(c) $|b| > \frac{1}{\sqrt{2}}$ (d) $|b| < \frac{1}{\sqrt{2}}$

Chord of Contact and Diameter

44. Equation of chord of the parabola $y^2 = 16x$ whose mid-point is (1, 1), is
(a) $x + y = 2$ (b) $x - y = 0$
(c) $8x + y = 9$ (d) $8x - y = 7$

45. The chord AB of the parabola $y^2 = 4ax$ cuts the axis of the parabola at C. If $A = (at_1^2, 2at_1)$, $B = (at_2^2, 2at_2)$ and $AC : AB = 1 : 3$, then
(a) $t_2 = 2t_1$ (b) $t_2 + 2t_1 = 0$
(c) $t_1 + 2t_2 = 0$ (d) None of these

46. AB is a chord of the parabola $y^2 = 4ax$ with vertex A, BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is
(a) a (b) $2a$
(c) $4a$ (d) $8a$

47. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord are
(a) $\{-1, 1\}$ (b) $\{-2, 2\}$
(c) $\left\{-2, \frac{1}{2}\right\}$ (d) $\left\{2, -\frac{1}{2}\right\}$

Miscellaneous

48. If a parabolic reflector is 20 cm in diameter and 5 cm deep, then the focus is [NCERT]
(a) (0, 5) (b) (5, 0)
(c) (0, –5) (d) (–5, 0)

49. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latusrectum. [NCERT]
(a) 19 sq units (b) 18 sq units
(c) 20 sq units (d) 17 sq units

50. Find the length of the line segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line segment makes an angle θ to the x-axis. [NCERT Exemplar]
(a) $\frac{2a \cos\theta}{\sin^2\theta}$ (b) $\frac{4a \cos\theta}{\sin^2\theta}$
(c) $\frac{4a \cos\theta}{3 \sin^2\theta}$ (d) None of these

51. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$, is
(a) $x^2 + y^2 - 2xy - 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$
(c) $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$
(d) $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

52. The parabola $y^2 = 8x$ and the circle $x^2 + y^2 = 2$
(a) have only two common tangents which are mutually perpendicular
(b) have only two common tangents which are parallel to each other
(c) have infinitely many common tangents
(d) does not have any common tangent

53. On the parabola $y = x^2$, the point atleast distance from the straight line $y = 2x - 4$ is
(a) (1, 1) (b) (1, 0)
(c) (1, −1) (d) (0, 0)

54. Angle between the tangents drawn to $y^2 = 4x$ at the points where it is intersected by the line $y = x - 1$ is equal to
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

55. The circle $x^2 + y^2 + 2\lambda x = 0$, $\lambda \in R$, touches the parabola $y^2 = 4x$ externally. Then,
(a) $\lambda > 0$ (b) $\lambda < 0$
(c) $\lambda > 1$ (d) None of these

56. The shortest distance between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 6x - 12y + 20 = 0$ is
(a) $4\sqrt{2} - 5$ (b) 0 (c) $3\sqrt{2} + 5$ (d) 1

57. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$, then
(a) $\lambda \in (0, 1)$ (b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$ (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. The shortest distance between the parabolas $y^2 = 4x$ and $y^2 = 2x - 6$ is
(a) 2 (b) $\sqrt{5}$
(c) 3 (d) None of these

2. A parabola is drawn with focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is
(a) $x^2 - 6x - 8y + 25 = 0$ (b) $y^2 - 8x - 6y + 25 = 0$
(c) $x^2 - 6x + 8y - 25 = 0$ (d) $x^2 + 6x - 8y - 25 = 0$

3. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is
(a) $x^2 + y^2 - 2xy - 4x - 4y - 4 = 0$
(b) $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$
(c) $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$
(d) $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

4. Axis of a parabola is $y = x$ and vertex and focus are at a distance $\sqrt{2}$ and $2\sqrt{2}$ respectively from the origin. Then, equation of the parabola is
(a) $(x - y)^2 = 8(x + y - 2)$ (b) $(x + y)^2 = 2(x + y - 2)$
(c) $(x - y)^2 = 4(x + y - 2)$ (d) $(x + y)^2 = 2(x - y + 2)$

5. If y_1, y_2 and y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$, then its area is
(a) $\frac{1}{2a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$
(b) $\frac{1}{4a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$
(c) $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$
(d) None of the above

6. The locus of a point on the variable parabola $y^2 = 4ax$, whose distance from focus is always equal to k, is equal to (a is parameter)
(a) $4x^2 + y^2 - 4kx = 0$ (b) $x^2 + y^2 - 4kx = 0$
(c) $2x^2 + 4y^2 - 8kx = 0$ (d) $4x^2 - y^2 + 4kx = 0$

7. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is
(a) $2\sqrt{3}$ (b) $4\sqrt{3}$
(c) $3\sqrt{2}$ (d) $2\sqrt{5}$

8. The ΔPQR of area A is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modules of the difference of the ordinates of the points Q and R is
(a) $\frac{A}{2a}$ (b) $\frac{A}{a}$
(c) $\frac{2A}{a}$ (d) $\frac{4A}{a}$

More Than One Correct Option

9. Let P be a point whose coordinates differ by unity and the point does not lie on any of the axes of reference. If the parabola $y^2 = 4x + 1$ passes through P, then the ordinate of P may be
(a) 3 (b) −1
(c) 5 (d) 1

10. Parabola $y^2 = 4x$ and the circle having it's centre at (6, 5) intersect at right angle. Possible point of intersection of these curves can be
(a) (9, 6) (b) $(2, \sqrt{8})$ (c) (4, 4) (d) $(3, 2\sqrt{3})$

Comprehension Based Questions

Passage

Directions (Q. Nos. 11-13) *Tangent to the parabola $y = x^2 + ax + 1$, at the point of intersection of y-axis also touches the circle $x^2 + y^2 = r^2$. Also, no point of the parabola is below the x-axis.*

11. The radius of circle when a attains its maximum value
(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) 1 (d) $\sqrt{5}$

12. The slope of the tangent when radius of the circle is maximum is
(a) 1 (b) 1 (c) 0 (d) 2

13. The minimum area bounded by the tangent and the coordinate axes is
(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Assertion and Reason

Directions (Q. Nos. 14-18) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

14. **Statement I** If normal at the ends of double ordinate $x = 4$ of parabola $y^2 = 4x$ meet the curve again at P and P' respectively, then $PP' = 12$ units.

Statement II If normal at t_1 of $y^2 = 4ax$ meet the parabola again at t_2, then $t_2 = t_1 - \frac{2}{t_1}$.

15. **Statement I** The lines from the vertex to the two extremities of a focal chord of the parabola $y^2 = 4ax$ are at an angle of $\frac{\pi}{2}$.

Statement II If extremities of focal chord of parabola are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, then $t_1t_2 = -1$.

16. **Statement I** The latusrectum of a parabola is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$, then the equation of directrix of the parabola is $4x - 3y + 8 = 0$.

Statement II If P be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then $(PM)^2 = (\text{latusrectum})(PN)$

17. **Statement I** Slope of tangents drawn from (4, 10) to parabola $y^2 = 9x$ are $\frac{1}{4}, \frac{9}{4}$.

Statement II Two tangents can be drawn to parabola from any point lying outside parabola.

18. **Statement I** In parabola $y^2 = 4ax$, the circle drawn taking focal radii as diameter touches y-axis.

Statement II The portion of the tangent intercepted between point of contact and directrix subtends 90° angle at focus.

Previous Years' Questions

19. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is **[AIEEE 2010]**
(a) $x = 1$ (b) $2x + 1 = 0$ (c) $x = -1$ (d) $2x - 1 = 0$

20. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then, the vertex of the parabola is at **[AIEEE 2008]**
(a) (2, 0) (b) (0, 2) (c) (1, 0) (d) (0, 1)

21. The equation of the tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is **[AIEEE 2007]**
(a) (0, 2) (b) (2, 4)
(c) (–2, 0) (d) (–1, 1)

22. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is **[AIEEE 2006]**
(a) $xy = \frac{35}{36}$ (b) $xy = \frac{64}{105}$ (c) $xy = \frac{105}{64}$ (d) $xy = \frac{3}{4}$

23. Let P be the point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of mid-point of PQ is **[AIEEE 2005]**
(a) $x^2 - 4y + 2 = 0$ (b) $x^2 + 4y + 2 = 0$
(c) $y^2 + 4x + 2 = 0$ (d) $y^2 - 4x + 2 = 0$

24. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then **[AIEEE 2004]**
(a) $d^2 + (2b + 3c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b - 3c)^2 = 0$ (d) $d^2 + (3b + 2c)^2 = 0$

Answers

Check Point 1

1. (b) **2.** (a) **3.** (a) **4.** (c) **5.** (b) **6.** (d) **7.** (a) **8.** (c) **9.** (b)

Check Point 2

1. (a) **2.** (c) **3.** (d) **4.** (a) **5.** (b) **6.** (c) **7.** (b) **8.** (b) **9.** (b) **10.** (d)
11. (c)

Round I

1. (a)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (c)	**10.** (b)
11. (c)	**12.** (a)	**13.** (a)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (b)
21. (b)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (d)	**27.** (a)	**28.** (a)	**29.** (c)	**30.** (d)
31. (d)	**32.** (a)	**33.** (d)	**34.** (b)	**35.** (b)	**36.** (c)	**37.** (a)	**38.** (d)	**39.** (c)	**40.** (b)
41. (d)	**42.** (c)	**43.** (b)	**44.** (d)	**45.** (b)	**46.** (c)	**47.** (a)	**48.** (b)	**49.** (b)	**50.** (b)
51. (c)	**52.** (a)	**53.** (a)	**54.** (d)	**55.** (a)	**56.** (a)	**57.** (a)			

Round II

1. (b)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (a)	**7.** (d)	**8.** (c)	**9.** (a,c)	**10.** (a,c)
11. (b)	**12.** (c)	**13.** (d)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (c)	**23.** (d)	**24.** (a)						

the Guidance

Round I

1. Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents parabola, if $h^2 = ab$.

2. Point (x, y) is the set of points equidistant from point (2, 3) and the line $3x + 4y - 2 = 0$. So, the given equation represents a parabola.

3. Since, the equation of latusrectum and equation of tangent both are parallel and they lie in the same side of the origin

$$\therefore \quad a = \left|\frac{-8+12}{\sqrt{1^2+1^2}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$\therefore$ Length of latusrectum $= 4a = 4(2\sqrt{2}) = 8\sqrt{2}$

4. Let any point on the parabola be $(at^2, 2at)$.

If the equation of parabola is $y^2 = 4ax$, then focus is $(a, 0)$.

Let the locus of a point be (α, β), if it is a mid-point.

$$\therefore \quad \alpha = \frac{at^2 + a}{2}, \quad \beta = \frac{2at + 0}{2}$$

$$\Rightarrow \quad 2\alpha = at^2 + a, \quad \beta = at$$

$$\therefore \quad 2\alpha = a\left(\frac{\beta}{a}\right)^2 + a$$

$$\Rightarrow \quad 2a\alpha = \beta^2 + a^2$$

$$\Rightarrow \quad \beta^2 = -a^2 + 2a\alpha$$

$$\Rightarrow \quad \beta^2 = \frac{4a}{2}\left(\alpha - \frac{a}{2}\right)$$

$\therefore$ The locus is $y^2 = \frac{4a}{2}\left(x - \frac{a}{2}\right)$.

The directrix is $X = -\frac{a}{2}$

$$\Rightarrow \quad x - \frac{a}{2} = -\frac{a}{2}$$

$$\Rightarrow \quad x = 0$$

5. The given equation of parabola is

$$y^2 = 8x$$

Here, $a = 2$

$\therefore$ Equation of directrix is $x + 2 = 0$

Let $P(x_1, y_1)$ be the point on the parabola.

$$\therefore \quad y_1^2 = 8x_1 \quad \ldots(i)$$

Also, $SP = 4 = PM$, where S and M are focus and point on directrix.

$$\therefore \quad x_1 + 2 = 4$$

$$\Rightarrow \quad x_1 = 2$$

On putting the value of x_1 in Eq. (i), we get

$$y_1^2 = 8 \times 2 = 16$$

$\Rightarrow$ $$y_1 = \pm 4$$

Hence, the coordinates of required points are (2, 4) and (2, – 4).

6. Since, the parabola $y^2 = 4ax$ passes through the point (3, 2).

$\therefore$ $$(2)^2 = 4a(3)$$

$\Rightarrow$ $$4 = 12a \Rightarrow a = \frac{1}{3}$$

$\therefore$ Length of latusrectum = $4a$

$$= 4 \times \frac{1}{3} = \frac{4}{3}$$

7. Given equation of parabola is $x^2 = -9y$, which is of the form $x^2 = -4ay$ *i.e.*, focus lies on the negative direction of y-axis

Here, $$4a = 9 \Rightarrow a = \frac{9}{4}$$

$\therefore$ Equation of directrix, $y = a$

$\Rightarrow$ $$y = \frac{9}{4}$$

and length of latusrectum = $4a = 9$

8. Since, the point $(9a, 6a)$ is bounded in the region formed by the parabola $y^2 = 16x$ and $x = 9$, then

$$y^2 - 16x < 0, \quad x - 9 < 0$$

$\Rightarrow$ $$36a^2 - 16 \cdot 9a < 0, \quad 9a - 9 < 0$$

$\Rightarrow$ $$36a(a-4) < 0, \quad a < 1$$

$$0 < a < 4, \quad a < 1 \Rightarrow 0 < a < 1$$

9. Given equation of parabola is rewritten as

$$169\{(x-1)^2 + (y-3)^2\} = (13)^2\left\{\left(\frac{5x-12y+17}{13}\right)^2\right\}$$

$\Rightarrow$ $$(x-1)^2 + (y-3)^2 = \left(\frac{5x-12y+17}{13}\right)^2$$

$\Rightarrow$ $$SP = PM$$

$\therefore$ Focus is (1, 3) and equation of directrix is

$$5x - 12y + 17 = 0$$

The distance of the focus from directrix $= \left|\frac{5-36+17}{\sqrt{25+144}}\right| = \frac{14}{13}$

$\therefore$ Length of latusrectum $= 2 \times \frac{14}{13} = \frac{28}{13}$

10. Given equation is

$$y^2 - 8y - x + 19 = 0$$

$\Rightarrow$ $$(y-4)^2 = x - 3$$

$\Rightarrow Y^2 = 4AX$, where $Y = y - 4$, $A = \frac{1}{4}$ and $X = x - 3$

$\therefore$ Focus is $(A, 0) = \left(\frac{1}{4}, 0\right) = \left(\frac{13}{4}, 4\right)$

Vertex is $(0, 0) = (3, 4)$

Directrix is $x = -A \Rightarrow x - 3 = -\frac{1}{4}$

$\Rightarrow$ $$x = \frac{11}{4}$$

11. If $x = a\sin^2 t \Rightarrow y^2 = 4a(a\sin^2 t)$

$\Rightarrow$ $$y = \pm 2a\sin t$$

$\therefore$ Option (c) is correct.

12. Given parametric equation of parabola is

$$x = t^2 + 1, \quad y = 2t + 1$$

$\Rightarrow$ $$x = \left(\frac{y-1}{2}\right)^2 + 1$$

$\Rightarrow$ $$(y-1)^2 = 4(x-1)$$

$\Rightarrow$ $$Y^2 = 4X$$

Vertex is (1, 1), length of latusrectum = 4

Clearly, equation of directrix is

$$X = -1 \Rightarrow x - 1 = -1 \Rightarrow x = 0$$

13. Since, the semi-latusrectum of a parabola is HM of segments of a focal chord.

$\therefore$ Semi-latusrectum $= \frac{2\,SP \cdot SQ}{SP + SQ} = \frac{2 \times 3 \times 2}{3+2} = \frac{12}{5}$

$\Rightarrow$ Latusrectum of the parabola $= 2 \times$ Semi-latusrectum $= \frac{24}{5}$

14. Since, focal chord of parabola $y^2 = ax$ is $2x - y - 8 = 0$

$\because$ This chord passes through focus *i.e.*, $\left(\frac{a}{4}, 0\right)$

$\therefore$ $$2 \cdot \frac{a}{4} - 0 - 8 = 0$$

$\Rightarrow$ $$a = 16$$

$\therefore$ Directrix is $x = -4 \Rightarrow x + 4 = 0$

15. Let $S \equiv y^2 - 4y + 9x + 13 = 0$

At point (– 2, 2),

$$S_1 = (2)^2 - 4(2) + 9(-2) + 13$$

$$= 4 - 8 - 18 + 13 = -9 < 0$$

Hence, given point lies inside the parabola.

16. Let A be the vertex of the parabola and AP is chord of parabola such that slope of AP is $\cot\alpha$.

Let coordinates of P be $(2t, t^2)$ which is a point on the parabola.

$\therefore$ $$\text{Slope of } AP = \frac{t}{2}$$

$\Rightarrow$ $$\cot\alpha = \frac{t}{2}$$

$\Rightarrow$ $$t = 2\cot\alpha$$

In ΔAPB, $AP = \sqrt{4t^2 + t^4}$

$$= t\sqrt{4 + t^2}$$

$$\therefore \quad AP = 2\cot\alpha\sqrt{4(1+\cot^2\alpha)}$$
$$= 2\cot\alpha\sqrt{4\,\text{cosec}^2\alpha} = 4\cot\alpha\,\text{cosec}\,\alpha$$
$$= 4\frac{\cos\alpha}{\sin\alpha}\text{cosec}\,\alpha = 4\cos\alpha\,\text{cosec}^2\alpha$$

17. **Hint** Given equation can be rewritten as
$$y^2 = \frac{4k}{4}\left(x - \frac{8}{k}\right).$$
The standard equation of parabola is
$$Y^2 = 4AX, \text{ where } A = \frac{k}{4}$$
$\therefore$ Equation of directrix is $X + \frac{k}{4} = 0$

18. Given focus, $S = (0, -3)$ and directrix $y - 3 = 0$. By definition of parabola,
$$PM^2 = PS^2$$
$$\therefore \quad \left[\frac{y-3}{\sqrt{1}}\right]^2 = (x-0)^2 + (y+3)^2$$
$$\Rightarrow \quad y^2 + 9 - 6y = x^2 + y^2 + 9 + 6y$$
$$\Rightarrow \quad x^2 = -12y$$

19. **Hint** Given vertex $(-3, 0)$ and directrix $x + 5 = 0$

Let the coordinate of focus be $5(a, 0)$.

We know vertex is the mid-point of directrix and focus.

$$\therefore \quad (-3, 0) = \left(\frac{-5+a}{2}, 0\right)$$
$$\Rightarrow \quad -3 = \frac{-5+a}{2} \Rightarrow a = -1$$
$\therefore$ Coordinate of focus is $(-1, 0)$.

By definition of parabola,
$$PM^2 = PS^2$$

20. Given, vertex $= (0, 4)$ and focus $= (0, 2)$

Here, we see that in both coordinates x-coordinate is zero. It means axis of parabola, is on y-axis and the parabola is open downward.

Let the equation of parabola be
$$(x-0)^2 = -4a(y-4) \quad \ldots(\text{i})$$
$\because a =$ distance between focus and vertex
$$= \sqrt{0^2 + (2-4)^2} = 2$$
On putting $a = 2$ in Eq. (i), we get
$$x^2 = -4 \times 2(y-4)$$
$$\Rightarrow \quad x^2 + 8y = 32 \text{ or } x^2 + 8y - 32 = 0$$

21. Given vertex $= (0, 0)$, point $= (2, 3)$ and axis $= x$-axis.

Since, point $(2, 3)$ lies in first quadrant and axis is x-axis.

Hence, equation of parabola will be of the form $y^2 = 4ax$, which passes through $(2, 3)$

i.e., $\quad (3)^2 = 4a \times (2) \Rightarrow a = \frac{9}{8}$

$\therefore$ Required equation of parabola is
$$y^2 = 4\left(\frac{9}{8}\right)x \Rightarrow y^2 = \frac{9}{2}x$$

22. Given equation of parabola is $y^2 = 4ax$. Since, $AB = 8a$, it means ordinate of A and B respectively $4a$ and $-4a$. General point on this parabola is $(at^2, 2at) \Rightarrow t = \pm 2$

So, $\quad at^2 = 4a$

$\therefore \quad OM = 4a, \quad AM = 4a$

So, $\quad \angle AOM = 45°$

$\therefore$ The $\angle AOB$ is $90°$.

23. The point of intersection between the curves $x^2 = 4(y+1)$ and $x^2 = -4(y+1)$ is $(0, -1)$.

The slopes of curve first and curve second at the point $(0, -1)$ are respectively
$$m_1 = \frac{2x}{4} = 0 \text{ and } m_2 = \frac{-2x}{4} = 0$$
$$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow \theta = 0°$$

24. Given equation of line is
$$x\cos\alpha + y\sin\alpha = p$$
or $\quad y = -x\cot\alpha + p\,\text{cosec}\,\alpha$

Here, $\quad m = -\cot\alpha$

We know that, the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$$= \left[\frac{a}{(-\cot\alpha)^2}, \frac{2a}{-\cot\alpha}\right]$$
$$= (a\tan^2\alpha, -2a\tan\alpha)$$

25. Since, the tangent to the parabola at point t_1 and t_2 are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$.

Also, tangents are perpendicular to the parabola therefore,

$$\frac{1}{t_1}\cdot\frac{1}{t_2} = -1$$

or $$t_1t_2 = -1$$

We also know that their point of intersection is

$$[at_1t_2, a(t_1+t_2)] \text{ or } [-a, a(t_1+t_2)]$$

$\therefore$ Point of intersection lie on directrix $x = -a$ or $x + a = 0$

26. Given that equation of parabola is $y^2 = 9x$

On comparing with $y^2 = 4ax$, we get $a = \frac{9}{4}$

Now, equation of tangent to the parabola $y^2 = 9x$ is

$$y = mx + \frac{9/4}{m} \quad \text{...(i)}$$

If this tangent passing through the point (4, 10), then

$$10 = 4m + \frac{9}{4m}$$

$\Rightarrow$ $$16m^2 - 40m + 9 = 0$$

$\Rightarrow$ $$(4m-9)(4m-1) = 0$$

$\Rightarrow$ $$m = \frac{1}{4}, \frac{9}{4}$$

On putting the values of m in Eq. (i),

$$4y = x + 36 \quad \text{and} \quad 4y = 9x + 4$$

$\Rightarrow$ $x - 4y + 36 = 0$ and $9x - 4y + 4 = 0$

27. The coordinates of end point of latusrectum are $(a, 2a)$ and $(a, -2a)$ *i.e.*, (3, 6) and $(3, -6)$.

The equation of directrix is $x = -3$.

The equation of tangents from the above points are $6y = 6(x+3)$ and $-6y = 6(x+3)$

$\Rightarrow$ $$x - y + 3 = 0$$

and $$x + y + 3 = 0$$

The intersection point is $(-3, 0)$.

The equation of the directrix of the parabola $y^2 = 12x$ is $x = -3$

$\Rightarrow$ Intersection point $(-3, 0)$ lies on the directrix.

28. Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$.

Tangents at A and B will intersect at the point C, whose coordinate is given by $\{at_1t_2, a(t_1+t_2)\}$. Clearly, ordinates of A, C and B are always in AP.

29. Given equation of curve is $y^2 + 2xy + x^2 + 2x + 3y + 1 = 0$. Here, $h^2 = ab$, therefore the given curve is a parabola. The position of the point $(1, -2)$ with respect to the parabola is obtained as

$$(-2)^2 + 2(1)(-2) + (1)^2 + 2(1) + 3(-2) + 1 = -2 < 0$$

Since, point is inside the parabola therefore no tangent can be drawn to the parabola.

30. It is clear from the figure, that only one common tangent is possible.

31. Given parabola is $y^2 = ax$

i.e., $$y^2 = 4\left(\frac{a}{4}\right)x \quad \text{...(i)}$$

Let point of contact is (x_1, y_1), then equation of tangent is

$$yy_1 = \frac{a}{2}(x + x_1)$$

Here, $$m = \frac{a}{2y_1} = \tan 45° \Rightarrow \frac{a}{2y_1} = 1 \Rightarrow y_1 = \frac{a}{2}$$

From Eq. (i), $$x_1 = \frac{a}{4}$$

$\therefore$ Point of contact is $\left(\frac{a}{4}, \frac{a}{2}\right)$.

32. As we know that distance from vertex to the parabola is equal to the focus and directrix.

$\therefore$ The tangent at the vertex divide in the ratio 1 : 1.

33. Let $P(at^2, 2at)$ any point on the parabola and focus is $(a, 0)$.

The equation of tangent at P is $yt = x + at^2$.

Since, it meets the directrix $x = -a$ at K.

Then, the coordinate of K is $\left(-a, \frac{at^2 - a}{t}\right)$.

$$\text{Slope of } SP = m_1 = \frac{2at}{a(t^2-1)}$$

$$\text{Slope of } SK = m_2 = \frac{a(t^2-1)}{-2at}$$

$\therefore$ $$m_1m_2 = \frac{2at}{a(t^2-1)}\cdot\frac{a(t^2-1)}{(-2at)} = -1$$

$\therefore$ $$\angle PSK = 90°$$

34. Since, tangent at P and Q on the parabola meet in T.

If the coordinates of P and Q are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively, then coordinates of T are $\{at_1t_2, a(t_1+t_2)\}$.

$\therefore \quad SP = a(1+t_1^2)$

$SQ = a(1+t_2^2)$

$ST^2 = a^2(1-t_1t_2)^2 + a^2(t_1+t_2)^2$

$= a^2(1+t_1^2+t_2^2+t_1^2t_2^2)$

$= a(1+t_1^2)\,a(1+t_2^2) = SP \cdot SQ$

Thus, SP, ST, SQ are in GP.

35. Let third tangent is tangent at vertices, then

$p_1 = |at_1t_2|, p_2 = at_1^2, p_3 = at_2^2$, clearly p_2, p_1, p_3 are in GP.

36. **Statement I** Given equation of parabola is

$y^2 = 4a(x+a)$ or $Y^2 = 4aX$

$\therefore$ Focus is $(a, 0)$

$\Rightarrow \quad x+a = a, \quad y = 0$

$\Rightarrow \quad x = 0, \quad y = 0$

$\therefore$ Focus is at origin (0, 0).

Statement II Given equation of line is $y = -\frac{lx}{m} - \frac{n}{m}$.

It will touch the parabola $y^2 = 4ax$, if

$$-\frac{n}{m} = \frac{a}{-\frac{l}{m}} \Rightarrow nl = am^2$$

$\therefore$ Both statements are true.

37. The equation of normal is

$y = mx - 8m - 4m^3 \qquad (\because y = mx - 2am - am^3)$

Since, it is passing through (2, 0).

$\therefore \quad 0 = 2m - 8m - 4m^3$

$\Rightarrow \quad m = 0$ and $2m^2 = -3$ (no real value exist)

Only one real value of m exist.

$\therefore$ One normal can be drawn.

38. Given that equation of parabola is $y^2 = 8x$.

$\Rightarrow \quad a = 2$

We know, if the normal at point $(at_1^2, 2at_1)$ is passing through the point on the parabola $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

Given point is (2, 4).

$\Rightarrow \quad at_1^2 = 2$

$\Rightarrow \quad t_1 = 1$

$\therefore \quad t_2 = -1 - \frac{2}{1} = -3$

The other end will be $(at_2^2, 2at_2)$ *i.e.*, $(18, -12)$

39. Equation of normal is $y = mx - \frac{m}{2} - \frac{m^3}{4}\left(a = \frac{1}{4}\right)$. It passes through $(c, 0)$.

$\therefore \quad 0 = cm - \frac{m}{2} - \frac{m^3}{2} \Rightarrow m = 0$

and $\quad \frac{m^2}{4} = c - \frac{1}{2} \Rightarrow c > \frac{1}{2}$

Then, all values of m are real.

40. Sum of ordinates of feet of normals drawn from a point is zero.

So, there arithmetic mean is zero.

41. Let the coordinates of P and Q are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. Then, the coordinates of R are $\{2a + a(t_1^2+t_2^2+t_1t_2), -at_1t_2(t_1+t_2)\}$.

Since, R lies on the parabola.

$\therefore \quad a^2t_1^2t_2^2(t_1+t_2)^2 = 4a[2a + a\{(t_1+t_2)^2 - t_1t_2\}]$

$\Rightarrow \quad (t_1+t_2)^2\{t_1^2t_2^2 - 4\} + 4(t_1t_2 - 2) = 0$

$\Rightarrow \quad t_1t_2 = 2$

$\Rightarrow \quad y_1y_2 = (2at_1)(2at_2) = 4a^2t_1t_2$

$\therefore \quad y_1y_2 = 8a^2$

42. Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then equation of tangent and normal at $P(at^2, 2at)$ are $ty = x + at^2$ and $y = -tx + 2at + at^3$, respectively.

Since, tangent and normal meet its axis at T and G.

$\therefore$ Coordinates of T and G are $(-at^2, 0)$ and $(2a + at^2, 0)$ respectively.

From definition of parabola

$SP = PM = a + at^2$

Now, $\quad SG = VG - VS = 2a + at^2 - a = a + at^2$

and $\quad ST = VS + VT = a + at^2$

Hence, $\quad SP = SG = ST$

43. Let $y = mx + \frac{1}{m}$ is a tangent to $y^2 = 4ax$.

Equation of normal to the parabola $x^2 = 4by$ at (x_1, y_1) is

$$y - y_1 = -\frac{2b}{x_1}(x - x_1)$$

and $\quad x_1^2 = 4by_1$

$\Rightarrow \quad y-\frac{x_1^2}{4b}=-\frac{2b}{x_1}(x-x_1)$

$\Rightarrow \quad y=-\frac{2b}{x_1}x+\frac{x_1^2}{4b}+2b$

On comparing with $y=mx+\frac{1}{m}$, we get

$$m=-\frac{2b}{x_1} \quad \text{...(i)}$$

$$\frac{x_1^2}{4b}+2b=\frac{1}{m} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{4b^2}{m^2 4b}+2b=\frac{1}{m}$$

$\Rightarrow \quad b+2bm^2=m$

$\Rightarrow \quad 2bm^2-m+b=0$

For real values of m, $D>0$

$\Rightarrow \quad 1-8b^2>0 \Rightarrow b^2<\frac{1}{8} \Rightarrow |b|<\frac{1}{2\sqrt{2}}$

44. Given equation of parabola is $y^2=16x$.

If (1, 1) is the mid-point of the chord, then its equation of chord is

$T=S_1$

$\therefore \quad y(1)-8(x+1)=1-16$

$\Rightarrow \quad y-8x-8=-15$

$\Rightarrow \quad 8x-y=7$

45. $\because A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ are such that $AC:AB=1:3$

$\therefore$ Coordinates of C are $\left(\frac{2at_1^2+at_2^2}{3}, \frac{4at_1+2at_2}{3}\right)$.

Point C lies on x-axis, then

$$\frac{4at_1+2at_2}{3}=0$$

$\Rightarrow \quad t_2+2t_1=0$

46. Given equation of parabola is $y^2=4ax$.

Let the coordinates of B are $(at^2, 2at)$.

Slope of $AB=\frac{2}{t}$

Since, BC is perpendicular to AB.

So, slope of $BC=-\frac{t}{2}$

Equation of BC is $y-2at=-\frac{t}{2}(x-at^2)$.

This line meets to the x-axis at point C.

Put $\quad y=0 \Rightarrow x=4a+at^2$

So, distance $CD=4a+at^2-at^2=4a$

47. Given equation of circle is $(x-6)^2+y^2=(\sqrt{2})^2$

$BC=\text{Radius}=\sqrt{2}$

The length of the tangent from S to B.

$\therefore \quad SB=\sqrt{(4-6)^2+0-2}=\sqrt{2^2-2}=\sqrt{2}$

From figure, ΔCBS is an isosceles triangle.

$\Rightarrow \quad \theta=45° \Rightarrow m=1 \quad (\because BC=BS)$

Similarly, for ΔCSD, $m=-1$

48. Taking vertex of the parabolic reflector at origin, x-axis along the axis of parabola. The equation of the parabola is $y^2=4ax$.

Given depth is 5 cm, diameter is 20 cm.

$\because$ Point $P(5, 10)$ lies on parabola.

$\therefore \quad (10)^2=4a(5) \Rightarrow a=5$

Clearly, focus is at the mid-point of given diameter

i.e., $\quad S=(5, 0)$

49. Clearly, latusrectum is a line perpendicular to the axes and passing through focus whose length is $4a$.

Given, $x^2=12y$, which is of the form $x^2=4ay$

$\Rightarrow \quad 4a=12=\text{Length of } ABC$

$\therefore \quad \text{Focus } C=(0, 3)$

Area of $\Delta OAB=\frac{1}{2}\times AB\times OC$

$=\frac{1}{2}\times 12\times 3=6\times 3=18$ sq units

50. Let any point (h, k) will satisfy

$$y^2 = 4ax \text{ i.e., } k^2 = 4ah \quad \ldots(i)$$

Let a line OP makes an angle θ from the x-axis

$\therefore$ In ΔOAP, $\sin\theta = \dfrac{PA}{OP}$

$$\sin\theta = \frac{k}{l}$$

$\Rightarrow \quad k = l\sin\theta$ and $\cos\theta = \dfrac{OA}{OP}$

$\Rightarrow \quad \cos\theta = \dfrac{h}{l} \Rightarrow h = l\cos\theta$

Hence, from Eq. (i), we get

$$l^2\sin^2\theta = 4a \times l\cos\theta \quad (\text{put } k = l\sin\theta, h = l\cos\theta)$$

$$\Rightarrow \quad l = \frac{4a\cos\theta}{\sin^2\theta}$$

51. Given that focus is $S(0, 0)$.

Let A is the vertex of parabola. Take any point Z on the directrix such that $AS = AZ$. Since, the given tangent $x - y + 1 = 0$ is parallel to the directrix.

Equation of directrix is $x - y + \lambda = 0$.

$\because$ A is the mid-point of SZ.

$\therefore \quad SZ = 2\,SA$

$$\Rightarrow \quad \frac{|0 - 0 + \lambda|}{\sqrt{1^2 + 1^2}} = 2 \times \frac{|0 - 0 + 1|}{\sqrt{1^2 + 1^2}}$$

$\Rightarrow \quad |\lambda| = 2 \Rightarrow \lambda = 2$

$\therefore$ Equation of directrix is $x - y + 2 = 0$.

Now, P be any point on the parabola.

$\therefore \quad SP = PM \Rightarrow SP^2 = PM^2$

$$\Rightarrow \quad (x-0)^2 + (y-0)^2 = \left(\frac{|x - y + 2|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$$

52. Let the equation $y = mx + c$ be the common tangents, so the curve $y^2 = 8x$ and $x^2 + y^2 = 2$.

Then, $\quad c = \dfrac{2}{m}$ and $c^2 = 2(1 + m^2)$

If $m^2 = t$, then

$$\frac{4}{t} = 2(1 + t) \Rightarrow t^2 + t - 2 = 0$$

$$\Rightarrow \quad (t + 2)(t - 1) = 0 \Rightarrow t = 1, -2$$

Thus, $\quad m = \pm 1 \quad (\because t \neq -2)$

Hence, tangents are $y = x + c$ and $y = -x + c$ which are perpendicular to each other.

53. Given equation of parabola is $y = x^2 \quad \ldots(i)$

Equation of straight line is $y = 2x - 4 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$x^2 - 2x + 4 = 0$$

Let $\quad z = x^2 - 2x + 4$

$\therefore \quad z' = 2x - 2$

For least value, $z' = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

z'' is positive at $x = 1$.

$\therefore$ It is minimum, putting $x = 1$ in Eq. (i), we get

$$y = 1$$

So, the required point at the least distance from the line is $(1, 1)$.

54. Since, the line $y = x - 1$ passes through focus $(1, 0)$

$\Rightarrow y = x - 1$ is a focal chord.

So, angle between tangent is $\dfrac{\pi}{2}$

55. Centre of circle must on negative x-axis for that λ must be positive as centre of circle is $(-\lambda, 0)$.

$\therefore$ Option (a) is correct.

56. Normal at a point $(m^2, -2m)$ on the parabola $y^2 = 4x$ is given by $y = mx - 2m - m^3$. If this is normal to the circle also, then it will passes through centre $(-3, 6)$ of the circle.

$\therefore \quad 6 = -3m - 2m - m^3 \Rightarrow m = -1$

Since, shortest distance between parabola and circle will occurs along common normal.

$\therefore$ Shortest distance = distance between $(m^2, -2m)$ and centre $(-3, 6)$ – radius of circle $= 4\sqrt{2} - 5$

57. Let any point on the line segment PQ is $R(\alpha, \beta)$, then

$$\alpha = \frac{\lambda(1) + 1}{\lambda + 1} = 1$$

and $\quad \beta = \dfrac{3\lambda + 1}{\lambda + 1} \quad (\because \lambda > 0 \text{ as } R \text{ is on segment } AB)$

A point is inside parabola $y^2 = 4x$, if

$$y^2 - 4x < 0$$

$$\Rightarrow \quad \left(\frac{3\lambda + 1}{\lambda + 1}\right)^2 - 4(1) < 0$$

$$\Rightarrow \quad \left(\frac{3\lambda + 1}{\lambda + 1} + 2\right)\left(\frac{3\lambda + 1}{\lambda + 1} - 2\right) < 0$$

$$\Rightarrow \quad (5\lambda + 3)(\lambda - 1) < 0$$

$$\Rightarrow \quad -\frac{3}{5} < \lambda < 1$$

So, $\quad 0 < \lambda < 1 \quad (\text{but } \lambda > 0)$

Round II

1. Shortest distance between two curves occured along the common normal.

Normal to $y^2 = 4x$ at $(m^2, 2m)$ is

$$y + mx - 2m - m^3 = 0$$

Normal to $y^2 = 2(x-3)$ at $\left(\frac{m^2}{2} + 3, m\right)$ is

$$y + m(x-3) - m - \frac{m^3}{2} = 0$$

Both normals are same, if $-2m - m^3 = -4m - \frac{1}{2}m^3$

$\Rightarrow \quad m = 0, \pm 2$

So, points will be (4, 4) and (5, 2) or $(4, -4)$ and $(5, -2)$.

Hence, shortest distance will be

$$\sqrt{(1+4)} = \sqrt{5}$$

2. $\because \quad y^2 - 12x - 4y + 4 = 0 \Rightarrow (y-2)^2 = 12x$

Its vertex is (0, 2) and $a = 3$,

its focus is (3, 2).

Hence, for the required parabola, focus is (3, 4),

vertex is (3, 2) and $a = 2$

Hence, the equation of the parabola is

$$(x-3)^2 = 4(2)(y-2)$$

or $\quad x^2 - 6x - 8y + 25 = 0$

3. Tangent at the vertex is

$$x - y + 1 = 0 \quad \ldots(i)$$

Therefore, equation of axis of the parabola is

$$x + y = 0 \quad \ldots(ii)$$

Now on solving Eqs. (i) and (ii), we get $A\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\therefore Z$ is (– 1, 1).

Now, directrix is

$$x - y + k = 0$$

But this passes through $Z(-1, 1)$, so

$$k = 2$$

$\therefore$ Equation of directrix is $x - y + 2 = 0$.

Therefore, by definition equation of parabola is given by

$$OP = PM$$

$\Rightarrow \quad OP^2 = PM^2$

$\Rightarrow \quad \left(\frac{x-y+2}{\sqrt{2}}\right)^2 = x^2 + y^2$

$\Rightarrow \quad (x - y + 2)^2 = 2x^2 + 2y^2$

$\Rightarrow \quad x^2 + y^2 + 4 - 2xy + 4x - 4y = 2x^2 + 2y^2$

$\Rightarrow \quad x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$

4. Since, distance of vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively.

$\therefore V(1, 1)$ and $F(2, 2)$ (*i.e.,* lying on $y = x$)

where, length of latusrectum

$$= 4a = 4\sqrt{2} \quad (\because a = \sqrt{2})$$

$\therefore$ By definition of parabola

$$PM^2 = (4a)(PN)$$

where, PN is length of perpendicular upon $x + y - 2 = 0$

(*i.e.,* tangent at vertex).

$\Rightarrow \quad \frac{(x-y)^2}{2} = 4\sqrt{2}\left(\frac{x+y-2}{\sqrt{2}}\right)$

$\Rightarrow \quad (x-y)^2 = 8(x+y-2)$

5. Let x_1, x_2 and x_3 be the abscissae of the points on the parabola whose ordinates are y_1, y_2 and y_3, respectively. Then, $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$. Therefore, the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix} \Rightarrow \Delta = \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

6. Let the point be $P(at^2, 2at)$.

Then, according to question, $SP = at^2 + a = k$ ...(i)

Let (α, β) be the moving point, then $\alpha = at^2$, $\beta = 2at$

$\therefore$ $$\frac{\alpha}{\beta} = \frac{t}{2}$$

and $$a = \frac{\beta^2}{4\alpha} \quad [\because \text{point } (\alpha, \beta) \text{ lies on } y^2 = 4ax]$$

On putting these values in Eq. (i), we get

$$\frac{\beta^2}{4\alpha}\left(1 + \frac{4\alpha^2}{\beta^2}\right) = k$$

$\Rightarrow$ $$\beta^2 + 4\alpha^2 = 4k\alpha$$

Hence, locus of a point is

$$4x^2 + y^2 - 4kx = 0$$

7. Chord through (2, 1) is $\frac{x-2}{\cos\theta} = \frac{y-1}{\sin\theta} = r$ ...(i)

On solving Eq. (i) with parabola $y^2 = x$, we have

$$(1 + r\sin\theta)^2 = 2 + r\cos\theta$$

$\Rightarrow$ $$\sin^2\theta\, r^2 + (2\sin\theta - \cos\theta)r - 1 = 0$$

This equation has two roots $r_1 = AC$ and $r_2 = -BC$.

Then, sum of roots $r_1 + r = 0$

$\Rightarrow$ $$2\sin\theta - \cos\theta = 0 \Rightarrow \tan\theta = \frac{1}{2}$$

$$AB = |r_1 - r_2|$$

$$= \sqrt{(r_1 + r_2)^2 - 4r_1r_2}$$

$$= \sqrt{4\frac{1}{\sin^2\theta}} = 2\sqrt{5} \quad \left(\because \tan\theta = \frac{1}{2} \Rightarrow \sin\theta = \frac{1}{\sqrt{5}}\right)$$

8. Difference of the ordinate,

$$d = \left|2at + \frac{2a}{t}\right| = 2a\left|t + \frac{1}{t}\right|$$

Now area, $A = \frac{1}{2}\begin{vmatrix} at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2\left(t + \frac{1}{t}\right)$

$\Rightarrow$ $$2a\left(t + \frac{1}{t}\right) = \frac{2A}{a}$$

9. Let point be $P = (\alpha, \alpha + 1)$, where $\alpha \neq 0, -1$

or $$P = (\alpha, \alpha - 1)$$

where, $\alpha \neq 0, 1$

$$(\alpha, \alpha + 1) \text{ is on } y^2 = 4x + 1$$

$$(\alpha + 1)^2 = 4\alpha + 1$$

$\Rightarrow$ $$\alpha^2 - 2\alpha = 0 \Rightarrow \alpha = 2 \quad (\because \alpha \neq 0)$$

Therefore, ordinate of P is 3.

$$(\alpha, \alpha - 1) \text{ is on } y^2 = 4x + 1$$

$$(\alpha - 1)^2 = 4\alpha + 1$$

$\Rightarrow$ $$\alpha^2 - 6\alpha = 0$$

$\Rightarrow$ $$\alpha = 6 \quad (\because \alpha \neq 0)$$

Therefore, ordinate of P is 3 and 5.

10. Let the possible point be $(t^2, 2t)$. Equation of tangent at this point is $yt = x + t^2$.

It must pass through (6, 5). (since, normal to circle always passes through its centre)

$\Rightarrow$ $$t^2 - 5t + 6 = 0 \Rightarrow t = 2, 3$$

$\therefore$ Possible points are (4, 4), (9, 6).

11. Since, no point of the parabola is below the x-axis,

$$D = a^2 - 4 \leq 0$$

Therefore, maximum value of a is 2.

Equation of parabola, when $a = 2$, is $y = x^2 + 2x + 1$. It intersects y-axis at (0, 1). Equation of the tangent at (0, 1) is $y = 2x + 1$.

Since, $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$

$$r = \frac{1}{\sqrt{5}}$$

12. Equation of the tangent at (0, 1) to the parabola $y = x^2 + ax + 1$ is $y - 1 = a(x - 0)$ or $ax - y + 1 = 0$

As it touches the circle,

$$r = \frac{1}{\sqrt{a^2+1}}$$

Radius is maximum when $a = 0$.
Therefore, equation of the tangent is $y = 1$.
Therefore, slope of the tangent is 0.

13. Equation of tangent is $y = ax + 1$. Intercepts are $-1/a$ and 1. Therefore, area of the triangle bounded by tangent and the axes is

$$\frac{1}{2}\left|-\frac{1}{a}\times 1\right| = \frac{1}{2|a|}$$

It is minimum when $a = 2$. Therefore, minimum area is 1/4.

14. End points of double ordinate $x = 4$ of parabola $y^2 = 4x$ are $(4, \pm 4)$.

$$\Rightarrow \quad t_1 = \pm 2 \Rightarrow t_2 = -t_1 - \frac{2}{t_1} = \pm 3$$

$\Rightarrow P(9, 6)$ and $P'(9, -6)$

$\therefore \quad PP' = 12$ units

15. If $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$, vertex $A \equiv (0, 0)$

$\therefore$ Slope of $AP \times$ Slope of AQ

$$= \left(\frac{2at_1 - 0}{at_1^2 - 0}\right) \times \left(\frac{2at_2 - 0}{at_2^2 - 0}\right)$$

$$= \frac{2}{t_1} \times \frac{2}{t_2} = \frac{4}{-1} = -4 \neq -1 \qquad (\because t_1 t_2 = -1)$$

$$\therefore \quad \angle PAQ \neq \frac{\pi}{2}$$

16. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then

$$(PM)^2 = (\text{latusrectum})\,(PN)$$

$$\Rightarrow \quad \left(\frac{3x + 4y - 4}{\sqrt{\{3^2 + 4^2\}}}\right)^2 = 4\left[\frac{4x + 3y + 7}{\sqrt{\{4^2 + (-3)^2\}}}\right]$$

$$\Rightarrow \quad Y^2 = 4\rho X$$

$$\therefore \quad \rho = 1, \quad Y = \frac{3x + 4y - 4}{5}, \quad X = \frac{4x - 3y + 7}{5}$$

$\therefore$ Directrix is $X + \rho = 0$

$$\Rightarrow \quad \frac{4x - 3y + 7}{5} + 1 = 0$$

or $\quad 4x - 3y + 12 = 0$

17. **Hint** Any tangent having slope m is

$$y = mx + \frac{l}{m} \text{ or } y = mx + \frac{914}{m}$$

18. Both the statements are true (see properties of focal chord), but statement II is not correct explanation of statement I.

19. We know that the locus of point P from which two perpendicular tangents are drawn to the parabola, is the directrix of the parabola.
Hence, the required locus is $x = -1$.

20. From the figure, it is clear that vertex of the parabola at (1, 0).

21. Perpendicular tangents can be drawn only from a point on the directrix, $x = -2$.

$\therefore$ The point is $(-2, 0)$.

22. $\frac{3y}{a^3} = x^2 + \frac{3x}{2a} - \frac{6}{a^2} = \left(x + \frac{3}{4a}\right)^2 - \frac{105}{16a^2}$

$$\Rightarrow \quad \left(x + \frac{3}{4a}\right)^2 = \frac{3}{a^3}\left(y + \frac{35a}{16}\right)$$

If (α, β) is the vertex, then $\alpha = -\frac{3}{4a}, \beta = -\frac{35a}{16}$

Hence, the locus of (α, β) is $xy = \frac{105}{64}$

23. The coordinates of P are (1, 0). A general point Q on $y^2 = 8x$ is $(2t^2, 4t)$. Mid-point of PQ is (h, k).

$\therefore \quad 2h = 2t^2 + 1 \qquad \ldots(i)$

and $\quad 2k = 4t \Rightarrow t = k/2$

Then, from Eq. (i), we get

$$2h = \frac{2k^2}{4} + 1$$

$$\Rightarrow \quad 4h = k^2 + 2$$

Hence, locus is $y^2 - 4x + 2 = 0$

24. The common chord is the line $x - y = 0$. But given line is

$$2bx + 3cy + 4d = 0$$

$$\therefore \quad d = 0, 2b + 3c = 0$$

15 Ellipse

JEE Main MILESTONE

- Ellipse
- Terms Related to an Ellipse
- Position of a Point with Respect to an Ellipse
- Auxiliary Circle
- Intersection of a Line and an Ellipse
- Condition of Tangency
- Tangent to the Ellipse
- Normal to the Ellipse
- Director Circle
- Number of Normal and Conormal Points
- Pair of Tangents
- Chord of Contact
- Diameter
- Some Useful Properties of an Ellipse

15.1 Ellipse

An ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the foci of the ellipse.

OR

The locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (*i.e.*, focus) in the same plane to its distance from a fixed straight line (*i.e.,* directrix) is always constant, which is always less than unity.

In Mathematics, an ellipse is a plane curve that results from the intersection of a cone by a plane in a way that produces a closed curve. Ellipse are closed curves and it is the bounded case of the conic sections.

Mathematically, $\frac{SP}{PM} = e$, where e is called **eccentricity** and $0 < e < 1$.

Standard Equations of an Ellipse

If the centre of the ellipse is origin and foci are on the x-axis or y-axis, then that types of equation are called standard equation of an ellipse. The two such possible orientations are shown below.

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a < b$

15.2 Terms Related to an Ellipse

Vertices

The point A and A', where the curve meets the line joining the foci S and S', are called the vertices of an ellipse.

Major and Minor Axes

Major axis is the one which lie along the line passing through focus and perpendicular to directrix and minor axis is the one which is perpendicular to major axis and passes through the mid-point of the foci.

Centre

Since, all chords passing through C are bisected at C. Hence, $C(0, 0)$ is the centre of an ellipse.

Ordinate and Double Ordinate

Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then, PN is called the ordinate of P and PNP' the double ordinate of P.

Latusrectum

The double ordinate passing through the focus is called latusrectum.

Note

- The second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse, if $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 - ab < 0$.
- The vertex divides the join of focus and point of intersection of directrix with axis internally and externally in the ratio of $e : 1$.
- Domain and range of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $[-a, a]$ and $[-b, b]$, respectively.

***Sample* Problem 1** *The equation of an ellipse whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$ and the directrix is $x - y + 3 = 0$, is*

(*a*) $7x^2 + 7y^2 + 2xy + 10x + 10y + 7 = 0$

(*b*) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

(*c*) $7x^2 + 7y^2 + 2xy - 10x + 10y + 7 = 0$

(*d*) *None of the above*

Interpret (*b*) Let $P(x, y)$ be any point on the ellipse whose focus is $S(-1, 1)$ and the directrix is $x - y + 3 = 0$. Draw PM perpendicular from $P(x, y)$ on the directrix $x - y + 3 = 0$. Then, by definition

$$SP = ePM \Rightarrow (SP)^2 = e^2(PM)^2$$

$$\Rightarrow \quad (x+1)^2 + (y-1)^2 = \frac{1}{4}\left\{\frac{x - y + 3}{\sqrt{2}}\right\}^2$$

$$\Rightarrow \quad 8(x^2 + y^2 + 2x - 2y + 2) = x^2 + y^2 + 9 - 2xy + 6x - 6y$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$

which is the required equation of an ellipse.

Terms Related to an Ellipse

	Fundamental Terms	**Ellipse** (Horizontal ellipse)	**Conjugate Ellipse** (Vertical ellipse)
(a)	Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a < b)$
(b)	Graph		
(c)	Centre	$C(0, 0)$	$C(0, 0)$

	Fundamental Terms	Ellipse (Horizontal ellipse)	Conjugate Ellipse (Vertical ellipse)
(d)	Vertices	$(\pm a, 0)$	$(0, \pm b)$
(e)	Length of major axis	$2a$	$2b$
(f)	Length of minor axis	$2b$	$2a$
(g)	Foci	$(\pm ae, 0)$	$(0, \pm be)$
(h)	Equations of directrices	$x = \pm\left(\frac{a}{e}\right)$	$y = \pm\left(\frac{b}{e}\right)$
(i)	Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \left(\frac{a}{b}\right)^2}$
(j)	Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
(k)	Ends of latusrectum	$\left(\pm ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{a^2}{b}, \pm be\right)$
(l)	Parametric equations	$\begin{cases} x = a\cos\alpha \\ y = b\sin\alpha \end{cases}$	$\begin{cases} x = a\cos\alpha \\ y = b\sin\alpha \end{cases}$
(m)	Parametric coordinates	$(a\cos\alpha, b\sin\alpha)$	$(a\cos\alpha, b\sin\alpha)$
(n)	Focal distance or radii	$\lvert SP\rvert = (a - ex_1)$ and $\lvert S'P\rvert = (a + ex_1)$	$\lvert SP\rvert = (b - ey_1)$ and $\lvert S'P\rvert = (b + ey_1)$
(o)	Sum of focal radii $\lvert SP\rvert + \lvert S'P\rvert$	$2a$	$2b$
(p)	Distance between foci	$2ae$	$2be$

Hot Spot

Determination of Eccentricity and its Application

In these types of problems sometimes eccentricity is already given, & we have to find other terms (as minor and major axis) and sometimes we have to find the eccentricity. These types of problems are directly or indirectly asked in various years of AIEEE (JEE Main) 2011, 2008, 2006, 2005, 2004, 2002. Following steps are given below can be followed to solve these types of problems.

Step I If equation of ellipse is given, then convert it in standard form and find eccentricity e by using the formula,

$$e = \sqrt{1 - \frac{b^2}{a^2}},\ a > b \text{ or } e = \sqrt{1 - \frac{a^2}{b^2}},\ b > a$$

Step II If e is given and any of the condition is given, then we can find a and b.

Step III To find other terms as foci, directrix etc., use the formula which gives relation between e and that term and simplifying it.

Above method can be explained by given illustrative examples.

***Sample* Problem 2** *The focus of an ellipse is at the origin. If the directrix is the line $x = 5$ and the eccentricity is $\frac{1}{3}$, then find the length of major axis.*

(a) $\frac{15}{4}$ (b) $\frac{15}{2}$

(c) $\frac{15}{8}$ (d) *None of these*

Interpret (a) **Step I** No need of step I.

Step II Here, e and other condition is given.

According to the given condition,

$$\frac{a}{e} - ae = 5$$

and $$e = \frac{1}{3}$$

$$\therefore \quad \frac{a}{1/3} - a \times \frac{1}{3} = 5$$

$$\Rightarrow \quad 3a - \frac{a}{3} = 5$$

$$\Rightarrow \quad \frac{8a}{3} = 5$$

$$\Rightarrow \quad a = \frac{15}{8}$$

Step III To find the length of major axis *i.e.*, $2a = 2 \times \frac{15}{8} = \frac{15}{4}$

***Sample* Problem 3** *The equation of ellipse having vertices* $(\pm 5, 0)$ *and foci* $(\pm 4, 0)$ *is* [NCERT]

(a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (b) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

(c) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (d) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

Interpret (c) Since, vertices $(\pm 5, 0)$ and foci $(\pm 4, 0)$ are on x-axis, as y-coordinate of both points is zero. Hence, equation of ellipse will be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

where, $a = 5$

and $ae = 4 \Rightarrow e = \frac{4}{5}$

$\because$ $b^2 = a^2(1 - e^2)$

$\therefore$ $b^2 = 25\left(1 - \frac{16}{25}\right)$

$\Rightarrow$ $b^2 = 9$

Put the values of $a^2 = 25$ and $b^2 = 9$ in Eq. (i), we get

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

***Sample* Problem 4** *The equation of ellipse having centre at* (0, 0) *major axis on y-axis passes through the points* (3, 2) *and* (1, 6) *is* [NCERT]

(a) $\frac{x^2}{40} + \frac{y^2}{10} = 1$ (b) $\frac{x^2}{10} + \frac{y^2}{40} = 1$

(c) $\frac{x^2}{20} + \frac{y^2}{30} = 1$ (d) *None of these*

Interpret (b) Since, major axis is along y-axis.
Hence, equation of ellipse will be of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \ldots(i)$$

Given that Eq. (i) passes through the points (3, 2) and (1, 6), *i.e.*, they will satisfy it.

$\therefore$ $\frac{3^2}{b^2} + \frac{2^2}{a^2} = 1$

$\Rightarrow$ $\frac{9}{b^2} + \frac{4}{a^2} = 1 \quad \ldots(ii)$

and $\frac{1^2}{b^2} + \frac{6^2}{a^2} = 1$

$\Rightarrow$ $\frac{1}{b^2} + \frac{36}{a^2} = 1 \quad \ldots(iii)$

On solving Eqs. (ii) and (iii), we get $a^2 = 40$, $b^2 = 10$

$\therefore$ Required equation of ellipse is

$$\frac{x^2}{10} + \frac{y^2}{40} = 1$$

15.3 Position of a Point with Respect to an Ellipse

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of an ellipse and $P(h, k)$ be any point in the plane of an ellipse, then

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

***Case* I** When $S_1 > 0$, point (x_1, y_1) lies outside the ellipse.

***Case* II** $S_1 = 0$, point (x_1, y_1) lies on the ellipse.

***Case* III** $S_1 < 0$, point (x_1, y_1) lies inside the ellipse.

***Sample* Problem 5** *The position of the point* $(1, -2)$ *relative to an ellipse* $5x^2 + 7y^2 = 120$ *is*

(a) *outside* (b) *inside*

(c) *on the ellipse* (d) *cannot say*

Interpret (b) The equation of an ellipse is $5x^2 + 7y^2 = 120$.

$$S_1 = 5(1)^2 + 7(-2)^2 - 120$$
$$= 5 + 28 - 120 = -87 < 0$$

So, the point $(1, -2)$ lies inside the ellipse.

15.4 Auxiliary Circle

The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse.

If the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then equation of an auxiliary circle is $x^2 + y^2 = a^2$.

Eccentric Angle of a Point

Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q Join CQ. The $\angle ACQ=\phi$ is called the eccentric angle of the point P on the ellipse.

***Sample* Problem 6** *If equation of ellipse is* $\frac{x^2}{16}+\frac{y^2}{9}=1$, *then equation of auxiliary circle is* [**NCERT**]

(a) $x^2+y^2=16$ (b) $x^2+y^2=9$

(c) $x^2+y^2=25$ (d) *None of these*

Interpret (a) Given equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Here, $a>b$

$\therefore$ Equation of auxiliary circle

$$x^2+y^2=a^2$$

$$\therefore \quad x^2+y^2=16$$

Check Point 1

1. Let P be a variable point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ with foci at S and S′. If A be the area of ΔPSS′, then maximum value of A is
   (a) 24 sq units (b) 12 sq units
   (c) 36 sq units (d) None of these
2. If p, q are the segments of a focal chord of an ellipse $b^2x^2+a^2y^2=a^2b^2$, then
   (a) $a^2(p+q)=2bpq$ (b) $b^2(p+q)=2apq$
   (c) $a(p+q)=2b^2pq$ (d) $b(p+q)=2a^2pq$
3. The equation of the ellipse whose foci are $(\pm 2, 0)$ and eccentricity $\frac{1}{2}$, is
   (a) $\frac{x^2}{12}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}+\frac{y^2}{12}=1$
   (c) $\frac{x^2}{16}+\frac{y^2}{8}=1$ (d) None of these
4. In an ellipse the distance between the foci is 8 and the distance between the directrices is 25. The length of major axis is
   (a) $10\sqrt{2}$ (b) $20\sqrt{2}$
   (c) $30\sqrt{2}$ (d) None of these
5. The distances from the foci of $P(x_1, y_1)$ on the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ are
   (a) $4\pm\frac{5}{4}y_1$ (b) $5\pm\frac{4}{5}x_1$
   (c) $5\pm\frac{4}{5}y_1$ (d) None of these
6. Find the equation of the ellipse whose foci are (2, 3), (−2, 3) and whose semi-minor axis is of length $\sqrt{5}$.
   (a) $5x^2+9y^2+54y+36=0$ (b) $5x^2+9y^2-54y+36=0$
   (c) $5x^2+9y^2-54y-36=0$ (d) None of these
7. The position of the point (4, − 3) relative to an ellipse $\frac{x^2}{28}+\frac{y^2}{20}=1$ is
   (a) inside (b) on the ellipse
   (c) outside (d) cannot say
8. The curve represented by $x=3(\cos t+\sin t)$ and $y=4(\cos t-\sin t)$ is
   (a) ellipse (b) parabola
   (c) hyperbola (d) circle

15.5 Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

and the given line be $y=mx+c$ ...(ii)

On eliminating y from Eqs. (i) and (ii), then

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$\Rightarrow \quad (a^2m^2+b^2)x^2+2mca^2x^2+c^2a^2-a^2b^2=0$$

It is a quadratic equation in x which gives two values of x.

Discriminant, $D=4m^2c^2a^2-4(a^2m^2+b^2)(c^2a^2-a^2b^2)$

$$=-a^2b^2c^2+a^4b^2m^2+a^2b^4$$

$$=-a^2b^2(c^2-a^2m^2+b^2)$$

(i) If $D>0$, *i.e.*, $c^2>a^2m^2+b^2$, then line intersect the ellipse at two points.

(ii) If $D=0$, *i.e.*, $c^2=a^2m^2+b^2$, then the line intersect the ellipse at one point.

(iii) If $D<0$ *i.e.*, $c^2<a^2m^2+b^2$, then line intersect the ellipse at an imaginary point *i.e.*, the line neither touches nor intersects the ellipse.

15.6 Condition of Tangency

The line $y = mx + c$ will be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point on the ellipse, if

$$c^2 = a^2m^2 + b^2$$

and point of contact is $\left(\pm \frac{a^2m}{c}, \pm \frac{b^2}{c}\right)$.

***Sample* Problem 7** *The line $2x + 3y = 12$ touches the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at the point* **[NCERT Exemplar]**

(a) (3, 2) (b) (2, 3) (c) (−3, −2) (d) (3, 4)

Interpret (a) Given equation of line can be rewritten as

$$y = \frac{-2x}{3} + \frac{12}{3} = \frac{-2x}{3} + 4$$

Here, $m = -\frac{2}{3}, c = 4$

Also, equation of ellipse can be rewritten as

$$\frac{x^2}{18} + \frac{y^2}{8} = 1$$

Here, $a^2 = 18, b^2 = 8$

$\therefore$ Point of contact is $\left(\pm \frac{a^2m}{2}, \pm \frac{b^2}{c}\right)$

$$= \left[\pm \frac{18 \times (-2/3)}{4}, \pm \frac{8}{4}\right] = (\pm 3, 2)$$

***Sample* Problem 8** *The straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2l^2 + b^2m^2$ is equal to*

(a) n (b) n^2 (c) n^3 (d) n^4

Interpret (b) The equation of given line is $lx + my + n = 0$.

or $y = -\frac{l}{m}x - \frac{n}{m}$.

This line will touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

$$\frac{n^2}{m^2} = \frac{a^2l^2}{m^2} + b^2 \Rightarrow a^2l^2 + b^2m^2 = n^2$$

15.7 Tangent to the Ellipse

A line which intersect the ellipse at only one point is called the tangent to the ellipse.

Equation of Tangent in Different Forms

(i) Point Form

The equation of tangent at the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$, if point lies on the curve.

(ii) Slope Form

The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope form is $y = mx \pm \sqrt{a^2m^2 + b^2}$ and points of contact are $\left(\mp \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$.

(iii) Parametric Form

The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos\theta, b \sin\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

Note The intersection points of two tangents of the ellipse at points $(a \cos\theta_1, b \sin\theta_1)$ and $(a \cos\theta_2, b \sin\theta_2)$ is

$$\left(\frac{a\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}\right).$$

***Sample* Problem 9** *The locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is*

(a) $(x^2 + y^2)^2 = ax^2 + by^2$ (b) $(x^2 + y^2) = a^2x^2 + b^2y^2$

(c) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ (d) *None of these*

Interpret (c) The equation of any tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } y = mx + \sqrt{a^2m^2 + b^2} \quad \text{...(i)}$$

Equation of line perpendicular to Eq. (i) and passing through (0, 0), is

$$y = -\frac{1}{m}x \Rightarrow m = -\frac{x}{y} \quad \text{(ii)}$$

On putting the value of m in Eq. (i), we get

$$y = -\frac{x^2}{y} + \sqrt{\left(a^2\frac{x^2}{y^2} + b^2\right)}$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

15.8 Normal to the Ellipse

A line which is perpendicular to the tangent of the ellipse is called the normal to the ellipse.

Equation of Normal in Different forms

(i) Point Form

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

(ii) Slope Form

The equation of normal of slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx \mp \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$ and points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right).$$

Note If a line $y = mx + c$ be a normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

$$c^2 = \frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$$

(iii) Parametric Form

The equation of normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2-b^2$$

***Sample* Problem 10** *The straight line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $\frac{a^2}{l^2}+\frac{b^2}{m^2}$ is equal to*

(a) $\frac{a^2-b^2}{n^2}$

(b) $\frac{(a^2-b^2)^2}{n^2}$

(c) $(a^2-b^2)^2$

(d) *None of the above*

Interpret (b) The equation of any normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2-b^2 \quad \ldots(i)$$

The straight line $lx + my + n = 0$ will be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\therefore \quad \frac{a\sec\theta}{l} = -\frac{b\,\text{cosec}\,\theta}{m} = \frac{a^2-b^2}{-n}$$

$$\Rightarrow \quad \cos\theta = \frac{-na}{l(a^2-b^2)}$$

and $$\sin\theta = \frac{nb}{m(a^2-b^2)}$$

Also, $$\sin^2\theta + \cos^2\theta = 1$$

$$\therefore \quad \left\{\frac{n^2b^2}{m^2(a^2-b^2)^2}\right\} + \left\{\frac{n^2a^2}{l^2(a^2-b^2)^2}\right\} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2}+\frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$$

15.9 Director Circle

The locus of the point of intersection of perpendicular tangents to an ellipse is known as its director circle.

The equation of the director circle is $x^2 + y^2 = a^2 + b^2$.

Clearly, it is a circle concentric to the ellipse and its radius is equal to $\sqrt{a^2+b^2}$.

Note The tangents drawn from any point on the director circle of a given ellipse are always at right angle.

***Sample* Problem 11** *The locus of the point of intersection of the perpendicular tangents to ellipse* **[NCERT Exemplar]**

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \text{ is}$$

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 = 5$ (d) $x^2 + y^2 = 13$

Interpret (d) The locus of the point of intersection of the perpendicular tangents to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a director circle and whose equation is given by $x^2 + y^2 = a^2 + b^2$

$\therefore$ The required equation of director circle is

$$x^2 + y^2 = 9 + 4 \Rightarrow x^2 + y^2 = 13$$

15.10 Number of Normal and Conormal Points

At a point on a given ellipse exactly one normal can be drawn. If there is a point not lying on a given ellipse, infinitely many lines can be drawn which intersect the ellipse. Out of these lines there are atmost four lines which are normal to the ellipse at the points where they cut the ellipse. Such points on the ellipse are called conormal points. Some points about the conormals points and various relation between their eccentric angle are as given below.

(i) The four normals can be drawn from a point to an ellipse.

(ii) **Conormal point** The point on the ellipse, the normal at which to the ellipse pass through a given point is called conormal point.

(iii) **Properties of eccentric angles of conormal points**

(a) The sum of the eccentric angles of the conormal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

(b) If $\theta_1, \theta_2, \theta_3$ and θ_4 be eccentric angles of four points on the ellipse, the normals at which are concurrent, then

(a) $\sum \cos(\theta_1+\theta_2)=0$ (b) $\sum \sin(\theta_1+\theta_2)=0$

(c) If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0,$$

then the normals at these points are concurrent.

(d) If the normals at four points $P(x_1, y_1), Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

***Sample* Problem 12** *If the normals at the points A, B, C and D of an ellipse meet in a point O, then $SA \cdot SB \cdot SC \cdot SD$ is equal to (where, S is one of the foci)*

(a) $\frac{b^2}{e^2}\cdot SO^2$ (b) $\frac{b^3}{e^3}\cdot SO^3$ (c) $\frac{2b^2}{e^2}\cdot SO^2$ (d) $\frac{b^2}{3e^2}\cdot 2SO^2$

Interpret (a) If S be the point $(-ae, 0)$, we have $SA = a + ex$, $SB = a + ex_2$, $SC = a + ex_3$ and $SD = a + ex_4$

$\therefore$ $SA \cdot SB \cdot SC \cdot SD = (a+ex_1)(a+ex_2)(a+ex_3)(a+ex_4)$

$$= a^4 + a^3e\sum x_1 + a^2e^2\sum x_1x_2 + ae^3\sum x_1x_2x_3 + e^4x_1x_2x_3x_4$$

$$= \frac{b^2}{e^2}\{(h+ae)^2+k\} \text{ (on substitution and simplification)}$$

$$= \frac{b^2}{e^2}\cdot SO^2$$

15.11 Pair of Tangents

The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

$$SS_1 = T^2$$

where, $S = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1,$

$S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

and $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

***Sample* Problem 13** *The equation of the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2+2y^2=5$ is* [NCERT Exemplar]

(a) $9x^2-4y^2-24xy+40y+30x-55=0$

(b) $9x^2-4y^2-24xy+40y-30x+55=0$

(c) $9x^2+4y^2+24xy+40y+30x-55=0$

(d) *None of the above*

Interpret (a) The combined equation of the pair of tangents drawn from (1, 2) to the ellipse $3x^2+2y^2=5$ is

$$(3x^2+2y^2-5)(3+8-5) = (3x+4y-5)^2 \qquad (\because SS_1 = T^2)$$

$$\Rightarrow (3x^2+2y^2-5)(6) = 9x^2+16y^2+25+24xy-40y-30x$$

$$\Rightarrow 9x^2-4y^2-24xy+40y+30x-55=0$$

15.12 Chord of Contact

Let PQ and PR be tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from any external point $P(h, k)$. Then, QR is called chord of contact of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose equation is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$.

***Sample* Problem 14** *If $\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$, then the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at*

(a) *focus* (b) *centre*

(c) *end of the major axis* (d) *end of minor axis*

Interpret (b) Let

$$P(a\cos\theta_1, b\sin\theta_1)$$

and $$Q(a\cos\theta_2, b\sin\theta_2)$$

be two points on the ellipse. Then,

$$m_1 = \text{slope of } OP = \frac{b}{a}\tan\theta_1$$

and $$m_2 = \text{slope of } OQ = \frac{b}{a}\tan\theta_2$$

$$\therefore \quad m_1 m_2 = \frac{b^2}{a^2}\tan\theta_1 \tan\theta_2 = \frac{b^2}{a^2}\times\left(-\frac{a^2}{b^2}\right) = -1$$

$$\therefore \quad \angle POQ = \frac{\pi}{2}$$

Hence, PQ makes a right angle at the centre of the ellipse.

Equation of Chord of the Ellipse whose Mid-point is (x_1, y_1)

The equation of chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose mid-point is (x_1, y_1) is $T = S_1$

where, $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

Note

- Equation of chord of the ellipse whose eccentric angles are θ and ϕ, is
$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$
- If the chord passes through $(ae, 0)$, then
$$\tan\frac{\phi_1}{2}\tan\frac{\phi_2}{2}=\frac{e-1}{e+1}$$
- If the chord passes through $(-ae, 0)$, then
$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right)=\frac{e+1}{e-1}$$

***Sample* Problem 15** *The locus of the middle points of chords of an ellipse, which passes through a fixed point, is*

(a) *an ellipse* (b) *a circle*
(c) *a hyperbola* (d) *a parabola*

Interpret (a) Let $P(x_1, y_1)$ be the middle point of any chord AB of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then equation of chord AB is

$$T = S_1$$

$$\Rightarrow \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$\Rightarrow \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \quad \ldots(i)$$

But it passes through a fixed point $Q(h, k)$, so its coordinates must satisfy Eq. (i).

$$\therefore \quad \frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

This can be rewritten as

$$\frac{\left(x_1-\frac{h}{2}\right)^2}{a^2}+\frac{\left(y_1-\frac{k}{2}\right)^2}{b^2}=\frac{1}{4}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Hence, locus of $P(x_1, y_1)$ is

$$\frac{\left(x-\frac{h}{2}\right)^2}{a^2}+\frac{\left(y-\frac{k}{2}\right)^2}{b^2}=\frac{1}{4}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Its obviously an ellipse with centre at $\left(\frac{h}{2}, \frac{k}{2}\right)$ and axes parallel to coordinates axes.

15.13 Diameter

A line through the centre of an ellipse is called a diameter of the ellipse.

The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = -\frac{b^2}{a^2 m}x$, which is passing through (0, 0).

Conjugate diameter Two diameters of an ellipse are said to be conjugate diameter, if each bisects all chords parallel to the other. The coordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$;
$P'(-a\cos\phi, -b\sin\phi)$;
$Q(-a\sin\phi, b\cos\phi)$;
$Q'(a\sin\phi, -b\cos\phi)$

If $y = m_1 x$ and $y = m_2 x$ be two conjugate diameters of an ellipse, then $m_1 m_2 = \frac{-b^2}{a^2}$.

***Sample* Problem 16** *If $2y = x$ and $3y + 4x = 0$ are the equation of a pair of conjugate diameters of an ellipse, then eccentricity of the ellipse is*

(a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{2}{5}}$
(c) $\sqrt{\frac{1}{3}}$ (d) $\sqrt{\frac{1}{2}}$

Interpret (c) Slope of $2y = x$ is $\frac{1}{2}$ (m_1 say) and slope of $3y + 4x = 0$ is $-\frac{4}{3}$ (m_2 say)

$$\therefore \quad m_1 m_2 = -\frac{b^2}{a^2}$$

$$\therefore \quad \left(\frac{1}{2}\right)\left(-\frac{4}{3}\right) = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{b^2}{a^2} = \frac{2}{3}$$

$$\because \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$\therefore \quad e = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

15.14 Some Useful Properties of an Ellipse

(i) The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.

(ii) If SM and $S'M'$ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \cdot S'M' = b^2$ and M, M' lie on the auxiliary circle.

(iii) If the tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis in T and minor axis in T', then

$$CN \cdot CT = a^2, \quad CN' \cdot CT' = b^2,$$

where, N and N' are the feet of the perpendiculars from P on the respective axes.

(iv) If SM and $S'M'$ be perpendiculars from the foci S and S' respectively upon a tangent to the ellipse, then CM and CM' are parallel to $S'P$ and SP, respectively.

(v) The common chord of an ellipse and a circle are equally inclined to the axes of the ellipse.

***Sample* Problem 17** *An ellipse slides between two lines at right angle to one another. Then, the locus of its centre is*

(a) *a circle* (b) *a parabola*
(c) *an ellipse* (d) *a hyperbola*

Interpret (a) Let the two lines be taken as the x-axis and y-axis, respectively.

Let $C(a, b)$ be the centre of the ellipse.

Let a and b be the semi-major and semi-minor axes of the ellipse.

Let $S(x_1, y_1)$ and $S'(x_2, y_2)$ be the foci of the ellipse.

Then, $\quad x_1 + x_2 = 2\alpha \quad$ ...(i)

and $\quad y_1 + y_2 = 2\beta \quad$ ...(ii)

Again, since x-axis and y-axis are the tangents to the ellipse,

$\therefore \quad y_1 y_2 = b^2 \quad$ ...(iii)

and $\quad x_1 x_2 = b^2 \quad$ ...(iv)

Also, $\quad S'S^2 = 4a^2e^2$

$\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = 4a^2e^2 \quad$ ...(v)

We have, to eliminate x_1, y_1, x_2, y_2 from the Eqs. (i), (ii), (iii), (iv) and (v).

Now, $\{(x_1 + x_2)^2 + (y_1 + y_2)^2\} - \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$

$$= 4(x_1x_2 + y_1y_2)$$

$\Rightarrow \quad 4\alpha^2 + 4\beta^2 - 4a^2e^2 = 4(b^2 + b^2) = 8b^2$

$\Rightarrow \quad \alpha^2 + \beta^2 - a^2e^2 = 2b^2$

$\Rightarrow \quad \alpha^2 + \beta^2 = 2b^2 + a^2e^2 = 2b^2 + a^2 - b^2 = a^2 + b^2$

Hence, the locus of (α, β) is

$$x^2 + y^2 = a^2 + b^2.$$

which is a circle.

Check Point **2**

1. On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangent is parallel to the line $8x = 9y$, is

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) None of these

2. An equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive end of the latusrectum is

(a) $x + ey + e^3a = 0$ (b) $x - ey - ae^3 = 0$

(c) $x - ey + e^3a = 0$ (d) $x + ey - e^3a = 0$

3. The circle $x^2 + y^2 = c^2$ contains the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

(a) $c < a$ (b) $c < b$ (c) $c > a$ (d) $c > b$

4. If Q be the point on the auxiliary circle corresponding to a point P on an ellipse. Then, the normals at P and Q meet on

(a) a fixed circle (b) an ellipse

(c) a hyperbola (d) None of these

5. If the tangent line to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts intercepts h and k from axes, then $\frac{a^2}{h^2} + \frac{b^2}{k^2}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

6. The locus of mid-points of chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that touch the circle $x^2 + y^2 = b^2$, is

(a) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^4} + \frac{y^2}{b^4}$ (b) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = b^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$

(c) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = a^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$ (d) None of these

7. The eccentric angles of the extremities of latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by

(a) $\tan^{-1}\left(\pm \frac{ae}{b}\right)$ (b) $\tan^{-1}\left(\pm \frac{be}{a}\right)$

(c) $\tan^{-1}\left(\pm \frac{b}{ae}\right)$ (d) $\tan^{-1}\left(\pm \frac{a}{be}\right)$

8. An ellipse is described by using endles string which is passed over two pins. If the axes are 6 cm and 4 cm, the necessary length of the string and distance between the pins respectively in cm, are

(a) $6, 2\sqrt{5}$ (b) $6, \sqrt{5}$

(c) $4, 2\sqrt{5}$ (d) None of these

9. An ellipse is sliding along the coordinate axes. If the foci of the ellipse are (1, 1) and (3, 3), then area of the director circle of the ellipse is

(a) 2π sq units (b) 4π sq units (c) 6π sq units (d) 8π sq units

10. If the normals at $P(\theta)$ and $Q\left(\frac{\pi}{2} + \theta\right)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the major axis at G and g respectivel,y the $PG^2 + Qg^2$ is equal to

(a) $b^2(1 - e^2)(2 - e^2)$ (b) $a^2(e^4 - e^2 + 2)$

(c) $a^2(1 + e^2)(2 + e^2)$ (d) $b^2(1 + e^2)(2 + e^2)$

WORKED OUT Examples

Example 1 *The equation of the ellipse where focus is* (2, 0) *and the directrix is* $x + y + 1 = 0$ *and eccentricity is equal to* $\frac{1}{\sqrt{2}}$, *is*

(a) $3x^2 + 3y^2 + 2xy + 18x + 2y + 15 = 0$
(b) $3x^2 + 3y^2 - 2xy + 18x + 2y + 15 = 0$
(c) $3x^2 + 3y^2 - 2xy - 18x - 2y + 15 = 0$
(d) *None of the above*

Solution (c) Let $S(2, 0)$ be the focus. Consider $P(x, y)$ be any point on the ellipse and PM be perpendicular from P on the directrix.

$\therefore \quad SP = e\,PM$ (by definition)

$\Rightarrow \quad SP^2 = e^2PM^2$

$$\Rightarrow \quad (x-2)^2 + (y-0)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left[\frac{x+y+1}{\sqrt{1+1}}\right]^2$$

$\Rightarrow \quad 4[(x-2)^2 + y^2] = (x + y + 1)^2$

$\Rightarrow \quad 4x^2 - 16x + 16 + 4y^2 = x^2 + y^2 + 1 + 2xy + 2x + 2y$

$\Rightarrow \quad 3x^2 + 3y^2 - 2xy - 18x - 2y + 15 = 0$

This is the required equation of an ellipse.

Example 2 *The foci of the ellipse*

$$25(x+1)^2 + 9(y+2)^2 = 225 \text{ are}$$

(a) $(-1, 2)$ *and* $(-1, -6)$ (b) $(1, 2)$ *and* $(1, 6)$
(c) $(1, 2)$ *and* $(-1, -6)$ (d) $(-1, 2)$ *and* $(1, 6)$

Solution (a) The given equation of an ellipse can be rewritten as

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

whose centre is $(-1, -2)$ and semi-major and semi-minor axes are 5 and 3 respectively *i.e.*,

$$b = 5 \quad \text{and} \quad a = 3$$

$\therefore$ The eccentricity of the ellipse is given by

$$a^2 = b^2(1 - e^2)$$

$$\Rightarrow \quad 9 = 25(1 - e^2) \quad \Rightarrow \quad e = \frac{4}{5}$$

On shifting the origin at $(-1, -2)$ the given equation reduces to

$$\frac{X^2}{9} + \frac{Y^2}{25} = 1 \quad \ldots(i)$$

where, $x = X - 1$ and $y = Y - 2$ ...(ii)

$\therefore$ Coordinates of the foci are $(X = 0, Y = \pm be)$ *i.e.*,

$$(-1, 2) \text{ and } (-1, -6).$$

Example 3 *The line* $lx + my + n = 0$ *cuts the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *in points whose eccentric angles differ by* $\frac{\pi}{2}$, *if* $a^2l^2 + b^2m^2$ *is equal to*

(a) n^2 (b) $2n^2$ (c) $2n$ (d) n

Solution (b) Suppose the line $lx + my + n = 0$ cuts the ellipse at $P(a\cos\theta, b\sin\theta)$ and $Q\left\{a\cos\left(\frac{\pi}{2} + \theta\right), b\sin\left(\frac{\pi}{2} + \theta\right)\right\}$. Then, these two points lie on the line

$$la\cos\theta + mb\sin\theta + n = 0$$

and $\quad -la\sin\theta + mb\cos\theta + n = 0$

$\Rightarrow \quad la\cos\theta + mb\sin\theta = -n$

and $\quad -la\sin\theta + mb\cos\theta = -n$

$\Rightarrow \quad (la\cos\theta + mb\sin\theta)^2 + (-la\sin\theta + mb\cos\theta)^2 = n^2 + n^2$

$\Rightarrow \quad l^2a^2 + m^2b^2 = 2n^2$

Example 4 *The product of the perpendiculars drawn from the foci upon any tangent to an ellipse*

(a) *depends upon foci*
(b) *is constant*
(c) *depends upon the tangent*
(d) *None of the above*

Solution (b) Let equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

Its foci are $S(ae, 0)$ and $S'(-ae, 0)$. The equation of tangent at any point $(a \cos\theta, b \sin\theta)$ to ellipse (i) is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \text{...(ii)}$$

Let the perpendicular from S and S' upon Eq. (ii) be SM and $S'N$, then

$$SM \cdot S'N = \frac{-1}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}(e^2\cos^2\theta - 1)$$

$$\Rightarrow \quad SM \cdot S'N = b^2 = \text{a constant.}$$

Example 5 *Minimum area of the triangle by any tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *with the coordinate axes is*

(a) $a^2 + b^2$ (b) $\frac{(a+b)^2}{2}$ (c) ab (d) $\frac{(a-b)^2}{2}$

Solution (c) Equation of tangent at $(a \cos\theta, b \sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$\therefore$ Coordinates of P and Q are $\left(\frac{a}{\cos\theta}, 0\right)$ and $\left(0, \frac{b}{\sin\theta}\right)$ respectively.

$$\therefore \quad \text{Area of } \Delta OPQ = \frac{1}{2}\left|\frac{a}{\cos\theta} \times \frac{b}{\sin\theta}\right| = \frac{ab}{|\sin 2\theta|}$$

$$\therefore \quad (\text{Area})_{\text{minimum}} = ab$$

Example 6 *The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse* $x^2 + 2y^2 = 6$ *which touches the ellipse* $x^2 + 4y^2 = 4$, *is*

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 6$

(c) $x^2 + y^2 = 9$ (d) *None of these*

Solution (c) The given equation of second ellipse can be rewritten as

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

Equation of tangent to this ellipse is

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \quad \text{...(i)}$$

Equation of the first ellipse can be rewritten as

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \text{...(ii)}$$

Let ellipse (i) meets the first ellipse at P and Q and the tangents at P and Q to the second ellipse intersected at (h, k), then Eq. (i) is the chord of contact of (h, k) with respect to the ellipse (ii) and thus, its equation is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \quad \text{...(iii)}$$

Since, Eqs. (i) and (iii) represent the same line.

$$\therefore \quad \frac{h/6}{\cos\theta/2} = \frac{k/3}{\sin\theta} = 1$$

$$\Rightarrow \quad h = 3\cos\theta \quad \text{and} \quad k = 3\sin\theta$$

Hence, locus is $x^2 + y^2 = 9$.

Example 7 *If a number of ellipses be described having the same major axis, but a variable minor axis, such that each one of the tangents at the ends of their latusrectum passes through the fixed point, then the point of concurrency is*

(a) (1, 1) (b) (0, 1)

(c) (1, 0) (d) *None of these*

Solution (b) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Since, major axis is fixed,

$\therefore a$ is variable.

Hence, e is variable. Now, the ends of the latusrectum are $\left(\pm ae, \pm\frac{b^2}{a}\right)$.

Hence, the tangents at these points are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\Rightarrow \quad \frac{\pm x \cdot ae}{a^2} + \frac{\pm y \cdot \frac{b^2}{a}}{b^2} = 1$$

$$\Rightarrow \quad \pm\frac{xe}{a} \pm \frac{y}{a} = 1$$

$$\Rightarrow \quad \pm xe \pm y = a$$

$$\Rightarrow \quad \pm ex \pm y - a = 0$$

Hence, there are 4 tangents.

Since, e is a variable.

$\therefore$ Each of the tangents passes through the point (0, 1) which is fixed.

Example 8 *An arch is the form of a semi ellipse, It is* 8 *m wide and and* 2 *m high of the centre. Find the heigh t of the arch at a point* 1.5 *m from one end.* **[NCERT]**

(a) 1.58 *m approx* (b) 1.56 *m approx*

(c) 1.62 *m approx* (d) *None of these*

Solution (b) Clearly, equation of ellipse takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Here, it is given $2a = 8$ and $b = 2 \Rightarrow a = 4, b = 2$

Put the values of a and b in Eq. (i), we get

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

Given, $AP = 1.5$ m

$\Rightarrow$ $OP = OA - AP = 4 - 1.5$

$\Rightarrow$ $OP = 2.5$ m

Let $PQ = k$

$\therefore$ Coordinate $Q = (2.5, k)$ will satisfy the equation of ellipse.

i.e., $$\frac{(2.5)^2}{16}+\frac{k^2}{4}=1$$

$$\Rightarrow \quad \frac{6.25}{16}+\frac{k^2}{4}=1$$

$$\Rightarrow \quad \frac{k^2}{4}=\frac{1}{1}-\frac{6.25}{16}=\frac{16-6.25}{16}$$

$$\Rightarrow \quad \frac{k^2}{4}=\frac{9.75}{16}$$

$$\Rightarrow \quad k^2=\frac{9.75}{4}$$

$$\Rightarrow \quad k^2=2.4375$$

$$\Rightarrow \quad k=1.56 \text{ m (approx)}$$

Example 9 *A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.* **[NCERT]**

(a) $\frac{x^2}{81}-\frac{y^2}{9}=1$ (b) $\frac{x^2}{9}+\frac{y^2}{3}=1$

(c) $\frac{x^2}{81}+\frac{y^2}{9}=1$ (d) $\frac{x^2}{9}+\frac{y^2}{81}=1$

Solution (c) Let AB be the rod of length 12 cm.

Let $OA = a$ and $OB = b$

Using by Pythagoras theorem in ΔOAB.

$(\text{Base})^2 + (\text{Perpendicular})^2 = (\text{Hypotenuse})^2$

$$\Rightarrow \quad a^2+b^2=(12)^2 \quad \text{...(i)}$$

$$\Rightarrow \quad a^2+b^2=144$$

Let $AP = 3$ cm $\Rightarrow PB = AB - AP = 12 - 3 = 9$ cm

$\Rightarrow$ $AP : BP = 3 : 9 = 1 : 3$

Let $P = (h, k)$ divides the points A and B internally in the ratio $m : n$, then

$$h=\frac{mx_2+nx_1}{m+n},\ k=\frac{my_2+ny_1}{m+n}$$

$$\therefore \quad h=\frac{1\times 0+3\times a}{1+3}\ h=\frac{3a}{4}$$

$$a=\frac{4h}{3}$$

and $$k=\frac{1\times b+3\times 0}{1+3}$$

$$k=\frac{b}{4}\Rightarrow b=4k$$

Put the values of $a=\frac{4h}{3}$ and $b = 4k$ in Eq. (i), we get

$$\left(\frac{4h}{3}\right)^2+(4k)^2=144$$

$$\Rightarrow \quad \frac{16h^2}{9}+16k^e=144$$

$$\Rightarrow \quad \frac{h^2}{9}+k^2=9\Rightarrow\frac{h^2}{81}+\frac{k^2}{9}=1$$

Hence, locus of a point $P(h, k)$ is

$$\frac{x^2}{81}+\frac{y^2}{9}=1$$

Example 10 *A man running a race course notes that sum of its distances from two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.* **[NCERT]**

(a) $\frac{x^2}{9}+\frac{y^2}{25}=1$ (b) $\frac{x^2}{25}+\frac{y^2}{9}=1$

(c) $\frac{x^2}{3}+\frac{y^2}{5}=1$ (d) *None of these*

Solution (b) Clearly, path traced by the man will be ellipse.

Given, $SP + S'P = 10$

i.e., $2a = 10 \Rightarrow a = 5$

Since, the coordinates of S and S' are $(c, 0)$ and $(-c, 0)$. Therefore, distance between S and S' is

$$2c = 8$$

$$\Rightarrow \quad c = 4$$

$$\because \quad c^2 = a^2 - b^2$$

$$\therefore \quad 16 = 25 - b^2$$

$$\Rightarrow \quad b^2 = 25 - 16$$

$$\Rightarrow \quad b^2 = 9$$

$$\Rightarrow \quad b = 3$$

Hence, equation of path (ellipse) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \quad \frac{x^2}{25}+\frac{y^2}{9}=1$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Basic Terms of Ellipse

1. If $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ $(abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0)$ represents an ellipse, if

(a) $h^2 = ab$ (b) $h^2 > ab$

(c) $h^2 < ab$ (d) None of these

2. The equation of the ellipse whose focus is (1, –1), the directrix of line $x - y - 3 = 0$ and eccentricity $\frac{1}{2}$ is [NCERT Exemplar]

(a) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

(b) $7x^2 + 2xy + 7y^2 + 7 = 0$

(c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

(d) None of the above

3. The length of the latusrectum of the ellipse $3x^2 + y^2 = 12$ is [NCERT Exemplar]

(a) 4 (b) 3

(c) 8 (d) $\frac{4}{\sqrt{3}}$

4. If e is the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a < b)$, then [NCERT Exemplar]

(a) $b^2 = a^2(1 - e^2)$

(b) $a^2 = b^2(1 - e^2)$

(c) $a^2 = b^2(e^2 - 1)$

(d) $b^2 = a^2(e^2 - 1)$

5. If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose foci are S and S', then $PS + PS'$ is equal to [NCERT Exemplar]

(a) 8 (b) 7

(c) 5 (d) 10

6. If the latusrectum of an ellipse is equal to half of minor axis, then its eccentricity is [NCERT Exemplar]

(a) $\frac{\sqrt{15}}{3}$ (b) $\frac{\sqrt{15}}{2}$

(c) $\frac{\sqrt{15}}{6}$ (d) $\frac{\sqrt{15}}{4}$

7. If the eccentricity of an ellipse is $\frac{5}{8}$ and the distance between its foci is 10, then length of latus rectum of the ellipse is [NCERT Exemplar]

(a) $\frac{39}{7}$ (b) $\frac{39}{4}$

(c) $\frac{39}{5}$ (d) $\frac{39}{8}$

8. The locus of a point which moves such that the sum of its distances from two fixed points is always a constant, is

(a) a straight line (b) a circle

(c) an ellipse (d) a hyperbola

9. Find the distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$. [NCERT Exemplar]

(a) –18 (b) 18

(c) 17 (d) 19

10. If equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{25} = 1$, then coordinate of the foci, eccentricity and the length of the latusrectum are respectively [NCERT]

(a) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{5}, \frac{7}{5}$

(b) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{5}, \frac{8}{5}$

(c) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{7}, \frac{8}{5}$

(d) None of the above

Equation of Ellipse and Intersection of Two Curves

11. If vertices and foci of an ellipse are $(0, \pm 13)$ and $(0, \pm 5)$ respectively, then the equation of an ellipse is [NCERT]

(a) $\frac{x^2}{144} + \frac{y^2}{169} = 1$ (b) $\frac{x^2}{169} + \frac{y^2}{144} = 1$
(c) $\frac{x^2}{12} + \frac{y^2}{13} = 1$ (d) None of these

12. Find the equation of an ellipse, if major axis on the x-axis and passes through the points (4, 3) and (6, 2). [NCERT]

(a) $\frac{x^2}{13} + \frac{y^2}{52} = 1$ (b) $\frac{x^2}{40} + \frac{y^2}{10} = 1$
(c) $\frac{x^2}{52} + \frac{y^2}{13} = 1$ (d) None of these

13. The curve with parametric equations $x = \alpha + 5\cos\theta$, $y = \beta + 4\sin\theta$ (where, θ is parameter) is
(a) a parabola (b) an ellipse
(c) a hyperbola (d) None of these

14. The curve represented by the equation $4x^2 + 16y^2 - 24x - 32y - 12 = 0$ is
(a) a parabola
(b) a pair of straight lines
(c) an ellipse with eccentricity $\frac{1}{2}$
(d) an ellipse with eccentricity $\frac{\sqrt{3}}{2}$

15. Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1), respectively. Then,
(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

16. The length of the axes of the conic $9x^2 + 4y^2 - 6x + 4y + 1 = 0$, are
(a) $\frac{1}{2}, 9$ (b) $3, \frac{2}{5}$ (c) $1, \frac{2}{3}$ (d) 3, 2

17. In an ellipse length of minor axis is 8 and eccentricity is $\frac{\sqrt{5}}{3}$. The length of major axis is
(a) 6 (b) 12 (c) 10 (d) 16

18. The sum of focal distance of any point on the ellipse with major and minor axes as $2a$ and $2b$ respectively, is equal to
(a) $2a$ (b) $2\frac{a}{b}$ (c) $2\frac{b}{a}$ (d) $\frac{b^2}{a}$

19. The eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$, whose distance from the centre of the ellipse is 2, is
(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{3}$ (d) $\frac{7\pi}{6}$

20. If e is eccentricity of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,(a > b)$ and e' is eccentricity of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a < b)$, then
(a) $e = e'$ (b) $ee' = 1$
(c) $\frac{1}{e^2} + \frac{1}{(e')^2} = 1$ (d) None of these

21. If the eccentricity of the two ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equal, then the value of $\frac{a}{b}$ is
(a) $\frac{5}{13}$ (b) $\frac{6}{13}$ (c) $\frac{13}{5}$ (d) $\frac{13}{6}$

22. If $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$ are the end points of a focal chord of an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, then which of the following is correct?
(a) $e = \frac{\sin\alpha - \sin\beta}{\sin(\alpha - \beta)}$ (b) $e = \frac{\cos\left(\frac{\alpha - \beta}{2}\right)}{\cos\left(\frac{\alpha + \beta}{2}\right)}$
(c) $\frac{e-1}{e+1} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ (d) None of these

23. The line $x = at^2$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the real points, if
(a) $|t| < 2$ (b) $|t| \le 1$
(c) $|t| > 1$ (d) None of these

Tangent and Normal to the Ellipse

24. The equation of tangents to the ellipse $3x^2 + 4y^2 = 5$, which are inclined at 30° to the x-axis, are
(a) $y = \sqrt{3}x \pm \frac{5}{2}$ (b) $y = \frac{1}{\sqrt{3}}x \pm \frac{5}{2}$
(c) $y = \frac{1}{\sqrt{3}}x \pm 1$ (d) None of these

25. The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$, is
(a) $\tan^{-1}(12/5)$ (b) $\tan^{-1}(6/\sqrt{5})$
(c) $\tan^{-1}(12/\sqrt{5})$ (d) $\tan^{-1}(6/5)$

26. If $x\cos\alpha + y\sin\alpha = p$ is a tangent to the ellipse, then
(a) $a^2\sin^2\alpha + b^2\cos^2\alpha = p^2$
(b) $a^2 + b^2\sin^2\alpha = p^2\operatorname{cosec}^2\alpha$
(c) $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$
(d) None of the above

27. Number of tangents from (7, 6) to ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is
(a) 0 (b) 1
(c) 2 (d) None of these

28. The distance of the centre of ellipse $x^2+2y^2-2=0$ to those tangents of the ellipse which are equally inclined from both the axes, is
(a) $\frac{3}{\sqrt{2}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{3}}{2}$

29. If the two tangents drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect perpendicularly at p, then locus of p is a circle $x^2+y^2=a^2+b^2$. Then, the circle is called
(a) auxiliary circle (b) director circle
(c) great circle (d) None of these

30. Equation of tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, which are perpendicular to the line $3x+4y=7$, are
(a) $4x-3y=\pm\sqrt{20}$ (b) $4x-3y=\pm\sqrt{12}$
(c) $4x-3y=\pm\sqrt{2}$ (d) $4x-3y=\pm 1$

31. If the normal at point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes in R and S respectively, then $PR:RS$ is equal to
(a) $a:b$ (b) $a^2:b^2$
(c) $b^2:a^2$ (d) $b:a$

32. On the ellipse $2x^2+3y^2=1$ the points at which the tangent is parallel to $4x=3y+4$, are
(a) $\left(\frac{2}{\sqrt{11}},\frac{1}{\sqrt{11}}\right)$ or $\left(-\frac{2}{\sqrt{11}},-\frac{1}{\sqrt{11}}\right)$
(b) $\left(-\frac{2}{\sqrt{11}},\frac{1}{\sqrt{11}}\right)$ or $\left(\frac{2}{\sqrt{11}},-\frac{1}{\sqrt{11}}\right)$
(c) $\left(-\frac{2}{5},-\frac{1}{5}\right)$
(d) $\left(-\frac{3}{5},-\frac{2}{5}\right)$ or $\left(\frac{3}{5},\frac{2}{5}\right)$

33. The locus of the point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is
(a) $x^2+y^2=9$ (b) $x^2+y^2=4$
(c) $x^2+y^2=13$ (d) $x^2+y^2=5$

34. If tangents are drawn to the ellipse $x^2+2y^2=2$, then the locus of the mid-point of the intercept made by the tangents between the coordinate axes is
(a) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$
(c) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (d) $\frac{x^2}{4}+\frac{y^2}{2}=1$

35. Tangent is drawn to the ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta,\sin\theta)\left[\text{where, }\theta\in\left(0,\frac{\pi}{2}\right)\right]$. Then, the value of θ such that sum of intercepts on axes made by this tangent is minimum, is
(a) $\pi/3$ (b) $\pi/6$
(c) $\pi/8$ (d) $\pi/4$

36. If $(\sqrt{3})bx+ay=2ab$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then eccentric angle ϕ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

37. If p is the length of the perpendicular from a focus upon the tangent at any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and r is the distance of P from the focus, then $\frac{2a}{r}-\frac{b^2}{p^2}$ is equal to
(a) -1 (b) 0 (c) 1 (d) 2

38. The tangent drawn from (α,β) to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=c^2$, then the locus of (α,β) is
(a) an ellipse (b) a circle
(c) a parabola (d) None of these

39. If the tangent at the point $\left(4\cos\phi,\frac{16}{\sqrt{11}}\sin\phi\right)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x=15$, then the value of ϕ is
(a) $\pm\pi/2$ (b) $\pm\pi/4$
(c) $\pm\pi/3$ (d) $\pm\pi/6$

40. A tangent at any point to the ellipse $4x^2+9y^2=36$ is cut by the tangent at the extremities of the major axis at T and T'. The circle on TT' as diameter passes through the point
(a) $(0,\sqrt{5})$ (b) $(\sqrt{5},0)$
(c) (2, 1) (d) $(0,-\sqrt{5})$

41. Product of the perpendicular from the foci upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a<b)$ is equal to
(a) $2a$ (b) a^2 (c) b^2 (d) ab^2

42. If the line $x\cos\alpha+y\sin\alpha=p$ be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then
(a) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=a^2-b^2$
(b) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=(a^2-b^2)^2$
(c) $p^2(a^2\sec^2\alpha+b^2\operatorname{cosec}^2\alpha)=a^2-b^2$
(d) $p^2(a^2\sec^2\alpha+b^2\operatorname{cosec}^2\alpha)=(a^2-b^2)^2$

43. The equation of the normal at the point (2, 3) on the ellipse $9x^2 + 16y^2 = 180$ is
(a) $3y = 8x - 10$ (b) $3y - 8x + 7 = 0$
(c) $8y + 3x + 7 = 0$ (d) $3x + 2y + 7 = 0$

44. Equation of the normal to the ellipse $4(x-1)^2 + 9(y-2)^2 = 36$, which is parallel to the line $3x - y = 1$, is
(a) $3x - y = \sqrt{5}$ (b) $3x - y = \sqrt{5} - 3$
(c) $3x - y = \sqrt{5} + 2$ (d) $3x - y = \sqrt{5}(\sqrt{5} + 1)$

45. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersect it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to
(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

46. In the normal at the end of latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity e, passes through one end of the minor axis, then
(a) $e^2(1 + e^2) = 0$ (b) $e^2(1 + e^2) = 1$
(c) $e^2(1 + e^2) = -1$ (d) $e^2(1 + e^2) = 2$

47. If $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ is normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m belonging to
(a) $(0, 1)$ (b) $(0, \infty)$
(c) R (d) None of these

48. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets the axes in G and g respectively, then $PG : Pg$ is equal to
(a) $a : b$ (b) $a^2 : b^2$
(c) $b^2 : a^2$ (d) $b : a$

49. If the normal at one end of the latusrectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the one end of the minor axis, then
(a) $e^4 - e^2 + 1 = 0$ (b) $e^2 - e + 1 = 0$
(c) $e^2 + e + 1 = 0$ (d) $e^4 + e^2 - 1 = 0$

50. The locus of the point of intersection of the tangents at the end points of the focal chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(b < a)$ is
(a) $x = \pm \frac{a^2}{\sqrt{a^2 - b^2}}$ (b) $y = \pm \frac{b^2}{\sqrt{a^2 - b^2}}$
(c) $x = \pm \frac{ab}{\sqrt{a^2 - b^2}}$ (d) None of these

51. The sum of the squares of the perpendiculars on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from two points on the minor axis each at a distance $\sqrt{a^2 - b^2}$ from the centre is
(a) $2a^2$ (b) $2b^2$ (c) $a^2 + b^2$ (d) $a^2 - b^2$

52. Angle between tangents drawn from the point (5, 4) to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, is
(a) 60° (b) 90° (c) 120° (d) 45°

Chord of Contact and Diameter of an Ellipse

53. Locus of the mid-points of the chord of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that chord is always touching the circle $x^2 + y^2 = c^2$, $(c < a, c < b)$ is
(a) $(b^2x^2 + a^2y^2)^2 = c^2(b^4x^2 + a^4y^2)$
(b) $(a^2x^2 + b^2y^2)^2 = c^2(a^4x^2 + b^4y^2)$
(c) $(b^2x^2 + a^2y^2)^2 = c^2(b^2x^4 + a^2y^4)$
(d) None of the above

54. Tangents are drawn from the point on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$, then all the chords of contact pass through a fixed point, whose coordinates are
(a) $\left(\frac{1}{5}, \frac{4}{5}\right)$ (b) $\left(\frac{4}{5}, \frac{1}{5}\right)$ (c) $\left(-\frac{4}{5}, -\frac{1}{5}\right)$ (d) $\left(\frac{4}{5}, -\frac{1}{5}\right)$

55. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to
(a) $\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{a^4}{b^4}$ (d) $-\frac{b^4}{a^4}$

56. If $lx + my + n = 0$ represents a chord of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ whose eccentric angles differ by 90°, then
(a) $a^2l^2 + b^2m^2 = n^2$ (b) $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$
(c) $a^2l^2 + b^2m^2 = 2n^2$ (d) None of these

57. The length of the common chord of the ellipse
$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$$
and the circle $(x-1)^2 + (y-2)^2 = 1$ is
(a) 2 (b) $\sqrt{3}$
(c) 4 (d) None of these

58. Equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, whose mid-point is (1, 1), is
(a) $25x+9y=36$ (b) $9x+25y=34$
(c) $9x-25y=34$ (d) None of these

59. Equation of the chord of contact of pair of tangents drawn to the ellipse $4x^2+9y^2=36$ from the point (m, n), where $mn=m+n$, m and n being non-zero positive integers is
(a) $2x+9y=18$ (b) $2x+2y=1$
(c) $4x+9y=18$ (d) None of these

Miscellaneous

60. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{4}+\frac{4}{7}y^2=1$ and having its centre at $\left(\frac{1}{2}, 2\right)$, is
(a) $\sqrt{5}$ (b) 3 (c) $\sqrt{12}$ (d) $\frac{7}{2}$

61. A common tangent to circle $x^2+y^2=16$ and an ellipse $\frac{x^2}{49}+\frac{y^2}{4}=1$ is
(a) $y=x+4\sqrt{5}$ (b) $y=x+\sqrt{53}$
(c) $y=\frac{2}{\sqrt{11}}x+\frac{4\sqrt{4}}{\sqrt{11}}$ (d) None of these

62. If a variable tangent of the circle $x^2+y^2=1$ intersects the ellipse $x^2+2y^2=4$ at points P and Q, then the locus of the point of intersection of tangent at P and Q is
(a) a circle of radius 2 units
(b) a parabola with focus as (2, 3)
(c) an ellipse with eccentricity $\frac{\sqrt{3}}{2}$
(d) None of the above

63. A man running round a race course notes that the sum of the distances of two flag posts from him is always 10 m and the distance between the flag posts is 8 m. The area of the path he encloses (in square metre) is
(a) 15π (b) 12π (c) 18π (d) 8π

64. The area of the quadrilateral formed by the tangents at the end points of latusrectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is
(a) 27/4 sq units (b) 9 sq units
(c) 27/2 sq units (d) 27 sq units

65. Equation of the circle passing through the intersection of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, is
(a) $x^2+y^2=a^2$ (b) $x^2+y^2=b^2$
(c) $x^2+y^2=\frac{a^2b^2}{a^2+b^2}$ (d) $x^2+y^2=\frac{2a^2b^2}{a^2+b^2}$

66. If a bar of given length moves with its extremities on two fixed straight lines at right angles, then the locus of any point on bar marked on the bar describes a/an
(a) circle (b) parabola (c) ellipse (d) hyperbola

67. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is
(a) 0 (b) 2 (c) 1 (d) 4

68. If θ and ϕ are eccentric angle of the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\theta-\phi$ is equal to
(a) $\pm\frac{\pi}{2}$ (b) $\pm\pi$
(c) 0 (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. Let (α, β) be a point from which two perpendicular tangents can be drawn to the ellipse $4x^2+5y^2=20$. If $F=4\alpha+3\beta$, then
(a) $-15 \le F \le 15$ (b) $F \ge 0$
(c) $-5 \le F \le 20$ (d) $F \le -5\sqrt{5}$ or $F \ge 5\sqrt{5}$

2. Three points A, B and C are taken on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentric angles θ, $\theta+\alpha$ and $\theta+2\alpha$, then
(a) the area of ΔABC is independent of θ
(b) the area of ΔABC is independent of α
(c) the maximum value of area is $\frac{\sqrt{3}}{4}ab$
(d) the maximum value of area is $\frac{3\sqrt{3}}{4}ab$

3. The point, at shortest distance from the line $x+y=7$ and lying on an ellipse $x^2+2y^2=6$, has coordinates
(a) $(\sqrt{2}, \sqrt{2})$ (b) $(0, \sqrt{3})$
(c) (2, 1) (d) $\left(\sqrt{5}, \frac{1}{\sqrt{2}}\right)$

4. A circle has the same centre as an ellipse and passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let P be any one of their points of intersection. If the major axis of the ellipse is 17 and the area of the ΔPF_1F_2 is 30, then the distance between the foci is

(a) 13 (b) 10
(c) 11 (d) None of these

5. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ and the circle $x^2+y^2=a^2$ at the points, where a common ordinate cuts them (on the same side of the x-axis). Then, the greatest acute angle between these tangents is given by

(a) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$
(c) $\tan^{-1}\left(\frac{2ab}{\sqrt{a-b}}\right)$ (d) $\tan^{-1}\left(\frac{2ab}{\sqrt{a+b}}\right)$

6. If the ellipse $\frac{x^2}{a^2-7}+\frac{y^2}{13-5a}=1$ is inscribed in a square of side length $\sqrt{2}a$, then a is equal to

(a) 6/5 (b) $(-\infty, -\sqrt{7}) \cup (\sqrt{7}, 13/5)$
(c) $(-\infty, -\sqrt{7}) \cup (13/5, \sqrt{7})$ (d) No such a exists

7. If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is inscribed in a rectangle whose length to breadth ratio is 2 : 1, then the area of the rectangle is

(a) $4\frac{a^2+b^2}{7}$ (b) $4\frac{a^2+b^2}{3}$
(c) $12\frac{a^2+b^2}{5}$ (d) $8\frac{a^2+b^2}{5}$

8. A parabola is drawn with focus is at one of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where, $a>b$) and directrix passing through the other focus and perpendicular to the major axes of the ellipse. If latusrectum of the ellipse and the parabola are same, then the eccentricity of the ellipse is

(a) $1-\frac{1}{\sqrt{2}}$ (b) $2\sqrt{2}-2$
(c) $\sqrt{2}-1$ (d) None of these

More Than One Correct Option

9. The points, where the normals to the ellipse $x^2+3y^2=37$ be parallel to the line $6x-5y+7=0$ is/are

(a) (5, 2) (b) (2, 5)
(c) (1, 3) (d) (–5, –2)

10. Let F_1 and F_2 be two foci of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P. Then,

(a) PN bisects $\angle F_1PF_2$
(b) PT bisects $\angle F_1PF_2$
(c) PT bisects angle $(180° - \angle F_1PF_2)$
(d) None of the above

11. If the tangent at the point $P(\theta)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x=15$, then θ is equal to

(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

Comprehension Based Questions

Passage

Directions (Q. Nos. 12-14) *The ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is such that it has the least area but contains the circle* $(x-1)^2+y^2=1$.

12. The eccentricity of the ellipse is

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{2}$ (d) None of these

13. Equation of auxiliary circle of ellipse is

(a) $x^2+y^4=65$ (b) $x^2+y^4=5$
(c) $x^2+y^4=45$ (d) None of these

14. Length of latusrectum of the ellipse is

(a) 2 units (b) 1 unit
(c) 3 units (d) 2.5 units

Assertion and Reason

Directions (Q. Nos. 15-21) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

15. **Statement I** The sum of focal distances of a point on the ellipse $9x^2+4y^2-18x-24y+9=0$ is 4.

Statement II The equation $9x^2+4y^2-18x-24y+9=0$ can be expressed as $9(x-1)^2+4(y-3)^2=36$.

16. **Statement I** The major and minor axes of the ellipse $5x^2 + 9y^2 - 54y + 36 = 0$ are 6 and 10, respectively.

Statement II The equation $5x^2 + 9y^2 - 54y + 36 = 0$ can be expressed as $5x^2 + 9(y-3)^2 = 45$.

17. **Statement I** The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by the line $x + y = b$ is $a^2 + 6ab - 7b^2 \geq 0$.

Statement II Equation of chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid-point (x_1, y_1), is $T = S_1$.

18. **Statement I** Tangents drawn from any point on the circle $x^2 + y^2 = 13$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are at right angles.

Statement II Equation of the auxiliary circle of the ellipse $\frac{x^2}{13} + \frac{y^2}{4} = 1$ is $x^2 + y^2 = 13$.

19. **Statement I** The area of the ellipse $2x^2 + 3y^2 = 6$ is more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

Statement II The length of semi-major axes of an ellipse is more than the radius of the circle.

20. **Statement I** If tangent at point P (in first quadrant) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$, meets corresponding directrix $x = a/e$ at point Q, then circle with minimum radius having PQ as chord passes through the corresponding focus.

Statement II PQ subtends right angle at corresponding focus.

21. **Statement I** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$. [AIEEE 2012]

Statement II If the line $y = mx + \frac{4\sqrt{3}}{m}, (m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

Previous Years' Questions

22. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is [JEE Main 2013]

(a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 6y + 7 = 0$
(c) $x^2 + y^2 - 6y - 5 = 0$ (d) $x^2 + y^2 - 6y + 5 = 0$

23. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ is semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is [AIEEE 2012]

(a) $4x^2 + y^2 = 4$ (b) $x^2 + 4y^2 = 8$
(c) $4x^2 + y^2 = 8$ (d) $x^2 + 4y^2 = 16$

24. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{2/5}$ is [AIEEE 2011]

(a) $5x^2 + 3y^2 - 48 = 0$ (b) $3x^2 + 5y^2 - 15 = 0$
(c) $5x^2 + 3y^2 - 32 = 0$ (d) $3x^2 + 5y^2 - 32 = 0$

25. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which is turn in inscribed in another ellipse that passes through the point (5, 0). Then, the equation of the ellipse is [AIEEE 2009]

(a) $x^2 + 12y^2 = 16$ (b) $4x^2 + 48y^2 = 48$
(c) $4x^2 + 64y^2 = 48$ (d) $x^2 + 16y^2 = 16$

26. A focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is 1/2, then length of semi-major axis is [AIEEE 2008]

(a) $\frac{5}{3}$ (b) $\frac{8}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

27. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then, its eccentricity is [AIEEE 2006]

(a) 1/2 (b) 4/5 (c) $1/\sqrt{5}$ (d) 3/5

28. If the angle between the lines joining the end points of minor axis of an ellipse with its foci is $\frac{\pi}{2}$, then the eccentricity of the ellipse is [AIEEE 2005]

(a) 1/2 (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $1/2\sqrt{2}$

29. An ellipse has OB as semi-minor axis, F and F' its foci and the $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is [AIEEE 2005]

(a) $1/\sqrt{3}$ (b) 1/4 (c) 1/2 (d) $1/\sqrt{2}$

30. The eccentricity of an ellipse, with centre at the origin, is 1/2 If one directrix is $x = 4$, the equation of the ellipse is [AIEEE 2004]

(a) $3x^2 + 4y^2 = 1$ (b) $3x^2 + 4y^2 = 12$
(c) $4x^2 + 3y^2 = 1$ (d) $4x^2 + 3y^2 = 12$

Answers

Check Point 1

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (c) **6.** (b) **7.** (c) **8.** (a)

Check Point 2

1. (b) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (b) **7.** (c) **8.** (d) **9.** (d) **10.** (b)

Round I

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (b)
11. (a)	**12.** (c)	**13.** (b)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (c)
21. (c)	**22.** (b)	**23.** (b)	**24.** (d)	**25.** (c)	**26.** (c)	**27.** (c)	**28.** (d)	**29.** (b)	**30.** (a)
31. (c)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (b)
41. (b)	**42.** (d)	**43.** (b)	**44.** (d)	**45.** (b)	**46.** (b)	**47.** (c)	**48.** (c)	**49.** (d)	**50.** (a)
51. (a)	**52.** (b)	**53.** (a)	**54.** (d)	**55.** (c)	**56.** (c)	**57.** (d)	**58.** (b)	**59.** (c)	**60.** (a)
61. (d)	**62.** (c)	**63.** (a)	**64.** (d)	**65.** (d)	**66.** (c)	**67.** (d)	**68.** (a)		

Round II

1. (a)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (a,b)	**10.** (a,c)
11. (c,d)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (a)
21. (b)	**22.** (a)	**23.** (d)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (d)	**30.** (b)

the Guidance

Round I

1. Equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents an ellipse, if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

and $h^2 < ab$

2. By definition of ellipse $SP = ePM$, where P is any point, S is focus and M is point on directrix.

$$\therefore \quad \sqrt{(x-1)^2 + (y+1)^2} = \frac{1}{2}\frac{|x-y-3|}{\sqrt{1^2+1^2}}$$

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y+1)^2} = \frac{1}{2\sqrt{2}}|x-y-3|$$

$$\Rightarrow \quad 8(x^2 + 1 - 2x + y^2 + 1 + 2y) = x^2 + y^2 + 9 - 2xy + 6y - 6x$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy - 10x + 10y + 7 = 0$$

3. Given equation of ellipse can be rewritten as

$$\frac{x^2}{4} + \frac{y^2}{12} = 1$$

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get $a^2 = 4, b^2 = 12$

Here, $b > a$

$\therefore$ Length of latusrectum $= \frac{2a^2}{b} = \frac{2 \times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

4. Given equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, (a < b)$$

$\therefore$ Eccentricity, $e = \sqrt{1 - \frac{a^2}{b^2}}$

$$\Rightarrow \quad e^2 = \frac{b^2 - a^2}{b^2}$$

$$\Rightarrow \quad a^2 = b^2(1 - e^2)$$

5. Given equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Here, $a^2 = 16, b^2 = 25$ *i.e.*, $b > a$

$\therefore$ $PS + PS' = 2b$ ($\because$ sum of focal radii $= 2b$)

$= 2 \times 5 = 10$

6. Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$.

According to the given condition,

Latusrectum of ellipse $= \frac{1}{2}$ (minor axis)

$$\Rightarrow \quad \frac{2b^2}{a} = \frac{1}{2}(b)$$

$\Rightarrow \quad 4b = a$

$\because \quad e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{b^2}{(4b)^2}}$

$= \sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{4}$

7. Given, $e = \frac{5}{8}$ and $2ae = 10$

$\therefore \quad 2a \times \frac{5}{8} = 10$

$\Rightarrow \quad a = 8$

$\because \quad b^2 = a^2(1-e^2) = 64\left(1-\frac{25}{64}\right) = 39$

$\therefore$ Length of latusrectum $= \frac{2b^2}{a} = \frac{2\times 39}{8} = \frac{39}{4}$

8. The locus of a point which moves such that the sum of its distances from two fixed points is always a constant is an ellipse.

9. Given equation of ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1$

Here, $a^2 = 36, b^2 = 20$

$\therefore$ Distance between directrices $= \frac{2a}{e}$

Now, $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{20}{36}} = \sqrt{\frac{16}{36}} = \frac{4}{6} = \frac{2}{3}$

$\therefore$ Distance between directrices $= \frac{2\times 6}{2/3} = 18$

10. Given equation of ellipse is

$$\frac{x^2}{4}+\frac{y^2}{25}=1$$

Here, $a^2 = 4, b^2 = 25$

i.e., $a < b$

$\therefore$ The major axis is along y-axis.

$\therefore \quad e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{4}{25}} = \frac{\sqrt{21}}{5}$

Here, major axis is along y-axis. So,

Foci $= (0, \pm e) + (0, \pm\sqrt{21})$

Length of latusrectum $= \frac{2a^2}{b} = \frac{2\times 4}{5} = \frac{8}{5}$

11. Given vertices $= (0, \pm 13)$ and foci $= (0, \pm 5)$

Here, we see that x-coordinate is 0, it means major axis is lies along y-axis.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, b > a$

Here, $b = 13$ and $be = 5$

$\Rightarrow \quad 13e = 5 \Rightarrow e = \frac{5}{13}$

$\because \quad a^2 = b^2(1-e^2) = 169\left(1-\frac{25}{169}\right) = 144$

$\Rightarrow \quad a^2 = 144$

$\therefore$ Equation of ellipse is $\frac{x^2}{144}+\frac{y^2}{169}=1$

12. Since, major axis is along x-axis. Hence, equation of ellipse will be of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Given that Eq.(i) passes through the points (4, 3) and (6, 2), *i.e.*, they will satisfy it

$\therefore \quad \frac{(4)^2}{a^2}+\frac{(3)^2}{b^2}=1 \Rightarrow \frac{16}{a^2}+\frac{9}{b^2}=1 \quad$ ...(ii)

and $\frac{(6)^2}{a^2}+\frac{(2)^2}{b^2}=1 \Rightarrow \frac{36}{a^2}+\frac{4}{b^2}=1 \quad$ ...(iii)

On multiplying Eq. (ii) by 4 and Eq. (iii) by 9, then subtracting, we get

$$\frac{64}{a^2}-\frac{324}{a^2} = 4-9 \Rightarrow -\frac{260}{a^2} = -5$$

$\Rightarrow \quad a^2 = \frac{260}{5} \Rightarrow a^2 = 52$

From Eq.(ii),

$$\frac{9}{b^2} = 1-\frac{16}{52} \Rightarrow \frac{9}{b^2} = \frac{52-16}{52}$$

$\Rightarrow \quad \frac{9}{b^2} = \frac{36}{52} \Rightarrow b^2 = \frac{9\times 52}{36} = 13$

Put the values of $a^2 = 52$ and $b^2 = 13$ in Eq. (i), we get

$$\frac{x^2}{52}+\frac{y^2}{13}=1$$

13. Given equation of curve is $x = \alpha + 5\cos\theta, y = \beta + 4\sin\theta$

or $\cos\theta = \frac{x-\alpha}{5}, \quad \sin\theta = \frac{y-\beta}{4}$

$\because \quad \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad \left(\frac{x-\alpha}{5}\right)^2 + \left(\frac{y-\beta}{4}\right)^2 = 1$

This represents the equation of an ellipse.

14. The given equation can be rewritten as

$4x^2 - 24x + 36 + 16y^2 - 32y + 16 - 36 - 16 - 12 = 0$

$\Rightarrow \quad (2x-6)^2 + (4y-4)^2 = 64$

$\Rightarrow \quad \frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$

This represents an ellipse.

Here, $a^2 = 16, b^2 = 4 \therefore e = \sqrt{1-\frac{4}{16}} = \frac{\sqrt{3}}{2}$

15. The position of the points (1, 2) and (2, 1) with respect to the circle $x^2 + y^2 = 9$ is given by $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$. Thus, both P and Q lie inside C.

The position of the points (1, 2) and (2, 1) with respect to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is given by $\frac{1^2}{9}+\frac{2^2}{4} = \frac{1}{9}+1 > 1$

and $\frac{2^2}{9}+\frac{1}{4}=\frac{16+9}{36}=\frac{25}{36}<1,$

P lies outside E and Q lies inside E. Thus, P lies inside C but outside E.

16. Given equation is $9x^2+4y^2-6x+4y+1=0$

$$\Rightarrow\ 9\left(x^2-\frac{2}{3}x+\frac{1}{3^2}\right)+4\left(y^2+y+\frac{1}{4}\right)+1-1-1=0$$

$$\Rightarrow\ \frac{\left(x-\frac{1}{3}\right)^2}{\left(\frac{1}{3}\right)^2}+\frac{\left(y+\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}=1 \qquad \text{(here, } a<b\text{)}$$

Length of major axis $=2b=2\left(\frac{1}{2}\right)=1$

Length of minor axis $=2a=2\left(\frac{1}{3}\right)=\frac{2}{3}$

17. Let the standard equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$.

Minor axis $=2b=8 \Rightarrow b=4$

and eccentricity $=e=\frac{\sqrt{5}}{3}$

Now, $b^2=a^2(1-e^2)$

$$\Rightarrow\ (4)^2=a^2\left(1-\frac{5}{9}\right)\Rightarrow 16=a^2\left(\frac{4}{9}\right)$$

$$\Rightarrow\ a^2=36 \Rightarrow a=6$$

Length of major axis $=2a=12$

18. We know that sum of focal distance of any point on the ellipse always equal to the length of major axis, *i.e.*, it is equal to $2a$.

19. Let point is $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ and let it's distance d from origin.

$$\therefore\ d=\sqrt{6\cos^2\theta+2\sin^2\theta}$$

$$\Rightarrow\ 2=\sqrt{2+4\cos^2\theta}$$

$$\Rightarrow\ 2+4\cos^2\theta=4$$

$$\Rightarrow\ \cos^2\theta=\frac{1}{2}$$

$$\Rightarrow\ \cos\theta=\pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow\ \theta=\frac{\pi}{4}$$

20. $e^2=\frac{a^2-b^2}{a^2}$ and $e'^2=\frac{b^2-a^2}{b^2}$

$$\Rightarrow\ \frac{1}{e^2}+\frac{1}{e'^2}=1$$

21. Equations of the ellipse are $\frac{x^2}{13^2}+\frac{y^2}{5^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and their eccentricity are

$$e=\sqrt{1-\frac{25}{169}}$$

and $e'=\sqrt{1-\frac{b^2}{a^2}}$

According to given condition, $e'=e$

$$\Rightarrow\ \sqrt{1-\left(\frac{b^2}{a^2}\right)}=\sqrt{1-\left(\frac{25}{169}\right)}$$

$$\Rightarrow\ \frac{b}{a}=\frac{5}{13} \qquad (\because a>0, b>0)$$

$$\Rightarrow\ \frac{a}{b}=\frac{13}{5}$$

22. If $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ are the end points of chord, then equation of chord is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

If it is a focal chord, it passes through $(ae, 0)$, so

$$e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow\ e=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

23. Putting $x=at^2$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$

we get $t^4+\frac{y^2}{b^2}=1$

i.e., $y^2=b^2(1-t^4)=b^2(1+t^2)(1-t^2)$

y is real, if $1-t^2\geq 0$

i.e., $|t|\leq 1$

24. Given equation of ellipse is

$$\frac{x^2}{\frac{5}{3}}+\frac{y^2}{\frac{5}{4}}=1$$

The equation of tangents in slope form is

$$y=mx\pm\sqrt{\frac{5}{3}m^2+\frac{5}{4}}$$

Slope of tangents are $\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

$$\therefore\ y=\pm\frac{1}{\sqrt{3}}x\pm\sqrt{\frac{5}{9}+\frac{5}{4}}$$

$$\Rightarrow\ y=\pm\frac{1}{\sqrt{3}}x\pm\frac{\sqrt{65}}{6}$$

25. The combined equation of the pair of tangents drawn from (1, 2) to the ellipse $3x^2+2y^2=5$ is $SS'=T^2$

$$\Rightarrow\ (3x^2+2y^2-5)[3(1)^2+2(2)^2-5]=[3x(1)+2y(2)-5]^2$$

$$\Rightarrow\ (3x^2+2y^2-5)(3+8-5)=(3x+4y-5)^2$$

$$\Rightarrow\ 9x^2-24xy-4y^2+40y+30x-55=0$$

This is the equation of pair of straight lines,

where, $a = 9,\ h = -12,\ b = -4$

The angle between these lines is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\Rightarrow \quad \tan\theta = \frac{2\sqrt{144 + 36}}{9 - 4} = \frac{2\sqrt{180}}{5} = \frac{12}{\sqrt{5}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$$

26. We have, $x\cos\alpha + y\sin\alpha = p$

$$\Rightarrow \quad y = -x\cot\alpha + p\,\text{cosec}\,\alpha$$

Since, above line is tangent to the ellipse.

$$\therefore \quad c^2 = a^2m^2 + b^2$$

$$\Rightarrow \quad p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha + b^2$$

$$\Rightarrow \quad a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$$

27. Let $S \equiv \frac{x^2}{16} + \frac{y^2}{25} - 1 = 0$

At point (7, 6), $S_1 > 0$. So, two tangents can be drawn from this point.

28. Equation of ellipse is $\frac{x^2}{2} + \frac{y^2}{1} = 1$. General equation of tangent to the ellipse of slope m is

$$y = mx \pm \sqrt{2m^2 + 1}$$

Since, this is equally inclined to axes, so $m = \pm 1$. Thus, tangents are

$$y = \pm x \pm \sqrt{2 + 1} = \pm x \pm \sqrt{3}$$

Distance of any tangent from origin

$$= \frac{|0 + 0 \pm \sqrt{3}|}{\sqrt{1^2 + 1^2}} = \frac{\sqrt{3}}{2}$$

29. The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.

Now, equation of director circle to an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 + b^2.$$

30. Let the equation of tangent, which is perpendicular to the line $3x + 4y = 7$ is $4x - 3y = \lambda$.

Since, it is a tangent to the ellipse.

$$\therefore \quad \lambda^2 = a^2m^2 + b^2$$

Here, $a^2 = 9, b^2 = 4$ and $m = \frac{4}{3}$

$$\therefore \quad \lambda^2 = 9 \times \left(\frac{4}{3}\right)^2 + 4 = 16 + 4$$

$$\Rightarrow \quad \lambda^2 = 20 \Rightarrow \lambda = \pm\sqrt{20}$$

$\therefore$ Equation of tangents are $4x - 3y = \pm\sqrt{20}$.

31. The equation of normal at point $P(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$$

The point of intersection with coordinate axes are $R\left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$ and $S\left(0, -\frac{a^2 - b^2}{a}\sin\theta\right)$.

$$\text{Now,} \quad RP^2 = \left[a\cos\theta - \left(\frac{a^2 - b^2}{a}\right)\cos\theta\right]^2 + b^2\sin^2\theta$$

$$= \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$\text{and} \quad RS^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$\therefore \quad RP^2 : RS^2 = b^4 : a^4$$

$$\Rightarrow \quad RP : PS = b^2 : a^2$$

32. We have, $a^2 = \frac{1}{2},\ b^2 = \frac{1}{3},\ m = \frac{4}{3}$

The required points are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

$$= \left(\pm\frac{\frac{1}{2}\times\frac{4}{3}}{\sqrt{\frac{1}{2}\times\frac{16}{9} + \frac{1}{3}}}, \mp\frac{\frac{1}{3}}{\sqrt{\frac{1}{2}\times\frac{16}{9} + \frac{1}{3}}}\right) = \left(\pm\frac{2}{\sqrt{11}}, \mp\frac{1}{\sqrt{11}}\right)$$

33. We know that, the locus of point of intersection of two perpendicular tangents drawn on the ellipse is $x^2 + y^2 = a^2 + b^2$, which is called director circle.

Given equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Here, $a^2 = 9,\ b^2 = 4$

$\therefore$ Locus is $x^2 + y^2 = a^2 + b^2$

$$\Rightarrow \quad x^2 + y^2 = 9 + 4 \Rightarrow x^2 + y^2 = 13$$

34. Let the point be $P(\sqrt{2}\cos\theta, \sin\theta)$ on $\frac{x^2}{2} + \frac{y^2}{1} = 1$.

$\therefore$ Equation of tangent at P is $\frac{x\sqrt{2}}{2}\cos\theta + y\sin\theta = 1$

whose intercept on coordinate axes are $A(\sqrt{2}\sec\theta, 0)$ and $B(0, \text{cosec}\,\theta)$.

$\therefore$ Mid-point of its intercept between axes is

$$\left(\frac{\sqrt{2}}{2}\sec\theta, \frac{1}{2}\,\text{cosec}\,\theta\right) = (h, k)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}\,h} \quad \text{and} \quad \sin\theta = \frac{1}{2k}$$

$$\text{Now,} \quad \cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{1}{2h^2} + \frac{1}{4k^2} = 1$$

The locus of mid-point M is $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$.

35. Equation of tangent at $(3\sqrt{3}\cos\theta, \sin\theta)$ to the ellipse $\frac{x^2}{27}+y^2=1$ is $\frac{x\cos\theta}{3\sqrt{3}}+y\sin\theta=1$.

It cuts intercepts on the coordinate axes.

$\therefore$ Sum of intercepts on axes is

$$3\sqrt{3}\sec\theta+\operatorname{cosec}\theta=f(\theta) \qquad \text{(say)}$$

On differentiating w.r.t. θ, we get

$$f'(\theta)=\frac{3\sqrt{3}\sin^3\theta-\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

For maxima and minima, put $f'(\theta)=0$

$$\Rightarrow \quad 3\sqrt{3}\sin^3\theta-\cos^3\theta=0$$

$$\Rightarrow \quad \tan^3\theta=\frac{1}{3\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6}$$

At $\theta=\frac{\pi}{3}$, $f''(\theta)>0$

$\therefore f(\theta)$ is minimum at $\theta=\frac{\pi}{6}$.

36. Given equation of tangent is $\frac{x}{a}\frac{\sqrt{3}}{2}+\frac{y}{b}\frac{1}{2}=1$ and equation of tangent at the point $(a\cos\phi, b\sin\phi)$ on the ellipse is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Both are same.

$$\therefore \quad \cos\phi=\frac{\sqrt{3}}{2}, \quad \sin\phi=\frac{1}{2} \Rightarrow \phi=\frac{\pi}{6}$$

37. The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

Length of perpendicular from the focus $(ae, 0)$ on the ellipse

$$=\left|\frac{e\cos\theta-1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\right|$$

$$=\left|\frac{ab(e\cos\theta-1)}{\sqrt{b^2\cos^2\theta+a^2(1-\cos^2\theta)}}\right|=\left|\frac{ab(e\cos\theta-1)}{\sqrt{a^2-a^2e^2\cos^2\theta}}\right|$$

$$\therefore \quad p=b\sqrt{\frac{1-e\cos\theta}{1+e\cos\theta}}$$

$$\Rightarrow \quad \frac{b^2}{p^2}=\frac{1+e\cos\theta}{1-e\cos\theta}$$

Now, $r^2=(ae-a\cos\theta)^2+b^2\sin^2\theta$

$$=a^2[(e-\cos\theta)^2+(1-e^2)\sin^2\theta]$$

$$=a^2[e^2\cos^2\theta-2e\cos\theta+1]=a^2(1-e\cos\theta)^2$$

$$\Rightarrow \quad r=a(1-e\cos\theta)$$

$$\therefore \quad \frac{2a}{r}-\frac{b^2}{p^2}=\frac{2}{1-e\cos\theta}-\frac{1+e\cos\theta}{1-e\cos\theta}=1$$

38. The chord of contact of (α,β) is $\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=1$. It touches the circle $x^2+y^2=c^2$.

$$\therefore \quad \frac{1}{\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}}}=c \Rightarrow \frac{\alpha^2}{\alpha^4}+\frac{\beta^2}{\beta^4}=\frac{1}{\chi^2}$$

Thus, the locus of (α,β) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$$

39. The equation of the tangent at $\left(4\cos\phi, \frac{16}{\sqrt{11}}\sin\phi\right)$ to the ellipse $16x^2+11y^2=256$ is

$$16(4\cos\phi)x+11\left(\frac{16}{\sqrt{11}}\sin\phi\right)y=256$$

$$\Rightarrow \quad 4x\cos\phi+\sqrt{11}y\sin\phi=16$$

This, touches the circle $(x-1)^2+y^2=4^2$.

$$\therefore \quad \left|\frac{4\cos\phi-16}{\sqrt{16\cos^2\phi+11\sin^2\phi}}\right|=4$$

$$\Rightarrow \quad (\cos\phi-4)^2=16\cos^2\phi+11\sin^2\phi$$

$$\Rightarrow \quad 4\cos^2\phi+8\cos\phi-5=0$$

$$\Rightarrow \quad (2\cos\phi-1)(2\cos\phi+5)=0$$

$$\Rightarrow \quad \cos\phi=\frac{1}{2} \Rightarrow \phi=\pm\frac{\pi}{3} \qquad \left(\because \cos\phi\neq\frac{5}{2}\right)$$

40. Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$.

Equation of the tangent at P is

$$\frac{x}{3}\cos\theta+\frac{y}{2}\sin\theta=1$$

which meets the tangents $x=3$ and $x=-3$ at the extremities of the major axis at $T\left\{3, \frac{2(1-\cos\theta)}{\sin\theta}\right\}$ and $T'\left\{-3, \frac{2(1+\cos\theta)}{\sin\theta}\right\}$.

Equation of circle on TT' as diameter is

$$(x-3)(x+3)+\left\{y-\frac{2(1-\cos\theta)}{\sin\theta}\right\}\left\{y-\frac{2(1+\cos\theta)}{\sin\theta}\right\}=0$$

$$\Rightarrow \quad x^2+y^2-\frac{4}{\sin\theta}y-5=0$$

which passes through $(\sqrt{5}, 0)$.

41. Given equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a<b)$$

It is a vertical ellipse with foci $(0, \pm be)$.

Equation of any tangent line to the above ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}$$

$\therefore$ Required product

$$= \left|\frac{-be + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}\right| \left|\frac{be + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}\right|$$

$$= \left|\frac{a^2m^2 + b^2 - b^2e^2}{m^2+1}\right|$$

$$= \left|\frac{a^2m^2 + b^2(1-e^2)}{m^2+1}\right|$$

$$= \left|\frac{a^2m^2 + a^2}{m^2+1}\right| = a^2 \qquad [\because a^2 = b^2(1-e^2)]$$

42. The equation of any normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2 \qquad \ldots(i)$$

The straight line $x\cos\alpha + y\sin\alpha = p$ will be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then Eq. (i) and $x\cos\alpha + y\sin\alpha = p$ will represent the same line.

$$\therefore \quad \frac{a\sec\phi}{\cos\alpha} = \frac{-b\operatorname{cosec}\phi}{\sin\alpha} = \frac{a^2-b^2}{p}$$

$$\Rightarrow \quad \cos\phi = \frac{ap}{(a^2-b^2)\cos\alpha}$$

and $$\sin\phi = \frac{-bp}{(a^2-b^2)\sin\alpha}$$

$$\because \quad \sin^2\phi + \cos^2\phi = 1$$

$$\Rightarrow \quad \frac{b^2p^2}{(a^2-b^2)^2\sin^2\alpha} + \frac{a^2p^2}{(a^2-b^2)^2\cos^2\alpha} = 1$$

$$\Rightarrow \quad p^2(b^2\operatorname{cosec}^2\alpha + a^2\sec^2\alpha) = (a^2-b^2)^2$$

43. The given equation can be rewritten as $\frac{x^2}{20} + \frac{y^2}{\frac{45}{4}} = 1$

On comparing the given equation with the standard equation, we get

$$a^2 = 20, \quad b^2 = \frac{45}{4}$$

$\therefore$ The equation of normal at the point (2, 3) is

$$\frac{x-2}{\frac{2}{20}} = \frac{y-3}{\left(\frac{12}{45}\right)}$$

$$\Rightarrow \quad 40(x-2) = 15(y-3)$$

$$\Rightarrow \quad 8x - 3y = 7 \quad \Rightarrow \quad 3y - 8x + 7 = 0$$

44. Equation of given ellipse is $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$.

Equation of normal to ellipse $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ is

$$3X\sec\theta - 2Y\operatorname{cosec}\theta = 5.$$

Slope of normal is $\frac{3}{2}\tan\theta$,

which is parallel to $3x - y = 1$, then $\frac{3}{2}\tan\theta = 3$

$$\Rightarrow \quad \tan\theta = 2$$

$$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}, \quad \cos\theta = \frac{1}{\sqrt{5}}$$

So, equation of normal is $3\sqrt{5}X - \sqrt{5}Y = 5$.

$$\because \quad X = x-1, \quad Y = y-2$$

$$\therefore \quad 3\sqrt{5}(x-1) - \sqrt{5}(y-2) = 5$$

$$\Rightarrow \quad \sqrt{5}(3x-y) = 5(\sqrt{5}+1)$$

$$\Rightarrow \quad 3x - y = \sqrt{5}(\sqrt{5}+1)$$

45. Equation of ellipse is $\frac{x^2}{14} + \frac{y^2}{5} = 1$.

Any point on the ellipse is $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$.

$\therefore$ Equation of normal at $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$ is

$$\sqrt{14}x\sec\theta - \sqrt{5}y\operatorname{cosec}\theta = 9.$$

It passes through $(a\cos 2\theta, b\sin 2\theta)$.

$$\therefore \quad \sqrt{14}\sqrt{14}\cos 2\theta\sec\theta - \sqrt{5}\sqrt{5}\sin 2\theta\operatorname{cosec}\theta = 9$$

$$\Rightarrow \quad 14\frac{\cos 2\theta}{\cos\theta} - 5\frac{\sin 2\theta}{\sin\theta} = 9$$

$$\Rightarrow \quad 14(2\cos^2\theta - 1) - 10\cos^2\theta = 9\cos\theta$$

$$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow \quad 18\cos^2\theta - 21\cos\theta + 12\cos\theta - 14 = 0$$

$$\Rightarrow \quad (3\cos\theta + 2)(6\cos\theta - 7) = 0$$

$$\Rightarrow \quad \cos\theta = -\frac{2}{3}, \quad \cos\theta \neq -\frac{7}{6}$$

46. Normal at $\left(ae, \frac{b^2}{a}\right)$ of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x - ae}{\frac{ae}{a^2}} = \frac{y - \frac{b^2}{a}}{\left(\frac{b^2}{a}/b^2\right)}$$

Since, it passes through $(0, -b)$, then

$$\frac{0 - ae}{\frac{ae}{a^2}} = \frac{-b - \frac{b^2}{a}}{\frac{1}{a}}$$

$$\Rightarrow \quad -a^2 = -a\left(b + \frac{b^2}{a}\right)$$

$$\Rightarrow \quad a^2 = ab + b^2$$

$$\Rightarrow \quad a^2 = ab + a^2 - a^2e^2 \quad (\because b^2 = a^2 - a^2e^2)$$

$$\Rightarrow \quad b = ae^2$$

$$\Rightarrow \quad b^2 = a^2e^4$$

$$\Rightarrow \quad a^2(1-e^2) = a^2e^4$$

$$\Rightarrow \quad 1 - e^2 = e^4 \Rightarrow e^2(e^2+1) = 1$$

47. The equation of normal to the given ellipse at $P(a\cos\theta, b\sin\theta)$ is

$$a\,x\sec\theta - by\,\text{cosec}\,\theta - a^2 = b^2$$

$$\Rightarrow \quad y = \left(\frac{a}{b}\tan\theta\right)x - \frac{a^2-b^2}{b}\sin\theta \quad \ldots\text{(i)}$$

Let $\frac{a}{b}\tan\theta = m$, then $\sin\theta = \frac{bm}{\sqrt{a^2+b^2m^2}}$

$\therefore$ From Eq. (i), we get

$$y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$$

$$\because \quad \frac{a}{b}\tan\theta \in R \;\Rightarrow\; m \in R$$

48. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then, the equation of the normal at P is

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$$

It meets the axes at $G\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$

and $\quad g\left(0, -\frac{a^2-b^2}{b}\sin\theta\right)$

$$\therefore \quad PG^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

$$PG^2 = \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta) \quad \ldots\text{(i)}$$

and $\quad Pg^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta) \quad \ldots\text{(ii)}$

$$\therefore \quad PG : Pg = b^2 : a^2$$

49. The coordinates of the extremity of the latusrectum which lies in the first quadrant are $\left(ae, \frac{b^2}{a}\right)$. The equation of the normal at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = (a^2-b^2)$$

$\therefore$ The equation of the normal at $\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow \quad ax - aey = e(a^2-b^2)$$

$$\Rightarrow \quad ax - aey = ea^2e^2$$

$$\Rightarrow \quad x - ey = ae^3$$

This passes through the extremity of the minor axis *i.e.*, $(0, -b)$

$$\therefore \quad 0 + eb - ae^3 = 0$$

$$\Rightarrow \quad b = ae^2$$

$$\Rightarrow \quad b^2 = a^2e^4$$

$$\Rightarrow \quad a^2(1-e^2) = a^2e^4$$

$$\therefore \quad e^4 + e^2 - 1 = 0$$

50. Since, locus of the point of intersection of the tangents at the end points of a focal chord is directrix.

$\therefore$ Required locus is $x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2-b^2}}$

51. $\because \quad e^2 = 1 - \frac{b^2}{a^2}$

$$\therefore \quad a^2 - b^2 = a^2e^2$$

So, the points on the minor axis at a distance $\sqrt{a^2-b^2}$ from the centre (0, 0) of the ellipse are $(0, \pm ae)$.

The equation of tangent at any point $(a\cos\theta, b\sin\theta)$ on the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$\therefore$ Required sum

$$= \left[\frac{\frac{ae\sin\theta}{b} - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right]^2 + \left[\frac{\frac{-ae\sin\theta}{b} - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right]^2$$

$$= \frac{(ae\sin\theta - b)^2 + (ae\sin\theta + b)^2}{(b^2\cos^2\theta + a^2\sin^2\theta)} \times a^2$$

$$= \frac{2a^2[(a^2-b^2)\sin^2\theta + b^2]}{b^2\cos^2\theta + a^2\sin^2\theta} = 2a^2$$

52. Equation of director circle of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

$$x^2 + y^2 = 25 + 16$$

$$\Rightarrow \quad x^2 + y^2 = 41$$

$\because$ The given point (5, 4) lies on the director circle, therefore the tangents are drawn from this point to the ellipse makes an angle 90°.

53. Let mid-point of the chord be (h, k), then equation of the chords be

$$\frac{hx^2}{a^2} + \frac{ky^2}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$

$$\Rightarrow \quad y = -\frac{b^2}{a^2}\cdot\frac{h}{k}x + \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\frac{b^2}{k} \quad \ldots\text{(i)}$$

Since, line (i) is touching the circle $x^2 + y^2 = c^2$.

$$\therefore \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\frac{b^4}{k^2} = c^2\left(1 + \frac{b^4h^2}{a^4k^2}\right)$$

Hence, locus is $(b^2x^2 + a^2y^2)^2 = c^2(b^4x^2 + a^4y^2)$.

54. Any point on the line $x - y - 5 = 0$ will be of the form $(t, t-5)$. Chord of contact of this point with respect to curve $x^2 + 4y^2 = 4$ is

$$tx + 4(t-5)y - 4 = 0$$

$$\Rightarrow \quad (-20y - 4) + t(x + 4y) = 0$$

which is a family of straight lines, each member of this family pass through point of intersection of straight lines $-20y - 4 = 0$ and $x + 4y = 0$ which is $\left(\frac{4}{5}, -\frac{1}{5}\right)$.

55. Chords of contact are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

and

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$$

Product of slopes $= -1$

$$\Rightarrow \quad \left(-\frac{x_1}{a^2}\cdot\frac{b^2}{y_1}\right)\left(-\frac{x_2}{a^2}\cdot\frac{b^2}{y_2}\right) = -1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

56. Equation of chord joining points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Now, $\quad \beta = \alpha + 90°$

$$\therefore \quad \frac{x}{a}\cos\left(\frac{2\alpha+90°}{2}\right) + \frac{y}{b}\sin\left(\frac{2\alpha+90°}{2}\right) = \frac{1}{\sqrt{2}}$$

Now, compare it with $lx + my = -n$, we get

$$\frac{\cos\left(\frac{2\alpha+90°}{2}\right)}{al} = \frac{\sin\left(\frac{2\alpha+90°}{2}\right)}{bm} = -\frac{1}{\sqrt{2}\,n}$$

$$\because \quad \cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \quad a^2l^2 + b^2m^2 = 2n^2$$

57. It is clear from the figure that the two curves do not intersect each other.

58. We know that, if (x_1, y_1) is the mid-point of the chord, then equation of chord is

$$T = S_1$$

$$\Rightarrow \quad \frac{xx_1}{25} + \frac{yy_1}{9} = \frac{x_1^2}{25} + \frac{y_1^2}{9}$$

$\because$ Point is (1, 1), then

$$\frac{x}{25} + \frac{y}{9} = \frac{1}{25} + \frac{1}{9}$$

$$\Rightarrow \quad 9x + 25y = 34$$

59. Given, $\quad m(n-1) = n$

n is divisible by $n - 1$.

$$\Rightarrow \quad n = 2 \Rightarrow m = 2$$

Hence, chord of contact of tangents drawn from (2, 2) to

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \text{ is}$$

$$\frac{2x}{9} + \frac{2y}{4} = 1$$

$$\Rightarrow \quad 4x + 9y = 18$$

60. Given equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{\frac{7}{4}} = 1$.

Here, $\quad a^2 = 4$ and $b^2 = \frac{7}{4}$

$$\therefore \quad b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad \frac{7}{4} = 4(1 - e^2)$$

$$\Rightarrow \quad e^2 = 1 - \frac{7}{16} = \frac{9}{16}$$

$$\Rightarrow \quad e = \frac{3}{4}$$

Thus, the foci are $\left(\pm\frac{3}{2}, 0\right)$.

The radius of required circle $= \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)^2 + (2-0)^2}$

$$= \sqrt{1+4} = \sqrt{5}$$

61. Let the equation of tangent to the circle $x^2 + y^2 = 16$ is

$$y = mx + 4\sqrt{1+m^2} \qquad (\because y = mx + a\sqrt{1+m^2})$$

and let the equation of tangent to the ellipse $\frac{x^2}{49} + \frac{y^2}{4} = 1$ is

$$y = mx + \sqrt{49m^2 + 4} \qquad (\because y = mx + \sqrt{a^2m^2 + b^2})$$

For common tangent,

$$4\sqrt{1+m^2} = \sqrt{49m^2 + 4}$$

$$\Rightarrow \quad 16 + 16m^2 = 49m^2 + 4$$

$$\Rightarrow \quad 12 = 33m^2$$

$$\Rightarrow \quad m^2 = \frac{12}{33}$$

$$\Rightarrow \quad m = \frac{2}{\sqrt{11}}$$

$$\therefore \quad y = \frac{2}{\sqrt{11}}x + 4\sqrt{1 + \frac{4}{11}}$$

$$= \frac{2}{\sqrt{11}}x + 4\sqrt{\frac{15}{11}}$$

62. Let point of intersection be (h, k). Then, equation of the line passing through P and Q is $hx + 2ky = 4$ (chord of contact).

Since, $hx + 2ky = 4$ touches $x^2 + y^2 = 1$, $\frac{16}{4k^2} = 1 + \frac{h^2}{4k^2}$

i.e., $4k^2 + h^2 = 16$. So, required locus is $4y^2 + x^2 = 16$, which is an ellipse of eccentricity $\frac{\sqrt{3}}{2}$ and length of latusrectum is 2 units.

63. Let P is the position of man and S, S' are position of flags, then

$$SP + S'P = 10 = 2a \Rightarrow a = 5$$

$$\therefore \quad SS' = 2ae = 8 \Rightarrow e = \frac{4}{5}$$

Now, $\quad e^2 = 1 - \frac{b^2}{a^2} \Rightarrow \frac{16}{25} = 1 - \frac{b^2}{25}$

$\Rightarrow \quad b^2 = 9 \Rightarrow b = 3$

Area of ellipse $= \pi ab = 15\pi$ sq m

64. The quadrilateral formed by the tangents at the end points of latusrectum is a rhombus. It is symmetrical about the axes.

So, total area is four times the area of the right angled triangle formed by the tangent and axes in the Ist quadrant.

Now, $\quad ae = \sqrt{a^2 - b^2}$

$\Rightarrow \quad ae = 2$

$\therefore$ Coordinates of one end point of latusrectum are $\left(2, \frac{5}{3}\right)$.

The equation of tangent at that point is $\frac{x}{\frac{9}{2}} + \frac{y}{3} = 1$. This equation meets the coordinate axes at a point $A\left(0, \frac{9}{2}\right)$ and $B(3, 0)$.

In ΔAOB, $\quad$ Area $= \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$

Total area of rhombus $ABCD = 4 \times$ Area of ΔAOB

$$= 4 \times \frac{27}{4} = 27 \text{ sq units}$$

65. Given equations of ellipse are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The point of the intersection of these ellipse are

$$\left(\pm \frac{ab}{\sqrt{a^2 + b^2}}, \pm \frac{ab}{\sqrt{a^2 + b^2}}\right) \text{ i.e., } P\left(\frac{ab}{\sqrt{a^2 + b^2}}, \frac{ab}{\sqrt{a^2 + b^2}}\right).$$

$\therefore$ The distance between $OP = r$

$$= \sqrt{\left(\frac{ab}{\sqrt{a^2 + b^2}} - 0\right)^2 + \left(\frac{ab}{\sqrt{a^2 + b^2}} - 0\right)^2}$$

$$= \frac{ab}{\sqrt{a^2 + b^2}} \sqrt{2}$$

$\therefore$ Equation of circle is

$$x^2 + y^2 = r^2$$

$$\Rightarrow \quad x^2 + y^2 = \frac{2a^2b^2}{a^2 + b^2}$$

66. Obviously, it is an ellipse, because the normal and tangent at point P of an ellipse bisect the internal and external angles between the focal distance of the point.

67. For the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, equation of director circle is $x^2 + y^2 = 25$. The director circle will cut the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ at four points. Hence, number of points is 4.

68. Let $y = m_1x$ and $y = m_2x$ be a pair of conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let $P(a \cos\theta, b \sin\theta)$ and $Q(a \cos\phi, b \sin\phi)$ be ends of these two diameters. Then,

$$m_1m_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{b \sin\theta - 0}{a \cos\theta - 0} \times \frac{b \sin\phi - 0}{a \cos\phi - 0} = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \sin\theta \sin\phi = -\cos\theta \cos\phi$$

$$\Rightarrow \quad \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = \pm \frac{\pi}{2}$$

Round II

1. (α, β) lies on the director circle of the ellipse *i.e.*, on $x^2 + y^2 = 9$.

So, we can assume

$$\alpha = 3\cos\theta, \quad \beta = 3\sin\theta$$

$\therefore \quad F = 12\cos\theta + 9\sin\theta = 3(4\cos\theta + 3\sin\theta)$

$\Rightarrow \quad -15 \le F \le 15$

2. $A \equiv (a\cos\theta, b\sin\theta)$

$B \equiv \{a\cos(\theta+\alpha), b\sin(\theta+\alpha)\}$

$C \equiv \{a\cos(\theta+2\alpha), b\sin(\theta+2\alpha)\}$

$\Delta \equiv$ Area of ΔABC

$$= \frac{1}{2}\begin{vmatrix} 1 & a\cos\theta & b\sin\theta \\ 1 & a\cos(a+\alpha) & b\sin(\theta+\alpha) \\ 1 & a\cos(\theta+2\alpha) & b\sin(\theta+2\alpha) \end{vmatrix}$$

$$= 2ab\sin^2\left(\frac{\alpha}{2}\right)\sin\alpha$$

$$\Delta(\alpha) = ab\sin\alpha(1-\cos\alpha)$$

$$= \frac{ab}{2}(2\sin\alpha - \sin 2\alpha)$$

$$\Delta'(\alpha) = 0$$

$\Rightarrow \quad \cos\alpha = 1$ or $\cos\alpha = -\frac{1}{2}$

$\cos\alpha = 1$ gives triangle $= 0$

$\cos\alpha = -\frac{1}{2}$ gives maximum value of triangle $= \frac{3\sqrt{3}}{4}ab$

3. The tangent at the point of shortest distance from the line $x + y = 7$ parallel to the given line.

Any point on the given ellipse is $(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$.

Equation of the tangent is

$$\frac{x\cos\theta}{\sqrt{6}} + \frac{y\sin\theta}{\sqrt{3}} = 1. \text{ It is parallel to } x + y = 7.$$

$\Rightarrow \quad \frac{\cos\theta}{\sqrt{6}} = \frac{\sin\theta}{\sqrt{3}}$

$\Rightarrow \quad \frac{\cos\theta}{\sqrt{2}} = \frac{\sin\theta}{1}$

$$= \frac{1}{\sqrt{3}}$$

The required point is (2, 1).

4. Let ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and circle $x^2 + y^2 = a^2e^2$

Radius of circle $= ae$

Point of intersection of circle and ellipse is

$$\left[\frac{a}{e}\sqrt{2e^2-1}, \frac{a}{e}(1-e^2)\right].$$

Now, area of ΔPF_1F_2

$$= \frac{1}{2}\begin{vmatrix} \frac{a}{e}\sqrt{2e^2-1} & \frac{a}{e}(1-e^2) & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = \frac{1}{2}\left|\frac{a}{e}(1-e^2)(2ae)\right| = 30$$

$\Rightarrow \quad a^2(1-e^2) = 30$ (given)

$\Rightarrow \quad a^2e^2 = a^2 - 30 = \left(\frac{17}{2}\right)^2 - 30 = \frac{169}{4}$

$\Rightarrow \quad 2ae = 13$

5.

Tangent to the ellipse at $P(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad \text{...(i)}$$

Tangent to the circle at $Q(a\cos\alpha, a\sin\alpha)$ is

$$\cos\alpha x + \sin\alpha y = a \quad \text{...(ii)}$$

Now, angle between tangents is θ,

then $\quad \tan\theta = \left|\dfrac{-\frac{b}{a}\cot\alpha - (-\cot\alpha)}{1 + \left(-\frac{b}{a}\cot\alpha\right)(-\cot\alpha)}\right|$

$$= \left|\frac{\cot\alpha\left(1 - \frac{b}{a}\right)}{1 + \frac{b}{a}\cot^2\alpha}\right| = \left|\frac{a-b}{a\tan\alpha + b\cot\alpha}\right|$$

$$= \left|\frac{a-b}{(\sqrt{a\tan\alpha} - \sqrt{b\cot\alpha})^2 + 2\sqrt{ab}}\right|$$

Now, the greatest value of the above expression is $\left|\frac{a-b}{2\sqrt{ab}}\right|$

when $\sqrt{a\tan\alpha} = \sqrt{b\cot\alpha}$

$\Rightarrow \quad \theta_{max} = \tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

6.

Since, sides of the square are tangent and perpendicular to each other, so two vertices lie on director circle.

$\Rightarrow \quad x^2 + y^2 = (a^2 - 7) + (13 - 5a)$

$= a^2$ ($\sqrt{2}a$ is side of the square)

$\Rightarrow \quad (a^2 - 7) + (13 - 5a) = a^2$

$\Rightarrow \quad a = \frac{6}{5}$

But for an ellipse to exist $a^2 - 7 > 0$ and $13 - 5a > 0$

$\Rightarrow \quad a \in (-\infty, -\sqrt{7})$

Hence, $\quad a \neq \frac{6}{5}$

Hence, no such a exists.

7. Since, mutually perpendicular tangents can be drawn from vertices of rectangle. So, all the vertices of rectangle should lie on director circle $x^2 + y^2 = a^2 + b^2$.

Let breadth $= 2l$ and length $= 4l$, then

area of rectangle $= 2l \times 4l = 8l^2$

$= \frac{8}{5}(a^2 + b^2)$

($\therefore$ In ΔOAB, $a^2 + b^2 = l^2 + 4l^2 \Rightarrow 5l^2 = a^2 + b^2$)

8. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Equation of the parabola with focus $S(ae, 0)$ and directrix $x + ae = 0$ is $y^2 = 4aex$.

Now, length of latusrectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

For the two latusrectum to be equal, we get

$$\frac{2b^2}{a} = 4ae$$

$\Rightarrow \quad \frac{2a^2(1 - e^2)}{a} = 4ae$

$\Rightarrow \quad 1 - e^2 = 2e$

$\Rightarrow \quad e^2 + 2e - 1 = 0$

Therefore, $\quad e = -\frac{2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

Hence, $\quad e = \sqrt{2} - 1$

9. Let (x_1, y_1) be a point, then $x_1^2 + 3y_1^2 = 37$ ...(i)

Equation of tangent at (x_1, y_1) is

$$xx_1 + 3yy_1 = 37$$

Slope of tangent $= -\frac{x_1}{3y_1}$

Then, $\quad$ slope of normal $= \frac{3y_1}{x_1} = \frac{6}{5}$ (given)

$\Rightarrow \quad x_1 = \frac{5y_1}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{25y_1^2}{4} + 3y_1^2 = 37$$

$\Rightarrow \quad y_1^2 = 4$

$\therefore \quad y_1 = \pm 2$

From Eq. (ii), $\quad x_1 = \pm 5$

$\therefore$ Required points are $(5, 2)$ and $(-5, -2)$.

10. $\because \frac{PF_1}{PF_2} = \frac{NF_1}{NF_2}$

$\therefore$ PN bisects the $\angle F_1PF_2$.

$\because$ Bisectors are perpendicular to each other.

$\therefore$ PT bisects the angle $(180° - \angle F_1PF_2)$.

11. Given equation of ellipse is $16x^2 + 11y^2 = 256$.

Equation of tangent at $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ is

$$16x(4\cos\theta) + 11y\left(\frac{16}{\sqrt{11}}\sin\theta\right) = 256$$

It touches $(x - 1)^2 + y^2 = 4^2$, if

$$\left|\frac{4\cos\theta - 16}{\sqrt{16\cos^2\theta + 11\sin^2\theta}}\right| = 4$$

$\Rightarrow \quad (\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

$\Rightarrow \quad 4\cos^2\theta + 8\cos\theta - 5 = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

12. Solving both equations, we have

$$\frac{x^2}{a^2}+\frac{1-(x-1)^2}{b^2}=1$$

$\Rightarrow \quad b^2x^2+a^2[1-(x-1)^2]=a^2b^2$

$\Rightarrow (b^2-a^2)x^2+2a^2x-a^2b^2=0$ ...(i)

For least area circle must touch the ellipse.

$\therefore$ Discriminant of Eq. (i) is zero

$\Rightarrow \quad 4a^4+4a^2b^2(b^2-a^2)=0$

$\Rightarrow \quad a^2+b^2(b^2-a^2)=0$

$\Rightarrow \quad a^2+b^2(-a^2e^2)=0$

$\Rightarrow \quad 1-b^2e^2=0$

$\Rightarrow \quad b=\frac{1}{e}$

Also, $\quad a^2=\frac{b^2}{1-e^2}=\frac{1}{e^2(1-e^2)} \qquad \left(\because b=\frac{1}{e}\right)$

$\Rightarrow \quad a=\frac{1}{e\sqrt{1-e^2}}$

Let S be the area of the ellipse. Then,

$$S=\pi ab=\frac{\pi}{e^2\sqrt{1-e^2}}=\frac{\pi}{\sqrt{e^4-e^6}}$$

Area is minimum, if $f(e)=e^4-e^6$ is maximum,

when $f'(e)=4e^3-6e^5=0$,

or $\quad e=\sqrt{2/3} \quad$ {which is point of maxima for $f(e)$}

S is least when $e=\sqrt{\frac{2}{3}}$

$\therefore \quad b=\frac{1}{\sqrt{2/3}}=\sqrt{\frac{3}{2}}$

and $\quad a=\frac{1}{\sqrt{\frac{2}{3}}\sqrt{1-\frac{2}{3}}}=\frac{3}{\sqrt{2}}$

$\therefore$ Equation of ellipse is $2x^2+6y^2=9$.

13. Equation of auxiliary circle of ellipse is $x^2+y^2=45$.

14. Length of latusrectum of ellipse is

$$\frac{2b^2}{a}=\frac{2\frac{9}{4}}{\frac{9}{2}}=1$$

15. $9x^2+4y^2-18x-24y+9=0$

$\Rightarrow \quad 9(x-1)^2+4(y-3)^2=36$

$\Rightarrow \quad \frac{(x-1)^2}{2^2}+\frac{(y-3)^2}{3^2}=1$

Here, $\quad b>a$

$\therefore$ Sum of focal distances of a point is $2b=6$.

16. $\because \quad 5x^2+9y^2-54y+36=0$

$\Rightarrow \quad 5x^2+9(y-3)^2=45$

$\Rightarrow \quad \frac{x^2}{3^2}+\frac{(y-3)^2}{(\sqrt{5})^2}=1$

$\therefore \quad$ Length of major axis $=2\times 3=6$

and length of minor axis $=2\times\sqrt{5}=2\sqrt{5}$

17. Let $(t, b-t)$ be a point on the line $x+y=b$, then equation of chord whose mid-point $(t, b-t)$ is

$$\frac{tx}{2a^2}+\frac{(b-t)y}{2b^2}-1=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}-1 \qquad \text{...(i)}$$

$(a, -b)$ lies on Eq. (i), then

$$\frac{ta}{2a^2}-\frac{b(b-t)}{2b^2}=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}$$

$\Rightarrow \quad t^2(a^2+b^2)-ab(3a+b)t+2a^2b^2=0$

$\because t$ is real.

$\therefore \quad B^2-4AC\geq 0$

$\Rightarrow \quad a^2b^2(3a+b)^2-4(a^2+b^2)2a^2b^2\geq 0$

$\Rightarrow \quad 9a^2+6ab+b^2-8a^2-8b^2\geq 0$

$\therefore \quad a^2+6ab-7b^2\geq 0$

18. The equation of the director circle of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is $x^2+y^2=13$.

Statement II is also true but it is not a correct explanation for statement I.

19. Given ellipse is $\frac{x^2}{3}+\frac{y^2}{2}=1$, whose area is $\pi\sqrt{3}\sqrt{2}=\pi\sqrt{6}$.

Circle is $x^2+y^2-2x+4y+4=0$ or $(x-1)^2+(y-2)^2=1$.

Its area is π. Hence, statement I is true.

Also, statement II is true but it is not the correct explanation of statement I.

Consider the ellipse $\frac{x^2}{25}+\frac{y^2}{1}=1$, whose area is 5π and circle $x^2+y^2=16$ whose area is 16π.

Also, here semi-major axis of ellipse $(=5)$ is more than the radius of the circle $(=4)$.

20. Statement II is true as it is one of the properties of the ellipse

Circle with minimum radius having PQ as chord when PQ is diameter of the circle, hence as shown in the figure it passes through the focus.

21. Standard equation of tangent with slope m to the parabola $y^2 = 16\sqrt{3}x$ is

$$y = mx + \frac{4\sqrt{3}}{m} \quad \ldots(i)$$

Standard equation of tangent with slope m to the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$ is

$$y = mx \pm \sqrt{2m^2 + 4} \quad \ldots(ii)$$

If a line L is a common tangent to both parabola and ellipse, then

L should be tangent to parabola, *i.e.*, its equation should be like Eq. (i)

L should be tangent to ellipse *i. e.*, its equation should be like Eq. (ii).

i.e., L must be like both of the Eqs. (i) and (ii).

Hence, comparing Eqs. (i) and (ii), we get

$$\frac{4\sqrt{3}}{m} = \pm\sqrt{2m^2 + 4}$$

On squaring, we get

$$m^2(2m^2 + 4) = 48$$

$$\Rightarrow \quad m^4 + 2m^2 - 24 = 0$$

$$\Rightarrow \quad (m^2 + 6)(m^2 - 4) = 0$$

$$\Rightarrow \quad m^2 = 4 \quad (\because m^2 \neq -6)$$

$$\Rightarrow \quad m = \pm 2$$

On substituting $m = \pm 2$ in the Eq. (i), we get the required equation of the common tangents as

$$y = 2x + 2\sqrt{3}$$

and $$y = -2x - 2\sqrt{3}$$

Hence, statement I is correct.

Statement II In statement II, we have already seen that, if the line $y = mx + \frac{4\sqrt{3}}{m}$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$, then it satisfies the equation $m^4 + 2m^2 - 24 = 0$.

Hence, statement II is also correct but it is not able to explain the statement I.

22. Given equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Here, $a = 4$, $b = 3$, $e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

$\therefore$ Foci is $(\pm ae, 0)$

$$= \left(\pm 4 \times \frac{\sqrt{7}}{4}, 0\right) = (\pm\sqrt{7}, 0)$$

$\therefore$ Radius of the circle,

$$r = \sqrt{(ae)^2 + b^2} = \sqrt{7 + 9} = \sqrt{16} = 4$$

Now, equation of circle is

$$(x - 0)^2 + (y - 3)^2 = 16$$

$$\therefore \quad x^2 + y^2 - 6y - 7 = 0$$

23. Diameter of circle $(x - 1)^2 + y^2 = 1$ is 2 units and that of circle $x^2 + (y - 2)^2 = 4$ is 4 units.

Semi-minor axis of ellipse, $b = 2$ units and semi-major axis of ellipse, $a = 4$ units.

Hence, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{16} = \frac{y^2}{4} = 1$$

$$\Rightarrow \quad x^2 + 4y^2 = 16$$

24. Given, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Passes through $P(-3, 1)$ and $e = \sqrt{\frac{2}{5}}$

$$\therefore \quad \frac{9}{a^2} + \frac{1}{b^2} = 1$$

and $$e^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

and $$\frac{2}{5} = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{27 + 5}{3a^2} = 1 \text{ and } \frac{b^2}{a^2} = \frac{3}{5}$$

$$\Rightarrow \quad a^2 = \frac{32}{3} \text{ and } b^2 = \frac{32}{5}$$

$\therefore$ Equation of ellipse is

$$\frac{3x^2}{32} + \frac{5y^2}{32} = 1 \Rightarrow 3x^2 + 5y^2 = 32$$

25. Let the equation of the required ellipse be $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$. But the ellipse passes through the point (2, 1).

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{b^2} = 1$$

$\Rightarrow \quad \frac{1}{b^2} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$

Hence, equation is

$$\frac{x^2}{16} + \frac{3y^2}{4} = 1 \Rightarrow x^2 + 12y^2 = 16$$

26. Since, $\frac{a}{e} - ae = 4$ and $e = \frac{1}{2}$

$\therefore \quad 2a - \frac{a}{2} = 4$

$\Rightarrow \quad \frac{3a}{2} = 4$

$\Rightarrow \quad a = \frac{8}{3}$

27. Since, $ae = 3, b = 4$

Now, $\quad \frac{16}{9} = \frac{b^2}{a^2e^2} = \frac{a^2(1-e^2)}{a^2e^2} = \frac{1-e^2}{e^2}$

$\Rightarrow \quad e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

28. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$

$\because \quad \angle FBF' = \frac{\pi}{2}$

$\therefore \quad \angle FBC = \frac{\pi}{4}$

$\therefore \angle CFB$ is also an angle of $\frac{\pi}{4}$.

$\Rightarrow \quad BC = CF \Rightarrow b = ae$

$\Rightarrow \quad b^2 = a^2e^2 \quad \ldots(i)$

We know, $\quad b^2 = a^2(1 - e^2)$

$\therefore \quad a^2e^2 = a^2(1 - e^2)$ [from Eq. (i)]

$\Rightarrow \quad e^2 = 1 - e^2$

$\Rightarrow \quad 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

29. $\angle FBF' = 90° \Rightarrow \angle OBF' = 45°$

$\Rightarrow \quad ae = b$

and $\quad e^2 = 1 - \frac{b^2}{a^2}$

$\Rightarrow \quad e^2 = 1 - \frac{a^2e^2}{a^2}$

$\Rightarrow \quad e^2 = 1 - e^2$

$\Rightarrow \quad 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

30. Since, $\frac{a}{e} = 4 \Rightarrow 2a = 4 \Rightarrow a = 2$

Now, $\quad b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{4}\right) = 3$

There fore, equation is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ or $3x^2 + 4y^2 = 12$

16 Hyperbola

JEE Main MILESTONE

- Hyperbola
- Conjugate Hyperbola
- Auxiliary Circle
- Position of a Point with Respect to a Hyperbola
- Intersection of a Line and a Hyperbola
- Tangent to the Hyperbola
- Normal to the Hyperbola
- Number of Normals and Conormal Points
- Equation of Pair of Tangents
- Chord of Contact
- Director Circle
- Diameter
- Asymptotes
- Rectangular Hyperbola

16.1 Hyperbola

A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola.

OR

A hyperbola is the locus of a point in a plane which moves in a plane in such a way that the ratio of its distance from a fixed point (*i.e.*, focus) to the distance from a fixed line (*i.e.*, directrix) is greater than unity .

Mathematically, $\frac{SP}{PM} = e$, where $e > 1$.

*In Mathematics a **hyperbola** is a type of smooth curve, lying in a plane, defined by its geometric properties or by equations for which it is the solution set . A hyperbola has two pieces, called connected components or branches, that are mirror images of each other and resemble two infinite bows. The hyperbola is one of the four kinds of conic section formed by the intersection of a plane and a cone.*

Standard Equation of Hyperbola

If the centre of the hyperbola is at the origin and foci are on the x-axis or y-axis, then that types of equation are called standard equation of an ellipse. The two such possible orientations are shown below.

(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Terms Related to Hyperbola

Vertices

The point A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.

Transverse and Conjugate Axes

Transverse axis is the one which lie along the line passing through the foci and perpendicular to the directrices and conjugate axis is the one which is perpendicular to the transverse axis and passes through the mid-point of the foci (*i.e.*, centre).

Centre

The mid-point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.

Focal Chord

A chord of a hyperbola which is passing through the focus is called a focal chord of the hyperbola.

Directrix

A line which is perpendicular to the axis and it lies between centre and vertex. The equation of directrix is $x = \pm \frac{a}{e}$.

Double Ordinates

If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then, QQ' is called a double ordinate of Q.

Latusrectum

The double ordinate passing through focus is called latusrectum.

Note

- The vertex divides the join of focus and the point of intersection of directrix with axis internally and externally in the ratio $e : 1$.
- Domain and range of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x \le -a$ or $x \ge a$ and $y \in R$ respectively.

***Sample* Problem 1** *The equation of the hyperbola whose directrix is $2x + y = 1$, focus is $(1, 2)$ and eccentricity is $\sqrt{3}$, is $7x^2 - 2y^2 + 12xy - 2x + 14y + k = 0$, then k is equal to*

(a) 0
(b) 22
(c) –22
(d) *None of the above*

Interpret (c) We know that, $SP^2 = e^2PM^2$

$$\Rightarrow \quad 5[(x-1)^2 + (y-2)^2] = 3(2x+y-1)^2$$

$$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 = 3(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$$

$$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$$

Thus, $k = -22$

16.2 Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Detailed Description of Hyperbola

	Fundamental Terms	Hyperbola	Conjugate Hyperbola
(a)	Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
(b)	Graph		
(c)	Centre	$C(0, 0)$	$C(0, 0)$
(d)	Vertices	$(\pm a, 0)$	$(0, \pm b)$
(e)	Length of transverse axis	$2a$	$2b$
(f)	Length of conjugate axis	$2b$	$2a$
(g)	Foci	$(\pm ae, 0)$	$(0, \pm be)$
(h)	Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
(i)	Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
(j)	Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
(k)	Ends of latusrectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
(l)	Parametric equations	$\begin{cases} x = a\sec\alpha \\ y = b\tan\alpha \end{cases}$ or $x=a\left(\frac{e^\theta+e^{-\theta}}{2}\right)$ and $y=\frac{e^\theta-e^{-\theta}}{2}$	$\begin{cases} x = a\tan\alpha \\ y = b\sec\alpha \end{cases}$
(m)	Parametric coordinates	$(a\sec\alpha, b\tan\alpha)$	$(a\tan\alpha, b\sec\alpha)$
(n)	Focal radi	$\lvert SP\rvert=(ex_1-a)$ and $\lvert S'P\rvert=(ex_1+a)$	$\lvert SP\rvert=(ey_1-b)$ and $\lvert S'P\rvert=(ey_1+b)$
(o)	Difference of focal radi = $\lvert SP\rvert-\lvert S'P\rvert$	$2a$	$2b$
(p)	Distance between foci	$2ae$	$2be$
(q)	Tangents at vertices	$x=a$ and $x=-a$	$x=b$ and $y=-b$

Sample **Problem 2** *If equation of ellipse is $16x^2-9y^2=576$, then foci, eccentricity and length of latusrectum is* [**NCERT**]

(a) $(\pm 10, 0), \frac{5}{3}, \frac{64}{5}$ (b) $(\pm 10, 0), \frac{5}{3}, \frac{64}{3}$

(c) $(\pm 20, 0), \frac{5}{3}, \frac{64}{3}$ (d) $(-10, 0), \frac{5}{3}, \frac{64}{3}$

Interpret (b) Given equation is $16x^2-9y^2=576$, divide it by 576, we get

$$\frac{16x^2}{576}-\frac{9y^2}{576}=\frac{576}{576}\Rightarrow\frac{x^2}{36}-\frac{y^2}{64}=1$$

Now, comparing $\frac{x^2}{36}-\frac{y^2}{64}=1$ with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$$a^2=36 \text{ and } b^2=64$$

$$\Rightarrow \quad a=6 \text{ and } b=8$$

$$\therefore \quad e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{64}{36}}=\sqrt{\frac{100}{36}}$$

$$\Rightarrow \quad e=\frac{10}{6}=\frac{5}{3}$$

Foci $=(\pm ae, 0)$

$$=(\pm 6\times\frac{5}{3}, 0)=(\pm 10, 0)$$

Length of latusrectum $= \dfrac{2b^2}{a}$

$$= \frac{2 \times 64}{6} = \frac{64}{3}$$

***Sample* Problem 3** *If vertices and foci of a hyperbola are* $(0, \pm 5)$ *and* $(0, \pm 8)$ *respectively, then equation of hyperbola is* [NCERT]

(a) $\dfrac{x^2}{39} - \dfrac{y^2}{25} = 1$ (b) $\dfrac{y^2}{25} - \dfrac{x^2}{39} = 1$

(c) $\dfrac{x^2}{39} + \dfrac{y^2}{25} = 1$ (d) *None of these*

Interpret (b) Since, vertices $(0, \pm 5)$ and foci $(0, \pm 8)$ lies on y-axis, as coefficient of X-axis is zero.

Hence, equation of hyperbola will be of the form

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

where it is given that $(0, \pm b) = (0, \pm 5)$ and foci $(0, \pm be) = (0, \pm 8)$

$$b = 5 \text{ and } be = \pm 8$$

$\because$ $$a^2 = b^2(e^2 - 1) = (be)^2 - b^2 = 64 - 25 = 39$$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$-\frac{x^2}{39} + \frac{y^2}{25} = 1$$

***Sample* Problem 4** *If foci is* $(\pm 3\sqrt{5}, 0)$ *and the length of latusrectum is* 8, *then equation of hyperbola is* [NCERT]

(a) $\dfrac{x^2}{25} + \dfrac{y^2}{20} = 1$ (b) $\dfrac{x^2}{20} - \dfrac{y^2}{25} = 1$

(c) $\dfrac{x^2}{25} - \dfrac{y^2}{20} = 1$ (d) *None of these*

Interpret (c) Since, foci $(\pm 3\sqrt{5}, 0)$ lies on x-axis, as y-coordinate is zero. Hence, equation of hyperbola will be of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

where it is given that foci $(\pm 3\sqrt{5}, 0) = (\pm ae, 0)$

$$ae = 3\sqrt{5}$$

$\Rightarrow$ $$a^2e^2 = 45 \Rightarrow a^2\left(1 + \frac{b^2}{a^2}\right) = 45 \quad \ldots(ii)$$

and length of latusrectum $= \dfrac{2b^2}{a} = 8$

$$b^2 = 4a$$

$\therefore$ From Eq.(ii), $$a^2\left(1 + \frac{4a}{a^2}\right) = 45$$

$\Rightarrow$ $$a^2 + 4a - 45 = 0$$

$\Rightarrow$ $$(a - 5)(a + 9) = 0$$

$\Rightarrow$ $$a = 5, -9$$

$\because$ a cannot be negative.

So, we take $a = 5$,

then $$b^2 = 4 \times 5 = 20$$

From Eq.(i),

$$\frac{x^2}{25} - \frac{y^2}{20} = 1$$

16.3 Auxiliary Circle

Let $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be the hyperbola, then equation of the auxiliary circle is $x^2 + y^2 = a^2$

Let $\angle QCN = \phi$. Here, P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \phi < 2\pi)$

Parametric Equations

The equation $x = a \sec\phi$ and $y = b \tan\phi$ are known as the parametric equations of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. This $(a \sec\phi, b \tan\phi)$ lies on the hyperbola for all values of ϕ.

***Sample* Problem 5** *If the equation of hyperbola is* $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, *then equation of auxiliary circle is*

(a) $x^2 + y^2 = 16$ (b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 = 25$ (d) *None of these*

Interpret (b) Given equation of hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

$\therefore$ Equation of auxiliary circle is $x^2 + y^2 = 9$

16.4 Position of a Point with Respect to a Hyperbola

Let the equation of a hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and coordinates of a point P are (h, k). Then,

Point P lies outside, if

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1 < 0$$

Point P lies on the hyperbola, if $\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$

and point P lies inside, if $\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1 > 0$

***Sample* Problem 6** *The position of the point* (5, – 4) *relative to the hyperbola* $9x^2 - y^2 = 1$, *is*

(a) *outside* (b) *on the hyperbola*
(c) *inside* (d) *None of these*

Interpret (c) The equation of hyperbola is $9x^2 - y^2 = 1$.

At point (5, – 4),

$$9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$$

So, the point (5, – 4) lies inside the hyperbola $9x^2 - y^2 = 1$.

Check Point **1**

1. The locus of the point $\left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$ is a hyperbola of eccentricity
(a) $\sqrt{3}$ (b) 3
(c) $\sqrt{2}$ (d) 2

2. The equation of the hyperbola whose directrix is x + 2y = 1, focus is (2, 1) and eccentricity 2, is
(a) $x^2 + 16xy - 11y^2 - 12x + 6y + 21 = 0$
(b) $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$
(c) $x^2 - 4xy - y^2 - 12x + 6y + 21 = 0$
(d) None of the above

3. The difference between the length 2a of transverse axis of a hyperbola of eccentricity e and the length of its latusrectum is
(a) $2a[3 - e^2]$ (b) $2a[2 - e^2]$
(c) $2a[e^2 - 1]$ (d) $a(2e^2 - 1)$

4. Which of the following does not represent hyperbola ?
(a) xy = 1 (b) $x^2 - y^2 = 5$
(c) (x – 1)(y – 3) = 3 (d) $x^2 - y^2 = 0$

5. The vertices of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$ are
(a) (6, 3), (–2, 3)
(b) (6, 3), (–6, 3)
(c) (–6, 3), (–6, –3)
(d) None of the above

6. Which of the following is independent of α in the hyperbola $\left(0 < \alpha < \frac{\pi}{2}\right) \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$?
(a) eccentricity (b) abscissa of foci
(c) directrix (d) vertex

7. The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represents hyperbola whose
(a) transverse axis is $4\sqrt{3}$ (b) conjugate axis is 4
(c) eccentricity is $\frac{\sqrt{19}}{\sqrt{3}}$ (d) centre is (–1, 2)

8. If the latusrectum of a hyperbola through one focus subtends 60° angle at the other focus, then its eccentricity is
(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{8}$ (d) $\sqrt{6}$

9. The equation of hyperbola, if vertices (0, ± 3) and foci (0, ± 5), is
(a) $\frac{y^2}{16} - \frac{x^2}{9} = 1$ (b) $\frac{y^2}{9} - \frac{x^2}{16} = 1$
(c) $\frac{y^2}{16} + \frac{x^2}{9} = 1$ (d) None of these

10. If equation of hyperbola is $\frac{x^2}{25} - \frac{y^2}{16} = 1$, then equation of auxiliary circle is
(a) $x^2 + y^2 = 41$ (b) $x^2 + y^2 = 16$
(c) $x^2 + y^2 = 50$ (d) $x^2 + y^2 = 25$

16.5 Intersection of a Line and a Hyperbola

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

and the given line be $y = mx + c$ ...(ii)

On eliminating y from Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow \quad (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0 \quad \text{...(iii)}$$

It is a quadratic equation in x which gives two values of x.

$\therefore D = 4m^2c^2a^4 - 4(a^2m^2 - b^2)a^2(b^2 + c^2) = -a^2m^2 + b^2 + c^2$

(i) If $D > 0$ *i.e.*, $c^2 > a^2m^2 - b^2$, then line intersect the hyperbola at two points.

(ii) If $D = 0$ *i.e.*, $c^2 = a^2m^2 - b^2$, then line touches the hyperbola.

(iii) If $D < 0$ *i.e.*, $c^2 < a^2m^2 - b^2$, then line neither touches nor intersect the hyperbola.

Condition of Tangency

The line $y = mx + c$ will touch a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

$$c^2 = a^2m^2 - b^2$$

and coordinates of point of contact are

$$\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

***Sample* Problem 7** *The line $x \cos\alpha + y \sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2 \cos^2\alpha - b^2 \sin^2\alpha$ is equal to*

(a) p (b) p^2 (c) $-p^2$ (d) $2p^2$

Interpret (b) The given line is $x \cos\alpha + y \sin\alpha = p$

or $y = -x \cot\alpha + p \operatorname{cosec}\alpha$

On comparing with $y = mx + c$, we get

$$m = -\cot\alpha, x = p \operatorname{cosec}\alpha$$

Since, the given line touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ then } c^2 = a^2m^2 - b^2$$

$$\therefore \quad p^2 \operatorname{cosec}^2\alpha = a^2 \cot^2\alpha - b^2$$

$$\Rightarrow \quad a^2 \cos^2\alpha - b^2 \sin^2\alpha = p^2$$

16.6 Tangent to the Hyperbola

A line which intersect the hyperbola at only one point is called the tangent to the hyperbola.

Equation of Tangent in Different Forms

(i) Point Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(ii) Slope Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the slope form is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

(iii) Parametric Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $(a \sec\theta, b \tan\theta)$ is $\frac{x}{a} \sec\theta - \frac{y}{b} \tan\theta = 1$

Note

- The tangents at the points $P(a \sec\theta_1, b \tan\theta_1)$ and $Q(a \sec\theta_2, b \tan\theta_2)$ intersect at the point

$$R\left\{\frac{a \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}, \frac{b \sin\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}\right\}.$$

- To a hyperbola, two tangents can be drawn from a point outside of the hyperbola.

***Sample* Problem 8** *If the line $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$, then θ is equal to*

(a) $\sin^{-1}\left(\frac{b}{m}\right)$ (b) $\sin^{-1}\left(\frac{b}{a}\right)$

(c) $\sin^{-1}\left(\frac{b}{ma}\right)$ (d) *None of these*

Interpret (c) Since, the point $(a \sec\theta, b \tan\theta)$ lies on the line

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$\therefore \quad b \tan\theta = am \sec\theta + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad (b \tan\theta - am \sec\theta)^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad b^2 \tan^2\theta + a^2m^2 \sec^2\theta - 2abm \tan\theta \sec\theta = a^2m^2 - b^2$$

$$\Rightarrow \quad a^2m^2 \tan^2\theta - 2abm \tan\theta \sec\theta + b^2 \sec^2\theta = 0$$

$$\Rightarrow \quad a^2m^2 \sin^2\theta - 2abm \sin\theta + b^2 = 0 \quad (\because \cos\theta \neq 0)$$

$$\therefore \quad \sin\theta = \frac{2abm \pm \sqrt{4a^2b^2m^2 - 4a^2b^2m^2}}{2a^2m^2} = \left(\frac{b}{am}\right)$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{b}{am}\right)$$

***Sample* Problem 9** *The equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$, which is perpendicular to the line $x - y + 4 = 0$ is*

(a) $x + y \pm 3\sqrt{3} = 0$ (b) $x + y \pm 3\sqrt{3} = 0$

(c) $x + y \pm 2\sqrt{3} = 0$ (d) $x - y \pm 3\sqrt{3} = 0$

Interpret (*b*) Let m be the solpe of the tangent. Since, the tangent is perpendicular to the line $x - y + 4 = 0$

$$\therefore \quad m \times 1 = -1$$

$$\Rightarrow \quad m = -1$$

Given equation of hyperbola is $x^2 - 4y^2 = 36$.

or $$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

Here, $$a^2 = 36, b^2 = 9$$

$\therefore$ The equation of tangents are

$$y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9} \qquad (\because y = mx \pm \sqrt{a^2m^2 - b^2})$$

$$= -x \pm \sqrt{27}$$

$$= -x \pm 3\sqrt{3}$$

$$\therefore \quad x + y \pm 3\sqrt{3} = 0$$

16.7 Normal to the Hyperbola

A line which is perpendicular to the tangent of the hyperbola is called the normal to the hyperbola.

Equation of Normal in Different Forms

(i) Point Form

The equation of normal at the point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

(ii) Slope Form

The equation of normal of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$ at the points

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp \frac{b^2m}{\sqrt{a^2 - b^2m^2}}\right)$$

Note

The line $y = mx + c$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = \frac{m^2(a^2 + b^2)^2}{a^2 - b^2m^2}$

(iii) Parametric Form

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$ is $ax \cos\theta + by \cot\theta = a^2 + b^2$

Note Four normals can be drawn to the hyperbola from any point outside to the hyperbola.

***Sample* Problem 10** *If the line $lx + my - n = 0$ will be a normal to the hyperbola, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{k}$, where k is equal to*

(a) n (b) n^2
(c) n^3 (d) *None of these*

Interpret (*b*) The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax \cos\phi + by \cot\phi = a^2 + b^2$$

$$\Rightarrow \quad ax \cos\phi + by \cot\phi - (a^2 + b^2) = 0 \qquad \text{...(i)}$$

The straight line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line.

$$\therefore \quad \frac{a \cos\phi}{l} = \frac{b \cot\phi}{m} = \frac{(a^2 + b^2)}{n}$$

$$\Rightarrow \quad \sec\phi = \frac{na}{l(a^2 + b^2)}$$

and $$\tan\phi = \frac{nb}{m(a^2 + b^2)}$$

$$\therefore \quad \sec^2\phi - \tan^2\phi = 1$$

$$\Rightarrow \quad \frac{n^2a^2}{l^2(a^2 + b^2)^2} - \frac{n^2b^2}{m^2(a^2 + b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

Thus, $$k = n^2$$

***Sample* Problem 11** *Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$*

(a) $(x^2 + y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2x^2y^2$
(b) $(x^2 + y^2)^2(a^2y^2 + b^2x^2) = (a^2 + b^2)^2x^2y^2$
(c) $(x^2 - y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2x^2y^2$
(d) *None of the above*

Interpret (*a*) Equation of normal at $P(a \sec\phi, b \tan\phi)$ is

$$ax \cos\phi + by \cot\phi = a^2 + b^2 \qquad \text{...(i)}$$

And equation of line perpendicular to Eq. (i) and passing through origin is

$$bx \cot \phi - ay \cos \phi = 0$$

$$\Rightarrow \quad bx - ay \sin \phi = 0 \quad \text{...(ii)}$$

$$\Rightarrow \quad \sin \phi = \frac{bx}{ay}$$

$$\therefore \quad \cos \phi = \sqrt{1 - \sin^2 \phi}$$

$$= \sqrt{1 - \frac{b^2x^2}{a^2y^2}} = \frac{\sqrt{a^2y^2 - b^2x^2}}{ay}$$

and $\cot \phi = \dfrac{\sqrt{a^2y^2 - b^2x^2}}{bx}$

From Eq. (i),

$$\frac{ax \times \sqrt{a^2y^2 - b^2x^2}}{ay} + \frac{by \times \sqrt{a^2y^2 - b^2x^2}}{bx} = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\sqrt{a^2y^2 - b^2x^2} = (a^2 + b^2)xy$$

$$\Rightarrow \quad (x^2 + y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2x^2y^2$$

16.8 Number of Normals and Conormal Points

There are exactly four lines passing through a given point such that they are normals to the hyperbola at the points where they intersect the hyperbola. Such points on the hyperbola are known as the **conormal points**.

Properties of Eccentric Angles of Conormal Points

(i) The sum of the eccentric angles of conormal points is an odd multiple of π.

(ii) If θ_1, θ_2, θ_3 and θ_4 are eccentric angles of four points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the normals at which are concurrent, then

(a) $\Sigma \cos(\theta_1 + \theta_2) = 0$

(b) $\Sigma \sin(\theta_1 + \theta_2) = 0$

(iii) If θ_1, θ_2 and θ_3 are the eccentric angle of three points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ such that $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$, then the normals at these points are concurrent.

(iv) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are concurrent, then

$$(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$$

16.9 Equation of Pair of Tangents

Let $P(x_1, y_1)$ be a point which lies outside the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then the equation of pair of tangents drawn from external point P to the hyperbola is given by

$$SS_1 = T^2$$

where, $S = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1$

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

and $T = \dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} - 1$

16.10 Chord of Contact

Let PA and PB be any two tangents to the hyperbola from a point $P(x_1, y_1)$, then AB is known as chord of contact and its equation is given by

$$T = 0$$

$$\Rightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

Equation of the Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, bisected at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

or $T = S_1$

***Sample* Problem 12** *From the points on the circle $x^2 + y^2 = a^2$ tangents are drawn to the hyperbola $x^2 - y^2 = a^2$, then the locus of the middle points of the chords of contact is the curve $(x^2 - y^2)^2 = k(x^2 + y^2)$, where k is equal to*

(a) a (b) a^3

(c) a^2 (d) *None of these*

Interpret (c) Since, any point of the circle $x^2+y^2=a^2$ is $(a\cos\theta, a\sin\theta)$, chord of contact of this point w.r.t. hyperbola $x^2-y^2=a^2$ is

$$x(a\cos\theta)-y(a\sin\theta)=a^2$$

or $$x\cos\theta-y\sin\theta=a \quad \text{...(i)}$$

If its mid-point be (h,k), then it is same as

$$T=S_1$$

$$\Rightarrow \quad hx-ky-a^2=h^2-k^2-a^2$$

$$\Rightarrow \quad hx-ky=h^2-k^2 \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$\frac{\cos\theta}{h}=\frac{\sin\theta}{k}=\frac{a}{(h^2-k^2)}$$

$$\therefore \quad (h^2-k^2)\cos\theta=ah \quad \text{...(iii)}$$

and $$(h^2-k^2)\sin\theta=ak \quad \text{...(iv)}$$

On squaring and adding Eqs. (iii) and (iv), we get

$$(h^2-k^2)^2=a^2h^2+a^2k^2$$

$$\Rightarrow \quad (h^2-k^2)^2=a^2(h^2+k^2)$$

Hence, the required locus is

$$(x^2-y^2)^2=a^2(x^2+y^2)$$

16.11 Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which are perpendicular to each other is called director circle. The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$.

Note For director circle of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, a must be greater than b. If $a<b$, then director circle $x^2+y^2=a^2-b^2$ does not exist.

***Sample* Problem 13** *The equation of the director circle of the hyperbola* $9x^2-16y^2=144$ *is*

(a) $x^2+y^2=7$ (b) $x^2+y^2=9$
(c) $x^2+y^2=16$ (d) $x^2+y^2=25$

Interpret (b) Given equation can be rewritten as $\frac{x^2}{16}-\frac{y^2}{9}=1$

$\therefore$ Required equation of director circle is

$$x^2+y^2=16-9$$

$$\therefore \quad x^2+y^2=9$$

16.12 Diameter

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.

Let $y=mx+c$ a system of parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords, then the equation of diameter of the hyperbola is $y=\frac{b^2x}{a^2m}$, which is passing through (0, 0).

Conjugate Diameter

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y=m_1x$ and $y=m_2x$ be conjugate diameters, then $m_1m_2=\frac{b^2}{a^2}$

***Sample* Problem 14** *If the lines* $lx+my+n=0$ *passes through the extremities of a pair of conjugate diameters of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *then* $a^2l^2-b^2m^2$ *is equal to*

(a) 0 (b) – 1
(c) 1 (d) 2

Interpret (a) The extremities of a pair of conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

According to the question, since extremities of a pair of conjugate diameters lie on $lx+my+n=0$

$$\therefore \quad l(a\sec\phi)+m(b\tan\phi)+n=0 \quad \text{...(i)}$$

$$\Rightarrow \quad l(a\tan\phi)+m(b\sec\phi)+n=0 \quad \text{...(ii)}$$

Then, from Eq. (i), $al\sec\phi+bm\tan\phi=-n$

or $$a^2l^2\sec^2\phi+b^2m^2\tan^2\phi+2ablm\sec\phi\tan\phi=n^2 \quad \text{...(iii)}$$

and from Eq. (ii), $al\tan\phi+bm\sec\phi=-n$

or $$a^2l^2\tan^2\phi+b^2m^2\sec^2\phi+2ablm\sec\phi\tan\phi=n^2 \quad \text{...(iv)}$$

Then, subtracting Eq. (iv) from Eq. (iii), we get

$$a^2l^2(\sec^2\phi-\tan^2\phi)+b^2m^2(\tan^2\phi-\sec^2\phi)=0$$

$$\Rightarrow \quad a^2l^2-b^2m^2=0$$

Check Point 2

1. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to the line $y = 2x$, is
(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 4y = 3$ (d) $3x - 4y = 2$

2. If the tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at $(2\sec\phi, 3\tan\phi)$ is parallel to the line $3x - y + 4 = 0$, then the value of ϕ is
(a) 45° (b) 60°
(c) 30° (d) 75°

3. A tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P and Q. The locus of the mid-point of PQ is
(a) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ (b) $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
(c) $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{2x^2y^2}{a^2b^2}$ (d) None of these

4. The equation of normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $(8, 3\sqrt{3})$ is
(a) $\sqrt{3}x + 2y = 25$ (b) $x + y = 25$
(c) $y + 2x = 25$ (d) $2x + \sqrt{3}y = 25$

5. The line $y = x + 2$ touches hyperbola $5x^2 - 9y^2 = 45$, the point of contact is
(a) (0, 2) (b) (3, 1) (c) $\left(-\frac{9}{2}, -\frac{5}{2}\right)$ (d) $\left(-\frac{9}{2}, \frac{5}{2}\right)$

6. The equation of director circle of the hyperbola $4x^2 - 9y^2 = 36$ is
(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 5$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 13$

7. Find the condition for the lines $Ax^2 + 2Hxy + By^2 = 0$ to be conjugate diameters of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
(a) $aA = bB$ (b) $a^2B = b^2A$
(c) $a^2A = b^2B$ (d) None of these

16.13 Asymptotes

An asymptote of any hyperbola is a straight line which touches in it two points at infinity. In other words asymptotes are the lines which are tangents to the curve at infinity.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm \frac{b}{a}x$.

Method to Determine the Asymptotes

Step I Determine the degree of the equation of the curve. Let it is n.

Step II If x^n is present in the equation of the curve, then there is no asymptote parallel to x-axis.

Step III If the term of x^n is missing, then equate to zero the coefficient of x^{n-1}, this will give the asymptote parallel to x-axis.

Similarly, we can find the asymptote parallel to y-axis.

Note

- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angle between the asymptotes are the coordinate axes.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is equal to $\frac{a^2b^2}{a^2 + b^2}$.

***Sample* Problem 15** *The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), its equation is $(2x + 3y - 8)(3x + 2y - 7) = k$, where k is equal to*
(a) 15 (b) 154
(c) 0 (d) *None of these*

Interpret (b) Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since, asymptotes passes through (1, 2), then

$$\lambda = -8 \quad \text{and} \quad \mu = -7$$

Thus, the equations of asymptotes are

$$2x + 3y - 8 = 0 \quad \text{and} \quad 3x + 2y - 7 = 0$$

Let the equation of hyperbola be

$$(2x + 3y - 8)(3x + 2y - 7) + v = 0 \qquad \ldots(i)$$

It passes through (5, 3), then

$$(10 + 9 - 8)(15 + 6 - 7) + v = 0 \Rightarrow 11 \times 14 + v = 0$$

$$\therefore \quad v = -154$$

On putting the value of v in Eq. (i), we get

$$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$$

This is the equation of required hyperbola.

16.14 Rectangular Hyperbola

A hyperbola whose asymptotes are at right angles to each other, is said to be rectangular hyperbola.

or

If the length of transverse and conjugate axes of any hyperbola are equal, then hyperbola is known as rectangular hyperbola.

The equation of rectangular hyperbola to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2-y^2=a^2$ and equation of the asymptotes are $y=\pm x$ *i.e.*, $y=x$ and $y=-x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

When the centre of any rectangular hyperbola be at the origin and its asymptotes coincide with the coordinate axes, then the equation of rectangular hyperbola is $xy=c^2$. Its eccentricity is $\sqrt{2}$.

(i) Rectangular Hyperbola of the Form $x^2-y^2=a^2$

(a) Asymptotes are perpendicular lines *i.e.*, $x \pm y = 0$

(b) Eccentricity, $e=\sqrt{2}$.

(c) Centre, (0, 0)

(d) Foci $(\pm\sqrt{2}\,a, 0)$

(e) Directrices, $x=\pm\frac{a}{\sqrt{2}}$

(f) Latusrectum $=2a$

(g) Point form, $x=x_1$, $y=y_1$

Equation of tangent, $xx_1-yy_1=a^2$

Equation of normal, $\frac{x_1}{x}+\frac{y_1}{y}=2$.

(h) Parametric form, $x=a\sec\theta$, $y=a\tan\theta$

Equation of tangent, $x\sec\theta-y\tan\theta=a$

Equation of normal, $\frac{x}{\sec\theta}+\frac{y}{\tan\theta}=2a$

(ii) Rectangular Hyperbola of the Form $xy=c^2$

(a) Asymptotes are perpendicular lines *i.e.*, $x=0$ and $y=0$

(b) Eccentricity, $e=\sqrt{2}$

(c) Centre,(0, 0)

(d) Foci are $S(\sqrt{2}c, \sqrt{2}c)$, $S_1(-\sqrt{2}c, -\sqrt{2}c)$

(e) Vertices, $V_1(c, c)$, $V_2(-c, -c)$

(f) Directrices, $x+y=\pm\sqrt{2}c$

(g) Latusrectum $=2\sqrt{2}c$

(h) Point form, $x=x_1$, $y=y_1$

Equation of tangent, $xy_1+yx_1=2c^2$

$\Rightarrow \qquad \frac{x}{x_1}+\frac{y}{y_1}=2$

Equation of normal, $xx_1-yy_1=x_1^2-y_1^2$

(i) Parametric form : $x=ct$, $y=\frac{c}{t}$

Equation of tangent, $x+yt^2=2ct$

Equation of normal, $t^2x-y=c\left(t^3-\frac{1}{t}\right)$

(iii) Properties of Rectangular Hyperbola $xy=c^2$

(a) Equation of the chord joining t_l and t_2 is $x+yt_1t_2-c(t_1+t_2)=0$.

(b) Equation of tangent at (x_1,y_1) is $xy_1+x_1y=2c^2$.

(c) Equation of tangent at $\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t}+yt=2c$.

(d) Point of intersection of tangents at t_1 and t_2 is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

(e) Equation of normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3-yt-ct^4+c=0$.

(f) Equation of normal to rectangular hyperbola at (x_1,y_1) is

$$x\,x_1-yy_1=x_1^2-y_1^2.$$

WORKED OUT Examples

Example 1 *The equation of hyperbola, having foci* $(\pm 5, 0)$ *and length of transverse axis is* 8, *is* [**NCERT**]

(*a*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (*b*) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(*c*) $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (*d*) *None of these*

Solution (*a*) Since, foci $(\pm 5, 0)$ lies on X-axis, as coordinate of Y is zero. Hence, equation of hyperbola will be of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

where it is given that foci $(\pm 5, 0) = (\pm ae, 0)$ and length of transverse axis is $2a = 8$

$\therefore$ $ae = 5$ and $2a = 8$

$\Rightarrow$ $a = 4$

$\Rightarrow$ $e = \frac{5}{4}$

$\therefore$ $b = a\sqrt{e^2 - 1} = 4\sqrt{\frac{25}{16} - 1}$

$b = 3$

Put $a^2 = 16$ and $b^2 = 9$ in Eq. (i), we get

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Example 2 *The equation* $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ *represents a hyperbola*

(*a*) *the length of whose transverse axis is* $4\sqrt{3}$

(*b*) *the length of whose conjugate axis is* 4

(*c*) *whose centre is* $(-1, 2)$

(*d*) *whose eccentricity is* $\sqrt{\frac{19}{3}}$

Solution (*d*) We have

$$16(x^2 - 2x) - 3(y^2 - 4y) = 44$$

$\Rightarrow$ $16(x-1)^2 - 3(y-2)^2 = 48$

$\Rightarrow$ $\frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$

This equation represents a hyperbola with eccentricity

$$e = \sqrt{1 + \frac{16}{3}} = \sqrt{\frac{19}{3}}$$

Example 3 *The equation of the transverse axis of the hyperbola* $(x-3)^2 + (y+1)^2 = (4x+3y)^2$ *is*

(*a*) $x + 3y = 0$ (*b*) $4x + 3y = 9$

(*c*) $3x - 4y = 13$ (*d*) $4x + 3y = 0$

Solution (*c*) $(x-3)^2 + (y+1)^2 = (4x+3y)^2$

$\Rightarrow$ $(x-3)^2 + (y+1)^2 = 25\left(\frac{4x+3y}{5}\right)^2$

$\Rightarrow$ $PS = 5PM$

$\Rightarrow$ Directrix is $4x + 3y = 0$ and focus $(3, -1)$.

So, equation of transverse axis is $y + 1 = \frac{3}{4}(x - 3)$

$\Rightarrow$ $3x - 4y = 13$

Example 4 *The locus of the point of intersection of the lines* $\sqrt{3}x - y - 4\sqrt{3}t = 0$ *and* $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (*where, t is a parameter*) *is a hyperbola whose eccentricity is*

(*a*) $\sqrt{3}$ (*b*) 2 (*c*) $\frac{2}{\sqrt{3}}$ (*d*) $\frac{4}{3}$

Solution (*b*) Eliminating t from the given two equations, we have

$$\frac{x^2}{16} - \frac{y^2}{48} = 1,$$

where eccentricity is $e = \sqrt{1 + \frac{48}{16}} = 2$

Example 5 *The equation of the hyperbola whose eccentricity is* $\sqrt{2}$ *and the distance between the foci is* 16, *taking transverse and conjugate axes of the hyperbola as x and y axes respectively, is*

(*a*) $x^2 - y^2 = 0$ (*b*) $x^2 - y^2 = 32$

(*c*) $x^2 - y^2 = 2$ (*d*) *None of these*

Solution (*b*) Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$.

$\therefore$ $2ae = 16 \Rightarrow 2a\sqrt{2} = 16 \Rightarrow a = 4\sqrt{2}$

Also, $b^2 = a^2(e^2 - 1) = 32(2 - 1) = 32$

Thus, $a^2 = 32$ and $b^2 = 32$

Hence, the required equation is

$$\frac{x^2}{32} - \frac{y^2}{32} = 1 \Rightarrow x^2 - y^2 = 32$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Basic Terms of Hyperbola

1. If equation $(10x-5)^2+(10y-4)^2=\lambda^2(3x+4y-1)^2$ represents a hyperbola, then
(a) $-2<\lambda<2$ (b) $\lambda>2$
(c) $\lambda<-2$ or $\lambda>2$ (d) $0<\lambda<2$

2. The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci is [NCERT Exemplar]
(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$ (d) None of these

3. If the equation of hyperbola is $9y^2-4x^2=36$, then vertices and length of latusrectum is [NCERT]
(a) $(0,+2), 9$ (b) $(0,\pm 2), 9$
(c) $(0,-2), 9$ (d) None of these

4. A general point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is
(a) $(a\sin\theta, b\cos\theta)$ (where, θ is parameter)
(b) $(a\tan\theta, b\sec\theta)$ (where, θ is parameter)
(c) $\left(a\frac{e^t+e^{-t}}{2}, b\frac{e^t-e^{-t}}{2}\right)$ (where, t is parameter)
(d) None of the above

5. If a point $(x,y)=(\tan\theta+\sin\theta, \tan\theta-\sin\theta)$, then locus of (x,y) is
(a) $(x^2y)^{2/3}+(xy^2)^{2/3}=1$ (b) $x^2-y^2=4xy$
(c) $(x^2-y^2)^2=16xy$ (d) $x^2-y^2=6xy$

6. Eccentricity of hyperbola $\frac{x^2}{k}+\frac{y^2}{k^2}=1 (k<0)$ is
(a) $\sqrt{1+k}$ (b) $\sqrt{1-k}$
(c) $\sqrt{1+\frac{1}{k}}$ (d) $\sqrt{1-\frac{1}{k}}$

7. The length of transverse axis of the hyperbola $3x^2-4y^2=32$ is
(a) $\frac{8\sqrt{2}}{\sqrt{3}}$ (b) $\frac{16\sqrt{2}}{\sqrt{3}}$ (c) $\frac{3}{32}$ (d) $\frac{64}{3}$

8. The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latusrectum is
(a) $2a(3-e^2)$ (b) $2a\,|2-e^2|$
(c) $2a(e^2-1)$ (d) $a(2e^2-1)$

9. If eccentricity of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is e and e' is the eccentricity of its conjugate hyperbola, then
(a) $e=e'$ (b) $ee'=1$
(c) $\frac{1}{e^2}+\frac{1}{(e')^2}$ (d) None of these

10. Any point on the hyperbola $\frac{(x+1)^2}{16}-\frac{(y-2)^2}{4}=1$ is of the form
(a) $(4\sec\theta, 2\tan\theta)$ (b) $(4\sec\theta-1, 2\tan\theta+2)$
(c) $(4\sec\theta-1, 2\tan\theta-2)$ (d) $(4\sec\theta-4, 2\tan\theta-2)$

11. The product of the perpendicular from two foci on any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is
(a) a^2 (b) b^2 (c) $-a^2$ (d) $-b^2$

Equation of Hyperbola and Intersection of Two Curves

12. The equation of the hyperbola whose foci are (6,4) and (−4,4) and eccentricity 2, is
(a) $\frac{4(x-1)^2}{25}+\frac{4(y-4)^2}{25}=1$ (b) $\frac{4(x+1)^2}{25}+\frac{4(y+4)^2}{75}=1$
(c) $\frac{4(x-1)^2}{75}-\frac{4(y-4)^2}{25}=1$ (d) $\frac{4(x-1)^2}{25}-\frac{4(y-4)^2}{75}=1$

13. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is [NCERT Exemplar]

(a) $x^2 - y^2 = 32$ (b) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

(c) $2x - 3y^2 = 7$ (d) None of these

14. Equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is [NCERT Exemplar]

(a) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$ (b) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$

(c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (d) None of these

15. The equation of hyperbola having vertices $(\pm 2, 0)$ and foci $(\pm 3, 0)$ is [NCERT]

(a) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (b) $\frac{x^2}{5} - \frac{y^2}{4} = 1$

(c) $\frac{x^2}{4} + \frac{y^2}{5} = 1$ (d) None of these

16. The equation of hyperbola having foci $(0, \pm\sqrt{10})$ and passing through the point (2, 3) is [NCERT]

(a) $x^2 + y^2 = 5$ (b) $y^2 - x^2 = 5$

(c) $x^2 - y^2 = 5$ (d) None of these

17. Equation $\frac{1}{r} = \frac{1}{8} + \frac{3}{8}\cos\theta$ represents

(a) a rectangular hyperbola (b) a hyperbola

(c) an ellipse (d) a parabola

18. The angle between lines joining the origin to the point of intersection of the line $\sqrt{3}x + y = 2$ and the curve $y^2 - x^2 = 4$ is

(a) $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\frac{\pi}{6}$

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{\pi}{2}$

Tangent and Normal to the Hyperbola

19. The straight line $x + y = \sqrt{2}p$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if

(a) $p^2 = 2$ (b) $p^2 = 5$ (c) $5p^2 = 2$ (d) $2p^2 = 5$

20. What is the slope of the tangent drawn to the hyperbola $xy = a, (a \neq 0)$ at the point $(a, 1)$?

(a) $\frac{1}{a}$ (b) $-\frac{1}{a}$ (c) a (d) $-a$

21. The locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{3} - \frac{y^2}{1} = 1$ is

(a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = 3$

(c) $x^2 - y^2 = 3$ (d) $x^2 + y^2 = 4$

22. Number of points from where perpendicular tangents to the curve $\frac{x^2}{16} - \frac{y^2}{25} = 1$ can be drawn, is

(a) 1 (b) 2

(c) 0 (d) None of these

23. The point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the product of whose slope is c^2, lies on the curve

(a) $y^2 - b^2 = c^2(x^2 + a^2)$ (b) $y^2 + a^2 = c^2(x^2 - b^2)$

(c) $y^2 + b^2 = c^2(x^2 - a^2)$ (d) $y^2 - a^2 = c^2(x^2 + b^2)$

24. Equation of the normal to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ perpendicular to the line $2x + y = 1$ is

(a) $\sqrt{21}(x - 2y) = 41$ (b) $x - 2y = 1$

(c) $\sqrt{41}(x - 2y) = 41$ (d) $\sqrt{21}(x - 2y) = 21$

25. The number of normals to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from an external point is

(a) 6 (b) 5 (c) 4 (d) 2

26. The equation of normal at $\left(at, \frac{a}{t}\right)$ to the hyperbola $xy = a^2$ is

(a) $xt^3 - yt + at^4 - a = 0$ (b) $xt^3 - yt - at^4 + a = 0$

(c) $xt^3 + yt + at^4 - a = 0$ (d) None of these

27. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$ be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\frac{a^2 + b^2}{a}$ (b) $-\left[\frac{a^2 + b^2}{a}\right]$ (c) $\frac{a^2 + b^2}{b}$ (d) $-\left[\frac{a^2 + b^2}{b}\right]$

28. The value of m, for which the line $y = mx + \frac{25\sqrt{3}}{3}$ is a normal to the conic $\frac{x^2}{16} - \frac{y^2}{9} = 1$, is

(a) $-\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) $-\frac{\sqrt{3}}{2}$ (d) None of these

Chord of Contact and Diameter of a Hyperbola

29. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $4x^2 - 9y^2 - 36 = 0$ are at right angles, then $\frac{x_1 x_2}{y_1 y_2}$ is equal to

(a) $\frac{9}{4}$ (b) $-\frac{9}{4}$ (c) $\frac{81}{16}$ (d) $-\frac{81}{16}$

30. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is
(a) $9x^2 - 8y^2 + 18x - 9 = 0$ (b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$ (d) $9x^2 - 8y^2 + 18x + 9 = 0$

31. If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then, the locus of the middle points of these chords is
(a) $y^2 = (x - a)x^3$ (b) $y^2(x - a) = x^3$
(c) $x^2(x - a) = x^3$ (d) None of these

32. The locus of middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is
(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 3y = 3$ (d) $3x - 4y = 2$

33. Equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$, which is bisected at the point (6, 2), is
(a) $16x - 75y = 418$ (b) $75x - 16y = 418$
(c) $25x - 4y = 400$ (d) None of these

Rectangular Hyperbola

34. If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ is equal to
(a) 5 (b) –5
(c) 4 (d) –4

35. The equation of a line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of the asymptotes is $3x - 4y - 6 = 0$, the equation of other asymptote is
(a) $4x - 3y + 17 = 0$ (b) $-4x - 3y + 17 = 0$
(c) $-4x + 3y + 1 = 0$ (d) $4x + 3y + 17 = 0$

36. The asymptotes of the hyperbola $xy = hx + ky$ are
(a) $x = k, y = h$ (b) $x = h, y = k$
(c) $x = h, y = h$ (d) $x = k, y = k$

37. Eccentricity of hyperbola whose asymptotes are $3x - 4y = 7$ and $4x + 3y = 8$, is
(a) $\sqrt{2}$
(b) 2
(c) not sufficient information
(d) None of the above

38. The product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes, is
(a) 1/2 (b) 2/3 (c) 3/2 (d) 2

39. The equation of the asymptotes of the hyperbola $3x^2 + 4y^2 + 8xy - 8x - 4y - 6 = 0$ is
(a) $3x^2 + 4y^2 + 8xy - 8x - 4y - 3 = 0$
(b) $3x^2 + 4y^2 + 8xy - 8x - 4y + 3 = 0$
(c) $3x^2 + 4y^2 + 8xy - 8x - 4y + 6 = 0$
(d) $4x^2 + 3y^2 + 2xy - x + y + 3 = 0$

40. If asymptotes of a hyperbola are at 90°, then
(a) eccentricity is $\sqrt{2}$
(b) eccentricity is 2
(c) eccentricity depends on equation of asymptotes
(d) None of the above

41. A hyperbola has the asymptotes $x + 2y = 3$ and $x - y = 0$ and passes through (2, 1). Its centre is
(a) (1, 2) (b) (2, 2)
(c) (1, 1) (d) (2, 1)

42. The angle between the asymptotes of the hyperbola $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$ is
(a) $\tan^{-1}(\pm 2)$ (b) $\tan^{-1}(\pm\sqrt{3})$
(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

43. The angle between the two asymptotes of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is
(a) $\pi - 2\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\pi - 2\tan^{-1}\left(\frac{3}{2}\right)$
(c) $2\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\pi - 2\tan^{-1}\left(\frac{4}{3}\right)$

44. If tangent and normal to a rectangular hyperbola $xy = c^2$ cut off intercepts a_1 and a_2 on one axis and b_1, b_2 on the other, then
(a) $a_1 = b_1$ (b) $a_2 = b_2$
(c) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ (d) $a_1a_2 + b_1b_2 = 0$

45. The length of the transverse axis of the rectangular hyperbola $xy = 18$ is
(a) 6 (b) 12
(c) 18 (d) 9

Miscellaneous

46. Tangents at any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the axes at A and B respectively. If the rectangle $OAPB$, where O is the origin is completed, then locus of point P is given by
(a) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ (b) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$
(c) $\frac{a^2}{y^2} - \frac{b^2}{x^2} = 1$ (d) None of these

47. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is
(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = y + 8$ (d) $y = x + 2$

Round II (Mixed Bag)

Only One Correct Option

1. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then, the eccentricity e of the hyperbola satisfies

(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$

(c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$

2. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. If locus of the mid-point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\left(\frac{e+1}{e-1}\right)$ (b) $\frac{e}{\sqrt{(e^2-1)}}$

(c) e (d) None of these

3. Consider the set of hyperbola $xy = k, k \in R$. Let e_1 be the eccentricity when $k = 4$ and e_2 be the eccentricity when $k = 9$, then $e_1 - e_2$ is equal to

(a) -1 (b) 0 (c) 2 (d) -3

4. With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

(a) less than 2 (b) 2

(c) 1/3 (d) None of these

5. Locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a/an

(a) ellipse (b) circle (c) hyperbola (d) parabola

6. The coordinates of a point on the hyperbola, $\frac{x^2}{24} - \frac{y^2}{18} = 1$, which is nearest to the line $3x + 2y + 1 = 0$ are

(a) (6, 3) (b) (–6, –3) (c) (6, –3) (d) (–6, 3)

7. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT \times ON$ is equal to

(a) e^2 (b) a^2

(c) b^2 (d) $\frac{b^2}{a^2}$

8. If tangents PQ and PR are drawn from variable point P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ (a > b)$ so that the fourth vertex S of parallelogram $PQSR$ lies on circumcircle of ΔPQR, then locus of P is

(a) $x^2 + y^2 = b^2$ (b) $x^2 + y^2 = a^2$

(c) $x^2 + y^2 = (a^2 - b^2)$ (d) None of these

9. Number of points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, is/are

(a) 0 (b) 2 (c) 3 (d) 4

More Than One Correct Option

10. The locus of the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

(a) director circle (b) $x^2 + y^2 = a^2$

(c) $x^2 + y^2 = a^2 - b^2$ (d) $x^2 + y^2 = a^2 + b^2$

11. A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$. The equation of the line is

(a) $9x + 3y - 8 = 0$ (b) $9x - 3y + 8 = 0$

(c) $9x + 3y + 8 = 0$ (d) $9x - 3y - 8 = 0$

12. If foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and eccentricity of the hyperbola is 2, then

(a) $a^2 + b^2 = 16$

(b) there is no director circle of the hyperbola

(c) centre of the director circle is (0, 0)

(d) length of latusrectum of the hyperbola $= 12$

Comprehension Based Questions

Passage

Directions (Q. Nos. 13 and 14) *Let the curves be*

$$x^2 - y^2 = 9, \quad P: y^2 = 4(x-5), \quad L: x = 9$$

On the basis of above information, answer the following questions.

13. If L is the chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$ (b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$ (d) $9x^2 - 8y^2 + 18x + 9 = 0$

14. If R is the point of intersection of the tangents of H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$
(c) $y = 7$ (d) $y = 9$

Assertion and Reason

Directions (Q. Nos. 15-20) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

15. **Statement I** $\frac{5}{3}$ and $\frac{5}{4}$ are the eccentricities of two conjugate hyperbolas.

Statement II If e and e_1 are the eccentricities of two conjugate hyperbolas, then $ee_1 > 1$.

16. **Statement I** A hyperbola whose asymptotes include a right angle is said to be equilateral hyperbola.

Statement II Eccentricity of an equilateral hyperbola is $\sqrt{2}$..

17. **Statement I** The line $bx - ay = 0$ will not meet the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,(a > b > 0)$

Statement II The line $y = mx + c$ does not meet the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 - b^2$

18. **Statement I** The equation of the director circle to the hyperbola $4x^2 - 3y^2 = 12$ is $x^2 + y^2 = 1$.

Statement II Director circle is the locus of the point of intersection of perpendicular tangents to a hyperbola.

19. **Statement I** Let $(2, \sqrt{2})$ be any point on hyperbola $x^2 - y^2 = 2$, then the product of distances of foci from P is equal to 6.

Statement II If S and S' be the foci, C the centre and P be any point on a hyperbola $x^2 - y^2 = a^2$, then $SP \cdot S'P = CP^2$

20. **Statement I** Asymptotes of hyperbola $3x + 4y = 2$ and $4x - 3y = 5$ are bisectors of transverse and conjugate axes of hyperbola.

Statement II Transverse and conjugate axes of hyperbola are bisectors of the asymptotes.

Previous Years' Questions

21. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by **[AIEEE 2011]**

(a) $-3x^2 + y^2 = 3$ (b) $x^2 - 3y^2 = 3$
(c) $3x^2 - y^2 = 3$ (d) $-x^2 + 3y^2 = 3$

22. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$. Which of the following remains constant when α varies? **[AIEEE 2007]**

(a) Directrix (b) Abscissae of vertices
(c) Abscissae of foci (d) Eccentricities

23. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is **[AIEEE 2005]**

(a) circle (b) ellipse
(c) parabola (d) hyperbola

Answers

Check Point 1

1. (c) **2.** (b) **3.** (b) **4.** (d) **5.** (a) **6.** (b) **7.** (c) **8.** (b) **9.** (b) **10.** (d)

Check Point 2

1. (a) **2.** (c) **3.** (a) **4.** (d) **5.** (c) **6.** (b) **7.** (c)

Round I

1. (c)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (b)
11. (b)	**12.** (d)	**13.** (a)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (c)	**19.** (d)	**20.** (b)
21. (a)	**22.** (c)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (d)	**30.** (b)
31. (b)	**32.** (a)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (a)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (a)
41. (c)	**42.** (a)	**43.** (c)	**44.** (d)	**45.** (b)	**46.** (a)	**47.** (d)			

Round II

1. (d)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (a,c)
11. (a,b,c,d)	**12.** (a,b,d)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (d)							

the Guidance

Round I

1. Given equation of hyperbola is

$$(10x-5)^2+(10y-4)^2=\lambda^2(3x+4y-1)^2$$

can be rewritten as

$$\frac{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{2}{5}\right)^2}}{\left|\frac{3x+4y-1}{5}\right|}=\left|\frac{\lambda}{2}\right|$$

This is of the form of $\frac{PS}{PM}=e$

where, P is any point on the hyperbola and S is a focus and M is the point of directrix.

Here, $\left|\frac{\lambda}{2}\right|>1 \Rightarrow |\lambda|>2$ $(\because e>1)$

$\Rightarrow$ $\lambda<-2$ or $\lambda>2$

2. Let equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Given, $\frac{2b^2}{a}=8$

$\Rightarrow$ $\frac{b^2}{a}=4$

and $2b=\frac{1}{2}(2ae)$

$\Rightarrow$ $2b=ae$

$\Rightarrow$ $4b^2=a^2e^2$

$\Rightarrow$ $4\left(\frac{b^2}{a^2}\right)=e^2$

$\Rightarrow$ $4(e^2-1)=e^2$ $[\because b^2=a^2(e^2-1)]$

$\Rightarrow$ $3e^2=4$

$\Rightarrow$ $e^2=\frac{4}{3}$

$\Rightarrow$ $e=\frac{2}{\sqrt{3}}$

3. Given equation is $9y^2-4x^2=36$, divide it by 36, we get

$$\frac{9y^2}{36}-\frac{4x^2}{36}=\frac{36}{36}$$

$\Rightarrow$ $\frac{y^2}{4}-\frac{x^2}{9}=1$

Now, comparing $\frac{y^2}{4}-\frac{x^2}{9}=1$ with $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, we get

$$b^2=4 \text{ and } a^2=9$$

$\Rightarrow$ $b=2$ and $a=3$

Here, in hyperbolic equation coefficient of y^2 is positive, so transverse axis is along Y-axis.

$$\text{Vertices}=(0,\pm b)=(0,\pm 2)$$

and latusrectum $=\frac{2a^2}{b}=\frac{2\times 9}{2}=9$

4. Now, taking option (c).

Let $x = a\dfrac{e^t + e^{-t}}{2} \Rightarrow \dfrac{2x}{a} = e^t + e^{-t}$ ...(i)

and $\dfrac{2y}{a} = e^t - e^{-t}$ ...(ii)

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

5. Put the value of $(x, y) \equiv (\tan\theta + \sin\theta, \tan\theta - \sin\theta)$ in the given option, we get the required result.

On putting the value of x and y in option (c), we get

$$[(\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2]^2$$
$$= 16(\tan\theta + \sin\theta) \times (\tan\theta - \sin\theta)$$
$$\Rightarrow \quad [\tan^2\theta + \sin^2\theta - \tan^2\theta - \sin^2\theta + 4\tan\theta\sin\theta]^2$$
$$= 16(\tan^2\theta - \sin^2\theta)$$
$$\Rightarrow \quad (4\tan\theta \cdot \sin\theta)^2 = 16(\tan^2\theta - \sin^2\theta)$$
$$\Rightarrow \quad 16\tan^2\theta\sin^2\theta = 16\tan^2\theta(1 - \cos^2\theta)$$
$$\Rightarrow \quad 16\tan^2\theta\sin^2\theta = 16\tan^2\theta\sin^2\theta$$

Hence, the option (c) satisfies.

6. Given equation can be rewritten as $\dfrac{y^2}{k^2} - \dfrac{x^2}{(-k)} = 1 (-k > 0)$

$$e^2 = 1 + \frac{(-k)}{k^2} = 1 - \frac{k}{k^2}$$
$$\Rightarrow \quad e = \sqrt{1 - \frac{1}{k}}$$

7. The given equation may be written as

$$\frac{x^2}{\frac{32}{3}} - \frac{y^2}{8} = 1$$
$$\Rightarrow \quad \frac{x^2}{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2} - \frac{y^2}{(2\sqrt{2})^2} = 1$$

On comparing the given equation with the standard equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get

$$a^2 = \left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2 \quad \text{and} \quad b^2 = (2\sqrt{2})^2$$

$\therefore$ Length of transverse axis of a hyperbola

$$= 2a = 2 \times \frac{4\sqrt{2}}{\sqrt{3}} = \frac{8\sqrt{2}}{\sqrt{3}}$$

8. Let equation of hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Length of transverse axis is $2a$ and

Length of latusrectum is $\dfrac{2b^2}{a}$.

Now, difference $E = \left|2a - \dfrac{2b^2}{a}\right| = \dfrac{2}{a}\left|2a^2 - a^2e^2\right|$

$\therefore$ Difference $= 2a\,|2 - e^2|$

9. Given equation of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and equation of conjugate hyperbola is $\dfrac{x^2}{b^2} - \dfrac{y^2}{a^2} = 1$

Since, e and e' are the eccentricities of the respective hyperbola, then

$$e^2 = 1 + \frac{b^2}{a^2}, (e')^2 = 1 + \frac{a^2}{b^2}$$
$$\therefore \quad \frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}$$

10. Any point on hyperbola $\dfrac{(x+1)^2}{16} - \dfrac{(y-2)^2}{4} = 1$ is of the form $(4\sec\theta - 1, 2\tan\theta + 2)$

11. Let equation of tangent to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

i.e., $mx - y + \sqrt{a^2m^2 - b^2} = 0$

$\therefore$ Required product

$$= \left|\frac{mae + \sqrt{a^2m^2 - b^2}}{\sqrt{m^2 + 1}}\right| \left|\frac{-mae + \sqrt{a^2m^2 - b^2}}{\sqrt{m^2 + 1}}\right|$$
$$= \left|\frac{a^2m^2 - b^2 - m^2a^2e^2}{m^2 + 1}\right|$$
$$= \left|\frac{m^2a^2(1 - e^2) - b^2}{m^2 + 1}\right| = \left|\frac{-m^2b^2 - b^2}{m^2 + 1}\right| \quad [\because b^2 = a^2(e^2 - 1)]$$
$$= b^2$$

12. Given, S (6,4) and $S'(-4,4)$ and eccentricity, $e = 2$

$$\therefore \quad SS' = \sqrt{(6+4)^2 + (4-4)^2} = 10$$

But $SS' = 2ae$

$$\therefore \quad 2a \times 2 = 10 \Rightarrow a = \frac{5}{2}$$

and we know that,

$$b^2 = a^2(e^2 - 1)$$
$$\Rightarrow \quad b^2 = \frac{25}{4}(4 - 1) = \frac{75}{4}$$

Centre of hyperbola is $\left[\dfrac{6 + (-4)}{2}, \dfrac{4 + 4}{2}\right] = (1,4)$

$\therefore$ Equation of hyperbola is $\dfrac{(x-1)^2}{\frac{25}{4}} - \dfrac{(y-4)^2}{\frac{75}{4}} = 1$

$$\Rightarrow \quad \frac{4(x-1)^2}{25} - \frac{4(y-4)^2}{75} = 1$$

13. Let equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(i)

According to the given condition,

$$2ae=16$$

and $e=\sqrt{2}$

$\Rightarrow$ $2a(\sqrt{2})=16$

$\Rightarrow$ $a=\frac{8}{\sqrt{2}}$

$\Rightarrow$ $a^2=32$

$\because$ $b^2=a^2(e^2-1)$

$=32(2-1)=32$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{32}-\frac{y^2}{32}=1$$

$\Rightarrow$ $x^2-y^2=32$

14. Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(i)$$

Given, $e=\frac{3}{2}$ and foci $=(\pm ae,0)=(\pm 2,0)$

$\therefore$ $e=\frac{3}{2}$ and $ae=2$

$\Rightarrow$ $a\times\frac{3}{2}=2\Rightarrow a^2=\frac{16}{9}$

$\because$ $b^2=a^2(e^2-1)$

$\therefore$ $b^2=\frac{16}{9}\left(\frac{9}{4}-1\right)$

$=\frac{16}{9}\times\frac{5}{4}=\frac{20}{9}$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{16/9}-\frac{y^2}{20/9}=1$$

$\Rightarrow$ $\frac{x^2}{4}-\frac{y^2}{5}=\frac{4}{9}$

15. Since, vertices $(\pm 2,0)$ and foci $(\pm 3,0)$ lies on X-axis, as coefficient of Y-axis is zero.

Hence, equation of hyperbola will be of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(i)$$

where, it is given that

$a=2$ and $ae=3$

$\Rightarrow$ $e=\frac{3}{2}$

$$b=a\sqrt{e^2-1}=2\sqrt{\frac{9}{4}-1}=\sqrt{5}$$

Put the values of $a^2=4$ and $b^2=5$ in Eq. (i), we get

$$\frac{x^2}{4}-\frac{y^2}{5}=1$$

16. Since, foci $(0,\pm\sqrt{10})$ lies on Y-axis, as x-coordinate is zero. Hence, equation of hyperbola will be of the form

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \quad ...(i)$$

where, it is given that foci $(0,\pm\sqrt{10})=(0,\pm be)\Rightarrow be=\sqrt{10}$

Also, hyperbola passes through the point (2, 3), *i.e.*, point will satisfy Eq. (i).

$\therefore$ $\frac{(3)^2}{a^2}-\frac{(2)^2}{b^2}=1$

$\Rightarrow$ $\frac{9}{a^2}-\frac{4}{b^2}=1$ ...(ii)

$\because$ $a^2=b^2e^2-b^2$

$\therefore$ $a^2+b^2=10$

$\Rightarrow$ $b^2=10-a^2$ ...(iii)

From Eq. (ii),

$$\frac{9}{a^2}-\frac{4}{10-a^2}=1\Rightarrow\frac{90-9a^2-4a^2}{a^2(10-a^2)}=1$$

$\Rightarrow$ $90-13a^2=10a^2-a^4$

$\Rightarrow$ $a^4-23a^2+90=0$

$\Rightarrow$ $a^4-18a^2-5a^2+90=0$

$\Rightarrow$ $a^2(a^2-18)-5(a^2-18)=0$

$\Rightarrow$ $(a^2-5)(a^2-18)=0$

$\Rightarrow$ $a^2=5$ or $a^2=18$

From Eq. (iii), if $a^2=5$

$\Rightarrow$ $b^2=10-5=5$

If $a^2=18$

$\Rightarrow$ $b^2=10-18=-8$ which is not possible.

So, $a^2=b^2=5$

Put $a^2=b^2=5$ in Eq. (i), we get

$$\frac{y^2}{5}-\frac{x^2}{5}=1\Rightarrow y^2-x^2=5$$

17. Given equation is $\frac{1}{r}=\frac{1}{8}+\frac{3}{8}\cos\theta$, it can be rewritten as $\frac{8}{r}=1+3\cos\theta$, which is the form of $\frac{1}{r}=1+e\cos\theta$

On comparing, we get

$$e=3>1$$

$\therefore$ Given equation represents a hyperbola

18. On homogenising $y^2-x^2=4$ with the help of the line $\sqrt{3}x+y=2$, we get

$$y^2-x^2=4\frac{(\sqrt{3}x+y)^2}{4}$$

$\Rightarrow$ $y^2-x^2=3x^2+y^2+2\sqrt{3}xy$

$\Rightarrow \quad 4x^2 + 2\sqrt{3}\,xy = 0$

$\therefore \quad \tan\theta = 2\dfrac{\sqrt{h^2 - ab}}{a+b}$

$\Rightarrow \quad \tan\theta = \dfrac{2\sqrt{3-0}}{4}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

19. Given equation of hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$.

Here, $a^2 = 9,\ b^2 = 4$

and equation of line is $y = -x + \sqrt{2}\,p$ ...(i)

If the line $y = mx + c$ touches the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$ ...(ii)

From Eq. (i), we get

$m = -1, c = \sqrt{2}p$

On putting these values in Eq. (ii), we get

$(\sqrt{2}p)^2 = 9(1) - 4 \Rightarrow 2p^2 = 5$

20. Given equation is $xy = a$

On differentiating, we get

$x\dfrac{dy}{dx} + y = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{y}{x}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(a,1)} = -\dfrac{1}{a}$

21. We know that the locus of the point of intersection of perpendicular tangents to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a circle $x^2 + y^2 = a^2 - b^2$

Thus, locus of the point of intersection of perpendicular tangents to the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 1$ is a circle

$x^2 + y^2 = 3 - 1$

$\Rightarrow \quad x^2 + y^2 = 2$

22. Director circle is set of points form where drawn tangents are perpendicular, in this case $x^2 + y^2 = a^2 - b^2$ (equation of director circle) *i.e.*, $x^2 + y^2 = -9$ is not a real circle, so there is no point from where tangents are perpendicular.

23. Let the slopes of the two tangents to the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be cm and $\dfrac{c}{m}$

The equations of tangents are

$y = cmx + \sqrt{a^2c^2m^2 - b^2}$ ...(i)

and $my - cx = \sqrt{a^2c^2 - b^2m^2}$ ...(ii)

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$(y - cmx)^2 - (my - cx)^2 = a^2c^2m^2 - b^2 - a^2c^2 + b^2m^2$

$\Rightarrow \quad (1 - m^2)(y^2 - c^2x^2) = -(1 - m^2)(a^2c^2 + b^2)$

$\Rightarrow \quad y^2 + b^2 = c^2(x^2 - a^2)$

24. Equation of normal to the hyperbola at the point $(5\sec\theta, 4\tan\theta)$ is

$5x\cos\theta + 4y\cot\theta = 25 + 16$ ...(i)

This line is perpendicular to the line $2x + y = 1$

$\therefore \quad m_1m_2 = -1$

$\Rightarrow \quad \left(\dfrac{-5\cos\theta}{4\cot\theta}\right)(-2) = -1$

$\Rightarrow \quad \sin\theta = -\dfrac{2}{5}$

$\therefore \quad \cos\theta = \sqrt{1 - \dfrac{4}{25}} = \mp\dfrac{\sqrt{21}}{5}$

and $\cot\theta = \mp\dfrac{\sqrt{21}}{2}$

From Eq. (i),

$5x\dfrac{\sqrt{21}}{5} - \dfrac{4y\sqrt{21}}{2} = 41$

$\Rightarrow \quad \sqrt{21}(x - 2y) = 41$

25. The maximum four normals can be drawn from an external point.

26. The equation of normal at point $\left(at, \dfrac{a}{t}\right)$ is $t^2x - y = a\left(t^3 - \dfrac{1}{t}\right)$

$\Rightarrow \quad xt^3 - yt - at^4 + a = 0$

27. Equation of the tangents at $P(a\sec\theta, b\tan\theta)$ is

$\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$

$\therefore$ Equation of the normal at P is

$ax + b\,\mathrm{cosec}\,\theta\, y = (a^2 + b^2)\sec\theta$ ...(i)

Similarly, the equation of normal at $Q(a\sec\phi, b\tan\phi)$ is

$ax + b\,\mathrm{cosec}\,\phi\, y = (a^2 + b^2)\sec\phi$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$y = \dfrac{a^2 + b^2}{b}\cdot\dfrac{\sec\theta - \sec\phi}{\mathrm{cosec}\,\theta - \mathrm{cosec}\,\phi}$

So, $k = y = \dfrac{a^2 + b^2}{b}\cdot\dfrac{\sec\theta - \sec\left(\dfrac{\pi}{2} - \theta\right)}{\mathrm{cosec}\,\theta - \mathrm{cosec}\left(\dfrac{\pi}{2} - \theta\right)}$

$= \dfrac{a^2 + b^2}{b}\cdot\dfrac{\sec\theta - \mathrm{cosec}\,\theta}{\mathrm{cosec}\,\theta - \sec\theta}$

$= -\left[\dfrac{a^2 + b^2}{b}\right]$

28. Given that, $y = mx + \frac{25\sqrt{3}}{3}$ ...(i)

and $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ...(ii)

Here, Eq. (i) is normal to Eq. (ii), then

$$\frac{(a^2+b^2)^2}{c^2} = \frac{a^2}{m^2} - \frac{b^2}{1}$$

$$\Rightarrow \quad \frac{(16+9)^2 \times 9}{625 \times 3} = \frac{16}{m^2} - \frac{9}{1}$$

$$\Rightarrow \quad \frac{16}{m^2} = 12 \Rightarrow m = \pm \frac{2}{\sqrt{3}}$$

29. The equation of hyperbola is

$$4x^2 - 9y^2 = 36$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{...(i)}$$

The equation of the chords of contact of tangents from (x_1, y_1) and (x_2, y_2) to the given hyperbola are

$$\frac{xx_1}{9} - \frac{yy_1}{4} = 1 \quad \text{...(ii)}$$

and $\frac{xx_2}{9} - \frac{yy_2}{4} = 1$ ...(iii)

Lines (ii) and (iii) are at right angles.

$$\frac{9}{4} \cdot \frac{x_1}{y_1} \times \frac{4}{9} \cdot \frac{x_2}{y_2} = -1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2} = -\left(\frac{9}{4}\right)^2 = -\frac{81}{16}$$

30. Let (h, k) be the point whose chord of contact w.r.t. hyperbola $x^2 - y^2 = 9$ is $x = 9$. We know that chord of (h, k) w.r.t. hyperbola $x^2 - y^2 = 9$ is $T = 0$

$$\Rightarrow \quad hx - ky - 9 = 0$$

But it is the equation of line $x = 9$. This is possible only when $h = 1, k = 0$.

Again, equation of pair of tangents is

$$T^2 = SS_1$$

$$\Rightarrow \quad (x-9)^2 = (x^2 - y^2 - 9)(1-9)$$

$$\Rightarrow \quad x^2 - 18x + 81 = (x^2 - y^2 - 9)(-8)$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

31. Equation of chord of hyperbola $x^2 - y^2 = a^2$ with mid-point as (h, k) is given by

$$xh - yk = h^2 - k^2$$

$$\Rightarrow \quad y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

This will touch the parabola $y^2 = 4ax$, if

$$-\left(\frac{h^2 - k^2}{k}\right) = \frac{a}{h/k}$$

$$\Rightarrow \quad ak^2 = -h^3 + k^2h$$

$\therefore$ Locus of the mid-point is $x^3 = y^2(x - a)$

32. Let (h, k) is mid-point of chord.

Then, its equation is

$$3hx - 2ky + 2(x+h) - 3(y+k) = 3h^2 - 2k^2 + 4h - 6k$$

$$\Rightarrow \quad x(3h+2) + y(-2k-3) = 3h^2 - 2k^2 + 2h - 3k$$

Since, this line is parallel to $y = 2x$.

$$\therefore \quad \frac{3h+2}{2k+3} = 2$$

$$\Rightarrow \quad 3h + 2 = 4k + 6$$

$$\Rightarrow \quad 3h - 4k = 4$$

Thus, locus of mid-point is $3x - 4y = 4$

33. Given equation of hyperbola is $25x^2 - 16y^2 = 400$. If $(6, 2)$ is the mid-point of the chord, then equation of chord is $T = S_1$.

$$\Rightarrow \quad 25(6x) - 16(2y) = (25)(36) - 16(4)$$

$$\Rightarrow \quad 75x - 16y = 418$$

34. Since, for a rectangular hyperbola, we have

(Coefficient of x^2) + (Coefficient of y^2) = 0

$$\lambda = -5$$

35. We know that asymptotes of rectangular hyperbola are mutually perpendicular, thus other asymptote should be $4x + 3y + \lambda = 0$

Also, intersection point of asymptotes is also the centre of the hyperbola.

Hence, intersection point of $4x + 3y + \lambda = 0$ and $3x - 4y - 6 = 0$ is $\left(\frac{18 - 4\lambda}{25}, \frac{-12\lambda - 96}{100}\right)$ and it should lie on the line $x - y - 1 = 0$

$$\therefore \quad \frac{18 - 4\lambda}{25} - \frac{-12\lambda - 96}{100} - 1 = 0$$

$$\Rightarrow \quad \lambda = 17$$

Hence, equation of other asymptote is $4x + 3y + 17 = 0$

36. Given that, $xy = hx + ky \Rightarrow (x-k)(y-h) = hk$

On shifting origin to (k, h) the above equation reduces to

$$XY = hk = c^2 \quad \text{(say)}$$

where, $x = X + k$ and $y = Y + h$

Then, the equation of the asymptotes are $X = 0$ and $Y = 0$

i.e.,

$$x = k, \quad y = h$$

37. Since, asymptotes $3x - 4y = 7$ and $4x + 3y = 8$ are perpendicular, therefore it is a rectangular hyperbola, so eccentricity is $\sqrt{2}$.

38. Given equation is $x^2 - 2y^2 - 2 = 0$, it can be rewritten as

$$\frac{x^2}{2} - \frac{y^2}{1} = 1$$

Here, $a^2 = 2, \; b^2 = 1$

We know that equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the product of length of perpendicular drawn from any point on the hyperbola to the asymptotes is

$$\frac{a^2b^2}{a^2+b^2}=\frac{2(1)}{2+1}=\frac{2}{3}$$

39. The equation of the asymptotes of the hyperbola

$$3x^2+4y^2+8xy-8x-4y-6=0$$

is $\quad 3x^2+4y^2+8xy-8x-4y+\lambda=0$

It should represent a pair of straight lines.

$\therefore \quad abc+2fgh-af^2-bg^2-ch^2=0$

$3\cdot4\cdot\lambda+2\cdot(-2)(-4)\ 4-3(-2)^2-4(-4)^2-\lambda(4)^2=0$

$\Rightarrow \quad 12\lambda+56-12-56-16\lambda=0$

$\Rightarrow \quad -4\lambda-12=0$

$\Rightarrow \quad \lambda=-3$

$\therefore$ Required equation is

$$3x^2+4y^2+8xy-8x-4y-3=0$$

40. Since, asymptotes are at 90°, it means that it is a rectangular hyperbola.

$\therefore$ Eccentricity is $\sqrt{2}$.

41. Equation of asymptotes are

$$x+2y=3 \quad \text{...(i)}$$

and $\quad x-y=0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x=1, y=1$$

$\therefore$ Centre of hyperbola is (1, 1) because asymptotes passes through the centre of the hyperbola.

42. Equation of asymptotes of the hyperbola are

$$x^2+2xy-3y^2=0$$

The angle between asymptotes is

$$\theta=\tan^{-1}\left[\frac{1-1(-3)}{1-3}\right]$$

$$=\tan^{-1}\left(\frac{1+3}{-2}\right)=\tan^{-1}(\pm2)$$

43. We know that angle between two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.

Equation of given hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Here, $\quad a=4$

and $\quad b=3$

$\therefore \quad$ Required angle $=2\tan^{-1}\left(\frac{3}{4}\right)$

44. $\because$ Tangent and normal are at 90°.

$\therefore$ Product of slopes is -1.

$\Rightarrow \quad \left(-\frac{b_1}{a_1}\right)\left(-\frac{b_2}{a_2}\right)=-1 \Rightarrow a_1a_2+b_1b_2=0$

45. The given equation of rectangular hyperbola is

$$xy=18 \quad \text{...(i)}$$

On comparing Eq. (i), with general equation of rectangular hyperbola $xy=\frac{a^2}{2}$, we get

$$\frac{a^2}{2}=18 \Rightarrow a^2=36 \Rightarrow a=6$$

$\therefore$ Length of the transverse axis of rectangular hyperbola is $2a=2\times6=12$

46. The equation of tangent is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

$\therefore$ Coordinates of A and B are $(a\cos\theta, 0)$ and $(0, -b\cot\theta)$ respectively.

Let coordinates of P are (h, k).

$\therefore \quad h=a\cos\theta, \quad k=-b\cot\theta$

$\Rightarrow \quad \frac{k}{h}=-\frac{b}{a\sin\theta} \Rightarrow \sin\theta=-\frac{bh}{ak}$

$\Rightarrow \quad \frac{b^2h^2}{a^2k^2}=\sin^2\theta \Rightarrow \frac{b^2h^2}{a^2k^2}+\frac{h^2}{a^2}=1$

$\Rightarrow \quad \frac{b^2}{k^2}+1=\frac{a^2}{h^2} \Rightarrow \frac{a^2}{h^2}-\frac{b^2}{k^2}=1$

Hence, the locus of P is $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$

47. Any point on parabola $y^2=8x$ is $(2t^2, 4t)$. The equation of tangent at that point is

$$yt=x+2t^2 \quad \text{...(i)}$$

Given that, $\quad xy=-1 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$y(yt-2t^2)=-1 \Rightarrow ty^2-2t^2y+1=0$$

$\because$ It is common tangent. It means they are intersect only at one point and the value of discriminant is equal to zero.

i.e., $\quad 4t^4-4t=0 \Rightarrow t=0, 1$

$\therefore$ The common tangent is $y=x+2$, (when $t=0$, it is $x=0$ which can touch $xy=-1$ at infinity only)

Round II

1. $\because$ PQ is the double ordinate. Let $MP = MQ = l$.

Given that ΔOPQ is an equilateral, then $OP = OQ = PQ$

$\Rightarrow \quad (OP)^2 = (OQ)^2 = (PQ)^2$

$\Rightarrow \quad \frac{a^2}{b^2}(b^2 + l^2) + l^2 = \frac{a^2}{b^2}(b^2 + l^2) + l^2 = 4l^2$

$\Rightarrow \quad \frac{a^2}{b^2}(b^2 + l^2) = 3l^2$

$\Rightarrow \quad a^2 = l^2\left(3 - \frac{a^2}{b^2}\right)$

$\Rightarrow \quad l^2 = \frac{a^2b^2}{(3b^2 - a^2)} > 0$

$\therefore \quad 3b^2 - a^2 > 0 \Rightarrow 3b^2 > a^2$

$\Rightarrow \quad 3a^2(e^2 - 1) > a^2$

$\Rightarrow \quad e^2 > 4/3 \;\therefore\; e > \frac{2}{\sqrt{3}}$

2. Equation of normal at $P(a\sec\phi, b\tan\phi)$ is

$ax\cos\phi + by\cot\phi = a^2 + b^2$.

Then, coordinates of L and M are

$$\left(\frac{a^2+b^2}{a}\cdot\sec\phi, 0\right) \text{ and } \left(0, \frac{a^2+b^2}{b}\tan\phi\right)$$

respectively.

Let mid-point of ML is $Q(h, k)$,

then $\quad h = \frac{(a^2+b^2)}{2a}\sec\phi$

$\therefore \quad \sec\phi = \frac{2ah}{(a^2+b^2)} \quad$ ...(i)

and $\quad k = \frac{(a^2+b^2)}{2b}\tan\phi$

$\therefore \quad \tan\phi = \frac{2bk}{(a^2+b^2)} \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sec^2\phi - \tan^2\phi = \frac{4a^2h^2}{(a^2+b^2)^2} - \frac{4b^2k^2}{(a^2+b^2)^2}$$

Hence, required locus is

$$\frac{x^2}{\left(\frac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\frac{a^2+b^2}{2b}\right)^2} = 1$$

Let eccentricity of this curve is e_1.

$\Rightarrow \quad \left(\frac{a^2+b^2}{2b}\right)^2 = \left(\frac{a^2+b^2}{2a}\right)^2(e_1^2 - 1)$

$\Rightarrow \quad a^2 = b^2(e_1^2 - 1)$

$\Rightarrow \quad a^2 = a^2(e^2 - 1)(e_1^2 - 1) \quad [\because b^2 = a^2(e^2 - 1)]$

$\Rightarrow \quad e^2e_1^2 - e^2 - e_1^2 + 1 = 1$

$\Rightarrow \quad e_1^2(e^2 - 1) = e^2$

$\Rightarrow \quad e_1 = \frac{e}{\sqrt{(e^2 - 1)}}$

3. $\because \quad xy = 4 \quad$ ...(i)

and $\quad xy = 9 \quad$ ...(ii)

Eqs. (i) and (ii) are the equations of rectangular hyperbolas.

$\therefore \quad e_1 = \sqrt{2}$ and $e_2 = \sqrt{2}$, then $e_1 - e_2 = 0$

4. Given hyperbola is

$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad$ ...(i)

$\therefore \quad e^2 = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow e = \frac{5}{3}$

Hence, its foci are $(\pm 5, 0)$.

The equation of the circle with (5, 0) as centre is

$(x - 5)^2 + y^2 = r^2 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we have

$16x^2 - 9[r^2 - (x - 5)^2] = 144$

$\Rightarrow \quad 25x^2 - 90x - 9r^2 + 81 = 0$

Since, the circle touches the hyperbola, above equation must have equal roots. Hence,

$90^2 - 4(25)(81 - 9r^2) = 0$

$\Rightarrow \quad 9 - (9 - r^2) = 0$

$\Rightarrow r = 0$, which is not possible.

Hence, the circle cannot touch at two points.

It can only be tangent at the vertex. Hence,

$r = 5 - 3 = 2$

5. Let the point be (h, k).

Then, equation of the chord of contact is $hx + ky = 4$

Since, $hx + ky = 4$ is tangent to $xy = 1$,

$\therefore \quad x\left(\dfrac{4-hx}{k}\right)=1$ has two equal roots,

$\Rightarrow \quad hx^2-4x+k=0 \Rightarrow hk=4$

$\therefore$ Locus of (h, k) is $xy = 4$

6.

Point P is nearest to the given line, if tangent at P is parallel to the given line.

Now, slope of tangent at $P(x_1, y_1)$ is

$\left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)}=\dfrac{18y_1}{24x_1}=\dfrac{3}{4}\dfrac{y_1}{x_1}$, which must be equal to $-\dfrac{3}{2}$

$\therefore \quad \dfrac{3}{4}\dfrac{y_1}{x_1}=-\dfrac{3}{2}$

$\Rightarrow \quad y_1=-2x_1 \quad$ ...(i)

Also, (x_1, y_1) lies on the curve. Hence,

$$\frac{x_1^2}{24}-\frac{y_1^2}{18}=1 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get two points (6, –3) and (–6, 3) of which (6, –3) is nearest.

7. Tangent at point P is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

It meets the x-axis at point $T(a\cos\theta, 0)$ and foot of perpendicular from P to x-axis is $N(a\sec\theta, 0)$.

From the diagram, we have

$$OT = a\cos\theta$$

and $\quad ON = a\sec\theta$

$\Rightarrow \quad OT \times ON = a^2$

8. Fourth vertex of parallelogram lies on circumcircle

$\Rightarrow$ Parallelogram is cyclic.

$\Rightarrow$ Parallelogram is a rectangle

$\Rightarrow$ Tangents are perpendicular

$\Rightarrow$ Locus of P is the director circle

i.e., $\quad x^2+y^2=a^2-b^2$

9. Director circle of circle $x^2+y^2=a^2$ is $x^2+y^2=2a^2$

The semi-transverse axis is $\sqrt{3}a$.

Radius of the circle is $\sqrt{2}a$.

Hence, director circle and hyperbola do not intersect.

10. Equation of any tangent in terms of slope m is

$$y=mx+\sqrt{(a^2m^2-b^2)}$$

It passes through (h, k), then

$$(k-mh)^2=a^2m^2-b^2$$

$\Rightarrow \quad m^2(h^2-a^2)-2mhk+k^2+b^2=0$

which is quadratic in m.

Let slopes of tangents are m_1 and m_2, then $m_1m_2=-1$

$\Rightarrow \quad \dfrac{k^2+b^2}{h^2-a^2}=-1$

$\Rightarrow \quad h^2+k^2=a^2-b^2$

Hence, required locus is $x^2+y^2=a^2-b^2$

which is director circle of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

11. The equation of tangent in terms of slope of $y^2=32x$ is

$$y=mx+\frac{8}{m} \quad \text{...(i)}$$

which is also tangent of the hyperbola $9x^2-9y^2=8$

i.e., $\quad x^2-y^2=\dfrac{8}{9}$

Then, $\quad \left(\dfrac{8}{m}\right)^2=\dfrac{8}{9}m^2-\dfrac{8}{9}$

$\Rightarrow \quad \dfrac{8}{m^2}=\dfrac{m^2}{9}-\dfrac{1}{9}$

$\Rightarrow \quad 72=m^4-m^2$

$\Rightarrow \quad m^4-m^2-72=0$

$\Rightarrow \quad (m^2-9)(m^2+8)=0$

$\Rightarrow \quad m^2=9, \quad (\because m^2+8\neq 0)$

$\Rightarrow \quad m=\pm 3$

From Eq. (i), we get

$$y=\pm 3x\pm\frac{8}{3}$$

$\Rightarrow \quad 3y=\pm 9x\pm 8$

$\Rightarrow \quad \pm 9x-3y\pm 8=0$

$\Rightarrow \quad 9x-3y+8=0,\ 9x-3y-8=0$

$-9x-3y+8=0,\ -9x-3y-8=0$

or $\quad 9x-3y+8=0,\ 9x-3y-8=0$

$\Rightarrow \quad 9x+3y-8=0$

and $\quad 9x+3y+8=0$

12. For the ellipse,

$$a=5 \text{ and } e=\sqrt{\frac{25-9}{25}}=\frac{4}{5}$$

$\therefore \quad ae=4$

Hence, the foci are (–4, 0) and (4, 0).

For the hyperbola, $ae = 4$, $e = 2$

$\therefore \quad a = 2$

$b^2 = 4(4-1) = 12$

$\Rightarrow \quad b = \sqrt{12}$

Length of latusrectum $= \dfrac{2b^2}{a} = 2 \times \dfrac{12}{2} = 12$

13. Let $R(h, k)$ be the point of intersection of the tangents to the extremities of the chord $L : x = 9$ to the hyperbola, then equation of L is $hx - ky = 9 \Rightarrow h = 1, k = 0$.

$\therefore$ Coordinates of R are (1, 0).

Equation of the pair of tangents from R to the hyperbola is

$$(x^2 - y^2 - 9)(1 - 9) = (x - 9)^2 \qquad (\because S_1 = T^2)$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

14. The equation of chord of contact of (1, 0) to the parabola is

$$y \times 0 = 2(x + 1) - 20 \Rightarrow x = 9$$

15. The equation of two conjugate hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and $$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$$\therefore \quad e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

and $$e_1^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2}$$

$$\therefore \quad \frac{1}{e^2} + \frac{1}{e_1^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

$$\Rightarrow \quad \frac{1}{(5/3)^2} + \frac{1}{(5/4)^2} = \frac{9}{25} + \frac{16}{25} = 1$$

and $$ee_1 = \frac{5}{3} \times \frac{5}{4} = \frac{25}{12} > 1$$

16. $\because \quad 2\tan^{-1}\left(\dfrac{b}{a}\right) = \dfrac{\pi}{2}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{4} \Rightarrow \frac{b}{a} = 1 \text{ or } a = b$$

Then, hyperbola convert in $x^2 - y^2 = a^2$

and $$e = \sqrt{\left(1 + \frac{a^2}{a^2}\right)} = \sqrt{2}$$

17. The intersection of line $bx - ay = 0$ and the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x^2}{a^2} - \left(\frac{b}{a}\right)^2 \frac{x^2}{b^2} = 1$$

$$\Rightarrow \quad 0 = 1$$

Hence, no solution exist.

But statement II is false.

18. Now, $\quad 4x^2 - 3y^2 = 12$

$$\Rightarrow \quad \frac{x^2}{3} - \frac{y^2}{4} = 1$$

Then, director circle is $x^2 + y^2 = 3 - 4 = -1$ which does not exist for existence $a > b$

19. First of all we will verify statement II.

Let $P(a \sec\theta, a \tan\theta)$ be any point on $x^2 - y^2 = a^2$, then

$$SP \cdot S'P = (ea \sec\theta - a)(ea \sec\theta + a)$$

$$= e^2a^2 \sec^2\theta - a^2$$

$$= 2a^2 \sec^2\theta - a^2 \qquad (\because e = \sqrt{2})$$

and $$CP^2 = a^2 \sec^2\theta + a^2 \tan^2\theta$$

$$= a^2 \sec^2\theta + a^2 \sec^2\theta - a^2$$

$$= 2a^2 \sec^2\theta - a^2$$

$$\Rightarrow \quad SP \cdot S'P = CP^2$$

$\therefore$ Statement II is true.

If we put $a = \sqrt{2}, \theta = \dfrac{\pi}{4}$, then statement I is verified.

20. Statement I is correct.

Also, statement II is true as asymptotes are perpendicular, they are bisectors of transverse and conjugate axes of hyperbola.

But statement II does not explain Statement I, as in hyperbolas other than rectangular hyperbolas asymptotes are not bisectors of transverse and conjugate axes.

21. Let equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where, $2ae = 4$ and $e = 2$

$\Rightarrow \quad a = 1$

$a^2e^2 = a^2 + b^2$

$\Rightarrow \quad 4 = 1 + b^2$

$\therefore \quad b^2 = 3$

Thus, equation of hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

$$\Rightarrow \quad 3x^2 - y^2 = 3$$

22. For $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the foci are $(\pm\sqrt{a^2 + b^2}, 0)$, here the foci are $(\pm 1, 0)$.

23. Since, $y = \alpha x + \beta$ is a tangent, then $\beta^2 = a^2\alpha^2 - b^2$. Hence, locus of P is a hyperbola $a^2x^2 - y^2 = b^2$.

17 Introduction to Three Dimensional Geometry

JEE Main MILESTONE

- Coordinate Axes and Coordinate Planes in Three Dimensional Space
- Distance between Two Points
- Section Formulae
- Area of a Triangle

17.1 Coordinate Axes and Coordinate Planes in Three Dimensional Space

Let $X'OX, Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O. O is the origin and the lines $X'OX, Y'OY$ and ZOZ' are called x-axis, y-axis and z-axis, respectively. These three lines are also called the rectangular axes of coordinates. The planes containing these three lines in pairs, determine three mutually perpendicular planes XOY, YOZ and ZOX.

In Mathematics, analytic geometry (Cartesian geometry) describes every point in three dimensional space by means of three coordinates, the position of any point in three dimensional space is given by an ordered triplet of real numbers, each number giving the distance of that point from the origin measured along the given axis, which is equal to the distance of that point from the plane determined by the other two axes.

The three planes divide space into eight cells called octants.

The following table show the signs of coordinates of points in various octants.

Octant Coordinate	Octant	x	y	z
$OXYZ$	I	+	+	+
$OX'YZ$	II	−	+	+
$OX'Y'Z$	III	−	−	+
$OXY'Z$	IV	+	−	+
$OXYZ'$	V	+	+	−
$OX'YZ'$	VI	−	+	−
$OX'Y'Z'$	VII	−	−	−
$OXY'Z'$	VIII	+	−	−

Coordinates of a Point in Space

The cartesian coordinates (x, y, z) of a point P in a space are the numbers at which the planes through P perpendicular to the axes out the axes. The coordinates of a point on x-axis are $(x, 0, 0)$, on y-axis are $(0, y, 0)$ and on z-axis are $(0, 0, z)$.

The standard equations of xy-plane, yz-plane and zx-plane are $z = 0, x = 0$ and $y = 0$, respectively.

Consider a point P in space whose position is given by triad (x,y,z) where, x, y,z are perpendicular distance from YZ-plane, ZX-plane and XY-plane respectively.

If we assume, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ unit vectors along OX, OY, OZ respectively, then position vector of point P is $x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ or simply (x, y, z).

***Sample* Problem 1** *A plane is parallel to yz-plane so it is perpendicular to* [NCERT]

(a) *x-axis* (b) *y-axis*
(c) *z-axis* (d) *None of these*

Interpret (a) If any plane is parallel to yz-plane, then it is perpendicular to x-axis.

***Sample* Problem 2** *In which of the following point lies in a fourth octant?*

(1, 2, 3), (4, –2,3), (4,–2,–5) and (4, 2, – 5) [NCERT]

(a) (1, 2,3) (b) (4,–2,3) (c) (4,–2,–5) (d) (4,2,–5)

Interpret (b)

Point	Octant
(1, 2, 3)	I (all the coordinates are positive)
(4,–2,3)	IV (y-coordinate is negative)
(4,–2,–5)	VIII (y and z-coordinates are negative)
(4,2,– 5)	V (z-coordinate is negative)

17.2 Distance between Two Points

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note

- The distance of a point $P(x, y, z)$ from the origin is $\sqrt{x^2 + y^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from x-axis is $\sqrt{y^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from y-axis is $\sqrt{x^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from z-axis is $\sqrt{x^2 + y^2}$.
- If $P(x, y, z)$ be a point in a space, then distance from yz, zx and xy-planes are respectively x, y and z

***Sample* Problem 3** *Find the coordinates of a point on Y-axis which are at a distance of $5\sqrt{2}$ from the point P(3, – 2, 5).* [NCERT]

(a) (0,–6,0) or (0,2,0) (b) (0,6,0) or (0,–2,0)
(c) (0,–6,0) or (1,2,0) (d) *None of these*

Interpret (a) Let any point on the Y-axis is $A(0, y, 0)$.

Given, distance between P and A, $PA = 5\sqrt{2}$

$$\Rightarrow \quad \sqrt{(3-0)^2 + (-2-y)^2 + (5-0)^2} = 5\sqrt{2}$$

On squaring both sides, we get

$$(3-0)^2 + (-2-y)^2 + (5-0)^2 = 50$$

$$\Rightarrow \quad 9 + 4 + y^2 + 4y + 25 = 50$$

$$\Rightarrow \quad y^2 + 4y + 38 - 50 = 0$$

$$\Rightarrow \quad y^2 + 4y - 12 = 0$$

$$\Rightarrow \quad (y + 6)(y - 2) = 0$$

$$\Rightarrow \quad y = -6, 2$$

Hence, the points on Y-axis are $(0, -6, 0)$ or $(0, 2, 0)$.

***Sample* Problem 4** *Which of the following pairs of points have a distance $\sqrt{43}$?* [NCERT]

(a) (2,3,5) and (4,3,1) (b) (–3,7,2) and (2, 4, –1)
(c) (–1,3,–4) and (1,–3,4) (d) (2,–1,3) and (–2,1,3)

Interpret (b) (a) Let the given points are $A(2, 3, 5)$ and $B(4, 3, 1)$.

$$\therefore \quad x_1 = 2,\ y_1 = 3,\ z_1 = 5$$
$$x_2 = 4,\ y_2 = 3,\ z_2 = 1$$

$\therefore$ Required distance

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\Rightarrow \quad AB = \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$$

$$= \sqrt{4 + 0 + 16} = \sqrt{20} = 2\sqrt{5}$$

(b) Let the given points are $A(-3, 7, 2)$ and $B(2, 4, -1)$.

Here, $x_1 = -3,\ y_1 = 7,\ z_1 = 2$

$x_2 = 2,\ y_2 = 4,\ z_2 = -1$

$\therefore$ Required distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{[2-(-3)]^2 + (4-7)^2 + (-1-2)^2}$$
$$= \sqrt{(2+3)^2 + (4-7)^2 + (-1-2)^2}$$
$$= \sqrt{25+9+9} = \sqrt{43}$$

(c) Let the given points are $A(-1, 3, -4)$ and $B(1, -3, 4)$.

$\therefore$ $x_1 = -1, y_1 = 3, z_1 = -4$
$x_2 = 1, y_2 = -3, z_2 = 4$

$\therefore$ Required distance, $AB = \sqrt{(1+1)^2 + (-3-3)^2 + (4+4)^2}$

$$= \sqrt{4+36+64} = \sqrt{104} = 2\sqrt{26}$$

(d) Let the given points are $A(2, -1, 3)$ and $B(-2, 1, 3)$.

$\therefore$ $x_1 = 2, y_1 = -1, z_1 = 3$
$x_2 = -2, y_2 = 1, z_2 = 3$

$\therefore$ Required distance

$$AB = \sqrt{(-2-2)^2 + (1+1)^2 + (3-3)^2} = \sqrt{4^2 + 2^2 + 0}$$
$$= \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

***Sample* Problem 5** *The equation of the set of points which are equidistant from the points* (1, 2, 3) *and* (3, 2, – 1) *is*

(a) $x - 3z = 0$ (b) $x - 2z = 0$
(c) $x - 4z = 0$ (d) $x + 2z = 0$

Interpret (b) Let the given points are A and B. Let $P(x, y, z)$ be any point equidistant from A and B.

$\therefore$ $PA = PB$,

i.e., distance between P and A = distance between P and B

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-3)^2 + (y-2)^2 + (z+1)^2}$$
$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = (x-3)^2 + (y-2)^2 + (z+1)^2$$
$$\Rightarrow x^2 + 1 - 2x + y^2 + 4 - 4y + z^2 + 9 - 6z$$
$$= x^2 + 9 - 6x + y^2 + 4 - 4y + z^2 + 1 + 2z$$
$$\Rightarrow 4x - 8z = 0 \Rightarrow x - 2z = 0$$

which is the required equation.

17.3 Section Formulae

(i) Section Formulae for Internal Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q internally in the ratio $m : n$. Then, the coordinates of R are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Note If R is the mid-point of PQ, then coordinates of R are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

(ii) Section Formulae for External Division

If P and Q are such that R divides the join of P and Q externally in the ratio $m : n$. Then, the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Note If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the mid-point of AB is $\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \frac{x_3 + y_3}{2}\right)$.

***Sample* Problem 6** *A point R with x-coordinate 4 lies on the line segment joining the points P*(2, – 3, 4) *and Q*(8, 0, 10). *Find the coordinates of the point R.* **[NCERT]**

(a) (4, –2, –6) (b) (4, 2, 6)
(c) (4, –2, 6) (d) *None of these*

Interpret (b) Let the point $R(x, y, z)$ divides PQ in the ratio $k : 1$.

$$\Rightarrow \frac{k \times 8 + 1 \times 2}{k+1} = 4 \quad \text{(given)}$$
$$\Rightarrow \frac{8k+2}{k+1} = 4$$
$$\Rightarrow 8k + 2 = 4k + 4$$
$$\Rightarrow 8k - 4k = 4 - 2$$
$$\Rightarrow 4k = 2 \Rightarrow k = \frac{1}{2} \Rightarrow k : 1 = 1 : 2$$

Hence, the point R divides PQ internally in the ratio 1 : 2. Therefore, y-coordinate of $R = \left(\frac{1 \times 0 + 2 \times (-3)}{1+2}\right) = \frac{-6}{3} = -2$ and z-coordinate of $R = \left(\frac{1 \times 10 + 2 \times 4}{1+2}\right) = \frac{10+8}{3} = \frac{18}{3} = 6$.

Hence, coordinate of R are (4, –2, 6).

17.4 Area of a Triangle

Let the vertices of a ΔABC be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

If A_{yz}, A_{zx}, A_{xy} be the projection of an area A on the coordinate planes yz, zx and xy respectively, then

Area, $$\Delta = \sqrt{A_{yz}^2 + A_{zx}^2 + A_{xy}^2}$$

where $$A_{yz} = \frac{1}{2}\left\| \begin{matrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{matrix} \right\|,$$

$$A_{zx} = \frac{1}{2}\begin{Vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{Vmatrix} \text{ and } A_{xy} = \frac{1}{2}\begin{Vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{Vmatrix}$$

Note Area of $\Delta ABC = \frac{1}{2}\begin{Vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{Vmatrix}$

Conditions for Collinearity

If A, B and C are collinear, then area of triangle should be zero

i.e., $\Delta = 0$

***Sample* Problem 7** *If there are three points $A(2, 3, 4)$, $B(-1, 2, -3)$ and $C(-4, 1, -10)$ in a space, then they are* **[NCERT Exemplar]**

(*a*) *collinear* (*b*) *non-collinear*
(*c*) *Do not say anything* (*d*) *None of these*

Interpret (*b*) Given, points are $A(2, 3, 4)$, $B(-1, 2, -3)$ and $C(-4, 1, -10)$.

Now, determine area of triangle.

$\therefore$ Area of triangle, $\Delta = \sqrt{\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2}$

where $\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{Vmatrix}$, $\Delta_{yz} = \frac{1}{2}\begin{Vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{Vmatrix}$

and $\Delta_{zx} = \frac{1}{2}\begin{Vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{Vmatrix}$

Now, $\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ -4 & 1 & 1 \end{Vmatrix} = \frac{1}{2}|[2(2-1) - 3(-1+4)] + 1(-1+8)]|$

$= \frac{1}{2}|[2 - 9 + 7]| = 0$

$\Delta_{yz} = \frac{1}{2}\begin{Vmatrix} 3 & 4 & 1 \\ 2 & -3 & 1 \\ 1 & -10 & 1 \end{Vmatrix} = \frac{1}{2}|[3(-3+10) - 4(2-1) + 1(-20+3)]|$

$= \frac{1}{2}|21 - 4 - 17| = 0$

$\Delta_{zx} = \frac{1}{2}\begin{Vmatrix} 4 & 2 & 1 \\ -3 & -1 & 1 \\ -10 & -4 & 1 \end{Vmatrix}$

$= \frac{1}{2}|[4(-1+4) - 2(-3+10) + 1(12-10)]|$

$= \frac{1}{2}|[12 - 14 + 2]| = 0$

$\therefore \quad \Delta = \sqrt{0^2 + 0^2 + 0^2} = 0$

Hence, given points are collinear.

Circles associated with a Triangle

(i) Centroid of a triangle

The centroid of a triangle is the point of intersection of its medians. The centroid divides the median in the ratio 2 : 1. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of a triangle, then the centroid of a triangle is

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(ii) Circumcentre of a Triangle

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the circumradius of the triangle.

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of a triangle, then the circumcentre of ΔABC is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

(iii) Incentre of a Triangle

The incentre of a triangle is the point of intersection of internal bisector of the angles. Also, it is a centre of a circle touching all the sides of a triangle.

$\therefore$ Coordinates of incentre

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}, \frac{az_1 + bz_2 + cz_3}{a+b+c}\right)$$

where a, b and c are the sides of a ΔABC.

(iv) Orthocentre of a triangle

It is the point of intersection of perpendiculars drawn from vertices on opposite sides (called altitudes) of a triangle and can be obtained by solving the equation of any two altitudes.

Here, O is the orthocentre, since $AE \perp BC, BF \perp AC, CD \perp AB$, then $OE \perp BC, OF \perp AC$ and $OD \perp AB$.

Solving any two we can get coordinate of O.

The orthocentre of ΔABC is

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}, \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

If a triangle is right angled triangle, then orthocentre is the point where right angle is formed.

(v) Excircle of a triangle

A circle touches one side outside the triangle and other two extended sides, then circle is known as excircle. Let ABC be a triangle, then there are three excircles with three excentres. Let I_1, I_2, I_3 be the centres of excircles opposite to vertices A, B and C, respectively. If vertices of triangle are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}, \frac{-az_1 + bz_2 + cz_3}{-a+b+c}\right)$$

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}, \frac{az_1 - bz_2 + cz_3}{a-b+c}\right)$$

$$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}, \frac{az_1 + bz_2 - cz_3}{a+b-c}\right)$$

Note Centroid G, Incentre I, Excentres I_1, I_2 and I_3, Orthocentre O, Circumcentre S are given by $\left(\frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}, \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}, \frac{m_1z_1 + m_2z_2 + m_3z_3}{m_1 + m_2 + m_3}\right)$.

where $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of ΔABC.

	m_1	m_2	m_3
G	1	1	1
I	$\sin A$	$\sin B$	$\sin C$
I_1	$-\sin A$	$\sin B$	$\sin C$
I_2	$\sin A$	$-\sin B$	$\sin C$
I_3	$\sin A$	$\sin B$	$-\sin C$
O	$\tan A$	$\tan B$	$\tan C$
S	$\sin 2A$	$\sin 2B$	$\sin 2C$

***Sample* Problem 8** *Find the orthocentre of the triangle formed by the points $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$.*

(a) $(1, -6, -6)$ (b) $(-1, 6, 6)$ (c) $(1, -6, 6)$ (d) $(1, 6, 6)$

Interpret (b) Now,

$$AB = \sqrt{(-1-0)^2 + (6-7)^2 + (6-10)^2} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(-4+1)^2 + (9-6)^2 + (6-6)^2} = \sqrt{18} = 3\sqrt{2}$$

$$AC = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2} = \sqrt{36} = 6$$

$$\therefore \quad AB^2 + BC^2 = 18 + 18 = 36 = AC^2$$

Hence, ΔABC is a right-angled triangle. So, orthocentre of a ΔABC is $(-1, 6, 6)$.

***Sample* Problem 9** *If the vertices of a triangle are $A(1, -1, -5)$, $B(3, 1, 3)$ and $C(9, 1, -3)$, then the circumcentre of a ΔABC is*

(a) $\left(\frac{13}{3}, \frac{1}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{13}{3}, \frac{1}{3}, -\frac{5}{3}\right)$

(c) $\left(-\frac{13}{3}, \frac{1}{3}, -\frac{5}{3}\right)$ (d) *None of these*

Interpret (b) Now, $AB = \sqrt{(3-1)^2 + (1+1)^2 + (3+5)^2}$

$$= \sqrt{4 + 4 + 64} = \sqrt{72}$$

$$BC = \sqrt{(9-3)^2 - (1-1)^2 + (-3-3)^2} = \sqrt{36 + 0 + 36} = \sqrt{72}$$

and $CA = \sqrt{(9-1)^2 + (1+1)^2 + (-3+5)^2} = \sqrt{64 + 4 + 4} = \sqrt{72}$

$\therefore \quad AB = BC = CA = \sqrt{72}$

Hence, ΔABC is an equilateral triangle. Since, in an equilateral triangle, circumcentre and centroid coincide it.

$\therefore$ Circumcentre of ΔABC = Centroid of ΔABC

$$= \left(\frac{1+3+9}{3}, \frac{-1+1+1}{3}, \frac{-5+3-3}{3}\right) = \left(\frac{13}{3}, \frac{1}{3}, -\frac{5}{3}\right)$$

WORKED OUT
Examples

Example 1 *If the origin is the centroid of ΔPQR with vertices $P(2a, 2, 6)$, $Q(-4, 3b, -10)$ and $R(8, 14, 2c)$, then find the values of a, b* and *c are respectively.* [**NCERT**]

(a) $-2, -\frac{16}{3}, 3$ (b) $2, \frac{16}{3}, -2$

(c) $-2, -\frac{16}{3}, 2$ (d) *None of these*

Solution (c) Centroid of the ΔPQR is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

i.e., $$\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)$$

Origin is the centroid *i.e.*, the coordinates of the centroid are (0, 0, 0), then

$$\frac{2a-4+8}{3}=0$$

$\Rightarrow$ $2a+4=0 \Rightarrow a=-2$

Also, $\frac{2+3b+14}{3}=0$

$\Rightarrow$ $3b+16=0$

$\Rightarrow$ $b=-\frac{16}{3}$

and $\frac{6-10+2c}{3}=0$

$\Rightarrow$ $2c-4=0 \Rightarrow c=2$

$\therefore$ $a=-2, b=\frac{-16}{3}, c=2$

Example 2 *Three vertices of a parallelogram ABCD are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.* [**NCERT**]

(a) $(-1, -2, 8)$ (b) $(1, -2, 8)$ (c) $(1, -2, -8)$ (d) $(1, 2, 8)$

Solution (b) Let *ABCD* be a parallelogram and coordinates of point $D(\alpha, \beta, \gamma)$ and the diagonals *AC* and *BD* intersect at *P*.

In a parallelogram diagonals bisect each other.

$\therefore$ Coordinates of mid-point of *BD*

= Coordinates of mid-point of *AC*

$$\Rightarrow \left(\frac{\alpha+1}{2}, \frac{\beta+2}{2}, \frac{\gamma-4}{2}\right)=\left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right)$$

$$\Rightarrow \left(\frac{\alpha+1}{2}, \frac{\beta+2}{2}, \frac{\gamma-4}{2}\right)=\left(\frac{2}{2}, 0, \frac{4}{2}\right)$$

$$\Rightarrow \left(\frac{\alpha+1}{2}, \frac{\beta+2}{2}, \frac{\gamma-4}{2}\right)=(1, 0, 2)$$

On comparing the corresponding coordinates, we get

$\frac{\alpha+1}{2}=1, \frac{\beta+2}{2}=0, \frac{\gamma-4}{2}=2$

$\Rightarrow$ $\alpha+1=2, \beta+2=0, \gamma-4=4$

$\Rightarrow$ $\alpha=1, \beta=-2, \gamma=8$

$\therefore$ Coordinates of point *D* are (1, –2, 8).

Example 3 *The area of triangle whose vertices are* (1, 2, 3), (2, 5, –1) *and* (–1, 1, 2) *is*

(a) 150 *sq units* (b) 145 *sq units*

(c) $\frac{\sqrt{155}}{2}$ *sq units* (d) $\frac{155}{2}$ *sq units*

Solution (c) Let $(1, 2, 3)=(x_1, y_1, z_1)$, $(2, , 5 -1)=(x_2, y_2, z_2)$ and $(-1, 1, 2)=(x_3, y_3, z_3)$

Now, $$\Delta_{xy}=\frac{1}{2}\left\|\begin{matrix}1 & 2 & 1\\ 2 & 5 & 1\\ -1 & 1 & 1\end{matrix}\right\|=\frac{1}{2}|[1(5-1)-2(2+1)+1(2+5)]|$$

$$=\frac{1}{2}|[4-6+7]|=\frac{5}{2}$$

$$\Delta_{yz}=\frac{1}{2}\left\|\begin{matrix}2 & 3 & 1\\ 5 & -1 & 1\\ 1 & 2 & 1\end{matrix}\right\|=\frac{1}{2}|[2(-1-2)-3(5-1)+1(10+1)]|$$

$$=\frac{1}{2}|[-6-12+11]|=\frac{7}{2}$$

$$\Delta_{zx}=\frac{1}{2}\left\|\begin{matrix}3 & 1 & 1\\ -1 & 2 & 1\\ 2 & -1 & 1\end{matrix}\right\|=\frac{1}{2}|[3(2+1)-1(-1-2)+1(1-4)]|$$

$$=\frac{1}{2}|[9+3-3]|=\frac{9}{2}$$

$$\therefore \text{Area of } \Delta=\sqrt{\Delta_{xy}^2+\Delta_{yz}^2+\Delta_{zx}^2}=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{7}{2}\right)^2+\left(\frac{9}{2}\right)^2}$$

$$=\sqrt{\frac{25}{4}+\frac{49}{4}+\frac{81}{4}}=\frac{\sqrt{155}}{2} \text{ sq units}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Cartesian Coordinates in Three Dimensions

1. A plane is parallel xy-plane, so it is perpendicular to
(a) z-axis (b) y-axis
(c) x-axis (d) None of these

2. The locus of a point for which $y = 0$, $z = 0$ is
(a) equation of x-axis (b) equation of y-axis
(c) equation of z-axis (d) None of these

3. The point $(-2, -3, -4)$ lies in the [NCERT]
(a) first octant (b) seventh octant
(c) second octant (d) eight octant

4. L is the foot of the perpendicular drawn from a point $P(3, 4, 5)$ on the xy-plane. The coordinates of point L are [NCERT Exemplar]
(a) (3, 0, 0) (b) (0, 4, 5)
(c) (3, 0, 5) (d) None of these

Distance between Two Points and Section Formulae

5. The distance of point $P(3, 4, 5)$ from the yz-plane is [NCERT Exemplar]
(a) 3 units (b) 4 units (c) 5 units (d) 550 units

6. What is the length of foot of perpendicular drawn from the point $P(3, 4, 5)$ on y-axis? [NCERT Exemplar]
(a) $\sqrt{41}$ (b) $\sqrt{34}$
(c) 5 (d) None of these

7. If the distance between the points $(a, 0, 1)$ and $(0, 1, 2)$ is $\sqrt{27}$, then the value of a is [NCERT Exemplar]
(a) 5 (b) ± 5
(c) -5 (d) None of these

8. If x-coordinate of a point P of line joining the points $Q(2, 2, 1)$ and $R(5, 2, -2)$ is 4, then the z-coordinate of P is
(a) -2 (b) -1 (c) 1 (d) 2

9. If A and B be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of points P such that $(PA)^2 + (PB)^2 = K^2$, where K is a constant. [NCERT]
(a) $2(x^2 + y^2 + z^2) + 4x + 14y + 4z + 109 - k^2 = 0$
(b) $2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - k^2 = 0$
(c) $x^2 + y^2 + z^2 + 4x + 14y + 4z + 109 - k^2 = 0$
(d) None of the above

10. Distance of the point (1, 2, 3) from the coordinate axes are
(a) 13, 10, 5 (b) $\sqrt{13}, \sqrt{10}, \sqrt{5}$
(c) $\sqrt{5}, \sqrt{13}, \sqrt{10}$ (d) $\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}}$

11. If the sum of the squares of the distance of a point from the three coordinate axes be 36, then its distance from the origin is
(a) 6 (b) $3\sqrt{2}$
(c) $2\sqrt{3}$ (d) None of these

12. The coordinates of a point which is equidistant from the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ are given by
(a) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (b) $\left(-\frac{a}{2}, -\frac{b}{2}, \frac{c}{2}\right)$
(c) $\left(\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ (d) $\left(-\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$

13. If $x^2 + y^2 = 1$, then the distance from the point $(x, y, \sqrt{1 - x^2 - y^2})$ to the origin is [NCERT Exemplar]
(a) 1 (b) -1 (c) 0 (d) 2

14. Three vertices of a parallelogram $ABCD$ are $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$. Find the fourth vertex D. [NCERT Exemplar]
(a) $(-4, -7, -6)$ (b) (4, 7, 6)
(c) $(4, 7, -6)$ (d) None of these

15. If a parallelopiped is formed by planes drawn through the points (5, 8, 10) and (3, 6, 8) parallel to the coordinate planes, then the length of diagonal of the parallelopiped is [NCERT Exemplar]
(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$

16. Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, – 4, 6) in the ratio 2 : 3 externally. [NCERT]
(a) (– 8, – 17, 3) (b) (– 8, 17, 3)
(c) (8, – 17, 3) (d) None of these

17. Find the ratio in which the YZ-plane divides the line segment formed by joining the points (– 2, 4, 7) and (3, – 5, 8). [NCERT]
(a) externally 2 : 3 (b) internally 2 : 3
(c) internally 3 : 2 (d) externally 3 : 2

18. Find the length of the medians of the triangle with vertices $A(0, 0, 6)$, $B(0, 4, 0)$ and $C(6, 0, 0)$. [NCERT]
(a) $7, 7, \sqrt{34}$ (b) $7, 8, \sqrt{34}$
(c) $7, 9, \sqrt{34}$ (d) None of these

19. Find the coordinates of the points which trisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$. [NCERT]
(a) (6, – 4, – 2), (8, – 10, 2) (b) (6, 4, – 2), (8, – 10, 2)
(c) (6, – 4, – 2), (8, 10, 2) (d) None of these

20. Find the centroid of a triangle, the mid-point of whose sides are $D(1, 2, -3)$, $E(3, 0, 1)$ and $F(-1, 1, -4)$. [NCERT Exemplar]
(a) (1, 1, 2) (b) (1, 1, – 2)
(c) (– 1, –1, –2) (d) (1, –1, –2)

21 The points $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ are vertices of a
(a) square (b) rhombus
(c) rectangle (d) None of these

22. The points (5, – 4, 2),(4,– 3, 1), (7, – 6, 4) and (8, – 7, 5) are the vertices of
(a) a rectangle (b) a square
(c) a parallelogram (d) None of these

23. If the coordinates of the vertices of a ΔABC are $A(-1, 3, 2)$, $B(2, 3, 5)$ and $C(3, 5, -2)$, then $\angle A$ is equal to
(a) 45° (b) 60°
(c) 90° (d) 30°

24. The mid-points of the sides of a triangle are (5, 7, 11), (0, 8, 5) and (2, 3, – 1). Then, the vertices are [NCERT Exemplar]
(a) (7, 2, 5), (3, 12, 17), (– 3, 4, – 7)
(b) (7, 2, 5), (3, 12, 17), (3, 4, 7)
(c) (7, 2, 5), (– 3, 12, 17), (– 3, – 4, – 7)
(d) None of the above

Area of Triangle and Volume of Tetrahedran

25. The area of the triangle, whose vertices are at the points (2, 1, 1), (3, 1, 2) and (– 4, 0, 1) is
(a) $\sqrt{19}$ (b) $\frac{1}{2}\sqrt{19}$
(c) $\frac{1}{2}\sqrt{38}$ (d) $\frac{1}{2}\sqrt{57}$

26. If vertices of a triangle are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$, then the area of a triangle is
(a) $\sqrt{6}$ (b) $2\sqrt{6}$
(c) $3\sqrt{6}$ (d) $4\sqrt{6}$

27. The triangle formed by the points
(0, 7, 10), (– 1, 6, 6), (– 4, 9, 6) is
(a) equilateral (b) isosceles
(c) right angled (d) right angled isosceles

28. The points (5, 2, 4), (6, – 1, 2) and (8, – 7, k) are collinear, if k is equal to
(a) – 2 (b) 2
(c) 3 (d) – 1

29. The point $A(1, -1, 3)$, $B(2, -4, 5)$ and $C(5, -13, 11)$ are [NCERT Exemplar]
(a) collinear (b) non-collinear
(c) Do not say anything (d) None of these

Circles associated with a Triangle

30. If orthocentre and circumcentre of a triangle are respectively (1, 1, 1) and (3, 2, 2), then the coordinates of its centroid is
(a) $\left(-\frac{7}{3}, \frac{5}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{7}{3}, \frac{5}{3}, \frac{5}{3}\right)$
(c) $\left(\frac{5}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (d) None of these

31. The incentre of triangle with vertices $A(1, 1, 2)$, $B(2, 1, 3)$ and $C(1, -1, 3)$ is
(a) $\left(\frac{3\sqrt{5}+\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{3\sqrt{5}-\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{5\sqrt{5}+3\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}\right)$
(b) $\left(\frac{1-2\sqrt{5}+\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{1+\sqrt{5}+\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{2+3\sqrt{5}+3\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}\right)$
(c) $\left(\frac{1+2\sqrt{5}+\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{1-\sqrt{5}-\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{2+3\sqrt{5}+3\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}\right)$
(d) None of the above

32. If the vertices of a triangle are $A(0, 4, 1)$, $B(2, 3, -1)$ and $C(4, 5, 0)$, then orthocentre of a ΔABC is
(a) (4, 5, 0) (b) (2, 3, – 1)
(c) (– 2, 3, – 1) (d) None of the above

Answers

1. (a)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (b)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (b)
11. (b)	**12.** (a)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (b)
21. (b)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (b)
31. (a)	**32.** (b)								

the **Guidance**

1. If a plane is parallel to xy-plane, then it is perpendicular to z- axis.

2. Since, in a x-axis, the coordinates of y and z are zero.

3. In a given point $(-2, -3, -4)$ all coordinates are negative. Hence, it lies in seventh octant.

4. Since, in a xy-plane, the z-coordinate will be zero. Hence, the coordinate of the foot of the point $L(3, 4, 0)$.

5. When we draw a perpendicular line from $P(3, 4, 5)$ on the yz-plane, the x-coordinate will be zero and the other coordinate y and z will be 4 and 5, *i.e.*, coordinate on yz-plane be $Q(0, 4, 5)$.

$\therefore$ Distance between P and $Q = PQ$

$$= \sqrt{(3-0)^2 + (4-4)^2 + (5-5)^2}$$

$$= \sqrt{3^2 + 0^2 + 0^2} = 3 \text{ units}$$

6. When we drawn perpendicular from the point $P(3, 4, 5)$ on y-axis, the x and z-coordinates will be zero and y-coordinate will be 4. *i.e.*, coordinate on y-axis $Q(0, 4, 0)$.

$\therefore$ Distance between P and $Q = PQ$

$$= \sqrt{(3-0)^2 + (4-4)^2 + (5-0)^2}$$

$$= \sqrt{3^2 + 0^2 + 5^2} = \sqrt{9+25} = \sqrt{34} \text{ units}$$

7. Given, distance between points $(a, 0, 1)$ and $(0, 1, 2)$ is $\sqrt{27}$.

$\therefore \sqrt{(a-0)^2 + (0-1)^2 + (1-2)^2} = \sqrt{27}$

$\Rightarrow \quad \sqrt{a^2 + 1^2 + 1^2} = \sqrt{27}$

$\Rightarrow \quad a^2 + 2 = 27$ ($\because$ squaring both sides)

$\Rightarrow \quad a^2 = 25 \Rightarrow a = \pm 5$

8. Suppose P divides QR in the ratio $\lambda : 1$. Then, coordinate of P are $\left(\frac{5\lambda+2}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$.

Since, the x-coordinate of P is 4.

i.e., $\quad \frac{5\lambda+2}{\lambda+1} = 4$

$\Rightarrow \quad \lambda = 2$

So, z-coordinate of P is $\frac{-2\lambda+1}{\lambda+1} = \frac{-4+1}{2+1} = -1$

9. Let the point P is (x, y, z).

Given, $\quad (PA)^2 + (PB)^2 = K^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = K^2$

$\Rightarrow \quad x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 - 10z + x^2 + 1 + 2x + y^2 + 9 - 6y + z^2 + 49 + 14z = K^2$

$\Rightarrow \quad 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 - K^2 = 0$

$\Rightarrow \quad 2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - K^2 = 0$

which is the required equation.

10. Let point be $P(x, y, z) = (1, 2, 3)$

$\therefore$ The distance from x-axis $= \sqrt{y^2 + z^2}$

$$= \sqrt{4+9} = \sqrt{13}$$

The distance from y-axis $= \sqrt{x^2 + z^2} = \sqrt{1^2 + 3^2}$

$$= \sqrt{1+9} = \sqrt{10}$$

The distance from z-axis $= \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2}$

$$= \sqrt{1+4} = \sqrt{5}$$

11. Let $P(x, y, z)$ be any point in the plane.

According to the given condition,

(distance from P to x-axis)^2 + (distance from P to y-axis)^2 + (distance from P to z-axis)$^2 = 36$

$\Rightarrow \quad (\sqrt{y^2 + z^2})^2 + (\sqrt{x^2 + z^2})^2 + (\sqrt{x^2 + y^2})^2 = 36$

$\Rightarrow \quad (y^2 + z^2) + (x^2 + z^2) + (x^2 + y^2) = 36$

$\Rightarrow \quad 2(x^2 + y^2 + z^2) = 36$

$\Rightarrow \quad x^2 + y^2 + z^2 = 18 \qquad \ldots(i)$

$\therefore$ The distance from origin to the point (x, y, z) is

$$= \sqrt{x^2 + y^2 + z^2} = \sqrt{18} \qquad [\because \text{from Eq. (i)}]$$

$$= 3\sqrt{2}$$

12. Let P be any point in a space.

According to the given condition,

$$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}=\sqrt{(x-a)^2+y^2+z^2}$$
$$=\sqrt{x^2+(y-b)^2+z^2}=\sqrt{x^2+y^2+(z-c)^2}$$

On squaring both sides, we get

$$x^2+y^2+z^2=(x-a)^2+y^2+z^2=x^2+(y-b)^2+z^2$$
$$=x^2+y^2+(z-c)^2$$
$$\Rightarrow \quad x^2+y^2+z^2=x^2+a^2-2ax+y^2+z^2$$
$$\Rightarrow \quad a^2-2ax=0 \Rightarrow x=\frac{a}{2}$$

Similarly, we can solve the other values, we get

$$y=\frac{b}{2}, z=\frac{c}{2}$$

13. $\therefore$ Distance from point $(x, y, \sqrt{1-x^2-y^2})$ to the origin.

$$=\sqrt{(x-0)^2+(y-0)^2+\sqrt{(1-x^2-y^2-0)^2}}$$
$$=\sqrt{x^2+y^2+1-x^2-y^2}$$
$$=\sqrt{1+1-(1)} \qquad (\because x^2+y^2=1 \text{ given})$$
$$=\sqrt{1}=1$$

14. Let the fourth vertex be (x, y, z).

We know that diagonals of a parallelogram are bisecting to each other.

i.e., mid-point of a diagonals are coinciding.

$\therefore$ Mid-point of diagonal AC = Mid-point of diagonal BD

$$\therefore \left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right)=\left(\frac{-1+x}{2}, \frac{-2+y}{2}, \frac{-1+z}{2}\right)$$
$$\Rightarrow \quad \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)=\left(\frac{-1+x}{2}, \frac{-2+y}{2}, \frac{-1+z}{2}\right)$$
$$\Rightarrow \quad \frac{3}{2}=\frac{-1+x}{2}, \frac{5}{2}=\frac{-2+y}{2}, \frac{5}{2}=\frac{-1+z}{2}$$
$$\Rightarrow \quad x=4, y=7, z=6$$

Hence, required point is (4, 7, 6).

15. The edge of a parallelopiped $=(3-5, 6-8, 8-10)$

$$=(-2, -2, -2)$$

$\therefore$ Length of diagonal $=\sqrt{(-2)^2+(-2)^2+(-2)^2}$

$$=\sqrt{4+4+4}=2\sqrt{3}$$

16. Let the point C divides the line externally in the ratio $2:3$.

Here, the ratio is $2:3$.

$\therefore \quad m=2, n=3$

The coordinates of point C

$$=\left[\left(\frac{mx_2-nx_1}{m-n}\right), \left(\frac{my_2-ny_1}{m-n}\right), \left(\frac{mz_2-nz_1}{m-n}\right)\right]$$
$$\Rightarrow \quad C=\left[\frac{2\times(1)-3\times(-2)}{(2-3)}, \frac{2\times(-4)-3\times 3}{(2-3)}, \frac{2\times 6-3\times 5}{(2-3)}\right]$$
$$=\left[\frac{2+6}{(-1)}, \frac{-8-9}{(-1)}, \frac{12-15}{(-1)}\right]$$
$$=(-8, 17, 3)$$

17. The given points are $A(-2, 4, 7)$ and $B(3, -5, 8)$.

Let the point $P(0, y, z)$ in YZ-plane divides AB in the ratio $k:1$, then

$$X\text{-coordinate of point } P=\frac{mx_2+nx_1}{m+n}$$
$$\frac{k\times 3+1\times(-2)}{k+1}=0 \ (\because x\text{-coordinate of } P \text{ is zero})$$
$$\Rightarrow \quad 3k-2=0$$
$$\Rightarrow \quad k=\frac{2}{3}$$
$$\Rightarrow \quad k:1=2:3$$

$\therefore YZ$-plane divides the segment internally in the ratio 2 : 3.

18. ABC is a triangle with vertices $A(0, 0, 6)$, $B(0, 4, 0)$ and $C(6, 0, 0)$.

Let points D, E and F are the mid-points of BC, AC and AB, respectively. So, AD, BE and CF will be the medians of the triangle.

$\Rightarrow$ Coordinates of point D

$$=\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right)=(3, 2, 0)$$

Coordinates of point $E=\left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right)=(3, 0, 3)$

and

coordinates of point $F=\left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right)=(0, 2, 3)$

Now, length of median AD = Distance between A and D

$$AD=\sqrt{(0-3)^2+(0-2)^2+(6-0)^2}$$
$$=\sqrt{9+4+36}=\sqrt{49}=7$$

Similarly, $BE=\sqrt{(0-3)^2+(4-0)^2+(0-3)^2}$

$$=\sqrt{9+16+9}=\sqrt{34}$$

and $\quad CF=\sqrt{(6-0)^2+(0-2)^2+(0-3)^2}$

$$=\sqrt{36+4+9}=\sqrt{49}=7$$

19. Let the point R_1 trisects the line PQ *i.e.,* it divides the line in the ratio 1 : 2.

$$\Rightarrow R_1 = \left[\frac{1\times 10 + 2\times 4}{1+2}, \frac{1\times(-16)+2\times 2}{1+2}, \frac{1\times 6 + 2\times(-6)}{1+2}\right]$$

$$= \left(\frac{10+8}{3}, \frac{-16+4}{3}, \frac{6-12}{3}\right) = \left(\frac{18}{3}, \frac{-12}{3}, \frac{-6}{3}\right)$$

$$= (6, -4, -2)$$

Again, let the point R_2 divides PQ internally in the ratio 2 : 1. Then,

$$\Rightarrow R_2 = \left[\frac{2\times 10 + 1\times 4}{2+1}, \frac{2\times(-16)+1\times 2}{2+1}, \frac{2\times 6 + 1\times(-6)}{1+2}\right]$$

$$= \left(\frac{20+4}{3}, \frac{-32+2}{3}, \frac{12-6}{3}\right) = \left(\frac{24}{3}, \frac{-30}{3}, \frac{6}{3}\right)$$

$$= (8, -10, 2)$$

Hence, required points are $(6, -4, -2)$ and $(8, -10, 2)$.

20. Given, mid-point of sides of a triangle are $D(1, 2, -3)$, $E(3, 0, 1)$ and $F(-1, 1, -4)$.

$$\therefore \text{Centroid of triangle} = \left(\frac{1+3-1}{3}, \frac{2+0+1}{3}, \frac{-3+1-4}{3}\right)$$

$$= \left(\frac{3}{3}, \frac{3}{3}, \frac{-6}{3}\right) = (1, 1, -2)$$

21. Given that, $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$.

Now, $AB = \sqrt{(7-5)^2 + (-4+1)^2 + (7-1)^2}$

$= \sqrt{4+9+36} = 7$

$BC = \sqrt{(1-7)^2 + (-6+4)^2 + (10-7)^2}$

$= \sqrt{36+4+9} = 7$

$CD = \sqrt{(-1-1)^2 + (-3+6)^2 + (4-10)^2}$

$= \sqrt{4+9+36} = 7$

$DA = \sqrt{(5+1)^2 + (-1+3)^2 + (1-4)^2}$

$= \sqrt{36+4+9} = 7$

$\therefore \quad AB = BC = CD = DA = 7,$

Also, $AC = \sqrt{(1-5)^2 + (-6+1)^2 + (10-1)^2}$

$= \sqrt{16+25+81} = \sqrt{122}$

$BD = \sqrt{(-1-7)^2 + (-4+3)^2 + (7-4)^2}$

$= \sqrt{64+1+9} = \sqrt{74}$

$\therefore$ Diagonal $AC \neq BD$

Hence, $ABCD$ is not a square, so it is a rhombus.

22. Let $A(5, -4, 2)$, $B(4, -3, 1)$, $C(7, -6, 4)$ and $D(8, -7, 5)$.

Then, $AB = \sqrt{(4-5)^2 + (-3+4)^2 + (1-2)^2}$

$= \sqrt{1+1+1} = \sqrt{3}$

$BC = \sqrt{(7-4)^2 + (-6+3)^2 + (4-1)^2}$

$= \sqrt{9+9+9} = 3\sqrt{3}$

$CD = \sqrt{(8-7)^2 + (-7+6)^2 + (5-4)^2}$

$= \sqrt{1+1+1} = \sqrt{3}$

$AD = \sqrt{(8-5)^2 + (-7+4)^2 + (5-2)^2}$

$= \sqrt{9+9+9} = 3\sqrt{3}$

$AC = \sqrt{(7-5)^2 + (-6+4)^2 + (4-2)^2}$

$= \sqrt{4+4+4} = 2\sqrt{3}$

$BD = \sqrt{(8-4)^2 + (-7+3)^2 + (5-1)^2}$

$= \sqrt{16+16+16} = 4\sqrt{3}$

Here, we see that opposite sides $AB = CD$, $BC = AD$ are equal and diagonals $AC \neq BD$.

Hence, given points are the vertices of a parallelogram.

23. $\because$ Vertices of ΔABC are $A(-1, 3, 2)$, $B(2, 3, 5)$ and $C(3, 5, -2)$.

$\Rightarrow \quad AB = \sqrt{9+0+9} = \sqrt{18}$

$CA = \sqrt{16+4+16} = 6$

and $\quad BC = \sqrt{1+4+49} = \sqrt{54}$

$\because \quad AB^2 + CA^2 = BC^2$

ΔABC is right angled triangle at A.

$\therefore \quad \angle A = 90°$

24. Let the vertices of a triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Since, D, E and F are the mid-points of AC, BC and AB.

$$\therefore \quad \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = (0, 8, 5)$$

$$\Rightarrow \quad x_1 + x_2 = 0, y_1 + y_2 = 16, z_1 + z_2 = 10 \quad \text{...(i)}$$

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right) = (2, 3, -1)$$

$$\Rightarrow \quad x_2 + x_3 = 4, y_2 + y_3 = 6, z_2 + z_3 = -2 \quad \text{...(ii)}$$

and $$\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right) = (5, 7, 11)$$

$$\Rightarrow \quad x_1 + x_3 = 10, y_1 + y_3 = 14, z_1 + z_3 = 22 \quad \text{...(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

$2(x_1 + x_2 + x_3) = 14, 2(y_1 + y_2 + y_3) = 36, 2(z_1 + z_2 + z_3) = 30$

$\Rightarrow x_1 + x_2 + x_3 = 7, y_1 + y_2 + y_3 = 18, z_1 + z_2 + z_3 = 15$ ...(iv)

On solving Eqs. (i), (ii), (iii) and (iv), we get

$$x_3 = 7, x_1 = 3, x_2 = -3;$$
$$y_3 = 2, y_1 = 12, y_2 = 4$$
and $$z_3 = 5, z_1 = 17, z_2 = -7$$

Hence, vertices of a triangle are

(7, 2, 5), (3, 12, 17), (–3, 4, –7).

25. $\because \quad \Delta_{yz} = \frac{1}{2}\begin{Vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{Vmatrix}, \Delta_{zx} = \frac{1}{2}\begin{Vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{Vmatrix}$

$$\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{Vmatrix}$$

$$\therefore \quad \Delta_{yz} = \frac{1}{2}\begin{Vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{Vmatrix} = \frac{1}{2}|[1(2-1) - 1(1-0) + 1(1-0)]|$$

$$= \frac{1}{2}|[1 - 1 + 1]| = \frac{1}{2}$$

$$\Delta_{zx} = \frac{1}{2}\begin{Vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & -4 & 1 \end{Vmatrix} = \frac{1}{2}|[1(3+4) - 2(2-1) + 1(-8-3)]|$$

$$= \frac{1}{2}|[7 - 2 - 11]| = 3$$

$$\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ -4 & 0 & 1 \end{Vmatrix}$$

$$= \frac{1}{2}|[2(1-0) - 1(3+4) + 1(0+4)]| = \frac{1}{2}$$

$\therefore$ Area of triangle, $\Delta = \sqrt{\Delta_{yz}^2 + \Delta_{zx}^2 + \Delta_{xy}^2}$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + (3)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + 9 + \frac{1}{4}}$$

$$= \frac{\sqrt{38}}{2} \text{ sq unit}$$

26. Now, $\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{Vmatrix} = \frac{1}{2}|1(0-2) + 1(2-0) + 1(4-0)|$

$$= \frac{1}{2}|-2 + 2 + 4| = 2$$

$$\Delta_{yz} = \frac{1}{2}\begin{Vmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{Vmatrix} = \frac{1}{2}|[-1(-1-1) - 2(0-2) + 1(0+2)]|$$

$$= \frac{1}{2}|[2 + 4 + 2]| = 4$$

$$\Delta_{zx} = \frac{1}{2}\begin{Vmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{Vmatrix} = \frac{1}{2}|[2(2-0) - 1(-1-1) + 1(0-2)]|$$

$$= \frac{1}{2}|[4 + 2 - 2]| = 2$$

$\therefore$ Area of triangle, $\Delta = \sqrt{\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2}$

$$= \sqrt{2^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$$

27. Let $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$.

Then, $$AB = \sqrt{(-1-0)^2 + (6-7)^2 + (6-10)^2}$$
$$= \sqrt{1 + 1 + 16} = \sqrt{18} = 3\sqrt{2}$$
$$BC = \sqrt{(-4+1)^2 + (9-6)^2 + (6-6)^2}$$
$$= \sqrt{9 + 9 + 0} = \sqrt{18} = 3\sqrt{2}$$
$$AC = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2}$$
$$= \sqrt{16 + 4 + 16} = \sqrt{36} = 6$$

Clearly, $AC^2 = AB^2 + BC^2$

Hence, triangle is right angled. Also, $AB = BC$

$\therefore$ Triangle is right angled isosceles.

28. Now, $\Delta_{xy} = \frac{1}{2}\begin{Vmatrix} 5 & 2 & 1 \\ 6 & -1 & 1 \\ 8 & -7 & 1 \end{Vmatrix}$

$$= \frac{1}{2}|[5(-1+7) - 2(6-8) + 1(-42+8)]|$$

$$= \frac{1}{2}|[30 + 4 - 34]| = 0$$

$$\Delta_{yz} = \frac{1}{2}\begin{Vmatrix} 2 & 4 & 1 \\ -1 & 2 & 1 \\ -7 & k & 1 \end{Vmatrix} = \frac{1}{2}|[2(2-k) - 4(-1+7) + 1(-k+14)]|$$

$$= \frac{1}{2}|[4 - 2k - 24 - k + 14]| = \frac{1}{2}|[-3k - 6]| = \frac{3k+6}{2}$$

$$\Delta_{zx} = \frac{1}{2}\begin{Vmatrix} 4 & 5 & 1 \\ 2 & 6 & 1 \\ k & 8 & 1 \end{Vmatrix} = \frac{1}{2}|[4(6-8) - 5(2-k) + 1(16-6k)]|$$

$$= \frac{1}{2}|[-8 - 10 + 5k + 16 - 6k]| = \frac{1}{2}|[-2 - k]|$$

$$= \frac{k+2}{2}$$

$\therefore$ For collinear, $\Delta = 0$

$$\therefore \quad \sqrt{0^2 + \left(\frac{3k+6}{2}\right)^2 + \left(\frac{k+2}{2}\right)^2} = 0$$

$$\Rightarrow \quad \frac{1}{4}[(3k+6)^2 + (k+2)^2] = 0$$

$$\Rightarrow \quad 9k^2 + 36 + 36k + k^2 + 4 + 4k = 0$$

$$\Rightarrow \quad 10k^2 + 40k + 40 = 0 \Rightarrow k^2 + 4k + 4 = 0$$

$$\Rightarrow \quad (k+2)^2 = 0 \Rightarrow k = -2$$

29. Now, $\Delta_{xy} = \frac{1}{2}\left\|\begin{matrix} 1 & -1 & 1 \\ 2 & -4 & 1 \\ 5 & -13 & 1 \end{matrix}\right\|$

$$= \frac{1}{2}|[1(-4+13)+1(2-5)+1(-26+20)]|$$

$$= \frac{1}{2}|[9-3-6]| = 0$$

$$\Delta_{yz} = \frac{1}{2}\left\|\begin{matrix} -1 & 3 & 1 \\ -4 & 5 & 1 \\ -13 & 11 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}|[-1(5-11)-3(-4+13)+1(-44+65)]|$$

$$= \frac{1}{2}|[6-27+21]| = 0$$

$$\Delta_{zx} = \frac{1}{2}\left\|\begin{matrix} 3 & 1 & 1 \\ 5 & 2 & 1 \\ 11 & 5 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}|[3(2-5)-1(5-11)+1(25-22)]|$$

$$= \frac{1}{2}|[-9+6+3]| = 0$$

$\therefore$ Area of triangle $= \sqrt{\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2}$

$$= \sqrt{0+0+0} = 0$$

Hence, given points are collinear.

30. We know that centroid divides the line segment joining orthocentre and circumcentre in the ratio 2 : 1. Since, the coordinates of orthocentre and circumcentre are (1, 1, 1) and (3, 2, 2), respectively.

$\therefore$ The coordinates of centroid are

$$\left(\frac{2\times3+1\times1}{2+1}, \frac{2\times2+1\times1}{2+1}, \frac{2\times2+1\times1}{2+1}\right)$$

$$= \left(\frac{7}{3}, \frac{5}{3}, \frac{5}{3}\right)$$

31. Current vertices of a triangle are $A(1, 1, 2)$, $B(2, 1, 3)$ and $C(1, -1, 3)$.

$\therefore AB = c = \sqrt{(2-1)^2 + (1-1)^2 + (3-2)^2} = \sqrt{1+0+1} = \sqrt{2}$

$BC = a = \sqrt{(1-2)^2 + (-1-1)^2 + (3-3)^2} = \sqrt{1+4+0} = \sqrt{5}$

$CA = b = \sqrt{(1-1)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{0+4+1} = \sqrt{5}$

$\because$ Incentre of triangle

$$= \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$$

$$= \left(\frac{\sqrt{5}\times1+\sqrt{5}\times2+\sqrt{2}\times1}{1+\sqrt{2}+\sqrt{5}}, \frac{\sqrt{5}\times1+\sqrt{5}\times1+\sqrt{2}\times-1}{1+\sqrt{2}+\sqrt{5}}, \frac{\sqrt{5}\times2+\sqrt{5}\times3+\sqrt{2}\times3}{1+\sqrt{2}+\sqrt{5}}\right)$$

$$= \left(\frac{3\sqrt{5}+\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{2\sqrt{5}-\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}, \frac{5\sqrt{5}+3\sqrt{2}}{1+\sqrt{2}+\sqrt{5}}\right)$$

32. Given vertices of a ΔABC are $A(0, 4, 1)$, $B(2, 3, -1)$ and $C(4, 5, 0)$.

Now, $AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$

$$= \sqrt{4+1+4} = 3$$

$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$

$$= \sqrt{4+4+1} = 3$$

and $CA = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$

$$= \sqrt{16+1+1} = 3\sqrt{2}$$

Here, we see that $(AB)^2 + (BC)^2 = (3)^2 + (3)^2$

$$= 18 = CA^2$$

$\therefore \Delta ABC$ is a right angled triangle at B.

We know that in a right angled triangle, the orthocentre is on right angled triangle.

Hence, required orthocentre is (2, 3, –1).

18 Limits and Derivatives

JEE Main MILESTONE

- Concept of Limit
- Limit
- Left Hand Limit
- Right Hand Limit
- Existence of Limit
- Algebra of Limits
- Evaluation of Algebraic Limit
- Evaluation of Trigonometric Limits
- Some Useful Expansion
- Derivative
- Derivative from First Principle
- Differentiation of Some Important Functions
- Algebra of Derivative of Functions

18.1 Concept of Limit

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$. Clearly, this function is defined for all x except $x = 2$ as it assumes the form $\dfrac{0}{0}$ (known as an indeterminate form) at $x = 2$. However, if $x \neq 2$, then

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

In The derivative is a measure of how a function changes as its input changes. Loosely speaking a derivative can be thought of as how much one quantity is changing in response to changes in some other quantity.

The following table exhibits the values of $f(x)$ at points which are close to 2 on its two sides, *viz*. left and right on the real line.

x	1.4	1.5	1.6	1.7	1.8	1.9	1.99	2
$f(x)$	3.4	3.5	3.6	3.7	3.8	3.9	3.99	$\frac{0}{0}$

x	2.01	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	4.01	4.1	4.2	4.3	4.4	4.5	4.6

The graph of this function is shown in the figure,

It is evident from the above table and the graph of $f(x)$ that as x increases and comes closer to 2 from lefth and side of 2, the values of $f(x)$ increases and come closer to 4. This is interpreted as

When x approaches to 2 from its left hand side, the function $f(x)$ tends to the limit 4.

If we use notation '$x \to 2^-$' to denote 2 from left hand side, the above statement can be restated as

$x \to 2^-$, $f(x) \to 4$ or $\lim_{x \to 2} f(x) = 4$ or left hand limit of $f(x)$ at $x = 2$ is 4.

Thus, $\lim_{x \to 2} f(x) = 4$ means that as x tends to 2 from left hand side, $f(x)$ is tending to 4.

From the above table as well as the graph of $f(x)$ shown in the figure we observe that as x decreases and comes closer to 2 from right hand side 2, the values of $f(x)$ decreases and come closer to 4. This is interpreted as

When x approaches to 2 from its right hand side, the function $f(x)$ tends to the limit 4.

Using the notation '$x \to 2^+$', to denote 'x tends to from right hand side' the above statement can be restated as $x \to 2^+$, $f(x) \to 4$ or $\lim_{x \to 2^+} f(x) = 4$ or right hand limit of $f(x)$ at $x = 2$ is 4.

Thus, $\lim_{x \to 2^+} f(x) = 4$ means that as x tends to 2 from right hand side, $f(x)$ is tending to 4.

If follows from the above discussion that for the function $f(x)$ is given by

$$f(x) = \frac{x^2 - 4}{x - 2}, \text{ we have}$$

(i) $\lim_{x \to 2^-} f(x) = 4$

(ii) $\lim_{x \to 2^+} f(x) = 4$

(iii) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

(iv) $f(2)$ does not exist *i.e.*, $f(x)$ is not defined at $x = 2$.

Note The following indeterminant forms are

$$\frac{0}{0}, \infty - \infty, \frac{\infty}{\infty}, 0 \times \infty, 1^\infty, \infty, 0^0, \infty^0$$

18.2 Limit

If $f(x)$ is a function of x such that, if x approaches to a constant value a, then the value of $f(x)$ also approaches to another constant k, then constant k is known as limit of $f(x)$ at $x = a$. Limit is defined as

$$\lim_{x \to a} f(x) = k$$

OR

A real number l is called the limit of the function f, if for every $\varepsilon > 0$, however small, there exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$, whenever $0 < |x - a| < \delta$

or we write

$$\lim_{x \to a} f(x) = l$$

18.3 Left Hand Limit

A function f is said to approach l as x approaches a from the left, if corresponding to an arbitrary positive number ε, there exists a positive number δ such that

$|f(x) - l| < \varepsilon$ whenever $a - \delta < x < a$.

It is written as $\lim_{x \to a^-} f(x) = l$ or $f(a - 0) = l$.

The working rule for finding the left hand limit is put $a - h$ for x in $f(x)$, where h is positive and very small and make h approach zero.

i.e., $$f(a - 0) = \lim_{h \to 0} f(a - h)$$

18.4 Right Hand Limit

A function f is said to approach l as x approaches a from right, if corresponding to an arbitrary positive number ε, there exists a positive number δ such that $|f(x) - l| < \varepsilon$ whenever $a < x < a + \delta$.

It is written as $\lim_{x \to a^+} f(x) = l$ or $f(a + 0) = l$.

The working rule for finding the right hand limit is 'put $a + h$ for x in $f(x)$, where h is positive and very small and make h approach zero.

i.e., $$f(a + 0) = \lim_{h \to 0} f(a + h)$$

***Sample* Problem 1** *Find the left hand limit of* $\dfrac{\frac{1}{x} + \frac{1}{2}}{x + 2}$ *at* $x = -2$.

(*a*) $\frac{1}{3}$ (*b*) $-\frac{1}{4}$ (*c*) $-\frac{1}{3}$ (*d*) $\frac{1}{4}$

Interpret (*b*) Let $L = \lim_{x \to -2^-} \dfrac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Put $x = -2 - h$

As $x \to -2$, then $h \to 0$

$$\therefore \quad L = \lim_{h \to 0} \frac{\frac{1}{-2-h} + \frac{1}{2}}{-h}$$

$$= \lim_{h \to 0} \frac{2 - 2 - h}{-2h(-2-h)} = \lim_{h \to 0} \frac{-h}{-2h(-2-h)} = \lim_{h \to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}$$

18.5 Existence of Limit

If both right hand limit and left hand limit exist and are equal, then their common value, evidently will be the limit of f as $x \to a$ *i.e.*,

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$, then $\lim_{x \to a} f(x) = l$,

If however, either both of these limits do not exist or both these limits exist but are not equal in value, then $\lim_{x \to a} f(x)$ does not exist.

***Sample* Problem 2** *Find the limit of the function,*

$$f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases} \text{ at } x = 3$$

(*a*) –1 (*b*) 1
(*c*) –3 (*d*) does not exist

Interpret (*d*) Given, $f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}$ at $x = 3$.

Now, $\text{LHL } \lim_{x \to 3^-} \dfrac{|x-3|}{x-3} = \lim_{x \to 3} \dfrac{-(x-3)}{x-3}$

$= \lim_{x \to 3} (-1) = -1$

$\text{RHL } \lim_{x \to 3^+} \dfrac{|x-3|}{x-3} = \lim_{x \to 3} \dfrac{(x-3)}{x-3}$

$\lim_{x \to 3} 1 = 1$

$\therefore$ LHL $\neq$ RHL

$= 1$ So, limit does not exist.

18.6 Algebra of Limits

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then,

(i) Limit of sum of two functions is sum of limits of the functions,
i.e., $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(ii) Limit of difference of two functions is difference of the limits of the functions
i.e., $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(iii) Limit of product of two functions is product of the limits of the functions
i.e., $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(iv) Limit of a quotient of two functions is quotient of the limits of the functions (wherenever denominator is non-zero)
i.e., $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

***Sample* Problem 3** *The value of* $\lim_{x \to 3} (x^4 + x^3 - 3x)$ *is*

(*a*) 99 (*b*) – 99
(*c*) – 85 (*d*) 85

Interpret (*a*) $\lim_{x \to 3} (x^4 + x^3 - 3x)$

$= \lim_{x \to 3} x^4 + \lim_{x \to 3} x^3 - 3 \lim_{x \to 3} x$

$= (3)^4 + (3)^3 - (3)(3)$

$= 81 + 27 - 9 = 99$

18.7 Evaluation of Algebraic Limit

Let $f(x)$ be an algebraic function (polynomial or rational) and a be any real number, then $\lim_{x \to a} f(x)$ is known as an algebraic limit.

e.g., $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$, $\lim_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$ etc.

The limit of algebraic functions can be find by the following methods.

(i) Method of Direct Substitution

$\lim_{x \to a} f(x)$ can be evaluated by method of direct substitution, if $f(x)$ exists finitely for $x = a$

e.g., $\lim_{x \to 1} \dfrac{x^2 + 2x + 3}{x^2 + x + 1} = \dfrac{(1)^2 + 2(1) + 3}{(1)^2 + 1 + 1} = \dfrac{6}{3} = 2$

***Sample* Problem 4** *The value of* $\lim_{x \to 4} \dfrac{4x + 3}{x - 2}$ *is* **[NCERT]**

(*a*) 19 (*b*) $\dfrac{19}{2}$
(*c*) $\dfrac{19}{3}$ (*d*) $-\dfrac{19}{2}$

Interpret (*b*) $\lim_{x \to 4} \dfrac{4x + 3}{x - 2} = \dfrac{4 \times 4 + 3}{4 - 2} = \dfrac{19}{2}$

(ii) Method of Factorisation

Consider $\lim_{x \to a} \dfrac{f(x)}{g(x)}$. If by putting $x = a$ the rational function $\dfrac{f(x)}{g(x)}$ takes the form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ etc., then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. In such a case we factorise the numerator and denominator and then, cancel out the common factor $(x - a)$. After cancelling out the common factor $(x - a)$, we again put $x = a$ in the given expression and see whether, we get a meaningful number or not. This process is repeated till we get a meaningful number.

***Sample* Problem 5** $\lim_{x\to 2} \frac{3x^2 - x - 10}{x^2 - 4}$ *is equal to* [NCERT]

(a) $\frac{11}{4}$ (b) $-\frac{11}{4}$

(c) $\frac{11}{5}$ (d) *None of these*

Interpret (a) $\lim_{x\to 2} \frac{3x^2 - x - 10}{x^2 - 4}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\lim_{x\to 2} \frac{3x^2 - (6-5)x - 10}{(x-2)(x+2)} = \lim_{x\to 2} \frac{3x^2 - 6x + 5x - 10}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{3x(x-2) + 5(x-2)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{3x+5}{x+2}$$

$$= \frac{3\times 2 + 5}{2+2} = \frac{6+5}{4} = \frac{11}{4}$$

(iii) Method of Rationalisation

Rationalisation method is used when, we have radical signs in an expression (like $\frac{1}{2}, \frac{1}{3}$ etc.,) and there exists a negative sign between two terms of an algebraic expression.

After rationalisation, the terms are factorised which on cancellation gives the required result.

***Sample* Problem 6** *The value of* $\lim_{x\to a} \frac{\sqrt{3x-a} - \sqrt{x+a}}{x-a}$ *is*

(a) $-\frac{1}{\sqrt{2}a}$ (b) $\frac{1}{\sqrt{2}a}$

(c) $\frac{1}{\sqrt{3}a}$ (d) $\frac{1}{\sqrt{2}}$

Interpret (b) $\lim_{x\to a} \frac{\sqrt{3x-a} - \sqrt{x+a}}{x-a}$

$$= \lim_{x\to a} \frac{\sqrt{3x-a} - \sqrt{x+a}}{x-a} \times \frac{\sqrt{3x-a} + \sqrt{x+a}}{\sqrt{3x-a} + \sqrt{x+a}}$$

$$= \lim_{x\to a} \frac{(3x-a) - (x+a)}{(x-a)(\sqrt{3x-a} + \sqrt{x+a})}$$

$$= \lim_{x\to a} \frac{2(x-a)}{(x-a)(\sqrt{3x-a} + \sqrt{x+a})}$$

$$= \frac{2}{\sqrt{3a-a} + \sqrt{a+a}} = \frac{2}{2\sqrt{2}a} = \frac{1}{\sqrt{2}a}$$

(iv) Evaluation of Limits Using Standard Results

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} (x^{n-1} + x^{n-2}a + x^{n-3}a + \ldots + a^{n-1})$$

(using expansion)

$$\Rightarrow \quad \lim_{x\to a} \frac{x^n - a^n}{x-a} = a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1}$$

$$\therefore \quad \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$$

Discuss 2 example of $\lim_{x\to\infty}$

Note When $x \to \infty$, then if k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by x^k and use the result $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where $x > 0$.

***Sample* Problem 7** $\lim_{x\to 0} \frac{(x+1)^5 - 1}{x}$ *is equal to* [NCERT]

(a) 5 (b) 4

(c) 3 (d) – 5

Interpret (a) $\lim_{x\to 0} \frac{(x+1)^5 - 1}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

Let $x + 1 = h$

When $x \to 0$, then $h \to 1$

$$\therefore \quad \lim_{h\to 1} \frac{h^5 - 1}{h-1} = \lim_{h\to 1} \frac{h^5 - 1^5}{h-1} = \lim_{h\to 1} 5(h)^{5-1} = 5 \times 1^4 = 5$$

***Sample* Problem 8** *Find the value of* $\lim_{x\to\infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$

(a) $-\frac{a}{d}$ (b) $\frac{a}{d}$

(c) $\frac{d}{a}$ (d) *None of these*

Interpret (b) When we put $x = \infty$ in the given expression, then the expression becomes $\frac{\infty}{\infty}$. So, here we divide numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a+0+0}{d+0+0} = \frac{a}{d}$$

18.8 Evaluation of Trigonometric Limits

In order to evaluate the said type of limits, we use the following standard results

(i) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (ii) $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$

(iii) $\lim_{\theta\to a} \frac{\sin(\theta - a)}{\theta - a} = 1$ (iv) $\lim_{\theta\to a} \frac{\tan(\theta - a)}{\theta - a} = 1$

(v) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$ (vi) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

***Sample* Problem 9** *The value of* $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$ *is* [NCERT]

(a) – 2 (b) 2 (c) 3 (d) – 3

Interpret (b) Given, $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

Let $x - \frac{\pi}{2} = h$

when $x \to \frac{\pi}{2}$, then $h \to 0$

$$\text{Therefore, given limit} = \lim_{h \to 0} \frac{\tan 2\left(\frac{\pi}{2} + h\right)}{h} = \lim_{h \to 0} \frac{\tan(\pi + 2h)}{h}$$

$$= \lim_{h \to 0} \frac{\tan 2h}{h} \qquad (\because \tan(\pi + \theta) = \tan\theta)$$

$$= \lim_{h \to 0} \frac{2 \tan 2h}{2h}$$

$$= 2 \times 1 = 2 \qquad \left(\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right)$$

18.9 Some Useful Expansions

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(ii) $a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$

(iii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, |x| < 1$

(iv) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots$

(v) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

(vi) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

(vii) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

***Sample* Problem 10** *Find the value of* $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$.

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

Interpret (b) $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$

$$= \lim_{x \to 0} \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots - 1}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2\left(-\frac{1}{2!} + \frac{x^2}{4!} - \ldots\right)}{x^2} = -\frac{1}{2!} = -\frac{1}{2}$$

18.10 Derivative

Suppose, f is a real valued function, the function defined by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Hence, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sometimes $f'(x)$ is denoted by $\frac{d}{dx}[f(x)]$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y w.r.t. x. It is also denoted by $D[f(x)]$. Further the derivative of f at $x = a$ is also denoted by

$$\left(\frac{df}{dx}\right)_a \text{ or } \left(\frac{df}{dx}\right)_{x=a}$$

Geometrical Meaning of a Derivative

Let $P\{x_0, f(x_0)\}$ and $Q\{x_0 + h, f(x_0 + h)\}$ be two points very near to each other on the curve $y = f(x)$. Draw PM and QN perpendiculars from P and Q on x-axis PL perpendicular from P on QN. Let the chord QP produced meet the x-axis at R and $\angle QPL = \angle QRN = \phi$. Now, in right angled ΔQPL.

$$\tan\phi = \frac{QL}{PL} = \frac{NQ - NL}{MN} = \frac{NQ - MP}{ON - OM}$$

$$= \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} \qquad \ldots(i)$$

When $h \to 0$, the point Q moving along the curve tends to P *i.e.*, $Q \to P$, then the chord PQ approaches the tangent line PT at the point P and then $\phi \to \psi$. Now, applying $\lim_{h \to 0}$ in Eq. (i) , then

$$\lim_{h \to 0} \tan\phi = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\tan\psi = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\Rightarrow \qquad f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

18.11 Derivative from First Principle

Let $f(x)$ is a function finitely differentiable at every point on the real number line. Then, its derivative is given by the first principle is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

***Sample* Problem 11** *Find the derivative of a function* $f(x) = \frac{x+1}{x-1}$ *by the first principle.*

(a) $\frac{2}{(x-1)^2}$ (b) $\frac{-2}{(x-1)^2}$ (c) $\frac{-2}{x-1}$ (d)

Interpret (b) Let $f(x) = \frac{x+1}{x-1}$

We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h} \qquad \left[\because f(x) = \frac{x+1}{x-1}\right]$$

$$= \lim_{h \to 0} \frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{(x+h-1)(x-1)h}$$

$$= \lim_{h \to 0} \frac{(x^2 - x + hx - h + x - 1) - (x^2 + hx - x + x + h - 1)}{(x+h-1)(x-1)h}$$

$$= \lim_{h \to 0} \frac{x^2 + hx - h - 1 - x^2 - hx - h + 1}{(x+h-1)(x-1)h}$$

$$= \lim_{h \to 0} \frac{-2h}{(x+h-1)(x-1)h} = \frac{-2}{(x+0-1)(x-1)} = \frac{-2}{(x-1)^2}$$

18.12 Differentiation of Some Important Functions

(i) $\frac{d}{dx}(c) = 0$, c is independent of x

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x, \left\{x \neq n\pi + \frac{\pi}{2}\right\}$

(vi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, \{x \neq n\pi\}$

(vii) $\frac{d}{dx}(\sec x) = \sec x \tan x, \left\{x \neq n\pi + \frac{\pi}{2}\right\}$

(viii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, \{x \neq n\pi\}$

(ix) $\frac{d}{dx}(a^x) = a^x \log_e a$ (x) $\frac{d}{dx}(e^x) = e^x$

(xi) $\frac{d}{dx}(\log x) = \frac{1}{x}, (x > 0)$ (xii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

***Sample* Problem 12** *Find the derivative of* $(ax^2 + b)^2$. **[NCERT]**

(a) $-4a^2x^3 + 4axb$ (b) $4a^2x^3 + 4axb$

(c) $4a^2x^3 - 4axb$ (d) *None of these*

Interpret (b) Let $f(x) = (ax^2 + b)^2 = a^2x^4 + b^2 + 2ax^2b$

On differentiating w.r.t. x, we get $\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(a^2x^4 + b^2 + 2ax^2b)$

$$= 4a^2x^3 + 0 + 4axb = 4a^2x^3 + 4axb$$

18.13 Algebra of Derivative of Functions

Let f and g be two functions such that their derivatives are defined in a common domain. Then,

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

i.e., $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

(ii) Derivative of difference of two functions is the difference of the derivatives of two functions.

i.e., $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x) \cdot g(x)] = g(x)\frac{d}{dx}f(x) + f(x) \cdot \frac{d}{dx}g(x)$$

(iv) Derivative of quotient of two functions is given by the following quotient rule (whenever the denominator is non-zero)

$$\therefore \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{[g(x)]^2}$$

***Sample* Problem 13** *Find the derivative of* $(3 \cot x + 4 \operatorname{cosec} x)$. **[NCERT]**

(a) $-3 \operatorname{cosec}^2 x - 4 \operatorname{cosec} x \cot x$

(b) $3 \operatorname{cosec}^2 x - 5 \operatorname{cosec} x \cot x$

(c) $-3 \operatorname{cosec}^2 x + 5 \operatorname{cosec} x \cot x$

(d) *None of the above*

Interpret (a) Let $y = 3 \cot x + 4 \operatorname{cosec} x$

On differentiating w.r.t x, we get $\frac{d}{dx}(y) = \frac{d}{dx}(3 \cot x + 4 \operatorname{cosec}(x)$

$$= \frac{3d}{dx}(\cot x) + 4\frac{d}{dx}(\operatorname{cosec} x) = -3 \operatorname{cosec}^2 x - 4 \operatorname{cosec} x \cot x$$

WORKED OUT Examples

Example 1 $\lim_{x \to 4} \frac{|x-4|}{x-4}$ *is equal to* [NCERT Exemplar]

(a) 3 (b) –2
(c) 1 (d) *does not exist*

Solution (d) Let $f(x) = \frac{|x-4|}{x-4}$

At $x = 4$,

$$\text{RHL} = \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4+h) = \lim_{h \to 0} \frac{|4+h-4|}{(4+h-4)}$$

$$= \lim_{h \to 0} \frac{(4+h-4)}{(4+h-4)} = 1$$

At $x = 4$, $\text{LHL} = \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4-h)$

$$= \lim_{h \to 0} \frac{|4-h-4|}{(4-h-4)}$$

$$= \lim_{h \to 0} \frac{-(4-h-4)}{(4-h-4)} = -1$$

$\because$ RHL $\neq$ LHL

$\therefore$ At $x = 4$, limit does not exist.

Note In an absolute function, sometimes it may be possible that LHL $\neq$ RHL, Please be careful about it.

Example 2 *Find the value of* $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$. [NCERT Exemplar]

(a) –1 (b) 0
(c) 1 (d) 3

Solution (b) Given, $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

$$= \lim_{x \to 0} \frac{(\sin 5x + \sin x) - 2\sin 3x}{x}$$

$$= \lim_{x \to 0} \frac{2\sin\frac{5x+x}{2}\cos\frac{5x-x}{2} - 2\sin 3x}{x}$$

$$\left(\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right)$$

$$= \lim_{x \to 0} \frac{2\sin 3x\cos 2x - 2\sin 3x}{x}$$

$$= \lim_{x \to 0} \frac{2\sin 3x(\cos 2x - 1)}{x}$$

$$= \lim_{x \to 0} 2\left(\frac{\sin 3x}{3x}\right) \times 3\left[\frac{-(1-\cos 2x)}{1}\right]$$

$$= -2 \times 1 \times 3(1 - \cos 0) = -6(1-1) = -6 \times 0 = 0$$

Example 3 *If* $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, *then find the value of k.* [NCERT Exemplar]

(a) $\frac{5}{3}$ (b) $-\frac{8}{3}$
(c) $\frac{8}{3}$ (d) *None of these*

Solution (c) Given, $\lim_{x \to 1} \frac{x^4 - 1^4}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$

$$\Rightarrow \quad 4(1)^{4-1} = \frac{3}{2}(k)^{3-2} \qquad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

$$\Rightarrow \quad 4 = \frac{3}{2}k$$

$$\Rightarrow \quad k = \frac{8}{3}$$

Example 4 *Let* $f(x) = x - [x]$; $x \in R$, *then* $f'\left(\frac{1}{2}\right)$ *is* [NCERT Exemplar]

(a) $\frac{3}{2}$ (b) 1
(c) 0 (d) –1

Solution (b) Given, $f(x) = x - [x]$

On differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = 1 - 0 \quad \Rightarrow \quad f'(x) = 1$$

($\because [\cdot]$ greatest integer function is constant function)

At $x = \frac{1}{2}$, $f'\left(\frac{1}{2}\right) = 1$

Example 5 *Let* $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \le x < 3 \end{cases}$, *the quadratic equation whose roots are* $\lim_{x \to 2^-} f(x)$ *and* $\lim_{x \to 2^+} f(x)$ *is* [NCERT Exemplar]

(a) $x^2 - 6x + 9 = 0$ (b) $x^2 - 7x + 8 = 0$
(c) $x^2 - 14x + 49 = 0$ (d) $x^2 - 10x + 21 = 0$

Solution (*d*) Given, $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \le x < 3 \end{cases}$

At $x = 2$, RHL $= \lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h)$

$= \lim_{h \to 0} 2(2 + h) + 3$

$= 2(2 + 0) + 3 = 4 + 3 = 7 = \alpha$ (say) $[\because f(x) = 2x + 3]$

LHL $= \lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h)$

$= \lim_{h \to 0} (2 - h)^2 - 1 = (2 - 0)^2 - 1$

$= 4 - 1 = 3 = \beta$ (say) $[\because f(x) = x^2 - 1]$

If a quadratic equation has roots α and β, then the equation is

$x^2 -$ (Sum of roots) $x +$ Product of roots $= 0$

i.e., $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e., $x^2 - (7 + 3)x + 7 \times 3 = 0 \Rightarrow x^2 - 10x + 21 = 0$

Example 6 *Find* $\lim_{x \to 0} f(x)$ *and* $\lim_{x \to 1} f(x)$, *where*

$$f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x + 1), & x > 0 \end{cases}$$

[**NCERT Exemplar**]

(*a*) 3, 5 (*b*) 3, 6
(*c*) 4, 7 (*d*) 3, –6

Solution (*b*) We have, $f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x + 1), & x > 0 \end{cases}$

At $x = 0$,

RHL $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} 3(0 + h + 1)$

$= 3(0 + 0 + 1) = 3 \times 1 = 3$

LHL $= \lim_{h \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$= \lim_{h \to 0} 2(0 - h) + 3 = 2(0 - 0) + 3 = 3$

$\Rightarrow$ RHL = LHL = 3

At $x = 1$, RHL $= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h)$

$= \lim_{h \to 0} 3(1 + h + 1)$ (putting $x = 1 + h$)

$= 3(1 + 0 + 1) = 6$

LHL $= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h)$ (putting $x = 1 - h$)

$= \lim_{h \to 0} 3(1 - h + 1) = 6$

$\therefore$ LHL = RHL = 6

Start Practice for JEE Main

Round I (Topically Divided Problems)

Limits

1. The value of $\lim_{x \to 1} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$ is [NCERT Exemplar]

(a) 20 (b) 21
(c) 15 (d) 17

2. The value of $\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$ is [NCERT Exemplar]

(a) $-\frac{8}{5}$ (b) $\frac{7}{5}$
(c) $\frac{8}{5}$ (d) None of these

3. The value of $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ is [NCERT Exemplar]

(a) 0 (b) 1
(c) –1 (d) 3

4. The value of $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$ is

(a) 8 (b) –8
(c) 7 (d) 5

5. The value of $\lim_{x \to \frac{1}{2}} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right)$ is [NCERT Exemplar]

(a) $\frac{7}{3}$ (b) $-\frac{7}{2}$
(c) $-\frac{7}{3}$ (d) $\frac{7}{2}$

6. Find n, if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$, $n \in N$. [NCERT Exemplar]

(a) 5
(b) 6
(c) 4
(d) 7

7. The value of $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$ is [NCERT Exemplar]

(a) $\frac{1}{2}$ (b) $-\frac{1}{4}$
(c) $\frac{1}{4}$ (d) $-\frac{1}{2}$

8. $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$ is equal to [NCERT Exemplar]

(a) 2 (b) –2
(c) 3 (d) –3

9. $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$ is equal to [NCERT Exemplar]

(a) –1 (b) 1
(c) 2 (d) –2

10. $\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$ is equal to [NCERT Exemplar]

(a) 3 (b) –2
(c) 2 (d) 4

11. $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ is equal to [NCERT Exemplar]

(a) $-\sqrt{a}\cos a$ (b) $\sqrt{a}\cos a$
(c) $-2\sqrt{a}\cos a$ (d) $2\sqrt{a}\cos a$

12. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ is equal to [NCERT Exemplar]

(a) 3 (b) 4
(c) 2 (d) 1

13. The value of $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ is [NCERT Exemplar]

(a) $\frac{1}{2\sqrt{3}}$ (b) $-\frac{1}{2\sqrt{2}}$
(c) $\frac{1}{2\sqrt{2}}$ (d) None of these

14. $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ is equal to [NCERT Exemplar]

(a) 1 (b) 2
(c) –1 (d) –2

15. $\lim_{x \to 0} \frac{x^2 \cos x}{1 - \cos x}$ is equal to [NCERT Exemplar]

(a) 2 (b) $\frac{3}{2}$
(c) $\frac{-3}{2}$ (d) 1

16. $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$ is equal to [NCERT Exemplar]

(a) n (b) 1
(c) $-n$ (d) 0

17. $\lim_{x \to 1} \frac{x^m - 1}{x^n - 1}$ is equal to [NCERT Exemplar]

(a) 1 (b) $\frac{m}{n}$
(c) $-\frac{m}{n}$ (d) $\frac{m^2}{n^2}$

18. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is equal to [NCERT Exemplar]

(a) $\frac{4}{9}$ (b) $\frac{1}{2}$
(c) $\frac{-1}{2}$ (d) –1

19. $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is equal to [NCERT Exemplar]

(a) $\frac{-1}{2}$ (b) 1
(c) $\frac{1}{2}$ (d) 1

20. $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is [NCERT Exemplar]

(a) 2 (b) 0 (c) 1 (d) –1

21. $\lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ is [NCERT Exemplar]

(a) $\frac{1}{10}$ (b) $\frac{-1}{10}$
(c) 1 (d) None of these

22. $\lim_{x \to 4} \frac{|x - 4|}{x - 4}$ is equal to

(a) 1 (b) –1
(c) does not exist (d) None of these

23. $\lim_{x \to 0} \frac{|\sin x|}{x}$ is [NCERT Exemplar]

(a) 1 (b) –1
(c) does not exist (d) None of these

24. $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ is [NCERT Exemplar]

(a) 2 (b) $\frac{1}{2}$
(c) $\frac{-1}{2}$ (d) $\frac{1}{4}$

25. Let $f(x) = \begin{cases} x + 2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$, find c, if $\lim_{x \to -1} f(x)$ exists. [NCERT Exemplar]

(a) –1 (b) 1 (c) 0 (d) 2

Derivatives

26. The derivative of $\left(x + \frac{1}{x}\right)^3$ is [NCERT Exemplar]

(a) $3x^2 + \frac{3}{x^4} + 3 + \frac{3}{x^2}$ (b) $3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}$
(c) $3x^2 + \frac{3}{x^4} + 3 - \frac{3}{x^2}$ (d) None of these

27. The derivative of $(3x + 5)(1 + \tan x)$ is [NCERT Exemplar]

(a) $3x \sec^2 x - 6 \sec^2 x - 3 + 3 \tan x$
(b) $3x \sec^2 x - 5 \sec^2 x - 3 - 3 \tan x$
(c) $3x \sec^2 x + 5 \sec^2 x + 3 + 3 \tan x$
(d) None of the above

28. The derivative of $\frac{x^5 - \cos x}{\sin x}$ is [NCERT Exemplar]

(a) $\frac{5x^4 \sin x - 1 - x^5 \cos x}{\sin^2 x}$ (b) $\frac{5x^4 \sin x + 1 - x^5 \cos x}{\sin^2 x}$
(c) $\frac{5x^4 \sin x - 1 - x^3 \cos x}{\sin^2 x}$ (d) None of these

29. The derivative of $(ax^2 + \cot x)(p + q \cos x)$ is [NCERT Exemplar]

(a) $ax^2 \cot x(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$
(b) $ax^2 \cot x(q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$
(c) $ax^2 \cot x(-q \sin x) + (p - q \cos x)(2ax + \operatorname{cosec}^2 x)$
(d) None of the above

30. The derivative of $(x^2 \sin x + \cos 2x)$ is [NCERT Exemplar]

(a) $x^2 \cos x + 2x \sin x - \sin 2x$
(b) $x^2 \cos x + 2x \sin x - 2 \sin 2x$
(c) $x^2 \cos x + x \sin x - 2 \sin 2x$
(d) $x^2 \cos x + 2x \sin x + 2 \sin 2x$

31. The derivative of $\sin^3 x \cos^3 x$ is [NCERT Exemplar]

(a) $\frac{3}{5} \sin^2 2x \cos 2x$ (b) $\frac{3}{4} \sin^2 2x \cos 2x$
(c) $\frac{3}{4} \sin 2x \cos^2 2x$ (d) $\frac{3}{2} \sin^2 2x \cos 2x$

32. Find the differentiation of $x^{2/3}$ by using first principle. [NCERT Exemplar]

(a) $\frac{2}{3}x^{-1/3}$ (b) $\frac{-2}{3}x^{-1/3}$

(c) $\frac{2}{3}x^{1/3}$ (d) None of these

33. Find the differentiation of $\cos(x^2+1)$ by using first principle. [NCERT Exemplar]

(a) $2x\sin(x^2+1)$ (b) $-2x\sin(x^2+1)$

(c) $-2x\sin(x^2-1)$ (d) None of these

34. Find the differentiation of $x\cos x$ by using first principle. [NCERT Exemplar]

(a) $-x\sin x+\cos x$ (b) $x\sin x+\cos x$

(c) $-x\cos x-\cos x$ (d) None of these

35. If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x=1$ is [NCERT Exemplar]

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

36. If $f(x)=\frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is [NCERT Exemplar]

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

37. If $y=\dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then $\frac{dy}{dx}$ is [NCERT Exemplar]

(a) $\frac{-4x}{(x^2-1)^2}$ (b) $\frac{-4x}{x^2-1}$

(c) $\frac{1-x^2}{4x}$ (d) $\frac{4x}{x^2-1}$

38. If $y=\frac{\sin x+\cos x}{\sin x-\cos x}$, then $\frac{dy}{dx}$ at $x=0$ is [NCERT Exemplar]

(a) -2 (b) 0

(c) $\frac{1}{2}$ (d) does not exist

39. If $y=\frac{\sin(x+9)}{\cos x}$, then $\frac{dy}{dx}$ at $x=0$ is [NCERT Exemplar]

(a) cos 9 (b) sin 9 (c) 0 (d) 1

40. If $f(x)=1+x+\frac{x^2}{2}+\ldots+\frac{x^{100}}{100}$, then $f'(1)$ is equal to [NCERT Exemplar]

(a) $\frac{1}{100}$ (b) 100

(c) does not exist (d) 0

Answers

1. (b)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (c)
11. (d)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (c)
21. (b)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (b)
31. (b)	**32.** (a)	**33.** (b)	**34.** (a)	**35.** (d)	**36.** (a)	**37.** (a)	**38.** (a)	**39.** (a)	**40.** (b)

the **Guidance**

1. $\lim_{x\to 1}\frac{(1+x)^6-1}{(1+x)^2-1}=\frac{(1+1)^6-1}{(1+1)^2-1}$

$$=\frac{2^6-1}{2^2-1}=\frac{64-1}{4-1}=\frac{63}{3}=21$$

2. $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}=\lim_{x\to\sqrt{2}}\frac{(x+\sqrt{2})(x-\sqrt{2})(x^2+2)}{(x-\sqrt{2})(x+4\sqrt{2})}$

$$=\lim_{x\to\sqrt{2}}\frac{(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})}$$

$$=\frac{(\sqrt{2}+\sqrt{2})(2+2)}{(\sqrt{2}+4\sqrt{2})}$$

$$=\frac{8\sqrt{2}}{5\sqrt{2}}=\frac{8}{5}$$

3. $\lim_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}=\lim_{x\to 1}\frac{(x-1)(x^6+x^5-x^4-x^3-x^2-x-1)}{(x-1)(x^2-2x-2)}$

$$=\lim_{x\to 1}\frac{x^6+x^5-x^4-x^3-x^2-x-1}{x^2-2x-2}$$

$$=\frac{1^6+1^5-1^4-1^3-1^2-1-1}{1^2-2(1)-2}$$

$$=\frac{-3}{-3}=1$$

4. $\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}=\lim_{x\to 2}\frac{x^2-4}{(\sqrt{3x-2}-\sqrt{x+2})}$

$$\times\frac{\sqrt{3x-2}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x-2} + \sqrt{x+2})}{(\sqrt{3x-2})^2 - (\sqrt{x+2})^2}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x-2} + \sqrt{x+2})}{(3x-2) - (x+2)}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x-2} + \sqrt{x+2})}{2x-4}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)(\sqrt{3x-2} + \sqrt{x+2})}{2(x-2)}$$

$$= \lim_{x \to 2} \frac{(x+2)(\sqrt{3x-2} + \sqrt{x+2})}{2}$$

$$= \frac{(2+2)(\sqrt{6-2} + \sqrt{2+2})}{2} = \frac{4(2+2)}{2} = 8$$

5. $\lim_{x \to 1/2} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right)$

$$= \lim_{x \to 1/2} \left(\frac{(8x-3)(2x+1) - (4x^2+1)}{(2x-1)(2x+1)} \right)$$

$$= \lim_{x \to 1/2} \left(\frac{16x^2 + 2x - 3 - 4x^2 - 1}{(2x-1)(2x+1)} \right)$$

$$= \lim_{x \to 1/2} \left(\frac{12x^2 + 2x - 4}{(2x-1)(2x+1)} \right)$$

$$= \lim_{x \to 1/2} \left(\frac{2(6x^2 + x - 2)}{(2x-1)(2x+1)} \right)$$

$$= 2 \lim_{x \to 1/2} \frac{6x^2 + 4x - 3x - 2}{(2x-1)(2x+1)}$$

$$= 2 \lim_{x \to 1/2} \frac{(3x+2)(2x-1)}{(2x-1)(2x+1)} = 2 \lim_{x \to 1/2} \frac{3x+2}{2x+1}$$

$$= 2 \left(\frac{3 \times \frac{1}{2} + 2}{2 \times \frac{1}{2} + 1} \right) = 2 \left(\frac{7/2}{1+1} \right) = \frac{7}{2}$$

6. Given, $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$

$$\Rightarrow \quad n(2)^{n-1} = 80 \quad \Rightarrow \quad n \times 2^{n-1} = 5 \times 2^4$$

$$\left[\because \lim_{x \to a} \left[\frac{x^n - a^n}{x-a} \right] = na^{n-1} \right]$$

$$\Rightarrow \quad n \times 2^{n-1} = 5 \times 2^{5-1}$$

On comparing, we get

$$n = 5$$

7. $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x} = \lim_{x \to 0} \frac{\sin^2 2x}{(2x)^2} \times \frac{(4x)^2}{\sin^2 4x} \times \frac{1}{4}$

$$= \frac{1}{4} \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \right)^2 \times \lim_{x \to 0} \left(\frac{4x}{\sin 4x} \right)^2$$

$$= \frac{1}{4} \times 1 \times 1 = \frac{1}{4} \qquad \left(\because \lim_{x \to a} \frac{\sin ax}{ax} = 1 \right)$$

8. $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} = \lim_{x \to 0} 2 \left(\frac{\sin x}{x} \right)^2 = 2 \times 1 = 2$

9. $\lim_{x \to 0} \frac{2 \sin x - \sin 2x}{x^3}$

$$= \lim_{x \to 0} \frac{2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right) - \left\{ 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots \right\}}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{2x^3}{3!} + \frac{2x^5}{5!} - \ldots \left(+ \frac{8x^3}{3!} - \frac{32x^5}{5!} + \ldots \right)}{x^3}$$

$$= \lim_{x \to 0} -\frac{2}{3!} + \frac{2}{5!} x^2 - \ldots \left(+ \frac{8}{3!} - \frac{32}{5!} x^3 - \ldots \right)$$

$$= -\frac{2}{3!} + 0 - \ldots \left(+ \frac{8}{3!} - 0 + \ldots \right) = -\frac{2}{6} + \frac{8}{6} = \frac{6}{6} = 1$$

10. Given, $\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$

Let $x - \frac{\pi}{6} = h$, when $x \to \frac{\pi}{6} \Rightarrow h \to 0$

$$\therefore \quad \lim_{h \to 0} \frac{\sqrt{3} \sin\left(\frac{\pi}{6} + h \right) - \cos\left(\frac{\pi}{6} + h \right)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3}\left(\sin \frac{\pi}{6} \cos h + \cos \frac{\pi}{6} \sin h \right) - \left(\cos \frac{\pi}{6} \cos h - \sin \frac{\pi}{6} \sin h \right)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3}\left(\frac{1}{2} \cos h + \frac{\sqrt{3}}{2} \sin h \right) - \left(\frac{\sqrt{3}}{2} \cos h - \frac{1}{2} \sin h \right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{3}}{2} \cos h + \frac{3}{2} \sin h - \frac{\sqrt{3}}{2} \cos h + \frac{1}{2} \sin h}{h}$$

$$= \lim_{h \to 0} \frac{2 \sin h}{h} = 2 \times 1 = 2$$

11. Given, $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

On multiplying and dividing by $\sqrt{x} + \sqrt{a}$, we get

$$\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \lim_{x \to a} \frac{(\sin x - \sin a)(\sqrt{x} + \sqrt{a})}{x - a}$$

Let $x - a = h$, when $x \to a \Rightarrow h \to 0$,

$$\therefore \lim_{h \to 0} \frac{[\sin(a+h) - \sin a][\sqrt{a+h} + \sqrt{a}]}{h}$$

$$= \lim_{h \to 0} \frac{2 \cos \frac{a+h+a}{2} \sin\left(\frac{a+h-a}{2} \right) (\sqrt{a+h} + \sqrt{a})}{h}$$

$$\left(\because \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \right)$$

$$= \lim_{h \to 0} \frac{2\cos\frac{2a+h}{2}\left(\sin\frac{h}{2}\right)(\sqrt{a+h}+\sqrt{a})}{2\left(\frac{h}{2}\right)}$$

$$= \frac{2\cos\left(\frac{2a+0}{2}\right)(\sqrt{a+0}+\sqrt{a})}{2} \qquad \left(\because \lim_{h \to 0} \frac{\sin h/2}{h/2} = 1\right)$$

$$= 2\sqrt{a}\cos a$$

12. Given, $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2} \qquad (\because \operatorname{cosec}^2 x - \cot^2 x = 1)$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x \to \frac{\pi}{6}} \frac{(\operatorname{cosec} x - 2)(\operatorname{cosec} x + 2)}{(\operatorname{cosec} x - 2)}$$

(by factorization)

$$= \lim_{x \to \frac{\pi}{6}} (\operatorname{cosec} x + 2) = \operatorname{cosec}\frac{\pi}{6} + 2 = 2 + 2 = 4$$

13. $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x} = \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{2}\cos^2\frac{x}{2}}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(1 - \cos^2\frac{x}{2}\right)}{\sin^2 x} = \lim_{x \to 0} \frac{\sqrt{2} \times \sin^2\frac{x}{2}}{\sin^2 x}$$

$$= \lim_{x \to 0} \sqrt{2} \times \frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2} \times \frac{x^2}{\sin^2 x} \times \frac{1}{4}$$

$$= \frac{\sqrt{2}}{4} \lim_{x \to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \frac{1}{2\sqrt{2}} \times 1 \times 1 = \frac{1}{2\sqrt{2}}$$

14. $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$

Put $x = \pi + h$ as $x \to \pi, h \to 0$

$$\therefore \lim_{h \to 0} \frac{\sin(\pi + h)}{h}$$

$$= \lim_{h \to 0} \frac{-\sin h}{h} = -(1) = -1$$

15. $\lim_{x \to 0} \frac{x^2 \cos x}{1 - \cos x}$

$$= \lim_{x \to 0} \frac{x^2 \cos x}{2\sin^2\frac{x}{2}} = \frac{1}{2} \lim_{x \to 0} \frac{\left(\frac{x}{2}\right)^2}{\sin^2\frac{x}{2}} \times 4 \lim_{x \to 0} \cos x$$

$$= 2 \times 1 \times 1 = 2$$

16. $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$

$$= \lim_{x \to 0} \frac{1 + nx + \frac{n(n-1)}{2}x^2 + \ldots - 1}{x}$$

$$= \lim_{x \to 0} \frac{nx\left(1 + \frac{n-1}{2}x + \ldots\right)}{x} = n(1 + 0 + \ldots) = n$$

17. $\lim_{x \to 1} \frac{x^m - 1}{x^n - 1}$

$$= \lim_{x \to 1} \frac{x^m - 1}{m} \times \frac{n}{(x^n - 1)} \times \frac{m}{n}$$

$$= 1 \times 1 \times \frac{m}{n} = \frac{m}{n} \qquad \left(\because \lim_{x \to 1} \frac{x^m - 1}{m} = 1\right)$$

18. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \to 0} \frac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$$= \lim_{\theta \to 0} \frac{\sin^2 2\theta}{(2\theta)^2} \times \frac{(3\theta)^2}{\sin 3\theta} \times \frac{4}{9} = 1 \times 1 \times \frac{4}{9} = \frac{4}{9}$$

19. We have, $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2\frac{x}{2}}{x\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)} = \lim_{x \to 0} \frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}} = \lim_{x \to 0} \frac{\tan\frac{x}{2}}{2 \cdot \frac{x}{2}}$$

$$= \frac{1}{2} \times 1 = \frac{1}{2} \qquad \left\{\because \lim_{x \to 0}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right) = 1\right\}$$

20. $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} \times \frac{\sqrt{x+1} \times \sqrt{1-x}}{\sqrt{x+1} + \sqrt{1-x}}$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{(\sqrt{x+1})^2 - (\sqrt{1-x})^2}$$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{x + 1 - (1-x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{2x} \times \lim_{x \to 0} (\sqrt{x+1} + \sqrt{1-x})$$

$$= \frac{1}{2} \times 1 \times (\sqrt{0+1} + \sqrt{1-0}) = \frac{1}{2} \times (2) = 1$$

21. $\lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{(x - 1)(2x + 3)}$

$$= \lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{(\sqrt{x} - 1)(\sqrt{x} + 1)(2x + 3)}$$

$$= \lim_{x \to 1} \frac{2x - 3}{(\sqrt{x} + 1)(2x + 3)} = \frac{2 - 3}{(1 + 1)(2 + 3)} = -\frac{1}{10}$$

22. We know that, $|x-4| = \begin{cases} x-4, & x \geq 4 \\ -(x-4), & x < 4 \end{cases}$

$$\text{LHL} = \lim_{x \to 4^-} \frac{|x-4|}{x-4}$$

$$= \lim_{x \to 4} \frac{-(x-4)}{x-4} = -1$$

$$\text{RHL} = \lim_{x \to 4^+} \frac{|x-4|}{x-4} = \lim_{x \to 4} \frac{x-4}{x-4} = 1$$

$\therefore$ LHL $\neq$ RHL

Hence, limit does not exist.

23. $\lim_{x \to 0} \frac{|\sin x|}{x}$

Since, $|\sin x| = \begin{cases} -\sin x, & x < 0 \\ \sin x, & x \geq 0 \end{cases}$

$\therefore$ $\text{LHL} = \lim_{x \to 0^-} \frac{-\sin x}{x} = -1$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$\therefore$ LHL $\neq$ RHL

Hence, limit does not exist.

24. $$\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \frac{2x + \frac{(2x)^3}{3} + \frac{(2x)^5}{15} + \ldots - x}{3x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}$$

$$= \lim_{x \to 0} \frac{x + \frac{(2x)^3}{3} + \frac{(2x)^5}{15} + \ldots}{2x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots}$$

$$= \lim_{x \to 0} \frac{1 + \frac{2^3 x^2}{3} + \frac{2^5 x^4}{15} + \ldots}{2 + \frac{x^2}{3!} - \frac{x^4}{5!} + \ldots} = \frac{1}{2}$$

25. Given, $f(x) = \begin{cases} x+2, & x \leq -1 \\ cx^2, & x > -1 \end{cases}$

Since, at $x = -1$, limit exists

$\therefore$ RHL = LHL

$\Rightarrow$ $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\Rightarrow$ $\lim_{h \to 0} f(-1+h) = \lim_{h \to 0} f(-1-h)$

$\Rightarrow$ $\lim_{h \to 0} c(-1+h)^2 = \lim_{h \to 0} -1-h+2$

$\Rightarrow$ $c(-1+0)^2 = +1-0$

$\Rightarrow$ $c = 1$

26. Let $y = \left(x + \frac{1}{x}\right)^3$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)^3$$

$$= \frac{d}{dx}\left\{x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)\right\}$$

$$= \frac{d}{dx}\left(x^3 + \frac{1}{x^3} + 3x + \frac{3}{x}\right) = 3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}$$

27. Let $y = (3x+5)(1+\tan x)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (3x+5)\frac{d}{dx}(1+\tan x) + (1+\tan x)\frac{d}{dx}(3x+5)$$

$$= (3x+5)(0+\sec^2 x) + (1+\tan x)(3)$$

$$= 3x\sec^2 x + 5\sec^2 x + 3 + 3\tan x$$

28. Let $y = \frac{x^5 - \cos x}{\sin x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x(5x^4 + \sin x) - (x^5 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + 1 - x^5 \cos x}{\sin^2 x}$$

29. Let $y = (ax^2 + \cot x)(p + q\cos x)$

$$\frac{dy}{dx} = (ax^2 + \cot x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \cot x)$$

$$= (ax^2 + \cot x)(0 - q\sin x) + (p + q\cos x)(2ax - \operatorname{cosec}^2 x)$$

$$= (ax^2 \cot x)(-q\sin x) + (p + q\cos x)(2ax - \operatorname{cosec}^2 x)$$

30. Let $y = x^2 \sin x + \cos 2x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x) + \frac{d}{dx}(\cos 2x)$$

$$= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) + (-2\sin 2x)$$

$$= x^2 \cos x + 2x \sin x - 2\sin 2x$$

31. Let $y = \sin^3 x \cos^3 x \times \frac{8}{8} = \frac{(\sin 2x)^3}{8}$

$$\frac{dy}{dx} = \frac{1}{8} 3(\sin^2 2x)\frac{d}{dx}(\sin 2x)$$

$$= \frac{3}{8}(\sin^2 2x) \times \cos 2x(2) = \frac{3}{4}\sin^2 2x \cos 2x$$

32. Let $f(x) = x^{2/3}$

We know by first principle, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore$ $f'(x) = \lim_{h \to 0} \frac{(x+h)^{2/3} - x^{2/3}}{h}$

$\therefore$ $$\lim_{h \to 0} \frac{\left(x^{2/3} + \frac{2}{3}x^{-1/3}h + \ldots\right) - x^{2/3}}{h} = \frac{2}{3x^{1/3}}$$

33. Let $f(x) = \cos(x^2 + 1)$

We know by first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{\cos[(x+h)^2 + 1] - \cos(x^2 + 1)}{h}$$

$$[\because f(x) = \cos(x^2 + 1)]$$

$$= \lim_{h \to 0} \frac{-2 \sin \frac{(x+h)^2 + 1 + x^2 + 1}{2} \sin \frac{(x+h)^2 + 1 - (x^2 + 1)}{2}}{h}$$

$$\left(\because \cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}\right)$$

$$= \lim_{h \to 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \sin \frac{(x+h)^2 - x^2}{2}}{h}$$

$$= \lim_{h \to 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \left[\sin \frac{(x+h)^2 - x^2}{2}\right]}{h \times \frac{(x+h)^2 - x^2}{2}} \times \left[\frac{(x+h)^2 - x^2}{2}\right]$$

$$= \lim_{h \to 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \times \frac{(x+h+x)(x+h-x)}{2}}{h}$$

$$= \lim_{h \to 0} \frac{-2 \sin \left[\frac{(x+h)^2 + x^2 + 2}{2}\right] (2x+h) \frac{h}{2}}{h}$$

$$= -2 \sin\left(\frac{2x^2 + 2}{2}\right) \times 2x \times \frac{1}{2} = -2x \sin(x^2 + 1)$$

34. Let $f(x) = x \cos x$

$\therefore$ We know by first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{(x+h)\cos(x+h) - x\cos x}{h} \quad [\because f(x) = x \cos x]$$

$$= \lim_{h \to 0} \frac{x\cos(x+h) + h\cos(x+h) - x\cos(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{x[\cos(x+h) - \cos x]}{h} + \frac{h\cos(x+h)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{-x2 \sin \frac{x+h+x}{2} \sin \frac{x+h-x}{2}}{h} + \cos(x+h)\right]$$

$$= \lim_{h \to 0} \left[\frac{-2x \sin \frac{2x+h}{2} \sin \frac{h}{2}}{2 \cdot \frac{h}{2}} + \cos(x+h)\right]$$

$$= -x \sin x + \cos x$$

35. Given, $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

On differentiating w.r.t. x, we get $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

At $x = 1$,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1}} - \frac{1}{2(1)^{3/2}} = \frac{1}{2} - \frac{1}{2} = 0$$

36. Given, $f(x) = \frac{x-4}{2\sqrt{x}}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\sqrt{x}(1) - (x-4)\frac{1}{2\sqrt{x}}}{2(\sqrt{x})^2}$$

$$\therefore \quad f'(1) = \frac{\sqrt{1}(1) - (1-4)\frac{1}{2\sqrt{1}}}{2(\sqrt{1})^2} = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4}$$

37. Given, $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} \Rightarrow y = \frac{x^2 + 1}{x^2 - 1}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

38. Given, $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= -\frac{(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

At $x = 0$

$$\frac{dy}{dx} = -\frac{-(\sin 0 - \cos 0)^2 - (\sin 0 + \cos 0)^2}{(\sin 0 - \cos 0)^2}$$

$$= \frac{-(0-1)^2 - (0+1)^2}{(0-1)^2} = \frac{-1-1}{1} = -2$$

39. Given, $y = \frac{\sin(x+9)}{\cos x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\cos x \cos(x+9) - \sin(x+9)\sin x}{(\cos x)^2}$$

$$= \frac{\cos(x+9+x)}{(\cos x)^2} = \frac{\cos(2x+9)}{(\cos x)^2}$$

At $x = 0$,

$$\frac{dy}{dx} = \frac{\cos 9}{(\cos 0)^2} = \cos 9$$

40. Given, $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$

On differentiating w.r.t. x, we get

$$f'(x) = 0 + 1 + \frac{2x}{2} + \ldots + \frac{100x^{99}}{100}$$

$$= 1 + x + x^2 + \ldots + x^{99}$$

$$\therefore \quad f'(x) = 1 + 1 + 1^2 + \ldots + 1^{99} = 100$$

19 Mathematical Reasoning

JEE Main MILESTONE

- Sentence
- Statement
- Open Statement
- Truth Table
- Logical Connectives or Sentencial Connectives
- Elementary Operations of Logic
- Converse, Inverse and Contrapositive of an Implication
- Tautology
- Contradiction

19.1 Sentence

A sentence is a relatively independent grammatical unit. It can stand alone or it can be combined with other sentences to form a text, a story etc.

There are following types of statements.

(i) Declarative/Assertive Sentence

A declarative sentence 'declares' or states a fact, arrangement or opinion. Declarative sentence can be either positive or negative.

e.g., Gopal Pandey is an intelligent student.

(ii) Imperative Sentence

The imperative commands (or sometimes requests). The imperative takes no subject as 'you' is the implied subject.

The imperative form ends with a period (.) or an exclamation point (!)

e.g., Please give me a cold drink.

(iii) Exclamatory Sentence

In this type of sentence, there is a sudden burst of feeling. The feelings may be full of joys, sorrows etc.

e.g., Hurrah ! We win hockey match.

(iv) Interrogative Sentence

A sentence that asks a question is an interrogative sentence. The interrogative form ends with a question mark (?).

e.g., Do you think that you win the prize today?

Reasoning is the foundation of mathematics. As science verifies through observation, mathematics relies on logic. Mathematical reasoning includes the kind of instinct exhibited by a good Engineer who finds solutions that work without worrying about the formal proof.

(v) Optative Sentence

A sentence that expresses a wish is an optative sentence.

e.g., Wish you best of luck !

19.2 Statement (Proposition)

We convey our daily views in the form of sentence which is a collection of words. This collection of words is called statements, if it has some sense. Therefore,

"A declarative sentence, whose truth or falsity can be decided is called a statement of logical sentence but the sentence should not be imperative, interrogative and exclamatory."

Statements are denoted by $p, q, r, ..$ etc.

e.g., "Delhi is the capital of India" is a statement, while "Do your work", is not a statement. There are following types of statements.

Note The working nature of statement in logic is same as nature of switch in circuit.

i.e., Switch — ON (1), OFF (0) and Statement — True (T), False (F)

(i) Simple Statement

A statement, if cannot be broken into two or more sentences, is a simple statement. The truth value of the simple statement does not explicitly depend on any other statement. Generally, small letters $p, q, r, \ldots$ denote simple statements.

(ii) Compound Statement

A statement formed by two or more simple statement by the words such as "and", "or", "not", "if then", "if and only if", then the resulting statement is called a compound statement. This is also called Logical connectives.

Note Two circuits can be connected either by series or parallel.

i.e., Two statements can be connected either by '$\wedge$' or '$\vee$'.

(iii) Substatements

Simple statements which when combined form a compound statement are called substatements, also called components.

Note

- A true statement is known as a valid statement.
- A false statement is known as an invalid statement.
- Imperative, exclamatory, interrogative, optative sentences are not statements.
- Mathematical identities are considered to be statements because they can either be true or false but not both.

***Sample* Problem 1** *Which of the following sentence is/are not statement?* **[NCERT]**

(a) *The sum of* 5 *and* 7 *is greater then* 10.
(b) *Answer this question*
(c) *The product of* (– 1) *and* 8 *is* 8.
(d) *All real numbers are complex numbers.*

Interpret (*b*)

(a) It is true that sum of 5 and 7 is greater than 10. Hence, it is a statement.
(b) It is an order. Hence, it is not a statement.
(c) It is true that sum of interior angles of a triangle is 180°. Hence, it is a statement.
(d) It is true that all real numbers are complex numbers. All real number can be expressed as $a + ib$. Hence, it is a statement.

19.3 Open Statement

A sentence which contains one or more variables such that when certain values are given to the variable it becomes a statement, is called an open statement.

eg, "He is a great man" is an open sentence because in this sentence "He" can be replaced by any person.

***Sample* Problem 2** *Which of the following statements is/are open statement(s)?*

(a) *Ram eats a mangoe.*
(b) *Krishna goes to school.*
(c) *He lives in India.*
(d) *Anil and Anuj they are good friend.*

Interpret (c) In a given options, only option (c) is an open statement, because in this sentence 'he' can be specify to any person.

19.4 Truth Table

A table that shows the relationship between the truth value of compound statement, $S(p, q, r, \ldots)$ and the truth values of its substatements $p, q, r, \ldots$ etc., is called the truth table of statement S.

(i) For a single statement p, number of rows = $2^1 = 2$

p
T
F

(ii) For two statements p and q, number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(iii) For three statements p, q and r,

Number of Rows $= 2^3 = 8$

p	q	r
T	F	F
T	F	T
T	T	F
T	T	T
F	F	F
F	F	T
F	T	F
F	T	T

Note If a compound statement has simply n substatements, then there are 2^n rows representing logical possibilities.

***Sample* Problem 3** *If there are 6 simple statements, then for making a table, the number of rows are*

(a) 32 (b) 64 (c) 16 (d) 128

Interpret (b) We know that, if compound statements has n substatements, then there are 2^n rows in a table.

Here, $n = 6$

$\therefore$ Total number of rows $= 2^6 = 64$

19.5 Logical Connectives or Sentencial Connectives

Two or more statements are combined to form a compound statement by using symbols. These symbols are called logical connectives.

Logical connectives are given below.

Words	Symbols
and	$\wedge$
or	$\vee$
implies that (if ..., then)	$\Rightarrow$
If and only if (implies and is implied by)	$\Leftrightarrow$

19.6 Elementary Operations of Logic

Formation of compound sentences from simple sentence using logical connectives are termed as elementary operation of logic. There are five such operations discussed below

(i) Negation (Inversion) of Statement

A statement which is formed by changing the truth value of a given statement by using word like 'no' or 'not' is called negation of a given statement. It is representalley the symbol '~'

e.g., Let p : the number 2 is greater than 7.

Then, $\sim p$: the number 2 is not greater than 7.

If p is statement, then negation of p is denoted by '$\sim p$'. The truth table for NOT is given by

p	$\sim p$
T	F
F	T

(ii) Conjunction

A compound sentence formed by two simple sentences p and q using connective "and" is called the conjunction of p and q and is represented by $p \wedge q$.

e.g., Let p : Ramesh is a student

and q : Ramesh belongs to allahabad. Then,

e.g., $p \wedge q \equiv$ Ramesh is a student and he belongs to Allahabad.

The truth table for operation 'and' is given by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Above conjunction can be explained through circuit.
Here, denote $1 \approx T, 0 \approx F$

$p \vee q$

A_1	A_2	Result (Bulb)
1	0	0
1	1	1
0	1	0
0	0	0

$\longrightarrow$

p	q	$p \wedge q$
T	F	F
T	T	T
F	T	F
F	F	F

Note

- The statement $p \wedge q$ is true, if both p and q are true.
- The statement $p \wedge q$ is false, if atleast one of p and q or both are false.

(iii) Disjunction (Alternation)

A compound sentence formed by two simple sentences p and q using connective "or" is called the disjunction of p and q and is represented by $p \vee q$.

e.g., Let p : Bus left early and q : My watch is going slow. Then, $p \vee q =$ Bus left early or my watch is going slow.

The truth table for 'operation' OR is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The above disjunction can be explained through circuit.

Here, denote $I \approx T, 0 \approx F$

A_1	A_2	Result (Bulb)
1	0	1
1	1	1
0	1	1
0	0	0

$\longrightarrow$

p	q	$p \vee q$
T	F	T
T	T	T
F	T	T
F	F	F

Note

- The statement $p \vee q$ is true, if atleast one of p and q or both are true.
- The statement $p \wedge q$ is false, if both p and q are false.

(iv) **Implication** (Conditional)

A compound sentence formed by two simple sentences p and q using connective "if ... then ..." is called the implication of p and q and represented by $p \Rightarrow q$ which is read as "p implies q". Here, p is called antecedent or hypothesis and q is called consequent or conclusion.

e.g., Let p : Train reaches in time.

and $q = I$ can attend the meeting. Then,

$p \Rightarrow q \equiv$ if train reaches in time, then I can attend the meeting.

The truth table for if ... then is given by

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It is clear from the truth table that column III is equal to column V

i.e., Statement $p \Rightarrow q$ is equivalent to $\sim p \vee q$.

(v) **Biconditional Statement**

Two simple sentences connected by the phrase "if and only if," form a biconditional statement. It is represented by the symbol '$\Leftrightarrow$'.

e.g., Let $p : \Delta ABC$ is an isosceles triangle

and q : two sides of a triangle are equal.

Then, $p \Leftrightarrow q : \Delta ABC$ is an isosceles triangle if and only if two sides of a triangle are equal.

The truth table for if and only if is given by

p	q	$p \Leftrightarrow q$	$\sim p$	$\sim q$	$\sim p \vee q$	$p \vee \sim q$	$(\sim p \vee q) \wedge (p \vee \sim q)$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T

Note

- It is clear form the truth table that column III is equal to column VIII. *i.e.*, statement $p \Leftrightarrow q$ is equivalent to $(\sim p \sim \vee q) \wedge (p \vee \sim q)$
- The statement $p \Leftrightarrow q$ has true, if either both are true or both are false.
- The statement $p \Leftrightarrow q$ has false, if exactly one of them is false.

Table for Basic Logical Connections

p	q	$\sim p$	$p \wedge q$	$q \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Hot Spot

Negation of a Statement

These types of problem are directly or indirectly asked in different years in AIEEE (JEE Main) from 2008 *to* 2012.

To solve such types of problem following steps are given.

Step I If the given problem is not in standard form, then converts it in standard form (mathematical form).

Step II Apply the negation operation on that standard form.

Step III Use other logical operation to simplify it and get the result.

Now, we explain the above method by giving illustrative example.

***Sample* Problem 4** *Find the negation of the statement* $(p \vee \sim q) \Rightarrow r$.

(a) $\sim (p \wedge \sim q) \Rightarrow \sim r$ (b) $p \wedge q \Rightarrow r$

(c) $\sim p \wedge q \Rightarrow r$ (d) *None of these*

Interpret (a) **Step I** Convert in standard form.

It is already in standard form.

Step II Applying the negation operation

$\therefore \quad \sim (p \vee \sim q) \Rightarrow \sim r$

Step III Use other logical operation

$$(\sim p \wedge (\sim \sim q)) \Rightarrow \sim r \equiv \sim p \wedge q \Rightarrow \sim r$$

***Sample* Problem 5** *Which of the following pairs of statements is/are negation of each other?* **[NCERT]**

I. *The number x is not a rational number*
The number x is not an irrational number

II. *The number x is a rational number*
The number is an irrational number

(a) I (b) II
(c) I *and* II (d) *None of these*

Interpret (c)

I. **The negation of the statement.** The number x is not a rational number *i.e.*, the number x is a rational number. The second statement is the same as x is not irrational. Hence, both statements are negations of each other.

II. **The negation** of the statement. The number x is not a rational number *i.e.*, the number x is not a rational number or we can say x is an irrational number. The second statement is same. Therefore, both are negations of each other.

***Sample* Problem 6** *If p : the unit digit of an integer is zero and q : it is divisible by 5, then the biconditional statement $p \Leftrightarrow q$ is*

(a) *The unit digit of an integer is zero, if and only if it is divisible by 5*
(b) *If the unit digit of an integer is zero, then it is divisible by 5*
(c) *The unit digit of an integer is zero or it is divisible by 5*
(d) *None of the above*

Interpret (a) Given, p : the unit digit of an integer is zero and q : it is divisible by 5.

The biconditional statement $p \Leftrightarrow q$ is "the unit digit of an integer is zero, if and only if it is divisible by 5".

***Sample* Problem 7** *Let p and q stand for statements*
p : Ram is tall.
q : Bimla is beautiful,
then the statement "Ram is not tall or Bimla is beautiful" stands for

(a) $p \vee q$ (b) $\sim p \vee q$ (c) $\sim p \wedge q$ (d) $p \wedge q$

Interpret (b) Given, p : Ram is tall and q : Bimla is beautiful

$\therefore$ Given, statement can be represented as $\sim p \vee q$

19.7 Converse, Inverse and Contrapositive of an Implication

If p and q are two statements, then here is a quick definition of their converse, inverse and contrapositive

Implied Statement : if then, Converse : if then , Inverse : if not then not, Contrapositive : if not , then not

Note

- The converse of $p \Rightarrow q$ is $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
- The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

e.g., If 2 + 2 = 4, then Jawahar Lal Nehru is the first prime minister of India.

Let $p \equiv 2 + 2 = 4$

and $q \equiv$ Jawahar Lal Nehru is the first prime minister of India. Bicondintial statement $p \Rightarrow q$

(i) The converse of $p \Rightarrow q$ is $q \Rightarrow p$
i.e., If Jawahar Lal Nehru is the first prime minister of India, then 2+2=4

(ii) The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
i.e., If Jawahar Lal Nehru is not the first prime minister of India, then 2+2=4,

(iii) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$ *i.e.*, if $2+2 \neq 4$,
then Jawahar Lal Nehru is not the first prime minister of India.

***Sample* Problem 8** *The contrapositive and converse of the statements "If the two lines are parallel, then they do not intersect in the same plane" is* **[NCERT]**

(a) *If the two straight lines intersect in a plane, then the lines are not parallel.*
If the two lines do not intersect in the same plane, then the two lines are parallel
(b) *If the two straight lines intersect in a plane, then the lines are parallel*
If the two lines do not intersect in the same plane, then the two lines are parallel
(c) *If the two straight lines intersect in a plane, then the lines are not parallel.*
If the two lines do not intersect in the same plane, then the two lines are not parallel
(d) *None of the above*

Interpret (a) **Contrapositive statement** If two straight lines intersect in a plane, then the lines are not parallel.

Converse Statement If two lines do not intersect in the same plane, then two lines are parallel.

19.8 Tautology

A **compound** statement is called a tautology, if it has truth value T whatever may be the truth value of its compounds.
e.g., Statement $(p \Rightarrow q) \wedge p \Rightarrow q$ is a tautology.
The truth table is prepared as follows

p	q	$p \Rightarrow q$	$p \Rightarrow q \wedge p$	$(p \Rightarrow q) \wedge p \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

19.9 Contradiction (Fallacy)

A compound statement is called contradiction, if its truth value is F whatever may be the truth value of its components. *e.g.,* Statement $\sim p \wedge p$ is a contradiction. The truth table is prepared as follows

p	$\sim p$	$\sim p \wedge p$
T	F	F
F	T	F

Note A statement which is neither a tautology nor a contradiction is a contigency.

***Sample* Problem 9** *The statement $p \vee \sim (p \wedge q)$ is a*

(a) *tautology*
(b) *contradiction*
(c) *contigency*
(d) *None of the above*

Interpret (a)

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Hence, all values of a given statement is true, so it is a tautology.

***Sample* Problem 10** *The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a*

(a) *tautology* (b) *contradiction*
(c) *contigency* (d) *None of these*

Interpret (b)

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Hence, all values of a given statement is false, so it is a contradiction.

***Sample* Problem 11** *The statement $p \wedge (q \Leftrightarrow r)$ is a*

(a) *tautology* (b) *contradiction*
(c) *contigency* (d) *None of these*

Interpret (c)

p	q	r	$q \Leftrightarrow r$	$p \wedge (q \Leftrightarrow r)$
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T
F	F	F	T	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	F

Algebra of Statement

(i) **Idempotent Laws**
For any statement p, we have
(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

(ii) **Commutative Laws**
For any two statements p and q, we have
(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

(iii) **Associative Laws**
For any three statements p, q and r, we have
(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(iv) **Distributive Laws**
For any three statements p, q and r, we have
(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(v) **Involution Laws**
For any statement p, we have $\sim (\sim p) \equiv p$

(vi) **De-morgan's Laws**
If p and q are any two statement, then
(a) $\sim (p \wedge q) = \sim p \vee \sim q$ (b) $\sim (p \vee q) = \sim p \wedge \sim q$

(vii) **Contrapositive Law**
For any two statements p and q, we have
$$p \Rightarrow q \equiv (\sim p) \Rightarrow (\sim q)$$

(viii) **Identity Law**
For any statement p, we have
$p \wedge T \equiv p$; where, T is a true statement.

***Sample* Problem 12** *The proposition $\sim (p \Leftrightarrow q)$ is equivalent to*

(a) $(p \vee \sim q) \wedge (q \wedge \sim p)$ (b) $(p \wedge \sim q) \vee (q \wedge \sim p)$
(c) $(p \wedge \sim q) \wedge (q \wedge \sim p)$ (d) *None of these*

Interpret (b) $\sim (p \Leftrightarrow q) \equiv \sim [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

$\equiv \sim (p \Rightarrow q) \vee (\sim (q \Rightarrow p))$

($\because$ by De-Morgan's law)

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

WORKED OUT
Examples

Example 1 *The negation of the statement "101 is not a multiple of 3" is* [NCERT Exemplar]

(a) 101 *is a multiple of* 3 (b) 101 *is a multiple of* 2
(c) 101 *is an odd number* (d) 101 *is an even number*

Solution (a) The negation of given statement is

"101 is a multiple of 3".

Example 2 *The converse of the statement "If sun is not shining, then sky is filled with clouds" is* [NCERT Exemplar]

(a) *if sky is filled with clouds, then the sun is not shining*
(b) *if sun is shining, then sky is filled with clouds*
(c) *if sky is clear, then sun is shining*
(d) *if sun is not shining, then sky is not filled with clouds*

Solution (a) The converse of the given statement is "If sky is filled with clouds, then the sun is not shining".

Example 3 *Which of the following statement is not a conjunction?* [NCERT Exemplar]

(a) *Ram and Shyam are friends*
(b) *Both Ram and Shyam are tall*
(c) *Both Ram and Shyam are enemies*
(d) *None of the above*

Solution (d) Here, we see that option (a), (b) and (c) are all conjunction statement.

Example 4 *The contrapositive and converse of the statement "I go to beach whenever it is a sunny day" is* [NCERT]

(a) (i) *If it is not a sunny day, then I do not go to beach.*
(ii) *If it is a sunny day, then I go to beach*
(b) (i) *If it is a sunny day, then I do not go to beach*
(ii) *If it is a sunny day, then I go to beach*
(c) (i) *If it is not a sunny day, then I go to beach.*
(ii) *If it is not a sunny day, then I go to beach.*
(d) *None of the above*

Solution (a) **Contrapositive statement** If it is not a sunny day, then I do not go to beach.

Converse statement If it is a sunny day, then I go to beach.

Example 5 *Suppose p : A natural number n is odd and q : natural number n is not divisible by 2, then the biconditional statement $p \Leftrightarrow q$ is* [NCERT]

(a) *A natural number n is odd, if and only if it is divisible by* 2
(b) *A natural number n is odd, if and only if it is not divisible by* 2
(c) *If a natural number n is odd, then it is not divisible by* 2
(d) *None of the above*

Solution (b) Given, p : A natural number n is odd and q : natural number n is not divisible by 2

The biconditional statement $p \Leftrightarrow q$ *i.e.*, "A natural number n is odd if and only if it not divisible by 2".

Example 6 *The statement $p \wedge (\sim q \vee r) \wedge (\sim r \vee q)$ is*

(a) *tautology*
(b) *contradiction*
(c) *contigency*
(d) *None of the above*

Solution (c)

p	q	r	$\sim q$	$\sim r$	$\sim q \vee r$	$\sim r \vee q$	$p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$
T	F	F	T	T	T	T	T
T	F	T	T	F	T	F	F
T	T	F	F	T	F	T	F
T	T	T	F	F	T	T	T
F	F	F	T	T	T	T	F
F	F	T	T	F	T	F	F
F	T	F	F	T	F	T	F
F	T	T	F	F	T	T	F

Start Practice for JEE Main

Round I (Topically Divided Problems)

Sentence and Statements

1. Which of the following is a statement? [NCERT Exemplar]

(a) x is a real number
(b) Switch off the fan
(c) 6 is a natural number
(d) Let me go

2. Which of the following is not a statement? [NCERT Exemplar]

(a) Smoking is injurious to health
(b) 2 + 2 = 4
(c) 2 is the only even prime number
(d) Come here

3. The negation of the statement

"72 is divisible by 2 and 3" is [NCERT Exemplar]

(a) 72 is not divisible by 2 or 72 is not divisible by 3
(b) 72 is not divisible by 2 and 72 is not divisible by 3
(c) 72 is divisible by 2 and 72 is not divisible by 3
(d) 72 is not divisible by 2 and 72 is divisible by 3

4. The contrapositive of the statement

"If 7 is greater than 5, then 8 is greater than 6" is [NCERT Exemplar]

(a) If 8 is greater than 6, then 7 is greater than 5
(b) If 8 is not greater than 6, then 7 is greater than 5
(c) If 8 is not greater than 6, then 7 is not greater than 5
(d) If 8 is greater than 6,then 7 is not greater than 5

5. The conditional statement of
"You will get a sweet dish after the dinner" is [NCERT Exemplar]

(a) If you take the dinner, then you will get a sweet dish
(b) If you take the dinner, you will get a sweet dish
(c) You get a sweet dish if and only if you take the dinner
(d) None of the above

6. Let R be the set of real numbers and $x \in R$. Then, $x + 3 = 8$ is

(a) open statement (b) a true statement
(c) false statement (d) None of these

7. Which of the following is not logically equivalent to the following proposition?

"A real number is either rational or irrational"

(a) If a number is neither rational nor irrational, then it is not real
(b) If a number is not a rational or not an irrational, then it is not real
(c) If a number is not real, then it is neither rational or irrational
(d) If a number is real, then it is rational or irrational

8. Which of the following is not a proposition ?

(a) $\sqrt{3}$ is a prime
(b) $\sqrt{2}$ is irrational
(c) Mathematics is interesting
(d) 5 is an even integer

9. Which of the following is an open statement

(a) x is a natural number
(b) Give me a glass of water
(c) Wish you best of luck
(d) Good morning to all

Elementary Operations of Logic

10. The negation of the "statement, if a quadrilateral is a square, then it is a rhombus" is

(a) if a quadrilateral is not a square, then it is a rhombus
(b) if a quadrilateral is a square, then it is not a rhombus
(c) a quadrilateral is a square and it is not a rhombus
(d) a quadrilateral is not a square and it is a rhombus

11. In the truth table for the statement $(\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow \sim p)$, the last column has the truth value in the following order
(a) TTTF (b) FTTF (c) TFFT (d) TTTT

12. If p, q and r are simple propositions with truth value true, false and true respectively, then the truth value of $((\sim p \vee q) \wedge \sim r) \Rightarrow p$
(a) true (b) false
(c) true, if r is false (d) true, if q is true

13. Let p and q be two statements. Then, $p \vee q$ is false, if
(a) p is false and q is true (b) both p and q are false
(c) both p and q are true (d) None of these

14. Let $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then, this law is known as
(a) Commutative law (b) Associative law
(c) De Morgan's law (d) Distributive law

15. Let p be the proposition Mathematics is interesting and let q be the proposition that Mathematics is difficult, then the symbol $p \wedge q$ means
(a) Mathematics is interesting implies that Mathematics is difficult.
(b) Mathematics is interesting implies and is implied by Mathematics is difficult.
(c) Mathematics is interesting and Mathematics is difficult.
(d) Mathematics is interesting or Mathematics is difficult.

16. If p : A man is happy.
q : A man is rich.
Then, the statement, "If a man is not happy, then he is not rich" is written as
(a) $\sim p \rightarrow \sim q$ (b) $\sim q \rightarrow p$
(c) $\sim q \rightarrow \sim p$ (d) $q \rightarrow \sim p$

17. Which of the following is the inverse of the proposition "If a number is a prime, then it is odd."
(a) If a number is not a prime, then it is odd
(b) If a number is not a prime, then it is not odd
(c) If a number is not odd then it is not a prime
(d) If a number is not odd, then it is a prime

18. The false statement in the following is
(a) $p \wedge (\sim p)$ is contradiction
(b) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
(c) $\sim (\sim p) \Leftrightarrow p$ is a tautology
(d) $p \vee (\sim p) \Leftrightarrow$ is a tautology

19. $\sim (p \vee (\sim q))$ is equal to
(a) $\sim p \vee q$ (b) $(\sim p) \wedge q$
(c) $\sim p \vee \sim p$ (d) $\sim p \wedge \sim p$

20. If p and q are two statements such that
p : the questions paper is easy
q : we shall pass,
then the symbolic statement $\sim p \rightarrow \sim q$ means
(a) If the question paper is easy, then we shall pass
(b) If the question paper is not easy, then we shall not pass
(c) The question paper is easy and we shall pass
(d) The question paper is easy or we shall pass

21. For the following three statements
p : 2 is an even number
q : 2 is a prime number.
r : Sum of two prime numbers is always even, then the symbolic statement $(p \wedge q) \rightarrow \sim r$ means
(a) 2 is an even and prime number and the sum of two prime numbers is always even
(b) 2 is an even and prime number and the sum of two prime numbers is not always even
(c) 2 is an even and prime number, then the sum of two prime numbers is not always even
(d) 2 is an even and prime number, then the sum of two prime numbers is always even

22. For two statements p and q
p : A quadrilateral is a parallelogram,
q : The opposite sides are parallel.
Then, the compound proposition, "A quadrilateral is a parallelogram if and only if the opposite sides are parallel" is represented by
(a) $p \vee q$ (b) $p \rightarrow q$
(c) $p \wedge q$ (d) $p \leftrightarrow q$

23. If p, q, r and s are four simple statements, such as
p : The school bus will come
q : I go to school
r : I shall meet my friends
s : I shall go out for a movie,
then the compound statement, "If the school bus does not come or I will not go to school, then I shall meet my friend and go out for a movie." is represented by
(a) $\sim (p \wedge q) \Rightarrow (r \wedge s)$
(b) $\sim (p \wedge \sim q) \Rightarrow (r \wedge s)$
(c) $\sim (p \wedge q) \Rightarrow (r \vee s)$
(d) None of these

24. If p = He is intelligent.
q = He is strong.
Then, symbolic form of statement
"It is wrong that he is intelligent or strong," is
(a) $\sim p \vee \sim p$ (b) $\sim (p \wedge q)$
(c) $\sim (p \vee q)$ (d) $p \vee \sim q$

25. Which of the following is true for the statements p and q?
(a) $p \wedge q$ is true when atleast one of p and q is true
(b) $p \rightarrow q$ is true when p is true and q is false
(c) $p \leftrightarrow q$ is true only when both p and q are true
(d) $\sim (p \vee q)$ is false only when both p and q are false

26. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively,
(a) T, F, F (b) F, F, F
(c) F, T, T (d) T, T, F

27. Negation of the statement $p \rightarrow (q \wedge r)$ is
(a) $\sim p \rightarrow \sim (q \vee r)$ (b) $\sim p \rightarrow \sim (q \wedge r)$
(c) $(q \wedge r) \rightarrow p$ (d) $p \wedge (\sim q \vee \sim r)$

28. $p \vee q$ is true when
(a) both p and q are true (b) p is true and q is false
(c) p is false and q is true (d) All of these

29. The statement $(\sim p \wedge q) \vee \sim q$ is
(a) $p \vee q$ (b) $p \wedge q$
(c) $\sim (p \vee q)$ (d) $\sim (p \wedge q)$

30. $\sim [\{\sim p) \wedge q]$ is equal to
(a) $p \vee (\sim q)$ (b) $p \vee q$
(c) $p \wedge (\sim q)$ (d) $\sim p \wedge \sim q$

31. $(\sim (\sim p)) \wedge q$ is equal to
(a) $\sim p \wedge q$ (b) $p \wedge q$
(c) $p \wedge \sim q$ (d) $\sim p \wedge \sim q$

32. Which of the following is true?
(a) $p \Rightarrow q \equiv \sim p \Rightarrow \sim q$
(b) $\sim (p \Rightarrow \sim q) \equiv \sim p \wedge q$
(b) $\sim (p \Rightarrow \sim q) \equiv \sim p \wedge q$
(c) $\sim (\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$
(d) $\sim (\sim p \Leftrightarrow q) \equiv [\sim (p \Rightarrow q) \wedge \sim (q \Rightarrow p)]$

33. Let truth values of p be F and q be T. Then, truth value of $\sim (\sim p \vee q)$ is
(a) T (b) F
(c) either T or F (d) neither T nor F

34. If statements p and r are false and q is true, then truth value of $\sim p \Rightarrow (q \wedge r) \vee r$ is
(a) T (b) F
(c) either T or F (d) neither T nor F

35. Which of the following is always true?
(a) $(p \Rightarrow q) \equiv \sim q \Rightarrow \sim p$
(b) $\sim (p \vee q) \equiv \sim p \vee \sim q$
(c) $\sim (p \Rightarrow q) \equiv p \wedge \sim q$
(d) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

36. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q and r are respectively.
(a) TFF (b) FFF
(c) FTT (d) TTF

37. The logically equivalent proposition of $p \Leftrightarrow q$ is
(a) $(p \wedge q) \vee (p \wedge q)$ (b) $(p \Rightarrow q) \wedge (q \Rightarrow p)$
(c) $(p \wedge q) \vee (q \Rightarrow p)$ (d) $(p \wedge q) \Rightarrow (p \vee q)$

38. The statement $p \Rightarrow (q \Rightarrow p)$ is equivalent to
(a) $p \Rightarrow (p \Rightarrow q)$ (b) $p \Rightarrow (p \vee q)$
(c) $p \Rightarrow (p \wedge q)$ (d) $p \Rightarrow (p \Leftrightarrow q)$

39. If p and q are simple propositions, then $p \Rightarrow \sim q$ is true when
(a) both p and q are true (b) both p and q are false
(c) p is false and q is true (d) None of these

40. Which of the following is logically equivalent to $\sim (\sim p \Rightarrow q)$?
(a) $p \wedge q$ (b) $p \wedge \sim q$
(c) $\sim p \wedge q$ (d) $\sim p \wedge \sim q$

41. $\sim (p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
(a) $\sim p$ (b) p
(c) q (d) $\sim q$

42. Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is
(a) $\sim (p \wedge r) \rightarrow \sim (r \vee q)$ (b) $(\sim p \vee \sim r) \vee (r \vee q)$
(c) $(p \wedge r) \wedge (r \wedge q)$ (d) $(p \wedge r) \wedge (\sim r \wedge \sim q)$

43. Which of the following is equivalent to $(p \wedge q)$?
(a) $p \rightarrow \sim q$ (b) $\sim (\sim p \wedge \sim q)$
(c) $\sim (p \rightarrow \sim q)$ (d) None of these

44. If p, q are true statements and r, s are false statements, then the truth value of $\sim ((p \wedge \sim r) \vee (\sim q \vee s))$ is
(a) true (b) false
(c) false for any p (d) None of these

45. If the compound statement $p \rightarrow (\sim p \vee q)$ is false, then the truth value of p and q are respectively
(a) T, T, (b) T, F
(c) F, T (d) F, F

46. Which of the following is always true?
(a) $(\sim p \vee \sim q) \equiv (p \wedge q)$
(b) $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
(c) $\sim (p \rightarrow \sim q) \equiv (p \wedge \sim q)$
(d) $\sim (p \leftrightarrow q) \equiv (p \rightarrow q) \rightarrow (q \rightarrow p)$

47. The negation of the compound statement $(p \vee q) \wedge r$ is
(a) $(\sim p \vee \sim q) \wedge \sim r$ (b) $(\sim p \wedge \sim q) \vee \sim r$
(c) $\sim (p \vee q) \Rightarrow r$ (d) $p \wedge q$

48. The negation of $(\sim p \wedge q) \vee (p \wedge \sim q)$ is
(a) $(p \vee \sim q) \vee (\sim p \vee q)$ (b) $(p \vee \sim q) \wedge (\sim p \vee q)$
(c) $(p \wedge \sim q) \wedge (\sim p \vee q)$ (d) $(p \vee \sim q) \wedge (p \vee \sim q)$

49. The negation of $(p \vee q) \wedge (p \vee \sim r)$ is
(a) $(\sim p \wedge \sim q) \vee (p \wedge \sim r)$ (b) $(\sim p \wedge \sim q) \wedge (\sim p \wedge r)$
(c) $(\sim p \wedge \sim q) \vee (\sim p \wedge r)$ (d) None of these

Tautology and Contradiction

50. A proposition is called a tautology, if it is
(a) always T
(b) always F
(c) sometimes T, sometimes F
(d) None of the above

51. If p and q are two statements, then statement $p \Rightarrow q \wedge \sim q$ is
(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) None of the above

52. Let p and q be two statements, then $(p \vee q) \vee \sim p$ is
(a) tautology (b) contradiction
(c) Both (a) and (b) (d) None of these

53. The statement $p \vee \sim p$ is
(a) tautology
(b) contradiction
(c) neither a tautology nor a contradiction
(d) None of the above

54. The statement $(p \Rightarrow q) \Leftrightarrow (\sim p \wedge q)$ is a
(a) tautology (b) contradiction
(c) neither (a) nor (b) (d) None of these

55. The proposition $\sim (p \Rightarrow q) \Rightarrow (\sim p \vee \sim q)$ is
(a) a Tautology (b) a Contradiction
(c) either (a) or (b) (d) neither (a) nor (b)

56. The negation of the compound proposition is $p \vee (\sim p \vee q)$
(a) $(p \wedge \sim q) \wedge \sim p$
(b) $(p \vee \sim q) \vee \sim p$
(c) $(p \wedge \sim q) \vee \sim p$
(d) None of the above

57. If p and q are two statements, then $\sim (p \wedge q) \vee \sim (q \Leftrightarrow p)$ is
(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) either tautology or contradiction

58. Let p and q be two statements. Then,
$$(\sim p \vee q) \wedge (\sim p \wedge \sim q) \text{ is a}$$
(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) both tautology and contradiction

59. The proposition $(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$ is
(a) contigency
(b) neither Tautology nor Contradiction
(c) contradiction
(d) tautology

60. If p and q are two statements, then
$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p) \text{ is a}$$
(a) contradiction (b) tautology
(c) neither (a) nor (b) (d) None of these

61. The proposition $S : (p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$ is
(a) a tautology (b) a contradiction
(c) either (a) or (b) (d) neither (a) nor (b)

62. The false statement in the following is
(a) $p \wedge (\sim p)$ is a contradiction
(b) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
(c) $\sim (\sim p) \Leftrightarrow p$ is a tautology
(d) $p \vee (\sim p)$ is a tautology

63. $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is
(a) a tautology
(b) a contradiction
(c) both a tautology and a contradiction
(d) neither a tautology nor a contradiction

64. The propositions $(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$ is
(a) Tautology and contradiction
(b) Neither tautology nor contradiction
(c) Contradiction
(d) Tautology

Round II (Mixed Bag)

Only One Option Correct

1. Statement $(p \wedge \sim q) \wedge (\sim p \vee q)$ is
(a) a tautology
(b) a contradiction
(c) both a tautology and a contradiction
(d) neither a tautology nor a contradiction

2. Which of the following is wrong statement?
(a) $p \rightarrow q$ is logically equivalent to $\sim p \vee q$
(b) If the truth values of p, q, r are T, F, T respectively, then the truth value of $(p \vee q) \wedge (q \vee r)$ is T
(c) $\sim (\vee q \vee q \vee r) \cong \sim p \wedge \sim q \wedge \sim r$
(d) The truth value of $p \wedge \sim (p \vee q)$ is always T

3. Which of the following statement is a contradiction?
(a) $(\sim p \vee \sim q) \vee (p \vee \sim q)$ (b) $(p \rightarrow q) \vee (p \wedge \sim q)$
(c) $(\sim p \wedge q) \wedge (\sim q)$ (d) $(\sim p \wedge q) \vee (\sim q)$

4. Which of the following is wrong?
(a) $p \vee \sim p$ is a tautology
(b) $\sim (\sim p) \leftrightarrow p$ is a tautology
(c) $p \wedge \sim p$ is a contradiction
(d) $((p \wedge q) \rightarrow q) \rightarrow p$ is a tautology

5. If p, q and r are simple propositions, then $(p \wedge q) \wedge (q \wedge r)$ is true, then
(a) p, q and r are true
(b) p, q are true and r is false
(c) p is true and q, r are false
(d) p, q and r are false

6. If $(q \wedge -r) \Rightarrow (q \vee r)$ is false, then p is
(a) True (b) False
(c) May be true and false (d) None of these

More Than One Correct Option

7. If p and q are simple proposition, then $(\sim p \wedge q) \vee (\sim q \wedge p)$ is false when p and q are respectively
(a) T, T (b) T, F
(c) F, F (d) F, T

8. If p, q and r are simple proposition, then $(\sim p \vee q) \Rightarrow r$ is true, when p, q and r are, respectively
(a) T, F, T (b) T, T, T
(c) T, F, F (d) F, F, F

9. Which of the following sentences is/are not statements? [NCERT]
(a) There are 35 days in a month.
(b) Mathematics is difficult.
(c) The square of a number is an even number.
(d) The side of a quadrilateral have equal lengths.

Comprehension Based Questions

Passage

Directions (Q. Nos. 10 and 11) *Let us define a truth table*

q	p	q	$\sim p$	$p \wedge q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$
F	T	T	T	F	F	T
T	T	F	F	F	T	F
F	F	T	T	F	T	F
T	F	F	T	F	T	F

On the basis of the above table, solve the following questions

10. The statement $\sim(\sim p \vee \sim q)$ is
(a) tautology (b) contradiction
(c) contigency (d) None of these

11. If $\sim(p \wedge q)$ is false, then corresponding values of p and q are, respectively
(a) T, T (b) T, F
(c) F, T (d) None of these

Assertion and Reason

Directions (Q. Nos. 12-21) *Each of these questions contans two statements : statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternatite choice, only one of which is the correct answer. You has to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I true, Statement II is false.
(d) Statement I is false, Statement II is true.

12. **Statement I** The statement $[p \wedge (p \to q)] \to q$ is a tautology.
Statement II If all truth values of a statement is true, then the statement is a tautology.

13. Let p : Ice is cold and q : blood is green be two statements, then
Statement I $p \vee q$: Ice is cold or blood is green.
Statement II $p \wedge q$: Ice is not cold or blood is green.

14. Let $S : \sim(p \Rightarrow q) \vee \sim(q \Rightarrow p)$
Statement I The statement S is logically equivalent to $(p \Leftrightarrow q)$
Statement II The logically equivalent proposition of $p \Leftrightarrow q$ is $(p \Rightarrow q) \wedge (q \Rightarrow p)$

15. If $p \to q$ be any conditional statement,
Statement I The converse of $p \to q$ is the statement $q \to p$.
Statement II The inverse of $p \to q$ is the statement $\sim q \to \sim p$.

16. Suppose p, q and r be any three statements.
Statement I The statement $p \to (q \to r)$ is a tautology.
Statement II $(p \wedge q) \to r$ and $p \to (q \to r)$ are identical.

17. Let p be the statement, "Mr A passed the examination", q be the statement, "Mr A is sad" and r be the statement "It is not true that Mr A passed therefore he is sad."
Statement I $r \equiv p \Rightarrow q$
Statement II The logical equivalent of $p \Rightarrow q$ is $\sim p \vee q$.

18. **Statement I** $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$
Statement II $\sim(p \leftrightarrow \sim q)$ is a tautology

19. **Statement I** $\sim(A \Leftrightarrow \sim \beta)$ is equivalent to $A \Leftrightarrow B$.
Statement II $A \vee (\sim(A \wedge \sim B))$ a tautology.

20. Consider
Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.
Statement II $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

21. Let p be the statement 'x is an irrational number', q be the statement 'y is a transcendental number' and r be the statement 'x is a rational number iff y is a transcendental number'.
Statement I r is equivalent to either q or p.
Statement II r is equivalent to $\sim(p \leftrightarrow \sim p)$.

Previous Years' Questions

22. The statement $p \to (q \to p)$ is equivalent to **[AIEEE 2008]**

(a) $p \to (p \leftrightarrow q)$ (b) $p \to (p \to q)$
(c) $p \to (p \vee q)$ (d) $p \to (p \wedge q)$

23. The negation of the statement

"If become a teacher, then I will open a school", is **[AIEEE 2012]**

(a) I will become a teacher and I will not open a school
(b) Either I will not become a teacher or I will not open a school
(c) Neither I will become a teacher nor I will open a school
(d) I will not become a teacher or I will open a school

24. The only statement among the following that is a tautology is **[AIEEE 2011]**

(a) $B \to [A \wedge (A \to B)]$
(b) $A \wedge (A \vee B)$
(c) $A \vee (A \wedge B)$
(d) $[A \wedge (A \to B)] \to B$

25. Consider the following statements
P : Suman is brilliant.
Q : Suman is rich.
R : Suman is honest.

The nagative of the statement. 'Suman is brilliant and dishonest, if and only if Suman is rich.' can be expressed as **[AIEEE 2011]**

(a) $\sim Q \leftrightarrow (P \sim R)$ (b) $\sim Q \leftrightarrow P \wedge R$
(c) $\sim (P \wedge \sim R) \leftrightarrow Q$ (d) $\sim P \wedge (Q \leftrightarrow \sim R)$

26. Let S be a non-empty subset of R. Consider the following statement

p : There is a rational number $x \in S$ such that $x > 0$.

Which of the following statements is the negation of the statement p? **[AIEEE 2010]**

(a) There is a rational number $x \in S$ such that $x \le 0$
(b) There is no rational number $x \in S$ such that $x \le 0$
(c) Every rational number $x \in S$ satisfies $x \le 0$
(d) $x \in S$ and $x \le 0 \Rightarrow x$ is not rational

Answers

Round I

1. (c)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (c)
11. (c)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (b)
21. (c)	**22.** (d)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (a)	**27.** (d)	**28.** (d)	**29.** (d)	**30.** (a)
31. (b)	**32.** (c)	**33.** (b)	**34.** (b)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (d)
41. (a)	**42.** (d)	**43.** (b)	**44.** (b)	**45.** (b)	**46.** (b)	**47.** (b)	**48.** (b)	**49.** (c)	**50.** (a)
51. (c)	**52.** (a)	**53.** (a)	**54.** (c)	**55.** (a)	**56.** (a)	**57.** (c)	**58.** (c)	**59.** (c)	**60.** (b)
61. (a)	**62.** (d)	**63.** (b)	**64.** (c)						

Round II

1. (b)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (a,b,c)	**8.** (a,b)	**9.** (b,c,d)	**10.** (c)
11. (a)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (d)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (b)
21. (*)	**22.** (c)	**23.** (a)	**24.** (d)	**25.** (a)	**26.** (c)				

the Guidance

Round I

1. (a) Here, without knowing the value of x, we do not know whether the given sentence is true or false. So, it is a sentence but not a statement.

(b) It is an imperative sentence and therefore it is not a statement

(c) It is a declarative sentence, which is clearly true. Therefore, it is a true statement

(d) It is an imperative sentence and therefore it is not a statement

2. In option (a), (b) and (c).

It is a declarative sentence, which is clearly true, therefore it is a true statement.

In option (d), it is an imperative sentence, therefore it is not a statement.

3. The negation of the given statement is "72 is not divisible by 2 or 72 is not divisible by 3".

4. The contrapositive of the given statement is "If 8 is not greater than 6, then 7 is not greater than 5.

5. The conditional statement of given statement is "If you take the dinner, then you will get a sweet dish".

6. Given, $x + 3 = 8$

It is an open statement.

7. Given proposition is not logically equivalent to "If a number is not a rational or not an irrational, then it is not real".

8. Mathematics is interesting is not a logical sentence. It may be interesting for some persons and may not be interesting for others.

$\therefore$ This is not a proposition.

9. "x is a natural number" is an open statement.

10. Let p and q be the propositions as given below

p : A quadrilateral is a square

q : A quadrilateral is a rhombus

The given proposition is $p \to q$

Now, $\sim (p \to q) \equiv p \wedge \sim q$

Therefore, the negation of the given proposition is

"A quadrilateral is a square and it is not a rhombus"

11. Let $S : (\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow \sim p)$

$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$	S
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

12. Let $S : ((\sim p \vee q) \wedge \sim r) \Rightarrow p$

$\Rightarrow \quad S : ((\sim T \vee F) \wedge \sim T) \Rightarrow T$

13.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

It is clear from the truth table that $p \vee q$ is false only when p and q are false.

14. By distributive law, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

15. Given, p : Mathematics is interesting

q : Mathematics is difficult

$\therefore$ $p \wedge q$. Mathematics is interesting and Mathematics is difficult.

16. $\because p$: A man is happy

and q : A man is rich.

'If a man is not happy, then he is not rich' is written as $\sim p \to \sim q$.

17. p : A number is a prime

q : It is odd.

We have, $p \Rightarrow q$

The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

i.e., if a number is not a prime, then it is not odd.

18. $p \Rightarrow q$ is logically equivalent to $\sim q \Rightarrow \sim p$

$\therefore$ $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology but not a contradiction.

19. $\sim (p \vee (\sim q)) \equiv \sim p \wedge \sim (\sim q) \equiv (\sim p) \wedge q$

20. Given, p : The question paper is easy and q : We shall pass given.

The symbolic representation of given option are

(a) : $p \to q$

(b) : $\sim p \to \sim q$

(c) : $p \wedge q$

(d) : $p \vee q$

21. Given, p : 2 is an even number

q : 2 is a prime number

r : sum of two prime numbers is always even.

The symbolic representation of given option (s) are

(a) : $p \wedge q \wedge r$

(b) : $p \wedge q \wedge \sim r$

(c) : $p \wedge q \Rightarrow \sim r$

(d) : $p \wedge q \Rightarrow r$

22. Given, p : A quadrilateral is a parallelogram.

q : The opposite sides are parallel.

"A quadrilateral is a parallelogram if and only if the opposite sides are parallel" is represented by $p \leftrightarrow q$.

23. (If the school bus does not come) $\vee$ (I will not go to school) $\rightarrow$ (I shall meet my friend) $\wedge$ (I shall go out for a movie)

$\Rightarrow \quad (\sim p) \vee (\sim q) \Rightarrow (r \wedge s)$

$\Rightarrow \quad \sim (p \wedge q) \Rightarrow (r \wedge s)$ (De-Morgan's Laws)

24. The symbolic form of given statement is $\sim (p \vee q)$.

25. (a) We know that, $p \wedge q$ is true when both p and q are true.

(b) We know that, $p \rightarrow q$ is false when p is true and q is false.

(c) We know that, $p \leftrightarrow q$ is true when either both p and q are true or both are false.

(d) If p and q both are false, then $p \vee q$ is false $\Rightarrow \sim (p \vee q)$ is true.

Hence, option (d) is true.

26.

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T

Hence, truth values of p, q and r are respectively T, F, F.

27. We know that, $\sim (p \rightarrow q) \cong p \wedge \sim q$

$\therefore \quad \sim (p \rightarrow \sim (q \wedge r) \cong p \wedge (\sim (q \wedge r))$

$\cong p \wedge (\sim q \vee \sim r)$ [By De-Morgan's laws]

28.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

29. $(\sim p \wedge q) \vee \sim q \equiv \sim q \vee (\sim p \wedge q)$ (by Commutative law)

$\equiv \sim q \vee (q \wedge q \sim p)$ (by Commutative law)

$\equiv \sim q \vee q\ (\sim q \vee \sim p)$ (by Distributive law)

$\equiv \sim (q \wedge p) \equiv \sim (p \wedge q)$

30. $\sim ((\sim p) \wedge q) \equiv \sim (\sim p) \vee \sim q \equiv p \vee (\sim q)$

31. $(\sim (\sim p)) \wedge q \equiv p \wedge q$

32. $\sim (p \Rightarrow q) \equiv p \wedge \sim q$

$\therefore \quad \sim (\sim p \Rightarrow \sim q) \equiv \sim p \wedge \sim (\sim q) \equiv \sim p \wedge q$

Thus, $\quad \sim (\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$

33.

p	q	$\sim p$	$\sim p \vee q$	$\sim (\sim p \vee q)$
F	T	T	T	F

34.

p	q	r	$\sim p$	$q \wedge r$	$(q \wedge r) \vee r$	$\sim p \Rightarrow (q \wedge r) \vee r$
F	T	F	T	F	F	F

35. Since, $p \Rightarrow q \equiv \sim p \vee q$

$\therefore \quad \sim (p \Rightarrow q) \equiv p \wedge \sim q$

36.

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T

37.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Rightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

38.

p	q	$q \Rightarrow p$	$p \Rightarrow (q \Rightarrow p)$	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T

Hence, $\quad p \Rightarrow (p \Rightarrow q) \equiv p \Rightarrow (p \vee q)$

39.

p	q	$\sim q$	$p \Rightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

40.

p	q	$\sim p$	$\sim q$	$\sim p \Rightarrow q$	$\sim (\sim p \Rightarrow q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Hence, $\quad \sim (\sim p \Rightarrow q) \equiv \sim p \wedge \sim q$

41.

p	q	$\sim p$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge q$	$\sim(p \vee q) \vee (\sim p \wedge q)$
T	T	F	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	F	T	T
F	F	T	F	T	F	T

$\therefore \quad \sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

42. We know that, $\sim(p \to q) \equiv p \wedge \sim q$

$\therefore \quad \sim((p \wedge r) \to (r \vee q)) \equiv (p \wedge r) \wedge [\sim(r \vee q)]$

$\equiv (p \wedge r) \wedge (\sim r \wedge \sim q)$

43.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$p \to \sim q$	$\sim(p \to \sim q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

44. $\sim((T \wedge \sim F) \vee (\sim T \vee F))$

$\Rightarrow \quad \equiv \sim((T \wedge T) \vee (F \vee F))$

$\Rightarrow \quad \equiv \sim((T) \vee (F)) \equiv \sim(T) \equiv F$

45. We know that, $p \to q$ is false only when p is true and q is false.

So, $p \to (\sim p \vee q)$ is false only when p is true and $(\sim p \vee q)$ is false.

But $(\sim p \vee q)$ is false if q is false because $\sim p$ is false.

Hence, $p \to (\sim p \vee q)$ is false when truth value of p and q are T and F respectively.

46. Since, $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$ and $\sim(p \wedge q) \equiv (\sim p \vee q)$

So, option (b) and (d) are not true.

$(p \to q) \equiv p \wedge \sim q$, so option (c) is not true.

Now, $\quad p \to q \sim p \vee q$

$\sim q \to \sim p \equiv [\sim(\sim q) \vee \sim p] \equiv q \vee \sim p \equiv \sim p \vee q$

$p \to q \equiv \sim q \to \sim p$

47. Let $S : (p \vee q) \wedge r$

$\Rightarrow \quad \sim S : \sim(p \vee q) \wedge r)$

$\Rightarrow \quad \sim S : \sim(p \vee q) \vee \sim r$ (De-Morgan's Laws)

$\therefore \quad \sim S : (\sim p \wedge \sim q) \vee \sim r$

48. Let $S : (\sim p \wedge q) \vee (p \wedge \sim q)$

$\Rightarrow \quad \sim S : \sim((\sim p \wedge q) \vee (p \wedge \sim q))$

$\Rightarrow \quad \sim S : \sim(\sim p \wedge q) \wedge \sim(p \wedge \sim q)$ (De-Morgan's Law)

$\therefore \quad \sim S : (p \vee \sim q) \wedge (\sim p \vee q)$

49. Let $S : (p \vee q) \wedge (p \vee \sim r)$

$\Rightarrow \quad \sim S : \sim(p \vee q) \vee \sim(p \vee \sim r)$

$\therefore \quad \sim S : (\sim p \wedge \sim q) \vee (\sim p \wedge r)$

50. In tautology it always true values.

51. By truth table

p	q	$\sim q$	$q \wedge \sim q$	$p \Rightarrow q \wedge \sim q$
T	T	F	F	F
T	F	T	F	F
F	T	F	F	T
F	F	T	F	T

Hence, it is neither a tautology nor contradiction.

52. By truth table

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

It is clear that $(p \vee q) \vee \sim p$ is a tautology.

53.

p	$\sim p$	$q \vee \sim p$
T	F	T
T	F	T
F	T	T
F	T	T

54.

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \wedge q$	$(p \Rightarrow q) \Leftrightarrow (\sim p \wedge q)$
T	T	F	T	F	F
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	F	F

55. By truth table

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$\sim p \vee \sim q$	$\sim(p \Rightarrow q) \Rightarrow (\sim p \vee \sim q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Hence, given proposition is a tautology.

56. Since, $\quad S : \sim(p \vee (\sim p \vee q))$

$\Rightarrow \quad S : \sim p \wedge \sim(\sim p \vee q)$ (De-Morgan's Law)

$\Rightarrow \quad S : \sim p \wedge (p \wedge \sim q)$ (De-Morgan's law)

57. By truth table

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$q \Leftrightarrow p$	$\sim(q \Leftrightarrow p)$	$\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$
T	T	T	F	T	F	F
T	F	F	T	F	T	T
F	T	F	T	F	T	T
F	F	F	T	T	F	T

It is clear that neither tautology nor contradiction.

58.

p	q	$\sim p$	$\sim q$	$(\sim p \vee q)$	$(\sim p \wedge \sim q)$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Hence, it is neither tautology nor contradiction.

59.

p	$\sim p$	$p \Rightarrow \sim p$	$\sim p \Rightarrow p$	$(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$
F	T	T	F	F
T	F	F	T	F

$\therefore$ Statements is contradiction.

60.

p	q	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

Hence, it is tautology.

61.

p	q	$p \Rightarrow q$	$(\sim p \vee q)$	$(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Since, all values of given proposition is true, hence it is a tautology.

62. $p \Rightarrow q$ is logically equivalent to $\sim q \Rightarrow \sim p$.

$\therefore \quad (p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

is a tautology but not a contradiction.

63. $(p \wedge \sim q) \wedge (\sim p \wedge q) = (p \wedge \sim q) \wedge (\sim q \wedge q) = f \wedge f = f$

(by using associative laws and commutative laws)

$\therefore \quad (p \wedge \sim q) \wedge (\sim p \wedge q)$ is a contradiction.

64.

p	$\sim p$	$p \Rightarrow \sim p$	$\sim p \Rightarrow p$	$(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$
T	F	F	T	F
F	T	T	F	F

Clearly, $(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$ is a contradiction.

Round II

1. The truth table of $(p \wedge \sim q) \wedge (\sim p \vee q)$ is as given below

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

The last column of the above truth table contains F only.

So, the given statement is a contradiction.

2. The truth tables of $p \to q$ and $\sim p \vee q$ are given below

p	q	$\sim p$	$p \to q$	$\sim (p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Clearly, truth tables of $p \to q$ and $\sim p \vee q$ are same.
So, $p \to q$ is logically equivalent to $\sim p \vee q$.
Hence, option (a) is correct.

If the truth value of p, q, r are T, F, T respectively, then the truth values of $p \vee q$ and $q \vee r$ are each equal to T. Therefore, the truth value of $(p \vee q) \wedge (q \vee r)$ is T.
Hence, option (b) is correct.

We know that, $\sim (p \vee q \vee r) \cong (\sim p \wedge \sim q \wedge \sim r)$
So, option (c) is correct.

If p is true and q is false, then $p \vee q$ is true. Consequently $\sim (p \vee q)$ is false and hence $p \wedge \sim (p \vee q)$ is false.
Hence, option (d) is wrong.

3. We consider following truth table.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$(\sim (p \vee q))$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	F	F	T	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	F	F	T	F

Clearly, last column of the above truth table contains only F.
Hence, $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction

4. The truth value of $\sim (\sim p) \leftrightarrow p$ as follow

p	$\sim p$	$\sim (\sim p)$	$\sim (\sim p) \to p$	$p \to \sim (\sim p)$	$\sim (\sim p) \leftrightarrow p$
T	F	T	T	T	T
F	T	F	T	T	T

Since, last column of above truth table contains only T.

Hence, $\sim (\sim p) \to p$ is a tautology.

5.

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \wedge (q \wedge r)$
T	F	F	F	F	F
T	F	T	F	F	F
T	T	F	T	F	F
T	T	T	T	T	T
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	T	F

6.

p	q	r	$\sim r$	$p \wedge \sim r$	$p \vee r$	$(p \wedge \sim r) \Rightarrow (q \vee r)$
T	F	F	T	T	F	F
T	F	T	F	F	T	T
T	T	F	T	T	T	T
T	T	T	F	F	T	T
F	F	F	T	F	F	T
F	F	T	F	F	T	T
F	T	F	T	F	T	T
F	T	T	F	F	T	T

7.

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$\sim q \wedge p$	$(\sim p \wedge q) \vee (\sim q \wedge p)$
T	T	F	F	F	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	T
F	F	T	T	F	F	F

8.

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \Rightarrow r$
T	F	F	F	F	F
T	F	T	F	F	T
T	T	F	F	T	F
T	T	T	F	T	T
F	F	F	T	T	F
F	F	T	T	T	T
F	T	F	T	T	F
F	T	T	T	T	T

9. (a) A month has 28 or 29 or 30 or 31 days. It is a false to say that a month has 35 days. So it is a statement.

(b) Mathematics may be difficult for one but may be easy for the other. Hence, it is not a statement.

(c) The square of a number may be even or it may be odd. Square of an odd number is always odd and square of an even number is always even.

e.g., $4^2 = 16$ (even) and $5^2 = 25$ (odd)

Hence, it not a statement.

(d) A quadrilateral may have equal lengths as it may be a rhombous or a square or the quadrilateral may have unequal sides (like parallelogram).

Hence, it is not a statement.

10. The values of $\sim(\sim p \vee \sim q)$ all are neither true nor false, so it is a contigency.

11. If $\sim(p \wedge q)$ is false, then corresponding values of p and q are respectively T and T.

12. Truth table for $[p \wedge (p \to q)] \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

All values of columns are true.

Therefore, $[p \wedge (p \to q)] \to q$

13.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p : Ice is cold q : Blood is green

$p \vee q$: Ice is cold or blood is green

$p \wedge q$: Ice is cold and blood is green

16. The truth table for $p \Leftrightarrow q$ is

p	q	$p \Leftrightarrow q$
T	F	F
T	T	T
F	F	T
F	T	F

Also, the truth table for $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is

p	q	$p \Rightarrow q$	$q \Rightarrow p$	S
T	F	F	T	F
T	T	T	T	T
F	F	T	T	F
F	T	T	F	F

Hence, we can say that

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$$

15. If $p \to q$ be any conditional statement, then

(i) The converse of $p \to q$ is the statement $q \to p$.

(ii) The inverse of $p \to p$ is the statement $\sim p \to \sim q$

(iii) The contrapositive of $p \to q$ is the statement $\sim q \to \sim p$

16.

p	q	r	$p \to r$	$p \wedge q$	$p \to (q \to r)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

From table columns of $p \to (q \to r)$ is not a tautology and column of $p \to (q \to r)$ and $(p \wedge q) \to r$ are identical.

Hence, Statement I is false, Statement II is true.

17. p : Mr A passed the examination.

q : Mr A is sad.

r : It is not true that Mr A passed therefore he is sad.

$\therefore \quad r : \sim p \Rightarrow q$

Hence, Statement I is false for Statement II.

The truth table for $p \Rightarrow q$ is

p	q	$p \Rightarrow q$
T	F	F
T	T	T
F	F	T
F	T	T

Also, the truth table for $\sim p \vee q$ is

p	q	$\sim p$	$\sim p \vee q$
T	F	F	F
T	T	F	T
F	F	T	T
F	T	T	T

Hence, we can say that

$$p \Rightarrow q \equiv \sim p \vee q$$

Logically equivalents

$$p \Rightarrow q \equiv \sim p \vee q$$

$$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

18.

p	q	$p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	T	T	F	T

19. **Statement I**

A	B	$\sim B$	$A \Leftrightarrow \sim B$	$\sim(A \Leftrightarrow \sim B)$	$A \Leftrightarrow B$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

$\therefore$ Statement I is true.

Statement II

A	B	$\sim B$	$(A \wedge \sim B)$	$\sim(A \wedge \sim B)$	$A \vee (\sim A \wedge \sim B)$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	F	T	T

Statement II is true but it is not a correct explanation of Statement I.

20. **Statement II** $(p \to q) \leftrightarrow (\sim q \to \sim p) = (p \to q) \leftrightarrow (p \to q)$

which is always true, so statement II is true.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q) \equiv p \wedge \sim q \wedge \sim p \wedge q)$

$$\equiv p \wedge \sim p \wedge \sim q \wedge q \equiv f \wedge f \equiv f$$

Hence, it is a fallacy statement. So, statement I is true.

Alternative Solution

Statement II $(p \to q) \leftrightarrow (\sim q \to \sim p)$

$\sim q \to \sim p$ ios contrapoaitive of $p \to q$ hence, $(p \to q) \leftrightarrow (p \to q)$ will be a tautology.

Statement II $(p \wedge \sim q)\ (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, it is a fallacy.

21. Here, p : x is an irrational number

q : y is a transcendental number.

r : x is a rational number iff y is a transcendental number

$\Rightarrow \quad r : \sim p \leftrightarrow q$

$$S_1 : r \equiv q \vee p \text{ and } S_2 : r \equiv \sim(p \leftrightarrow \sim q)$$

p	q	$\sim p$	$\sim q$	$r : \sim p \leftrightarrow q$	$q \vee p$	$(p \leftrightarrow \sim q)$	$\sim(p \leftrightarrow \sim q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	T	T	F
F	T	T	F	T	T	T	F
F	F	T	T	F	F	F	T

It is clear from the table that r is not equivalent to either of the statements.

Hence, none of the given options is correct.

22.

q	p	$q \to p$	$p \to (q \to p)$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	T	F	T

$\therefore$ Statement $p \to (q \to p)$ is equivalent to $p \to (p \vee q)$

23. Let us assume that p : 'I become a teacher' and q : I will open a school. Negation of $(p \to q)$ is $\sim(p \to q) \equiv p \wedge \sim q$ which means that 'I will become a teacher and I will not open a school'.

24.

A	B	$A \vee B$	$A \wedge B$	$A \vee (A \vee B)$	$A \vee (A \wedge B)$	$A \to B$	$A \wedge (A \to B)$	$A \wedge (A \to B) \to B$	$B \to (A \wedge (A \to B))$
T	T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	F	T	T
F	T	T	F	F	F	T	F	T	F
F	F	F	F	F	F	T	F	T	T

$\because$ The truth value of all the elements of the column $A \wedge (A \to B) \to B$ $\therefore$ $A \wedge (A \to B) \to B$ is a tautology.

25. Suman is brilliant and dishonest, if and onlyif Suman is rich, is expressed as, $\quad Q \leftrightarrow (P \wedge \sim R)$

$\therefore$ Negation of it will be,

$$\sim Q \leftrightarrow (P \wedge \sim R)$$

26. P : There is rational number $x \in S$ such that $x > 0$

$\sim P$: Every rational number $x \in S$ satisfies $x \leq 0$

20 Statistics

JEE Main MILESTONE

- Representation of Data
- Measures of Central Tendency
- Mean
- Geometric Mean (GM)
- Harmonic Mean (HM)
- Median
- Mode
- Emperical Relationship between Mean, Median and Mode
- Positional Measures
- Measures of Dispersion
- Range
- Quartile Deviation
- Mean Deviation
- Standard Deviation

20.1 Representation of Data

The data once collected, must be arranged or organised in a way so that definite inferences can be made out from it. The most common organisational schemes are Tabular representation Graphical representation.

In tabular representation, the data can be organised as discrete frequency or continuous frequency distribution. Tabular representation provides both economy of representation and organisation. On the other hand in case of Graphical representation the most common types in which data can be represented are : Histogram, Pie diagram or an Ogive. The benefit of Graphical representation is that the data is often better understood and inferences/results can be made out of it easily and quickly.

Statistics is concerned with scientific method for collecting, organising, summarising, presenting and analysing data as well as with drawing valid conclusions and making reasonable decisions on the basis of such analysis.

20.2 Measures of Central Tendency

Generally, average of a distribution is a measure of the middle or expected value of distribution, such values are known as measures of central tendency. These are five types of central tendency
(i) Mean (ii) Geometric mean (iii) Harmonic mean (iv) Median (v) Mode.

20.3 Mean

The sum of all the observations is divided by the number of observations is called mean and it is denoted by $\bar{x}$.

(i) Mean of Ungrouped or Individual Data

If $x_1, x_2, x_3, \ldots, x_n$ are n observations, then mean by

(a) **Direct method**

$$\bar{x} = \frac{x_1 + x_2 + \ldots. + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(b) **Shortcut method**

$$\bar{x} = A + \frac{1}{n}\sum_{i=1}^{n} d_i$$

where, A = Assumed mean and $d_i = x_i - A$

(ii) Mean of Grouped or Continuous Data

If $x_1, x_2, x_3, \ldots, x_n$ are n observations whose corresponding frequencies are $f_1, f_2, f_3, \ldots, f_n$, then mean by

(a) **Direct method**

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

(b) **Shortcut method**

$$\bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

where A = Assumed mean and $d_i = x_i - A$

***Sample* Problem 1** *The arithmetic mean of the marks from the following table is*

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	12	18	27	20	17	6

(a) 20 (b) 28
(c) 2800 (d) 100

Interpret (b)

Marks	Class mark x	f	fx
0-10	5	12	60
10-20	15	18	270
20-30	25	27	675
30-40	35	20	700
40-50	45	17	765
50-60	55	6	330
		$\Sigma f = 100$	$\Sigma fx = 2800$

$$\therefore \quad \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{2800}{100} = 28$$

Note Class marks, since the grouped data represents the frequencies included between a specific class interval. So, we actually do not know how many times a particular observation lying in between that class limit is repeated. So, assume that the given frequency is the frequency of mid-point of that particular class interval. This mid-point is known as class mark.

Thus, class mark of an interval

$$= \frac{(\text{lower limit of interval} + \text{upper limit of interval})}{2}$$

Weighted Mean

If corresponding weight of $x_1, x_2, \ldots, x_n$ are $w_1, w_2, \ldots, w_n$ respectively, then

$$\text{Weighted mean} = \frac{w_1 x_1 + w_2 x_2 + \ldots.. + w_n x_n}{w_1 + w_2 + \ldots. + w_n}$$

$$= \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

***Sample* Problem 2** *The weighted mean of the first n natural numbers, the weights being the corresponding numbers, is*

(a) $\frac{n+1}{2}$ (b) $\frac{n+2}{2}$
(c) $\frac{2n+1}{3}$ (d) *None of these*

Interpret (c) First n natural numbers are $1, 2, 3, \ldots, n$; whose corresponding weights are $1, 2, 3, \ldots, n$ respectively.

$\therefore$ Weight mean

$$= \frac{1 \times 1 + 2 \times 2 + \ldots.. + n \times n}{1 + 2 + \ldots.. + n}$$

$$= \frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + \ldots + n}$$

$$= \frac{n(n+1)(2n+1)}{\frac{6n(n+1)}{2}} = \frac{2n+1}{3}$$

Combined Mean

If two sets of observations are given, then the combined mean of both the sets can be calculated by the following formula

$$\bar{x}_{12} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

where, $\bar{x}_1$ = mean of first set of observations
n_1 = number of observations in first set
$\bar{x}_2$ = mean of second set of observations
n_2 = number of observations in second set

Properties of Mean

(i) Algebraic sum of the deviations of a set of values from their mean is zero.

(ii) The sum of the squares of the deviations of a set of values is minimum when taken about mean.

(iii) Mean is affected by the change or shifting of origin and scale.

***Sample* Problem 3** *The average salary of male employees in a firm was ₹ 5200 and that of females was ₹ 4200. The mean salary of all the employees was ₹ 5000. The percentage of male and female employees are respectively*

(a) 80, 20 (b) 20, 80 (c) 60, 40 (d) 52, 48

Interpret (a) Let $x_1 = 5200$, $x_2 = 4200$, $\bar{x} = 5000$

Also, we know that $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$\Rightarrow \quad 5000\,(n_1 + n_2) = 5200\,n_1 + 4200\,n_2$

$\Rightarrow \quad \dfrac{n_1}{n_2} = \dfrac{4}{1}$

$\therefore$ The percentage of male employees in the firm

$$= \frac{4}{4+1} \times 100 = 80\%$$

and the percentage of female employees in the firm

$$= \frac{1}{4+1} \times 100 = 20\%$$

20.4 Geometric Mean (GM)

The nth root of the product of the values is called geometric mean.

(i) Geometric Mean for Ungrouped Data

If $x_1, x_2, x_3, \ldots, x_n$ be n observations, then geometric mean is given by $G = (x_1 \cdot x_2 \cdot x_3 \ldots x_n)^{1/n}$

$$\Rightarrow \quad \log G = \frac{1}{n}(\log x_1 + \log x_2 + \log x_3 + \ldots + \log x_n)$$

$$\Rightarrow \quad \log G = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

$$\Rightarrow \quad G = \text{antilog}\left[\frac{1}{n}\sum_{i=1}^{n} \log x_i\right]$$

(ii) Geometric Mean for Grouped Data

If $x_1, x_2, \ldots, x_n$ be n observations whose corresponding frequencies are $f_1, f_2, \ldots, f_n$, then geometric mean is given by

$$\text{GM} = (x_1^{f_1} \cdot x_2^{f_2} \ldots x_n^{f_n})^{1/N} = \text{antilog}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right]$$

***Sample* Problem 4** *The geometric mean of the numbers $3, 3^2, 3^3, \ldots, 3^n$ is*

(a) $3^{2/n}$ (b) $3^{(n-1)/2}$ (c) $3^{n/2}$ (d) $3^{(n+1)/2}$

Interpret (d) $\therefore \text{GM} = (3 \cdot 3^2 \ldots 3^n)^{1/n}$

$$= 3^{\frac{1+2+\ldots+n}{n}}$$

$$= 3^{\frac{n(n+1)}{2n}} = 3^{\frac{n+1}{2}}$$

20.5 Harmonic Mean (HM)

The harmonic mean of any series is the reciprocal of the arithmatic mean of the reciprocals of the observations.

(i) Harmonic Mean for Ungrouped Data

If $x_1, x_2, \ldots x_n$ be n observations, then harmonic mean is given by

$$\text{HM} = \frac{n}{\dfrac{1}{x_1} + \dfrac{1}{x_2} + \ldots + \dfrac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \dfrac{1}{x_i}}$$

(ii) Harmonic Mean for Grouped Data

If $x_1, x_2, \ldots x_n$ be n observations, whose corresponding frequencies are $f_1, f_2, \ldots, f_n$, then harmonic mean is given by

$$\text{HM} = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \dfrac{f_i}{x_i}}$$

***Sample* Problem 5** *Find the harmonic mean of $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}$, occurring with frequencies 1, 2, 3, …, n, respectively.*

(a) $\dfrac{n-1}{3-n}$ (b) $\dfrac{n+1}{3+n}$

(c) $\dfrac{n+1}{3-n}$ (d) *None of these*

Interpret (b) We know that,

$$\text{Harmonic mean} = \frac{\sum f}{\sum\left(\dfrac{f}{x}\right)}$$

$$\therefore \quad \sum f = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

and

$$\sum \frac{f}{x} = \frac{1}{1/2} + \frac{2}{2/3} + \frac{3}{3/4} + \ldots + \frac{n}{n/(n+1)}$$

$$= 2 + \frac{3 \times 2}{2} + \frac{4 \times 3}{3} + \ldots + \frac{n(n+1)}{n}$$

$$= 2 + 3 + 4 + \ldots + n + (n+1)$$

which is an arithmetic progression with $a = 2$ and $d = 1$.

By the formula of sum of n term of an AP, $\sum \frac{f}{x} = \left[\frac{n}{2}\{2a + (n-1)d\}\right]$,

we have

$$= \frac{n}{2}\{2 \times 2 + n - 1\}$$

$$\therefore \quad = \frac{n}{2}(3 + n)$$

$$\therefore \quad \text{Harmonic mean} = \frac{\frac{n(n+1)}{2}}{\frac{n(3+n)}{2}}$$

$$= \frac{n(n+1) \times 2}{n(3+n) \times 2} = \frac{n+1}{3+n}$$

20.6 Median

A set of observations whenever arranged in order of magnitude (ascending or descending), then the middlemost or central value gives the median. It divides the observation into two equal parts in such a way that the number of observations smaller than median is equal to the number greater than it.

(i) Median for Ungrouped or Individual Data

Suppose n observations are arranged in ascending or descending order.

(a) If n is an odd number, then

$$\text{Median} = \text{Value of}\left(\frac{n+1}{2}\right)\text{th term}$$

(b) If n is an even number, then

$$\text{Median} = \frac{\text{Value of}\left(\frac{n}{2}\right)\text{th term} + \text{Value of}\left(\frac{n}{2}+1\right)\text{th term}}{2}$$

(ii) Median for Discrete Frequency Data

First arrange the data in ascending or descending order and find cumulative frequency. Now find $\frac{N}{2}$, where $N = \Sigma f_i$. See the cumulative frequency just greater than $\frac{N}{2}$. The corresponding value of x is median.

(iii) Median for Continuous Data

In this case, the class corresponding to the cumulative frequency just greater than $\frac{N}{2}$ is called the median class and the value of median is obtained by the following formula.

$$\text{Median} = l + \left(\frac{\frac{N}{2} - c}{f}\right) \times h$$

where, l = lower limit of median class

f = frequency of median class

h = size of median class

c = cumulative frequency of class before median class

Note **Cumulative frequency** The cumulative frequency of a value is its frequency plus the frequencies of all smaller values *e.g.*,

x	f	cf
1	4	4
2	6	10 (4 + 6)
3	4	14 (4 + 6 + 4)

***Sample* Problem 6** *From the data given, the median of the average deposit balance of saving for the branch during March 1982 is*

Average deposit balance (₹)	Number of deposit
Less than 100	26
100-200	68
200-300	145
300-400	242
400-500	188
500-600	65
600-700	16

(a) 356 (b) 300

(c) 56.2 (d) 356.2

Interpret (d)

Average deposit balance (₹)	f	Cumulative frequency Cf
Less than 100	26	26
100- 200	68	94
200-300	145	239
300-400	242	481
400-500	188	669
500-600	65	734
600-700	16	750

$$\frac{N}{2} = \frac{750}{2} = 375$$

The frequency just greater than 375 is 481.

$\therefore$ Median class is 300–400.

$$\text{Median} = l + \frac{\left(\frac{N}{2} - c\right)}{f} \times h$$

$$= 300 + \frac{375 - 239}{242} \times 100$$

$$= 300 + 56.2 = 356.2$$

20.7 Mode

Mode is the value of the variate which occurs most frequency.

(i) Mode for Individual Data

The value which is repeated maximum number of times is called mode of the series.

(ii) Mode for Discrete Frequency Data

In this case, mode is the value of the variate corresponding to the maximum frequency.

(iii) Mode for Continuous Data

The class which has maximum frequency is called modal class or group. The mode is given by the formula

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

where, l = lower limit of modal class

h = size of modal class

f_1 = frequency of modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

Note In the symmetrical distribution, the values of mean, median and mode coincide.

***Sample* Problem 7** *The mode of the following distribution is*

Class interval	Frequency
0-10	5
10-20	8
20-30	7
30-40	12
40-50	28
50-60	20
60-70	10
70-80	10

(a) 46 (b) 6.66
(c) 46.67 (d) *None of these*

Interpret (c) Here, maximum frequency is 28. Thus, the class 40 – 50 is the modal class.

$$\therefore \text{ Mode} = 40 + \frac{10(28-12)}{(2\times 28 - 12 - 20)} = 40 + 6.666 = 46.67 \text{ (approx.)}$$

20.8 Emperical Relationship between Mean, Median and Mode

A distribution in which the values of mean, median and mode coincide, is known symmetrical and if these values are not equal, then the distribution is said asymmetrical or skewed. In a moderately skewed distribution, there is a relation amongst mean, median and mode which is as follows

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median})$$

***Sample* Problem 8** *If in a frequency distribution, the mean and median are 20 and 21 respectively, then its mode is approximately*

(a) 24 (b) 23
(c) 25 (d) *None of these*

Interpret (b) $\because$ Mean – Mode = 3 (Mean – Median)

$\therefore$ 20 – Mode = 3(20 – 21)

$\Rightarrow$ 20 – Mode = –3 $\Rightarrow$ Mode = 23

Check Point 1

1. If a variable take values 0, 1, 2, ...,n with frequencies $q^n, \frac{n}{1}q^{n-1}p, \frac{n(n-1)}{1\cdot 2}q^{n-2}p^2, \ldots, p^n$, where p + q = 1, then the mean is
(a) np (b) nq
(c) n (p + q) (d) None of these

2. For a certain frequency table which has been partly reproduced here, the arithmetic mean was found to be ₹ 28.07

Income (in ₹)	15	20	25	30	35	40
Number of Workers	8	12	?	16	3	10

If the total number of workers is 75, then the missing frequencies are
(a) 14, 15 (b) 15, 14
(c) 13, 16 (d) 12, 17

3. The median wage of the following distribution is (in ₹)

Wages (in ₹)	2000-3000	3000-4000	4000-5000	5000-6000	6000-7000
Number of workers	3	5	20	10	5

(a) 4000 (b) 4575
(c) 4675 (d) None of these

4. If the mean of the numbers 27 + x, 31+ x, 89 + x, 107 + x, 156 + x is 82, then the mean of 130 + x, 126 + x, 68 + x, 50 + x, 1+ x is
(a) 75 (b) 157
(c) 82 (d) 80

5. The mode of the following distribution is

Class Interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	5	8	7	12	28	20	10

(a) 46.67 (b) 40 (c) 44.67 (d) 45

6. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70, 80 were wrongly read as 40, 20 and 50, respectively. The correct mean is
(a) 48 (b) 82.5
(c) 50 (d) 80

7. The weighted mean of first n natural numbers whose weights are equal to the squares of corresponding numbers is
(a) $\frac{n+1}{2}$ (b) $\frac{3n(n+1)}{2(2n+1)}$
(c) $\frac{(n+1)(2n+1)}{6}$ (d) $\frac{n(n+1)}{2}$

8. The mean deviation from the median is
(a) greater than the measured from any other value
(b) less than that measured from any other value
(c) equal to that measured from any other value
(d) maximum if all observations are positive

20.9 Positional Measures

These are the values of the variable which divide the total frequency into number of equal parts *e.g.,* Median divides the total frequency into two equal parts.

Some of these are as follows

(i) Quartiles

Those values of the variable which divides total distribution into four equal parts are known as quartiles. Thus, there will be three quartiles dividing the total distribution in four equal parts.

Three quartiles are Q_1 (lower quartile), Q_2 (median), Q_3 (upper quartile).

	Individual or Discrete Series	Continuous Series	Formula
Q_r	$\frac{r(n+1)}{2}$th term	$\frac{rn}{4}$th term	$Q_r = l + \frac{h}{f}\left(\frac{rn}{4} - c\right)$

(ii) Deciles

Those values of variables which divides the total distribution into ten equal parts, are known as deciles.

(iii) Percentiles

Those values of variables which divides the total distribution into hundred parts, are known as percentiles.

20.10 Measures of Dispersion

A measure of dispersion is designed to state the extent to which the individual observations vary from their average. These are following types of dispersion are

(i) Range (ii) Quartile deviation (iii) Mean deviation (iv) Standard deviation (v) Variance

20.11 Range

Range is the difference between two extreme observations of distribution. If L and S are largest and smallest observations respectively, then

$$\text{Range} = L - S \quad \text{and} \quad \text{coefficient of range} = \frac{L-S}{L+S}$$

Note In a grouped frequency distribution, the range is calculated by subtracting the lower limit of lower class intervals from the upper limit of the highest class.

20.12 Quartile Deviation

Quartile deviation or semi-interquartile range QD is given by

$$\text{QD} = \frac{1}{2}(Q_3 - Q_1)$$

and coefficient of $\text{QD} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$

where, Q_1 and Q_3 are the first and third quartiles of the distribution respectively.

***Sample* Problem 9** *The quartile deviation of daily wages of in (₹) of 11 persons given below*

140, 145, 130, 165, 160, 125, 150, 170, 175, 120, 180

(a) 22.5 (b) 25
(c) 20 (d) 21.2

Interpret (c) The given data in ascending order of magnitude is 120, 125, 130, 140, 145, 150, 160, 165, 170, 175, 180

Here, $Q_1 = \left(\frac{n+1}{4}\right)$ th term $= \frac{11+1}{4} = \frac{12}{4} = 3$rd

$= 130$

$Q_3 = \frac{3(n+1)}{4}$th $= \frac{3 \times (11+1)}{4} = 9$th

$= 170$

$\therefore \quad \text{QD} = \frac{Q_3 - Q_1}{2} = \frac{170 - 130}{2} = \frac{40}{2}$

$= 20$

20.13 Mean Deviation

Mean deviation of a distribution is the arithmetic mean of the absolute deviation of the terms of the distribution from its statistical mean (AM, median or mode).

(i) For Ungrouped or Individual Series

If $x_1, x_2, \ldots, x_n$ are n observations, then mean deviation from average, A (usually mean, median, mode) is

$$\text{MD} = \frac{\sum_{i=1}^{n} |x_i - A|}{n} = \frac{\sum_{i=1}^{n} |d_i|}{n}$$

where, $d_i = x_i - A$

(ii) For Grouped Distribution

If $x_1, x_2, \ldots, x_n$ are n observations whose corresponding frequencies are $f_1, f_2, \ldots, f_n$, then mean deviation from average A (usually mean, median or mode) is

$$\text{MD} = \frac{\sum_{i=1}^{n} f_i |x_i - A|}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i |d_i|}{\sum_{i=1}^{n} f_i}$$

$$\text{Coefficient of MD} = \frac{\text{Mean deviation}}{\text{Corresponding average}}$$

Note

- Algebraic sum of the deviations of a set of values from their arithmetic mean is zero.
- Mean deviation is least when taken from median.

***Sample* Problem 10** *Find the mean deviation about median for the following data.*

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of girls	6	8	14	16	4	2

[NCERT]

(a) 10.34 (b) 10.50 (c) 10.40 (d) 10.37

Interpret (a)

Class	f_i	cf	Mid value x_i	$\lvert x_i - M \rvert$	$f_i \lvert x_i - M \rvert$
0-10	6	6	5	$\lvert 5 - 27.86 \rvert = 22.86$	137.16
10-20	8	(14)C	15	$\lvert 15 - 27.86 \rvert = 12.86$	102.88
(20-30)	(14)	28	25	$\lvert 25 - 27.86 \rvert = 2.86$	40.04
30-40	16	44	35	$\lvert 35 - 27.86 \rvert = 7.14$	114.24
40-50	4	48	45	$\lvert 45 - 27.86 \rvert = 17.14$	68.56
50-60	2	50	55	$\lvert 55 - 27.86 \rvert = 27.14$	54.28
Total	$\Sigma f_i = 50$				517.16

$$\frac{N}{2} = \frac{50}{2} = 25$$

$$\Rightarrow \quad C = 14, f = 14, l = 20, h = 10$$

$$\text{Median } M = l + \frac{\frac{N}{2} - C}{f} \times h$$

$$= 20 + \frac{25 - 14}{14} \times 10$$

$$= 20 + \frac{11 \times 10}{14}$$

$$= 20 + 7.86 = 27.86$$

$$\therefore \text{ Mean deviation about median} = \frac{\Sigma f_i |x_i - M|}{\Sigma f_i}$$

$$= \frac{517.16}{50} = 10.34$$

20.14 Standard Deviation

The standard deviation of a variate is the square root of the arithmetic mean of the squares of all deviations of the values of the variate x from the arithmatic mean of the observations and is denoted by σ.

(i) Standard Deviation for Individual Series

If $x_1, x_2, x_3, \ldots, x_n$ are n observations, then

$$\text{SD } (\sigma) = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} \quad \text{or} \quad (\sigma) = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - (\bar{x})^2}$$

or $$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} d_i\right)^2}$$

where, $d_i = x_i - \bar{x}$

(ii) Standard Deviation for Frequency Distribution

If $x_1, x_2, x_3, \ldots, x_n$ are n observations and corresponding frequencies are $f_1, f_2, \ldots, f_n$ respectively, then

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}} \quad \text{or} \quad (\sigma) = \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - (\bar{x})^2$$

or $$\sigma = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i^2\right) - \frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i\right)^2}$$

where, $d_i = x_i - \bar{x}$ and $N = \Sigma f_i$

Properties of Standard Deviation

(i) It is independent of origin.

(ii) It is dependent of change of scale.

(iii) It is not less than mean deviation from mean.

(iv) Let n_1 and n_2 be the sizes of two series. Their means and standard deviations are $\bar{x}_1, \bar{x}_2$ and σ_1, σ_2 respectively. Let $\bar{x}$ denote the combined mean of two series, that is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$\therefore$ The combined standard deviation of two series is given by

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

or $$\sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

where, $d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$

(v) $\sigma = \frac{3(\text{Quartile deviation})}{2}$

(vi) $\sigma = \frac{5(\text{Mean deviation})}{4}$

***Sample* Problem 11** *The mean and standard deviation of 20 observations are found to be 10 and 2. On rechecking, it was found that an observation 8 was incorrect. Find the correct standard deviation if wrong item is omitted.* **[NCERT]**

(a) 1.88 (b) 1.99

(c) 1.95 (d) *None of these*

Interpret (*b*) Given, $\bar{x} = 10$ and $\sigma = 2, n = 20$

$\Rightarrow \quad \frac{\Sigma x_i}{20} = 10$

$\Rightarrow \quad \Sigma x_i = 20 \times 10$

$\Rightarrow \quad \Sigma x_i = 200$

If observation 8 is omitted, then $\Sigma x_i = 200 - 8 = 192$

Now, remaining number of observation = 19

$\Rightarrow$ Correct mean $= \frac{\Sigma x_i}{n} = \frac{192}{19} = 10.10$

Again, $\sigma = 2 \Rightarrow \sigma^2 = 4$

$\Rightarrow \quad \frac{\Sigma x_i^2}{n} - (\bar{x})^2 = 4 \Rightarrow \frac{\Sigma x_i^2}{20} - (10)^2 = 4$

$\Rightarrow \quad \Sigma x_i^2 = (4 + 100) \times 20 = 104 \times 20 = 2080$

If observation 8 is omitted, then $\Sigma x_i^2 = 2080 - 64 = 2016$

Now, correct standard deviation $\sigma = \sqrt{\frac{2016}{19} - \left(\frac{192}{19}\right)^2}$

$$= \sqrt{\frac{2016 \times 19 - (192)^2}{19 \times 19}}$$

$$= \frac{1}{19}\sqrt{38304 - 36864}$$

$$= \frac{1}{19}\sqrt{1440} = \frac{37.95}{19} = 1.99$$

Hot Spot

Variance

The variance of a variate is the square of standard deviation and it is denoted by σ^2 or Var (x).

$$\text{Var}(x) = \sigma^2 = (\text{standard deviation})^2$$

Coefficient of dispersion $= \frac{\sigma}{\bar{x}}$

These types of problem are directly asked in two consecutive years of AIEEE (JEE Main) 2013, 2012 and 2006. To solve such types of problems following steps are given below.

Step I Firstly write the given data, if it is given.

Step II If either change of origin or change of scale is given, then apply it.

Step III If other condition is given, then apply it.

Step IV Simplify it. Use the formula of variance.

***Sample* Problem 12** *Find the variance for the following frequency distribution*

Class	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	2	3	5	10	3	5	2

(a) 2280 (b) 2276

(c) 2275 (d) *None of the above*

Interpret (*b*) **Step I** Write the given data.

Class	Frequency (f_i)	Mid-value x_i	Deviation from mean $d_i = \frac{x_i - A}{h}$, $A = 105$	d_i^2	$f_i d_i$	$f_i d_i^2$
0-30	2	15	– 3	9	– 6	18
30-60	3	45	– 2	4	– 6	12
60-90	5	75	– 1	1	– 5	5
90-120	10	105	0	0	0	0
120-150	3	135	1	1	3	3
150-180	5	165	2	4	10	20
180-210	2	195	3	9	6	18
Total	30				2	76

Step II Use the formula and simplify it.

$$\text{Variance} = \left[\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2\right] \times h^2 = \left[\frac{76}{30} - \left(\frac{2}{30}\right)^2\right] \times (30)^2$$

$$= \left[\frac{76}{30} - \frac{4}{30 \times 30}\right] \times 900$$

$$= \left(\frac{2280 - 4}{900}\right) \times 900$$

$$= \frac{2276}{900} \times 900 = 2276$$

Effect of Average and Dispersion on Change of Origin and Scale

	Change of origin	Change of scale
Mean	Dependent	Dependent
Median	Not dependent	Dependent
Mode	Not dependent	Dependent
Standard deviation	Not dependent	Dependent
Variance	Not dependent	Dependent

***Sample* Problem 13** *The mean and standard deviation of marks obtained by 50 students of a class in three subjects, Mathematics, Physics and Chemistry are given below*

Subject	Mathematics	Physics	Chemistry
Mean	42	32	40.9
Standard deviation	12	15	20

Which of these three subjects shows the highest variability in marks and which shows the lowest? [**NCERT**]

(a) *Chemistry, Mathematics* (b) *Mathematics, Physics*
(c) *Chemistry, Physics* (d) *None of these*

Interpret (a) Here, $n = 50$

For Mathematics,

Coefficient of variation (CV) $= \frac{\sigma}{\bar{x}} \times 100$

$$= \frac{12}{42} \times 100 = \frac{2}{7} \times 100 = \frac{200}{7}$$

$$= 28.57 \quad \text{...(i)}$$

For Physics, $\text{CV} = \frac{\sigma}{\bar{x}} \times 100 = \frac{15}{32} \times 100$

$$= \frac{1500}{32} = 46.87 \quad \text{...(ii)}$$

For Chemistry, $\text{CV} = \frac{\sigma}{\bar{x}} \times 100$

$$= \frac{20}{40.9} \times 100 = \frac{2000}{40.9} = 48.89 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we have

CV of Chemistry > CV of Physics > CV of Mathematics

$\therefore$ Chemistry shows the highest variability and Mathematics shows the least variability.

Check Point **2**

1. Let s be the standard deviation of n observations. Each of the n observations is multiplied by a constant c. Then, the standard deviation of the resulting number is
(a) s (b) cs
(c) $s\sqrt{c}$ (d) None of these

2. The mean deviation from the mean of the AP a, a + d, a + 2d, ..., a + 2nd is
(a) $n(n+1)d$ (b) $\frac{n(n+1)d}{2n+1}$
(c) $\frac{n(n+1)d}{2n}$ (d) $\frac{n(n-1)d}{2n+1}$

3. The variance of 6, 8, 10, 12, 14 is
(a) 7 (b) 8
(c) 12 (d) 16

4. If MD is 12, the value of SD will be
(a) 75 (b) 12
(c) 24 (d) None of these

5. The mean of 5 observations is 4 and their variance is 5.2. If three If there observations are 1, 2 and 6, then the other two are
(a) 2 and 9 (b) 3 and 8
(c) 4 and 7 (d) 5 and 6

6. Let r be the range and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ be the standard deviation of a set of observations $x_1, x_2, \ldots, x_n$, then
(a) $S \le r\sqrt{\frac{n}{n-1}}$ (b) $S = r\sqrt{\frac{n}{n-1}}$
(c) $S \ge r\sqrt{\frac{n}{n-1}}$ (d) None of these

7. The mean and standard deviation of six observations are 8 and 4, respectively. If each observation is multiplied by 3, then find the mean and new standard deviation of the resulting observations.
(a) 24, 12 (b) 24,144 (c) 24, 150 (d) 12, 12

8. The sum and sum of squares corresponding to length x (in cm) and weight (y) (in g) of 50 plant products are given below

$$\sum_{i=1}^{50} x_i = 212, \sum_{i=1}^{50} x_i^2 = 902.8$$

and $$\sum_{i=1}^{50} y_i = 261, \sum_{i=1}^{50} y_i^2 + 1457.6$$

Which is more varying, the length or weight?
(a) weight (b) length
(c) cannot be determined (d) None of these

WORKED OUT Examples

Example 1 *Mean of 9 observations is 100 and mean of 6 observations is 80, then the mean of 15 observations is*

(*a*) 92 (*b*) 29
(*c*) 1380 (*d*) 15

Solution (*a*) $n_1 = 9, \quad \bar{x}_1 = 100$ and $n_2 = 6, \quad \bar{x}_2 = 80$

$$\therefore \quad \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$= \frac{9 \times 100 + 6 \times 80}{9 + 6}$$

$$= \frac{1380}{15} = 92$$

Example 2 *Mean and mode of the following data are respectively*

Class	f_i
0-10	22
10-20	38
20-30	46
30-40	35
40-50	20

(*a*) 24.56, 24.26 (*b*) 24.26, 24.56
(*c*) 39.55, 16.1 (*d*) *None of these*

Solution (a)

Class	x_i	f_i	$x_i f_i$
0-10	5	22	110
10-20	15	38	570
20-30	25	46	1150
30-40	35	35	1225
40-50	45	20	900
		161	3955

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{3955}{161} = 24.56 \text{ (approx.)}$$

Median = 24.26 (as solved in example 1)

$$\therefore \quad \text{Mode} = 3\,(24.46) - 2\,(24.56)$$

$$= 24.26$$

Example 3 *Median, for the following distribution is*

Wages (₹)	Number of labour
0-10	22
10-20	38
20-30	46
30-40	35
40-50	20

(*a*) 24.46 (*b*) 14.80
(*c*) 34.21 (*d*) *None of these*

Solution (*a*)

Class	f	cf
0-10	22	22
10-20	38	60
20-30	46	106
30-40	35	141
40-50	20	161

Here, $\frac{N}{2} = \frac{161}{2} = 80.5$

Median class is 20 – 30.

$$\therefore \quad \text{Median} = l + \left(\frac{\frac{N}{2} - c}{f}\right) h$$

$$= 20 + \left(\frac{80.5 - 60}{46} \times 10\right) = 24.46$$

Example 4 *The median of distribution* 83, 54, 78, 64, 90, 59, 67, 72, 70, 73 *is*

(*a*) 71 (*b*) 70
(*c*) 72 (*d*) *None of these*

Solution (*a*) On arranging in ascending order, we get

54, 59, 64, 67, 70, 72, 73, 78, 83, 90

$n = 10$ (even number)

$$\therefore \text{Median} = \frac{\text{Value of } \frac{10}{2} \text{ th term} + \text{Value of } \left(\frac{10}{2} + 1\right) \text{th term}}{2}$$

$$= \frac{\text{Value of 5th term} + \text{Value of 6th term}}{2} = \frac{70 + 72}{2} = 71$$

Example 5 *The mean and standard deviation of a group of 100 observations were found to be 20 and 3, respectively. Later on it was found that three observations were incorrect, which are recorded by 21, 21 and 18. Find the mean and standard deviation, if the incorrect observation are omitted.* **[NCERT]**

(*a*) 3.05 (*b*) 3.03
(*c*) 3.09 (*d*) *None of these*

Solution (*b*) Given, $n = 100, \bar{x} = 20, \sigma = 3$

$\because \quad \bar{x} = 20$

$\Rightarrow \quad \dfrac{\Sigma x}{100} = 20$

$\Rightarrow \quad \Sigma x_i = 100 \times 20$

$\Rightarrow \quad \Sigma x_i = 2000$

Now, incorrect observations 21, 21 and 18 are omitted, then correct sum is

$$\Sigma x_i = 2000 - 21 - 21 - 18 = 2000 - 60 = 1940$$

Now, correct mean of remaining 97 observations is

$$\bar{x} = \frac{1940}{97} = 20$$

Again, $\quad \sigma = 3 \Rightarrow \sqrt{\dfrac{\Sigma x_i^2}{n} - (\bar{x})^2} = 3$

$\Rightarrow \quad \dfrac{\Sigma x_i^2}{100} - (20)^2 = 9$

$\Rightarrow \quad \dfrac{\Sigma x^2}{100} = 9 + 400$

$\Rightarrow \quad \Sigma x^2 = 409 \times 100 = 40900$

Now, correct Σx^2 is

$$\Sigma x^2 = 40900 - (21)^2 - (21)^2 - (18)^2$$
$$= 40900 - 441 - 441 - 324$$
$$= 40900 - 1206 = 39694$$

Now, correct SD for remaining 97 observations is

$$\sigma = \sqrt{\frac{39694}{97} - \left(\frac{1940}{97}\right)^2}$$
$$= \sqrt{409.2 - (20)^2} = \sqrt{409.2 - 400}$$
$$= \sqrt{9.2} = 3.03$$

Example 6 *Find the mean deviation about mean for the data in following tables*

Income per day	Number of persons
0-100	4
100-200	8
200-300	9
300-400	10
400-500	7
500-600	5
600-700	4
700-800	3

[NCERT]

(*a*) 358, 157.92 (*b*) 359, 158.92
(*c*) 358, 158.20 (*d*) *None of these*

Solution (*a*)

Class	(f_i)	Mid value x_i	$d_i = \dfrac{x_i - A}{B}$, $A = 350$, $h = 100$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0-100	4	50	– 3	–12	308	1232
100-200	8	150	– 2	–16	208	1664
200-300	9	250	– 1	–9	108	972
300-400	10	350	0	0	8	80
400-500	7	450	1	7	92	644
500-600	5	550	2	10	192	960
600-700	4	650	3	12	292	1168
700-800	3	750	4	12	392	1176
Total	$\Sigma f_i = 50$			4		7896

$$\text{Mean } \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} \times h = 350 + \frac{4}{50} \times 100 = 350 + 8$$

$$\bar{x} = 358$$

$$\text{Mean deviation about the mean} = \frac{\Sigma f_i \mid x_i - \bar{x} \mid}{\Sigma f_i} = \frac{7896}{50} = 157.92$$

Example 7 *The mean life of a sample of 60 bulbs was 650 h and the standard deviation was 8 h. A second sample of 80 bulbs has a mean life of 660 h and standard deviation 7 h. Find the over all standard deviation.* **[NCERT Exemplar]**

(*a*) 8.97 (*b*) 8.98
(*c*) 8.94 (*d*) *None of these*

Solution (*c*) Given, $n_1 = 60, \bar{x}_1 = 650, \sigma_1 = 8,$

$n_2 = 80, \bar{x}_2 = 660, \sigma_2 = 7$

$$\therefore \text{ Combined SD} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

$$= \sqrt{\frac{60 \times 64 + 80 \times 49}{60 + 80} + \frac{60 \times 80\,(650 - 660)^2}{(60 + 80)^2}}$$

$$= \sqrt{\frac{3840 + 3920}{140} + \frac{(4800 \times 100)}{(140)^2}}$$

$$= \sqrt{\frac{7760}{140} + \frac{480000}{19600}}$$

$$= \sqrt{\frac{776}{14} + \frac{4800}{196}}$$

$$= \sqrt{55.42 + 24.49}$$

$$= \sqrt{79.91} = 8.94$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Mean, Geometric and Harmonic Mean

1. A statistical measure which cannot be determined graphically is
(a) median (b) mode
(c) harmonic mean (d) mean

2. The arithmetic mean of an AP and the mean of first and last term of an AP are
(a) equal
(b) unequal
(c) square of each other
(d) None of the above

3. The mean height of 25 male workers in a factory is 61 cm and the mean height of 35 female workers in the same factory is 58 cm. The combined mean height of 60 workers in the factory is
(a) 59.25 (b) 59.5
(c) 59.75 (d) 58.75

4. The average of the squares of the numbers 0, 1, 2, 3, 4,..., n is
(a) $\frac{1}{2}n(n+1)$ (b) $\frac{1}{6}n(2n+1)$
(c) $\frac{1}{6}(n+1)(2n+1)$ (d) $\frac{1}{6}n(n+1)$

5. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70 and 80 were wrongly read as 40, 20 and 50, respectively. The correct mean is
(a) 48 (b) $82\frac{1}{2}$
(c) 50 (d) 80

6. The AM of n numbers of a series is $\bar{x}$. If the sum of first $(n-1)$ terms is k, then the nth number is
(a) $\bar{x} - k$ (b) $n\bar{x} - k$
(c) $\bar{x} - nk$ (d) $n\bar{x} - nk$

7. The mean of the series $x_1, x_2, \ldots, x_n$ is $\bar{x}$. If x_2 is replaced by λ, then the new mean is
(a) $\bar{x} - x_2 + \lambda$ (b) $\frac{\bar{x} - x_2 - \lambda}{n}$
(c) $\frac{(n-1)\bar{x} + \lambda}{n}$ (d) $\frac{n\bar{x} - x_2 + \lambda}{n}$

8. For a certain frequency table which has been partly reproduced here, the arithmetic mean was found to be ₹ 28.07.

Income (in ₹)	Number of workers
15	8
20	12
25	?
30	16
35	?
40	10

If the total number of workers is 75, then missing frequencies are respectively
(a) 14, 15 (b) 15, 14 (c) 13, 16 (d) 12, 17

9. If the average of the numbers 148, 146, 144, 142, ... in AP, be 125, then the total numbers in the series will be
(a) 18 (b) 24 (c) 30 (d) 48

10. The mean of the values of 1, 2, 3, ..., n with respectively frequencies $x, 2x, 3x, \ldots, nx$ is
(a) $\frac{n}{2}$ (b) $\frac{1}{3}(2n+1)$
(c) $\frac{1}{6}(2n+1)$ (d) $\frac{n}{2}$

11. The mean of n items is $\bar{x}$. If the first term is increased by 1, second by 2 and so on, then the new mean is
(a) $\bar{x} + n$ (b) $\bar{x} + \frac{n}{2}$
(c) $\bar{x} + \frac{n+1}{2}$ (d) None of these

12. If the mean of n observations $1^2, 2^2, 3^2, \ldots, n^2$ is $\frac{46n}{11}$, then n is equal to
(a) 11 (b) 12
(c) 23 (d) 22

13. The weighted AM of first n natural numbers whose weights are equal to the corresponding numbers is equal to
(a) $2n + 1$ (b) $\frac{1}{2}(2n + 1)$
(c) $\frac{1}{3}(2n + 1)$ (d) $\frac{2n + 1}{6}$

14. The AM of the series 1, 2, 4, 8, 16, …, 2^n is
(a) $\frac{2^n - 1}{n}$ (b) $\frac{2^{n+1} - 1}{n + 1}$
(c) $\frac{2^n + 1}{n}$ (d) $\frac{2^n - 1}{n + 1}$

15. In a class of 50 students, 10 have failed and their average marks are 28. The total marks obtained by the entire class are 2800. The average marks of those who have passed, are
(a) 43 (b) 53
(c) 63 (d) 70

16. If $\bar{x}_1$ and $\bar{x}_2$ are the means of two distributions such that $\bar{x}_1 < \bar{x}_2$ and $\bar{x}$ is the mean of the combined distribution, then
(a) $\bar{x} < \bar{x}_1$ (b) $\bar{x} > \bar{x}_2$
(c) $\bar{x} = \frac{\bar{x}_1 + \bar{x}_2}{2}$ (d) $\bar{x}_1 < \bar{x} < \bar{x}_2$

17. If the mean of the set of numbers $x_1, x_2, \ldots, x_n$ is $\bar{x}$, then the mean of the numbers $x_i + 2i, 1 \le i \le n$ is
(a) $\bar{x} + 2n$ (b) $\bar{x} + n + 1$
(c) $\bar{x} + 2$ (d) $\bar{x} + n$

18. The mean of 30 given numbers, when it is given that the mean of 10 of them is 12 and the mean of the remaining 20 is 9, is equal to
(a) 11 (b) 10 (c) 9 (d) 5

19. Mean of 100 observations is 45. If it was later found that two observations 19 and 31 were incorrectly recorded as 91 and 13. The correct mean is
(a) 44 (b) 45
(c) 44.46 (d) 45.54

20. The mean of a set of observations is $\bar{x}$. If each observation is divided by, $\alpha \neq 0$ and then is increased by 10, then the mean of the new set is
(a) $\frac{\bar{x}}{\alpha}$ (b) $\frac{\bar{x} + 10}{\alpha}$
(c) $\frac{\bar{x} + 10\alpha}{\alpha}$ (d) $\alpha\bar{x} + 10$

21. The mean of n observations is $\bar{x}$. If one observation x_{n+1} is added, then the mean remains same. The value of x_{n+1} is
(a) 0 (b) 1
(c) n (d) $\bar{x}$

22. In a class of 100 students, the average amount of pocket money is ₹ 35 per student. If the average is ₹ 25 for girls and ₹ 50 for boys, then the number of girls in the class is
(a) 20 (b) 40 (c) 60 (d) 80

23. If the sum of deviations of a number of observations about 4 is 30 and that about 3 is 40. Then, mean of the observations is
(a) 7 (b) 10
(c) 11 (d) None of these

24. If a variate x is expressed as a linear function of two variates u and v in the form $x = au + bv$, then mean $\bar{x}$ of x is
(a) $a\bar{u} + b\bar{v}$ (b) $\bar{u} + \bar{v}$
(c) $\bar{b}u + \bar{a}v$ (d) None of these

25. The average salary of male employees in a firm was ₹ 520 and that of females was ₹ 420. The mean salary of all the employees was ₹ 500. The percentage of male employees is
(a) 80% (b) 60% (c) 40% (d) 20%

26. The AM of n observations is M. If the sum of $(n - 4)$ observations is a, then the mean of remaining four observations is
(a) $\frac{nM - a}{4}$ (b) $\frac{nM + a}{2}$
(c) $\frac{nM - a}{2}$ (d) $nM + a$

27. The mean of the n observations $x_1, x_2, x_3, \ldots, x_n$ be $\bar{x}$. Then, the mean of n observations $2x_1 + 3, 2x_2 + 3, 2x_3 + 3, \ldots, 2x_n + 3$ is
(a) $3\bar{x} + 2$ (b) $2\bar{x} + 3$
(c) $\bar{x} + 3$ (d) $2\bar{x}$

28. The mean age of a combined group of men and women is 25 yr. If the mean age of the group of men is 26 yr and that of the group of women is 21 yr, then the percentage of men and women in the group are respectively
(a) 60, 40 (b) 80, 20
(c) 20, 80 (d) 40, 60

29. The AM of ${}^{2n+1}C_0, {}^{2n+1}C_1, {}^{2n+1}C_2, \ldots, {}^{2n+1}C_n$ is
(a) $\frac{2^n}{n}$ (b) $\frac{2^n}{n + 1}$
(c) $\frac{2^{2n}}{n}$ (d) $\frac{2^{2n}}{(n + 1)}$

30. The mean of the values $0, 1, 2, 3, \ldots, n$ with the corresponding weights ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ respectively is

(a) $\frac{2^n}{(n+1)}$ (b) $\frac{2^{n+1}}{n(n+1)}$

(c) $\frac{n+1}{2}$ (d) $\frac{n}{2}$

31. If a variable takes values 0, 1, 2, ..., n with frequencies $q^n, \frac{n}{1}q^{n-1}p, \frac{n(n-1)}{1\cdot 2}q^{n-2}p^2, \ldots p^n$, where $p + q = 1$, then the mean is

(a) np (b) nq

(c) $n(p + q)$ (d) None of these

32. The GM of the numbers $2, 2^2, 2^3, \ldots 2^n$ is

(a) $2^{2/n}$ (b) $2^{n/2}$

(c) $2^{n-1/2}$ (d) $2^{n+1/2}$

33. Let G_1 and G_2 be the ... geometric means of two ... series $x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n$. If G is the geometric mean of $\frac{x_i}{y_i}$, $i = 1, 2, \ldots, n$. Then, G is equal to

(a) $G_1 - G_2$ (b) $\frac{\log G_1}{\log G_2}$

(c) $\frac{G_1}{G_2}$ (d) $\log\left(\frac{G_1}{G_2}\right)$

34. Geometric mean of first group of 4 observations is 8 and that of second group of 3 observations is 1024. Then, geometric mean of all the 7 observations is

(a) 64 (b) 32

(c) 128 (d) None of these

35. If the observations 2, 4, 8 and 16 occur 8, 6, 4 and 2 times respectively, then the geometric mean of the observations is

(a) 8 (b) $4\sqrt{2}$

(c) 4 (d) None of these

36. The harmonic mean of the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ is

(a) $\frac{1}{n+1}$ (b) $\frac{2}{n+1}$

(c) $\frac{3}{n+1}$ (d) None of these

37. If harmonic mean of first 5 observations is $\frac{5}{2}$ and harmonic mean of another 5 observations is $\frac{9}{2}$, then harmonic mean of all 10 observations is

(a) 7 (b) $\frac{45}{14}$

(c) $\frac{101}{36}$ (d) None of these

38. Geometric mean of 10 observations is 8. If geometric mean of first six observations is 4, then geometric mean of last four observations is

(a) $16\sqrt{2}$ (b) $8\sqrt{2}$

(c) 16 (d) None of these

39. Two possible observations have arithmetic mean 3 and geometric mean $2\sqrt{2}$. If each observation is multiplied by 2, then harmonic mean will be

(a) $\frac{16}{3}$ (b) $\frac{8}{3}$

(c) 12 (d) None of these

40. Geometric mean of first group of 5 observations is 8 and that of second group of 4 observations is $128\sqrt{2}$. Then, grouped geometric mean is

(a) 64 (b) $32\sqrt{2}$

(c) 32 (d) None of these

Median and Mode

41. Which of the following is not a measure of central tendency?

(a) Mean (b) Median

(c) Mean-deviation (d) Mode

42. For dealing with qualitative data the best average is

(a) AM (b) GM (c) median (d) mode

43. The mean weight of 9 items is 15. If one more item is added to the series, the mean becomes 16. The value of 10th item is

(a) 35 (b) 30 (c) 25 (d) 20

44. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}, \alpha - 3, \alpha - 2, \alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha + 5\ (\alpha > 0)$, then the median is

(a) $\alpha - \frac{5}{4}$ (b) $\alpha - \frac{1}{2}$ (c) $\alpha - 2$ (d) $\alpha + \frac{5}{4}$

45. Coefficient of skewness for the values

Median = 18.8, $Q_1 = 14.6$, $Q_3 = 25.2$ is

(a) 0.2 (b) 0.5

(c) 0.7 (d) None of these

46. Median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3, \ldots, {}^{2n}C_n$ (where, n is even) is

(a) ${}^{2n}C_{\frac{n}{2}}$ (b) ${}^{2n}C_{\frac{n+1}{2}}$

(c) ${}^{2n}C_{\frac{n-1}{2}}$ (d) None of these

47. If the sum of 11 consecutive natural numbers is 2761, then the middle number is

(a) 249 (b) 250

(c) 251 (d) 252

48. If the mode of the data is 18 and the mean is 24, then median is
(a) 18 (b) 24 (c) 22 (d) 21

49. If in a moderately asymmetrical distribution mode and mean of the data are 6λ and 9λ respectively, then median is
(a) 8λ (b) 7λ (c) 6λ (d) 5λ

50. If the median of 21 observations is 40 and if the observations greater than the median are increased by 5, then the median of the new data will be
(a) 40 (b) 45
(c) $4 + \frac{50}{21}$ (d) $45 - \frac{50}{21}$

51. In a moderately skewed distribution the values of mean and median are 5 and 6, respectively. The value of mode in such a situation is approximately equal to
(a) 8 (b) 11
(c) 16 (d) None of these

Quartile Deviation and Mean Deviation

52. Consider the following statements
I. The values of median and mode can be determined graphically.
II. Mean, median and mode have the same unit.
III. Range is the best measure of dispersion.
Which of these is/are correct?
(a) Only I (b) Only II
(c) Both II and III (d) None of these

53. The quartile deviation of daily wages (in ₹) of 7 persons given below 12, 7, 15, 10, 17, 19 and 25 is
(a) 14.5 (b) 5
(c) 9 (d) 4.5

54. The quartile deviation for the following data is

x	2	3	4	5	6
f	3	4	8	4	1

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

55. If two variables x and y are such that $2y + 5 = 3x$ and Quartile Deviation (QD) of x is 8, then (QD) of y is
(a) 2 (b) 8
(c) 4 (d) None of these

56. Find the quartile deviation of the following distribution.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	10	20	35	25	10

(a) 8.20 (b) 8.25
(c) 8.30 (d) None of these

57. For a series the value of mean deviation is 15, the most likely value of its quartile deviation is
(a) 12.5 (b) 11.6
(c) 13 (d) 9.7

58. The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is [NCERT Exemplar]
(a) 2 (b) 2.57 (c) 3 (d) 3.75

59. When tested, the lives (in hours) of 5 bulbs were noted as follows
1357, 1090, 1666, 1494, 1623
The mean deviations (in hours) from their mean is [NCERT Exemplar]
(a) 178 (b) 179
(c) 220 (d) 356

60. Following are the marks obtained by 9 students in a Mathematics test
50, 69, 20, 33, 53, 39, 40, 65, 59
The mean deviation from the median is [NCERT Exemplar]
(a) 9 (b) 10.5
(c) 12.67 (d) 14.76

61. Find the mean deviation about the mean of the set of first n natural numbers when n is an odd number. [NCERT Exemplar]
(a) $\frac{n^2 + 1}{4n}$ (b) $\frac{n^2 - 1}{4n}$
(c) $\frac{n^2 - 3}{5n}$ (d) None of these

62. Find the mean deviation from the median of the following data [NCERT Exemplar]

Class interval	0-6	6-12	12-18	18-24	24-30
Frequency	4	5	3	6	2

(a) 7.08 (b) 7
(c) 7.1 (d) 7.05

Standard Deviation

63. The standard deviation of the data 6, 5, 9, 13, 12, 8, 10 is [NCERT Exemplar]
(a) $\sqrt{\frac{52}{7}}$ (b) $\frac{52}{7}$
(c) $\sqrt{6}$ (d) 6

64. The mean of 100 observations is 50 and their standard deviation is 5. The sum of all squares of all the observations is [NCERT Exemplar]
(a) 50000 (b) 250000
(c) 252500 (d) 255000

65. Let a, b, c, d and e be the observations with mean m and standard deviation S. The standard deviation of the observations $a + k, b + k, c + k, d + k, e + k$ is [NCERT Exemplar]

(a) S (b) kS
(c) $S + k$ (d) $\frac{S}{k}$

66. The SD of 15 items is 6 and if each item is decreases by 1, then standard deviation will be

(a) 5 (b) 7 (c) $\frac{91}{15}$ (d) 6

67. If SD of X is S, then SD of the variable $\mu = \frac{aX + b}{c}$, where a, b and c are constants, is

(a) $\left|\frac{c}{a}\right|\sigma$ (b) $\left|\frac{a}{c}\right|\sigma$
(c) $\left|\frac{b}{c}\right|\sigma$ (d) $\frac{c^2}{a^2}\sigma$

68. If $\bar{x}$ is the arithmetic mean of n independent variates $x_1, x_2, x_3, \ldots, x_n$ each of the standard deviation σ, then variance $(\bar{x})$ is

(a) $\frac{\sigma^2}{n}$ (b) $\frac{n\sigma^2}{2}$
(c) $\frac{(n+1)\sigma^2}{3}$ (d) None of these

69. Let x_1, x_2, x_3, x_4 and x_5 be the observations with mean m and standard deviation S. The standard deviation of the observations

kx_1, kx_2, kx_3, kx_4 and kx_5 is [NCERT Exemplar]

(a) $k + S$ (b) $\frac{S}{k}$
(c) kS (d) S

70. Let $x_1, x_2, \ldots, x_n$ be n observations. Let $w_i = lx_i + k$ for $i = 1, 2, \ldots, n$, where l and k are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of w_i's is 55 and standard deviation of w_i's is 15, the values of l and k should be [NCERT Exemplar]

(a) $l = 1.25, k = -5$ (b) $l = -1.25, k = 5$
(c) $l = 2.5, k = -5$ (d) $l = 2.5, k = 5$

71. The variance of the first n natural numbers is

(a) $\left(\frac{n^2 - 1}{12}\right)$ (b) $\frac{n(n^2 - 1)}{12}$
(c) $\left(\frac{n^2 + 1}{12}\right)$ (d) $\frac{n(n^2 + 1)}{12}$

72. Standard deviations for first 10 natural numbers is [NCERT Exemplar]

(a) 5.5 (b) 3.87
(c) 2.97 (d) 2.87

73. Consider the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. If 1 is added to each number, then variance of the numbers so obtained is [NCERT Exemplar]

(a) 6.5 (b) 2.87
(c) 3.87 (d) 8.25

74. The following information relates to a sample of size 60 : $\Sigma x^2 = 18000$, $\Sigma x = 960$.

The variance is [NCERT Exemplar]

(a) 6.63 (b) 16
(c) 22 (d) 44

75. Coefficient of variation of two distributions are 50 and 60 and their arithmetic means are 30 and 25 respectively. Difference of their standard deviation is [NCERT Exemplar]

(a) 0 (b) 1 (c) 1.5 (d) 2.5

76. If the variance of $1, 2, 3, 4, 5, \ldots, 10$ is $\frac{99}{12}$, then the standard deviation of $3, 6, 9, 12, \ldots, 30$ is

(a) $\frac{297}{4}$ (b) $\frac{3}{2}\sqrt{33}$
(c) $\frac{3}{2}\sqrt{99}$ (d) $\sqrt{\frac{99}{12}}$

77. The mean and standard deviation of some data for the time taken to complete a test are calculated with the following results

Number of observations = 25, mean = 18.2 s, standard deviation = 3.25 s.

Further, another set of 15 observations $x_1, x_2, \ldots, x_{15}$, also in seconds, is now available and we have $\sum_{i=1}^{15} x_i = 279$ and $\sum_{i=1}^{15} x_i^2 = 5524$. Find the standard deviation based on all 40 observations. [NCERT Exemplar]

(a) 3.80 (b) 3.87
(c) 3.70 (d) None of these

78. If for a distribution $\Sigma(x - 5) = 3$, $\Sigma(x - 5)^2 = 43$ and the total number of items is 18, find the mean and standard deviation. [NCERT Exemplar]

(a) 94, 1.53
(b) 93, 1.53
(c) 93, 1.55
(d) None of the above

79. The mean and variance of n values of a variable x are 0 and σ^2, respectively. If the variable $y = x^2$, then mean of y is

(a) σ (b) σ^2
(c) 1 (d) None of these

80. The variance of the numbers 2, 3, 11 and x is $\frac{49}{4}$. Find the value of x.

(a) 6, $\frac{14}{3}$ (b) 6, $\frac{14}{5}$
(c) 6, $\frac{16}{3}$ (d) None of these

81. If runs of two players A and B in 10 cricket matches are such that player A has mean 50 and variance 36 and player B has mean 60 and variance 81 of runs, then the player more consistent in runs is

(a) A
(b) B
(c) Both are equally consistent
(d) None of the above

82. Mean and standard deviation of 100 observations were found to be 40 and 10, respectively. If at the time of calculation two observations were wrongly taken as 30 and 70 in place of 3 and 27, respectively, find the correct standard deviation. [NCERT Exemplar]

(a) 10.20 (b) 10.24
(c) 10.29 (d) 10.27

83. Two samples of sizes 100 and 150 have means 45 and 55 and standard deviations 7 and 12, respectively. Find the mean and standard deviation of the combined sample.

(a) 11.30 (b) 11.45
(c) 11.40 (d) 11.50

Round II (Mixed Bag)

Only One Correct Option

1. The marks of some students were listed out of 75. The SD of marks was found to be 9. Subsequently the marks were raised to a maximum of 100 and variance of new marks was calculated. The new variance is

(a) 81 (b) 122
(c) 144 (d) None of these

2. A batsman scores sums in 10 innings 38, 70, 48, 34, 42, 55, 46, 63, 54 and 44, then the deviation from median is

(a) 8.6 (b) 6.4
(c) 9.6 (d) 10.6

3. Consider any set of 201 observations $x_1, x_2, \ldots, x_{200}, x_{201}$. It is given that $x_1 < x_2 < \ldots < x_{200} < x_{201}$. Then, the mean deviation of this set of observations about a point k is minimum when k is equal to

(a) $(x_1 + x_2 + \ldots + x_{200} + x_{201}) / 201$
(b) x_1
(c) x_{101}
(d) x_{201}

4. In a college of 300 students every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers are

(a) atleast 30 (b) atmost 20
(c) exactly 25 (d) None of these

5. If the mean of a set of observations $x_1, x_2, \ldots, x_{10}$ is 20, then the mean of $x_1 + 4, x_2 + 8, \ldots, x_{10} + 40$ is

(a) 34 (b) 38 (c) 40 (d) 42

6. If $\sum_{i=1}^{18} (x_i - 8) = 9$ and $\sum_{i=1}^{18} (x_i - 8)^2 = 45$, then the standard deviation of $x_1, x_2, \ldots, x_{18}$ is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$
(c) $\frac{3}{2}$ (d) None of these

7. If the standard deviation of $x_1, x_2, \ldots, x_n$ is 3.5, then the standard deviation of $-2x_1 - 3, -2x_2 - 3, \ldots, -2x_n - 3$ is

(a) -7 (b) -4
(c) 7 (d) 1.75

8. The mean deviation from mean of the observation $a, a + d, a + 2d, \ldots, a + 2nd$ is

(a) $\frac{n(n+1)d^2}{3}$ (b) $\frac{n(n+1)}{2} d^2$
(c) $a + \frac{n(n+1)d^2}{2}$ (d) None of these

9. The first of two samples has 100 items with mean 15 and SD = 3. If the whole group has 250 items with mean 15.6 and SD = $\sqrt{13.44}$, the SD of the second group is

(a) 4 (b) 5
(c) 6 (d) 3.52

10. If the mean of five observations $x, x+2, x+4, x+6$ and $x+8$ is 11, then the mean of last three observations is

(a) 13 (b) 15
(c) 17 (d) None of these

11. The median of 19 observations of a group is 30. If two observations with values 8 and 32 are further included, then the median of the new group of 21 observations will be

(a) 28 (b) 30
(c) 32 (d) 34

12. If a variable x takes values x_i such that $a \le x_i \le b$, for $i = 1, 2, \ldots n$, then

(a) $a^2 \le \text{var}(x) \le b^2$ (b) $a \le \text{var}(x) \le b$

(c) $\frac{a^2}{4} \le \text{var}(x)$ (d) $(b - a)^2 \ge \text{var}(x)$

13. If the arithmetic mean of two unequal positive real numbers a and b $(a > b)$ be twice as their geometric mean, then $a : b$ is equal to

(a) $(2 + \sqrt{3}) : (\sqrt{2} + 3)$ (b) $(2 - \sqrt{3}) : (2 + \sqrt{3})$

(c) $(2 + \sqrt{3}) : (2 - \sqrt{3})$ (d) None of these

14. Following is the statement of marks obtained by two students A and B in 10 examination papers.

Paper	1	2	3	4	5	6	7	8	9	10
Marks scored by *A*	44	80	76	48	52	72	68	56	60	54
Marks scored by *B*	48	75	54	60	63	69	72	51	57	66

If the consistency of performance is the criterion for awarding a prize, find the prize winner among A and B.

(a) student A

(b) student B

(c) either students or A and B

(d) None of the above

15. There are 60 students in a class. The following is the frequency distribution of the marks obtained by the students in a test

Marks	0	1	2	3	4	5
Frequency	$x-2$	x	x^2	$(x+1)^2$	$2x$	$x+1$

where, x is a positive integer. Find the mean and standard deviation of the marks.

(a) 2.9, 1.14 (b) 2.8, 1.12

(c) 2.8, 1.18 (d) None of these

More Than One Correct Option

16. The variable x takes two values x_1 and x_2 with frequencies f_1 and f_2, respectively. If σ denotes the standard deviation of x, then

(a) $\sigma^2 = \frac{f_1x_1^2 + f_2x_2^2}{f_1 + f_2} - \left(\frac{f_1x_1 + f_2x_2}{f_1 + f_2}\right)^2$

(b) $\sigma^2 = \frac{f_1f_2}{(f_1 + f_2)^2}(x_1 - x_2)^2$

(c) $\sigma^2 = \frac{(x_1 - x_2)^2}{f_1 + f_2}$

(d) None of the above

17. Which of the following is a correct statement?

(a) The sum of the deviations from arithmetic mean is zero

(b) Computation of arithmetic mean is based on all observations

(c) It gives no importance to extreme values

(d) None of the above

Comprehension Based Questions

Passage

Directions (Q. Nos. 18 and 19) *Let us consider the series $x_1, x_2, \ldots, x_n$ whose mean is $\bar{x}$ and variance is σ^2.*
On the basis of above information, answer the following questions.

18. If x_i is replaced by x_i', then new mean is

(a) $\bar{x} - x_i + x_i'$ (b) $\frac{(n-1)\bar{x} + x_i'}{n}$

(c) $\frac{n\bar{x} - x_i + x_i'}{n}$ (d) None of these

19. If 5 is added in each observation, then the new variance is

(a) σ^2 (b) $\sigma^2 + 5$

(c) $\sigma^2 - 5$ (d) None of these

Assertion and Reason

Directions (Q. Nos. 20-25) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

20. Suppose two groups of scores A and B are such that $A = (x, x+2, x+4)$ and $B = (x-2, x+2, x+6)$

Statement I Group B has more variability than group A.

Statement II The value of mean for group B is more than that of group A.

21. The algebraic sum of deviation from their mean is defined as $\sum_{i=1}^{n} (x_i - \bar{x})$

Statement I The algebraic sum of the deviations of 20 observations measured from 30 is 2. The mean value of the observations is 30.

Statement II The sum of deviation from their mean is zero.

22. If n is a natural number, then

Statement I The mean of the squares of first n natural number is $\frac{(n+1)(2n+1)}{6}$.

Statement II $\Sigma n = \frac{n(n+1)}{2}$

23. Standard deviation is not depend on change of origin.

Statement I The standard deviation of variable $\frac{ax+b}{c}$ is $\left|\frac{a}{c}\right|\sigma$.

Statement II The standard deviation of a linear equation is $\sigma \times |\text{coefficient of } x|$.

24. **Statement I** If μ is the mean of a distribution, then $\Sigma f_i(y_i - \mu)$ is equal to 0.

Statement II The mean of the square of first n natural numbers is $\frac{1}{6}n(2n+1)$.

25. **Statement I** The variance of first n natural numbers is $\frac{n^2-1}{12}$.

Statement II The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

Previous Years' Questions

26. If the mean deviations about the median of the numbers $a, 2a, \ldots, 5a$ is 50, then $|a|$ is equal to **[AIEEE 2011]**

(a) 3 (b) 4 (c) 5 (d) 2

27. For two data sets, each of size 5, the variance are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is **[AIEEE 2010]**

(a) $\frac{5}{2}$ (b) $\frac{11}{2}$ (c) 6 (d) $\frac{13}{2}$

28. If the mean deviation of number $1, 1+d, 1+2d, \ldots, 1+100d$ from their mean is 255, then the d is equal to **[AIEEE 2009]**

(a) 10.0 (b) 20.0
(c) 10.1 (d) 20.2

29. The mean of the numbers a, b, 8, 5 and 10 is 6 and the variance is 6.80. Then, which one of the following gives possible values of a and b? **[AIEEE 2008]**

(a) $a = 3, b = 4$ (b) $a = 0, b = 7$
(c) $a = 5, b = 2$ (d) $a = 1, b = 6$

30. The average marks of boys in a class is 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is **[AIEEE 2007]**

(a) 40% (b) 20%
(c) 80% (d) 60%

Answers

Check Point 1

1. (a) **2.** (b) **3.** (c) **4.** (a) **5.** (a) **6.** (c) **7.** (b) **8.** (b)

Check Point 2

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (c) **6.** (a) **7.** (a) **8.** (a)

Round I

1. (c)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (b)
11. (c)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (d)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (c)
21. (d)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (d)
31. (a)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (a)	**39.** (a)	**40.** (c)
41. (c)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (c)	**48.** (c)	**49.** (a)	**50.** (a)
51. (a)	**52.** (a)	**53.** (d)	**54.** (d)	**55.** (d)	**56.** (b)	**57.** (a)	**58.** (b)	**59.** (a)	**60.** (c)
61. (b)	**62.** (b)	**63.** (a)	**64.** (c)	**65.** (a)	**66.** (d)	**67.** (b)	**68.** (a)	**69.** (c)	**70.** (a)
71. (a)	**72.** (d)	**73.** (d)	**74.** (d)	**75.** (a)	**76.** (b)	**77.** (b)	**78.** (b)	**79.** (b)	**80.** (a)

Round II

1. (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (c)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (a)
11. (b)	**12.** (d)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (a,b)	**17.** (a,b)	**18.** (c)	**19.** (a)	**20.** (c)
21. (d)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (c)

the Guidance

Round I

1. A statistical measure which cannot be determined graphically is harmonic mean.

2. Let $a, a+d, a+2d, \ldots, a(n-1)d$ be the n terms in AP.

$$\therefore \quad \text{Mean} = \frac{a + (a+d) + \ldots + \{a + (n-1)d\}}{n}$$

$$\Rightarrow \quad \text{Mean} = \frac{n}{2}\left\{\frac{2a + (n-1)d}{n}\right\}$$

$$\Rightarrow \quad \text{Mean} = \frac{a + \{a + (n-1)d\}}{2}$$

$$\therefore \quad \text{Mean} = \begin{pmatrix}\text{Mean of first and last term} \\ \text{of the given AP}\end{pmatrix}$$

3. Since, $n_1 = 25, n_2 = 35, \bar{x}_1 = 61, \bar{x}_2 = 58$

$$\because \quad \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\therefore \quad \bar{x} = \frac{(25)(61) + (35)(58)}{25 + 35} = \frac{1525 + 20230}{60} = \frac{3555}{60} = 59.25$$

4. $$\text{Mean} = \frac{0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2}{(n+1)}$$

$$= \frac{n(n+1)(2n+1)}{6(n+1)}$$

$$= \frac{1}{6}n(2n+1)$$

5. Sum of 100 items $= 49 \times 100 = 4900$

Sum of items added $= 60 + 70 + 80 = 210$

Sum of items replaced $= 40 + 20 + 50 = 110$

New sum $= 4900 - 110 + 210 = 5000$

$$\therefore \quad \text{New mean} = \frac{5000}{100} = 50$$

6. Let the n numbers be $x_1, x_2, \ldots, x_n$. Then,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad \bar{x} = \frac{x_1 + x_2 + \ldots + x_{n-1} + x_n}{n}$$

$$\Rightarrow \quad \bar{x} = \frac{k + x_n}{n} \qquad (\because x_1 + x_2 + \ldots + x_{n-1} = k)$$

$$\therefore \quad x_n = n\bar{x} - k$$

7. We have, $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_{n-1} + x_n}{n}$

$$\Rightarrow \quad n\bar{x} = x_1 + x_2 \ldots + x_{n-1} + x_n$$

Let $\bar{y}$ be the new mean when x_2 is replaced by λ. Then,

$$\bar{y} = \frac{x_1 + \lambda + x_3 + \ldots + x_{n-1} + x_n}{n}$$

$$\Rightarrow \quad \bar{y} = \frac{(x_1 + x_2 + \ldots + x_n) - x_2 + \lambda}{n}$$

$$\therefore \quad \bar{y} = \frac{n\bar{x} - x_2 + \lambda}{n}$$

8. $\because 8 + 12 + f_1 + 16 + f_2 + 10 = 75$

$$\Rightarrow \quad f_1 + f_2 = 29 \qquad \ldots\text{(i)}$$

and $120 + 240 + 25f_1 + 480 + 35f_2 + 400 = 28.07 \times 75$

$$\Rightarrow \quad 1240 + 25f_1 + 35f_2 = 2105.25$$

$$\Rightarrow \quad 5f_1 + 7f_2 = 173.25 \qquad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$f_1 = 15 \quad \text{and} \quad f_2 = 14$$

9. Given series is 148, 146, 144, 142, ... whose first term and common difference is

$$a = 148, d = (146 - 148) = -2$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = 125 \qquad \text{(given)}$$

$$\Rightarrow \quad 125n = \frac{n}{2}[2 \times 148 + (n-1) \times (-2)]$$

$$\Rightarrow \quad n^2 - 24n = 0 \quad \Rightarrow \quad n(n-24) = 0$$

$$\Rightarrow \quad n = 24 \qquad (n \neq 0)$$

10. $$\therefore \text{Mean} = \frac{1 \times x + 2 \times 2x + 3 \times 3x + \ldots n \times nx}{x + 2x + 3x + \ldots + nx}$$

$$= \frac{x(1^2 + 2^2 + 3^2 \ldots + n^2)}{x(1 + 2 + 3 + \ldots + n)} = \frac{\dfrac{n(n+1)(2n+1)}{6}}{\dfrac{n(n+1)}{2}} = \frac{2n+1}{3}$$

11. Let $x_1, x_2, \ldots x_n$ be the n items.

Given, $$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} \qquad \ldots\text{(i)}$$

$$\therefore \quad \text{New mean} = \frac{(x_1 + 1) + (x_2 + 2) + \ldots + (x_n + n)}{n}$$

$$= \frac{(x_1 + x_2 + \ldots + x_n) + (1 + 2 + \ldots + n)}{n}$$

$$= \bar{x} + \frac{n(n+1)}{2n} \qquad [\because \text{from Eq. (i)}]$$

$$= \bar{x} + \frac{(n+1)}{2}$$

12. According to the given condition,

$$\frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n} = \frac{46n}{11}$$

$$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6n} = \frac{46n}{11}$$

$\Rightarrow \quad 11(2n^2 + 3n + 1) = 276n$

$\Rightarrow \quad 22n^2 - 243n + 11 = 0$

$\Rightarrow \quad (n - 11)(22n - 1) = 0 \Rightarrow n = 11$

13. The required mean is

$$\bar{X} = \frac{1\cdot 1 + 2\cdot 2 + 3\cdot 3 + \ldots + n\cdot n}{1 + 2 + 3 + \ldots + n}$$

$$= \frac{\Sigma n^2}{\Sigma n} = \frac{n(n+1)\dfrac{(2n+1)}{6}}{\dfrac{n(n+1)}{2}} = \frac{2n+1}{3}$$

14. The required AM is

$$\bar{X} = \frac{1 + 2 + 2^2 + 2^3 + \ldots + 2^n}{n+1}$$

$$= \frac{1(2^{n+1} - 1)}{(2-1)} \cdot \frac{1}{(n+1)} = \frac{2^{n+1} - 1}{n+1}$$

15. $\because$ Total marks of 10 failed students $= 28 \times 10 = 280$

and total marks of 50 students $= 2800$

$\therefore$ Total marks of 40 passed students $= 2800 - 280 = 2520$

$\therefore$ Average marks of 40 passed students $= \dfrac{2520}{40} = 63$

16. Let n_1 and n_2 be the number of observations in two groups having means $\bar{x}_1$ and $\bar{x}_2$, respectively. Then,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Now, $\quad \bar{x} - \bar{x}_1 = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} - \bar{x}_1$

$\Rightarrow \quad \bar{x} - \bar{x}_1 = \dfrac{n_2(\bar{x}_2 - \bar{x}_1)}{n_1 + n_2} > 0 \quad [\because \bar{x}_2 > \bar{x}_1 (\text{given})]$

$\Rightarrow \quad \bar{x} > \bar{x}_1 \quad \ldots(\text{i})$

Also, $\quad \bar{x} - \bar{x}_2 = \dfrac{n_1(\bar{x}_1 - \bar{x}_2)}{n_1 + n_2} < 0 \quad [\because \bar{x}_2 > \bar{x}_1 (\text{given})]$

$\Rightarrow \quad \bar{x} < \bar{x}_2 \quad \ldots(\text{ii})$

From Eqs. (i) and (ii), we have $\bar{x}_1 < \bar{x} < \bar{x}_2$

17. $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$

$\Rightarrow \quad \displaystyle\sum_{i=1}^{n} x_i = n\bar{x}$

Let $M = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} (x_i + 2i) = \dfrac{1}{n}\sum_{i=1}^{n} x_i + 2(1 + 2 + \ldots + n)$

$\Rightarrow \quad M = \dfrac{1}{n} n\bar{x} + 2\dfrac{n(n+1)}{2}$

$\therefore \quad M = \bar{x} + (n+1)$

18. Given that, $n_1 = 10, \bar{x}_1 = 12, n_2 = 20, \bar{x}_2 = 9$

$\therefore \quad \bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \dfrac{10 \times 12 + 20 \times 9}{10 + 20}$

$= \dfrac{120 + 180}{30} = \dfrac{300}{30} = 10$

19. Total of corrected observations

$= 4500 - (91 + 13) + (19 + 31) = 4446$

$\therefore \quad \text{Mean} = \dfrac{4446}{100} = 44.46$

20. Let $x_1, x_2, x_3, \ldots x_n$ be n observations. Then,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$\therefore$ New mean, $\bar{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\frac{x_i}{\alpha} + 10\right)$

$$= \frac{1}{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{n}\cdot(10n)$$

$$= \frac{1}{\alpha}\bar{x} + 10 = \frac{\bar{x} + 10\alpha}{\alpha}$$

21. Let $x_1, x_2, x_3, \ldots, x_n$ be n observations.

$\therefore \quad \bar{x} = \dfrac{\Sigma x_i}{n} \quad \ldots(\text{i})$

New mean $= \dfrac{\Sigma x_i + x_{n+1}}{n+1}$

According to the question, $\bar{x} = \dfrac{\Sigma x_i + x_{n+1}}{n+1}$

$\Rightarrow \quad (n+1)\bar{x} = n\bar{x} + x_{n+1}$

$\Rightarrow \quad x_{n+1} = \bar{x}$

22. Let the number of girls in the class $= y$

$\therefore$ Number of boys in the class $= 100 - y$

Now, $\bar{x}_1 = 25, \; n_1 = y, \; \bar{x}_2 = 50, \; n_2 = 100 - y$

and $\quad \bar{x} = 35, \; n_1 + n_2 = 100$

$\therefore \quad 35 = \dfrac{25 \times y + 50 \times (100 - y)}{100}$

$\Rightarrow \quad 3500 = 25y + 5000 - 50y$

$\Rightarrow \quad 25y = 1500 \Rightarrow y = 60$

$\therefore$ Number of girls in the class $= 60$

23. Given that, $\quad \displaystyle\sum_{i=1}^{n} (x_i - 4) = 30$

and $\quad \displaystyle\sum_{i=1}^{n} (x_i - 3) = 40$

i.e., $\quad \Sigma x_i - 4n = 30 \quad \ldots(\text{i})$

and $\quad \Sigma x_i - 3n = 40 \quad \ldots(\text{ii})$

On solving Eqs. (i) and (ii), we get

$n = 10$ and $\Sigma x_i = 70$

$\therefore \quad \text{Mean} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \dfrac{1}{10}(70) = 7$

24. Since, $x = au + bv$

$\therefore \quad \Sigma x = a\Sigma u + b\Sigma v$

$\Rightarrow \quad n\bar{x} = a(n\bar{u}) + b(n\bar{v})$

$\Rightarrow \quad \bar{x} = a\bar{u} + b\bar{v}$

25. Since, $\bar{x}_1 = 520$, $\bar{x}_2 = 420$ and $\bar{x} = 500$

Also, we know $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$\Rightarrow \quad 500(n_1 + n_2) = 520n_1 + 420n_2$

$\Rightarrow \quad 20n_1 = 80n_2$

$\Rightarrow \quad n_1 : n_2 = 4 : 1$

Hence, the percentage of male employees in the firm

$$= \left(\frac{4}{4+1}\right) \times 100 = 80\%$$

26. Let $x_1, x_2, \ldots, x_n$ be n observations.

$$\therefore \quad M = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$= \frac{x_1 + x_2 + \ldots + x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} + x_n}{n}$$

$\Rightarrow nM = a + x_{n-3} + x_{n-2} + x_{n-1} + x_n$

$$\Rightarrow \quad \frac{nM - a}{4} = \frac{x_{n-3} + x_{n-2} + x_{n-1} + x_n}{4}$$

27. Given that, $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$

$\Rightarrow \quad n\bar{x} = x_1 + x_2 + \ldots\ldots + x_n$

Now, required mean $= \dfrac{2x_1 + 3 + \ldots + 2x_n + 3}{n}$

$$= \frac{2(x_1 + x_2 + \ldots + x_n) + 3n}{n}$$

$$= \frac{2n\bar{x} + 3n}{n} = 2\bar{x} + 3$$

28. The formula for combined mean is

$$\bar{X} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1 + n_2}$$

We are given $\bar{X} = 25$, $\bar{X}_1 = 26$, $\bar{X}_2 = 21$. Let $n_1 + n_2 = 100$ and n_1 denotes men and n_2 denotes women.

$$n_2 = 100 - n_1$$

$$\therefore \quad 25 = \frac{26n_1 + 21(100 - n_1)}{100} \Rightarrow n_1 = 80$$

So, $n_2 = 20$

Hence, the percentage of men and women is 80 and 20, respectively.

29. ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1}$

$$= 2^{2n+1}$$

Now, ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1}$,

${}^{2n+1}C_1 = {}^{2n+1}C_{2n} \ldots {}^{2n+1}C_r = {}^{2n+1}C_{2n-r+1}$

So, sum of first $(n + 1)$ terms = sum of last $(n + 1)$ terms

$\Rightarrow \quad {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

$$\Rightarrow \quad \frac{{}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n}{n+1} = \frac{2^{2n}}{(n+1)}$$

30. Required mean $= \dfrac{0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n}$

$$= \frac{n \cdot 2^{n-1}}{2^n} = \frac{n}{2}$$

31. Required mean,

$$\Rightarrow \quad \bar{x} = \frac{0 \cdot q^n + 1 \cdot \frac{n}{1} q^{n-1}p + 2 \cdot \frac{n(n-1)}{2!} q^{n-2}p^2 + \ldots + n \cdot p^n}{q^n + \frac{n}{1} q^{n-1}p + \frac{n(n-1)}{2!} q^{n-2}p^2 + \ldots + p^n}$$

$$\Rightarrow \quad \bar{x} = \frac{0 \cdot {}^nC_0 q^n p^0 + 1 \cdot {}^nC_1 q^{n-1}p + \ldots + n \cdot {}^nC_n q^0 p^n}{{}^nC_0 q^n p^0 + {}^nC_1 q^{n-1}p^1 + \ldots + {}^nC_n q^{n-n}p^n}$$

$$\Rightarrow \quad \bar{x} = \frac{\sum_{r=0}^{n} r \cdot {}^nC_r q^{n-r}p^r}{\sum_{r=0}^{n} {}^nC_r q^{n-r}p^r} = \frac{\sum_{r=0}^{n} r \cdot \frac{n}{r}\, {}^{n-1}C_{r-1} q^{n-r} p \cdot p^{r-1}}{\sum_{r=0}^{n} {}^nC_r q^{n-r}p^r}$$

$$\Rightarrow \quad \bar{x} = \frac{np\left[\sum_{r=1}^{n} {}^{n-1}C_{r-1} p^{r-1} q^{(n-1)-(r-1)}\right]}{\sum_{r=0}^{n} {}^nC_r q^{n-r}p^r}$$

$$\Rightarrow \quad \bar{x} = \frac{np(q+p)^{n-1}}{(q+p)^n}$$

$$\therefore \quad \bar{x} = np \qquad (\because q + p = 1)$$

32. $\text{GM} = (2^1 \cdot 2^2 \cdot 2^3 \ldots 2^2)^{1/n}$

$\Rightarrow \quad \text{GM} = (2^{1+2+3\ldots+n})^{1/n}$

$$\Rightarrow \quad \text{GM} = \left\{2^{\frac{n(n+1)}{2}}\right\}^{1/n}$$

$$\text{GM} = \{2\}^{\frac{n+1}{2}}$$

33. Given, $G_1 = (x_1 \cdot x_2 \ldots x_n)^{1/n}$

and $G_2 = (y_1 \cdot y_2 \ldots y_n)^{1/n}$

$$\therefore \quad G = \left(\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \ldots \frac{x_n}{y_n}\right)^{1/n} = \frac{(x_1 \cdot x_2 \ldots x_n)^{1/n}}{(y_1 \cdot y_2 \ldots y_n)^{1/n}} = \frac{G_1}{G_2}$$

34. By using combined geometric mean formula, we get

$$\text{GM} = \sqrt[(n_1+n_2)]{x_1^{n_1} \times x_2^{n_2}}$$

$$= \sqrt[7]{8^4 \times 1024^3} = \sqrt[7]{2^{12} \times 2^{30}} = 2^{42/7} = 2^6$$

$\Rightarrow \quad \text{GM} = 64$

35. $\because$ We know that,

Geometric mean $= \sqrt[n]{x_1 \times x_2 \times \ldots \times x_n}$

$$= \sqrt[20]{2^8 \times 4^6 \times 8^4 \times 16^2}$$

$$= \sqrt[20]{2^8 \times 2^{12} \times 2^{12} \times 2^8}$$

$$= \sqrt[20]{2^{40}} = 2^{\frac{40}{20}} = 2^2 = 4$$

36. We know that,

$$\text{Harmonic mean} = \frac{n}{\Sigma 1/x} = \frac{n}{1+2+3+\ldots+n}$$

$$= \frac{n}{\frac{n(n+1)}{2}} = \frac{2}{n+1}$$

37. Let H_1 and n_1 are the harmonic mean and number of observations of first group and H_2 and n_2 are the harmonic mean and number of observations of another group.

$\therefore$ By using combined harmonic mean formula, we get

$$\text{Combined harmonic mean} = \frac{n_1+n_2}{\frac{n_1}{H_1}+\frac{n_2}{H_2}} = \frac{5+5}{\frac{5}{\frac{5}{2}}+\frac{5}{\frac{9}{2}}}$$

$$= \frac{10}{\frac{28}{9}} = \frac{90}{28} = \frac{45}{14}$$

38. We are given, here geometric mean of 10 observations is 8 and geometric mean of first six observations is 4, then we have to find the geometric mean of last four observations. Let G be the GM of last four observations.

$\therefore$ By using grouped geometric mean,

i.e., $8^{10} = 4^6 \times G^4$

$\Rightarrow$ $G^4 = \frac{8^{10}}{4^6} = \frac{2^{30}}{2^{12}} = 2^{18}$

$\Rightarrow$ $G = 2^{18/4} = 2^4\sqrt{2} = 16\sqrt{2}$

39. Since, we know that

$$(\text{GM})^2 = \text{AM} \times \text{HM}$$

$$(2\sqrt{2})^2 = 3 \times \text{HM}$$

$$\Rightarrow \quad \text{HM} = \frac{8}{3}$$

Also, given here that, if each observation is multiplied by 2, then

$$\text{HM} = \frac{8}{3} \times 2 = \frac{16}{3}$$

40. Suppose that, G_1 is the GM of first 5 observations $= 8$ and G_2 is the GM of last 4 observations $= 128\sqrt{2}$

Also, let G be the grouped mean

$$= 9\sqrt{(G_1^{n_1} \times G_2^{n_2})} = (G_1^5 \times G_2^4)^{1/9}$$

$$= [8^5 \times (128\sqrt{2})^4]^{1/9} = \left[(2^3)^5 \times \left(2^{7+\frac{1}{2}}\right)^4\right]^{1/9}$$

$$= (2^{15} \times 2^{30})^{1/9} = 2^{\frac{45}{9}} = 2^5 = 32$$

41. Mean deviation is not a measure of central tendency.

42. For dealing with qualitative data, the best average is median.

43. Total weight of 9 items $= 15 \times 9 = 135$

and total weight of 10 items $= 16 \times 10 = 160$

$\therefore$ Weight of 10th item $= 160 - 135 = 25$

44. Since, $\alpha > 0$ (given), arrange the data in ascending order.

$$\alpha - \frac{7}{2}, \alpha - 3, \alpha - \frac{5}{2}, \alpha - 2, \alpha - \frac{1}{2}, \alpha + \frac{1}{2}, \alpha + 4, \alpha + 5$$

$$\text{Median} = \frac{\alpha - 2 + \alpha - \frac{1}{2}}{2} = \frac{2\alpha - \frac{5}{2}}{2} = \alpha - \frac{5}{4}$$

45. $\text{Coefficient of skewness} = \frac{Q_3 + Q_1 - 2\,(\text{median})}{Q_3 - Q_1}$

$$= \frac{25.2 + 14.6 - 2\,(18.8)}{25.2 - 14.6} = \frac{2.2}{10.6} = 0.20$$

46. ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, \ldots, {}^{2n}C_n$ is odd number of binomial coefficients (when n is even) and middle binomial coefficient is ${}^{2n}C_{n/2}$ which is required median.

47. Let the first natural number be x

According to the question,

$$(x) + (x+1) + (x+2) + (x+3) + (x+4) + (x+5) + (x+6) + (x+7) + (x+8) + (x+9) + (x+10) = 2761$$

$\Rightarrow$ $11x + 55 = 2761$

$\Rightarrow$ $x = \frac{2761 - 55}{11} = \frac{2706}{11} = 246$

$\therefore$ Middle number $= x + 5 = 246 + 5 = 251$

48. Since, Mode $= 3$ (Median) $- 2$ (Mean)

$\therefore$ $18 = 3$ (Median) $- 2(24) \Rightarrow$ Median $= 22$

49. $\because$ Mode $= 3$ (Median) $- 2$ (Mean)

$\therefore$ $6\lambda = 3$ (Median) $- 18\lambda$

$\Rightarrow$ Median $= 8\lambda$

50. If the observations greater than the median are increased by 5, then C_n median will not be affected.

51. Given that, mean $= 5$, median $= 6$

For a moderately skewed distribution, we have

Mode $= 3$ median $- 2$ mean

$\Rightarrow$ Mode $= 3\,(6) - 2\,(5) = 8$

52. The values of median and mode can be determined graphically is the correct statement.

53. Firstly, arrange the data in ascending order

i.e., 7, 10, 12, 15, 17, 19, 25

$\therefore$ $Q_1 = \text{Size of}\left(\frac{n+1}{4}\right)\text{th item}$

$= \text{Size of 2nd item} = 10$

$Q_3 = \text{Size of}\left[\frac{3(n+1)}{4}\text{ th}\right]\text{item}$

$= \text{Size of 6th item} = 19$

$\therefore$ Quartile deviation $= \frac{Q_3 - Q_1}{2} = \frac{19 - 10}{2} = 4.5$

54. Here, $N = 20$

$$Q_1 = \frac{N+1}{4} = \left(\frac{21}{4}\right)\text{th} = 3$$

Similarly, $Q_3 = 3\left(\frac{N+1}{4}\right)$ th observation

$$= 3\left(\frac{21}{4}\right) = \frac{63}{4} \text{ th} = 5$$

$$\therefore \quad QD = \frac{1}{2}(Q_3 - Q_1) = \frac{1}{2}(5-3) = 1$$

55. Given that,

$$2y + 5 = 3x$$

$$\Rightarrow \quad y = \frac{3x}{2} - \frac{5}{2}$$

Therefore, QD of $y = \left|\frac{3}{2}\right| \times$ QD of $x = \frac{3}{2} \times 8 = 12$

$[\because y = bx + c \Rightarrow QD(y) = |b|\, QD(x)]$

56. From given data, we make the following table

Class Interval (CI)	Frequency (f)	Cumulative frequency (cf)
0-10	10	10
10-20	20	30
20-30	35	65
30-40	25	90
40-50	10	100

Here, $\frac{N}{4} = \frac{100}{4} = 25$ and $\frac{3N}{4} = \frac{3 \times 100}{4} = 74$

$$\therefore \quad Q_1 = L_1 + \frac{\frac{N}{4} - cf}{f} \times i = 10 + \frac{25-10}{20} \times 10$$

$$= 10 + 7.5 = 17.5$$

and $$Q_3 = L_1 + \frac{\frac{3N}{4} - cf}{f} \times i = 30 + \frac{75-65}{25} \times 10$$

$$= 30 + 4 = 34$$

Hence, quartile deviation $= \frac{Q_3 - Q_1}{2} = \frac{34 - 17.5}{2} = 8.25$

57. Since, $MD = \frac{4}{5}\sigma$, $QD = \frac{2}{3}\sigma$

$$\therefore \quad \frac{MD}{QD} = \frac{6}{5}$$

$$\Rightarrow \quad QD = \frac{5}{6}(MD) = \frac{5}{6}(15) = 12.5$$

58. Mean, $$\bar{x} = \frac{3 + 10 + 10 + 4 + 7 + 10 + 5}{7} = \frac{49}{7}$$

$$= 7$$

$\therefore$ Mean deviation, $MD = \sum_{i=1}^{7} \frac{|x_i - \bar{x}|}{n}$

$$= \frac{\left[\begin{array}{l} |3-7| + |10-7| + |10-7| + |4-7| + |7 \\ \quad -7| + |10-7| + |5-7| \end{array}\right]}{7}$$

$$= \frac{4+3+3+3+0+3+2}{7} = \frac{18}{7} = 2.57$$

59. $\therefore$ Mean, $$\bar{x} = \frac{1357 + 1090 + 1666 + 1494 + 1623}{5}$$

$$= \frac{7230}{5} = 1446$$

$\therefore$ Mean deviation, $MD = \sum_{i=1}^{5} |x_i - \bar{x}|$

$$= \frac{\begin{array}{l} |1357 - 1446| + |1090 - 1446| + |1666 \\ \quad -1446| + |1494 - 1446| + |1623 - 1446| \end{array}}{5}$$

$$= \frac{89 + 356 + 220 + 48 + 177}{5} = \frac{890}{5} = 178$$

60. Firstly, given marks are arranged in ascending order 20, 33, 39, 40, 50, 53, 59, 65, 69

$\therefore$ Median, $M_d = \left(\frac{9+1}{2}\right)$ th term = 5th term = 50

$\therefore$ Mean deviation, $MD = \frac{\sum_{i=1}^{9} |x_i - M_d|}{n}$

$$= \frac{\begin{array}{l} |20-50| + |33-50| + |39-50| + |40-50| + |50 \\ \quad -50| + |53-50| + |59-50| + |65-50| + |69-50| \end{array}}{9}$$

$$= \frac{30 + 17 + 11 + 10 + 0 + 3 + 9 + 15 + 19}{9}$$

$$= \frac{114}{9} = 12.67$$

61. Now, mean of first n natural numbers,

$$\bar{x} = \frac{1 + 2 + \ldots + n}{n}$$

$$= \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\therefore \quad \text{Mean deviation} = \frac{\Sigma |x_i - \bar{x}|}{n}$$

$$= \frac{\left[\begin{array}{l} \left|\frac{n+1}{2} - 1\right| + \left|\frac{n+1}{2} - 2\right| \ldots + \left|\frac{n+1}{2} - \frac{n+1}{2}\right| \\ \quad + \left|\frac{n+1}{2} - \left(\frac{n+1}{2} + 1\right)\right| + \left|\left(\frac{n+1}{2}\right) - \left(\frac{n+1}{2} + 2\right)\right| \\ \quad + \ldots + \left|\frac{n+1}{2} - \left(\frac{n+1}{2} + \frac{n-1}{2}\right)\right| \end{array}\right]}{n}$$

$$= \frac{\frac{n-1}{2} + \frac{n-3}{2} + \ldots + 0 + 1 + 2 + \ldots + \frac{n-1}{2}}{n}$$

$$= \frac{\left\{\frac{n}{2} + \frac{n}{2} + \ldots + (n-1) \text{ term}\right\} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{n-1}{2} \text{ term}}{n}$$

$$= \frac{\frac{n(n-1)}{2 \times 2} + \frac{(n-1)}{2 \times 2}}{n} = \frac{n^2 - n + n - 1}{4n}$$

$$= \frac{n^2 - 1}{4n}$$

62.

Class interval	Mid value x_i	f	cf	$\lvert x - M_d \rvert$	$f \lvert x - M_d \rvert$
0-6	3	4	4	11	44
6-12	9	5	9	5	25
12-18	15	3	12	1	3
18-24	21	6	18	7	42
24-30	27	2	20	13	26
Total		20			140

Now, $\frac{N}{2} = \frac{20}{2} = 10$, which lies in the interval 12-18.

$$l = 12, cf = 9, f = 3$$

$$\therefore \quad M_d = 12 + \frac{10-9}{3} \times 6 \quad \left(\because M_d = l + \frac{\frac{N}{2} - cf}{f} \times h \right)$$

$$= 12 + 2 = 14$$

$$\therefore \quad \text{Mean deviation} = \frac{\Sigma f_i \mid x_i - M_d \mid}{N} = \frac{140}{20} = 7$$

63. **Hint** Now, Mean $\bar{x} = \frac{6+5+9+13+12+8+10}{7} = \frac{63}{7} = 9$

$\therefore$ Standard deviation, $\sigma = \sqrt{\frac{(x_i - \bar{x})^2}{n}}$

64. **Hint** $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - (\bar{x})^2}$

65. We know that, if any constant is added in each observation, then standard deviation remain same.

$\therefore$ The standard deviation of the observations $a + k, b + k, c + k, d + k, e + k$ is S.

66. We know that, standard deviation is independent of change of origin.

Hence, if each item is decreased by 1, then standard deviation will be same *i.e.*, 6.

67. We know that,

$$\text{var}(aX + b) = a^2 \text{var}(X)$$

$$\therefore \quad \text{var}\left(\frac{aX+b}{c}\right) = \left(\frac{a}{c}\right)^2 \text{var}(X) = \frac{a^2}{c^2}\sigma^2$$

$$\therefore \quad \text{SD} = \sqrt{\text{var}\left(\frac{aX+b}{c}\right)} = \left|\frac{a}{c}\right|\sigma$$

68. We have, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\therefore \quad \text{var}(\bar{x}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} \text{var}(x_i) + 2\sum_{i \neq j}^{n} \text{cov}(x_i, x_j)\right]$$

$$= \frac{1}{n^2}[n\sigma^2] = \frac{\sigma^2}{n}$$

[$\because$ x_i and x_j are independent variable, therefore $\text{cov}(x_i, x_j) = 0$]

69. Standard deviation is dependent on charge of scale. Therefore, the standard deviation of $kx_1, kx_2, kx_3, kx_4, kx_5$ is kS.

70. Given, $w_i = lx_i + k$

$$M(x_i) = \bar{x} = 48, \sigma(x_i) = 12$$

$$M(\bar{w}) = 55 \text{ and } \sigma(w) = 15$$

$$M(w_i) = lM(x_i) + M(k)$$

$$55 = l \times 48 + k \quad \ldots\text{(i)}$$

and $\quad \sigma(w_i) = l\sigma(x_i) + \sigma(k)$

$$\Rightarrow \quad 15 = l(12) + 0$$

$$\Rightarrow \quad l = \frac{15}{12} = 1.25$$

$\therefore$ From Eq. (i),

$$55 = 1.25 \times 48 + k$$

$$\Rightarrow \quad k = 55 - 60$$

$$\therefore \quad k = -5$$

71. $$\bar{x} = \frac{1+2+3+\ldots+n}{n} = \frac{(n+1)}{2}$$

Variance, $\sigma^2 = \frac{\Sigma(x_i)^2}{n} - (\bar{x})^2 = \frac{\Sigma n^2}{n} - \left(\frac{n+1}{2}\right)^2$

$$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$$

72. **Hint** Now mean, $\bar{x} = \frac{1+2+\ldots+10}{10}$

$$= \frac{\frac{10(10+1)}{2}}{10} = \frac{11}{2} \quad \left[\because \Sigma n = \frac{n(n+1)}{2}\right]$$

$$\therefore \quad \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - (\bar{x})^2}$$

73. We know SD of first 10 natural number is 2.87.

i.e., $\quad \sigma = 2.87$

Variance, $\sigma^2 = 8.24$

We know that, variance is independent of change of origin.

Hence, after adding 1 on each observation new variance will same *i.e.*, 8.25

74. Given, $\Sigma x^2 = 18000$, $\Sigma x = 960$ and $n = 60$

$$\because \quad \sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = \frac{18000}{60} - \left(\frac{960}{60}\right)^2$$

$$= 300 - (16)^2 = 300 - 256 = 44$$

75. Given, coefficient of variation, $C_1 = 50$

and coefficient of variation, $C_2 = 60$

$$\bar{x}_1 = 30 \text{ and } \bar{x}_2 = 25$$

$$\because \quad C = \frac{\sigma}{\bar{x}} \times 100$$

$$\therefore \quad 50 = \frac{\sigma_1}{30} \times 100$$

$\Rightarrow \quad \sigma_1 = 15$ and $60 = \frac{\sigma_2}{25} \times 100$

$\Rightarrow \quad \sigma_2 = 15$

$\therefore$ Required difference, $\sigma_1 - \sigma_2 = 15 - 15 = 0$

76. Given, $\sigma_{10}^2 = \frac{99}{12} = \frac{33}{4}$

$\Rightarrow \quad \sigma_{10} = \frac{\sqrt{33}}{2}$

SD of required series $= 3\sigma_{10} = \frac{3\sqrt{33}}{2}$

77. $n_1 = 25, \bar{x}_1 = 18.2, \sigma_1 = 3.25, n_2 = 15$

$$\sum_{i=1}^{15} x_i = 279 \text{ and } \Sigma x_i^2 = 5524$$

Now, $\bar{x}_2 = \frac{\sum_{i=1}^{15} x_i}{n_2} = \frac{279}{15} = 18.6$

$$\sigma_2 = \sqrt{\frac{\Sigma x_i^2}{n_2} - (\bar{x}_2)^2} = \sqrt{\frac{5524}{15} - (18.6)^2}$$

$$= \sqrt{368.27 - 345.96} = \sqrt{22.31} = 4.72$$

Now, $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$$= \frac{25 \times 18.2 + 15 \times 18.6}{25 + 15}$$

$$= \frac{455 + 279}{40} = \frac{734}{40} = 18.35$$

$\therefore \quad d_1 = \bar{x}_1 - \bar{x}_{12}$

$= 18.2 - 18.35 = -0.15$

$d_2 = \bar{x}_2 - \bar{x}_{12} = 18.6 - 18.35 = 0.25$

$\therefore$ Combined standard deviation

$$\sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}}$$

$$= \sqrt{\frac{25 \times (3.25)^2 + 15 \times (4.72)^2 + 25 \times (-0.15)^2 + 15 \times (0.25)^2}{25 + 15}}$$

$$= \sqrt{\frac{264.06 + 334.176 + 0.5625 + 0.9375}{40}}$$

$$= \sqrt{\frac{599.736}{40}} = \sqrt{14.9934} = 3.87$$

78. Given, $\Sigma(x - 5) = 3$ and $\Sigma(x - 5)^2 = 43, n = 18$

$\therefore \quad$ Mean $= 5 + \frac{\Sigma(x-5)}{18} = 5 + \frac{3}{18} = 5 + \frac{1}{6}$

$$= \frac{31}{6} = 5.17$$

Now, $\Sigma(x - 5)^2 = 43$

$\Rightarrow \quad \Sigma x^2 - \Sigma 10x + \Sigma 25 = 43$

$\Rightarrow \quad \Sigma x^2 - 10\{\Sigma(x - 5 + 5)\} + 18 \times 25 = 43$

$\Rightarrow \quad \Sigma x^2 - 10\Sigma(x - 5) - 10\ \Sigma 5 + 450 = 43$

$\Rightarrow \quad \Sigma x^2 - 10(3) - 10 \times 18 \times 5 + 450 = 43$

$\Rightarrow \quad \Sigma x^2 - 30 - 900 + 450 = 43$

$\Rightarrow \quad \Sigma x^2 = 523$

and $\quad \Sigma(x - 5) = 3$

$\Rightarrow \quad \Sigma x - 5\Sigma 1 = 3$

$\Rightarrow \quad \Sigma x = 3 + 5 \times 18 = 93$

$\because \quad \sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{523}{18} - \left(\frac{93}{18}\right)^2}$

$$= \sqrt{29.06 - (5.17)^2} = \sqrt{29.06 - 26.73}$$

$$= \sqrt{2.33} = 1.53$$

79. Given that,

mean $= 0$ and variance $= \sigma^2$ (for x)

Therefore, variance $= \sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$\Rightarrow \quad \sigma^2 = \frac{\Sigma x^2}{n} - 0 \quad \Rightarrow \sigma^2 = \frac{\Sigma x^2}{n}$

Now, $\quad y = x^2$

$\Rightarrow \quad \Sigma y = \Sigma x^2$

$\therefore \quad \bar{y} = \frac{\Sigma x^2}{n} = \sigma^2$

80. From given data, we make the following table

x	x^2
2	4
3	9
11	121
x	x^2
$\Sigma x = 16 + x$	$\Sigma x^2 = 134 + x^2$

But we know that, variance $= \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$\Rightarrow \quad \frac{134 + x^2}{4} - \left(\frac{16 + x}{4}\right)^2 = \frac{49}{4}$ (given)

$\Rightarrow \quad \frac{134 + x^2}{4} - \frac{(256 + x^2 + 32x)}{16} = \frac{49}{4}$

$\Rightarrow \quad \frac{3x^2 - 32x + 280}{16} = \frac{49}{4}$

$\Rightarrow \quad 280 + 3x^2 - 32x = \frac{49}{4} \times 16$

$\Rightarrow \quad 280 + 3x^2 - 32x = 196$

$\Rightarrow \quad 3x^2 - 32x + 84 = 0$

$\Rightarrow \quad (x - 6)(3x - 14) = 0$

$\Rightarrow \quad x = 6, x = \frac{14}{3}$

Therefore, the values of x are 6 and $\frac{14}{3}$.

81. Given that,

mean and variance of players A and B is

$$\bar{x}_1 = 50, \sigma_1^2 = 36 \text{ or } \sigma_1 = 6$$

$$\bar{x}_2 = 60, \sigma_2^2 = 81 \text{ or } \sigma_2 = 9$$

From the following formula, we can check consistency

coefficient of variation $= \dfrac{\text{SD}}{\text{Mean}} \times 100$

Therefore, coefficient of variation of player,

$$A = \frac{\sigma_1}{\bar{x}_1} \times 100 = \frac{6}{50} \times 100 = 12\%$$

and coefficient of variation of player,

$$B = \frac{\sigma_2}{\bar{x}_2} \times 100 = \frac{9}{60} \times 100 = 15\%$$

Since, coefficient of variation of player A is less, hence it is more consistent in comparision to player B.

82. $\therefore$ New mean, $\bar{x} = \dfrac{100 \times 40 + 3 + 27 - 30 - 70}{100}$

$$= \frac{4000 - 70}{100} = \frac{3930}{100}$$

$$= 39.3$$

$\because$ $\Sigma x^2 = N(\sigma^2 + \bar{x}^2)$

$\therefore$ $\text{SD} = 100(100 + 1600)$

$$= 170000$$

New, $\Sigma x^2 = 170000 - (30)^2 - (70)^2 + (3)^2 + (27)^2$

$$= 170000 - 900 - 4900 + 9 + 729$$

$$= 164938$$

$\therefore$ New, $\text{SD} = \sqrt{\dfrac{\text{New } \Sigma x^2}{N} - (\text{New } \bar{x})^2}$

$$= \sqrt{\frac{164938}{100} - (39.3)^2}$$

$$= \sqrt{1649.38 - 1544.49} = \sqrt{104.89}$$

$$= 10.24$$

83. **Hint** combined mean

$$(\bar{x}) = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{100 \times 45 + 150 \times 55}{100 + 150}$$

$$= \frac{4500 + 8250}{250} = 51$$

$\therefore$ $d_1^2 = (\bar{x} - \bar{x}_1)^2 = (51 - 45)^2 = 36$

and $d_2^2 = (\bar{x} - \bar{x}_2)^2 = (51 - 55)^2 = 16$

Therefore, combined SD $(\sigma) = \sqrt{\dfrac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$

Round II

1. Given, $\sigma = 9$

Let a student obtains x marks out of 75. Then, his marks out of 100 are $\dfrac{4x}{3}$. Each observation is multiply by $\dfrac{4}{3}$.

$\therefore$ New SD, $\sigma = \dfrac{4}{3} \times 9 = 12$

Hence, variance is $\sigma^2 = 144$

2. The ascending order of the given data are

$$34, 38, 42, 44, 46, 48, 54, 55, 63, 70$$

Hence, Median, $M = \dfrac{46 + 48}{2} = 47$

$\therefore$ Median deviation

$$= \frac{\Sigma|x_i - M|}{n} = \frac{\Sigma|x_i - 47|}{n}$$

$$= \frac{13 + 9 + 5 + 3 + 1 + 1 + 7 + 8 + 16 + 23}{10} = 8.6$$

3. Given that, $x_1 < x_2 < x_3 < \ldots < x_{201}$

$\therefore$ Median of the given observation $= \left(\dfrac{201 + 1}{2}\right)$ th item

$$= x_{101}$$

Now, deviation will be minimum of taken from the median.

$\therefore$ Mean deviation will be minimum, if $k = x_{101}$.

4. Let n be the number of newspapers which are read.

Then, $60n = (300) \times 5$

$\Rightarrow$ $n = 25$

5. Mean $= \dfrac{1}{10}[(x_1 + x_2 + \ldots + x_{10}) + (4 + 8 + \ldots + 40)]$

$$= \frac{1}{10}(x_1 + x_2 + \ldots + x_{10}) + \frac{4}{10}(1 + 2 + \ldots + 10)$$

$$= 20 + \frac{4 \times 10 \times 11}{10 \times 2} = 42$$

6. Let $d_i = x_i - 8$

$\therefore$ $\sigma_x^2 = \sigma_d^2 = \dfrac{1}{18}\Sigma d_i^2 - \left(\dfrac{1}{8}\Sigma d_i\right)^2$

$$= \frac{1}{18} \times 45 - \left(\frac{9}{18}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4} \Rightarrow \sigma_x^2 = \frac{3}{2}$$

7. If $d_i = \dfrac{x_i - A}{h}$, then $\sigma_x = |h|\sigma_d$

Now, $-2x_i - 3 = \dfrac{x_i + \frac{3}{2}}{-\frac{1}{2}}$

Here, $h = -\dfrac{1}{2}$

$\therefore$ $\sigma_d = \dfrac{1}{|h|}\sigma_x = 2 \times 3.5 = 7$

8. $\bar{x} = \frac{1}{2n+1}[a + (a + d) + \ldots + (a + 2nd)]$

$= \frac{1}{2n+1}[(2n+1)a + d(1 + 2 + \ldots + 2n)]$

$= a + d\frac{2n}{2} \cdot \frac{(2n+1)}{2n+1} = a + nd$

$\therefore$ MD from mean $= \frac{1}{2n+1}\Sigma|x_i - \bar{x}|$

$= \frac{1}{2n+1}2|d|(1 + 2 + \ldots + n) = \frac{n(n+1)|d|}{(2n+1)}$

9. We know, $\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$

where, $d_1 = m_1 - a, d_2 = m_2 - a$, a being the mean of the whole group.

$\therefore \quad 15.6 = \frac{100 \times 15 + 150 \times m_2}{250}$

$\Rightarrow \quad m_2 = 16$

Thus, $13.44 = \frac{\left[\begin{array}{l}(100 \times 9 + 150 \times \sigma^2) + \\ 100 \times (0.6)^2 + 150 \times (0.4)^2\end{array}\right]}{250}$

$\Rightarrow \quad \sigma = 4$

10. Since, $\frac{x + (x+2) + (x+4) + (x+6) + (x+8)}{5} = 11$

$\Rightarrow \quad \frac{5x + 20}{5} = 11 \Rightarrow x = 7$

$\therefore$ Mean of the last three values $= \frac{11 + 13 + 15}{3} = 13$

11. Since, there are 19 observations. So, the middle term is 10th. After including 8 and 32, *i.e.*, 8 will come before 30 and 32 will come after 30.

Here, new median will remain 30.

12. Since, SD < Range

$\Rightarrow \quad \sigma \le (b - a)$

$\Rightarrow \quad \sigma^2 \le (b - a)^2$

13. According to the question,

We have given that,

Arithmetic mean = 2 × Geometric mean

$\Rightarrow \quad \frac{a + b}{2} = 2\sqrt{ab}$

$\Rightarrow \quad a + b = 4\sqrt{ab} \quad \ldots(i)$

We know that, $(a - b)^2 = (a + b)^2 - 4ab$

On substituting the values of $(a + b)$ from Eq. (i), we get

$(a - b)^2 = (4\sqrt{ab})^2 - 4ab = 12ab$

$\Rightarrow \quad a - b = 2\sqrt{3}\sqrt{ab} \quad \ldots(ii)$

On multiplying Eqs. (i) and (ii) simultaneously, we get

$a + b = 4\sqrt{ab}$

$a - b = 2\sqrt{3}\sqrt{ab}$

On adding above equations, we get

$2a = \sqrt{ab}(4 + 2\sqrt{3})$

$\Rightarrow \quad 2a = 2\sqrt{ab}(2 + \sqrt{3})$

$\Rightarrow \quad a = \sqrt{ab}(2 + \sqrt{3})$

On putting the value of a in Eq. (i), we get

$\sqrt{ab}(2 + \sqrt{3}) + b = 4\sqrt{ab}$

$\Rightarrow \quad b = 4\sqrt{ab} - \sqrt{ab}(2 + \sqrt{3})$

$\Rightarrow \quad b = \sqrt{ab}(4 - 2 - \sqrt{3})$

$= \sqrt{ab}(2 - \sqrt{3})$

$\therefore$ Required ratio

$a : b = \frac{\sqrt{ab}(2 + \sqrt{3})}{\sqrt{ab}(2 - \sqrt{3})}$

$= (2 + \sqrt{3}) : (2 - \sqrt{3})$

14. For consistency of performance we need to know the coefficient of variation of marks obtained by two students *A* and *B*. Therefore, from given data, we make the following table

Coefficient of variation of marks obtained by students *A*

x	$d = x - A, A = 72$	d^2
44	−28	784
80	+8	64
76	+4	16
48	−24	576
52	−20	400
72 = *A*	0	0
68	−4	16
56	−16	256
60	−12	144
54	−18	324
	$\Sigma d = -110$	$\Sigma d^2 = 2580$

Hence, $\sigma_A = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} = \sqrt{\frac{2580}{10} - \left(\frac{-110}{10}\right)^2}$

$= \sqrt{258 - 121} = \sqrt{137} = 11.7$

$\therefore \quad \bar{x}_A = A + \frac{\Sigma d}{n}$

$= 72 - \frac{110}{10} = 61$

Therfore, coefficient of variation

$= \frac{SD_A}{\bar{x}_A} \times 100$

$= \frac{11.7}{61} \times 100 = 19.18$

Again, from given data, we make the following table

Coefficient of variation of marks obtained by student *B*

y	$d = y - A, A = 69$	d^2
48	−21	441
75	+6	36
54	−15	225
60	−9	81
63	−6	36
69 = A	0	0
72	+3	9
51	−18	324
57	−12	144
66	−3	9
	$\Sigma d = -75$	$\Sigma d^2 = 1305$

$$\therefore \quad \sigma_B = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{1305}{10} - \left(\frac{-75}{10}\right)^2}$$

$$= \sqrt{130.5 - 56.25} = \sqrt{74.25} = 8.6168$$

Now, $\bar{x}_B = A + \frac{\Sigma d}{n} = 69 - \frac{75}{10}$

$$= \frac{690 - 75}{10} = 61.5$$

$\therefore$ Coefficient of variation $= \frac{\sigma_B}{\bar{x}_B} \times 100$

$$= \frac{8.6168}{61.5} \times 100 = 14.01$$

Since, coefficient of variation is less in case of Student *B*.

Therefore, *B* is more consistent.

15. First find x as given total number of students is 60 *i.e.*,

$$x - 2 + x + x^2 + (x+1)^2 + 2x + x + 1 = 60$$

$$\Rightarrow \quad 5x - 2 + x^2 + x^2 + 1 + 2x + 1 = 60$$

$$\Rightarrow \quad 2x^2 + 7x = 60$$

$$\Rightarrow \quad 2x^2 + 7x - 60 = 0$$

$$\Rightarrow \quad 2x^2 + 15x - 8x - 60 = 0$$

$$\Rightarrow \quad x(2x + 15) - 4(2x + 15) = 0$$

$$\Rightarrow \quad (2x + 15)(x - 4) = 0$$

$$\Rightarrow \quad x = -\frac{15}{2}, x = 4$$

$$\Rightarrow \quad x = 4 \qquad (\because x \text{ is positive})$$

Marks (x_i)	Frequency (f_i)	x_i^2	$f_i x_i$	$f_i x_i^2$
0	2	0	0	0
1	4	1	4	4
2	16	4	32	64
3	25	9	75	225
4	8	16	32	128
5	5	25	25	125
Total	60		168	546

$$\text{Mean } \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{168}{60} = 2.8$$

$$\therefore \quad \text{Variance } \sigma^2 = \sqrt{\frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2} = \sqrt{\frac{546}{60} - (2.8)^2}$$

$$= \sqrt{9.1 - 7.84} = \sqrt{1.26} = 1.12$$

16. Since, $\sigma_x^2 = \left\{\frac{f_1x_1^2 + f_2x_2^2}{f_1 + f_2} - \left(\frac{f_1x_1 + f_2x_2}{f_1 + f_2}\right)^2\right\}$

$$= \frac{1}{(f_1 + f_2)}\left\{f_1x_1^2 + f_2x_2^2 - \frac{(f_1x_1 + f_2x_2)^2}{f_1 + f_2}\right\}$$

$$= \frac{1}{(f_1 + f_2)^2}(f_1^2x_1^2 + f_1f_2x_1^2 + f_2f_1x_2^2 + f_2^2x_2^2 - f_1^2x_1^2 - f_2^2x_2^2 - 2f_1f_2x_1x_2)$$

$$= \frac{f_1f_2}{(f_1 + f_2)^2}(x_1 - x_2)^2$$

17. Let $x_1, x_2, x_3, \ldots, x_n$ be n observations.

$\therefore$ Sum of deviations from AM is

$$d = \sum_{i=1}^{n} (x_i - \bar{x})$$

$$\Rightarrow \quad d = \sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} (1)$$

$$\Rightarrow \quad d = n\bar{x} - n\bar{x} = 0$$

18. The sum of the distribution is $n\bar{x}$.

The sum of new distribution $= n\bar{x} - x_i + x_i'$

$\therefore$ Mean of the new distribution $= \frac{n\bar{x} - x_i + x_i'}{n}$

19. As we know that variance is independent of change of origin, so the new variance will be same.

20. Since, $A = (x, x + 2, x + 4)$

and $B = (x - 2, x + 2, x + 6)$

$\therefore$ Mean of $A = \frac{x + x + 2 + x + 4}{3} = x + 2$

and mean of $B = \frac{x - 2 + x + 2 + x + 6}{3} = x + 2$

Hence, group *B* has more variability than group *A*.

21. Since, deviation $= \Sigma(x_i - A)$

$$\Rightarrow \quad 2 = \Sigma x_i - 20 \times 30$$

$$\Rightarrow \quad \Sigma x_i = 602$$

$$\therefore \quad \text{Mean} = \frac{602}{20} = 30.1$$

22. Required mean $= \frac{\Sigma n^2}{n}$

$$= \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$$

Statement II is also a true statement.

Hence, option (c) is correct.

23. Since, $\sigma\left(\frac{ax+b}{c}\right)=\left|\frac{a}{c}\right|\sigma$

Hence, both statements are true and statement II is a correct explanation for statement I.

24. μ = Mean $=\frac{\Sigma f_i \cdot y_i}{\Sigma f_i}$

Now, $\Sigma f_i(y_i-\mu)=\Sigma f_i y_i-\mu\Sigma f_i$

$$=\mu\Sigma f_i-\mu\Sigma f_i=0$$

Statement I is true. Again, the mean of the square of first n natural numbers

$$=\frac{\Sigma n^2}{n}=\frac{n(n+1)(2n+1)}{n\cdot 6}$$

$$=\frac{(n+1)(2n+1)}{6}$$

$\therefore$ Statement II is false.

25. Variance = $(\text{SD})^2=\frac{\Sigma x_i^2}{n}-\left(\frac{\Sigma x_i}{n}\right)^2$

$$=\frac{n(n+1)(2n+1)}{6n}-\left\{\frac{n(n+1)}{2n}\right\}^2$$

$$=\frac{(n+1)(2n+1)}{6}-\frac{(n+1)^2}{4}$$

$$=(n+1)\left(\frac{4n+2-3n-3}{12}\right)$$

$$=\frac{n^2-1}{12}$$

Statement I is true; statement II is also true; statement II is correct explanation for statement I.

26. Median of $a, 2a, 3a, 4a, \ldots, 50a$ is $\frac{25a+26a}{2}=(25.5)\,a$

Mean deviation about median

$$=\frac{\sum_{i=1}^{50}|x_i-\text{median}|}{n}$$

$\Rightarrow$ $50=\frac{1}{50}\{2|a|\cdot(0.5+1.5+2.5+\ldots+24.5)\}$

$\Rightarrow$ $2500=2|a|\cdot\frac{25}{2}(2\times 0.5+24\times 1)=2|a|\cdot\frac{25}{2}(25)$

$\Rightarrow$ $|a|=4$

27. $\because \sigma_x^2=4$ and $\sigma_y^2=5$

Also, $\bar{x}=2$ and $\bar{y}=4$

Now, $\frac{\Sigma x_i}{5}=2 \Rightarrow \Sigma x_i=10;\ \Sigma y_i=20$

and $\sigma_x^2=\left(\frac{1}{5}\Sigma x_i^2\right)-(\bar{x})^2$

$\Rightarrow$ $\Sigma x_i^2=5(4+4)$

$\Rightarrow$ $\Sigma x_i^2=40$

and $\sigma^2 y=\left(\frac{1}{5}\Sigma y_i^2\right)-(\bar{y})^2$

and $\Sigma y_i^2=5(5+16)$

$\Sigma y_i^2=105$

Combined mean,

$$\sigma_z^2=\frac{1}{10}(\Sigma x_i^2+\Sigma y_i^2-\left(\frac{\bar{x}+\bar{y}}{2}\right)^2$$

$$=\frac{1}{10}(40+105)-9=\frac{145-90}{10}=\frac{55}{10}=\frac{11}{2}$$

28. $(\bar{x})=\frac{\text{Sum of quantities}}{n}=\frac{\frac{n}{2}(a+l)}{n}$

$$=\frac{1}{2}(1+1+100\,d)=1+50\,d$$

$\therefore$ $\text{MD}=\frac{1}{n}\sum|x_i-\bar{x}|$

$\Rightarrow$ $255=\frac{1}{101}(50\,d+49\,d+48\,d+\ldots+d+0+d+\ldots+50\,d)$

$$=\frac{2d}{101}\left(\frac{50\times 51}{2}\right)$$

$\Rightarrow$ $d=\frac{255\times 101}{50\times 51}=10.1$

29. According to the given condition,

$$6.80=\frac{\left[\begin{array}{l}(6-a)^2+(6-b)^2+(6-8)^2\\ \quad+(6-5)^2+(6-10)^2+(6-6)^2\end{array}\right]}{5}$$

$\Rightarrow$ $34=(6-a)^2+(6-b)^2+4+1+16$

$\Rightarrow$ $(6-a)^2+(6-b)^2=13=9+4$

$\Rightarrow$ $(6-a)^2+(6-b)^2=3^2+2^2$

$\Rightarrow$ $a=3,\ b=4$

30. Let the number of boys and girls be x and y.

$\therefore$ $52x+42y=50(x+y)$

$\Rightarrow$ $52x+42y=50x+50y$

$\Rightarrow$ $2x=8y \Rightarrow x=4y$

$\therefore$ Total number of students in the class

$$=x+y=4y+y=5y$$

$\therefore$ Required percentage of boys $=\left(\frac{4y}{5y}\times 100\right)\%=80\%$

21 Fundamentals of Probability

JEE Main MILESTONE

- Basic Definition
- Event
- Important Events
- Probability
- Addition Theorem of Probability

21.1 Basic Definition

(i) Experiment

An operation which results in some well defined outcomes is called an experiment.

(ii) Random Experiment

An experiment whose outcome cannot be predicted with certainty is called a random experiment. In other words, if an experiment is performed many times under similar conditions and the outcome of each time is not the same, then this experiment is called a random experiment.

e.g., Throwing of an unbiased die.

Probability is a measure or estimation of how likely it is that something will happen or that a statement is true. The higher the degree of probability, the more likely the event is to happen or in a longer series of samples, the greater the number of times such event is expected to happen.

(iii) Sample Space

The set of all possible outcomes of a random experiment is called the sample space for that experiment. It is usually denoted by S.

e.g., When two coins are tossed, then the sample space is

$$S = \{(H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2)\}$$

(iv) Sample Point/Event Point

Each element of the sample space is called a sample point or an event point.

e.g., When a dice is thrown, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, where 1, 2, 3, 4, 5 and 6 are the sample points.

Coin In a unbiased coin there are two sides, one side is Head (H) and other side is Tail (T).

Die In a die there are six faces, the number on this faces are marked $\{1, 2, 3, 4, 5, 6\}$.

Card A pack of cards has 52 cards. There are four suits Spade (♠ black face), Heart (♥red face), Diamond (♦ red face) and Club (♣ black face). Each having 13 cards. In 13 cards of each suit there are 4 face cards namely, Ace, King, Queen and Jack, so there are all 16 face cards (4 Ace, 4 King, 4 Queen and 4 Jacks.)

21.2 Event

An event is associated with a subset of sample space. Events can be classified into various types on the basis of the elements they have.

Types of Events

(i) **Impossible and sure events** The emtpy set ϕ is called an impossible event. For example, getting two heads in tossing a unbiased coin one time.

The event which is certain to occur is said be the sure event.

e.g., In tossing a die any number will come is a sure event.

(ii) **Simple event** If an event E has only one sample point of a sample space, it is called a simple (or elementary) event.

e.g. In the experiment of tossing a coin twicely getting both head *i.e.*, $E = \{H, H\}$.

(iii) **Compound event** If an event has more than one sample point, it is called a compound event.

e.g., In the experiment of tossing a coin thrice the events.

E : Exactly one head appeared

F : Atleast one head appeared

G : Atmost one head appeared, etc., are all compound events. As

$E = \{HTT, THT, TTH\}$

$F = \{HTT, THT, TTH, HHT, THH, HTH, HHH\}$

$G = \{HTT, THT, TTH, TTT\}$

Each of the above subsets contain more than one sample point. Hence, they are all compound events.

Algebra of Events

Let A, B and C be events associated with an experiment whose sample space is S.

(i) Complementary Event

For every event A, there corresponds another event A' or A^c or $\overline{A}$ called the complementary event to A. It is also called the event not A.

e.g., Let $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ be a sample space and let.

$$A = \{HTH, HHT, THH\} \text{ be a subset of } S,$$

then complementary of A is defined as

$$\overline{A} = \{HHH, HTT, THT, TTH, TTT\}$$

(ii) Event A or B

It is the set of all those elements, which are either in A or B or in both. And it is denoted by the symbol $A \cup B$.

e.g., Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$

(iii) Events A and B

It is the set of all those elements, which are common in A and B. And it is denoted by the symbol $A \cap B$.

e.g., Let $A = \{2, 3, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \cap B = \{2, 3, 5\}$.

(iv) Event A but not B

It is the set of all those elements, which are in A but not in B. And it is denoted by the symbol $A - B$ or $A \cap B'$.

e.g., Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, then $A - B = \{1, 3\}$.

***Sample* Problem 1** *A coin is tossed, if it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.* **[NCERT]**

(a) $\{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$

(b) $\{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H3, H5\}$

(c) $\{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5\}$

(d) *None of the above*

Interpret (a) Let H and T represent a head and tail.

Let red ball represent R_1, R_2 and black ball represent B_1, B_2 and B_3.

The sample space is

$$S = \{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$$

21.3 Important Events

Equally Likely Events

Outcomes are said to be equally likely when we have no reason to believe that one is more likely to occur than the other *e.g.*, When an unbiased die is thrown all the six faces, 1, 2, 3, 4, 5, 6 are equally likely to come up.

Exhaustive Events

A set of events is said to be exhaustive, if one of them must necessarily happen every time the experiment is performed.

e.g., When a die is thrown, events 1, 2, 3, 4, 5, 6 form an exhaustive set of events.

Note We can say that the total number of elementary events of a random experiment is called the exhaustive number of cases.

Mutually Exclusive Events

Two or more events are said to be mutually exclusive if one of them occurs, others cannot occur. Thus, if two or more events are said to be mutually exclusive, if no two of them can occur together.

Hence, $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive if and only if $A_i \cap A_j = \phi, \forall i \neq j$.

e.g., When a die is thrown, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Let A is an event of occurrence of number greater than 4, *i.e.*, $\{5, 6\}$.

B is an event of occurrence of an odd number $\{1, 3, 5\}$.

C is an event of occurrence of an even number $\{2, 4, 6\}$.

Here, events B and C are mutually exclusive but the events A and B or A and C are not mutually exclusive.

***Sample* Problem 2** *Three coins are tossed once. Let A denote the event 'three heads show,' B denote the event 'two heads and one tail show', C denote the event 'three tails show' and D denote the event 'a head shows on the first toss', which events are mutually exclusive?* [**NCERT**]

(a) $A \cap B, A \cap D$

(b) $A \cap C, B \cap D$

(c) $B \cap C, C \cap D$

(d) *None of these*

Interpret (c) When three coins are tossed, then there are $2^3 = 8$ possible outcomes.

i.e., $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

A = Three heads shows = $\{HHH\}$

B = Two heads and one tail show = $\{HHT, HTH, THH\}$

C = Three tails show = $\{TTT\}$

D = A head shows on the first toss

$= \{HHH, HHT, HTH, HTT\}$

Here,
$$A \cap B = \phi$$
$$A \cap C = \phi$$
$$B \cap C = \phi$$
$$C \cap D = \phi$$
$$A \cap B \cap C = \phi$$

Here, A and B, A and C, B and C, C and D; A, B and C are mutually exclusive.

***Sample* Problem 3** *Two dice are thrown. The events A, B and C are as follows* [**NCERT**]

A : *Getting an even number on the first die.*

B : *Getting an odd number on the first die.*

C : *Getting the sum of the numbers on the dice* ≤ 5.

Which of the following statement is false?

(a) *A and B are mutually exclusive*

(b) *A and B are mutually exclusive and exhaustive*

(c) $A = B'$

(d) *Aand C are mutually exclusive*

Interpret (d) If two dice are thrown, then total number of possible outcomes, $S = 6 \times 6 = 36$ which are as follows

↓ \ →	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

A = getting an even number on the first die

$= \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

B = getting an odd number on the first die

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$C = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$

Here, B' = Event getting an odd number on the first die.

(a) True

$\because$ A = getting an even number on the first die

B = getting an odd number on the first die

$\Rightarrow A \cap B = \phi$

$\therefore A$ and B are mutually exclusive events.

(b) True

$\therefore A \cup B = S$ *i.e.*, exhaustive. Also, $A \cap B = \phi$

(c) True

$\therefore$ B = getting an odd number on the first die

$\Rightarrow B'$ = getting an even number on first die = A

$\therefore$ $A = B'$

(d) False

$\because A \cap C = \{(2,1), (2,2), (2,3), (4,1)\} \neq \phi$, so A and C are not mutually exclusive.

21.4 Probability

The probability of an event E to occur is the ratio of the number of cases in its favour to the total number of cases.

$$\therefore P(E) = \frac{\text{Number of cases favourable to event } E}{\text{Total number of cases}}$$
$$= \frac{n(E)}{n(S)}$$

Probability of non-occurrence of event E is

$$P(\overline{E}) = 1 - P(E)$$

and $0 \le P(E) \le 1$. If $P(E) = 1$, then event E is known as certain event and if $P(E) = 0$, then E is known as impossible event.

Odds in favour and odds against an event If a is the number of cases favourable to the event E, b is the number of cases favourable to the event E'. (*i.e.*, number of cases against to E). Then, odds in favour of E are $a : b$ and odds against of E are $b : a$. Then, $P(E) = \frac{a}{a+b}$ and $P(\overline{E}) = \frac{b}{a+b}$.

Thus, odds in favour of an event $E = \frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(\overline{E})}$

and odds against an event $E = \frac{b}{a} = \frac{P(\overline{E})}{P(E)}$

Hot Spot 1

Problems Related to Basic Probability

In this types of problems, sample space of an event are given and we have to find out probability of given events. These types of problems are directly and indirectly asked in AIEEE (JEE Main) 2010, 2007, 2006, 2003.

Following steps are given below

Step I Firstly write the given sample space.

Step II Using the formula of probability $= \frac{\text{Favourable cases}}{\text{Total number of cases}}$

Step III Simplify it and get the result.

***Sample* Problem 4** *4 cards are drawn from a well shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?* **[NCERT]**

(a) $\frac{283}{20825}$

(b) $\frac{286}{20825}$

(c) $\frac{286}{2087}$

(d) *None of the above*

Interpret (*b*) **Step I** Write the sample space.

Cards (52)
- Club (13)
- Diamond (13)
- Heart (13)
- Spade (13)

Total number of ways selecting 4 cards out of 52 cards, $n(s) = {}^{52}C_4$

If E be the event obtaining 3 diamonds and 2 spade, then

$$n(E) = {}^{13}C_3 \times {}^{13}C_1$$

Step II Using the formula $P = \frac{n(E)}{n(S)}$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4}$

Step III Simplification $P = \frac{13!}{10!3!} \times 13 \times \frac{4! \times 48!}{52!} = \frac{286}{20825}$

***Sample* Problem 5** *There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?* **[NCERT]**

(a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{1}{7}$ (d) $\frac{1}{5}$

Interpret (*b*) Let E be the event that woman is selected.

$\therefore$ $n(E) = 6$

Number of sample space $n(S)$ = Total number of persons in the council

$$= 4 + 6 = 10 \quad \therefore P = \frac{n(E)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

***Sample* Problem 6** *Three coins are tossed once. Find the probability of getting atleast 2 heads and atmost 2 heads.* **[NCERT]**

(a) $\frac{1}{4}, \frac{7}{8}$ (b) $\frac{1}{2}, \frac{7}{8}$ (c) $\frac{1}{5}, \frac{7}{8}$ (d) $\frac{1}{3}, \frac{7}{8}$

Interpret (*b*) If three coins are tossed, then total number of possible outcomes $= 2^3 = 8$, which are

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(i) If E be the event of getting atleast 2 heads.

(atleast means we can consider the cases having 2 heads and higher than 2)

$\Rightarrow$ $n(E) = 2$ heads + 3 heads $= 3 + 1 = 4$

$\Rightarrow$ Probability of getting atleast 2 heads $P = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(ii) If E be the event of getting atmost 2 heads.

$\Rightarrow$ $n(E) = 2$ heads + 1 head + 0 head

(atmost means we cannot consider the cases of having heads higher than 2) $= 3 + 3 + 1 = 7$

$\Rightarrow$ Probability of getting atmost 2 heads $P = \frac{n(E)}{n(S)} = \frac{7}{8}$

21.5 Addition Theorem of Probability

(i) If A and B be any two events in a sample space S, then the probability of occurrence of atleast one of the events A and B is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note

- If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0 \therefore P(A \cup B) = P(A) + P(B)$
- Two events A and B are mutually exclusive iff $P(A \cup B) = P(A) + P(B)$.
- $1 = P(S) = P(A \cup A') = P(A) + P(A')$ $(\because A \cap A' = \phi)$ or $P(A') = 1 - P(A)$.

(ii) If A, B and C are any three events in a sample space S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Note

- If A, B, C are mutually exclusive events, then $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$, $A \cap B \cap C = \phi$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

- If A and B are any two events, then $(A - B) \cap (A \cap B) = \phi$ and $A = (A - B) \cap (A \cap B)$

$$\therefore \quad P(A) = P(A - B) + P(A \cap B) = P(A \cap B') + P(A \cap B)$$

or $\quad P(A) - P(A \cap B) = P(A - B) = P(A \cap B')$

$$(\because A - B = A \cap B')$$

Similarly, $P(B) - P(A \cap B) = P(B - A = P(B \cap A')$

(iii) **General form of addition theorem of probability**

If $A_1, A_2, \ldots, A_n$ are n sample space, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<i} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Note For any number of (finite or infinite) mutually exclusive events, $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Relation between Set Theory and Set Probability

		Set Theory	Set Probability
1.	When two events A and B occurs	(i) n (atleast one of the events occurs) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (ii) n (exactly one of the event occur), $n(A \cap \bar{B})\, n(\bar{A} \cup B) = n(A) + n(B) - 2n(A \cap B)$	(i) P (atleast one of the events occurs) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (ii) p (exactly one of the events occurs) $P(A \cap \bar{B}) + P(\bar{A} \cap B) = P(A) + P(B) - 2P(A \cap B)$
2.	When three events A, B and C occurs	(i) n (atleast one of the vents occur), $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$ (ii) n (atleast two of the events occur) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 2n(A \cap B \cap C)$ (iii) n (exactly two of the events occur) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$ (iv) n (exactly one of the events occur) $= n(A) + n(B) + n(C) - 2n(B \cap C) - 2n(C \cap A) - 2n(A \cap B) + 3n(A \cap B \cap C)$	(i) P (atleast one of the events occurs) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$ (ii) P (atleast two of the events occur) $= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$ (iii) P (exactly two of the events occur) $= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$ (iv) P (exactly one of the events occur) $= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Important Results

Let A, B and C are three events.

(i) The probability of occurrence of atleast one of the events A, B and C is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

(ii) P(atleast two of A, B and C occur)

$$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$

(iii) P(exactly two of A, B and C occur)

$$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$$

(iv) P(exactly one of A, B and C occur)

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(C \cap A) + 3P(A \cap B \cap C)$$

(v) **Booley's Inequality** If events $E_1, E_2, \ldots, E_n$ are associated with a random experiment, then

(a) $P(\bigcap_{i=1}^{n} E_i) \geq \sum_{i=1}^{n} P(E_i) - (n - 1)$

(b) $P(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$

Hot Spot 2

Problems Related to Set Theoretical Probability

In this type of problems, probability of some events are given and we have to find out probability of other events by addition theorem and results based on it of probability under the given conditions. These types of problems are directly asked in JEE 2008, 2005.

Following steps are given below.

Step I Firstly, write the given probability or find the probability of an event.

Step II Use the addition theorem of probability under the given condition.

Step III Simplify it and get the required result.

***Sample* Problem 7** *In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is* 0.8 *and the probability of passing the second examination is* 0.7. *The probability of passing atleast one of them is* 0.95. *What is the probability of passing both?* [**NCERT**]

(*a*) 0.55 (*b*) 0.58
(*c*) 0.59 (*d*) 0.60

Interpret (*a*) **Step I** Write the given probability.

Here, $P(\text{I}) = 0.8, P(\text{II}) = 0.7$

$P(\text{atleast one of I and II}) = P(\text{I} \cup \text{II}) = 0.95$

Step II Use the addition theorem of probability.

Now, $P(\text{I} \cup \text{II}) = P(\text{I}) + P(\text{II}) - P(\text{I} \cap \text{II})$

$\therefore \quad 0.95 = 0.8 + 0.7 - P(\text{I} \cap \text{II})$

Step III Simplify it.

$\Rightarrow \quad 0.95 = 1.5 - P(\text{I} \cap \text{II})$

$\Rightarrow \quad P(\text{I} \cap \text{II}) = 1.5 - 0.95 = 0.55$

***Sample* Problem 8** *In a class of* 60 *students,* 30 *opted for NCC,* 32 *opted for NSS and* 24 *opted for both NCC and NSS. Following statements are given below.* [**NCERT**]

I. *The probability that the student opted for NCC or NSS is* $\frac{19}{30}$.

II. *The probability that the student has opted for neither NCC nor NSS is* $\frac{11}{30}$.

Choose the correct statements.

(*a*) *Only* I (*b*) *Only* II
(*c*) *Both* I *and* II (*d*) *None of these*

Interpret (*c*)

(i) Let A and B denote the students in NCC and NSS, respectively.

Here, $n(A) = 30, n(B) = 32, n(A \cap B) = 24$

($\because$ 24 students opted for both NCC and NSS *i.e.*, they are common in both)

$\therefore \quad P(A) = \frac{30}{60}, P(B) = \frac{32}{60}$ and $P(A \cap B) = \frac{24}{60}$

P (student opted for NCC or NSS)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [$\because P(A \cup B) = P(A \text{ or } B)$]

$$= \frac{30}{60} + \frac{32}{60} - \frac{24}{60} = \frac{30 + 32 - 24}{60} = \frac{62 - 24}{60} = \frac{38}{60} = \frac{19}{30}$$

(ii) P (student has opted neither NCC nor NSS)

$= 1 - P$ (student opted for NCC or NSS)

$$= 1 - \frac{19}{30} = \frac{30 - 19}{30} = \frac{11}{30}$$

WORKED OUT
Examples

Example 1 *A coin is tossed and a die is rolled. The probability that the coin shows a head and the die shows 3, is*

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$

(c) $\frac{11}{12}$ (d) $\frac{1}{11}$

Solution (*b*) Since, die has 6 outcomes and coin has two outcomes.

$\therefore$ Total number of outcomes $= 6 \times 2 = 12$

Favourable ways = 1

$\therefore$ Required probability $= \frac{1}{12}$

Example 2 *What is the probability that in the toss of two dice we obtain an even sum or a sum less than 5?*

(a) $\frac{5}{9}$ (b) $\frac{1}{9}$

(c) $\frac{1}{6}$ (d) $\frac{1}{2}$

Solution (*a*) A = Event of obtaining an even sum

$= \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), \ldots, (6, 2), (6, 4), (6, 6)\}$

B = Event of obtaining a sum less than 5

$= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$

$A \cap B = \{(1, 1), (1, 3), (2, 2), (3, 1)\}$

$\therefore$ $n(A) = 18$ and $n(B) = 6$, $n(A \cap B) = 4$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{1}{2} + \frac{1}{6} - \frac{1}{9}$$

$$= \frac{9 + 3 - 2}{18} = \frac{5}{9}$$

Example 3 *From 4 children, 2 women and 4 men, 4 are selected. The probability that there are exactly 2 children among the selected, is*

(a) $\frac{48}{90}$ (b) $\frac{3}{7}$

(c) $\frac{9}{17}$ (d) *None of these*

Solution (*b*) Total number of selection $= {}^{10}C_4 = 210$

2 children can be selected in three combinations.

(i) 2 children and 2 women $= {}^4C_2 \times {}^2C_2 = 6$

(ii) 2 children and 2 men $= {}^4C_2 \times {}^4C_2 = 36$

(iii) 2 children, 1 man and 1 woman

$= {}^4C_2 \times {}^4C_1 \times {}^2C_1 = 48$

Total favourable cases = 48 + 36 + 6 = 90

$\therefore$ Required probability $= \frac{90}{210} = \frac{9}{21} = \frac{3}{7}$

Example 4 *From a pack of 52 playing cards, three cards are drawn at random. The probability of drawing a king, a queen and a knave is*

(a) $\frac{1}{52}$ (b) $\frac{16}{5525}$

(c) $\frac{6}{525}$ (d) $\frac{64}{5525}$

Solution (*b*) $n(S) = {}^{52}C_3$

$n(E)$ = number of selecting 3 cards out of these one is king, one is queen and one is knave.

$= {}^4C_1 \times {}^4C_1 \times {}^4C_1 = 64$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{64}{{}^{52}C_3}$

$$= \frac{64 \times 1 \times 2 \times 3}{52 \times 51 \times 50} = \frac{16}{5525}$$

Example 5 *A box contain 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box. What is the probability that* **[NCERT]**

(i) *all will be blue?*

(ii) *atleast one will be green?*

(a) $\frac{{}^{10}C_5}{{}^{60}C_5}, 1 - \frac{{}^{50}C_5}{{}^{60}C_5}$ (b) $\frac{{}^{20}C_5}{{}^{60}C_5}, 1 - \frac{{}^{30}C_5}{{}^{60}C_5}$

(c) $\frac{{}^{20}C_5}{{}^{60}C_5}, 1 - \frac{{}^{30}C_4}{{}^{60}C_5}$ (d) $\frac{{}^{30}C_5}{{}^{60}C_5}$

Solution (*b*)

10 Red marbles
20 Blue marbles
30 Green marbles

Total number of ways selecting 5 marbles out of 60 marbles = ${}^{60}C_5$

(i) If E be the event all marbles will be blue, then $n(E) = {}^{20}C_5$

$\therefore$ Required probability $= \dfrac{{}^{20}C_5}{{}^{60}C_5}$

(ii) P (atleast one will be green) $= 1 - P$(no will be green)

$= 1 - \dfrac{{}^{30}C_5}{{}^{60}C_5}$

Example 6 *The probability that a student will pass the final examination in both English and Hindi* 0.5 *and the probability of passing neither is* 0.1. *If the probability of passing the English examination is* 0.75. *What is the probability of passing the Hindi examination?* **[NCERT]**

(*a*) 0.70 (*b*) 0.60

(*c*) 0.65 (*d*) 0.64

Solution (*c*) Let H and E denote the students passing in Hindi and English examination.

Here, $P(H \cap E) = 0.5$ (given)

$P(H' \cap E') = P(H \cup E)' = 0.1$

(the probability of passing in both English and Hindi is $P(H \cap E)$, then probability of passing neither will be $P(H \cup E)'$ because it is complementary to 1st event).

$\therefore \quad P(H \cup E) = 1 - P(H \cup E)' = 1 - 0.1 = 0.9$

Also, $P(E) = 0.75$

$\Rightarrow \quad P(H \cup E) = P(H) + P(E) - P(H \cap E)$

$\Rightarrow \quad 0.9 = P(H) + 0.75 - 0.5$

$\Rightarrow \quad 0.9 = P(H) + 0.25$

$\Rightarrow \quad P(H) = 0.9 - 0.25 = 0.65$

Example 7 *One mapping (function) is selected at random from all the mappings of the set* $A = \{1, 2, 3, \ldots, n\}$ *into itself. The probability that the mapping selected is one to one ?* **[NCERT]**

(*a*) $\dfrac{n!}{n^{n-1}}$ (*b*) $\dfrac{n!}{n^n}$

(*c*) $\dfrac{n!}{2n^n}$ (*d*) *None of these*

Solution (*b*) Total number of mappings from a set A having elements into itself is n^n.

And the total number of one to one mapping is $n!$.

$\therefore$ Required probability $= \dfrac{n!}{n^n}$

Example 8 *Out of* 100 *students, two sections of* 40 *and* 60 *are formed. If you and your friend are among the* 100 *students, what is the probability that both enter the same section?* **[NCERT]**

(*a*) $\dfrac{15}{33}$ (*b*) $\dfrac{17}{33}$

(*c*) $\dfrac{16}{33}$ (*d*) *None of these*

Solution (*b*) Let the two sections have A and B are of 40 and 60 students, respectively.

When both enter the same section

There are two cases,

(I) Either both are in section A or

(II) Either both are in section B.

Case I If both are in section A, then 40 students out of 100 students can be selected = ${}^{100}C_{40}$ ways.

And 38 students can be selected out of 98 students = ${}^{98}C_{38}$ ways

$$\therefore \quad \text{Probability} = \frac{{}^{98}C_{38}}{{}^{100}C_{40}}$$

$$= \frac{98!}{38!\,60!} \times \frac{40!\,60!}{100!} \qquad \left[\because {}^{n}C_r = \frac{n!}{r!(n-r)!}\right]$$

$$= \frac{98! \times 40 \times 39 \times 38!}{38! \times 100 \times 99 \times 98!}$$

$$= \frac{40 \times 39}{100 \times 99} = \frac{2}{5} \times \frac{13}{33} = \frac{26}{165} \qquad \ldots\text{(i)}$$

Case II If both are in section B, then 60 students out of 100 students can be selected = ${}^{100}C_{60}$ ways

58 students out of 98 students can be selected = ${}^{98}C_{58}$

$$\therefore \text{Probability} = \frac{{}^{98}C_{58}}{{}^{100}C_{60}} = \frac{98!}{58!\,40!} \times \frac{60!\,40!}{100!}$$

$$= \frac{98! \times 60 \times 59 \times 58! \times 40!}{58! \times 40! \times 100 \times 99 \times 98!}$$

$$= \frac{60 \times 59}{100 \times 99} = \frac{3}{5} \times \frac{59}{99} = \frac{59}{5 \times 33} = \frac{59}{165} \qquad \ldots\text{(ii)}$$

Hence, required probability that students are either in section A or B

$$= \frac{26}{165} + \frac{59}{165} = \frac{85}{165} = \frac{17}{33} \qquad \text{[from Eqs. (i) and (ii)]}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Sample Space

1. A coin is tossed and a die is thrown. In coin, H and T occur and in die, a number from 1 to 6 may occur. The sample space is [NCERT]

(a) $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

(b) $\{1H, 2H, 3H, 4H, 5H, 6H, T1, T2, T3, T4, T5, T6\}$

(c) $\{H, T, 1, 2, 3, 4, 5, 6\}$

(d) None of the above

2. A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment. [NCERT]

(a) $\{RW, WW\}$ (b) $\{RW, WW, WR\}$

(c) $\{RW, RR\}$ (d) None of these

3. Which of the following cannot be valid assignments of probabilities for outcomes of sample space

$$S = \{W_1, W_2, W_3, W_4, W_5, W_6, W_7\}?$$

Assignment	W_1	W_2	W_3	W_4	W_5	W_6	W_7
(I)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(II)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(III)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(IV)	–0.1	0.2	0.3	0.4	–0.2	0.1	0.3
(V)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

[NCERT]

(a) I, II (b) I, III

(c) II, III, IV (d) None of these

Basic Probability

4. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. Find the probability that the sum of the numbers that turn up is 3. [NCERT]

(a) $-\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

5. A letter is selected at random from the word ASSASSINATION. Find the probability that letter is (i) a vowel (ii) a consonant. [NCERT]

(a) $\frac{5}{11}, \frac{7}{11}$ (b) $\frac{6}{13}, \frac{6}{13}$

(c) $\frac{6}{13}, \frac{7}{13}$ (d) $\frac{6}{11}, \frac{7}{11}$

6. What is the probability that when one die is thrown, the number appearing on top is even?

(a) 1/6 (b) 1/3

(c) 1/2 (d) None of these

7. A pair of dice thrown, if 5 appears on atleast one of the dice, then the probability that the sum is 10 or greater, is

(a) $\frac{11}{36}$ (b) $\frac{2}{9}$ (c) $\frac{3}{11}$ (d) $\frac{1}{12}$

8. In a college, 25% of the boys and 10% of the girls offer Mathematics. The girls constitute 60% of the total number of students. If a students is selected at random and is found to be studying Mathematics. The probability that the student is a girl is

(a) 1/6 (b) 3/8 (c) 5/8 (d) 5/6

9. If birth to a male child and birth to a female child are equal probable, then what is the probability that atleast one of the three children born to a couple is male ?

(a) $\frac{4}{5}$ (b) $\frac{7}{8}$ (c) $\frac{8}{7}$ (d) $\frac{1}{2}$

10. A and B toss a coin alternately till one of them tosses heads and wins the game, their respective probability of winning are

(a) $\frac{1}{4}$ and $\frac{3}{4}$ (b) $\frac{1}{2}$ and $\frac{1}{2}$
(c) $\frac{1}{3}$ and $\frac{2}{3}$ (d) $\frac{1}{5}$ and $\frac{4}{5}$

11. The probability that a leap year will have 53 Fridays or 53 Saturdays, is

(a) $\frac{2}{7}$ (b) $\frac{3}{7}$ (c) $\frac{4}{7}$ (d) $\frac{1}{7}$

12. In a non-leap year, the probability of having 53 Tuesdays or 53 Wednesdays is [NCERT Exemplar]

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$
(c) $\frac{3}{7}$ (d) None of these

13. Without repetition of the numbers, four digit numbers are formed with the numbers 0, 2, 3, 5. The probability of such a number divisible by 5 is [NCERT Exemplar]

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$
(c) $\frac{1}{30}$ (d) $\frac{1}{4}$

14. 6 boys and 6 girls sit in a row randomly. The probability that all 6 girls sit together, is [NCERT Exemplar]

(a) $\frac{1}{64}$ (b) $\frac{1}{8}$
(c) $\frac{1}{132}$ (d) None of these

15. Three coins are tossed together, then the probability of getting atleast one head is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$
(c) $\frac{1}{8}$ (d) $\frac{7}{8}$

16. If any four numbers are selected and they are multiplied, then the probability that the last digit will 1,3,5 or 7, is

(a) 4/625 (b) 18/625
(c) 16/625 (d) None of these

17. If a coin be tossed n times, then probability that the head comes odd times is

(a) 1/2 (b) $1/2^n$
(c) $1/2^{n-1}$ (d) None of these

18. The probability that the same number appear on throwing three dice simultaneously, is

(a) 1/6 (b) 1/36
(c) 5/36 (d) None of these

19. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die. [NCERT Exemplar]

(a) $\frac{5}{9}$ (b) $\frac{4}{9}$
(c) $\frac{2}{9}$ (d) None of these

20. A die is rolled three times. The probability of getting a larger number than the previous number each time is

(a) $\frac{15}{216}$ (b) $\frac{5}{54}$
(c) $\frac{13}{216}$ (d) $\frac{1}{18}$

21. The probability of having atleast one tail in 4 throws with a coin, is

(a) $\frac{15}{16}$ (b) $\frac{1}{16}$
(c) $\frac{1}{4}$ (d) 1

22. Fifteen coupons are numbered 1 to 15. Seven coupons are selected at random, one at a time with replacement. The probability that the largest number appearing on a selected coupon be 9, is

(a) $\left(\frac{1}{15}\right)^7$ (b) $\left(\frac{8}{18}\right)^7$
(c) $\left(\frac{3}{5}\right)^7$ (d) None of these

23. A fair coin is tossed repeatedly. If the tail appears on first four tosses, then the probability of the head appearing on the fifth toss is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{32}$ (c) $\frac{31}{32}$ (d) $\frac{1}{5}$

24. A five digit number is formed by writing the digits 1, 2, 3, 4, 5, in a random order without repetitions. Then, the probability that the number is divisible by 4, is

(a) 3/5 (b) 18/5
(c) 1/5 (d) 6/5

25. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that atleast one letter is in its proper envelope. [NCERT]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{1}{5}$ (d) None of these

26. The number lock of a suitcase has four wheels, each labelled with 10-digits *i.e.*, from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase? [**NCERT**]

(a) $\frac{1}{5040}$ (b) $\frac{3}{5040}$
(c) $\frac{7}{5040}$ (d) None of these

27. Four candidates A, B, C and D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that C will be selected? [**NCERT**]

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$
(c) $\frac{5}{9}$ (d) $\frac{4}{9}$

28. An insurance salesman sells policies to 5 men, all of identical age and in good health. The probability that a man of this particular age will be alive after 30 yr is 2/3. The probability that after the lapse of 30 yr all the five persons will be alive, is

(a) $\frac{1}{16}$ (b) $\frac{16}{81}$
(c) $\frac{32}{243}$ (d) None of these

29. Twenty five coins are tossed simultaneously. The probability that the fifth coin will fall with head upwards, is

(a) $\frac{5}{25}$ (b) $\frac{5}{2^{25}}$
(c) $\frac{1}{2}$ (d) None of these

30. If odds against solving a question by three students are 2 : 1, 5 : 2 and 5 : 3 respectively, then probability that the question is solved only by one student is

(a) 31/56 (b) 24/56
(c) 25/56 (d) None of these

31. In $x = 33^n$, n is a positive integral value, then what is the probability that x will have 3 at its units place?

(a) 1/3 (b) 1/4 (c) 1/5 (d) 1/2

32. Two persons A and B take turns in throwing a pair of dice. The first person to through 9 from both dice will be awarded the prize. If A throws first, then the probability that B wins the game, is

(a) $\frac{9}{17}$ (b) $\frac{8}{17}$
(c) $\frac{8}{9}$ (d) $\frac{1}{9}$

33. The accompanying Venn diagram shows three events, A, B and C and also the probabilities of the various intersections [for instance, $P(A \cap B) = .07$]. Find $P(B \cap C)$ and probability of exactly one of the three occurs. [**NCERT Exemplar**]

(a) 0.15, 0.15 (b) 0.15, 0.51
(c) 0.15, 0.50 (d) None of these

34. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one-by-one. The probability that minimum of the two number is less than 4, is

(a) $\frac{1}{15}$ (b) $\frac{14}{15}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$

35. A five digit number is chosen at random. The probability that all the digit are distinct and digits at odd places are odd and digits at even place are even, is

(a) $\frac{1}{60}$ (b) $\frac{2}{75}$
(c) $\frac{1}{50}$ (d) $\frac{1}{75}$

36. A three digit number, which is a multiple of 11, is chosen at random. Probability that the number so chosen is also a multiple of 9, is equal to

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$
(c) $\frac{1}{100}$ (d) $\frac{9}{100}$

37. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with these three vertices is equilateral is equal to

[**NCERT Exemplar**]

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$
(c) $\frac{1}{10}$ (d) $\frac{1}{20}$

38. If from each of three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black ball will be drawn, is

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$
(c) $\frac{1}{32}$ (d) $\frac{3}{16}$

39. There are four machines and it is known that exactly two of them are faulty. They are tested, one-by-one in a random order till both the faulty machines are identified. Then, the probability that only two tests are needed, is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

40. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5, is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{7}$

(c) $\frac{1}{8}$ (d) $\frac{1}{49}$

41. If n integers taken at random are multiplied together, then the probability that the last digit of the product is 1,3, 7 or 9 is

(a) $2^n/5^n$ (b) $4^n - 2^n/5^n$

(c) $4^n/5^n$ (d) None of these

42. There are 9999 tickets bearing numbers 0001, 0002, ... , 9999. If one ticket is selected from these tickets at random, the probability that the number on the ticket will consists of all different digits, is

(a) $\frac{5040}{9999}$ (b) $\frac{5000}{9999}$

(c) $\frac{5030}{9999}$ (d) None of these

43. Three ships A, B and C sail from England to India. If the ratio of their arriving safely are 2 : 5, 3 : 7 and 6 : 11, respectively, then the probability of all the ships for arriving safely is

(a) 18/595 (b) 6/17 (c) 3/10 (d) 2/7

44. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

(a) $\frac{16}{81}$ (b) $\frac{1}{81}$

(c) $\frac{80}{81}$ (d) $\frac{65}{81}$

45. Thirty two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players, the better ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up, respectively, is

(a) 16/31 (b) 1/2

(c) 17/31 (d) None of these

46. The probability of solving a question by three students are 1/2, 1/4, 1/6, respectively. Probability of question being solved will be

(a) 33/48 (b) 35/48

(c) 31/48 (d) 37/48

47. The probabilities of winning a race by three persons A, B and C are 1/2, 1/4 and 1/4, respectively. They run two races. The probability of A winning the second race when B wins the first race is

(a) 1/3 (b) 1/2 (c) 1/4 (d) 2/3

Use of Permutation and Combination in Probability

48. A letter is taken out at random from 'ASSISTANT' and another is taken out from 'STATISTICS'. The probability that they are the same letters is

(a) $\frac{1}{45}$ (b) $\frac{13}{90}$

(c) $\frac{19}{90}$ (d) None of these

49. A bag contains 5 white and 3 black balls and 4 balls are successively drawn out and not replaced. The probability that they are alternately of different colours, is

(a) $\frac{1}{196}$ (b) $\frac{2}{7}$ (c) $\frac{13}{56}$ (d) $\frac{1}{7}$

50. Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive.

[NCERT Exemplar]

(a) $\frac{186}{190}$ (b) $\frac{187}{190}$

(c) $\frac{188}{190}$ (d) $\frac{18}{{}^{20}C_3}$

51. While shuffling a pack of 52 playing cards, 2 are accidentally dropped. Find the probability that the missing cards to be of different colours.

[NCERT Exemplar]

(a) $\frac{29}{52}$ (b) $\frac{1}{2}$ (c) $\frac{26}{51}$ (d) $\frac{27}{51}$

52. Seven persons are to be seated in a row. The probability that two particular persons sit next to each other is [NCERT Exemplar]

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$

(c) $\frac{2}{7}$ (d) $\frac{1}{2}$

53. A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel is

[NCERT Exemplar]

(a) $\frac{1}{3}$ (b) $\frac{4}{11}$ (c) $\frac{2}{11}$ (d) $\frac{3}{11}$

54. If a committee of 3 is to be chosen from a group of 38 people of which you are a member. What is the probability that you will be on the committee?
(a) $({}^{38}C_3)$ (b) $({}^{37}C_2)$
(c) $({}^{37}C_2)/({}^{38}C_3)$ (d) 666/8436

55. A bag contains 5 brown and 4 white socks. A man pulls out two socks. The probability that these are of the same colour, is
(a) 5/108 (b) 18/108 (c) 30/108 (d) 48/108

56. If the letters of the word ALGORITHM are arranged at random in a row, what is the probability the letters GOR must remain together as a unit? [NCERT Exemplar]
(a) $\frac{3}{72}$ (b) $\frac{1}{72}$ (c) $\frac{5}{72}$ (d) $\frac{7}{72}$

57. Among 15 players, 8 are batsmen and 7 are bowlers. The probability that a team is chosen of 6 batsmen and 5 bowlers, is
(a) $\frac{{}^{8}C_6 \times {}^{7}C_5}{{}^{15}C_{11}}$ (b) $\frac{{}^{8}C_6 + {}^{7}C_5}{{}^{15}C_{11}}$
(c) $\frac{15}{28}$ (d) None of these

58. Two cards are drawn without replacement from a well-shuffled pack. The probability that one of them is an ace of heart, is
(a) $\frac{1}{25}$ (b) $\frac{1}{26}$
(c) $\frac{1}{52}$ (d) None of these

59. In a certain lottery 10000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize, if you buy two tickets? [NCERT]
(a) $\frac{{}^{9090}C_2}{{}^{10000}C_2}$ (b) $\frac{{}^{9990}C_2}{{}^{10000}C_2}$
(c) $\frac{{}^{9900}C_2}{{}^{10000}C_2}$ (d) None of these

60. One urn contains two black balls (labelled B_1 and B_2) and one white ball. A second urn contains one black ball and two white balls (labelled W_1 and W_2). Suppose, the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then, a second ball is chosen at random from the same urn without replacing the first ball. What is the probability that two balls of opposite colour are chosen? [NCERT Exemplar]
(a) $\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $\frac{2}{5}$ (d) None of these

61. A box contains 3 white and 2 red balls. If we draw one ball and without replacing the first ball, the probability of drawing red ball in the second draw is
(a) $\frac{8}{25}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{21}{25}$

62. In a lottery three were 90 tickets numbered 1 to 90. Five tickets were drawn at random. The probability that two of the tickets drawn numbers 15 and 89, is
(a) 2/801 (b) 2/623 (c) 1/267 (d) 1/623

63. A basket contains 5 apples and 7 oranges and another basket contains 4 apples and 8 oranges. One fruit is picked out from each basket. The probability that the fruits are both apples or both oranges, is
(a) 24/144 (b) 56/144
(c) 68/144 (d) 76/144

64. Out of 30 consecutive integers, 2 are chosen at random. The probability that their sum is odd, is
(a) $\frac{14}{29}$ (b) $\frac{16}{29}$
(c) $\frac{15}{29}$ (d) $\frac{10}{29}$

65. There are 10 prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top 10 contestants in a Mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8th contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C is
(a) 1/4 (b) 1/3
(c) 1/12 (d) 1/10

Addition Theorem of Probability

66. If $P(A \cup B) = P(A \cap B)$ for any two events A and B, then [NCERT Exemplar]
(a) $P(A) = P(B)$ (b) $P(A) > P(B)$
(c) $P(A) < P(B)$ (d) None of these

67. Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9. [NCERT Exemplar]
(a) 0.556 (b) 0.557
(c) 0.559 (d) None of these

68. In a large metropolitan area, the probabilities are 0.87, 0.36, 0.30 that a family (randomly chosen for a sample survey) owns a colour television set, a black and white television set or both kinds of sets. What is the probability that a family owns either anyone or both kinds of sets? [NCERT Exemplar]
(a) 0.95 (b) 0.93
(c) 0.98 (d) None of these

69. A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card. [NCERT Exemplar]

(a) $\frac{5}{13}$ (b) $\frac{7}{13}$
(c) $\frac{25}{52}$ (d) None of these

70. Given two events A and B. If odds against A are as 2 : 1 and those in favour of $A \cup B$ are as 3 : 1, then
(a) $1/2 \le P(B) \le 3/4$ (b) $5/12 \le P(B) \le 3/4$
(c) $1/4 \le P(B) \le 3/5$ (d) None of these

71. In shuffling a pack of playing cards, four are accidently dropped. The probability that missing cards should be one from each suit, is
(a) $\frac{1}{256}$ (b) $\frac{1}{270725}$
(c) $\frac{2197}{20825}$ (d) None of these

72. If A and B are any two events, then $P(\overline{A} \cap B)$ is equal to
(a) $P(\overline{A})\,P(\overline{B})$
(b) $1 - P(A) - P(B)$
(c) $P(A) + P(B) - P(A \cap B)$
(d) $P(B) - P(A \cap B)$

73. If $P(A) = 1/3$, $P(B) = 1/2$ and $P(A \cup B) = 5/6$, then events A and B are
(a) mutually exclusive
(b) independent as well as mutually exhaustive
(c) independent
(d) dependent only on A

74. Given, $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find $P(A$ or $B)$, if A and B are mutually exclusive events. [NCERT]
(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

75. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$. Find (i) $P(E \text{ or } F)$ (ii) P (not E and not F). [NCERT]
(a) $\frac{1}{8}, \frac{7}{8}$ (b) $\frac{5}{8}, \frac{3}{8}$
(c) $\frac{1}{7}, \frac{6}{7}$ (d) None of these

76. In class XI of a school, 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology. [NCERT]
(a) 0.5 (b) 0.6
(c) 0.65 (d) None of these

77. From the employees of a company, 5 persons are selected to represent them in managing committee of the company. Particulars of five persons are as follows

S.No.	Name	Sex	Age (in years)
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokes person. What is the probability that the spokes person will be either male or over 35 yr? [NCERT]
(a) $\frac{4}{5}$ (b) $\frac{1}{5}$
(c) $\frac{3}{5}$ (d) None of these

78. If $P(A \cap B) = \frac{1}{3}$, $P(A \cup B) = \frac{5}{6}$ and $P(A) = \frac{1}{2}$, then which one of the following is correct?
(a) A and B are independent events
(b) A and B are mutually exclusive events
(c) $P(A) = P(B)$
(d) None of the above

79. If A and B are arbitrary events, then
(a) $P(A \cap B) \ge P(A) + P(B)$ (b) $P(A \cup B) \le P(A) + P(B)$
(c) $P(A \cap B) = P(A) + P(B)$ (d) None of these

80. An experiment yields 3 mutually exclusive and exhaustive events A, B and C. If $P(A) = 2P(B) = 3P(C)$, then $P(A)$ is equal to
(a) $\frac{1}{11}$ (b) $\frac{2}{11}$
(c) $\frac{3}{11}$ (d) $\frac{6}{11}$

81. If the probabilities for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is [NCERT Exemplar]
(a) >.5 (b) .5
(c) ≤ .5 (d) 0

82. The probability that atleast one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\overline{A}) + P(\overline{B})$ is [NCERT Exemplar]
(a) 0.4 (b) 0.8
(c) 1.2 (d) 1.6

83. If A and B are events of the same experiments with $P(A) = 0.2$, $P(B) = 0.5$, then maximum value of $P(A' \cap B)$ is
(a) 0.2 (b) 0.5
(c) 0.63 (d) 0.25

84. A man and his wife appear for an interview for two posts. The probability of the man's selection is 1/5 and that of his wife's selection is 1/7. The probability that atleast one of them is selected, is

(a) $\frac{9}{35}$ (b) $\frac{12}{35}$

(c) $\frac{2}{7}$ (d) $\frac{11}{35}$

85. Given two mutually exclusive events A and B such that $P(A) = 0.45$ and $P(B) = 0.35$, $P(A \cap B)$ is equal to

(a) $\frac{63}{400}$ (b) 0.8

(c) $\frac{63}{200}$ (d) 0

Round II (Mixed Bag)

Only One Correct Option

1. Three letters are written to there different persons and addresses on the three envelopes are also written. Without looking at the addresses, the letters are kept in these envelopes. The probability that all the letters are not placed into their right envelopes is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$

2. A natural number x is chosen at random from the first 100 natural numbers. The probability that $x + \frac{100}{x} > 50$ is

(a) $\frac{1}{10}$ (b) $\frac{11}{50}$

(c) $\frac{11}{20}$ (d) None of these

3. If four digit numbers greater than 5000 are randomly formed from the digits 0, 1, 3, 5 and 7, what is the probability of forming number divisible by 5 when

(i) the digits are repeated ?

(ii) the repetition of digits is not allowed ?

(a) $\frac{30}{83}, \frac{3}{8}$ (b) $\frac{34}{83}, \frac{1}{8}$

(c) $\frac{33}{83}, \frac{3}{8}$ (d) None of these

4. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that

I. All the three balls are white

II. All the three balls are red

III. One ball is red and two balls are white

	I	II	III
(a)	$\frac{5}{143}$	$\frac{29}{143}$	$\frac{40}{143}$
(b)	$\frac{5}{143}$	$\frac{28}{143}$	$\frac{40}{143}$
(c)	$\frac{7}{143}$	$\frac{28}{143}$	$\frac{40}{143}$

(d) None of the above

5. A class consists of 80 students, 25 of them are girls and 55 are boys. If 10 of them are rich and the remaining are poor and also 20 of them are intelligent, then the probability of selecting an intelligent rich girl is

(a) 5/128

(b) 25/128

(c) 5/512

(d) None of the above

6. Three critics review a book. Odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4, respectively, for the three critics. The probability that majority are in favour of the book is

(a) 35/49 (b) 125/343

(c) 164/343 (d) 209/343

More Than One Correct Option

7. If A and B are two events such that $P(A) = 3/4$ and $P(B) = 5/8$, then

(a) $P(A \cup B) \geq 3/4$

(b) $P(A' \cap B) \leq 1/4$

(c) $3/8 \leq P(A \cap B) \leq 5/8$

(d) None of the above

8. The probability that a 50 yr old man will be alive at 60 is 0.83 and the probability that a 45 yr old woman will be alive at 55 is 0.87. Then,

(a) the probability that both will be alive is 0.7221

(b) atleast one of them will alive is 0.9779

(c) atleast one of them will alive is 0.8230

(d) the probability that both will be alive is 0.6320

9. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

(a) $p_1 = 1/9$

(b) $p_1 = 1/16$

(c) $p_2 = 1/3$

(d) $p_2 = 1/4$

Comprehension Based Questions

Passage

Directions (Q. Nos. 10-12) *In an objective paper, there are two sections of* 10 *questions each. For 'section* 1*', each question has five options and only one option is correct and 'section* 2*' has four options with multiple answers and marks for a question in this section is awarded only if he ticks all correct answers. Marks for each question in 'section* 1*' is* 1 *and in 'section* 2*' is* 3*. (There is no negative marking.)*

10. If a candidate attempts only two questions by guessing, one from 'section 1' and one from 'section 2', the probability that he scores in both questions is
(a) 74/75 (b) 1/25 (c) 1/15 (d) 1/75

11. If a candidate in total attempts four questions all by guessing, then the probability of scoring 10 marks is
(a) $1/15\,(1/15)^3$ (b) $4/5\,(1/15)^3$
(c) $1/15\,(14/15)^3$ (d) None of these

12. The probability of getting a score less than 40 by answering all the questions by guessing in this paper is
(a) $(1/75)^{10}$ (b) $1-(1/75)^{10}$
(c) $(74/75)^{10}$ (d) None of these

Assertion and Reason

Directions (Q. Nos. 13-16) *Each of these questions contains two statements : Statement* I *(Assertion) and Statement* II *(Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) and (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

13. **Statement I** 20 persons are sitting in a row. Two of these persons are selected at random. The probability that two selected persons are not together is 0.7.

Statement II If A is an event, then $P(\text{not } A) = 1 - P(A)$.

14. **Statement I** If A and B are two events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$, then $\frac{1}{6} \le P(A \cap B) \le \frac{1}{2}$.

Statement II $P(A \cup B) \le \max\{P(A), P(B)\}$

and $P(A \cap B) \ge \min\{P(A), P(B)\}$.

15. **Statement I** If A, B, C be three mutually independent events, then A and $B \cup C$ are also independent events.

Statement II Two events A and B are independent if and only if $P(A \cap B) = P(A)\,P(B)$.

16. **Statement I** The probability of drawing either an ace or a king from a pack of card in a single draw is 2/13.

Statement II For two events A and B which are not mutually exclusive,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Previous Years' Questions

17. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours, is **[AIEEE 2010]**
(a) $\frac{1}{3}$ (b) $\frac{2}{7}$
(c) $\frac{1}{21}$ (d) $\frac{2}{23}$

18. A die is thrown. Let A be the event that the number obtained is greater than 3 and B be the event that the number obtained is less than 5. Then, $P(A \cup B)$ is **[AIEEE 2008]**
(a) $\frac{2}{5}$ (b) $\frac{3}{5}$
(c) 0 (d) 1

19. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only, if the first misses the target. The probability that the target is hit by the second plane, is **[AIEEE 2007]**
(a) 0.06 (b) 0.14
(c) 0.32 (d) 0.7

20. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for complement of event A. Then, events A and B are **[AIEEE 2005]**
(a) mutually exclusive and independent
(b) independent but not equally likely
(c) equally likely but not independent
(d) equally likely and mutually exclusive

Answers

Round I

1. (a)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (c)
11. (b)	**12.** (b)	**13.** (d)	**14.** (c)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (b)
21. (a)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (c)
31. (b)	**32.** (b)	**33.** (b)	**34.** (d)	**35.** (d)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (a)
41. (a)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (b)
51. (c)	**52.** (c)	**53.** (a)	**54.** (c)	**55.** (d)	**56.** (b)	**57.** (a)	**58.** (b)	**59.** (b)	**60.** (a)
61. (b)	**62.** (a)	**63.** (d)	**64.** (c)	**65.** (a)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (b)	**70.** (b)
71. (c)	**72.** (d)	**73.** (a)	**74.** (c)	**75.** (b)	**76.** (b)	**77.** (a)	**78.** (a)	**79.** (b)	**80.** (d)
81. (b)	**82.** (c)	**83.** (b)	**84.** (d)	**85.** (d)					

Round II

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (c)	**6.** (d)	**7.** (a,b,c)	**8.** (a,b)	**9.** (a,c)	**10.** (d)
11. (b)	**12.** (b)	**13.** (d)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (b)

the **Guidance**

Round I

1. The sample space

$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

where, H and T represent head and tail of a coin.

2. Here, all three white balls are identical. So, all are denoted by W and red ball is denoted by R.

$$\boxed{\begin{matrix} R, \\ W, W, W \end{matrix}}$$

The sample space is $S = \{RW, WW, WR\}$

3. I. Sum of all probabilities

$= 0.1 + 0.01 + 0.05 + 0.03 + 0.01 + 0.2 + 0.6 = 1$

Hence, the assignment is valid, as sum is 1.

II. Valid, as sum is 1 (same as part (I)).

III. Sum of all probabilities

$= 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 + 0.7 = 2.8$

Hence, assignment is not valid, as sum is $2.8 > 1$.

IV. Not valid, as probabilities of any event can't be negative.

V. Not valid, as $\frac{15}{14} > 1$, which is not possible.

4.

Total number of possible cases in coin = 2

Number of possible cases in die = 6

The sum is 3 only if 1 turns up on coin and 2 turns up on die *i.e.,* only possibility is (1, 2).

$\therefore$ Required probability

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{1}{2 \times 6} = \frac{1}{12}$$

5. Given word is A S S A S S I N A T I O N

Number of vowels $= 3(A) + 2(I) + 1(O) = 6$

Number of consonant = 7

Total number of letters = 13

(i) P (Vowels) $= \frac{6}{13}$

(ii) P (Consonant) $= \frac{7}{13}$

6. Number of favourable cases = 3

Required probability $= \frac{3}{6} = \frac{1}{2}$

7. Favourable cases of getting 10 or greater than 10, if 5 appears on atleast one of dice.

$= \{(5, 6), (6, 5), (5, 5)\}$

Number of favourable cases = 3

Total number of cases = 36

$\therefore$ Required probability $= \frac{3}{36} = \frac{1}{12}$

8. Let total number of students be 100 in which 60% girls and 40% boys.

Number of boys = 40, number of girls = 60

25% of boys offer Mathematics $= \frac{25}{100} \times 40 = 10$ boys

10% of girls offer Mathematics $= \frac{10}{100} \times 60 = 6$ girls

It means, 16 students offer Mathematics.

$\therefore$ Required probability $= \frac{6}{16} = \frac{3}{8}$

9. $S = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG\}$

and $E = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG\}$

$n(E) = 7$ and $n(S) = 8$

$\therefore$ $P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}$

10. A and B toss a coin alternately till one of them tosses heads and win the game, their respectively probabilities of winning are $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

11. In a leap year, there are 366 days in which 52 weeks and two days. The combination of 2 days may be : Sun-Mon, Mon-Tue, Tue-Wed, Wed-Thu, Thu-Fri, Fri-Sat, Sat-Sun.

$P(53 \text{ Fri}) = \frac{2}{7}$; $P(53 \text{ Sat}) = \frac{2}{7}$

and $P(53 \text{ Fri and } 53 \text{ Sat}) = \frac{1}{7}$

$\therefore$ $P(53 \text{ Fri or Sat}) = P(53 \text{ Fri}) + P(53 \text{ Sat}) - P(53 \text{ Fri and Sat})$

$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{3}{7}$

12. In a non-leap year, there are 365 days in which 52 weeks and one day extra.

It may be Mon, Tue, Wed, Thu, Fri, Sat, Sun.

$\therefore$ Required probability

$= P(53 \text{ Tuesday}) + P(53 \text{ Wednesday}) - P(53 \text{ Tuesday} \cap 53 \text{ Wednesday})$

$= \frac{1}{7} + \frac{1}{7} - 0 = \frac{2}{7}$

13. The number of ways is 4!. Any number divisible by 5 is possible only when 0 or 5 in the unit place.

In a unit place, there are two ways of selecting a number (0 or 5). In rest of three places, three numbers are arranged in 3! ways.

$\therefore$ Number of favourable ways $= 3! \times 2 - 3!$

($\because$ here, we subtract those cases in which 0 is in thousand place, because it becomes three digit number)

$= 12 - 6 = 6$

$\therefore$ Required probability $= \frac{6}{4!} = \frac{6}{24} = \frac{1}{4}$

14. Assuming all six girls as one unit.

$\therefore$ Required probability $= \frac{7!\,6!}{12!} = \frac{1}{132}$

15. P (atleast on head) $= 1 - P$ (zero head)

$= 1 - P$ (all three tails) $= 1 - \frac{1}{8} = \frac{7}{8}$

16. Total number of digits in any number at the unit place is 10.

$\therefore$ $n(S) = 10$

To get the last digit in product is 1, 3, 5 or 7, it is necessary the last digit in each number must be 1, 3, 5 or 7.

$n(A) = 4$

$\therefore$ $P(A) = \frac{4}{10} = \frac{2}{5}$

Hence, required probability $= \left(\frac{2}{5}\right)^4 = \frac{16}{625}$

17. A coin is tossed n times.

$\therefore$ Total number of ways $= 2^n$

If head comes odd times, then favourable ways $= 2^{n-1}$.

$\therefore$ Required probability of getting odd times head $= \frac{2^{n-1}}{2^n}$

$= \frac{1}{2}$

18. Total number of ways $= 6 \times 6 \times 6$

Favourable number of ways $= 6$

$\therefore$ Required probability $= \frac{6}{6 \times 6 \times 6} = \frac{1}{36}$

19. Let the probability of getting an even number is p and then probability of getting an odd number is $2p$.

$\therefore$ Required probability

$$= \frac{P(4) + P(5) + P(6)}{P(1) + P(2) + P(3) + P(4) + P(5) + P(6)}$$

$$= \frac{p + 2p + p}{2p + p + 2p + p + 2p + p}$$

$$= \frac{4p}{9p} = \frac{4}{9}$$

20. The total number of ways $= 6^3 = 216$

If the second number is i $(i > 1)$, then the total number of favourable ways

$$= \sum_{i=1}^{5} (i - 1)(6 - i) = 20$$

$\therefore$ Required probability $= \frac{20}{216} = \frac{5}{54}$

21. Probability of no tail in four throws $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

Probability of atleast one tail $= 1 - \frac{1}{16} = \frac{15}{16}$

22. Probability in each case $= \frac{9}{15} = \frac{3}{5}$

Required probability (with replacement) $= \left(\frac{3}{5}\right)^7$

23. The event that the fifth toss results in a head is independent of the event that the first four tosses result in tails.

$\therefore$ Probability of the required event $= \frac{1}{2}$

24. The number is divisible by 4, if last two digits are 12, 24, 32 and 52.

Remaining three place can be filled by 3! ways.

$\therefore$ Favourable cases $= 3! \times 4$

Required probability $= \frac{3! \times 4}{5!}$

$$= \frac{3! \times 4}{5 \times 4 \times 3!} = \frac{1}{5}$$

25. Let the envelope be denoted by E_1, E_2, E_3 and the corresponding letters by L_1, L_2, L_3. 1 letter in correct envelope and 2 in wrong envelope may be put as

$(E_1L_1, E_2L_3, E_3L_2), (E_1L_3, E_2L_2, E_3L_1)$ and (E_1L_2, E_2L_1, E_3L_3)

Two letters or consequently all the letters are in correct envelope may be put in one way

i.e., (E_1L_1, E_2L_2, E_3L_3)

$\therefore$ Number of cases $= 3! = 6$

Number of favourable cases $= 4$

Probability that atleast one letter in its proper envelope $= \frac{4}{6} = \frac{2}{3}$

26. Since, repetition is not allowed.

So, first place can be filled in 10 ways.

Second place can be filled in 9 ways.

Third place can be filled in 8 ways.

and fourth place can be filled in 7 ways.

Hence, by fundamental principle of counting, total number of ways

$$= 10 \times 9 \times 8 \times 7 = 5040$$

In which only 1 sequence can open the lock.

$\therefore$ Number of favourable cases $= 1$

$\therefore$ Required probability $= \frac{1}{5040}$

27. Let the probability of D selected is x.

Then, $P(C) = 2x$

$P(B) = 2x$

and $P(A) = 4x$

Now, $P(A) + P(B) + P(C) + P(D) = 1$

($\because$ sum of all probabilities is equal to one)

$\therefore$ $4x + 2x + 2x + x = 1$

$\Rightarrow$ $9x = 1 \Rightarrow x = \frac{1}{9}$

$\therefore$ $P(C \text{ selected}) = 2x = 2 \times \frac{1}{9} = \frac{2}{9}$

28. Probability that one person is alive $= \frac{2}{3}$

Probability that all five are alive $= \left(\frac{2}{3}\right)^5 = \frac{32}{243}$

29. There is no condition on other coins only 5th coin will fall head upwards. So, probability $= \frac{1}{2}$

30. The probabilities of solving the question by these three students are 1/3, 2/7 and 3/8, respectively.

$\therefore$ $P(A) = \frac{1}{3}; P(B) = \frac{2}{7}; P(C) = \frac{3}{8}$

Then, probability of question solved by only one student is

$$P(A\overline{B}\,\overline{C} \text{ or } \overline{A}B\overline{C} \text{ or } \overline{A}\,\overline{B}C) = P(A)P(\overline{B})P(\overline{C}) + P(\overline{A})P(B)P(\overline{C}) + P(\overline{A})P(\overline{B})P(C)$$

$$= \frac{1}{3} \times \frac{5}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{2}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{5}{7} \times \frac{3}{8}$$

$$= \frac{25 + 20 + 30}{168} = \frac{75}{168} = \frac{25}{56}$$

31. Given that, $x = 33^n$

where, n is a positive integral value.

Here, only four digits may be at the unit place *i.e.,* 1, 3, 7, 9.

$\therefore$ $n(S) = 4$

Let E be the event of getting 3 at its units place.

$n(E) = 1$

$\therefore$ $P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$

32. The probability of throwing 9 with two dice $= \frac{4}{36} = \frac{1}{9}$

$\therefore$ The probability of not throwing 9 with two dice

$= 1 - \frac{1}{9} = \frac{8}{9}$

If A is to win, he should throw 9, in Ist or 3rd or 5th and so on.

If B is to win, he should throw, 9 in 2nd, 4th and so on.

B can get the chance only when A should not get it.

$$\therefore \quad B\text{'s chances} = \left(\frac{8}{9}\right) \cdot \frac{1}{9} + \left(\frac{8}{9}\right)^3 \cdot \frac{1}{9} + \ldots = \frac{\frac{8}{9} \times \frac{1}{9}}{1 - \left(\frac{8}{9}\right)^2} = \frac{8}{17}$$

33. From Venn diagram, it is clear that $P(B \cap C) = 0.15$ and P(exactly one of the three occurs)

$$= 0.13 + 0.10 + 0.28 = 0.51$$

34. Total ways $= 6 \times 5 = 30$

Favourable events

= The minimum of the two numbers is less than 4.

$n(E) = 6 \times 4 = 24$

[we can select one from {1, 2, 3, 4} and other from (1, 2, 3, 4, 5, 6)]

$\therefore$ Required probability $= \frac{24}{30} = \frac{4}{5}$

35. Total number of 5- digit numbers

$$= 9 \times 10 \times 10 \times 10 \times 10 = 90000$$

Number of favourable numbers

$$= 5 \times 5 \times 4 \times 4 \times 3 = 12000$$

Thus, required probability $= \dfrac{1200}{90000} = \dfrac{1}{75}$

36. Three digit numbers multiple of 11 are 110, 121, ..., 990 (81 numbers). Now, number also divisible by 9 are divisible by 99. So, numbers are 198, 297, ..., 990 (9 numbers).

So, required probability $= \dfrac{9}{81} = \dfrac{1}{9}$

37. We can choose three vertices out of 6 in ${}^6C_3 = 20$ ways. Chosen vertices can form an equilateral triangle in just two ways *viz*. $A_1A_3A_5$ and $A_2A_4A_6$.

$\therefore$ Required probability $= \dfrac{2}{20} = \dfrac{1}{10}$

38. P(2 white and 1 black)

$= P(W_1W_2B_3$ or $W_1B_2W_3$ or $B_1W_2W_3)$

$= P(W_1)P(W_2)P(B_3) + P(W_1)P(B_2)P(W_3)$

$+ P(B_1)P(W_2)P(W_3)$

$$= \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{32}(9 + 3 + 1) = \frac{13}{32}$$

39. The probability that only two tests are needed = (Probability that the first tested machine is faulty) × (Probability that the second tested machine is faulty given the first machine tested is faulty) $= \dfrac{2}{4} \times \dfrac{1}{3} = \dfrac{1}{6}$

40. $7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401, \ldots$

Therefore, for $7^r, r \in N$ the number ends at unit place 7, 9, 3, 1, 7,

$\therefore$ $7^m + 7^n$ will be divisible by 5, if it end at 5 or 0.

But it cannot end at 5.

Also, it cannot end at 0.

For this m and n should be as follows

	m	n
1	$4r$	$4r + 2$
2	$4r + 1$	$4r + 3$
3	$4r + 2$	$4r$
4	$4r + 3$	$4r + 1$

For any given value of m, there will be 25 values of n. Hence, the probability of the required event is $\dfrac{100 \times 25}{100 \times 100} = \dfrac{1}{4}$.

41. In any number of last digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, last digit of each number can be chosen in 10 ways. Thus, exhaustive number of ways is 10^n. If the last digit be 1, 3, 7 or 9, then none of the numbers can be even or end in 0 or 5. Thus, we have a choice of four digits, *viz*. 1, 3, 7 or 9 with which each of n numbers should end. So, favourable number of ways is 4^n. Hence, the required probability is

$$\frac{4^n}{10^n} = \left(\frac{2}{5}\right)^n$$

42. Total number of cases = 9999

Favourable cases $= 10 \times 9 \times 8 \times 7 = 5040$

$\therefore$ Probability $= \dfrac{5040}{9999}$

43. We have, ratio of the ships A, B and C for arriving safely are 2 : 5, 3 : 7 and 6 : 11, respectively. Therefore, the probability of ship A for arriving safely is $2/(2 + 5) = 2/7$.

Similarly, for B the probability is $3/(3 + 7) = 3/10$ and for C the probability is $C = 6/(6 + 11) = 6/17$.

Therefore, the probability of all the ships for arriving safely is $(2/7) \times (3/10) \times (6/17) = 18/595$.

44. Let A = getting not less than 2 and not greater than 5.

$\Rightarrow$ $A = \{2, 3, 4, 5\} \Rightarrow P(A) = \dfrac{4}{6}$

But die is rolled four times, therefore the probability in getting four throws

$$= \left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{4}{6}\right) = \frac{16}{81}$$

45. For ranked 1 and 2 players to be winners and runners up respectively, they should not be paired with each other in any round. Therefore, the required probability is

$$30/31 \times 14/15 \times 6/7 \times 2/3 = 16/31.$$

46. We have, $P(A) = \dfrac{1}{2}, P(B) = \dfrac{1}{4}$ and $P(C) = \dfrac{1}{6}$

$$P(\overline{A}) = \frac{1}{2}, P(\overline{B}) = \frac{3}{4}, P(\overline{C}) = \frac{5}{6}$$

Then, the probability that the problem is not solved is

$$P(\overline{A})\, P(\overline{B})P(\overline{C}) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) = \frac{5}{16}$$

Hence, probability that problem is solved is

$$1 - 5/16 = 11/16 = \frac{33}{48}$$

47. The probability of winning of A the second race is 1/2 (since both events are independent).

48. ASSISTANT : A A I N SSS TT

STATISTICS : A C II SSS TTT

Same letters can be A, I, S, T

Probability of choosing, $A = \dfrac{{}^2C_1}{{}^9C_1} \times \dfrac{{}^1C_1}{{}^{10}C_1} = \dfrac{1}{45}$

Probability of choosing, $I = \frac{{}^1C_1}{{}^9C_1} \times \frac{{}^2C_1}{{}^{10}C_1} = \frac{1}{45}$

Probability of choosing, $S = \frac{{}^3C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{1}{10}$

Probability of choosing, $T = \frac{{}^2C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{1}{15}$

So, total probability $= \frac{19}{90}$

49. Required probability $= P(WBWB) + P(BWBW)$

$$= \left(\frac{{}^5C_1 \times {}^3C_1 \times {}^4C_1 \times {}^2C_1}{{}^8C_1 \times {}^7C_1 \times {}^6C_1 \times {}^5C_1}\right) + \left(\frac{{}^5C_1 \times {}^3C_1 \times {}^4C_1 \times {}^2C_1}{{}^8C_1 \times {}^7C_1 \times {}^6C_1 \times {}^5C_1}\right)$$

$$= \frac{1}{14} + \frac{1}{14} = \frac{2}{14} = \frac{1}{7}$$

50. When we select three consecutive numbers from 1 to 20. There are 18 combination are possible

(*i.e.*, 1, 2, 3, ; 2, 3, 4, . . . ; 18, 19, 20)

$\therefore$ Probability of selecting three consecutive numbers

$$= \frac{18}{{}^{20}C_3} = \frac{18}{\frac{20 \times 19 \times 18}{3 \times 2 \times 1}} = \frac{3}{190}$$

$\therefore$ Required probability $= 1 -$ Probability of not selecting three consecutive numbers

$$= 1 - \frac{3}{190} = \frac{187}{190}$$

51. Probability that the missing cards to be different colours

$$= \frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2}$$

($\because$ in a pack of cards, there are two colours each of them are 26 cards)

$$= \frac{26 \times 26}{\frac{52 \times 51}{2 \times 1}} = \frac{26}{51}$$

52. Total seven persons are seated in a row by ${}^7P_1$ ways. Two particular persons are seated next to each other, *i.e.*, we consider two persons as a single unit.

$\therefore$ These two persons and rest five persons are seated in a row by 6! ways but these two persons can be arranged in 2! ways.

$\therefore$ Required probability $= \frac{2! \times 6!}{{}^7P_1} = \frac{2! \times 6!}{7!}$

$$= \frac{2}{7}$$

53. In a word PROBABILITY, vowels are (O, A, I) and consonants are (P, R, B, L, T, Y).

$\therefore$ Required probability $= \frac{{}^3C_1}{{}^9C_1}$

$$= \frac{3}{9} = \frac{1}{3}$$

54. From the given condition, it is clear that a particular person is always in a committee of 3 persons. It means, we have to select 2 person out of 37 person.

$\therefore$ Required probability $= \frac{{}^{37}C_2}{{}^{38}C_3}$

55. Total number of socks $= 5 + 4 = 9$

The number of ways to select 2 socks out of 9 $= {}^9C_2$

Number of ways to select both brown socks $= {}^5C_2$

And number of ways to select both white socks $= {}^4C_2$

$\therefore P$(either both brown or white) $= \frac{{}^5C_2 + {}^4C_2}{{}^9C_2}$

$$= \frac{\frac{5!}{3! \cdot 2!} + \frac{4!}{2! \cdot 2!}}{\frac{9!}{7! \cdot 2!}} = \frac{10 + 6}{36}$$

$$= \frac{16}{36} \times \frac{3}{3} = \frac{48}{108}$$

56. In a word ALGORITHM, letters are (A, L, G, O, R, I, T, H, M)

$\therefore$ The total number of ways, $n(S) = 9!$

The number of favourable ways, $n(F) = 7! \times 3!$

($\because$ GOR consider as one unit and along with six letters, they can be arranged in 7!).

$\therefore$ Required probability $= \frac{n(F)}{n(S)} = \frac{7!}{9!} = \frac{1}{9 \times 8} = \frac{1}{72}$

57. Total number of ways of selecting 11 players $= {}^{15}C_{11}$

Favourable cases $= {}^8C_6 \times {}^7C_5$

$\therefore$ Required probability $= \frac{{}^8C_6 \times {}^7C_5}{{}^{15}C_{11}}$

58. There are two conditions arise.

(i) When first is an ace of heart and second one is non-ace of heart, the probability $= \frac{1}{52} \times \frac{51}{51} = \frac{1}{52}$

(ii) When first is non-ace of heart and second one is an ace of heart, the probability $= \frac{51}{52} \times \frac{1}{51} = \frac{1}{52}$

$\therefore$ Required probability $= \frac{1}{52} + \frac{1}{52} = \frac{1}{26}$

59. Number of ways in which 2 tickets can be selected out of 10000 tickets $= {}^{10000}C_2$

Number of ways in which 2 tickets without prizes can be selected out of 9990 tickets

$$= {}^{9990}C_2$$

$\therefore$ Required probability $= \frac{{}^{9990}C_2}{{}^{10000}C_2}$

60. $\underbrace{B_1, B_2, W}_{\text{Urn}(U_1)}$ $\underbrace{B, W_1, W_2}_{\text{Urn}(U_2)}$

***Case* I** Selection from first urn probability

$$= \left[\frac{{}^2C_1}{{}^3C_1} \times \frac{1}{{}^2C_1}\right] = \left[\frac{2}{3} \times \frac{1}{2}\right] = \frac{1}{3}$$

***Case* II** Selection from second urn Probability

$$= \left[\frac{1}{{}^3C_1} \times \frac{{}^2C_1}{{}^2C_1}\right] = \left[\frac{1}{3} \times 1\right] = \frac{1}{3}$$

P (two balls are of opposite colour) $= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

61. There are two cases arise.

***Case* I** If Ist ball is white, then

Probability $= \frac{{}^3C_1}{{}^5C_1} \times \frac{{}^2C_1}{{}^4C_1} = \frac{6}{20} = \frac{3}{10}$

***Case* II** If Ist ball is red, then

Probability $= \frac{{}^2C_1}{{}^5C_1} \times \frac{{}^1C_1}{{}^4C_1} = \frac{2}{20} = \frac{1}{10}$

$\therefore$ Required probability $= \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$

62. Out of 90 tickets, two tickets already considered, instead of selecting 5 tickets we have to select only 3 tickets out of 88 tickets.

$$\therefore \text{ Required probability} = \frac{{}^{88}C_3}{{}^{90}C_5} = \frac{\dfrac{88 \times 87 \times 86}{3 \times 2 \times 1}}{\dfrac{90 \times 89 \times 88 \times 87 \times 86}{5 \times 4 \times 3 \times 2 \times 1}}$$

$$= \frac{5 \times 4}{90 \times 89} = \frac{2}{801}$$

63. P (selecting an apple from both baskets)

$= P$ (apple from first basket) $\cdot P$ (apple from second basket)

$= \frac{{}^5C_1}{{}^{12}C_1} \cdot \frac{{}^4C_1}{{}^{12}C_1}$

P (selecting a orange from both baskets)

$= P$(orange from first basket) $\cdot P$(orange from second basket)

$= \frac{{}^7C_1}{{}^{12}C_1} \cdot \frac{{}^8C_1}{{}^{12}C_1}$

$$\text{Required probability} = \frac{{}^5C_1 {}^4C_1}{{}^{12}C_1 {}^{12}C_1} + \frac{{}^7C_1 {}^8C_1}{{}^{12}C_1 {}^{12}C_1}$$

$$= \frac{20 + 56}{144} = \frac{76}{144}$$

64. The sum of the selected numbers is odd, if exactly one of them is even and one is odd.

$\therefore$ Favourable number of cases $= {}^{15}C_1 \cdot {}^{15}C_1$

$\therefore$ Required probability $= \frac{{}^{15}C_1 \cdot {}^{15}C_1}{{}^{30}C_2} = \frac{15}{29}$

65. $n(S) = {}^{10}C_7 = 120$

$n(A) = {}^5C_4 \times {}^3C_2 \times {}^2C_1$

$P(E) = \frac{5 \times 3 \times 2}{120} = \frac{1}{4}$

66. Given, $P(A \cup B) = P(A \cap B)$

This is possible only when $P(A) = P(B)$.

67. Here, $n(S) = 1000$

Let E = Event of selecting an integer is a multiple of 2

$= \{2, 4, 8, \ldots, 1000\}$

$\therefore \quad n(E) = 500$

and F = Event of selecting an integer is a multiple of 9

$= \{9, 18, \ldots 999\}$

$\therefore \quad n(F) = 111$

and $E \cap F$ = Event of selecting an integer is a multiple of 2 and 9

$= \{18, 36, \ldots, 990\}$

$\therefore \quad n(E \cap F) = 55$

$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{500}{1000}$

and $\quad P(F) = \frac{n(F)}{n(S)} = \frac{111}{1000}$

and $\quad P(E \cap F) = \frac{n(E \cap F)}{n(S)} = 55$

$\therefore \quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$= \frac{500}{1000} + \frac{111}{1000} - \frac{55}{100}$$

$$= \frac{556}{1000} = 0.556$$

68. Let E_1 = Event of selecting a colour television set

E_2 = Event of selecting a black and white television set

$\therefore \quad P(E_1) = 0.87, P(E_2) = 0.36$

and $\quad P(E_1 \cap E_2) = 0.30$

$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.87 + 0.36 - 0.30 = 0.93$

69. Let E_1 = Event of getting a king card.

E_2 = Event of getting a heart card

and E_3 = Event of getting a red card.

$\therefore \quad P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

$- P(E_1 \cap E_2) - P(E_2 \cap E_3)$

$- P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52}$$

$$= \frac{43}{52} - \frac{16}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

70. $P(A) = \frac{1}{3}, P(A \cup B) = \frac{3}{4}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$

$\Rightarrow \quad \frac{3}{4} \le \frac{1}{3} + P(B) \Rightarrow \frac{5}{12} \le P(B)$

Again, we have $B \subseteq A \cup B$

$\therefore \quad P(B) \le P(A \cup B) = \frac{3}{4}$

Hence, $5/12 \le P(B) \le 3/4$.

71. Total cases $= {}^{52}C_4$

Favourable cases $= ({}^{13}C_1)^4$

So, probability $= \frac{({}^{13}C_1)^4}{{}^{52}C_4}$

$$= \frac{13 \times 13 \times 13 \times 13 \times 1 \times 2 \times 3 \times 4}{52 \times 51 \times 50 \times 49} = \frac{2197}{20825}$$

72. $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$P(\bar{A} \cap B)$

73. We know, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad \frac{5}{6} = \frac{1}{3} + \frac{1}{2} - P(A \cap B)$

$\Rightarrow \quad \frac{5}{6} = \frac{5}{6} - P(A \cap B) \Rightarrow P(A \cap B) = 0$

$\therefore$ Events A and B are mutually exclusive.

74. $\because A$ and B are mutually exclusive events

$\Rightarrow \quad A \cap B = \phi \Rightarrow P(A \cap B) = 0$

$\therefore \quad P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{5} + \frac{1}{5} - 0 = \frac{4}{5}$$

75. Given, $P(E) = \frac{1}{4}, P(F) = \frac{1}{2}, P(E \text{ and } F) = \frac{1}{8}$ and $P(E \cap F) = \frac{1}{8}$

(i) $P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$

($\because A$ or B means $A \cup B$)

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{2+4-1}{8} = \frac{5}{8}$$

(ii) $P(\text{not } E \text{ and not } F)$

$= P(E' \cap F')$ [$\because$ De-Morgan's law $E' \cap F' = (E \cup F)'$]

$= P(E \cup F)' = 1 - P(E \cup F) = 1 - \frac{5}{8} = \frac{3}{8}$

76. Let M and B denote the students of Mathematics and Biology.

Given, $P(M) = 40\%, P(B) = 30\%$

$P(M \cap B) = 10\%$

($\because$ 10% of the class study both subjects *i.e.*, they are common in both subjects)

$\therefore \quad P(M \text{ or } B) = P(M \cup B)$

$= P(M) + P(B) - P(M \cap B)$

$$= \frac{40}{100} + \frac{30}{100} - \frac{10}{100} = \frac{60}{100} = 60\% = 0.6$$

77. Let A be the event that selected person is male or B be the event that selected person is over 35 yr.

$\therefore \quad P(A) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5}$

$P(B) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{5} + \frac{2}{5} - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$$

78. Given that, $P(A \cap B) = \frac{1}{3}, \quad P(A \cup B) = \frac{5}{6},$

and $P(A) = \frac{1}{2}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad \frac{5}{6} = \frac{1}{2} + P(B) - \frac{1}{3}$

$\Rightarrow \quad P(B) = \frac{5}{6} + \frac{1}{3} - \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

$\therefore \quad P(A) \times P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} = P(A \cap B)$

This shows that A and B are independent events.

79. We know,

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$

[$\because P(A \cap B) \ge 0$]

80. Clearly, $P(A \cup B \cup C) = 1$

$\Rightarrow \quad P(A) + P(B) + P(C) = 1$

$\Rightarrow \quad P(A) + \frac{1}{2}P(A) + \frac{1}{3}P(A) = 1$

$\Rightarrow \quad \frac{11}{6}P(A) = 1$

$\Rightarrow \quad P(A) = \frac{6}{11}$

81. Given, $P(A) = 0.2$ and $P(B) = 0.3$

Here, $P(A \cap B) = 0$

$\therefore$ Required probability $= P(A) + P(B) - P(A \cap B)$

$= 0.2 + 0.3 - 0$

$= 0.5$

82. Given, $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$

$\because \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 1 - P(\bar{A}) + 1 - P(\bar{B}) - P(A \cap B)$

$\therefore \quad 0.6 = 1 - P(\bar{A}) + 1 - P(\bar{B}) - 0.2$

$\Rightarrow \quad P(\bar{A}) + P(\bar{B}) = 2 - 0.8 = 1.2$

83. $P(A') = 1 - P(A) = 0.8, P(A' \cap B)$ will maximum, if $B \subseteq A'$ in which case $A' \cap B = B$. So,

$$P(A' \cap B) = P(B) = 0.5$$

84. $P(A \cup B) = P(A) + P(B) - P(AB)$

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

(A and B are independents events)

$$= \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \frac{11}{35}$$

85. $P(A) = 0.45$,

$P(B) = 0.35$ (events are mutually exclusive)

$P(A \cap B) = 0$

Round II

1. Favourable ways $= 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 2$

Total ways $= 3!$

$\therefore$ Probability $= \frac{2}{3!} = \frac{1}{3}$

2. $x + \frac{100}{x} > 50$

$\Rightarrow \quad x^2 + 100 > 50x \quad (\because x \in N)$

$\Rightarrow \quad x^2 - 50x + 100 > 0 \Rightarrow (x - 25)^2 > 525$

$\Rightarrow \quad x - 25 < -\sqrt{525}$ or $x - 25 > \sqrt{525}$

$\Rightarrow \quad x < 25 - 22.9$ or $x > 25 + 22.9$

$\Rightarrow \quad x \le 2$ or $x \ge 48$

Hence, the number of favourable cases $= 2 + 53 = 55$

Thus, required probability $= \frac{55}{100} = \frac{11}{20}$

3. ***Case* I When digits are repeated.**

Total four digit numbers which are greater than 5000 from the digits 0, 1, 3, 5,7 *i.e.*, in thousand place, the number will be 5 or 7 and rest of three places any five numbers will come

$\boxed{5}\ \underline{5} \times \underline{5} \times \underline{5} - 1 = 124$

(here, one is subtracted because one case is possible in which all the three places are occupied by the number zero.)

$\boxed{7}\ \underline{5} \times \underline{5} \times \underline{5} = 125$

$\therefore$ Total four digit numbers, $n(S) = 124 + 125 = 249$

Total four digit numbers which are divisible by 5.

i.e., In unit place number will be 0 or 5 and thousand place the number will be 5 to 7. And rest of the two places any five numbers will be come.

$\boxed{5}\ \underline{5} \times \underline{5}\ \boxed{0} - 1 = 24$

$\boxed{7}\ \underline{5} \times \underline{5}\ \boxed{0} = 25$

$\boxed{5}\ \underline{5} \times \underline{5}\ \boxed{5} = 25$

$\boxed{7}\ \underline{5} \times \underline{5}\ \boxed{5} = 25$

Number of favourable cases, $n(E)$

$= 24 + 25 + 25 + 25 = 99$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{99}{249} = \frac{33}{83}$

***Case* II When digits are not repeated.**

Total four digit numbers which are greater than 5000 from the digits 0, 1, 3, 7, 5 *i.e.*, In thousand place, the number will be 5 or 7. Since, the digit are not repeated, then rest of the place may be filled by 4, 3, 2.

5 $\quad \underline{4} \times \underline{3} \times \underline{2} = 24$

7 $\quad \underline{4} \times \underline{3} \times \underline{2} = 24$

Total four digit number $n(S) = 24 + 24 = 48$

Total four digit numbers which are divisible by 5

$\boxed{5}\ \underline{3} \times \underline{2}\ \boxed{0} = 6$

$\boxed{7}\ \underline{3} \times \underline{2}\ \boxed{0} = 6$

$\boxed{7}\ \underline{3} \times \underline{2}\ \boxed{5} = 6$

Favourable number of cases $n(E) = 6 + 6 + 6 = 18$

$\therefore$ Required probability $= \frac{18}{48} = \frac{3}{8}$

4. I. P(All three balls are white)

$$= \frac{{}^5C_3}{{}^{13}C_3} = \frac{\frac{5!}{2!3!}}{\frac{13!}{10! \times 3!}}$$

$$= \frac{5! \times 10!}{2! \times 13!} = \frac{5 \times 4 \times 3}{13 \times 12 \times 11} = \frac{5}{143}$$

II. P(All the three balls are red)

$$= \frac{{}^8C_3}{{}^{13}C_3} = \frac{\frac{8!}{5!3!}}{\frac{13!}{3! \times 10!}}$$

$$= \frac{8! \times 10!}{5! \times 13!} = \frac{8 \times 7 \times 6}{13 \times 12 \times 11} = \frac{28}{143}$$

III. P(One ball is red and two balls are white)

$$= \frac{{}^8C_1\ {}^5C_2}{{}^{13}C_3} = \frac{8 \times 10}{\frac{13 \times 12 \times 11}{3 \times 2}} = \frac{40}{143}$$

5. Total number of the students is 80. Total number of girls is 25. Total number of boys is 55. There are 10 rich, 70 poor, 20 intelligent students in the class. Therefore, required probability is

$$\frac{1}{4} \times \frac{1}{8} \times \frac{25}{80} = \frac{5}{512}$$

(I) (R) (G)

6. The probability that the first critic favours the book is

$$P(E_1) = \frac{5}{5+2} = \frac{5}{7}$$

The probability that the second critic favours the book is

$$P(E_2) = \frac{4}{4+3} = \frac{4}{7}$$

The probability that the third critic favours the book is

$$P(E_3) = \frac{3}{3+4} = \frac{3}{7}$$

Majority will be in favour of the book, if atleast two critics favour the book. Hence, the probability is

$$P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1)P(E_2)P(\bar{E}_3) + P(E_1)P(\bar{E}_2)P(E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1)P(E_2)P(E_3)$$

$$= \frac{5}{7} \times \frac{4}{7} \times \left(1 - \frac{3}{7}\right) + \frac{5}{7} \times \left(1 - \frac{4}{7}\right) \times \frac{3}{7} + \left(1 - \frac{5}{7}\right) \times \frac{4}{7} \times \frac{3}{7} + \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{209}{343}$$

7. $\because A \subseteq A \cup B$

$\Rightarrow \quad P(A) \le P(A \cup B) \Rightarrow P(A \cup B) \ge \frac{3}{4}$

Also, $\quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$\ge P(A) + P(B) - 1 = \frac{3}{4} + \frac{5}{8} - 1 = \frac{3}{8}$$

Now, $\quad A \cap B \subseteq B$

$\Rightarrow \quad P(A \cap B) \le P(B) = \frac{5}{8}$

$\therefore \quad \frac{3}{8} \le P(A \cap B) \le \frac{5}{8} \quad$ ...(i)

and $\quad P(A \cap B') = P(A) - P(A \cap B)$

$\Rightarrow \quad \frac{3}{4} - \frac{5}{8} \le P(A \cap B') \le \frac{3}{4} - \frac{3}{8}$

$\Rightarrow \quad \frac{1}{8} \le P(A \cap B') \le \frac{3}{8}$

$\therefore \quad P(A \cap B) = P(B) - P(A' \cap B) \quad$ [using Eq. (i)]

$\Rightarrow \quad \frac{3}{8} \le P(B) - P(A' \cap B) \le \frac{5}{8}$

$\Rightarrow \quad 0 \le P(A' \cap B) \le \frac{1}{4}$

8. The probability that both will be alive for 10 yr, hence, *i.e.*, the probability that the man and his wife both will be alive 10 yr, hence is $0.83 \times 0.87 = 0.7221$. The probability that atleast one of them will be alive is $1 - P$ (that none of them remains alive 10 yr hence)

$$= 1 - (1 - 0.83)(1 - 0.87) = 1 - 0.17 \times 0.13$$

$$= 0.9779$$

9. Let p_1 and p_2 be the chances of happening of the first and second events, respectively. Then, according to the given conditions, we have

$$p_1 = p_2^2$$

and $\quad \frac{1 - p_1}{p_1} = \left(\frac{1 - p_2}{p_2}\right)^3$

Hence,

$$\frac{1 - p_2^2}{p_2^2} = \left(\frac{1 - p_2}{p_2}\right)^3 \Rightarrow p_2(1 + p_2) = (1 - p_2)^2$$

$\Rightarrow \quad 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3}$

$\therefore \quad p_1 = \frac{1}{9}$

10. Let p_1 be the probability of being an answer correct from section 1. Then, $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then, $p_2 = 1/15$. Hence, the required probability is $1/5 \times 1/15 = 1/75$.

11. Scoring 10 marks from four questions can be done in $3 + 3 + 3 + 1 = 10$ ways, so as to answer three questions from section 2 and 1 question from section 1 correctly. Hence, the required probability is

$$\frac{{}^{10}C_3\ {}^{10}C_1}{{}^{20}C_4} \frac{1}{5} \left(\frac{1}{15}\right)^3$$

12. To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10}$ $(1/15)^{10}$. Hence, probability of getting a score less than 40 is

$$1 - \left(\frac{1}{5}\right)^{10} \left(\frac{1}{15}\right)^{10} = 1 - \left(\frac{1}{75}\right)^{10}$$

13. The number of ways of selecting two persons out of 20 is ${}^{20}C_2 = 190$.

The number of ways in which two selected persons together is 19.

$\therefore \quad$ Required probability $= 1 - \frac{19}{190} = 0.9$

14. We have,

$$P(A \cup B) \ge \max\{P(A), P(B)\} = \frac{2}{3} \text{ or } P(A \cup B) \ge \frac{2}{3}$$

Now, $P(A \cap B) = P(A) + P(B) - P(A \cup B) \ge P(A) + P(B) - 1$

$$= \frac{1}{2} + \frac{2}{3} - 1 = \frac{1}{6} \text{ or } P(A \cap B) \ge \frac{1}{6} \quad \text{...(i)}$$

and $\quad P(A \cap B) \le \min\{P(A), P(B)\} = \frac{1}{2}$

$$P(A \cap B) \le \frac{1}{2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{6} \le P(A \cap B) \le \frac{1}{2}$$

15. $P\{A \cap (B \cap C)\} = P(A \cap B \cap C) = P(A)P(B)P(C)$

$\therefore \quad P[A \cap (B \cup C)]$

$= P[(A \cap B) \cup (A \cap C)]$

$= P[(A \cap B) + (A \cap C)] - P[(A \cap B) \cap (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A)P(B) + P(A)(P(C) - P(A)P(B)P(C)$

$= P(A)[P(B) + P(C) - P(B)P(C)]$

$= P(A)P(B \cup C)$

Therefore, A and $B \cup C$ are independent events.

16. Clearly, both are correct but statement II is not the correct explanation for statement I.

17. Total number of cases $= {}^9C_3 = 84$

Number of favourable cases $= {}^3C_1 \cdot {}^4C_1 \cdot {}^2C_1 = 24$

$\therefore \quad p = \frac{24}{84} = \frac{2}{7}$

18. Since, $\quad A = \{4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$

$\therefore \quad A \cap B = \{4\}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = 1$

19. Let the events be

A = Ist aeroplane hit the target

B = IInd aeroplane hit the target

and their corresponding probabilities are

$P(A) = 0.3$

and $\quad P(B) = 0.2$

$\Rightarrow \quad P(\overline{A}) = 0.7$

and $\quad P(\overline{B}) = 0.8$

$\therefore$ Required probability

$= P(\overline{A})P(B) + P(\overline{A})P(\overline{B})P(\overline{A})P(B) + \ldots$

$= (0.7)(0.2) + (0.7)(0.8)(0.7)(0.2)$

$+ (0.7)(0.8)(0.7)(0.8)(0.7)(0.2) + \ldots$

$= 0.14\,[1 + (0.56) + (0.56)^2 + \ldots]$

$= 0.14\left(\frac{1}{1 - 0.56}\right)$

$= \frac{0.14}{0.44} = 0.32$

20. Since, $\quad 1 - P(A \cup B) = \frac{1}{6}$

$\Rightarrow \quad P(\overline{A}) - P(B) + \frac{1}{4} = \frac{1}{6}$

$\Rightarrow \quad P(B) = \frac{1}{3}$

and $\quad P(A) = \frac{3}{4}$

Now, $\quad P(A \cap B) = \frac{1}{4} = P(A)P(B)$

Hence, the events A and B are independent events but not equally likely.

PART - II

Chapters from Class 12th Syllabus

22 Matrix

JEE Main MILESTONE

- Matrix
- Algebra of Matrices
- Types of Matrices
- Equality of Two Matrices
- Trace of a Matrix
- Transpose of a Matrix
- Conjugate of a Matrix
- Conjugate Transpose of a Matrix
- Special Types of Matrices
- Minors and Cofactors
- Adjoint of a Square Matrix
- Elementary Row Transformations
- Elementary Matrix

22.1 Matrix

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix (plural matrices) of order $m \times n$.

This type of array is enclosed by [ ] or ( ) or | | | |.

Each of mn numbers of a matrix is known as element of a matrix. A matrix is generally denoted by $A, B, C, \ldots$ etc., and its element is denoted by a_{ij}, where a_{ij} belongs to the ith row and jth column and is called (i, j)th element of the matrix $A = [a_{ij}]$.

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

e.g., $A = \begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 6 \\ 4 & 8 & -12 \end{bmatrix}$ is a matrix of order 3×3.

British Mathematician ARTHUR CAYLEY was the first person to introduce the concept of matrices. After that, the physiat Heisenberg used matrices as a tool to explain his famous Quantum Principle. The study of matrices was originated while solving different types of linear problems.

Note

- A matrix is only an arrangement of numbers, it has no definite value
  e.g., $[7] \neq 7$
- The elements of matrix may be scalar or vector quantity.

22.2 Algebra of Matrices

Addition and Subtraction of Two Matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices whose orders are same, then

$$A + B = [a_{ij} + b_{ij}], \forall\, i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Also, $A - B = A + (-B)$

e.g., If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Then, $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Properties of Matrix Addition

Let A, B and C are three matrices of same order, then

(i) Matrix addition is commutative, *i.e.*, $A + B = B + A$

(ii) Matrix addition is associative, *i.e.*,

$$(A + B) + C = A + (B + C)$$

(iii) If O is a null matrix of order $m \times n$ and $A + O = A = O + A$, then O is known as additive identity.

(iv) If for each matrix $A = [a_{ij}]_{m \times n}$, a matrix $(-A)$ is such that

$$A + (-A) = O = (-A) + A,$$

then matrix $(-A)$ is known as additive inverse of A.

(v) Matrix addition follows cancellation law,

i.e., $A + B = A + C \Rightarrow B = C$ (left cancellation law)

and $B + A = C + A \Rightarrow B = C$ (right cancellation law)

Note Two matrices are said to be conformable for addition or subtraction, if they are of the same order.

Sample **Problem 1** *If* $A = \begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$, *then the value of* $A - B$ *is* [NCERT]

(a) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$ (b) $\begin{bmatrix} (a-b)^2 & (b-c)^2 \\ (a+c)^2 & (a+b)^2 \end{bmatrix}$

(c) $\begin{bmatrix} (a-b)^2 & (b+c)^2 \\ (a-c)^2 & (a+b)^2 \end{bmatrix}$ (d) *None of these*

Interpret (b) Since, $A = \begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix}$

and $B = \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

$\therefore$ $A - B = \begin{bmatrix} a^2+b^2-2ab & b^2+c^2-2bc \\ a^2+c^2+2ac & a^2+b^2+2ab \end{bmatrix}$

$$= \begin{bmatrix} (a-b)^2 & (b-c)^2 \\ (a+c)^2 & (a+b)^2 \end{bmatrix}$$

Scalar Multiplication

Let $A = [a_{ij}]$ be any $m \times n$ matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA.

Thus, if $A = [a_{ij}]_{m \times n}$, then $kA = [ka_{ij}]_{m \times n}$.

e.g., If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$,

then $2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \\ 2 & 6 & 2 \end{bmatrix}$

Properties of Scalar Multiplication

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices and λ and μ are two scalars, then

(i) $\lambda(A + B) = \lambda A + \lambda B$ (ii) $(\lambda + \mu)A = \lambda A + \mu A$

(iii) $(\lambda)\mu A = \mu(\lambda A) = \lambda(\mu A)$ (iv) $(-\lambda)A = -(\lambda A) = \lambda(-A)$

Multiplication of Two Matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that the number of columns of A is equal to the number of rows of B, then a matrix $C = [c_{ij}]_{m \times p}$ of order $m \times p$ is known as product of matrices A and B, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

On multiplication of matrices is denoted by $C = AB$

e.g., If $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 1 & 7 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2\times5+1\times1 & 2\times2+1\times7 \\ 3\times5+5\times1 & 3\times2+5\times7 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 11 \\ 20 & 41 \end{bmatrix}$$

Note In multiplication of two matrices A and B, the order roles an important role.

Properties of Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{n \times p}$ and $C = [c_{ij}]_{p \times m}$ are three matrices, then

(i) Generally, matrix multiplication is not commutative, *i.e.*, $AB \neq BA$

(ii) Matrix multiplication is associative,

i.e., $A(BC) = (AB)C$

(iii) Matrix multiplication is distributive over matrix addition.

$$A(B + C) = AB + AC$$

(iv) If A is a $m \times n$ matrix and I_n is an identity matrix of order $n \times n$ and I_m is an identity matrix of order $m \times m$, then

$$I_m A = A \text{ and } AI_n = A$$

In particular, if A is a square matrix of order n, then

$$AI_n = I_n A = A$$

(v) $AB = O$ does not necessarily imply that $A = O$ or $B = O$ or both A and B are O.

e.g., If $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq O$

But $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Sample Problem 2 *The value of* $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} [4 \quad 5 \quad 2] \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ *is*

(a) $\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$ (b) $[3 \; -6 \; 9]$ (c) $[0 \; 0 \; 0]$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Interpret (a) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} [4 \quad 5 \quad 2] \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 5 & 2 \\ -8 & -10 & -4 \\ 12 & 15 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$$

Sample Problem 3 *If* $P = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ *and* $Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, *then the value of PQ is equal to* [NCERT Exemplar]

(a) PQ (b) P

(c) Q (d) QP

Interpret (d) $PQ = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$= \begin{bmatrix} ax+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+by+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+cz \end{bmatrix} = \begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & cz \end{bmatrix}$$

$$PQ = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} ax+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+by+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+cz \end{bmatrix} = \begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & cz \end{bmatrix} = PQ$$

Hence, $PQ = QP$

22.3 Types of Matrices

(i) Row Matrix

A matrix which has only one row and any number of columns, is called a row matrix.

e.g., $A = [27 \quad 85 \quad 1 \quad 4]_{1 \times 4}$ is a row matrix.

(ii) Column Matrix

A matrix is said to be a column matrix, if it has only one column and any number of rows.

e.g., $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$ is a column matrix.

(iii) Rectangular Matrix

A matrix in which number of rows is not equal to the number of columns or *vice-versa* is called a rectangular matrix.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a rectangular matrix of order 2×3.

(iv) Square Matrix

A matrix in which number of rows is equal to the number of columns, is called a square matrix. The elements a_{ij} of a square matrix $A = [a_{ij}]_{m \times m}$ for which $i = j$ *i.e*, the elements $a_{11}, a_{22}, \ldots, a_{mm}$ are called the diagonal elements and the line along which they lie is called the principal diagonal or leading diagonal of the matrix.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3 \times 3}$

is a square matrix of order 3 in which diagonal elements are 1, 2, 1.

(v) Null Matrix

A matrix of order $m \times n$ whose all elements are zero, is called a null matrix of order $m \times n$.

It is denoted by O.

e.g., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are two null matrices of order 2×2 and 2×3, respectively.

(vi) Diagonal Matrix

A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero and diagonal elements are not all equal.

If $d_1, d_2, d_3, \ldots, d_n$ are elements of principal diagonal of a diagonal matrix of order $n \times n$, then matrix is denoted as diag $[d_1, d_2, \ldots, d_n]$.

e.g., $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is a diagonal matrix which is denoted by $A = \text{diag}\,[a, b, c]$.

Note The number of zeroes in a diagonal matrix is given by $n^2 - n$, where n is an order of the matrix.

(vii) Triple Diagonal Matrix

A square matrix A is said to be a triple diagonal matrix, if all its elements are zero except possibly for those lying on the principle diagonal, the diagonal immediately above as well as below the principle diagonal.

e.g., $\begin{bmatrix} 5 & -3 & 0 \\ -3 & 4 & -3 \\ 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{bmatrix}$

(viii) Scalar Matrix

A square matrix $A = [a_{ij}]$ is said to be scalar matrix, if

(a) $a_{ij} = 0, \forall\ i \neq j$

(b) $a_{ij} = k, \forall\ i = j$, where $k \neq 0$

i.e., a diagonal matrix is said to be a scalar matrix, if the elements of principle diagonal are same.

e.g., $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix.

(ix) Unit Matrix

A square matrix $A = [a_{ij}]$ is said to be a unit matrix or identity matrix, if

(a) $a_{ij} = 0, \forall\ i \neq j$ (b) $a_{ij} = 1, \forall\ i = j$

i.e., A diagonal matrix, whose elements of principle diagonal are equal to 1 and all remaining elements are zero, is known as unit or identity matrix. It is denoted by I.

e.g., $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

(x) Upper Triangular Matrix

A square matrix $A = [a_{ij}]$ is known as upper triangular matrix, if

$a_{ij} = 0, \forall\ i > j.$

e.g., $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix.

(xi) Lower Triangular Matrix

A square matrix $A = [a_{ij}]$ is known as lower triangular matrix, if $a_{ij} = 0, \forall\ i < j$.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ is a lower triangular matrix.

(xii) Submatrices of a Matrix

A matrix B obtained by deleting the row (s) or Column (s) or both of a matrix A is said to be a submatrix of A *i.e.,* the matrix B constituted by the array of elements, which are left after deleting some rows or columns or both of matrix A is called submatrix of A.

(a) **Principle Submatrix** A square submatrix B of a square matrix A is called a principle submatrix, if the diagonal elements of B are aslo diagonal elements of A.

(b) **Leading Submatrix** A principle square submatrix B is said to be a leading submatrix of a square matrix A if it is obtained by deleting only some of the last rows and the corresponding columns such that the leading elements (*i.e.,* a_{11}) is not lost.

(xiii) Horizontal Matrix

Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

e.g., $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}$ is a horizontal matrix.

(xiv) Vertical Matrix

Any matrix in which the number of rows is more than the number of columns is called column matrix.

e.g., $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ is a column matrix.

22.4 Equality of Two Matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal, if

(i) Number of rows in A is equal to number of rows in B.

(ii) Number of columns in A is equal to number of columns in B.

(iii) $a_{ij} = b_{ij}, \forall\ i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Sample Problem 4 *If*

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix},$$

then the value of z is

(a) 1 (b) 2
(c) 3 (d) 4

Interpret (a) Given, $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$3x = x + 4$$
$$\Rightarrow \quad 2x = 4$$
$$\Rightarrow \quad x = 2$$
and
$$3y = 6 + x + y$$
$$\Rightarrow \quad 2y = 6 + x$$
$$\Rightarrow \quad y = \frac{6+x}{2}$$

On putting the value of x, we get

$$y = \frac{6+2}{2} = \frac{8}{2} = 4$$

Now, $3z = -1 + z + w,$

$$2z = -1 + w$$
$$z = \frac{-1+w}{2} \quad \ldots(i)$$

Now, $3w = 2w + 3$

$$\Rightarrow \quad w = 3$$

On putting the value of w in Eq. (i), we get

$$z = \frac{-1+3}{2} = \frac{2}{2} = 1$$

Hence, the values of x, y, z and w are 2, 4, 1 and 3.

22.5 Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then, the sum of all diagonal elements of A is called the trace of A and is denoted by $\text{tr}(A)$.

Thus, $\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$

e.g., $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$, then $\text{tr}(A) = 2 + 3 + 4 = 9$

Properties of Trace of a Matrix

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are two square matrix of order n, then

(i) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

(ii) $\text{tr}(AB) = \text{tr}(BA)$

(iii) $\text{tr}(\lambda A) = \lambda\,\text{tr}(A)$, where λ is a scalar.

(iv) $\text{tr}(A') = \text{tr}(A)$

Check Point 1

1. If $U = [2 \quad -3 \quad 4]$, $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $X = [0 \quad 2 \quad 3]$ and $Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then the value of $UV + XY$ is
   (a) 20 (b) $[-20]$ (c) -20 (d) $[20]$

2. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then A^2 is equal to
   (a) A (b) $-A$ (c) 2A (d) $-2A$

3. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, then AB is equal to
   (a) B (b) A (c) O (d) I

4. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2 + 2AB$, then the values of a and b are respectively
   (a) 1, -2 (b) 1, 2
   (c) -1, 2 (d) -1, -2

5. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then
   (a) $AB = \begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ (b) $AB = [-2 \quad -1 \quad 4]$
   (c) $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (d) AB does not exist

6. Matrix A has m rows and (n + 5) columns, matrix B has m rows and (11 – n) columns. If both AB and BA exist, then
   (a) AB and BA are square matrix
   (b) AB and BA are of order 8×8 and 3×13 respectively
   (c) AB = BA
   (d) None of the above

7. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$, then matrix

   $D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$ such that A + B – D = O, is

   (a) $\begin{bmatrix} -2 & 0 \\ 3 & 1 \\ 9 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 1 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$
   (c) $\begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$ (d) None of these

8. Matrix X satisfying the equations $2X - Y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$, $2Y + X = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ is
   (a) $\begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ (d) None of these

9. The values of x, so that $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$ is/are
   (a) ± 2 (b) 0
   (c) $\dfrac{-7 \pm \sqrt{35}}{2}$ (d) $\dfrac{-9 \pm \sqrt{53}}{2}$

10. If $A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then A^{100} is equal to
   (a) $\begin{bmatrix} 1 & 0 \\ 25 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{bmatrix}$ (d) None of these

22.6 Transpose of a Matrix

If $A = [a_{ij}]_{m \times n}$ is a matrix of order $m \times n$, then the transpose of A can be obtained by changing all rows to columns and all columns to rows *i.e.*, transpose of $A = [a_{ji}]_{n \times m}$. It is denoted by A', A^T or A^t.

e.g., If $A = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 9 & 4 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 6 \\ 3 & 9 \\ 5 & 4 \end{bmatrix}$

Properties of Transpose

If A and B are two matrices and k is a scalar, then

(i) $(A')' = A$
(ii) $(A + B)' = A' + B'$
(iii) $(kA)' = kA'$
(iv) $(AB)' = B'A'$ (reversal law)

Note If A, B and C are any three matrices conformable for multiplication, then $(ABC)' = C'B'A'$

***Sample* Problem 5** *If* $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, *then the value of* $(A + 2B)'$ *is*

(a) $\begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & -4 \\ 6 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 6 \\ -4 & 5 \end{bmatrix}$

Interpret (*b*) Given, $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow \quad A = (A')' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} \quad [\because (A')' = A]$

and $\quad 2B = 2\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

$\therefore \quad A + 2B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$

$\Rightarrow \quad (A + 2B)' = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$

22.7 Conjugate of a Matrix

The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$.

e.g., If $A = \begin{bmatrix} 1+2i & 2-3i \\ 4-5i & 5+6i \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i \\ 4+5i & 5-6i \end{bmatrix}$

Properties of Conjugate

If A and B are two matrices and k is a scalar, then

(i) $\overline{(\overline{A})} = A$ (ii) $\overline{(A + B)} = \overline{A} + \overline{B}$
(iii) $\overline{(kA)} = k\overline{A}$ (iv) $\overline{AB} = \overline{A} \cdot \overline{B}$

22.8 Conjugate Transpose of a Matrix

The transpose of the conjugate of a matrix A is called conjugate transpose of A and is denoted by A^{Θ} or A^{*}.

$$A^{\Theta} = \text{Conjugate of } A' = (\overline{A'})$$

The transpose of the conjugate of A is the same as the conjugate of the transpose of A.

e.g., If $$A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & \alpha+\beta i & 3i \\ 2 & -5 & 4-i \end{bmatrix},$$

then $$A^{\Theta} = (\overline{A'}) = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & \alpha-\beta i & -5 \\ 5+9i & -3i & 4+i \end{bmatrix}$$

Properties of Conjugate Transpose

(i) For a matrix A, $(\overline{A'}) = (\overline{A})'$

(ii) $(A^{\Theta})^{\Theta} = A$

(iii) If A and B are two matrices, then $(A+B)^{\Theta} = A^{\Theta} + B^{\Theta}$

(iv) $(kA)^{\Theta} = kA^{\Theta}$, where k is any scalar.

(v) $(AB)^{\Theta} = B^{\Theta}A^{\Theta}$

22.9 Special Types of Matrices

(i) Nilpotent Matrix

If $A^k = O$ and $A^{k+1} \neq O$, where k is a positive integer and O is a null matrix, then A is called nilpotent matrix, k is called the index of the nilpotent matrix A.

(ii) Periodic Matrix

If $A^{k+1} = A$, where k is a positive integer, then A is known as periodic matrix, k is known as period of matrix A.

For $k = 1$, $A^2 = A$.

(iii) Idempotent Matrix

If $A^2 = A$, then square matrix A is known as idempotent matrix.

(iv) Involutory Matrix

If $A^2 = I$, where I is an identity matrix, then A is called an involutory matrix.

(v) Symmetric Matrix

If for a square matrix A, $A' = A$, then A is known as symmetric matrix.

e.g., If $$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix},$$

then $$A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \Rightarrow A' = A$$

Thus, matrix A is a symmetric matrix.

(vi) Skew-symmetric Matrix

A square matrix $A = [a_{ij}]_{m \times m}$ is said to be a skew-symmetric matrix, if

(a) $a_{ij} = -a_{ji}, \forall\ i, j$

(b) Each element of diagonal is zero.

OR

A square matrix A is said to be a skew-symmetric matrix, if $A' = -A$.

e.g., If $$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix},$$

then $$A' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & -3 \\ 3 & -3 & 0 \end{bmatrix} = -A$$

Hence, A is a skew-symmetric matrix.

Note Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrix.

i.e., $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

***Sample* Problem 6** *If A and B are symmetric matrices of same order, then AB − BA is a* [NCERT]

(a) *skew-symmetric matrix* (b) *symmetric matrix*

(c) *zero matrix* (d) *identity matrix*

Interpret (a) Given, A and B are symmetric matrices

$\Rightarrow$ $A' = A$ and $B' = B$

$(AB - BA)' = (AB)' - (BA)'$ $[\because (A-B)' = A' - B']$

$= B'A' - A'B'$ $[\because (AB)' = B'A']$

$= BA - AB$

$= -(AB - BA)$ $(\because A' = A \text{ and } B' = B)$

$\Rightarrow$ $(AB - BA)' = -(AB - BA)$

Thus, $(AB - BA)$ is a skew-symmetric matrix.

(vii) Orthogonal Matrix

If the product of a square matrix and its transpose A' is an identity matrix, then matrix A is said to be an orthogonal matrix.

i.e., $$AA' = I = A'A$$

e.g., Let $$A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$\therefore$ $$A' = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

Also, $$AA' = \frac{1}{2\sqrt{2}} \cdot \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AA' = I$$

Similarly, $A'A = I$

Thus, A is an orthogonal matrix.

Note If A and B are two orthogonal matrices, then AB will also be an orthogonal matrix.

***Sample* Problem 7** *If* $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, *then the matrix A is*

(a) *symmetric matrix* (b) *skew-symmetric matrix*
(c) *identity matrix* (d) *orthogonal matrix*

Interpret (*d*) Here,

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \quad A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}' \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\alpha)(\cos\alpha) + (\sin\alpha)(\sin\alpha) & (\cos\alpha)(\sin\alpha) + (-\sin\alpha)(\cos\alpha) \\ (\sin\alpha)(\cos\alpha) + (\cos\alpha)(-\sin\alpha) & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cdot\cos\alpha - \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha - \sin\alpha\cdot\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

So, A is an orthogonal matrix.

(viii) Hermitian Matrix

A square matrix A such that $A^{\Theta} = \overline{A}' = A$, then A is known as hermitian matrix.

e.g., Let $$A = \begin{bmatrix} \alpha & \lambda + i\mu \\ \lambda - i\mu & \beta \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} \alpha & \lambda - i\mu \\ \lambda + i\mu & \beta \end{bmatrix}$$

$$A^{\Theta} = (\overline{A}') = \begin{bmatrix} \alpha & \lambda + i\mu \\ \lambda - i\mu & \beta \end{bmatrix} = A$$

$\therefore$ $A^{\Theta} = A$

Thus, A is a hermitian matrix.

(a) Every diagonal element of a hermitian matrix must be real.

(b) A hermitian matrix over the set of real numbers is actually a real symmetric matrix.

(ix) Skew-hermitian Matrix

A square matrix A, such that $A^{\Theta} = -A$, then A is known as skew-hermitian matrix.

e.g., Let $$A = \begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix}$$

$$\Rightarrow \quad A^{\Theta} = (\overline{A}') = \begin{bmatrix} -2i & 2+3i \\ -2+3i & i \end{bmatrix}$$

$$= -\begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix} = -A \therefore A^{\Theta} = -A$$

Thus, A is a skew-hermitian matrix.

(a) If A is a skew-hermitian matrix, then $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii} = 0$ *i.e.*, a_{ii} must be purely imaginary or zero.

(b) A skew-hermitain matrix over the set of real numbers is actually a real symmetric matrix.

(x) Unitary Matrix

If for a square matrix A, $AA^{\Theta} = I$, then A is known as unitary matrix.

(xi) Singular and Non-singular Matrix

If A is a square matrix and $|A| = 0$, then A is known as singular matrix.

e.g., Let $$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

and $$|A| = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = 6 - 6 = 0$$

$\therefore$ A is a singular matrix.

If A is a square matrix and $|A| \neq 0$, then A is known as non-singular matrix.

e.g., Let $$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 4 - 6 = -2 \neq 0$$

$\therefore$ A is a non-singular matrix.

***Sample* Problem 8** *If* $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, *then A is a/an*

(a) *idempotent matrix* (b) *nilpotent matrix*
(c) *involutory matrix* (d) *periodic matrix*

Interpret (a) Since, $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, A is an idempotent matrix.

***Sample* Problem 9** *If* $A = \begin{bmatrix} ab & b^2 \\ -a^2 & ab \end{bmatrix}$, *then A is a/an*

(a) *idempotent matrix*
(b) *nilpotent matrix*
(c) *involutory matrix*
(d) *None of the above*

Interpret (b) Since, $A = \begin{bmatrix} ab & b^2 \\ -a^2 & ab \end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\Rightarrow \quad A^2 = O$

Thus, A is a nilpotent matrix of order 2.

***Sample* Problem 10** *The number of values of x for which the matrix* $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ *is singular, is*

(a) 0 (b) 1 (c) 2 (d) 3

Interpret (c) let $(A) = |A| = \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$

$\Rightarrow \quad -(x-3)^2 x = 0 \quad \Rightarrow x = 0, 3$

$\therefore$ Number of solutions is 2.

Check Point 2

1. If A is symmetric as well as skew-symmetric matrix, then A is
   (a) diagonal matrix (b) null matrix
   (c) triangular matrix (d) None of these
2. If $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then A is a/an
   (a) idempotent matrix (b) involutory matrix
   (c) nilpotent matrix (d) None of these
3. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then AA' is a/an
   (a) symmetric matrix (b) skew-symmetric matrix
   (c) orthogonal matrix (d) None of these
4. If A is a square matrix, then
   (a) $A + A'$ is symmetric (b) $A + A'$ is skew-symmetric
   (c) $A - A'$ is symmetric (d) None of the above
5. If A is an orthogonal matrix, then A^{-1} equals
   (a) A (b) A^T
   (c) A^2 (d) None of these
6. If A is a 3×3 skew-symmetric matrix, then trace of A is equal to
   (a) -1 (b) 7
   (c) $|A|$ (d) None of these
7. If A is a skew-symmetric matrix and n is an odd positive integer, then A^n is a
   (a) symmetric matrix (b) skew-symmetric matrix
   (c) diagonal matrix (d) None of these
8. A skew-symmetric matrix S satisfies the relation $S^2 + I = 0$, where I is an unitary matrix, then S is an
   (a) idempotent matrix (b) involutory matrix
   (c) orthogonal matrix (d) None of these
9. If AB = A and BA = B, then which of the following is not true?
   (a) A is idempotent matrix (b) B is idempotent matrix
   (c) A^T is idempotent matrix (d) None of these
10. In which of the following type of matrix there always exists an inverse?
   (a) Idempotent matrix (b) Orthogonal matrix
   (c) Involutory matrix (d) None of these

22.10 Minors and Cofactors

(i) Minors

Let $A = [a_{ij}]$ is a square matrix of order n. Then, the minor M_{ij} of a_{ij} in A is the determinant of a square submatrix of order $(n-1)$ obtained by removing the ith row and jth column of A.

Consider a determinant of order 3, such that

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then, the minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Also, the minor a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$ and similar other cases.

(ii) Cofactors

Let $A = [a_{ij}]$ is a square matrix of order n. Then, the cofactor C_{ij} of a_{ij} in A is $(-1)^{i+j}$ times M_{ij}, where M_{ij} is the minor of a in A.

$\therefore \quad C_{ij} = (-1)^{i+j} M_{ij}$

Consider the following determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then, cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Also, the cofactor of a_{12} is $c_{12} = (-1)^{1+2} M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and similar other cases.

Note Minors and cofactors are defined only for the elements of square matrix.

22.11 Adjoint of a Square Matrix

Let $A = [a_{ij}]_{n \times n}$ is a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A is denoted by adj(A) and is defined as the transpose of the cofactor matrix.

***Sample* Problem 11** *The adjoint of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} \text{ is}$$

(a) $\begin{bmatrix} 6 & 15 & -15 \\ -3 & 0 & 0 \\ 0 & -16 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -16 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} -15 & 6 & 15 \\ 0 & -3 & 0 \\ 5 & 0 & -16 \end{bmatrix}$ (d) *None of these*

Interpret (b) If C be the matrix of cofactors of the element in $|A|$. Then,

$$C = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

Here,

$\therefore \quad c_{11} = \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15, \; c_{23} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$c_{12} = -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0, \; c_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15$

$c_{13} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = -10, \; c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0$

$c_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, \; c_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5$

$c_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3, \; \therefore C = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$

$\therefore \quad$ Adj A = Transpose of $C = C'$

$$= \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -16 & 0 & 5 \end{bmatrix}$$

Note Adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of the off diagonal elements.

Properties of Adjoint of a Square Matrix

Properties of adjoint of a square matrix is an important topic for JEE (Main) examination point of view. Generally question are seen from this topic. The level of question is from moderate to typical.

In these types of questions, generally we have given a square matrix and asking the adjoint of that matrix in moderate level. i.e., based on property of adjoint.

(i) $(\text{adj } A)A = A(\text{adj } A) = |A| \cdot I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$, if $|A| \neq 0$

(iii) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(iv) If $|A| = 0$, then $(\text{adj } A)A = A(\text{adj } A) = 0$

(v) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where A is a non-singular matrix

(vi) $\text{adj}(A^T) = (\text{adj } A)^T$

(vii) Adjoint of a diagonal matrix is a diagonal matrix.

(viii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(ix) $\text{adj}(A^m) = (\text{adj } A)^m, M \in N$

(x) $\text{adj}(KA) = K^{n-1}(\text{adj } A)^m, K \in R$

(xi) $\text{adj}(In) = In$

(xii) $\text{adj } 0 = 0$

(xiii) Adjoint of symmetric matrix is a symmatric matrix.

(xiv) A is singular $\Rightarrow |\text{adj } A| = 0$

(xv) Adjoint of triangular matrix is a triangular matrix.

***Sample* Problem 12** *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *then adj (adj A) is equal to*

(a) A (b) I

(c) O (d) *None of these*

Interpret (a) We know that,

$\text{adj}(\text{adj } A) = |A|^{n-2} A$, if $|A| \neq 0$

Since, $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Here, $n = 3$ (3 order matrix)

$\therefore \quad |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) + 3(2-0) + 4(-2-0)$

$= 1 \neq 0$

$\therefore$ A is non-singular.

$$\text{adj}(\text{adj } A) = |A|^{3-2} \cdot A = A$$

Inverse of a Matrix

Inverse of a Matrix is an important topic for JEE (Main) examination and a small practice can help you in solving such problems very quickly as the level of questions is easy to average.

While solving problems based on inverse of a matrix, always keep in mind.

(i) Adjoint of a square matrix

(ii) Properties of Adjoint of a square matrix.

If two square matrices of same order are A and B, for which

$$AB = BA = I_n$$

Then, B is known as inverse of A,

i.e., $\quad A^{-1} = B$

If $|A| \neq 0$ *i.e.*, A is non-singular, then

$$A^{-1} = \frac{\text{adj } A}{|A|}.$$

Properties of Inverse of a Matrix

If A, B and C are three matrices of same order and $|A| \neq 0$, $|B| \neq 0$ and $|C| \neq 0$, then

(i) (a) $AB = AC \Rightarrow B = C$ (left cancellation law)

(b) $BA = CA \Rightarrow B = C$ (right cancellation law)

(ii) (a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(iii) $(A^T)^{-1} = (A^{-1})^T$

(iv) $(kA)^{-1} = \frac{1}{k}A^{-1}$, if $k \neq 0$

(v) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(vi) If A is a non-singular matrix, then

$$|A^{-1}| = |A|^{-1} \Rightarrow |A^{-1}| = \frac{1}{|A|}$$

(vii) If A is a symmetric matrix, then A^{-1} is also a symmetric matrix.

(viii) A square matrix is invertible iff it is non-singular and every invertiable matrix possesses a unique inverse.

***Sample* Problem 13** *The inverse of* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *is*

(a) $\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (d) *None of these*

Interpret (*b*) Given, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = 0(2-3) - 1(1-9) + 2(1-6)$

$= 0 + 8 - 10 = -2 \neq 0$

Now, $C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$

$C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1;$

$C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5, \quad C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -1$

$C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$

$\therefore$ Matrix of cofactors, $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$\therefore \quad \text{adj}(A) = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

Hence, $\quad (A)^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

22.12 Elementary Row Transformations

Any one of the following operations on a matrix is called an elementary row (or column) transformation.

(i) Interchanging any two rows (or columns). This transformation is indicated by

$$R_i \leftrightarrow R_j \text{ (or } C_i \leftrightarrow C_j)$$

(ii) Multiplication of the elements of any row (or column) by a non-zero scalar quantity. This transformation is indicated as

$$R_i \leftrightarrow kR_i \text{ (or } C_i \leftrightarrow kC_i)$$

(iii) Addition of a constant multiple of the elements of any row to the corresponding element of any other row. This transformation is indicated as

$$R_i \to R_i + kR_j.$$

Note Two matrices are said be equivalent, if one is obtained from the other by elementary transformations. The symbol $\approx$ is used for equivalence.

22.13 Elementary Matrix

A matrix obtained from an identity matrix by a single elementary operation is called an elementary matrix.

Inverse of a Matrix using Elementary Row Transformation

Let $A = IA$

If matrix A (LHS) is reduced to I by elementary row transformation, then suppose I (RHS) is reduced to P and not change A in RHS *i.e.*, after transformation, we get $I = PA$, then P is the inverse of A,

$\therefore$ $P = A^{-1}$

***Sample* Problem 14** *The inverse of* $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ *by elementary row transformation is*

(a) $\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$

(c) $-\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ (d) *None of these*

Interpret (c) Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$. We know that, $A = IA$

$$\therefore \quad \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{using } R_1 \to R_1 + R_2 - R_3)$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -5 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -3 & -3 & 4 \end{bmatrix} A \quad (\text{using } R_2 \leftrightarrow R_3)$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 3/5 & 3/5 & -4/5 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$(\text{using } R_2 \to -\frac{1}{5}R_2 \text{ and } R_3 \to \frac{-1}{5}R_3)$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 1/5 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$(\text{using } R_2 \to R_2 - 2R_3 \text{ and } R_1 \to R_1 - 4R_3)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$(\text{using } R_1 \to R_1 - R_2)$$

$$\Rightarrow \quad I_3 = -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \quad (\because AA^{-1} = I_3)$$

Check Point 3

1. If A, B are two $n \times n$ non-singular matrices, then
   (a) AB is a non-singular (b) AB is singular
   (c) $(AB)^{-1} = A^{-1}B^{-1}$ (d) $(AB)^{-1}$ does not exist

2. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj (A) is equal to
   (a) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$, then A^{-1} equals
   (a) $-\frac{1}{8}\begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$ (b) $-\frac{1}{8}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$
   (c) $\frac{1}{8}\begin{bmatrix} -1 & -3 \\ -2 & 2 \end{bmatrix}$ (d) None of these

4. The inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ is
   (a) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ (d) None of these

5. If a 3×3 matrix A has its inverse equal to A, then A^2 is equal to
   (a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

6. For two unimodular complex numbers z_1 and z_2, then $\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1}\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$ is equal to
   (a) $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (d) None of these

7. If $A^3 = 0$, then $I + A + A^2$ is equal to
   (a) $I - A$ (b) $I + A^{-1}$
   (c) $(I - A)^{-1}$ (d) None of these

8. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$, then the value of $|\text{adj } A|$ is equal to
   (a) 5 (b) 0
   (c) 1 (d) None of these

9. If P is non-singular matrix, then the value of adj (P^{-1}) in terms of P is
   (a) $\frac{P}{|P|}$ (b) $P|P|$
   (c) P (d) None of these

10. If A and B are two non-singular matrices of the same order such that $B^r = I$ for some positive integer $r > 1$, then $A^{-1}B^{r-1}A - A^{-1}B^{-1}A$ is equal to
   (a) I (b) 2I
   (c) 0 (d) $-I$

Solution of Simultaneous Linear Equations

Solution of simultaneous linear equation is the most important topic of this chapter and also JEE (Main) examination point of views. Generally, question are seen from this topic. The level of question is from moderate to typical. But a small practice can help you in solving such problems very quickly.

While solving problems based on solution of simultaneous linear equation, always keep in mind the results on homogeneous and non-homogeneous equation separately. Also, do not confuse between them.

Let the n simultaneous linear equations in n unknowns are

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

then system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1} \quad \text{or } AX = B$$

where, matrix A is **called coefficient matrix**, matrix X is called **variable matrix** and matrix B is called column matrix of given constants.

Solution of Non-homogeneous System of Equations

If $B \neq 0$, then given system of equations $AX = B$ is non-homogeneous.

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(\text{adj } A) \cdot B = O$, then the system of equations is consistent and has an infinite number of solutions.

Solution of Homogeneous System of Equations

If $B = 0$, then given system of equation $AX = B$ is homogeneous.

(i) If $|A| \neq 0$, the system of equations has only trivial solution and it has one solution.

(ii) If $|A| = 0$, the system of equations has non-trivial solution and it has infinite solutions.

(iii) If number of equations < number of unknowns, then it has non-trivial solution.

Sample **Problem 15** *The system of linear equations* $3x - y - 2z = 2$, $2y - z = -1$ *and* $3x - 5y = 3$ *has*

(a) *a unique solution*
(b) *an infinitely many solutions*
(c) *no solution*
(d) *None of the above*

Interpret (c) The given system is $3x - y - 2z = 2$, $0x + 2y - z = -1$ and $3x - 5y + 0z = 3$. Which can be written as $Ax = B$, *where*

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, \; x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Here, $$|A| = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} = 3(0-5) + 1(0+3) - 2(0-6)$$

$$= -15 + 3 + 12 = 0$$

$\therefore A$ is a singular matrix.

Therefore, nothing can be said about consistency as yet. So, we compute (adj A) B. Cofactors of A are

$A_{11} = -5$, $A_{12} = -3$, $A_{13} = -6$, $A_{21} = 10$, $A_{22} = 6$, $A_{23} = 12$, $A_{31} = 5$, $A_{32} = -3$, $A_{33} = 6$,

$$\text{adj}(A) = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}^{T} = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore \; (\text{adj}A)\, B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -10 & +15 \\ -6 & -6 & +9 \\ -12 & -12 & +18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix}$$

$$\neq 0$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equation is inconsitent *i.e.*, it has no solution.

WORKED OUT Examples

Example 1 *If* $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, *then* A^{-1} *is equal to*

(a) A (b) I
(c) 0 (d) *None of these*

Solution (a) Since, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\Rightarrow \quad |A| = 1$ and $\text{adj}(A) = A$

$\therefore \quad A^{-1} = \dfrac{\text{adj}(A)}{|A|} = \dfrac{A}{1} = A$

Example 2 *Elements of a matrix A of order* 10×10 *are defined as* $a_{ij} = \omega^{i+j}$ *(where,* ω *is cube root of unity), then tr(A) of the matrix is*

(a) 0 (b) ω
(c) 3 (d) *None of these*

Solution (d) $\text{tr}(A) = (a_{11} + a_{22} + \ldots + a_{10}a_{10})$

$= (\omega^2 + \omega^4 + \omega^6 + \ldots + \omega^{20})$

$= \omega^2(1 + \omega^2 + \omega^4 + \ldots + \omega^{18})$

$= \omega^2[(1 + \omega^2 + \omega) + \ldots + (1 + \omega + \omega^2) + 1]$

$= \omega^2 [0 + \ldots + 0 + 1] = \omega^2$

Example 3 *The values of x for which the matrix* $A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix}$ *is singular, are*

(a) $0, \dfrac{3 \pm \sqrt{205}}{2}$ (b) $0, \dfrac{3}{2}$
(c) $0, \pm \dfrac{\sqrt{205}}{2}$ (d) *None of these*

Solution (a) Matrix $A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix}$ will be singular, if

$$|A| = \begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$$

$\Rightarrow \quad (x+1)(x^2 - 4x - 12 - 2) + 3(-5x + 30 - 8) + 4(-5 - 4x - 8) = 0$

$\Rightarrow \quad (x+1)(x^2 - 4x - 14) + 3(-5x + 22) + 4(-4x - 13) = 0$

$\Rightarrow \quad x(x^2 - 3x - 49) = 0$

$\Rightarrow \quad x = 0, \dfrac{3 \pm \sqrt{205}}{2}$

Example 4 *If* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, *then* A^n *is equal to*

(a) $\begin{bmatrix} -\cos n\theta & \sin n\theta \\ -\sin n\theta & \sin n\theta \end{bmatrix}$ (b) $\begin{bmatrix} \sin n\theta & \sin n\theta \\ -\sin n\theta & \sin n\theta \end{bmatrix}$,

(c) $\begin{bmatrix} \sin n\theta & -\sin n\theta \\ \sin n\theta & \sin n\theta \end{bmatrix}$ (d) *None of these*

Solution (b) Since, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

$\therefore \quad A^2 = A \cdot A$

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Similarly, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

Example 5 *If* $A = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}$, *then* $\det(AA^T)$ *is equal to*

(a) 0 (b) 7 (c) 2 (d) 3

Solution (a) Since, $A = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}$ and $A^T = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$

$\therefore \quad AA^T = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$

$= \begin{bmatrix} a^2 + p^2 & ab + pq & ac + pr \\ ab + qp & b^2 + q^2 & bc + qr \\ ac + pr & bc + qr & c^2 + r^2 \end{bmatrix}$

$$\Rightarrow \qquad \det(AA^T) = \begin{vmatrix} a^2+p^2 & ab+pq & ac+pr \\ ab+qp & b^2+q^2 & bc+qr \\ ac+pr & bc+qr & c^2+r^2 \end{vmatrix}$$

$$= \begin{vmatrix} a & p & 0 \\ b & q & 0 \\ c & r & 0 \end{vmatrix} \begin{vmatrix} a & b & c \\ p & q & r \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Example 6 *If* $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, *then* $E(\alpha)\cdot E(\beta)$ *is equal to* [**NCERT**]

(a) $E(\alpha-\beta)$ (b) $E(\alpha+\beta)$
(c) $E(\alpha)$ (d) $E(\beta)$

Solution (b) Since, $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\therefore \qquad E(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

and $$E(\beta) = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$E(\alpha)\cdot E(\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\therefore \qquad E(\alpha)\cdot E(\beta) = E(\alpha+\beta)$$

Example 7 *If* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *and* $A\,(\text{adj } A) = \lambda I$, *then the value of* λ *is*

(a) 1 (b) 2
(c) 3 (d) 4

Solution (a) Since, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

and $$\text{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \qquad A\cdot(\text{adj } A) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad A\cdot(\text{adj } A) = I$$

But $$A\cdot(\text{adj } A) = \lambda I$$

$$\Rightarrow \qquad \lambda = 1$$

Example 8 *If* $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, *then* $(A^{-1})^3$ *is equal to*

(a) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (b) $\frac{1}{27}\begin{bmatrix} -1 & 26 \\ 0 & 27 \end{bmatrix}$
(c) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (d) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

Solution (a) Here, $|A| = 3$, $\text{adj } A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$,

$$\therefore \qquad A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \qquad (A^{-1})^3 = \frac{1}{27}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}^3$$

$$= \frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$$

Example 9 *If* $\text{adj}(B) = A$, $|P| = |Q| = 1$, *then* $\text{adj}(Q^{-1}BP^{-1})$ *is equal to*

(a) PQ (b) QAP
(c) PAQ (d) $PA^{-1}Q$

Solution (c) $\text{adj}(Q^{-1}BP^{-1}) = \text{adj } Q^{-1}\cdot \text{adj } B\cdot \text{adj } P^{-1}$

$$= \frac{P}{|P|} A \cdot \frac{Q}{|Q|}$$

$$= PAQ$$

Example 10 *If* $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, *then* $A^2 + A$ *is equal to* [**NCERT Exemplar**]

(a) $A+I$ (b) A^2+I
(c) $A(A+I)$ (d) None of these
(*where, I is* 3×3 *unit matrix*)

Solution (a) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$,

$$A^2 = A \times A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+(-1) & -1+0+(-1) \\ 2+2+0 & 0+1+3 & -2+3+3 \\ 0+2+0 & 0+1+1 & 0+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\therefore \qquad A + I = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A^2 + A = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1+0 & -2+(-1) \\ 4+2 & 4+1 & 4+3 \\ 2+0 & 2+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix}$$

$$A(A+1) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Example 11 *If A and B are square matrices of the same order such that AB = BA, then AB^n is equal to*

(a) A^nB (b) AB^n
(c) AB (d) B^nA
where, $n \notin N$

Solution (d) Given that, $AB = BA$ ...(i)

We want to prove that $AB^n = B^nA$ ...(ii)

For $n = 1$, Eq. (ii) is obviously true.

Let Eq. (ii) be true for a positive integer $n = m$.

i.e., $AB^m = B^mA$ ...(iii)

Then, for $n = m + 1$, $AB^{m+1} = A(B^mB) = (AB^m)B$

(associative law of matrix multiplication)

$= (B^mA)B$ [using Eq. (iii)]

$= B^m(AB) = B^m(BA)$ [using Eq. (i)]

$= (B^mB)A = B^{m+1}A$

Hence, by induction Eq. (ii) is true for all $n \in N$.

Example 12 *If* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ *is such that $A^2 = I$, then* [NCERT]

(a) $1 + \alpha^2 + \beta\gamma = 0$ (b) $1 - \alpha^2 + \beta\gamma = 0$
(c) $1 - \alpha^2 - \beta\gamma = 0$ (d) $1 + \alpha^2 - \beta\gamma = 0$

Solution (c)

(i) Find A^2 and use the given equation $A^2 = I$ as given in question.

(ii) Equate the respective elements of A^2 and I to find out the correct option.

Given, $A^2 = I$

$\therefore$ $AA = I$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \alpha\gamma - \gamma\alpha & \gamma\beta + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing the corresponding elements, we have

$$\alpha^2 + \beta\gamma = 1 \Rightarrow \alpha^2 + \beta\gamma - 1 = 0$$

$$\Rightarrow 1 - \alpha^2 - \beta\gamma = 1$$

Example 13 *If* $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$. *Then, the value of x is* [NCERT]

(a) $\pm 2\sqrt{3}$ (b) $\pm \sqrt{3}$ (c) ± 4 (d) $\pm 4\sqrt{3}$

Solution (d) Here, $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

$$\Rightarrow [x \quad -5 \quad -1] \left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} \right) = O$$

$$\Rightarrow [x \quad -5 \quad -1] \begin{bmatrix} x + 0 + 2 \\ 0 + 8 + 1 \\ 2x + 0 + 3 \end{bmatrix} = O$$

$$\Rightarrow [x(x+2) + (-5)(9) + (-1)(2x+3)] = O$$

$$\Rightarrow x^2 - 48 = 0 \Rightarrow x^2 = 48$$

$$\Rightarrow x = \pm\sqrt{48} = \pm 4\sqrt{3}$$

Example 14 *Find the values of x, y and z, if the matrix* $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ *satisfies the equation $A'A = I$ are* [NCERT]

(a) $\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}$ (b) $\pm\frac{1}{2}, \pm\frac{1}{6}, \pm\frac{1}{3}$
(c) $\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{5}}$ (d) *None of these*

Solution (a) Given, $A'A = I$

$$\therefore \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}' \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 + x^2 + x^2 & 0 + xy - xy & 0 - xz + xz \\ 0 + yx - yx & 4y^2 + y^2 + y^2 & 2yz - yz - yz \\ 0 - zx + zx & 2yz - yz - yz & z^2 + z^2 + z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing the corresponding elements, we have

$$2x^2 = 1, \quad 6y^2 = 1, \quad 3z^2 = 1 \Rightarrow x^2 = \frac{1}{2},$$

$$y^2 = \frac{1}{6}, \quad z^2 = \frac{1}{3}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}, \quad y = \pm\frac{1}{\sqrt{6}}, \quad z = \pm\frac{1}{\sqrt{3}}$$

Example 15 *If A and B are square matrices of the same order such that AB = BA, then $(AB)^n$ is equal to* [**NCERT**]

(a) AB (b) A^nB
(c) B^nA (d) A^nB^n
(*where*, $n \in N$)

Solutoin (*d*) Here, given that $AB = BA$ ...(i)

We want to prove that

$$(AB)^n = A^nB^n \quad \text{...(ii)}$$

For $n = 1$, Eq. (ii) is obviously true. [$\because$ from Eq. (i)]

Let Eq. (ii) be true for a positive integer $n = m$

i.e., $(AB)^m = A^mB^m$ ...(iii)

Then, for $n = m + 1$,

$$(AB)^{m+1} = (AB)^m (AB) = (A^mB^m)(AB) \quad \text{[using Eq. (iii)]}$$

$$= A^m (B^mA)B = A^m (AB^m)B$$

$$(\because AB^n = B^n A \text{ for all } n \in N \text{ whenever } AB = BA)$$

$$= (A^mA)(B^mB) = A^{m+1}B^{m+1}$$

Hence, by induction Eq. (ii) is true for all $n \in N$.

Example 16 *If* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, *then A^k is equal to*

where n is any positive integer. [**NCERT**]

(a) $\begin{bmatrix} 1+2^n & -4^n \\ n & 1-2^n \end{bmatrix}$ (b) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

(c) $\begin{bmatrix} 1+2n & -4n \\ 1 & 1-2n \end{bmatrix}$ (d) *None of these*

Solution (*b*) We are required to prove that for all $n \in N$.

$$P(n) = A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

Let $n = 1$

$$P(1) : A^1 = \begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \quad \text{...(i)}$$

$\therefore$ which is true for $n = 1$.

Let the result be true for $n = k$.

$$P(k) = A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \quad \text{...(ii)}$$

Let $n = k + 1$

Then, $P(k+1) : A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$

$$= \begin{bmatrix} 2k+3 & -4k+4 \\ k+1 & -2k-1 \end{bmatrix}$$

Now, LHS $= A^{k+1} = A^k A^1$

$$= \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

[using Eqs. (i) and (ii)]

$$= \begin{bmatrix} (1+2k)\cdot 3 + (-4k)\cdot 1 & (1+2k)\cdot(-4) + (-4k)\cdot(-1) \\ k\cdot 3 + (1-2k)\cdot 1 & k\cdot(-4) + (1-2k)\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$ whenever it is true for $n = k$. So, by principle of Mathematical Induction, it is true for all $n \in N$.

Example 17 *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *then $(aI + bA)^n$ is equal to*

where, I is the identity matrix of order 2 and $n \in N$. [**NCERT**]

(a) $a^nI + na^{-1}\, bA$ (b) $a^nI + na^{n-1}\, bA$

(c) $a^nI + nab^{n-1}A$ (d) *None of these*

Solution (*b*) Given, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We shall prove the result by principle of Mathematical Induction.

Let $P(n) : (aI + bA)^n = a^nI + na^{n-1}\, bA$

Putting $n = 1$

$$P(1) : (aI + bA)^1 = a^1I + 1\cdot a^{1-1}bA$$

$$aI + bA = aI + bA$$

which is true for $n = 1$.

Let the result be true for $n = k$.

$\therefore$ $P(k) : (aI + bA)^k = a^kI + k.\, a^{k-1}bA$ ...(i)

Now, we prove that the result is true for $n = k + 1$

$\therefore$ $P(k+1) : (aI + bA)^{k+1} = a^{k+1}I + (k+1)\cdot a^{k+1-1}bA$

Now, taking LHS $= (aI + bA)^{k+1}$

$$= (aI + bA)^k (aI + bA)$$

$$(\because a^{x+y} = a^x \times a^y)$$

$$= (a^kI + k.\, a^{k-1}bA)(aI + bA)$$

[$\because$ using Eq. (i) putting $(aI + bA)^k = a^kI + k\, a^{k-1}bA$]

$$= a^{k+1}I + ka^kIbA + a^kIbA + ka^{k-1}b^2A^2$$

$$= a^{k+1}I + ka^kb(IA) + a^kb((IA) + ka^{k-1}b^2(A^2)$$

$$\left(\because AI = A \text{ and } A^2 = AA \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0\right)$$

$$= a^{k+1}I + ka^kbA + a^kbA + ka^{k-1}b^2 \times 0$$

$$= a^{k+1}I + (k+1).\, a^kbA$$

Therefore, the result is true for $n = k + 1$ whenever it is true for $n = k$. So, by principle of Mathematical Induction, it is true for all $n \in N$.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Matrix and Algebra of Matrix

1. A 2×2 matrix, $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+2j)^2}{2}$ is [NCERT]

(a) $\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}_{2\times 2}$ (b) $\begin{bmatrix} 9/2 & 25/2 \\ 8 & 8 \end{bmatrix}_{2\times 2}$

(c) $\begin{bmatrix} 8 & 9/2 \\ 25/2 & 8 \end{bmatrix}_{2\times 2}$ (d) None of these

2. If $A = \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$,

$B = \frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$,

then $A - B$ is equal to [NCERT Exemplar]

(a) I (b) 0

(c) $2I$ (d) $\frac{1}{2}I$

3. If A and B are two matrices of the order $3 \times m$ and $3 \times n$, respectively and $m = n$, then the order of matrix $(5A - 2B)$ is [NCERT Exemplar]

(a) $m \times 3$ (b) 3×3

(c) $m \times n$ (d) $3 \times n$

4. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to [NCERT Exemplar]

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. If matrix $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = 1$, if $i \neq j = 0$, if $i = j$, then A^2 is equal to [NCERT Exemplar]

(a) I (b) A

(c) O (d) None of these

6. If A is matrix of order $m \times n$ and B is a matrix such that AB' and $B'A$ are both defined, then order of matrix B is [NCERT Exemplar]

(a) $m \times m$ (b) $n \times n$

(c) $n \times m$ (d) $m \times n$

7. If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to [NCERT Exemplar]

(a) A (b) $I - A$

(c) $I + A$ (d) $3A$

8. For any two matrices A and B, we have [NCERT Exemplar]

(a) $AB = BA$ (b) $AB \neq BA$

(c) $AB = O$ (d) None of these

9. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then which of the following is correct? [NCERT Exemplar]

(a) $(A + B) \cdot (A - B) = A^2 - B^2$

(b) $(A + B) \cdot (A - B) = A^2 - B^2$

(c) $(A + B) \cdot (A - B) = I$

(d) None of the above

10. If A is square matrix such that $A^2 = A$, then $(A + I)^3$ is equal to [NCERT Exemplar]

(a) $A + 1$ (b) $7A + I$

(c) $3A + I$ (d) $A - I$

11. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, then $(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is equal to [NCERT Exemplar]

(a) A

(b) I

(c) $I + A$

(d) None of the above

12. If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, then the value of k, so that $A^2 = kA - 2I$ is [NCERT]

(a) 1 (b) −1
(c) 2 (d) None of these

13. If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$, then the value of $A^3 - 6A^2 + 7A$ is equal to [NCERT Exemplar]

(a) I (b) 0 (c) $-2I$ (d) $2I$

14. If $2X + 3Y = \begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$ and $3x + 2y = \begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$, then the value of x and y is [NCERT]

(a) $\begin{bmatrix}-5 & 0\\ -1 & 4\end{bmatrix}$ and $\begin{bmatrix}2 & 0\\ -1 & -1\end{bmatrix}$

(b) $\begin{bmatrix}5/2 & 0\\ 1 & 0\end{bmatrix}$ and $\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}4 & 1\\ 3/2 & 2\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\ 1 & 5/2\end{bmatrix}$

(d) $\begin{bmatrix}\frac{2}{5} & \frac{-12}{5}\\ \frac{-11}{5} & 3\end{bmatrix}$ and $\begin{bmatrix}\frac{2}{5} & \frac{13}{5}\\ \frac{14}{5} & -2\end{bmatrix}$

15. If $\begin{bmatrix}x+y & 2x+z\\ x-y & 2z+w\end{bmatrix} = \begin{bmatrix}4 & 7\\ 0 & 10\end{bmatrix}$, then the values of x, y, z, w are

(a) 2, 2, 3, 4 (b) 2, 3, 1, 2
(c) 3, 3, 0, 1 (d) None of these

16. If $A=\begin{bmatrix}0 & 2\\ 3 & -4\end{bmatrix}$ and $kA=\begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}$, then the values of k, a, b are respectively

(a) − 6, − 12, − 18 (b) − 6, 4, 9
(c) − 6, − 4, − 9 (d) − 6, 12, 18

17. If $A=[a_{ij}]_{n\times n}$ be a diagonal matrix with diagonal element all different and $B=[b_{ij}]_{n\times n}$ be some another matrix. Let $AB=[c_{ij}]_{n\times n}$, then c_{ij} is equal to

(a) $a_{jj}\ b_{ij}$ (b) $a_{ii}\ b_{ij}$ (c) $a_{ij}\ b_{ij}$ (d) $a_{ij}\ b_{ji}$

18. Assuming that the sum and product given below are defined, which of the following is not true for matrices?

(a) $A + B = B + A$
(b) $AB = AC$ does not imply $B = C$
(c) $AB = O$ implies $A = O$ or $B = O$
(d) $(AB)' = B'A'$

19. If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and I is the unit matrix of order 2, then A^2 equals

(a) $4A - 3I$ (b) $3A - 4I$
(c) $A - I$ (d) $A + I$

20. If $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$, then

(a) $A^2 = I$ (b) $A^2 = -A$
(c) $B^2 = -I$ (d) $B^2 = I$

21. If $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$, if $Q_1=\frac{1}{2}(A+A')$ and $Q_2=\frac{1}{2}(A-A')$, then Q_1, Q_2 is equal to

(a) I_3 (b) O_3
(c) A (d) A^2

22. If $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$, then the sum of the elements of the matrix $3X - 4Y$ is equal to

(a) 14 (b) 16
(c) 12 (d) 25

23. For the matrix $A=\begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 2 & 1 & 0\end{bmatrix}$, which of the following is correct?

(a) $A^3 + 3A^2 - I = O$ (b) $A^3 - 3A^2 - I = O$
(c) $A^3 + 2A^2 - I = O$ (d) $A^3 - A^2 + I = O$

24. If $P=\begin{bmatrix}i & 0 & -i\\ 0 & -i & i\\ -i & i & 0\end{bmatrix}$ and $Q=\begin{bmatrix}-i & i\\ 0 & 0\\ i & -i\end{bmatrix}$, then PQ is equal to

(a) $\begin{bmatrix}-2 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}$ (b) $\begin{bmatrix}2 & -2\\ -1 & 1\\ -1 & 1\end{bmatrix}$

(c) $\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

25. A square matrix P satisfies $P^2 = I - P$, where I is the identity matrix. If $P^n = 5I - 8P$, then n is equal to

(a) 4 (b) 5
(c) 6 (d) 7

26. Let $A=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and

$B=\begin{bmatrix}\cos^2\theta & \sin\phi\cos\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$, then $AB = O$, if

(a) $\theta = n\phi,\ n = 0, 1, 2, \ldots$
(b) $\theta + \phi = n\pi,\ n = 0, 1, 2, \ldots$
(c) $\theta = \phi + (2n+1)\frac{\pi}{2},\ n = 0, 1, 2, \ldots$
(d) $\theta = \phi + n\frac{\pi}{2},\ n = 0, 1, 2, \ldots$

Types and Equality of Matrices

27. The matrix $P=\begin{bmatrix}0&0&4\\0&4&0\\4&0&0\end{bmatrix}$ is a [NCERT Exemplar]

(a) square matrix (b) diagonal matrix
(c) unit matrix (d) None of these

28. If $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then which one of the following holds for all $n\geq 1$, by the principle of Mathematical Induction

(a) $A^n = nA + (n-1)I$ (b) $A^n = 2^{n-1}\ A + (n-1)I$
(c) $A^n = nA - (n-1)I$ (d) $A^n = 2^{n-1}A - (n-1)I$

29. If $D = \text{diag}\ [d_1, d_2, d_3, \ldots, d_n]$, where $d_i \neq 0$, $\forall\ i = 1, 2, \ldots, n$, then D^{-1} is equal to

(a) O (b) I_n
(c) diag $[d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1}]$ (d) None of these

30. Which of the given values of x and y make the following pairs of matrices equal $\begin{bmatrix}3x+7&5\\y+1&2-3x\end{bmatrix}, \begin{bmatrix}0&y-2\\8&4\end{bmatrix}$? [NCERT]

(a) $x=\frac{-1}{3}, y=7$ (b) Not possible to find
(c) $y=7, x=\frac{-2}{3}$ (d) $x=\frac{-1}{3}, y=\frac{-2}{3}$

31. If $\begin{bmatrix}2x+y&4x\\5x-7&4x\end{bmatrix}=\begin{bmatrix}7&7y-13\\y&x+6\end{bmatrix}$, then the value of $x+y$ is [NCERT Exemplar]

(a) $x=3, y=1$ (b) $x=2, y=3$
(c) $x=2, y=4$ (d) $x=3, y=3$

32. If $A=\begin{bmatrix}a+4&3b\\8&-6\end{bmatrix}$ and $B=\begin{bmatrix}2a+2&b^2+2\\8&b^2-5b\end{bmatrix}$ such that $A=B$, then the values of a and b is [NCERT Exemplar]

(a) $a=2$ and $b=1,2$ (b) $a=1$ and $b=3$
(c) $a=-2$ and $b=\frac{1}{2}, -1$ (d) None of these

33. The non-zero values of x satisfying the matrix equation

$$x=\begin{bmatrix}2x&2\\3&x\end{bmatrix}+2\begin{bmatrix}8&5x\\4&4x\end{bmatrix}=2\begin{bmatrix}(x^2+8)&24\\(10)&6x\end{bmatrix}\text{ is}$$

[NCERT Exemplar]

(a) 1 (b) 2 (c) 3 (d) 4

34. If $A=\begin{bmatrix}\alpha&0\\1&1\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\5&1\end{bmatrix}$, then value of α for which $A^2=B$ is

(a) 1 (b) -1
(c) 4 (d) No real values

Trace, Transpose, Conjugate and Conjugate Transpose of a Matrix

35. If the trace of the matrix

$$A=\begin{bmatrix}x-1&0&2&5\\3&x^2-2&4&1\\-1&-2&x-3&1\\2&0&4&x^2-6\end{bmatrix}$$

is 0, then x is equal to

(a) $\{2, 3\}$ (b) $\{-2, -3\}$
(c) $\{-3, 2\}$ (d) $\{1, 2\}$

36. If $A+2B=\begin{bmatrix}2&-4\\1&6\end{bmatrix}$, $A'+B'=\begin{bmatrix}1&2\\0&-1\end{bmatrix}$, then A is equal to

(a) $\begin{bmatrix}0&4\\3&-8\end{bmatrix}$ (b) $\begin{bmatrix}1&-4\\-1&7\end{bmatrix}$ (c) $\begin{bmatrix}0&-4\\3&8\end{bmatrix}$ (d) $\begin{bmatrix}1&4\\1&-7\end{bmatrix}$

37. If a square matrix A is such that $AA^T = I = A^TA$, then $|A|$ is equal to

(a) 0 (b) ± 1
(c) ± 2 (d) None of these

38. Let $M=[a_{uv}]_{n\times n}$ be a matrix, where

$$a_{uv}=\sin(\theta_u-\theta_v)+i\cos(\theta_u-\theta_v),$$

then M is equal to

(a) $\overline{M}$ (b) $-\overline{M}$ (c) $\overline{M}^T$ (d) $-\overline{M}^T$

Special Types of Matrices

39. If A and B are matrices of same order, then $(AB'-BA')$ is a [NCERT Exemplar]

(a) skew-symmetric matrix (b) null matrix
(c) symmetric matrix (d) unit matrix

40. If A is a skew-symmetric matrix of order 3, then matrix A^3 is

(a) skew-symmetric matrix (b) symmetric matrix
(c) diagonal matrix (d) None of these

41. For what value of x, the matrix

$$\begin{bmatrix}3-x&2&2\\2&4-x&1\\-2&-4&-1-x\end{bmatrix}$$

is singular

(a) $x=1,2$ (b) $x=0,2$ (c) $x=0, 1$ (d) $x=0, 3$

42. If a, b and c are all different from zero such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, then the matrix

$$A=\begin{bmatrix}1+a&1&1\\1&1+b&1\\1&1&1+c\end{bmatrix}\text{ is}$$

(a) symmetric
(b) non-singular
(c) can be written as sum of a symmetric and a skew-symmetric matrix
(d) All of the above

43. If $w \neq 1$ is a cube root of unity, then

$$A = \begin{bmatrix} 1+2w^{100}+w^{200} & w^2 & 1 \\ 1 & 1+w^{100}+2w^{200} & w \\ w & w^2 & 2+w^{100}+2w^{200} \end{bmatrix}$$

(a) A is singular (b) $|A| \neq 0$
(c) A is symmetric (d) None of these

44. If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, then x is equal to
(a) 4 (b) 3
(c) -4 (d) -3

45. For any 2×2 matrix A, if $A\,(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ is equal to
(a) 0 (b) 10
(c) 20 (d) 100

46. The matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is
(a) unitary (b) orthogonal
(c) nilpotent (d) involutory

47. If $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ is the sum of a symmetric matrix B and skew-symmetric matrix, C, then B is
(a) $\begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & -2 \\ -2 & 5 & -2 \\ 2 & 2 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 6 & 6 & 7 \\ -6 & 2 & -5 \\ -7 & 5 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 6 & -2 \\ 2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

48. Matrix $A = \begin{bmatrix} 1 & 0 & -k \\ 2 & 1 & 3 \\ k & 0 & 1 \end{bmatrix}$ is invertible for
(a) $k = 1$ (b) $k = -1$
(c) $k = \pm 1$ (d) None of these

49. If A is a singular matrix, then $A\,\text{adj}(A)$
(a) is a scalar matrix
(b) is a zero matrix
(c) is an identity matrix
(d) is an orthogonal matrix

50. If $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$, then $A \cdot (\text{adj } A)$ is equal to
(a) A (b) $|A|$
(c) $|A|\,I$ (d) None of these

51. If A is a skew-symmetric matrix of odd order, then $|\text{adj } A|$ is equal to
(a) 0 (b) n
(c) n^2 (d) None of these

Adjoint of a Square Matrix

52. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = (\text{adj } A)$ and $C = 5A$, then $\frac{|\text{adj } B|}{|C|}$ is equal to
(a) 5 (b) 25 (c) -1 (d) 1

53. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then $\det[\text{adj } (\text{adj } A)]$ is equal to
(a) 12^4 (b) 13^4
(c) 14^4 (d) None of these

54. If A be a square matrix of order n and if $|A| = D$ and $|\text{adj } A| = D'$, then
(a) $DD' = D^2$ (b) $DD' = D'^{-1}$
(c) $DD' = D^n$ (d) None of these

55. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, then adj A is equal to
(a) A (b) A'
(c) $3A$ (d) $3A'$

Minor and Cofactor and Inverse of a Matrix

56. If $A = [a_{ij}]$ is a 4×4 matrix and C_{ij} is the cofactor of the element a_{ij} in $|A|$, then the expression $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$ is equal to
(a) 0 (b) -1
(c) 1 (d) $|A|$

57. If $A^2 - A + I = 0$, then the inverse of A is
(a) $A - I$ (b) $I - A$
(c) $A + 1$ (d) A

58. The element in the first row and third column of the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is
(a) -2 (b) 0 (c) 1 (d) 7

59. $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ and $A^{-1} = \lambda$ (adj A), then λ is equal to

(a) $-\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $-\frac{1}{3}$ (d) $\frac{1}{6}$

60. The multiplicative inverse of matrix $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ is

(a) $\begin{bmatrix} 4 & -1 \\ -7 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -1 \\ 7 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -7 \\ 7 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$

61. If $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $AX = B$, then X is equal to

(a) $[0 \quad 7]$ (b) $\frac{1}{3}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ (c) $\frac{1}{3}[5 \quad 7]$ (d) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

62. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. If $AX = B$, then X is equal to

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

63. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$G(y) = \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix}$,

then $[F(x) \cdot G(y)]^{-1}$ is equal to

(a) $F(x)\, G(-y)$ (b) $F^{-1}(x)\, G^{-1}(y)$

(c) $G^{-1}(y)\, F^{-1}(x)$ (d) $G(-y)\, F(-x)$

64. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then

(a) $a = 2,\ c = 1/2$ (b) $a = 1,\ c = -1$

(c) $a = -1,\ c = 1$ (d) $a = \frac{1}{2},\ c = \frac{1}{2}$

65. If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and $AB = I$, then B is equal to

(a) $\cos^2\frac{\theta}{2} \cdot A$ (b) $\cos^2\frac{\theta}{2} \cdot A^T$

(c) $\cos^2\theta \cdot I$ (d) $\sin^2\frac{\theta}{2} \cdot A$

Elementary Matrix

66. By elementary transformation method, the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ is

(a) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ (d) None of these

Solution of Simultaneous Linear Equations

67. The system of equations

$$(a\alpha + b)x + ay + bz = 0$$
$$(b\alpha + c)x + by + cz = 0$$
$$(a\alpha + b)y + (b\alpha + c)z = 0$$

has a non-trival solution, if

(a) a, b, c are in AP

(b) a, b, c are in GP

(c) a, b, c are in HP

(d) α is not a root of $ax^2 + 2bx + c = 0$

68. The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has no solution, if α is

(a) not -2 (b) 1

(c) -2 (d) either -2 or 1

69. The number of solutions of the system of equations $x_2 - x_3 = 1$, $-x_1 + 2x_3 = -2$ and $x_1 - 2x_2 = 3$ is

(a) zero (b) one

(c) two (d) infinite

70. The values of x, y, z in order, if the system of equations $3x + y + 2z = 3$, $2x - 3y - z = -3$ and $x + 2y + z = 4$ has unique solution, are

(a) 2, 1, 5 (b) 1, 1, 1

(c) 1, -2, -1 (d) 1, 2, -1

71. The values of α for which the system of equations

$$x + y + z = 1$$
$$x + 2y + 4z = \alpha$$
$$x + 4y + 10z = \alpha^2$$

is consistent, are

(a) 1, -2 (b) -1, 2

(c) 1, 2 (d) None of these

Miscellaneous

72. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then

(a) Adj A is a zero matrix (b) Adj $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(c) $A^{-1} = A$ (d) $A^2 = I$

73. Total number of possible matrices of order 3×3 with each entry 2 or 0 is [NCERT Exemplar]

(a) 9 (b) 27 (c) 81 (d) 512

74. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and $A^{-1} = A'$, then the value of α . [NCERT Exemplar]

(a) $\in R$ (b) $\in R \sim \left\{\frac{\pi}{2}\right\}$ (c) 0 (d) $\frac{\pi}{2}$

75. If I is a unit matrix of order 10, then the determinant of I is equal to

(a) 10 (b) 1 (c) 1/10 (d) 9

76. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then

(a) $A^3 = 9A$ (b) $A^3 = 27A$

(c) $A + A = A^2$ (d) A^{-1} exist

77. If A and B are square matrices of order 3, then

(a) $\text{adj}(AB) = \text{adj } A + \text{adj } B$

(b) $(A + B)^{-1} = A^{-1} + B^{-1}$

(c) $AB = 0 \Rightarrow |A| = 0$ or $|B| = 0$

(d) $AB = 0 \Rightarrow |A| = 0$ and $|B| = 0$

78. If $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

(a) $a = \cos 2\theta, b = \sin 2\theta$ (b) $a = 1, b = 0$

(c) $a = \sin 2\theta, b = \cos 2\theta$ (d) $a = 1, b = 1$

79. Let A be an orthogonal non-singular matrix of order n, then the determinant of matrix '$A - I_n$' *i.e.*, $|A - I_n|$ is equal to

(a) $|I_n - A|$ (b) $|A|$

(c) $|A|\,|I_n - A|$ (d) $(-1)^n |A|\,|I_n - A|$

Round II (Mixed Bag)

Only One Correct Option

1. If $A(\pi/6) = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $B(\pi/4) = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$[A(\pi/6) \cdot B(\pi/4)]^{-1}$ is equal to

(a) $\begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 1 & 0 \\ 3 & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (d) None of these

2. The system of equations $x + y + 3z = 1$, $2x + y + 2z = 3$ and $3x + 2y + 5z = 3$ have

(a) unique solution (b) infinite solution

(c) inconsistent (d) None of these

3. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then $\det(\text{adj}(A))$ is

(a) $(14)^2$ (b) $(13)^2$ (c) $(14)^3$ (d) $(13)^3$

4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ and $A^n = I$, then value of n is equal to

(a) 2 (b) 4 (c) 6 (d) 3

5. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2005} P$ is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

6. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $\det A^3 = 125$, then α is equal to

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 5

7. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix}$, then which of the following is incorrect.

(a) $A(\alpha, \beta)' = A(-\alpha, \beta)$
(b) $A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$
(c) adj $(A(\alpha, \beta) = e^{+\beta} A(-\alpha, -\beta)$
(d) $(A(\alpha, \beta)' = A(\alpha, -\beta)$

8. Let $\alpha = \frac{\pi}{5}$ and $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $B = A + A^2 + A^3 + A^4$ is

(a) singular (b) non-singular
(c) symmetric (d) $|B| = 1$

9. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where a, b and c are real positive number, $abc = 1$ and $A^T A = I$, then which of the following is/are true.

(a) $a + b + c = 5$
(b) $a^2 + b^2 + c^2 = 6$
(c) $ab + bc + ca = 2$
(d) $a^3 + b^3 + c^3 = 4$

10. If adj $B = A$ and $|P| = |Q| = 1$, then adj $(Q^{-1} BP^{-1})$ is equal

(a) APQ (b) PAQ
(c) B (d) A

11. If $i = \sqrt{-1}$, $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$, then which of the following matrix is idempotent

(a) $\begin{bmatrix} a & i \\ i & -b \end{bmatrix}$ (b) $\begin{bmatrix} b & i \\ i & a \end{bmatrix}$

(c) $\begin{bmatrix} a & i \\ i & b \end{bmatrix}$ (d) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

12. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and if $A^6 = KA - 205I$, then

(a) $K = 11$ (b) $K = 22$
(c) $K = 33$ (d) $K = 44$

13. Let A is a 3×3 matrix and $A = [a_{ij}]_{3\times 3}$. If for every column matrix X, if $X' AX = 0$ and $a_{23} = -2009$, then $a_{32} = \ldots$

(a) 2009 (b) −2009
(c) 0 (d) 2008

14. The system of simultaneous equations $kx + 2y - z = 1$ $(k-1)y - 2z = 2$ and $(k+2)z = 3$ has a unique solution, if k is equal to

(a) -2 (b) -1 (c) 0 (d) 1

15. Matrix M_r is defined as $M_r = \begin{bmatrix} r & r-1 \\ r-1 & r \end{bmatrix}$, $r \in N$ value of det $(M_1) + \det(M_2) + \det(M_3) + \ldots + \det(M_{2007})$ is

(a) 2007 (b) 2008
(c) 2008^2 (d) 2007^2

16. The matrices

$$P = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \text{ and } Q = \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \\ 12 & -5 & m \\ -8 & 1 & 5 \end{bmatrix}$$

are such that $PQ = I$, an identity matrix. Solving the equation $\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$, the value of y comes out to be -3. Then, the value of m is equal to

(a) 27 (b) 7
(c) -27 (d) -7

17. If $A = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & -2 \\ 0 & -2 \end{bmatrix}$, then $(A + B)^{-1}$ is equal to

(a) is a skew-symmetric matrix
(b) $A^{-1} + B^{-1}$
(c) does not exist
(d) None of the above

18. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix reflection in the line

(a) $x = 1$ (b) $x + y = 1$
(c) $y = 1$ (d) $x = y$

19. If I_n is the identity matrix of order n, then $(I_n)^{-1}$ is equal to

(a) does not exist (b) I_n
(c) 0 (d) nI_n

20. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & x \end{bmatrix}$ is an idempotent matrix, then x is equal to

(a) -5 (b) -1
(c) -3 (d) -4

21. If $A = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$, then A^{-1} is equal to

(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

22. Matrix A such that $A^2 = 2A - I$, where I is the identity matrix. Then, for $n \geq 2$, A^n is equal to

(a) $nA - (n-1)I$ (b) $nA - I$
(c) $2^{n-1}A - (n-1)I$ (d) $2^{n-1}A - I$

23. If $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, then $I + 2A + 3A^2 + \ldots\infty$

(a) $\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 2 \\ -3 & -8 \end{bmatrix}$

24. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $A^T A = AA^T = I$, then xy is equal to

(a) -1 (b) 1
(c) 2 (d) -2

25. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 1 \end{bmatrix}$, then α is equal to

(a) -2 (b) 5 (c) 2 (d) -1

26. If A is skew-symmetric and $B = (I - A)^{-1}(I + A)$, then B is

(a) singular (b) symmetric
(c) skew-symmetric (d) orthogonal

27. If $A = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then $I + A + A^2 + \ldots\infty$ is equal to

(a) $\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$ (b) $\frac{2}{7}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$
(c) $\frac{2}{7}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ (d) undefined

28. If $A = \begin{bmatrix} \frac{-1+i\sqrt{3}}{2i} & \frac{-1-i\sqrt{3}}{2i} \\ \frac{1+i\sqrt{3}}{2i} & \frac{1-i\sqrt{3}}{2i} \end{bmatrix}$, $i = \sqrt{-1}$ and $f(x) = x^2 + 2$, then $f(A)$ is equal to

(a) $\left(\frac{5 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\left(\frac{3 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $(2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

More Than One Correct Option

29. If $A = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then which of the following are true?

(a) $|A^T| = 1$ (b) $|A^{-1}| = 1$
(c) $A^{-1} = \text{adj } A$ (d) $|AA^T| = 10$

30. If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then

(a) $\alpha = \pm\frac{1}{\sqrt{2}}$ (b) $\alpha = \pm\frac{1}{\sqrt{3}}$
(c) $\beta = \pm\frac{1}{\sqrt{6}}$ (d) $\alpha = \pm\sqrt{3}\beta$

31. The values of λ and μ for which the system of equations.

$x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have no solution are

(a) $\lambda = 3$ (b) $\mu = 10$
(c) $\lambda \neq 3$ (d) $\lambda \neq 10$

Comprehension Based Questions

Passage

Directions (Q. Nos. 32-36) *A and B are two matrices of same order* 3×3, *where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 8 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 8 \\ 7 & 2 & 9 \end{bmatrix}$$

32. The value of adj (adj A) is equal to

(a) $-A$ (b) $4A$
(c) $8A$ (d) $16A$

33. The value of $|\text{adj (adj } A)|$ is equal to

(a) 9 (b) 1
(c) 25 (d) 81

34. The value of $|\text{adj } (B)|$ is equal to

(a) 24 (b) 24^2
(c) 24^3 (d) 8^2

35. The value of $|\text{adj } (AB)|$ is equal to

(a) 24 (b) 24^2
(c) 24^3 (d) 65

36. The value of $|(\text{adj (adj (adj (adj } A))))|$ is equal to

(a) 2^4 (b) 2^9
(c) 1 (d) 2^{19}

Assertion and Reason

Directions (Q. Nos. 37-50) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

37. **Statement I** The matrix $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix.

Statement II If A and B are orthogonal, then AB is also orthogonal.

38. **Statement I** If A is a skew-symmetric matrix of order 3×3, then $\det(A) = 0$ or $|A| = 0$.

Statement II If A is square matrix, then $\det(A) = \det(A') = \det(-A')$

39. **Statement I** The inverse of $A = \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}$ does not exist.

Statement II The matrix A is non-singular.

40. **Statement I** If a matrix of order 2×2, commutes with every matrix of order 2×2, then it is scalar matrix.

Statement II A scalar matrix of order 2×2 commutes with every 2×2 matrix.

41. **Statement I** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

Statement II $A = [a_{ij}]$ is square matrix such that $a_{ij} = 0, \forall\ i \neq j$, then A is called diagonal matrix.

42. Let A be a square matrix of order 3 satisfying $AA' = I$ to

Statement I $A'A = I$
Statement II $(AB)' = B'A'$

43. **Statement I** If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $\text{adj}(\text{adj}\, A) = A$.

Statement II $|\text{adj}(\text{adj}\, A)| = |A|^{(n-1)^2}$, A be n rowed non-singular matrix.

44. **Statement I** If $A = \begin{bmatrix} 2 & 1+2i \\ 1-2i & 7 \end{bmatrix}$, then det (A) is real.

Statement II If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, a_{ij} being complex numbers when $i \neq j$, then det (A) is always real.

45. **Statement I** If A is a matrix of order 2×2, then $|\text{adj}\, A| = |A|$ because

Statement II $|A| = |A^T|$

46. **Statement I** If $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ and $|A| = 0$ because

Statement II The value of the determinant of a skew-symmetric matrix is zero.

47. Let A and B be two symmetric matrices of order 3.

Statement I $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement II AB is symmetric matrix, if matrix multiplication of A with B is commutative.

48. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix.

Define $\text{Tr}(A)$ = sum of diagonal elements of A and $|A|$ = determinant of matrix A.

Statement I $\text{Tr}(A) = 0$

Statement II $|A| = 1$

49. Let A be 2×2 matrix.

Statement I $\text{adj}(\text{adj}\, A) = A$

Statement II $|\text{adj}\, A| = A$

50. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by Tr (A), the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement I If $A \neq I$ and $A \neq -I$, then $\det(A) = -1$

Statement II If $A \neq I$ and $A \neq -I$, then $\text{Tr}(A) \neq 0$

Previous Years' Questions

51. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A| = 4$, then α is equal to [JEE Main 2013]

(a) 4 (b) 11
(c) 5 (d) 0

52. Let P and Q be 3×3 matrices $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to [AIEEE 2012]

(a) -2 (b) 1 (c) 0 (d) -1

53. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $u_1 + u_2$ is equal to [AIEEE 2012]

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

54. The number of values of k for which the linear equations $4x + ky + 2z = 0$, $kx + 4y + z = 0$ and $2x + 2y + z = 0$ posses a non-zero solution is [AIEEE 2011]

(a) 2 (b) 1
(c) zero (d) 3

55. Consider the system of linear equations

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 1$$

The system has [AIEEE 2010]

(a) infinite number of solutions
(b) exactly 3 solutions
(c) a unique solution
(d) no solution

56. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is [AIEEE 2010]

(a) less than 4 (b) 5
(c) 6 (d) atleast 7

57. Let A be a square matrix all of whose entries are integers. Then, which one of the following is true? [AIEEE 2008]

(a) If $\det(A) = \pm 1$, then A^{-1} need not exist
(b) If $\det(A) = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers
(c) If $\det(A) \neq \pm 1$, then A^{-1} exists and all its entries are non-integers
(d) If $\det(A) = \pm 1$, then A^{-1} exists and all its entries are integers

58. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $\det(A^2) = 25$, then $|\alpha|$ is [AIEEE 2007]

(a) 1 (b) $\frac{1}{5}$
(c) 5 (d) 5^2

59. If A and B are 3×3 matrices such that $A^2 - B^2 = (A - B)(A + B)$, then [AIEEE 2006]

(a) either A or B is zero matrix
(b) either A or B is unit matrix
(c) $A = B$
(d) $AB = BA$

60. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $a, b, \in N$, then [AIEEE 2006]

(a) there exists exactly one B such that $AB = BA$
(b) there exists infinitely many B's such that $AB = BA$
(c) there cannot exist any B such that $AB = BA$
(d) there exist more than but finite number of B's such that $AB = BA$

Answers

Check Point 1

1. (d)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (b)

Check Point 2

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (d)	**10.** (a)

Check Point 3

1. (a)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (c)

Round I

1. (b)	**2.** (d)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (b)
11. (c)	**12.** (a)	**13.** (c)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (c)	**30.** (b)
31. (b)	**32.** (a)	**33.** (d)	**34.** (d)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (d)	**39.** (a)	**40.** (a)
41. (d)	**42.** (d)	**43.** (a)	**44.** (c)	**45.** (b)	**46.** (c)	**47.** (a)	**48.** (c)	**49.** (b)	**50.** (c)
51. (a)	**52.** (d)	**53.** (c)	**54.** (c)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (d)	**59.** (a)	**60.** (d)
61. (b)	**62.** (d)	**63.** (d)	**64.** (b)	**65.** (b)	**66.** (a)	**67.** (b)	**68.** (c)	**69.** (a)	**70.** (d)
71. (c)	**72.** (c)	**73.** (d)	**74.** (a)	**75.** (b)	**76.** (a)	**77.** (c)	**78.** (b)	**79.** (c)	

Round II

1. (a)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (d)	**10.** (b)
11. (c)	**12.** (d)	**13.** (a)	**14.** (b)	**15.** (d)	**16.** (d)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (c)
21. (a)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (d)	**27.** (c)	**28.** (d)	**29.** (a,b,c)	**30.** (a,c,d)
31. (a,b)	**32.** (a)	**33.** (b)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (c)	**38.** (b)	**39.** (c)	**40.** (a)
41. (c)	**42.** (b)	**43.** (c)	**44.** (a)	**45.** (b)	**46.** (a)	**47.** (c)	**48.** (a)	**49.** (d)	**50.** (d)
51. (b)	**52.** (b)	**53.** (d)	**54.** (b)	**55.** (c)	**56.** (d)	**57.** (d)	**58.** (b)	**59.** (d)	**60.** (d)

the Guidance

Round I

1. Here, $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$, where $a_{ij}=\dfrac{(i+2j)^2}{2}$

$$a_{11}=\frac{(1+2)^2}{2}=\frac{9}{2},\ a_{12}=\frac{(1+4)^2}{2}=\frac{25}{2},$$

$$a_{21}=\frac{(2+2)^2}{2}=8$$

and $$a_{22}=\frac{(2+4)^2}{2}=18$$

Hence, the required matrix is $A=\begin{bmatrix} 9/2 & 25/2 \\ 8 & 8 \end{bmatrix}_{2\times 2}$

2. Given that, $$A=\frac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$$

and $$B=\frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix},$$

then $$A-B=\frac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$$

$$-\frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$=\frac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x)+\cos^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right)-\tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right)-\sin\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x)+\tan^{-1}(\pi x) \end{bmatrix}=\frac{1}{\pi}\begin{bmatrix} \dfrac{\pi}{2} & 0 \\ 0 & \dfrac{\pi}{2} \end{bmatrix}$$

$$\left\{\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2} \text{ and } \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right\}$$

$$=\begin{bmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\frac{1}{2}I$$

3. Matrix B is of the order $3 \times n$. Therefore, matrix 2B is also of the same order.

Matrix A is of the order $3 \times m$ *i.e.*, $3 \times n$ $(\because m = n)$

Therefore, matrix $5A$ is also of the same order.

Now, both the matrices $5A$ and $2B$ are of the order $3 \times n$.

Thus, matrix $(5A - 2B)$ is well defined and is of the order $3 \times n$.

4. Given that, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. Given that,

$$a_{ij} = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases},$$

then $A = [a_{ij}]_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

6. Given that the order of $A = m \times n$

Since, AB' and $B'A$ are both defined.

Then, for AB', B' must be of the order like as $n \times p$ ...(i)

But for $B'A$, B' must be of the order like as $q \times m$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

the required order of B' is $n \times m$.

$\Rightarrow$ Order of matrix B is $m \times n$.

7. Given that, $A^2 = I$ ...(i)

Now, we have

$$(A - I)^3 + (A + I)^3 - 7A = (A)^3 - I^3 - 3AI(A - I) + (A)^3 + I^3 + 3A \cdot I(A + I) - 7A$$

$$= 2(A)^3 - 3A(A - I) + 3A(A + I) - 7A$$

$$= 2(A^2 \cdot A) - 3A(A - I - A - I) - 7A$$

$$= 2(I \cdot A) - 3A(-2I) - 7A \quad \text{[from Eq. (i)]}$$

$$= 2A + 6A \cdot I - 7A$$

$$= 2A + 6A - 7A = 8A - 7A = A$$

8. (a) $AB = BA$ is only possible when A and B are both square matrices.

(b) $AB \neq BA$ is not always possible condition because some times $AB = BA$ is possible when A and B are both square matrices.

(c) $AB = 0$ is not always true, this condition is possible when ($A = 0$ or $B = 0$) or (A and B both are zero matrix) or (both are non-zero matrices)

$\therefore$ Option (d) is correct.

9. Given that,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and $B^2 = B \cdot B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

and $(A + B)(A - B) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence, $(A + B)(A - B) \neq A^2 - B^2$

10. Given that, $A^2 = A$ ...(i)

Now, we have

$$(I + A)^3 = (I)^3 + (A)^3 + 3I \cdot A(I + A)$$

$$= I + A^2 \cdot A + 3A(I + A) \quad (\because I^3 = I, I \cdot A = A)$$

$$= I + A \cdot A + 3A(I + A) \quad (\because A^2 = A)$$

$$= I + A^2 + 3(A \cdot I + A^2)$$

$$= I + A + 3(A + A) \quad (\because A^2 = A)$$

$$= I + A + 3(2A) = I + A + 6A = 7A + I$$

11. Here, $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$, where $t = \tan\left(\frac{\alpha}{2}\right)$

Now, $$\cos\alpha = \frac{1 - \tan^2\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \frac{1 - t^2}{1 + t^2}$$

and $$\sin\alpha = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \frac{2t}{1 + t^2}$$

Now, we have

$$(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -t \\ +t & 0 \end{pmatrix}\right] \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-t^2+2t^2}{1+t^2} & \frac{-2t+t(1-t^2)}{1+t^2} \\ \frac{-t(1-t^2)+2t}{1+t^2} & \frac{2t^2+1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+t^2}{1+t^2} & \frac{-2t+t-t^3}{1+t^2} \\ \frac{-t+t^3+2t}{1+t^2} & \frac{2t^2+1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+t^2}{1+t^2} & \frac{-t(1+t^2)}{1+t^2} \\ \frac{t(1+t^2)}{1+t^2} & \frac{1+t^2}{1+t^2} \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \quad \text{... (i)}$$

Now, $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+1 & -t+0 \\ t+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \quad \text{...(ii)}$$

On putting the value of t in both equations, we get

$$\begin{bmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{bmatrix}$$

$$\therefore \quad (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = I + A$$

12. Given, $A^2 = kA - 2I \Rightarrow AA = kA - 2I$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$3k - 2 = 1 \Rightarrow k = 1$$
$$-2k = -2 \Rightarrow k = 1$$
$$4k = 4 \Rightarrow k = 1$$
$$-4 = -2k - 2 \Rightarrow k = 1$$

Hence, $k = 1$

13. $A^2 = A \times A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$+ 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix}$$

$$+ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore \quad A^3 - 6A^2 + 7A = -2I$$

14. Given, $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ ...(i)

and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$ ...(ii)

On multiplying Eq. (i) by 2, Eq. (ii) by 3 and then subtracting, we get

$$2(2X + 3Y) - 3(3X + 2Y) = 2\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$$

$$\Rightarrow 4X + 6Y - 9X - 6Y = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix}$$

$$\Rightarrow -5X = \begin{bmatrix} 4-6 & 6+6 \\ 8+3 & 0-15 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix}$$

$$\Rightarrow X = -\frac{1}{5}\begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 & -12/5 \\ -11/5 & 3 \end{bmatrix}$$

Then, from Eq. (i),

$$3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2X = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2\begin{bmatrix} 2/5 & -12/5 \\ -11/5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\frac{4}{5} & 3+\frac{24}{5} \\ 4+\frac{22}{5} & 0-6 \end{bmatrix} = \begin{bmatrix} 6/5 & 39/5 \\ 42/5 & -6 \end{bmatrix}$$

$$\therefore \quad Y = \frac{1}{3}\begin{bmatrix} 6/5 & 39/5 \\ 42/5 & -6 \end{bmatrix} = \begin{bmatrix} 2/5 & 13/5 \\ 14/5 & -2 \end{bmatrix}$$

15. Since, $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$

$\Rightarrow \quad x+y=4 \quad$ ...(i)

$x-y=0 \quad$ ...(ii)

$2x+z=7 \quad$ ...(iii)

and $\quad 2z+w=10 \quad$ ...(iv)

On solving these equations, we get

$$x=2, \quad y=2, \quad z=3, \quad w=4$$

16. $\because \quad kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$\Rightarrow \quad k\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$\Rightarrow \quad 2k=3a, \quad 3k=2b, \quad -4k=24$

$\Rightarrow \quad a=\frac{2k}{3}, \quad b=\frac{3k}{2}, \quad k=-6$

$\therefore \quad a=-4, \quad b=-9, \quad k=-6$

17. $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$ (in general)

And in a diagonal matrix non-diagonal elements are zero *i.e.*,

$$a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ a_{ii}, & \text{if } i = j \end{cases}$$

So, $\quad c_{ij} = a_{ii}\, b_{ij}$

18. If $AB = O$, then A and B may be equal to O individually. It is not necessary in any condition.

19. $\because \quad A^2 = A \cdot A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

and $\quad 4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$\therefore \quad A^2 = 4A - 3I$

20. $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$B^2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

21. $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$Q_1 = \frac{1}{2} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, $\quad Q_2 = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$\therefore \quad Q_1 Q_2 = O_3$

22. $X + Y + X - Y = 2X$

$$\therefore 2X = (X+Y) + (X-Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \therefore X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again, $\quad 2Y = (X+Y) - (X-Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$3x - 4y = 3\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} - 4\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -1 & 8 \end{bmatrix}$$

Sum of element of $3x - 4y = 7 + 0 - 1 + 8 = 14$

23. Given, $\quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 7 & 9 & 3 \\ 15 & 19 & 6 \\ 9 & 12 & 4 \end{bmatrix}$

Hence, $\quad A^3 - 3A^2 - I = O$

24. Since, $\quad P = \begin{bmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} -i & i \\ 0 & 0 \\ i & -i \end{bmatrix}$

$\therefore \quad PQ = \begin{bmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{bmatrix}\begin{bmatrix} -i & i \\ 0 & 0 \\ i & -i \end{bmatrix}$

$$= \begin{bmatrix} -i^2 - i^2 & i^2 + i^2 \\ i^2 & -i^2 \\ i^2 & -i^2 \end{bmatrix} = \begin{bmatrix} 1+1 & -1-1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$$

25. $\because \quad P^3 = P(I - P) \qquad (\because P^2 = I - P)$

$= PI - P^2 = PI - (I - P) = P - I + P = 2P - I$

Now, $\quad P^4 = P \cdot P^3$

$\Rightarrow \quad P^4 = P(2P - I)$

$\Rightarrow \quad P^4 = 2P^2 - P \Rightarrow P^4 = 2I - 2P - P$

$\Rightarrow \quad P^4 = 2I - 3P$ and $P^5 = P(2I - 3P)$

$\Rightarrow \quad P^5 = 2P - 3(I - P)$

$\Rightarrow \quad P^5 = 5P - 3I$

Also, $\quad P^6 = P(5P - 3I)$

$\Rightarrow \quad P^6 = 5P^2 - 3P \Rightarrow P^6 = 5(I - P) - 3P$

$\Rightarrow \quad P^6 = 5I - 8P$

So, $\quad n = 6$

Alternate Solution

$\because \quad P^n = 5I - 8P = 5(I - P) - 3P$

$= P(5P - 3I) \qquad (\because P^2 = I - P)$

$= P(2P - 3P^2) = P^2(2I - 3P)$

$= P^2[2(I - P) - P]$

$= P^2[2P^2 - P] = P^3[2P - I]$

$= P^4[I - P] = P^4 \cdot P^2 = P^6$

$\Rightarrow \quad n = 6$

26. $AB = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \sin\theta\cos\phi\cos\theta\sin\phi & \cos^2\theta\sin\phi\cos\phi + \sin^2\phi\sin\theta\cos\theta \\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\sin\phi\cos\phi & \cos\theta\sin\theta\sin\phi\cos\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \sin\phi\cos\theta\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$

$\because \quad AB = O$

$\Rightarrow \quad \cos(\theta - \phi) = 0$

$\Rightarrow \quad \cos(\theta - \phi) = \cos(2n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{2} + \phi, \quad$ where $n = 0, 1, 2, \ldots$

27. Given matrix, $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a square and scalar matrix.

28. $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\therefore \quad A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

$nA = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix}, (n-1)I = \begin{bmatrix} n-1 & 0 \\ n & n-1 \end{bmatrix}$

$nA - (n-1)I = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$

29. Given, $\quad D = \text{diag}\,(d_1, d_2, d_3, \ldots, d_n)$

$\Rightarrow \quad D^{-1} = \text{diag}\,(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$

30. If two matrices are equal, then their corresponding elements are equal. So, to find the values of x and y, equate the elements and frame the equations and solve them.

According to the question,

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

By definition of equality of matrices, we have

$3x + 7 = 0 \quad \ldots(i)$

$5 = y - 2 \quad \ldots(ii)$

$y + 1 = 8 \quad \ldots(iii)$

$2 - 3x = 4 \quad \ldots(iv)$

From Eq. (ii), $\quad y = 7$

From Eq. (i), $\quad 3x + 7 = 0 \Rightarrow x = \dfrac{-7}{3}$

From Eq. (iv), $\quad 2 - 3x = 4 \quad \Rightarrow x = \dfrac{-2}{3}$

Since, x can have only one value at a time. Hence, it is not possible to find the values of x and y for which the given matrices are equal.

So, correct option is (b).

Note Sometimes on solving an equation, we get more than one values of the variables. This means that such a matrix does not exist.

31. Given that, $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

By equality of two matrices,

$4x = x + 6$

$\Rightarrow \quad 3x = 6 \Rightarrow x = 2$ and $2x + y = 7$

$\Rightarrow \quad 2(2) + y = 7$

$\Rightarrow \quad y = 7 - 4$

$\therefore \quad y = 3$

32. By condition,

$A = B$

$$\begin{bmatrix} a+4 & 3b \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

By equality of two matrices,

$a + 4 = 2a + 2$ and $3b = b^2 + 2$

$\Rightarrow \quad a = 2$ and $b^2 - 3b + 2 = 0$

$\Rightarrow \quad b^2 - 2b - b + 2 = 0$

$\Rightarrow \quad b(b-2) - 1(b-2) = 0$

$\Rightarrow \quad (b-2)(b-1) = 0$

$\therefore \quad b = 1, 2$

33. Given that,

$$x\begin{bmatrix}2x & 2\\ 3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\ 4 & 4x\end{bmatrix}=2\begin{bmatrix}(x^2+8) & 24\\ 10 & 6x\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2x^2 & 2x\\ 3x & x^2\end{bmatrix}+\begin{bmatrix}16 & 10x\\ 8 & 8x\end{bmatrix}=\begin{bmatrix}(2x^2+16) & 48\\ 20 & 12x\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2x^2+16 & 2x+10x\\ 3x+8 & x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

By equality of two matrices,

$$3x+8=20$$
$$\Rightarrow \quad 3x=12$$
$$\Rightarrow \quad x=4$$

34. Given that, $A=\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}$

$$\Rightarrow \quad A^2=\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\alpha^2+0 & 0+0\\ \alpha+1 & 0+1\end{bmatrix}=\begin{bmatrix}\alpha^2 & 0\\ \alpha+1 & 1\end{bmatrix}$$

Also, $B=A^2$ (given)

$$\Rightarrow \quad \begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}=\begin{bmatrix}\alpha^2 & 0\\ \alpha+1 & 1\end{bmatrix}$$

Clearly, this is not satisfied by any real value of α.

35. Trace of matrix is defined as

$$\sum_{i=1}^{n} a_{ii}=2x^2+2x-12=0 \Rightarrow x=-3,2$$

36. $A+B=(A'+B')'=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$

$$\therefore \quad A=2(A+B)-(A+2B)=\begin{bmatrix}2 & 0\\ 4 & -2\end{bmatrix}-\begin{bmatrix}2 & -4\\ 1 & 6\end{bmatrix}=\begin{bmatrix}0 & 4\\ 3 & -8\end{bmatrix}$$

37. Given matrix A is a square matrix

and $AA^T=I=A^TA$

$$\Rightarrow \quad |AA^T|=|I|=|A^TA|$$
$$\Rightarrow \quad |A||A^T|=1=|A^T||A|$$
$$\Rightarrow \quad |A|^2=1 \qquad (\because A\cdot A^T=|A|^2)$$
$$\Rightarrow \quad |A|=\pm 1$$

38. Given, $M=[a_{uv}]_{n\times n}=[\sin(\theta_u-\theta_v)+i\cos(\theta_u-\theta_v)]$

$$\Rightarrow \quad \overline{M}=[\sin(\theta_u-\theta_v)-i\cos(\theta_u-\theta_v)]$$
$$\Rightarrow \quad (\overline{M})^T=[\sin(\theta_v-\theta_u)-i\cos(\theta_v-\theta_u)]$$
$$=[-\sin(\theta_u-\theta_v)-i\cos(\theta_u-\theta_v)]$$
$$=-[\sin(\theta_u-\theta_v)+i\cos(\theta_u-\theta_v)]=-M$$

39. Let $P=AB'-BA'$ be any matrix.

Now, $P'=(AB'-BA')'=(AB)'-(BA')'$

$$=(B')'(A')-(A')'(B') \qquad [\because (AB)'=B'A']$$
$$[\therefore (A')'=A']$$
$$=-[AB'-BA']=-P$$

$\because \quad P'=-P$

P is a skew-symmetric matrix.

40. Since, $A'=-A$

$\therefore \quad A^3=AAA$ and $(A^3)'=A'A'A'=-A^3$

Hence, matrix A^3 is a skew-symmetric matrix.

41. Since, the given matrix is singular,

$$\Rightarrow \quad \begin{vmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ -2 & -4 & -1-x\end{vmatrix}=0$$

$$R_2\to R_2+R_3 \Rightarrow \begin{vmatrix}3-x & 2 & 2\\ 2 & -x & -x\\ -2 & -4 & -1-x\end{vmatrix}=0$$

$$\Rightarrow \quad x\begin{vmatrix}3-x & 2 & 2\\ 0 & 1 & 1\\ 2 & 4 & 1+x\end{vmatrix}=0$$

$$\Rightarrow \quad x(x-3)^2=0 \Rightarrow x=0,3$$

42. Note that A is symmetric. Next, we have

$$|A|=abc\begin{vmatrix}1+\frac{1}{a} & \frac{1}{b} & \frac{1}{c}\\ \frac{1}{a} & 1+\frac{1}{b} & \frac{1}{c}\\ \frac{1}{a} & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix}$$

$$=abc\begin{vmatrix}1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c}\\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c}\\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix} \quad (C_1\to C_1+C_2+C_3)$$

$$=abc\begin{vmatrix}1 & \frac{1}{b} & \frac{1}{c}\\ 1 & 1+\frac{1}{b} & \frac{1}{c}\\ 1 & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix}=abc\begin{vmatrix}1 & \frac{1}{b} & \frac{1}{c}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=abc\neq 0$$

43. We have, $A=\begin{vmatrix}1+2w+w^2 & w^2 & 1\\ 1 & 1+w^2+2w & w\\ w & w^2 & 2+w+2w^2\end{vmatrix}$

$$|A|=w\begin{vmatrix}w & w & 1\\ 1 & 1 & w\\ w & w & -w\end{vmatrix}=0$$

Hence, A is singular.

44. A is symmetric $\Rightarrow A^T=A$

$$\Rightarrow \quad \begin{bmatrix}3 & 2x+3\\ x-1 & x+2\end{bmatrix}=\begin{bmatrix}3 & x-1\\ 2x+3 & x+2\end{bmatrix}$$

$$\Rightarrow \quad x-1=2x+3 \Rightarrow x=-4$$

45. Since, $A[\text{adj}(A)]=|A|\cdot I$

$$\therefore \quad \begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=|A|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}|A| & 0\\ 0 & |A|\end{bmatrix}$$

$$\therefore \quad |A|=10$$

46. We have, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore$ Matrix A is nilpotent.

47. Given, $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$

and symmetric matrix, $B = \frac{A + A'}{2}$

$$\therefore \quad B = \frac{1}{2}\left\{\begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$$

48. Since, A is invertible.

$$\therefore \quad |A| \neq 0 \Rightarrow \begin{vmatrix} 1 & 0 & -k \\ 2 & 1 & 3 \\ k & 0 & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad 1(1-0) + k(0-k) \neq 0$$

$$\Rightarrow \quad 1 - k^2 \neq 0 \Rightarrow k \neq \pm 1$$

49. Given that, A is a singular matrix.

$$\therefore \quad |A| = 0$$

$$\because \quad A \text{ adj } A = |A| = 0$$

$\therefore$ A adj A is a zero matrix.

50. Given, $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

$$\Rightarrow \quad |A| = 1$$

$$\therefore \quad A \text{ adj } (A) = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\, I$$

51. Since, A is a skew-symmetric matrix. Therefore,

$$A^T = -A \Rightarrow |A^T| = |-A|$$

$$\Rightarrow \quad |A| = (-1)^n |A|$$

Also, n is odd.

$$\therefore \quad 2|A| = 0$$

$$\Rightarrow \quad |A| = 0$$

Thus, $|\text{adj } A| = |A|^2 = 0$

52. Since, $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$

$$\therefore \quad B = \text{adj}(A) = \begin{bmatrix} 3 & 1 & 1 \\ -6 & -2 & 3 \\ -4 & -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj}(B) = \begin{bmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \quad |\text{adj } (B)| = \begin{vmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{vmatrix} = 625$$

Given that, $C = 5A$

$$\Rightarrow \quad |C| = 5^3 |A| = 125 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{vmatrix} = 625$$

Hence, $\frac{|\text{adj}(B)|}{|C|} = \frac{625}{625} = 1$

53. $|A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 14$

$$\because \quad [\text{adj}(\text{adj } A)] = |A|^{n-2} A = 14^{3-2} A = 14A$$

$$\therefore \quad |\text{adj}(\text{adj } A)| = |14A| = 14^3 |A| = 14^4$$

54. $|A|\,|\text{adj}(A)| = |A|^n$ for order n

$$\Rightarrow \quad DD' = D^n$$

55. Given, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$\therefore \quad B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj } A = (B)' = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = 3A'$$

56. $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} = |A|$

57. $A^2 - A + I = 0 \Rightarrow I = A - A^2 \Rightarrow I = A(I - A)$

$$\Rightarrow \quad A^{-1} = I - A$$

58. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

and $\quad \text{adj } A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $\quad A^{-1} = \frac{1}{|A|} \text{adj } A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

So, required element $= A_{13}^{-1} = 7$

59. Given that, $\quad A^{-1} = \lambda \, (\text{adj } A)$

On comparing with $A^{-1} = \frac{1}{|A|} \text{adj}(A)$, we get

$$\lambda = \frac{1}{|A|}$$

Now, $\quad |A| = \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6 \Rightarrow \lambda = -\frac{1}{6}$

60. Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

Let B be the multiplicative inverse of A, then

$$AB = I \Rightarrow B = A^{-1} = \frac{1}{8-7}\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

61. Given, $\quad A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore \quad A^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Also, $\quad AX = B \quad$ (given)

$\Rightarrow \quad X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3+2 \\ 6+1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

62. Since, $\quad A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

Now, $\quad |A| = 1(0-2) + 1(2-3) + 2(4-0) = 5$

$\therefore \quad A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}$

Now, $\quad A^{-1}B = \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

63. $\{F(x) \cdot G(y)\}^{-1} = G^{-1}(y) \cdot F^{-1}(x)$, if $|F(x)| \neq 0, |G(x)| \neq 0$

Here, $\quad |f(x)| = 1, |G(y)| = 1$

Also, $\quad F^{-1}(x) = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(-x)$

and $\quad G^{-1}(y) = \begin{bmatrix} \cos y & 0 & -\sin y \\ 0 & 1 & 0 \\ \sin y & 0 & \cos y \end{bmatrix} = G(-y)$

$\Rightarrow \quad [F(x) \cdot G(y)]^{-1} = G^{-1}(y) F^{-1}(x) = G(-y) \; F(-x)$

64. We must have $AA^{-1} = I$

$(3,1)$th entry of $AA^{-1} = I = (1,3)$th entry of AA^{-1}

$\Rightarrow \quad 3 \times \frac{1}{2} + a \times (-4) + 1 \times \frac{5}{2} = 0 = 0 \times \frac{1}{2} + 1 + c + 2 \times \frac{1}{2}$

$\Rightarrow \quad -4a + 4 = 0$ and $c + 1 = 0$

$\Rightarrow \quad a = 1$ and $c = -1$

65. Given, $\quad AB = I \;\Rightarrow\; B = A^{-1}$

Now, $\quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}}{1 + \tan^2\frac{\theta}{2}} = \frac{A^T}{\sec^2\frac{\theta}{2}} = \cos^2\frac{\theta}{2} A^T$

66. Let $\quad A = IA$

$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -3 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A$$

Applying $R_2 \to -R_2$ and $R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} A$$

$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

67. The given system equations will have a non-trivial solution, if

$$\Delta = \begin{vmatrix} a\alpha + b & a & b \\ b\alpha + c & b & c \\ 0 & a\alpha + b & b\alpha + c \end{vmatrix} = 0$$

$\Rightarrow \quad -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

68. For no solution or infinitely many solutions

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0 \Rightarrow \alpha = 1, \alpha = -2$$

But for $\alpha = 1$, clearly there are infinitely many solutions and when we put $\alpha = -2$ in given system of equations and adding them together LHS $\neq$ RHS $\Rightarrow$ No solutions.

69. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix}$$

$$= 0 + 2 - 2 = 0$$

$$\Rightarrow \quad |A| = 0$$

Now, $(\text{adj } A)\, B = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 4-4+6 \\ 2-2+3 \\ 2-2-3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} \neq O$$

$\therefore$ This system of equation is inconsistent, so it has no solution.

70. Given equations are

$$3x + y + 2z = 3 \quad \ldots(i)$$
$$2x - 3y - z = -3 \quad \ldots(ii)$$
and
$$x + 2y + z = 4 \quad \ldots(iii)$$

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$= 3(-3+2) - 1(2+1) + 2(4+3)$$
$$= -3 - 3 + 14 = 8$$

$$\text{adj } A = \begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}^T = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Now, $X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow x = 1, \quad y = 2, \quad z = -1$$

71. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

For consistent, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = 0$

$$\Rightarrow \quad 1(20-16) - 1(10-4) + 1(4-2) = 0$$
$$\Rightarrow \quad 4 - 6 + 2 = 0$$
$$\Rightarrow \quad 0 = 0 \text{ and } (\text{adj } A)\, B = O$$
$$\Rightarrow \quad \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad \begin{bmatrix} 4 - 6\alpha + 2\alpha^2 \\ -6 + 9\alpha - 3\alpha^2 \\ 2 - 3\alpha + \alpha^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad 2\alpha^2 - 6\alpha + 4 = 0, \quad -3\alpha^2 + 9\alpha - 6 = 0$$

and $\alpha^2 - 3\alpha + 2 = 0$

Now, $2\alpha^2 - 6\alpha + 4 = 0$

$$\Rightarrow \quad (2\alpha - 2)(\alpha - 2) = 0 \Rightarrow \alpha = 1, 2$$

Similarly, from other equations we also get the same value.

72. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $|A| = -1$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Hence, $A^{-1} = A$

73. As order of 3×3 matrix contains 9 elements. Each element can be selected in 2 ways (it can be either 0 or 2).

Hence, all the nine entries can be chosen in $2^9 = 512$ ways.

(by the multiplication principle)

$\therefore$ Required number of matrices is 512.

74. Given that, $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore$ Transpose of A *i.e.*, $A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ ... (i)

$$\text{adj}(A) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$|A| = \cos^2\alpha + \sin^2\alpha = 1$$

$$\therefore \quad A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}{1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$A' = A^{-1}, \forall\, \alpha \in R$$

75. Determinant of unit matrix of any order is 1.

76. Given that,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3A$$

$$\therefore \quad A^3 = A \cdot A^2 = A \cdot 3A = 3A^2 = 3 \cdot 3A = 9A$$

77. $AB = 0 \Rightarrow |A| = 0$ or $|B| = 0$

78. $AA^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore\ a = 1, b = 0$

79. $\because A \cdot A^T = I_n \Rightarrow A - I_n = A - AA^T = A(I_n - A^T)$

$$\Rightarrow \quad |A - I_n| = |A(I_n - A^T)|$$
$$= |A|\,|I_n - A^T| = |A|\,|I_n - A|$$

Round II

1. $\because \quad (AB)^{-1} = B^{-1} \cdot A^{-1}$

$\therefore \quad [A(\pi/6) \cdot B(\pi/4)]^{-1} = [B(\pi/4)]^{-1}[A(\pi/6)]^{-1}$

It is known that $[A(\pi/6)]^{-1} = A(-\pi/6)$

$\therefore \quad [A(\pi/6) \cdot B(\pi/4)]^{-1} = B(-\pi/4) \cdot A(-\pi/6)]$

$$= \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & 0 & \sin\left(-\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) & 0 \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) & \left\{\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)\right\} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. The coefficient matrix for the linear equations is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

where, $|A| = 0$, *i.e.*, the coefficient matrix is singular. And D_1 and D_2 are not singular. Hence, the given equation is inconsistent.

3. $\because \quad |\text{adj}(A)| = |A|^{n-1}$

$\therefore \quad |A| = 14$

$\therefore \quad |\text{adj}(A)| = (|A|)^2 = (14)^2$

4. $\because$ Square of involutary matrix is equal to identity matrix.

$$A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^n$$

On comparing, $n = 2$.

5. P is orthogonal matrix as $P^T P = I$

$$Q^{2005} = (PAP^T)(PAP^T)\ldots(PAP^T) = PA^{2005}P^T$$

$\therefore \quad P^T Q^{2005} P = P^T \cdot PA^{2005}P^T \cdot P = A^{2005}$

Now, $\quad A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

6. $125 = \det(A^3) = (\det A)^3 = (\alpha^2 - 4)^3$

$\Rightarrow \quad \alpha^2 - 4 = 5$

$\Rightarrow \quad \alpha^2 = 9 \Rightarrow \alpha = \pm 3$

7. We have,

$$A(\alpha, \beta)' = A(-\alpha, \beta)$$

Also, $\quad A(\alpha, \beta) \cdot A(-\alpha, -\beta) = 1$

$\Rightarrow \quad A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$

Next $\quad \text{adj}\, A(\alpha, \beta) = |A(\alpha, \beta)|\, A(\alpha, \beta)^{-1} = e^{\beta} A(-\alpha, -\beta)$

8. We have,

$$A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

We have, $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha = 0$

$\therefore \quad 5\alpha = \pi$

and $\quad \sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha = 4\cos\frac{\pi}{5}\cos\frac{\pi}{10} = a$ (say)

Thus, $\quad B = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$

$\therefore B$ is symmetric.

$$|B| = a^2 = 16\cos^2\frac{\pi}{5}\cos^2\frac{\pi}{10} > 0$$

$\therefore B$ is non-singular.

9. Here,

$$A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

So, $A^T=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, interchanging row and columns.

$$\therefore \quad A^TA=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}^2=A^2$$

$\therefore \quad |A^TA|=|A^2|$ but $|A^TA|=I$ (given)

Now, $\quad I=|A^2|\Rightarrow 1=|A|^2$

Now, $\quad |A|=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $R_1\to R_1+R_2+R_3$, $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$

$$=-(a^3+b^3+c^3-3abc)=-(a^3+b^3+c^3-3)$$

$(\because abc=1)$

$\therefore \quad |A|^2=1\Rightarrow(a^3+b^3+c^3-3)^2=1 \quad$ ...(i)

As a, b, c are positive, $\dfrac{a^3+b^3+c^3}{3}>\sqrt[3]{a^3b^3c^3}=abc=1$

$\therefore \quad a^3+b^3+c^3>3$

$a^3+b^3+c^3-3=1 \quad$ [from Eqs. (i)]

$\therefore \quad a^3+b^3+c^3=4$

10. $\text{adj}\,(Q^{-1}\ BP^{-1})=\text{adj}\,(Q^{-1}B)\{^{-1}=\text{adj}\,p^{-1.}\cdot\text{adj}\,(Q^{-1}B)$

$$=\text{adj}\,P^{-1.}\cdot\text{adj}\,B\cdot Q^{-1}$$

11. $\begin{bmatrix} a & i \\ i & b \end{bmatrix}\cdot\begin{bmatrix} a & i \\ i & b \end{bmatrix}=\begin{bmatrix} a^2-1 & (a+b)i \\ (a+b)i & b^2-1 \end{bmatrix}$

$\therefore \quad a^2-1=a,\ a+b=1$

$$a=\frac{1\pm\sqrt{5}}{2},\ b=\frac{1\mp\sqrt{5}}{2}$$

12. $A^2=4A-5I$

$A^3=11A-20I$

$A^6=A^{3.}\cdot A^3=44A-205I$

13. Let $X=\begin{bmatrix} X_1 \\ X_2 \\ X^3 \end{bmatrix}\therefore X'\cdot A\cdot X=0$

$\Rightarrow \quad a_{11}X_1^2+a_{22}X_2^2a_{33}X_3^2+(a_{12}+a_{21})X_1X_2$

$$+(a_{13}+a_{31})X_1X_3+(a_{23}+a_{32})X_2X_3=0$$

This is true X_i

$$a_{11}=a_{22}=a_{33}=0$$
$$a_{12}+a_{21}=0$$
$$a_{13}+a_{31}=0$$
$$a_{23}+a_{32}=0$$

14. For unique solution,

$$\begin{vmatrix} k & 2 & -1 \\ 0 & k-1 & -2 \\ 0 & 0 & k+2 \end{vmatrix}\neq 0$$

$\Rightarrow \quad k(k-1)(k+2)\neq 0\Rightarrow k\neq 0$, 1 or -2

15. $\det(M_r)=\begin{vmatrix} r & r-1 \\ r-1 & r \end{vmatrix}=2r-1$

$$\sum_{r=1}^{2007}\det(M_r)=2\sum_{r=1}^{2007}r-2007$$

$$=2\times\frac{2007\times 2008}{2}-2007=(2007)^2$$

16. $\because PQ=I \ \Rightarrow\ P^{-1}=Q$

Now, the system in matrix notation is $PX=B$

$\therefore \quad X=P^{-1}B=QB$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \\ 13 & -5 & m \\ -8 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad y=\dfrac{1}{9}(13-5+5m)$

$\Rightarrow \quad -27=8+5m \quad$ (given $y=-3$)

$\therefore \quad m=-7$

17. Now, $\quad A+B=\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}+\begin{bmatrix} -2 & -2 \\ 0 & -2 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow \quad |A+B|=\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}=2-4=-2$

Also, $\quad \text{adj}\,(A+B)=\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}$

$$\therefore \quad (A+B)^{-1}=-\frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ 1 & -1/2 \end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix} 4 & -4 \\ -2 & 3 \end{bmatrix}\text{ and }B^{-1}=\frac{1}{4}\begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \quad A^{-1}+B^{-1}=\begin{bmatrix} 1 & -1 \\ -1/2 & 3/4 \end{bmatrix}+\begin{bmatrix} -1/2 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

$$=\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$$

$\therefore \quad (A+B)^{-1}\neq A^{-1}+B^{-1}$

18. $\because \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} y \\ x \end{bmatrix}=\begin{bmatrix} X \\ Y \end{bmatrix}$

Then, $\quad X=y$ and $Y=x$ i.e., $y=x$

19. $\because \quad I_n=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ : & : & : & : & : \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, |I_n|=1$

$$\text{adj}\,(I_n)=I_n\ \therefore\ (I_n)^{-1}=I_n$$

20. Since, A is an idempotent matrix, therefore $A^2 = A$

$$\Rightarrow \begin{bmatrix} 2 & -2 & -16-4x \\ -1 & 3 & 16+4x \\ 4+x & -8-2x & -12+x^2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & x \end{bmatrix}$$

On comparing, $16 + 4x = 4 \Rightarrow x = -3$

21. $\because$ $|A| = 1$

and $$A^c = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$\text{adj } A = (A^c)' = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(-x)$$

22. Given, $A^2 = 2A - I$

Now, $A^3 = A^2 \cdot A = 2A^2 - IA$

$= 2A^2 - A = 2(2A - I) - A$

$= 3A - 2I = 3A - (3-1)I$

... ...

... ...

$A^n = nA - (n-1)I$

23. $$A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore$ $I + 2A + 3A^2 + \ldots = I + 2A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$$

24. A is orthogonal. Each row is orthogonal to the other rows.

$R_1 \cdot R_3 = 0 \Rightarrow x + 4 + 2y = 0$

and $R_2 \cdot R_3 = 0 \Rightarrow 2x + 2 - 2y = 0$

On solving, we get $x = -2, y = -1$

$\therefore$ $xy = 2$

25. Here, $AA^{-1} = I$

If R_1 of A is multiplied by C_3 of A^{-1}, we get

$2 - \alpha + 3 = 0 \Rightarrow \alpha = 5$

26. Now, $BB^T = (1-A)^{-1}(I+A)(I+A)^T[(I-A)^{-1}]^T$

$= (1-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$= (1-A)^{-1}(I-A)(I+A)(I+A)^{-1} = I \cdot I = I$

$\Rightarrow B$ is an orthogonal matrix.

27. $\therefore$ $I + A + A^2 + \ldots \infty = (I - A)^{-1}$

$$= \begin{bmatrix} 2 & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \frac{4}{7}\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

28. Thus, $$A = \begin{bmatrix} \frac{\omega}{i} & \frac{\omega^2}{i} \\ \frac{-\omega^2}{i} & \frac{-\omega}{i} \end{bmatrix} = \frac{\omega}{i}\begin{bmatrix} 1 & \omega \\ -\omega & -1 \end{bmatrix}$$

$\therefore$ $$A^2 = -\omega^2\begin{bmatrix} 1-\omega^2 & 0 \\ 0 & 1-\omega^2 \end{bmatrix}$$

$$= -\begin{bmatrix} -\omega^2+\omega^4 & 0 \\ 0 & -\omega^2+\omega^4 \end{bmatrix}$$

$$= \begin{bmatrix} -\omega^2+\omega & 0 \\ 0 & -\omega^2+\omega \end{bmatrix}$$

$f(x) = x^2 + 2$

$\therefore$ $$f(A) = A^2 + 2 = \begin{bmatrix} -\omega^2+\omega & 0 \\ 0 & -\omega^2+\omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= [-\omega^2 + \omega + 2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (3 + 2\omega)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

29. $$A^T = \text{Transpose of } A = \begin{bmatrix} \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $$|A^T| = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} = 1$$

Also, $|A^T| = |A|$. So, $|A| = 1$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \text{adj}A$$

$$A^{-1} = \text{adj}A = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $$|A^{-1}| = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

30. $A \cdot A^T = I$

$\therefore$ $2\beta^2 = \gamma^2, \alpha^2 = \beta^2 + \gamma^2$

$\therefore$ $$\frac{\alpha^2}{3} = \frac{\beta^2}{1} = \frac{\gamma^2}{2} = K$$

and $$\frac{\alpha^2}{3} = \frac{\beta^2}{1} = \frac{\gamma^2}{2} = K$$

31. The required conditions are $|A| = 0$ and (adj A) $B = 0$

$\therefore$ $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0 \text{ and } \begin{bmatrix} 2\lambda-6 & 2-\lambda & 1 \\ -\lambda+3 & \lambda-1 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $(2\lambda - 6) + (3 - \lambda) + 0 = 0$

and $0 \cdot 6 - 10 + \mu = 0$

$\Rightarrow$ $\lambda = 3, \mu = 10$

Solutions (Q. Nos. 32-36)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 8 \end{vmatrix} = 1(24-24) - 2(16-20) + 3(12-15) = -1$$

$$|B| = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 3 & 8 \\ 7 & 2 & 9 \end{vmatrix} = 3(27-16) - 2(18-56) + 5(4-21) = 24$$

32. adj (adj A) $= |A|^{n-2} A = |A|\, A = -A$

33. $|\text{adj (adj } A)| = |A|^{(n-1)^2} = |A|^4 = (-1)^4 = 1$

34. $|\text{adj } B| = |B|^{n-1} = |B|^2 = 24^2$

35. $AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 8 \end{bmatrix}\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 8 \\ 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 28 & 14 & 48 \\ 40 & 21 & 70 \\ 83 & 44 & 145 \end{bmatrix}$

$\therefore \quad |AB| = \begin{vmatrix} 28 & 14 & 48 \\ 40 & 21 & 70 \\ 83 & 44 & 145 \end{vmatrix}$

Applying $C_1 \to C_1 - 2C_2, C_3 \to C_3 - 3C_2$

Then, $|AB| = \begin{vmatrix} 0 & 14 & 6 \\ -2 & 21 & 7 \\ -5 & 44 & 13 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 6 \\ -2 & 0 & 7 \\ -5 & 5 & 13 \end{vmatrix}$

$(C_2 \to C_2 - 3C_3)$

$= 0 + 2(-52-30) - 5(-28)$

$= -164 + 140 = -24$

$\therefore \quad |\text{adj } AB| = |AB|^{n-1} = |AB|^2 = (24)^2$

36. $|(\text{adj (adj (adj (adj } A))))| = |\text{adj (adj } (-A))|$

$= |-A|^{(n-1)^2} = |-A|^4 = (|A|)^4 = (-1)^4 = 1$

37. $\because \quad AA' = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ A is orthogonal.

Also, if A and B are orthogonal, then AB is orthogonal.

38. Let $A = \begin{bmatrix} 0 & -c & b \\ c & 0 & a \\ -b & -a & 0 \end{bmatrix}$ and $A = -A'$

$\therefore \quad \det(A) = \det(-A')$

$= -\det(A') = -\det(A)$

$\therefore \quad \det(A) = 0$

$\because \quad \det(A') = \det(-A')$ is not true.

$\therefore \quad \det(-A') = (-1)^3 \det(A') = -\det A'$

39. $\because |A| = \begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} = 15 - 12 = 3 \neq 0$

$\therefore$ A is non-singular matrix.

$\therefore$ A^{-1} is exist.

40. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$

Also, $AB = BA$ (given)

$\Rightarrow \quad \begin{bmatrix} ax+bz & ay+bu \\ cx+dz & cy+du \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy \\ az+cu & bz+du \end{bmatrix}$

On comparing, we get

$ax + bz = ax + cy$

$\Rightarrow \quad bz = cy$

$\Rightarrow \quad \frac{z}{c} = \frac{y}{b} = \lambda$ (say)

$\therefore \quad y = b\lambda, \quad z = c\lambda$ ...(i)

and $ay + bu = bx + dy$

$\Rightarrow \quad ab\lambda + bu = bx + bd\lambda$ [from Eq. (i)]

$\Rightarrow \quad a\lambda + u = x + d\lambda = k$ (say)

For $\lambda = 0, \quad y = 0, \quad z = 0, \quad u = k, \quad x = k$

Then, $B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ = scalar matrix

Also, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Then, $AB = BA = \begin{bmatrix} ak & bk \\ ck & dk \end{bmatrix} = kA$

41. If $A = [a_{ij}]_{n \times n}$ is a square matrix such that $a_{ij} = 0$ for $i \neq j$, then A is called diagonal matrix. Thus, the given statement is true and $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

42. **Statement I** Given, $AA' = I \Rightarrow |A|^2 = I$, A is an invertible matrix. $\Rightarrow A' = A^{-1} \Rightarrow A'A = A^{-1}A = I$

Hence, option (b) is correct.

43. Since, adj adj (adj adj A) $= |A|^{n-2} A$

Here, $n = 3$

$\therefore \quad$ adj (adj A) $= |A|\, A$

Now, $|A| = 3(-3+4) + 3(2) + 4(-2) = 1$

$\therefore \quad$ adj (adj A) $= A$

Hence, option (b) is correct.

44. Clearly, $|A| = \begin{bmatrix} 2 & 1+2i \\ 1-2i & 7 \end{bmatrix} = 9$ which is real, so statement I is correct.

45. $\therefore |\text{adj } A| = |A|^{n-1} = |A|$

46. Because the value of the determinants of skew-symmetric matrix of odd order is zero.

47. Given, $A^T = A$ and $B^T = B$

Statement I $[A(BA)]^T = (BA)^T \cdot A^T = (A^T B^T) A^T$

$= (AB) A = A (BA)$

$\therefore$ $A(BA)$ is symmetric matrix.

Similarly, $(AB) A$ is symmetric matrix.

$\therefore$ statement I is true.

Also, Statement II is true, as if A and B is symmetric.

$\Rightarrow (AB)$ is symmetric, iff $(AB = BA)$

i.e., AB is commutative.

48. A satisfies $A^2 - \text{Tr}(A) \cdot A + (\det A) I = 0$

On comparing with $A^2 - I = 0$, we get

$$\text{Tr}(A) = 0, \ |A| = -1$$

Aliter

Let $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ a, b, c, d \neq 0$$

Now, $$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\Rightarrow$ $$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$\Rightarrow$ $$a^2 + bc = 1, \ bc + d^2 = 1$$

and $$ab + bd = ac + cd = 0$$

Also, $$c \neq 0 \text{ and } b \neq 0$$

$\Rightarrow$ $$a + d = 0$$

$\therefore$ $$\text{Tr}(A) = a + d = 0$$

and $$|A| = ad - bc = -a^2 - bc = -1$$

49. **Statement I** $\text{adj}(\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$

Statement II $|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

50. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\therefore$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(\because A^2 = I)$

$\Rightarrow$ $$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$b(a + d) = 0, \ c(a + d) = 0$$

and $$a^2 + bc = 1, bc + d^2 = 0$$

$\Rightarrow$ $$a^2 = 1, d = -1, b = c = 0$$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A \neq I, \ A \neq -I$$

$$\det(A) = -1$$

Statement I is true.

Statement II $\text{Tr}(A) = 1 - 1 = 0$. Statement II is false.

51. Given, $$P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$$

$\therefore$ $$|P| = 1(12 - 12) - \alpha(4 - 6) + 3(4 - 6) = 2\alpha - 6$$

$\because$ $$P = \text{adj}(A) \text{(given)}$$

$\therefore$ $$|P| = |\text{adj } A| = |A|^2 = 16 \quad (\because |\text{adj } A| = |A|^{n-1})$$

$\therefore$ $$2\alpha - 6 = 16 \Rightarrow 2\alpha = 22 \Rightarrow \alpha = 11$$

52. On subtracting the given equation, we get

$\Rightarrow$ $$P^3 - P^2 Q = Q^3 - Q^2 P$$

$\Rightarrow$ $$P^2(P - Q) = Q^2(Q - P)$$

$\Rightarrow$ $$(P - Q)(P^2 + Q^2) = 0$$

$\Rightarrow$ $$|P^2 + Q^2| = 0$$

53. Since, both Au_1 and Au_2 are given, hence adding them, we get

$$Au_1 + Au_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 0 \\ 0 + 1 \\ 0 + 0 \end{bmatrix}$$

$\Rightarrow$ $$A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since, A is non-singular matrix, *i.e.*, $|A| \neq 0$, hence multiplying both sides by A^{-1} (from RHS), we get

$$A^{-1} A(u_1 + u_2) = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

Now, $$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 + 0 = 1$$

$\therefore$ Adjoint of $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore$ $$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad (\because |A| = 1)$$

Hence, from Eq. (i), we get

$$u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$u_1 + u_2 = \begin{bmatrix} 1 + 0 + 0 \\ -2 + 1 + 0 \\ 1 - 2 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

54. Since, equation has non-zero solution.

$\Rightarrow \quad \Delta = 0$

$\therefore \quad \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 4(4-2) - k(k-2) + 2(2k-8) = 0$

$\Rightarrow \quad 8 - k^2 + 2k + 4k - 16 = 0$

$\Rightarrow \quad k^2 - 6k + 8 = 0$

$\Rightarrow \quad (k-2)(k-4) = 0$

$\Rightarrow \quad k = 2, 4$

$\therefore$ Number of values of k is two.

55. The given system of linear equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -8 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to -R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

Clearly, the given system of equations has no solution.

Aliter Subtracting the addition of first two equations from third equation, we get $0 = -5$, which is an absurd result. Hence, the given system of equation has no solution.

56. Consider $\begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}$. By placing 1 in any one of the 6 * position and 0 elsewhere. We get, 6 non-singular matrices.

Similarly, $\begin{pmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{pmatrix}$ gives atleast one non-singular.

57. As $\det(A)^2 = \pm 1$, A^{-1} exists

and $\quad A^{-1} = \dfrac{1}{\det(A)}(\text{adj}\, A) = \pm(\text{adj}\, A)$

All entries in adj (A) are integers.

$\therefore A^{-1}$ has integer entries.

58. $\det(A^2) = (\det A)^2 = \begin{vmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{vmatrix}^2 = 25$

$\Rightarrow \quad (25\alpha)^2 = 25$

$\Rightarrow \quad \alpha^2 = \dfrac{1}{25}$

$\Rightarrow \quad |\alpha| = \dfrac{1}{5}$

59. $A^2 - B^2 = (A-B)(A+B) = A^2 + AB - BA - B^2$

$\therefore \quad AB = BA$

60. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

$\Rightarrow \quad AB = BA$

$\Rightarrow \quad a = b = 1, 2, 3, \ldots$

23 Determinant

JEE Main MILESTONE

- Minors and Cofactors of a Determinant
- Sarrus Rule
- Product of Determinants
- Differentiation of Determinant
- Integration of Determinant
- Applications of Determinant in Geometry

Determinants arise in connection with simultaneous linear algebraic equations expressions, which are very complicated and lengthy can be easily handled, if they are expressed as determinants.

An expression which is related to a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n_1} & a_{n_2} & \dots & a_{nn} \end{bmatrix}$$

is known as a determinant of A. It is denoted by $\det(A)$, Δ or $|A|$ and written as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

OR

Every square matrix can be associated to an expression or a number which is known as its determinant.

The determinant is a value associated with a square matrix. It can be computed from the entries of the matrix by a specific arithmetic expression, while other ways to determine its value exist as well. The determinant provides important information when the matrix is that of the coefficients of a system of linear equations or when it corresponds to a linear transformation.

Note

- An arrangement of numbers is known as matrix and a matrix having equal number of rows and columns is known as **square matrix**.
- Matrix has no definite value while each determinant has a definite value.
- Determinant of a matrix which is not a square matrix, cannot be find out.

Value of a Determinant of Order 1

If $A = [a_{11}]$ is a square matrix of order 1×1,

then $|A| = a_{11}$

Value of a Determinant of Order 2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2×2, then corresponding determinant

is $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Value of a Determinant of Order 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3,

whose corresponding determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\therefore \quad |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

e.g., If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= (2-1) - 2(3-1) + 3(3-2)$$

$$= 1 - 2(2) + 3 = 4 - 4 = 0$$

Note The determinant of order greater than 2 can be expanded along any row or column.

23.1 Minors and Cofactors of a Determinant

(i) Minors

Let $A = [a_{ij}]$ is a square matrix of order n, then minor of a_{ij} is a determinant of a square matrix of order $(n-1)$ which is M_{ij}. M_{ij} is obtained on leaving ith row and jth column of A.

$\therefore$ Minors of elements a_1, a_2 and a_3 of determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ are } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

respectively.

e.g., If $A = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$, then

$$M_{11} = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 6 - 4 = 2$$

$$M_{12} = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -9 + 2 = -7$$

$$M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 12 - 4 = 8$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = 6 + 12 = 18$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3 - 6 = -3$$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -4 - 4 = -8$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2 - 6 = -8$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} = -1 + 9 = 8$$

and $$M_{33} = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 2 + 6 = 8$$

(ii) Cofactors

The cofactor C_{ij} of a_{ij} in Δ is equal to $(-1)^{i+j}$ times the determinant of order $(n-1)$ obtained by leaving ith row and jth column of Δ.

It follows that, cofactor of an element a_{ij} of determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ is } C_{ij}.$$

$\Rightarrow \quad C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of a_{ij}.

Cofactor of $a_{12} = C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

e.g., If $A = \begin{vmatrix} 4 & -7 \\ -3 & 2 \end{vmatrix}$, then cofactors are

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 2$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-3) = 3$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 7$$

and $$C_{22} = (-1)^{2+2} M_{22} = M_{22} = 4$$

Sample Problem 1 *If* $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$, *then its value is* [NCERT]

(a) – 5 (b) – 7 (c) – 10 (d) – 12

Interpret (d) $\because \quad |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

It can be observed that in the second row, two entries are zero. Thus, we expand the determinant along the second row for easier calculation.

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix}$$

$$= \{3 \times (-5) - 3 \times (-1)\} = -15 + 3 = -12$$

23.2 Sarrus Rule

Write down the three rows of the determinant and rewrite the first two rows. The three diagonals sloping down to the right given the three positive terms and the three diagonals sloping down to the left given the three negative terms.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}$$

Negative terms	Positive terms
$a_{13}a_{22}a_{31}$	$a_{11}a_{22}a_{33}$
$a_{23}a_{32}a_{11}$	$a_{21}a_{32}a_{13}$
$a_{33}a_{12}a_{21}$	$a_{31}a_{12}a_{23}$
Sum $= N$	Sum $= P$

$\therefore \quad \Delta = P - N$

Note This method is not applicable for determinants of order greater than 3.

Sample Problem 2 *The value of determinant* $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ *using Sarrus rule, is*

(*a*) 230
(*b*) 166
(*c*) 64
(*d*) *None of the above*

Interpret (*a*) Let $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$

$$\begin{array}{ccc} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \\ 3 & 2 & 5 \\ 9 & -1 & 4 \end{array}$$

Negative terms	Positive terms
-10	15
36	135
-90	16
N $= -64$	P $= 166$

$\therefore \quad \Delta = P - N = 166 + 64 = 230$

Hot Spot 1

Properties of a Determinant

Properties of a determinant is the most important topic of this chapter. Generally questions are seen in AIEEE (JEE Main) from this topic. The level of the questions sometimes simple based on the use the properties and sometimes, it is from moderate to typical.

In moderate level question, generally questions are asked on the concept of sequence and series, logarithm, cube root of unity, polynomial etc.

For solving this types of question, we use the above given concepts and then properties of determinants.

(i) The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

e.g., $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

(ii) If two adjacent rows (or columns) of a determinant are interchanged, then the value of the determinant, so obtained is the negative of the value of the original determinant, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

e.g., $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

(iii) If two rows (or columns) of a determinant are identical, then its value is zero, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = 0$$

e.g., $\begin{vmatrix} a & a & c \\ b & b & a \\ c & c & b \end{vmatrix} = 0$

Here, first and second columns are same.

(iv) Each element of a row (or column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant, *i.e.*,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

e.g., $$\begin{vmatrix} 2 & 4 & 6 \\ 4 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

(v) If any two rows (or columns) of a determinants are proportional, then its value is zero, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

e.g., $$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

(vi) If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of the two or more determinants, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + c_1 & a_{22} + c_2 & a_{23} + c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_1 & c_2 & c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

e.g., $$\begin{vmatrix} a & b & c \\ d + e & f + g & h + i \\ j & k & l \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & f & h \\ j & k & l \end{vmatrix} + \begin{vmatrix} a & b & c \\ e & g & i \\ j & k & l \end{vmatrix}$$

(vii) If each element of a row (or column) of a determinant is multiplied by a constant k and then added to (or subtracted) from the corresponding elements of any other row (or column), then the value of the determinant remains the same, *i.e.*,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + ka_{21} & a_{32} + ka_{22} & a_{33} + ka_{23} \end{vmatrix}$$

e.g., $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g + ak & h + bk & i + ck \end{vmatrix}$$

(viii) If each element of a row (or column) of a determinant is zero, then its value is zero.

(ix) If r rows (or r columns) become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

(x) The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero, *i.e.*, if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

(xi) The sum of the products of elements of any row (or column) of a determinant (where $|A|_{n \times n} = \Delta$) with the cofactors of the corresponding elements of same row (or column) is equal to Δ.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$$

(xii) If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ a & b & c \end{vmatrix}$, then

(a) $$\sum_{x=1}^{n} \Delta(x) = \begin{vmatrix} \sum_{x=1}^{n} f_1(x) & \sum_{x=1}^{n} f_2(x) & \sum_{x=1}^{n} f_3(x) \\ \sum_{x=1}^{n} g_1(x) & \sum_{x=1}^{n} g_2(x) & \sum_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$$

(b) $$\prod_{x=1}^{n} \Delta(x) = \begin{vmatrix} \prod_{x=1}^{n} f_1(x) & \prod_{x=1}^{n} f_2(x) & \prod_{x=1}^{n} f_3(x) \\ \prod_{x=1}^{n} g_1(x) & \prod_{x=1}^{n} g_2(x) & \prod_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$$

Note

- In $\Delta = |a_{ij}|$ is a determinant of order n, then the value of the determinant $|A_{ij}|$, where A_{ij} is the cofactor of a_{ij}, is Δ^{n-1}.
- If $A = B + C$, then it is not necessary that
$$\det(A) = \det(B) + \det(C)$$
- If A is a square matrix of order $n \times n$, then
$$\det(kA) = k^n (\det A)$$
- If A, B and C are three square matrix such that ith row of A is equal to the sum of ith row of B and C and remaining rows of A, B and C are same, then
$$\det(A) = \det(B) + \det(C)$$
- $\det(A^n) = (\det A)^n$, where n is a positive integer.
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- If the rows and columns are interchange, Δ is unchanged *i.e.*, $|A^T| = |A|$.
- Product of two determinants *i.e.*,
$$|AB| = |A||B| = |BA| = |AB^T| = |A^T B| = |A^T B^T|$$

Sample Problem 3 *The value of the determinant* $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$ *is* [**NCERT**]

(a) 1 (b) xy^2 (c) abc (d) 0

Interpret (d) Let $A = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$, we get $A = \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0$

($\because$ the two columns ($C_2 = C_3$) of the determinants are identical).

Sample Problem 4 *The value of the determinant*

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \text{ is}$$ [**NCERT**]

(a) abc (b) 0
(c) 1 (d) *None of these*

Interpret (b) Let $A = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$$

[$\because$ each element of C_1 (first column) is zero].

Sample Problem 5 *Let A be a square matrix of order* 3×3, *then* $|kA|$ *is equal to* [**NCERT**]

(a) $k|A|$ (b) $k^2|A|$
(c) $k^3|A|$ (d) $3k|A|$

Interpret (c) A is a square matrix of order 3×3.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Then, $kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix} \therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$

Taking out common factor k from each row, we get

$$= k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\therefore$ $|kA| = k^3|A|$

Sample Problem 6 *If* ω *is a cube root of unity, then the vlaue of determinant* $\begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$ *is*

(a) 0 (b) $3\omega^2$
(c) $-3\omega^2$ (d) ω^2

Interpret (c) $\because$ $1 + \omega + \omega^2 = 0$

$$\therefore \begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & \omega^2 & -\omega \\ 1+\omega+\omega^2 & \omega & -\omega^2 \\ \omega^2+2\omega & \omega & -\omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \omega^2 & -\omega \\ 0 & \omega & -\omega^2 \\ \omega-1 & \omega & -\omega^2 \end{vmatrix} \quad (\because 1+\omega+\omega^2 = 0)$$

$= (\omega - 1)(-\omega^4 + \omega^2) = (\omega - 1)(-\omega + \omega^2)$

$= -\omega^2 + \omega^3 + \omega - \omega^2 = -\omega^2 + 1 + \omega - \omega^2$

$= -\omega^2 - \omega^2 - \omega^2 = -3\omega^2$

Check Point **1**

1. $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b+c-a & c+a-b & a+b-c \end{vmatrix}$ is equal to

(a) abc (b) a + b + c
(c) ab + bc + ca (d) 0

2. If $\Delta_1 = \begin{vmatrix} 7 & x & 2 \\ -5 & x+1 & 3 \\ 4 & x & 7 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} x & 2 & 7 \\ x+1 & 3 & -5 \\ x & 7 & 4 \end{vmatrix}$, then $\Delta_1 - \Delta_2 = 0$ for

(a) x = 2 (b) all real x
(c) x = 0 (d) None of these

3. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then k is equal to

(a) 0 (b) 1
(c) 2 (d) 3

4. If a + b + c = 0, then one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

(a) x = 1
(b) x = 2
(c) $x = a^2 + b^2 + c^2$
(d) x = 0

5. The value of the determinant $\begin{vmatrix} {}^5C_0 & {}^5C_3 & 1 \\ {}^5C_1 & {}^5C_4 & 1 \\ {}^5C_2 & {}^5C_5 & 1 \end{vmatrix}$ is

(a) 0 (b) $-6!$
(c) 80 (d) None of these

6. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+bk+ck^2)(a+bk^2+ck)$, then k is equal to

(a) 1 (b) -1
(c) ω (d) $-\omega$

7. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = k(a+b+c)^3$, then k is equal to

(a) 0 (b) 1 (c) 2 (d) 3

8. If a, b and c are cube roots of unity, then $\begin{vmatrix} e^a & e^{2a} & e^{3a}-1 \\ e^b & e^{2b} & e^{3b}-1 \\ e^c & e^{2c} & e^{3c}-1 \end{vmatrix}$ is equal to

(a) 0 (b) e
(c) e^2 (d) e^3

23.3 Product of Determinants

Let two determinants are

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Then, $\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Sample Problem 7 *If* $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$ $= (a^3 + b^3 + c^3 + kabc)^2$, *then the value of k is*

(a) 3 (b) -3 (c) 4 (d) 0

Interpret (b) We have, $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc - a^2) + b(ac - b^2) + c(ab - c^2)]^2$$

$$= [a^3 + b^3 + c^3 - 3abc]^2$$

$\therefore \quad k = -3$

23.4 Differentiation of Determinant

(i) Let $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$

where $a_{ij}(x)$ is a differentiable function, then

$$\frac{d}{dx}\Delta(x) = \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a'_{21}(x) & a'_{22}(x) & a'_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a'_{31}(x) & a'_{32}(x) & a'_{33}(x) \end{vmatrix}$$

(ii) Let $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

where $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ are constants, then

$$\frac{d}{dx}\Delta(x) = \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Sample Problem 8 *If* $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, *then the value of* $f'(x)$ *at* $x = 0$ *is*

(a) -2 (b) 2
(c) 0 (d) *None of these*

Interpret (c) $f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix}$

$$+ \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$$

$$\Rightarrow f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0 + 0 + 0 = 0$$

23.5 Integration of Determinant

Let $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\therefore \int \Delta(x)\, dx = \begin{vmatrix} \int a_{11}(x)\, dx & \int a_{12}(x)\, dx & \int a_{13}(x)\, dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note This formula is applicable only, if their is variable only in one row or column, otherwise expand the determinant and then integrate.

Sample **Problem 9** *If* $f(x) = \begin{vmatrix} x^3 & \sin x \\ 1 & 2 \end{vmatrix}$, *then the value of* $\int_{-a}^{a} f(x)\, dx$ *is*

(a) 0 (b) 1
(c) 2 (d) *None of these*

Interpret (a) $\int_{-a}^{a} f(x)\, dx = \begin{vmatrix} \int_{-a}^{a} x^3\, dx & \int_{-a}^{a} \sin x\, dx \\ 1 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0$$

23.6 Applications of Determinant in Geometry

(i) Area of Triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then

$$\text{Area of triangle} = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Sample **Problem 10** *The area of the triangle with vertices* $(-2, -3)$, $(3, 2)$ *and* $(-1, -8)$ *is* [NCERT]

(a) $\frac{15}{2}$ (b) $\frac{47}{2}$
(c) 15 (d) *None of these*

Interpret (c) Required area $= \left| \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \right|$

$$= \left| \frac{1}{2} |-2(2+8) + 3(3+1) + 1(-24+2)| \right|$$

$$= \left| \frac{1}{2} |-20 + 12 - 22| \right| = \left| \frac{1}{2} |-30| \right| = 15 \text{ sq units}$$

(ii) Condition of Collinearity of Three Points

Let three points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then these points will be collinear, if

$$\text{Area of } \Delta ABC = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Sample **Problem 11** *The points* $(a, b+c)$, $(b, c+a)$ *and* $(c, a+b)$ *are*

(a) *form a triangle* (b) *collinear*
(c) *meaningless* (d) *None of these*

Interpret (b) $\because \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} \quad (C_1 \to C_1 + C_2)$$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

$$= 0 \quad (\because \text{two columns are similar})$$

$\therefore$ Points are collinear.

(iii) Equation of Straight Line Passing Through Two Points

Let two points are $A(x_1, y_1)$ and $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then equation of line is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Sample **Problem 12** *The equation of straight line passing through the points* $A(3, 1)$ *and* $B(9, 3)$ *is* [NCERT]

(a) $x - 2 = 0$ (b) $x - 3y = 0$
(c) $y + 2 = 0$ (d) $y - 2 = 0$

Interpret (b) Let $P(x, y)$ be any point on the line joining $A(3, 1)$ and $B(9, 3)$. Then, the points A, B and P are collinear. Therefore, the area of ΔABP will be zero.

$$\therefore \quad \frac{1}{2} \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2} |3(3-y) - 1(9-x) + 1(9y - 3x)| = 0$$

$$\Rightarrow \quad 9 - 3y - 9 + x + 9y - 3x = 0$$

$$\Rightarrow \quad 6y - 2x = 0$$

$$\Rightarrow \quad x - 3y = 0$$

Hence, the equation of the line joining the given points is $x - 3y = 0$.

Hot Spot 2

Solution Of System of Linear Equations

Solution of system of linear equation is important topic for JEE Main (AIEEE) examination and a small practice can help you in solving such problems very quickly as the level of questions is easy to average.

While solving problems based on solution of system of linear equation, always keep in mind that system of linear equation is homogeneous or non-homogeneous.

In solution of system of non-homogeneous equation in two or three variables we always use cramer's rule.

Let $a_1x + b_1y + c_1z = d_1$,

$a_2x + b_2y + c_2z = d_2$

and $a_3x + b_3y + c_3z = d_3$

is a system of linear equations.

A set of values of variables x, y and z which simultaneously satisfy these three equations is called a solution.

Consistent

If the system of equations has a unique solution or infinite many solution, then the system of equations is known as consistent.

Inconsistent

If the system of equations has no solution, then system of equations is known as inconsistent.

Trivial and Non-trivial Solution

If the value of all variables of system of equations is zero, *i.e.*, $x = 0$, $y = 0$ and $z = 0$, then solution is known as trivial solution.

If system of equations has infinite many solution, then the solution is known as non-trivial solution.

Homogeneous and Non-homogeneous System

If $d_1 = d_2 = d_3 = 0$ in given system of equations, then system of equations is said to be homogeneous otherwise it is said to be non-homogeneous system.

Cramer's Rule

A system of simultaneous linear equations can be solved by Cramer's rule named for the Swiss Mathematician Gabriel Cramer.

(i) The solution of the system of linear equations

$$a_1x + b_1y = c_1$$

and $$a_2x + b_2y = c_2$$

is given by $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$ where,

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

provided that $D \neq 0$.

(ii) Let the determinant of coefficient of equations

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2$$

and $a_3x + b_3y + c_3z = d_3$ is D. $\therefore D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If $D \neq 0$, then $\dfrac{x}{D_1} = \dfrac{y}{D_2} = \dfrac{z}{D_3} = \dfrac{1}{D}$

where, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$;

and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Conditions for Consistency

(i) If $D \neq 0$, then system of equation is consistent and has a unique solution given by

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}$$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then system of equation is consistent with infinitely many solution.

(iii) If $D = 0$ and atleast one of D_1, D_2 and D_3 is non-zero, then system of equation is inconsistent.

Sample **Problem 13** *The number of values of k, for which the system of equations*

$$(k + 1)x + 8y = 4k$$

$$kx + (k + 3)y = 3k - 1$$

has no solution, is

(*a*) infinite (*b*) 1 (*c*) 2 (*d*) 3

Interpret (*b*) Given system of linear equation is non-homogeneous equations.

Step I First we find D, D_1 and D_2.

$$D = \begin{vmatrix} (k+1) & 8 \\ k & (k+3) \end{vmatrix}, D_1 = \begin{vmatrix} 4k & 8 \\ 3k-1 & (k+3) \end{vmatrix}$$

and $$D_2 = \begin{vmatrix} k+1 & 4k \\ k & 3k-1 \end{vmatrix}$$

Step II Calculate all three determinants.

$$D = (k+1)(k+3) - 8k = (k-1)(k-3)$$
$$D_1 = 4k(k+3) - 8(3k-1) = 4(k-1)(k-2)$$
$$D_2 = (k+1)(3k-1) - 4k^2 = -(k-1)^2$$

Step III For inconsistent or no solution, $D = 0$

$$D = (k-1)(k-3) = 0 \Rightarrow k = 1, 3$$

and atleast one of the D_1 and D_2 is non-zero.

So, for $k = 1$

$$D_1 = D_2 = 0 \text{ and for } k = 3$$
$$D_1 \neq 0, D_2 \neq 0$$

Step IV So, for $k = 3$. The system of equations has no solution.

Step V Required number of values of k is 1.

Solution of Homogeneous System of Equations

Let
$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
and
$$a_3x + b_3y + c_3z = 0$$
be a homogeneous system of linear equations. Then,

(i) System of equations has a unique trivial

$$x = y = z = 0, \text{ if } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

(ii) System of equations has non-trivial solution, if $D = 0$

Sample Problem 14 *The system of linear equations* $x + y + z = 2$, $2x + y - z = 3$ and $3x + 2y + kz = 4$ *has a unique solution, if k is not equal to*

(a) 4 (b) -4 (c) 0 (d) 3

Interpret (c) **Step I** The given system of equations are

$$x + y + z = 2$$
$$2x + y - z = 3 \text{ and } 3x + 2y + kz = 4$$

Step II This system has a unique solution, if

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$$\Rightarrow \quad 1(k+2) - 1(2k+3) + 1(4-3) \neq 0$$
$$\Rightarrow \quad k + 2 - 2k - 3 + 1 \neq 0$$
$$\Rightarrow \quad -k - 0 \neq 0 \Rightarrow k \neq 0$$

Check Point 2

1. If in x, y and z the system of linear equations

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

has non-zero solution, then θ is equal to

(a) $n\pi$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $\frac{n\pi}{3}$ (d) $\frac{n\pi}{2}$

2. If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, then $\int_0^{\pi/4} \Delta(x)\,dx$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 0 (d) $-\frac{1}{4}$

3. The system of equations

$$2x - y + z = 0$$
$$x - 2y + z = 0 \text{ and } \lambda x - y + 2z = 0,$$

has infinite number of non-trivial solution for

(a) $\lambda = 1$ (b) $\lambda = 5$
(c) $\lambda = -5$ (d) no real value of λ

4. If $\Delta(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ \alpha & \alpha^2 & \alpha^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}[\Delta(x)]$ at $x = 0$ is

(a) -1 (b) 0 (c) 1 (d) 2

5. Let a, b and c be positive real numbers. The following system of equations in x, y and z

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{-x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ has,}$$

(a) no solution (b) unique solution
(c) infinitely many solutions (d) finitely many solutions

6. The system of simultaneous equations $kx + 2y - z = 1$, $(k-1)y - 2z = 2$ and $(k+2)z = 3$ have a unique solution, if k is equal to

(a) -2 (b) -1 (c) 0 (d) 1

7. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$ and $cx + cy + z = 0$ where a, b and c are non-zero, non-unity, has a non-trivial solution, then the value of $\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}$ is

(a) 0 (b) 1 (c) -1 (d) $\frac{abc}{a^2 + b^2 + c^2}$

8. If u, v and w be functions of x and if suffixes denote differentiation with respect to x, then $\frac{d}{dx}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$ is equal to

(a) $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$ (b) 0
(c) 1 (d) None of these

WORKED OUT Examples

Example 1 $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$ *is equal to*

[**NCERT**]

(a) $(1 - x^3)$ (b) 1

(c) $(1 - x^3)^2$ (d) $2(1 - x^3)^2$

Solution (c) $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$

(using $C_1 \to C_1 + C_2 + C_3$)

Take out $(1 + x + x^2)$ common from C_1, we get

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x(1-x) \\ 0 & x(x-1) & 1-x^2 \end{vmatrix}$$

Take out $(1 - x)$ common from R_2 and same from R_3, we get

$$= (1+x+x^2)(1-x)(1-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

Expanding along C_1, we get

$$= (1+x+x^2)(1-x)(1-x)[(1\times(1+x) - (-x)(x)]$$

$$= (1+x+x^2)(1-x)(1-x)(1+x+x^2)$$

$$= [(1-x^3)(1-x^3)] = (1-x^3)^2$$

$[\because 1 - x^3 = (1-x)(1+x+x^2)]$

Example 2 $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ *is equal to*

[**NCERT**]

(a) $1 - a^2 - b^2 - c^2$ (b) $a^2 + b^2 + c^2$

(c) $1 + a^2 + b^2 + c^2$ (d) $a^2b^2c^2$

Solution (c)

(i) Take common factors a, b and c out from R_1, R_2 and R_3, respectively.

(ii) Operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ followed by multiplying C_1 by a, C_2 by b and C_3 by c.

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors a, b and c from R_1, R_2 and R_3 respectively, we get

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix} = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

On multiply and divide C_1 by a, C_2 by b and C_3 by c and take common $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ out from C_1, C_2 and C_3 respectively, we get

$$= abc \times \frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we get

$$= -1\times(-c^2) + 1[1(a^2+1) + 1(b^2)]$$

$$= (a^2 + b^2 + c^2 + 1) = 1 + a^2 + b^2 + c^2$$

Example 3 *If* $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = kabc$, *then the value of k is*

[**NCERT Examplar**]

(a) 1 (b) 2

(c) 3 (d) 4

Solution (*d*) We have, $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2a & -2a \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \quad [\text{applying } R_1 \to R_1 - (R_2 + R_3)]$$

$$= -2a \begin{vmatrix} 0 & 1 & 1 \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \quad (\text{taking } -2a \text{ common from } R_1)$$

$$= (-2a)(-bc - bc) \quad (\text{expanding along } R_1)$$

$\therefore \quad k = 4$

Example 4 *The value of determinant*

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix} \text{ is}$$

(*a*) *independent of* α (*b*) *independent of* β
(*c*) *independent of* α *and* β (*d*) *None of these*

Solution (*a*) $\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix}$$

(applying $R_3 \to R_3 - R_1 \cos\beta + R_2 \sin\beta$)

$$= (1+\sin\beta-\cos\beta)(\cos^2\alpha + \sin^2\alpha)$$

$$= 1+\sin\beta-\cos\beta$$

which is independent of α.

Example 5 $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ *is equal to*

(*a*) $abc(a+b+c)^2$ (*b*) $2abc(a+b+c)^2$
(*c*) $2abc(a+b+c)^3$ (*d*) $2abc(a+b+c)$

Solution (*c*) Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$(C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3)$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$[R_3 \to R_3 - (R_1 + R_2)]$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

$$= \frac{(a+b+c)^2}{ab} \cdot ab \cdot 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2ab(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$= 2ab(a+b+c)^2 \{(b+c)(c+a) - ab\}$$

$$= 2abc(a+b+c)^2(a+b+c)$$

$$= 2abc(a+b+c)^3$$

Example 6 *Using Cramer's rule, the solution of system of linear equations*

$$2x - y = 17 \quad \text{and} \quad 3x + 5y = 6 \text{ is}$$

(*a*) $x = 7, y = 3$ (*b*) $x = 7, y = -3$
(*c*) $x = -7, y = 3$ (*d*) $x = -7, y = -3$

Solution (*b*) Let $D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 10 + 3 = 13 \Rightarrow D \neq 0$

Now, $D_1 = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix} = 85 + 6 = 91$

and $D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix} = 12 - 51 = -39$

By Cramer's rule $x = \dfrac{D_1}{D} = \dfrac{91}{13} = 7$

and $y = \dfrac{D_2}{D} = -\dfrac{39}{13} = -3$

Example 7 $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ *is equal to*

[NCERT Exemplar]

(*a*) abc

(*b*) $\left(1 + \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$

(*c*) $abc\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$

(*d*) $abc\left(1 + \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$

Solution (*d*) Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$(C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1)$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Example 8 *If m is a positive integer*

$\Delta_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$, *then the value of* $\sum_{r=0}^{m} \Delta_r$ *is*

(*a*) 1 (*b*) 0
(*c*) 2 (*d*) *None of these*

Solution (*b*) $\Delta_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$

$$\therefore \quad \sum_{r=0}^{m} \Delta_r = \begin{vmatrix} \sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m} {}^mC_r & \sum_{r=0}^{m} 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$$= \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$= 0$ ($\because$ two rows are same)

Example 9 *Let a, b* and *c are positive and unequal, then the value of determinant* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ *is*

(*a*) *negative* (*b*) *positive*
(*c*) 0 (*d*) *None of these*

Solution (*a*) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$

$(C_1 \to C_1 + C_2 + C_3)$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$$

$= (a+b+c)\{(c-b)(b-c) - (a-c)(a-b)\}$
$= (a+b+c)\{bc - c^2 - b^2 + bc - a^2 + ab + ac - bc\}$
$= (a+b+c)\{-a^2 - b^2 - c^2 + ab + bc + ca\}$

$\Rightarrow \Delta = -\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

$\because$ *a, b* and *c* are positive and unequal. Then,
$a+b+c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

$\therefore \quad \Delta < 0$

$\therefore$ The value of determinant is negative.

Example 10 *The roots of the equation*

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0 \text{ are}$$

(*a*) 1, 2 (*b*) –1, 2 (*c*) 1, –2 (*d*) –1, –2

Solution (*b*) $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x+1 & 1 & 1 \\ x+1 & x-1 & 1 \\ x+1 & 1 & x-1 \end{vmatrix} = 0 \quad (C_1 \to C_1 + C_2 + C_3)$

$\Rightarrow (x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

$\Rightarrow (x+1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-2 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = 0 \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$\Rightarrow (x+1)(x-2)^2 = 0$

$\Rightarrow x = -1, 2$

Example 11 *If* $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$ *and* a, b, c *are all distinct, then abc (ab + bc + ca) is equal to*

(a) $a+b+c$ (b) abc

(c) 0 (d) *None of these*

Solution (a) Given, $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - 1 \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc(a-b)(b-c)(c-a)(ab+bc+ca) - (a-b)(b-c)(c-a)(a+b+c) = 0$$

$$\Rightarrow abc(ab+bc+ca) - (a+b+c) = 0$$

$$\Rightarrow abc(ab+bc+ca) = (a+b+c)$$

Example 12 *If a, b and c are in AP, then determinant* $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ *is* [**NCERT**]

(a) *zero* (b) 1

(c) x (d) $2x$

Solution (a) Let $A = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2(2b-a-c) \\ x+4 & x+5 & x+2c \end{vmatrix}$$

(using $R_2 \to 2R_2 - R_1 - R_3$)

But a, b *and* c are in AP. Using $2b = a + c$, we get

$$A = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

Since, all elements of R_2 are zero.

Example 13 $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$ *is equal to* [**NCERT**]

(a) $\sin\alpha \cdot \sin\beta \cdot \sin\gamma$

(b) $\cos\alpha \cdot \cos\beta \cdot \cos\gamma$

(c) $\sin(\alpha+\beta+\gamma) \cdot \cos(\alpha+\beta+\gamma)$

(d) 0

Solution (d) Firstly, we use the relation $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and then we expressed the determinant as sum of two determinants and then apply the operations separately.

Let $A = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$[\because \cos(A+B) = \cos A \cos B - \sin A \sin B]$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix} + \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\sin\delta \end{vmatrix}$$

Taking common $\cos\delta$ from C_3 in first determinant and taking common $-\sin\delta$ from C_3 in second determinant.

$$= \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix} - \sin\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta & \sin\beta \\ \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

$$= \cos\delta \cdot 0 - \sin\delta \cdot 0 = 0$$

Since, C_2 and C_3 in first determinant C_1 and C_3 in second determinant are identical so values of determinants are zero.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Determinant and its Properties

1. Determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is [NCERT]

(a) Independent of θ (b) Dependent of θ
(c) Dependent of θ and x (d) None of these

2. $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$ is equal to [NCERT]

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$
(c) 0 (d) 1

3. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is [NCERT Exemplar]

(a) 3 (b) ± 3 (c) ± 6 (d) 6

4. If $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0$, then

(a) a is one of the cube roots of unity
(b) b is one of the cube roots of unity
(c) $\left(\frac{a}{b}\right)$ is one of the cube roots of unity
(d) $\left(\frac{a}{b}\right)$ is one of the cube roots of -1

5. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then [NCERT]

(a) $|3A| = 3|A|$ (b) $|3A| = 9|A|$
(c) $|3A| = 27|A|$ (d) None of these

6. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ is equal to [NCERT]

(a) $a+b+c$ (b) abc
(c) 0 (d) None of these

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$ is equal to [NCERT]

(a) $a^2b^2c^2$ (b) $2a^2b^2c^2$
(c) $4a^2b^2c^2$ (d) $8a^2b^2c^2$

8. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$ is equal to [NCERT]

(a) $(x-y)(y-z)(z-x)(x+y+z)$
(b) $(x-y)(y-z)(z-x)(xy+yz+zx)$
(c) $(x+y)(y+z)(z+x)(xy+yz+zx)$
(d) None of the above

9. $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$ is equal to [NCERT]

(a) $k(y+k)$
(b) $k^2(y+k)$
(c) $k^2(3y+k)$
(d) $k^2(y+3k)$

10. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ is equal to [NCERT]

(a) $2(a+b+c)^2$
(b) $(a+b+c)^3$
(c) $2(a+b+c)^3$
(d) $(a+b+c)^2$

11. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ is equal to [NCERT]

(a) $(5x-4)(4+x)^2$ (b) $(5x+4)(4-x)^2$
(c) $(5x-4)^2(4+x)$ (d) $(5x+4)^2(4-x)$

12. If $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = k(x+y+z)^3$,

then the value of k is [NCERT]

(a) 1 (b) 2 (c) 4 (d) 8

13. $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ is equal to [NCERT]

(a) $-(x^3+y^3)$ (b) (x^3+y^3)

(c) $-3(x^3+y^3)$ (d) $-2(x^3+y^3)$

14. $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ is equal to [NCERT]

(a) 0 (b) 1

(c) $x+y$ (d) xy

15. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$ is equal to [NCERT]

(a) $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$

(b) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

(c) $(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

(d) None of the above

16. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$ is equal to [NCERT]

where p is any scalar.

(a) $(1+pxyz)(x-y)(y-z)(z-x)$

(b) $(1-pxyz)(x+y)(y+z)(z+x)$

(c) $(1-pxyz)(x-y)(y-z)(z-x)$

(d) $pxyz(x-y)(y-z)(z-x)$

17. If

$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = k(a+b+c)(ab+bc+ca)$,

then the value of k is [NCERT]

(a) 1 (b) 2

(c) 3 (d) 9

18. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$ is equal to [NCERT]

(a) 0

(b) $10p$

(c) 1

(d) None of the above

19. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is [NCERT Exemplar]

(a) $a^3+b^3+c^3$ (b) $3bc$

(c) $a^3+b^3+c^3-3abc$ (d) None of these

20. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) $abc(b-c)(c-a)(a-b)$

(b) $(b-c)(c-a)(a-b)$

(c) $(a+b+c)(b-c)(c-a)(a-b)$

(d) None of the above

21. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is [NCERT Exemplar]

(a) $9x^2(x+y)$ (b) $9y^2(x+y)$

(c) $3y^2(x+y)$ (d) $7x^2(x+y)$

22. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then the value of $\Delta + \Delta_1$ is [NCERT Exemplar]

(a) $\Delta \cdot \Delta_1$ (b) 1

(c) 0 (d) None of these

23. The value of $\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$ is [NCERT Exemplar]

(a) 1 (b) 32 (c) $\tan\theta$ (d) 0

24. $\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) $(x+p)(x^2+px-2q^2)$

(b) $(x-p)(x^2+px-2q^2)$

(c) $(x-p)(x^2-px+2q^2)$

(d) None of the above

25. If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then Δ is equal to [NCERT Exemplar]

(a) $a+b+c$ (b) abc

(c) $\frac{1}{2}(a+b+c)$ (d) 0

26. If $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$, then the other roots are [NCERT Exemplar]

(a) 1, 3 (b) 0, 2 (c) – 1, 1 (d) 2, 4

27. If the determinant $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then the value of $\sin\theta$ is [NCERT Exemplar]

(a) $\frac{1}{3}$ or 1 (b) $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{3}}{2}$

(c) 0 or $\frac{1}{2}$ (d) None of these

28. Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then [NCERT Exemplar]

(a) $\Delta_1 = -\Delta$ (b) $\Delta \neq \Delta_1$

(c) $\Delta - \Delta_1 = 0$ (d) None of these

29. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) $2x^3y^3z^3$ (b) $2x^2y^2z^2$

(c) $4x^2y^2z^2$ (d) $x^3y^3z^3$

30. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) $(x + y + z)(xy + yz + zx)$

(b) $3(xy + yz + zx)$

(c) $3(x + y + z)(xy + yz + zx)$

(d) $3(x + y + z)$

31. $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) xyz (b) $x^2y^2z^2$

(c) 0 (d) $(x + y + z)$

32. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) $(a - 1)$ (b) $(a - 1)^2$ (c) $(a - 1)^3$ (d) $(a - 1)^4$

33. $\begin{vmatrix} 1 & 5 & \pi \\ \log_e e & 5 & \sqrt{5} \\ \log_{10} 10 & 5 & e \end{vmatrix}$ is equal to

(a) $\sqrt{\pi}$ (b) e (c) 1 (d) 0

34. The value of $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$ is

(a) $4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$ (b) $3\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

(c) $2\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$ (d) None of these

35. The value of determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ is

(a) 0 (b) $2abc$

(c) $a^2b^2c^2$ (d) None of these

36. The arbitrary constant on which the value of the determinant $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \cos(p-d)a & \cos pa & \cos(p-d)a \\ \sin(p-d)a & \sin pa & \sin(p-d)a \end{vmatrix}$ does not depend, is

(a) α (b) p (c) d (d) a

37. If a, b and c are unequal what is the condition that the value of the determinant, $\Delta \equiv \begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix}$ is 0?

(a) $1 + abc = 0$

(b) $a + b + c + 1 = 0$

(c) $(a - b)(b - c)(c - a) = 0$

(d) None of the above

38. The value of x obtained from the equation $\begin{vmatrix} x+\alpha & \beta & \gamma \\ \gamma & x+\beta & \alpha \\ \alpha & \beta & x+\gamma \end{vmatrix} = 0$ will be

(a) 0 and $-(\alpha + \beta + \gamma)$ (b) 0 and $(\alpha + \beta + \gamma)$

(c) 1 and $(\alpha - \beta - \gamma)$ (d) 0 and $(\alpha^2 + \beta^2 + \gamma^2)$

39. A factor of $\Delta(x) = \begin{vmatrix} x^3+1 & 2x^4+3x^2 & 3x^5+4x \\ 2 & 5 & 7 \\ 3 & 14 & 19 \end{vmatrix}$ is

(a) x (b) $(x - 1)^2$

(c) $(x + 1)^2$ (d) None of these

40. The roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ are

(a) $-1, -2$ (b) $-1, 2$ (c) $1, -2$ (d) $1, 2$

41. If $a+b+c=0$, then the solution of the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$ is

(a) 0 (b) $\pm\frac{3}{2}(a^2+b^2+c^2)$

(c) $0, \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$ (d) $0, \pm\sqrt{a^2+b^2+c^2}$

42. If a, b and c are positive integers, then the determinant $\Delta=\begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by

(a) x^3 (b) x^2

(c) $(a^2+b^2+c^2)$ (d) None of these

43. If x, y and z are different from zero and $\Delta=\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix}=0$, then the value of the expression $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$ is

(a) 0 (b) -1 (c) 1 (d) 2

44. Determinant $\begin{vmatrix} 1 & x & y \\ 2 & \sin x+2x & \sin y+2y \\ 3 & \cos x+3x & \cos y+3y \end{vmatrix}$ is equal to

(a) $\sin(x-y)$ (b) $\cos(x-y)$

(c) $\cos(x+y)$ (d) $xy\sin(x-y)$

45. The value of the determinant $\begin{vmatrix} a^2 & a & 1 \\ \cos(nx) & \cos(n+1)x & \cos(n+2)x \\ \sin(nx) & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$ is independent of

(a) n (b) a

(c) x (d) None of these

46. If $D_r=\begin{vmatrix} 2^{r-1} & 3^{r-1} & 4^{r-1} \\ x & y & z \\ 2^n-1 & (3^n-1)/2 & (4^n-1)/3 \end{vmatrix}$, then the value of $\sum_{r=1}^{n} D_r$ is equal to

(a) 1 (b) -1

(c) 0 (d) None of these

47. For the values of A, B, C and P, Q, R the value of $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$ is

(a) 0 (b) $\cos A\cos B\cos C$

(c) $\sin A\sin B\sin C$ (d) $\cos P\cos Q\cos R$

48. The value of $\sum_{n=1}^{N} U_n$, if $U_n=\begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$, is

(a) 0 (b) 1

(c) -1 (d) None of these

49. If $f(x)=\begin{vmatrix} 1 & 1 & 1 \\ 2x & (x-1) & x \\ 3x(x-1) & (x-1)(x-2) & x(x-1) \end{vmatrix}$, then $f(50)$ is equal to

(a) 0 (b) 1 (c) 100 (d) -100

50. The value of the determinant $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$ is

(a) $a^3\left(1-\frac{2}{a}\right)$ (b) $a^3\left(1+\frac{3}{a}\right)$

(c) $a^3\left(1-\frac{3}{a}\right)$ (d) $a^3\left(1+\frac{2}{a}\right)$

Differentiation and Integration of Determinant

51. The value of λ, if $ax^4+bx^3+cx^2+50x+d$

$$=\begin{vmatrix} x^3-14x^2 & -x & 3x+\lambda \\ 4x+1 & 3x & x-4 \\ -3 & 4 & 0 \end{vmatrix}, \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) 3

52. If $\Delta(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}[\Delta(x)]$ at $x=0$ is

(a) -1 (b) 0

(c) 1 (d) dependent of a

53. If $\Delta_1=\begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2=\begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

(a) $\Delta_1=3(\Delta_2)^2$ (b) $\frac{d}{dx}(\Delta_1)=3\Delta_2$

(c) $\frac{d}{dx}(\Delta_1)=2\Delta_2$ (d) $\Delta_1=3\Delta_2^{3/2}$

54. If $\Delta(x)=\begin{vmatrix} f(x)+f(-x) & 0 & x^4 \\ 3 & f(x)-f(-x) & \cos x \\ x^4 & 2x & f(x)f(-x) \end{vmatrix}$,

where, $f(x)$ is a real valued function of x, then the value of $\int_{-2}^{2} x^4\,\Delta(x)$

(a) depends upon the function $f(x)$
(b) is 4
(c) is – 4
(d) is zero

55. If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, then $\int_0^{\pi/2} \Delta(x)\, dx$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 0 (d) $-\frac{1}{2}$

Application of Determinant in Geometry

56. The equation of the line joining (1, 2) and (3, 6) using determinants is [NCERT]

(a) $y = x$ (b) $2y = x$
(c) $2x = y$ (d) $x = 3y$

57. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq units. The value of k will be [NCERT Exemplar]

(a) 9 (b) 3 (c) –9 (d) 6

58. If the coordinates of the vertices of an equilateral triangle with sides of length a are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is equal to [NCERT Exemplar]

(a) $\frac{a^4}{4}$ (b) $\frac{3a^2}{4}$ (c) $\frac{5a^4}{4}$ (d) $\frac{3a^4}{4}$

Solution of System of Linear Equations

59. If the system of equations $x + ky - z = 0$, $3x - ky - z = 0$ and $x - 3y + z = 0$ has non-zero solution, then k is equal to

(a) – 1 (b) 0 (c) 1 (d) 2

60. The system of equations $3x - 2y + z = 0$, $\lambda x - 14y + 15z = 0$, $x + 2y - 3z = 0$ has a solution other than $x = y = z = 0$, then λ is equal to

(a) 1 (b) 2 (c) 3 (d) 5

61. If the system of equations $2x + 3y + 5 = 0$, $x + ky + 5 = 0$ and $kx - 12y - 14 = 0$ be consistent, then value of k is

(a) $-2, \frac{12}{5}$ (b) $-1, \frac{1}{5}$
(c) $-6, \frac{17}{5}$ (d) $6, -\frac{12}{5}$

62. If the equations $2x + 3y + 1 = 0$, $3x + y - 2 = 0$ and $ax + 2y - b = 0$ are consistent, then

(a) $a - b = 2$ (b) $a + b + 1 = 0$
(c) $a + b = 3$ (d) $a - b - 8 = 0$

Miscellaneous

63. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ is (θ is real number) [NCERT Exemplar]

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\sqrt{2}$ (d) $\frac{2\sqrt{3}}{4}$

64. If $x, y \in R$, then the determinant $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies in the interval [NCERT Exemplar]

(a) $[-\sqrt{2}, \sqrt{2}]$ (b) $[-1, 1]$
(c) $[-\sqrt{2}, 1]$ (d) $[-1, -\sqrt{2}]$

65. Let $A = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, where $0 \le \theta \le 2\pi$, then the range of $|A|$ is [NCERT]

(a) (2, 4) (b) [2, 4]
(c) [2, 4) (d) All of these

66. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is [NCERT Exemplar]

(a) 0 (b) 2 (c) 1 (d) 3

67. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to [NCERT Exemplar]

(a) 0 (b) –1
(c) 1 (d) None of these

68. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to [NCERT Exemplar]

(a) 0 (b) –1
(c) 2 (d) 3

69. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then [NCERT Exemplar]

(a) $f(a) = 0$ (b) $f(b) = 0$
(c) $f(0) = 0$ (d) $f(1) = 0$

70. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these number is [NCERT Exemplar]

(a) 4 (b) 5 (c) – 4 (d) 9

71. In a ΔABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then ΔABC is a isosceles triangle. [NCERT Exemplar]

(a) Equilateral triangle (b) Right isosceles triangle
(c) Scalene triangle (d) Isosceles triangle

72. If $a_1, a_2, a_3, \ldots, a_r$ are in GP, then the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is [NCERT Exemplar]

(a) dependent on r (b) independent of r
(c) dependent on a (d) None of these

73. In a third order determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consists of sum of four terms. Then, it can be decomposed into n determinant, where n has the value

(a) 1 (b) 9 (c) 16 (d) 24

74. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is

(a) 0 (b) 1
(c) $\log_e xyz$ (d) None of these

75. If $f(\alpha) = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix}$, then $f(\sqrt[3]{3})$ is equal to

(a) 1 (b) – 4 (c) 4 (d) 2

76. If the matrix M_r is given by $M_r = \begin{bmatrix} r & r-1 \\ r-1 & r \end{bmatrix}$, $r = 1, 2, 3, \ldots$, then the value of

$$\det(M_1) + \det(M_2) + \ldots + \det(M_{2008}) \text{ is}$$

(a) 2007 (b) 2008 (c) $(2008)^2$ (d) $(2007)^2$

77. $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix}$, where $a = i$, $b = \omega$, $c = \omega^2$, then Δ is equal to

(a) i (b) $-\omega^2$ (c) ω (d) $-i$

78. If $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+2 & x+3 & x+4 \\ x+a & x+b & x+c \end{vmatrix} = 0$, then a, b, c are in

(a) AP (b) HP
(c) GP (d) None of these

79. If a, b and c are respectively the pth, qth and rth terms of an AP, then $\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix}$ is equal to

(a) 1 (b) – 1 (c) 0 (d) pqr

80. If a, b and c are in AP, then the value of $\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$ is

(a) $x - (a + b + c)$ (b) $9x^2 + a + b + c$
(c) $a + b + c$ (d) 0

81. Let $[x]$ represent the greatest integer less than or equal to x, then the value of the determinant $\begin{vmatrix} [e] & [\pi] & [\pi^2-6] \\ [\pi] & [\pi^2-6] & [e] \\ [\pi^2-6] & [e] & [\pi] \end{vmatrix}$ is

(a) – 8 (b) 8
(c) 10 (d) None of these

82. If $n = 3k$ and 1, ω, ω^2 are the cube roots of unity, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ has the value

(a) 0 (b) 1
(c) – 1 (d) None of these

83. The value of the determinant $\begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$ is

(a) 2 (10! 11!) (b) 2 (10! 13!)
(c) 2 (10! 11! 12!) (d) 2 (11! 12! 13!)

84. The coefficient of x in $f(x) = \begin{vmatrix} x & 1+\sin x & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$, $-1 < x \le 1$, is

(a) 1 (b) –2 (c) –1 (d) 0

85. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) 1 (d) $3\sqrt{3}$

86. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$, then the value of

$$\begin{vmatrix} b_2c_3 - b_3c_2 & c_2a_3 - c_3a_2 & a_2b_3 - c_3b_2 \\ b_3c_1 - b_1c_3 & c_3a_1 - c_1a_3 & a_3b_1 - a_1b_3 \\ b_1c_2 - b_2c_1 & c_1a_2 - c_2a_1 & a_1b_2 - a_2b_1 \end{vmatrix} \text{ is}$$

(a) 5 (b) 25 (c) 125 (d) 0

87. If $x^a y^b = e^m$, $x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively

(a) $\frac{\Delta_1}{\Delta_3}$ and $\frac{\Delta_2}{\Delta_3}$ (b) $\frac{\Delta_2}{\Delta_1}$ and $\frac{\Delta_3}{\Delta_1}$

(c) $\log\left(\frac{\Delta_1}{\Delta_3}\right)$ and $\log\left(\frac{\Delta_2}{\Delta_3}\right)$ (d) e^{Δ_1/Δ_3} and e^{Δ_2/Δ_3}

88. If $c < 1$ and the system of equations $x + y - 1 = 0$, $2x - y - c = 0$ and $-bx + 3by - c = 0$ is consistent, then the possible real values of b are

(a) $b \in \left(-3, \frac{3}{4}\right)$ (b) $b \in \left(-\frac{3}{4}, 4\right)$

(c) $b \in \left(-\frac{3}{4}, 3\right)$ (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. If $y = \sin px$ and y_n is the nth derivative of y, then $\Delta = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is equal to

(a) 0 (b) dependent of x
(c) non-zero constant (d) None of these

2. If x, y, z are integers in AP, lying between 1 and 9 and $x51$, $y41$ and $z31$ are three digited numbers, then value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x + y + z$ (b) $x - y + z$
(c) 0 (d) None of these

3. If a, b, c are sides of a ΔABC and A, B, C are angles opposite a, b, c, then $\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$ gives

(a) Δ = area of triangle (b) Δ = perimeter of triangle
(c) $\Delta = \Sigma a^2$ (d) None of these

4. If p, q, r are real, then $\Delta = \begin{vmatrix} p & 2-i & i+1 \\ 2+i & q & 3+i \\ 1-i & 3-i & r \end{vmatrix}$ is always

(a) real (b) imaginary
(c) 0 (d) None of these

5. If $\Delta = \begin{vmatrix} my + nz & mq - nr & mb + nc \\ kz - mx & kr - mp & kb - ma \\ nx + ky & np + kq & na + kb \end{vmatrix}$, then Δ ia equal to

(a) 0 (b) $\Delta \neq 0$
(c) $\Delta = f(x, y, z)$ (d) None of these

6. The system of equations $x + 2y + 3z = 4$, $2x + 3y + 4z = 5$, $3x + 4y + 5z = 6$ has

(a) infinitely many solutions (b) no solution
(c) unique solution (d) None of these

7. If $\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$, then Δ is

(a) an integer (b) a natural number
(c) an irrational number (d) an imaginary number

8. All the values of t for which the system of equations.

$$(t-1)x + (3t+1)y + 2tz = 0$$
$$(t-1)x + (4t-2)y + (t+3)z = 0$$
$$2x + (3t+1)y + 3(t-1)z = 0$$

has a non-trivial solution, are

(a) 1 or 2 (b) 2 or 4
(c) 0 or 3 (d) None of these

9. If $a, b, c \in R$, then the determinant

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \\ bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \end{vmatrix}$$

is always non-negative. Also, find the condition for the determinant to be zero.

(a) $a + b + c = 0$ or $a = b = c$
(b) only $a + b + c = 0$
(c) only $a = b = c$
(d) None of the above

10. If $a, b \in R$, then

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$ [NCERT]

(a) $\le 27a^2b^2$ (b) $\ge 27a^2b^2$
(c) $\ge 9a^2b^2$ (d) None of these

11. The maximum value of

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$ is

(a) – 2 (b) – 6
(c) 6 (d) 2

12. If $2s = a + b + c$ and $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}$

$= ks^3(s-a)(s-b)(s-c)$,

then the numerical quantity k should be
(a) 1 (b) 2 (c) 3 (d) 4

13. If $A = \begin{vmatrix} 23 & 1+i & -i \\ 1-i & -31 & 4-5i \\ i & 4+5i & 17 \end{vmatrix}$, then det (A) is

(a) complex number with positive real part
(b) complex number with negative imaginary part
(c) pure imaginary
(d) real

14. $\begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{(n+1)}P_{(n+1)} & {}^{(n+2)}P_{(n+2)} \\ {}^nC_n & {}^{(n+1)}C_{(n+1)} & {}^{(n+2)}C_{(n+2)} \end{vmatrix}$ is equal to

(a) $n\,(n!)$ (b) $(n+1)\,(n+1)!$
(c) $(n+2)\,(n+2)!$ (d) $(n^2+n+1)n!$

15. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, then the constant term is

(a) 0 (b) 1
(c) – 1 (d) 2

16. If a, b and c are cube roots of unity, then $\begin{vmatrix} e^a & e^{2a} & e^{3a}-1 \\ e^b & e^{2b} & e^{3b}-1 \\ e^c & e^{2c} & e^{3c}-1 \end{vmatrix}$ is equal to

(a) 0 (b) e
(c) e^2 (d) e^3

17. $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$ and the vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then abc is equal to

(a) 2 (b) – 1
(c) 1 (d) 0

18. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value – 1. Then,

(a) C is empty
(b) B and C have the same number of elements
(c) $A = B \cup C$
(d) B has twice as many elements as C

19. $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & y+z \end{vmatrix} = 0$, if

(a) x, y and z are in AP (b) x, y and z are in GP
(c) x, y and z are in HP (d) xy, yz and zx are in AP

20. If ω is an imaginary cube root of unity, then the value of $\begin{vmatrix} a & b\omega^2 & a\omega \\ b\omega & c & b\omega^2 \\ c\omega^2 & a\omega & c \end{vmatrix}$ is

(a) $a^3 + b^3 + c^2 - 3abc$ (b) $a^2b - b^2c$
(c) 0 (d) $a^2 + b^2 + c^2$

21. If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$, then θ is equal to

(a) $\frac{7\pi}{24}, \frac{11\pi}{24}$ (b) $\frac{5\pi}{24}, \frac{7\pi}{24}$
(c) $\frac{11\pi}{24}, \frac{\pi}{24}$ (d) $\frac{\pi}{24}, \frac{7\pi}{24}$

22. Let $px^4 + qx^3 + rx^2 + sx + t = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix}$,

where p, q, r, s and t are constants, then t is equal to
(a) 0 (b) 1 (c) 2 (d) –1

23. If α, β and γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 0 (b) – 2
(c) 2 (d) 4

24. If $\Delta_r = \begin{vmatrix} 2r & x & N(N+1) \\ 6r^2-1 & y & N^2(2N+3) \\ 4r^3-2Nr & z & N^3(N+1) \end{vmatrix}$, where $N \in$ natural number. Then, $\sum_{t=1}^{N} \Delta_r$ is equal to

(a) N (b) N^2
(c) zero (d) None of these

25. If a, b and c are pth, qth and rth terms of an HP, then $\begin{vmatrix} bc & ac & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) term containing a, b, c, p, q and r
(b) a constant
(c) zero
(d) None of the above

26. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(a) 0 (b) -1
(c) 1 (d) no real values

27. If [ ] denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0$; $0 \le y < 1$; $1 \le z < 2$, then the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

28. Let $\Delta = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & 1+x_1y_3 \\ 1+x_2y_1 & 1+x_2y_2 & 1+x_2y_3 \\ 1+x_3y_1 & 1+x_3y_2 & 1+x_3y_3 \end{vmatrix}$, then value of Δ is

(a) $x_1x_2x_3 + y_1y_2y_3$
(b) $x_1x_2x_3y_1y_2y_3$
(c) $x_2x_3y_2y_3 + x_3x_1y_3y_1 + x_1x_2y_1y_2$
(d) 0

29. If $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$, where $a \ne 0$, $b \ne 0$, $c \ne 0$, then $a^{-1} + b^{-1} + c^{-1}$ is

(a) 4 (b) -3
(c) -2 (d) -1

30. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then $f(2x) - f(x)$ is equal to

(a) ax (b) $ax(2a + 3x)$
(c) $ax(2 + 3x)$ (d) None of these

31. If $\Delta_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2+n+1 & n^2+n \\ 2r-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{r=1}^{n} \Delta_r = 56$, then n is equal to

(a) 4 (b) 6
(c) 7 (d) 8

32. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then $\Delta(x)$ is a polynomial of degree

(a) 2 (b) 3
(c) atmost 2 (d) atmost 3

33. If $\alpha, \beta, \gamma \in R$, then the determinant

$$\Delta = \begin{vmatrix} (e^{i\alpha}+e^{-i\alpha})^2 & (e^{i\alpha}-e^{-i\alpha})^2 & 4 \\ (e^{i\beta}+e^{-i\beta})^2 & (e^{i\beta}-e^{-i\beta})^2 & 4 \\ (e^{i\gamma}+e^{-i\gamma})^2 & (e^{i\gamma}-e^{-i\gamma})^2 & 4 \end{vmatrix} \text{ is}$$

(a) independent of α, β and γ
(b) dependent on α, β and γ
(c) independent of α, β
(d) independent of α, γ

More Than One Correct Option

34. The determinant $\Delta = \begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by

(a) x (b) x^2
(c) x^3 (d) x^4

35. If $A + B + C = \pi$, $e^{i\theta} = \cos\theta + \sin\theta$ and

$$z = \begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}, \text{ then}$$

(a) $\text{Re}(z) = 4$ (b) $\text{lm}(z) = 0$
(c) $\text{Re}(z) = -4$ (d) $\text{lm}(z) = -1$

36. The determinant $\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if

(a) a, b, c are in AP
(b) a, b, c are in GP
(c) a, b, c are in HP
(d) α is the root of $ax^2 + 2bx + c = 0$

37. Let $f(x) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$, where the symbols have their usual meanings. The $f(x)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n + 1)!$
(c) $n!$ (d) None of these

38. Eliminating a, b, c from $x = \dfrac{a}{b-c}$, $y = \dfrac{b}{c-a}$, $z = \dfrac{c}{a-b}$, we get

(a) $\begin{vmatrix} 1 & -x & x \\ 1 & -y & y \\ 1 & -z & z \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & -x & x \\ 1 & 1 & -y \\ 1 & z & 1 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = 0$ (d) None of these

39. If T_p, T_q, T_r are the pth, qth and rth terms of an AP, then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ cannot be equal to

(a) 1 (b) – 1
(c) 0 (d) $p + q + r$

40. Suppose x, y, z are positive and none of x, y, z is 1. If $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \sin(x+y) & -\cos(x+y) & \sin^2 z \end{vmatrix}$, then Δ is independent of

(a) x (b) y
(c) y and z only (d) z

41. If $S_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2 - 1 & y & n^2(2n+3) \\ 4r^3 - 2nr & z & n^3(n+1) \end{vmatrix}$, then the value $\sum_{r=1}^{n} S_r$ is independent of

(a) x (b) y (c) n (d) z

42. Let $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 \Delta(x)dx = -16$, where a, b, c, d are in AP, then common difference of the AP is

(a) 1 (b) 2
(c) – 2 (d) None of these

43. The equation $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = 0$ has

(a) exactly two distinct roots
(b) one pair of equal real roots
(c) modulus of each root 1
(d) three pairs of equal roots

Comprehension Based Questions

Passage I

Directions (Q. Nos. 44-46) *Consider the determinant* $\Delta = \begin{vmatrix} p & q & r \\ x & y & z \\ l & m & n \end{vmatrix}$ *M_{ij} denotes the minor of an element in ith row and jth column and C_{ij} denotes the cofactor of an element in ith row and jth column.*

44. The value of $p \cdot C_{21} + q \cdot C_{22} + r \cdot C_{23}$ is

(a) 0 (b) $-\Delta$
(c) Δ (d) Δ^2

45. The value of $x \cdot C_{21} + y \cdot C_{22} + z \cdot C_{23}$ is

(a) 0 (b) $-\Delta$
(c) Δ (d) Δ^2

46. The value of $q \cdot M_{12} - y \cdot M_{22} + m \cdot M_{32}$ is

(a) 0 (b) $-\Delta$ (c) Δ (d) Δ^2

Passage II

Directions (Q. Nos. 47-49) *A 3×3 determinant has its entries as either 1 or – 1. The number of such determinants is $2^9 = 512$. We will call a 3×3 determinant with entires 1 or – 1 as minus special, if product of elements of any rows or any columns is – 1 for example* $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$ *is a minus special determinant.*

47. The number of 3×3 minus special determinants must be

(a) 10 (b) 12
(c) 16 (d) 18

48. The number of $n \times n$ minus special determinants must be

(a) 2^{n-1} (b) $2^{(n-1)^2}$
(c) $\dfrac{13n^2 - 37n + 26}{2}$ (d) 2^n

49. The minimumn value of a 3×3 minus special determinant is

(a) – 6 (b) – 4
(c) – 2 (d) 0

Directions (Q. Nos. 50-56) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

50. **Statement I**

$$\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$ is independent of θ.

Statement II If $f(\theta) = c$, then $f(\theta)$ is independent of θ.

51. **Statement I** If $f(x) = \begin{vmatrix} (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \\ (1+x)^{41} & (1+x)^{42} & (1+x)^{43} \end{vmatrix}$,

then coefficient of x in $f(x)$ is zero.

Statement II If $F(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n$, then $A_1 = F'(0)$, where dash denotes the differential coefficient.

52. **Statement I** If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$,

then $\Delta'(x) \neq \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$

Statement II $\frac{d}{dx}\{f(x)\,g(x)\} \neq \frac{d}{dx} f(x) \frac{d}{dx} g(x)$

53. **Statement I** If a, b, c are even natural numbers, then $\Delta = \begin{vmatrix} a-1 & a & a+1 \\ b-1 & b & b+1 \\ c-1 & c & c+1 \end{vmatrix}$ is an even natural number.

Statement II Sum and product of two even natural numbers is also an even natural number.

54. Let x, y, z are three integers lying between 1 and 9 such that $x51$, $y41$ and $z31$ are three digit numbers.

Statement I The value of the determinant $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is zero.

Statement II The value of a determinant is zero, if the entries in any two rows (or columns) of the determinant are correspondingly proportional.

Previous Years' Questions

55. Let a, b and c be such that $(b+c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then the value of n is **[AIEEE 2009]**

(a) zero (b) any even integer
(c) any odd integer (d) any integer

56. Let a, b and c be any real numbers. Suppose that there are real numbers x, y and z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE 2008]**

(a) 1 (b) 2
(c) -1 (d) 0

57. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $xy \neq 0$, then D is divisible by **[AIEEE 2007]**

(a) both x and y
(b) x but not y
(c) y but not x
(d) niether x nor y

58. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree **[AIEEE 2005]**

(a) 0 (b) 1
(c) 2 (d) 3

Answers

Check Point 1

1. (d)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (c)	**7.** (b)	**8.** (a)

Check Point 2

1. (a)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (a)

Round I

1. (a)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (b)
11. (b)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (d)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (c)	**20.** (d)
21. (b)	**22.** (c)	**23.** (d)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (c)
31. (c)	**32.** (c)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (a)	**39.** (b)	**40.** (b)
41. (c)	**42.** (d)	**43.** (d)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (a)	**48.** (a)	**49.** (a)	**50.** (b)
51. (c)	**52.** (b)	**53.** (a)	**54.** (d)	**55.** (a)	**56.** (c)	**57.** (b)	**58.** (d)	**59.** (c)	**60.** (d)
61. (c)	**62.** (a)	**63.** (a)	**64.** (a)	**65.** (d)	**66.** (c)	**67.** (a)	**68.** (a)	**69.** (c)	**70.** (c)
71. (d)	**72.** (b)	**73.** (d)	**74.** (a)	**75.** (b)	**76.** (c)	**77.** (a)	**78.** (a)	**79.** (c)	**80.** (d)
81. (a)	**82.** (a)	**83.** (c)	**84.** (b)	**85.** (a)	**86.** (b)	**87.** (d)	**88.** (c)		

Round II

1. (a)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (c)
21. (a)	**22.** (a)	**23.** (a)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (b)	**30.** (c)
31. (c)	**32.** (c)	**33.** (a)	**34.** (a,b,c,d)	**35.** (b,c)	**36.** (b,d)	**37.** (a,c)	**38.** (b,c)	**39.** (a,c)	**40.** (a,b,d)
41. (a,b,c,d)	**42.** (b,c)	**43.** (b,c,d)	**44.** (a)	**45.** (c)	**46.** (b)	**47.** (c)	**48.** (b)	**49.** (b)	**50.** (a)
51. (a)	**52.** (d)	**53.** (d)	**54.** (d)	**55.** (c)	**56.** (a)	**57.** (a)	**58.** (c)		

the **Guidance**

Round I

1. Let $A = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Expanding to corresponding first row, we get

$$A = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= -x^3.$$

Hence, A is independent of θ.

2. Given, determinant $= \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding corresponding to R_1, we get

$$= \cos\alpha\cos\beta(\cos\alpha\cos\beta - 0) - \cos\alpha\sin\beta(-\cos\alpha\sin\beta - 0) - \sin\alpha(-\sin^2\beta\sin\alpha - \cos^2\beta\,\alpha)$$

$$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha[\sin^2\beta + \cos^2\beta]$$

$$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha[\sin^2\beta + \cos^2\beta]$$

$$= \cos^2\alpha(1) + \sin^2\alpha[1]$$

$$= \cos^2\alpha + \sin^2\alpha \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= 1$$

3. Given $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

On expanding both determinants,

$$2x^2 - 40 = 18 + 14$$

$$\Rightarrow \quad 2x^2 = 40 + 32 = 72 \Rightarrow x^2 = 36$$

$$\Rightarrow \quad x = \pm 6$$

4. We have, $\Delta = \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0$

$$\Rightarrow \quad \Delta = a(a^2 - 0) - b(0 - b^2) = a^3 + b^3$$

$\Rightarrow \quad a^3 + b^3 = 0 \Rightarrow \left(\frac{a}{b}\right)^3 = -1$

$\therefore \left(\frac{a}{b}\right)$ is one of the cube roots of -1.

5. Given, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

It can be observed that in the first column, two entries are zero. Thus, we expand the matrix A along the first column (C_1) for finding $|A|$.

$\therefore \quad |A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

$= 1(1 \times 4 - 0 \times 2) = 4$

$\therefore \quad 27|A| = 27 \times 4 = 108$ ...(i)

Now, $3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\therefore \quad |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$

It can be observed that in the first column, two entries are zero. Thus, we expand the determinant along the first column (C_1) for easier calculation.

$|3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} = 3(3 \times 12 - 0 \times 6)$

$\Rightarrow \quad |3A| = 3(36) = 108$ ...(ii)

From Eqs. (i) and (ii), we get

$$|3A| = 27|A|$$

Note If A is a square matrix of order n, then $|kA| = k^n |A|$

6. Let $A = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+ac+bc \\ 1 & ca & bc+ba+ca \\ 1 & ab & ca+cb+ab \end{vmatrix}$

(using $C_3 \to C_3 + C_2$)

Taking $ab + bc + ca$ common from C_3, we get

$= (ab + bc + ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} = (ab + bc + ca) \times 0 = 0$

(since, the two columns C_1 and C_3 are identical)

7. (i) Take out factors a, b, c from R_1, R_2 and R_3, respectively.

(ii) Take out factors a, b, c from C_1, C_2 and C_3, respectively. It will change the determinant into easiest form which you will be able to solve.

$= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

(taking out factors a from R_1, b from R_2 and c from R_3)

$= (abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

(taking out factorn a form C_1, from C_2 and C from C_3)

$= a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{vmatrix}$ (using $R_1 \to R_1 + R_2$ and $R_2 \to R_2 - R_3$)

Expanding corresponding to first row R_1, we get

$= a^2b^2c^2\left[0\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\right]$

$= a^2b^2c^2[0 - 0 + 2(0 + 2)] = 4a^2b^2c^2$

8. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$

(using $R_1 \to xR_1$ and $R_2 \to yR_2$ and $R_3 \to zR_3$)

$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$ (take out xyz common from C_3)

$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding corresponding to C_3, we get

$= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$

$= [(y^2 - x^2)(z^3 - x^3) - (z^2 - x^2)(y^3 - x^3)]$

$= (y + x)(y - x)(z - x)(z^2 + x^2 + xz)$

$- (z + x)(z - x)(y - x)(y^2 + x^2 + xy)$

$= (y - x)(z - x)[(y + x)(z^2 + x^2 + xz) - (z + x)(y^2 + x^2 + xy)]$

$= (y - x)(z - x)[yz^2 + yx^2 + xyz + xz^2 + x^3$

$+ x^2z - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y]$

$= (y - x)(z - x)[yz^2 - zy^2 + xz^2 - xy^2]$

$= (y - x)(z - x)[yz(z - y) + x(z^2 - y^2)]$

$= (y - x)(z - x)[yz(z - y) + x(z - y)(z + y)]$

$= (y - x)(z - x)[(z - y)(xy + yz + zx)]$

$= (x - y)(y - z)(z - x)(xy + yz + zx)$

9. $= \begin{vmatrix} y + k & y & y \\ y & y + k & y \\ y & y & y + k \end{vmatrix} = \begin{vmatrix} 3y + k & y & y \\ 3y + k & y + k & y \\ 3y + k & y & y + k \end{vmatrix}$

(using $C_1 \to C_1 + C_2 + C_3$)

$= (3y + k)\begin{vmatrix} 1 & y & y \\ 1 & y + k & y \\ 1 & y & y + k \end{vmatrix}$

[take out $(3y + k)$ common from C_1]

$$= (3y + k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_3, we get

$$= (3y + k)\,(1 \times k \cdot k) = k^2\,(3y + k)$$

10. Firstly apply $R_1 \to R_1 + R_2 + R_3$ and then you will be able to take a term common from R_1 and then solve it.

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

(using $R_1 \to R_1 + R_2 + R_3$)

Take out $(a + b + c)$ common from R_1, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

(using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

Expanding along R_1, we get

$$= (a + b + c)\,\{1(-b-c-a)\,(-c-a-b)\}$$

$$= (a + b + c)\,[-(b + c + a) \times (-)\,(c + a + b)]$$

$$= (a + b + c)\,(a + b + c)\,(a + b + c) = (a + b + c)^3$$

11. Firstly apply $C_1 \to C_1 + C_2 + C_3$ and then you will be able to take a term common from C_1 and then solve it.

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$$

(using $C_1 \to C_1 + C_2 + C_3$)

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

[take out $(5x + 4)$ common from C_1].

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x+4 & 0 \\ 0 & 0 & -x+4 \end{vmatrix}$$

Expanding along C_1, we get

$$= (5x + 4)\,\{1\,(4 - x)\,(4 - x)\}$$

$$= (5x + 4)\,(4 - x)^2$$

12. $$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

(using $C_1 \to C_1 + C_2 + C_3$)

Take out $2(x + y + z)$ common from C_1, we get

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

(using $R \to R_1 - R_i$, $R_3 \to R_3 - R_1$)

Take out $(x + y + z)$ common from R_2 and R_3, we get

$$= 2(x+y+z)\,(x+y+z)\,(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we get

$$= 2(x + y + z)^3\,[(1)\,(1 - 0)] = 2(x + y + z)^3$$

$$= k(x + y + z)^3$$

$\therefore \quad k = 2$

13. Let $A = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$$= \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix} \text{(using } C_1 \to C_1 + C_2 + C_3)$$

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$$

[taking out $2(x + y)$ common from C_1]

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along R_3, we get

$$= 2(x + y) \times 1(-x^2 + y(x - y))$$

$$= 2(x + y)(-x^2 + xy - y^2)$$

$$= -2(x + y)(x^2 - xy + y^2) = -2\,(x^3 + y^3)$$

14. Let $A = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$

(using $R_2 \to xR_2 - R_1$ and $R_3\ R_3 - R_1$)

Expanding along C_1, we get $A = 1\,(yx - 0) = xy$

15. Firstly, operate $C_3 \to C_3 + C_1$ and take common $(\alpha + \beta + \gamma)$ from C_3 and then easy to expand the determinant.

$$\text{Let } A = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\gamma+\beta \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$

(using $C_3 \to C_3 + C_1$)

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

[taking out $(\alpha + \beta + \gamma)$ common from C_1]

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_3, we get

$= (\alpha + \beta + \gamma)\,[(\beta - \alpha)\,(\gamma^2 - \alpha^2) - (\gamma - \alpha)\,(\beta^2 - \alpha^2)]$

$= (\alpha + \beta + \gamma)\,[(\beta - \alpha)\,(\gamma - \alpha)\,(\gamma + \alpha) - (\gamma - \alpha)(\beta - \alpha)(\beta + \alpha)]$

$= (\alpha + \beta + \gamma)\,(\beta - \alpha)\,(\gamma - \alpha)\,[(\gamma + \alpha - \beta - \alpha]$

$= (\alpha + \beta + \gamma)\,(\gamma - \beta)\,(\beta - \alpha)\,(\gamma - \alpha)$

$= (\alpha + \beta + \gamma)\,(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \alpha)$

16. Firstly, we write the given determinant as sum of two determinants, since all elements of C_3 ae expressed as sum of two terms and then apply operations in both determinant separately.

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

In the first determinant, interchange C_1 and C_3; C_2 and C_3 and in the second determinant take out p from C_3, we get

$$= (-1)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + p\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(taking out x from R_1, y from R_2 and z from R_3 in the second determinant)

$$= (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_1, we get

$= (1 + pxyz)\,[(y - x)(z^2 - x^2) - (z - x)(y^2 - x^2)]$

$= (1 + pxyz)\,[(y - x)(z - x)(z + x) - (y - x)\,(z - x)\,(y + x)]$

$= (1 + pxyz)\,(y - x)(z - x)[(z + x) - (y + x)]$

$= (1 + pxyz)\,(y - x)(z - x)[(z + x - y - x)]$

$= (1 + pxyz)[(y - x)(z - x)(z - y) = (1 + pxyz)(x - y)(y - z)(z - x)$

17. $$\text{Let } A = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

(using $C_1 \to C_1 + C_2 + C_3$)

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

[taking out $(a + b + c)$ common from C_1]

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

Expanding along C_1, we get

$= (a + b + c)[(2b + a)(2c + a) - (a - b)\,(a - c)]$

$= (a + b + c)[(4bc + 2ab + 2ac + a^2 - a^2 + ac + ba - bc)]$

$= (a + b + c)(3ab + 3bc + 3ac) = 3(a + b + c)(ab + bc + ca)$

18. $$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

(using $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$)

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(using } R_3 \to R_3 - 3R_2\text{)}$$

Expanding along C_1, we get $= 1\,(1 \times 1 - 0) = 1$

19. $$\text{Let } \Delta = \begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix} \quad \text{(using operation, } C_2 \to C_2 + C_3\text{)}$$

$$= \begin{vmatrix} a-b & a+b+c & a \\ b-c & a+b+c & b \\ c-a & a+b+c & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a-b & 1 & a \\ b-c & 1 & b \\ c-a & 1 & c \end{vmatrix}$$

(using operations, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (a+b+c)\begin{vmatrix} a-b & 1 & a \\ 2b-a-c & 0 & b-a \\ c-2a+b & 0 & c-a \end{vmatrix}$$

Expanding along C_2

$= -(a+b+c)\{(2b-a-c)(c-a)-(b-a)(c-2a+b)\}$

$= -(a+b+c)\{(2bc-2ba-ac+a^2-c^2+ac-bc$
$+2ab-b^2+ac-2a^2+ab\}$

$= -(a+b+c)(-a^2-b^2-c^2+ab+bc+ca)$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= a^3+b^3+c^3-3abc \quad [\because a^3+b^3+c^3-3abc=(a+b+c)$
$(a^2+b^2+c^2-ab-bc-ca)]$

20. Let $\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

$= \begin{vmatrix} b(b-a) & (a-c) & c(b-a) \\ a(-a+b) & (a-b) & b(b-a) \\ c(b-a) & (c-a) & a(b-a) \end{vmatrix}$

$= (b-a)^2 \begin{vmatrix} b & (b-c) & c \\ a & (a-b) & b \\ c & (c-a) & a \end{vmatrix}$ (using operation $C_2 \to C_2 + C_3$)

$= (b-a)^2 \begin{vmatrix} b & b & c \\ a & a & b \\ c & c & a \end{vmatrix}$ ($\because C_1$ and C_2 are identical columns)

$= 0$

21. Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

(using operation $R_1 \to R_1 + R_2 + R_3$)

$\Delta = \begin{vmatrix} 3(x+y) & 3(x+y) & 3(x+y) \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$\Delta = 3(x+y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

(using operation $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$\Delta = 3(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix}$

Expanding along R_1

$\Delta = 3(x+y)\{2y^2+y^2\} = 3(x+y)(3y^2)$

$\Delta = 9y^2(x+y)$

22. We have, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$

$= \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$ (interchanging C_1 and C_2)

$= (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta \Rightarrow \Delta_1 + \Delta = 0$

23. Applying $C_1 \to C_1 - C_2 - C_3$, we have

$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$

24. Applying $C_1 \to C_1 - C_2$, we have

$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$

$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$ (applying $R_1 \to R_1 + R_2$)

Expanding along C_1, we have

$\Delta = (x-p)(px+x^2-2q^2) = (x-p)(x^2+px-2q^2)$

25. Interchanging rows and columns, we get

$\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

Taking '-1' common from R_1, R_2 and R_3, we get

$\Delta = (-1)^3\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta$

$\Rightarrow \quad 2\Delta = 0$ or $\Delta = 0$

26. Applying $R_1 \to (R_1 + R_2 + R_3)$, we get

$\Delta = \begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$

Taking $(x+4)$ common from R_1, we get

$\Delta = (x+4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$\Delta = (x+4)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}$

Expanding along R_1,

$$\Delta = (x+4)[(x-1)(x-3)-0]$$

Thus, $\Delta = 0$ implies

$$x = -4, 1, 3$$

27. Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

$\Rightarrow \quad 2[5(2+7\sin 3\theta) - 10(\cos 2\theta + 4\sin 3\theta)] = 0$

$\Rightarrow \quad 2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$

$\Rightarrow \quad 2 - 2\cos 2\theta - \sin 3\theta = 0$

$\Rightarrow \quad \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $(2\sin\theta - 1) = 0$ or $(2\sin\theta + 3) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

28. Since, $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

29. Take x^2, y^2 and z^2 common from C_1, C_2 and C_3 respectively, then its is easy to evaluate the determinant.

Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix} = (x^2y^2z^2)\begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

(taking x^2, y^2 and z^2 common from C_1, C_2 and C_3 respectively)

$$= (x^3y^3z^3)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

(taking x, y and z common from R_1, R_2 and R_3 respectively)

$$= (x^3y^3z^3)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \qquad (\text{using } R_3 \to R_3 - R_2)$$

Expanding along C_1, we get

$$= (x^3y^3z^3).(-1).\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -(x^3y^3z^3)(-1-1) = 2x^3y^3z^3$$

Hence, $\Delta = 2x^3y^3z^3$

30. Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & -x+y & -x+z \end{vmatrix}$$

[taking $(x+y+z)$ common from C_1]

$$= x+y+z\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (x+y+z)\cdot 1\begin{vmatrix} 2y+x & x-y \\ x-z & 2z+x \end{vmatrix}$$

Expanding along C_1, we get

$= (x+y+z)[(2y+x)(2z+x) - (x-z)(x-y)]$

$= (x+y+z)[(4yz + 2xy + 2xz + x^2) - (x^2 - xy - xz + yz)]$

$= (x+y+z)(3xy + 3yz + 3zx)$

Hence, $\quad \Delta = 3(x+y+z)(xy+yz+zx)$

31. Firstly, multiply R_1 by x, R_2 by y and R_3 by z and take xyz common from C_1 and C_2 and then it is easy to simplify.

Let $\quad \Delta = \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$

On applying $R_1 \to R_1(x)$, $R_2 \to R_2(y)$ and $R_3 \to R_3(z)$, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+xz \\ yz^2x^2 & xyz & yz+yx \\ zx^2y^2 & xyz & xz+yz \end{vmatrix}$$

($\because R_1, R_2$ and R_3 are multiplied by x, y and z respectively, therefore we divide the determinant by xyz also)

$$\Rightarrow \quad \Delta = \frac{1}{xyz}(xyz)^2\begin{vmatrix} yz & 1 & xy+xz \\ zx & 1 & yz+yx \\ xy & 1 & xz+yz \end{vmatrix}$$

(taking xyz common from C_1 and C_2)

$$\Rightarrow \quad \Delta = xyz\begin{vmatrix} yz & 1 & xy+yz+zx \\ zx & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix} \qquad (\text{using } C_3 \to C_3 + C_1)$$

$$\Rightarrow \quad \Delta = xyz(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

(taking $xy + yz + zx$ common from C_3)

$$\Rightarrow \quad \Delta = xyz(xy+yz+zx)\cdot 0 = 0$$

($\because C_2$ and C_3 are indentical)

32. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & 1-a & 0 \\ 3-a^2-2a & 2-2a & 0 \end{vmatrix}$

(using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= (1-a)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ -(a^2+2a-3) & -2(a-1) & 0 \end{vmatrix}$

[taking common $(1-a)$ from R_2]

$= (1-a)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ -(a+3)(a-1) & -2(a-1) & 0 \end{vmatrix}$

$= -(1-a)(a-1)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ a+3 & 2 & 0 \end{vmatrix}$

[taking common $-(a-1)$ from R_3]

Expanding along C_3, we get

$= (a-1)(a-1)[1\{2(1+a)-(a+3)\} + 0 + 0]$

$= (a-1)(a-1)(a-1) = (a-1)^3$

33. Let $\Delta = \begin{vmatrix} 1 & 5 & \pi \\ \log_e e & 5 & \sqrt{5} \\ \log_{10} 10 & 5 & e \end{vmatrix}$

$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 5 & \pi \\ 1 & 5 & \sqrt{5} \\ 1 & 5 & e \end{vmatrix} = 5\begin{vmatrix} 1 & 1 & \pi \\ 1 & 1 & \sqrt{5} \\ 1 & 1 & e \end{vmatrix}$ $\quad (\because \log_a a = 1)$

$= 0$ $\quad$ ($\because$ two columns are identical)

34. $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_3$

$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

[applying $R_3 \to R_3 - (R_1 - 2R_2)$]

$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

35. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ $\quad$ Applying $C_1 \to C_1 - C_2$

$= \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 4\begin{vmatrix} 1 & (a^x-a^{-x})^2 & 1 \\ 1 & (b^x-b^{-x})^2 & 1 \\ 1 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$

($\because$ two columns are identical)

36. Applying $C_3 \to C_3 - C_1$, we get

$\Delta = \begin{vmatrix} 1 & \alpha & \alpha^2-1 \\ \cos(p-d)a & \cos pa & 0 \\ \sin(p-d)a & \sin pa & 0 \end{vmatrix}$

$= (\alpha^2-1)\{-\cos pa \sin(p-d)a + \sin pa \cos(p-d)a\}$

$= (\alpha^2-1)\sin\{-(p-d)a + pa\}$

$\Rightarrow \quad \Delta = (\alpha^2-1)\sin da$

which is independent of p.

37. Given determinant, $\Delta = \begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = 0$

On splitting the determinant into two determinants, we get

$\Delta = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow \quad (1+abc)[1(bc^2-cb^2) - a(c^2-b^2) + a^2(c-b)] = 0$

$\Rightarrow \quad (1+abc)[(a-b)(b-c)(c-a)] = 0$

Since, a, b and c are different, the second factor cannot be zero.

Hence, $1 + abc = 0$

38. We have, $\begin{vmatrix} x+\alpha & \beta & \gamma \\ \gamma & x+\beta & \alpha \\ \alpha & \beta & x+\gamma \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad \begin{vmatrix} x+\alpha+\beta+\gamma & \beta & \gamma \\ x+\alpha+\beta+\gamma & x+\beta & \alpha \\ x+\alpha+\beta+\gamma & \beta & x+\gamma \end{vmatrix} = 0$

$\Rightarrow \quad (x+\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 1 & x+\beta & \alpha \\ 1 & \beta & x+\gamma \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$\Rightarrow \quad (x+\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 0 & x & \alpha-\gamma \\ 0 & 0 & x \end{vmatrix} = 0$

$\Rightarrow \quad (x+\alpha+\beta+\gamma)(x^2-0) = 0$

$\Rightarrow \quad x = 0$ or $x = -(\alpha+\beta+\gamma)$

39. Since, $\Delta(1) = 0$ and $\Delta'(1) = 0$, so $(x-1)^2$ is a factor of $\Delta(x)$.

40. We have, $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} 0 & 6 & 15 \\ 0 & -2-2x & 5(1-x^2) \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

($R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$)

$$\Rightarrow \quad 3\cdot 2\cdot 5\begin{vmatrix} 0 & 1 & 1 \\ 0 & -(1+x) & 1-x^2 \\ 1 & x & x^2 \end{vmatrix} = 0$$

(taking common 3 from R_1, 2 from C_2, 5 from C_3)

$$\Rightarrow \quad (1+x)\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1-x \\ 1 & x & x^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1+x)(2-x) = 0$$

$$\Rightarrow \quad x+1=0 \text{ or } x-2=0$$

$$\Rightarrow \quad x = -1, 2$$

41. Given, $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad (a+b+c-x)\begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$$\Rightarrow \quad (a+b+c-x)\begin{vmatrix} 1 & 0 & 0 \\ c & b-x-c & a-c \\ b & a-b & c-x-b \end{vmatrix} = 0$$

$$\Rightarrow \quad (a+b+c-x)\,[1(b-x-c)(c-x-b)-(a-c)(a-b)] = 0$$

$$\Rightarrow (a+b+c-x)[bc-xb-b^2-xc+x^2+bx - c^2+cx+bc-(a^2-ab-ac+bc)] = 0$$

$$\Rightarrow \quad (a+b+c-x)[x^2-a^2-b^2-c^2+ab+bc+ca] = 0$$

$$\Rightarrow \quad x = a+b+c \quad \text{or} \quad x^2 = a^2+b^2+c^2+ab+bc+ca$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x^2 = a^2+b^2+c^2+\frac{1}{2}(a^2+b^2+c^2)$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$$

42. We have, $\Delta = \dfrac{1}{abc}\begin{vmatrix} a^3+ax & a^2b & a^2c \\ ab^2 & b^3+bx & b^2c \\ c^2a & c^2b & c^3+xc \end{vmatrix}$

Taking a, b and c common in columns Ist, IInd and IIIrd, we get

$$\Delta = \begin{vmatrix} a^2+x & a^2 & a^2 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 1 & 0 \\ b^2 & x & 0 \\ c^2 & 0 & x \end{vmatrix}$$

$$= x(x-b^2)(a^2+b^2+c^2+x)$$

43. $\begin{vmatrix} a & b-y & c-z \\ a-x & y & c-z \\ a-x & b-y & c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$\Rightarrow \quad \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ 0 & -y & z \end{vmatrix} = 0$$

$$\Rightarrow \quad a(yz) + x(bz - yz + cy - yz) = 0$$

$$\Rightarrow \quad ayz + bzx + cyx = 2xyz$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

44. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 2 & \sin x + 2x & \sin y + 2y \\ 3 & \cos x + 3x & \cos y + 3y \end{vmatrix}$

$$= \begin{vmatrix} 1 & x & y \\ 0 & \sin x & \sin y \\ 0 & \cos x & \cos y \end{vmatrix} \quad (R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$= \sin x \cos y - \cos x \sin y = \sin(x-y)$$

45. Let $\Delta = \begin{vmatrix} a^2 & a & 1 \\ \cos(nx) & \cos(n+1)x & \cos(n+2)x \\ \sin(nx) & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$

Since, $\cos(nx) + \cos(n+2)x = 2\cos(n+1)x\cos x$

and $\sin(nx) + \sin(n+2)x = 2\sin(n+1)x\cos x$

Applying $C_1 \to C_1 - 2\cos x \cdot C_2 + C_3$

$$\therefore \quad \Delta = \begin{vmatrix} a^2 - 2a\cos x + 1 & a & 1 \\ 0 & \cos(n+1)x & \cos(n+2)x \\ 0 & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

$$= (a^2 - 2a\cos x + 1)\,[\cos(n+1)x\sin(n+2)x - \cos(n+2)x\sin(n+1)x]$$

$$= (a^2 - 2a\cos x + 1)\sin x$$

$\therefore$ Δ is independent of n.

46. We have, $D_r = \begin{vmatrix} 2^{r-1} & 3^{r-1} & 4^{r-1} \\ x & y & z \\ 2^n - 1 & (3^n-1)/2 & (4^n-1)/3 \end{vmatrix}$

$$\Rightarrow \quad \sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} 3^{r-1} & \sum_{r=1}^{n} 4^{r-1} \\ x & y & z \\ 2^n - 1 & (3^n-1)/2 & (4^n-1)/3 \end{vmatrix}$$

$$\Rightarrow \quad \sum_{r=1}^{n} D_r = \begin{vmatrix} 2^n - 1 & (3^n-1)/2 & (4^n-1)/3 \\ x & y & z \\ 2^n - 1 & (3^n-1)/2 & (4^n-1)/3 \end{vmatrix}$$

$$\Rightarrow \quad \sum_{r=1}^{n} D_r = 0 \qquad (\because \text{two rows are same})$$

47. $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$= \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos A\cos Q + \sin A\sin Q & \cos A\cos R + \sin A\sin R \\ \cos B\cos P + \sin B\sin P & \cos B\cos Q + \sin B\sin Q & \cos B\cos R + \sin B\sin R \\ \cos C\cos P + \sin C\sin P & \cos C\cos Q + \sin C\sin Q & \cos C\cos R + \sin C\sin R \end{vmatrix}$$

This determinants can be rewritten as 8 determinants and the value of each of these 8 determinants is zero.

i.e., $\cos P\cos Q\cos R \begin{vmatrix} \cos A & \cos A & \cos A \\ \cos B & \cos B & \cos B \\ \cos C & \cos C & \cos C \end{vmatrix} = 0$

Similarly, other determinants can be shown zero.

48. $\sum_{n=1}^{N} U_n = \begin{vmatrix} \Sigma n & 1 & 5 \\ \Sigma n^2 & 2N+1 & 2N+1 \\ \Sigma n^3 & 3N^2 & 3N \end{vmatrix}$

$$= \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left\{\frac{N(N+1)}{2}\right\}^2 & 3N^2 & 3N \end{vmatrix}$$

$$= \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 5 \\ 4N+2 & 2N+1 & 2N+1 \\ 3N(N+1) & 3N^2 & 3N \end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$= \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 4N+2 & 2N+1 & 4N+2 \\ 3N(N+1) & 3N^2 & 3N(N+1) \end{vmatrix}$$

$= 0$ ($\because$ two columns are identical)

49. Let $f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & (x-1) & x \\ 3x(x-1) & (x-1)(x-2) & x(x-1) \end{vmatrix}$

$$= (x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$= (x-1)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 2x & -2 & x \end{vmatrix} = (x-1)[-2x+2x] = 0$$

$\therefore \quad f(x) = 0$

$\Rightarrow \quad f(50) = 0$

50. Operating $R_1 \to R_1 + R_2 + R_3$, we get the determinant

$$= \begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = (3+a)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

Operate $C_3 \to C_3 - C_1, C_2 \to C_2 - C_1 = (3+a)\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix}$

$$(3+a)a^2 = a^3\left(1 + \frac{3}{a}\right)$$

51. We have,

$$ax^4 + bx^3 + cx^2 + 50x + d = \begin{vmatrix} x^3 - 14x^2 & -x & 3x+\lambda \\ 4x+1 & 3x & x-4 \\ -3 & 4 & 0 \end{vmatrix}$$

On differentiating w.r.t. x, we get

$$4ax^3 + 3bx^2 + 2cx + 50 = \begin{vmatrix} 3x^2 - 28x & -1 & 3 \\ 4x+1 & 3x & x-4 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} x^3 - 14x^2 & -x & 3x+\lambda \\ 4 & 3 & 1 \\ -3 & 4 & 0 \end{vmatrix}$$

Now, put $x = 0$, we get

$$50 = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \lambda \\ 4 & 3 & 1 \\ -3 & 4 & 0 \end{vmatrix}$$

$\Rightarrow \quad 50 = 25\lambda \Rightarrow \lambda = 2$

52. We have, $\Delta(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$

$$\frac{d^n}{dx^n}[\Delta(x)] = \begin{vmatrix} \frac{d^n}{dx^n}x^n & \frac{d^n}{dx^n}\sin x & \frac{d^n}{dx^n}\cos x \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$$

($\because$ differentiation of R_2 and R_3 are zero)

$$= \begin{vmatrix} n! & \sin\left(x + \frac{n\pi}{2}\right) & \cos\left(x + \frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$$

$$\Rightarrow \quad [\Delta^n(x)]_{x=0} = \begin{vmatrix} n! & \sin\left(0 + \frac{n\pi}{2}\right) & \cos\left(0 + \frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$$

$$= \begin{vmatrix} n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$$

$= 0$ $\quad$ ($\because R_1$ and R_2 are identical)

53. We have, $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} = x^3 - 3abx + ab^2 + a^2b$

$\Rightarrow \quad \frac{d}{dx}\Delta_1 = 3(x^2 - ab)$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} = x^2 - ab$

$\therefore \quad \frac{d}{dx}(\Delta_1) = 3(x^2 - ab) = 3\,\Delta_2$

54. $\Delta(-x) = \begin{vmatrix} f(-x) + f(x) & 0 & x^4 \\ 3 & f(-x) - f(x) & \cos x \\ x^4 & -2x & f(-x)\,f(x) \end{vmatrix}$

$\begin{vmatrix} f(x) + f(-x) & 0 & x^4 \\ 3 & f(x) - f(-x) & \cos x \\ x^4 & -2x & f(x)\,f(-x) \end{vmatrix} = -\Delta(x)$

So, $\Delta(x)$ is an odd function.

$\Rightarrow x^4\Delta(x)$ is an odd function

$\Rightarrow \quad \int_{-2}^{2} x^4\,\Delta(x)\,dx = 0$

55. Given, $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$

Applying $C_3 \to C_3 + C_2 - C_1$

$$= \begin{vmatrix} 1 & \cos x & 0 \\ 1 + \sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix}$$

$$= \cos x - \cos x(1 + \sin x)$$

$$= -\cos x \sin x = -\frac{1}{2}\sin 2x$$

$\therefore \quad \int_0^{\pi/2} \Delta x\,dx = -\frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx$

$$= -\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{2}$$

56. If using the relation $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ for finding the equation of lineMake (x_1, y_1) as (1, 2) and (x_2, y_2) as (3, 6) and (x_3, y_3) as (x, y) solve to get equation of line.

Let $P(x, y)$ be any point on the line joining $A(1, 2)$ and $B(3, 6)$. If the points A, B and P are collinear, then the area of ΔABP will be zero.

$\therefore \quad \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}[1(6 - y) - 2(3 - x) + 1(3y - 6x)] = 0$

$\Rightarrow \quad 6 - y - 6 + 2x + 3y - 6x = 0$

$\Rightarrow \quad 2y - 4x = 0$

$\Rightarrow \quad y = 2x$

Hence, the equation of the line joining the given points is $y = 2x$.

57. Required area = 9 sq units

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & K & 1 \end{vmatrix} = 9$

$\Rightarrow \quad -K(-3 - 3) = 18$

$-k(-6) = 18$

$\Rightarrow \quad 6K = 18$

$\Rightarrow \quad K = 3$

58. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then

$$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \ldots(i)$$

Also, we know that if a be the length of an equilateral triangle, then

$$\text{area} = \frac{\sqrt{3}}{4}a^2 \quad \ldots(ii)$$

$\therefore$ From Eqs. (i) and (ii), we get

$$\frac{\sqrt{3}}{4}a^2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}a^2 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

On squaring both sides, we get

$$\frac{3}{4}a^4 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

59. It has a non-zero solution, if

$$\begin{vmatrix} 1 & k & -1 \\ 3 & -k & -1 \\ 1 & -3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-k - 3) - k(3 + 1) - 1(-9 + k) = 0$

$\Rightarrow \quad -6k + 6 = 0$

$\Rightarrow \quad k = 1$

60. Given system of equations are

$$3x - 2y + z = 0$$
$$\lambda x - 14y + 15z = 0$$
and $$x + 2y - 3z = 0$$

The system of equations has infinitely many (non-trivial) solutions, if $\Delta = 0$.

$$\Rightarrow \quad \Delta = \begin{vmatrix} 3 & -2 & 1 \\ \lambda & -14 & 15 \\ 1 & 2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(42 - 30) - \lambda(6 - 2) + 1(-30 + 14) = 0$$
$$\Rightarrow \quad 36 - 4\lambda - 16 = 0$$
$$\Rightarrow \quad \lambda = 5$$

61. The homogeneous linear system of equations is consistent *i.e.*, possesses trivial solution, if

$$\Delta \equiv \begin{vmatrix} 2 & 3 & 5 \\ 1 & k & 5 \\ k & -12 & -14 \end{vmatrix} \neq 0$$

$$\Rightarrow 2(-14k + 60) - 3(-14 - 5k) + 5(-12 - k^2) \neq 0$$
$$\Rightarrow \quad 5k^2 + 13k - 102 \neq 0$$
$$\Rightarrow \quad (5k - 17)(k + 6) \neq 0$$
$$\Rightarrow \quad k \neq -6, \frac{17}{5}$$

62. Since, the given equations are consistent.

$$\therefore \quad \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ a & 2 & -b \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(-b + 4) - 3(-3b + 2a) + 1(6 - a) = 0$$
$$\Rightarrow \quad -2b + 8 + 9b - 6a + 6 - a = 0$$
$$\Rightarrow \quad 7b - 7a = -14$$
$$\Rightarrow \quad a - b = 2$$

63. Given, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 + \cos\theta & 1 & 1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin\theta & 0 \\ 1 + \cos\theta & -\cos\theta & -\cos\theta \end{vmatrix} = 1(-\sin\theta\cos\theta)$$
$$= -\frac{1}{2}\sin 2\theta$$

For maximum value of Δ, $\sin 2\theta$ will be minimum.

$$\therefore \quad \Delta = -\frac{1}{2}(-1) = \frac{1}{2} \quad (\because \text{minimum value of } \sin x \text{ is } -1)$$

64. Applying $R_3 \to R_3 - \cos y\, R_1 + \sin y\, R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta = (\sin y - \cos y)(\cos^2 x + \sin^2 x)$$
$$= (\sin y - \cos y) = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y\right]$$
$$= \sqrt{2}\left[\cos\frac{\pi}{4}\sin y - \sin\frac{\pi}{4}\cos y\right] = \sqrt{2}\sin\left(y - \frac{\pi}{4}\right)$$

Hence, $$-\sqrt{2} \le \Delta \le \sqrt{2}$$

65. $A = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$$= 1(1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$$
$$= 2(1 + \sin^2\theta)$$

Since, the maximum and minimum value of $\sin^2\theta$ is 1 and 0.

$$\therefore \quad |A| \in [2, 4]$$

66. Given, $$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} \sin x + 2\cos x & \sin x + 2\cos x & \sin x + 2\cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow \quad (\sin x + 2\cos x)\begin{vmatrix} 1 & 1 & 1 \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

[taking common $(\sin x + 2\cos x)$ from R_1]

$$\Rightarrow \quad (\sin x + 2\cos x)\begin{vmatrix} 1 & 0 & 0 \\ \cos x & \sin x - \cos x & 0 \\ \cos x & 0 & \sin x - \cos x \end{vmatrix} = 0$$

(using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$\Rightarrow \quad (\sin x + 2\cos x)[1(\sin x - \cos x)^2] = 0$$
$$\Rightarrow \quad \sin x + 2\cos x = 0 \text{ or } (\sin x - \cos x)^2 = 0$$
$$\Rightarrow \quad \tan x = -2 \text{ or } \tan x = 1$$

For $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, $\tan x \neq -2$.

But $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$

Hence, only one solution exists.

67. Given A, B and C are the angles of a triangle.

$$\because \quad A + B + C = \pi$$
$$\Rightarrow \quad A + B = \pi - C$$

Now, let $\cos(A + B) = \cos(\pi - C) = -\cos C$

$$\Rightarrow \quad \cos A\cos B - \sin A\sin B = -\cos C$$
$$\Rightarrow \quad \cos A\cos B + \cos C = \sin A\sin B \quad \text{...(i)}$$

Similarly, $\cos A\cos C + \cos B = \sin A\sin C$ ...(ii)

and $\sin(B + C) = \sin(\pi - A) = \sin A$ ...(iii)

Now, let $\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Expanding along R_1, we get

$\Delta = -1(1-\cos^2 A) + \cos C(\cos C + \cos A \cos B)$
$+ \cos B(\cos B + \cos A \cos C)$

$= -\sin^2 A + \cos C(\sin A \sin B) + \cos B(\sin A \sin C)$

[using Eqs. (i) and (ii) and $\cos^2 A + \sin^2 A = 1$]

$= -\sin^2 A + \sin A(\sin B \cos C + \cos B \sin C)$

$= -\sin^2 A + \sin A \sin(B + C)$

[$\because \sin(x+y) = \sin x \cos y + \cos x \sin y$]

$= -\sin^2 A + \sin^2 A = 0$ [using Eq. (iii)]

68. Given, $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$

$f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t - \cos t & 0 & 2t-1 \\ \sin t - \cos t & 0 & t-1 \end{vmatrix}$

Expanding along C_2

$f(t) = -t\{(2\sin t - \cos t)(t-1) - (2t-1)(\sin t - \cos)\}$

$= -t\{2t\sin t - t\cos t - 2\sin t + \cos t - 2t\sin t$
$+ 2t\cos t + \sin t - \cos t\}$

$= -t(t\cos t - \sin t)$

$\Rightarrow \quad f(t) = t\sin t - t^2 \cos t$

$\Rightarrow \quad \dfrac{f(t)}{t^2} = \dfrac{\sin t}{t} - \cos t$

$\Rightarrow \quad \lim\limits_{t\to 0} \dfrac{f(t)}{t^2} = \lim\limits_{t\to 0} \dfrac{f(t)}{t^2}$

$= \lim\limits_{t\to 0}\left\{\dfrac{\sin t}{t} - \cos t\right\} \left(\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1\right)$

$= \lim\limits_{t\to 0}\dfrac{\sin t}{t} = \lim\limits_{t\to 0}\cos t$

$= 1 - \cos 0 = 1 - 1 = 0$

69. Given, $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

Put $x = 0$

$f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$

Expanding along R_1

$f(0) = a(0 + bc) - b(ac) = abc - abc = 0$

70. Given that, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

Expanding along C_1

$(2a^2 + 4) - 2(-4a - 20) = 86$

$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$

$2a^2 + 8a - 42 = 0$

$\Rightarrow \quad a^2 + 4a - 21 = 0$

$\Rightarrow \quad a^2 + 7a - 3a - 21 = 0$

$\Rightarrow \quad a(a+7) - 3(a+7) = 0$

$\Rightarrow \quad (a+7)(a-3) = 0$

$\therefore \quad a = -7, 3$

$\therefore$ Sum of these number $= 3 - 7 = -4$

71. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \quad (R_3 \to R_3 - R_2)$

$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix}$

$(C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1)$

Expanding along R_1, we get

$\Delta = (\sin B - \sin A)(\sin^2 C - \sin^2 B)$
$- (\sin C - \sin B)(\sin^2 B - \sin^2 A)$

$= (\sin B - \sin A)(\sin C - \sin B)(\sin C - \sin A) = 0$

$\Rightarrow$ Either $\sin B - \sin A = 0$ or $\sin C - \sin B$ or $\sin C - \sin A = 0$

$\Rightarrow \quad A = B$ or $B = C$ or $C = A$

i.e., ΔABC is isosceles.

72. If a be the first term and R be the common ratio of a GP, then rth term, $a_r = aR^{r-1}$. Use this relation and simplify it.

Let $\Delta = \begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$

Let a and R be the first term and common ratio of GP. Then,

$a_r = aR^{r-1}$

$\therefore \quad \Delta = \begin{vmatrix} aR^r & aR^{r+4} & aR^{r+8} \\ aR^{r+6} & aR^{r+10} & aR^{r+14} \\ aR^{r+10} & aR^{r+16} & aR^{r+20} \end{vmatrix}$

$= (aR^r)(ar^{r+6})(aR^{r+10}) \begin{vmatrix} 1 & R^4 & R^8 \\ 1 & R^4 & R^8 \\ 1 & R^6 & R^{10} \end{vmatrix}$

(taking common aR^r, aR^{r+6}, aR^{r+10} from R_1, R_2, and R_3 respectively)

$= (aR^r)(aR^{r+6})(aR^{r+10})[0]$ ($\because R_1$ and R_2 are identical)

$= 0.$

Hence, given determinant is independent of r.

73. Since, the first column consists of sum of two terms, second column consists of sum of three terms and third column consists of sum of four terms.

$\therefore \quad n = 2 \times 3 \times 4 = 24$

74. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$= 1(1 - \log_z y \log_y z) - \log_x y (\log_y x - \log_y z \log_z x) + \log_x z (\log_y x \log_z y - \log_z x)$

$= (1 - \log_z z) - \log_x y (\log_y x - \log_y z \log_z x) + \log_x z (\log_y x \log_z y - \log_z x)$

$= (1 - 1) - (1 - \log_x y \log_y x) + (\log_x z \log_z x - 1) = 0$ (since, $\log_x y \log_y x = 1$)

$= 0 - (1 - 1) + (1 - 1) = 0$

75. Given, $f(\alpha) = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix}$

$= 1(\alpha^3 - 1) - \alpha(\alpha^2 - \alpha^2) + \alpha^2(\alpha - \alpha^4)$

$= \alpha^3 - 1 - 0 + \alpha^3 - \alpha^6$

$\Rightarrow \quad f(\sqrt[3]{3}) = 3 - 1 - 0 + 3 - 3^2$

$= 6 - 10 = -4$

76. $\because \det(M_r) = r^2 - (r-1)^2 = 2r - 1$

$\therefore \det(M_1) + \det(M_2) + \ldots + \det(M_{2008})$

$= 1 + 3 + 5 + \ldots + 4015$

$= \frac{2008}{2}[2 + (2008 - 1)2]$

$= 2008(2008) = (2008)^2$

77. We have,

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix}$$

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_2$

$$= \begin{vmatrix} a & a & a+b+c \\ 0 & a & 2a+b \\ 0 & a & a+b \end{vmatrix}$$

$= a[a^2 + ab - 2a^2 - ab]$

$= -a^3 = i$ ($\because a = i$, given)

78. Given that, $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+2 & x+3 & x+4 \\ x+a & x+b & x+c \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} -1 & -1 & x+3 \\ -1 & -1 & x+4 \\ a-b & b-c & x+c \end{vmatrix} = 0$

(using $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$\Rightarrow \begin{vmatrix} 0 & 0 & -1 \\ -1 & -1 & x+4 \\ a-b & b-c & x+c \end{vmatrix} = 0 \quad (R_1 \to R_1 - R_2)$

$\Rightarrow \quad (-1)(-b + c + a - b) = 0$

$\Rightarrow \quad 2b - a - c = 0$

$\Rightarrow \quad a + c = 2b$

$\therefore$ a, b and c are in AP.

79. Let the first term and common difference of an AP are A and D, respectively.

$\therefore \quad a = A + (p-1)D, \quad b = A + (q-1)D,$

and $\quad c = A + (r-1)D$

Now, $\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} = \begin{vmatrix} A + (p-1)D & p & 1 \\ A + (q-1)D & q & 1 \\ A + (r-1)D & r & 1 \end{vmatrix}$

Applying $C_1 \to C_1 - DC_2 + DC_3$

$$= \begin{vmatrix} A & p & 1 \\ A & q & 1 \\ A & r & 1 \end{vmatrix} = A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} = 0$$

($\because$ two columns are identical)

80. Let $\Delta = \begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} x+2 & 1 & x+a \\ x+4 & 1 & x+b \\ x+6 & 1 & x+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$\Rightarrow \quad \Delta = \begin{vmatrix} x+2 & 1 & x+a \\ 2 & 0 & b-a \\ 4 & 0 & c-a \end{vmatrix}$

$= -1(2c - 2a - 4b + 4a)$

$\Rightarrow \quad \Delta = 2(2b - c - a)$ ...(i)

Since, a, b and c are in AP.

$\therefore \quad b = \frac{a+c}{2}$ [from Eq. (i)]

$\Delta = 2(a + c - c - a) = 0$

81. $\begin{vmatrix} [e] & [\pi] & [\pi^2 - 6] \\ [\pi] & [\pi^2 - 6] & [e] \\ [\pi^2 - 6] & [e] & [\pi] \end{vmatrix}$

$$= \begin{vmatrix} 2 & 3 & 3 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix}$$

$= 2(9 - 4) - 3(9 - 6) + 3(6 - 9)$

$= 10 - 9 - 9 = -8$

82. Given that, $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix}$$

($\because$ if n is multiple of 3, then $1+\omega^n+\omega^{2n}=0$)

$= 0$

83. Let $\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix} = 10!11!12! \begin{vmatrix} 1 & 11 & 11\times 12 \\ 1 & 12 & 12\times 13 \\ 1 & 13 & 13\times 14 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 10!11!12! \begin{vmatrix} 1 & 11 & 11\times 12 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix}$$

$= (10!11!12!)\,(50-48) = 2\cdot(10!11!12!)$

84. Given, $f(x) = \begin{vmatrix} x & 1+\sin x & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$

$$= x\{-2(1+x^2)\} - (1+\sin x)(-2x^2) + \cos x\{1+x^2-x^2\log(1+x)\}$$

$$= -2x - 2x^3 + 2x^2 + 2x^2\sin x + \cos x\{1+x^2-x^2\log(1+x)\}$$

$\therefore$ Coefficient of x in $f(x) = -2$

85. In ΔABC, given $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

$\Rightarrow \quad 1(c^2-ab) - a(c-a) + b(b-c) = 0$

$\Rightarrow \quad a^2+b^2+c^2-ab-bc-ca = 0$

$\Rightarrow \quad 2a^2+2b^2+2c^2-2ab-2bc-2ac = 0$

$\Rightarrow \quad (a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca) = 0$

$\Rightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2 = 0$

Here, sum of squares of three numbers can be zero, if and only if $a = b = c$.

$\Rightarrow \Delta ABC$ is an equilateral triangle.

$\Rightarrow \quad \angle A = \angle B = \angle C = 60°$

$\therefore \sin^2 A + \sin^2 B + \sin^2 C = \sin^2 60° + \sin^2 60° + \sin^2 60°$

$$= \left(\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right) = \frac{9}{4}$$

86. Let $A \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$ ...(i)

$\therefore \begin{vmatrix} b_2c_3-b_3c_2 & c_2a_3-c_3a_2 & c_2b_3-c_3b_2 \\ b_3c_1-b_1c_3 & c_3a_1-c_1a_3 & a_3b_1-a_1b_3 \\ b_1c_2-b_2c_1 & c_1a_2-c_2a_1 & a_1b_2-a_2b_1 \end{vmatrix}$

$|\text{adj}\,A| = (5)^{3-1}$ [from Eq. (i)]

$= 5^2 = 25$ ($\because |\text{adj}\,A| = |A|^{n-1}$)

87. Given that, $x^a y^b = e^m$, $x^c y^d = e^n$

and $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\Rightarrow \quad a\log x + b\log y = m$

$\Rightarrow \quad c\log x + d\log y = n$

By Cramer's rule $\log x = \dfrac{\Delta_1}{\Delta_3}$ and $\log y = \dfrac{\Delta_2}{\Delta_3}$

$\Rightarrow \quad x = e^{\Delta_1/\Delta_3}$ and $y = e^{\Delta_2/\Delta_3}$

88. Since system of equations is consistent.

$\therefore \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -c \\ -b & 3b & -c \end{vmatrix} = 0$

$\Rightarrow \quad c + bc - 6b + b + 2c + 3bc = 0$

$\Rightarrow \quad 3c + 4bc - 5b = 0$

$\Rightarrow \quad c = \dfrac{5b}{3+4b}$

But

$\Rightarrow \quad c < 1 \Rightarrow \dfrac{5b}{3+4b} < 1$

$\Rightarrow \quad \dfrac{b-3}{3+4b} < 0$

$\Rightarrow \quad b \in \left(-\dfrac{3}{4}, 3\right)$

Round II

1. $\Delta = \begin{vmatrix} \sin px & p\cos px & -p^2\sin px \\ -p^3\cos px & p^4\sin px & p^5\cos px \\ -p^6\sin px & -p^7\cos px & p^8\sin px \end{vmatrix}$

Taking $-p^6$ common from R_3, we find that R_1 and R_3 become identical

$\therefore \quad \Delta = 0$

2. $\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 100x+50+1 & 100y+40+1 & 100z+30+1 \\ x & y & z \end{vmatrix}$

(operating $R_2 - 100R_3 - 10R_1$)

$$= \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = x - 2y + z = 0$$

3. Using sine formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

$$\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$$

(taking a common both from R_1 and C_1)

$$= \begin{vmatrix} 1 & bk & ck \\ bk & 1 & \cos A \\ ck & \cos A & 1 \end{vmatrix} = \begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

$$= 1(1-\cos^2 A) - \sin B(\sin B - \cos A \sin C)$$
$$+ \sin C(\sin B \cos A - \sin C)$$
$$= \sin^2 A - \sin^2 B - \sin^2 C + 2\sin B \cos A \sin C = 0$$

4. $\overline{\Delta} = \Delta$

$\Rightarrow \Delta$ is real

5. $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \times \begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix} = 0$

6. Since, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

($\because$ entries in each row are in AP with common difference 1)

7. Taking $\sqrt{6}$ common from C_1, we get

$$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ \sqrt{2} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{3} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$$

Operating $R_2 - \sqrt{2}R_1, R_3 - \sqrt{3}R_1$, we get

$$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ 0 & \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ 0 & \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix}$$

$$= \sqrt{6}\begin{vmatrix} \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix} (C_2 - \sqrt{2}iC_1) = \sqrt{6}(-3\sqrt{6} + 2\sqrt{6})$$

8. For non-trival solution $\begin{vmatrix} t-1 & 3t+1 & 2t \\ t-1 & 4t-2 & t+3 \\ 2 & 3t+1 & 3(t-1) \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} t-1 & 3t+1 & 2t \\ 0 & t-3 & 3-t \\ 3-t & 0 & t-3 \end{vmatrix} \quad (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$\Rightarrow \quad (t-3)^2 \begin{vmatrix} t-1 & 3t+1 & 2t \\ 0 & -1 & 1 \\ 1 & 0 & t-3 \end{vmatrix} = 0$$

$$\Rightarrow \quad (t-3)^2[(t-1)(1) - (3t+1)(-1) + 2t(1)] = 0$$

$$\Rightarrow (t-3)^3[t-1+3t+1+2t] = 0 \Rightarrow t = 0 \text{ or } t = 3$$

9. We can write Δ as product of two determinants as follows

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta_1^2$$

where, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Thus, $\Delta \geq 0$

Also, $\Delta_1 = (a+b+c)\,\Delta_2$, where

$$\Delta_2 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -\frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

$\because \quad \Delta = 0 \Leftrightarrow a+b+c = 0$

or $\quad b - c = 0 = c - a = a - b$

10. Using $C_1 \to C_1 - bC_3, C_2 \to C_2 + aC_3$, we get

$$\Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2+b^2 \end{vmatrix}$$

Using $C_3 \to C_3 + 2bC_1$, we get

$$\Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2a \\ b & -a & 1-a^2+b^2 \end{vmatrix}$$

$$\Delta = (1+a^2+b^2)^3$$

Since, AM $\geq$ GM,

$$\frac{1+a^2+b^2}{3} \geq (a^2b^2)^{1/3}$$

11. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = (1+\sin^2 x + \cos^2 x + 4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 1 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

Using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$f(x) = 2(1 + 2\sin 2x)$$

$$\max f(x) = 6$$

$$\min f(x) = -2$$

12. Let $\alpha = s - a, \beta = s - b, \gamma = s - c$,

then determinant

$$= \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix} = 2\alpha\beta\gamma(\alpha+\beta+\gamma)^2$$

(standard determinant)

$$= 2(s-a)(s-b)(s-c)s^3 \Rightarrow k = 2$$

13. $\overline{A} = \begin{vmatrix} 23 & 1-i & i \\ 1+i & -31 & 4+5i \\ -i & 4-5i & 17 \end{vmatrix}$

$$(\overline{A})^T = A$$

$\Rightarrow$ $\det(\overline{A})^T = \det(A)$

$\Rightarrow$ $\det(\overline{A}) = \det(A)$

$\Rightarrow$ $\det(\overline{A}) = \det(A)$

$\Rightarrow$ $\det(A)$ is real.

14. Let $\Delta = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} n & 1 & 1 \\ n! & n \cdot n! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix}$$

$(C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1)$

$= (n+1)(n+1)! - n \cdot n!$

$= [(n+1)^2 - n]n!$

$(n^2 + n + 1) \cdot n!$

15. Put $x = 0$, then we get $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1$

16. $\Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$

$$= e^a \cdot e^b \cdot e^c \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$= 0$, since $a + b + c = 0$

17. $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1 + abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow$ $1 + abc = 0$, since the determinant $\neq 0$

18. If we interchange any two rows of a determinant in the set B, its value becomes $-1$and hence, it is in C. Likewise, for every determinant in C, there is corresponding determinant in B.

$\therefore$ B and C have the same number of elements

19. Applying $C_1 \to C_1 - pC_2 - C_3$

$$\begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2 + 2yp + z) & xp + y & yp + z \end{vmatrix} = 0$$

$\Rightarrow$ $0 = (xp^2 + 2yp + z)(y^2 - xz) \Rightarrow y^2 = xz$

20. $\Delta = \begin{vmatrix} a(1+\omega) & b\omega^2 & a\omega \\ b(\omega + \omega^2) & c & b\omega^2 \\ c(\omega^2 + 1) & a\omega & c \end{vmatrix}$ $(C_1 \to C_1 + C_3)$

$$= \begin{vmatrix} -a\omega^2 & b\omega^2 & a\omega \\ -b & c & b\omega^2 \\ -c\omega & a\omega & c \end{vmatrix} = \omega^2 \cdot \omega \begin{vmatrix} -a & b & a\omega^2 \\ -b & c & b\omega^2 \\ -c & a & c\omega^2 \end{vmatrix}$$

$$= -\omega^2 \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} = 0$$

21. Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow$ $\sin^2\theta + \cos^2\theta + 1 + 4\sin 4\theta = 0$ (expanding along R_3)

$\Rightarrow$ $\sin 4\theta = -\frac{1}{2}$

$\Rightarrow$ $\theta = \frac{7\pi}{24}, \frac{11\pi}{24}$

22. Put $x = 0$ in the given equation, we get

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 12 = 0$$

23. Since, $\alpha + \beta + \gamma = 0$

Applying $C_1 \to C_1 + C_2 + C_3$ in the given determinant.

$\therefore$ $\Delta = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$

24. Now, $\sum_{r=1}^{N} 2r = N(N+1)$

$$\sum_{r=1}^{N} (6r^2 - 1) = 6N\left[\frac{(N+1)(2N+1)}{6}\right] - N$$

$$= 2N^3 + 3N^2$$

$$\sum_{r=1}^{N} (4r^3 - 2Nr) = \frac{4N^2(N+1)^2}{4} - \frac{2N \cdot N(N+1)}{2}$$

$$= N^3(N+1)$$

$\therefore$ $\sum_{r=1}^{N} \Delta_r = \begin{vmatrix} N(N+1) & x & N(N+1) \\ 2N^3 + 3N^2 & y & N^2(2N+3) \\ N^3(N+1) & z & N^3(N+1) \end{vmatrix} = 0$

25. Let A be the first term and D be the common difference of the corresponding AP, then

$$\frac{1}{a} = A + (p-1)D, \frac{1}{b} = a + (q-1)D$$

and $\frac{1}{c} = A + (r-1)D$

Now, $\Delta = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

$[R_1 \to R_1 - D(R_2) - (A - D)\, R_3]$

$= 0$

26. Using Cramer's rule, $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 1 + a^3 = 0 \quad \therefore \quad a = -1$

27. Since, $-1 \le x < 0$

$\therefore \quad [x] = -1$

Also, $0 \le y < 1 \Rightarrow [y] = 0$ and $1 \le z < 2 \Rightarrow [z] = 1$

$\therefore$ Given determinant becomes

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

28. We can write $\Delta = \Delta_1 + y_1\Delta_2$, where

$$\Delta_1 = \begin{vmatrix} 1 & 1 + x_1y_2 & 1 + x_1y_3 \\ 1 & 1 + x_2y_2 & 1 + x_2y_3 \\ 1 & 1 + x_3y_2 & 1 + x_3y_3 \end{vmatrix}$$

and $$\Delta_2 = \begin{vmatrix} x_1 & 1 + x_1y_2 & 1 + x_1y_3 \\ x_2 & 1 + x_2y_2 & 1 + x_2y_3 \\ x_3 & 1 + x_3y_2 & 1 + x_3y_3 \end{vmatrix}$$

In Δ_1, use $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ so that ,

$$\Delta_1 = \begin{vmatrix} 1 & x_1y_2 & x_1y_3 \\ 1 & x_2y_2 & x_2y_3 \\ 1 & x_3y_2 & x_3y_3 \end{vmatrix} = 0$$

($\because C_2$ and C_3 are proportional)

In Δ_2, use $C_2 \to C_2 - y_2C_1$ and $C_3 \to C_3 - y_3C_1$ to get

$$\Delta_2 = \begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{vmatrix} = 0 \quad (\because C_2 \text{ and } C_3 \text{ are identical})$$

$\therefore \quad \Delta = 0$

29. Take a, b and c common from R_1, R_2, R_3 respectively

$$\therefore \quad \Delta = abc\begin{vmatrix} \frac{1}{a} + 1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} + 1 & \frac{1}{b} + 2 & \frac{1}{b} \\ \frac{1}{c} + 1 & \frac{1}{c} + 1 & \frac{1}{c} + 3 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\left(3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1 + \frac{1}{b} & 2 + \frac{1}{b} & \frac{1}{b} \\ 1 + \frac{1}{c} & 1 + \frac{1}{c} & 3 + \frac{1}{c} \end{vmatrix}$$

Now, applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$ and on expanding, we get

$$\Delta = 2\,abc\left[3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right] = 0$$

$\because \quad a \ne 0, \quad b \ne 0, \quad c \ne 0$

$\therefore \quad a^{-1} + b^{-1} + c^{-1} = -3$

30. $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

Applying $R_3 \to R_3 - xR_2$

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ 0 & 0 & a + x \end{vmatrix} = (a + x)\,(a^2 + ax)$$

$\Rightarrow \quad f(x) = a(a + x)^2$

$\therefore \quad f(2x) = a(a + 2x)^2$

$\Rightarrow \quad f(2x) - f(x) = ax(2a + 3x)$

31. Putting $r = 1, 2, 3, \ldots, n$ and using the formula

$$\Sigma 1 = n \quad \text{and} \quad \Sigma r = \frac{(n+1)n}{2}$$

$$\Sigma(2r - 1) = 1 + 3 + 5 + \ldots = n^2$$

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} n & n & n \\ n(n+1) & n^2 + n + 1 & n^2 + n \\ n^2 & n^2 & n^2 + n + 1 \end{vmatrix} = 56$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & n \\ 0 & 1 & n^2 + n \\ -n - 1 & -n - 1 & n^2 + n + 1 \end{vmatrix} = 56$$

$\Rightarrow \quad n(n + 1) = 56$

$\Rightarrow \quad n^2 + n - 56 = 0$

$\Rightarrow \quad (n + 8)(n - 7) = 0$

$\Rightarrow \quad n = 7 \ (n \ne -8)$

32. Let $f(x) = a_0x^2 + a_1x + a_2$

and $g(x) = b_0x^2 + b_1x + b_2$

Also, $h(x) = c_0x^2 + c_1x + c_2$

Then, $$\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ 2a_0x + a_1 & 2b_0x + b_1 & 2c_0x + c_1 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix}$$

$$= x\begin{vmatrix} f(x) & g(x) & h(x) \\ 2a_0 & 2b_0 & 2c_0 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix}$$

$$= 0 + 2\begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix}$$

$$= 2\,[(b_1c_0 - b_0c_1)\, f(x) - (a_1c_0 - a_0c_1)\, g(x) + (a_1b_0 - a_0b_1)\, h(x)]$$

Hence, degree of $\Delta(x) \le 2$

33. Given, $\Delta = \begin{vmatrix} (e^{i\alpha}+e^{-i\alpha})^2 & (e^{i\alpha}-e^{-i\alpha})^2 & 4 \\ (e^{i\beta}+e^{-i\beta})^2 & (e^{i\beta}-e^{-i\beta})^2 & 4 \\ (e^{i\gamma}+e^{-i\gamma})^2 & (e^{i\gamma}-e^{-i\gamma})^2 & 4 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} 4 & (e^{i\alpha}-e^{-i\alpha})^2 & 4 \\ 4 & (e^{i\beta}-e^{-i\beta})^2 & 4 \\ 4 & (e^{i\gamma}-e^{-i\gamma})^2 & 4 \end{vmatrix}$$

$= 0$ ($\because$ two columns are same)

Hence, it is independent of α, β and γ.

34. Let $a \neq 0$, then on applying $C_1 \to aC_1$, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} a^3+ax^2 & ab & ac \\ a^2b & b^2+x^2 & bc \\ a^2c & bc & c^2+x^2 \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Delta = \frac{1}{a}\begin{vmatrix} a(a^2+b^2+c^2+x^2) & ab & ac \\ b(a^2+b^2+c^2+x^2) & b^2+x^2 & bc \\ c(a^2+b^2+c^2+x^2) & bc & c^2+x^2 \end{vmatrix}$$

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x^2)\begin{vmatrix} a & ab & ac \\ b & b^2+x^2 & bc \\ c & bc & c^2+x^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1, C_3 \to C_3 - cC_1$

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x^2)\begin{vmatrix} a & 0 & 0 \\ b & x^2 & 0 \\ c & 0 & x^2 \end{vmatrix}$$

$\therefore \quad \Delta = (a^2+b^2+c^2+x^2)\,x^4 \quad (\because a \neq 0)$

Now, if $a = 0$, then $\Delta = 0$

Also, it can be easily seen that Δ is divisible by x, x^2, x^3 and x^4.

35. $$z = e^{iA}e^{iB}e^{iC}\begin{vmatrix} e^{iA} & e^{-i(C+A)} & e^{-i(B+A)} \\ e^{-i(C+B)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$\Rightarrow \quad z = -\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

$[\because e^{i(A+B+C)} = e^{i\pi} = \cos\pi + i\sin\pi = -1]$

Applying $R_1 \to R_1 + R_3, R_2 \to R_2 + R_3$

$$\Rightarrow \quad z = -\begin{vmatrix} 0 & -2e^{iB} & 0 \\ -2e^{iA} & 0 & 0 \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

$\Rightarrow \quad z = 2e^{iB}\{2e^{i(A+C)}\}$

$\Rightarrow \quad z = 4e^{i(A+B+C)} = 4e^{i\pi} = -4$

36. Since, given that

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - (\alpha R_1 + R_2)$, we get

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+2b\alpha+c) \end{vmatrix}$$

$\Rightarrow \quad \Delta = (b^2-ac)(a\alpha^2+2b\alpha+c) = 0$

$\Rightarrow \quad b^2 = ac$ or $a\alpha^2+2b\alpha+c = 0$

$\Rightarrow$ a, b and c are in GP or α is the root of the equation

$$ax^2+2bx+c = 0$$

37. $\because \quad f(x) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$

$$= \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix} \quad (\because {}^nP_n = n!, {}^nC_n = 1)$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

Then, $f(x) = \begin{vmatrix} n & 1 & 2 \\ n! & n\cdot n! & (n^2+3n+1)\,n! \\ 1 & 0 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 \\ n\cdot n! & (n^2+3n+1)n! \end{vmatrix} = n!(n^2+n+1)$$

38. $\because \quad x = \dfrac{a}{b-c}, \quad y = \dfrac{b}{c-a}, \quad z = \dfrac{c}{a-b}$

or $-a+bx-cx = 0, \quad -ay-b+cy = 0, \quad az-bz-c = 0$

Now, on eliminating a, b and c, we get

$$\begin{vmatrix} -1 & x & -x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1)^3\begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = 0$$

Also, on applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1 & -x & x \\ 1 & 1 & -y \\ 1 & z & 1 \end{vmatrix} = 0$$

39. $T_p = a + (p-1)d$

Similarly, T_q and T_r

$$\begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} pd & qd & rd \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} -d & -d & -d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

40. $\Delta = \dfrac{1}{(\log x)(\log y)}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \sin(x+y) & -\cos(x+y) & \sin^2 z \end{vmatrix} = 0$

Using $\left[\log_a b = \dfrac{\log b}{\log a}\right]$

41. $\sum_{r=1}^{n} S_r = \begin{vmatrix} \sum_{r=1}^{n} 2r & x & n(n+1) \\ \sum_{r=1}^{n}(6r^2-1) & y & n^2(2n+3) \\ \sum_{r=1}^{n}(4r^3-2nr) & z & n^3(n+1) \end{vmatrix}$

$= \begin{vmatrix} n(n+1) & x & n(n+1) \\ n^2(2n+3) & y & n^2(2n+3) \\ n^3(n+1) & z & n^3(n+1) \end{vmatrix} = 0$

(as C_1 and C_3 are same)

42. Using $R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$, we get

$$\Delta = -2k^2$$

$\therefore \quad \int_0^2 \Delta(x)\,dx = \int_0^2 -2k^2\,dx = -2x^2[x]_0^2$

$= -4k^2 = -16$

$\therefore \quad k = 2, -2$

43. Using $R_1 \to R_1 + R_2 + R_3, \Delta = (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

$\therefore \quad \Delta = (1+x+x^2)^2(x-1)^2$

$\therefore \quad \Delta = 0$ has roots $\omega, \omega, \omega^2, \omega^2, 1, 1$

44. Since, p, q and r the entries of first row and C_{21}, C_{22} and C_{23} are cofactors of second row

$\therefore \quad P \cdot C_{21} + q \cdot C_{22} + r \cdot C_{23} = 0$

45. Since, x, y and z are the entries of second raw and C_{21}, C_{22} and C_{23} are cofactors of second row.

$\therefore x \cdot C_{21} + y \cdot C_{22} + 3 \cdot C_{23} = \Delta$

46. $q \cdot M_{12} - y \cdot M_{22} + m \cdot M_{32}$

$= -q \cdot C_{12} - y \cdot C_{22} - m \cdot C_{32}$

$= -(q \cdot C_{12} + y \cdot C_{22} + m \cdot C_{32})$

$= -\Delta$

47. The first row can either be filled in ${}^3C_3$ ways (–1 alt all the blaces) or ${}^3C_1$ ways (–1 at one place)

$\Rightarrow$ First row can be filled in ${}^3C_3 + {}^3C_1$ ways corresponding to each such case the second row can also be filled in ${}^3C_3 + {}^3C_1$ ways. But third row elements have got to be fixed automatically.

$\Rightarrow$ Number of minus special determinants

$= ({}^3C_3 + {}^3C_1)({}^3C_3 + {}^3C_1) = 16$

48. Following the solution of Q. 10. The number of $n \times n$ minus special determinants $= ({}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots)^{n-1}$

(choose first $n = 1$ rows the last row has to be automatically fixed)

$= (2^{n-1})^{n-1} = 2^{(n-1)^2}$

$\Rightarrow$ (b) is correct

49. Since on expanding. Determinant = sum of six terms each of which is –1or –1

$\Rightarrow$ Minimum can not be less than –6.

But –6 is possible only in one case

$$a_1b_2c_3 = -1$$
$$a_2b_3c_1 = -1$$
$$a_3b_1c_2 = -1$$
$$a_1b_3c_2 = -1$$
$$a_3b_2c_1 = +1$$
$$a_2b_1c_3 = +1$$

These equations can not exist simultaneously because if we multiply all of them, we get $1 = -1$ a contradiction.

We now observed that there is a determinant whose value is –4.

Take $D = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$\Rightarrow$ Minimum value is –4

50. Let $f(\theta) = \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$\therefore \quad f'(\theta) = \begin{vmatrix} -\sin(\theta+\alpha) & -\sin(\theta+\beta) & -\sin(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$+ \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$+ \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ 0 & 0 & 0 \end{vmatrix}$

$= 0 + 0 + 0 = 0$

$\Rightarrow \quad f'(\theta) = 0$

$\Rightarrow \quad f(\theta) = c$

51. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$

$\therefore \quad f'(x) = 0 + a_1 + 2a_2x + \ldots$

or $f'(0) = a_1$

$\therefore \quad a_1 = \begin{vmatrix} 21 & 22 & 23 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 31 & 32 & 33 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 41 & 42 & 43 \end{vmatrix}$

$= 0 + 0 + 0 = 0$

52. $\because \quad \frac{d}{dx} f(x)\, g(x) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

$\Rightarrow \quad \frac{d}{dx} f(x) g(x) \neq \frac{d}{dx} f(x) \frac{d}{dx} g(x)$

Given, $\quad \Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = f_1(x) g_2(x) - f_2(x) g_1(x)$

$\therefore \quad \frac{d}{dx} \{\Delta(x)\} = \{f_1'(x) g_2(x) + g_2'(x) f_1(x)\}$

$$- \{f_2(x) g_1'(x) + g_1(x) f_2'(x)\}$$

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

$$\neq \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

53. $\Delta = \begin{vmatrix} a-1 & a & a+1 \\ b-1 & b & b+1 \\ c-1 & c & c+1 \end{vmatrix} = \begin{vmatrix} 0 & a & a+1 \\ 0 & b & b+1 \\ 0 & c & c+1 \end{vmatrix} \quad (C_1 \to C_1 + C_3 - 2C_2)$

$\therefore \Delta = 0$, which is not a natural number.

54. $\because \Delta = \begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$

$$= \begin{vmatrix} 5 & 4 & 3 \\ 100x + 51 & 100y + 41 & 100z + 31 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} \qquad (R_2 = R_2 - 100R_3 - 10R_1)$$

which is zero provided x, y and z are in AP.

55. $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$$

$$= [1 + (-1)^{n+2}] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

This is equal to zero only, if $n + 2$ is an odd *i.e.*, n is an odd integer.

56. Given equations are $x - cy - bz = 0$

$$cx - y + az = 0$$

and $\quad bx + ay - z + 0$

For non-zero solution,

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow \quad 1 - a^2 - c^2 - abc - abc - b^2 = 0$

$\therefore \quad a^2 + b^2 + c^2 = 2abc = 1$

57. Applying $R_3 \to R_3 - R_1, R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$$

which is divisible by both x and y.

58. Applying $C_1 \to C_1 + C_2 + C_3$ and put $a^2 + b^2 + c^2 = -2$

$$f(x) = \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 1 & 1 + b^2x & (1 + c^2)x \\ 1 & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$$= \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 0 & 1 - x & 0 \\ 0 & 0 & 1 - x \end{vmatrix} = (x - 1)^2$$

24 Relation and Function

JEE Main MILESTONE

- Relation
- Inverse Relation
- Composition of Relations
- Congruence Modulo m
- Constants and Variables
- Function
- Domain and Range of a Real Function
- Real Valued Function
- Bounded Function
- Equality of Functions
- Algebra of Functions
- Explicit and Implicit Functions
- Types of Functions
- Composite Functions
- Inverse Function
- Classification of Functions
- Even and odd Functions
- Periodic Functions

24.1 Relation

Let A and B be two non-empty sets, then relation R from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R *i.e.,* aRb. If $(a, b) \in R$, then we write it as aRb.

If $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R.

Total number of relations Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then, $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

The relation we are going to study here is an abstraction of relations we see in our everyday life such as those between parent and child, between an object and its owner, among name, social security number, address and telephone number etc.

Function of a real variable is a mathematical function whose domain is the real line. More loosely, a function of a real variable is sometimes taken to mean any function whose domain is a subset of the real line.

Domain and Range of a Relation

Let R be a relation from a set A to set B. Then, the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, $\text{Dom}(R) = \{a : (a, b) \in R\}$

and $\text{Range}(R) = \{b : (a, b) \in R\}$

Note Domain of a relation from A to B is a subset of A and its range is a subset of B.

e.g., If R is a relation from set $A = \{2, 4, 5\}$ to set $B = \{1, 2, 3, 4, 6, 8\}$ defined by $x\,Ry$ such that x divides y, then

(i) $2R2$, $2R4$, $2R6$, $2R8$, $4R4$ $4R8$ are the possible relations from set A to set B

Also, $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$ is the set of ordered pairs.

(ii) Domain $(R) = \{2, 2, 2, 2, 4, 4\} \equiv \{2, 4\}$

Range $(R) = \{2, 4, 6, 8, 4, 8\} \equiv \{2, 4, 6, 8\}$

Important Types of Relations

(i) **Void Relation** A relation R in a set A is called empty or void relation, if no element of A is related to any element of A, *i.e.*, $R = \phi \subset A \times A$

(ii) **Universal Relation** A relation R in a set A is called universal relation, if each element of A is related to every element of A, *i.e.*, $R = A \times A$.

Note Void relation and universal relation are some times called trivial relations.

(iii) **Identity Relation** The relation I_A on A is called the identity relation, if every element of A is related to itself only. Hence, mathematically, the relation

$$I_A = \{(a, a) : a \in A\}$$

is called the identity relation on A.

If $A = \{1, 2, 3\}$ be any set, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$. Also, the set of ordered pairs represented by $\{(1, 1), (3, 3)\}$ is not an identity relation because of the absence of $(2, 2)$.

(iv) **Reflexive Relation** A relation R on a set A is said to be reflexive, if every element of A is related to itself.

Thus, if R is reflexive, then $(a, a) \in R, \forall a \in A$.

A relation R on a set A is not reflexive, if there exists an element $a \in A$ such that $(a, a) \notin R$.

e.g., (a) If $A = \{1, 2, 3\}$ be a set.

Then, $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A.

But $R' = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive, because for $2 \in A$, $(2, 2) \notin R$

(b) Let L be the set of all lines in a plane. Then, relation R on L defined by $(l_1, l_2) \in R$ such that l_1 is parallel to l_2 is reflexive, since every line is parallel to itself.

Note The universal relation on a non-void set A is reflexive.

(v) **Symmetric Relation** A relation R on a set A is said to be a symmetric relation iff

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$$

i.e., $$aRb \Rightarrow bRa, \forall a, b \in A$$

Also, A relation R on a set A is not a symmetric relation if there are atleast two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$

e.g.,

(a) Let $A = \{1, 2, 3, 4\}$ be any set and let R_1 and R_2 be relations on A_1 such that

$$R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$$

and $$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$$

Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \in R_2$.

(b) Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R$, x is perpendicular to y. Then, R is a symmetric on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$ *e.g.,* if $(L_2, L_1) \in R$ automatically implies $(L_2, L_1) \in R$

Note

(i) The identity and the universal relations on a non-void set are symmetric relations.

(ii) A reflexive relation on a set A is not necessarily symmetric.

(iii) The relation R on a set A is symmetric iff $R = R^{-1}$.

(vi) **Transitive Relation** Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R$. $(a, c) \in R$ for all $a, b, c \in A$.

i.e., $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,c$ for all $a, b, c \in A$.

e.g.,

(a) On the set N of natural numbers, the relation R defined by xRy, such that x is less than y is transitive, because for any $x, y, z \in N$

$$x < y \text{ and } y < z \Rightarrow x < z$$

i.e., xRy and $yRz \Rightarrow xRz$

(b) Let L be the set of all straight lines in a plane. Then, the relation is parallel to on L is a transitive relation, because for any $l_1, l_2, l_3 \in L$

$l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3$

Note The identity and the universal relations on a non-void set are transitive.

(vii) **Anti-symmetric Relation** Let A be any set. A relation R on set A is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

Note

(i) By definition, if $(a, b) \in R$ and $(b, a) \notin R$, then also R is an anti-symmetric relation.

(ii) The identity relation on a set A is an anti-symmetric relation.

Hot Spot

Equivalence Relation

This topic is very important for JEE Main. Generally questions are asked from this topic, the nature of this question is moderate.

A relation R on a set A is said to be an equivalence relation on A iff

(a) it is reflexive, *i.e.*, $(a, a) \in R, \forall a \in A$.

(b) it is symmetric, *i.e.*, $(a, b) \in R$.

$\Rightarrow (b, a) \in R, \ \forall a, b \in A$

(c) it is transitive, *i.e.*, $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow (a, c) \in R, \forall a, b, c \in A$

To solve such type of problem following steps are given below.

Step I Firstly we write the given relation.

Step II Check relation reflexive, symmetric and transitive one by one.

Step III If all three relations are satisfied, then it is one one and if atleast one of them is not satisfied, then it is not an equivalence relation,

***Sample* Problem 1** *Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R$, such that line l_1 is parallel to line l_2, then R is*

(a) *anti-symmetric* (b) *not a transitive*

(c) *equivalence relation* (d) *None of these*

Interpret (b) **Step I** Write the given relation. Let L be the given set of all lines in a plane and it is defined by $l_1, l_2 \in L$ such that $l_1 \| l_2$

Step II Check all the three relations.

Reflexive For each line $l \in L$, we have

$$l \| l \ \Rightarrow \ (l, l) \in R, \forall l \in L \ \Rightarrow \ R \text{ is reflexive}$$

Symmetric Let $l_1, l_2 \in R$ such that $(l_1, l_2) \in R$, Then,

$$(l_1, l_2) \in R \ \Rightarrow \ l_1 \| l_2 \ \Rightarrow \ l_2 \| l_1 \ \Rightarrow \ (l_2, l_1) \in R$$

So, R is symmetric on L.

Transitive Let $l_1, l_2, l_3 \in L$ such that $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$. Then,

$$(l_1, l_2) \in R \text{ and } (l_2, l_3) \in R \ \Rightarrow \ l_1 \| l_2 \text{ and } l_2 \| l_3 \ \Rightarrow \ l_1 \| l_3$$
$$\Rightarrow \ (l_1, l_3) \in R$$

So, R is transitive on L.

Step III If all three relations are satisfied, then it is an equivalence relation.

Hence, R is a reflexive, symmetric and transitive, thus R is an equivalence relation on L.

Properties of Equivalence Relations

(i) The inverse of an equivalence relation is also an equivalence relation, *i.e.*, if R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation, on a set..

(iii) The intersection of two equivalence relations on a set is an equivalence relation on the set.

24.2 Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then, the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

i.e., If $(a, b) \in R$ then $(b, a) \in R^{-1}$

Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

24.3 Composition of Relations

Let R and S be two relations from set A to B and B to C respectively, then we can define relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of R and S.

But $\qquad RoS \neq SoR$

Also, $\qquad (SoR)^{-1} = R^{-1}oS^{-1}$

If $A = \{1, 2, 3\}, B = \{a, b, c, d\}, C = \{p, q, r, s\}$ be three sets, such that

$R = \{(1, a), (2, c), (1, c), (2, d)\}$ is a relation from A to B and $S = \{(a, s), (b, q), (c, r)\}$ is relation from B to C. Then, SoR is a relation from A to C given by $SoR = \{(1, s), (2, r), (1, r)\}$

24.4 Congruence Modulo m

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b \pmod{m}$

Thus, $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m

e.g., $18 \equiv 3 \pmod{5}$ because $18 - 3 = -15$ which is divisible by 5.

Similarly, $3 \equiv 13 \pmod{2}$ because $3 - 13 = -10$ which is divisible by 2.

But $25 \neq 2 \pmod{4}$ because 4 is not a divisor of $25 - 2 = 23$

Check Point 1

1. Let $R = \{(x, y) : x^2 + y^2 = 1, x. y \in R\}$ be a relation in R. The relation R is
   (a) reflexive (b) symmetric
   (c) transitive (d) anti-symmetric
2. The void relation on a set A is
   (a) reflexive (b) symmetric and transitive
   (c) reflexive and symmetric (d) reflexive and transitive
3. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$. Then, R^{-1} is
   (a) $\{(3, 3), (3, 1), (5, 2)\}$ (b) $\{(1, 3), (2, 5), (3, 3)\}$
   (c) $\{(1, 3), (5, 2)\}$ (d) None of these
4. A relation from P to Q is
   (a) a universal set of $P \times Q$ (b) $P \times Q$
   (c) an equivalent set of $P \times Q$ (d) a subset of $P \times Q$
5. For real numbers x and y, we write $xRy \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then, the relation R is
   (a) reflexive (b) symmetric
   (c) transitive (d) None of these
6. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
   (a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
   (b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
   (c) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
   (d) None of the above
7. Let $R = \{(a, a)\}$ be a relation on a set A. Then, R is
   (a) symmetric
   (b) anti-symmetric
   (c) symmetric and anti-symmetric
   (d) neither symmetric nor anti-symmetric

24.5 Constants and Variables

Those quantities which are unaltered under any mathematical operation are called constants.

Those quantities which are altered under any mathematical operation and can be opted any value, are known as variables.

Variables are of two types

(i) Independent variable

(ii) Dependent variable

(i) Independent Variable

Those variables which can be opted any value are called independent variable.

(ii) Dependent Variable

Those variables whose values depend upon the other variables, are called dependent variable.

e.g., Let $y = x^2 + 7x + 2$

Here, x is an independent variable whereas y is a dependent variable. Since, the value of y depends upon the value of x.

24.6 Function

Let A and B be two non-empty sets. Then, a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that

(i) all elements of set A are associated to elements in set B.

(ii) an element of set A is associated to a unique element in set B.

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as 'map' (or mapping), 'correspondence' are used as synonyms for 'function'. If f is a function from a set A to set B, then we write $f : A \to B$ or $A \to B$.

If an element $a \in A$ is associated to an element $b \in B$, then b is called the 'f-image of a' or 'image of a under f' or 'the value of the function f at a'. Also, a is called the pre-image of b under the function f. We write it as $b = f(a)$

(i) all the independent elements (inputs) in A must have their images in set B.

(ii) to each and every independent element in A there corresponds one and only one image in B.

(iii) every function is a relation but every relation may or may not be a function.

Note

- Function cannot be multivalued.
- If A and B have m and n elements respectively, then the number of functions from A to B is n^m.

Vertical Parallel Line Test (VPL Test)

This is a geometrical test to check whether a relation is a function or not.

"If we draw a vertical parallel line *i.e.,* any line parallel to y-axis, then, if this line intersects the graph of the expression in more than one point, then the expression is a relation else, if it intersects at only one point, the expression is a function."

In figure (a), the vertical parallel line intersects the curve at two points thus the expression is a relation whereas in figure (b), the vertical parallel line intersects the curve at one point. Thus, the expression is a function.

***Sample* Problem 3** *Let $A = \{2, 4, 6, 8\}$ and $B = \{s, t, u, v, w\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures.*

then which of the following is correct?

(a) *f_1 and f_2 are fucntions* (b) *f_3 and f_4 are functions*
(c) *f_1 and f_4 are functions* (d) *f_2 and f_3 are functions*

Interpret (c) Now, see that f_1 is not function from set A to set B, since there is an element $6 \in A$ which is not associated to any element of B but f_2 and f_3 are the functions from A to B, because under f_2 and f_3 each elements in A is associated to a unique element in B. But f_4 is not function from A to B because an elements $8 \in A$ is associated to two elements u and w in B.

24.7 Domain and Range of a Real Function

Domain

Generally, real functions in calculus are described by some formula and their domains are not explicily stated. In such cases, in order to find the domain of a function f(say) we find all real numbers x for which $f(x)$ is a real valued.

In other words, determining the domain of a function means finding all real numbers x for which $f(x)$ is real. *e.g.*, If $f(x) = \sqrt{2 - x}$, then $f(x)$ is real for $x \le 2$. For $x > 2$, $f(x)$ is not real. So, domain of $f(x)$ is the set of all real numbers less than or equal to 2 *i.e.*, $(-\infty, 2]$.

Rules for Finding Domain

(i) Expression under even root (*i.e.*, square root, fourth root) ≥ 0.

(ii) Denominator $\ne 0$.

(iii) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$, while domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

***Sample* Problem 4** *The domain of the function $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is*

(a) $[-1, 2) \cup [3, \infty)$ (b) $(-1, 2) \cup [3, \infty)$
(c) $[-1, 2] \cup [3, \infty)$ (d) *None of these*

Interpret (a) $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{x-2}} = \dfrac{\sqrt{(x+1)(x-3)(x-2)}}{x-2}$

$f(x)$ is defined, if $(x+1)(x-3)(x-2) \ge 0$

or $\{x - (-1)\}(x-2)(x-3) \ge 0$

$\Rightarrow$ $-1 \le x \le 2$ or $x \ge 3$

Also, $x - 2 \ne 0 \Rightarrow x \ne 2$

$\therefore$ Domain of $f = \{[-1, 2] \cup [3, \infty)\} - \{2\}$

$= [-1, 2) \cup [3, \infty)$

Codomain

The set of all possible outcomes for the function f is called the codomain of the function f.

e.g., Let $f : \{a, b, c\} \to \{\alpha, \beta, \gamma, \delta\}$ is a function, that the codomain of f is $\{\alpha, \beta, \gamma, \delta\}$.

Range

The range of a function $f(x)$ is the set of values of $f(x)$ which it attains at points in its domain. For a real function the codomain is always a subset of R. So, range of a real function f is the set of all points y such that $y = f(x)$, where $x \in \text{dom } f(x)$.

Rules for Finding Range

First of all find the domain of $y = f(x)$.

(i) If domain $\in$ finite number of points

$\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(ii) If domain $\in R$ or $R - \{$some finite points$\}$

Then, express x in terms of y. From this find y for x to be defined. (*i.e.,* find the value of y for which x exists).

(iii) If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

Note

(i) Range is the subset of codomain.

(ii) Range can never exceed codomain for a given function.

***Sample* Problem 5** *The range of the function* $y = \frac{x}{1+x^2}$ *is*

(a) $\left[0, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\left[-\frac{1}{2}, 0\right]$ (d) *None of these*

Interpret (b) $y = \frac{x}{1+x^2}$, which is defined for all real x.

$\therefore$ Domain of $y = (-\infty, \infty)$

We have, $y = \frac{x}{1+x^2} \Rightarrow y + yx^2 = x$

$\Rightarrow$ $x^2y - x + y = 0$

For x to be real,

$$1 - 4y^2 \geq 0, \quad y \neq 0$$

$\Rightarrow$ $(1-2y)(1+2y) \geq 0, \quad y \neq 0$

$\Rightarrow$ $(2y-1)(2y+1) \leq 0, \quad y \neq 0$

$\Rightarrow$ $-\frac{1}{2} \leq y \leq \frac{1}{2}, \quad y \neq 0$

Also, if $y = 0$, then $x = 0$ so that $y = 0$ is also in the range.

$\therefore$ Range of $y = \left[-\frac{1}{2}, \frac{1}{2}\right]$

24.8 Real Valued Function

All those functions of which domain and codomain are subsets of R are called real valued functions.

24.9 Bounded Function

A function 'f' is said to be bounded if $|f(x)| \leq m$, for some finite 'm' for every x in domain of f.

24.10 Equality of Functions

Two functions f and g are said to be equal, iff

(i) the domain of f = domain of g.

(ii) the codomain of f = the codomain of g, and

(iii) $f(x) = g(x)$ for every x belonging to their common domain.

If two functions f and g are equal, then we write $f = g$.

e.g., If $A = \{1, 2\}$, $B = \{10, 13,\}$ $f : A \to B$, $f(x) = x^2 + 9$ and $g = A \to B$, $g(x) = 3x + 7$, then $f = g$ because domains and codomains of both f and g are same also

$$f(1) = 10 = g(1), f(2) = 13 = g(2)$$

24.11 Algebra of Functions

Given function $f : D_1 \to R$ and $g : D_2 \to R$, we describe function $f + g$, $f - g$, fg and f/g as follows

$f + g : D \to R$ is a function defined by

$$(f+g)(x) = f(x) + g(x),$$

$f - g : D \to R$ is a function defined by

$$(f-g)(x) = f(x) - g(x)$$

$fg : D \to R$ is a function defined by

$$(fg)(x) = f(x)\, g(x)$$

$\frac{f}{g} : C \to R$ is a function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \; g(x) \neq 0,$$

where $D = D_1 \cap D_2$ and $C = \{x \in D : g(x) \neq 0\}$

***Sample* Problem 6** *If* $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$ *and* $g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$, *then the function f/g is*

(a) $\begin{cases} x - 1, & x < 3 \\ \frac{x-4}{x-3}, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$ (b) $\begin{cases} x - 1, & x < 3 \\ \frac{x-4}{x-3}, & 3 \leq x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$

(c) $\begin{cases} \frac{x-1}{x-2}, & x < 3 \\ x - 4, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$ (d) None of these

Interpret (a) We redefine the function $f(x)$ and $g(x)$ in the intervals as shown below.

$$f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & 3 \leq x < 4, \\ x - 4 & x \geq 4 \end{cases}$$

$$g(x) = \begin{cases} x - 3, & x < 3 \\ x - 3, & 3 \leq x < 4 \\ x^2 + 2x + 2 & x \geq 4 \end{cases} \Rightarrow \frac{f(x)}{g(x)} = \begin{cases} x - 1, & x < 3 \\ \frac{x-4}{x-3}, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$$

24.12 Explicit and Implicit Functions

A function is said to be an explicit function, if the dependent variable y is expressible completely in terms of independent variable.

A function is said to be an implicit function, if the dependent variable is not directly expressible in terms of the independent variable x.

e.g., $y = 3e^x - x^2 + 2, x \in R$ is an explicit function while $ye^{x^2} + xe^{-y^2} = e^{a^2}$ is an implicit function.

24.13 Types of Functions

(i) One One Function

A function $f: A \to B$ is said to one one, if different elements in A have different images in B.

This type of functions are known as injective functions.

e.g., Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Consider $f: A \to B$, $f(x) = 2x$. Then, $f(1) = 2; f(2) = 4$ and $f(3) = 6$. Clearly, f is a one one function from A to B such that different elements in A have different images in B.

How to prove whether a function is injective (one to one)?

If we are able to prove that for two same images (outputs) in codomain, there existed two same pre-images (inputs) in domain, the function is one to one. But if at all there existed two different pre-images (inputs) for two same images (outputs), the function is not one to one.

So, let us follow the following steps to trace this type of function.

Step I Let us consider two same images (outputs) of a function $y = f(x)$, such that $y_1 = y_2$

$\Rightarrow \quad f(x_1) = f(x_2)$

Step II Solve the expression obtained in Step I

i.e., $f(x_1) = f(x_2)$

till you get any relation between x_1 and x_2

Step III If the relation you get is $x_1 = x_2$

$\Rightarrow$ The function is injective.

Geometrical Test for Injectivity HPL Test

Geometrically, if we draw a horizontal parallel line (HPL), *i.e.,* any line which is parallel to x-axis ($y = a$), then if this line intersects the graph of the function in atleast two points, then the function is not a one to one function else, if it intersects at only one point, the function is one to one, *i.e.,* injective.

e.g.,

(a) One to One Function

(b) Many to One Function

Differential Test for Injectivity

If the function is a strictly monotonic, *i.e.,* strictly increasing or strictly decreasing throughout its domain, then the function is one to one. The figure (a) above represents a strictly monotonic.

For a strictly monotonic $\frac{dy}{dx} > 0$ or < 0 throughout

(a) for strictly increasing function $\frac{dy}{dx} > 0$

(b) for strictly decreasing function $\frac{dy}{dx} < 0$.

***Sample* Problem 7** *Let $f: R \to R$ defined by $f(x) = x^3 + ax^2 + 3x + 100$, then the values of a for which f is a one one function, is*

(a) $-2 \le a \le 3$ (b) $-2 \le a \le 2$

(c) $-3 \le a \le 3$ (d) *None of these*

Interpret (c) $f(x) = x^3 + ax^2 + 3x + 100 \Rightarrow f'(x) = 3x^2 + 2ax + 3$.

For $f(x)$ to be one-one. $f'(x) \ge 0$ or ≤ 0.

But $f(x)$ is a quadratic expression and coefficient of $x^2 > 0$

So, that $f'(x) \ge 0 \Rightarrow D \le 0 \Rightarrow 4a^2 - 36 \le 0 \Rightarrow a^2 \le 9$

$\Rightarrow -3 \le a \le 3$.

(ii) Many One Function

If let $f: A \to B$. If two or more than two elements of set A have the same image in B, then f is said to be many one.

e.g., Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$

where $f: A \to B$ given by $f(x) = x^2$

Here, $f(-1) = 1; f(1) = 1; f(2) = 4$ and $f(3) = 9$.

Clearly, two elements 1 and -1 have the same image $1 \in B$.

So, f is a many one function.

Sample **Problem 8** *If* $f: R \to R$ *be defined as* $f(x) = \dfrac{x^2 + 4x + 7}{x^2 + x + 1}$, *then* $f(x)$ *is*

(a) *one one*
(b) *many one*
(c) *many many*
(d) *None of the above*

Interpret (b) For the given function $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1^2 + 4x_1 + 7}{x_1^2 + x_1 + 1} = \frac{x_2^2 + 4x_2 + 7}{x_2^2 + x_2 + 1}$$

$$\Rightarrow \quad (x_1 - x_2)(2x_1 + 2x_2 + 1 + x_1x_2) = 0.$$

Here, one of the solution of $2x_1 + 2x_2 + 1 + x_1x_2 = 0$ is $x_1 = 0$ and $x_2 = -1/2$.

Hence, $f(0) = f\left(-\frac{1}{2}\right) = 7$, so that $f(x)$ is many one.

Remark Any function will be either one one or many one.

(iii) Onto Function

If the function $f: A \to B$ is such that each and every element in B is the f image of atleast one element in A, then f is a function of A onto B. It is also called **Surjective** function.

OR

If each element in codomain have atleast one pre-image in the set of domain, then function is onto.

e.g., Let $A = \{-1, 1, 2, -2\}, B = \{1, 4\}$ and $f: A \to B$, be a function defined by $f(x) = x^2$. Then, f is onto because

$$f(A) = \{f(-1), f(1), f(2), f(-2)\} = \{1, 4\} = B.$$

Note If range is same as codomain, then function is onto.

(iv) Into Function

If the function $f: A \to B$ is such that there is atleast one element in B which is not f image of any element in A, then f is called an into function.

OR

If there is even single element in the set of codomain which does not have its pre-image, then it is known as into function.

e.g., Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$. Consider $f: A \to B$; $f(x) = x - 2$. Then, $f(2) = 0$; $f(3) = 1$, $f(5) = 3$ and $f(7) = 5$.

Clearly, f is a function from A to B. Now, there exists an element $7 \in B$, having no pre-image in A. So, f is an into function.

Note If range of a function f is proper subset of codomain, then function f is into.

One One Onto Function

A function which is one one function and onto function is called one one onto function.

It is also called bijective function.

Sample **Problem 9** *Let* $f: (-\infty, \infty) \to [2, \infty)$ *be a function defined by* $f(x) = \sqrt{x^2 - 2a + a^2}$, $a \in R$, *then the value of a for which f is onto*

(a) 1 (b) $\sqrt{3}$ (c) $1 \pm \sqrt{3}$ (d) $1 \pm \sqrt{5}$

Interpret (d) For f to be onto range of the function should be $[2, \infty)$. So, $a^2 - 2a = 4$

$$\Rightarrow \quad a = 1 \pm \sqrt{5}.$$

24.14 Composite Functions

Let $f: A \to B$ and $g: B \to C$ be two functions. Then, a function $gof: A \to C$ defined by

$$(gof)(x) = gf(x)\}, \text{ for all } x \in A$$

is called the composition of f and g.

Let A, B and C be three non-void sets and let $f: A \to B$, $g: B \to C$ be two functions. Since, f is a function from A to B, therefore for each $x \in A$ there exists a unique element $f(x) \in B$. Again, since g is a function from B to C, therefore corresponding to $f(x) \in B$ there exists a unique element $g\{f(x)\} \in C$. Thus, for each $x \in A$ there exists a unique element $g\{f(x)\} \in C$.

It follows from the above discussion that f and g when considered together define a new function from A to C. This function is called the composition of f and g and is denoted by gof. We define it formally as follows :

Note

- *gof* is defined only if for each $x \in A, f(x)$ is an element of *g* so that we can take its *g*-image. Hence, for the composition *gof* to exist, the range of *f* must be a subset of the domain of *g*.
- gof exists iff the range of *f* is a subset of domain of *g*. Similarly, *fog* exists if range of *g* is a subset of domain of *f*.

Properties of Composition of Functions

(i) If *f* and *g* are even functions $\Rightarrow$ *fog* is an even function.

(ii) If *f* is an odd and *g* is an even function $\Rightarrow$ *fog* is an odd function.

(iii) If *f* is an even and *g* is an odd function $\Rightarrow$ *fog* is an even function.

(iv) The composition of functions is not commutative.

i.e., $fog \neq gof$

(v) The composition of functions is associative

i.e., If *f, g, h* are any three functions such that $(fog)\,oh$ and $fo\,(goh)$ both exist, then

$$(fog)\,oh = fo\,(goh)$$

(vi) The composition of two bijective function is also a **bijective function**.

(vii) The composition of any function with the **identity function** is the function itself.

i.e., If $f : A \to B$, then

$$foI_A = I_B of = f$$

(viii) If *f, g* and *h* be functions from *R* to *R* then,

$$(f + g)\,oh = foh + goh$$

and $$(f \cdot g)oh = (foh)\cdot(goh)$$

***Sample* Problem 10** *If* $f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0 \\ 2 - x, & \text{if } x < 0 \end{cases}$, *then* $(fof)(x)$ *is*

(a) $|x|$ (b) $4 - |x|$

(c) $4 + |x|$ (d) *None of these*

Interpret (c) $f(x) = 2 + |x|$

$$(fof)(x) = 2 + |f(x)| = 2 + 2|(x)| = 4 + |x|$$

24.15 Inverse Function

Let $f : A \to B$ is a bijective function, then there exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall x \in A$ and $y \in B$, then *g* is called inverse function of *f*.

Hence, $$g = f^{-1} : B \to A$$

Properties of Inverse Function

(i) Inverse of bijective function is unique.

(ii) Inverse of bijective function is also bijective function.

(iii) If $f : A \to B$ is bijective function and $g : B \to A$ is inverse of *f*, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity function of sets *A* and *B*, respectively.

(iv) If $f : A \to B$ and $g : B \to A$ are two bijective functions, then $gof : A \to C$ is also bijective function and $(gof)^{-1} = f^{-1}og^{-1}$.

(v) $fog \neq gof$, but if $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$ and $(fog)(x) = (gof)(x) = x$.

Note If the inverse of *f* exists, then *f* is called an invertible function.

***Sample* Problem 11** *The inverse of the function* $f(x) = \log(x^2 + 3x + 1)$; $x \in [1, 3]$ *assuming it to be an onto function, is*

(a) $\dfrac{-3 - \sqrt{5 + 4e^x}}{2}$ (b) $\dfrac{-3 \pm \sqrt{5 + 4e^x}}{2}$

(c) $\dfrac{-3 + \sqrt{5 + 4e^x}}{2}$ (d) *None of these*

Interpret (c) Given, $f(x) = \ln(x^2 + 3x + 1)$

$$\therefore \quad f'(x) = \frac{2x + 3}{(x^2 + 3x + 1)} > 0 \quad \forall\ x \in [1, 3],$$

which is a strictly increasing function. Thus, $f(x)$ is injective, given that $f(x)$ is onto. Hence, the given function $f(x)$ is invertible.

Now, $$f\{f^{-1}(x)\} = x \Rightarrow \ln((f^{-1}(x))^2 + 3(f^{-1}(x)) + 1) = x$$

$$\Rightarrow \quad \{f^{-1}(x)\}^2 + 3\{f^{-1}(x)\} + 1 - e^x = 0$$

$$\therefore \quad f^{-1}(x) = \frac{-3 \pm \sqrt{9 - 4\cdot 1(1 - e^x)}}{2}$$

$$= \frac{-3 \pm \sqrt{(5 + 4e^x)}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{-3 + \sqrt{(5 + 4e^x)}}{2} \qquad [\because f^{-1}(x) \in [1, 3]]$$

Hence, $$f^{-1}(x) = \frac{-3 + \sqrt{(5 + 4e^x)}}{2}$$

Sample Problem 12 *The inverse of the function*

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \leq 4 \\ 8\sqrt{x}, & x > 4 \end{cases} \text{ is}$$

(a) $\begin{cases} 1/x, & x < 1 \\ \sqrt{x}, & 1 \leq x \leq 16 \\ x^2/64, & x > 16 \end{cases}$ (b) $\begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \leq x \leq 16 \\ x^2/64, & x > 16 \end{cases}$

(c) $\begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \leq x \leq 16 \\ x^2/64, & x > 16 \end{cases}$ (d) *None of these*

Interpret (*d*) Given, $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ...(i)

$\therefore$ $x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 4 \\ y^2/64, & y^2/64 > 4 \end{cases}$

$\Rightarrow$ $f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$ [from Eq. (i)]

Hence, $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 4 \\ x^2/64, & x > 4 \end{cases}$

24.16 Classification of Functions

(i) Constant Function

A function which does not change as its parameters vary, *i.e.,* the function whose rate of change is zero. In short, we can say that a constant function is a function that always gives or returns the same value.

Domain $x \in (-\infty, \infty)$ *i.e.,* $x \in \mathbb{R}$

Range $y \in \{c\}$

Nature (a) Many of One [HPL Test]

(b) Into Function for codomain as a Set of Real Number, *i.e.,* $\mathbb{R}$

(c) Neither monotonically increasing nor monotonically decreasing in its domain

Note The function graph of a one dimensional constant function is a straight line.

(ii) Algebraic Functions

A. Polynomial Function

A polynomial function is the function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + .. + a_n$$

The value of n must be a non-negative integer, *i.e.,* it must be a whole numbers and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$. The degree of the polynomial is the highest value for n, where $a_n \ne 0$, *i.e., n.*

Domain The entire number line, *i.e.,* $x \in \mathbb{R}$

Range Varies from function to function

Special Cases

***Case* I** If n is even.

Let $y = x^2$

Range $y \in [0, \infty)$

Nature (a) Many to One [HPL Test]

(b) Onto function for codomain as a set of Non-negative real numbers.

(c) Monotonically Decreasing for all $x \in (-\infty, 0)$

(d) Monotonically increasing for all $x \in (0, \infty)$

(e) Even function

***Case* II** If n is odd.

Let $y = x^3$

Range $y \in \mathbb{R}$

Nature (a) One to One [HPL Test]

(b) Onto function for codomain as a set of real numbers, *i.e.,* $\mathbb{R}$

(c) Monotonically increasing throughout its domain

(d) Odd function

B. Rational Function

A rational function is the function of the form

$$y = \frac{f(x)}{g(x)} = \frac{a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \ldots + b_m}$$

the value of n must be a non-negative integer, *i.e.,* it must be a hole number and $a_i \in \mathbb{R}, b_j \in \mathbb{R}$ for all $1 \le i \le n$ and $1 \le j \le m$ respectively.

Also, $g(x) \ne 0$

$\because$ $g(x) = 0$ will give all those points where the function is discontinuous.

Domain $x \in \mathbb{R} - \{x : x = \alpha, \forall (x) = 0\}$

Range Varies from function to function

Special Cases

***Case* I** If n is even.

Let $y = \frac{1}{x^2}$

Domain $x \in \mathbb{R} - \{0\}$

Range $y \in (0, \infty)$

Nature (a) Many to one [HPL Test]

(b) One Function for Codomain as a set of positive real numbers.

(c) Monotonically increasing for all $x \in (-\infty, 0)$

(d) Monotonically decreasing for all $x \in (0, \infty)$

(e) Even function

***Case* II** If n is odd.

Let $y = \frac{1}{x^3}$

Domain $x \in \mathbb{R} - \{0\}$

Range $y \in \mathbb{R} - \{0\}$

Nature (a) One to One [HPL Test]

(b) Into function for codomain as a set of real numbers *i.e.*, $\mathbb{R}$

(c) Monotonically decreasing throughout its domain

(d) Odd function

Sample **Problem 13** *The range of the function* $f(x) = \frac{2x-2}{x^2-2x+3}$ *is*

(a) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{\sqrt{2}}, 0\right]$

(c) $\left[-\frac{1}{2}, 0\right]$ (d) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Interpret (*d*) Here, $f(x) = \frac{2x-2}{x^2-2x+3}$

Let $y = f(x)$ *i.e.*, $y = \frac{2x-2}{x^2-2x+3}$

$\Rightarrow$ $yx^2 - 2(y+1)x + 3y + 2 = 0$

which is a quadratic in x. For above quadratic to have real roots, $\Delta > 0$

$\Rightarrow$ $(y+1)^2 - y(3y+2) \geq 0 \Rightarrow y^2 \leq \frac{1}{2}$

$\Rightarrow$ $-\frac{1}{\sqrt{2}} \leq y \leq \frac{1}{\sqrt{2}}$

Hence, the range of $f(x)$ is $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

C. Irrational Function

An Irrational Function is the function of the form

$$y = \{f(x)\}^{p/q}$$

where, p and q are integers, have no common factor and also $q \neq 0, q > 0$.

Domain Varies from function to function

Special Cases

***Case* I** If q is even.

For q being an even number, the function

$$y = \{f(x)\}^{p/q}$$

is defined if and only if

$f(x) \geq 0$ provided $\left(\frac{p}{q}\right)$ is positive

else $f(x) > 0$ provided $\left(\frac{p}{q}\right)$ is negative

Let $y = \sqrt{x}$

Domain $x \in [0, \infty)$

Range $y \in [0, \infty)$

Nature (a) One to one [HPL Test]

(b) Into Function for codomain as a set of real numbers.

(c) Monotonically increasing throughout its domain

***Case* II** If q is odd

For q being an odd number, the function

$$y = \{f(x)\}^{p/q}$$

is defined if and only if

$x \in R$ provided $\left(\frac{p}{q}\right)$ is positive

else $x \in \mathbb{R} - \{x : x = \alpha, \forall f(\alpha) = 0\}$

Let $y = \sqrt[3]{x}$

Domain $x \in \mathbb{R}$

Range $y \in \mathbb{R}$

Nature (a) One to one [HPL Test]

(b) Onto Function for codomain as a set of real Numbers *i.e.*, $\mathbb{R}$

(c) Monotonically increasing throughout its domain.

D. Identity Function

Identity function is the form of $y = f(x) = x$

which is a straight line passing through origin and having slope unity.

Domain $x \in (-\infty, \infty)$ *i.e.*, $x \in \mathbb{R}$

Range $y \in (-\infty, \infty)$, *i.e.*, $y \in \mathbb{R}$

Nature (a) One to one [HPL Test]

(b) Onto function for codomain as a set of real number *i.e.*, $\mathbb{R}$

(c) Monotonically increasing throughout its domain

(d) Odd function.

E. Reciprocal Function

The reciprocal function is the inverse of an identify function, such that

$$y = f(x) = \frac{1}{x}$$

which is an equation of a rectangular hyperbola whose asymptotes (here x-axis and y-axis) intersect at right angles.

Domain $x \in \mathbb{R} - \{0\}$

Range $y \in \mathbb{R} - \{0\}$

Nature (a) One to one [HPL Test]

(b) Into Function for codomain as a set of real numbers, *i.e.*, $\mathbb{R}$

(c) Monotonically decreasing throughout its domain.

(d) Odd function

(iii) Trigonometric Function

A. Sine Function

Since function is the form of $y = \sin x$

Domain $x \in \mathbb{R}$

Range $y \in [-1, 1]$

Nature (a) Many to one [HPL Test]

One to one in $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Onto function for codomain = $[-1, 1]$

(c) Monotonically increasing for all

$$x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$$

(d) Monotonically decreasing for all

$$x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$$

(e) Odd function

(f) Periodic with period 2π

B. Cosine Function

Cosine function is the form of $y = \cos x$

Domain $x \in \mathbb{R}$

Range $y \in [-1, 1]$

Nature (a) Many to one [HPL Test]

One to one in $x \in [0, \pi]$

(b) Onto function for codomain = $[-1, 1]$

(c) Monotonically increasing for all $x \in (2n\pi - \pi, 2n\pi)$

(d) Monotonically decreasing for all $x \in (2n\pi, 2n\pi + \pi)$

(e) Even function

(f) Periodic with period 2π

C. Tangent Function

Tangent function is the form of $y = \tan x$

By definition, for any right angled triangle, $\tan x$ represents the ratio of perpendicular to that of its base. **Geometrically**, $\tan x$ means the slope of the curve at any point (x, y) on the curve $y = f(x)$.

Domain $x \in R - \left\{(2n + 1)\frac{\pi}{2}\right\}$ where, $n \in I$

Range $y \in \mathbb{R}$

Nature (a) Many to one [HPL Test]

One to one in $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(b) Onto function for codomain as set of real numbers

(c) Monotonically increasing for all $x \in \mathbb{R}$

(d) Odd function

(e) Periodic with period π

D. Cosecant Function

Cosecant function is the form of $y = \operatorname{cosec} x$

By definition, cosecant function is the reciprocal of sine function.

Domain $x \in \mathbb{R} - \{n\pi\}$ where, $n \in 1$

Range $y \in (x, -1) \cup [1, x)$

Nature (a) Many to one [HPL Test]

One to one in $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Into function for codomain as a set of real numbers

(c) Monotonically decreasing for all

$$x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$$

(d) Monotonically increasing for all

$$x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$$

(e) Odd function

(f) Periodic with period 2π

E. Secant Function

Secant function is the form of $y = \sec x$

By definition, secant function is the reciprocal of cosine function.

Domain $x \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$

where, $n \in I$

Range $y \in (-\infty, -1] \cup [1, \infty]$

Nature (a) Many-to-one [HPL Test]

(b) Into function for codomain as a set of real numbers.

(c) Monotonically decreasing for all $x \in (2n\pi - \pi, 2n\pi)$

(d) Monotonically increasing for all $x \in (2n\pi, 2n\pi + \pi)$

(e) Even Function

(f) Periodic with Period 2π

F. Cotangent Function

Cotangent function is the form of $y = \cot x$

By definition cotangent function is the reciprocal of tangent function.

Domain $x \in \mathbb{R} = \{n\pi\}$ where, $n \in I$

Range $y \in \mathbb{R}$

Nature (a) Many to one [HPL Test] one to one in $x \in (0, \pi)$

(b) Onto function for codomain as set of real numbers

(c) Monotonically decreasing for all $x \in \mathbb{R}$

(d) Odd Function

(e) Periodic with period π

***Sample* Problem 14** *The range of the function* $f(x) = 3\sin x + 8\cos\left(x - \frac{\pi}{3}\right) + 5$ *is*

(a) $[-5, 5]$

(b) $[5 - \sqrt{73 + 24\sqrt{3}}, 5 + \sqrt{73 + 24\sqrt{3}}]$

(c) $[5 - \sqrt{73 + \sqrt{3}}, 5 + \sqrt{5 + \sqrt{3}}]$

(d) *None of the above*

Interpret (b) Here, $f(x) = 3\sin x + 8\cos\left(x - \frac{\pi}{3}\right) + 5$

$$= 3\sin x + 4(\cos x + \sqrt{3}\sin x) + 5$$

$$= (3 + 4\sqrt{3})\sin x + 4\cos x + 5$$

Put $3 + 4\sqrt{3} = r\cos\theta$ and $4 = r\sin\theta$, so that

$$r = \sqrt{73 + 24\sqrt{3}}$$

and $$\theta = \tan^{-1}\frac{4}{3 + 4\sqrt{3}}$$

$\Rightarrow$ $$f(x) = \sqrt{73 + 24\sqrt{3}}\sin(x + \theta) + 5$$

$\Rightarrow$ Range of $f(x)$ is $[5 - \sqrt{73 + 24\sqrt{3}}, 5 + \sqrt{73 + 24\sqrt{3}}]$

(iv) Inverse Circular/Trigonometric Functions

A. Arc Sine Function

Arc sine function is the form of $y = \sin^{-1} x$

The inverse sine is the **multivalued function**.

Because inverse sine of 0.5 will mean either 30°, 150°, 390°, 510°, ... etc., and all are valid.

But as we know that for any function to possess an inverse, it must be bijective, so if we see the graph of sine function it is a one to one function $\forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principle range for the sine function to possess an inverse. Hence, hereafter if it is asked for inverse of sine 0.5 our answer will be 30° unless asked for more general values.

Note that in the notation, is the sine the superscript −1 denotes the inverse function, not the multiplicative inverse.

Domain $x \in [-1, 1]$

Range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Nature (a) One to one [HPL Test]
(b) Onto Function
(c) Monotonically increasing throughout its domain
(d) Odd function

Note Inverse Function are symmetric in the line $y = x$

B. Arc Cosine Function

Arc cosine function is the form of $y = \cos^{-1} x$

The inverse cosine is the multivalued function.

Because inverse cosine of 0.5 will mean either $-60°$, $60°$ $300°$, $420°$, ... etc. and all are valid.

But, as we know that for any function to possess an inverse, it must be bijective, so if we see the graph of cosine function it is a one to one function $\forall\, x \in [0, \pi]$, which is the principle range for the cosine function to possess an inverse. Hence, hereafter if it is asked for inverse of cosine 0.5 our answer will be $60°$ unless asked for more general values.

Domain $x \in [-1, 1]$

Range $x \in [0, \pi]$

Nature (a) One to one [HPL Test]
(b) Onto Function
(c) Monotonically decreasing throughout its domain
(d) Even Function
$\because \cos^{-1}(-x) = \cos^{-1}(x)$

C. Arc Tangent Function

Arc tangent function is the form of $y = \tan^{-1} x$

The inverse tangent is the multivated function.

Because inverse tangent of 1 will mean either $45°$, $225°$, $405°$, ... etc., and all are valid.

But as we know that for any function to possess an inverse, to must be bijective, so if we see the graph of tangent function, it is a one to one function $\forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principle range for the tangent function to possess an inverse. Hence, hereafter if it is asked for inverse of tangent 1 our answer will be only $45°$ unless asked for more general values.

Domain $x \in R$,

Range $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Nature (a) One to one [HPL Test]
(b) Onto function
(c) Monotonically increasing throughout its domain
(d) Neither Even nor Odd

D. Arc Cosecant Function

Arc cosecant function is the form of $y = \text{cosec}^{-1}(x)$

By definition,

$$\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

Domain $x \in [-\infty, -1]\ (1, \infty]$

Range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Nature (a) One to one [HPL Test[
(b) Onto Function
(c) Monotonically decreasing throughout its domain
(d) Odd Function

E. Arc Secant Function

Arc secant function is the form of $y = \sec^{-1}(x)$

By definition, $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$

Domain $x \in [-\infty, -1] \cup (1, \infty]$

Range $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Nature (a) One of one [HPL Test]
(b) Onto Function
(c) Monotonically increasing throughout its domain
(d) Even Function

F. Arc Cotangent Function

Arc cotangent function is the form of $y = \cot^{-1}(x)$

By definition,

$$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$$

Domain $x \in R$

Range $y \in (0, \pi)$

Nature (a) One to one [HPL Test]
(b) Onto Function
(c) Monotonically decreasing throughout its domain
(d) Neither even nor an Odd Function

***Sample* Problem 15** *The domain of the function* $f(x) = \sin^{-1}\sqrt{4 - x^2}$ *is*

(a) $[-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$ (b) $[-2, 1] \cup [1, 2]$
(c) $[-2, -1] \cup [0, 2]$ (d) *None of these*

Interpret (a) $f(x) = \sin^{-1}\sqrt{4 - x^2}$

for $f(x)$ to be defined $0 \le 4 - x^2 \le 1 \Rightarrow x^2 = 4 \le 0$

and $x^2 - 3 \ge 0$

$\Rightarrow x \in [-2, 2]$ and $x \in [-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

$\Rightarrow x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$

So, domain $(f) : x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$

(v) Logarithmic Function

A function is the form of

$$f(x) = \log_a x$$

is a logarithmic function, the value of the function depends upon the value of a. For $0 < a < 1$, the function is decreasing, and for $a > 1$, the function is increasing

Domain $x > 0$, *i.e.*, $x \in (0, \infty)$

Range $y \in R$

Special Cases

***Case* I** If $0 < a < 1$

Nature (a) One to One [HPL Test]
(b) Onto Function for Codomain as a set of real numbers
(c) Monotonically decreasing throughout its domain.

***Case* II** If $a > 1$

Nature (a) One to one [HPL Test]
(b) Onto Function for Codomain as a set of real numbers
(c) Monotonically increasing throughout its domain.

***Sample* Problem 16** *The domain of the function* $f(x) = \log(-2 + 3x - x^2)$ *is*

(a) $[1, 2]$ (b) $(0, 2]$
(c) $(1, 2)$ (d) $[0, 2]$

Interpret (c) $f(x) = \log(-2 + 3x - x^2)$

for $f(x)$ to be defined $-2 + 3x - x^2 > 0$

$\Rightarrow x^2 - 3x + 2 < 0$

$\Rightarrow (x - 1)(x - 2) < 0$

$\Rightarrow x \in (1, 2)$

So, domain $(f) : x \in (1, 2)$

(vi) Exponential Function

A function is the form of

$$f(x) = a^x$$

is an exponential function. It is the inverse of logarithmic function. The value of the function depends upon the value of a.

For $0 < a < 1$, the function is decreasing, and for $a > 1$, the function is increasing

Domain $x \in R$

Range $y \in R^+$

Special Cases

***Case* I** If $0 < a < 1$

Nature (a) One to one [HPL Test]
(b) Onto Function for Codomain as a set of positive real numbers
(c) Monotonically decreasing throughout its domain.

***Case* II** If $a > 1$

Nature (a) One to one [HPL Test]

(b) Onto Function for Codomain as a set of positive real numbers.

(c) Monotonically increasing throughout its domain.

(vii) Some Special Functions

A. Absolute Valued Function

A function is the form of

$$Y - |x| - \begin{cases} -x & x < 0 \\ x & x \le 0 \end{cases}$$

Domain $x \in R$

Range $y \in R^+ + \{0\} = [0, \infty)$

No part of the curve line below x-axis

Nature (a) Many to one [HPL Test]

(b) Onto Function for codomain = $R^+ + \{0\}$

(c) Monotonically increasing for all $x > 0$

(d) Monotonically decreasing for all $x < 0$

(e) Even function

(f) Continuous throughout

(g) Non-Differentiable at $x = 0$

Properties of Modulus Function

(i) For any real number x, $\sqrt{x^2} = |x|$.

(ii) If a, b are positive real numbers, then

(a) $x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$

(b) $x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$

(c) $a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(iii) $|x \pm y| \le |x| + |y|$

(iv) $|x + y| \ge ||x| - |y||$

B. Signum Function

A function is the form of

$$y = \frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Domain $x \in R$

Range $y \in \{-1, 0, 1\}$

Nature (a) Many to one [HPL Test]

(b) Into function for codomain as the set of Real Numbers

(c) Neither monotonically increasing nor monotonically decreasing.

(d) Odd Function

(e) Discontinuous at $x = 0$

Note Such like functions also comes in the category of Piecewise constant functions.

C. Greatest Integer Function (Floor Function)

A function is the form of

$$y = f(x) = [x]$$

by definition, $[x]$ also known as Integral Part function is equal to Greatest Integer less than or equal to x.

It is also known as Floor Function.

Domain $x \in R$

Range $y \in I$

Nature (a) Many to one [HPL Test]

(b) Into Function for codomain as the set of real numbers

(c) Discontinuous at all Integral values of x.

Graph	Values of x	$y = [x]$
	$-4 \le x < -3$	–4
	$-3 \le x < -2$	–3
	$-2 < x < -1$	–2
	$-1 \le x < 0$	–1
	$0 \le x < 1$	0
	$1 \le x < 2$	1
	$2 \le x < 3$	2
	$3 \le x < 4$	3

Properties of Greatest Integer Function

If n is an integer and x is any real number between n and $n + 1$, then

(i) $[-n] = -[n]$

(ii) $[x + n] = [x] + n$

(iii) $[-x] = -[x] - 1$, x is not an integer.

(iv) $[x + y] \geq [x] + [y]$

(v) $[x] > n \Rightarrow x \geq n + 1$

(vi) $[x] < n \Rightarrow x < n$

(vii) $[x + y] = [x] + [y + x - [x]]$, for all $x, y \in R$

(viii) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$

Sample **Problem 17** *The domain of the function* $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$ *is*

(a) $(-\infty, -2) \cup [4, \infty)$ (b) $(-\infty, -2] \cup [4, \infty)$

(c) $(-\infty, -2) \cup (4, \infty)$ (d) *None of these*

Interpret (a) $f(x)$ is defined, if $[x]^2 - [x] - 6 > 0$

$\Rightarrow$ $([x] - 3)([x] + 2) > 0$

$\Rightarrow$ $[x] < -2$ or $[x] > 3$

But $[x] < -2 \Rightarrow [x] = -3, -4, -5, \ldots$

$\therefore$ $x < -2$

Also, $[x] > 3 \Rightarrow [x] = 4, 5, 6, \ldots$

$\therefore$ $x \geq 4$

Domain of $f = (-\infty, -2) \cup [4, \infty)$

D. Least Integer Function/Ceiling Function

A function is the form of

$$y = f(x) = (x) = \text{lint}\,(x)$$

by definition, $[x]$ is equal to least integer greater than or equal to x. It is also known as Ceiling Function.

Domain $x \in R$

Range $y \in I$

Nature (a) Many to One [HPL Test]

(b) Into function for codomain as the set of real numbers

(c) Discontinuous at all Integral values of x.

E. Fractional Part Function

A function is the form of

$$y = f(x) = \{x\}$$

By definition, mathematically, $\{x\} = x - [x]$

it always returns the fractional part of the variable and is always positive.

Domain $x \in R$

Range $y \in [0, 1)$

Nature (a) Many to one [HPL Test]

(b) Into Function for Codomain as the set of real numbers

(c) Monotonically increasing throughout

(d) Discontinuous at all integral values of x

(e) Periodic with period unity

Properties of Fractional Part Function

P1 If $0 \leq x < 1$, then $\{x\} = x$

P2 If $x \in 1$, then $\{x\} = 0$

P3 if $x \notin 1$ and $x > 0$, then $\{-x\} = 1 - \{x\}$

F. Dirichlet Function

Let c and d be real numbers (usually) taken as $c = 1$ and $d = 0$). The Dirichlet function is defined by

$$y = f(x) = \begin{cases} c & \text{for } x \text{ rational} \\ d & \text{for } x \text{ irrational} \end{cases}$$

and is discontinuous everywhere.

The Dirichlet function can be written analytically as

$$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to +\infty} \cos^{2n} (m!\,\pi x) \right\}$$

Domain $x \in R$

Range $y \in \{c, d\}$ usually $\{0, 1\}$

Nature (a) Many-to-one [HPL Test]

(b) Into function for codomain as the set of real Numbers

(c) Neither increasing nor decreasing

(d) Not a periodic function

(e) Discontinuous everywhere

***Sample* Problem 18** *The solution of the equation* $\{x+1\}+2x=4[x+1]-6$ *are [·] and {} denote integral and fractional part function respectively.*

(a) 1, 8/3 (b) 1, 2
(c) 2, 8/3 (d) *None of these*

Interpret (a) $\{x+1\}+2x=4[x+1]-6$

$\Rightarrow$ $x+1-[x+1]+2x=4[x+1]-6$

$\Rightarrow$ $3x+1=5[x+1]-6$

$\Rightarrow$ $3x+1=5([x]+1)-6$

$\Rightarrow$ $3x=5[x]-2$ ...(i)

$\Rightarrow$ $3([x]+\{x\})=5[x]-2$

$\Rightarrow$ $3\{x\}=2[x]-2$ ...(ii)

Now, $0\le\{x\}<1$

$\Rightarrow$ $0\le 3\{x\}<3$

$\Rightarrow$ $0\le 2[x]-2<3$

$\Rightarrow$ $2\le 2[x]<5$

$\Rightarrow$ $1\le[x]<\frac{5}{2}$ $\Rightarrow$ $[x]=1,2$

From Eq. (i), we get $x=1$, when $[x]=1$

and $x=\frac{8}{3}$, when $[x]=2$

Hence, $x=1,\frac{8}{3}$ are the required solutions.

Check Point **2**

1. Domain of the function $f(x)=\sqrt{2-2x-x^2}$ is
(a) $-\sqrt{3}\le x\le\sqrt{3}$
(b) $-1-\sqrt{3}\le x\le -1+\sqrt{3}$
(c) $-2\le x\le 2$
(d) $-2+\sqrt{3}\le x\le -2-\sqrt{3}$

2. Domain of the function $f(x)=\frac{x-3}{(x-1)\sqrt{x^2-4}}$ will be
(a) (1, 2) (b) $(-\infty,-2]\cup[2,\infty)$
(c) $(-\infty,-2)\cup(1,\infty)$ (d) $(-\infty,\infty)-\{1,\pm 2\}$

3. Range of the function $f(x)=\frac{x^2+34x-71}{x^2+2x-7}$ is
(a) [5, 9] (b) $(-\infty,5]\cup[9,\infty)$
(c) (5, 9) (d) None of these

4. The domain of the function $y=\frac{1}{\sqrt{|x|-x}}$ is
(a) $(-\infty, 0)$ (b) $(-\infty, 0]$
(c) $[-\infty, -1)$ (d) $(-\infty, \infty)$

5. If [ ] denotes the greatest integer function, then the domain of the real valued function $\log_{\left[x+\frac{1}{2}\right]}|x^2-x-2|$ is
(a) $\left[\frac{3}{2},\infty\right)$ (b) $\left[\frac{3}{2},2\right)\cup(2,\infty)$
(c) $\left(\frac{1}{2},2\right)\cup(2,\infty)$ (d) None of these

6. Let $2\sin^2 x+3\sin x-2>0$ and $x^2-x-2<0$ (x is in radian), then x lies in the interval
(a) $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (b) $\left(-1,\frac{5\pi}{6}\right)$
(c) (–1, 2) (d) $\left(\frac{\pi}{6},2\right)$

7. The range of the function $f(x)=\frac{1+x^2}{x^2}$ is equal to
(a) [0, 1] (b) (0, 1)
(c) $(1,\infty)$ (d) $(1,\infty)$

8. The function $f:R\to R$ defined by $f(x)=(x-1)(x-2)(x-3)$ is
(a) one one but not onto (b) onto but not one one
(c) both one one and onto (d) neither one one nor onto

9. If $f:R\to R$ is given by $f(x)=x^2+3$, then $f^{-1}(x)$ is equal to
(a) $x^{1/3}-3$ (b) $x^{1/3}+3$
(c) $(x-3)^{1/3}$ (d) $x+3^{1/3}$

10. If $f:R\to R$ be defined as $f(x)=x+\sin x+7$, then f(x) is
(a) one one (b) onto
(c) many one (d) into

11. Let $f:(-\infty,1]\to(-\infty,1]$ such that $f(x)=x(2-x)$, then $f^{-1}(x)$ is
(a) $1+\sqrt{1-x}$ (b) $1\pm\sqrt{1-x}$
(c) $1-\sqrt{1-x}$ (d) None of these

12. $f:[-3\pi,2\pi]\to[-1,1]$ defined by $f(x)=\sin 3x$ is
(a) into (b) onto
(c) one one (d) None of these

13. If $g(x)=x^2+x-2$ and $\frac{1}{2}gof(x)=2x^2-5x+2$, then the value of f(x) is
(a) $2x-3$ (b) $2x+3$
(c) $2x^2+3x+1$ (d) $2x^2-3x-1$

14. If $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, x<0\\ 0, x=0\\ 1, x>0\end{cases}$, then, $\forall x$ the value of $f[g(x)]$ is
(a) x (b) 1
(c) f(x) (d) g(x)

24.17 Even and Odd Functions

A real function $f(x)$ is an even function, if $f(-x) = f(x)$
e.g., $f(x) = \cos x$ is an even function.
A real function $f(x)$ is an odd function, if $f(-x) = -f(x)$
e.g., $f(x) = \sin x$ or x^3 are odd functions.

Properties of Even and Odd Functions

(i) Product of two odd functions or two even functions is an even function.
(ii) Product of an even function and an odd function is an odd function.
(iii) Each function can be expressed as the sum of an even function and an odd function.
Such that

$$f(x) = \frac{1}{2}\underbrace{\{f(x) - f(-x)\}}_{\text{Even function}} + \frac{1}{2}\underbrace{\{f(x) - f(-x)\}}_{\text{odd function}}$$

(iv) Derivative of an even function is an odd function and derivative of an odd function is an even function.
(v) Square of an even or an odd function is always an even function.
(vi) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.
(vii) The graph of an even function is symmetrical about y-axis.

***Sample* Problem 19** *The functions* $f_1(x) = \log(x + \sqrt{x^2+1})$ *and* $f_2(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ *are respectively.*

(a) *odd and even function*
(b) *even and odd function*
(c) *odd and odd function*
(d) *even and even function*

Interpret (d) (i) $f_1(x) = \log(x + \sqrt{x^2+1})$

$$\Rightarrow \quad f_1(-x) = \log(-x + \sqrt{x^2+1})$$
$$= \log\left(\frac{1}{x + \sqrt{x^2+1}}\right) = \log(x + \sqrt{x^2+1})^{-1}$$
$$= -\log(x + \sqrt{x^2+1}) = -f_1(x)$$

So, $f_1(x)$ is an odd function

(ii) We have, $f_2(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$\therefore \quad f_2(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right) = -x\left(\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right) = -x\left(\frac{1 - a^x}{1 + a^x}\right)$$
$$= x\left(\frac{a^x - 1}{a^x + 1}\right) = f_2(x)$$

So, $f_2(x)$ is an even function.

Even and Odd Extension

A function $f(x)$ defined on the interval $[0, a]$ can be extended to $[-a, a]$, so that $f(x)$ becomes an even or odd function on the interval $[-a, a]$. If this extension is an even function, it is called even extension and if this extension is an odd function, it is called odd extension.

Let g be the extension. Then, for even extension, we define

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \in [0, a] \\ f(-x), & \text{if} \quad x \in [-a, 0] \end{cases}$$

and for odd extension, we define

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \in [0, a] \\ -f(x), & \text{if} \quad x \in [-a, 0] \end{cases}$$

***Sample* Problem 20** *Let the function* $f(x) = x^2 + x + \sin x - \cos x$ *be defined on the interval* $[0, 1]$, *then odd and even extensions of* $f(x)$ *in the interval* $[-1, 1]$ *are respectively*

(a) $\begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ -x^2 + x + \sin x + \cos x & -1 < x < 0 \end{cases}$

and $\begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ x^2 - x - \sin x - \cos x & -1 \le x < 0 \end{cases}$

(b) $\begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ x^2 - x - \sin x - \cos x & -1 \le x < 0 \end{cases}$

and $\begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ -x^2 + x + \sin x + \cos x & -1 < x < 0 \end{cases}$

(c) *does not exist*
(d) *None of the above*

Interpret (a) Odd extension

To make $f(x)$ an odd function in the interval $[-1, 1]$, we define $f(x)$ as follows

$$f(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ -f(-x) & -1 \le x < 0 \end{cases}$$
$$= \begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ -(x^2 - x - \sin x - \cos x) & -1 \le x < 0 \end{cases}$$
$$= \begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ -x^2 + x + \sin x + \cos x & -1 \le x < 0 \end{cases}$$

Even extension

To make $f(x)$ an even function in the interval $[-1, 1]$, we define $f(x)$ as follows

$$f(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ f(-x) & -1 \le x < 0 \end{cases}$$
$$= \begin{cases} x^2 + x + \sin x - \cos x & 0 \le x \le 1 \\ x^2 - x - \sin x - \cos x & -1 \le x < 0 \end{cases}$$

24.18 Periodic Function

A function $f: X \to Y$ is said to be a periodic function, if there exist a positive real number T such that $f(x+T) = f(x)$, $\forall x \in X$. The least value of T is called the fundamental period of function. In general, the fundamental period (principal period) is called the period of a function. Graphically graph gets repeated after each period of the function.

Standard Results on Some Periodic Functions

S. No.	Function	Period
1.	$\sin^n x, \cos^n x, \sec^n x, \text{cosec}^n x$	π (if n is an even number) 2π (If n is an odd number)
2.	$\tan^n x, \cot^n x$	π
3.	$\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert, \lvert\sec x\rvert, \lvert\text{cosec}\, x\rvert, \lvert\cot x\rvert$	π
4.	$x - [x]$	1
5.	$\sin^{-1}(\sin x), \cos^{-1}(\cos x)$ $\text{cosec}^{-1}(\text{cosec}\, x), \sec^{-1}(\sec x)$	2π
6.	$\tan^{-1}(\tan x), \cot^{-1}(\cot x)$	π

Properties of Periodic Functions

(i) If $f(x)$ is a periodic function with fundamental period T, then $\frac{1}{f(x)}$ will also be a periodic function with same fundamental period T.

(ii) If $f(x)$ is a periodic function with period T, then $f(ax+b)$, $a \neq 0$ is also a periodic function with fundamental period $\frac{T}{|a|}$.

(iii) If $f(x)$ is a periodic function with period T, then $af(x)+b$ is also a periodic function with same fundamental period T.

(iv) Let $f(x)$ and $g(x)$ be two periodic functions such that period of $f(x) = \frac{m}{n}$, where $m, n \in N$ and m, n are coprime and period of $g(x) = \frac{r}{s}$, where $r, s \in N$ and r, s are coprime. Then, $(f+g)(x)$ is periodic with period T given by

$$T = \frac{\text{LCM of } (m, r)}{\text{HCF of } (n, s)}$$

provided that there does not exist a positive number $k < T$ for which $f(k+x) = g(x)$ and $g(k+x) = f(x)$, else k will be the period of $(f+g)(x)$.

This result is also true for functions $\frac{f}{g}$, $f - g$ and fg.

(v) If $f(x)$ is a periodic function with period T and $g(x)$ is a monotonic function, then $g[f(x)]$ is also a periodic function with same period T as that of $f(x)$.

(vi) If $f_1(x)$, $f_2(x)$ and $f_3(x)$ are periodic functions with periods T_1, T_2 and T_3 respectively also, if $h(x) = af_1(x) \pm bf_2(x) \pm cf_3(x)$. then, period of

$$\{h(x)\} = \begin{cases} \text{LCM of } \{T_1, T_2, T_3\}, & \text{if } h(x) \text{ is an odd function} \\ \frac{1}{2}\text{ LCM of } \{T_1, T_2, T_3\}, & \text{if } h(x) \text{ is an even function} \end{cases}$$

***Sample* Problem 21** *If $f(x) = \sin x + \cos x$ is a periodic function, then 'a' is a*

(a) *irrational number* (b) *rational number*
(c) *natural number* (d) *None of these*

Interpret (b) Period of $\sin x = 2\pi$ and period of $\cos ax = \frac{2\pi}{a}$

$\therefore$ Period of $\sin x + \cos ax = \text{LCM of } \frac{2\pi}{1}$ and $\frac{2\pi}{a}$

$$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of 1 and } a} = \frac{2\pi}{\lambda}$$

where, λ is the HCF of 1 and a.

Since, λ is the HCF of 1 and a, $\frac{1}{\lambda}$ and $\frac{a}{\lambda}$ should be both integers.

Suppose $\frac{1}{\lambda} = m$ and $\frac{a}{\lambda} = n$, then $\frac{\frac{a}{\lambda}}{\frac{1}{\lambda}} = \frac{n}{m}$, where $n, m \in I$, *i.e.*, $a = \frac{n}{m}$

Hence, a is rational number with period = 2

***Sample* Problem 22** *Which of the following function has period π?*

(a) $\tan 3x + \cos \frac{5x}{2}$

(b) $f(x) = \cos x + \{x\}$

(c) $f(x) = \cos(\cos x) + \cos(\sin x)$

(d) None of the above

[*here* $\{x\}$ *denotes the fractional part of* x]

Interpret (a) Let $f(x) = \tan 3x + \cos \frac{5x}{2}$

Period of $\tan 3x = \frac{\pi}{3}$ and period of $\cos \frac{5x}{2} = \frac{2\pi}{5/2} = \frac{4\pi}{5}$

$\therefore$ Period of $f(x) = \frac{\text{LCM of } \pi \text{ and } 4\pi}{\text{HCF of 3 and 5}} = 4\pi$

(b) $f(x) = \cos x + \{x\} = \cos x + x - [x]$

$\cos x$ is periodic with period 2π and $x - [x]$ is periodic with period 1. But LCM of 2π and 1 does not exist. Hence, $f(x)$ is not periodic.

(c) $f(x) = \cos(\cos x) + \cos(\sin x)$

$\because$ $\cos\{\cos(\pi + x)\} = \cos(-\cos x) = \cos(\cos x)$

and $\cos\{\sin(\pi + x)\} = \cos(-\sin x) = \cos(\sin x)$

$\therefore$ $\cos(\cos x)$ and $\cos(\sin x)$ are periodic functions with period π. But

$\cos\{\cos(\pi/2 + x)\} = \cos(-\sin x) = \cos(\sin x)$

and $\cos\{\sin(\pi/2 + x)\} = \cos(\cos x)$

$\therefore$ $f(x)$ is periodic with period $\frac{\pi}{2}$.

WORKED OUT
Examples

Example 1 *Let R is relation from* $(11, 12, 13)$ *to* $\{8, 10, 12\}$ *defined by* $y = x - 3$. *Then,* R^{-1} *is*

(a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 18), (13, 10)\}$
(c) $\{(10, 13), (8, 11)\}$ (d) *None of these*

Solution (a) R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by

$$y = x - 3 \Rightarrow x - y = 3.$$

$\therefore$ $R = \{(11, 8), (13, 10)\}$

Hence, $R^{-1} = \{(8, 11), (10, 13)\}$

Example 2 *Let L denotes the set of all straight lines in a plane and a relation R be defined by* $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. *Then, R is*

(a) *reflexive* (b) *symmetric*
(c) *transitive* (d) *None of these*

Solution (b) Here, $\alpha R \beta \Leftrightarrow \alpha \perp \beta$

$\therefore$ $\alpha \perp \beta \Leftrightarrow \beta \perp \alpha$

Hence, R is symmetric.

Example 3 *Suppose that* $g(x) = 1 + \sqrt{x}$ *and* $f\{g(x)\} = 3 + 2\sqrt{x} + x$, *then* $f(x)$ *is*

(a) $1 + 2x^2$ (b) $2 + x^2$ (c) $1 + x$ (d) $2 + x$

Solution (b) Given, $g(x) = 1 + \sqrt{x}$ and $f\{g(x)\} = 3 + 2\sqrt{x} + x$ ...(i)

$\Rightarrow$ $f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$

$\therefore$ $f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$

$\therefore$ $f(x) = 2 + x^2$

Example 4 *Let* $f(x) = ax + b$ *and* $g(x) = cx + d$, $a \neq 0, c \neq 0$. *Assume* $a = 1, b = 2$, *if* $(fog)(x) = (gof)(x)$ *for all x. What can you say about c and d?*

(a) *c and d both arbitrary* (b) $c = 1$ *and d is arbitrary*
(c) *c is arbitrary and* $d = 1$ (d) $c = 1, d = 1$

Solution (b) Now, $(fog)(x) = f\{g(x)\} = a(cx + d) + b$

and $(gof)(x) = g\{f(x)\} = c(ax + b) + d$

Since, $cx + d + 2 = cx + 2c + d$ (given)

Hence, $c = 1$ and d is arbitrary.

Example 5 *If* $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$ *for* $x \in R$, *then* $f(2002)$ *is equal to*

(a) 1 (b) 2
(c) 3 (d) 4

Solution (a) Given, $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$

$\Rightarrow$ $f(x) = \dfrac{\cos^2 x + \sin^2 x(1 - \cos^2 x)}{\sin^2 x + \cos^2 x(1 - \sin^2 x)}$

$\Rightarrow$ $f(x) = \dfrac{\sin^2 x + \cos^2 x - \sin^2 x \cdot \cos^2 x}{\sin^2 x + \cos^2 x - \sin^2 x \cdot \cos^2 x}$

$\Rightarrow$ $f(x) = 1$

$\therefore$ $f(2002) = 1$

Example 6 *Let* $g(x) = 1 + x - [x]$ *and* $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0, \forall x, \\ 1, & x > 0 \end{cases}$

then $f\{g(x)\}$ *is equal to*

(a) x (b) 1
(c) $f(x)$ (d) $g(x)$

Solution (b) $g(x) = 1 + n - n = 1$ (put $x = n \in Z$)

and $g(x) = 1 + n + k - n = 1 + k$ (put $x = n + k$)

(where, $n \in Z, 0 < k < 1$)

Now, $f\{g(x)\} = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

Clearly, $g(x) > 0, \forall x$

So, $f\{g(x)\} = 1, \forall x$

Example 7 *If* $f(x) = 2[x] + \cos x$, *then* $f : R \to R$ *is*

(a) *one one and onto* (b) *one one and into*
(c) *many one and into* (d) *many one and onto*

Solution (c) Since, $f(x) = 2[x] + \cos x$

$$= \begin{cases} \cos x, & 0 \le x < 1 \\ 2 + \cos x, & 1 \le x < 2 \\ 4 + \cos x, & 2 \le x < 3 \end{cases}$$

Since, $\cos x < 1$ and $2 + \cos x > 1$.

$\therefore$ $f(x)$ never gives the value one.

Hence, $f(x)$ is into.

If $0 < \alpha < \pi - 3$, then $f(\pi - \alpha) = f(\pi + \alpha)$

So, $f(x)$ is not one one.

Example 8 *Let $f : R \sim \{n\} \to R$ be a function defined by $f(x) = \dfrac{x-m}{x-n}$, where $m \neq n$. Then,*

(a) *f is one one onto* (b) *f is one one into*

(c) *f is many one onto* (d) *f is many one into*

Solution (b) For any $x, y \in R$, we have

$$f(x) = f(y) \Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n} \Rightarrow x = y$$

So, f is one one.

Let $\alpha \in R$ such that $f(x) = \alpha$

$$\therefore \quad \frac{x-m}{x-n} = \alpha \Rightarrow x = \frac{m - n\alpha}{1-\alpha}$$

Clearly, $x \in R$ for $\alpha = 1$

So, f is not onto.

Example 9 *If $f(x) = x^2$ and $g(x) = 2^x$, then what is the solution set of $fog(x) = gof(x)$?*

(a) $\{0\}$ (b) $\{2\}$

(c) $\{0, 2\}$ (d) *None of these*

Solution (c) $\because$ $fog(x) = f\{g(x)\} = f(2^x) = (2^x)^2 = 2^{2x}$

and $gof(x) = g\{f(x)\} = g(x^2) = 2^{x^2}$

$\because$ $fog(x) = fog(x) \Rightarrow 2^{2x} = 2^{x^2}$

$\Rightarrow$ $2x = x^2 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2$

$\therefore$ Solution set is $\{0, 2\}$.

Example 10 *Domain of $f(x) = y = \sqrt{\log_3 \{\cos(\sin x)\}}$ is*

(a) $\left\{\dfrac{n\pi}{2} : n \in I\right\}$ (b) $\{2n\pi : n \in I\}$

(c) $\{n\pi : n \in I\}$ (d) *None of these*

Solution (c) Here 'y' would assume real values if $\log_3 \{\cos(\sin x)\} \geq 0$

$\Rightarrow$ $\cos(\sin x) \geq 1$

$\Rightarrow$ $\cos(\sin x) = 1$ as $-1 \leq \cos\theta \leq 1$

$\Rightarrow$ $\sin x = 0$

$\Rightarrow$ $x = n\pi, n \in I$.

Hence, the domain consists of the multiples of π

i.e., Domain $(y) = \{n\pi : n \in I\}$.

Example 11 *The range of $f(x) = y = \dfrac{e^x}{[x]+1}$, $x \geq 0$ is*

(a) $[1, \infty)$ (b) $(0, \infty)$

(c) $[1, \infty) - \{0\}$ (d) *None of these*

Solution (a) Here, $y = \dfrac{e^x}{[x]+1}$, $x \geq 0$

Clearly, $1 + [x] \geq 1 \quad \forall \; x \in [0, \infty)$

Also $e^x \geq 1 \quad \forall \; x \in [0, \infty)$

For $x \in [0, 1)$; $y = e^x$

For $x \in [1, 2)$; $y = \dfrac{e^x}{2}$

For $x \in [2, 3)$; $y = \dfrac{e^x}{3}$

And so on. Clearly, all values which are greater than or equal to one, would be assumed by y. Hence, range $(y) \in [1, \infty)$.

Alternative Method

From the above mentioned information, graph of the function can be drawn. Thus, the range of $f(x)$ is $[1, \infty)$.

Example 12 *Let $f : R \to R$ and $g : R \to R$ be functions defined by*

$$f(x) = \begin{cases} 2x, & x < 1 \\ 2x^2 - 1, & x \geq 1 \end{cases} \text{and } g(x) = \begin{cases} x+2, & x < 0 \\ 2x, & x \geq 0 \end{cases}, \text{ then } (fog)(x) \text{ is}$$

(a) *does not exist*

(b) $\begin{cases} 2x+4, & x < -1 \\ 2x^2 + 8x + 7, & -1 \leq x < 0 \\ 4x, & 0 \leq x < \dfrac{1}{2} \\ 8x^2 - 1, & x \geq \dfrac{1}{2} \end{cases}$

(c) $\begin{cases} 2x+4, & x \leq -1 \\ 2x^2 + 8x + 7, & -1 < x \leq 0 \\ 4x, & 0 < x \leq \dfrac{1}{2} \\ 8x^2 - 1, & x > \dfrac{1}{2} \end{cases}$

(d) *None of the above*

Solution (b) $(f+g) : R \to R$ and $(fg) : R \to R$ are functions defined by $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. To find $(f+g)(x)$ and $(fg)(x)$, we rewrite $f(x)$ and $g(x)$ as under

$$f(x) = \begin{cases} 2x, & x < 0 \\ 2x, & 0 \leq x < 1, \\ 2x^2 - 1, & x \geq 1 \end{cases} \quad g(x) = \begin{cases} x+2, & x < 0 \\ 2x, & 0 \leq x < 1, \\ 2x, & x \geq 1 \end{cases}$$

$$\text{Now, } (fog)x = f[g(x)] = \begin{cases} 2g(x), & g(x) < 1 \\ 2\{g(x)\}^2 - 1, & g(x) \geq 1 \end{cases}$$

$$= \begin{cases} 2(x+2) & x+2<1 & x<0 \\ 2(2x) & 2x<1 & x\geq 0 \\ 2(x+2)^2-1 & x+2\geq 1 & x<0 \\ 2(2x)^2-1 & 2x\geq 1 & x\geq 0 \end{cases}$$

$$= \begin{cases} 2x+4 & x<-1 & x<0 \\ 4x & x<\frac{1}{2} & x\geq 0 \\ 2x^2+8x+7 & x\geq -1 & x<0 \\ 8x^2-1 & x\geq \frac{1}{2} & x\geq 0 \end{cases}$$

$$= \begin{cases} 2x+4 & x<-1 \\ 4x & 0\leq x<\frac{1}{2} \\ 2x^2+8x+7 & -1\leq x<0 \\ 8x^2-1 & x\geq \frac{1}{2} \end{cases}$$

$$= \begin{cases} 2x+4 & x<-1 \\ 2x^2+8x+7 & -1\leq x<0 \\ 4x & 0\leq x<\frac{1}{2} \\ 8x^2-1 & x\geq \frac{1}{2} \end{cases}$$

Note To find *fog*, we first take first interval of $g(x)$ and take its range. If it is subset of any interval of $f(x)$, then directly apply formula of $f(x)$ on $g(x)$ otherwise break it into 2 intervals and similarly do the same for the other intervals of $g(x)$.

Example 13 *Let* $f(x) = \left[\frac{f(x)+f(-x)}{g(x)-g(-x)}\right]^n$, *then* $F(x)$ *is even and odd according as n is*

(a) *even and odd* (b) *odd and even*
(c) *even and even* (d) *odd and odd*

Solution (a) Let $h(x) = f(x) + f(-x)$ and $k(x) = g(x) - g(-x)$

Then, $h(-x) = f(-x) + f(x) = h(x)$

and $k(-x) = g(-x) - g(x) = -k(x)$

$\Rightarrow$ $h(x)$ is even while $k(x)$ is odd.

Thus, $f(x) = \left[\frac{h(x)}{k(x)}\right]^n$ and $f(-x) = \left[\frac{h(-x)}{k(-x)}\right]^n = (-1)^n \left[\frac{h(x)}{k(x)}\right]^n = (-1)^n f(x)$

Hence, $f(x)$ is an even function when n is even and an odd function when n is odd.

Example 14 *Let* $f(x) = 2\cos^2 x + \sqrt{3}\cdot\sin 2x + 1$. *Then, its domain and range ,so that f is an invertible function*

(a) $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$ *and* $[0, 4]$

(b) $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ *and* $[0, 2]$

(c) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ *and* $[0, 4]$

(d) *None of the above*

Solution (a) $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1 = 2\sin\left(2x + \frac{\pi}{6}\right) + 2$

or $2\cos\left(2x + \frac{\pi}{3}\right) + 2$

Let X = Domain of $f = \left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$ or $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$

and Y = Range of $f = [0, 4]$.

Then, $f : X \to Y$ defined by $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1$ is both one one and onto.

Hence, $f^{-1}(x)$ will exist and domain of $f^{-1} = y \cdot$ Range of $f^{-1} = x$

Hence, $f^{-1} : Y \to X$ is a function defined by

$$f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\frac{x-2}{2} - \frac{\pi}{6}\right) \text{ or } \frac{1}{2}\left(\cos^{-1}\frac{x-2}{2} + \frac{\pi}{3}\right)$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Relation and Equivalence Relation

1. Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as follows
$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$, then R is **[NCERT Exemplar]**
(a) reflexive and symmetric
(b) symmetric and transitive
(c) reflexive and transitive
(d) a equivalence relation

2. If $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, then the range of f and g respectively. **[NCERT Exemplar]**
(a) (5, 3) and (2, 3)
(b) (5, 6) and (3, 2)
(c) (2, 3) and (5, 6)
(d) (2, 3) and (6, 5)

3. Let R be a relation on the set N of natural numbers defined by nRm, if n divides m. Then, R is **[NCERT Exemplar]**
(a) reflexive and symmetric
(b) transitive and symmetric
(c) equivalence
(d) reflexive, transitive but not symmetric

4. Relation R in set N of natural numbers defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$ are **[NCERT Exemplar]**
(a) reflexive
(b) symmetric
(c) not reflexive not symmetric
(d) not reflexive not symmetric not transitive

5. Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(x, y) : y \text{ is divisible by } x\}$ are **[NCERT]**
(a) reflexive and transitive
(b) reflexive and symmetric
(c) symmetric and transitive
(d) None of the above

6. Relation R in the set Z of all integers defined as $R = \{(x, y) : x - y \text{ is an integer}\}$ is/are **[NCERT]**
(a) reflexive and symmetric
(b) only transitive
(c) symmetric but not transitive
(d) an equivalence relation

7. The relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is **[NCERT]**
(a) reflexive (b) symmetric
(c) transitive (d) None of these

8. The relation R in R defined by $R = \{(a, b) : a \le b^3\}$ is **[NCERT]**
(a) reflexive (b) symmetric
(c) transitive (d) None of these

9. The relation R in the set A of all the books in a library of a college, given by $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$ is a/an **[NCERT]**
(a) reflexive
(b) reflexive and symmetric
(c) equivalence relation
(d) None of the above

10. The relation R in the set $A = \{1, 2, 3, 4, 5\}$ is given by $R = \{(a, b) : |a - b| \text{ is even}\}$, is a/an **[NCERT]**
(a) equivalence relation (b) only transitive
(c) reflexive and symmetric (d) None of these

11. The relation R in the set $A = \{x \in Z : 0 \le x \le 12\}$. Given by, $R = \{(a, b) : |a - b| \text{ is a multiple 4}\}$ is a/an **[NCERT]**
(a) only reflexive (b) not reflexive
(c) equivalence relation (d) None of these

12. The example of relation which are reflexive and transitive but not symmetric is **[NCERT]**
(a) $R = \{(a, b) : a^3 \ge b^3\}$ (b) $R = \{(a, b) : a \ge b\}$
(c) $R = \{(a, b) : a^2 \ge b\}$ (d) None of these

13. The relation R in the set A of points in a plane, given by $R = \{(P, Q)$: distance of the point P from the origin is same as the distance of the point Q from the origin$\}$, is a/an [NCERT]

(a) equivalence relation (b) only symmetric
(c) not reflexive (d) None of these

14. The relation R, defined in the set of A all triangles as $R = \{(T_1 T_2) : T_1$ is similar to $T_2\}$. is a/an [NCERT]

(a) reflexive and symmetric (b) only transitive
(c) equivalence relation (d) None of these

15. Let L be the set of all lines in xy-plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$, then R is a/an [NCERT]

(a) reflexive (b) symmetric
(c) transitive (d) equivalence relation

16. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer. [NCERT]

(a) R is reflexive and symmetric but not transitive
(b) R is reflexive and transitive but not symmetric
(c) R is symmetric and transitive but not reflexive
(d) R is an equivalence relation

17. Given a non-empty set X. Consider $P(X)$, which is the set of all subset of X. Defined the relation R in $P(X)$ as follows

For subsets A and B in $P(X)$, ARB if and only if $A \subset B$, then R is [NCERT]

(a) an equivalence relation on $P(X)$
(b) not an equivalence relation on $P(X)$
(c) a anti-symmetric
(d) None of the above

18. Let $A = \{1, 2, 3\}$. Then, number of relations containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is [NCERT]

(a) 1 (b) 2 (c) 3 (d) 4

19. Let $A = \{1, 2, 3\}$. Then, number of equivalence relations containing $(1, 2)$ is [NCERT]

(a) 1 (b) 2 (c) 3 (d) 4

Domain and Range of Real Function

20. If D is the domain of the function $\sec^{-1}(\log x)$, then D contains

(a) $\left(-\infty, \frac{1}{e}\right)$ (b) $e - 1$
(c) 1 (d) None of these

21. The function $f(x)$ is defined in $[0,1]$, then the domain of definition of the function $f[\log(1 - x^2)]$ is

(a) $x \in \{0\}$
(b) $x \in [-\sqrt{1+e}, -1] \cup [1 + \sqrt{1+e}]$
(c) $x \in (-\infty, \infty)$
(d) None of the above

22. The domain of definition of $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(a) $R - \{-1, -2\}$ (b) $(-2, \infty)$
(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

23. The domain of the function $f(x) = \log_{0.2} \log_{0.5} \log_{0.25} x$ is equal to

(a) $(1, \infty)$ (b) $\left(\frac{1}{4}, \infty\right)$
(c) $\left(\frac{1}{4}, 1\right)$ (d) None of these

24. The domain of the function $f(x) = \dfrac{1}{1 - \tan x}$ is

(a) $R \sim \left\{n\pi + \frac{\pi}{2}\right\}$ (b) $R \sim \left\{n\pi + \frac{\pi}{4}\right\}$
(c) R (d) None of these

25. Domain of the function $f(x) = \sqrt{\dfrac{1}{\sin x} - 1}$ is

(a) $\bigcup_{n \in I}\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ (b) $\bigcup_{n \in I}(2n\pi, (2n+1)\pi)$
(c) $\bigcup_{n \in I}[(2n-1)\pi, 2n\pi]$ (d) None of these

26. Domain of $f(x) = \sin^{-1}[2 - 4x^2]$ ($[\cdot]$ denotes the greatest integer function) is

(a) $[-1, 1]$ (b) $(-2, 2)$
(c) $\left[-\frac{\sqrt{3}}{2}, 0\right) \cup \left(0, \frac{\sqrt{3}}{2}\right]$ (d) None of these

27. The domain of definition of the function $y(x)$ given by equation $2^x + 2^y = 2$ is

(a) $0 < x \le 1$ (b) $0 \le x \le 1$
(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

28. The domain of the function $F(x) = \dfrac{1}{[x]} + \sqrt{2x - x^2}$ is

(a) $[1, 2]$ (b) $[0, 2]$ (c) $[0, 1]$ (d) $[1, 0]$

29. Range of the function $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}, x \in R$ is

(a) $(1, \infty)$ (b) $\left(1, \frac{11}{7}\right]$
(c) $\left(1, \frac{7}{3}\right]$ (d) $\left(1, \frac{7}{5}\right]$

30. Domain and Range of $f(x) = x^3 + 3x^2 + 20x - \sin x$ are respectively

(a) R, R (b) R, R^+
(c) R^+, R (d) None of these

31. If $f(x) = [x^2] - [x]^2$, where $[\cdot]$ denotes the greatest integer function and $x \in [0, 2]$, then the set of values of $f(x)$ is

(a) $\{-1, 0\}$ (b) $\{-1, 0, 1\}$
(c) $\{0\}$ (d) $\{0, 1, 2\}$

32. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $\left(0, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

33. If $0 < x < \frac{\pi}{2}$ and $\log_{\sin x} \tan x \geq 0$, then range of x is equal to

(a) $\left(0, \frac{\pi}{2}\right]$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) None of these

34. If $f(x) = (x + 1)^2 - 1, (x \geq -1)$. Then, the set $S = \{x : f(x) = f^{-1}(x)\}$ is

(a) $\left\{0, -1, -\frac{3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$

(b) $\{0, 1, -1\}$

(c) $\{0, -1\}$

(d) empty set

Types of Mapping (Function)

35. The function $f : R \to R$ defined by $f(x) = \frac{x}{x^2 + 1}$, $\forall x \in R$ is **[NCERT Exemplar]**

(a) one one and onto (b) neither one one nor onto

(c) one one (d) on to

36. Let R be the set of real numbers and $f : R \to R$ be the function defined by $f(x) = 4x + 5$, then f is **[NCERT Exemplar]**

(a) injective (b) surjective

(c) invertible (d) None of these

37. Set A has 3 elements and the set B has 4 elements. Then, the number of injective mappings that can be defined from A to B is **[NCERT Exemplar]**

(a) 144 (b) 12

(c) 24 (d) 64

38. Let $f : R \to R$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and –3, respectively are **[NCERT Exemplar]**

(a) $\phi, \{4, -4\}$ (b) $\{3, -3\}, \phi$

(c) $\{4, -4\}, \phi$ (d) $\{4, -4\}, \{2, -2\}$

39. The greatest integer function $f : R \to R$, given by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is **[NCERT]**

(a) one one onto (b) one one

(c) on to (d) neither one one nor onto

40. The modulus function $f : R \to R$, given by $f(x) = |x|$, where $|x|$ is x, if x is non-negative and $|x|$ is $-x$, if x is negative is **[NCERT]**

(a) neither one one nor onto

(b) one one onto

(c) one one but not onto

(d) None of the above

41. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B, then f is **[NCERT]**

(a) one one (b) many one

(c) one one onto (d) None of these

42. Let $f : N \to N$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in N$. Then, the function f is **[NCERT]**

(a) onto (b) one one

(c) bijective (d) None of these

43. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$ is **[NCERT]**

(a) f is only one one (b) f is one one but not onto

(c) f is one one and onto (d) None of these

44. Let the function $f : R \to R$ be defined by $f(x) = \cos x$, $\forall x \in R$, then f is **[NCERT Exemplar]**

(a) one one and onto (b) only one one

(c) only onto (d) neither one one nor onto

45. Which of the following is a bijective function on their ranges?

(a) $f(x) = (x - 2)^2, x \in R$ (b) $f(x) = x^3, x \in R^+$

(c) $f(x) = \sin x, 0 \leq x \leq \pi$ (d) None of these

46. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then, the number of onto functions from E to F is

(a) 14 (b) 16 (c) 12 (d) 8

47. If $f : \left[0, \frac{\pi}{2}\right] \to [0, \infty)$ be a function defined by $y = \sin\left(\frac{x}{2}\right)$, then f is

(a) injective (b) surjective

(c) bijective (d) None of these

48. Let A be a set containing m distinct elements, then the total number of distinct functions from A to itself is

(a) m^m (b) m^2 (c) $m!$ (d) $2^m - 1$

49. The function $f : R \to R$ defined by $f(x) = |(x - 1)(x - 2)|$ is

(a) one one onto (b) one one into

(c) many one onto (d) many one into

50. Let $f:(e,\infty)\to R$ be defined by $f(x)=\log[\log(\log x)]$, then f is
(a) one one onto (b) many one onto
(c) one one into (d) many one into

51. Which of the following functions are bijective?
(a) $f: Z \to Z$ defined by $y = x+2$
(b) $f: Z \to Z$ defined by $y = 2x$
(c) $f: R \to R^+$ defined by $y = \sqrt{x}$
(d) $f: R \to R$ defined by $y = x+|x|$

52. The function $f: R' \to (1, e)$ defined by $f(x)=\dfrac{x^2+e}{x^2+1}$ is
(a) one one but not onto (b) onto but not one one
(c) both one one and onto (d) neither one one nor onto

53. If $f(x)=x^3+3x^2+12x-2\sin x$, where $f: R\to R$, then
(a) $f(x)$ is many one and onto
(b) $f(x)$ is one one and onto
(c) $f(x)$ is one one and into
(d) $f(x)$ is many one and into

Real Function

54. Let $f, g: R\to R$ be two functions defined as $f(x)=|x|+x$ and $g(x)=|x|-x, \forall x\in R$. Then, fog and gof are respectively. [NCERT Exemplar]
(a) $\begin{cases}0, & x>0\\ -4x, & x<0\end{cases}$ and 0
(b) $\begin{cases}0, & x>0\\ 2x, & x<0\end{cases}$ and 0
(c) 0 and $\begin{cases}0, & x>0\\ -4x, & x<0\end{cases}$
(d) None of these

55. Let $f: R\to R$ be defined by $f(x)=\dfrac{x}{\sqrt{1+x^2}}$, then $(fofof)(x)$ is [NCERT Exemplar]
(a) $\dfrac{x}{\sqrt{1+x^2}}$ (b) $\dfrac{x}{\sqrt{1+3x^2}}$
(c) x (d) 1

56. If $f(x)=\{4-(x-7)^3\}$, then $f^{-1}(x)$ is [NCERT Exemplar]
(a) $7+(4-x)^{1/3}$ (b) $7+(4-x)^3$
(c) $7+(x-4)^{1/3}$ (d) $7+(x-4)^3$

57. Let $f:\{1,3,4\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,3\}$ be given by $f:\{(1,2),(3,5),(4,1)\}$ and $g:\{(1,3),(2,3),(5,1)\}$, then gof is [NCERT]
(a) $\{(3,1),(1,3),(3,4)\}$
(b) $\{(1,3),(3,1),(4,3)\}$
(c) $\{(1,1),(2,2),(3,3),(4,4)\}$
(d) None of the above

58. If $f(x)=\dfrac{(4x+3)}{(6x-4)}, x\neq\dfrac{2}{3}$, then the value of $(fof)(x)$ is, for all $x\neq\dfrac{2}{3}$. [NCERT]
(a) $\dfrac{1}{x}$ (b) x^2 (c) x (d) $\dfrac{1}{x^2}$

59. Let $f: x\to y$ be an invertible function, then f has [NCERT]
(a) unique inverse (b) not a unique inverse
(c) only two inverse (d) None of these

60. Let $f: R-\left\{-\dfrac{4}{3}\right\}\to R$ be a function defined as $f(x)=\dfrac{4x}{3x+4}, x\neq-\dfrac{4}{3}$. The inverse of f is the map g: Range $f\to R-\left\{-\dfrac{4}{3}\right\}$ is given by [NCERT]
(a) $g(y)=\dfrac{3y}{3-4y}$ (b) $g(y)=\dfrac{4y}{4-3y}$
(c) $g(y)=\dfrac{4y}{3-4y}$ (d) $g(y)=\dfrac{3y}{4-3y}$

61. Let $A=\{-1,0,1,2\}, B=\{-4,-2,0,2\}$ and $f, g: A\to B$ be the function defined by $f(x)=x^2-x, x\in A$ and $g(x)=2\left|x-\dfrac{1}{2}\right|-1, x\in A$, then f and g [NCERT]
(a) does not exist (b) are not equal
(c) are equal (d) None of these

62. Let $f: R\to R$ be the Signum function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$ and $g: R\to R$ be the greatest integer function given by $g(x)=[x]$ is greatest integer less than or equal to x. Then, fog and gof do not coincide in [NCERT]
(a) (0, 1) (b) [0, 1]
(c) [0, 1) (d) (0, 1]

63. If $X=\{0,1,2,3,4\}, Y=\{2,3,4,5\}$ and $x\in X, y\in Y$, then which of the following are functions in the given set?
(a) $f_1=\{(x,y): y=x+1\}$ (b) $f_2=\{(x,y): x+y>5\}$
(c) $f_3=\{(x,y): y\le x\}$ (d) None of these

64. If $f(x)=\dfrac{a^x+a^{-x}}{2}$ and $f(x+y)+f(x-y)=kf(x)f(y)$, then k is equal to
(a) 2 (b) 4
(c) −2 (d) None of these

65. If $f:[1,\infty)\to[2,\infty)$ is given by $f(x)=x+\dfrac{1}{x}$, then $f^{-1}(x)$ equal to
(a) $\dfrac{x+\sqrt{x^2-4}}{2}$ (b) $\dfrac{x}{1+x^2}$ (c) $\dfrac{x-\sqrt{x^2-4}}{2}$ (d) $1+\sqrt{x^2-4}$

66. Which of the following cannot have their inverse defined?

(a) $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$ defined by $y=\sin^3 x$

(b) $f:R\to R^+$ defined by $y=e^x$

(c) $f:R^+\to R$ defined by $y=\log|x|$

(d) $f:R\to R^+$ defined by $y=e^{|x|}$

67. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x)=2^{x(x-1)}$, then $f^{-1}(x)$ is equal to

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1+\sqrt{1+4\log_2 y})$

(c) $\frac{1}{2}(1-\sqrt{1-4\log_2 x})$ (d) not defined

68. If $f:\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\to[-1,1]$ is defined by $f(x)=\sin x$, then $f^{-1}(x)$ is given by

(a) $\sin^{-1}x$ (b) $\pi+\sin^{-1}x$

(c) $\pi-\sin^{-1}x$ (d) None of these

69. If $f(x+y,x-y)=xy$, then $f(x,y)$ is equal to

(a) xy (b) x^2-y^2 (c) $\frac{x^2-y^2}{4}$ (d) $x+y$

70. If $f(x)=|x-1|$, then

(a) $f(x^2)=[f(x)]^2$ (b) $f(x+y)=f(x)+f(y)$

(c) $f(|x|)=|f(x)|$ (d) None of these

71. If $f(x)=lx+m$ and $f(x+2)-f(x)=2$, then

(a) $l=1,m=0$ (b) $l=0,m=1$

(c) $l=0,m=0$ (d) $l=1,m\in R$

72. If $af(x)+bf\left(\frac{1}{x}\right)=x+\frac{5}{x},(a\neq b)$, then $f(x)$ is equal to

(a) $\frac{1}{a^2-b^2}\left(x+\frac{1}{x}\right)$

(b) $\frac{1}{a^2-b^2}\left[x(5a-b)+\frac{1}{x}(5b-a)\right]$

(c) $\frac{1}{a^2-b^2}\left[x(a-5b)+\frac{1}{x}(5a-b)\right]$

(d) None of the above

73. If $f(x)=\cos(\log x)$, then

$f(x^2)f(y^2)-\frac{1}{2}\left\{f\left(\frac{x^2}{y^2}\right)+f(x^2y^2)\right\}$ has the value

(a) -2 (b) -1

(c) $\frac{1}{2}$ (d) None of these

74. The function $f(x)=\log\left(\frac{1+x}{1-x}\right)$ satisfies the equation

(a) $f(x+2)-2f(x+1)+f(x)=0$

(b) $f(x+1)+f(x)=f(x[x+1])$

(c) $f(x_1)\cdot f(x_2)=f(x_1+x_2)$

(d) $f(x_1)\cdot f(x_2)=f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$

75. If $f(x)=\frac{x-1}{x+1}$, then $f(2x)$ is equal to

(a) $\frac{f(x)+1}{f(x)+3}$ (b) $\frac{3f(x)+1}{f(x)+3}$

(c) $\frac{f(x)+3}{f(x)+1}$ (d) $\frac{f(x)+3}{3f(x)+1}$

76. Two functions f and g are said to commute if $(fog)(x)=(gof)(x)$, for all x, which of the following function are commute?

(a) $f(x)=x^3, g(x)=x+1$

(b) $f(x)=\sqrt{x}, g(x)=\cos x$

(c) $f(x)=x^m, g(x)=x^n, m\neq n; m,n\in I$

(d) $f(x)=x-1, g(x)=x^2+1$

77. Let $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$, then for all x, $f[g(x)]$ is equal to

(a) x (b) 1

(c) $f(x)$ (d) $g(x)$

78. If $g[f(x)]=|\sin x|$ and $f[g(x)]=(\sin\sqrt{x})^2$, then

(a) $f(x)=\sin^2 x, g(x)=\sqrt{x}$

(b) $f(x)=\sin x, g(x)=|x|$

(c) $f(x)=x^2, g(x)=\sin\sqrt{x}$

(d) f and g cannot be determined

79. If $f(x)\,f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$, $\forall x\in R-\{0\}$, where $f(x)$ be a polynomial function and $f(5)=126$, then $f(3)$ is equal to

(a) 28 (b) 26

(c) 27 (d) 25

80. If $fog=|\sin x|$ and $gof=\sin^2\sqrt{x}$ then $f(x)$ and $g(x)$ are

(a) $f(x)=\sqrt{\sin x}, g(x)=x^2$

(b) $f(x)=|x|, g(x)=\sin x$

(c) $f(x)=\sqrt{x}, g(x)=\sin^2 x$

(d) $f(x)=\sin\sqrt{x}, g(x)=x^2$

81. If f is a function such that $f(0)=2$, $f(1)=3$ and $f(x+2)=2f(x)-f(x+1)$ for every real x, then $f(5)$ is

(a) 7 (b) 13

(c) 1 (d) 5

82. Let $f:\left[\frac{\pi}{3},\frac{2\pi}{3}\right]\to[0,4]$ be a function defined as $f(x)=\sqrt{3}\sin x-\cos x+2$. Then, $f^{-1}(x)$ is given by

(a) $\sin^{-1}\left(\frac{x-2}{2}\right)-\frac{\pi}{6}$ (b) $\sin^{-1}\left(\frac{x-2}{2}\right)+\frac{\pi}{6}$

(c) $\frac{2\pi}{3}+\cos^{-1}\left(\frac{x-2}{2}\right)$ (d) None of these

Even and Odd Function

83. If 'f' is a real valued function not identically zero, satisfying $f(x+y)+f(x-y)=2f(x)\cdot f(y)$, $\forall x, y \in R$, then $f(x)$ is
(a) odd (b) even
(c) neither even nor odd (d) cannot say

84. If $f(x)$ is a function that is odd and even simultaneously, then $f(3)-f(2)$ is equal to
(a) 1 (b) –1 (c) 0 (d) 2

85. $f(x)=\dfrac{\cos x}{\left[\dfrac{x}{\pi}\right]+\dfrac{1}{2}}$, (where x is not an integral multiple of π and $[\times]$ denote the greatest integer function), is
(a) an odd function (b) an even function
(c) neither odd nor even (d) cannot be determined

Periodic Function

86. The period of the function $f(x)=\sin\left(\sin\dfrac{x}{5}\right)$ is
(a) 2π (b) $2\pi/5$ (c) 10π (d) 5π

87. If $f(x+2)+f(x)=f(x+1)$, then
(a) $f(x+1)=f(x)$
(b) $f(x+1)=2f(x)$
(c) $f(x+2)=2f(x)$
(d) $f(x)$ is a periodic function

88. $\sin ax+\cos ax$ and $|\sin x|+|\cos x|$ are periodic functions of same fundamental period, then a equals
(a) 0 (b) 1 (c) 2 (d) 4

89. The function $g(x)=a|\sin x|+a^2|\cos x|+f(a)$ has period equal to $\dfrac{\pi}{2}$, then a is equal to
(a) 2 (b) 3 (c) 1 (d) 4

90. Which of the following is a periodic function?
(a) $x-[x]+\sin x$ (b) $\sin[x]$
(c) $x\sin x$ (d) $(x-[x])\cos(\pi x)$

91. The period of the function $f(x)=|\sin x|-|\cos x|$ is equal to
(a) $\dfrac{\pi}{2}$ (b) π (c) 2π (d) 3π

92. The period of the function $f(x)=\dfrac{|\sin x|+|\cos x|}{|\sin x-\cos x|}$ is
(a) $\dfrac{\pi}{4}$ (b) 2π
(c) $\dfrac{\pi}{2}$ (d) π

93. Let $f(x)=\tan^{-1}x+\cot^{-1}x$, $g(x)=\sin^{-1}x+\cos^{-1}x$, then
(a) $f(x)$ and $g(x)$ are periodic functions
(b) $f(x)$ is periodic function but $g(x)$ is not a periodic function
(c) $f(x)$ and $g(x)$ are non-periodic functions
(d) $f[g(x)]$ is periodic function

Round II (Mixed Bag)

Only One Correct Option

1. If $f:R\to R$ be a function satisfying
$f(2x+3)+f(2x+7)=2, \forall x\in R$, then period of $f(x)$ is
(a) 2 (b) 4 (c) 8 (d) 16

2. The range of the function
$$f(x)=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right],$$
(where [·] denotes the greatest integer function) is
(a) $\left\{\dfrac{\pi}{2}\right\}$ (b) $\{\pi\}$
(c) $\left\{-\dfrac{1}{2},0\right\}$ (d) $\left(0,\dfrac{\pi}{2}\right)$

3. The domain of function
$$f(x)=\sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}+\frac{1}{x^2-36} \text{ is}$$
(a) $(-\infty, 0)-\{-6\}$ (b) $(0, \infty)-\{1, 6\}$
(c) $(1, \infty)-\{6, -6\}$ (d) $[1, \infty)-\{6\}$

4. If $f(x)$ and $g(x)$ are periodic functions with periods 7 and 11 respectively. Then, the period of
$$F(x)=f(x)\,g\left(\frac{x}{5}\right)-g(x)f\left(\frac{x}{3}\right) \text{ is}$$
(a) 177 (b) 222 (c) 433 (d) 1155

5. If $f(2x+3y, 2x-7y)=20x$, then $f(x, y)$ equals
(a) $7x-3y$ (b) $7x+3y$ (c) $3x-7y$ (d) $3x+7y$

6. If $f(x)=-\dfrac{x|x|}{1+x^2}$, then $f^{-1}(x)$ equals
(a) $\sqrt{\dfrac{|x|}{1-|x|}}$ (b) $-(\text{sgn } x)\sqrt{\dfrac{|x|}{1-|x|}}$
(c) $-\sqrt{\dfrac{x}{1-x}}$ (d) None of these

7. The function $f(x)=\lambda|\sin x|+\lambda^2|\cos x|+g(\lambda)$ has period equal to $\dfrac{\pi}{2}$, then λ is
(a) 2 (b) 1
(c) 3 (d) None of these

8. Which of the following is not an odd function?
(a) $g(x) - g(-x)$ (b) $\{g(x) - g(-x)\}^3$
(c) $\log\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right)$ (d) $xg(x) \cdot g(-x) + \tan(\sin x)$

9. Let $f : N \to R;\ f(x) = 2x - 1,\ g : R \to R;\ g(x) = \frac{x^2}{2}$, then $(gof)(0)$, is
(a) 1 (b) 1/2
(c) −1/2 (d) None of these

10. Total number of solutions of the equations $2^x\,|2 - |x|| = 1$ are
(a) 2 (b) 1
(c) 0 (d) None of these

11. If $1 + (x^2 - 1) + (x^2 - 1)^2 + (x^2 - 1)^3 + \ldots$ up to ∞ terms $= \frac{1}{2 - x^2}$ then x belongs to
(a) $(0, \sqrt{2})$ (b) $(-\sqrt{2}, \sqrt{2})$
(c) $(-\sqrt{2}, \sqrt{2}) - \{0\}$ (d) None of these

12. If $f(x) = \cos^{-1}\left(\frac{2 - |x|}{4}\right) + [\log(3 - x)]^{-1}$, then its domain is
(a) $[-2, 6]$ (b) $[-6, 2) \cup (2, 3)$
(c) $[-6, 2]$ (d) $[-2, 2) \cup (2, 3]$

13. Let $f : R \to R$ such that $f\{x - f(y)\} = f\{f(y)\} + xf(y) + f(x) - 1\ \forall x, y \in R$. Then, the value of $|\{f(16)\}|$ is
(a) 125 (b) 126 (c) 127 (d) 128

14. If $f\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} - 4\left(x^2 + \frac{1}{x^2}\right) + 13$, then the value of $f(2 + \sqrt{3})$ is
(a) 9 (b) 18 (c) 27 (d) 0

More Than One Correct Option

15. If $f : R \to R$ where $f(x) = ax + \cos x$ is an invertible function, then
(a) $a \in (-2, 1] \cup [1, 2)$ (b) $a \in [-2, 2]$
(c) $a \in (-\infty, -1] \cup [1, \infty)$ (d) $a \in [-1, 1]$

16. Let $f : A \to B$ be a function defined by $f(x) = \sqrt{1 - x^2}$
(a) $f(x)$ is one one if $A = [0, 1]$
(b) $f(x)$ is onto if $B = [0, 1]$
(c) $f(x)$ is one one if $A = [-1, 0]$
(d) $f(x)$ is into if $B = [-1, 1]$

17. Among the following which is/are periodic functions
(a) $\frac{|x|}{x}$
(b) $\sin \pi x + \{x\}$ where $\{\}$ to decimal part of x
(c) $\cos x + \tan \pi x$
(d) $\max\left\{\sin x, \frac{1}{2}\right\}$

18. The possible values of 'a' for which the function $f(x) = e^{x - [x]} + \cos ax$ (where $[\cdot]$ denotes the greatest integer function) is periodic with finite fundamental period is
(a) π (b) 2π
(c) 3π (d) 1

Comprehension Based Questions

Passage

Directions (Q. Nos. 19-21) *Let* $f(x + y) + f(x - y) = 2f(x)f(y)$ $\forall x, y \in R$ and $f(0) \neq 0$. Then,

19. The function $f(x)$ is
(a) even (b) odd
(c) both even and odd (d) neither even nor odd

20. The value of $f(2)$, if $f(1) = k$ is
(a) k (b) $2k^2 + 1$ (c) $2k^2 - 1$ (d) $-k$

21. The value of $f(3) - 2kf(2)$, if $f(1) = k$ is
(a) k (b) $-k$ (c) $2k$ (d) $-2k$

Assertion and Reason

Directions (Q. Nos. 22-29) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

22. **Statement I** The graph of $f(x)$ is symmetrical about the line $x = 1$, then, $f(1 + x) = f(1 - x)$.
Statement II Even functions are symmetric about the y-axis.

23. Let $f(1 + x) = f(1 - x)$ and $f(4 + x) = f(4 - x)$
Statement I $f(x)$ is periodic with period 6
Statement II 6 is not necessarily fundamental period of $f(x)$

24. **Statement I** $f : A \to B$ and $g : B \to C$ are two function then $(gof)^{-1} = f^{-1}og^{-1}$.
Statement II $f : A \to B$ and $g : B \to C$ are bijections then f^{-1} and g^{-1} are also bijections.

25. **Statement I** Range of $f(x) = |x|(|x| + 2) + 3$ is $[3, \infty)$
Statement II If a function $f(x)$ is defined $\forall x \in R$ and for $x \geq 0$ if $a \leq f(x) \leq b$ and $f(x)$ is even function than range of $f(x)$ is $[a, b]$.

26. Let $f(x) = \sin^{-1}\dfrac{x-3}{2} - \log_{10}(4-x)$

Statement I The domain of $f(x)$ is $[1, 3]$

Statement II $\sin^{-1} x$ is defined for $|x| \le 1$ and $\log_{10} x$ is defined for $x > 0$

27. If $f : R \to R$ and $g : R \to R$ be two mappings such that

Statement I $f(x) = \sin x$ and $g(x) = x^2$, then $fog \ne gof$.

Statement II $(fog)\,x = f(x)\,g(x) = (gof)x$.

28. **Statement I** Let n be a fixed positive integer and a relation R be defined in I (the set of all integers) as follows : aRb iff $\dfrac{n}{(a-b)}$, then $a - b$ is divisible by n.

Then, relation is an equivalence.

Statement II If R and R' are symmetric relation, then the relation $R \cap R'$ is not symmetric.

29. **Statement I** $f(x) = |x-2| + |x-3| + |x-5|$ is an odd function for all values of x lie between 3 and 5.

Statement II For odd function $f(-x) = f(x)$.

Previous Years' Questions

30. If $A = \sin^2 x + \cos^2 x$, then for all real x [AIEEE 2011]

(a) $\dfrac{13}{16} \le A \le 1$ (b) $1 \le A \le 2$

(c) $\dfrac{3}{4} \le A \le \dfrac{13}{16}$ (d) $\dfrac{3}{4} \le A \le 1$

31. Consider the following relations

$R = \{(x, y) \mid x$ and y are real numbers and $x = wy$ for some rational number $w\}$;

$S = \left\{\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \middle| \; m, n, p$ and q are integers such that $n, q \ne 0$ and $qm = pn\right\}$. Then, [AIEEE 2010]

(a) R is an equivalence relation but S is not an equivalence relation.

(b) neither R nor S is an equivalence relation.

(c) S is an equivalence relation but R is not an equivalence relation.

(d) R and S both are equivalence relations.

32. For real x, let $f(x) = x^3 + 5x + 1$, then [AIEEE 2009]

(a) f is one one but not onto R

(b) f is onto R but not oneone

(c) f is oneone and onto R.

(d) f is neither one one nor onto R

33. Let R be the real line. Consider the following subsets of the plane $R \times R$.

$$S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$$

and $$T = \{(x, y): x - y \text{ is an integer}\}$$

Which one of the following is true? [AIEEE 2008]

(a) T is an equivalence relation on R but S is not.

(b) Neither S nor T is an equivalence relation on R.

(c) Both S and T are equivalence relations on R.

(d) S is an equivalence relation on R but T is not.

34. The largest interval lying in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\dfrac{x}{2} - 1\right) + \log(\cos x)$ is defined, is [AIEEE 2007]

(a) $[0, \pi]$ (b) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (d) $\left[0, \dfrac{\pi}{2}\right)$

35. Let W denotes the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W$: the words x and y have atleast one letter in common$\}$. Then, R is [AIEEE 2006]

(a) reflexive, symmetric but not transitive

(b) reflexive, symmetric and transitive

(c) reflexive, not symmetric and transitive

(d) not reflexive, symmetric and transitive

36. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is [AIEEE 2005]

(a) an equivalence relation

(b) reflexive and symmetric

(c) reflexive and transitive

(d) only reflexive

Answers

Check Point 1

1. (b)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (c)

Check Point 2

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (d)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (c)
11. (c)	**12.** (b)	**13.** (a)	**14.** (b)						

Round I

1. (a)	**2.** (c)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (d)	**9.** (c)	**10.** (a)
11. (c)	**12.** (a)	**13.** (a)	**14.** (c)	**15.** (d)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (d)
21. (a)	**22.** (d)	**23.** (c)	**24.** (d)	**25.** (b)	**26.** (c)	**27.** (d)	**28.** (a)	**29.** (c)	**30.** (a)
31. (d)	**32.** (d)	**33.** (d)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (a)
41. (a)	**42.** (a)	**43.** (c)	**44.** (d)	**45.** (b)	**46.** (a)	**47.** (a)	**48.** (a)	**49.** (d)	**50.** (a)
51. (a)	**52.** (c)	**53.** (b)	**54.** (a)	**55.** (b)	**56.** (a)	**57.** (b)	**58.** (c)	**59.** (a)	**60.** (b)
61. (a)	**62.** (d)	**63.** (d)	**64.** (a)	**65.** (a)	**66.** (d)	**67.** (b)	**68.** (c)	**69.** (c)	**70.** (d)
71. (d)	**72.** (c)	**73.** (d)	**74.** (b)	**75.** (b)	**76.** (c)	**77.** (b)	**78.** (a)	**79.** (a)	**80.** (c)
81. (b)	**82.** (b)	**83.** (b)	**84.** (c)	**85.** (a)	**86.** (c)	**87.** (d)	**88.** (d)	**89.** (c)	**90.** (d)
91. (b)	**92.** (d)	**93.** (b)							

Round II

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (d)	**10.** (d)
11. (c)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (a.c)	**16.** (a,b,c,d)	**17.** (b,d)	**18.** (a,b,c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (d)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (d)
31. (c)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (c)				

the Guidance

Round I

1. R is reflexive and symmetric but not transitive, since for $(1, 0) \in R$ and $(0, 3) \in R$ whereas $(1, 3) \notin R$.

2. The range of $f = \{2, 3\}$ and the range of $g = \{5, 6\}$.

3. Since, n divides n, $\forall n \in N$, R is reflexive, R is not symmetric, since, for $3, 6 \in N, 3R6 \neq 6R3$.

R is transitive, since for n, m, r whenever n/m and $m/r \Rightarrow n/r$, *i.e.*, n divides m and m divides r, then n will divide r.

4. Here, $A = N$, the set of natural numbers and

$$R = \{(x, y) : y = x + 5, x < 4\}$$
$$= \{(x, x+5) : x \in N \text{ and } x < 4\} = \{(1, 6), (2, 7), (3, 8)\}$$

(a) For reflexive $(x, x) \in R$, $\forall x$ putting $y = x$, $x \neq y + 5$

$\Rightarrow$ $(1, 1) \notin R$. So, R is not reflexive.

(b) For symmetrical $(x, y) \in R \Rightarrow (y, x) \in R$ putting $y = x + 5$, then $x \neq y + 5 \Rightarrow (1, 6) \in R$ but $(6, 1) \notin R$. So, R is not symmetric.

(c) For transitivity $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$ if $y = x + 5$, $z = y + 5$, then $z \neq x + 5$. Since, $(1, 6) \in R$ and there is no order pair in R which has 6 as the first element same in the case for $(2, 7)$ and $(3, 8)$. So, R is not transitive.

5. (a) For reflexive, we know that x is divisible by x for all $x \in A$.

$\therefore$ $(x, x) \in R$ for all $x \in R$. So, R is reflexive.

(b) For symmetry, we observe that 6 is divisible by 2. This means that $(2, 6) \in R$ but $(6, 2) \notin R$. So, R is not symmetric.

(c) For transitivity, let $(x, y) \in R$ and $(y, z) \in R$, then z is divisible by x.

$\Rightarrow \quad (x, z) \in R$

For example, 2 is divisible by 1, 4 is divisible by 2
So, 4 is divisible by 1. So, R is transitive.

6. (a) For reflexive put $y = x, x - x = 0$ which is an integer for all $x \in Z$. So, R is reflexive on Z.

(b) For symmetry let $(x, y) \in R$, then $(x - y)$ is an integer λ and also $y - x = -\lambda$

$[\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\therefore$ $y - x$ is an integer $\Rightarrow (y, x) \in R$. So, R is symmetric.

(c) For transitivity let $(x, y) \in R$ and $(y, z) \in R$

$\therefore$ $x - y$ = integer and $y - z$ = integers, then $x - z$ is also an integer

$\therefore$ $(x, z) \in R$. So, R is transitive.

Hence, R is an equivalence relation.

7. Let $A = \{1, 2, 3, 4, 5, 6\}$

A relation R is defined on set A is

$R = \{a, b) : b = a + 1\}$. Therefore,
$R = \{(1, 2), (2, 3) (3, 4), (4, 5), (5, 6)\}$
Now, $\quad 6 \in A$ but $(6, 6) \notin R$.
Therefore, R is not reflexive.
It can be observed that $(1, 2) \in R$ but $(1, 2) \notin R$. Therefore, R is not symmetric.
Now, $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$. Therefore, R is not transitive.

Hence, R is neither reflexive nor symmetric nor transitive.

8. Given that, $R = \{(a, b): a \le b^3\}$

It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \in R$ as $\frac{1}{2} < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

So, R is not reflexive

Now, $\quad (1, 2) \in R$ (as $1 < 2^3 = 8$)

But $\quad (2, 1) \notin R$ (as $2^3 > 1$)

So, R is not symmetric

We have $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$ as $3 < \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$

But $\quad \left(3, \frac{6}{5}\right) \notin R$ as $3 > \left(\frac{6}{5}\right)^3$

Therefore, R is not transitive.
Hence, R is neither reflexive nor symmetric nor transitive.

9. Here, set A is the set of all books in the library of a college.
and $R = \{(x, y): x$ and y have the same number of pages$\}$
Now, R is reflexive, since $\{x, x\} \in R$ as x and x has the same number of pages.

Let $\quad \{x, y\} \in R$

$\Rightarrow$ x and y have the same number of pages.

$\Rightarrow$ y and x have the same number of pages.

$\Rightarrow$ $(y, x) \in R$. So, R is symmetric. Now,

let $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ x and y have the same number of pages y and z have the same number of pages.

$\Rightarrow$ x and z have the same number of pages $\Rightarrow$ $(x, z) \in R$

Therefore, R is transitive. Hence, R is an equivalence relation.

10. Given that, $A = \{1, 2, 3, 4, 5\}$, $\quad R = \{(a, b) : |a - b|$ is even$\}$

Let $a \in A \Rightarrow |a - a| = 0$ (which is even), $\forall$ a
So, R is reflexive.
Let $(a, b) \in R \Rightarrow |a - b|$ is even $\Rightarrow |-(b - a)| = |b - a|$ is also even

$\Rightarrow \quad (b, a) \in R$. So, R is symmetric.

Now, $\quad$ let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ $(a - b)$ is even and $(b - c)$ is even

$\Rightarrow$ $(a - c) = (a - b) + (b - c)$ is even

(sum of two even integers is even)

$\Rightarrow \quad |a - c|$ is even $\Rightarrow (a, c) \in R$.

So, R is transitive. Hence, R is an equivalence relation.

11. $R = \{(a, b) : |a - b|$ is a multiple of $4\}$

For any element $a \in A$, we have $(a, a) \in R$ as $|a - a| = 0$ is a multiple of 4. Therefore, R is reflexive.

Now, let $(a, b) \in R$

$\Rightarrow$ $|a - b|$ is a multiple of 4,

$|b - a| = |-(a - b)| = |a - b|$ is multiple of 4

$\Rightarrow$ $(b, a) \in R$ Therefore, R is symmetric.
Now, let $(a, b), (b, c) \in R$

So, $|a - b|$ and $|b - c|$ is multiple of 4.

$\Rightarrow$ $|a - c|$ is also multiple of 4 $\Rightarrow$ $(a, c) \in R$

Therefore, R is transitive.

Hence, R is an equivalence relation.

12. Define a relation R in R as

$$R = \{(a, b) : a^3 \ge b^3\}$$

Clearly $(a, a) \in R$ as $a^3 = a^3$. Therefore, R is reflexive.

Now, $\quad (2, 1) \in R$ (as $2^3 \ge 1^3$)

But $\quad (1, 2) \notin R$ (as $1^3 < 2^3$)

Therefore, R is not symmetric.

Now, let $(a, b), (b, c) \in R$

$\Rightarrow a^3 \ge b^3$ and $b^3 \ge c^3 \Rightarrow a^3 \ge c^3 \Rightarrow (a, c) \in R$. Therefore, R is transitive.

Hence, relation R is reflexive and transitive but not symmetric.

13. Here, $R = \{(P, Q)$: distance of point P from the origin is same as the distance of point Q from the origin$\}$. Clearly, $(P, P) \in R$, since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

Therfore, R is reflexive.

Now, let $(P, Q) \in R$.

$\Rightarrow$ The distance of point P from the origin is same as the distance of point Q from the origin.

$\Rightarrow$ The distance of point Q from the origin is same as the distance of point P from the origin.

$\Rightarrow \quad (Q, P) \in R$

Therefore, R is symmetric.

Now, let $\quad (P, Q), (Q, S) \in R$

The distance of points P and Q from the origin is same and also the distance of points Q and S from the origin is same.

$\Rightarrow$ The distance of points P and S from the origin is same.

$\Rightarrow \quad (P, S) \in R$

Therefore, R is transitive. Therefore, R is an equivalence relation.

14. Here, $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$

R is reflexive; since every triangle is similar to itself.

Further, if $(T_1, T_2) \in R$, then T_1 is similar to T_2.

$\Rightarrow \quad T_2$ is similar to $T_1 \Rightarrow (T_2, T_1) \in R$

Therefore, R is symmetric.

Now, let $\quad (T_1, T_2), (T_2, T_3) \in R$.

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3.

$\Rightarrow T_1$ is similar to $T_3 \Rightarrow (T_1, T_3) \in R$

Therefore, R is transitive. Thus, R is an equivalence relation.

15. Here, $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$

R is reflexive as any line L_1 is parallel to itself *i.e.*, $(L_1, L_1) \in R$.

Now, let $(L_1, L_2) \in R$

$\Rightarrow L_1$ is parallel to $L_2 \Rightarrow L_2$ is parallel to $L_1 \Rightarrow (L_2, L_1) \in R$

Therefore, R is symmetric.

Now, let $(L_1, L_2), (L_2, L_3) \in R$

$\Rightarrow L_1$ is parallel to L_2, also L_2 is parallel to L_3.

$\Rightarrow L_1$ is parallel to L_3

$\Rightarrow (L_1, L_3) \in R$. Therefore, R is transitive.

Hence, R is an equivalence relation.

16. Here, $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$

Since, $(a, a) \in R$., for every $a \in \{1, 2, 3, 4\}$. Therefore, R is reflexive.

Now, since $(1, 2) \in R$ but $(2, 1) \notin R$. Therefore, R is not symmetric.

Also, it is observed that $(a, b), (b, c) \in R$

$\Rightarrow (a, c) \in R$. For all $a, b, c \in \{1, 2, 3, 4\}$

Therefore, R is transitive. Hence, R is reflexive and transitive but not symmetric.

17. Since, every set is a subset of itself, ARA for all $A \in P(X)$.

Therefore, R is not reflexive.

Let $\quad ARB \Rightarrow A \subset B$

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

Therefore, R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow \quad A \subset C \Rightarrow ARC$. Therefore, R is transitive.

Hence, R is not an equivalence relation since, it is not symmetric.

18. This is because relation R is reflexive as $(1, 1), (2, 2), (3, 3) \in R$.

Relations R is symmetric since $(1, 2), (2, 1) \in R$ and $(1, 3), (3, 1) \in R$.

But relation R is not transitive as $(3, 1), (1, 2) \in R$ but $(3, 2) \notin R$.

Now, if we add any one of the two pairs $(3, 2)$ and $(2, 3)$ (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

19. It is given that $A = \{1, 2, 3\}$.

An equivalence relation is reflexive, symmetric and transitive.

The smallest equivalence relation containing $(1, 2)$ is given by,

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Now, we are left with only four pairs *i.e.*, $(2, 3), (3, 2), (1, 3)$, and $(3, 1)$.

If we add any one pair [say $(2, 3)$] to R_1, then for symmetry we must add $(3, 2)$.

Also, for transitivity we are required to add $(1, 3)$ and $(3, 1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1, 2)$ is two.

20. $\sec^{-1}x$ is defined for $x \in R \sim (-1, 1)$.

Thus, $\quad \log x \in R \sim (-1, 1)$

$\Rightarrow \quad x \in R^+ \sim \left(\frac{1}{e}, e\right)$

21. Since, $\quad 0 \le \log(1 - x^2) \le 1$

$\Rightarrow \quad 1 \le 1 - x^2 \le e$

Now, $\quad 1 - x^2 \le e, \forall x \in R$

But $1 - x^2 \ge 1$ is possible only when $x = 0$.

22. For domain of $f(x)$, we should have $x > -3$

and $\quad x^2 + 3x + 2 \ne 0$

$\Rightarrow \quad (x + 1)(x + 2) \ne 0$

$\Rightarrow \quad x \ne -1, -2$

Hence, domain $= (-3, \infty) - \{-1, -2\}$.

23. We have, $f(x) = \log_{0.2} \log_{0.5} \log_{0.25} x$

$$\log_{(1/2)} \log_{(1/4)} x > 0 \Rightarrow 0 < \log_{1/4} x < 1$$

$$\Rightarrow \quad 1 > x > \frac{1}{4} \Rightarrow \frac{1}{4} < x < 1$$

24. Domain of tan$x$ is $R \sim \left\{n\pi + \frac{\pi}{2}\right\}$ but here tan $x \neq 1$. So, $x \neq n\pi + \frac{\pi}{4}$ also. So, domain is

$$R \sim \left\{n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right\} \qquad \text{(where } n \in I\text{)}$$

25. Given function is defined for

$$\frac{1-\sin x}{\sin x} \geq 0$$

or $$\frac{(\sin x - 1)(\sin x - 0)}{\sin^2 x} \leq 0$$

$\therefore$ $$0 < \sin x \leq 1$$

Which is true for $x \in \bigcup_{n \in I} \{2n\pi, (2n+1)\pi\}$

26. For function to be defined, we must have $-1 \leq 2 - 4x^2 < 2$

$\therefore$ $2 - 4x^2 < 2 \Rightarrow x \in R - \{0\}$

and $2 - 4x^2 \geq -1 \Rightarrow x^2 \leq \frac{3}{4}$

$\Rightarrow$ $$x \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] \sim \{0\}$$

27. We have, $2^x > 0, \forall x \in R$

Thus, $2^x = 2 - 2^y < 2, \forall x \in R$

As 2^x is an increasing function, we get $-\infty < x < 1$.

28. Domain of $\frac{1}{[x]}$ is $(-\infty, 0) \cup [1, \infty)$ and for domain of

$$2x - x^2 \geq 0 \Rightarrow x(x-2) \leq 0$$

$\Rightarrow$ $$0 \leq x \leq 2$$

So, domain of $f(x) = \frac{1}{[x]} + \sqrt{2x - x^2}$ is [1, 2].

29. Let $$y = \frac{x^2 + x + 2}{x^2 + x + 1}$$

$\Rightarrow$ $$x^2y + xy + y - x^2 - x - 2 = 0$$

$\Rightarrow$ $$(y-1)x^2 + (y-1)x + (y-2) = 0$$

$\therefore$ $D \geq 0$ (as $x \in R$) (but $y \neq 1$)

$\Rightarrow$ $(y-1)^2 - 4(y-1)(y-2) \geq 0$ $(\because b^2 - 4ac \geq 0)$

$\Rightarrow$ $(y-1)(y-1-4y+8) \geq 0$

$\Rightarrow$ $(y-1)(7-3y) \geq 0$

$\Rightarrow$ $$1 < y \leq \frac{7}{3}$$

30. $f(x) = x^3 + 3x^2 + 20x - \sin x$

$f'(x) = 3x^2 + 6x + 20 - \cos x = 3\left[(x+1)^2 + \frac{17}{3}\right] - \cos x$

$\Rightarrow$ $f'(x) > 0, \forall x \in R$

$\Rightarrow$ $f(x)$ is on increasing function for all $x \in R$.

$\Rightarrow$ Range of $f(x)$ is the set of real numbers and domain of $f(x)$ is also set of real numbers.

31. $$f(x) = \begin{cases} 0, & x \in [0, \sqrt{2}) \cup \{2\} \\ 1, & x \in [\sqrt{2}, \sqrt{3}) \\ 2, & x \in [\sqrt{3}, 2) \end{cases}$$

32. $f(x) = (1+b^2)x^2 + 2bx + 1$

$$= (1+b^2)\left[x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2} - \frac{b^2}{(1+b^2)^2} + \frac{1}{1+b^2}\right]$$

$$= (1+b^2)\left[\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{(1+b^2)^2}\right]$$

$\Rightarrow$ $f(x)$ is minimum when $x + \frac{b}{1+b^2} = 0$

$\therefore$ $$x = -\frac{b}{1+b^2}$$

Hence, minimum value of $f(x)$

$$= (1+b^2)\left[0 + \frac{1}{(1+b^2)^2}\right] = \frac{1}{1+b^2}$$

$\therefore$ $m(b) = \frac{1}{1+b^2}$ and it has a range (0, 1].

33. Here, base is less than 1. (as $\sin x \leq 1$)

So, $\tan x \leq 1$ for $\log_{\sin x} \tan x \geq 0$

$\Rightarrow$ $$0 < x \leq \frac{\pi}{4}$$

34. $f(x) = (x+1)^2 - 1$

$\Rightarrow$ $(x+1)^2 = f(x) + 1$

$\Rightarrow$ $x + 1 = \pm\sqrt{f(x)+1}$

$\Rightarrow$ $x + 1 = \sqrt{f(x)+1}$ as $x + 1 \geq 0$

$\therefore$ $x = -1 + \sqrt{f(x)+1}$

$\Rightarrow$ $f^{-1}(y) = -1 + \sqrt{y+1}$

or $f^{-1}(x) = -1 + \sqrt{x+1}$

Now, $f^{-1}(x) = f(x)$

$\Rightarrow$ $(x+1)^2 - 1 = -1 + \sqrt{x+1}$

$\Rightarrow$ $(x+1)^2 = \sqrt{x+1} \Rightarrow x = 0, -1$

35. For $x_1, x_2 \in R$, consider

$$f(x_1) = f(x_2)$$

$\Rightarrow$ $$\frac{x_1}{x_1^2 + 1} = \frac{x_2}{x_2^2 + 1}$$

$\Rightarrow$ $x_1x_2^2 + x_1 = x_2x_1^2 + x_2$

$\Rightarrow$ $x_1x_2(x_2 - x_1) = x_2 - x_1$

$\Rightarrow$ $x_1 = x_2$ or $x_1x_2 = 1$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence, f is not one one. Also, f is not onto for if so then for $1 \in R$ $\exists x \in R$ such that $f(x) = 1$

which gives $\frac{x}{x^2+1}=1$. But there is no such x in the domain R, since the equation $x^2-x+1=0$ does not give any real value of x.

36. Here, the function $f:R\to R$ is defined as $f(x)=4x+5=y$ (say). Then,

$$4x=y-5 \quad \text{or} \quad x=\frac{y-5}{4}$$

This leads to a function $g:R\to R$ defined as

$$g(y)=\frac{y-5}{4}.$$

Therefore, $(gof)(x)=g(f(x))=g(4x+5)$

$$=\frac{4x+5-5}{4}=x$$

or $gof=I_R$

Similarly, $(fog)(y)=f\{g(y)\}$

$$=f\left(\frac{y-5}{4}\right)=4\left(\frac{y-5}{4}\right)+5=y$$

or $fog=I_R$

Hence, f is invertible and $f^{-1}=g$ which is given by

$$f^{-1}(x)=\frac{x-5}{4}$$

37. The total number of injective mappings from the set containing 3 elements into the set containing 4 elements is ${}^4P_3=4!=24$.

38. Since, for $f^{-1}(17)=x$

$\Rightarrow$ $f(x)=17$ or $x^2+1=17$

$\Rightarrow$ $x=\pm 4$ or $f^{-1}(17)=\{4,-4\}$

and for $f^{-1}(-3)=x\Rightarrow f(x)=-3 \Rightarrow x^2+1=-3$

$\Rightarrow$ $x^2=-4$

and hence $f^{-1}(-3)=\phi$.

39. Here, $f:R\to R$ is given by, $f(x)=[x]$

It is seen that $f(1.2)=[1.2]=1$, $f(1.9)=[1.9]=1$

Therefore, $f(1.2)=f(1.9)$ but $1.2\neq 1.9$

Therefore, f is not one one.

Now, consider $0.7\in R$. It is known that $f(x)=[x]$ is always an integer.

Thus, there does not exist any element $x\in R$ such that $f(x)=0.7$.

Therefore, f is not onto.

Hence, the greatest integer function is neither one one nor onto.

40. Here, $f:R\to R$ is given by $f(x)=|x|=\begin{cases} x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$

It is seen that $f(-1)=|-1|=1$, $f(1)=|1|=1$

Therefore, $f(-1)=f(1)$ but $-1\neq 1$

Therefore, f is not one one.

Now, consider $-1\in R$

It is known that $f(x)=|x|$ is always non-negative.

Thus, there does not exist any element x in domain R such that

$$f(x)=|x|=-1.$$

Therefore, f is not onto.

Hence, the modulus functions is neither one one nor onto.

41. Given that, $A=\{1,2,3\}$ and $B=\{4,5,6,7\}$

Now, $f:A\to B$ is defined as $f=\{(1,4),(2,5),(3,6)\}$

Therefore, $f(1)=4$, $f(2)=5$, $f(3)=6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one one.

42. Here, $f:N\to N$ is defined as, $f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd}\\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n\in N$. It can be observed that

$f(1)=\frac{1+1}{2}=1$ and $f(2)=\frac{2}{2}=1$ (by definition of f)

$\therefore$ $f(1)=f(2)$, where $1\neq 2$.

Therefore, f is not one one. Consider a natural number n in co domain N.

Case I When n is odd.

Therefore, $n=2r+1$ for some $r\in N$.

Then, there exists $4r+1\in N$ such that

$$f(4r+1)=\frac{4r+1+1}{2}=2r+1.$$

Therefore, f is onto.

Case II When n is even

Therefore, $n=2r$ for some $r\in N$

Then, there exists $4r\in N$ such that $f(4r)=\frac{4r}{2}=2r$

Therefore, f is onto. Hence, f is not a bijective function.

43. Here, $A=R-\{3\}, B=R-\{1\}$

and $f:A\to B$ is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$

Let $x,y\in A$ such that $f(x)=f(y)$

$\Rightarrow$ $\frac{x-2}{x-3}=\frac{y-2}{y-3}\Rightarrow (x-2)(y-3)=(y-2)(x-3)$

$\Rightarrow$ $xy-3x-2y+6=xy-3y-2x+6$

$\Rightarrow$ $-3x-2y=-3y-2x$

$\Rightarrow$ $3x-2x=3y-2y\Rightarrow x=y$

Therefore, f is one one. Let $y\in B=R-\{1\}$. Then, $y\neq 1$

The function f is onto if there exists $x\in A$ such that $f(x)=y$.

Now, $f(x)=y$

$\Rightarrow$ $\frac{x-2}{x-3}=y\Rightarrow x-2=xy-3y\Rightarrow x(1-y)=-3y+2$

$\Rightarrow$ $x=\frac{2-3y}{1-y}\in A$ $[y\neq 1]$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

Therefore, f is onto. Hence, function f is one one and onto.

44. Given, $f : R \to R$ defined by $f(x) = \cos x$

For one one,

Let $x_1, x_2 \in R$ be any two elements, such that $f(x_1) = f(x_2) \Rightarrow \cos x_1 = \cos x_2$

Since, $f(0) = f(2\pi) = 1$ but $1 \neq 2\pi$

$\therefore$ f is not one one.

For onto,

Let $y \in R$ be any element, such that

$$f(x) = y \Rightarrow \cos x = y$$

For $y = 2 \in R$, there is no $x \in R$ such that $\cos x = 2$

$\therefore$ Element $2 \in R$ does not have pre-image in R.

$\therefore$ f is not onto. Therefore, f is neither one one nor onto.

45. $f(x) = (x-2)^2$ is many one function (a parabola). $f(x) = x^3$ is an increasing function (one one function). Here, codomain is same as range because 'on their ranges' mean codomain is same as range.

46. For an onto function, the image set = Range. Now, for each element of the domain E, there exists two possible images 1 and 2 in the codomain.

Thus, total number of functions $= 2^4 = 16$ (since, there are 4 elements in domain). Two of these functions are not onto (all the four elements have the same image either 1 and 2). Thus, total number of onto functions $= 16 - 2 = 14$.

47. We have, $y = \sin\frac{x}{2}$

and $\quad 0 \le x \le \frac{\pi}{2} \Rightarrow 0 \le \frac{x}{2} \le \frac{\pi}{4}$

$\Rightarrow \quad 0 \le \sin\frac{x}{2} \le \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(0, \frac{1}{\sqrt{2}}\right) \subset [0, \infty)$

So, function is not surjective but function is injective as for any $0 \le x \le \frac{x}{2}$, $\sin\frac{\pi}{2}$ gives unique image.

48. Each element in set A has m options in it's images, so number of functions $= \underbrace{m \times m \times \ldots \times m}_{m\text{-times}} = m^m$

49. $f(x) = |(x-1)(x-2)|$ is always non-negative. So, $f(x)$ is into function.

Now, $\quad f(x) = \begin{cases} (x-1)(x-2), & x \le 1, x \ge 2 \\ -(x-1)(x-2), & 1 < x < 2 \end{cases}$

$\therefore$ Function is decreasing in $(-\infty, 1)$ and $\left(\frac{3}{2}, 2\right)$ and increasing in $\left(1, \frac{3}{2}\right)$ and $(2, \infty)$.

50. $f(x) = \log[\log(\log x)], x \in (e, \infty)$

For $\quad x > e \Rightarrow \log x > 1 \Rightarrow \log(\log x) > 0$

$\Rightarrow \quad -\infty < \log[\log(\log x)] < \infty$

$\Rightarrow$ f is onto function.

51. (a) Z means set of integers, clearly $y = x + 2$ is bijective as for integral values of x we get all integral values of y and for every x we have, different value of y.

(b) $y = 2x$ is not bijective as $y(x)$ gives only even values, so function is into function.

(c) Function cannot have images for negative values of x.

(d) $y = x + |x| = \begin{cases} 2x, & x \ge 0 \\ 0, & x < 0 \end{cases}$

$\Rightarrow$ $y(x) \ge 0$, while $Y \in R$, into function.

$\therefore$ Only the option (a) is bijective function.

52. $f(x) = \frac{x^2 + e}{x^2 + 1}$

$$f'(x) = \frac{2x(x^2+1) - 2x(x^2+e)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3 - 2ex}{(x^2+1)^2}$$

$$= \frac{2x - 2xe}{(x^2+1)^2} = \frac{2x(1-e)}{(x^2+1)^2} < 0$$

$f'(x) < 0$, $f(x)$ is decreasing

Hence, f is one one function.

$x \to 0, f(x) \to e$

$x \to \infty, f(x) \to e$

Hence, range $= (1, e) =$ co domain

53. $\quad f(x) = x^3 + 3x^2 + 12x - 12x - 2\sin x$

$\quad f'(x) = 3x^2 + 6x + 12 - 2\cos x$

$f'(x) = 0$ for all values of x

$\therefore$ $f(x)$ is an increasing function, so $f(x)$ is one one and onto

54. Here, $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0, & \text{if } x \ge 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

Therefore, gof gets defined as

For $x \ge 0$, $\quad (gof)(x) = g\{f(x)\} = g(2x) = 0$

and for $x < 0$, $\quad (gof)(x) = g\{f(x)\} = g(0) = 0$

Consequently, we have $(gof)(x) = 0, \forall x \in R$

Similarly, fog gets defined as

For $x \geq 0$, $(fog)(x) = f\{g(x)\} = f(0) = 0$

and for $x < 0$, $(fog)(x) = f\{g(x)\} = f(-2x) = -4x$

i.e., $(fog)(x) = \begin{cases} 0, & x > 0 \\ -4x, & x < 0 \end{cases}$

55. Given that, $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

$$(fof)x = \{f(f(x))\}$$

$$= f\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\dfrac{x}{\sqrt{1+x^2}}}{\sqrt{1+\dfrac{x^2}{(1+x)^2}}}$$

$$= \frac{x}{\sqrt{1+2x^2}}$$

Now, $(fofof)x = \{f(fof)x\}x$

$$= f\left(\frac{x}{\sqrt{1+2x^2}}\right) = \frac{\dfrac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\dfrac{x^2}{1+2x^2}}}$$

$$= \frac{x}{\sqrt{1+3x^2}}$$

56. Given that, $f(x) = \{4 - (x-7)^3\}$

$\Rightarrow$ Let $y = \{4 - (x-7)^3\}$

$\Rightarrow$ $(x-7)^3 = 4 - y$

$\Rightarrow$ $(x-7) = (4-y)^{1/3}$

$\Rightarrow$ $x = 7 + (4-y)^{1/3} = f^{-1}(y)$

$\therefore$ $f^{-1}(x) = 7 + (4-x)^{1/3}$

57. The functions $f:\{1,3,4\} \to \{1,2,5\}$ and $g:\{1,2,5\} \to \{1,3\}$ are defined as $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$.

$\therefore$ $gof(1) = g\{f(1)\} = g(2) = 3$ $[\because f(1) = 2 \text{ and } g(2) = 3]$

$gof(3) = g\{f(3)\} = g(5) = 1$ $[\because f(3) = 5 \text{ and } g(5) = 1]$

$gof(4) = g\{f(4)\} = g(1) = 3$ $[\because f(4) = 1 \text{ and } g(1) = 3]$

$\therefore$ $gof = \{(1,3),(3,1),(4,3)\}$

58. It is given that $f(x) = \dfrac{(4x+3)}{(6x-4)}, x \neq \dfrac{2}{3}$.

$$(fof)(x) = f(f(x)) = f\left(\frac{(4x+3)}{(6x-4)}\right)$$

$$= \frac{4\left(\dfrac{4x+3}{6x-4}\right)+3}{6\left(\dfrac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Therefore, $fof(x) = x$, for all $x \neq \dfrac{2}{3} \Rightarrow fof = I$

Hence, the given function f is invertible and the inverse of f is itself.

59. Let $f: X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have

$\Rightarrow$ $fog_1(y) = I_y(y) = fog_2(y) \Rightarrow f\{g_1(y)\} = f\{g_2(y)\}$

$\Rightarrow$ $g_1(y) = g_2(y)$ (f is invertible $\Rightarrow f$ is one one)

$\Rightarrow$ $g_1 = g_2$ (g is one one)

Hence, f has a unique inverse.

60. Given that, $f: R - \left\{-\dfrac{4}{3}\right\} \to R$ is defined as $f(x) = \dfrac{4x}{3x+4}$

Let y be an arbitrary element of range f.

Then, there exists $x \in R - \left\{-\dfrac{4}{3}\right\}$ such that $y = f(x)$

$\Rightarrow$ $y = \dfrac{4x}{3x+4} \Rightarrow 3xy + 4y = 4x$

$\Rightarrow$ $x(4-3y) = 4y$

$\Rightarrow$ $x = \dfrac{4y}{4-3y}$

Let us define g : Range $f \to R - \left\{-\dfrac{4}{3}\right\}$ as $g(y) = \dfrac{4y}{4-3y}$.

Now, $(gof)(x) = g(f(x)) = g\left(\dfrac{4x}{3x+4}\right)$

$$= \frac{4\left(\dfrac{4x}{3x+4}\right)}{4-3\left(\dfrac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

and $(fog)(y) = f\{g(y)\} = f\left(\dfrac{4y}{4-3y}\right)$

$$= \frac{4\left(\dfrac{4y}{4-3y}\right)}{3\left(\dfrac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

Therefore, $gof = I_{R-\left\{-\frac{4}{3}\right\}}$ and $fog = I_R$

Thus, g is the inverse of f *i.e.*, $f^{-1} = g$

Hence, the inverse of f is the map g : Range $f \to R - \left\{-\dfrac{4}{3}\right\}$, which is given by $g(y) = \dfrac{4y}{4-3y}$.

61. Given that, $A = \{-1, 0, 1, 2\}, B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g : A \to B$ are defined by

$f(x) = x^2 - x, x \in A$ and $g(x) = 2\left|x - \dfrac{1}{2}\right| - 1, x \in A$.

It is observed that $f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$

and $g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$

$\Rightarrow \quad f(-1) = g(-1)$

Next $\quad f(0) = (0)^2 - 0 = 0$

and $\quad g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\Rightarrow \quad f(0) = g(0)$

Next, $\quad f(1) = (1)^2 - 1 = 1 - 1 = 0$

and $\quad g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\Rightarrow \quad f(1) = g(1)$

Next, $\quad f(2) = (2)^2 - 2 = 4 - 2 = 2$

and $\quad g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$

$\Rightarrow \quad f(2) = g(2)$

$\therefore \; f(a) = g(a), \forall\, a \in A$. Hence, the functions f and g are equal.

62. It is given that $f : R \to R$ is defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

Also, $g : R \to R$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x. Now, let $x \in (0, 1]$. Then, we have

$$[x] = 1 \text{ if } x = 1 \text{ and } [x] = 0 \text{ if } 0 < x < 1$$

$$\therefore \quad fog(x) = f\{g(x)\} = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0, 1) \end{cases}$$

$$= \begin{cases} 1, & \text{if } x = 1 \\ 0, & x \in (0, 1) \end{cases}$$

$$gof(x) = g\{f(x)\} = g(1) = [1] = 1 \qquad (\because x > 0)$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

But $\quad fog(1) \neq gof(1)$

Hence, fog and gof do not coincide in $(0, 1]$.

63. $f_1 = \{(1,2), (2,3), (3,4), (4,5)\}$

$f_2 = \{(1,5), (2,4), (2,5), (3,3), (3,4), (3,5), (4,2), (4,3), (4,4), (4,5)\}$

$f_3 = \{(2,2), (3,2), (3,3), (4,2), (4,3), (4,4)\}$

f_1 is not a function for corresponding to 0, there is no image

f_2 is not a function $x = 2$, we have two images 4 and 5

or for $x = 3$, we have three images 3, 4 and 5.

or for $x = 4$, we have images 2, 3, 4 and 5.

f_3 is not a function for same reason as of f_2.

64. On putting $y = 0$ in the given relation, we get

$$f(x) + f(x) = kf(x)\, f(0)$$

$\Rightarrow \quad 2f(x) = kf(x) \qquad [\because f(0) = 1]$

$\Rightarrow \quad k = 2$

65. Given, $\quad y = f(x) = x + \frac{1}{x}$

$\Rightarrow \quad xy = x^2 + 1 \;\Rightarrow\; x^2 - xy + 1 = 0$

$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\Rightarrow \quad f^{-1}(y) = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}$

Since, $f^{-1} : [2, \infty) \to [1, \infty)$, we get

$$y = f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

66. (a) Since, $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow -1 \le \sin^3 x \le 1$, same codomain and range. So, function is onto, also one one.

(b) $y = e^x$, range is R^+ so function is onto and also increasing so function is one one.

(c) $y = \log|x|$ as $x \in R^+$, so $|x| = x \Rightarrow y = \log x$, range is R, function increasing so function is one one.

So, in first three alternatives their inverse are defined. In last option (d) $x \in R \;\Rightarrow\; |x| \ge 0$ or $e^{|x|} \ge 1$. So, range is proper subset of codomain. Function is into. So, inverse is not defined.

67. Let $y = 2^{x(x-1)}$ where $y \ge 1$ as $x \ge 1$

$\Rightarrow \quad \log_2 y = x(x-1)$

$\Rightarrow \quad x^2 - x - \log_2 y = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$

But as $x \ge 1$

$\Rightarrow \quad x = \frac{1 + \sqrt{1 + 4\log_2 y}}{2}$

68. Obviously f is one one and onto, thus f^{-1} exists. Since,

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}, \; \frac{\pi}{2} \le \pi - \sin^{-1} x \le \frac{3\pi}{2}$$

69. Let $x + y = u$, $x - y = v \Rightarrow x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$

$\therefore \quad f(u, v) = \frac{u^2 - v^2}{4} \Rightarrow f(x, y) = \frac{x^2 - y^2}{4}$

70. $f(x) = |x - 1|$, $\quad f(x^2) = |x^2 - 1| \neq [f(x)]^2$

Now, $\quad f(y) = |y - 1|$

$\therefore \quad f(x) + f(y) \neq f(x + y)$

and $\quad f(|x|) = ||x| - 1| \neq |f(x)|$

None of options (a), (b) and (c) are correct.

71. $\therefore \; f(x + 2) = l(x + 2) + m$

and $\quad f(x) = lx + m$

$\Rightarrow \quad f(x + 2) - f(x) = 2l$

$\Rightarrow \quad 2l = 2 \Rightarrow l = 1$

But m can take any real values.

72. $af(x) + bf\left(\frac{1}{x}\right) = x + \frac{5}{x}$ ...(i)

$af\left(\frac{1}{x}\right) + bf(x) = \frac{1}{x} + 5x$ ...(ii)

On multiplying Eq. (i) by a and Eq. (ii) by b and then subtract, we get

$$(a^2 - b^2) f(x) = ax + \frac{5a}{x} - \frac{b}{x} - 5bx$$

$$\Rightarrow \quad f(x) = \frac{1}{(a^2 - b^2)}\left[x(a - 5b) + \frac{1}{x}(5a - b)\right]$$

73. We have, $f(x) = \cos(\log x)$

$\therefore \quad f(x^2) = \cos(\log x^2)$

$f(y^2) = \cos(\log y^2)$

$$f\left(\frac{x^2}{y^2}\right) = \cos\left\{\log\left(\frac{x^2}{y^2}\right)\right\}$$

and $f(x^2y^2) = \cos\{\log(x^2y^2)\}$

$$\therefore \quad f(x^2)f(y^2) - \frac{1}{2}\left[f\left(\frac{x^2}{y^2}\right) + f(x^2y^2)\right]$$

$$= \cos\{\log(x^2)\} \cdot \cos\{\log(y^2)\} - \frac{1}{2}\left[\cos\log\left(\frac{x^2}{y^2}\right) + \cos\log(x^2y^2)\right]$$

$$= \cos(\log(x^2))\cos(\log(y^2)) - \cos\{\log(x^2)\} \cdot \cos\{\log(y^2)\}$$

$$= 0$$

74. Let $f(x) = \log\{g(x)\}$

$\therefore \quad f(x_1) + f(x_2) = \log\{g(x_1)\} + \log\{g(x_2)\}$

$= \log\{g(x_1) \cdot g(x_2)\}$

75. Given that, $f(x) = \frac{x-1}{x+1} \Rightarrow f(2x) = \frac{2x-1}{2x+1}$ ...(i)

Also, $x f(x) + f(x) = x - 1$

$\Rightarrow \quad x = \frac{f(x)+1}{1-f(x)}$ ...(ii)

From Eqs. (i) and (ii),

$$f(2x) = \frac{2\left(\frac{f(x)+1}{1-f(x)}\right) - 1}{2\left(\frac{f(x)+1}{1-f(x)}\right) + 1} \Rightarrow f(2x) = \frac{3f(x)+1}{f(x)+3}$$

76. (a) $f(x) = x^3$, $g(x) = x + 1$

$f[g(x)] = (x+1)^3$, $g[f(x)] = x^3 + 1$

(b) $f(x) = \sqrt{x}$, $g(x) = \cos x$

$f[g(x)] = \sqrt{\cos x}$, $g[f(x)] = \cos\sqrt{x}$

(c) $f(x) = x^m$, $g(x) = x^n$

$f[g(x)] = (x^n)^m$, $g[f(x)] = (x^m)^n$

(d) $f(x) = (x-1)$, $g(x) = x^2 + 1$

$f[g(x)] = (x^2 + 1) - 1 = x^2$

$g[f(x)] = (x-1)^2 + 1$

So, function $f(x) = x^m$, $g(x) = x^n$ are commute.

77. $g(x) = 1 + x - [x] = 1 + \{x\}$ ...(i)

where $\{x\}$ is the fractional part of x.

Thus, $g(x) \geq 1$ $(\because 0 \leq \{x\} < 1)$

Now, $f[g(x)] = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

From Eq. (i), we get $f[g(x)] = 1$

78. When $f(x) = \sin^2 x$

and $g(x) = \sqrt{x}$

$\therefore \quad (fog)(x) = f[(g(x)] = f(\sqrt{x}) = (\sin\sqrt{x})^2$

when $f(x) = \sin x$, $g(x) = |x|$

$\therefore \quad (fog)(x) = f[g(x)] = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2}$

$= \sin|x| \neq |\sin x|$

79. $f(x) = 1 \pm x^n$ or $f(5) = 1 \pm 5^n$

$\Rightarrow \quad 126 = 1 + 5^n$

$\Rightarrow \quad \pm 5^n = 125 \Rightarrow \pm 5^n = 5^3$

$n = 3$

$f(3) = 1 + 3^3 = 28$

80. $fog = f\{g(x)\} = |\sin x| = \sqrt{\sin^2 x}$

Also, $gof = g\{f(x)\} = \sin^2\sqrt{x}$

Obviously, $\sqrt{\sin^2 x} = \sqrt{g(x)}$ and $\sin^2\sqrt{x} = \sin^2\{f(x)\}$

i.e., $g(x) = \sin^2 x$ and $f(x) = \sqrt{x}$.

81. $x = 0 \Rightarrow f(2) = 2f(0) - f(1) = 2 \times 2 - 3 = 1$

$\Rightarrow \quad x = 1 \Rightarrow f(3) = 6 - 1 = 5$

$x = 2 \Rightarrow f(4) = 2f(2) - f(3) = 2 \times 1 - 5 = -3$

$x = 3 \Rightarrow f(5) = 2f(3) - f(4) = 2(5) - (-3) = 13$

Hence, (b) is correct answer.

82. $f(x) = \sqrt{3}\sin x - \cos x + 2 = 2\sin\left(x - \frac{\pi}{6}\right) + 2$

Since, $f(x)$ is one one and onto, f is invertible.

Now, $fof^{-1}(x) = x \Rightarrow 2\sin\left(f^{-1}(x) - \frac{\pi}{6}\right) + x$

$$\Rightarrow \quad \sin\left(f^{-1}(x) - \frac{\pi}{6}\right) = \frac{x}{2} - 1$$

$$\Rightarrow \quad f^{-1}(x) = \sin^{-1}\left(\frac{x}{2} - 1\right) + \frac{\pi}{6}$$

Because $|(x/2) - 1| \leq 1$ for all $x \in [0, 4]$.

Hence, (b) is the correct answer.

83. $f(x+y)+f(x-y)=2f(x)\,f(y)$

$$f(0)=2f^2(0)$$

$$f(0)\neq 0,\quad f(0)=1$$

$$f(y)+f(-y)=2f(0)\,f(y)$$

$$f(y)+f(-y)=2f(y)$$

$$f(-y)=f(x)$$

$$f(-x)=f(x),\ \text{even}$$

84. $f(x)=0,\ \forall\, x\in R \ \Rightarrow\ f(3)-f(2)=0$

Hence, (c) is correct answer.

85. $f(-x)=\dfrac{\cos(-x)}{\left[-\dfrac{x}{\pi}\right]+\dfrac{1}{2}}=\dfrac{\cos x}{-\left[\dfrac{x}{\pi}\right]-1+\dfrac{1}{2}}$

$$\left(\text{as } x\neq n\pi \ \Rightarrow\ \frac{x}{\pi}\notin I,\ \text{so as}\left[-\frac{x}{\pi}\right]=-\left[\frac{x}{\pi}\right]-1\right)$$

$=\dfrac{\cos x}{\left[\dfrac{x}{\pi}\right]+\dfrac{1}{2}}=-f(x)\ \Rightarrow\ f(x)$ is an odd function.

Hence, (a) is correct answer.

86. $f(x)=\sin\left\{\sin\left(\dfrac{x}{5}\right)\right\}$

$$\Rightarrow\ f(x+10\pi)=\sin\left\{\sin\left(\frac{x+10\pi}{5}\right)\right\}$$

$$\Rightarrow\ f(x+10\pi)=\sin\left\{\sin\left(2\pi+\frac{x}{5}\right)\right\}=\sin\left\{\sin\left(\frac{x}{5}\right)\right\}=f(x)$$

Therefore, the period of $f(x)$ is 10π.

87. $f(x+2)+f(x)=f(x+1)$ ...(i)

$\Rightarrow\ f(x+3)+f(x+1)=f(x+2)$ ...(ii)

(putting $x=x+1$)

On adding Eqs. (i) and (ii), we get

$$f(x+3)=-f(x)$$

$$\Rightarrow\ f(x+6)=-f(x+3)$$

$$\Rightarrow\ f(x)=f(x+6)$$

So, $f(x)$ is a periodic function.

88. Fundamental period of $|\sin x|+|\cos x|$ is $\dfrac{\pi}{2}$. Fundamental period of $\sin ax+\cos ax$ is $\dfrac{2\pi}{a}$. Hence, $a=4$

89. If $a=1, g(x)=|\sin x|+|\cos x|+f(1)$, then period of $g(x)=\dfrac{\pi}{2}$

90. Let $f(x)=(x-[x])\cos(\pi x)$

Period of $x-[x]=\{x\}$ is 1 and period of $\cos(\pi x)$ is 2. So, period of $\{x-[x]\}\cos(\pi x)$ is 2.

91. The period of $|\sin x|$ is π and also period of $|\cos x|$ is π. So, period of $|\sin x|-|\cos x|$ is π.

92. Since, period of $|\sin x|+|\cos x|$ is $\dfrac{\pi}{2}$ and period of $|\sin x-\cos x|$ is π so period of $\dfrac{|\sin x|+|\cos x|}{|\sin x-\cos x|}$ is π.

93. $\therefore\ f(x)=\dfrac{\pi}{2}$ for all x

But $\qquad g(x)=\dfrac{\pi}{2},\ -1\le x\le 1$

So, $g(x)$ is not a periodic function.

Round II

1. Given, $\quad f(2x+3)+f(2x+7)=2$ ...(i)

On replacing x by $x+1$ and x by $x+2$ in Eq. (i), we get

$f(2x+5)+f(2x+9)=2$ ...(ii)

and $\quad f(2x+7)+f(2x+11)=2$ ...(iii)

Now, on subtracting Eq. (i) from Eq. (iii), we get

$$f(2x+11)=f(2x+3)$$

$\therefore$ Period of the function $=8$

2. $\because\ \left[x^2-\dfrac{1}{2}\right]=\left[x^2+\dfrac{1}{2}-1\right]=\left[x^2+\dfrac{1}{2}\right]-1$

$$\therefore\quad f(x)=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left(\left[x^2+\frac{1}{2}\right]-1\right)$$

$$\because\quad \left(x^2+\frac{1}{2}\right)\ge\frac{1}{2}$$

$$\left[x^2+\frac{1}{2}\right]=0 \text{ or } 1$$

If $\qquad \left[x^2+\dfrac{1}{2}\right]=0$

Then, $f(x)=\sin^{-1}(0)+\cos^{-1}(0-1)=0+\pi=\pi$ and for

$$\left[x^2+\frac{1}{2}\right]=1$$

$$f(x)=\sin^{-1}1+\cos^{-1}(1-1)$$

$$=\sin^{-1}1+\cos^{-1}0=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Therefore, range of $f(x)$ is $\{\pi\}$.

3. $f(x)=\sqrt{\log_{0.4}\left(\dfrac{x-1}{x+5}\right)}+\dfrac{1}{x^2-36}$

$$\Rightarrow\quad f(x)=f_1+f_2$$

For domain of f_1

$$f_1=\frac{1}{x^2-36}$$

$$x^2-36\neq 0\Rightarrow x\neq\pm 6$$

For domain of f_2

$$f_2 = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$$

It is defined, if

$$\log_{0.4}\left(\frac{x-1}{x+5}\right) \geq 0 = \log_{0.4}(1)$$

and $\quad \dfrac{x-1}{x+5} > 0$

$\Rightarrow \quad 0 < \dfrac{x-1}{x+5} \leq 1$

i.e., $\quad \dfrac{x-1}{x+5} > 0$

$\Rightarrow \quad x \in (-\infty, -5) \cup (1, \infty)$

and $\quad \dfrac{x-1}{x+5} - 1 \leq 0$

$\Rightarrow \quad \dfrac{-6}{x+5} \leq 0 \quad \Rightarrow \quad \dfrac{6}{x+5} \geq 0$

$\Rightarrow \quad x + 5 \geq 0 \quad \Rightarrow \quad x \geq -5$

$\therefore \quad$ Domain of $f_2 = (1, \infty)$

Hence, domain of $f(x) = (1, \infty) - \{6, -6\}$

4. Period of $f(x)$ is 7.

$\therefore$ Period of $f\left(\dfrac{x}{3}\right)$ is $\dfrac{7}{1/3} = 21$ and period of $g(x)$ is 11.

$\therefore$ Period of $g\left(\dfrac{x}{5}\right)$ is $\dfrac{11}{1/5} = 55$

$\therefore \quad T_1 =$ Period of $f(x)\, g\left(\dfrac{x}{5}\right) = 7 \times 55 = 385$

and $\quad T_2 =$ Period of $g(x) f\left(\dfrac{x}{3}\right) = 11 \times 21 = 231$

$\therefore \quad$ Period of $F(x) = \text{LCM}\{T_1, T_2\}$

$= \text{LCM}\{385, 231\}$

$= 7 \times 11 \times 3 \times 5 = 1155$

5. Let $2x + 3y = A$ and $2x - 7y = B$

Then, $\quad 7A + 3B = 20x$

$\therefore \quad f(A, B) = 7A + 3B$

$\therefore \quad f(x, y) = 7x + 3y$

6. Let $\quad y = f(x) = \begin{cases} \dfrac{-x^2}{1+x^2}, \text{ if } x \geq 0 \\ \dfrac{x^2}{1+x^2}, \text{ if } x < 0 \end{cases}$

$$f^{-1}(y) = \begin{cases} \sqrt{\dfrac{-y}{1+y}}, & y < 0 \\ -\sqrt{\dfrac{y}{1-y}}, & y > 0 \end{cases} \Rightarrow f^{-1}(x) = \begin{cases} \sqrt{\dfrac{-x}{1+x}}, & x < 0 \\ -\sqrt{\left(\dfrac{x}{1-x}\right)}, & x > 0 \end{cases}$$

$$\therefore \quad f^{-1}(x) = -\begin{cases} \sqrt{\left(\dfrac{-x}{1+x}\right)}, & x < 0 \\ \sqrt{\left(\dfrac{-x}{1-x}\right)}, & x > 0 \end{cases}$$

$$= -(\text{sgn } x)\sqrt{\frac{|x|}{1-|x|}}$$

7. Since, period of $|\sin x| + |\cos x| = \pi/2$ it is possible when $\lambda = 1$

8. (a) Let $h(x) = g(x) - g(-x)$

$h(-x) = g(-x) - g(x) = -h(x)$ odd

(b) odd

(c) $f(x) = \log(x^2 - x + 1)$

$f(-x) = \log(x^2 - x + 1)$

(d) Let $f(x) = xg(x)\, g(-x) + \tan(\sin x)$

$\Rightarrow \quad f(-x) = -f(x)$

9. $\quad f(x) = 2x - 1$

$\Rightarrow \quad g(x) = \dfrac{x^2}{2}$

$g\{f(0)\}$ is undefined as $0 \notin N$.

10. $|2 - |x| = 2^{-x}$

$\therefore$ Total number of solutions are 3.

11. $\quad \dfrac{1}{1-(x^2-1)} = \dfrac{1}{2-x^2}$

$\Rightarrow \quad -1 < x^2 - 1 \leq 1 \Rightarrow 0 < x^2 \leq 2$

$\Rightarrow \quad x \in (-\sqrt{2}, \sqrt{2}) - \{0\}.$

12. $f(x) = \cos^{-1}\left(\dfrac{2-|x|}{4}\right) + \dfrac{1}{\log_e(3-x)}$

$-1 \leq \dfrac{2-|x|}{4} \leq 1$ and $3 - x > 0$ and $3 - x \neq 1$

By solving the above inequality, we get

$$x \in [6, 2) \cup (2, 3]$$

13. Putting $x = f(y) = 0$, then $f(0) = f(0) + 0 + f(0) - 1 \Rightarrow f(0) = 1$

Putting $x = f(y)$

We get, $f(0) = f(x) + x^2 + f(x) - 1 \Rightarrow f(x) = 1 - \dfrac{x^2}{2}$

$$f(16) = 1 - \frac{256}{2} = 1 - 128 = -127$$

$$|f(16)| = 127$$

14. If $f(x) = x^3 + \frac{1}{x^3} - 4\left(x^2 + \frac{1}{x^2}\right) + 13$

$$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) - 4\left[\left(x + \frac{1}{x}\right)^2 - 2\right] + 13$$

$$= \left(x + \frac{1}{x}\right)^3 - 4\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) + 21$$

Then, $f(2 + \sqrt{3}) = 4^3 - 4 \times 4^2 - 3(4) + 21 = -12 + 21 = 9$

15. Let $f(x) = ax + \cos x$

$f'(x) = a - \sin x \Rightarrow \quad a \in (-\infty, -1] \cup [1, \infty)$

16. $f(x) = \sqrt{1 - x^2}$

Hence, options are obvious from the above diagram.

17. (a) $f(x) = \frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$; non-periodic

(b) $f(x) = \sin \pi x + \{x\}$

Period of $\sin \pi x$ is 2

Period of $\sin \pi \{x\}$ is 1

Hence, f is periodic

(c) $f(x) = \cos x + \tan \pi x$

Period of $\cos x$ is 2π and period of $\tan \pi x$ is 1

$\Rightarrow$ f is non-periodic.

(d) Similarly, $f(x) = \max\left\{\sin x, \frac{1}{2}\right\}$

$$= \frac{2 \sin x + 1}{4} + \frac{|2 \sin x - 1|}{4} \text{ is periodic.}$$

18. Since, the period of $e^{x - [x]}$ is 1 and period of $\cos ax$ is $\frac{2\pi}{a}$.

$\therefore$ Period of $f(x) = \text{LCM of } \left\{1, \frac{2\pi}{a}\right\}$

i.e., period of $f(x)$ is possible only when a is irrational of type π.

(a) If $a = \pi$

Then, period of $f(x) = \text{LCM of } \{1, 2\} = 2$

(b) If $a = 2\pi$

Then, period of $f(x) = \text{LCM of } \{1, 1\} = 1$

(c) If $a = 3\pi$

Then, period of $f(x) = \text{LCM of } \left\{1, \frac{2}{3}\right\}$

$$= \text{LCM of } \left\{\frac{2}{2}, \frac{2}{3}\right\} = \frac{2}{1} = 2$$

19. Put $x = 0 = y$ in given functional equation

We get, $f(0) = 1$

Put $x = 0$ in the given functional equation, we get $f(-y) = f(y)$

20. Put $x = 1, y = 1$ in $f(x + y) + f(x - y) = 2f(x)\, f(y)$

$f(2) + f(0) = 2f^2(1) \Rightarrow \quad f(2) + 2k^2 - 1$

21. $f(3) = 4f^3(1) - 3f(1) = 4k^3 - 3k$

Now, $f(3) - 2kf(2) = 4k^3 - 3k - 2k[2k^2 - 1]$

$= 4k^3 - 3k - 4k^3 + 2k = -k$

22. Graph of $f(x)$ is symmetric about the line $x = 0$ if $f(-x) = f(x)$, *i.e.*, if $f(0 - x) = f(0 + x)$

$\therefore$ Graph of $y = f(x)$ is symmetric about $x = 1$, if

$f(1 + x) = f(1 - x)$.

Hence, (a) is the correct answer.

23. $f(1 + x) = f(1 - x) \Rightarrow f(2 - x) = f(x)$ ...(i)

and $f(4 + x) = f(4 - x)$

$\Rightarrow$ $f(8 - x) = f(x)$ ...(ii)

From Eqs. (i) and (ii), we have

$f(8 - x) = f(2 - x)$ replace x by $2 - x$

$\Rightarrow$ $f(8 - (2 - x)) = f(2 - (2 - x))$

$\Rightarrow$ $f(6 + x) = f(x)$

$\Rightarrow$ f is periodic but 6 need not be the fundamental periodic.

24. **Assertion** $f : A \to B,\ g : B \to C$ are two functions, then $(gof)^{-1} \neq f^{-1}og^{-1}$ (since, function need not posses inverses.

Reason Bijective functions are invertibles.)

25. Let $|x| = \alpha, \quad \alpha \geq 0$

$\alpha(\alpha + 2) + 3 = \alpha^2 + 2\alpha + 3 = (\alpha + 1)^2 + 2 > 2$

$\Rightarrow$ $(\alpha + 1)^2 + 2 \geq 3 \Rightarrow f(x) \in [3, \infty)$.

26. For $\sin^{-1} \frac{x - 3}{2}$ to be defined, we must have $-1 \leq \frac{x - 3}{2} \leq 1$.

$\log_{10}(4 - x)$ is defined for $4 - x > 0$

27. Since, $(fog)x = f\{g(x)\} = f(x^2) = \sin x^2$

and $(gof)x = g\{f(x)\} = g(\sin x) = \sin^2 x$

$\therefore$ $fog \neq gof$

28. **Statement I.** (a) ***R* is reflexive**

Let $a \in I$, then $a - a = 0$ which is divisible by n.

$\therefore$ $aRa, \forall\, a \in I$

(b) ***R* is symmetric**

Let $a, b \in I$

$\therefore$ $a - b = nk, k \in I$

$\Rightarrow$ $(b - a) = -kn$, where $-k \in I$

i.e., $aRb = bRa$

(c) ***R* is transitive**

Now, $aRb \Rightarrow a - b = k_1 n$ and bRc

$\Rightarrow$ $b - c = k_2 n, k_1, k_2 \in I$

Now, $a - c = (a - b) + (b - c) = k_1 n + k_2 n = (k_1 + k_2)n$

Statement II. As $R \cap R'$ are not disjoint, there is atleast one ordered pair, say (a, b) in $R \cup R'$ but $(a, b) \in R \cap R' \Rightarrow (a, b) \in R$ and $(a, b) \in R$.

Since, R and R' are symmetric.

Hence, $R \cap R'$ is symmetric.

Hence, both statements are true but statement II is not a correct explanation of statement I.

29. Here, $f(x) = \begin{cases} -3x + 10, & \forall\, x \le 2 \\ -x + 6, & \forall\, 2 < x \le 3 \\ x, & \forall\, 3 < x \le 5 \\ 3x - 10, & \forall\, x > 5 \end{cases}$

$\therefore \quad f(x) = x, \forall\, 3 < x < 5 \Rightarrow f(-x) = -x = -f(x)$

30. $A = \sin^2 x + \cos^4 x$

$\Rightarrow \quad A = 1 - \cos^2 x + \cos^4 x$

$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$

$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$ ...(i)

where, $0 \le \left(\cos^2 x - \frac{1}{2}\right)^2 \le \frac{1}{4}$ ...(ii)

$\therefore \quad \frac{3}{4} \le A \le 1$

31. Since, the relation R is defined as $R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$.

(i) **Reflexive** $xRx \Rightarrow x = wx$

$\therefore \quad w = 1 \in$ Rational number

So, the relation R is reflexive

(ii) **Symmetric** $xRy \not\Rightarrow yRx$ as $0R1 \Rightarrow 0 \cdot (1)$

but $1R0 \Rightarrow 1 = w \cdot (0)$

which is not true for any rational number.

So, the relation R is not symmetric.

Thus, R is not equivalence relation.

Now, for the relation S is defined as

$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \in \text{integers such that } n, q \ne 0 \text{ and } qm = pn\right\}$

(i) **Reflexive** $\frac{m}{n} R \frac{m}{n} \Rightarrow mn = mn$ (true)

Hence, the relation S is reflexive.

(ii) **Symmetric** $\frac{m}{n} R \frac{p}{q} \Rightarrow mq = np$

$\Rightarrow np = mq \Rightarrow \frac{p}{q} R \frac{m}{n}$

Hence, the relation S is symmetric.

(iii) **Transitive** $\frac{m}{n} R \frac{p}{q}$ and $\frac{p}{q} R \frac{r}{s}$

$mq = np$ and $ps = rq$

$\Rightarrow \quad mq \cdot ps = np \cdot rq \Rightarrow ms = nr$

$\Rightarrow \quad \frac{m}{n} = \frac{r}{s} \Rightarrow \frac{m}{n} R \frac{r}{s}$

So, the relation S is transitive.

Hence, the relation S is equivalence relation.

32. Given, $f(x) = x^3 + 5x + 1$

Now, $f'(x) = 3x^2 + 5 > 0, \forall\, x \in R$

$\therefore$ $f(x)$ is strictly increasing function.

Hence, $f(x)$ is one one function.

Clearly, $f(x)$ is a continuous function and also increasing on R.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} = \infty$

$\therefore f(x)$ takes every value between $-\infty$ and ∞.

Thus, $f(x)$ is onto function.

33. Since, $(1, 2) \in S$ but $(2, 1) \notin S$

$\therefore S$ is not symmetric.

Hence, S is not an equivalence relation.

Given, $T = \{(x, y) : (x - y) \in I\}$

Now, $x - x = 0 \in I$, it is reflexive relation.

Again, now $(x - y) \in I$

$\Rightarrow y - x \in I$, it is symmetric relation.

Let $x - y = I_1$ and $y - z = I_2$

Now, $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$

$\therefore T$ is also transitive.

Hence, T is an equivalence relation.

34. $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$

Since, 4^{-x^2} is defined for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$\cos^{-1}\left(\frac{x}{2} - 1\right)$ is defined, if $-1 \le \frac{x}{2} = 1 \le 1$

$\Rightarrow \quad 0 \le x \le 4$

and $\log(\cos x)$ is defined, if $\cos x > 0$

$\Rightarrow \quad x \in \left[0, \frac{\pi}{2}\right)$

35. Let $W = \{\text{CAT, TOY, YOU, } \ldots\}$

Clearly, R is reflexive and symmetric but not transitive.

(since, ${}_{CAT}R_{TOY}, {}_{TOY}R_{YOU} \not\Rightarrow {}_{CAT}R_{YOU}$)

36. Since, for every element of A, there exist a element $(3, 3), (6, 6), (9, 9), (12, 12) \in R \Rightarrow R$ is reflexive relation.

Now, $(6, 12) \in R$ but $(12, 6) \notin R$. So, it is not a symmetric relation.

Also, $(3, 6), (6, 12) \in R \Rightarrow (3, 12) \in R \Rightarrow R$ is transitive.

25 Inverse Trigonometric Functions

JEE Main MILESTONE

- Inverse Trigonometric Function
- Graphs of Inverse Trigonometric Functions
- Domain and Range of Inverse Trigonometric Functions
- Properties of Inverse Trigonometric Functions

25.1 Inverse Trigonometric Function

If a function is one-one and onto from A to B, then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x)$, then g is called the inverse function of f, denoted by $x = g(y)$.

$\because$ $$g = f^{-1} \therefore x = f^{-1}(y)$$

Consider the sine function with domain R and range $[-1, 1]$. It is clear that this is not a one-one and onto (bijection), so it is not invertible. So, we have to restrict the domain of it in such a way that it becomes one-one, then it would become invertible.

Suppose we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and codomain $[-1, 1]$, then it is a bijection and therefore, invertible, such types of trigonometric functions are called inverse trigonometric functions.

In Mathematics, the inverse trigonometric functions (occasionally as cyclometric functions) are the inverse functions of the trigonometric functions with suitably restricted domains. They are used for computing the angle, from any of its trigonometric ratios. These functions have a wide range of use in Navigation, Physics, Engineering etc.

Principal Value of Inverse Trigonometric Functions

Since, sine is a many valued function, so for defining the inverse function, we restrict as one-one onto function *i.e.,* we consider domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is called the principal value of $\sin^{-1}$ function.

Principal Value of Inverse Trigonometric Functions

S. No.	Functions	Interval of principal value
1.	$\sin^{-1} x$	$-\pi/2 \le y \le \pi/2$, where $y = \sin^{-1} x$
2.	$\cos^{-1} x$	$0 \le y \le \pi$, where $y = \cos^{-1} x$
3.	$\tan^{-1} x$	$-\pi/2 < y < \pi/2$, where $y = \tan^{-1} x$
4.	$\text{cosec}^{-1} x$	$-\pi/2 \le y \le \pi/2$, where $y = \text{cosec}^{-1} x$, $y \ne 0$
5.	$\sec^{-1} x$	$0 \le y \le \pi$, where $y = \sec^{-1} x$, $y \ne \pi/2$
6.	$\cot^{-1} x$	$0 < y < \pi$, where $y = \cot^{-1} x$

Note If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of the function.

***Sample* Problem 1** $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ *is equal to* [NCERT]

(a) π (b) $-\frac{\pi}{3}$

(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

Interpret (b) Let $\tan^{-1}\sqrt{3} = x \Rightarrow \tan x = \sqrt{3} \Rightarrow \tan x = \tan\frac{\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (principal interval)

Let $\sec^{-1}(-2) = y$

$\Rightarrow \quad \sec y = -2$

$\Rightarrow \quad \sec y = -\sec\frac{\pi}{3}$

$\Rightarrow \quad \sec y = \sec\left(\pi - \frac{\pi}{3}\right)$ [$\because \sec(\pi - \theta = -\sec\theta)$]

$\Rightarrow \quad \sec y = \sec\left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad y = \frac{2\pi}{3} \in [0, \pi] - \left(\frac{\pi}{2}\right)$ (principal interval)

$\therefore \tan^{-1}\sqrt{3} - \sec^{-1}(-2) = x - y = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

***Sample* Problem 2** $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ *is equal to* [NCERT]

(a) $3\pi/4$ (b) $\pi/4$

(c) $\pi/6$ (d) $-3\pi/4$

Interpret (a) Let $\tan^{-1}(1) = x \Rightarrow \tan x = 1 = \tan\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$

where principal value $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \tan^{-1}(1) = \frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$

$\Rightarrow \quad \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$

[$\because \cos(\pi - \theta) = -\cos\theta$]

$\Rightarrow y = \frac{2\pi}{3}$, where principal value $y \in [0, \pi]$

$\therefore \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z \Rightarrow \sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow z = -\frac{\pi}{6}$, where principal value $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \quad \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= x + y + z = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$

$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

25.2 Graphs of Inverse Trigonometric Functions

(i) $\sin\theta = x \Rightarrow \theta = \sin^{-1} x$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in [-1, 1]$

Note $\sin^{-1} x$ is an increasing function in $[-1, 1]$.

(ii) $\operatorname{cosec}\theta = x \Rightarrow \theta = \operatorname{cosec}^{-1} x$, where

$\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

and $x \in (-\infty, -1] \cup [1, \infty)$

Note $\operatorname{cosec}^{-1} x$ is a decreasing function in $(-\infty, -1)$. It also decreases in $[1, \infty)$.

(iii) $\cos\theta = x \Rightarrow \theta = \cos^{-1} x$, where $\theta \in [0, \pi]$ and $x \in [-1, 1]$

Note $\cos^{-1} x$ is a decreasing function in $[-1, 1]$.

(iv) $\sec\theta = x \Rightarrow \theta = \sec^{-1} x$, where $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and

$$x \in (-\infty, -1] \cup [1, \infty)$$

Note $\sec^{-1} x$ is an increasing function in $(-\infty, -1]$. It also increases in $[1, \infty)$.

(v) $\tan\theta = x \Rightarrow \theta = \tan^{-1} x$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x \in (-\infty, \infty)$

Note $\tan^{-1} x$ is an increasing function in R.

(vi) $\cot\theta = x \Rightarrow \theta = \cot^{-1} x$, where $\theta \in (0, \pi)$ and $x \in (-\infty, \infty)$

Note $\tan^{-1} x$ is a decreasing function in R.

25.3 Domain and Range of Inverse Trigonometric Functions

Suppose we defined a sine inverse trigonometric function as $\sin^{-1}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Here, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[-1, 1]$ are called domain and range of sine inverse trigonometric function.

Domain and Range of Inverse Trigonometric Functions

S. No.	Functions	Domain	Range
1.	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1} x$	R	$(-\pi/2, \pi/2)$
4.	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
6.	$\cot^{-1} x$	R	$(0, \pi)$

***Sample* Problem 3** *Range of* $\tan^{-1}\left(\frac{2x}{1+x^2}\right)$ *is*

(a) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$ (d) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Interpret (a) $(1 + x^2) \geq 2|x| \Rightarrow \frac{2|x|}{1+x^2} \leq 1 \Rightarrow -1 \leq \frac{2x}{1+x^2} \leq 1$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

***Sample* Problem 4** *If* $f(x) = \sec^{-1}x + \tan^{-1} x$, *then* $f(x)$ *is real for*

(a) $x \in [-1, 1]$ (b) $x \in [-1, 1]$

(c) $x \in (-\infty, -1] \cup [1, -\infty)$ (d) $x \in (-\infty, -1) \cup (1, \infty]$

Interpret (c) Given, $f(x) = \sec^{-1} + \tan^{-1} x$

Here, domain of $\sec^{-1}x = R$ and

domain of $\tan^{-1} x = (-\infty, -1] \cup [1, \infty]$

$\therefore$ $D\{f(x)\} = D(\sec^{-1}x) \cap D(\tan^{-1} x)$

$= \{(-\infty, -1] \cup [1, \infty)\} \cap R$

$= (-\infty, -1) \cup [1, \infty)$

25.4 Properties of Inverse Trigonometric Functions

Self Adjusting Property

(i) $\sin^{-1} (\sin \theta) = \theta; \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1} (\cos \theta) = \theta; \forall \theta \in [\theta, \pi]$

(iii) $\tan^{-1} (\tan \theta) = \theta; \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv) $\text{cosec}^{-1} (\text{cosec}\ \theta) = \theta; \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \neq 0$

(v) $\sec^{-1} (\sec \theta) = \theta; \forall\ \theta \in [0, \pi], \theta \neq \frac{\pi}{2}$

(vi) $\cot^{-1} (\cot \theta) = \theta; \forall\ \theta \in (0, \pi)$

(vii) $\sin (\sin^{-1} x) = x, \forall x \in [-1, 1]$

(viii) $\cos (\cos^{-1} x) = x, \forall x \in [-1, 1]$

(ix) $\tan (\tan^{-1} x) = x, \forall x \in R$

(x) $\text{cosec} (\text{cosec}^{-1} x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(xi) $\sec (\sec^{-1} x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(xii) $\cot (\cot^{-1} x) = x, \forall x \in R$

***Sample* Problem 5** *The value of* $\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$ *is* [NCERT]

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) *None of these*

Interpret (b) $\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$

$= \tan^{-1}\left[2 \cos\left\{2 \sin^{-1}\left(\sin \frac{\pi}{6}\right)\right\}\right]$ $\left(\because \sin \frac{\pi}{6} = \frac{1}{2}\right)$

$= \tan^{-1}\left[2 \cos\left(2 \times \frac{\pi}{6}\right)\right] = \tan^{-1}\left[2 \cos \frac{\pi}{3}\right]$

$= \tan^{-1}\left[2 \times \frac{1}{2}\right] = \tan^{-1} (1)$ $\left(\because \cos \frac{\pi}{3} = \frac{1}{2}\right)$

$= \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4}$ $\left(\because \tan \frac{\pi}{4} = 1\right)$

***Sample* Problem 6** *The value of* $\sin^{-1} (\sin 5)$ *is*

(a) 5 (b) $5 - 2\pi$

(c) $5 - \pi$ (d) *None of these*

Interpret (b) Here, $\theta = 5$ rad

Clearly, 5 rad does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But $2\pi - 5$ and $5 - 2\pi$ both lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that

$\sin (5 - 2\pi) = \sin [-(2\pi - 5)] = -\sin (-5) = \sin 5$

$\therefore$ $\sin^{-1} (\sin 5) = \sin^{-1} [\sin (5 - 2\pi)] = 5 - 2\pi$

Negative Arguments

(i) $\sin^{-1} (-x) = -\sin^{-1} (x), \forall x \in [-1, 1]$

(ii) $\cos^{-1} (-x) = \pi - \cos^{-1} (x), \forall x \in [-1, 1]$

(iii) $\tan^{-1} (-x) = -\tan^{-1} x, \forall x \in R$

(iv) $\text{cosec}^{-1} (-x) = -\text{cosec}^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1} (-x) = \pi - \sec^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1} (-x) = \pi - \cot^{-1} x, \forall x \in R$

***Sample* Problem 7** $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ *is equal to* [NCERT]

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) 1

Interpret (d) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\sin \frac{\pi}{6}\right)\right]$ $\left(\because \sin \frac{\pi}{6} = \frac{1}{2}\right)$

$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left\{\sin\left(-\frac{\pi}{6}\right)\right\}\right]$ $[\because \sin (-x) = -\sin x]$

$= \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin \frac{\pi}{2} = 1$

Reciprocal Arguments

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \forall x > 0 \\ -\pi + \cot^{-1} x, & \forall\ x < 0 \end{cases}$

Inverse Sum Identities

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \forall x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

Conversion of One Inverse to Another Inverse Trigonometric Function

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(\sqrt{1+x^2})$

$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

***Sample* Problem 8** *If $\cos^{-1} x - \sin^{-1} x = 0$, then the value of x is*

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) 1 (d) *None of these*

Interpret (b) $\cos^{-1} x - \sin^{-1} x = 0$

$\Rightarrow \quad \frac{\pi}{2} - 2\sin^{-1} x = 0 \quad \left(\because \cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{\pi}{4} = \sin^{-1} x \Rightarrow x = \sin\left(\frac{\pi}{4}\right) \Rightarrow x = \frac{1}{\sqrt{2}}$

Hot Spot

Sum and Difference of Inverse Trigonometric Function

In this type of problem, we find the number of solution of a variable included in the given condition. When we find the number of solution from given condition, we use the inverse trigonometric formulae at initial and intermediate stage. At last we get the desired solution.

These types of problems were directly or indirectly asked in various years in AIEEE (JEE Main) 2008, 2005, 2002.

(i) $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \text{ or if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y < 0, \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \text{ or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Note

- If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$, then
$$x^2 + y^2 + z^2 + 2xyz = 1$$
- If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then
$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

(iii) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(iv) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

Note

- If $\sin^{-1} x + \sin^{-1} y = \theta$, then $\cos^{-1} x + \cos^{-1} y = \pi - \theta$
- If $\cos^{-1} x + \cos^{-1} y = \theta$, then $\sin^{-1} x + \sin^{-1} y = \pi - \theta$

(v) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(vi) $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

(vii) If $x_1, x_2, x_3, \ldots, x_n \in R$, then

$$\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right)$$

where, $S_1 = x_1 + x_2 + \ldots + x_n = \Sigma x_1$

$S_2 = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n = \Sigma x_1x_2$

$S_3 = \Sigma x_1x_2x_3 \ldots$ and so on.

Note

- If $\tan^{-1} x + \tan^{-1} y = \dfrac{\pi}{2}$, then $xy = 1$
- If $\cot^{-1} x + \cot^{-1} y = \dfrac{\pi}{2}$, then $xy = 1$

Step I Firstly, we convert the given function in standard form of inverse trigonometric function.

Step II Check their domains in respective intervals.

Step III Use the appropriate formula for sum and difference of inverse trigonometric function.

Step IV Simplify it and get the result.

***Sample* Problem 9** *Find the solution of* $\tan^{-1}\dfrac{x+1}{x-1} + \cot^{-1}\dfrac{x}{x-1} = \tan^{-1}(-7)$.

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) –2

Interpret (*b*) **Step I** Firstly convert the function into standard form.

i.e., $$\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$$

Step II Given function is defined in their domain.

Step III Using the formula sum of inverse trigonometric function.

$$\therefore \quad \tan^{-1}\left[\frac{\dfrac{x+1}{x-1} + \dfrac{x-1}{x}}{1 - \dfrac{(x+1)}{(x-1)} \times \dfrac{x-1}{x}}\right] = \tan^{-1}(-7)$$

Step IV Simplify $\tan^{-1}\left[\dfrac{\dfrac{x^2 + x + x^2 + 1 - 2x}{x(x-1)}}{1 - \dfrac{(x+1)}{x}}\right] = \tan^{-1}(-7)$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2x^2 - x + 1}{-(x-1)}\right] = \tan^{-1}(-7)$$

$$\Rightarrow \quad -\frac{(2x^2 - x + 1)}{x-1} = -7$$

$$\Rightarrow \quad 2x^2 - x + 1 = 7x - 7 \Rightarrow 2(x^2 - 4x + 4) = 0$$

$$\Rightarrow \quad 2(x-2)^2 = 0 \Rightarrow x = 2$$

***Sample* Problem 10** *If* $\sin^{-1}\dfrac{1}{3} + \sin^{-1}\dfrac{2}{3} = \sin^{-1} x$, *then the value of x is*

(*a*) $\dfrac{\sqrt{5}+\sqrt{2}}{9}$ (*b*) $\dfrac{\sqrt{5}+4\sqrt{2}}{9}$

(*c*) $\dfrac{\sqrt{5}+2\sqrt{2}}{9}$ (*d*) *None of these*

Interpret (*b*) $\sin^{-1}\dfrac{1}{3} + \sin^{-1}\dfrac{2}{3} = \sin^{-1} x$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right) = \sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\cdot\frac{\sqrt{5}}{3} + \frac{2}{3}\cdot\frac{\sqrt{8}}{3}\right) = \sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right) = \sin^{-1} x$$

$$\Rightarrow \quad x = \frac{\sqrt{5}+4\sqrt{2}}{9}$$

***Sample* Problem 11** *The simplified form of* $\tan\left(\sin^{-1}\dfrac{3}{5} + \cot^{-1}\dfrac{3}{2}\right)$ *is* **[NCERT]**

(*a*) $\dfrac{17}{6}$ (*b*) $\dfrac{17}{5}$

(*c*) $\dfrac{17}{4}$ (*d*) *None of these*

Interpret (*a*) $\tan\left(\sin^{-1}\dfrac{3}{5} + \cot^{-1}\dfrac{3}{2}\right)$

$$= \tan\left[\tan^{-1}\frac{\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}} + \tan^{-1}\frac{2}{3}\right]$$

$$\left(\because \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} \text{ and } \cot^{-1}\frac{x}{y} = \tan^{-1}\frac{y}{x}\right)$$

$$= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{\left(\frac{3}{4}+\frac{2}{3}\right)}{1-\frac{3}{4}\times\frac{2}{3}}\right] = \tan\left[\tan^{-1}\frac{\frac{17}{12}}{\frac{1}{2}}\right]$$

$$\left[\because \tan^{-1}x = \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{17}{6}\right] = \frac{17}{6}$$

Inverse Trigonometric Ratios of Multiple Angles

(i) $2\sin^{-1}x$

$$= \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), & \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$$

(iii) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1), & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), & \text{if } -1 \le x \le 0 \end{cases}$$

(iv) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), & \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), & \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

(v) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1 < x \le 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

(vi) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

(vii) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

(viii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x < 0 \end{cases}$$

Sample **Problem 12** *The simplified form of* $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$, $a > 0$; $\frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$ *is* **[NCERT]**

(*a*) $3\tan^{-1}\left(\frac{x}{a}\right)$

(*b*) $2\tan^{-1}\left(\frac{x}{a}\right)$

(*c*) $3\tan^{-1}\left(\frac{a}{x}\right)$

(*d*) *None of the above*

Interpret (*a*) Let $x = a\tan\theta$

$$\Rightarrow \quad \frac{x}{a} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left[\frac{3a^2(a\tan\theta)-(a\tan\theta)^3}{a^3-3a(a\tan\theta)^2}\right]$$

$$= \tan^{-1}\left[\frac{a^3(3\tan\theta-\tan^3\theta)}{a^3(1-3\tan^2\theta)}\right]$$

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$\left(\because \tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right) \quad \text{[from Eq. (i)]}$$

WORKED OUT

Examples

Example 1 *The value of* $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ *is equal to* **[NCERT Exemplar]**

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{3\pi}{2}$

(*c*) $\frac{5\pi}{2}$ (*d*) $\frac{7\pi}{2}$

Solution (*a*) $\cos^{-1}\left(\cos\frac{3\pi}{2}\right) = \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{2}\right)\right)$

$$= \cos^{-1}\left(-\cos\frac{\pi}{2}\right) = \pi - \cos^{-1}\left(\cos\frac{\pi}{2}\right)$$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Example 2 *If* $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1} x\right) = 0$, *then x is equal to* **[NCERT Exemplar]**

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$ (*c*) 0 (*d*) 1

Solution (*b*) Given, $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1} x\right) = 0$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1} x = \cos^{-1} 0$$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1} x = \frac{\pi}{2}$$

Above equation becomes an identity, if

$$x = \frac{2}{5} \qquad \left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$$

Example 3 *The value of* $\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ *is*

(*a*) 1 (*b*) 2

(*c*) 0 (*d*) *None of these*

Solution (*a*) $\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

$$= \sin\left[-\tan^{-1}(\sqrt{3}) + \pi - \cos^{-1}\frac{\sqrt{3}}{2}\right]$$

$$[\because \tan^{-1}(-x) = -\tan^{-1} x \text{ and } \cos^{-1}(-x) = \pi - \cos^{-1} x]$$

$$= \sin\left[-\frac{\pi}{3} + \pi - \frac{\pi}{6}\right] = \sin\frac{\pi}{2} = 1$$

Example 4 *If* $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = 4$, *then x is equal to*

(*a*) tan 2 (*b*) tan 4

(*c*) tan (1/4) (*d*) tan 8

Solution (*d*) Put $x = \tan\theta$, we get

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1} x$$

$$\therefore \quad \frac{1}{2}\tan^{-1} x = 4$$

$$\Rightarrow \quad \tan^{-1} x = 8$$

$$\Rightarrow \quad x = \tan 8$$

Example 5 $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$, *then x is equal to* **[NCERT]**

(*a*) $0, \frac{1}{2}$ (*b*) $1, \frac{1}{2}$

(*c*) *zero* (*d*) $\frac{1}{2}$

Solution (*c*) Given, $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

$$\Rightarrow \quad -2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1}(1-x) \Rightarrow -2\sin^{-1} x = \cos^{-1}(1-x)$$

$$\left[\because \sin^{-1}(1-x) + \cos^{-1}(1-x) = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cos(-2\sin^{-1} x) = 1 - x \quad \text{(multiplying both sides by } \cos x)$$

$$\Rightarrow \quad \cos(2\sin^{-1} x) = 1 - x \qquad [\because \cos(-x) = +\cos x]$$

$$\Rightarrow \quad [1 - 2\sin^2(\sin^{-1} x)] = 1 - x \qquad (\because \cos 2x = 1 - 2\sin^2 x)$$

$$\Rightarrow \quad 1 - 2[\sin(\sin^{-1} x)]^2 = 1 - x \qquad [\because \sin^2 x = (\sin x)^2]$$

$$\Rightarrow \quad 1 - 2x^2 = 1 - x \Rightarrow 2x^2 - x = 0$$

$$\Rightarrow \quad x(2x-1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } 2x - 1 = 0 \Rightarrow x = 0 \text{ or } \frac{1}{2}$$

But $x = \frac{1}{2}$ does not satisfy the given equation, so $x = 0$.

Example 6 *The value of*

$$\left[\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}\right] \text{ is}$$

[NCERT Exemplar]

(a) $\frac{b}{a}$ (b) $\frac{2}{a}$

(c) $\frac{2b}{a}$ (d) *None of these*

Solution (c) Let $\cos^{-1}\left(\frac{a}{b}\right)=\theta \Rightarrow \cos\theta=\frac{a}{b}$

$$\therefore \tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$$

$$=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$=\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}+\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}$$

$$=\frac{\left[1+\tan\left(\frac{\theta}{2}\right)\right]^2+\left[1-\tan\left(\frac{\theta}{2}\right)\right]^2}{1-\tan^2\left(\frac{\theta}{2}\right)}$$

$$=2\cdot\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}=2\cdot\frac{\cos^2\left(\frac{\theta}{2}\right)+\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}$$

$$=2\cdot\frac{1}{\cos\theta}=2\cdot\frac{1}{\frac{a}{b}}=\frac{2b}{a}$$

Example 7 *The value of* $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)$ *is*

(a) $\tan^{-1}x$ (b) $\tan x$

(c) $\cot x$ (d) $\operatorname{cosec}^{-1}x$

Solution (a) $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)$

$$=2\tan^{-1}\left[\operatorname{cosec}\left\{\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}\right\}-\tan\left\{\tan^{-1}\left(\frac{1}{x}\right)\right\}\right]$$

$$=2\tan^{-1}\left[\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] \qquad (\text{put } x=\tan\theta)$$

$$=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right]$$

$$=2\tan^{-1}\tan\frac{\theta}{2}=2\cdot\frac{\theta}{2}=\theta=\tan^{-1}x$$

Example 8 *In* ΔABC, $\angle C=\frac{\pi}{2}$

and $$\sin^{-1}x=\sin^{-1}\left(\frac{ax}{c}\right)+\sin^{-1}\left(\frac{bx}{c}\right),$$

where a, b *and* c *are the sides of triangle, then the total number of different values of* x *is*

(a) 0 (b) 1

(c) 2 (d) 3

Solution (d) $\angle C=\frac{\pi}{2} \Rightarrow a^2+b^2=c^2$

Clearly, $|x|\le 1$, also

$$\frac{a^2x^2}{c^2}+\frac{b^2x^2}{c^2}=x^2\le 1$$

Now, $$\sin^{-1}x=\sin^{-1}\left(\frac{ax}{c}\right)+\sin^{-1}\left(\frac{bx}{c}\right)$$

$$\Rightarrow \sin^{-1}x=\sin^{-1}\left(\frac{ax}{c}\sqrt{1-\frac{b^2x^2}{c^2}}+\frac{bx}{c}\sqrt{1-\frac{a^2x^2}{c^2}}\right)$$

$$\Rightarrow x=\frac{ax}{c^2}\sqrt{c^2-b^2x^2}+\frac{bx}{c^2}\sqrt{c^2-a^2x^2}$$

$$\Rightarrow x=0$$

$$\Rightarrow c^2=a\sqrt{c^2-b^2x^2}+b\sqrt{c^2-a^2x^2}$$

$$\Rightarrow c^4=a^2(c^2-b^2x^2)+b^2(c^2-a^2x^2)+2ab\sqrt{c^2-b^2x^2}\sqrt{c^2-a^2x^2}$$

$$\Rightarrow c^4=(a^2+b^2)c^2-2a^2b^2x^2+2ab\sqrt{(c^2-b^2x^2)(c^2-a^2x^2)}$$

$$\Rightarrow abx^2=\sqrt{c^4-c^2(a^2+b^2)x^2+a^2b^2x^4} \qquad (\because a^2+b^2=c^2)$$

$$\Rightarrow a^2b^2x^4=c^4-c^4x^2+a^2b^2x^4$$

$$\Rightarrow x^2=1$$

$$\Rightarrow x=1,-1$$

Total number of different values of x are three namely

$$x=0,\ 1,\ -1$$

Example 9 *The value of* $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$ *is*

(a) $\pi-x$ (b) $\pi-\frac{x}{2}$

(c) $\pi-\frac{x}{3}$ (d) *None of these*

Solution (b) $$\cot^{-1}\left[\frac{\left(\cos\frac{1}{2}x-\sin\frac{1}{2}x\right)+\left(\cos\frac{1}{2}x+\sin\frac{1}{2}x\right)}{\left(\cos\frac{1}{2}x-\sin\frac{1}{2}x\right)-\left(\cos\frac{1}{2}x+\sin\frac{1}{2}x\right)}\right]$$

$$=\cot^{-1}\left(-\cot\frac{1}{2}x\right)$$

$$=\cot^{-1}\cot\left(\pi-\frac{1}{2}x\right)=\pi-\frac{1}{2}x$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Domain and Range

1. The principal value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is [NCERT Exemplar]

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{5}$
(c) $\frac{\pi}{10}$ (d) $-\frac{\pi}{10}$

2. Which one of the following is correct? [NCERT Exemplar]

(a) $\tan 1 > \tan^{-1} 1$ (b) $\tan 1 < \tan^{-1} 1$
(c) $\tan 1 = \tan^{-1} 1$ (d) None of these

Properties of Inverse Trigonometric Functions

3. If $\cos^{-1} x > \sin^{-1} x$, then [NCERT Exemplar]

(a) $x < 0$ (b) $-1 < x < 0$
(c) $0 \le x < \frac{1}{\sqrt{2}}$ (d) $-1 \le x < \frac{1}{\sqrt{2}}$

4. If $\theta = \tan^{-1} a$, $\phi = \tan^{-1} b$ and $ab = -1$, then $(\theta - \phi)$ is equal to

(a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) None of these

5. The principal value of $\sin^{-1}(\sin 10)$ is

(a) 10 (b) $10 - 3\pi$
(c) $3\pi - 10$ (d) None of these

6. If $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1} x$, then x is equal to

(a) 0
(b) 1
(c) -1
(d) None of the above

7. If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$, then $\sum_{i=1}^{20} x_i$ is equal to

(a) 20 (b) 10
(c) 0 (d) None of these

8. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is [NCERT Exemplar]

(a) $[1, 2]$ (b) $[-1, 1]$
(c) $[0, 1]$ (d) None of these

9. The domain of the function $\cos^{-1}(2x - 1)$ is [NCERT Exemplar]

(a) $[0, 1]$ (b) $[-1, 1]$ (c) $(-1, 1)$ (d) $[0, \pi]$

10. If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x is equal to [NCERT Exemplar]

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

11. The value of $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

12. The value of $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\}$ is

(a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

13. The value of the expression $2\sec^{-1} 2 + \sin^{-1}\left(\frac{1}{2}\right)$ is [NCERT Exemplar]

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$
(c) $\frac{7\pi}{6}$ (d) 1

14. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ is equal to [NCERT Exemplar]

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$

(c) $\frac{3\pi}{5}$ (d) π

15. The value of $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ is [NCERT Exemplar]

(a) $\frac{25}{24}$ (b) $\frac{25}{7}$

(c) $\frac{24}{25}$ (d) $\frac{7}{24}$

16. The value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ is [NCERT Exemplar]

(a) 2 (b) 1

(c) 0 (d) –1

17. The simplified form of $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$ is [NCERT Exemplar]

(a) 0 (b) –1

(c) 1 (d) 2

18. The value of

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right) \text{ is}$$

[NCERT Exemplar]

(a) $\frac{\pi}{5}$ (b) $\frac{3\pi}{5}$

(c) $-\frac{\pi}{12}$ (d) $\frac{\pi}{12}$

19. $\sin^{-1} x + \sin^{-1}\frac{1}{x} + \cos^{-1} x + \cos^{-1}\frac{1}{x}, x \notin \pm 1$ is equal to

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) None of these

20. The simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$, where $x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$ is [NCERT Exemplar]

(a) $\tan^{-1}\frac{4}{3} - x$

(b) $\tan^{-1}\frac{1}{3} - x$

(c) $\tan^{-1}\frac{4}{3} + x$

(d) None of the above

21. The value of x for which $\cos^{-1}(\cos 4) > 3x^2 - 4x$ is

(a) $\left(0, \frac{2 + \sqrt{6\pi - 8}}{3}\right)$

(b) $\left(\frac{2 - \sqrt{6\pi - 8}}{3}, 0\right)$

(c) $(-2, 2)$

(d) $\left(\frac{2 - \sqrt{6\pi - 8}}{3}, \frac{2 + \sqrt{6\pi - 8}}{3}\right)$

22. If $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$, then x is equal to

(a) $\pm\frac{5}{3}$ (b) $\pm\frac{\sqrt{5}}{3}$

(c) $\pm\frac{5}{\sqrt{3}}$ (d) None of these

23. If $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x \geq 0$, then the smallest interval in which θ lies, is given by

(a) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$ (b) $-\frac{\pi}{4} \leq \theta \leq 0$

(c) $0 \leq \theta \leq \frac{\pi}{4}$ (d) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

24. If $\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2}$ and $\tan^{-1} x - \tan^{-1} y = 0$, then $x^2 + xy + y^2$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{3}{2}$ (d) $\frac{1}{8}$

25. The solution of $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$ is

(a) $\frac{1}{6}$ (b) -1

(c) $\left(\frac{1}{6}, -1\right)$ (d) None of these

26. The value of $\sin\{2\tan^{-1}(.75)\}$ is equal to [NCERT Exemplar]

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin 1.5

27. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in]0, 1[$, then the value of x is

(a) 0 (b) $\frac{a}{2}$ (c) a (d) $\frac{2a}{1-a^2}$

28. If we consider only the principal values of the inverse trigonometric functions, then the value of $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right)$ is

(a) $\sqrt{\frac{29}{3}}$ (b) $\frac{29}{3}$ (c) $\sqrt{\frac{3}{29}}$ (d) $\frac{3}{29}$

29. If $\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)=\frac{\pi}{2}$, then x is equal to

(a) $\sqrt{ab}$ (b) $\sqrt{2ab}$ (c) $2ab$ (d) ab

30. If $\tan^{-1}\left(\frac{x-1}{x+2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$, then x is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}}$ (c) $\pm\sqrt{\frac{5}{2}}$ (d) $\pm\frac{1}{2}$

31. The value of the expression $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is

[NCERT Exemplar]

(a) $2+\sqrt{5}$ (b) $\sqrt{5}-2$ (c) $\frac{\sqrt{5}+2}{2}$ (d) $5+\sqrt{2}$

32. If $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then the value of

$$\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right) \text{ is}$$

(a) $\frac{x}{2}$ (b) $2x$

(c) $3x$ (d) x

33. The value of

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right) \text{ is}$$

(a) 0

(b) 1

(c) $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z$

(d) None of the above

34. If $\tan\theta+\tan\left(\frac{\pi}{3}+\theta\right)+\tan\left(-\frac{\pi}{3}+\theta\right)=a\tan 3\theta$, then a is equal to

(a) 1/3 (b) 1

(c) 3 (d) None of these

35. If $x^2+y^2+z^2=r^2$, then

$$\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right) \text{ is equal to}$$

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) None of these

36. If in a ΔABC, $\angle A=\tan^{-1}2$ and $\angle B=\tan^{-1}3$, then $\angle C$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) None of these

37. The value of the expression $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos(\tan^{-1}2\sqrt{2})$ is

[NCERT Exemplar]

(a) $\frac{13}{15}$ (b) $\frac{14}{15}$

(c) $\frac{11}{15}$ (d) None of these

38. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then value of x is

(a) a (b) b (c) $\frac{a+b}{1-ab}$ (d) $\frac{a-b}{1+ab}$

39. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$ is equal to

(a) $\frac{2a}{1+a^2}$ (b) $\frac{1-a^2}{1+a^2}$

(c) $\frac{2a}{1-a^2}$ (d) None of these

40. The solution of $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ is

[NCERT Exemplar]

(a) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ (b) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$

(c) $\frac{\pi}{2}+\frac{1}{2}\cos^{-1}x^2$ (d) None of these

41. If $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x)$, then the value of x is

[NCERT Exemplar]

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) None of these

42. $\cos^{-1}\{\cos 2\cot^{-1}(\sqrt{2}-1)\}$ is equal to

(a) $\sqrt{2}-1$ (b) $\frac{\pi}{4}$

(c) $\frac{3\pi}{4}$ (d) None of these

43. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots+\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

is equal to

[NCERT Exemplar]

(a) $\frac{a_1-a_n}{1+a_1a_n}$ (b) $\frac{a_n-a_1}{1+a_1a_n}$

(c) $\frac{a_n+a_1}{1-a_1a_n}$ (d) None of these

44. The value of $\cos(2\cos^{-1}x+\sin^{-1}x)$ at $x=\frac{1}{5}$ is

(a) 1 (b) 3 (c) 0 (d) $-\frac{2\sqrt{6}}{5}$

45. Solution of the equation $\cot^{-1} x + \sin^{-1} \frac{1}{\sqrt{5}} = \frac{\pi}{4}$ is

(a) $x = 3$ (b) $x = \frac{1}{\sqrt{5}}$

(c) $x = 0$ (d) None of these

46. If $3 \sin^{-1} \frac{2x}{1+x^2} - 4 \cos^{-1} \frac{1-x^2}{1+x^2}$ $+ 2 \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{3}$,

then value of x is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$

(c) 1 (d) None of these

47. The sum of the infinite series

$\cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \ldots$ is

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) None of these

48. If $\alpha = \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{1}{3}$ and $\beta = \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{1}{3}$, then

(a) $\alpha < \beta$ (b) $\alpha = \beta$

(c) $\alpha > \beta$ (d) None of these

49. If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then the value of $a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c^2)}$ will be

(a) $2abc$ (b) abc (c) $\frac{1}{2} abc$ (d) $\frac{1}{3} abc$

50. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

(a) $x^2 + y^2 = z^2$ (b) $x^2 + y^2 + z^2 = 0$

(c) $x^2 + y^2 + z^2 = 1 - 2xyz$ (d) None of these

51. If $\cos^{-1} \sqrt{p} + \cos^{-1} \sqrt{1-p} + \cos^{-1} \sqrt{1-q} = \frac{3\pi}{4}$, then the value of q is

(a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

52. If a, b and c be positive real numbers and the value of

$\theta = \tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}}$

$+ \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}}$, then $\tan \theta$ is equal to

(a) 0 (b) 1

(c) $\frac{a+b+c}{abc}$ (d) None of these

53. $\tan^{-1} \frac{c_1 x - y}{c_1 y + x} + \tan^{-1} \frac{c_2 - c_1}{1 + c_2 c_1} + \tan^{-1} \frac{c_3 - c_2}{1 + c_3 c_2}$

$+ \ldots + \tan^{-1} \frac{1}{c_n}$ is equal to

(a) $\tan^{-1} \left(\frac{y}{x}\right)$ (b) $\tan^{-1} (yx)$

(c) $\tan^{-1} \left(\frac{x}{y}\right)$ (d) $\tan^{-1} (x - y)$

54. $\sum_{m=1}^{n} \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2}\right)$ is equal to

(a) $\tan^{-1} \left(\frac{n^2 + n}{n^2 + n + 2}\right)$ (b) $\tan^{-1} \left(\frac{n^2 - n}{n^2 - n + 2}\right)$

(c) $\tan^{-1} \left(\frac{n^2 + n + 2}{n^2 + n}\right)$ (d) None of these

55. $\sin^{-1} [x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ is equal to

(a) $\sin^{-1} x + \sin^{-1} \sqrt{x}$ (b) $\sin^{-1} x - \sin^{-1} \sqrt{x}$

(c) $\sin^{-1} \sqrt{x} - \sin^{-1} x$ (d) None of these

56. The number of real solutions of $\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is

[NCERT Exemplar]

(a) 0 (b) 1 (c) 2 (d) ∞

57. If $\cos^{-1} \alpha + \cos^{-1} \beta + \cos^{-1} \gamma = 3\pi$, then $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$ is equal to

[NCERT Exemplar]

(a) 0 (b) 1 (c) 6 (d) 12

58. The number of real solutions of the equation $\sqrt{1 + \cos 2x} = \sqrt{2} \cos^{-1} (\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is

[NCERT Exemplar]

(a) 0 (b) 1 (c) 2 (d) infinite

59. $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$ is equal to

[NCERT Exemplar]

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Miscellaneous

60. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then the value of $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}}$ is

(a) 0 (b) 1 (c) 2 (d) 3

61. $\cos^{-1} \left\{\frac{1}{2} x^2 + \sqrt{1-x^2} \sqrt{1 - \frac{x^2}{4}}\right\} = \cos^{-1} \frac{x}{2} - \cos^{-1} x$

holds for

(a) $|x| \le 1$ (b) $x \in R$ (c) $0 \le x \le 1$ (d) $-1 \le x \le 0$

62. The value of $\sum_{r=0}^{\infty} \tan^{-1} \left(\frac{1}{1 + r + r^2}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{4}$

(c) $\frac{\pi}{4}$ (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. If $f(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\sqrt{1-x^2}\right), -\frac{1}{2} \le x \le 1$, then $f(x)$ is equal to

(a) $\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(x)$ (b) $\sin^{-1} x - \frac{\pi}{6}$

(c) $\sin^{-1} x + \frac{\pi}{6}$ (d) None of these

2. If $x \in (0, 1)$, then the value of $\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is equal to

(a) $-\frac{\pi}{2}$ (b) zero (c) $\frac{\pi}{2}$ (d) π

3. The greatest and the least values of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ are respectively

(a) $-\frac{\pi}{2}, \frac{\pi}{2}$ (b) $-\frac{\pi^3}{8}, \frac{\pi^3}{8}$

(c) $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ (d) None of these

4. Sum of infinite terms of the series

$\cot^{-1}\left(1^2 + \frac{3}{4}\right) + \cot^{-1}\left(2^2 + \frac{3}{4}\right) + \cot^{-1}\left(3^2 + \frac{3}{4}\right) + \ldots$ is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}(2)$

(c) $\tan^{-1} 3$ (d) None of these

5. If x_1, x_2, x_3 and x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$, then $\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4$ is equal to

(a) β (b) $\frac{\pi}{2} - \beta$ (c) $\pi - \beta$ (d) $-\beta$

6. If θ and ϕ are the roots of the equation $8x^2 + 22x + 5 = 0$, then

(a) both $\sin^{-1} \theta$ and $\sin^{-1} \phi$ are equal

(b) both $\sec^{-1} \theta$ and $\sec^{-1} \phi$ are real

(c) both $\tan^{-1} \theta$ and $\tan^{-1} \phi$ are real

(d) None of the above

7. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then

$$x^4 + y^4 + z^4 + 4x^2y^2z^2 = k(x^2y^2 + y^2z^2 + z^2x^2)$$

where, k is equal to

(a) 1 (b) 2

(c) 4 (d) None of these

8. The number of triplets (x, y, z) satisfying $\sin^{-1} x + \cos^{-1} y + \sin^{-1} z = 2\pi$, is

(a) 0 (b) 2 (c) 1 (d) infinite

9. The value of a for which $ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$ has a real solution, is

(a) $-\frac{2}{\pi}$ (b) $\frac{2}{\pi}$

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

10. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$ and $f(1) = 2$, $f(p+q) = f(p) \cdot f(q), \forall\ p, q \in R$, then

$x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{(x+y+z)}{x^{f(1)} + y^{f(2)} + z^{f(3)}}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

11. The sum of the infinite series

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right) + \sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right) + \ldots + \ldots + \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{(n-1)}}{\sqrt{\{n(n+1)\}}}\right) + \ldots \text{ is}$$

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

12. The value of $\tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x - \sin\theta}\right)$ is

(a) 2θ (b) θ

(c) $\theta/2$ (d) independent of θ

13. If $[\cot^{-1} x] + [\cos^{-1} x] = 0$, where x is a non-negative real number and [.] denotes the greatest integer function, then complete set of values of x is

(a) $(\cos 1, 1]$ (b) $(\cot 1, 1)$

(c) $(\cos 1, \cot 1)$ (d) None of these

More Than One Correct Option

14. If $a \le \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \le b$, then

(a) $a = 0$ (b) $b = \frac{\pi}{2}$

(c) $a = \frac{\pi}{4}$ (d) $b = \pi$

15. If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} + \ldots\right) + \cos^{-1}(1 + b + b^2 + \ldots) = \frac{\pi}{2}$, then

(a) $b = \frac{2a-3}{3a}$ (b) $b = \frac{3a-2}{2a}$

(c) $a = \frac{3}{2-3b}$ (d) $a = \frac{2}{3-2b}$

Comprehension Based Questions

Passage

Directions (Q. Nos. 16-18)

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{x_r - x_{r-1}}{1 + x_{r-1}x_r}\right) = \sum_{r=1}^{n} (\tan^{-1} x_r - \tan^{-1} x_{r-1})$$

$$= \tan^{-1} x_n - \tan^{-1} x_0, \forall n \in N$$

On the basis of above information, answer the following questions.

16. The sum to infinite terms of the series $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{9}\right) + \ldots + \tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right) + \ldots.$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) None of these

17. The value of $\operatorname{cosec}^{-1}\sqrt{5} + \operatorname{cosec}^{-1}\sqrt{65} + \operatorname{cosec}^{-1}\sqrt{325} + \ldots.\infty$ is

(a) π (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

18. The sum of infinite terms of the series $\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \ldots.$ is

(a) $\frac{\pi}{2}$ (b) $\cot^{-1} 2$
(c) $\tan^{-1} 2$ (d) None of these

Assertion and Reason

Directions (Q. Nos. 19-23) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

19. **Statement I** If $x < 0$, $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Statement II $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in R$

20. Let us define $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$.

Statement I The value of $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$ is $\frac{\pi}{4}$.

Statement II If $x > 0$, $y > 0$, then $\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y-x}{y+x}\right) = \frac{\pi}{4}$

21. **Statement I** The only solution of $\sin^{-1} 2x + \sin^{-1} 3x = \frac{\pi}{3}$ is $x = \sqrt{\left(\frac{3}{76}\right)}$.

Statement II Sum of two negative angles cannot be positive.

22. Let us consider $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ and $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

Statement I $\operatorname{cosec}^{-1}\left(\frac{3}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right) - 2\cot^{-1}\left(\frac{1}{7}\right) - \cot^{-1}(7)$ is equal to $\cot^{-1} 7$.

Statement II $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$

23. **Statement I** $\tan\left[\cos^{-1}\left(\frac{1}{\sqrt{82}}\right) - \sin^{-1}\left(\frac{5}{\sqrt{26}}\right)\right]$ is equal to $\frac{29}{3}$.

Statement II $\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 = \frac{51}{50}$, if $x = \frac{1}{5\sqrt{2}}$

Previous Years' Questions

24. If x, y and z are in AP and $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are also in AP, then **[JEE Main]**

(a) $x = y = z$ (b) $2x = 3y = 6z$
(c) $6x = 3y = 2z$ (d) $6x = 4y = 3z$

25. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is **[AIEEE 2008]**

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$
(c) $\frac{3}{17}$ (d) $\frac{4}{17}$

26. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then value of x is **[AIEEE 2007]**

(a) 1 (b) 3 (c) 4 (d) 5

27. If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[AIEEE 2005]**

(a) $-4\sin^2\alpha$
(b) $4\sin^2\alpha$
(c) 4

(d) $2 \sin^2 \alpha$

28. The trigonometric equation $\sin^{-1} x = 2 \sin^{-1} a$, has a solution for [AIEEE 2003]

(a) $-\frac{1}{\sqrt{2}} < a < \frac{1}{\sqrt{2}}$

(b) all real values of a

(c) $|a| \le \frac{1}{2}$

(d) $|a| \ge \frac{1}{\sqrt{2}}$

29. If $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x$ is equal to [AIEEE 2002]

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ (c) $\tan \alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

30. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to [AIEEE 2002]

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

Answers

Round I

1. (d)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (b)
11. (c)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (d)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (c)	**27.** (d)	**28.** (d)	**29.** (a)	**30.** (c)
31. (b)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (b)	**38.** (d)	**39.** (c)	**40.** (a)
41. (b)	**42.** (c)	**43.** (b)	**44.** (d)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (a)	**49.** (a)	**50.** (c)
51. (d)	**52.** (a)	**53.** (c)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (d)	**60.** (a)
61. (a)	**62.** (a)								

Round II

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (c)	**10.** (c)
11. (c)	**12.** (b)	**13.** (b)	**14.** (a,d)	**15.** (a,c)	**16.** (a)	**17.** (d)	**18.** (d)	**19.** (d)	**20.** (a)
21. (a)	**22.** (d)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (d)

the Guidance

Round I

1. $\cos\left(\frac{33\pi}{5}\right) = \cos\left(6\pi + \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$

$= \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = \sin\left(-\frac{\pi}{10}\right)$

$= \sin^{-1}\sin\left(-\frac{\pi}{10}\right) = -\frac{\pi}{10}$

2. 1 rad $> 45°$

$\Rightarrow \quad \tan 1° > \tan 45° \Rightarrow \tan 1 > 1$

Also, $\quad \tan^{-1} 1 = \frac{\pi}{4} < 1$

Hence, $\quad \tan 1 > \tan^{-1} 1$

3. $\cos^{-1} x, \sin^{-1} x$ are real, if $-1 \le x \le 1$

But $\quad \cos^{-1} x > \sin^{-1} x \Rightarrow 2\cos^{-1} x > \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} x = \frac{\pi}{4}$

$\therefore \quad \cos(\cos^{-1} x) < \cos\frac{\pi}{4} \Rightarrow x < \frac{1}{\sqrt{2}}$

The common values are $-1 \le x < \frac{1}{\sqrt{2}}$.

4. Given that, $\theta = \tan^{-1} a$ and $\phi = \tan^{-1} b$

and $\quad ab = -1$

$\therefore \quad \tan\theta \tan\phi = ab = -1$

$\Rightarrow \quad \tan\theta = -\cot\phi$

$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{2} + \phi\right)$

$\Rightarrow \quad \theta - \phi = \frac{\pi}{2}$

5. $\sin^{-1}(\sin 10) = \sin^{-1}\{\sin(3\pi - 10)\} = 3\pi - 10$

6. $\because \quad \cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}\frac{4}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1} x$

$\Rightarrow \quad \cos^{-1} x = 0 \quad \Rightarrow \quad x = \cos 0 = 1$

$\therefore \quad x = 1$

7. Since, $\quad -\frac{\pi}{2} < \sin^{-1} x \le \frac{\pi}{2}$

$\therefore \quad \sin^{-1} x_i = \frac{\pi}{2}, \quad 1 \le i \le 20$

$\Rightarrow \quad x_i = 1, \quad 1 \le i \le 20$

Thus, $\quad \sum_{i=1}^{20} x_i = 20$

8. Given, $f(x) = \sin^{-1}\sqrt{x-1}$

For domain of $f(x) \quad -1 \le \sqrt{x-1} \le 1$

$\Rightarrow \quad 0 \le (x-1) \le 1 \Rightarrow 1 \le x \le 2$

$\therefore \quad x \in [1, 2]$

9. For domain of $\cos^{-1}(2x-1)$, $-1 \le (2x-1) \le 1$

$\Rightarrow \quad 0 \le 2x \le 2 \quad \Rightarrow 0 \le x \le 1 \quad \therefore x \in [0, 1]$

10. Given, $3\tan^{-1} x + \cot^{-1} x = \pi$

$\therefore \quad 2\tan^{-1} x + (\tan^{-1} x + \cot^{-1} x) = \pi$

$\Rightarrow \quad 2\tan^{-1} x + \frac{\pi}{2} = \pi \qquad \left(\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right)$

$\Rightarrow \quad 2\tan^{-1} x = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1} x = \frac{\pi}{4} = \tan^{-1} 1 \Rightarrow x = 1$

11. $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \cos\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \cos\cos^{-1}\sqrt{1 - \frac{3}{4}}$

$= \cos\cos^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}$

12. $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\} = \tan\left\{\pi - \cos^{-1}\left(\frac{2}{7}\right) - \frac{\pi}{2}\right\}$

$= \tan\left\{\frac{\pi}{2} - \cos^{-1}\left(\frac{2}{7}\right)\right\} = \tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$

$= \tan\left\{\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)\right\} = \frac{2}{3\sqrt{5}}$

13. $\because 2\sec^{-1} 2 + \sin^{-1}\left(\frac{1}{2}\right)$

$= 2\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2} \qquad \left(\because \sec^{-1} x = \cos^{-1}\frac{1}{x}\right)$

$= \cos^{-1}\frac{1}{2} + \left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2}\right)$

$= \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$

14. Given, $\quad \tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5} \qquad \ldots(i)$

$\therefore \cot^{-1} x + \cot^{-1} y = \frac{\pi}{2} - \tan^{-1} x + \frac{\pi}{2} - \tan^{-1} y$

$= \pi - (\tan^{-1} x + \tan^{-1} y)$

$= \pi - \left(\frac{4\pi}{5}\right) \qquad [\because \text{from Eq.(i)}]$

$= \frac{\pi}{5}$

15. $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] = \cot\left[\cot^{-1}\frac{7}{\sqrt{(25)^2 - (7)^2}}\right]$

$\left(\because \cot^{-1}\frac{x}{y} = \cot^{-1}\frac{x}{\sqrt{y^2 - x^2}}\right)$

$= \frac{7}{\sqrt{625 - 49}} = \frac{7}{\sqrt{576}} = \frac{7}{24}$

16. $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} + \frac{\pi}{3}\right)\right] + \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left(-\cot\frac{\pi}{3}\right) + \cos^{-1}\left[\cos\frac{\pi}{6}\right]$

$\left[\because \tan\left(\frac{\pi}{2} + \theta\right) = -\tan\theta \text{ and } \cos(2\pi + \theta) = \cos\theta\right]$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right] + \cos^{-1}\left[\cos\frac{\pi}{6}\right]$

$\left[\because \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta\right]$

$= \tan^{-1}\left[-\tan\frac{\pi}{6}\right] + \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] + \frac{\pi}{6}$

$[\because \tan(-\theta) = -\tan\theta]$

$= -\frac{\pi}{6} + \frac{\pi}{6} = 0$

17. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right] = \cos\left[\pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$

$= \cos\left[\pi - \frac{\pi}{6} + \frac{\pi}{6}\right] = \cos\pi = -1$

18. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(-\sin\frac{\pi}{2}\right)$

$[\because \sin(-\theta) = -\sin\theta]$

$$= \frac{-\pi}{6} + \frac{\pi}{3} - \tan^{-1}\left(\sin\frac{\pi}{2}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= \frac{\pi}{6} - \tan^{-1}(1) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$$

19. $\sin^{-1} x + \sin^{-1}\frac{1}{x} + \cos^{-1} x + \cos^{-1}\frac{1}{x} = f(x)$

Domain of $f(x) = \pm 1$ but according to question $x \neq \pm 1$.

20. Let $E = \cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

Let $\cos\theta = \frac{3}{5}$ ...(i)

Then, $\sin\theta = \frac{4}{5}$ ...(ii)

On dividing Eq. (ii) by Eq.(i), we get

$$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\frac{4}{3} \quad \text{...(iii)}$$

$$\therefore \quad E = \cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$$

$$= \cos^{-1}(\cos\theta\cos x + \sin\theta\sin x)$$

[using Eqs. (i) and (ii)]

$$= \cos^{-1}[\cos(\theta - x)]$$

$$= \theta - x = \tan^{-1}\frac{4}{3} - x \quad \text{[from Eq. (iii)]}$$

21. Now, $\cos^{-1}(\cos 4) = \cos^{-1}\{\cos(2\pi - 4)\} = 2\pi - 4$

$$\Rightarrow \quad 2\pi - 4 > 3x^2 - 4x$$

$$\Rightarrow \quad 3x^2 - 4x - (2\pi - 4) < 0$$

$$\Rightarrow \quad \frac{2 - \sqrt{6\pi - 8}}{3} < x < \frac{2 + \sqrt{6\pi - 8}}{3}$$

22. Let $\cot^{-1}\frac{1}{2} = \phi \Rightarrow \frac{1}{2} = \cot\phi$

$$\Rightarrow \quad \sin\phi = \frac{1}{\sqrt{1 + \cot^2\phi}} = \frac{2}{\sqrt{5}}$$

Let $\cos^{-1} x = \theta \Rightarrow \sec\theta = \frac{1}{x}$

$$\Rightarrow \quad \tan\theta = \sqrt{\sec^2\theta - 1}$$

$$\Rightarrow \quad \tan\theta = \sqrt{\frac{1}{x^2} - 1}$$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{1 - x^2}}{x}$$

Now, $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$

$$\Rightarrow \quad \tan\left(\tan^{-1}\frac{\sqrt{1 - x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \frac{\sqrt{1 - x^2}}{x} = \frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad \sqrt{(1 - x^2)5} = 2x$$

On squaring both sides, we get

$$(1 - x^2)5 = 4x^2$$

$$\Rightarrow \quad 9x^2 = 5$$

$$\Rightarrow \quad x = \pm\frac{\sqrt{5}}{3}$$

23. We have, $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x$

$$= \frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x$$

Since, $0 \le x \le 1$, therefore $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

24. $\because \quad \tan^{-1} x - \tan^{-1} y = 0 \Rightarrow x = y$

Also, $\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2} \Rightarrow 2\cos^{-1} x = \frac{\pi}{2}$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}} \Rightarrow x^2 = \frac{1}{2}$$

Hence, $x^2 + xy + y^2 = 3x^2 = \frac{3}{2}$

25. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$$\Rightarrow \quad \frac{3x + 2x}{1 - 6x^2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 5x = 1 - 6x^2$$

$$\Rightarrow \quad 6x^2 + 5x - 1 = 0 \Rightarrow x = -1, \frac{1}{6}$$

But when $x = -1$,

$\tan^{-1} 2x = \tan^{-1}(-2) < 0$

and $\tan^{-1} 3x = \tan^{-1}(-3) < 0$

This, value will not satisfy the given equation.

Hence, $x = \frac{1}{6}$

26. $\sin\{2\tan^{-1}(0.75)\}$

$$= \sin\left[\tan^{-1}\left(\frac{2(0.75)}{1 - (0.75)^2}\right)\right]$$

$$= \sin\left\{\tan^{-1}\left(\frac{1.5}{1 - 0.5625}\right)\right\}$$

$$= \sin\left[\tan^{-1}\left(\frac{1.5}{0.4375}\right)\right] = \sin\left[\tan^{-1}\left(\frac{24}{7}\right)\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{24}{\sqrt{(24)^2 + (7)^2}}\right)\right]$$

$$\left[\because \tan^{-1}\left(\frac{x}{y}\right) = \sin^{-1}\frac{x}{\sqrt{x^2 + y^2}}\right]$$

$$= \sin\left(\sin^{-1}\frac{24}{\sqrt{476+49}}\right)$$

$$= \sin\left(\sin^{-1}\frac{24}{25}\right) = \frac{24}{25} = 0.96$$

27. Given, $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\therefore \quad 2\tan^{-1} a + 2\tan^{-1} a = 2\tan^{-1} x$

$\Rightarrow \quad 4\tan^{-1} a = 2\tan^{-1} x$

$\Rightarrow \quad 2\tan^{-1} a = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\frac{2a}{1-a^2} = \tan^{-1} x$

$\Rightarrow \quad x = \frac{2a}{1-a^2}$

28. $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right)$

$= \tan(\tan^{-1} 7 - \tan^{-1} 4)$

$= \tan\left[\tan^{-1}\left(\frac{7-4}{1+28}\right)\right] = \frac{3}{29}$

29. $\because \quad \tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}} = \tan\frac{\pi}{2}$

$\Rightarrow \quad 1 - \frac{ab}{x^2} = 0$

$\Rightarrow \quad x^2 = ab \Rightarrow x = \sqrt{ab}$

30. We have, $\tan^{-1}\frac{x-1}{x+2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$

$\Rightarrow \quad \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right] = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{2x(x+2)}{4x+5} = 1$

$\Rightarrow \quad 2x^2 + 4x = 4x + 5$

$\Rightarrow \quad x = \pm\sqrt{\frac{5}{2}}$

31. $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right) = \sqrt{\frac{1-\cos\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)}{1+\cos\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)}}$

$= \sqrt{\frac{1-\frac{2}{\sqrt{5}}}{1+\frac{2}{\sqrt{5}}}} = \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$

$= \frac{\sqrt{(\sqrt{5})^2-2^2}}{\sqrt{5}+2} = \frac{\sqrt{5-4}}{\sqrt{5}+2}$

$= \frac{1}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2} = \frac{\sqrt{5}-2}{(\sqrt{5})^2-2^2}$

$= \frac{\sqrt{5}-2}{5-4} = \sqrt{5}-2$

32. $\tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)$

$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{\frac{6\tan x}{1+\tan^2 x}}{5+\frac{3(1-\tan^2 x)}{1+\tan^2 x}}\right)$

$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{6\tan x}{8+2\tan^2 x}\right)$

$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\tan x}{4+\tan^2 x}\right)$

$= \tan^{-1}\left(\frac{\frac{\tan x}{4}+\frac{3\tan x}{4+\tan^2 x}}{1-\frac{3\tan^2 x}{4(4+\tan^2 x)}}\right) \quad \left(\text{as } \left|\frac{\tan x}{4}\cdot\frac{3\tan x}{4\tan^2 x}\right| < 1\right)$

$= \tan^{-1}\left(\frac{16\tan x + \tan^3 x}{16+\tan^2 x}\right) = \tan^{-1}(\tan x) = x$

33. $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$

$= \cot^{-1} y - \cot^{-1} x + \cot^{-1} z - \cot^{-1} y + \cot^{-1} x - \cot^{-1} z$

$= 0$

34. $\tan\theta + \tan\left(\frac{\pi}{3}+\theta\right) + \tan\left(-\frac{\pi}{3}+\theta\right) = a\tan 3\theta$

$\Rightarrow \quad \tan\theta + \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} + \frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta} = a\tan 3\theta$

$\Rightarrow \quad \tan\theta + \frac{8\tan\theta}{1-3\tan^2\theta} = a\tan 3\theta$

$\Rightarrow \quad \frac{3(3\tan\theta-\tan^3\theta)}{1-3\tan^2\theta} = a\tan 3\theta$

$\Rightarrow \quad 3\tan 3\theta = a\tan 3\theta$

$\Rightarrow \quad a = 3$

35. Given that, $x^2 + y^2 + z^2 = r^2$

Now, $\tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$

$$= \tan^{-1}\left[\frac{\frac{xy}{zr} + \frac{yz}{xr} + \frac{xz}{yr} - \frac{xyz}{r^3}}{1 - \left(\frac{x^2 + y^2 + z^2}{r^2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{xy}{zr} + \frac{yz}{xr} + \frac{xz}{yr} - \frac{xyz}{r^3}}{1 - \frac{r^2}{r^2}}\right] = \tan^2 \infty = \frac{\pi}{2}$$

36. Given that, $\angle A = \tan^{-1} 2, \angle B = \tan^{-1} 3$

We know that, $\angle A + \angle B + \angle C = \pi$

$\Rightarrow \quad \tan^{-1} 2 + \tan^{-1} 3 + \angle C = \pi$

$\Rightarrow \quad \tan^{-1}\left(\frac{2+3}{1-2\times 3}\right) + \angle C = \pi$

$\Rightarrow \quad \tan^{-1}(-1) + \angle C = \pi$

$\Rightarrow \quad \frac{3\pi}{4} + \angle C = \pi$

$\Rightarrow \quad \angle C = \frac{\pi}{4}$

37. Let $E = \sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2})$

Now, $2\tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) = \tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}$

$$\left(\because 2\tan^{-1}\theta = \tan^{-1}\frac{2\theta}{1-\theta^2}\right)$$

$$= \tan^{-1}\frac{3}{4} = \sin^{-1}\frac{\frac{3}{4}}{\sqrt{1+\left(\frac{3}{4}\right)^2}} \quad \left[\because \tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right]$$

$$= \sin^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right) = \sin^{-1}\frac{3}{5}$$

and $\tan^{-1}(2\sqrt{2}) = \cos^{-1}\frac{1}{\sqrt{1+(2\sqrt{2})^2}} = \cos^{-1}\left(\frac{1}{3}\right)$

$$\left[\because \tan^{-1} x = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$\therefore \quad E = \sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2})$

$$= \sin\left(\sin^{-1}\frac{3}{5}\right) + \cos\left(\cos^{-1}\frac{1}{3}\right) = \frac{3}{5} + \frac{1}{3} = \frac{9+5}{15} = \frac{14}{15}$$

38. $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow \quad 2\tan^{-1} a - 2\tan^{-1} b = 2\tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\frac{a-b}{1+ab} = \tan^{-1} x \Rightarrow x = \frac{a-b}{1+ab}$

39. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

$$= \tan\left[\frac{1}{2}\cdot 2\tan^{-1} a + \frac{1}{2}\cdot 2\tan^{-1} a\right]$$

$$= \tan(2\tan^{-1} a)$$

$$= \tan\left[\tan^{-1}\left(\frac{2a}{1-a^2}\right)\right]$$

$$= \frac{2a}{1-a^2}$$

40. $\therefore \tan^{-1}\left[\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right]$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right) \quad (\because \text{put } x^2 = \cos 2\theta)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \left(\tan\frac{\pi}{4}\right)(\tan\theta)}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right] \quad \left[\because \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$$\left(\because x^2 = \cos 2\theta = \Rightarrow \theta = \frac{1}{2}\cos^{-1} x^2\right)$$

41. We have, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x$

$\Rightarrow \quad \sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

42. $\cos^{-1}[\cos\{2\cot^{-1}(\sqrt{2}-1)\}]$

$$= \cos^{-1}[\cos\{2(67.5°)\}]$$

$$= \cos^{-1}\{\cos(135°)\} = 135° = \frac{3\pi}{4}$$

43. Given, $a_1, a_2, a_3, a_4, \ldots, a_n$ is an arithmetic progression, then $d = a_2 - a_1 = a_4 - a_3 = \ldots = a_n - a_{n-1}$

$$\therefore \quad \tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{a_4-a_3}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)\right]$$

$$= \tan[\tan^{-1} a_2 - \tan^{-1} a_1 + \tan^{-1} a_3 - \tan^{-1} a_2 + \tan^{-1} a_4 - \tan^{-1} a_3 + \ldots + \tan^{-1} a_n - \tan^{-1} a_{n-1}]$$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} x - \tan^{-1} y\right]$$

$$= \tan[\tan^{-1} a_n - \tan^{-1} a_1]$$

$$= \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)\right] = \frac{a_n-a_1}{1+a_1a_n}$$

44. $\cos(2\cos^{-1} x + \sin^{-1} x)$

$$= \cos[2(\cos^{-1} x + \sin^{-1} x) - \sin^{-1} x]$$

$$= \cos(\pi - \sin^{-1} x) = -\cos(\sin^{-1} x)$$

$$= -\cos\left[\sin^{-1}\left(-\frac{1}{5}\right)\right] \qquad \left(\because x = \frac{1}{5}\right)$$

$$= -\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right) = -\frac{2\sqrt{6}}{5}$$

45. $\because \quad \cot^{-1} x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2} = \tan^{-1} 1$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1} 1 - \tan^{-1}\frac{1}{2}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$$

$$\Rightarrow \quad x = 3$$

46. $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$

On putting $x = \tan\theta$, we get

$$3\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) - 4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + 2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{\pi}{3}$$

$$\Rightarrow 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$$

$$\Rightarrow \quad 3(2\theta) - 4(2\theta) + 2(2\theta) = \frac{\pi}{3}$$

$$\Rightarrow \quad 6\theta - 8\theta + 4\theta = \frac{\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6} \Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \tan\frac{\pi}{6} \Rightarrow x = \frac{1}{\sqrt{3}}$$

47. Let $S_\infty = \cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \ldots$

$$\therefore \quad T_n = \cot^{-1} 2n^2$$

$$= \tan^{-1}\frac{1}{2n^2}$$

$$= \tan^{-1}\left(\frac{2}{4n^2}\right) = \tan^{-1}\left[\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right]$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$$\therefore \quad S_n = \sum_{n=1}^{\infty}\{\tan^{-1}(2n+1) - \tan^{-1}(2n-1)\}$$

$$= \tan^{-1}\infty - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

48. Since, $\alpha = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{1}{3}\right)$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{1}{9}} + \frac{1}{3}\sqrt{1-\frac{16}{25}}\right)$$

$$\Rightarrow \quad \alpha = \sin^{-1}\left(\frac{8\sqrt{2}}{15} + \frac{3}{15}\right) = \sin^{-1}\left(\frac{8\sqrt{2}+3}{15}\right)$$

Since, $\frac{8\sqrt{2}+3}{15} < 1$

$$\therefore \quad \alpha < \frac{\pi}{2}$$

Now, $\beta = \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{1}{3}\right)$

$$\Rightarrow \quad \beta = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right) + \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)$$

$$= \pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{1}{3}\right) = \pi - \alpha$$

$$\Rightarrow \quad \beta > \alpha \qquad \left(\because \alpha < \frac{\pi}{2}\right)$$

49. Let $\sin^{-1} a = A, \quad \sin^{-1} b = B, \quad \sin^{-1} c = C$

$\therefore \quad \sin A = a, \quad \sin B = b, \quad \sin C = c$ ...(i)

and $A + B + C = \pi$

Then, $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$ ...(ii)

$\Rightarrow \quad \sin A\cos A + \sin B\cos B + \sin C\cos C$

$= 2\sin A \sin B \sin C$

$\Rightarrow \sin A\sqrt{(1-\sin^2 A)} + \sin B\sqrt{(1-\sin^2 B)}$

$+ \sin C\sqrt{(1-\sin^2 C)} = 2\sin A \sin B \sin C$ ...(iii)

$\Rightarrow a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c^2)} = 2abc$

50. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1}(xy - \sqrt{1-y^2}\sqrt{1-x^2}) = \pi - \cos^{-1} z$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi - \cos^{-1} z)$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

On squaring both sides, we get

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 = 1 - 2xyz$

51. Let $\alpha = \cos^{-1}\sqrt{p}, \beta = \cos^{-1}\sqrt{1-p}$

and $\gamma = \cos^{-1}\sqrt{1-q}$

$\Rightarrow \cos\alpha = \sqrt{p}, \quad \cos\beta = \sqrt{1-p}$

and $\cos\gamma = \sqrt{1-q}$

Therefore,

$\sin\alpha = \sqrt{1-p}, \sin\beta = \sqrt{p}$ and $\sin\gamma = \sqrt{q}$

The given equation may be written as,

$$\alpha + \beta + \gamma = \frac{3\pi}{4}$$

$$\Rightarrow \alpha + \beta = \frac{3\pi}{4} - \gamma$$

$$\Rightarrow \cos(\alpha+\beta) = \cos\left(\frac{3\pi}{4} - \gamma\right)$$

$\Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \cos\left\{\pi - \left(\frac{\pi}{4} + \gamma\right)\right\} = -\cos\left(\frac{\pi}{4} + \gamma\right)$$

$$\Rightarrow \sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p} = -\left(\frac{1}{\sqrt{2}}\sqrt{1-q} - \frac{1}{\sqrt{2}}\sqrt{q}\right)$$

$$\Rightarrow 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q$$

$$\Rightarrow q = \frac{1}{2}$$

52. $\theta = \tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}}$

$+ \tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$

Let $s^2 = \dfrac{a+b+c}{abc}$

Hence, $\theta = \tan^{-1}\sqrt{a^2s^2} + \tan^{-1}\sqrt{b^2s^2} + \tan^{-1}\sqrt{c^2s^2}$

$= \tan^{-1}(as) + \tan^{-1}(bs) + \tan^{-1}(cs)$

$$= \tan^{-1}\left[\frac{as + bs + cs - abcs^3}{1 - abs^2 - acs^2 - bcs^2}\right]$$

Hence, $\tan\theta = \left[\dfrac{s[(a+b+c) - abcs^2]}{1-(ab+bc+ca)s^2}\right]$

$$= \left[\frac{s\{(a+b+c) - (a+b+c)\}}{1 - s^2(ab+bc+ca)}\right] = 0$$

53. $\tan^{-1}\left(\dfrac{c_1x - y}{c_1y + x}\right) + \tan^{-1}\left(\dfrac{c_2 - c_1}{1 + c_2c_1}\right) + \tan^{-1}\left(\dfrac{c_3 - c_2}{1 + c_3c_2}\right)$

$+ \ldots + \tan^{-1}\dfrac{1}{c_n}$

$$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{1}{c_1}}{1 + \frac{x}{y}\cdot\frac{1}{c_1}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_1} - \frac{1}{c_2}}{1 + \frac{1}{c_1c_2}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_2} - \frac{1}{c_3}}{1 + \frac{1}{c_2c_3}}\right)$$

$+ \ldots + \tan^{-1}\dfrac{1}{c_n}$

$= \tan^{-1}\dfrac{x}{y} - \tan^{-1}\dfrac{1}{c_1} + \tan^{-1}\dfrac{1}{c_1} - \tan^{-1}\dfrac{1}{c_2} + \tan^{-1}\dfrac{1}{c_2}$

$- \tan^{-1}\dfrac{1}{c_3} + \ldots + \tan^{-1}\dfrac{1}{c_{n-1}} - \tan^{-1}\dfrac{1}{c_n} + \tan^{-1}\dfrac{1}{c_n}$

$= \tan^{-1}\left(\dfrac{x}{y}\right)$

54. We have, $\displaystyle\sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{m^4 + m^2 + 2}\right)$

$$= \sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{1 + (m^2+m+1)(m^2-m+1)}\right)$$

$$= \sum_{m=1}^{n}\tan^{-1}\left(\frac{(m^2+m+1) - (m^2-m+1)}{1 + (m^2+m+1)(m^2-m+1)}\right)$$

$$= \sum_{m=1}^{n}[\tan^{-1}(m^2+m+1) - \tan^{-1}(m^2-m+1)]$$

$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}7 - \tan^{-1}3)$

$+ (\tan^{-1}13 - \tan^{-1}7) + \ldots + [\tan^{-1}(n^2+n+1)$

$- \tan^{-1}(n^2-n+1)]$

$$= \tan^{-1}\frac{n^2+n+1-1}{1 + (n^2+n+1)\cdot 1} = \tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right)$$

55. Let $x = \sin\theta$ and $\sqrt{x} = \sin\phi$, where $x \in [0, 1]$

$\Rightarrow \theta, \phi \in [0, \pi/2]$

$\Rightarrow \theta - \phi \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

Now, $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

$= \sin^{-1}(\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta})$

$= \sin^{-1}(\sin\theta\cos\phi - \sin\phi\cos\theta)$

$= \sin^{-1}\sin(\theta - \phi) = \theta - \phi = \sin^{-1}(x) - \sin^{-1}(\sqrt{x})$

56. Clearly, $x(x+1) \geq 0$ and $x^2 + x + 1 \leq 1$

$\Rightarrow \quad x(x+1) = 0 \Rightarrow x = 0, -1$

when $x = 0$,

LHS $= \tan^{-1} 0 + \sin^{-1} 1 = \frac{\pi}{2}$

when $x = -1$,

LHS $= \tan^{-1} 0 + \sin^{-1} \sqrt{1-1+1} = 0 + \sin^{-1}(1) = \frac{\pi}{2}$

Thus, the number of solutions is 2.

57. Given, $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\Rightarrow \quad \cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi$

$\Rightarrow \quad \alpha = \cos\pi, \ \beta = \cos\pi, \gamma = \cos\pi$

$\Rightarrow \quad \alpha = -1, \beta = -1, \gamma = -1$

$\therefore \quad \alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$= -1(-1-1) - 1(-1-1) - 1(-1-1)$

$= 2 + 2 + 2 = 6$

58. Given, $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$

$\therefore \quad \sqrt{2\cos^2 x} = \sqrt{2}\, x$

$\Rightarrow \quad \sqrt{2}\,|\cos x| = \sqrt{2}\, x$

For $x \in \left[\frac{\pi}{2}, \pi\right]$, $|\cos x| = -\cos x$

$-\sqrt{2}\cos x = \sqrt{2}\, x$

$\Rightarrow \quad -\cos x = x$

$\Rightarrow \quad \cos x = -x$

Hence, no solution exist

59. $4\tan^{-1}\frac{1}{5} = 2\left[2\tan^{-1}\frac{1}{5}\right]$

$$= 2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} = 2\tan^{-1}\frac{5}{12}$$

$$= \tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}} = \tan^{-1}\frac{120}{119}$$

So, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239}$

$$= \tan^{-1}\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\times\frac{1}{239}}$$

$$= \tan^{-1}\frac{(120\times 239) - 119}{(119\times 239) + 120}$$

$$= \tan^{-1}\frac{28561}{28561} = \tan^{-1} 1 = \frac{\pi}{4}$$

60. As we know that,

$$|\sin^{-1} x| \leq \frac{\pi}{2}$$

$\therefore$ Given relation,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

is possible only when

$$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$

$\Rightarrow \quad x = y = z = 1$

$$\therefore \ x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}}$$

$$= 1 + 1 + 1 - \frac{9}{1+1+1}$$

$$= 3 - \frac{9}{3} = 0$$

61. Since, $0 \leq \cos^{-1}\left(\frac{x^2}{2} + \sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right) \leq \frac{\pi}{2}$

Because $\cos^{-1} x$ is in first quadrant when x is positive

and $\quad \cos^{-1}\frac{x}{2} - \cos^{-1} x \geq 0$

So, $\quad \cos^{-1}\frac{x}{2} \geq \cos^{-1} x$

Also, $\quad \left|\frac{x}{2}\right| \leq 1, \ |x| \leq 1$

$\Rightarrow \quad |x| \leq 1$

62. $\because \tan^{-1}\left[\frac{1}{1+r+r^2}\right] = \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)$

$= \tan^{-1}(r+1) - \tan^{-1}(r)$

$$\therefore \ \sum_{r=0}^{n}[\tan^{-1}(r+1) - \tan^{-1}(r)]$$

$= \tan^{-1}(n+1) - \tan^{-1}(0)$

$= \tan^{-1}(n+1)$

$$\Rightarrow \ \sum_{r=0}^{\infty}\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Round II

1. Let $x = \sin\theta$, where $-\frac{1}{2} \le x \le 1 \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

Then, $f(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\sqrt{1-x^2}\right)$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$$

$$= \sin^{-1}\left\{\sin\left(\theta - \frac{\pi}{6}\right)\right\}$$

$$= \theta - \frac{\pi}{6} = \sin^{-1}x - \frac{\pi}{6} \qquad \left(\because \theta - \frac{\pi}{6} \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]\right)$$

2. Let $y = \tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$. As $x \in (0, 1)$, $\theta \in \left(0, \frac{\pi}{4}\right)$ and $\frac{\pi}{2} - 2\theta \in (0, \pi/2)$.

Therefore,

$$y = \tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - 2\theta\right)\right\} + \cos^{-1}(\cos 2\theta)$$

$$= \frac{\pi}{2} - 2\theta + 2\theta = \frac{\pi}{2}$$

3. We have, $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$$= (\sin^{-1}x + \cos^{-1}x)^3 - 3\sin^{-1}x\cos^{-1}x(\sin^{-1}x + \cos^{-1}x)$$

$$= \frac{\pi^3}{8} - 3(\sin^{-1}x\cos^{-1}x)\frac{\pi}{2}$$

$$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1}x + \frac{3\pi}{2}(\sin^{-1}x)^2$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x\right]$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{16} - \frac{\pi^2}{16}\right]$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$$

$\therefore$ The least value is $\frac{\pi^3}{32}$. $\left[\because \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \left(\frac{3\pi}{4}\right)^2\right]$

$\therefore$ The greatest value is $\frac{\pi^3}{32} + \frac{9\pi^2}{16} \times \frac{3\pi}{2} = \frac{7\pi^3}{8}$

4. Here, $T_n = \cot^{-1}\left(n^2 + \frac{3}{4}\right)$

$$= \tan^{-1}\left(\frac{4}{4n^2+3}\right) = \tan^{-1}\left[\frac{1}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$$

$$= \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$$

$\therefore \quad S_\infty = T_\infty^{-1} - \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \quad S_\infty = \cot^{-1}\left(\frac{1}{2}\right) \Rightarrow S_\infty = \tan^{-1}(2)$

5. We have, $\Sigma x_1 = \sin 2\beta$, $\Sigma x_1 x_2 = \cos 2\beta$, $\Sigma x_1 x_2 x_3 = \cos\beta$

and $x_1 x_2 x_3 x_4 = -\sin\beta$

$\therefore \quad \tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4$

$$= \tan^{-1}\left(\frac{\Sigma x_1 - \Sigma x_1 x_2 x_3}{1 - \Sigma x_1 x_2 + x_1 x_2 x_3 x_4}\right)$$

$$= \tan^{-1}\left(\frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta}\right)$$

$$= \tan^{-1}\left[\frac{(2\sin\beta - 1)\cos\beta}{\sin\beta(2\sin\beta - 1)}\right] = \tan^{-1}(\cot\beta)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \beta\right)\right\} = \frac{\pi}{2} - \beta$$

6. $8x^2 + 22x + 5 = 0 \Rightarrow x = -\frac{1}{4}, -\frac{5}{2}$

$\because \quad -1 < -\frac{1}{4} < 1$ and $-\frac{5}{2} < -1$

$\therefore \sin^{-1}\left(-\frac{1}{4}\right)$ exists but $\sin^{-1}\left(-\frac{5}{2}\right)$ does not exist.

$\sec^{-1}\left(-\frac{5}{2}\right)$ exists but $\sec^{-1}\left(-\frac{1}{4}\right)$ does not exist.

$\tan^{-1}\left(-\frac{1}{4}\right)$ and $\tan^{-1}\left(-\frac{5}{2}\right)$ both exist.

7. We have, $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}z$

$\Rightarrow \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = z$

$\Rightarrow \quad x^2(1-y^2) = z^2 + y^2(1-x^2) - 2yz\sqrt{(1-x^2)}$

$\Rightarrow \quad (x^2 - z^2 - y^2)^2 = 4y^2z^2(1-x^2)$

$\Rightarrow x^4 + y^4 + z^4 - 2x^2z^2 + 2y^2z^2 - 2x^2y^2$
$+ 4x^2y^2z^2 - 4y^2z^2 = 0$

$\Rightarrow \quad x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$

$\therefore \quad k = 2$

8. The given equation is satisfied only when

$$x = 1,$$
$$y = -1, \quad z = 1$$

9. Here, $\quad x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$

But $\quad -1 \leq (x^2 - 2x + 2) \leq 1$

which is possible only when

$$x^2 - 2x + 2 = 1$$

$\Rightarrow \quad x = 1$

Then, $\quad a(1)^2 + \sin^{-1}(1) + \cos^{-1}(1) = 0$

$\Rightarrow \quad a + \frac{\pi}{2} + 0 = 0$

$\Rightarrow \quad a = -\frac{\pi}{2}$

10. $\because \quad -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$

and $\quad -\frac{\pi}{2} \leq \sin^{-1} z \leq \frac{\pi}{2}$

Given that, $\quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

which is possible only when

$$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$

$\Rightarrow \quad x = y = z = 1$

Put $\quad p = q = 1$

Then, $\quad f(2) = f(1)f(1) = 2 \cdot 2 = 4$

and put $p = 1, q = 2$

Then, $\quad f(3) = f(1)f(2) = 2 \cdot 2^2 = 8$

$$\therefore \quad x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x + y + z}{x^{f(1)} + y^{f(2)} + z^{f(3)}}$$
$$= 1 + 1 + 1 - \frac{3}{1 + 1 + 1}$$
$$= 3 - 1 = 2$$

11. $\because \quad T_r = \sin^{-1}\left[\frac{\sqrt{r} - \sqrt{(r-1)}}{\sqrt{r(r+1)}}\right]$

$$= \tan^{-1}\left[\frac{\sqrt{r} - \sqrt{(r-1)}}{1 + \sqrt{r}\sqrt{(r-1)}}\right]$$

$$S_n = \sum_{r=1}^{n} \tan^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{1 + \sqrt{r}\sqrt{r-1}}\right)$$

$$= \sum_{r=1}^{n} \{\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{(r-1)}\}$$

$$= \tan^{-1}\sqrt{n} - \tan^{-1}\sqrt{0}$$
$$= \tan^{-1}\sqrt{n} - 0$$

$\therefore \quad S_\infty = \tan^{-1}\infty = \frac{\pi}{2}$

12. $\tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x - \sin\theta}\right)$

$$= \tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \tan^{-1}\left(\frac{x - \sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left[\frac{\frac{x\cos\theta}{1 - x\sin\theta} - \frac{x - \sin\theta}{\cos\theta}}{1 + \left(\frac{x\cos\theta}{1 - x\sin\theta}\right)\left(\frac{x - \sin\theta}{\cos\theta}\right)}\right]$$

$$= \tan^{-1}\left(\frac{x\cos^2\theta - x + \sin\theta + x^2\sin\theta - x\sin^2\theta}{\cos\theta - x\cos\theta\sin\theta + x^2\cos\theta - x\cos\theta\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{-x\sin^2\theta - x + \sin\theta + x^2\sin\theta - x\sin^2\theta}{\cos\theta - 2x\cos\theta\sin\theta + x^2\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{-2x\sin^2\theta + \sin\theta + x^2\sin\theta}{\cos\theta - 2x\cos\theta\sin\theta + x^2\cos\theta}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\theta(-2x\sin\theta + 1 + x^2)}{\cos\theta(1 - 2x\sin\theta + x^2)}\right\}$$

$$= \tan^{-1}(\tan\theta) = \theta$$

13. $\because \quad 0 \leq \cos^{-1} x \leq \pi$

and $\quad 0 < \cot^{-1} x < \pi$

Given, $\quad [\cot^{-1} x] + [\cos^{-1} x] = 0$

$\Rightarrow \quad [\cot^{-1} x] = 0$ and $[\cos^{-1} x] = 0$

$\Rightarrow \quad 0 < \cot^{-1} x < 1$ and $0 \leq \cos^{-1} x < 1$

$\therefore \quad x \in (\cot 1, \infty)$ and $x \in (\cos 1, 1)$

$\Rightarrow \quad x \in (\cot 1, 1)$

14. We have,

$$\tan^{-1} x + \cot^{-1} x + \sin^{-1} x = \frac{\pi}{2} + \sin^{-1} x \quad \text{...(i)}$$

Since, $\quad -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$

$\Rightarrow \quad 0 \leq \frac{\pi}{2} + \sin^{-1} x \leq \pi$

$\Rightarrow \quad 0 \leq \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \leq \pi$

[from Eq. (i)]

$\therefore \quad a = 0$ and $b = \pi$

15. The given relation is possible, when

$$a - \frac{a^2}{3} + \frac{a^3}{9} + \ldots = 1 + b + b^2 + \ldots$$

Also, $\quad -1 \leq a - \frac{a^2}{3} + \frac{a^3}{9} + \ldots \leq 1$

and $\quad -1 \le 1 + b + b^2 + \ldots \le 1$

$\Rightarrow \quad |b| < 1 \Rightarrow |a| < 3$ and $\dfrac{a}{1+\frac{a}{3}} = \dfrac{1}{1-b}$

$\Rightarrow \dfrac{3a}{a+3} = \dfrac{1}{a-b}$, there are infinitely many solutions.

$\Rightarrow \quad 3a - 3ab = a + 3 \Rightarrow 2a - 3ab = 3$

$\Rightarrow \quad b = \dfrac{2a-3}{3a}$ and $a = \dfrac{3}{2-3b}$

16. $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{2^{r-1}}{1+2^{2r-1}}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{2^r - 2^{r-1}}{1+2^r \cdot 2^{r-1}}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \{\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})\}$

$= \lim\limits_{n\to\infty} (\tan^{-1} 2^n - \tan^{-1} 2^0) = \tan^{-1} 2^\infty - \tan^{-1} 1$

$= \tan^{-1}\infty - \tan^{-1} 1 = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$

17. $\text{cosec}^{-1}\sqrt{5} + \text{cosec}^{-1}\sqrt{65} + \text{cosec}^{-1}\sqrt{325} + \ldots\infty$

$= \cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \ldots\infty$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \cot^{-1}(2r^2) = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{2}{4r^2}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{2}{1+4r^2-1}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left[\dfrac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right]$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \{\tan^{-1}(2r+1) - \tan^{-1}(2r-1)\}$

$= \lim\limits_{n\to\infty} \{\tan^{-1}(2n+1) - \tan^{-1} 1\}$

$= \tan^{-1}\infty - \tan^{-1} 1 = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$

18. $\because \quad 3 = 1 + 1 + 1^2$

$7 = 1 + 2 + 2^2$

$13 = 1 + 3 + 3^2$

$\ldots \ldots \ldots \ldots \ldots$

$\ldots \ldots \ldots \ldots \ldots$

$\therefore \quad \cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \ldots$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \cot^{-1}(1 + r + r^2)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{1}{1+r+r^2}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left[\dfrac{(r+1)-r}{1+r(r+1)}\right]$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} [\tan^{-1}(r+1) - \tan^{-1} r]$

$= \lim\limits_{n\to\infty} [\tan^{-1}(n+1) - \tan^{-1} 1]$

$= \tan^{-1}\infty - \dfrac{\pi}{4} = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$

19. If $x < 0$, $\tan^{-1}\left(\dfrac{1}{x}\right) = -\pi + \cot^{-1} x$

$\tan^{-1} x + \tan^{-1}\dfrac{1}{x} = \tan^{-1} x - \pi + \cot^{-1} x = -\pi + \dfrac{\pi}{2} = -\dfrac{\pi}{2}$

Statement I is false but statement II is true.

20. $\tan^{-1}\left(\dfrac{3}{4}\right) + \tan^{-1}\left(\dfrac{1}{7}\right) = \tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \dfrac{\pi}{4}$

$= \tan^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y-x}{x+y}\right) = \dfrac{\pi}{4}$

Both statement I and statement II are true and statement II is correct explanation for statement I.

21. $\because \quad \sin^{-1} 2x + \sin^{-1} 3x = \dfrac{\pi}{3}$

$\Rightarrow \quad \dfrac{\pi}{2} - \cos^{-1} 2x + \dfrac{\pi}{2} - \cos^{-1} 3x = \dfrac{\pi}{3}$

$\Rightarrow \quad \cos^{-1} 2x + \cos^{-1} 3x = \dfrac{2\pi}{3}$

$\Rightarrow \quad \cos^{-1}\{6x^2 - \sqrt{1-(2x)^2}\sqrt{1-(3x)^2}\} = \dfrac{2\pi}{3}$

$\Rightarrow \quad 6x^2 - \sqrt{(1-13x^2+36x^4)} = -\dfrac{1}{2}$

$\Rightarrow \quad \left(6x^2 + \dfrac{1}{2}\right)^2 = 1 - 13x^2 + 36x^4$

$\Rightarrow \quad 19x^2 = \dfrac{3}{4} \Rightarrow x = \pm\sqrt{\dfrac{3}{76}}$

But sum of two negative numbers cannot be $\dfrac{\pi}{3}$.

$\therefore \quad x = \sqrt{\dfrac{3}{76}}$ is the only solution.

22. $\sin^{-1}\left(\dfrac{2}{3}\right) + \cos^{-1}\left(\dfrac{2}{3}\right) - \tan^{-1} 7 - \cot^{-1} 7 - \cot^{-1}\left(\dfrac{1}{7}\right)$

$= \dfrac{\pi}{2} - \dfrac{\pi}{2} - \cot^{-1}\left(\dfrac{1}{7}\right) = -\tan^{-1} 7$

23. $\tan\left[\cos^{-1}\left(\dfrac{1}{\sqrt{82}}\right) - \sin^{-1}\left(\dfrac{5}{\sqrt{26}}\right)\right] = \tan(\tan^{-1} 9 - \tan^{-1} 5)$

$= \tan\tan^{-1}\left(\dfrac{9-5}{1+9\times 5}\right) = \dfrac{2}{23}$

In Statement II, put $\cot^{-1} x = y \Rightarrow x = \cot y$

$$\text{LHS} = (\cot y \cos y + \sin y)^2$$

$$= \frac{(\cos^2 y + \sin^2 y)^2}{\sin^2 y} = 1 + \cot^2 y = 1 + x^2$$

$$= 1 + \frac{1}{50} = \frac{51}{50} \qquad \left(\because x = \frac{1}{5\sqrt{2}}\right)$$

24. Since, x, y and z are in AP.

$\therefore \quad 2y = x + z$

Also, $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are in AP.

$\therefore \quad 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \quad \frac{x+z}{1-y^2} = \frac{x+z}{1-xz} \Rightarrow y^2 = xz$$

Since, x, y and z are in AP as well as in GP.

$\therefore \quad x = y = z$

25. Since, $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$$\cot\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

$$= \cot\tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{1}{2}}\right]$$

$$= \cot\tan^{-1}\left[\frac{\left(\frac{17}{12}\right)}{\left(\frac{1}{2}\right)}\right] = \cot\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{6}{17}$$

26. Since, $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right) \Rightarrow x = 3$$

27. Given, $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$$\Rightarrow \quad \frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha$$

$$\Rightarrow \quad 2\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = 2\cos\alpha - xy$$

On squaring both sides, we get

$$4(1-x^2)\left(\frac{4-y^2}{4}\right) = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$$

$$\Rightarrow \quad (1-x^2)(4-y^2) = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$$

$$\Rightarrow \quad 4 - y^2 - 4x^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$$

$$\Rightarrow \quad 4x^2 + y^2 - 4xy\cos\alpha = 4 - 4\cos^2\alpha$$

$$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$$

28. $\because$ The range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Given, $\sin^{-1} x = 2\sin^{-1} a$

Since, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$$\Rightarrow \quad -\frac{\pi}{2} \le 2\sin^{-1} a \le \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1} a \le \frac{\pi}{4}$$

$$\Rightarrow \quad \sin\left(-\frac{\pi}{4}\right) \le a \le \sin\frac{\pi}{4} \Rightarrow -\frac{1}{\sqrt{2}} \le \alpha \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |a| \le \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$$

29. We know that,

$$\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} \qquad \text{...(i)}$$

and given that

$$\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2\cot^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} + x$$

$$\Rightarrow \quad \sqrt{\cos\alpha} = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow \quad \sqrt{\cos\alpha} = \frac{\cot\frac{x}{2} - 1}{1 + \cot\frac{x}{2}}$$

$$\Rightarrow \quad \sqrt{\cos\alpha} = \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

$$\Rightarrow \quad \cos\alpha = \frac{1-\sin x}{1+\sin x}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-\sin x}{1+\sin x} \Rightarrow \sin x = \tan^2\left(\frac{\alpha}{2}\right)$$

30. $\because \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \quad \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

26 Limits, Continuity and Differentiability

JEE Main MILESTONE

- Limits
- Indeterminate Forms
- Fundamental Theorems on Limits
- Squeeze Play Theorem (Sandwich Theorem)
- Some Useful Expansions
- Some Standard Results on Limits
- Exponential Limits
- Evaluation of Algebraic Limits
- Use of Newton-Leibnitz's Formula in Evaluating the Limits
- Continuity
- Continuity of Composition of Function
- Applications of Continuity
- Discontinuity
- Different kinds of Discontinuity
- Properties of Continuous Functions
- Relation between Derivability and Continuity
- Properties of Differentiable Function
- Continuity and Differentiability of Different Functions

26.1 Limits

Let $y = f(x)$ be a given function defined in the neighbourhood of $x = a$ but not necessarily at the point $x = a$. The limiting behaviour of the function in the neighbourhood of $x = a$ when $|x - a|$ is small, is called the limit of the function when x approaches a and we write this as $\lim_{x \to a} f(x)$.

Let $\lim_{x \to a} f(x) = l$. It would simply mean that when we approach the point $x = a$ from the values which are just greater than or just smaller than $x = a$, $f(x)$ would have a tendency to move closer to the value l. This is same as saying, "Difference between $f(x)$ and l can be made as small as we like by suitably choosing x in the neighbourhood of $x = a$". Mathematically, we write this as, $\lim_{x \to a} f(x) = l$, which is equivalent to saying that $|f(x) - l| < \varepsilon$, $\forall x$ such that $0 < |x - a| < \delta$, and ε depends on δ, where ε and δ are sufficiently small positive numbers.

In Mathematics, a limit is the value that a function or sequence "approaches" as the input or index approaches some value. Limits are essential to calculus (and mathematical analysis in general) and are used to define continuity, derivatives and integrals.

It should be clear that the limit of $f(x)$ at $x = a$ would exist if and only if, $f(x)$ is well defined in the neighbourhood of $x = a$ (not necessarily at $x = a$) and has a unique behaviour in the neighbourhood of $x = a$.

Note Normally, students have the perception that limit should be a finite number. But it is not really so. It is quite possible that $f(x)$ had infinite limit as $x \to a$. If $\lim_{x \to a} f(x) = \infty$, it would simply mean that functions has tendency to assume very large positive values in the neighbourhood of $x = a$, for example $\lim_{x \to 0} \frac{1}{|x|} = \infty$.

Geometric Interpretation of Limits

Geometrically, the limit statement $\lim_{x \to c} f(x) = L$ means that the height of the graph of $y = f(x)$ approaches L as x approaches c.

For instance, the figure (a) represents graph of the function $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ is a line. It is a hole at $(1, 3)$ because the function $f(x)$ is not defined at $x = 1$ in its domain and the points (x, y) on the graph approach this hole as x approaches 1 from either side.

Also, fig. (b) and (c) show the graphs of two functions that do not posses a limit as x approaches c.

The limit does not exist in fig. (b) because $f(x)$ approaches 5 as x approaches c from the right and approaches a different value, 3 as x approaches c from the left. The function in fig. (c) has no limit as x approaches c because the values of $f(x)$ increase without bound as x gets close to c and do not approach any (finite) number. These type of functions are sometimes said to have an infinite limit as x approaches c.

Right Hand Limit

A function f is said to approach l as x approaches a from right, if corresponding to an arbitrary positive number ε, there exists a positive number δ, such that $|f(x) - l| < \varepsilon$ whenever $a < x < a + \delta$.

It is written as $\lim_{x \to a^+} f(x) = l$ or $f(a + 0) = l$.

The working rule for finding the right hand limit is 'put $a + h$ for x in $f(x)$, where h is positive and very-very small and make h approach zero.

i.e., $$f(a + 0) = \lim_{h \to 0} f(a + h)$$

Left Hand Limit

A function f is said to approach l as x approaches a from the left, if corresponding to an arbitrary positive number ε, there exists a positive number δ such that

$|f(x) - l| < \varepsilon$, whenever $a - \delta < x < a$.

It is written as $$\lim_{x \to a^-} f(x) = l$$

or $$f(a - 0) = l.$$

The working rule for finding the left hand limit is put $a - h$ for x in $f(x)$, where h is positive and very-very small and make h approach zero.

i.e., $$f(a - 0) = \lim_{h \to 0} f(a - h)$$

Hot Spot 1

Existence of Limit

Existence of limit is the most important topic for JEE (Main) examination and a small practice can help you in solving such problems very quickly as the level of question is Medium to easy.

Generally, in this type questions ask that whether limit exists or not or to determine the limit of given function.

If both right hand limit and left hand limit exist and are equal, then their common value, evidently will be the limit of f as $x \to a$ *i.e.*,

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$,

then $\lim_{x \to a} f(x) = l$,

If however, either both of these limits do not exist or both these limits exist but are not equal in value,

then $\lim_{x \to a} f(x)$ does not exist.

while we solving such types of problems we follow the steps given below

Step I First we calculate the left hand limit of given function.

Step II Now, we calculate the right hand limit of given function.

Step III Now, we check either both limit exist or not or both are equal. If both limit are equal, then we say limit exist other wise does not exist.

Sample Problem 1 *For what values of m does the* $\lim_{x \to 2} f(x)$ *exist, when* $f(x) = \begin{cases} mx - 3, & \text{when } x < 2 \\ \dfrac{x}{m}, & \text{when } x \geq 2 \end{cases}$

(a) $\frac{1}{2}, 1$ (b) $-\frac{1}{2}, 2$

(c) $-1, 1$ (d) *None of these*

Interpret (b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (mx - 3) = 2m - 3$;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{m} = \frac{2}{m}$$

$\lim_{x \to 2} f(x)$ exists when $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$$\Rightarrow \quad 2m - 3 = \frac{2}{m} \Rightarrow 2m^2 - 3m - 2 = 0$$

$$\therefore \quad m = -\frac{1}{2}, 2$$

Sample Problem 2 $\lim_{x \to 2} \dfrac{x-2}{|x-2|}$ *is equal to*

(a) 1 (b) –1

(c) 0 (d) *does not exist*

Interpret (d) $\because \ |x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ (x-2), & \text{if } x < 2 \end{cases}$

Now, $\text{RHL} = \lim_{x \to 2^+} \left(\dfrac{x-2}{x-2}\right)$

$= \lim_{x \to 2^+} 1 = 1$

and $\text{LHL} = \lim_{x \to 2^-} \left(\dfrac{x-2}{-(x-2)}\right)$

$= -\lim_{x \to 2^-} \left(\dfrac{x-2}{x-2}\right)$

$= -1$

Since, $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$

Therefore, the limit does not exist.

Uniqueness of Limit If $\lim_{x \to a} f(x)$ exists, then it is unique.

There cannot be two distinct numbers l_1 and l_2 such that when x tends to a, the function $f(x)$ tends to both l_1 and l_2.

26.2 Indeterminate Forms

If a function $f(x)$ takes any of the following forms at $x = a$; $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^0, \infty^0, 1^\infty$, then $f(x)$ is said to be indeterminate at $x = a$ and these forms are called indeterminate forms.

26.3 Fundamental Theorems on Limits

If $f(x)$ and $g(x)$ be two functions of x such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

(i) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
(sum and difference rule)

(ii) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$, where k is a fixed real number.
(constant multiple rule)

(iii) $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$ (product rule)

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a} g(x) \neq 0$ (quotient rule)

(v) $\lim_{x \to a} [f(x)]^{g(x)} = \left[\lim_{x \to a} f(x)\right]^{\lim_{x \to a} g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$

(vi) $\lim_{x \to a} g[f(x)] = g\left[\lim_{x \to a} f(x)\right]$, provided g is continuous function at $x = \lim_{x \to a} f(x)$

(vii) $\lim_{x \to a} \log f(x) = \log\left[\lim_{x \to a} f(x)\right]$, provided $\lim_{x \to a} f(x) > 0$

(viii) $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)}$

(ix) If $f(x) \le g(x)$ for every x excluding a, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

(x) If $f(x) \le g(x) \le h(x)$ for every x excluding a, then $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{n \to a} g(x) = l$.

(xi) $\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)|$

(xii) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(xiii) $\lim_{x \to a} (mx + c) = ma + c$

(xiv) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

(xv) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

26.4 Squeeze Play Theorem (Sandwich Theorem)

If $f(x) \le g(x) \le h(x), \ \forall x \in (\alpha, \beta) - \{a\}$

and $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$

then, $\lim_{x \to a} g(x) = l$, where $a \in (\alpha, \beta)$.

Sample Problem 3 $\lim_{x \to \infty} \frac{[x]}{x}$, *where* [·] *is greatest integer function, is equal to*

(a) 0 (b) 1

(c) –1 (d) *None of these*

Interpret (*b*) We have, $x - 1 < [x] \le x$

$\Rightarrow \quad 1 - \frac{1}{x} < \frac{[x]}{x} \le 1$

Now, $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right) = 1$

Therefore, by Sandwich theorem $\lim_{x \to \infty} \frac{[x]}{x} = 1$

26.5 Some Useful Expansions

Many limits can be evaluated very easily by applying expansion series. Some of the standard expansions are

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(ii) $a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$

(iii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, |x| < 1$

(iv) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots$

(v) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, x \in R$

(vi) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, x \in R$

(vii) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(viii) $\sin^{-1} x = x + \frac{1^2}{3!} x^3 + \frac{1^2 \cdot 3^2}{5!} x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \ldots, |x| < 1$

(ix) $\cos^{-1} x = x - \frac{x^3}{6} + \ldots; |x| < 1$

(x) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots; |x| < 1$

(xi) $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}$

(xii) $(1 + x)^{1/x} = e\left\{1 - \frac{x}{2} + \frac{11}{24} x^2 - \ldots \infty\right\}$

(xiii) $\sec x = 1 + \frac{x^2}{2} + \frac{5}{12} x^4 + \frac{61}{720} x^6 + \ldots \infty; |x| < \frac{\pi}{2}$

(xiv) $\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360} x^3 + \frac{31}{15120} x^5 + \ldots \infty; 0 < |x| < \pi$

(xv) $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2}{945} x^5 - \ldots \infty; 0 < |x| < \pi$

(xvi) $\tan^{-1} x = \pm \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \infty;$

$$\begin{cases} + & \text{if } x \ge 1 \\ - & \text{if } x \le -1 \end{cases}$$

(xvii) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \infty, x \in R$

(xviii) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \infty; x \in R$

(xix) $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots \infty; |x| < \frac{\pi}{2}$

26.6 Standard Results on Limits

To evaluate trigonometric limits, reduce the terms of the function in terms of $\sin\theta$ and $\cos\theta$. Remove the positive and negative signs in between two terms *i.e.,* express the function in product form. Arrange terms and take help of the following standard results.

Trigonometric Limits

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(ii) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(iii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$

(vi) $\lim_{x\to 0} \cos x = 1$

(vii) $\lim_{x\to a} \frac{\sin(x-a)}{x-a} = 1$

(viii) $\lim_{x\to a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a, |a| \le 1$

(x) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a, |a| \le 1$

(xi) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

(xii) $\lim_{x\to \infty} \frac{\sin x}{x} = \lim_{x\to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x\to \infty} \frac{\sin 1/x}{1/x} = 1$

Sample **Problem 4** $\lim_{x\to 0} \frac{e^{\tan x} - e^x}{\tan x - x}$ *is equal to*

(*a*) 1 (*b*) $\frac{1}{2}$ (*c*) $\frac{1}{3}$ (*d*) 0

Interpret (*b*) Since, $e^{\tan x} = 1 + \tan x + \frac{\tan^2 x}{2!} + \ldots$

But we know that,

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

$$\therefore \quad e^{\tan x} = \left[1 + \left\{x + \frac{x^3}{3} + \ldots\right\} + \frac{1}{2}\left\{x + \frac{x^3}{3} + \ldots\right\}^2 + \frac{1}{6}\left\{x + \frac{x^3}{3} + \ldots\right\}^3 + \ldots\right]$$

$$= 1 + \left(x + \frac{x^3}{3} + \ldots\right) + \frac{1}{2}\left(x^2 + \frac{2}{3}x^4 + \ldots\right) + \frac{1}{6}(x^3 + \ldots)$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(\frac{1}{3} + \frac{1}{6}\right) + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{1}{2}x^3 + \ldots$$

and $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$$\therefore \quad e^{\tan x} - e^x = x^3\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots$$

$$= \frac{x^3}{6} + \text{Higher powers of } x$$

Similarly, $\tan x - x = \frac{x^3}{3} + \frac{2}{15}x^5 + \text{Higher powers of } x$

$$\therefore \quad \lim_{x\to 0} \frac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x\to 0} \frac{\frac{x^3}{6} + \text{Higher powers of } x}{\frac{x^3}{3} + \text{Higher powers of } x}$$

$$= \frac{1}{2}$$

Limits of Polynomial and Rational Functions

(i) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(iii) If $\lim_{x\to \infty} \frac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ and $f(x)$ and $g(x)$ are both polynomials of x. Then, we divide numerator and denominator by the highest power of x and put 0 for $\frac{1}{x}$.

If m and n are positive integers and $a_0, b_0 \ne 0$ are non-zero real numbers, then

$$\lim_{x\to \infty} \frac{a_0x^m + a_1x^{m-1} + \ldots}{b_0x^n + b_1x^{n-1} + \ldots} = \begin{cases} \frac{a_0}{b_0}, & \text{when } m = n \\ 0, & \text{when } m < n \\ \infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} > 0 \\ -\infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} < 0 \end{cases}$$

Note When $x \to \infty$, then if k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by x^k and use the result $\lim_{x\to \infty} \frac{1}{x^n} = 0$, where $x > 0$.

26.7 Exponential Limits

We use the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$

(i) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

(iii) $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda$, where $(\lambda \neq 0)$

Exponential Limits of the Form 1^∞

To evaluate the exponential form 1^∞, we use following results,

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,

then $\lim_{x\to a} [1 + f(x)]^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$

or when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

Then, $\lim_{x\to a} [f(x)]^{g(x)} = \lim_{x\to a} [1 + f(x) - 1]^{g(x)}$

$$= e^{\lim_{x\to a} [f(x) - 1]\, g(x)}$$

(i) $\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$ (ii) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

(iii) $\lim_{x\to 0} (1 + \lambda x)^{\frac{1}{x}} = e^\lambda$ (iv) $\lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

(v) $\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{does not exist}, & a < 0 \end{cases}$

Logarithmic Limits

We use the series $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty$,

where $-1 < x \le 1$ and expansion is true only, if base is e.

(i) $\lim_{x\to 0} \frac{\log(1 + x)}{x} = 1$

(ii) $\lim_{x\to e} \log_e x = 1$

(iii) $\lim_{x\to 0} \frac{\log_e (1 - x)}{x} = -1$

(iv) $\lim_{x\to 0} \frac{\log_a (1 + x)}{x} = \log_a e, a > 0, \neq 1$

(v) If $\lim_{x\to a} f(x)$ exists and positive, then

$$\lim_{x\to a} [f(x)]^{\phi(x)} = e^{\lim_{x\to a} \phi(x) \log f(x)}$$

26.8 Evaluation of Algebraic Limits

Let $f(x)$ be an algebraic function (polynomial or rational) and a be any real number, then $\lim_{x\to a} f(x)$ is known as an algebraic limit.

e.g., $\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$, $\log_{x\to 0} \frac{\sqrt{1-x} - \sqrt{1-x}}{x}$ etc.

The limit of algebraic functions can be find by the following methods.

(i) Method of Direct Substitution

$\lim_{x\to a} f(x)$ can be evaluated by method of direct substitution, if $f(x)$ exists finitely for $x = a$.

e.g., $\lim_{x\to 1} \frac{x^2 + 2x + 3}{x^2 + x + 1} = \frac{(1)^2 + 2(1) + 3}{(1)^2 + 1 + 1}$

$$= \frac{6}{3} = 2$$

(ii) Method of Factorisation

Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$. If by putting $x = a$ the rational function $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}, \frac{\infty}{\infty}$ etc, then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. In such a case we factorise the numerator and denominator and then, cancel out the common factor $(x - a)$. After cancelling out the common factor $(x - a)$, we again put $x = a$ in the given expression and see whether, we get a meaningful number or not. This process is repeated till we get a meaningful number.

Sample Problem 5 $\lim_{x\to 1} \frac{x^3 - x^2 \log x + \log x - 1}{x^2 - 1}$

is equal to

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) $\frac{3}{2}$

Interpret (d) The given limit $= \lim_{x\to 1} \frac{(x^3 - 1) - (x^2 - 1) \log x}{x^2 - 1}$

$$= \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1) - (x-1)(x+1) \log x}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)[x^2 + x + 1 - (x+1) \log x]}{(x-1)(x+1)}$$

$$= \frac{1^2 + 1 + 1 - (1+1) \log 1}{(1+1)} = \frac{3}{2}$$

(iii) Method of Rationalisation

Rationalisation method is used when, we have radical signs in an expression (like $\frac{1}{2}, \frac{1}{3}$ etc) and there exists a negative sign between two terms of an algebraic expression.

After rationalisation, the terms are factorised which on cancellation gives the required result.

Sample Problem 6 $\lim_{x \to \pm\infty} x(\sqrt{x^2+k} - x), k > 0$ *is equal to*

(a) $\frac{k}{2}, -\infty$ (b) $k, 0$

(c) $2k, \infty$ (d) *None of these*

Interpret (a) $\lim_{x \to \pm\infty} x(\sqrt{x^2+k} - x) \frac{(\sqrt{x^2+k}+x)}{(\sqrt{x^2+k}+x)}$

$$= \lim_{x \to \pm\infty} \frac{x(x^2+k-x^2)}{(\sqrt{x^2+k}+x)} = \lim_{x \to \pm\infty} \frac{xk}{\left(|x|\sqrt{\left(1+\frac{k}{x^2}\right)}\right)+x}$$

Here, we have to consider two cases.

(i) When $x \to \infty; |x| = x$

Then, the given limit $= \lim_{x \to \infty} \frac{xk}{x\sqrt{\left(1+\frac{k}{x^2}\right)}+x}$

$$= \lim_{x \to \infty} \frac{xk}{x\left(\sqrt{\left(1+\frac{k}{x^2}\right)}+1\right)} = \frac{k}{2}$$

(ii) When $x \to -\infty; |x| = -x$

Then, we have $\lim_{x \to -\infty} \frac{xk}{-x\sqrt{\left(1+\frac{k}{x^2}\right)}+x}$

$$\lim_{x \to -\infty} \frac{xk}{x\left(-\sqrt{\left(1+\frac{k}{x^2}\right)}+1\right)} = \frac{k}{-1^- + 1} = \frac{k}{0^-} \to -\infty$$

(iv) Evaluation by Application of Standard Results

In this method, we use some standard results of limits.

Sample Problem 7 $\lim_{x \to 1} \frac{1+x^{1/3}}{1+x^{1/5}}$ *is equal to*

(a) $\frac{3}{5}$ (b) $-\frac{5}{3}$ (c) $\frac{5}{3}$ (d) $-\frac{3}{5}$

Interpret (c) Limit $= \lim_{x \to -1} \frac{x^{1/3}-(-1)}{x^{1/5}-(-1)} = \frac{5}{3}$

Note $\lim_{x \to -1} \frac{(x^{1/5})^5-(-1)}{x^{1/5}-(-1)} = 5(-1)^4$

$$\lim_{x \to 1} \frac{(x^{1/3})^3-(-1)}{x^{1/3}-(-1)} = 3(-1)^2$$

Sample Problem 8 $\lim_{x \to e} \frac{\log x - 1}{x - e}$ *is equal to*

(a) e (b) $\frac{1}{e}$ (c) 1 (d) 0

Interpret (b) Above limit $= \lim_{x \to e} \frac{\ln x/e}{e\left(\frac{x}{e}-1\right)} = \lim_{t \to 1} \frac{\ln t}{(t-1)\,e}$

(putting $x/e = t$)

Put $t = 1 + y = \lim_{y \to 0} \frac{\ln(1+y)}{ey} = \frac{1}{e}$

(v) Method of Substitution

In order to evaluate $\lim_{x \to a} f(x)$ we may substitute $x = a + h$ or $a - h$, so that as $x - a, h \to 0$

Thus, $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a \pm h)$

This method is applied to bring the limit at zero as the most of formulae are given as $\lim_{x \to a}$

Sample Problem 9 $\lim_{x \to -1^+} \frac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}$ *is equal to*

(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{2\pi}}$ (d) $\sqrt{\frac{\pi}{2}}$

Interpret (c) Put $\cos^{-1} x = \theta \Rightarrow x = \cos\theta$

$\therefore$ Above limit $= \lim_{\theta \to \pi} \frac{\sqrt{\pi}-\sqrt{\theta}}{\sqrt{1+\cos\theta}} = \lim_{\theta \to \pi} \frac{\pi-\theta}{\sqrt{1+\cos\theta}} \cdot \frac{1}{\sqrt{\pi}+\sqrt{\theta}}$

$$= \lim_{\theta \to \pi} \frac{\pi-\theta}{\sqrt{2\cos^2\theta/2}} \lim_{\theta \to \pi} \frac{1}{\sqrt{\pi}+\sqrt{\theta}}$$

$$= \lim_{k \to 0} \frac{k}{\sqrt{2\cos^2\left(\frac{\pi}{2}-\frac{k}{2}\right)}} \cdot \frac{1}{2\sqrt{\pi}}$$

(putting $\theta = \pi - k$) $= \lim_{k \to 0} \frac{k/2}{\frac{\sqrt{2}}{2}\sin\frac{k}{2}} \cdot \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}$

Hot Spot 2

L' Hospital's Rule

'L' Hospital Rule is the most important topic for JEE (Main) Examination and a small practice can help you in solving such problems very quickly as the level of question very quickly as the level of question is medium to easy.

While solving the problem based on L' Hospital Rule, always keep in mind the Indeterminate form.

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) both are continuous at $x = a$

(iii) both are differentiable at $x = a$

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided that $g(a) \neq 0$

Above rule is also applicable, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x)$, $g'(x)$ satisfy all the conditions embeded in L'Hospital's rule. We can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$

While we solving the such types of problems we follow the steps below.

Step I Whenever, we solve the limit based problem. We see that the problem form indeterminate form or not.

Step II If form, then we check the form $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$

Step III If the form either both of these form, we apply L' Hospital rule *i.e.*, differentiate the function Nr and Dr separately. Until the forms $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$, resolve in determinate form.

Step IV Finaly, we apply the limit and get the desired result.

Sample **Problem 10** $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$ *is equal to*

(a) $-\frac{1}{2}e$ (b) $\frac{e}{2}$ (c) e (d) $2e$

Interpret (a) Here, $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$ is of the form $\frac{0}{0}$ because

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

To evaluate the given limit first we shall obtain an expansion for $(1+x)^{1/x}$ in ascending powers of x.

Let $y = (1+x)^{1/x}$. Then,

$$\log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

$$= 1 + z, \text{ where } z = -(x/2) + x^2/3 - \dots$$

$$\therefore\ y = e^{1+z} = e\cdot e^z = e\cdot\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)^2 + \dots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8}x^2 + \text{terms containing powers of } x \text{ higher than 3}\right]$$

$$= e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \dots\right]$$

Now, $$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \dots\right] - e}{x}$$

$$= \lim_{x\to 0} \frac{e\left[-\frac{1}{2}x + \frac{11}{24}x^2 + \dots\right]}{x}$$

$$= \lim_{x\to 0} e\left[-\frac{1}{2} + \frac{11}{24}x + \dots\right] = -\frac{1}{2}e$$

Sample **Problem 11** $\lim_{x \to 0} \frac{\log \log (1 - x^2)}{\log \log \cos x}$ *is equal to*

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) ∞

Interpret (b) We have, $\lim_{x \to 0} \frac{\log \log (1 - x^2)}{\log \log \cos x}$ (form ∞/∞)

$$= \lim_{x \to 0} \frac{\frac{1}{\log 1 - x^2)} \cdot \frac{1}{1 - x^2} \cdot (-2x)}{\frac{1}{\log \cos x} \cdot \frac{1}{\cos x} \cdot (-\sin x)}$$

$$= 2 \lim_{x \to 0} \frac{x \cos x \log \cos x}{\sin x \cdot (1 - x^2) \log (1 - x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1 - x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)} \quad \text{(from 0/0)}$$

$$= 2 \lim_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\frac{1}{1 - x^2} \cdot (-2x)}$$

$$= 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1 - x^2}{\cos x} \right) = 1$$

Note We have, seen that in both cases when the form is ∞/∞ or 0/0 the rule of evaluating the limit by differentaiting the numerator and denominator separately holds good. Also, we can easily convert the form ∞/∞ to the form 0/0 and *vice-versa*. Therefore, at every storage, we should not carefully that which form will be more suitable to evaluate the limit most quickly. Moreover, in some cases it will be necessary to convert the form ∞/∞ to the form 0/0, otherwise the process of differentiating the numerator and the denominator would never terminate.

Sample **Problem 12** $\lim_{x \to \pi/2} \tan x \cdot \log_e \sin x$ *is equal to*

(a) 0

(b) $\frac{1}{2}$

(c) 1

(d) *None of the above*

Interpret (a) Above limit $= \lim_{x \to \pi/2} \frac{\log_e \sin x}{\cos x}$ $\left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \cdot \cos x}{-\operatorname{cosec}^2 x} \quad \text{(using L' Hospital's rule)}$$

$$= \lim_{x \to \pi/2} (-\sin x \cdot \cos x) = 0$$

Important Results

(i) $\lim_{x \to a} \frac{\sin f(x)}{f(x)} = 1 \left(\text{provided} \lim_{x \to a} f(x) = 0 \right)$

(ii) $\lim_{x \to a} \cos f(x) = 1 \left(\text{provided} \lim_{x \to a} f(x) = 0 \right)$

(iii) $\lim_{x \to a} \frac{\tan f(x)}{f(x)} = 1 \left(\text{provided} \lim_{x \to a} f(x) = 0 \right)$

(iv) $\lim_{x \to a} [1 + f(x)]^{1/f(x)} = e \left(\text{provided} \lim_{x \to a} f(x) = 0 \right)$

(v) $\lim_{x \to a} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e \left(\text{provided} \lim_{x \to a} f(x) = \infty \right)$

(vi) $\lim_{x \to a} \frac{e^{f(x)} - 1}{f(x)} = 1 \left(\text{provided} \lim_{x \to a} f(x) = 0 \right)$

(vii) $\lim_{x \to a} \frac{b^{f(x)} - 1}{f(x)} = \log b \ (b > 0)$

(viii) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

26.9 Use of Newton–Leibnitz's Formula in Evaluating the Limits

Let us consider the definite integral

$$I(x) = \int_{\phi(x)}^{\psi(x)} f(t)\, dt$$

Newton-Leibnitz's formula states that

$$\frac{d}{dx}[I(x)] = f[\psi(x)]. \left\{ \frac{d}{dx} \psi(x) \right\} - f[\phi(x)] \left\{ \frac{d}{dx} \phi(x) \right\}$$

Some Rules to Find the Limits

(i) Limit at $x = a$ cannot exist only, if $y = f(x)$ is not defined in both neighbourhood of a. *e.g.*, $\lim_{x \to a} \sqrt{x}$ does not exist as $\sqrt{x}$ is not defined in the left neighbourhood of $x = a$.

(ii) If on putting $x = a$, $f(x)$ is determinate, then normally limit is equal to $f(a)$.*e.g.*, $\lim_{x \to a} \sqrt{x} = \sqrt{a}$. But $\lim_{x \to 2} \frac{[x]}{x} \neq 1$ as $[x]$ changes it's characteristics at $x = 2$.

Here, $\lim_{x \to 2^-} \frac{[x]}{x} = \frac{1}{2}$ and $\lim_{x \to 2^+} \frac{[x]}{x} = 1$, so limit does not exist.

Check Point **1**

1. $\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x-1)}}{x-1}$

(a) exists and it equals $\sqrt{2}$ (b) exists and it equals $-\sqrt{2}$
(c) does not exists (d) None of these

2. The value of $\lim_{x \to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x}$ is

(a) 1/2 (b) 0
(c) –1 (d) 2/3

3. $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$ is equal to

(a) 1/2 (b) 0
(c) 1 (d) None of these

4. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$, where [x] denotes the greatest integer ≤ x, then $\lim_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0
(c) –1 (d) None of these

5. $\lim_{n \to \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$ is equal to

(a) 2 (b) 3
(c) 5 (d) None of these

6. $\lim_{x \to 0} \frac{3^{2x} - 2^{3x}}{x}$ is equal to

(a) $\log \frac{3}{2}$ (b) 1
(c) $\log \frac{9}{8}$ (d) 0

7. $\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{3x+1} - \sqrt{5x-1}}$ is equal to

(a) 4 (b) –4
(c) 2 (d) None of these

8. $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$ is equal to

(a) sin a (b) 0
(c) cos a (d) None of these

9. If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ be finite, then the value of a is

(a) 2 (b) –2
(c) 1 (d) 0

10. $\lim_{x \to \infty} x^5 \left[\frac{1}{x^3}\right]$ is equal to; where [·] denotes the greatest integer function.

(a) 0 (b) $\frac{3}{5}$
(c) 1 (d) None of these

26.10 **Continuity**

The geometrical significance of continuity is that if the function is continuous, its graph does not bear a break otherwise it is discontinuous. The point where the graph of the function breaks is called the point of discontinuity. Thus, a function is continuous at a point, if limit exist at this point and equal to the value of the function at this point.

e.g., Graphs of functions $\sin x$, x and e^x etc., are continuous while $\tan x$ and $\sec x$ etc., are discontinuous.

Continuous function

Discontinuous function

Cauchy's Definition of continuity

A function f is said to be continuous at a point a of its domain Differentiate form every $\varepsilon > 0$, there exists a $\delta > 0$ (dependent on ε) such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Heine's Definiton of Continuity

A function f is said to be continuous at a point a of its domain D, if for every sequence $< a_n >$ of the points in D converging to a, then the sequence < fan > converges to $f(a)$ *i.e.*, $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$.

Types of Continuity

(i) Continuity at a point
(ii) Continuity in a an open interval
(iii) Continuity in a closed interval

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if and only if it satisfies the given condition

(i) $f(a)$ exists (a lies in the domain of f)

(ii) $\lim_{x\to a} f(x)$ exist *i.e.*, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

or

RHL = LHL

(iii) $\lim_{x\to a} f(x) = f(a)$.

Geometrical Interpretation of Continuity at a Point

As we can see in the fig. (a) that the graph of the function suffers a gap at $x = a$, *i.e.*, while drawing the graph of this function $y = f(x)$ with a pen, you will have to pick up your pen a bit at $x = a$ and then start plotting all points greater than a.

Now, it can be easily seen from the graph, that the limit of $f(x)$ as x approaches a from left equals the limit of $f(x)$ as x approaches a from the right.

i.e., $$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

Since, the value of $f(x)$ at $x = a$ is c that is why we say that the function is discontinuous at $x = a$ and $x = a$ is the point of discontinuity.

Now, if this point (a, c) is moved in such a way that it is now positioned at (a, b), as shown in figure (b), then the gap at (a, b) will be filled so as to maintain the continuity of the graph.

i.e., for the graph of the function to be continuous, we must have

Limit = Value

$$\Rightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Continuity in an Open Interval

A function $f(x)$ is said to be continuous in an open interval $]a, b[$ or $a < x < b$, if it is continuous at each point of the interval.

Continuity from Left and from Right

Let $f(x)$ be a function defined *o* an open interval I and let $a \in I$, then f is continuous from the left at a, if $\lim_{x\to a^-} f(x)$ exists and is equal to $f(a)$. Similarly, $f(x)$ is said to be continuous from the right at a, if $\lim_{x\to a^+} f(x)$ exists and is equal to $f(a)$.

Hot Spot 3 Continuity in a Closed Interval

Continuity in a closed interval is the most important topic for JEE Main Examination point of views. Generally, questions are seen from this topic in examination. The level of question is from moderate to typical.

While solving problems based on continuity in a closed interval, always keep in mind the concept of continuity of the function at a point. *i.e.*, (LHL = RHL = VF)

Let $f(x)$ be a function on defined on the closed interval $[a, b]$. Then, $f(x)$ is said to be continuous on the closed interval $[a, b]$, if it is

(i) continuous from the right at a

(ii) continuous from the left at b and

(iii) continuous on the open interval $]a, b[$.

Note

(i) A function is said to be discontinuous on an open interval, if there is atleast one point of the interval where the function is discontinuous.

(ii) All polynomials, trigonometrical functions, exponential and logarithmic functions are continuous in their domain.

(iii) We never talk about continuity/discontinuity at point at which we can't approach from either side of the point. These points are called isolated points.

Sample Problem 13 *The continuity of the function $f(x) = [x^2] - [x]^2$, $\forall x \in R$ at the end points of the interval $[-1, 0]$, where [·] denotes the greatest integer function, is*

(a) *$f(x)$ is continuous at the end points of the interval $[-1, 0]$*

(b) *$f(x)$ is not continuous at the end points of the interval $[-1, 0]$*

(c) *$f(x)$ is continuous only at $x = 0$*

(d) *None of the above*

Interpret (b) Continuity at $x = -1$

$$f(-1) = [(-1)^2] - [-1]^2$$

$$= [1] - (-1)^2 = 1 - 1 = 0$$

$$\text{RHL} = \lim_{x\to -1^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$$

So, $f(-1) \neq \text{RHL}$

Continuity at $x = 0$

$$f(0) = [(0)^2] - [0]^2 = 0 - 0 = 0$$

$$\text{LHL} = \lim_{x \to 0^-} \{[x^2] - [x^2]\} = 0 - 1 = -1$$

So, $f(0) \neq \text{LHL}$

Hence, the function is not continuous at the end points of the interval $[-1, 0]$.

Sample Problem 14 *A function f is defined as follows*

$$f(x) = \begin{cases} 1 & , \text{ when } -\infty < x < 0 \\ 1 + \sin x & , \text{ when } 0 \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \dfrac{\pi}{2}\right)^2 & , \text{ when } \dfrac{\pi}{2} \le x < \infty \end{cases}$$

Continuity of $f(x)$ is

(a) *$f(x)$ is continuous at $x = \dfrac{\pi}{2}$*

(b) *$f(x)$ is continuous at $x = 0$*

(c) *$f(x)$ is discontinuous at $x = 0$*

(d) *$f(x)$ is continuous over the whole real number*

Interpret (a) Continuity at $x = 0$

LHL at $x = 0$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1) = 1$

RHL at $x = 0$ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + \sin x) = 1$

$f(0) = 1 + \sin 0 = 1$

$= \text{LHL} = \text{RHL} = f(0)$ so $f(x)$ is continuous at $x = 0$.

Continuity at $x = \dfrac{\pi}{2}$

LHL at $x = \dfrac{\pi}{2} = \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (1 + \sin x) = 1 + 1 = 2$

RHL at $x = \dfrac{\pi}{2} = \lim_{x \to \frac{\pi}{2}^+} f(x) = 2 + \left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\right)^2 = 2$

$$f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$$

$\therefore$ $\text{LHL} = \text{RHL} = f\left(\dfrac{\pi}{2}\right)$

So, $f(x)$ is continuous at $x = \left(\dfrac{\pi}{2}\right)$

Hence, $f(x)$ is continuous over the whole real number.

Sample Problem 15 *Let* $f(x) = \begin{cases} \left[\tan\left(\dfrac{\pi}{4} + x\right)\right]^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases}$.

For what value of k is $f(x)$ continuous at $x = 0$?

(a) 1 (b) e (c) $\dfrac{1}{e}$ (d) e^2

Interpret (d) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[\tan\left(\dfrac{\pi}{4} + x\right)\right]^{1/x}$

$$= \lim_{x \to 0} \left(\frac{1 + \tan x}{1 - \tan x}\right)^{1/4}$$

$$\left[\lim_{x \to 0} \{(1 + \tan x)^{1/\tan x}\}\right]^{\tan x/x} \left[\lim_{x \to 0} \{(1 - \tan x)^{-1/\tan x}\}^{\tan x/x}\right]$$

$= e \times e = e^2$

Since, $f(x)$ is continuous at $x = 0$, $\lim_{x \to a} f(x) = f(0)$

$\Rightarrow$ $e^2 = k$.

Hence, $f(x)$ is continuous at $x = 0$ when $k = e^2$

Sample Problem 16 *$f(x) = x^n$, where n is a positive integer or any positive rational, is*

(a) *continuous* (b) *discontinuous*

(c) *cannot be determined* (d) *None of these*

Interpret (a) Let $x = c$, be any value of x.

Then, $f(c + 0) = \lim_{h \to 0} (c + h)^n = c^n$ and $f(c - 0) = \lim_{h \to 0} (c - h)^n = c^n$

Also, $f(c) = c^n$

$\therefore$ $f(c + 0) = f(c - 0) = f(c)$

Hence, $f(x) = x^n$ is continuous for $x = c$

But c is any value of x. Therefore, $f(x) = x^n$ is continuous for all values of x.

Sample Problem 17 *The function $f(x) = a[x + 1] + b[x - 1]$, where $[x]$ is the greatest integer function, then the condition for which $f(x)$ is continuous at $x = 1$, is*

(a) $a + b = 0$ (b) $a - b = 0$

(c) $a = 2b$ (d) *None of these*

Interpret (a) $f(x)$ is continuous at $x = 1$

$\Rightarrow$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow$ $\lim_{x \to 1^-} a[x + 1] + b[x - 1] = \lim_{x \to 1^+} a[x + 1] + b[x - 1]$

$\Rightarrow$ $a - b = 2a + 0 \cdot b$

$\Rightarrow$ $a + b = 0$

26.11 Continuity of Composition of Function

If f is continuous at $x = c$ and g is continuous at $x = f(c)$, then the composite $g[f(x)]$ is continuous at $x = c$, *e.g.*, $f(x) = \dfrac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous at $x = 0$, hence

the composite $(gof)(x) = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ will also be continuous at $x = 0$.

Sample Problem 18 *The points of discontinuity of* $y = \dfrac{1}{u^2 + u - 2}$, *where* $u = \dfrac{1}{x-1}$, *is*

(a) $\frac{1}{2}, 1, \frac{3}{2}$ (b) $1, \frac{3}{2}, 2$

(c) $\frac{1}{2}, 1, 2$ (d) $1, 2, 3$

Interpret (c) The function $u = f(x) = \dfrac{1}{x-1}$ is discontinuous at the point $x = 1$. ...(i)

The function $y = g(x) = \dfrac{1}{u^2 + u - 2} = \dfrac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$

when $u = -2$, $\dfrac{1}{x-1} = -2$

$$\Rightarrow \qquad x = \frac{1}{2}$$

$$u = 1 \qquad \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$$

Hence, the composite function $y = g[f(x)]$ is discontinuous at three points $x = \frac{1}{2}, 1, 2$

26.12 Applications of Continuity

Intermediate Value Theorem

Suppose $f(x)$ is continuous on an interval I and a and b are any two points of I. Then, if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.

Extreme Value Theorem

If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$ and $m \le f(x) \le M$ for every other x in I.

26.13 Discontinuity

A function f is said to be discontinuous at a point a in its domain, if there is a break in the graph of the function at $x = a$. Then, the point $x = a$ is called a point of discontinuity.

Since, for the function to be continuous, we must have

$$L = R = V \text{ at } x = a$$

i.e., $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

But if any one of the conditions is false, then the function will be continuous at $x = a$. Hence, the discontinuity may arise due to any of the following situations

(i) $L \ne R$

i.e., $\lim\limits_{x \to a^-} f(x) \ne \lim\limits_{x \to a^+} f(x)$

$\Rightarrow$ The limit of function at $x = a$ does not exist.

(ii) $L = R \ne V$

i.e., $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

$\Rightarrow$ The limit of the function exists but is not equal to the value of the function at $x = a$

26.14 Different Kinds of Discontinuity

(i) Removable Discontinuity

A function $f(x)$ is said to have a removable discontinuity at a point α, if $f(\alpha + 0), f(\alpha - 0)$ and $f(\alpha)$ exist but $f(\alpha + 0) = f(\alpha - 0) \ne f(\alpha)$. The function can be made continuous by defining it in such a way that $\lim\limits_{x \to a} f(x) = f(a)$.

e.g., $f(x) = [\sin x]$, where $x \in (0, \pi)$ has a removable discontinuity at $x = \dfrac{\pi}{2}$

$$\because \qquad \lim_{x \to \frac{\pi}{2}} [\sin x] = 0 \text{ but } f\left(\frac{\pi}{2}\right) = 1$$

To remove this we redefine $f(x)$ as follows

$$f(x) = \begin{cases} [\sin x], & x \in \left(0, \frac{\pi}{2}\right)\left(\frac{\pi}{2}, \pi\right) \\ 0, & x = \frac{\pi}{2} \end{cases}$$

Now, $f(x)$ is continuous for $x \in (0, \pi)$.

(ii) Discontinuity of First Kind

A function $f(x)$ is said to have a discontinuity of the first kind or ordinary discontinuity at α, if $f(\alpha + 0)$ and $f(\alpha - 0)$ both exist but are not equal.

(iii) Discontinuity of Second Kind

A function $f(x)$ is said to have a discontinuity of the second kind at α, if none of the limits $f(\alpha + 0)$ and $f(\alpha - 0)$ exist.

(iv) Infinite Discontinuity

If either or both of $f(\alpha + 0)$ and $f(\alpha - 0)$ be infinite ($+\infty$ or $-\infty$), then $f(x)$ has infinite discontinuity at $x = \alpha$.

Sample Problem 19 $f(x) = \begin{cases} \dfrac{a^{[x]+x} - 1}{[x] + x}, & x \neq 0 \\ \log_e a, & x = 0 \end{cases}$, $a > 0$ *at* $x = 0$ *is*

(a) *continuous* (b) *discontinuous*
(c) *cannot be determined* (d) *None of these*

Interpret (b) $\text{RHL} = \lim\limits_{h \to 0} \dfrac{a^{[h]+h} - 1}{[h] + h} = \lim\limits_{h \to 0} \dfrac{a^h - 1}{h} = \log_e a$

$$\text{LHL} = \lim_{h \to 0} \frac{a^{[-h]-h} - 1}{[-h] - h}$$

$$= \lim_{h \to 0} \frac{a^{-h-1} - 1}{-1 - h} \qquad \{\because [-h] = -1\}$$

$$= \frac{a^{-1} - 1}{-1} = 1 - a^{-1}$$

$\because$ $\text{RHL} \neq \text{LHL} = f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$

Sample Problem 20 *Continuity of the function* $[\cos x]$ *at* $x = \dfrac{\pi}{2}$, *where* $[\cdot]$ *denotes the greatest integer function is*

(a) *continuous at* $x = \dfrac{\pi}{2}$ (b) *discontinuous at* $x = \dfrac{\pi}{2}$
(c) *does not exist* (d) *None of these*

Interpret (b) $\text{LHL} = \lim\limits_{x \to \frac{\pi}{2}^-} [\cos x] = 0$

$$\text{RHL} = \lim_{x \to \frac{\pi}{2}^+} [\cos x] = -1$$

$$f\left(\frac{\pi}{2}\right) = \left[\cos \frac{\pi}{2}\right] = 0$$

Clearly, $\text{LHL} \neq \text{RHL}$

So, the function is discontinuous at $x = \dfrac{\pi}{2}$

Note

(i) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

e.g., $f(x) = x$ and $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(ii) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

e.g., $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

(iii) **Continuity of an inverse function** If the function $y = f(x)$ is defined, continuous and strictly monotonic on the interval x, then there exist a single valued inverse function $x = \varphi(y)$ defined, continuous and also strictly monotonic in the range of the function $y = f(x)$.

26.15 Properties of Continuous Functions

(i) If f and g are continuous functions, then
(a) $f \pm g$ and fg are continuous.
(b) cf is continuous, where c is a constant.
(c) $\dfrac{f}{g}$ is continuous at those points, where $g(x) \neq 0$.

(ii) If f is continuous in $[a, b]$, then it is bounded in $[a, b]$ *i.e.*, there exist m and M such that

$$m \leq f(x) \leq M, \forall x \in [a, b]$$

where, m and M are called minimum and maximum values of $f(x)$ respectively in the interval $[a, b]$.

(iii) If f is continuous in $[a, b]$, then f assumes atleast once every value between minimum and maximum values of $f(x)$.

Thus, $a \leq x \leq b \Rightarrow m \leq f(x) \leq M$ or range of $f(x) = [m, M]$, if $x \in [a, b]$.

(iv) If f is continuous in its domain, then $|f|$ is also continuous in its domain.

(v) If f is continuous at a and $f(a) \neq 0$, then there exist an open interval $(a - \delta, a + \delta)$ such that for all $x \in (a - \delta, a + \delta)$, $f(x)$ has the same sing as $f(a)$.

(vi) If f is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there exists atleast one solution of the quation $f(x) = 0$ in the open interval (a, b).

(vii) If f is continuous on $[a, b]$ and maps $[a, b]$ into $[a, b]$, then for some $x \in [a, b]$, we have $f(x) = x$.

(viii) If f is continuous in domain D, then $\dfrac{1}{f}$ is also continuous in $D - \{x : f(x) = 0\}$.

(ix) A function $f(x)$ is said to be everywhere continuous, if it is continuous on the entire real line $(-\infty, \infty)$.

Integral function of a continuous function is a continuous function.

Check Point 2

1. For a real number y, let [y] denotes the greatest integer less than or equal to y, then the function $f(x) = \frac{\tan[\pi(x-\pi)]}{1+[x]^2}$ is
   (a) discontinuous at some x
   (b) continuous at all x but the derivative f′(x) does not exist for some x
   (c) f′(x) exists ∀ x but f′′(x) does not exist for some x
   (d) f′′(x) exists for all x

2. If $f(x) = x(\sqrt{x} + \sqrt{x+1})$, then
   (a) f(x) is continuous but not differentiable at x = 0
   (b) f(x) is differentiable at x = 0
   (c) f(x) is not differentiable at x = 0
   (d) None of the above

3. The set of all points where the function $f(x) = \frac{x}{1+|x|}$ is differentiable, is
   (a) $(-\infty, \infty)$ (b) $[0, \infty)$
   (c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$

4. If $f(x) = \frac{a\cos x - \cos bx}{x^2}$, $x \neq 0$ and f(0) = 4 is continuous at x = 0, then the ordered pair (a, b) is
   (a) (± 1, 3) (b) (1, ± 3)
   (c) (– 1, – 3) (d) (1, 3)

5. If the function $f(x) = \begin{cases} \frac{x^2-(a+2)x+a}{x-2}, & \text{for } x \neq 2 \\ 2, & \text{for } x = 2 \end{cases}$ is continuous at x = 2, then
   (a) a = 0 (b) a = 1
   (c) a = – 1 (d) None of these

Hot Spot 4

Differentiability

Differentiability is also the most important topic for JEE Main Examination. Generally, questions are seen from this topic and also the level of question is from moderate to typical leve.

While solving problems based on differentiability, always keep in mind the concept of continuity and curve sketching.

Let $f(x)$ be a real valued function defined on an open interval (a, b), where $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$ iff $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

This limit is called the derivative or differential coefficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\frac{d}{dx}[f(x)]_{x=c}$

Thus, $f(x)$ is differentiable at $x = c$.

$$\Leftrightarrow \quad \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely.}$$

Differentiability of a Function at a Point

The function $f(x)$ is differentiable at a point P, iff there exists a unique tangent at point P.

In other words, $f(x)$ is differentiable at a point P, iff the curve does not have P as a corner point *i. e.*, the function is not differentiable at those points on which function has holes or sharp edges.

Let us consider the function $f(x) = |x - 1|$

It is not differentiable at $x = 1$. Since, $f(x)$ has sharp edges at $x = 1$.

Progressive Derivative/Right Hand Derivative

Let $f(x)$ be a function and a be a point in the domain of f. Then, right hand derivative is denoted by $Rf'(a)$ and defined as

$$\text{RHD} = Rf'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Regressive Derivative/Left Hand Derivative

Let $f(x)$ be a function and a be a point in the domain of f. Then, left hand derivative is denoted by $Lf'(a)$ and defined as

$$\text{LHD} = Lf'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

Mathematically A function $f(x)$ is said to be differentiable at a point a in its domain, if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist finitely or if and only if

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

i.e., Left hand derivative (LHD) = Right hand derivative (RHD)

or $\qquad Lf'(a) = Rf'(a)$

If $Lf'(a) \neq Rf'(a)$, then function $f(x)$ is not differentiable at $x = a$.

Note A necessary condition for existence of a finite derivative is that the function must be continuous. But it is not the sufficient condition.

Let $f(x)$ has a finite derivative $f'(x_0)$ at $x = x_0$

$$\therefore \quad f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \ldots(i)$$

$$\text{Now, } f(x_0 + h) - f(x_0) = \left\{\frac{f(x_0 + h) - f(x_0)}{h}\right\} h \qquad (h \neq 0)$$

Taking limits on both sides when $h \to 0$, we have

$$\lim_{h \to 0} \{f(x_0 + h) - f(x_0)\} = \lim_{h \to 0} \left\{\frac{f(x_0 + h) - f(x_0)}{h}\right\} \lim_{h \to 0} (h)$$

$$\Rightarrow \quad \lim_{h \to 0} \{f(x_0 + h) - f(x_0)\} = f'(x_0) \cdot 0 \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad \lim_{h \to 0} (x_0 + h) - f(x_0) = 0$$

$$\Rightarrow \quad \lim_{h \to 0} f(x_0 + h) = f(x_0)$$

Hence, $f(x)$ will be continuous at $x = x_0$, if it is differentiable at the same point.

But the converse may not Hold True.

Sample **Problem 21** *The function*

$$f(x) = \begin{cases} \dfrac{|x|(3e^{1/|x|} + 4)}{2 - e^{1/|x|}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \textit{is,}$$

(a) *differentiable at* $x = 0$ (b) *non-differentiable at* $x = 0$

(c) *not continuous at* $x = 0$ (d) *cannot be determined*

Interpret (a) The given function may be written as

$$f(x) = \begin{cases} \dfrac{-x(3e^{-1/x} + 4)}{2 - e^{1/x}}, & x < 0 \\ 0, & x = 0 \\ \dfrac{x(3e^{1/x} + 4)}{2 - e^{1/x}}, & x > 0 \end{cases}$$

For continuity, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x(3e^{-1/x} + 4)}{2 - e^{-1/x}}$

$$= \lim_{x \to 0^-} \frac{-x(3 + 4e^{1/x})}{2e^{1/x} - 1} = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x(3e^{1/x} + 4)}{2 - e^{1/x}}$$

$$= \lim_{x \to 0^+} \frac{x(3 + 4e^{-1/x})}{2e^{-1/x} - 1} = 0$$

Since, $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$, $f(x)$ is continuous at $x = 0$.

For differentiability,

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h(3e^{-1/h} + 4)}{h(2 - e^{-1/h})}$$

$$= \lim_{h \to 0^-} \frac{-(3 + 4e^{-1/h})}{2e^{1/h} - 1} = 3$$

and $\qquad f'(0^+) = \lim_{h \to 0^+} \frac{h(3e^{1/h} + 4)}{h(2 - e^{-1/h})}$

$$\lim_{h \to 0^+} \frac{(3 + 4e^{-1/h})}{2e^{-1/h} - 1} = -3$$

Since, $f'(0^-) \neq f'(0^+)$, $f(x)$ is not differentiable at $x = 0$

Geometrical Interpretation of Differentiability

Geometrically, we interpret $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ as the slope of the tangent to the graph of $y = f(x)$ at the point $[x, f(x)]$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point. In other words, a function is not differentiable at a corner point of a curve, *i.e.*, a point, where the curve suddenly changes direction. See the following graphs

$f(x) = |x - a| + b$

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} x^2, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases}$$

Note that if a function is discontinuous at the point $x = a$, then there is necessarily a break at that point in the graph of function. So, every differentiable function is continuous but the converse is not true. That is a continuous function need not be differentiable.

Differentiability in an Interval

A function $y = f(x)$ defined on an **open interval** (a, b) is said to be differentiable in an open interval (a, b), if it is differentiable at each point of (a, b).

A function $y = f(x)$ defined on a **closed interval** $[a, b]$ is said to be differentiable in closed interval $[a, b]$, if it is differentiable at each point of an open interval (a, b)

and $$\lim_{x \to a^+} \frac{f(x) - f(a)}{(x - a)}$$

and $$\lim_{x \to b^-} \frac{f(x) - f(b)}{(x - b)} \text{ both exist.}$$

OR

A function f is said to be a differentiable function, if it is differentiable at every point of its domain.

Sample Problem 22 *If* $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$, *then* $f(x)$ *is*

(a) continuous on $[-1, 1]$
(b) differentiable on $(-1, 0) \cup (0, 1)$
(c) both (a) and (b)
(d) None of the above

Interpret (a) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$ Clearly, $f(x)$ is continuous on $[-1, 1]$.

The domain of $f(x)$ is $[-1, 1]$.

For $x \neq 0$, $x \neq \pm 1$, we have

$$f'(x) = \frac{1}{2\sqrt{1 - \sqrt{1 - x^2}}} \cdot \frac{x}{\sqrt{1 - x^2}}$$

Since, $f(x)$ is not defined on the right side of $x = 1$ and on the left side of $x = -1$. Also, $f'(x) \to \infty$ when $x \to +1$ or $x \to -1$. So, we check the differentiability at $x = 0$.

Now, $$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0 - h) - f(0)}{h} = \frac{-1}{\sqrt{2}}$$

Similarly, $$Rf'(0) = \frac{1}{\sqrt{2}}$$

Hence, $f(x)$ is not differentiable at $x = 0$.

Causes of non-derivability

(i) a corner, where function is continuous but LHD and RHD differ and are finite.

(ii) a cusp, where the slope approaches ∞ from one side and $-\infty$ from the other.

(iii) a vertical tangent where the slope approaches ∞ from both sides or approaches $-\infty$ from the both sides.

(iv) discontinuity

26.16 Relation between Derivability and Continuity

(i) If $f'(a)$ exists, then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$. In general a function f is derivable at x, then f is continuous at x. *i.e.,* if $f(x)$ is derivable for every point of its domain of definition, then it is continuous in that domain. The converse of the above result is need not be true *e.g.,* The functions $f(x) = |x|$ and $g(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ x, & x = 0 \end{cases}$ both are continuous at $x = 0$ but not derivable at $x = 0$.

(ii) Let $f_+'(a) = \lambda$ and $f_-'(a) = \mu$, where λ and μ are finite, then

(a) $\lambda = \mu \Rightarrow f$ is derivable at $x = a \Rightarrow f$ is continuous at $x = a$.

(b) $\lambda \neq \mu \Rightarrow f$ is not derivable at $x = a$ but f is continuous at $x = a$. If a function f not differentiable but is continuous at $x = a$, it geometrically implies a sharp corner or kink at $x = a$.

(c) If f is not continuous at $x = a$, then it is not differentiable at $x = a$.

26.17 Properties of Differentiable Functions

(i) Every polynomial function is differentiable at each $x \in R$.

(ii) The exponential function a^x, $a > 0$ is differentiable at each $x \in R$.

(iii) Every constant function is differentiable at each $x \in \mathbb{R}$.

(iv) The logarithmic function is differentiable at each point in its domain.

(v) Trigonometric and inverse-trigonometric functions are differentiable in their respective domains.

(vi) The sum, difference, product and quotient of two differentiable functions is differentiable.

(vii) The composition of differentiable function is a differentiable function.

(viii) Absolute functions are always continuous throughout but not differentiable at their critical point.

e.g., $|x - a|$ is continuous throughout but not differentiable at $x = a$.

26.18 Continuity and Differentiability of Different Functions

Type of Functions	Curve	Domain and Range	Continuity and Differentiability
Identity function	$f(x) = x$	Domain $= R$, Range $=]-\infty, \infty[$	Continuous and differentiable in their domain
Exponential function	$f(x) = a^x$, $a > 0$, $a \neq 1$	Domain $= R$, Range $=]0, \infty[$	
Logarithmic function	$f(x) = \log_a x$; $x, a > 0$ and $a \neq 1$	Domain $= (0, \infty)$, Range $= R$	
Root function	$f(x) = \sqrt{x}$	Domain $= [0, \infty)$, Range $= [0, \infty)$	Continuous and differentiable in $(0, \infty)$
Greatest integer function	$f(x) = [x]$	Domain $= R$, Range $= I$	Other than integral value it is continuous and differentiable
Least integer function	$f(x) = (x)$	Domain $= R$, Range $= I$	
Fractional part function	$f(x) = \{x\} = x - [x]$	Domain $= R$, Range $= [0, 1)$	
Signum function	$f(x) = \frac{\lvert x \rvert}{x} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$	Domain $= R$, Range $= \{-1, 0, 1\}$	Continuous and differentiable everywhere except at $x = 0$
Constant function	$f(x)=c$	Domain $=R$, Range $=\{c\}$ and where $c \to$ constant	
Polynomial function		Domain $=R$	Continuous and differentiable everywhere.
Sine function	$y = \sin x$	Domain $= R$, Range $= [-1, 1]$	Continuous and differentiable in their domain
Cosine function	$y = \cos x$	Domain $= R$, Range $= [-1, 1]$	
Tangent function	$y = \tan x$	Domain $= R, = R - \left[(2n+1)\frac{\pi}{2}\right]$, Range $= R$	
Cosecant function	$y = \operatorname{cosec} x$	Domain $= R - n\pi$, Range $= (-\infty, -1] \cup [1, \infty)$	
Secant function	$y = \sec x$	Domain $= R - \left\{(2n+1)\frac{\pi}{2}\right\}$, Range $= (-\infty, -1] \cup [1, \infty)$	
Cotangent function	$y = \cot x$	Domain $= R - \{n\pi\}$, Range $= R$	
Arc sine function	$y = \sin^{-1} x$	Domain $= [-1, 1]$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Continuous and differentiable in their domain
Arc cosine function	$y = \cos^{-1} x$	Domain $= [-1, 1]$, Range $= [0, \pi]$	
Arc tangent function	$y = \tan^{-1} x$	Domain $= R$, Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
Arc cosecant function	$y = \operatorname{cosec}^{-1} x$	Domain $= (-\infty, 1] \cup [1, \infty)$, Range $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$	
Arc secant function	$y = \sec^{-1} x$	Domain $= (-\infty, -1] \cup [1, \infty)$, Range $= [0, \pi] - \left\{\frac{\pi}{2}\right\}$	
Arc cotangent function	$y = \cot^{-1} x$	Domain $= R$, Range $= (0, \pi)$	

Hot Spot 5

Problems on Functional Equations

Problems on functional questions is an important topic of this chapter. Sometimes question are seen from this topic. The level of question is from moderate to typical.

While solving problems based on this topic, always keep in mind the concept of

(i) Limits
(ii) Continuity
(iii) Differentiability
(iv) Function

In this type of problems, firstly we simplify the given condition and then use the derivative formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Further, if it is needed, then we use the integration and simplify it.

Sample Problem 24 *If* $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ *for all* $x, y \in (-1, 1)$ *and* $\lim_{x \to 0} \frac{f(x)}{x} = 2$, *then* $f\left(\frac{1}{\sqrt{3}}\right)$ *and* $f'(1)$ *are*

(a) $\frac{\pi}{3}$ *and* 1 (b) $\frac{\pi}{6}$ *and* $\frac{1}{2}$ (c) $\frac{\pi}{2}$ *and* 0 (d) $\frac{\pi}{4}$ *and* $\sqrt{3}$

Interpret (a) $f(x) = f(y) = f\left(\frac{x+y}{1-xy}\right)$ ...(i)

Putting $x = y = 0$, we get $f(0) = 0$

Putting $y = -x$, we get $f(x) + f(-x) = f(0)$

$\Rightarrow$ $f(-x) = -f(x)$ ...(ii)

Also, $\lim_{x \to 0} \frac{f(x)}{x} = 2$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ ...(iii)

$$= \lim_{h \to 0} \frac{f(x+h) + f(-x)}{h}$$

[using Eq. (ii), $f(-x) = -f(x)$]

$$f'(x) = \lim_{h \to 0} \frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$$ [using Eq. (i)]

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \left(\frac{1}{1+xh+x^2}\right)$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \lim_{h \to 0} \frac{1}{1+xh+x^2}$$

$$\left(\text{using } \lim_{x \to 0} \frac{f(x)}{x} = 2\right)$$

$$\Rightarrow \quad f'(x) = 2 \times \frac{1}{1+x^2}$$

$$\Rightarrow \quad f'(x) = \frac{2}{1+x^2}$$

On integrating both sides, $f(x) = 2\tan^{-1}(x) + C$

where $f(0) = 0 \Rightarrow C = 0$. Thus, $f(x) = 2\tan^{-1} x$

Hence,

$$f\left(\frac{1}{\sqrt{3}}\right) = 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$$

and $f'(1) = \frac{2}{1+1^2} = \frac{2}{2} = 1$

Sample Problem 25 *Let* $f : R \to R$ *is a function satisfies condition* $f(x + y^3) = f(x) + [f(y)]^3$ *for all* $x, y \in R$. *If* $f'(0) \geq 0$, *then* $f(10)$ *is*

(a) 8 (b) 9
(c) 0 *or* 10 (d) 2 *or* 4

Interpret (c) Given, $f(x + y^3) = f(x) + [f(y)]^3$ ...(i)

and $f'(0) \geq 0$ ...(ii)

Replacing x, y by 0

$f(0) = f(0) + f(0)^3 \Rightarrow f(0) = 0$ ...(iii)

Also, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$ ...(iv)

Let $I = f'(0) = \lim_{h \to 0} \frac{f[0 + (h^{1/3})^3] - f(0)}{(h^{1/3})^3}$

$$= \lim_{h \to 0} \frac{f[(h^{1/3})^3]}{(h^{1/3})^3}$$

$$= \lim_{h \to 0} \left[\frac{f(h^{1/3})}{(h^{1/3})}\right]^3 = I^3$$

$$I = I^3$$

or $\quad l = 0, 1, -1$ as $f'(0) \geq 0$

$\therefore \quad f'(0) = 0, 1 \quad \ldots(v)$

Thus,

$$f'(x) = \lim_{h \to 0} f \frac{(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f[x + (h^{1/3})^3] - f(x)}{(h^{1/3})^3}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) - [f(h^{1/3})]^3 - f(x)}{(h^{1/3})^3} \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \left[\frac{f(h^{1/3})}{(h^{1/3})}\right] = [f'(0)]^3$$

$$\Rightarrow \quad f'(x) = 0, 1 \quad \text{[as } f'(0) = 0, 1 \text{ using Eq. (v)]}$$

On integrating both sides, $f(x) = c$ or $x + c$ as $f(0) = 0 \Rightarrow f(x) = 0$ or x.

Thus, $f(10) = 0$ or 10.

Check Point 3

1. If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then f(x) is differentiable on
   (a) $[-1, 1]$ (b) $R - \{-1, 1\}$
   (c) $R - (-1, 1)$ (d) None of these
2. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ and if f(x) is differentiable at $x = 0$, then
   (a) $a = b = c = 0$
   (b) $a = 0, b = 0, c \in R$
   (c) $b = c = 0, a \in R$
   (d) $c = 0, a = 0, b \in R$
3. The set of points where the function $f(x) = \sqrt{1 - e^{-x^2}}$ is differentiable, is
   (a) $(-\infty, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$
   (c) $(-1, \infty)$ (d) None of these
4. Let $f(x + y) = f(x) f(y), \forall x, y \in R$, $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ equals
   (a) 6 (b) 3
   (c) 5 (d) None of these
5. If $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)} \forall x, y \in R, y \neq 0$ and $f(t) \neq 0$, if $t \neq 0$ and $f'(1) = 3$, then f(x), is
   (a) x
   (b) x^2
   (c) x^3
   (d) Constant

WORKED OUT

Examples

Example 1 $\lim_{x \to 0} \frac{\sqrt{1-\cos x}}{x}$ *is equal to*

(a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) 0 (d) *does not exist*

Solution (d) $\sqrt{1-\cos x} = \begin{cases} -\sqrt{2}\sin\frac{x}{2}, & x<0 \\ \sqrt{2}\sin\frac{x}{2}, & x \geq 0 \end{cases}$

Therefore, $\lim_{x \to 0^-} \frac{\sqrt{1-\cos x}}{x} = \lim_{x \to 0^-} \frac{-\sqrt{2}\sin\frac{x}{2}}{x} = -\frac{1}{\sqrt{2}}$.

$$\lim_{x \to 0^+} \frac{\sqrt{1-\cos x}}{x} = \lim_{x \to 0^+} \frac{\sqrt{2}\sin\frac{x}{2}}{x} = \frac{1}{\sqrt{2}}$$

Since, $\lim_{x \to 0^-} \frac{\sqrt{1-\cos x}}{x} \neq \lim_{x \to 0^+} \frac{\sqrt{1-\cos x}}{x}$,

$\lim_{x \to 0} \frac{\sqrt{1-\cos x}}{x}$ does not exist.

Example 2 $\lim_{n \to \infty} \frac{[x]+[2x]+[3x]+\ldots+[nx]}{n^2}$, *where* $[\cdot]$ *denotes the greatest integer function, is equal to*

(a) $\frac{x}{2}$ (b) x

(c) $2x$ (d) *None of these*

Solution (a) We know that, $x-1<[x] \leq x$

$$\Rightarrow \quad x+2x+\ldots+nx-n < \sum_{r=1}^{n}[rx] \leq x+2x+\ldots+nx$$

$$\Rightarrow \quad \frac{x \cdot n(n+1)}{2} - n < \sum_{r=1}^{n}[rx] \leq \frac{x \cdot n(n+1)}{2}$$

$$\Rightarrow \quad \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum_{r=1}^{n}[rx] \leq \frac{x}{2}\left(1+\frac{1}{n}\right).$$

Now, $\lim_{n \to \infty} \frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$

and $\lim_{n \to \infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Using Sandwich theorem, we find that

$$\lim_{x \to \infty} \frac{[x+[2x]+\ldots+[nx]]}{n^2} = \frac{x}{2}$$

Alternative Solution

We know that $[rx] = rx - \{x_r\}$ for $r = 1, 2, 3, \ldots, n$ and $0 \leq \{x_r\} < 1$ for each r,

Also, $\sum_{r=1}^{n}[rx] = \sum_{r=1}^{n}[(rx)-\{x_r\}] = x\sum_{r=1}^{n} r - \sum_{r=1}^{n}\{x_r\} = x\frac{n(n+1)}{2} - k$

where, $k<n$ (since each $\{x_r\}<1$).

Hence, $\lim_{n \to \infty} \frac{1}{n^2}\sum_{r=1}^{n}[rx] = \lim_{n \to \infty}\sum_{r=1}^{n}\left[\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{k}{n^2}\right] = \frac{x}{2}$

Example 3 $\lim_{x \to 3} \frac{[x]-3}{(x-3)}$, *is equal to*

(a) 0 (b) 2

(c) 3 (d) *does not exist*

Solution (d) $\lim_{x \to 3} \frac{[x]-3}{x-3}$

Towards the right of $x=3, [x]=3$

$\Rightarrow$ $[x]-3=0$, in the right neighbourhood of $x=3$

$$\Rightarrow \quad \lim_{x \to 3+0} \frac{[x]-3}{x-3} = 0$$

Towards the left of $x=3, [x]=2$

$\Rightarrow$ $[x]-3=-1$, in the left neighbourhood of $x=3$

$$\Rightarrow \quad \lim_{x \to 3-0} \frac{[x]-3}{x-3} = \lim_{x \to 3-0} \frac{-1}{x-3} = \infty$$

Thus, $\lim_{x \to 3} \frac{[x]-3}{x-3}$ does not exist.

Example 4 *Redefine the function* $f(x) = [x] + [-x]$ *in such a way that it becomes continuous for*

(a) $x \in (0, 2)$ (b) $x \in (0, 1)$

(c) $x \in R$ (d) *None of these*

Solution (a) Here, $\lim_{x \to 1} f(x) = -1$ but $f(1) = 0$

Hence, $f(x)$ has a removable discontinuity at $x=1$

To remove this we define $f(x)$ as follows

$$f(x) = [x] + [-x], x \in (0, 1) \cup (1, 2)$$
$$= -1, \quad x = 1.$$

Now, $f(x)$ is continuous for $x \in (0, 2)$

Example 5 $\lim_{x \to y} \frac{x^y - y^x}{x^x - y^y}$, *is equal to*

(a) $\frac{1 + \log y}{\log y}$ (b) $\frac{1 - \log y}{1 + \log y}$

(c) $\frac{1 - \log y}{\log (ey)}$ (d) $\log (ex)$

Solution (c) $\lim_{x \to y} \frac{x^y - y^x}{x^x - y^y}$; $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to y} \frac{yx^{y-1} - y^x \log y}{x^x \log (ex)} = \frac{1 - \log y}{\log (ey)}$$

Example 6 *Let* $f(x) = \frac{|x^3 - 6x + 11x - 6|}{x^3 - 6x^2 + 11x - 6}$, *then the set of points ' a' where* $\lim_{x \to a} f(x)$ *does not exist.*

(a) $a = 2, 3$ (b) $a = 1, 2$

(c) $a = -1, 1$ (d) $a = 0, 1$

Solution (a) We write, $f(x) = \left(\frac{|x-1|}{x-1}\right)\left(\frac{|x-2|}{x-2}\right)\left(\frac{|x-3|}{x-3}\right)$

$$= \begin{cases} -1, & x < 1 \\ 1, & 1 < x < 2 \\ -1 & 2 < x < 3 \\ 1, & x > 3 \end{cases}$$

Therefore, the limits exists at all points except at $x = 1, 2, 3$. For example, at $x = 1$

$$\lim_{x \to 1^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 1$$

Since, $\lim_{x \to 1} f(x) \neq \lim_{x \to 1^+} f(x)$

Since, $\lim_{x \to 1} f(x)$ does not exist.

Similarly, $\lim_{x \to a} f(x)$ does not exist when $a = 2, 3$

Example 7 *The values of a and b, So that* $\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{\sin^3 x}$ *may be equal to* 1.

(a) $a = \frac{5}{2}, b = \frac{3}{2}$ (b) $a = \frac{3}{2}, b = \frac{1}{2}$

(c) $a = \frac{-3}{2}, b = -\frac{1}{2}$ (d) $a = -\frac{5}{2}, b = -\frac{3}{2}$

Solution (d) We write,

$$\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{\sin^3 x} = \left[\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{x^3}\right] \left[\lim_{x \to 0} \frac{x}{\sin x}\right]^3$$

$$= \lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{1 + a \cos x - ax \sin x - b \cos x}{3x^2} \quad \text{(using L'Hospital rule)}$$

The denominator being 0 for $x = 0$, the expression will lead to a finite limit if and only if the numerator is also zero for $x = 0$. This happens when

$$1 + a - b = 0. \quad \ldots\text{(i)}$$

Assuming that Eq. (i) is satisfied, we have

$$\lim_{x \to 0} \frac{1 + (a - b) \cos x - ax \sin x}{3x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{(b - 2a) \sin x - ax \cos x}{6x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{(b - 3a) \cos x + ax \sin x}{6} = \frac{b - 3a}{6}$$

As given, $\frac{b - 3a}{6} = 1 \Rightarrow b - 3a = 6$ ...(ii)

From Eqs. (i) and (ii), we get $a = -\frac{5}{2}$ and $b = -\frac{3}{2}$

Alternative solution

We write,

$$\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{x\left[1 + a\left(1 - \frac{x^2}{2} + \frac{x^2}{24} - \ldots\right)\right] - b\left(x - \frac{x^3}{6} + \ldots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{x(1 + a - b) + x^3 \frac{b}{6} - \frac{a}{2} + \text{terms of order } x^4}{x^3} = 1$$

(given)

$$\Rightarrow \quad 1 + a - b = 0 \quad \text{and} \quad \frac{b}{6} - \frac{a}{2} = 1$$

$$\Rightarrow \quad a = -\frac{5}{2}, b = -\frac{3}{2}.$$

Example 8 $\lim_{x \to 4} \frac{(\cos \alpha)^x - (\sin \alpha)^x - \cos 2\alpha}{(x - 4)}$, $\alpha \in \left(0, \frac{\pi}{2}\right)$ *is equal to*

(a) $\cos^4 \alpha \cdot \log (\cos \alpha) - \sin^4 \alpha \cdot \log (\sin \alpha)$

(b) $\sin^4 \alpha \cdot \log (\cos \alpha) - \cos^4 \alpha \cdot \log (\sin \alpha)$

(c) $\sin^4 \alpha \cdot \log (\cos \alpha) + \cos^4 \alpha \cdot \log (\sin \alpha)$

(d) *None of the above*

Solution (a) $\lim_{x \to 4} \frac{(\cos \alpha)^x - (\sin \alpha)^x - \cos 2\alpha}{(x - 4)}$, $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 4} \frac{(\cos \alpha)^x - (\sin \alpha)^x - (\cos^2 \alpha - \sin^2 \alpha)(\cos^2 \alpha + \sin^2 \alpha)}{(x - 4)}$$

$$= \lim_{x \to 4} \frac{(\cos \alpha)^x - (\sin \alpha)^x - \cos^4 \alpha + \sin^4 \alpha}{(x - 4)}$$

$$= \lim_{x \to 4} \frac{(\cos \alpha)^4 [(\cos \alpha)^{x-4} - 1] - \sin^4 \alpha [(\sin \alpha)^{x-4} - 1]}{(x - 4)}$$

$$= \cos^4 \alpha \cdot \lim_{x \to 4} \frac{(\cos \alpha)^{x-4} - 1}{x - 4} - \sin^4 \alpha \cdot \lim_{x \to 4} \frac{(\sin \alpha)^{x-4} - 1}{x - 4}$$

$$= \cos^4 \alpha \cdot \ln (\cos \alpha) - \sin^4 \alpha . \ln (\sin \alpha)$$

Example 9 $\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin^{-1}x}{\sin^3 x}$, *is equal to*

(a) 0 (b) $-\dfrac{1}{2}$

(c) $-\dfrac{1}{3}$ (d) $\dfrac{1}{3}$

Solution (b) $\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin^{-1}x}{\sin^3 x}=\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin^{-1}x}{x^3\left(\dfrac{\sin^3 x}{x^3}\right)}$

$$=\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3}$$

$$=\lim_{x\to 0}\frac{\tan^{-1}x-\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}}{x^3}=\lim_{x\to 0}\frac{\tan^{-1}\left(\dfrac{x-\dfrac{x}{\sqrt{1-x^2}}}{1+\dfrac{x^2}{\sqrt{1-x^2}}}\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{\tan^{-1}\left(\dfrac{x\sqrt{1-x^2}-x}{x^2+\sqrt{1-x^2}}\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{\tan^{-1}\left(\dfrac{x\sqrt{1-x^2}-x}{x^2+\sqrt{1-x^2}}\right)(x\sqrt{1-x^2}-x)}{x^3\cdot\left(\dfrac{x\sqrt{1-x^2}-x}{x^2+\sqrt{1-x^2}}\right)(x^2+\sqrt{1-x^2})}$$

$$=\lim_{x\to 0}\frac{\sqrt{1-x^2}-1}{x^2(x^2+\sqrt{1-x^2})}$$

$$=\lim_{x\to 0}\frac{1-x^2-1}{x^2(x^2+\sqrt{1-x^2})(\sqrt{1-x^2}+1)}=-\frac{1}{2}$$

Example 10 *If*

$$\lim_{x\to\infty}(\sqrt{x^4+ax^3+3x^2+bx+2})-\sqrt{x^4+2x^3-cx^2+3x-d}=4,$$

then the value of a, b, c and d is

(a) $a=1, b=2, c=3$ *and* $d=4$

(b) $a=2, b=3, c=5$ *and* $d=0$

(c) $a=2, b=4, c=5$ *and* $d\in R$

(d) $a=2, c=5$ *and* $b, d\in R$

Solution (d) Given that ,

$$4=\lim_{x\to\infty}(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d})$$

$$=\lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+2+d}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$$

Since, the limit is finite, the degree of the numerator must be at the most $2\Rightarrow a-2=0$ *i.e.*, $a=2$

Hence, $4=\lim_{x\to\infty}\dfrac{(3+c)+\dfrac{b-3}{x}+\dfrac{2+d}{x^2}}{\sqrt{1+\dfrac{a}{x}+\dfrac{3}{x^2}+\dfrac{b}{x^3}+\dfrac{2}{x^4}}+\sqrt{1+\dfrac{2}{x}-\dfrac{c}{x^2}+\dfrac{3}{x^3}-\dfrac{d}{x^4}}}$

$$=\frac{3+c}{2}.$$

Therefore, $c=5$

Hence, $a=2, c=5$ and b, d are any real numbers.

Example 11 *Let* $f(x+y)=f(x)+f(y)$ *for all x and y. If the function* $f(x)$ *is continuous at* $x=0$, *then* $f(x)$ *is continuous*

(a) *only at* $x=0$

(b) *at* $x\in R-\{0\}$

(c) *for all x*

(d) *None of the above*

Solution (d) We are given that

$f(x+y)=f(x)+f(y)$; for all x and y.

Since, $f(x)$ is continuous at $x=0$, we have $\lim_{x\to 0}f(x)=f(0)$.

To show that $f(x)$ is continuous at any point a, we shall prove that $\lim_{x\to a}f(x)=f(a)$

$\Rightarrow$ $\lim_{h\to 0}f(a+h)=f(a)$

Indeed, $\lim_{h\to 0}f(a+h)=\lim_{h\to 0}[f(a)+f(h)]$

$$=f(a)+\lim_{h\to 0}f(h)=f(a)+f(0)$$

$$=f(a+0)=f(a)$$

Example 12 *Let* $f(x)=\dfrac{1}{x-1}$, *then the points of discontinuity of the composite function* $y=f[f\{f(x)\}]$, *are*

(a) $x=\dfrac{1}{2}, 1$ *and* 2 (b) $x=0, \dfrac{1}{2}, 1$

(c) $x=0, 1, 2$ (d) $x=1, \dfrac{3}{2}, 2$

Solution (d) We know that $f(x)=\dfrac{1}{x-1}$ is discontinuous at $x=1$

For, $x\neq 1$, $f\{f(x)\}=\dfrac{1}{\dfrac{1}{x-1}-1}=\dfrac{x-1}{2-x}$ is discontinuous at $x=2$

For $x\neq 1$ and 2; $f[f\{f(x)\}]=\dfrac{1}{\dfrac{x-1}{2-x}-1}$

$=\dfrac{2-x}{2x-3}$, which is discontinuous at $x=\dfrac{3}{2}$

Hence, the points of discontinuity are $x=1, x=\dfrac{3}{2}$ and $x=2$

Example 13 *A function $f: R \to R$ satisfies the equation $f(x+y) = f(x)\, f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Then, $f'(x)$ is equal to*

(a) $f(x)$ (b) $2f(x)$

(c) $\frac{1}{2} f(x)$ (d) *None of these*

Solution (b) We are given that,

$$f(x+y) = f(x)\, f(y) \quad \ldots(i)$$

and
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2 \quad \ldots(ii)$$

Now,
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x+0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)\, f(h) - f(x)\, f(0)}{h} \quad \text{[by Eq. (i)]}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2f(x), \quad \text{[by Eq. (ii)]}$$

Example 14 *Let* $f(x) = \begin{cases} -4, & -4 \le x < 0 \\ x^2 - 4, & 0 \le x \le 4 \end{cases}$,

then $g(x) = f(|x|) + |f(x)|$ is

(a) *continuous as well as differentiable at $x = 0, 2$*

(b) *continuous but not differentiable at $x = 0, 2$*

(c) *Neither continuous nor differentiable at $x = 0, 2$*

(d) *Continuous as well as differentiable at $x = 0$ and continuous at $x = 0$ but not differentiable at $x = 2$*

Solution (d) $-4 \le x \le 4 \Rightarrow 0 \le |x| \le 4$

$$\Rightarrow \quad |f(x)| = \begin{cases} |-4| & -4 \le x < 0 \\ |x^2 - 4| & 0 \le x \le 4 \end{cases}$$

i.e.,
$$|f(x)| = \begin{cases} 4 & -4 \le x < 0 \\ 4 - x^2 & 0 \le x < 2 \\ x^2 - 4 & 2 \le x \le 4 \end{cases}$$

and
$$f(|x|) = x^2 - 4, -4 \le x \le 4$$

$$\Rightarrow \quad g(x) = \begin{cases} x^2 & -4 \le x < 0 \\ 0 & 0 \le x < 2 \\ 2x^2 - 8 & 2 \le x \le 4 \end{cases}$$

At $x = 0$, $g(x)$ is continuous as well as differentiable.
At $x = 2$, $g(x)$ is continuous but not differentiable.

Example 15 *Function* $f(x) = \begin{cases} |x+1|, & x < -2 \\ 2x + 3, & -2 \le x < 0 \\ x^2 + 3 & 0 \le x < 3 \\ x^3 - 15, & x \ge 3 \end{cases}$ *is*

(a) *continuous at all points in R.*

(b) *continuous only at $x = -2$*

(c) *continuous at all points in R except at $x = -2$*

(d) *None of the above*

Solution (c) We rewrite $f(x)$ as $f(x) = \begin{cases} -x - 1, & x < -2 \\ 2x + 3, & -2 \le x < 0 \\ x^2 + 3 & 0 \le x < 3 \\ x^3 - 15, & x \ge 3 \end{cases}$

As we can see, $f(x)$ is defined as a polynomial function in each of the intervals $(-\infty, -2), (-2, 0), (0, 3)$ and $(3, \infty)$. Therefore, it is continuous in each of these four intervals.

At the point $x = -2$.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to 2^-} (-x - 1) = 1$$

and
$$\lim_{x \to -2^+} f(x) = \lim_{x \to 2^+} (2x + 3) = -1,$$

Therefore, $\lim_{x \to -2} f(x)$ does not exist.

Thus, $f(x)$ is discontinuous at $x = -2$. At the point $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 3,$$

Thereofre, $f(x)$ is continuous at $x = 0$. At the point $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) = 12,$$

Therefore, $f(x)$ is continuous at $x = 3$.

So, we conclude that $f(x)$ is continuous at all points in R except at $x = -2$

Example 16 $\lim_{x \to 1} (\log_2 2x)^{\log_x 5}$ *is equal to*

(a) $\log_2 5$ (b) $e^{\log_2 5}$ (c) e (d) 0

Solution (b) $\lim_{x \to 1} (\log_2 2x)^{\log_x 5}$

$$= \lim_{x \to 1} (\log_2 2 + \log_2 x)^{\log_x 5}$$

$$= \lim_{x \to 1} \left[(1 + \log_2 x)^{\frac{1}{\log_2 x}} \right]^{\log_2 x \log_x 5}$$

$$= e^{\lim_{x \to 1} \log_2 5} = e^{\log_2 5}$$

Example 17 $\lim_{x \to 0} \frac{\tan[e^2]\, x^2 - \tan[-e^2]\, x^2}{\sin^2 x}$ *is equal to*

(a) 15 (b) 14

(c) 13 (d) 12

Solution (a) $\lim_{x \to 0} \frac{\tan[e^2]\, x^2 - \tan[-e^2]\, x^2}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{\tan 7x^2 + \tan 8x^2}{\sin^2 x} \quad (\because 7 < e^2 < 8)$$

$$= \lim_{x \to 0} \frac{\frac{\tan 7x^2}{7x^2} 7 + \frac{\tan 8x^2}{8x^2} 8}{\left(\frac{\sin x}{x}\right)^2}$$

$$= \frac{7 + 8}{1} = 15$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Problem Based on Limits

1. $\lim_{m \to \infty} \left(\cos \frac{x}{m}\right)^m$ is equal to

(a) 0 (b) e

(c) $\frac{1}{e}$ (d) 1

2. The value of $\lim_{x \to 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x}$ is

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{6}{5}$ (d) $\frac{5}{6}$

3. If $\lim_{x \to \infty} \left[\frac{x^3 + 1}{x^2 + 1} - (ax + b)\right] = 2$, then

(a) $a = 1$ and $b = 1$ (b) $a = 1$ and $b = -1$

(c) $a = 1$ and $b = -2$ (d) $a = 1$ and $b = 2$

4. $\lim_{n \to \infty} \left(\frac{n^2 - n + 1}{n^2 - n - 1}\right)^{n(n-1)}$ is equal to

(a) e (b) e^2

(c) e^{-1} (d) 1

5. If $x_n = \frac{1 - 2 + 3 - 4 + 5 - 6 + \ldots - 2n}{\sqrt{n^2 + 1} + \sqrt{4n^2 - 1}}$, then $\lim_{n \to \infty} x_n$ is equal to

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$

(c) $\frac{2}{3}$ (d) 1

6. $\lim_{x \to \infty} \frac{(x + 1)^{10} + (x + 2)^{10} + \ldots + (x + 100)^{10}}{x^{10} + 10^{10}}$ is equal to

(a) 0 (b) 1

(c) 10 (d) 100

7. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

(a) $-\pi$ (b) π

(c) $\pi/2$ (d) 1

8. The value of $\lim_{x \to \frac{1}{\alpha}} \frac{\sin(cx^2 + bx + a)}{x\alpha - 1}$ (where α and β are the roots of $ax^2 + bx + c = 0$) is

(a) $c\left(\frac{\beta - \alpha}{\alpha\beta}\right)$ (b) $\left(\frac{\beta - \alpha}{\alpha}\right)$

(c) $\frac{1}{\alpha}$ (d) $\frac{c}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$

9. $\lim_{x \to 0} \frac{\cos(\sin x) - 1}{x^2}$ is equal to

(a) 1 (b) -1

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

10. The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is

(a) 1 (b) 2 (c) 3 (d) 4

11. $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$ [given that $f'(2) = 6$ and $f'(1) = 4$]

(a) does not exist (b) is equal to $-\frac{3}{2}$

(c) is equal to $\frac{3}{2}$ (d) is equal to 3

12. $\lim_{n \to 0} \frac{\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{n^2}\right)}{\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\ldots\left(1 - \frac{1}{n}\right)}$ is equal to

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) 2 (d) -2

13. $\lim_{x \to 1} (1 - x^2) \log_{(x^2)} 2$ is equal to

(a) $\log 2$ (b) $\log 3$

(c) $-\log 2$ (d) $-\log 3$

14. The value of
$$\lim_{x \to \frac{\pi}{2}} \tan^{-1}\left(\frac{\sin(a \tan^3 x + b \tan^2 x + c \tan x)}{a \tan^3 x + b \tan^2 x + c \tan x}\right) \text{ is}$$
(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

15. The value of $\lim_{x \to 0} \left(\frac{e^{x \log (2^x - 1)} - (2^x - 1)^x \sin x}{e^{x \log x}}\right)^{1/x}$ is
(a) e (b) $\frac{1}{e} \log 2$
(c) $e \log 2$ (d) None of these

16. If $\lim_{x \to 0} (1 + ax + bx^2)^{2/x} = e^3$, then
(a) $a = 3, b = 0$ (b) $a = \frac{3}{2}, b \neq 1$
(c) $a = \frac{3}{2}, b = 4$ (d) $a = 2, b = 3$

17. $\lim_{x \to \infty} \frac{e^{x^2} - 1}{e^{x^2} + 1}$ is equal to
(a) 0 (b) 1
(c) –1 (d) does not exist

18. $\lim_{n \to \infty} \frac{1^3 + 2^3 + \ldots + n^3}{(n^2 + 1)^2}$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) $\frac{3}{2}$ (d) None of these

19. $\lim_{x \to 1^-} \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2}$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) $\frac{2}{3}$ (d) None of these

20. $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{2}{3}$ (d) 0

21. $\lim_{x \to 0} (1 + \sin x)^{1/x^2}$ is equal to
(a) 0 (b) 1
(c) $\frac{1}{2}$ (d) does not exist

22. $\lim_{n \to \infty} \left\{\frac{1}{1 - n^2} + \frac{2}{1 - n^2} + \ldots + \frac{n}{1 - n^2}\right\}$ is equal to
(a) 0 (b) $-\frac{1}{2}$
(c) $\frac{1}{2}$ (d) 1

23. $\lim_{x \to 0} \frac{e^{\sin x} - \sin x - 1}{x^2}$ is equal to
(a) $\frac{1}{3}$ (b) 1
(c) e (d) $\frac{1}{2}$

24. $\lim_{x \to \frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ is equal to
(a) 1 (b) $\frac{\pi}{2}$
(c) $\frac{1}{16}$ (d) 0

25. $\lim_{x \to \infty} \left\{\frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{1 + |x|^3}\right\}$ is
(a) 2 (b) 1
(c) –1 (d) does not exist

26. If $\lim_{x \to 5} \frac{x^\lambda - 5^\lambda}{x - 5} = 500$, then positive values of λ is
(a) 3 (b) 4 (c) 5 (d) 6

27. $\lim_{n \to \infty} n \cos\left(\frac{\pi}{4n}\right) \cdot \sin\left(\frac{\pi}{4n}\right)$ is equal to
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{9}$ (d) $\frac{\pi}{3}$

28. $\lim_{x \to 0} \frac{1 - \cos 3x}{x(3^x - 1)}$ is equal to
(a) $\frac{9}{2}$ (b) $\frac{9}{(2 \log 3)}$
(c) $\frac{9 \log 3}{2}$ (d) 1

29. If $f(x)$ is continuous and $f\left(\frac{9}{2}\right) = \frac{2}{9}$, then $\lim_{x \to 0} f\left(\frac{1 - \cos 3x}{x^2}\right)$ is equal to
(a) $\frac{9}{2}$ (b) $\frac{2}{9}$
(c) 0 (d) $-\frac{9}{2}$

30. If $f(x) = \begin{cases} \frac{\sin [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x \to 0} f(x)$ equals
(a) 1 (b) 0
(c) –1 (d) does not exist

31. $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$ is equal to

(a) 1 (b) −1
(c) 0 (d) does not exist

32. The value of $\lim_{x \to 1} \left(\tan \frac{\pi x}{4}\right)^{\tan \frac{\pi x}{2}}$ is

(a) e^{-2} (b) e^{-1} (c) e (d) 1

33. $\lim_{x \to 0} (\cos x)^{1/x^4}$

(a) 0 (b) 1
(c) e (d) limit does not exist

34. $\lim_{x \to 0} (1^{\operatorname{cosec}^2 x} + 2^{\operatorname{cosec}^2 x} + \ldots + n^{\operatorname{cosec}^2 x})^{\sin^2 x}$ is

(a) 1 (b) $\frac{1}{n}$ (c) n (d) 0

Problems Based on Continuity

35. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$ is [NCERT]

(a) continuous everywhere except at $x = 1$
(b) continuous everywhere except at $x = 0$
(c) discontinuous everywhere
(d) None of the above

36. The values of a and b, so that the function defined by $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$, are [NCERT]

(a) $a = 1, b = \frac{1}{2}$ (b) $a = \frac{3}{2}, b = \frac{5}{2}$
(c) $a = 1, b \in R$ (d) $a = b + \frac{2}{3}$

37. For what value of λ is the function $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0, 1$ is [NCERT]

(a) $\lambda = 0$ (b) $\lambda = 1$ (c) $\lambda = 0, 1$ (d) $\lambda \in R$

38. Function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$ is [NCERT]

(a) continuous at $x = 0$
(b) discontinuous at $x = 0$
(c) discontinuous at every point
(d) continuous at every point

39. For what value of k, function $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \ne \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$. [NCERT]

(a) 1 (b) 3
(c) R (d) 6

40. The values of a and b. Such that the function defined by $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$ is a continuous function is [NCERT]

(a) $a = 1$ and $b = 2$ (b) $a = 2$ and $b = 1$
(c) $a, b \in R$ (d) None of these

41. $f(x) = \begin{cases} |x| \cos\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is [NCERT Exemplar]

(a) discontinuous at $x = 0$ (b) continuous at $x = 0$
(c) does not exist (d) None of these

42. $f(x) = \begin{cases} |x - a| \sin \frac{1}{x - a}, & \text{if } x \ne 0 \\ 0, & \text{if } x = a \end{cases}$ is [NCERT]

(a) continuous at $x = a$ (b) discontinuous at $x = a$
(c) differentiable at $x = a$ (d) None of these

43. $f(x) = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \frac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ is [NCERT]

(a) discontinuous at $x = 1$ (b) discontinuous at $x = 2$
(c) continuous at $x = 1$ (d) None of these

44. For what value of k, $f(x) = \begin{cases} \frac{2^{x+2}}{4^x - 16}; & x \ne 2 \\ k; & x = 2 \end{cases}$ is continuous at $x = 2$ [NCERT]

(a) 1 (b) $\frac{3}{2}$
(c) 2 (d) $\frac{1}{2}$

45. For what value of k, the function $f(x) = \begin{cases} \frac{\sqrt{1 + kx} - \sqrt{1 - kx}}{x}, & \text{if } -1 \le x < 0 \\ \frac{2x + 1}{x - 2}, & \text{if } 0 \le x \le 1 \end{cases}$ is continuous at $x = 0$ [NCERT]

(a) $\frac{1}{2}$ (b) 1
(c) $-\frac{3}{2}$ (d) $-\frac{1}{2}$

46. For what value of k, the function $f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & x \ne 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$ [NCERT]

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{3}{2}$ (d) no value

47. The value of a and b such that the function f defined by $f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$ is continuous function at $x=4$, are [NCERT]

(a) $a=1$ and $b=-1$ (b) $a=-1$ and $b=1$
(c) $a=0$ and $b=-1$ (d) $a=1$ and $b=0$

48. If $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq 2\\ k, & x=2\end{cases}$ is continuous at $x=2$, then the value of k is [NCERT]

(a) 1 (b) 3
(c) 6 (d) 7

49. If $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, x\neq\dfrac{\pi}{4}$, then the value of $f\left(\dfrac{\pi}{4}\right)$, so that $f(x)$ becomes continuous at $x=\dfrac{\pi}{4}$ is [NCERT]

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$
(c) 1 (d) 9

50. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{if } x<0\\ a & \text{if } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0\end{cases}$ for what value of a, f is continuous at $x=0$? [NCERT]

(a) 2 (b) 4
(c) 6 (d) 8

51. The number of points at which the function $f(x)=\dfrac{1}{x-[x]}$ ([·] denotes, the greatest integer function) is not continuous is [NCERT Exemplar]

(a) 1 (b) 2
(c) 3 (d) None of these

52. For the function $f(x)=\dfrac{\log_e(1+x)-\log_e(1-x)}{x}$ to be continuous at $x=0$, the value of $f(0)$ should be

(a) -1 (b) 0
(c) -2 (d) 2

53. If $f(x)=(\sin x)^{\frac{1}{\pi-2x}}$, $x\neq\dfrac{\pi}{2}$ is continuous at $x=\dfrac{\pi}{2}$, then $f\left(\dfrac{\pi}{2}\right)$ is

(a) 0 (b) 1
(c) e (d) None of these

54. If the function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}, (x\neq 0)$ is continuous at each point of its domain, then the value of $f(0)$ is

(a) 2 (b) $\dfrac{1}{3}$
(c) $\dfrac{2}{3}$ (d) $-\dfrac{1}{3}$

55. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\left\{1-\tan\left(\frac{x}{2}\right)\right\}(1-\sin x)}{\left\{1+\tan\left(\frac{x}{2}\right)\right\}(\pi-2x)^3}$ is equal to

(a) $\dfrac{1}{8}$ (b) 0
(c) $\dfrac{1}{32}$ (d) ∞

56. If $f(x)=\dfrac{\sin 2x+A\sin x+B\cos x}{x^3}$ is continuous at $x=0$, then the values of A and B also $f(0)$ are

(a) $A=-2, B=0$ and $f(0)=-1$
(b) $A=0, B=-2$ and $f(0)=1$
(c) $A=1, B=-1$ and $f(0)=0$
(d) None of the above

57. If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{for } -1\le x<0\\ 2x^2+3x-2, & \text{for } 0\le x\le 1\end{cases}$ is continuous at $x=0$, then k is equal to

(a) -4 (b) -3
(c) -2 (d) -1

58. If $f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\left(\frac{x}{a}\right)\log\left(1+\frac{x^2}{3}\right)}, & x\neq 0\\ 9(\log 4)^3, & x=0\end{cases}$ is a continuous function at $x=0$, then the value of a is equal to

(a) 3 (b) 1 (c) 2 (d) 0

59. The function $f(x)=\begin{cases}\dfrac{\sin^3(\sqrt{x})\log(1+3x)}{(\tan^{-1}\sqrt{x})^2(e^{5\sqrt{x}}-1)x}, & x\neq 0\\ a, & x=0\end{cases}$ is continuous in [0, 1], if

(a) $a=0$ (b) $a=\dfrac{3}{5}$ (c) $a=2$ (d) $a=\dfrac{5}{3}$

60. Let $f(x)=\begin{cases}\dfrac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+4}-2}, & x\neq 0\\ a, & x=0\end{cases}$, then the value of a in order that $f(x)$ may be continuous at $x=0$ is

(a) -8 (b) 8 (c) -4 (d) 4

61. If $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$, then

(a) $f(x)$ is an odd function
(b) $f(x)$ is continuous at $x = \frac{1}{2}$
(c) $f(x)$ is continuous at $x = 0$
(d) $f(x)$ is a periodic function

62. If $f(x) = \begin{cases} (\cos x + 3 \sin x)^{5 \operatorname{cosec} x}, & x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \{0\} \\ \lambda, & x = 0 \end{cases}$

is continuous at $x = 0$, then λ will be

(a) e^{15} (b) e^{2}
(c) 15 (d) 1

63. The function $f(x) = \dfrac{\ln(1 + ax) - \ln(1 - bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$, so that it continuous at $x = 0$, is

(a) $a - b$ (b) $a + b$
(c) $\ln a - \ln b$ (b) $\ln a + \ln b$

Problems Based on Discontinuity

64. Let $f(x) = \begin{cases} 1 + x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$, then the points of discontinuity of $g(x) = f[f(x)]$ is/are

(a) only at $x = 1$ (b) $x = 2, 3$
(c) only at $x = 3$ (d) $x = 1, 2$

65. $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$ is [NCERT]

(a) continuous at $x = -3$ and discontinuous at $x = 3$
(b) continuous at $x = -3, 3$
(c) discontinuous at $x = -3, 3$
(d) continuous at $x = 3$ and discontinuous at $x = 3$

66. The function f given by $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$, is [NCERT]

(a) discontinuous at $x = 0$
(b) continuous at $x = 0$
(c) continuous everywhere
(d) None of the above

67. Given $f(0) = 0$ and $f(x) = \dfrac{1}{(1 - e^{-1/x})}$ for $x \ne 0$. Then, only one of the following statements on $f(x)$ is true. That is $f(x)$ is

(a) continuous at $x = 0$
(b) not continuous at $x = 0$
(c) both continuous and differentiable at $x = 0$
(d) not defined at $x = 0$

68. The function $f(x) = |x| + \dfrac{|x|}{x}$ is

(a) continuous at the origin
(b) discontinuous at the origin because $|x|$ is discontinuous there
(c) discontinuous at the origin because $\dfrac{|x|}{x}$ is discontinuous there
(d) discontinuous at the origin because both $|x|$ and $\dfrac{|x|}{x}$ are discontinuous there

69. Let $f(x) = \begin{cases} x^2 \cos e^{1/x} & \text{when } x \ne 0 \\ 1 & \text{when } x = 0 \end{cases}$. Then, $f(x)$ is

(a) discontinuous at $x = 0$
(b) continuous but not differentiable at $x = 0$
(c) differentiable at $x = 0$
(d) $\lim_{x \to 0} f(x)$ exist

70. Let $f(x) = \begin{cases} \sqrt{1 + x^2}, & x < \sqrt{3} \\ \sqrt{3}x - 1, & \sqrt{3} \le x < 4 \\ [x], & 4 \le x < 5 \\ |1 - x|, & x \ge 5 \end{cases}$, where $[x]$ is the greatest integer $\le x$. The number of points of discontinuity of $f(x)$ is R is

(a) 3 (b) 0 (c) infinite (d) 1

Problems Based on Differentiability

71. The function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$ is differentiable at $x = 0$, then $f'(0)$ is [NCERT]

(a) $\frac{1}{2}$ (b) 2
(c) 1 (d) 0

72. $f(x) = x|x|$ is [NCERT]

(a) discontinuous at $x = 0$ (b) not differentiable at $x = 0$
(c) differentiable at $x = 0$ (d) None of these

73. If $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ Then, [NCERT Exemplar]

(a) $f'(0^+)$ and $f'(0^-)$ do not exist
(b) $f'(0^+)$ exists but $f'(0^-)$ does not exist
(c) $f'(0^+) = f'(0^-)$
(d) None of the above

74. The function $f(x) = |x^3|$ is

(a) differentiable everywhere
(b) continuous but not differentiable at $x = 0$
(c) not a continuous function
(d) None of the above

75. The set of points of differentiability of the function $f(x) = \begin{cases} \dfrac{\sqrt{x+1}-1}{(x)^{3/2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

(a) R (b) $[0, \infty]$
(c) $(0, \infty)$ (d) $R - \{0\}$

76. If $f(x) = \dfrac{x}{1+|x|}$ for $x \in R$, then $f'(0)$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

77. The number of points of non-differentiability for the function $f(x) = |x| + |\cos x| + \tan\left(x + \dfrac{\pi}{4}\right)$ in the interval $(-2, 2)$ is

(a) 1 (b) 2
(c) 3 (d) 4

78. Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - |x|, & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$ be a real-valued function. Then, the set of points where $f(x)$ is not differentiable is

(a) $\{0\}$ (b) $\{0, 1\}$
(c) $\{1\}$ (d) null set

79. For the function $f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, the derivative from the right, $f'(0^+)$... and the derivative from the left $f'(0^-)$ are

(a) $\{0, 1\}$ (b) $\{1, 0\}$ (c) $\{1, 1\}$ (d) 0, 0

Problems Based on Continuity and Differentiability

80. The function $f(x) = |x-1| + |x-2|$ is [NCERT]

(a) continuous and differentiable everywhere
(b) continuous at $x = 1, 2$ but differentiable anywhere
(c) continuous everywhere but not differentiable at $x = 1, 2$
(d) None of the above

81. The function $y = |\sin x|$ is continuous for any x but it is not differentiable at

(a) $x = 0$ only
(b) $x = \pi$ only
(c) $x = k\pi$ (k is an positive integer) only
(d) $x = 0$ and $x = k\pi$ (k is an integer)

82. The function $f(x) = \begin{cases} e^{2x} - 1, & x < 0 \\ ax + \dfrac{bx^2}{2} - 1, & x \geq 0 \end{cases}$ is continuous and differentiable for

(a) $a = 1, b = 2$ (b) $a = 2, b = 1$
(c) $a = 2$, any b (d) any a, $b = 4$

83. Let $f(x) = \begin{cases} x^n \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is continuous but not differentiable at $x = 0$, if

(a) $n \in (0, 1)$ (b) $n \in [1, \infty)$
(c) $(-\infty, 0)$ (d) $n = 0$

84. Let $f(x) = \begin{cases} \dfrac{1}{|x|} & \text{for } |x| \geq 1 \\ ax^2 + b & \text{for } |x| < 1 \end{cases}$. If $f(x)$ is continuous and differentiable everywhere, then

(a) $a = \dfrac{1}{2}, b = -\dfrac{3}{2}$ (b) $a = -\dfrac{1}{2}, b = \dfrac{3}{2}$
(c) $a = 1, b = -1$ (d) $a = b = 1$

Miscelleneous

85. If $f(x) = x, x \leq 1$ and $f(x) = x^2 + bx + c, x > 1$ and $f'(x)$ exists finitely for all $x \in R$, then

(a) $b = -1, c \in R$ (b) $c = 1, b \in R$
(c) $b = 1, c = -1$ (d) $b = -1, c = 1$

86. If $f(x), g(x)$ be differential functions and $f(1) = g(1) = 2$, then $\lim\limits_{x \to 1} \dfrac{f(1)g(x) - f(x)g(1) - f(1) + g(1)}{g(x) - f(x)}$ is equal to

(a) 0 (b) 1
(c) 2 (d) None of these

87. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim\limits_{x \to 0} \dfrac{f(x^2) - f(x)}{f(x) - f(0)}$ is

(a) 1 (b) 0
(c) -1 (d) 2

88. Let $f : R \to R$ be a differentiable function and $f(1) = 4$. Then, the value of $\lim\limits_{x \to 1} \int_4^{f(x)} \dfrac{2t}{x-1}\, dt$, if $f'(1) = 2$ is

(a) 16 (b) 8 (c) 4 (d) 2

89. If $f(2) = 4$ and $f'(2) = 1$, then $\lim\limits_{x \to 2} \dfrac{x\, f(2) - 2f(x)}{x-2}$ is equal to

(a) 2 (b) -2 (c) 1 (d) 3

90. Let $f : R \to R$ be a function defined by $f(x) = \max(x, x^3)$. The set of all points where $f(x)$ is not differentiable, is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$
(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

91. Let $f : R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then, $\lim\limits_{x \to 0} \left\{\dfrac{f(1+x)}{f(1)}\right\}^{1/x}$ equals

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

92. If $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2)=2$, then $y=f(x)$ is equal to
(a) $2x+1$ (b) $2x$
(c) $2x+2$ (d) Constant

93. If $f(9)=9$, $f'(9)=4$, then $\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$, equals
(a) 2 (b) 4
(c) 6 (d) 0

94. If $G(x)=-\sqrt{25-x^2}$ then $\lim_{x\to 1}\frac{G(x)-G(1)}{x-1}$ has the value
(a) $\frac{1}{24}$ (b) $\frac{1}{5}$ (c) $-\sqrt{24}$ (d) $\frac{1}{2\sqrt{6}}$

95. $\lim_{x\to 1}\frac{x\sin\{x\}}{x-1}$, where $\{x\}$ denotes the fractional part of x, is equal to
(a) -1 (b) 0
(c) 1 (d) does not exist

96. If $(x+y)=f(x)\,f(y)$ for all $x, y\in R$, $f(5)=2$ and $f'(0)=3$. Then, $f'(5)$ equals
(a) 6 (b) 5
(c) 4 (d) 3

97. Let $f:[1,10]\to Q$ be a continuous function and $f(1)=10$, then $f(10)$ is equal to
(a) $\frac{1}{10}$ (b) 10
(c) 1 (d) Cannot be obtained

98. If α is a repeated root of $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{\sin(ax^2+bx+c)}{(x-\alpha)^2}$ is
(a) 0 (b) a (c) b (d) c

99. If $f(a)=2$, $f'(a)=1$, $g(a)=-1$, $g'(a)=2$, then the value of $\lim_{x\to a}\frac{g(x)\ f(a)-g(a)\ f(x)}{x-a}$ is
(a) -5 (b) $\frac{1}{5}$
(c) 5 (d) 1

100. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB=AC$ and h is the altitude from A to BC, then the ΔABC has perimeter $P=2[\sqrt{(2hr-h^2)}+\sqrt{2hr}]$ and area $A=\ldots$ Also, $\lim_{h\to 0}\frac{A}{P^3}$
(a) $\frac{1}{r}$ (b) $\frac{1}{64r}$ (c) $\frac{1}{128r}$ (d) $\frac{1}{2r}$

Round II (Mixed Bag)

Only One Correct Option

1. If f is strictly increasing function, then $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is equal to
(a) 0 (b) 1 (c) -1 (d) 2

2. $\lim_{x\to -\infty}\frac{x^5\tan\left(\frac{1}{\pi x^2}\right)+3|x|^2+7}{|x|^3+7|x|+8}$ is equal to
(a) $-\frac{1}{\pi}$ (b) 0
(c) ∞ (d) does not exist

3. Let $f(x)=3x^{10}-7x^8+5x^6-21x^3+3x^2-7$, then the value of $\lim_{h\to 0}\frac{f(1-h)-f(1)}{h^3+3h}$ is
(a) $\frac{53}{3}$ (b) $\frac{22}{3}$ (c) 13 (d) $\frac{22}{13}$

4. $\lim_{n\to\infty}\sin[\pi\sqrt{n^2+1}]$ is equal to
(a) ∞ (b) 0
(c) does not exist (d) None of these

5. The function defined by
$$f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4}, & x<1\end{cases}$$ is
(a) continuous at $x=1$
(b) continuous at $x=3$
(c) differentiable at $x=1$
(d) All of the above

6. If $f(x)=\begin{cases}e^x, & x\le 0\\ |1-x|, & x>0\end{cases}$, then
(a) $f(x)$ is differentiable at $x=0$
(b) $f(x)$ is continuous at $x=0, 1$
(c) $f(x)$ is differentiable at $x=1$
(d) None of the above

7. If $x>0$ and g is a bounded function, then $\lim_{x\to\infty}\frac{f(x)\cdot e^{nx}+g(x)}{e^{nx}+1}$ is
(a) 0
(b) $f(x)$
(c) $g(x)$
(d) None of the above

8. If $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$ is equal to

(a) does not exist (b) $-3/2$
(c) $3/2$ (d) 3

9. If $f(x) = \cot^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ and $g(x) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}, 0 < a < \frac{1}{2}$ is

(a) $\frac{3}{2(1 + a^2)}$ (b) $\frac{3}{2(1 + x^2)}$
(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

10. $\lim_{x \to 0} |x|^{[\cos x]}$, where [.] is the greatest integer function, is

(a) 1 (b) does not exist
(c) 0 (d) None of these

11. $\lim_{x \to \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

12. $\lim_{x \to 0} \frac{\log(a + x) - \log a}{x} + k \lim_{x \to 0} \frac{\log x - 1}{x - e} = 1$, then

(a) $k = e\left(1 - \frac{1}{a}\right)$
(b) $k = e(1 + a)$
(c) $k = e(2 - a)$
(d) the equality is not possible

13. Let $f : R \to R$ be a differentiable function having $f(2) = 6, f'(2) = \frac{1}{48}$. Then, $\lim_{x \to 2} \frac{\int_6^{f(x)} 4t^3 dt}{x - 2}$ is equal to

(a) 12 (b) 18 (c) 24 (d) 36

14. The value of $\lim_{n \to \infty} \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)$ is

(a) 1 (b) $\frac{\sin x}{x}$
(c) $\frac{x}{\sin x}$ (d) None of these

15. Let $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = \sin x + \cos x$, then points of discontinuity of $f\{g(x)\}$ in $(0, 2\pi)$ is

(a) $\left\{\frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (b) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$
(c) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$ (d) $\left\{\frac{5\pi}{4}, \frac{7\pi}{3}\right\}$

16. The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1} x, & |x| \le 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$ is

(a) $R - \{-1\}$ (b) R
(c) $R - \{-3\}$ (d) None of these

17. The value of $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}; (a, b, c > 0)$ is

(a) $(abc)^3$ (b) abc
(c) $(abc)^{1/3}$ (d) None of these

18. $f(x) = [x] + \sqrt{\{x\}}$, where [.] and {.} denotes the greatest integer function and fractional part respectively, then

(a) $f(x)$ is continuous but non-differentiable at $x = 1$
(b) $f(x)$ is differentiable at $x = 1$
(c) $f(x)$ is discontinuous at $x = 1$
(d) None of the above

19. The value of the constant α and β such that $\lim_{x \to \infty} \left(\frac{x^2 + 1}{x + 1} - \alpha x - \beta\right) = 0$ are respectively

(a) (1, 1) (b) (−1, 1)
(c) (1, −1) (d) (0, 1)

20. $f(x) = [\sin x] + [\cos x], x \in [0, 2\pi]$, where [.] denotes the greatest integer function. Total number of points, where $f(x)$ is non-differentiable is equal to

(a) 2 (b) 3 (c) 5 (d) 4

21. $\lim_{x \to \frac{\pi}{4}} \frac{\int_0^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}}$ is equal to

(a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$
(c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4f(2)$

22. Let f be twice differentiable function satisfying $f(1) = 1, f(2) = 4, f(3) = 9$, then

(a) $f''(x) = 2, \forall x \in (R)$
(b) $f'(x) = 5, f(x)$ for some $x \in (1, 3)$
(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$
(d) None of the above

23. If $\lim_{x \to 0} \frac{[(a - n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n + 1}{n}$
(c) n (d) $n + \frac{1}{n}$

24. The limit of the following is

$$\lim_{x \to 3} \frac{\sqrt{1 - \cos(x^2 - 10x + 21)}}{(x - 3)}$$

(a) $-(2)^{3/2}$ (b) $(2)^{1/2}$
(c) $(2)^{-3/2}$ (d) 3

25. The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k an integer, is
(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

26. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then, g is
(a) onto, if f is onto
(b) one-one, if f is one-one
(c) continuous, if f is continuous
(d) differentiable, if f is differentiable

27. $\lim_{n \to \infty} {}^nC_x \left(\frac{m}{n}\right)^x \left(1 - \frac{m}{n}\right)^{n-x}$ is equal to
(a) 0 (b) 1 (c) $\frac{m^x \cdot e^{-m}}{x!}$ (d) $\frac{m^x \cdot e^m}{x!}$

28. If $\{x\}$ and $[x]$ are the fractional part function and greatest integer functions of x respectively, then $\lim_{x \to [a]} \frac{e^{\{x\}} - \{x\} - 1}{\{x\}^2}$ is equal to
(a) 0 (b) 1/2
(c) $e - 2$ (d) does not exist

29. $k = \lim_{x \to \infty} \left(\frac{\sum_{k=1}^{1000} (x+k)^m}{x^m + 100^{1000}} \right)$ is $(m > 101)$
(a) 10 (b) 10^2 (c) 10^3 (d) 10^4

30. If $\sum_{r=1}^{k} \cos^{-1} \beta_r = \frac{k\pi}{2}$ for any $k \geq 1$ and $A = \sum_{r=1}^{k} (\beta_r)^r$, then $\lim_{x \to A} \frac{(1+x)^{1/3} - (1-2x)^{1/4}}{x + x^2}$ is equal to
(a) 0 (b) $\frac{1}{2}$
(c) $\frac{\pi}{2}$ (d) $\frac{5}{6}$

31. If $\lim_{x \to 0} \frac{x^a \sin^b x}{\sin(x^c)}$, $a, b, c \in R \sim \{0\}$ exists and has non-zero value, then
(a) a, b, c are in AP (b) a, b, c are in GP
(c) a, b, c are in HP (d) None of these

32. Let $f : R \to R$ be a function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{3}$, $f(0) = 3$ and $f'(0) = 3$, then
(a) $\frac{f(x)}{x}$ is differentiable in R
(b) $f(x)$ is continuous but not differentiable in R
(c) $f(x)$ is continuous in R
(d) $f(x)$ is bounded in R

33. If $f(x) = \frac{\tan \pi[(2x - 3\pi)^3]}{1 + [2x - 3\pi]^2}$ ([·] denotes the greatest integer function), then
(a) $f(x)$ is continuous in R
(b) $f(x)$ is continuous in R but not differentiable in R
(c) $f'(x)$ exists everywhere but $f'(x)$ does not exist at some $x \in R$
(d) None of the above

34. If $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 2, & x \text{ is irrational} \end{cases}$, then
(a) $f(x)$ is continuous in $R \sim I$
(b) $f(x)$ is continuous in $R \sim Q$
(c) $f(x)$ is continuous in R but not differentiable in R
(d) $f(x)$ is neither continuous nor differentiable in R

35. If $f(x)$ is a twice differentiable function, then between two consecutive roots of the equation $f'(x) = 0$, there exists
(a) atleast one root of $f(x) = 0$
(b) atmost one root of $f(x) = 0$
(c) exactly one root of $f(x) = 0$
(d) atmost one root of $f''(x) = 0$

36. The values of p and q so that $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \leq 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$ are
(a) $p = 3$ and $q = 5$ (b) $p = 5$ and $q = 3$
(c) $p = 2$ and $q = 4$ (d) $p = 4$ and $q = 2$

More Than One Correct Option

37. If the function $f(x) = \begin{cases} x + a^2\sqrt{2} \sin x, & 0 \leq x < \pi/4 \\ x \cot x + b, & \pi/4 \leq x < \pi/2 \\ b \sin 2x - a \cos 2x, & \pi/2 \leq x k \leq \pi \end{cases}$

is continuous in the interval $[0, \pi]$, then the values of (a, b) are
(a) $(-1, -1)$ (b) $(0, 0)$ (c) $(-1, 1)$ (d) $(1, 1)$

38. If $\lim_{n \to \infty} \left(an - \frac{1 + n^2}{1 + n} \right) = b$, a finite number, then
(a) $a = 1$ (b) $a = 0$ (c) $b = 1$ (d) $b = -1$

39. A function $f(x)$ is defined as $f(x) = \frac{A \sin x + \sin 2x}{x^3}$, $(x \neq 0)$. If the function is continuous at $x = 0$, then
(a) $A = -2$ (b) $f(0) = -1$
(c) $A = 1$ (d) $f(0) = 1$

40. If $f(x) = |\log x|$, then
(a) $f'(1^+) = 1$ (b) $f'(1^-) = -1$
(c) $f'(1) = 1$ (d) $f'(1) = -1$

41. The function $y = \sin^{-1}(\cos x)$ is not differentiable at
(a) $x = \pi$ (b) $x = -2\pi$
(c) $x = 2\pi$ (d) $x = \frac{\pi}{2}$

42. If $f(x) = x^3 \operatorname{sgn}(x)$, then
(a) f is differentiable at $x = 0$
(b) f is continuous but not differentiable at $x = 0$
(c) $f'(0^+) = 0$
(d) $f'(0^-) = 1$

43. $f(x)$ is differentiable function and $[f(x) \cdot g(x)]$ is differentiable at $x = a$, then
(a) $g(x)$ must be differentiable at $x = a$
(b) if $g(x)$ is discontinuous, then $f(a) = 0$
(c) $f(a) \neq 0$, then $g(x)$ must be differentiable
(d) nothing can be said

44. If $x + |y| = 2y$, then y as a function of x is
(a) defined for all real x
(b) continuous at $x = 0$
(c) differentiable for all x
(d) such that $\frac{dy}{dx} = \frac{1}{3}$ for $x < 0$

45. The set of points where the function $f(x) = \frac{x}{(1 + |x|)}$ is differentiable, is
(a) $(-\infty, \infty)$ (b) $[0, \infty)$
(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$

46. If $f(x) = \min\{1, x^2, x^3\}$, then
(a) $f(x)$ is continuous everywhere
(b) $f(x)$ is continuous and differentiable everywhere
(c) $f(x)$ is not and differentiable at two points
(d) $f(x)$ is not differentiable at one point

47. If $f(x) = \cos\left[\frac{\pi}{x}\right]\cos\left(\frac{\pi}{2}(x-1)\right)$, where $[\cdot]$ denotes the greatest integer function, then $f(x)$ is continuous at
(a) $x = 0$ (b) $x = 1$
(c) $x = 2$ (d) None of these

48. If $f(x) = x + |x| + \cos[\pi^2]x$ and $g(x) = \sin x$, where $[\cdot]$ denotes the greatest integer function, then
(a) $f(x) + g(x)$ is continuous everywhere
(b) $f(x) + g(x)$ is differentiable everywhere
(c) $f(x) + g(x)$ is differentiable everywhere
(d) $f(x) + g(x)$ is continuous but not differentiable at $x = 0$

49. Let $f(x + y) = f(x) + f(y) + 2xy - 1\ \forall x, y \in R$. If $f(x)$ is differentiable and $f'0) = \sin\phi$, then
(a) $f(x) < 0, \forall x \in R$ (b) $f(x) > 0, \forall x \in R$
(c) $f(x) \geq \frac{3}{4}, \forall x \in R$ (d) $-1 \leq f(x) \leq 1, \forall x \in R$

50. If $\lim_{x \to 0} \frac{a \sin x + be^x + \cos x - 2}{x^2} = 6$, then
(a) $a = -7$ (b) $a + b = 0$
(c) $b + c = 2$ (d) $b - c = 12$

51. If $f(x) = \min(\tan x, \cot x)$, then
(a) $f(x)$ is discontinuous at $x = 0, \frac{\pi}{4}$ and $\frac{5\pi}{4}$
(b) $f(x)$ is continuous at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$
(c) $\int_0^{\pi/2} f(x)\, dx = 2 \ln \sqrt{2}$
(d) $f(x)$ is periodic with period π

52. Let $F(x) = 1 + f(x) + [f(x)]^2 + [f(x)]^3$ where $f(x)$ is an increasing differentiable function, $\forall x \in R$ and $F(x) = 0$ has a positive root, then
(a) $F(x)$ is an increasing function
(b) $F(0) < 0$
(c) $f(0) < -1$
(d) $F'(0) > 0$

53. Let $f(x) = \begin{cases} 1 + \frac{2x}{\lambda}, & 0 \leq x < 1 \\ \lambda x, & 1 \leq x < 2 \end{cases}$, if $\lim_{x \to 1} f(x)$ exists, then λ is
(a) -2 (b) -1
(c) 1 (d) 2

54. Let $h(x) = \min\{x, x^2\}$ for every real number x. Then,
(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all $x > 1$
(d) h is not differentiable at two values of x

Comprehension Based Question

Passage I

Directions (Q. Nos. 55 and 56) If $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 1$ and $g(0) = \lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$, then

On the basis of above information, answer the following questions

55. The value of $f(x)$ is
(a) x (b) x^2
(c) $3x$ (d) None of these

56. The value of $g(0)$ is
(a) $\log_e 2$
(b) $\frac{1}{2}\log_e 2$
(c) $2\log_e 2$
(d) $\log_e\left(\frac{1}{2}\right)$

Passage II

Directions (Q. Nos. 57 and 58) *Let* $f(x)=\begin{cases} x+a, & x<0 \\ |x-1|, & x\geq 0\end{cases}$

and $g(x)=\begin{cases} x+1, & x<0 \\ (x-1)^2|+b & x\geq 0\end{cases}$

where, a and *b are non-negative real numbers.*

On the basis of above information, answer the following questions

57. The value of a, if $(gof)x$ is continuous for all real x, is
(a) –1 (b) 0 (c) 1 (d) 2

58. The value of b, if $gof(x)$ is continuous for all real x, is
(a) –1 (b) 0 (c) 1 (d) 2

Assertion and Reason

Directions (Q. Nos. 59 -76) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

59. **Statement I** The function $f(x)=(3x-1)|4x^2-12x+5|\cos \pi x$ is differentiable at $x=\frac{1}{2}$ and $\frac{5}{2}$.

Statement II $\cos(2n+1)\frac{\pi}{2}=0, \forall\ n\in I$.

60. **Statement I** $f(x)=|\log x|$ is differentiable at $x=1$.

Statement II Both $\log x$ and $-\log x$ are differentiable at $x=1$.

61. **Statement I** The set of all points where the function $f(x)=\begin{cases} 0, & x=0 \\ \dfrac{x}{1+e^{1/x}}, & x\neq 0\end{cases}$ is differentiable is $(-\infty, \infty)$.

Statement II $Lf'(0)=1$, $Rf'(0)=0$ and $f'(x)=\dfrac{1+e^{1/x}+e^{1/x}\dfrac{1}{x}}{(1+e^{1/x})^2}$, which exists when $x\neq 0$.

62. **Statement I** $f(x)=\begin{cases} 3-x^2, & x>1 \\ x^3+1, & x\leq 1\end{cases}$, then $f(x)$ is differentiable at $x=1$

Statement II A function $y=f(x)$ is said to have a derivative, if

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$$

63. **Statement I** $\lim\limits_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist

Statement II LHL = 1 and RHL = – 1.

64. Consider the function $f(x)=(|x|-|x-1|)^2$

Statement I $f(x)$ is continuous everywhere but not differentiable at $x=0$ and 1.

Statement II $f'(0^-)=0$, $f'(0^+)=-4$, $f'(1^-)=4$, $f'(1^+)=0$.

65. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous $g(0)\neq 0$, $g'(0)=0$, $g''(0)\neq 0$ and $f(x)=g(x)\sin x$.

Statement I $\lim\limits_{x\to 0}[g(x)\cot x-g(x)\operatorname{cosec} x]=f''(0)$, and

Statement II $f'(0)=g(0)$

66. **Statement I** $f(x)=\sqrt{(x-2)}+\sqrt{(2-x)}$ is continuous at $x=2$.

Statement II $f(x)$ is a point function.

67. **Statement I** The function $f(x)=\{x\}$, where {·} denotes the fractional part function, is discontinuous at $x=1$.

Statement II $\lim\limits_{x\to 1-}f(x)\neq\lim\limits_{x\to 1+}f(x)$

68. **Statement I** The function $f(x)=\dfrac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}}$ is continuous everywhere if $f(0)=2$.

Statement II For continuous function

$$f(0)=\lim_{x\to 0}f(x)$$

69. **Statement I** If $f(x)$ is continuous, then $|f(|x|)|$ is also continuous.

Statement II If $|f(x)|\leq|x|, \forall\ x\in R$, then $|f(x)|$ is continuous at $x=0$.

70. **Statement I** If $a_1, a_2, \ldots, a_n>0$, then

$$\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\ldots+a_n^{\frac{1}{x}}}{n}\right)^{nx}=a_1a_2\ldots a_n$$

Statement II If $\lim\limits_{x\to a}f(x)=1$, $\lim\limits_{x\to a}g(x)=\infty$, then $\lim\limits_{x\to a}(f(x))^{g(x)}=e^{\lim\limits_{x\to a}g(x)\{f(x)-1\}}$

71. **Statement I** $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos 2x}}{x}$ does not exist.

Statement II $|\sin x|=\begin{cases}\sin x, & 0<x<\pi/2 \\ -\sin x, & -\pi/2<x<0\end{cases}$

72. **Statement I** $f(x) = \frac{1}{\{x\}}$ is discontinuous for integral values of x, where {} denotes the fractional part function.

Statement II For integral values of x, $f(x)$ is not defined.

73. Let f be a function satisfying $f(x)f(y) = f(x) + f(y) + f(xy) - 2$ for all $x, y \in R$ and $f(2) = 5$, then

Statement I $\lim_{x \to 4} f(x) = 17$

Statement II $f(x) = x^2 + 1$

74. Consider the function $f(x) = |x-2| + |x-5|, x \in R$.

Statement I $f'(4) = 0$

Statement II f is continuous in $[2, 5]$ and differentiable in $(2, 5)$ and $f(2) = f(5)$.

75. Let $f: R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$.

Statement I $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement II $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in R$.

76. Let $f(x) = x|x|$ and $g(x) = \sin x$

Statement I gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement II gof is twice diffeerntiable at $x = 0$.

Previous Years' Questions

77. $\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to **[JEE Main 2013]**

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (2) 2

78. If $f: R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is **[AIEEE 2012]**

(a) continuous for every real x
(b) discontinuous only at $x = 0$
(c) discontinuous only at non-zero integral values of x
(d) continuous only at $x = 0$

79. $\lim_{x \to 2} \left(\frac{\sqrt{1 - \{\cos 2(x-2)\}}}{x-2}\right)$ is equal to **[AIEEE 2011]**

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) $1/\sqrt{2}$ (d) does not exist

80. The values of p and q for which the function

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$$

is continuous for all x in R, are **[AIEEE 2011]**

(a) $p = \frac{5}{2}, q = \frac{1}{2}$ (b) $p = -\frac{3}{2}, q = \frac{1}{2}$
(c) $p = \frac{1}{2}, q = \frac{3}{2}$ (d) $p = \frac{1}{2}, q = -\frac{3}{2}$

81. If $f: (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f(0) = 1$. Let $g(x) = [f\{2f(x) + 2\}]^2$. Then, $g'(0)$ is equal to **[AIEEE 2010]**

(a) 4 (b) -4
(c) 0 (d) -2

82. Let $f: R \to R$ be a positive increasing function with $\lim_{x \to \infty} \frac{f(3x)}{f(x)} = 1$. Then, $\lim_{x \to \infty} \frac{f(2x)}{f(x)}$ is equal to **[AIEEE 2010]**

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3

83. Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1}, & \text{if } x \ne 1 \\ 0 & \text{, if } x = 1 \end{cases}$

Then, which one of the following is true? **[(AIEEE 2008]**

(a) f is differentiable at $x = 1$ but not at $x = 0$.
(b) f is neither differentiable at $x = 0$ nor at $x = 1$.
(c) f is differentiable at $x = 0$ and $x = 1$.
(d) f is differentiable at $x = 0$ but not at $x = 1$.

84. Let $f: R \to R$ be function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Then, which of the following is true? **[AIEEE 2007]**

(a) $f(x) \ge 1$ for all $x \in R$
(b) $f(x)$ is not differentiable at $x = 1$
(c) $f(x)$ is differentiable everywhere
(d) $f(x)$ is not differentiable

85. The set of points, where $f(x) = \frac{x}{1 + |x|}$ is differentiable, is **[AIEEE 2006]**

(a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$
(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

86. $\lim_{n \to \infty} \left(\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \ldots + \frac{n}{n^2}\sec^2 1\right)$ is equal to **[AIEEE 2005]**

(a) $\frac{1}{2}\tan 1$ (b) $\tan 1$ (c) $\frac{1}{2}\operatorname{cosec} 1$ (d) $\frac{1}{2}\sec 1$

Answers

Check Point 1

1. (c)	**2.** (d)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (a)

Check Point 2

1. (d)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (a)

Check Point 3

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)

Round I

1. (d)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (d)
11. (b)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (d)
31. (a)	**32.** (b)	**33.** (a)	**34.** (c)	**35.** (a)	**36.** (d)	**37.** (d)	**38.** (a)	**39.** (d)	**40.** (b)
41. (b)	**42.** (a)	**43.** (c)	**44.** (d)	**45.** (d)	**46.** (b)	**47.** (a)	**48.** (d)	**49.** (a)	**50.** (d)
51. (d)	**52.** (d)	**53.** (b)	**54.** (b)	**55.** (c)	**56.** (a)	**57.** (c)	**58.** (a)	**59.** (b)	**60.** (c)
61. (c)	**62.** (a)	**63.** (b)	**64.** (d)	**65.** (a)	**66.** (a)	**67.** (b)	**68.** (d)	**69.** (a)	**70.** (d)
71. (d)	**72.** (c)	**73.** (a)	**74.** (a)	**75.** (c)	**76.** (b)	**77.** (c)	**78.** (a)	**79.** (a)	**80.** (c)
81. (d)	**82.** (c)	**83.** (a)	**84.** (b)	**85.** (d)	**86.** (c)	**87.** (c)	**88.** (a)	**89.** (a)	**90.** (d)
91. (c)	**92.** (c)	**93.** (b)	**94.** (d)	**95.** (d)	**96.** (a)	**97.** (b)	**98.** (b)	**99.** (c)	**100.** (c)

Round II

1. (c)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (a)
11. (b)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (d)	**18.** (a)	**19.** (c)	**20.** (c)
21. (a)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (d)
31. (d)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (a)	**37.** (b,d)	**38.** (a,c)	**39.** (a,b)	**40.** (a,b)
41. (a,b,c)	**42.** (a,c)	**43.** (b,c)	**44.** (a,b,d)	**45.** (a,b,c,d)	**46.** (a,d)	**47.** (b,c)	**48.** (a,c)	**49.** (b,c)	**50.** (a,b,c,d)
51. (c,d)	**52.** (a,b,c,d)	**53.** (b,d)	**54.** (a,c,d)	**55.** (a)	**56.** (b)	**57.** (c)	**58.** (b)	**59.** (a)	**60.** (d)
61. (d)	**62.** (d)	**63.** (c)	**64.** (a)	**65.** (b)	**66.** (d)	**67.** (a)	**68.** (a)	**69.** (b)	**70.** (a)
71. (a)	**72.** (a)	**73.** (a)	**74.** (a)	**75.** (a)	**76.** (c)	**77.** (d)	**78.** (c)	**79.** (d)	**80.** (b)
81. (b)	**82.** (a)	**83.** (d)	**84.** (c)	**85.** (b)	**86.** (a)				

the Guidance

Round I

1. $$\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m = \lim_{m\to\infty}\left[1+\left(\cos\frac{x}{m}-1\right)\right]^m$$

$$= \lim_{m\to\infty}\left[1-\left(1-\cos\frac{x}{m}\right)\right]^m$$

$$= \lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m$$

$$= e^{\lim\limits_{m\to\infty} m\left[-2\sin^2\frac{x}{2m}\right]}$$

$$= e^{\lim\limits_{m\to\infty} -2\left(\frac{\sin\frac{x}{2m}}{x/2m}\right)^2\left(\frac{x^2}{4m^2}\right)m}$$

$$= e^{-2\lim\limits_{m\to\infty}\frac{x^2}{4m}} = e^0 = 1$$

2. $$\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x\sin 5x}{x^2\sin 3x}$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 x}{x^2}\right)\frac{\left(\frac{\sin 5x}{x}\right)}{\left(\frac{\sin 3x}{x}\right)}$$

$$= \lim_{x\to 0}2\left(\frac{\sin x}{x}\right)^2\times\frac{5\lim\limits_{x\to 0}\left(\frac{\sin 5x}{5x}\right)}{3\lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2\times 5}{3} = \frac{10}{3} \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

3. Given that, $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$

$$\Rightarrow \quad \lim_{x\to\infty}\left[\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}\right]=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left[\frac{x(1-a)-b-\frac{a}{x}+\frac{(1-b)}{x^2}}{1+\frac{1}{x^2}}\right]=2$$

This limit will exist, if

$$1-a=0 \quad \text{and} \quad b=-2$$

$$\Rightarrow \quad a=1 \quad \text{and} \quad b=-2$$

4. $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$

$$=\lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$$

$$=\lim_{n\to\infty}\frac{[n(n-1)]^{n(n-1)}\left[1+\frac{1}{n(n-1)}\right]^{n(n-1)}}{[n(n-1)]^{n(n-1)}\left[1-\frac{1}{n(n-1)}\right]^{n(n-1)}}$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}}=\frac{e}{e^{-1}}=e^2 \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\right]$$

5. $\lim_{n\to\infty}\frac{1-2+3-4+5-6+\ldots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$

$$=\lim_{n\to\infty}\frac{n\left[\left(\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\ldots\right)-2\right]}{n\left[\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}\right]}$$

$$=\frac{-2}{1+2}=\frac{-2}{3}$$

6. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\ldots+(x+100)^{10}}{x^{10}+10^{10}}$

$$=\lim_{x\to\infty}\frac{x^{10}\cdot\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\ldots+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}$$

$$=\lim_{x\to\infty}\frac{\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\ldots+\left(1+\frac{100}{x}\right)^{10}\right]}{\left[1+\frac{10^{10}}{x^{10}}\right]}=100$$

7. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$

$$=\lim_{x\to 0}\left\{\frac{\cos(\pi\cos^2 x)\cdot\pi\cdot 2\cos x(-\sin x)}{2x}\right\}$$

(using L′ Hospital's rule)

$$=\lim_{x\to 0}\pi\cos(\pi\cos^2 x)\cdot\cos x\cdot\left(\frac{-\sin x}{x}\right)$$

$$=\pi(-1)\cdot 1\cdot(-1)=\pi$$

8. $\because$ Roots of $ax^2+bx+c=0$ are α and β, so roots of $cx^2+bx+a=0$ are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

$$\therefore \quad cx^2+bx+a=c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)$$

Now, $\lim_{x\to\frac{1}{\alpha}}\frac{\sin\left[c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right]}{\alpha c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}\cdot c\left(x-\frac{1}{\beta}\right)$

$$=\frac{c}{\alpha}\lim_{x\to\frac{1}{\alpha}}\left[\frac{\sin\left\{c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right\}}{c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}\right]\cdot\lim_{x\to\frac{1}{\alpha}}\left(x-\frac{1}{\beta}\right)$$

$$=\frac{c}{\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

9. $\lim_{x\to 0}\frac{\cos(\sin x)-1}{x^2}=-\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2}$

$$=-\lim_{x\to 0}\frac{2\sin^2\left(\sin\frac{x}{2}\right)}{x^2}=-2\times\frac{1}{4}=-\frac{1}{2}$$

10. $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2^2\left(\frac{x}{2}\right)^2}\cdot\frac{(e^x-\cos x)}{x^{n-2}}$

$$=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\lim_{x\to 0}\frac{(e^x-\cos x)}{x^{n-2}}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{(e^x-\cos x)}{x^{n-2}} \qquad (\text{for } n=1 \text{ the limit } = 0)$$

Applying L′ Hospital's rule,

$$=\frac{1}{2}\lim_{x\to 0}\frac{e^x+\sin x}{(n-2)x^{n-3}}$$

So, if $n=3$, the limit is $\frac{1}{2(n-2)}$ which is finite.

If $n=4$, the limit is infinite.

11. $\lim\limits_{h\to 0}\dfrac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$$=\lim_{h\to 0}\frac{\dfrac{f(2h+2+h^2)-f(2)}{(2+2h)}\times(2+2h)}{\dfrac{f(h-h^2+1)-f(1)}{(1-2h)}\times(1-2h)}$$

$$=\lim_{h\to 0}\frac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}$$

$$=\frac{f'(2)}{f'(1)}\cdot\frac{2}{1}=\frac{6\times 2}{4\times 1}=3$$

12. $\lim\limits_{n\to 0}\dfrac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right)}$

$$=\lim_{n\to 0}\frac{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right)}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right)}$$

$$=\lim_{n\to 0}\frac{3}{2}\times\frac{4}{3}\times\ldots\times\frac{n+1}{n}=\lim_{n\to 0}\frac{n+1}{2}=\frac{1}{2}$$

13. $\lim\limits_{x\to 1}(1-x^2)\log_{x^2}2=\lim\limits_{x\to 1}\dfrac{1-x^2}{\log_2 x^2}=\lim\limits_{x\to 1}\dfrac{-2x}{\frac{1}{x^2}\cdot\log_2 e\cdot 2x}$

$$=\lim_{x\to 1}\frac{-2x^2}{2\log_2 e}=-\frac{2\log 2}{2}=-\log 2$$

14. Now, $\lim\limits_{x\to\frac{\pi}{2}}\tan^{-1}\left\{\dfrac{\sin(a\tan^3 x+b\tan^2 x+c\tan x)}{a\tan^3 x+b\tan^2 x+c\tan x}\right\}$

$\therefore \lim\limits_{x\to\frac{\pi}{2}}\tan x$ is ∞ or $-\infty$ as x approaches $\frac{\pi}{2}$ from left or right, respectively.

$$-1\ge\sin(a\tan^3 x+b\tan^2 x+c\tan x)\le 1$$

But $a\tan^3 x+b\tan^2 x+c\tan x$ is either approaches ∞ or approaches $-\infty$.

So, $\dfrac{\sin(a\tan^3 x+b\tan^2 x+c\tan x)}{a\tan^3 x+b\tan^2 x+c\tan x}$ approaches 0 as x approaches $\frac{\pi}{2}$. Hence, limit is 0.

15. $\lim\limits_{x\to 0}\left\{\dfrac{e^{x\log(2^x-1)}-(2^x-1)^x\sin x}{e^{x\log x}}\right\}^{1/x}$

$$=\lim_{x\to 0}\left\{\frac{(2^x-1)^x-(2^x-1)^x\sin x}{x^x}\right\}^{1/x}$$

$$=\lim_{x\to 0}\frac{2^x-1}{x}\lim_{x\to 0}(1-\sin x)^{1/x}$$

$$=\log 2\cdot e^{-1}=\frac{\log 2}{e}$$

16. $\lim\limits_{x\to 0}(1+ax+bx^2)^{2/x}=\lim\limits_{x\to 0}(1+ax+bx^2)^{\frac{1}{ax+bx^2}\cdot\frac{2(ax+bx^2)}{x}}$

$$=e^{\lim\limits_{x\to 0}\frac{2(ax+bx^2)}{x}}=e^3$$

$\Rightarrow$ $a=\dfrac{3}{2}$, b any real number.

Hence, (c) is the correct answer.

17. This is of the form ∞/∞.

$$\lim_{x\to\infty}\frac{e^{x^2}-1}{e^{x^2}+1}=\lim_{x\to\infty}\frac{1-e^{-x^2}}{1+e^{-x^2}}=\lim_{x\to\infty}\frac{1-(1/e)^{x^2}}{1+(1/e)^{x^2}}$$

$$=\frac{1-0}{1+0}=1$$

18. $\lim\limits_{n\to\infty}\dfrac{1^3+2^3+\ldots+n^3}{(n^2+1)^2}=\lim\limits_{n\to\infty}\dfrac{n^2(n+1)^2}{4(n^2+1)^2}$

$$=\lim_{n\to\infty}\frac{n^4\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{4n^4\left(1+\frac{2}{n^2}+\frac{1}{n^4}\right)}=\frac{1}{4}$$

19. Put $\cos^{-1}x=y$ and $x\to 1^-\Rightarrow y\to 0$

$$\lim_{x\to 1^-}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}=\lim_{y\to 0}\frac{1-\sqrt{\cos y}}{y^2}$$

Now, rationalizing numerator

$$\lim_{y\to 0}\frac{(1-\cos y)}{y^2(1+\sqrt{\cos y})}$$

$$\lim_{y\to 0}\frac{1-\cos y}{y^2}=\lim_{y\to 0}\frac{1}{1+\sqrt{\cos y}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

20. $\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$

$$=\lim_{x\to 0}\frac{x\left\{2x+\frac{8x^3}{3}+\frac{64x^5}{15}+\ldots\right\}-2x\left\{x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right\}}{4\sin^4 x}$$

$$=\lim_{x\to 0}\frac{x^4\left\{\left(\frac{8}{3}-\frac{2}{3}\right)+\text{terms containing positive power of }x\right\}}{4\sin^4 x}$$

$$=\frac{1}{4}\cdot 2\times\frac{1}{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^4}=\frac{1}{2}\times\frac{1}{1}=\frac{1}{2}$$

21. $\lim\limits_{x\to 0}(1+\sin x)^{1/x^2}$ (1^∞ form)

$$=e^{\lim\limits_{x\to 0}\left(\frac{\sin x}{x^2}\right)}=e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{\sin x}{x}\right)}$$

$$=\begin{cases}0, & \text{when } x\to 0 \text{ from left,}\\ \infty, & \text{when } x\to 0 \text{ from right.}\end{cases}$$

Thus, the given limit does not exist.

22. $\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+\ldots+\frac{n}{1-n^2}\right)$

$$=\lim_{n\to\infty}\frac{1+2+3+\ldots+n}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{\Sigma n}{1-n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2(1-n^2)}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2\left[\frac{1}{n^2}-1\right]}=-\frac{1}{2}$$

23. $\lim_{x\to 0}\frac{e^{\sin x}-\sin x-1}{x^2}$

$$=\lim_{x\to 0}\frac{\left(1+\frac{\sin x}{1!}+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots\right)-\sin x-1}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\left(\frac{1}{2!}+\frac{\sin x}{3!}+\ldots\right)=\frac{1}{2}$$

24. $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$

Let $x=t+\frac{\pi}{2}$

If $x\to\frac{\pi}{2}\Rightarrow t\to 0$

$$\lim_{t\to 0}\frac{\sin t-\tan t}{-8t^3}=\lim_{t\to 0}\frac{t-\frac{t^3}{1}+\ldots-t-\frac{t^3}{3}\ldots}{-8t^3}$$

$$=\lim_{t\to 0}\frac{-\frac{t^3}{2}}{-8t^3}=\frac{1}{16}$$

25. $\lim_{x\to\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$

Let $L=\lim_{x\to-\infty}\frac{x^3}{1+|x|^3}\left[x\sin\left(\frac{1}{x}\right)+\frac{1}{x}\right]$

$$=\lim_{x\to-\infty}\frac{x^3}{|x|^3}\left[\frac{1}{1+\frac{1}{|x|x}}\right]\left[x\sin\left(\frac{1}{x}\right)+\frac{1}{x}\right]\qquad\ldots(i)$$

Now, $\lim_{x\to-\infty}\frac{x^3}{|x|^3}=\lim_{x\to-\infty}\frac{x^3}{-x^3}=-1\quad(\because |x|=-x \text{ for } x<0)$

$$\lim_{x\to-\infty}\frac{1}{|x|x}=0$$

and $\lim_{x\to-\infty}x\sin\left(\frac{1}{x}\right)=\lim_{y\to 0^-}\frac{\sin y}{y}=1\qquad\left(\text{where } y=\frac{1}{x}\right)$

$\therefore$ From Eq. (i), $L=-1\left(\frac{1}{1+0}\right)(1+0)=-1$

26. $\lim_{x\to 5}\frac{x^\lambda-5^\lambda}{x-5}=\lambda\cdot 5^{\lambda-1}=500$

$\Rightarrow\qquad\lambda=4$

27. $\lim_{n\to\infty}n\cdot\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)=\lim_{n\to\infty}\frac{n}{2}\times\sin\frac{2\pi}{4n}$

$$=\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\times\frac{\pi}{2}\times\frac{1}{2}=\frac{\pi}{4}$$

28. $\lim_{x\to 0}\frac{1-\cos 3x}{x(3^x-1)}=\lim_{x\to 0}\frac{3\sin 3x}{x\cdot 3^x\log 3+3^x-1}$

$$=\lim_{x\to 0}\frac{9\cos 3x}{\log 3\,(x\cdot 3^x\log 3+3^x)+3^x\log 3}$$

$$=\lim_{x\to 0}\frac{9\cos 3x}{x\cdot 3^x(\log 3)^2+2\cdot 3^x\log 3}=\frac{9}{2\log 3}$$

29. $\lim_{x\to 0}f\left(\frac{1-\cos 3x}{x^2}\right)=f\left(\lim_{x\to 0}\frac{1-\cos 3x}{x^2}\right)=f\left(\frac{9}{2}\right)=\frac{2}{9}.$

30. The given function can be restarted as

$$f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & \text{if } x\in(-\infty,0)\cup[1,\infty]\\ 0, & \text{if } x\in[0,1]\end{cases}$$

$\therefore\qquad\lim_{x\to 0^-}f(x)=\lim_{h\to 0}\frac{\sin(-h)}{[-h]}$

$$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}=\sin 1$$

and $\qquad\lim_{x\to 0^+}f(x)=\lim_{h\to 0}0=0$

$\because\qquad\lim_{x\to 0^-}f(x)\neq\lim_{x\to 0^+}f(x)$

$\therefore\ \lim_{x\to 0}f(x)$ does not exist.

31. $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$

$$=\lim_{x\to 0}\frac{2\sin x(1-\cos x)(1+\cos x)}{x^3(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{2\sin^3 x}{x^3}\times\frac{1}{1+\cos x}=2\times(1)^3\times\frac{1}{1+1}=1.$$

32. $\lim_{x\to 0}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$ $\qquad(1^\infty \text{ form})$

$$=e^{\lim_{x\to 1}\left(\tan\frac{\pi x}{4}-1\right)\tan\frac{\pi x}{2}}=e^{\lim_{x\to 1}\left(\frac{\sin\frac{\pi x}{4}-\cos\frac{\pi x}{4}}{\cos\frac{\pi x}{4}}\right)\frac{2\sin\frac{\pi x}{4}\cos\frac{\pi x}{4}}{\cos\frac{\pi x}{2}}}$$

$$=e^{\lim_{x\to 1}\frac{2\sin^2\frac{\pi x}{4}-2\sin\frac{\pi x}{4}\cos\frac{\pi x}{4}}{\cos\frac{\pi x}{2}}}=e^{\lim_{x\to 1}\left(\frac{1-\cos\frac{\pi x}{2}-\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}\right)}$$

$$=e^{\lim_{x\to 1}\frac{2\sin^2\frac{\pi x}{2}\times\frac{\pi}{2}-\cos\frac{\pi x}{2}\times\frac{\pi}{2}}{-\sin\frac{\pi x}{2}\times\frac{\pi}{2}}}=e^{-1}$$

33. $\lim_{x \to 0} (\cos x)^{\frac{1}{x^4}} = e^{\lim_{x \to 0} \frac{\cos x - 1}{x^4}}$

$$= e^{\lim_{x \to 0} \frac{-2\sin\frac{x}{2}}{x/2} \cdot \frac{\sin\frac{x}{2}}{x/2} \times \frac{1}{4x^2}} = e^{-\infty} = 0$$

34. Let $y = \text{cosec}^2 x$; Required limit $= \lim_{y \to \infty} (1^y + 2^y + \ldots n^y)^{1/y}$

$(0/\infty$ form)

$$= \lim_{y \to \infty} (n^y)^{1/y} \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \ldots + \left(\frac{n-1}{n}\right)^y + 1\right]^{1/y}$$

$$= \lim_{y \to \infty} n \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \ldots + \left(\frac{n-1}{n}\right)^y + 1\right]^{1/y}$$

$$= n \cdot 1^0 = n \cdot 1 = n$$

35. Here, $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

For $x < 0$, $f(x) = 2x$; $0 < x < 1$, $f(x) = 0$ and $x > 1$, $f(x) = 4x$ are polynomial and constant function, so it is continuous in the given interval. So, we have to check the continuity at $x = 0, 1$.

At $x = 0$, $\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x)$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$\therefore \lim_{h \to 0} [2(0 - h)] = \lim_{h \to 0} (-2h) = -2 \times 0 = 0$,

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$$

Also, $f(0) = 0$

$\therefore$ LHL = RHL = $f(0)$

Thus, $f(x)$ is continuous for all values of x.

At $x = 1$, $\text{LHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (0) = 0$,

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x)$$

Putting $x = 1 + h$ as $x \to 1^+$ when $h \to 0$

$$\lim_{h \to 0} [4(1 + h)] = \lim_{h \to 0} (4 + 4h) = 4 + 4 \times 0 = 4$$

$\therefore$ LHL $\neq$ RHL. Thus, $f(x)$ is continuous everywhere except at $x = 1$.

36. Here, $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

$$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$$

Putting $x = 3 - h$ as $x \to 3^-$ when $h \to 0$

$\therefore \lim_{h \to 0} [a(3 - h) + 1] = \lim_{h \to 0} (3a - ah + 1) = 3a + 1$

$$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$$

Putting $x = 3 + h$ as $x \to 3^+$ when $h \to 0$

$\therefore \lim_{h \to 0} [b(3 + h) + 3] = \lim_{h \to 0} (3b + bh + 3) = 3b + 3$

Also, $f(3) = 3a + 1$ $[\because f(x) = ax + 1]$

Since, $f(x)$ is continuous at $x = 3$.

$\therefore$ LHL = RHL = $f(3)$

$\Rightarrow 3a + 1 = 3b + 3 \Rightarrow 3a = 3b + 2 \Rightarrow a = b + \frac{2}{3}$

37. Here, $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$

At $x = 0$, $\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda(x^2 - 2x)$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$\therefore \lim_{h \to 0} \lambda[(0 - h)^2 - 2(0 - h)] = \lim_{h \to 0} [\lambda(h^2 + 2h)] = 0$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$$

Putting $x = 0 + h$ as $x \to 0^+$ when $h \to 0$

$\therefore \lim_{h \to 0} [4(0 + h) + 1] = \lim_{h \to 0} [4h + 1] = 0 + 1 = 1$

$\therefore$ LHL $\neq$ RHL. Thus, $f(x)$ is not continuous at $x = 0$ for any value of λ.

At $x = 1$, $\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4x + 1)$

Putting $x = 1 - h$ as $x \to 1^-$ when $h \to 0$

$\therefore \lim_{h \to 0} [4(1 - h) + 1] = \lim_{h \to 0} [5 - 4h] = 5 - 0 = 5$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x + 1)$$

Putting $x = 1 + h$ as $x \to 1^+$ when $h \to 0$

$\therefore \lim_{h \to 0} [4(1 + h) + 1] = \lim_{h \to 0} (5 + 4h) = 5 + 0 = 5$

Also, $f(1) = 4 \times 1 + 1 = 5$ $[\because f(x) = 4x + 1]$

Thus, $f(x)$ is continuous at $x = 1$ for all values of λ.

38. Here, $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 \sin\frac{1}{x}$$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$\therefore \lim_{h \to 0} (0 - h)^2 \sin\frac{1}{(0 - h)} = \lim_{h \to 0} \left(-h^2 \sin\frac{1}{h}\right)$

$[\because \sin(-\theta) = -\sin\theta]$

$= -0 \times \sin(\infty) = 0$ $(\because -1 \le \sin x \le 1, \forall x \in R)$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 \sin\frac{1}{x}$$

Putting $x = 0 + h$ as $x \to 0^+$ when $h \to 0$

$\therefore \lim_{h \to 0} (0 + h)^2 \sin\frac{1}{0 + h} = \lim_{h \to 0} h^2 \sin\frac{1}{h} = 0 \times \sin(\infty)$

$(\because -1 \le \sin x \le 1, \forall x \in R)$

$\therefore$ LHL = RHL = $f(0)$. Thus, $f(x)$ is continuous at $x = 0$.

39. Here, $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

$\therefore$ $$\text{LHL} = \lim_{x \to \frac{\pi}{2}^-} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{k\cos x}{\pi - 2x}$$

Putting $x = \dfrac{\pi}{2} - h$ as $x \to \dfrac{\pi}{2}$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{k\sin h}{2h}$$

$$= \lim_{h \to 0} \frac{k}{2} \times \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

$$\text{RHL} = \lim_{x \to \pi^+/2} f(x) = \lim_{x \to \pi^+/2} \frac{k\cos x}{\pi - 2x}$$

Putting $x = \dfrac{\pi}{2} + h$ as $x \to \dfrac{\pi^+}{2}$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-k\sin h}{-2h}$$

$$= \lim_{h \to 0} \frac{k}{2} \times \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2}$$

$$\left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

Also, $f\left(\dfrac{\pi}{2}\right) = 3$. Since, $f(x)$ is continuous at $x = \dfrac{\pi}{2}$.

$$\therefore \quad \text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right) = \frac{k}{2} = 3 \Rightarrow k = 6$$

40. Here, $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$

At $x = 2$, $\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (5) = 5$

$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$$

Putting $x = 2 + h$ as $x \to 2^+$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} [a(2 + h) + b] = \lim_{h \to 0} (2a + ah + b) = 2a + b$$

Also, $f(2) = 5$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore$ $\text{LHL} = \text{RHL} = f(2) \Rightarrow 2a + b = 5$ ...(i)

At $x = 10$, $\text{LHL} = \lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$

Putting $x = 10 - h$ as $x \to 10^-$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} [a(10 - h) + b] = \lim_{h \to 0} (10a - ah + b) = 10a + b$$

$$\text{RHL} = \lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} (21) = 21$$

Also, $f(10) = 21$

Since, $f(x)$ is continuous at $x = 10$

$\text{LHL} = \text{RHL} = f(10) \Rightarrow 10a + b = 21$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$$8a = 16 \Rightarrow a = 2$$

Put $a = 2$ in Eq. (i), we get

$$2 \times 2 + b = 5 \Rightarrow b = 1$$

41. We want to check the continuity at $x = 0$.

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} |-h| \cos\left(\frac{1}{-h}\right)$$

$$= \lim_{h \to 0} h\cos\left(\frac{1}{h}\right) = 0 \times \cos\frac{1}{0} = 0$$

$$(\because -1 \le \cos x \le 1, \ \forall x \in R)$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} (0 + h)$$

$$= \lim_{h \to 0} |h| \cos\left(\frac{1}{h}\right) = \lim_{h \to 0} h\cos\frac{1}{h} = 0 \times \cos\frac{1}{0} = 0$$

and $f(0) = 0$

Thus, LHL = RHL = $f(0) = 0$. Hence, function is continuous at $x = 0$.

42. We want to check the continuity of the function at $x = a$.

$$\text{LHL} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$$

$$= \lim_{h \to a} |a - h - a| \sin\left[\frac{1}{(a - h - a)}\right]$$

$$= \lim_{h \to 0} |-h| \sin\left(\frac{-1}{h}\right) = -\lim_{h \to 0} h\sin\frac{1}{h} = 0 \times \sin\frac{1}{0} = 0$$

$$(\because -1 \le \sin x \le 1, \forall x \in R)$$

$$\text{RHL} = \lim_{x \to a^+} f(x) = \lim_{h \to 0} (a + h)$$

$$= \lim_{h \to 0} |a + h - a| \sin\left[\frac{1}{(a + h - a)}\right]$$

$$= \lim_{h \to 0} h\sin\left(\frac{1}{h}\right) = 0 \times \sin\frac{1}{0} = 0 \quad \text{and} \quad f(a) = 0$$

Thus, LHL = RHL = $f(a) = 0$. Hence, the function is continuous at $x = a$.

43. We want to check the continuity of the function at $x = 1$.

We need to prove that LHL = RHL = $f(1)$

$$f(1) = \frac{1^2}{2} = \frac{1}{2}$$

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} \frac{(1 - h)^2}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h)$$

$$= \lim_{h \to 0} 2(1+h)^2 - 3(1+h) + \frac{3}{2}$$

$$= 2 - 3 + \frac{3}{2} = \frac{1}{2}$$

Thus, LHL = RHL = $f(1) = \frac{1}{2}$. Hence, function is continuous at $x = 1$.

44. Given, $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & x \neq 2 \\ k, & x = 2 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{4(2^x - 4)}{(2^x)^2 - 4^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{4}{2^x + 4}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Continuity at $x = 2$. We have, $f(2) = k$

Case I $x < 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left[\frac{4}{2^x + 4}\right] = \lim_{h \to 0} \left[\frac{4}{2^{(2-h)} + 4}\right]$$

(by putting $x = 2 - h$) as $x \to 2^{-1}$, then $h \to 0$

$$= \left[\frac{4}{4+4}\right] = \frac{1}{2}$$

Case II $x > 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left[\frac{4}{2^x + 4}\right] = \lim_{h \to 0} \left[\frac{4}{2^{(2+h)} + 4}\right]$$

(by putting $x = 2 + h$) as $x \to 2^+$, then $h \to 0$

$$= \left[\frac{4}{4+4}\right] = \frac{1}{2}$$

So, function f is continuous at $x = 2$, if

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2) \Rightarrow \frac{1}{2} = \frac{1}{2} = k \Rightarrow k = \frac{1}{2}$$

45. Given, that the function is continuous .

At $x = 0$, LHL $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{1-kh} - \sqrt{1+kh})}{-h} \times \frac{(\sqrt{1-kh} + \sqrt{1+kh})}{(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$= \lim_{h \to 0} \frac{(1-kh) - (1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})} = \frac{2k}{2} = k$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{2h + 1}{h - 2} = \frac{-1}{2}$$

Since, $f(x)$ is continuous so, $k = -\frac{1}{2}$

46. $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

We have to check the continuity at $x = 0$.

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{-h}{|-h| + 2(-h)^2}$$

$$= \lim_{h \to 0} \frac{-h}{h + 2h^2} = \lim_{h \to 0} \frac{-1}{1 + 2h} = -1$$

$f(0) = k$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{h}{|h| + 2h^2}$$

$$= \lim_{h \to 0} \frac{1}{1 + 2h} = 1$$

Since, LHL ≠ RHL, function will remain discontinuous at $x = 0$.

∴ For no value of k, the function is continuous at $x = 0$.

47. $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$

Given that the functions is continuous at $x = 4$

$$\text{LHL} = \text{RHL} = f(4) \quad \text{...(i)}$$

Now, $f(4) = a + b$ ...(ii)

$$\text{LHL} = \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4 - h)$$

$$= \lim_{h \to 0} \frac{4 - h - 4}{|4 - h - 4|} + a = \lim_{h \to 0} \frac{-h}{h} + a = a - 1 \quad \text{...(iii)}$$

$$\text{RHL} = \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4 + h)$$

$$= \lim_{h \to 0} \frac{4 + h - 4}{|4 + h - 4|} + b = \lim_{h \to 0} \frac{h}{h} + b = b + 1 \quad \text{...(iv)}$$

∴ From Eq. (i), $a - 1 = b + 1 \Rightarrow a - b = +2$

From Eqs. (ii) and (iii), $a + b = a - 1 \Rightarrow b = -1$

From Eqs. (ii) and (iv), $a + b = b + 1 \Rightarrow a = 1$

Thus, $a = 1$ and $b = -1$

48. Given, $f(2) = k$

$$\text{Now, } \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+5)(x-2)^2}{(x-2)^2}$$

$$= \lim_{x \to 2} (x + 5) = 7$$

As f is continuous at $x = 2$, we have

$$\lim_{x \to 2} f(x) = f(2)$$

$$\Rightarrow \quad k = 7$$

49. Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

Therefore, $\lim\limits_{x \to \frac{\pi}{4}} f(x) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)}{(\sqrt{2}\cos x + 1)} \cdot \frac{(\sqrt{2}\cos x + 1)}{(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} \cdot (\sin x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left(\frac{\cos x + \sin x}{\sqrt{2}\cos x + 1}\right) \cdot (\sin x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\cos x + \sin x)}{\sqrt{2}\cos x + 1} \sin x = \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

Thus, $\lim\limits_{x \to \frac{\pi}{4}} f(x) = \dfrac{1}{2}$

If we define $f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$, then $f(x)$ will become continuous at $x = \dfrac{\pi}{4}$. Hence, for f to be continuous at $x = \dfrac{\pi}{4}$, $f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$.

50. Here, $f(0)$ = a left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1 - \cos 4x}{x^2}$$

$$= \lim_{x \to 0^-} \frac{2\sin^2 2x}{x^2}$$

$$= \lim_{2x \to 0^-} 8\left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8$$

and right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4},$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{(\sqrt{16 + \sqrt{x}} + 4)(\sqrt{16 + \sqrt{x}} - 4)}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{16 + \sqrt{x} - 16}$$

$$= \lim_{x \to 0^+} (\sqrt{16 + \sqrt{x}} + 4) = 8$$

Thus, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 8$. Hence, f is continuous at $x = 0$ only if $a = 8$.

51. $x - [x] = 0$ when x is an integer,

so that $f(x)$ is discontinuous for all $x \in I$, *i.e.*, $f(x)$ is discontinuous at infinite number of points.

Hence, (d) is the correct answer.

52. Given, $f(x) = \dfrac{\log_e(1+x) - \log_e(1-x)}{x}$

Now, $\lim\limits_{x \to 0} \dfrac{\log\left(\dfrac{1+x}{1-x}\right)}{x} = \lim\limits_{x \to 0} \dfrac{2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots \infty\right)}{x}$

$$= \lim_{x \to 0} 2\left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \ldots \infty\right) = 2$$

Since, it is continuous.

$\therefore \quad f(0) = \lim\limits_{x \to 0} f(x) = 2$

53. Given $f(x)$ is continuous at $x = \dfrac{\pi}{2}$

$\therefore \quad f\left(\dfrac{\pi}{2}\right) = \lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\frac{1}{\pi - 2x}}$ (1^∞ form)

$$= e^{\lim\limits_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\pi - 2x}} = e^{\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{-2}} = e^0 = 1$$

54. Since, $f(x)$ is continuous.

$\therefore \quad \lim\limits_{x \to 0}\left(\dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}\right) = f(0)$ $\left(\dfrac{0}{0} \text{ form}\right)$

Applying L' Hospital's rule

$$f(0) = \lim_{x \to 0} \frac{\left(2 - \frac{1}{\sqrt{1 - x^2}}\right)}{\left(2 + \frac{1}{1 + x^2}\right)} = \frac{\left(2 - \frac{1}{\sqrt{1}}\right)}{\left(2 + \frac{1}{1}\right)} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$$

55. Put $x = \dfrac{\pi}{2} - h$ as $x \to \dfrac{\pi}{2}, h \to 0$

$\therefore$ Given limit

$$= \lim_{h \to 0} \frac{1 - \tan\left(\frac{\pi}{4} + \frac{h}{2}\right)}{1 + \tan\left(\frac{\pi}{4} + \frac{h}{2}\right)} \cdot \frac{(1 - \cos h)}{(2h)^3}$$

$$= \lim_{h \to 0} \tan\frac{h}{2} \frac{2\sin^2\frac{h}{2}}{8h^3}$$

$$= \lim_{h \to 0} \frac{1}{4} \cdot \frac{\tan\frac{h}{2}}{\frac{h}{2} \times 2} \times \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \times \frac{1}{4}$$

$$= \lim_{x \to 0} \frac{1}{32} \cdot \left(\frac{\tan\frac{h}{2}}{\frac{h}{2}}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = \frac{1}{32}$$

56. As $f(x)$ is continuous at $x = 0$,

$\therefore \lim_{x \to 0} f(x) = f(0)$ and both $f(0)$ and $\lim_{x \to 0} f(x)$ are finite.

$$\Rightarrow \quad f(0) = \lim_{x \to 0} \frac{\sin 2x + A \sin x + B \cos x}{x^3}$$

As denominator $\to 0$, when $x \to 0$.

Numerator should also $\to 0$, when $x \to 0$

which is possible only, if

$\Rightarrow \sin 2(0) + A \sin (0) + B \cos (0) = 0$

$\Rightarrow B = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} \frac{\sin 2x + A \sin x}{x^3}$$

$$\Rightarrow f(0) = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{2 \cos x + A}{x^2}\right) = \lim_{x \to 0} \left(\frac{2 \cos x + A}{x^2}\right)$$

Again, we can see that denominator $\to 0$ as $x \to 0$

$\therefore$ Numerator should also approach 0 as $x \to 0$

$\Rightarrow 2 + A = 0 \Rightarrow A = -2$

$$\Rightarrow f(0) = \lim_{x \to 0} \left(\frac{2 \cos x - 2}{x^2}\right) = \lim_{x \to 0} \left(\frac{-4 \sin^2 x/2}{x^2}\right)$$

$$= \lim_{x \to 0} \left(\frac{-\sin^2 x/2}{x^2/4}\right) = -1$$

So, we get $A = -2, B = 0$ and $f(0) = -1$

57. $$\text{LHL} = \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x \to 0^-} \frac{2kx}{x(\sqrt{1+kx} + \sqrt{1-kx})} = k$$

$$\text{RHL} = \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$$

Since, it is given that $f(x)$ is continuous.

$\therefore \quad \text{LHL} = \text{RHL} \Rightarrow k = -2$

58. We have, $$f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin(x/a) \log[1 + (x^2/3)]}, & x \neq 0 \\ 9 (\log 4)^3, & x = 0 \end{cases}$$

For $f(x)$ to be continuous at $x = 0$, we should have

$$\lim_{x \to 0} f(x) = f(0) = 9 (\log 4)^3$$

$$\therefore \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)^3 \frac{\dfrac{x}{a}}{\sin\left(\dfrac{x}{a}\right)} \cdot \frac{ax^2}{\log\left(1 + \dfrac{x^2}{3}\right)}$$

$$= (\log 4)^3 a \lim_{x \to 0} \left(\frac{x^2}{\dfrac{1}{3}x^2 - \dfrac{1}{18}x^4 + \ldots}\right)$$

$\Rightarrow 9 (\log 4)^3 = 3a(\log 4)^3$

$\Rightarrow \quad a = 3$

59. We have, $$f(x) = \begin{cases} \dfrac{\sin^3(\sqrt{x}) \log(1 + 3x)}{x(\tan^{-1}\sqrt{x})^2(e^{5\sqrt{x}} - 1)x} & , x \neq 0 \\ a & , x = 0 \end{cases}$$

For continuity in [0, 1], $f(0) = f(0^+)$, otherwise it is discontinuous.

$$\therefore \quad \lim_{x \to 0^+} \frac{\sin^3 (\sqrt{x}) \log (1 + 3x)}{x(\tan^{-1}\sqrt{x})^2(e^{5\sqrt{x}} - 1)} = a$$

$$\Rightarrow \lim_{x \to 0^+} \left[\frac{3}{5} \frac{\sin^3(\sqrt{x})}{(\sqrt{x})^3} \cdot \frac{(\sqrt{x})^2}{(\tan^{-1}\sqrt{x})^2} \times \frac{\log(1+3x)}{3x} \cdot \frac{5\sqrt{x}}{(e^{5\sqrt{x}} - 1)}\right] = 0$$

$$\Rightarrow \quad a = \frac{3}{5}$$

60. Given, $$f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 4} - 2}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

be continuous at $x = 0$, then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 4} - 2}$$

$$= \lim_{x \to 0} \frac{\cos 2x - 1}{\sqrt{x^2 + 4} - 2}$$

$$= \lim_{x \to 0} \left[\frac{-2 \sin 2x}{\dfrac{2x}{\sqrt{x^2 + 4}}}\right] \quad \text{(by L' Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{-\sin 2x \sqrt{x^2 + 4}}{x}$$

$$= -\lim_{x \to 0} \left[2 \cos 2x \sqrt{x^2 + 4} + \sin 2x \frac{x}{\sqrt{x^2 + 4}}\right]$$

(by L' Hospital's rule)

$$= -[2\sqrt{4}] = -4$$

Thus, $\lim_{x \to 0} f(x) = -4$

For function to be continuous at $x = 0$.

$$f(0) = \lim_{x \to 0} f(x)$$

$$\Rightarrow \quad a = -4$$

61. $f(x)$ is an even function as when x is rational, $-x$ is also rational, similarly when x is irrational, $-x$ is also irrational. $f(x)$ is not continuous at $x = \frac{1}{2}$ as we cannot find limit of $f(x)$ at 1/2 (not clear whether $f(x) = x$ is to be used or $f(x) = -x$ is to be used).

$f(x)$ is continuous at $x = 0$, here $\lim_{x \to 0} f(x) = 0$ (because in the neighbourhood of 0 whether we use x or $-x$ to find out the limit, it is zero).

62. $$\lambda = \lim_{x \to 0} f(x) = e^{\lim_{x \to 0} (\cos x + 3 \sin x - 1) \frac{5}{\sin x}}$$

$$= e^{15}.$$

63. For $f(x)$ to be continuous at $x = 0$

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{\ln(1 + ax) - \ln(1 - bx)}{x}$$

$$\left[\text{using } \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1\right] = a + b$$

64. $g(x) = f[f(x)] = \begin{cases} 1 + f(x), & 0 \le f(x) \le 2 \\ 3 - f(x), & 2 < f(x) \le 3 \end{cases}$

$$g(x) = \begin{cases} 2 + x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 4 - x, & 2 < x \le 3 \end{cases}$$

Here, $g(x)$ changes the inequality sign at $x = 1$ and $x = 2$. Thus, to check the continuity of $g(x)$ at $x = 1$ and $x = 2$.

At $x = 1$, $\text{LHL} = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (2 + x) = 3$

$$\text{RHL} = \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x) = 1$$

$\therefore$ $\text{LHL} \ne \text{RHL} \Rightarrow g(x)$ is discontinuous at $x = 1$

At $x = 2$, $\text{LHL} = \lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x) = 0$

$$\text{RHL} = \lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} (4 - x) = 2$$

$\therefore$ $\text{LHL} \ne \text{RHL} \Rightarrow g(x)$ is discontinuous at $x = 2$.

65. Polynomial function is always continuous. So, we have to check the continuity only at $x = -3$ and 3.

Here, $f(x) = \begin{cases} |x| + 3, \text{if} & x \le -3 \\ -2x, \text{ if} & -3 < x < 3 \\ 6x + 2, \text{if} & x \ge 3 \end{cases}$

For $x < -3$, $f(x) = |x| + 3$;

$-3 < x < 3$, $f(x) = -2x$ and $x > 3$, $f(x) = 6x + 2$

is a polynomial function, so it is continuous in a given interval. So, we have to check the continuity only at $x = -3$ and 3.

At $x = -3$, $\text{LHL} = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (|x| + 3)$

Putting $x = (-3 - h)$ as $x \to -3^-$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} [|-3 - h| + 3] = \lim_{h \to 0} (6 + h) = 6 + 0 = 6$$

$$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-2x)$$

Putting $x = -3 + h$ as $x \to -3^+$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} -2(-3 + h) = \lim_{h \to 0} (6 - 2h) = 6 - 2 \times 0 = 6$$

Also, $f(-3) = |-3| + 3 = 3 + 3 = 6$ $[\because f(x) = |x| + 3]$

$\therefore$ $\text{LHL} = \text{RHL} = f(-3)$

Thus, $f(x)$ is continuous at $x = -3$.

At $x = 3$, $\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$

Putting $x = 3 - h$ as $x \to 3^-$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} -2(3 - h) = \lim_{h \to 0} (-6 + 2h) = -6 + 2 \times 0 = -6$$

$$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$$

Putting $x = 3 + h$ as $x \to 3^+$ when $h \to 0$

$$\therefore \quad \lim_{h \to 0} [6(3 + h) + 2] = \lim_{h \to 0} (18 + 6h + 2)$$

$$= \lim_{h \to 0} (20 + 6h) = 20 + 6 \times 0 = 20$$

$\therefore$ $\text{LHL} \ne \text{RHL}$. Thus, $f(x)$ is discontinuous at $x = 3$.

66. The left hand limit of f at $x = 0$ is given by

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0 - 1}{0 + 1} = -1$$

Similarly, $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x \to 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} = \lim_{x \to 0^+} \frac{1 - e^{\frac{-1}{x}}}{1 + e^{\frac{-1}{x}}}$$

$$= \frac{1 - 0}{1 + 0} = 1$$

Thus, $\lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$, therefore, $\lim_{x \to 0} f(x)$ does not exist. Hence, f is discontinuous at $x = 0$.

67. $$\lim_{x \to 0} \frac{1}{1 - e^{-1/x}} = \lim_{x \to 0} \frac{1}{1 - e^{-1/0}} = \frac{1}{1 - \frac{1}{e^{\infty}}} = 1$$

and $f(0) = 0$

$$\Rightarrow \quad \lim_{x \to 0} \frac{1}{1 - e^{-1/x}} \ne f(0)$$

$\therefore$ Function is not continuous at $x = 0$.

68. Since, $|x|$ is discontinuous at $x = 0$ and $\frac{|x|}{x}$ is also discontinuous at $x = 0$.

$\therefore$ $f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

69. $f(x) = \begin{cases} x^2 \cos e^{1/x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = 0$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = 0$$

and $\lim_{x \to 0} f(x) = 0$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0} f(x)$

$\Rightarrow$ $f(x)$ is discontinuous function at $x = 0$.

70. LHL $= \lim_{x \to \sqrt{3}^-} f(x) = 2$

$$\text{RHL} = \lim_{x \to \sqrt{3}^+} = f(x) = 2$$

$$\lim_{x \to \sqrt{3}} f(x) = 2$$

Hence, $f(x)$ is continuous at $x = \sqrt{3}$

$$\text{LHL} = \lim_{x \to 4^-} f(x) = 4\sqrt{3} - 1$$

$$\text{RHL} = \lim_{x \to 4^+} f(x) = 4$$

Hence, $f(x)$ is discontinuous at $x = 4$

$$\text{LHL} = \lim_{x \to 5^-} f(x) = 4$$

$$\text{RHL} = \lim_{x \to 5^+} f(x) = 4$$

$$\lim_{x \to 5} f(x) = 4$$

Hence, $f(x)$ is continuous at $x = 5$.

71. We have, $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$$\text{LHD at } (x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0 - h) - f(0)}{0 - h - 0} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$\Rightarrow$ LHD (at $x = 0$) $= 0 \times$ (an oscillating number between -1 and 1) $= 0$

$$\text{RHD (at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$\Rightarrow$ RHD (at $x = 0$) $= 0 \times$ (an oscillating number between -1 and 1) $= 0$

$\therefore$ LHD at $(x = 0)$ = RHD (at $x = 0$) $= 0$

So, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

72. We may rewrite f as $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$\text{Now, } Lf'(0) = \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{-h^2 - 0}{h} = \lim_{h \to 0^-} -h = 0$$

$$\text{Now, } Rf'(0) = \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0^-} h = 0$$

Since, the left hand derivative and right hand derivative both are equal, hence f is differentiable at $x = 0$.

73. As for the derivative $f'(0)$ we have,

$$f'(0^+) = \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

which doesn't exist.

Similarly, the limit $f'(0^-)$ doesn't exist.

74. The range of the function x^3 is $(-\infty, \infty)$ and the range of $f(x)$ is $[0, \infty)$, f is clearly differentiable except possibly at the point $x = 0$.

Now, clearly by definition $Rf'(0) = Lf'(0) = 0$

So that, f is differentiable at $x = 0$ and hence everywhere.

Hence, (a) is the correct answer.

Alternative solution

Here, $f(x) = \begin{cases} -x^3, & x < 0 \\ x^3, & x > 0 \end{cases}$, so that $f'(x) = \begin{cases} -3x^2, & x < 0 \\ 3x^2, & x > 0 \end{cases}$

$\Rightarrow$ the function is differentiable everywhere including $x = 0$.

75. The given function is differentiable at all points in domain except possibly at $x = 0$.

$$\text{Now, } Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h + 1} - 1}{h^{3/2}} = \lim_{h \to 0} \frac{h}{h^{3/2}(\sqrt{h + 1} + 1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{h}(\sqrt{h + 1} + 1)} \to \infty$$

So, the function is not differentiable at $x = 0$.

Hence, the required set is $(0, \infty)$.

76. Let $x < 0$

Then, $|x| = -x$

$$\therefore \quad f'(x) = \frac{d}{dx}\left(\frac{x}{1 - x}\right)$$

$$= \frac{(1 - x) - x(-1)}{(1 - x)^2} = \frac{1}{(1 - x)^2}$$

$$\Rightarrow \quad [f'(x)]_{x=0} = 1$$

Again, let $x > 0$, then $|x| = x$

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x}\right) = \frac{(1+x)-x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$\Rightarrow \quad [f'(x)]_{x=0} = 1$

$\Rightarrow \quad f'(0) = 1$

77. We have, $f(x) = |x| + |\cos x| + \tan\left(x + \frac{\pi}{4}\right)$

$|x|$ is not differentiable at $x = 0$, $|\cos x|$ is not differentiable at points $x = n\pi + \frac{\pi}{2}$ and $\tan\left(x + \frac{\pi}{4}\right)$ is not differentiable at point where $x + \frac{\pi}{4} = n\pi + \frac{\pi}{2}$ or $x = n\pi + \frac{\pi}{4}$. In the interval $(-1, 2)$ there are 3 points of non-differentiability.

78. Given, $f(x) = \begin{cases} (x-1)^2 \sin\frac{1}{x-1} - |x|, & x \neq 1 \\ -1, & x = 1 \end{cases}$

We know that $|x|$ is not differentiable at $x = 0$

$(x-1)^2 \sin\frac{1}{x-1} - |x|$ is not differentiable at $x = 0$.

At all other values of x, $f(x)$ is differentiable.

The required set of points is $\{0\}$.

79. $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$Rf'(0) = f'(0^+) = \lim_{h \to 0} \frac{\frac{h}{1+e^{1/h}} - 0}{h} = \lim_{h \to 0} \frac{1}{1+e^{1/h}} = 0$$

$$Lf'(0) = f'(0^-) = \lim_{h \to 0} \frac{\frac{-h}{1+e^{-1/h}} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{1}{1+\frac{1}{e^{1/h}}} = \frac{1}{1+0} = 1$$

80. Yes, we have the continuous function $f(x) = |x-1| + |x-2|$, which is continuous at all $x \in R$ but differentiable at all x except 1, 2.

Here, $\quad f(x) = |x-1| + |x-2|$

$$= \begin{cases} -(x-1)-(x-2), & \text{if } x < 1 \\ x-1-(x-2), & \text{if } 1 \le x \le 2 \\ x-1+x-2, & \text{if } x > 2 \end{cases}$$

i.e., $\quad f(x) = \begin{cases} -2x+3, & \text{if } x < 1 \\ 1, & \text{if } 1 \le x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

Since, polynomial function is continuous, so it is clear that f is continuous at all except possible at 1, 2.

Now, we have to check the continuity at 1, 2

At $x = 1$, $\quad f(1) = 1$,

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-2x+3) = -2+3 = 1$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 = 1$$

$\Rightarrow \quad f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow$ f is continuous at $x = 1$

At $x = 2$, $\quad f(2) = 1$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1) = 1$$

$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x-3) = 2 \times 2 - 3 = 1$$

$\Rightarrow \quad f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

So, f is continuous at $x = 2$. Thus, f is continuous at all $x \in R$.

Also, $\quad f'(x) = \begin{cases} -2, \text{if } x < 1 \\ 0, \text{ if } 1 < x < 2 \\ 2, \text{ if } x > 2 \end{cases}$

We check the differentiability at 1, 2

$$Lf'(1) = \lim_{h \to 0^-} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0^-} \frac{-2(1-h)+3-1}{-h} = -2$$

and $\quad Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0$

$\therefore \quad L\,f'(1) \neq R\,f'(1) \Rightarrow f$ is not differentiable at 1.

Further,

$\Rightarrow \quad L\,f'(2) = \lim_{h \to 0^-} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0^+} \frac{1-1}{-h} = 0$

and $\quad Rf'(2) = \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0^+} \frac{2(2+h)-3-1}{h} = 2$$

$\therefore \quad L\,f'(2) \neq R\,f'(2) \Rightarrow f$ is not differentiable at 2.

Thus, we see that $f(x) = |x-1| + |x-2|$ is continuous everywhere and differentiable also at all $x \in R$ except at 1, 2.

81. It can be easily seen from the graph of $f(x) = |\sin x|$ that it is continuous everywhere but not differentiable at integer multiples of π and at $x = 0$.

82. Given, f is continuous at $x = 0$.

$\therefore \quad f(0^-) = f(0^+) = f(0) = -1$

Also, it is differentiable at $x = 0$.

$\therefore \quad Lf'(0) = Rf'(0)$

$$\Rightarrow \quad \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{e^{-2h}-1+1}{-h}\right) = \lim_{h\to 0}\left(\frac{ah+\frac{bh^2}{2}-1+1}{h}\right)$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{e^{-2h}}{-h}\right) = \lim_{h\to 0}\left(a+\frac{bh}{2}\right) \quad \left(\because \lim_{x\to 0}\frac{e^{-2x}}{x} = -2\right)$$

$$\Rightarrow \quad 2 = a + 0$$

$\Rightarrow$ $a = 2$, b is any number.

83. Since, $f(x)$ is continuous at $x = 0$, therefore

$$\lim_{x\to 0} f(x) = f(0) \quad \Rightarrow \quad \lim_{x\to 0} x^n \sin\frac{1}{x} = 0, \quad \forall n > 0$$

$f(x)$ is differentiable at $x = 0$ if $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ exists finitely

$$\Rightarrow \quad \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \text{ exists finitely}$$

$$\Rightarrow \quad \lim_{x\to 0} x^{n-1}\sin\frac{1}{x} \text{ exists finitely}$$

$$\Rightarrow \quad n-1>0 \quad \Rightarrow \quad n>1$$

Hence, $f(x)$ is continuous but not differentiable at $x = 0$, if $n \in (0, 1)$.

84. The given function is clearly continuous at all points except possibly at $x = \pm 1$. As $f(x)$ is an even function, so we need to check its continuity only at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{x\to 1^-}(ax^2+b) = \lim_{x\to 1^+}\frac{1}{|x|}$$

$$\Rightarrow \quad a + b = 1 \qquad \ldots(i)$$

Clearly, $f(x)$ is differentiable for all x, except possible at $x = \pm 1$. As $f(x)$ is an even function, so we need to check its differentiability at $x = 1$ only.

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \quad \lim_{x\to 1^-}\frac{ax^2+b-1}{x-1} = \lim_{x\to 1^+}\frac{\frac{1}{|x|}-1}{x-1}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{ax^2-a}{x-1} = \lim_{x\to 1}\frac{-1}{x}$$

$$\Rightarrow \quad 2a = -1 \quad \Rightarrow \quad a = -\frac{1}{2}$$

On putting $a = -\frac{1}{2}$ in Eq. (i), we get $b = \frac{3}{2}$.

Hence, (b) is the correct answer.

85. $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$

$f(x)$ is differentiable at $x = 1$

So, $\quad 2 + b = 1 \quad \Rightarrow \quad b = -1$

Hence, it is obviously $f(x)$ is continuous at $x = 1$.

Hence, $\quad$ LHL = RHL

$$1 = 1 + b + c \quad \Rightarrow \quad c = 1$$

86. $\lim_{x\to 1}\frac{f(1)\,g(x) - f(x)\,g(1) - f(1) + g(1)}{g(x) - f(x)} \quad \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 1}\frac{f(1)\,g'(x) - g(1)f'(x)}{g'(x) - f'(x)} \quad \text{(using L' Hospital's rule)}$$

$$= \frac{2g'(1) - 2f'(1)}{g'(1) - f'(1)} = 2$$

87. Here, $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)} = \lim_{x\to 0}\frac{f'(x^2)\,2x - f'(x)}{f'(x)} = \frac{-f'(0)}{f'(0)} = -1$

88. We have,

$$\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{x-1}\,dt = \lim_{x\to 1}\frac{\int_1^{f(x)} 2t\,dt}{x-1} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L'-hospital,'s rule

$$= \lim_{x\to 1}\frac{2f(x)\,f'(x)}{1-0} = 2f(1)\cdot f'(1)$$

[given, $\because f(1) = 4$ and $f'(1) = 2$]

$$= 2\cdot 4\cdot 2 = 16$$

89. Given, $f(2) = 4$ and $f'(2) = 1$

Then, $\lim_{x\to 2}\frac{x\,f(2) - 2f(x)}{x-2} = \lim_{x\to 2}\frac{4x - 2f(x)}{x-2}$

$$= \lim_{x\to 2}\frac{4 - 2f'(x)}{1} \quad \text{(by L' Hospital's rule)}$$

$$= 4 - 2\times 1 = 2$$

90. $f(x) = \max\{x, x^3\}$. Considering the graph separately.

$y = x^3$ and $y = x$

Now, $f(x) = \begin{cases} x & \text{in } (-\infty, -1] \\ x^3 & \text{in } [-1, 0] \\ x & \text{in } [0, 1] \\ x^3 & \text{in } [1, \infty] \end{cases}$

The point of consideration are

$$f'(-1^-) = 1 \text{ and } f'(-1^-) = 3$$
$$f'(-0^-) = 0 \text{ and } f'(-0^-) = 1$$
$$f'(1^-) = 1 \text{ and } f'(1^-) = 3$$

Hence, f is not differentiable at $-1, 0, 1$.

Note $y = x^3$ is odd order parabola and all curves $y = x^n$ (if is odd) and $y = x$ always intersect at (1, 1) and (–1, 1).

91. $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x) - \log f(1)]}$

$$= e^{\lim_{x\to 0}\frac{f'(1+x)/f(1+x)}{1}} = e^{\frac{f'(1)}{f(1)}} = e^{\frac{6}{3}} = e^2$$

92. Given, $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ ...(i)

$\because$ $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right) - f\left(\frac{3x+0}{3}\right)}{h}$$

$$f'(x) = \lim_{h\to 0}\left\{\frac{\frac{2+f(3x)+f(3h)}{3} - \frac{2+f(3x)+f(0)}{3}}{h}\right\}$$

[using Eq. (i)]

$$f'(x) = \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} = \lim_{h\to 0}\frac{f(0+3h)-f(0)}{3h}$$

$$= f'(0)$$

$\therefore$ $f'(x) = f'(0) = 2$

$\therefore$ $\int f'(x)\cdot dx = 2\int dx$

$\left[\because f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}\right.$ putting $x = y = 0$

$\Rightarrow$ $\left.f(0) = \frac{0+2f(0)}{3} \Rightarrow f(0) = 2\right]$

$\Rightarrow$ $f(x) = 2x + C$

Putting $f(0) = C = 2 \therefore f(x) = 2x + 2$

93. Given that, $f(9) = 9, f'(9) = 4$

Then, $$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$$

$$= \lim_{x\to 9}\frac{\{\sqrt{f(x)}-3\}\{\sqrt{f(x)}+3\}}{(\sqrt{x}-3)(\sqrt{x}+3)}\frac{\sqrt{x}+3}{\sqrt{f(x)}+3}$$

$$= \lim_{x\to 9}\frac{f(x)-9}{x-9}\cdot\left[\frac{3+3}{3+3}\right] = \lim_{x\to 9}\frac{f(x)-f(9)}{x-9}\cdot 1 = f'(9) = 4$$

94. $$\lim_{x\to 1}\frac{-\sqrt{25-x^2}-(-\sqrt{24})}{x-1}$$

$$= \lim_{x\to 1}\frac{\sqrt{24}-\sqrt{25-x^2}}{x-1}\times\frac{\sqrt{24}+\sqrt{25-x^2}}{\sqrt{24}+\sqrt{25-x^2}}$$

$$\lim_{x\to 1}\frac{x^2-1}{(x-1)[\sqrt{24}+\sqrt{25-x^2}]} = \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

95. $$\lim_{x\to 1-0}\{x\} = \lim_{x\to 1-0}(x-[x]) = 1-0 = 1$$

$$\lim_{x\to 1+0}\{x\} = \lim_{x\to 1+0}(x-[x]) = 1-1 = 0$$

$\therefore$ $$\lim_{x\to 1-0}\frac{x\sin\{x\}}{x-1} = \lim_{x\to 1-0}\frac{x}{x-1}\sin\{x\} = -\infty$$

$$\lim_{x\to 1+0}\frac{x\sin\{x\}}{\{x\}}\frac{x-1}{x-1} = 1\times 1\times 1 = 1$$

Since, LHL $\neq$ RHL.

96. $$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{f(5)\,f(h)-f(0)\,f(5)}{h}$$

$$= \lim_{h\to 0} f(5)\left\{\frac{f(0+h)-f(0)}{h}\right\}$$

$$= f(5)\cdot f'(0) = 2\times 3 = 6$$

97. As $f(x)$ is continuous in $[1,10]$, $f(x)$ will attain all values between $f(1)$ and $f(10)$. As $f(x)$ takes rational values for all x and there are innumerable irrational values between $f(1)$ and $f(10)$ which implies that $f(x)$ can take rational values for all x. If $f(x)$ has a constant values at all points between $x = 1$ and $x = 10$, *i.e.*, $f(x)$ must be a constant function. As $f(1) = 10$, we get

$$f(x) = 10,\ \forall x\in[1,10] \Rightarrow f(10) = 10$$

98. $$\lim_{x\to\alpha}\frac{\sin(ax^2+bx+c)}{(x-\alpha)^2} = \lim_{x\to\alpha}\frac{\sin a(x-\alpha)(x-\alpha)}{(x-\alpha)^2}$$

$$= \lim_{x\to\alpha}\frac{\sin a(x-\alpha)^2}{a(x-\alpha)^2}\times a = a$$

99. $$\lim_{x\to a}\frac{g(x)\,f(a)-g(a)\,f(x)}{x-a}$$

$$\lim_{h\to 0}\frac{g(a+h)\,f(a)-g(a)\,f(a+h)}{h} \quad [\text{for } x = a+h]$$

$$\lim_{h\to 0}\frac{g(a+h)\,f(a)-g(a)\,f(a)+g(a)\,f(a)-g(a)\,f(a+h)}{h}$$

$$= \lim_{h\to 0} f(a)\left[\frac{g(a+h)-g(a)}{h}\right] - g(a)\left[\frac{f(a+h)-f(a)}{h}\right]$$

$$= f(a)\,g'(a) - g(a)\,f'(a) = 2\times 2-(-1)\times 1 = 5$$

100. In ΔABC, $AB = AC$

$AD \perp BC$ (D is mid-point of BC)

Let, r = radius of circumcircle

$\Rightarrow$ $OA = OB = OC = r$

Now, $BD = \sqrt{BO^2 - OD^2}$

$$= \sqrt{r^2-(h-r)^2} = \sqrt{2r-h^2}$$

$\Rightarrow$ $BC = 2\sqrt{2rh-h^2}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\times BC\times AD = h\sqrt{2rh-h^2}$

Also, $$\lim_{h\to 0}\frac{A}{p^3} = \frac{h\sqrt{2rh-h^2}}{8(\sqrt{2rh-h^2}+\sqrt{2hr})^3}$$

$$= \lim_{h\to 0}\frac{h^{3/2}\sqrt{2r-h^2}}{8h^{3/2}(\sqrt{2r-h}+\sqrt{2r})^3}$$

$$= \lim_{h\to 0}\frac{\sqrt{2r-h}}{8[\sqrt{2r-h}+\sqrt{2r}]^3}$$

$$= \frac{\sqrt{2}r}{8(\sqrt{2}r+\sqrt{2}r)^3} = \frac{\sqrt{2}r}{8\cdot 8\cdot 2r\cdot\sqrt{2}r} = \frac{1}{128r}$$

Round II

1. $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)} \quad \text{(using L' Hospital's rule)}$$

$$=-1+\lim_{x\to 0}\frac{2xf'(x^2)}{f'(x)}=-1$$

[$\because f'(0)\neq 0$ as f is strictly increasing function]

2. We have, $\lim_{x\to -\infty}\frac{x^5\tan\left(\frac{1}{\pi x^2}\right)+3|x|^2+7}{|x|^3+7|x|+8}$

$$=\lim_{x\to -\infty}\frac{x^5\tan\left(\frac{1}{\pi x^2}\right)+3x^2+7}{-x^3-7x+8}$$

$(\because x<0=|x|=-x)$

$$=\lim_{x\to -\infty}\frac{x^2\tan\left(\frac{1}{\pi x^2}\right)+\frac{3}{x}+\frac{7}{x^3}}{-1-\frac{7}{x^2}+\frac{8}{x^3}}$$

On dividing the numerator and denominator by x^3,

$$=\lim_{x\to -\infty}\frac{\frac{1}{\pi}\cdot\frac{\tan\left\{\frac{1}{(\pi x^2)}\right\}}{\frac{1}{(\pi x^2)}}+\frac{3}{x}+\frac{7}{x^3}}{-1-\frac{7}{x^2}+\frac{8}{x^3}}$$

$$=\frac{\frac{1}{\pi}\cdot 1+0+0}{-1-0+0}=-\frac{1}{\pi}$$

$$\left(\because x\to -\infty\Rightarrow\frac{1}{\pi x^2}\to 0 \text{ and } \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right)$$

3. $\lim_{h\to 0}\frac{f(1-h)-f(1)}{h^3+3h}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}\cdot\frac{-1}{h^2+3}$

$$=f'(1)\cdot\left(\frac{-1}{3}\right)=\frac{53}{3}$$

4. $\lim_{n\to\infty}\sin\left\{n\pi\left(1+\frac{1}{n^2}\right)^{1/2}\right\}$

$$=\lim_{n\to\infty}\sin\left\{n\pi\left(1+\frac{1}{2n^2}-\frac{1}{8n^4}+\ldots\infty\right)\right\}$$

$$=\lim_{n\to\infty}\sin\left\{n\pi+\frac{\pi}{2n}-\frac{\pi}{8n^3}+\ldots\infty\right\}$$

$$=\lim_{n\to\infty}(-1)^n\sin\pi\left(\frac{1}{2n^2}-\frac{1}{8n^4}+\ldots\infty\right)=0$$

5. Since, $|x-3|=x-3$, if $x\geq 3$ and $|x-3|=-x+3$, if $x<3$.

$\therefore$ The given function can be defined as

$$f(x)=\begin{cases}\frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4}, & x<1\\ 3-x, & 1\leq x<3\\ x-3, & x\geq 3\end{cases}$$

$$\Rightarrow \quad f'(x)=\begin{cases}\frac{x}{2}-\frac{3}{2}, & x<1\\ -1, & 1\leq x<3\\ 1, & x\geq 3\end{cases}$$

At $x=1$,

$$\text{LHD}=f'(1)=-1 \text{ and RHD}=f'(1)=-1$$

6. $f(x)=\begin{cases}e^x, & x\leq 0\\ 1-x, & 0<x\leq 1\\ x-1, & x>1\end{cases}$

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{1-h-1}{h}=-1$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$$

So, it is not differentiable at $x=0$.

Similarly, it is not differentiable at $x=1$ but it is continuous at $x=0$ and 1.

7. Given, $x>0$ and g is a bounded function. Then,

$$\lim_{x\to\infty}\frac{f(x)\cdot e^{nx}+g(x)}{e^{nx}+1}=\lim_{n\to\infty}\frac{f(x)}{1+\left(\frac{1}{e^{nx}}\right)}=\frac{g(x)}{e^{nx}+1}$$

$$=\frac{f(x)}{1+0}+\frac{\text{finite}}{\infty}=f(x)$$

8. $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$$=\lim_{h\to 0}\frac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}=\frac{6\times 2}{4\times 1}=3$$

9. $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

On putting $x=\tan\theta$ in both equations, we get

$$f(\theta)=\cot^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$\Rightarrow \quad f(\theta)=\cot^{-1}(\tan 3\theta)$$

$$\Rightarrow \quad f(\theta)=\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-3\theta\right)\right\}=\frac{\pi}{2}-3\theta$$

$$\therefore \quad f'(\theta)=-3 \qquad \ldots\text{(i)}$$

and $$g(\theta) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta$$

$\therefore$ $$g'(\theta) = 2$$

Now, $$\lim_{x\to a}\left\{\frac{f(x)-f(a)}{g(x)-g(a)}\right\} = \lim_{x\to a}\left\{\frac{f(x)-f(a)}{x-a}\right\} \times \frac{1}{\lim\limits_{x\to a}\left\{\frac{g(x)-g(a)}{x-a}\right\}}$$

$$= f'(x)\cdot\frac{1}{g'(x)} = -3\times\frac{1}{2} = -\frac{3}{2}$$

10. $$\text{RHL} = \lim_{x\to 0^+} = \lim_{x\to 0^+}(x)^0 = 1$$

$$\text{LHL} = \lim_{x\to 0^-} = \lim_{x\to 0^-}(-x)^0 = \lim_{x\to 0^-} 1 = 1$$

$\because$ RHL = LHL

$\therefore$ $$\lim_{x\to 0}|x|^{[\cos x]} = 1$$

11. $$\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right] = \lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}$$

12. Let $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

Therefore, given function $= f'(a) + kf'(e) = 1$

$\Rightarrow$ $$\frac{1}{a}+\frac{k}{e} = 1 \Rightarrow k = e\left(\frac{a-1}{a}\right)$$

13. $$\lim_{x\to 2}\frac{\int_6^{f(x)} 4t^3 dt}{x-2} = \lim_{x\to 2}\frac{4\{f(x)\}^3\times f'(x)}{1} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= 4\{f(2)\}^3\times f'(2) = 18$$

14. We know that,

$$\cos A\cdot\cos 2A\cdot\cos 4A\ldots\cos 2^{n-1}A = \frac{\sin 2^n A}{2^n\sin A}$$

Take $A = \frac{x}{2^n}$, then

$$\cos\left(\frac{x}{2^n}\right)\cdot\cos\left(\frac{x}{2^n-1}\right)\ldots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) = \frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$\therefore \lim_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\ldots\cos\left(\frac{x}{2^{n-1}}\right)\cdot\cos\left(\frac{x}{2^n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)} = \lim_{n\to\infty}\frac{\sin x}{x}\cdot\frac{x/2^n}{\sin(x/2^n)} = \frac{\sin x}{x}$$

15. $$f\{g(x)\} = \begin{cases} 1, & 0<x<3\pi/4 \text{ or } 7\pi/4<x<2\pi \\ 0, & x = 3\pi/4, 7\pi/4 \\ -1, & 3\pi/4<x<7\pi/4 \end{cases}$$

Clearly, $[f\{g(x)\}]$ is not continuous at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

16. $$f(x) = \begin{cases} \frac{1}{2}(-x-1), & x<-1 \\ \tan^{-1}x, & -1\le x\le 1 \\ \frac{1}{2}(x+1), & x>1 \end{cases}$$

$$f'(x) = \begin{cases} -\frac{1}{2}, & x<-1 \\ \frac{1}{1+x^2}, & -1<x<1 \\ \frac{1}{2}, & x>1 \end{cases}$$

$\therefore$ $$f'(-1-0) = -\frac{1}{2}$$

$$f'(-1+0) = \frac{1}{1+(-1+0)^2} = \frac{1}{2}$$

$$f'(1-0) = \frac{1}{1+(1-0)^2} = \frac{1}{2}$$

and $$f'(1+0) = \frac{1}{2}$$

$\therefore$ $f'(-1)$ does not exist.

$\therefore$ Domain of $f'(x) = R - \{-1\}$

17. Let $$y = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$$

$$\Rightarrow \log y = \lim_{x\to 0}\frac{2}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right)$$

$$= 2\lim_{x\to 0}\frac{\log(a^x+b^x+c^x)-\log 3}{x}$$

Apply L'Hospital's rule,

$$= 2\lim_{x\to 0}\frac{\frac{a^x\log a+b^x\log a+c^x\log a}{a^x+b^x+c^x}}{1}$$

$$\log y = \log(abc)^{2/3} \Rightarrow y = (abc)^{2/3}$$

18. $f(1) = 1$, $f(1-0) = 1-1+1 = 1$, $f(1+0) = 1+0 = 1$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$\Rightarrow f'(1-0) = \lim_{h\to 0}\frac{(1-1)+\sqrt{1-h}-1}{-h}$$

$$= \lim_{h\to 0}\frac{1-h-1}{-h(1+\sqrt{1-h})} = \frac{1}{2}$$

$$\Rightarrow f'(1+0) = \lim_{h\to 0}\frac{1+\sqrt{h-1}}{h} = +\infty$$

Thus, $f(x)$ is continuous for all x but non-differentiable at all integral values of x.

19. $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-\alpha x-\beta\right)=0$

$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2(1-\alpha)-x(\alpha+\beta)+1-b}{x+1}=0$

$\therefore \quad 1-\alpha=0,\ \alpha+\beta=0 \Rightarrow \alpha=1, \beta=-1$

20. $[\sin x]$ is non-differentiable at $x=\frac{\pi}{2}, \pi, 2\pi$ and $[\cos x]$ is non-differentiable at $x=0, \frac{\pi}{2}, \frac{3\pi}{2}$ and 2π.

Thus, $f(x)$ is definitely non-differentiable at $x=\pi, \frac{3\pi}{2}, 0$.

Also, $f\left(\frac{\pi}{2}\right)=1, f\left(\frac{\pi}{2}-0\right)=0, f(2\pi)=1,$

$f(2\pi-0)=-1$

Thus, $f(x)$ is also non-differentiable at $x=\frac{\pi}{2}$ and 2π.

21. $\lim_{x\to\pi/4}\frac{\int_2^{\sec^2 x} f(t)}{x^2-\frac{\pi^2}{16}}dt=\frac{\lim_{x\to\frac{\pi}{4}} f(\sec^2 x)\,2\sec x\cdot \sec x\tan x}{2x}$

$=\frac{2f(2)}{\pi/4}=\frac{8f(2)}{\pi}$

22. Let $g(x)=f(x)-x^2$

$\therefore$ $g(x)$ has atleast real roots which are $x=1, 2$ and 3.

$\Rightarrow$ $g'(x)$ has atleast two real roots in $x\in(1, 3)$.

$\Rightarrow$ $g''(x)$ has atleast one real root in $x\in(1, 3)$.

$\Rightarrow$ $f''(x)-2\cdot 1=0$

Hence, $f''(x)=2$ for atleast one $x\in(1, 3)$.

23. Since, $\lim_{x\to 0}\left[(a-n)n-\frac{\tan x}{x}\right]\cdot\frac{\sin nx}{x}=0$

$\Rightarrow \quad [(a-n)(n-1)]n=0 \quad \Rightarrow \quad (a-n)n=1$

$\Rightarrow \quad a=n+\frac{1}{n}$

24. $\lim_{x\to 3}\frac{\sqrt{1-\cos(x^2-10x+21)}}{(x-3)}$

$=\lim_{x\to 3}\frac{\sqrt{2}\sin\frac{(x-3)(x-7)}{2}}{(x-3)}$

$=\lim_{x\to 3}\frac{\sqrt{2}\sin\frac{(x-3)(x-7)}{2}}{\frac{(x-3)\cdot(x-7)}{2}}\cdot\frac{(x-7)}{2}$

$=\lim_{x\to 3}(x-7)\cdot\lim_{x\to 3}\frac{\sin\frac{(x-3)(x-7)}{2}}{\frac{(x-3)(x-7)}{2}}\times\frac{1}{\sqrt{2}}$

$=-(2)^{3/2}$

25. If x is just less than k, then $[x]=k-1$

$\therefore \quad f(x)=(k-1)\sin\pi x$

LHD of $\quad f(x)=\lim_{x\to k}\frac{(k-1)\sin\pi x-k\sin\pi k}{x-k}$

$=\lim_{x\to k}\frac{(k-1)\sin\pi x}{x-k}$, where $x=k-h$

$=\lim_{h\to 0}\frac{(k-1)\sin\pi(k-h)}{-h}=(k-1)(-1)^k\pi$

26. Let $h(x)=|x|$. Then, $g(x)=|f(x)|=h\{f(x)\}$

Since, composition of two continuous functions is continuous and g is continuous, if f is continuous.

27. $\lim_{n\to\infty}{}^nC_x\left(\frac{m}{n}\right)^x\left(1-\frac{m}{n}\right)^{n-x}=\lim_{n\to\infty}\frac{n!}{x!(n-x)!}\cdot\frac{m^x}{n^x}\cdot\frac{\left(1-\frac{m}{n}\right)^n}{\left(1-\frac{m}{n}\right)^x}$

$=\lim_{n\to\infty}\frac{m^x}{x!}\frac{\left(1-\frac{m}{n}\right)^n}{n^x}\left\{\frac{n!}{(n-x)!\left(1-\frac{m}{n}\right)^x}\right\}$

$=\lim_{n\to\infty}\frac{m^x}{x!}\cdot\left(1-\frac{m}{n}\right)^n\frac{n(n-1)(n-2)\ldots(n-x+1)}{n^x\left(1-\frac{m}{n}\right)^x}$

$=\lim_{n\to\infty}\frac{m^x}{x!}\left(1-\frac{m}{n}\right)^n\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1+\frac{1-x}{n}\right)}{\left(1-\frac{m}{n}\right)^x}$

$=\frac{m^x}{x!}\cdot\frac{e^{-m}\cdot 1}{1^x}=\frac{m^x\cdot e^{-m}}{x!}$

28. LHL $=\lim_{x\to[a]^-}\frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}=\lim_{h\to 0}\frac{e^{\{[a]-h\}}-\{[a]-h\}-1}{\{[a]-h\}^2}$

$=\lim_{h\to 0}\frac{e^{1-h}-(1-h)-1}{(1-h)^2}=e-2$

and RHL $=\lim_{x\to[a]^+}\frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}=\lim_{h\to 0}\frac{e^{\{[a]+h\}}-\{[a]+h\}-1}{\{[a]+h\}^2}$

$=\lim_{h\to 0}\frac{e^h-h-1}{h^2}=P$ (say) ...(i)

On replacing h by $-h$ in P, then

$P=\lim_{h\to 0}\frac{e^{-h}+h-1}{h^2}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2P=\lim_{h\to 0}\frac{e^h+e^{-h}-2}{h^2}=\lim_{h\to 0}\frac{(e^h-1)^2}{e^h\cdot h^2}=\frac{(1)^2}{1}=1$

$\Rightarrow \quad P=\frac{1}{2}$

$\therefore$ RHL $=\frac{1}{2}$ [from Eq. (i)]

Hence, LHL $\neq$ RHL

29. On dividing the numerator and denominator by x^m. Then,

$$k = \lim_{x \to \infty} \left\{ \frac{\sum_{k=1}^{1000} \left(1+\frac{k}{x}\right)^m}{1+\frac{10^{1000}}{x^m}} \right\}$$

$$= \frac{1+1+1+\ldots \text{upto 1000 times}}{1+0}$$

$$= 1000 = 10^3$$

30. Given, $\sum_{r=1}^{k} \cos^{-1}\beta_r = \frac{k\pi}{2}$

which is possible only when

$$\cos^{-1}\beta_1 = \cos^{-1}\beta_2 = \cos^{-1}\beta_3 = \ldots = \cos^{-1}\beta_k = \frac{\pi}{2}$$

$\therefore \quad \beta_1 = \beta_2 = \beta_3 = \ldots. = \beta_k = 0$

Hence, $\quad A = \sum_{r=1}^{l} (\beta_r)^r = 0$

$$\therefore \lim_{x \to 0} \frac{(1+x)^{1/3} - (1-2x)^{1/4}}{x+x^2}$$

$$= \lim_{x \to 0} \frac{\left(\frac{(1+x)^{1/3} - (1)^{1/3}}{(1+x)-1}\right) - \left(\frac{(1-2x)^{1/4} - (1)^{1/4}}{(1-2x)-1} \times (-2)\right)}{(1+x)}$$

$$\frac{\frac{1}{3} + 2 \times \frac{1}{4}}{1+0} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

31. $$\lim_{x \to 0} x^a \frac{\left\{\frac{\sin x}{x}\right\}^b}{\left\{\frac{\sin(x^c)}{x^c}\right\}} \cdot \frac{x^b}{x^c} = \lim_{x \to 0} x^{a+b-c} \frac{\left\{\frac{\sin x}{x}\right\}^b}{\left\{\frac{\sin(x^c)}{x^c}\right\}}$$

The above limit has non-zero value only when

$$a + b - c = 0$$

32. Given, $f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{3}$

On replacing x by $3x$ and y by zero.

Then, $\quad f(x) = \frac{f(3x) + f(0)}{3}$

$\Rightarrow \quad f(3x) - 3f(x) = -f(0) \quad \ldots(i)$

and $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(3x) + f(3h)}{3} - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(3x) + f(3h) - 3f(x)}{3h}$$

$$= \lim_{h \to 0} \frac{f(3h) - f(0)}{3h} \quad \text{[from Eq. (i)]}$$

$$= f'(0) = 3$$

$\therefore \quad f(x) = 3x + C$

$\Rightarrow \quad f(0) = 0 + C = 3 \quad \Rightarrow \quad C = 3$

Then, $\quad f(x) = 3x + 3$

Hence, $f(x)$ is continuous and differentiable everywhere.

33. $\because [(2x - 3\pi)^3] = \text{Integer}$ (say n)

$\therefore \quad \tan n\pi = 0$

and $\quad 1 + [2x - 3\pi]^2 \neq 0$

Thus, $\quad f(x) = 0$

Hence, $f(x)$ is continuous and differentiable for all $x \in R$.

34. Since, $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 2, & x \text{ is irrational} \end{cases}$

$\because$ Real number = Rational number + Irrational number. If x is rational, then $f(x) = 1$ and if x is irrational, then $f(x) = 2$.

Hence, $f(x)$ is not continuous in R.

35. If $b - 1 \le x \le b$, from Rolle's theorem

$$f(b-1) = f(b) = 0$$

and $\quad f'(x) = 0$

Let $\quad x = c$

Such that $\quad b - 1 \le c \le b$

Hence, $f(x) = 0$ has atmost one root.

36. Given function,

$$f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$$

Now,

$$\text{LHD} = Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - h(1)}{-h}$$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 + 3(1-h) + p - (1+3+p)}{-h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h + 3 - 3h + p - 4 - p}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h}{-h} = \lim_{h \to 0} -h + 5 = 5$$

$$\text{RHD} = Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{q(1+h) + 2 - (1+3+p)}{h}$$

$$= \lim_{h \to 0} \frac{q + qh + 2 - 4 - p}{h}$$

$$= \lim_{h \to 0} \frac{(q-p) + qh - 2}{h}$$

$$= \lim_{h \to 0} q + \frac{(q-p-2)}{h}$$

Since, the given function $f(x)$ is differentiable.

At $x = 1$, so both hands must be exist.

$\therefore \quad q - p - 2 = 0 \quad$ ...(i)

and $\quad Lf'(1) = Rf'(1)$

$\Rightarrow \quad 5 = q \Rightarrow q = 5 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$p = q - 2 = 5 - 2 = 3 \quad \text{...(iii)}$$

$\therefore$ Required values are, $p = 3$ and $q = 5$.

37. At $x = \frac{\pi}{4}$,

$$\text{LHL} = \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} (x + a^2\sqrt{2}\sin x) = a^2 + \frac{\pi}{4}$$

$$\text{RHL} = \lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} (x \cot x + b) = b + \frac{\pi}{4}$$

Since, $f(x)$ is continuous at $x = \frac{\pi}{4}$.

$\therefore \quad a^2 + \frac{\pi}{4} = b + \frac{\pi}{4} \quad \therefore \quad a^2 = b \quad$ ...(i)

At $\quad x = \frac{\pi}{2}$,

$$\text{LHL} = \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (x \cot x + b) = b$$

$$\text{RHL} = \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} (b \sin 2x - a \cos 2x) = a$$

Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$\therefore \quad a = b \quad$ ...(ii)

From Eqs. (i) and (ii), $a = 0, 1$ and $b = 0, 1$

$\therefore \quad (a, b) = (0, 0), (1, 1)$

38. $\lim_{n \to \infty} \left(an - \frac{1+n^2}{1+n} \right) = \lim_{n \to \infty} \frac{n^2(a-1) + an - 1}{1-n}$

If limit exist, so $n^2(a-1)$ should be zero.

$\therefore \quad a = 1$

$\therefore \quad a = b = 1$

39. Let $g(x) = A \sin x + \sin 2x$ and $h(x) = x^3$.

Since, $\quad g(0) = 0$ and $h(0) = 0$

So, $\quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{A \cos x + 2 \cos 2x}{3x^2}$

Again, $\quad \lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} f(x)$ exists

So, $\quad A + 2 = 0$ or $A = -2$

Hence, $\quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-2 \cos x + 2 \cos 2x}{3x^2}$

$$= \lim_{x \to 0} \frac{2 \sin x - 4 \sin 2x}{6x} = \frac{2}{6} - \frac{4}{3} = -1$$

Hence, $\quad A = -2$ and $f(0) = -1$

40. $f'(1+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{|\log(1+h)| - |\log 1|}{h}$

$$= \lim_{h \to 0} \frac{|\log(1+h)|}{h}$$

$$= \lim_{h \to 0} \frac{\log(1+h)}{h} = 1$$

$$f'(1-) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{|\log(1-h)| - |\log 1|}{-h}$$

$$= \lim_{h \to 0} \frac{|\log(1-h)|}{-h} = -1$$

41. $y' = -\frac{\sin x}{\sqrt{1 - \cos^2 x}} = -\frac{\sin x}{|\sin x|}$

So, the functions is not differentiable at the points where $\sin x = 0$ *i.e.*, $x = k\pi$, where $k \in I$

42. $f(x) = \begin{cases} -x^3, & x < 0 \\ 0, & x = 0 \\ x^3, & x > 0 \end{cases}$

At $x = 0$, LHL $= 0$, RHL $= 0$

RHD $= 0$, RHD $= 0$

43. $\left\{ \frac{d}{dx} [f(x)\, g(x)] \right\}_{x=a} = f'(a)\, g(a) + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \cdot f(a)$

If $f(a) \neq 0 \Rightarrow g'(a)$ must exist.

Also, if $g(a)$ is discontinuous, $f(a)$ must be 0 for $f(x) \cdot g(x)$ to be differentiable.

44. Given that, $x + |y| = 2y$

If $y < 0$, then $x - y = 2y$

$\Rightarrow \quad y = \frac{x}{3}$

$\Rightarrow \quad x < 0$

If $y = 0$, then $x = 0$.

If $y > 0$, then $x + y = 2y$

$\Rightarrow \quad y = x \Rightarrow x > 0$

Thus, we can define $f(x) = y = \begin{cases} x/3, & x < 0 \\ x, & x \geq 0 \end{cases}$

Continuity at $x = 0$,

$$\text{LHL} = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} -\frac{h}{3} = 0$$

$$\text{RHL} = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} h = 0$$

$$f(0) = 0$$

As $\quad \text{LHL} = \text{RHL} = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Differentiability at $x = 0$,

$$Lf' = \frac{1}{3}; Rf' = 1$$

As $Lf' \neq Rf' \Rightarrow$ $f(x)$ is not differentiable at $x = 0$.

But for $x < 0$, $\frac{dy}{dx} = \frac{1}{3}$. Thus, (a), (b) and (d) are correct.

45. The given function is,

$$f(x) = \frac{x}{1 + |x|} = \begin{cases} \frac{x}{1 - x}, & x < 0 \\ \frac{x}{1 + x}, & x \geq 0 \end{cases}$$

For $x < 0$, $f'(x) = \frac{1(1 - x) - (-1)x}{(1 - x)} = \frac{1}{(1 - x)^2}$, (defined $\forall\, x < 0$)

For $x > 0$, $f'(x) = \frac{1(1 + x) - 1(x)}{(1 + x)^2} = \frac{1}{(1 + x)^2}$, (defined $\forall\, x > 0$)

For $x = 0$,

$$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{\frac{-h}{1 + h} - 0}{-h} = 1$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{h}{1 + h} - 0}{h} = 1$$

$$Lf'(0) = Rf'(0)$$

$\Rightarrow$ f is differentiable at $x = 0$.

Hence, f is differentiable in $(-\infty, \infty)$.

46. From graph, $f(x)$ is continuous everywhere but not differentiable at $x = 1$.

which is discontinuous at $x = \frac{1}{2}, 1, \frac{3}{2}$.

47. $f(x) = \cos\left[\frac{\pi}{x}\right] \cos\left\{\frac{\pi}{2}(x - 1)\right\} = \sin\left(\frac{\pi}{2}x\right) \cos\left[\frac{\pi}{x}\right]$

Since, $f(0)$ is undefined.

$\therefore$ f is discontinuous at $x = 0$.

$$f(1^-) = \lim_{x \to 1^-} \cos\left[\frac{\pi}{x}\right] \sin\left(\frac{\pi}{2}x\right) \text{ as } x \to 1^-, \frac{\pi}{x} \to \pi^+$$

$$\therefore \quad \left[\frac{\pi}{x}\right] = 3$$

$$\therefore \quad f(1^-) = \lim_{x \to 1} \cos 3 \sin\left(\frac{\pi x}{2}\right) = \cos 3$$

$$f(1^+) = \lim_{x \to 1^+} \cos\left[\frac{\pi}{x}\right] \sin\left(\frac{\pi}{2}\right)$$

as $\quad x \to 1^+; \frac{\pi}{x} \to \pi$ still $\left[\frac{\pi}{x}\right] = 3$

$\therefore$ $f(1^+) = \cos 3$, also $f(1) = \cos 3$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Similarly, $f(x)$ is continuous at $x = 2$

48. $f(x) + g(x) = x + |x| + \cos[\pi^2]x + \sin x$

not differentiable at $x = 0$

$f(x)\, g(x) = x \sin x + |x| + \sin x \cdot \cos[\pi^2]x$, continuous and differentiable everywhere.

49. Put $x = y = 0 \quad \Rightarrow \quad f(0) = 1$

Now, $\quad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$$

$$= 2x + \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2x + f'(0) \quad [\text{since, } f(0) = 1]$$

or $\quad f'(x) = 2x + \sin\phi$

$\therefore \quad f(x) = x^2 + x \sin\phi + c$

but $\quad f(0) = 1 \quad \Rightarrow \quad c = 1$

$$\therefore \quad f(x) = x^2 + x \sin\phi + 1 = \left(x + \frac{\sin\phi}{2}\right)^2 + \left(1 - \frac{\sin^2\phi}{4}\right)$$

$$\geq 1 - \frac{\sin^2\phi}{4} \geq \frac{3}{4} > 0$$

50. $$\lim_{x \to 0} \frac{a\left(x - \frac{x^3}{6} + \ldots\right) + b\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} \ldots\right) + c\left(1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots\right) - 2}{x^2} = 6$$

$$\Rightarrow \quad b + c - 2 = 0, \quad a + b = 0, \quad \frac{b}{2} - \frac{c}{2} = 6$$

$$\Rightarrow \quad a = -7, b = 7, c = -5$$

51. Clearly, from graph

$$I = \int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/2} \cot x\,dx$$

$$= [\ln|\sec x|]_0^{\pi/4} + [\ln|\sin x|]_{\pi/4}^{\pi/2}$$

$$= 2\ln\sqrt{2}$$

Obviously, the period of $\tan x$ and $\cot x$ is π.

52. $F'(x) = [1 + 2f(x)] + 3[f(x)]^2 f'(x) > 0$,

So, $F(x)$ is increasing

So, $F(0) < 0$

$\Rightarrow \quad [1 + f(0)][1 + \{f(0)\}^2] < 0$

$\Rightarrow \quad f(0) < -1.$

53. LHL = RHL

$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \quad \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h)$

$\Rightarrow \quad \lim_{x \to 0} \left\{1 + \frac{2(1-h)}{\lambda}\right\} = \lim_{h \to 0} \lambda(1+h)$

$\Rightarrow \quad 1 + \frac{2}{\lambda} = \lambda$

$\Rightarrow \quad \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda + 1) = 0$

$\therefore \quad \lambda = -1, 2$

54. If $x < x^2$

Then, $\quad h(x) = x, x(x-1) > 0$

$\therefore \quad x > 1$ and $x < 0$

$\therefore \quad h'(x) = 1$

and if $\quad x^2 < x,\ x(x-1) < 0$

$\Rightarrow \quad 0 < x < 1$

Then, $\quad h(x) = x^2$

55. Since, $f(x + y) = f(x) + f(y)$, for all $x, y \in R$

$\Rightarrow f(x) = xf(1)$, for all $x \in R$

$\Rightarrow f(x) = x$ for all $x \in R$ $\quad [\because f(1) = 1$, given$]$

56. Now, $\quad g(0) = \lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$

$$= \lim_{x \to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{(2^{\tan x - \sin x} - 1) \cdot 2^{\sin x}}{x^2 \sin x}$$

$$= \lim_{x \to 0} \frac{(2^{\tan x - \sin x} - 1)}{(\tan x - \sin x)} \cdot \frac{\tan x - \sin x}{x^2 \sin x} \cdot 2^{\sin x}$$

$$= \log_e 2 \cdot \frac{1}{2} \cdot 2^0 \qquad \left(\because \lim_{x \to 0} \frac{\tan x - \sin x}{x^2 \sin x} = \frac{1}{2}\right)$$

$$= \frac{1}{2}\log_e 2$$

57. Given, $(gof)\,x$ is continuous.

$\therefore$ $g(x)$ and $f(x)$ are also continuous.

Then, $\quad f(0) = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$\Rightarrow \quad 1 = \lim_{h \to 0} (0 - h + a) = a$

$\therefore \quad a = 1$

58. $g(0) = \lim_{x \to 0^-} g(x) = \lim_{h \to 0} g(0 - h)$

$= \lim_{h \to 0} (0 - h + 1)$

$\Rightarrow \quad 1 + b = 1$

$\Rightarrow \quad b = 0$

59. Statement I is correct as though $|4x^2 - 12x + 5|$ is non-differentiable at $x = \frac{1}{2}$ and $\frac{5}{2}$ but $\cos \pi x = 0$ those points.

So, $f'\left(\frac{1}{2}\right)$ and $f'\left(\frac{5}{2}\right)$ exists.

60. $f(x) = -\log x, x < 1 = \log x, \quad x \geq 1$

$$f'(x) = \begin{cases} -1/x, & x < 1 \\ \frac{1}{x}, & x \geq 1 \end{cases}$$

$\therefore \quad f'(1^-) = -1$ and $f'(1^+) = 1$

Hence, $f(x)$ is not differentiable.

61. $f(x) = \begin{cases} \frac{x}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\therefore \quad Rf'(0) = f'(0^+) = \lim_{h \to 0} \frac{\frac{h}{1 + e^{1/h}} - 0}{h} = \lim_{h \to 0} \frac{1}{1 + e^{1/h}} = 0$$

$$Lf'(0) = f'(0^-) = \lim_{h \to 0} \frac{\frac{-h}{1 + e^{-1/h}} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{1}{1 + \frac{1}{e^{1/h}}} = \frac{1}{1 + 0} = 1$$

$\therefore \quad f'(0^+) = 0$ and $f'(0^-) = 1$

62. $f(x) = \begin{cases} 3 - x^2, & x > 1 \\ x^3 + 1, & x \le 1 \end{cases}$

$\therefore \quad f'(x) = 3x^2 = 3 \quad \text{at} \quad x = 1^-$

$f'(x) = -2x = -2 \quad \text{at} \quad x = 1^+$

63. Let $\quad f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$

$\therefore \quad \text{LHL} = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0 - h)$

$= \lim\limits_{h \to 0} \dfrac{e^{-1/h} - 1}{e^{-1/h} + 1} = \dfrac{e^{-\infty} - 1}{e^{-1} + 1} = \dfrac{0 - 1}{0 + 1} = -1$

and $\quad \text{RHL} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} \dfrac{e^{1/h} - 1}{e^{1/h} + 1}$

$\left(\text{form} \dfrac{\infty}{\infty}, e^{\infty} = \infty\right)$

(divide above and below by $e^{1/h}$)

$\therefore \quad \text{RHL} = \lim\limits_{h \to 0} \dfrac{1 - e^{-1/h}}{1 + e^{-1/h}} = \dfrac{1 - e^{-\infty}}{e^{-\infty}} = \dfrac{1 - 0}{1 + 0} = 1$

$\therefore \quad \text{LHL} \ne \text{RHL}$

Hence, $\lim\limits_{x \to 0} \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right)$ does not exist.

64. $f(x) = (|x| - |x - 1|)^2$

$\therefore \quad f(x) = \begin{cases} (-x + x - 1)^2 = 1, & x < 0 \\ (x + x - 1)^2 = (2x - 1)^2, & 0 \le x \le 1 \\ (x - x + 1)^2 = 1, & x > 1 \end{cases}$

$\therefore \quad f'(x) = 0 \Rightarrow f'(0^-) = -4$

$f'(x) = 2(2x - 1) \cdot 2 = -4 \Rightarrow f'(0^+) = 0$

$f'(x) = 4(2x - 1) = 4 \Rightarrow f'(1^-) = 4$

$f'(x) = 0 \Rightarrow f'(1^+) = 0$

65. $f(x) = g(x) \sin x$

$f''(x) = g''(x) \sin x + 2g'(x) \cos x + g(x) \sin x$

$f''(0) = 2g'(0) = 0$

$f'(x) = g'(x) \sin x + g(x) \cos x$

$f'(0) = g(0)$

For statement I, $\quad \lim\limits_{x \to 0} \dfrac{g(x) \cos x - g(x)}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{g'(x) \cos x - g(x) \sin x}{\cos x} = g'(0) = 0 = f^2(0)$

For statement II $f'(0) = g(0)$

66. $\sqrt{x - 2}$ is defined for $x \ge 2$

and $\sqrt{(2 - x)}$ is defined for $x \le 2$

$\therefore$ $f(x)$ is defined only when $x = 2$

$\therefore \quad f(2) = 0$

Hence, $f(x)$ is discontinuous at $x = 2$

And $f(x)$ is a point function.

($\because$ domain and range consist of one value only)

67. It is clear from figure that

$f(x)$ is discontinuous at $x = 1$.

$\because \quad \lim\limits_{x \to 1^-} f(x) \ne \lim\limits_{x \to 1^+} f(x)$

68. $f(0) = \text{RHL} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(h)$

$= \lim\limits_{h \to 0} \dfrac{(27 - 2h)^{1/3} - 3}{9 - 3(243 + 5h)^{1/5}}$

$= \dfrac{1}{3} \lim\limits_{h \to 0} \dfrac{\dfrac{(27 - 2h)^{1/3} - (27)^{1/3}}{(27 - 2h) - 27} \times -2h}{\dfrac{(243)^{1/5} - (243 + 5h)^{1/5}}{243 - (243 + 5h)} \times -5h}$

$= \dfrac{1}{3} \cdot \dfrac{2}{5} \cdot \dfrac{\frac{1}{3}(27)^{-2/3}}{\frac{1}{5}(243)^{-4/5}} = \dfrac{2}{15} \cdot \dfrac{5}{3} \cdot \dfrac{3^{-2}}{3^{-4}} = \dfrac{2}{9} \cdot 9 = 2$

69.

Also, $|f(x)| \le |x|$

$\Rightarrow$ $|f(0)| \le |0|$

$\Rightarrow$ $|f(0)| \le 0$

$\therefore$ $|f(0)| = 0$

$\Rightarrow$ $f(0) = 0$

$\because$ $|f(x)| \le |x|$

$$\lim_{x \to 0} |f(x)| \le \lim_{x \to 0} |x|$$

$\Rightarrow$ $\lim_{x \to 0} |f(x)| \le 0$

But $\lim_{x \to 0} |f(x)| \ge 0$

$\therefore$ $\lim_{x \to 0} |f(x)| = 0 = |f(0)|$

RHL = VF

Hence, $|f(x)|$ is continuous at $x = 0$.

70. Statement II is correct. Put $x = \frac{1}{y}$, then

$$\lim_{x \to \infty} (a_1^{1/x} + a_2^{1/x} + \ldots + a_n^{1/x})^{nx}$$

$$= \lim_{y \to 0} \left(\frac{a_1^y + a_2^y + \ldots + a_n^y}{n} \right)^{\frac{n}{y}}$$

$$= e^{\lim_{y \to 0} \left(\frac{a_1^y + a_2^y + \ldots + a_n^y - n}{n} \right) \cdot \frac{n}{y}}$$

(using statement II)

$$= e^{\lim_{y \to 0} \left(\frac{a_1^y - 1}{y} + \frac{a_2^y - 1}{y} + \ldots + \frac{a_n^y - 1}{y} \right)}$$

$$= e^{\log a_1 + \log a_2 + \ldots + \log a_n}$$

$$= e^{\log(a_1 a_2 \ldots a_n)} = a_1 a_2 \ldots a_n$$

71. Since, $\lim_{x \to 0} \frac{\sqrt{1 - \cos 2x}}{x} = \sqrt{2} \lim_{x \to 0} \frac{|\sin x|}{x}$

$$= \sqrt{2} \begin{cases} \lim_{x \to 0} \frac{\sin x}{x}, & 0 < x < \pi/2 \\ -\lim_{x \to 0} \frac{\sin x}{x}, & -\pi/2 < x < 0 \end{cases}$$

$$= \sqrt{2} \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & -\pi/2 < x < 0 \end{cases}$$

Hence, LHL $\ne$ RHL

72. If $x \in I$, then $\{x\} = 0$

$\therefore$ $f(x)$ is discontinuous for integral values of x.

73. Put $y = \frac{1}{x}$ in given relation and taking $f(1) = 2$, we get

$$f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$\Rightarrow$ $f(x) = \pm x^n + 1$

$\because$ $f(2) = 5 \Rightarrow f(x) = x^2 + 1$

$\therefore$ $\lim_{x \to 4} f(x) = 17$

74. We know that,

$$|x - 2| = \begin{cases} x - 2, & x \ge 2 \\ 2 - x, & x < 2 \end{cases}$$

and $$|x - 5| = \begin{cases} x - 5, & x \ge 5 \\ 5 - x, & x < 5 \end{cases}$$

$\Rightarrow$ $$|x - 2| = \begin{cases} 2 - x, & x < 2 \\ x - 2, & 2 \le x \le 5 \\ x - 2, & x > 5 \end{cases}$$

and $$|x - 5| = \begin{cases} 5 - x, & x < 2 \\ 5 - x, & 2 \le x \le 5 \\ x - 5, & x > 5 \end{cases}$$

$\Rightarrow$ $$f(x) = |x - 2| + |x - 5|$$

$$= \begin{cases} (2 - x) + (5 - x), & x < 2 \\ (x - 2) + (5 - x), & 2 \le x \le 5 \\ (x - 2) + (x - 5), & x > 5 \end{cases}$$

$$= \begin{cases} 7 - 2x, & x < 2 \\ 3, & 2 \le x \le 5 \\ 2x - 7, & x > 5 \end{cases}$$

Now, we can draw the graph of f very easily.

From the above graph, we can analyse all the required things.

Statement I $f'(4) = 0$

It is obviously clear that, f is constant around $x = 4$, hence, $f'(4) = 0$.

Hence, Statement I is correct.

Statement II It can be clearly seen that

(i) f is continuous, $\forall\ x \in [2, 5]$

(ii) f is differential, $\forall\ x \in (2, 5)$

(iii) $f(2) = f(5) = 3$

Hence, statement II is also correct but obviously not a correct explanation of statement I.

75. $\because f(x) = \dfrac{1}{e^x + \frac{2}{e^x}}$

Using AM $\ge$ GM,

$$\frac{e^x + \frac{2}{e^x}}{2} \ge \left(e^x \cdot \frac{2}{e^x} \right)^{1/2}, \quad \text{as } e^x > 0$$

$$\Rightarrow \quad e^x + \frac{2}{e^x} \geq 2\sqrt{2}$$

$$\Rightarrow \quad 0 < \frac{1}{e^x + \frac{2}{e^x}} \leq \frac{1}{2\sqrt{2}}$$

$$\therefore \quad 0 < f(x) \leq \frac{1}{2\sqrt{2}}, \quad \forall x \in R$$

Hence, statement II is true and statement I as for some c.

$$\therefore \quad f(c) = \frac{1}{3}$$

Aliter

$$f(x) = \frac{1}{e^x + 2e^{-x}} = \frac{e^x}{e^{2x} + 2}$$

$$\Rightarrow \quad f'(x) = \frac{(e^{2x} + 2)\, e^x - 2e^{2x} \cdot e^x}{(e^{2x+2})^2}$$

$$\Rightarrow \quad f'(x) = 0 \quad \Rightarrow \quad e^{2x} + 2 = 2e^{2x}$$

$$\Rightarrow \quad e^{2x} = 2 \quad \Rightarrow \quad e^x = \sqrt{2}$$

Maximum value of $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

$$0 < f(x) \leq \frac{1}{2\sqrt{2}}, \quad \forall \; x \in R$$

Since, $\quad 0 < \frac{1}{3} < \frac{1}{2\sqrt{2}}$ for some $c \in R$

$$f(c) = \frac{1}{3}$$

76. $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \geq 0 \end{cases}$$

$$\therefore \quad (gof)'(x) = \begin{cases} -2x\cos x^2, & x < 0 \\ 2x\cos x^2, & x \geq 0 \end{cases}$$

Clearly, $L(gof)''(0) = 0 = R(gof)'(0)$

$\therefore gof$ is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$

Now, $\quad (gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2 \sin x^2, & x < 0 \\ 2\cos x^2 - 4x^2 \sin x^2, & x \geq 0 \end{cases}$

$\therefore \quad L(gof)''(0) = -2$ and $R(gof)'(0) = 2$

$\therefore \quad L(gof)''(0) \neq R(gof)''(0)$

$\therefore$ $gof(x)$ is not twice differentiable at $x = 0$

77. Let $I = \lim_{x \to 0} \frac{(1 - \cos^2 x)}{x^2} \cdot \frac{(3 + \cos x)}{1} \cdot \frac{x}{\tan 4x}$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x \to 0} (3 + \cos x) \cdot \lim_{x \to 0} \frac{4x}{4\tan 4x}$$

$$= 2 \cdot (1)^2 \cdot (3 + \cos 0) \cdot \frac{1}{4}$$

$$= 2 \cdot 1 \cdot (3 + 1) \cdot \frac{1}{4} = 2 \cdot 4 \cdot \frac{1}{4} = 2$$

78. Now, $\cos x$ is continuous, $\forall x \in R \Rightarrow \cos \pi \left(x - \frac{1}{2}\right)$ is also continuous, $\forall \; x \in R$.

Hence, the continuity of f depends upon the continuity of $[x]$, then $[x]$ is discontinuous, $\forall \; x \in I$.

So, we should check the continuity of f at $x = n, \forall \; n \in I$

LHL at $x = n$ is given by

$$f(n^-) = \lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x] \cos \pi \left(x - \frac{1}{2}\right)$$

$$= (n - 1) \cos \frac{(2n - 1)\pi}{2} = 0$$

RHL at $x = n$ is given by

$$f(n^+) = \lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] \cos \pi \left(x - \frac{1}{2}\right)$$

$$= (n) \cos \frac{(2n - 1)\pi}{2} = 0$$

Also, value of the function at $x = n$ is

$$f(n) = [n] \cos \pi \left(n - \frac{1}{2}\right) = (n) \cos \frac{(2n - 1)\pi}{2} = 0$$

$$\therefore \quad f(n^+) = f(n^-) = f(n)$$

Hence, f is continuous at $x = n, \forall \; n \in I$.

79. $\lim_{x \to 2} \frac{\sqrt{1 - \cos 2(x - 2)}}{(x - 2)}$

$$\Rightarrow \quad \lim_{x \to 2} \frac{\sqrt{2 - \sin^2 (x - 2)}}{(x - 2)} = \frac{\sqrt{2}\,|\sin (x - 2)|}{(x - 2)}$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{\sqrt{2}\,|\sin (x - 2)|}{(x - 2)}$$

RHL at $x = 2$,

$$\lim_{h \to 0} \frac{\sqrt{2}\,|\sin (2 + h - 2)|}{(2 + h) - 2} = \lim_{h \to 0} \frac{\sqrt{2}\,|\sin h|}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{\sqrt{2}\sin h}{h} = \sqrt{2}$$

LHL at $x = 2$,

$$\lim_{h \to 0} \frac{\sqrt{2}\,|\sin(2 - h - 2)|}{(2 - h) - 2} = \lim_{h \to 0} \frac{\sqrt{2}\,|\sin(-h)|}{-h}$$

$$\lim_{h \to 0} \frac{\sqrt{2}\sin h}{-h} = -\sqrt{2}$$

So, the limit does not exist.

80. Here, $f(x) = \begin{cases} \frac{\sin(p + 1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{\frac{3}{2}}}, & x > 0 \end{cases}$

Since, $f(x)$ is continuous for $x \in R$.

So, the function is continuous at $x = 0$.

RHL at $x = 0$,

$$\lim_{h \to 0} \frac{\sqrt{h + h^2} - \sqrt{h}}{h^{3/2}} = \lim_{h \to 0} \frac{\sqrt{h}\,\{\sqrt{h+1} - 1\}}{h\sqrt{h}}$$

$$\Rightarrow \lim_{h \to 0} \left(\frac{\sqrt{h+1} - 1}{h} \times \frac{\sqrt{h+1} + 1}{\sqrt{h+1} + 1} \right) = \lim_{h \to 0} \frac{(h+1) - 1}{h\{\sqrt{h+1} + 1\}}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{1}{\sqrt{h+1} + 1} = \frac{1}{2} \qquad \text{...(i)}$$

LHL at $x = 0$,

$$\lim_{h \to 0} \frac{\sin(p+1)(-h) + \sin(-h)}{-h} = \lim_{h \to 0} \frac{\sin(p+1)h}{h} + \frac{\sin h}{h}$$

$$\Rightarrow \quad (p+1) + 1 = (p+2) \qquad \text{...(ii)}$$

$$f(0) = q \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{1}{2} = q = p + 2$$

$$\therefore \quad p = -\frac{3}{2}, q = \frac{1}{2}$$

81. We have,

$$f : (-1, 1) \longrightarrow R$$

$$f(0) = -1 \qquad f'(0) = 1$$

$$g(x) = [f\{2f(x) + 2\}]^2$$

$$\Rightarrow \quad g'(x) = 2[f\{2f(x) + 2\}] \times f'\{2f(x) + 2\} \times 2f'(x)$$

$$\Rightarrow \quad g'(0) = 2[f\{2f(0) + 2\}] \times f'\{2f(0) + 2\} \times 2f'(0)$$

$$= 2[f(0)] \times f'(0) \times 2f'(0)$$

$$= 2 \times (-1) \times 1 \times 2 \times 1 = -4$$

82. Since, $f(x)$ is a positive increasing function.

$$\therefore \quad 0 < f(x) < f(2x) < f(3x)$$

$$\Rightarrow \quad 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \quad \lim_{x \to \infty} 1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le \lim_{x \to \infty} \frac{f(3x)}{f(x)}$$

By Sandwich theorem, $\lim_{x \to \infty} \frac{f(2x)}{f(x)} = 1$

83. Now, $f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h \to 0} \frac{(1 - h - 1) \cdot \sin\left(\frac{1}{1 - h - 1}\right) - 0}{-h}$$

$$= \lim_{h \to 0} \sin\left(-\frac{1}{h}\right) = -\lim_{h \to 0} \sin\frac{1}{h}$$

and $f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{(1 + h - 1)\ \sin\left(\frac{1}{1 + h - 1}\right) - 0}{h} = \lim_{h \to 0} \sin\frac{1}{h}$$

$$\therefore \quad f'(1^-) \ne f'(1^+)$$

$\therefore$ f is not differentiable at $x = 1$.

Again, now

$$f'(0) = \lim_{h \to 0} \frac{(0 + h + 1) \cdot \sin\left(\frac{1}{0 + h + 1}\right) - \sin 1}{-h}$$

$$= \lim_{h \to 0} \frac{\left[-\left\{(h+1)\cos\left(\frac{1}{h+1}\right) \times \left(\frac{1}{(h+1)^2}\right)\right\} + \sin\left(\frac{1}{h+1}\right) \right]}{-1}$$

(using L′ Hospital's rule)

$$= \cos 1 - \sin 1$$

and $f'(0^+) = \lim_{h \to 0} \frac{(0 + h - 1) \cdot \sin\left(\frac{1}{0 + h - 1}\right) - \sin 1}{h}$

$$= \lim_{h \to 0} \frac{(h-1)\cos\left(\frac{1}{h-1}\right)\left\{\frac{-1}{(h-1)^2}\right\} + \sin\left(\frac{1}{h-1}\right)}{1}$$

(using L′ Hospital's rule)

$$= \cos 1 - \sin 1 \Rightarrow f'(0^-) = f'(0^+)$$

Hence, f is differentiable at $x = 0$.

84. $f(x) = \min\{x + 1, |x| + 1\} = x + 1, \forall\, x \in R$

Hence, $f(x)$ is differentiable everywhere.

85. Since, $f(x) = \frac{x}{1 + |x|} = \frac{g(x)}{h(x)}$ (say)

It is clear that $g(x)$ and $h(x)$ are differentiable on $(-\infty, \infty)$ and $(-\infty, 0) \cup (0, \infty)$.

Now, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x}{1 + |x|} - 0}{x} = 1$

Hence, $f(x)$ is differentiable on $(-\infty, \infty)$.

86. $\lim_{n \to \infty} \frac{1}{n}\left[\frac{1}{n}\sec^2\left(\frac{1}{n}\right)^2 + \frac{2}{n}\sec^2\left(\frac{2}{n}\right)^2 + \ldots + \frac{n}{n}\sec^2\left(\frac{n}{n}\right)^2\right]$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right) \sec^2\left(\frac{r}{n}\right)^2 = \int_0^1 x \sec^2(x^2)\, dx$$

Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{1}{2}dt$

$$\therefore \quad A = \frac{1}{2}\int_0^1 \sec^2 t\, dt = \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}\tan 1$$

27 Differentiation

JEE Main MILESTONE

- Differentiation of a Function
- Standard Derivatives
- Differentiation of a Function with Respect to Another Function
- Differentiation by Substitution
- Differentiation of Determinants

Differentiation is a method to compute the rate at which a dependent variable y changes with respect to the change in the independent variable x. This rate of change is called the derivative of y with respect to x.

Geometrical Meaning of Derivative

Let us consider the function $f(x)$ and the corresponding curve $y = f(x)$. Clearly, line joining two points $M_0(x, y)$ and $M_1(x + \Delta x, y + \Delta y)$ on the curve will be the secant to the curve and the slope of this secant is given by $\tan \phi = \frac{\Delta y}{\Delta x}$ (where, ϕ is the angle made by the secant with the positive direction of the x-axis). In the limiting case when $\Delta x \to 0$ the point M_1 approaches M_0 and the secant joining these two points will become the tangent at M_0 whose slope will be given by $\tan \alpha = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ which means that slope of the tangent to the curve $y = f(x)$ at any argument is equal to the value of the derivative at that argument.

In calculus, the derivative is a measure of how a function changes as its input changes. Loosely speaking, a derivative can be thought of as how much one quantity is changing in response to changes in some other quantity; for example, the derivative of the position of a moving object with respect to time is the object's instantaneous velocity.

Geometrically, a function is not differentiable in the following cases

27.1 Differentiation of a Function

Let $f(x)$ is a function differentiable in an interval $[a, b]$. That is at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function $g:[a, b] \to R$, such that, $\forall\, x \in [a, b]$, $g(x) = f'(x)$.

This new function is said to be differentiation (differential coefficient) of the function $f(x)$ with respect to x and it is denoted by $\frac{df(x)}{dx}$ or $Df(x)$ or $f'(x)$.

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiation from First Principle

Let $f(x)$ is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Sample Problem 1 *The derivative of the function* $\cos ax$ *at x is*

(a) $\sin ax$ (b) $-\sin ax$ (c) $-a \sin ax$ (d) $a \cos ax$

Interpret (c) Let $y = \cos ax$, we have

$$\Delta y = \cos a(x + \Delta x) - \cos ax = -2 \sin\left(ax + \frac{a}{2}\Delta x\right) \sin \frac{a\Delta x}{2}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{-2 \sin\left(ax + \frac{a}{2}\Delta x\right) \sin \frac{a\Delta x}{2}}{\Delta x}$$

Hence, $$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= -2 \lim_{\Delta x \to 0} \sin\left(ax + \frac{a}{2}\Delta x\right) \lim_{\Delta x \to 0} \frac{\sin \frac{a}{2}\Delta x}{\Delta x}$$

$$= -a \sin ax$$

In particular, if $a = 1$, then $y = \cos x$ and $y' = -\sin x$

Sample Problem 2 *The derivative of* $\tan^{-1} x$ *with respect to x is*

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$ (c) $\frac{x}{1+x^2}$ (d) $\frac{2x}{1+x^2}$

Interpret (a) Let $\tan^{-1} x = \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \quad x = \tan\theta \quad \text{and} \quad \tan^{-1}(x + h) = \theta + \Delta\theta \qquad \ldots(i)$$

$$\Rightarrow \quad x + h = \tan(\theta + \Delta\theta)$$

Let $$\lim_{h \to 0} \frac{\tan^{-1}(x + h) - \tan^{-1} x}{h} = L$$

$$L = \lim_{h \to 0} \frac{\theta + \Delta\theta - \theta}{h} = \lim_{h \to 0} \frac{\Delta\theta}{h} \qquad \text{[from Eqs. (i) and (ii)]}$$

$$= \lim_{h \to 0} \frac{\Delta\theta}{\tan(\theta + \Delta\theta) - \tan\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta\theta}{\tan(\theta + \Delta\theta) - \tan\theta}$$

$$= \lim_{\Delta\theta \to 0} \frac{\Delta\theta \cos(\theta + \Delta\theta) \cos\theta}{\sin \Delta\theta} = \cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1 + x^2}$$

27.2 Standard Derivatives

(i) Algebraic Functions

(a) $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}\{f(x)\}^n = n\{f(x)\}^{n-1} \frac{d}{dx} f(x)$

(b) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -nx^{-n-1}$, $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\}^n = \frac{-n}{\{f(x)\}^{n-1}} \frac{d}{dx} f(x)$

(c) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \frac{d}{dx} f(x)$

(ii) Exponential Functions

(a) $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx} e^{\{f(x)\}} = [e^{\{f(x)\}}] \frac{d}{dx} f(x)$

(b) $\frac{d}{dx}(a^x) = a^x \log_e a$, $\frac{d}{dx} a^{\{f(x)\}} = [a^{\{f(x)\}} \log_e a] \frac{d}{dx} f(x)$

(iii) Logarithmic Functions

(a) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, $\frac{d}{dx}\{\log_e f(x)\} = \frac{1}{f(x)} \frac{d}{dx} f(x)$

(b) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$, $\frac{d}{dx}\{\log_a f(x)\} = \frac{1}{f(x) \log_e a} \frac{d}{dx} f(x)$

(iv) Trigonometric Functions

(a) $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x)$

(b) $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x)$

(c) $\frac{d}{dx}(\tan x) = \sec^2 x$, $\frac{d}{dx} \tan f(x) = \sec^2 f(x) \frac{d}{dx} f(x)$

(d) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2(x)$, $\frac{d}{dx} \cot f(x) = -\operatorname{cosec}^2 f(x) \frac{d}{dx} f(x)$

(e) $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\frac{d}{dx} \sec f(x) = \sec f(x) \tan f(x) \frac{d}{dx} f(x)$

(f) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\frac{d}{dx} \operatorname{cosec} f(x) = -\operatorname{cosec} f(x) \cot f(x) \frac{d}{dx} f(x)$

(v) Inverse Circular Functions

(a) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}; |x| < 1$

$\frac{d}{dx}[\sin^{-1} f(x)] = \frac{1}{\sqrt{1 - \{f(x)\}^2}} \frac{d}{dx} f(x); |f(x)| < 1$

(b) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}; |x| < 1$

$\frac{d}{dx}[\cos^{-1} f(x)] = -\frac{1}{\sqrt{1-\{f(x)\}^2}}\frac{d}{dx} f(x); |f(x)| < 1$

(c) $\frac{d}{dx}\tan^{-1} x = \left(\frac{1}{1+x^2}\right); x \in R$

$\frac{d}{dx}[\tan^{-1} f(x)] = \frac{1}{1+\{f(x)\}^2}\frac{d}{dx} f(x); f(x) \in R$

(d) $\frac{d}{dx}\cot^{-1} x = -\left(\frac{1}{1+x^2}\right); x \in R$

$\frac{d}{dx}[\cot^{-1} f(x)] = -\frac{1}{1+\{f(x)\}^2}\frac{d}{dx} f(x); f(x) \in R$

(e) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$

$\frac{d}{dx}[\sec^{-1} f(x)] = \frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}}\frac{d}{dx} f(x); |f(x)| > 1$

(f) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\left(\frac{1}{|x|\sqrt{x^2-1}}\right); |x| > 1$

$\frac{d}{dx}[\text{cosec}^{-1} f(x)] = -\frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}}\frac{d}{dx} f(x); |f(x)| > 1$

Hot Spot 1

Rule for Differentiation

Rule for differentiation is an important topic for JEE (Main) examination point of view and a small practice can help you in solving such problems very quickly as the level of questions is easy to average. While solving problems based on rule for differentiation, always keep in mind.

(i) Product Rule of differentiation
(ii) Quotient Rule of differentiation
(iii) Chain Rule
(iv) Derivative of parametric function
(v) Relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$

1. The derivative of a constant function is zero, *i.e.*, $\frac{d}{dx}(c) = 0$

2. The derivative of constant times a function is constant times the derivative of the function, *i.e.*, $\frac{d}{dx}\{c \cdot f(x)\} = c\frac{d}{dx}\{f(x)\}$

3. The derivative of the sum or difference of two functions is the sum or difference of their derivatives, *i.e.*,
$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}$$

4. **Product Rule of Differentiation** The derivative of the product of two functions
$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx}\{g(x)\} + g(x) \cdot \frac{d}{dx}\{f(x)\}$$
$$= \begin{pmatrix}\text{First}\\ \text{function}\end{pmatrix} \times \begin{pmatrix}\text{Derivative of}\\ \text{second function}\end{pmatrix} + \begin{pmatrix}\text{Second}\\ \text{function}\end{pmatrix} \times \begin{pmatrix}\text{Derivative of}\\ \text{first function}\end{pmatrix}$$

5. **Quotient Rule of Differentiation** The derivative of the quotient of two functions
$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot \frac{d}{dx}\{f(x)\} - f(x) \cdot \frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$$
$$= \frac{(\text{Denominator} \times \text{Derivative of Numerator}) - (\text{Numerator} \times \text{Derivative of Denominator})}{(\text{Denominator})^2}$$

6. **Derivative of a Function of a Function** (Chain Rule) If y is a differentiable function of t and t is a differentiable function of x, *i.e.*, $y = f(t)$ and $t = g(x)$, then
$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$, then
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

7. **Derivative of Parametric Functions** Sometimes x and y are separately given as functions of a single variable t (called a parameter), *i.e.*, $x = f(t)$ and $y = g(t)$. In this case,
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$$
and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dv}$$

8. If $\frac{d}{dx}F(x) = f(x)$, then $\frac{d}{dx}F(ax+b) = af(ax+b)$

9. **Derivative of Composite Function** If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(fog)(x) = f\{g(x)\}$ is differentiable at x and $(fog)'(x) = f'\{g(x)\} \cdot g'(x)$.

10. **Relation between *dy/dx* and *dx/dy*** If the inverse functions f and g are defined by $y = f(x)$ and $x = g(y)$ and if $f'(x)$ exists and $f'(x) \neq 0$, then $g'(y) = \frac{1}{f'(x)}$. This result can also be written as, if $\frac{dy}{dx}$ exists and $\frac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = 1\Big/\left(\frac{dy}{dx}\right) \quad \text{or} \quad \frac{dy}{dx}\cdot\frac{dx}{dy} = 1$$

or $$\frac{dy}{dx} = 1\Big/\left(\frac{dx}{dy}\right), \text{ where } \left[\frac{dx}{dy} \neq 0\right]$$

Note Differentiation of an even function is an odd function and differentiation of an odd function is an even function.

Sample Problem 3 *Derivative of* $\cos x^3 \cdot \sin^2(x^5)$ *with respect to x is* [**NCERT**]

(a) $10x^4 (\sin x^5)(\cos x^5)(\cos x^3) - 3x^2 \sin x^3 \cdot \sin^2 x^5$

(b) $5x^3 (\sin x^5)(\cos x^5)(\cos x^3) - 3x^2 \sin x^3 \cdot \sin^2 x^5$

(c) $5x^3 (\sin x^5)(\cos x^5)(\cos x^3) - 6x^2 \sin x^3 \cdot \sin^2 x^5$

(d) *None of the above*

Interpret (a) Since, the given function is a product of two different functions

So, using the product rule *i.e.*, $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

Let $$y = \cos x^3 \sin^2(x^5)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos x^3 \sin^2(x^5)\}$$

$$= \cos x^3 \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}(\cos x^3)$$

[using product rule, $\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u$]

$$= (\cos x^3)(2\sin x^5)\frac{d}{dx}(\sin x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

[using chain rule, $\frac{d}{dx}f\{g(x)\} = f'(x)\frac{d}{dx}g(x)$]

$$= (\cos x^3)(2\sin x^5)(\cos x^5)\frac{d}{dx}(x^5) + \sin^2(x^5)(-\sin x^3)(3x^2)$$

(using chain rule)

$$= (\cos x^3)(2\sin x^5)(\cos x^5)(5x^4) - \sin^2(x^5)(\sin x^3)(3x^2)$$

$$= 10x^4(\sin x^5)(\cos x^5)(\cos x^3) - 3x^2 \sin x^3 \sin^2 x^5$$

Sample Problem 4 *Derivative of* $\frac{\sin(ax+b)}{\cos(cx+d)}$ *with respect to x is* [**NCERT**]

(a) $a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\cdot\sec(cx+d)$

(b) $a\cos(ax+b)\sin(cx+d) + c\sin(ax+b)\tan(cx+d)\cdot\sec(cx+d)$

(c) $a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\cdot\cos(cx+d)$

(d) *None of the above*

Interpret (a) Since, the given function is in u/v form.

So, applying the division rule

i.e., $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$$

Let $$y = \frac{\sin(ax+b)}{\cos(cx+d)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sin(ax+b)}{\cos(cx+d)}\right]$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\{\sin(ax+b)\} - \sin(ax+b)\frac{d}{dx}\{\cos(cx+d)\}}{\{\cos(cx+d)\}^2}$$

(by quotient rule)

$$= \frac{\cos(cx+d)\cos(ax+b)\cdot(a+0) + \sin(ax+b)\sin(cx+d)(c+0)}{\cos^2(cx+d)}$$

$$\left[\begin{array}{l}\text{chain rule}\\ \frac{d}{dx}\sin(ax+b) = \cos(ax+b)\frac{d}{dx}(ax+b)\\ \qquad = \cos(ax+b)\times(a\times 1+0)\\ \frac{d}{dx}\cos(cx+d) = -\sin(cx+d)\frac{d}{dx}(cx+d)\\ \qquad = -\sin(cx+d)\times(c\times 1+0)\end{array}\right]$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b)}{\cos^2(cx+d)} + \frac{c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + \frac{c\sin(ax+b)\sin(cx+d)}{\cos(cx+d)\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Sample Problem 5 *If* $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, *then the value of* $\frac{dy}{dx}$ *is* [**NCERT**]

(a) $\frac{y}{x}$ (b) $\frac{x}{y}$ (c) $-\frac{y}{x}$ (d) $-\frac{x}{y}$

Interpret (c) Given, $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$

i.e., $x = a^{\frac{1}{2}\sin^{-1}t}$ and $y = a^{\frac{1}{2}\cos^{-t}t}$ $[\because (a^b)^c = a^{bc}]$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a^{\frac{1}{2}\sin^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}\sin^{-1}t\right) \quad \left(\because \frac{d}{dx}a^x = a^x\log a\right)$$

$$= a^{\frac{1}{2}\sin^{-1}t}\log a\left(\frac{1}{2\sqrt{1-t^2}}\right) = \frac{a^{\frac{1}{2}\sin^{-1}t}\log a}{2\sqrt{1-t^2}}$$

and $$\frac{dy}{dt} = a^{\frac{1}{2}\cos^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}\cos^{-1}t\right) \quad \text{(using chain rule)}$$

$$= a^{\frac{1}{2}\cos^{-1}t}\log a\left(\frac{-1}{2\sqrt{1-t^2}}\right) = \frac{-a^{\frac{1}{2}\cos^{-1}t}\log a}{2\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-a^{\frac{1}{2}\cos^{-1}t}}{a^{\frac{1}{2}\sin^{-1}t}} = -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}} = -\frac{y}{x}$$

Alternate Method

$$x = \sqrt{a^{\sin^{-1} t}} \quad \text{...(i)}$$

and $$y = \sqrt{a^{\cos^{-1} t}} \quad \text{...(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\Rightarrow \quad xy = \sqrt{a^{\sin^{-1} t}} \times \sqrt{a^{\cos^{-1} t}}$$

$$\Rightarrow \quad xy = \sqrt{a^{\sin^{-1} t} \cdot a^{\cos^{-1} t}}$$

$$\Rightarrow \quad xy = \sqrt{a^{\sin^{-1} t + \cos^{-1} t}} \left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad xy = \sqrt{a^{\pi/2}}$$

On differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \left[\frac{d}{dx}(\text{constant}) = 0\right]$$

Sample Problem 6 *If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then $\frac{dy}{dx}$ is equal to*

(a) $\tan\frac{\theta}{2}$ (b) $\cot\frac{\theta}{2}$ (c) $\tan\theta$ (d) $\cot\theta$

Interpret (a) Given, $x = a(\theta + \sin\theta)$

$$\therefore \quad \frac{dx}{d\theta} = a(1 + \cos\theta) \text{ and } y = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = a(\sin\theta)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1 + \cos\theta)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

Hot Spot 2

Differentiation of Implicit Functions

Differentiation of an implicit function is an important topic for JEE (Main) examination and a little practice can help you in solving such problems very quickly as the level of questions is easy to average.

Sometimes the level of question is from moderate to typical.

(i) In given implicit function, we solve this for $\frac{dy}{dx}$ directly.

(ii) **Moderate level** Sometime in question we asked $\frac{dy}{dx}$ at some particular point of x.

So, after getting $\frac{dy}{dx}$, we put the given particular value of x in $\frac{dy}{dx}$ and get the desired result

If the relation between the variables x and y is given by an equation containing both of the form $\phi(x, y) = 0$ and this equation is not immediately solvable for y *i.e.*, it is not possible to express y as a function of x in the form $y = \phi(x)$, then y is called an implicit function of x. In such a case to find $\frac{dy}{dx}$ differentiate both sides of the given relation w.r.t. x keeping in mind that the derivative of $f(y)$ w.r.t. x is $\frac{d\phi}{dy} \cdot \frac{dy}{dx}$.

e.g., $$\frac{d}{dx}(\tan y) = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(y^4) = 4y^3 \cdot \frac{dy}{dx}$$

Shortcut for Implicit Functions For implicit function, put $\frac{d}{dx}\{f(x, y)\} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where $\frac{\partial f}{\partial x}$ is a partial differential of given function with respect to x and $\frac{\partial f}{\partial y}$ means partial differential of given function with respect to y.

Sample Problem 7 *If $\sin^2 y + \cos(xy) = k$, then $\frac{dy}{dx}$ is equal to* [NCERT]

(a) $\frac{y\sin(xy)}{\sin 2y - x\sin(xy)}$ (b) $\frac{y\cos(xy)}{\sin 2y - x\sin(xy)}$

(c) $\frac{y\sin(xy)}{\sin 2y - x\cos(xy)}$ (d) *None of these*

Interpret (a) Given, $\sin^2 y + \cos xy = k$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(k)$$

$$\Rightarrow \quad \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$$

$$\Rightarrow \quad 2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\frac{d}{dx}(xy) = 0$$

$$\left[\text{using chain rule, } \frac{d}{dx}f\{g(x)\} = f'(x)\frac{d}{dx}g(x)\right]$$

$$\Rightarrow \quad \sin 2y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$(\because \sin 2x = 2\sin x \cdot \cos x)$

$$\Rightarrow \quad \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} = y \sin xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \sin (xy)}{\sin 2y - x \sin (xy)}$$

Sample Problem 8 *If* $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$, *then* $\frac{dy}{dx}$ *is equal to* **[NCERT Exemplar]**

(a) $\frac{x}{y} \cdot \frac{\sqrt{1-x^3}}{\sqrt{1-y^3}}$ (b) $\frac{x^2\sqrt{1-x^3}}{y^2\sqrt{1-y^3}}$

(c) $\frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$ (d) $\frac{x^2\sqrt{1-x^6}}{y^2\sqrt{1-y^6}}$

Interpret (c) Here, $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Let $x^3 = \sin\theta, y^3 = \sin\phi$, then we get

$$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a^3(\sin\theta - \sin\phi)$$

$$\cos\theta + \cos\phi = a^3(\sin\theta - \sin\phi)$$

$$2\cos\left(\frac{\theta+\phi}{2}\right) \cdot \cos\left(\frac{\theta-\phi}{2}\right) = a^3\left[2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$\cos\left(\frac{\theta-\phi}{2}\right) = a^3 \sin\left(\frac{\theta-\phi}{2}\right) = \cot\left(\frac{\theta-\phi}{2}\right) = a^3$$

$$\theta - \phi = 2\cot^{-1}(a^3) \quad \text{or} \quad \sin^{-1}x^3 - \sin^{-1}y^3 = 2\cot^{-1}(a^3),$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^6}} \cdot 3x^2 - \frac{1}{\sqrt{1-y^6}} \cdot 3y^2 \cdot \frac{dy}{dx} = 0$$

Hence, $$\frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$$

Sample Problem 9 *If* $y\cos x + x\cos y = \pi$, *then the value of* $\left.\frac{dy}{dx}\right|_{\text{at } (x=0)}$ *is*

(a) 0 (b) 1 (c) –1 (d) *None of these*

Interpret (b) Given that,

$$y\cos x + x\cos y = \pi \quad \ldots(i)$$

Put $x = 0$ in given equation, we get

$$y\cos 0 + 0 = \pi \Rightarrow y = \pi$$

Now, differentiation w.r.t. x, of Eq. (i),

$$\frac{dy}{dx}\cos x - \sin x \cdot y - x\sin y\frac{dy}{dx} + \cos y = 0$$

$$\frac{dy}{dx} = \frac{y\sin x - \cos y}{\cos x - x\sin y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\text{at } (x=0)} = \frac{\pi \cdot \sin 0 - \cos\pi}{\cos 0 - 0 \cdot \sin\pi} = \frac{0-(-1)}{1-0} = \frac{1}{1} = 1$$

27.3 Differentiation of a Function with Respect to Another Function

If $y = f(x)$ and $z = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ w.r.t. $g(x)$ *i.e.*, $\frac{dy}{dz}$ use the following formula.

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

Note (i) *If y and z are functions of x, then*

$$\frac{d}{dx}(y^z) = zy^{z-1}\frac{dy}{dx} + (y^z \log y)\frac{dz}{dx}$$

(ii) *If $y = z^n$ and $z = f(x)$, then*

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = nz^{n-1}\frac{dz}{dx}$$

Sample Problem 10 *Differentiation of* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *with respect to* $\tan^{-1}x$, *where* $x \in (-1, 1)$ *is*

(a) 1 (b) 2

(c) $\frac{1}{1+x}$ (d) $\frac{1}{2}$

Interpret (b) $$\frac{\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)}{\frac{d}{dx}\tan^{-1}(x)} = \frac{\frac{d}{dx}(2\tan^{-1}x)}{\frac{d}{dx}(\tan^{-1}x)}$$

$$= \frac{2\frac{d}{dx}(\tan^{-1}x)}{\frac{d}{dx}(\tan^{-1}x)} = 2$$

Sample Problem 11 *Differentiation of* $\log\tan x$ *with respect to* $\sin^{-1}(e^x)$ *is*

(a) $\frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x \cdot \cos x}$ (b) $\frac{\sin x \cdot \cos x}{e^{-x}\sqrt{1-e^{2x}}}$

(c) $\frac{e^{x}\sqrt{1-e^{2x}}}{\sin x \cdot \cos x}$ (d) *None of these*

Interpret (a) $$\frac{d(\ln\tan x)}{d\{\sin^{-1}(e^x)\}} = \frac{\frac{d}{dx}(\ln\tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x \sec^2 x}{e^x \cdot \frac{1}{\sqrt{1-e^{2x}}}}$$

$$= \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin \cdot \cos x}$$

Hot Spot 3 Logarithmic Differentiation

Logarithmic differentiation is the most important topic of this chapter from JEE (Main) examination point of view. Generally questions are seen from this topic. The level of questions is from moderate to typical.

The functions which can be evaluated by using logarithmic differentiation are of type

$$y = [f(x)]^{g(x)}$$

or $$y = f_1(x)\, f_2(x)\, f_3(x) \ldots$$

or $$y = \frac{f_1(x)\, f_2(x)\, f_3(x) \ldots}{g_1(x)\, g_2(x)\, g_3(x) \ldots}$$

For this type of function take the logarithm and then differentiate.

Note Shortcut method for differentiating functions of the form $[f(x)]^{g(x)}$ is

(a) $$\frac{dy}{dx} = \begin{bmatrix} \text{Differentiate treating} \\ f(x) \text{ as a variable } (x^n) \text{ and} \\ g(x) \text{ as constant} \end{bmatrix} + \begin{bmatrix} \text{Differentiate treating} \\ g(x) \text{ as a variable } (a^x) \\ \text{and } f(x) \text{ as constant} \end{bmatrix}$$

(b) $$\frac{dy}{dx} = [f(x)]^{g(x)} \cdot \frac{d}{dx}[g(x) \log f(x)]$$

Sample Problem 12 *If $x^y \cdot y^x = 1$, then the value of $\frac{dy}{dx}$ is*

(a) $-\frac{(y + x \log y)}{x + y \log x} \cdot \frac{y}{x}$ (b) $\frac{(y + x \log y)}{(x + y \log x)} \cdot \frac{y}{x}$

(c) $-\frac{(x + y \log y)}{(y + x \log x)} \cdot \frac{y}{x}$ (d) *None of these*

Interpret (a) Taking ln on both sides,

$$y \ln x + x \ln y = \ln 1$$

On differentiating both sides, we get

$$y \cdot \frac{d}{dx}(\ln x) + \left\{\frac{d}{dx} y\right\} \cdot \ln x + \left\{\frac{d}{dx} x\right\} \cdot \ln y + x\left\{\frac{d}{dx} \ln y\right\} = 0$$

or $$y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} + 1 \cdot \ln y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 0$$

$$\left[\ln x + \frac{x}{y}\right] \cdot \frac{dy}{dx} = -\left[\frac{y}{x} + \ln y\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(y + x \ln y)}{(x + y \ln x)} \cdot \frac{y}{x}$$

Sample Problem 13 *If $y = x^{(\log x)^{\log \log x}}$, then $\frac{dy}{dx}$ is equal to*

(a) $\frac{y \log y}{x \log x}(2 \log \log x + 1)$

(b) $\frac{x \log x}{y \log y}(2 \log \log x + 1)$

(c) $\frac{2y \log y}{x \log x}(\log \log x + 1)$

(d) *None of the above*

Interpret (a) $y = x^{(\log x)^{\log \log x}}$

On taking log both sides, we get

$$\log y = (\log x)^{\log \log x} \cdot \log x$$

Again, on taking log both sides, we get

$$\log \log y = (\log \log x) \log \log x + \log \log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{\log y} \cdot \frac{1}{y}\frac{dy}{dx} = 2 \log \log x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \log y}{x \log x}(2 \log \log x + 1)$$

Sample Problem 14 *If $y = (\cot x)^{\sin x}$, then the value of $\frac{dy}{dx}$ is*

(a) $-\operatorname{cosec} x (\cot x)^{\sin x - 1} + (\cot x)^{\sin x} \cdot \log (\cot x) \cdot \cos x$

(b) $-\operatorname{cosec} x (\cot x)^{\sin x} + (\cot x)^{\sin x - 1} \cdot \log (\cot x) \cdot \cos x$

(c) $(\operatorname{cosec} x) \cdot (\cot x)^{\sin x - 1} - (\cot x)^{\sin x} \cdot \log \cot (x) \cdot \cos x$

(d) *None of the above*

Interpret (a) $$\frac{dy}{dx} = \underbrace{(\sin x)(\cot x)^{\sin x - 1}\frac{d}{dx}(\cot x)}_{\substack{\text{(differentiating cot } x \text{ keeping sin } x \\ \text{constant similar to } x^n)}} + \underbrace{(\cot x)^{\sin x} \cdot \log (\cot x) \cdot \frac{d}{dx}(\sin x)}_{\substack{\text{(differentiating sin } x \text{ keeping cot } x \\ \text{constant similar to } a^x)}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sin x (\cot x)^{\sin x - 1}(-\operatorname{cosec}^2 x) + (\cot x)^{\sin x} \cdot \log (\cot x)(\cos x)$$

$$= -\operatorname{cosec} x (\cot x)^{\sin x - 1} + (\cot x)^{\sin x} \cdot \log (\cot x) \cdot \cos x$$

Check Point 1

1. Let $f(x) = \frac{3x+2}{5x-3}$, then $\left[\frac{d}{dx}\{f^{-1}(x)\}\right]_{x=f(1)}$ is equal to

(a) $-\frac{19}{4}$ (b) $-\frac{4}{19}$ (c) -19 (d) None of these

2. If $y = (\log_e x)[\log_x(\log_e x)]$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x\log_x \log_x x}$ (b) $\frac{1}{x\log_e x}$

(c) 0 (d) None of these

3. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ is equal to

(a) $\tan^2\theta$ (b) $|\sec\theta|$ (c) $\sec^2\theta$ (d) None of these

4. If $y = \log\left(\frac{1+x}{1-x}\right)^{1/4} - \frac{1}{2}\tan^{-1}x$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x^2}{1-x^4}$ (b) $\frac{2x^2}{1-x^4}$

(c) $\frac{x^2}{2(1-x^4)}$ (d) None of these

5. If $y = x^2\log x + \frac{2}{\sqrt{x}}$, then $\frac{dy}{dx}$ is equal to

(a) $x + 2\log x - \frac{1}{\sqrt{x}}$ (b) $x + 2x\log x - \frac{1}{x^{3/2}}$

(c) $x + 2x\log x - \frac{2}{x^{3/2}}$ (d) None of these

6. If $f(x) = (\log_{\cot x}\tan x)(\log_{\tan x}\cot x)^{-1} + \tan^{-1}\left(\frac{4x}{4-x^2}\right)$, then $f'(2)$ is equal to

(a) 2 (b) 0 (c) $\frac{1}{2}$ (d) -2

7. If $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2at}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{a(1-t^2)}{2t}$ (b) $\frac{a(t^2-1)}{2t}$ (c) $\frac{2(t^2+1)}{2t}$ (d) $\frac{a(t^2-1)}{t}$

8. If $y = x^{\sqrt{x}}$, then $\frac{dy}{dx}$ is equal to

(a) $x^{\sqrt{x}}\left(\frac{2+\log x}{2\sqrt{x}}\right)$ (b) $x^{\sqrt{x}}\left(\frac{2+\log x}{\sqrt{x}}\right)$

(c) $\frac{2+\log x}{2\sqrt{x}}$ (d) None of these

9. If $y = \frac{1+\sin x-\cos x}{1+\sin x+\cos x}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{1-\cos x}$ (b) $1+\cos x$ (c) $1-\cos x$ (d) $\frac{1}{1+\cos x}$

10. If $x = e^{-t^2}$ and $y = \tan^{-1}(2t+1)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{-e^{t^2}}{2t(2t^2+2t+1)}$ (b) $\frac{e^{t^2}}{2t(t^2+2t+1)}$

(c) $\frac{-e^{t^2}}{t(2t^2+2t+1)}$ (d) None of these

27.4 Differentiation by Substitution

Substitution is useful to reduce the function into simple form. For problems involving inverse trigonometric functions, first try for a suitable substitution to simplify it and then differentiate it. If no such substitution is found, then differentiate directly. *Some standard substitutions are given below.*

	Expressions	Substitutions
1.	$\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
2.	$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
3.	$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
4.	$\frac{a+x}{a-x}$ or $\frac{a-x}{a+x}$	$x = a\tan\theta$
5.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$
6.	$\frac{2x}{1+x^2}$ or $\frac{2x}{1-x^2}$	$x = \tan\theta$
7.	$a\sin x + b\cos x$	$a = r\cos\alpha, b = r\sin\alpha$
8.	$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x = \alpha\sin^2\theta + \beta\cos^2\theta$
9.	$\sqrt{2ax-x^2}$	$x = a(1-\cos\theta)$

Differential Coefficient Using Inverse Trigonometrical Substitutions

Sometimes the given function can be deducted with the help of inverse trigonometrical substitution and then find the differential coefficient is very easy.

(i) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1} x = \cos^{-1}(2x^2-1)$ or $\cos^{-1}(1-2x^2)$

(iii) $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) \\ \tan^{-1}\left(\dfrac{2x-x^2}{1}\right) \\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \end{cases}$

(iv) $3\sin^{-1} x = \sin^{-1}(3x-4x^3)$

(v) $3\cos^{-1} x = \cos^{-1}(4x^3-3x)$

(vi) $3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(vii) $\cos^{-1} x + \sin^{-1} x = \pi/2$

(viii) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(ix) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$

(x) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$

(xi) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{(1-x^2)(1-y^2)}]$

(xii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

Important Trigonometrical Formulae

The following formulae will be helpful to solve the problems involving inverse trigonometric function etc.

(i) $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(iv) $\dfrac{1 \pm \tan x}{1 \mp \tan x} = \tan\left(\dfrac{\pi}{4} \pm x\right)$

Sample Problem 15 *The value of*

$$\frac{d}{dx}\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] \text{ is}$$

[NCERT Exemplar]

(a) $\dfrac{-x}{\sqrt{1-x^4}}$ (b) $\dfrac{x}{\sqrt{1-x^4}}$ (c) $\dfrac{x}{\sqrt{1-x^2}}$ (d) $\dfrac{-x}{\sqrt{1-x^2}}$

Interpret (a) Assume $x^2 = \cos 2\theta$ to simplify and then use the relation $1+\cos 2\theta = 2\cos^2\theta$ and $1-\cos 2\theta = 2\sin^2\theta$ to simplify the relation.

On putting $x^2 = \cos 2\theta$, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\left[\because \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta \qquad \left(\begin{array}{l} \because 0 < x^2 < 1 \Rightarrow 0 < \cos 2\theta < 1 \\ \Rightarrow 0 < 2\theta < \pi/2 \\ \Rightarrow 0 < \theta < \pi/4 \\ \Rightarrow \pi/4 < \pi/4+\theta < \pi/2 \end{array}\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \quad [\because \cos 2\theta = x^2 \Rightarrow \theta = (1/2)\cos^{-1}x^2]$$

On differentiating w.r. t. x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\pi/4) + \frac{1}{2}\frac{d}{dx}(\cos^{-1}x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{2}\frac{(-1)}{\sqrt{1-x^4}}\frac{d}{dx}(x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}\times\frac{2x}{\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

27.5 Differentiation of Determinants

If $\quad y = \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$

Then, $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

i.e., to differentiate a determinant, differentiate one row (or column) at a time, keeping other unchanged.

Sample Problem 16 *If* $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, *then* $f'(x)$ *is equal to*

(a) $6x^2$
(b) $6x$
(c) 0
(d) *None of the above*

Interpret (a) $\because f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

$$\therefore f'(x) = \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & 2x & x^3 \\ 1 & 2 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & 3x^2 \\ 1 & 2x & 6x \\ 0 & 2 & 6 \end{vmatrix}$$

$$= 1(12x^2 - 6x^2) + 6x(2x - 2x) - 2(6x^2 - 3x^2) + 6(2x^2 - x^2)$$

$$= 6x^2 + 0 - 6x^2 + 6x^2 = 6x^2$$

Hot Spot 4

Successive Differentiation

Successive differentiation is the most important topic for JEE (Main) examination. Generally questions are seen from this topic. The level of questions is from moderate to typical.

If the function $y = f(x)$ be differentiated with respect to x, then the result $\frac{dy}{dx}$ or $f'(x)$, so obtained is a function of x (may be a constant).

Hence, $\frac{dy}{dx}$ can again be differentiated with respect of x.

The differential coefficient of $\frac{dy}{dx}$ with respect to x is written as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ or $f''(x)$. Again, the differential coefficient of $\frac{d^2y}{dx^2}$ with respect to x is written as

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \text{ or } f'''(x) \ldots$$

Here, $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ are respectively known as first, second, third, ... order differential coefficients of y with respect to x. These are alternatively denoted by $f'(x), f''(x), f'''(x), \ldots$ or $y_1, y_2, y_3, \ldots$ respectively.

The process of differentiating a function more than once is called successive differentiation.

Note $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ but $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{d\theta^2}}{\frac{d^2x}{d\theta^2}}$ Relation between $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) \div \frac{dy}{dx}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) \div \frac{dy}{dx}$$

$$= -\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^3 = -\frac{(y'')}{(y')^3}$$

Sample Problem 17 *If* $y = Ae^{mx} + Be^{nx}$, *then the value of* $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx}$ *is equal to* [NCERT]

(a) $-y$ (b) 0 (c) mny (d) $-mny$

Interpret (b) Given, $y = A\,e^{mx} + B\,e^{nx}$ ...(i)

On differentiating twicely w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx) = A\,e^{mx}\,m + B\,e^{nx}\,n \quad \ldots\text{(ii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = m^2\,Ae^{mx} + n^2Be^{nx} \quad \ldots\text{(iii)}$$

Using Eqs. (i), (ii) and (iii), we have

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= m^2\,Ae^{mx} + n^2\,Be^{nx} - (m+n)\{Ame^{mx} + n\,Be^{nx}\} + mn\{Ae^{mx} + Be^{nx}\}$$

i.e., $$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$

Sample Problem 18 *If* $y = (\sin^{-1}x)^2 + k\sin^{-1}x$, *then* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ *is equal to*

(a) 0 (b) 1 (c) 2 (d) y

Interpret (c) Here, $y = (\sin^{-1}x)^2 + k\sin^{-1}x$.

On differentiating both sides w.r.t.x, we have

$$\frac{dy}{dx} = 2\frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{k}{\sqrt{1-x^2}} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y + k^2$$

On differentiating this w.r.t.x, we get

$$(1-x^2)\,2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx} \Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

Sample **Problem 19** *If* $y = e^{\sin^2 x}$, *then* $\frac{d^2x}{dy^2}$ *in terms of x is*

(a) $-(2\operatorname{cosec}^2 2x \cdot \cot 2x + \operatorname{cosec} 2x) \cdot e^{-2\sin^2 x}$

(b) $(2\operatorname{cosec}^2 2x \cdot \cot 2x + \operatorname{cosec} 2x) \cdot e^{2\sin^2 x}$

(c) $(2\operatorname{cosec}^2 2x \cdot \cot 2x - \operatorname{cosec} 2x)\, e^{\sin^2 x}$

(d) *None of the above*

Interpret (a) Here, $y = e^{\sin^2 x}$.

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \sin 2x \cdot e^{\sin^2 x} \Rightarrow \frac{dx}{dy} = \operatorname{cosec} 2x \cdot e^{-\sin^2 x}$$

On differentiating w.r.t.y, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dy}(\operatorname{cosec} 2x \cdot e^{-\sin^2 x}) = \frac{d}{dx}(\operatorname{cosec} 2x \cdot e^{-\sin^2 x})\frac{dx}{dy}$$

$$= (-2\operatorname{cosec} 2x \cot 2x\, e^{-\sin^2 x} - e^{-\sin^2 x})\operatorname{cosec} 2x \cdot e^{-\sin^2 x}$$

$$= -(2\operatorname{cosec}^2 2x \cot 2x + \operatorname{cosec} 2x)\, e^{-2\sin^2 x}.$$

Leibnitz Theorem

If u and v are functions of x such that their nth derivative exist, then

$$D^n(u \cdot v) = {}^nC_0(D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + {}^nC_3(D^{n-3}u)(D^3v) + \ldots + {}^nC_r D^{n-r}u \cdot D^r v + \ldots + {}^nC_n(D^n v)$$

nth Derivative of Some Functions

(i) $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(iii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

(iv) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(v) $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(vi) $\frac{d^n}{dx^n}(a^x) = a^x(\log a)^n$

(vii) (a) $\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx + c + n\phi)$

(b) $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx + c + n\phi)$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Check Point **2**

1. The derivative of $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ is

(a) $\frac{x}{\sqrt{1-x^4}}$ (b) $-\frac{x}{\sqrt{1-x^4}}$

(c) $-\frac{1}{2\sqrt{1-x^2}}$ (d) None of these

2. If $y = \sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$, then the value of $\frac{dy}{dx}$ is

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $-\frac{x}{\sqrt{1-x}}$ (c) $-\frac{x}{\sqrt{1-x^2}}$ (d) None of these

3. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = 1$, is equal to

(a) 0 (b) -1 (c) 1 (d) 2

4. If $y = \frac{e^{2x}\cos x}{x\sin x}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{e^{2x}\{(2x-1)\cot x - x\operatorname{cosec}^2 x\}}{x^2}$

(b) $\frac{e^{2x}\{(2x+1)\cot x - x\operatorname{cosec}^2 x\}}{x^2}$

(c) $\frac{e^{2x}\{(2x-1)\cot x + x\operatorname{cosec}^2 x\}}{x^2}$

(d) None of the above

5. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{6x^2-2x+2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

(b) $\frac{6x^2-2x+2}{(x^2+1)^2}\sin^2\left(\frac{2x-1}{x^2+1}\right)^2$

(c) $\frac{-2x^2-2x+2}{(x^2+1)^2}\sin^2\left(\frac{2x-1}{x^2+1}\right)$

(d) $\frac{-2x^2+2x+2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

6. If $x = a\sin\theta$, $y = b\cos\theta$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{a}{b^2}\sec^2\theta$ (b) $-\frac{b}{a}\sec^2\theta$ (c) $-\frac{b}{a^2}\sec^2\theta$ (d) $\frac{-b}{a^2}\sec^3\theta$

7. If $y = A\cos nx + B\cos nx$, then $\frac{d^2y}{dx^2}$ is equal to

(a) n^2y (b) $-y$ (c) $-n^2y$ (d) None of these

8. If $x = \log p$ and $y = \frac{1}{p}$, then

(a) $\frac{d^2y}{dx^2} - 2p = 0$ (b) $\frac{d^2y}{dx^2} + y = 0$

(c) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ (d) $\frac{d^2y}{dx^2} - y = 0$

WORKED OUT
Examples

Example 1 *If $f(x) = |x-1|$ and $g(x) = f[f\{f(x)\}]$, then for $x > 2$, $g'(x)$ is equal to*

(a) -1, *if* $2 \le x < 3$ (b) 1, if $2 \le x < 3$

(c) 1, *for all* $x > 2$ (d) *None of these*

Solution (a) We have, $f(x) = x - 1$ $(\because x > 2)$

$$f[f(x)] = f(x-1) = |(x-1)-1| = |x-2|$$

$$g(x) = f[f\{f(x)\}] = f(x-2)$$

$$= |(x-2)-1| = |x-3| = \begin{cases} x-3, & \text{if } x \ge 3 \\ -x+3, & \text{if } 2 \le x < 3 \end{cases}$$

$$\therefore \quad g'(x) = \begin{cases} 1, & \text{if } x \ge 3 \\ -1, & \text{if } 2 \le x < 3 \end{cases}$$

Example 2 *If* $y = \sin^{-1}\left[\dfrac{\log x^2}{1+(\log x)^2}\right]$, *then* $\dfrac{dy}{dx}$ *is equal to*

(a) $\dfrac{2}{1+\log x}$ (b) $\dfrac{2}{x[1+(\log x)]}$

(c) $\dfrac{2}{x[1+(\log x)^2]}$ (d) $\dfrac{2}{1+(\log x)^2}$

Solution (c) We have, $y = \sin^{-1}\left[\dfrac{\log x^2}{1+(\log x)^2}\right]$

$$= \sin^{-1}\left[\frac{2\log x}{1+(\log x)^2}\right]$$

Let $\log x = \tan\phi$

Then, $y = \sin^{-1}\left(\dfrac{2\tan\phi}{1+\tan^2\phi}\right) = 2\phi$

$$= 2\tan^{-1}(\log x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{1+(\log x)^2}\cdot\frac{1}{x} = \frac{2}{x[1+(\log x)^2]}$$

Example 3 *If* $f(x) = \cos x \cos 2x \cos 4x \cos 8x \cos 16x$, *then* $f'\left(\dfrac{\pi}{4}\right)$ *is*

(a) $\sqrt{2}$ (b) $\dfrac{1}{\sqrt{2}}$

(c) 1 (d) $\dfrac{\sqrt{3}}{2}$

Solution (a)

$$f(x) = \frac{2\sin x\cdot\cos x\cdot\cos 2x\cdot\cos 4x\cdot\cos 8x\cdot\cos 16x}{2\sin x}$$

$$= \frac{\sin 32x}{2^5\sin x}$$

$$\therefore \quad f'(x) = \frac{1}{32}\cdot\frac{32\cos 32x\cdot\sin x - \cos x\cdot\sin 32x}{\sin^2 x}$$

$$\Rightarrow \quad f'\left[\frac{\pi}{4}\right] = \frac{32\times\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\times 0}{32\left[\frac{1}{\sqrt{2}}\right]^2} = \sqrt{2}$$

Example 4 *If* $x = at^2$ *and* $y = 2at$, *then* $\dfrac{d^2y}{dx^2}$ *at* $t = 2$ *is*

(a) $-\dfrac{1}{16a}$ (b) $\dfrac{1}{16a}$

(c) $\dfrac{1}{16}$ (d) $\dfrac{1}{a}$

Solution (a) We have, $x = at^2$ and $y = 2at$

$$\Rightarrow \quad \frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{t}$$

Now, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{t^2}\cdot\dfrac{dt}{dx} = -\dfrac{1}{t^2}\cdot\dfrac{1}{2at} = -\dfrac{1}{2at^3}$

At $t = 2$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{2a(2)^3} = -\dfrac{1}{16a}$

Example 5 *The derivative of* $y = (1-x)(2-x)\ldots(n-x)$ *at* $x = 1$ *is equal to*

(a) 0 (b) $(-1)(n-1)!$

(c) $n!-1$ (d) $(-1)^{n-1}(n-1)!$

Solution (b) $\dfrac{dy}{dx} = -[(2-x)(3-x)\ldots(n-x) + (1-x)(3-x)$

$$\ldots(n-x) + \ldots(1-x)(2-x)\ldots(n-1-x)]$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = -[(n-1)! + 0 + \ldots + 0]$$

$$= (-1)(n-1)!$$

Example 6 *If $y = a\cos(\log x) - b\sin(\log x)$, then the value of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y$ is*

(a) 0 (b) 1 (c) 2 (d) 3

Solution (a) $y = a\cos(\log x) - b\sin(\log x)$

On differentiating w.r.t. x, we get

$$\therefore \quad \frac{dy}{dx} = a\frac{[-\sin(\log x)]}{x} - \frac{b\cos(\log x)}{x}$$

$$= -\frac{[a\sin(\log x) + b\cos(\log x)]}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -[a\sin(\log x) + b\cos(\log x)]$$

Again, on differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\left[\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}\right] = -\frac{y}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Example 7 *Let f and g be differentiable function satisfying $g'(a) = 2$, $g(a) = b$ and $fog = I$ (identity function). Then, $f'(b)$ is equal to*

(a) $\frac{1}{2}$ (b) 2

(c) $\frac{2}{3}$ (d) *None of these*

Solution (a) Since, $fog = I \Rightarrow fog(x) = x$ for all x

$$\Rightarrow \quad f'\{g(x)\}\, g'(x) = 1 \text{ for all } x$$

$$\Rightarrow \quad f'\{g(a)\} = \frac{1}{g'(a)} = \frac{1}{2}$$

$$\Rightarrow \quad f'(b) = \frac{1}{2} \qquad [\because g(a) = b]$$

Example 8 *If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, then $\frac{dy}{dx}$ is equal to*

(a) $\frac{7}{1+x^2}$ (b) $\frac{4}{1+x^2}$

(c) $\frac{1}{x}$ (d) *None of these*

Solution (b) We have, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Let $x = \tan\theta$

Then, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$$= 2\theta + 2\theta = 4\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{4}{1+x^2}$$

Example 9 *If $x^2 + y^2 = t - \frac{1}{t}$, $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $x^3y\frac{dy}{dx}$ is equal to*

(a) 1 (b) 2

(c) 3 (d) 4

Solution (a) $x^4 + y^4 = \left(t - \frac{1}{t}\right)^2 + 2 = (x^2 + y^2)^2 + 2$

$$\Rightarrow \quad x^2y^2 = -1$$

$$\Rightarrow \quad y^2 = -\frac{1}{x^2}$$

On differentiating, we get

$$2y\frac{dy}{dx} = \frac{2}{x^3}$$

$$\Rightarrow \quad x^3y\frac{dy}{dx} = 1$$

Example 10 *If $y = \frac{1}{4}u^4$ and $u = \frac{2}{3}x^3 + 5$, then $\frac{dy}{dx}$ is equal to*

(a) x^2 (b) $2x^3 + 15$

(c) $\frac{2}{27}x^2$ (d) *None of these*

Solution (d) We have, $y = \frac{1}{4}u^4$

$$\therefore \quad \frac{dy}{du} = \frac{1}{4}\cdot 4u^3 = u^3$$

and $u = \frac{2}{3}x^3 + 5$

$$\therefore \quad \frac{du}{dx} = \frac{2}{3}\cdot 3x^2 = 2x^2$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{2}{3}x^3 + 5\right)^3 (2x^2)$$

$$= \frac{2}{27}x^2(2x^3 + 15)^3$$

Example 11 *If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then, the value of $[f(10)]^2 - [\phi(10)]^2$ is*

(a) 0 (b) 9

(c) 41 (d) *None of these*

Solution (b) $\frac{d}{dx}\{[f(x)]^2 - [\phi(x)]^2\} = 2[f(x)\cdot f'(x) - \phi(x)\cdot\phi'(x)]$

$$= 2[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)]$$

$$= 0 \quad [\because f'(x) = \phi(x) \text{ and } \phi'(x) = f(x)]$$

$$\Rightarrow \quad [f(x)]^2 - [\phi(x)]^2 = \text{constant}$$

$$\therefore \quad [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2$$

$$= [f(3)]^2 - [f'(3)]^2$$

$$= 25 - 16 = 9$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Rule for Differentiation

1. $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ is equal to

(a) 1 (b) $\frac{x^2+1}{x^2-4}$ (c) $\frac{x^2-1}{x^2-4}$ (d) $e^x \cdot \frac{x^2-1}{x^2-4}$

2. Derivative of $2\sqrt{\cot(x^2)}$ with respect to x is [NCERT]

(a) $\frac{x \operatorname{cosec}^2(x^2)}{2\sqrt{\cot(x^2)}}$ (b) $\frac{-2x \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$

(c) $\frac{-x \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$ (d) None of these

3. Derivative of $\sqrt{e^{\sqrt{x}}}$ with respect to x is [NCERT]

(a) $\frac{e^{\sqrt{x}}}{2\sqrt{xe^{\sqrt{x}}}}$ (b) $\frac{4e^{\sqrt{x}}}{\sqrt{xe^{\sqrt{x}}}}$

(c) $\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$ (d) $\frac{e^{\sqrt{x}}}{\sqrt{e^{\sqrt{x}}}}$

4. If $x = 4t$, $y = \frac{4}{t}$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) t^2 (b) $\frac{1}{t}$ (c) $-\frac{1}{t}$ (d) $-\frac{1}{t^2}$

5. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\tan\theta$ (b) $\sec\theta$ (c) $-\tan\theta$ (d) $\cot\theta$

6. Derivative of $\sqrt{\tan\sqrt{x}}$ with respect to x is [NCERT Exemplar]

(a) $\frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}$ (b) $\frac{\sec^2\sqrt{x}}{4\sqrt{x}\cdot\tan\sqrt{x}}$

(c) $\frac{\sec^2\sqrt{x}}{\sqrt{x}\sqrt{\tan\sqrt{x}}}$ (d) $\frac{4\sec^2\sqrt{x}}{\sqrt{x}\sqrt{\tan\sqrt{x}}}$

7. If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, then the value of $\left(\frac{dy}{dx}\right)_{\text{at } \theta = \frac{\pi}{3}}$ is [NCERT Exemplar]

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{3}}$

8. Derivative of $\log[\log(\log x^5)]$ with respect to x is [NCERT Exemplar]

(a) $\frac{1}{x\log x\cdot\log(\log x^5)}$ (b) $\frac{1}{x\log(\log x^5)}$

(c) $\frac{5}{x\log(\log x^5)}$ (d) None of these

9. If $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{t^2+1}{t^2-1}$ (b) $\frac{1+t^2}{1-t^2}$ (c) $\frac{1-t^2}{1+t^2}$ (d) $\frac{t^2-1}{t^2+1}$

10. If $x = \frac{1+\log t}{t^2}$ and $y = \frac{3+2\log t}{t}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{1}{t}$ (b) -1 (c) 1 (d) $-t$

11. The value of $\frac{d}{dx}\left[\left(\frac{\tan^2 2x - \tan^2 x}{1-\tan^2 2x\tan^2 x}\right)\cot 3x\right]$ is

(a) $\tan 2x\tan x$ (b) $\tan 3x\tan x$

(c) $\sec^2 x$ (d) $\sec x\tan x$

12. If $y = \log^n x$, where $\log^n$ means log log log... (repeated n times), then $x\log x\log^2 x\log^3 x\ldots\log^{n-1} x\log^n x\frac{dy}{dx}$ is equal to

(a) $\log x$ (b) $\log^n x$

(c) $\frac{1}{\log x}$ (d) 1

13. If $f(x) = \sqrt{1+\cos^2(x^2)}$, then the value of $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is

(a) $\frac{\sqrt{\pi}}{6}$ (b) $-\sqrt{\frac{\pi}{6}}$ (c) $\frac{1}{\sqrt{6}}$ (d) $\frac{\pi}{\sqrt{6}}$

14. If $f(x) = (\log_{\cot x}\tan x)(\log_{\tan x}\cot x)^{-1} + \tan^{-1}\frac{4x}{4-x^2}$, then $f'(2)$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 1 (d) -1

15. If $f(x) = \log_x(\log_e x)$, then $f'(x)$ at $x = e$ is equal to

(a) 1 (b) 2 (c) 0 (d) $\frac{1}{e}$

16. Let $f(x) = e^x$, $g(x) = \sin^{-1}x$ and $h(x) = f[g(x)]$, then $\frac{h'(x)}{h(x)}$ is equal to

(a) $e^{\sin^{-1}x}$ (b) $\frac{1}{\sqrt{1-x^2}}$
(c) $\sin^{-1}x$ (d) $\frac{1}{(1-x^2)}$

17. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$. If $F(x) = (hogof)(x)$, then $F''(x)$ is equal to

(a) $a\,\text{cosec}^3 x$ (b) $2\cot x^2 - 4x^2\,\text{cosec}^2 x^2$
(c) $2x\cot x^2$ (d) $-2\,\text{cosec}^2 x$

Differentiation of an Implicit Function

18. If $xy + y^2 = \tan x + y$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{\sec^2 x}{x+2y}$ (b) $\frac{y-\sec^2 x}{x+2y}$
(c) $\frac{\sec^2 x - y}{x+2y-1}$ (d) None of these

19. If $\sin^2 x + \cos^2 y = 1$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{\sin 2x}{\sin 2y}$ (b) $\frac{\sin^2 y}{\sin 2x}$ (c) $\frac{\sin^2 x}{\sin^2 y}$ (d) $-\frac{\sin^2 y}{\sin^2 x}$

20. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x < 1$, then $\frac{dy}{dx}$ is [NCERT]

(a) $\frac{1}{(1+x)}$ (b) $\frac{1}{(1+x)^2}$ (c) $\frac{1}{1+x^2}$ (d) $-\frac{1}{(1+x)^2}$

21. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, then $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is [NCERT]

(a) $-c$ (b) $-\frac{c}{a}$
(c) $-\frac{a}{c}$ (d) $-abc$

22. If $\cos y = x\cos(a+y)$, with $\cos a \neq 1$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{\sin a}{\cos^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$
(c) $\frac{\cos^2(a+y)}{\sin a}$ (d) $\frac{\cos^2(a+y)}{\cos a}$

23. If $x\sin(a+y) + \sin a\cos(a+y) = 0$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\frac{\cos^2(a+y)}{\cos a}$
(c) $\frac{\sin^2(a+y)}{\cos a}$ (d) $\frac{\cos^2(a+y)}{\sin a}$

24. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$
(c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

Differentiation of a Function with Respect to Another Function

25. Differential coefficient of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$, when $x \neq 0$, is [NCERT Exemplar]

(a) 1 (b) 2
(c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

Logarithmic Differentiation

26. Derivative of $\cos x \cdot \cos 2x \cdot \cos 3x$ with respect to x is [NCERT]

(a) $-\tan x - 2x\tan 2x - 3x\tan 3x$
(b) $-\cos x \cdot \cos 2x \cdot \cos 3x(\tan x + 2x\tan 2x + 3x\tan 3x)$
(c) $\cos x \cdot \cos 2x \cdot \cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$
(d) None of the above

27. If $y^x = x^y$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{y - x\log y}{x - y\log x}$ (b) $-\left(\frac{y - x\log y}{x - y\log x}\right)$
(c) $-\frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$ (d) $\frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$

28. If $xy = e^{(x-y)}$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{y(x-1)}{x(1+y)}$ (b) $\frac{x(x-1)}{y(1+y)}$

(c) $\frac{x(1+x)}{y(1-y)}$ (d) $\frac{y(1+x)}{x(1-y)}$

29. If $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$, then the value of $f'(1)$ is [NCERT]

(a) 0 (b) 20

(c) 93 (d) 120

30. Derivative of $(\log x)^{\log x}$ with respect to x is [NCERT]

(a) $(\log x)^{\log x}\left\{\frac{1}{x} + x\log(\log x)\right\}$

(b) $(\log x)^{\log x}\{x + x\log(\log x)\}$

(c) $(\log x)^{\log x}\left\{x + \frac{\log(\log x)}{x}\right\}$

(d) $(\log x)^{\log x}\left\{\frac{1}{x} + \frac{\log(\log x)}{x}\right\}$

31. If $y^x = e^{y-x}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{(1+\log y)}{y\log y}$ (b) $\frac{(1+\log y)^2}{y\log y}$

(c) $\frac{1+\log y}{(\log y)^2}$ (d) $\frac{(1+\log y)^2}{\log y}$

32. If $x = e^{x/y}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{x-y}{x\log x}$ (b) $\frac{y-x}{\log x}$

(c) $\frac{y-x}{x\log x}$ (d) $\frac{x-y}{\log x}$

33. If $y = (\cos x)^{(\cos x)^{\cos x \cdots \infty}}$, then $\frac{dy}{dx}$ is equal to [NCERT Exemplar]

(a) $\frac{y\tan x}{y\log\cos x - 1}$ (b) $\frac{y^2\tan x}{y\log\cos x - 1}$

(c) $\frac{y\tan x}{1 + y\log\cos x}$ (d) None of these

34. If $x^m \cdot y^n = (x+y)^{m+n}$, then which of the option is correct? [NCERT Exemplar]

(a) $\frac{dy}{dx} = -\frac{x}{y}$ and $\frac{d^2y}{dx^2} = -x$

(b) $\frac{dy}{dx} = \frac{x}{y}$ and $\frac{d^2y}{dx} = -y$

(c) $\frac{dy}{dx} = \frac{y}{x}$ and $\frac{d^2y}{dx^2} = 0$

(d) None of the above

35. If $x = \exp\left\{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right\}$, then $\frac{dy}{dx}$ is equal to

(a) $2x[1 + \tan(\log x)] + x\sec^2(\log x)$

(b) $x[1 + \tan(\log x)] + \sec^2(\log x)$

(c) $2x[1 + \tan(\log x)] + x^2\sec^2(\log x)$

(d) $2x[1 + \tan(\log x)] + \sec^2(\log x)$

36. If $\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n} = f(x)$,

then $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$ is equal to

(a) $\frac{f'(x)}{f(x)}$ (b) $\frac{f(x)}{f'(x)}$ (c) $-\frac{f'(x)}{f(x)}$ (d) 0

Differentiation by Substitution

37. If $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{-1}{1+x^2}$ (b) $\frac{-2}{1+x^2}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{2}{1+x^2}$

38. If $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$, then $\frac{dy}{dx}$ is equal to [NCERT]

(a) $\frac{-1}{1+x^2}$ (b) $\frac{1}{1+x^2}$ (c) $\frac{-2}{1+x^2}$ (d) $\frac{2}{1+x^2}$

39. Differential coefficient of

$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$ is [NCERT]

(a) $\sin\frac{x}{2} - \cos\frac{x}{2}$ (b) $\cos\frac{x}{2} - \sin\frac{x}{2}$

(c) $\sin\frac{x}{2} + \cos\frac{x}{2}$ (d) None of these

40. If $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1$, then $\frac{dy}{dx}$ is [NCERT]

(a) 0 (b) $\frac{\pi}{2}$ (c) 1 (d) 4

41. Differentiate $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$ with respect to x is [NCERT Exemplar]

(a) 0 (b) 1 (c) −1 (d) 2

42. Differentiate $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

and $\frac{a}{b}\tan x > -1$, then $\frac{dy}{dx}$ is [NCERT Exemplar]

(a) 0 (b) −1

(c) 1 (d) None of these

43. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then $\frac{dy}{dx}$ is equal to **[NCERT Exemplar]**

(a) $\sqrt{\frac{1-x^2}{1-y^2}}$ (b) $\sqrt{\frac{1-y^2}{1-x^2}}$ (c) $-\sqrt{\frac{1-y^2}{1-x^2}}$ (d) $-\frac{y}{x}$

44. Derivative of $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$ with respect to x is **[NCERT Exemplar]**

(a) $\frac{1}{\sqrt{x}(1+x)}$ (b) $\frac{-2}{\sqrt{x}(1+x)}$

(c) $\frac{-1}{2\sqrt{x}(1+x)}$ (d) None of these

45. If $y=\tan^{-1}(\sec x-\tan x)$, then $\frac{dy}{dx}$ is equal to

(a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

46. If $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{y}$ (b) $-\frac{x}{y}$ (c) $\frac{y}{x}$ (d) $-\frac{y}{x}$

47. $\frac{d}{dx}\left[\sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$ is equal to

(a) -1 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) 1

48. $\frac{d}{dx}[\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2})]$ is equal to

(a) $\frac{1}{2\sqrt{x(1-x)}}-\frac{1}{\sqrt{1-x^2}}$

(b) $\frac{1}{\sqrt{1-\{x\sqrt{1-x}-\sqrt{x(1-x^2)}\}^2}}$

(c) $\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x(1-x)}}$

(d) $\frac{1}{\sqrt{x(1-x)(1-x)^2}}$

49. If $y=\tan^{-1}\left[\frac{\log(e/x^2)}{\log(ex^2)}\right]+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2}$ is equal to

(a) 2 (b) 1

(c) 0 (d) -1

50. If $y=\left[\tan^{-1}\frac{1}{1+x+x^2}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\ldots+\text{ upto } n \text{ terms}\right]$, then $y'(0)$ is equal to

(a) $\frac{-1}{(n^2+1)}$ (b) $\frac{-n^2}{(n^2+1)}$

(c) $\frac{n^2}{(n^2+1)}$ (d) None of these

Differentiation of Determinants

51. If $f(x)=\begin{vmatrix} x^3 & x^2 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant, then $\frac{d^3f(x)}{dx^3}$ is

(a) proportional to x^2 (b) proportional to x

(c) proportional to x^3 (d) a constant

Successive Differentiation

52. If $y=3\cos(\log x)+4\sin(\log x)$, then $x^2y_2+xy_1$ is equal to **[NCERT]**

(a) y (b) $-xy$

(c) $-y$ (d) 0

53. If $y=500e^{7x}+600e^{-7x}$, then the value of $\frac{d^2y}{dx^2}$ is **[NCERT]**

(a) y (b) $7y$

(c) $40y$ (d) $49y$

54. If $e^y(x+1)=1$, then $\frac{d^2y}{dx^2}$ is equal to **[NCERT]**

(a) y (b) $\frac{dy}{dx}$

(c) $-y$ (d) $\left(\frac{dy}{dx}\right)^2$

55. If $y=(\tan^{-1}x)^2$, then $(x^2+1)^2y_2+2x(x^2+1)y_1$ is equal to **[NCERT]**

(a) 0 (b) 1

(c) 2 (d) 4

56. If $y=e^{a\cos^{-1}x}$, $-1\le x\le 1$, then which of these are correct? **[NCERT]**

(a) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+d^2y=0$

(b) $(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-a^2y=0$

(c) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

(d) None of the above

57. If $x=\sin t$ and $y=\sin pt$, then $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}$ is equal to **[NCERT Exemplar]**

(a) $-y$ (b) y (c) py (d) $-p^2y$

58. If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$, then $\frac{d^2y}{dx^2}$ is equal to

(a) x (b) $-x$
(c) $-y$ (d) y

59. If variables x and y are related by the equation $x = \int_0^y \frac{1}{\sqrt{1+9u^2}}\,du$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\sqrt{1+9y^2}$ (b) $\frac{1}{1+9y^2}$
(c) $9y$ (d) $\frac{1}{9}y$

60. If $f(x) = x^n + 4$, then the value of

$$f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \ldots + \frac{f^n(1)}{n!} \text{ is}$$

(a) 2^{n-1} (b) $2^n + 4$
(c) $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ (d) None of these

61. Let $f(x) = x^{2n}, n \in N$, then $f^n(x)$, *i.e.*, nth derivative of $f(x)$ with respect to x is

(a) $\frac{n!}{2n!} \cdot x^{-n}$ (b) $\frac{n!}{2n!} \cdot x^n$
(c) $\frac{2n!}{n!} \cdot x^{-n}$ (d) $\frac{2n!}{n!} \cdot x^n$

Miscellaneous

62. Derivative of $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x is [NCERT]

(a) $(x \cot x + \log \sin x) + \frac{1}{2\sqrt{x - x^2}}$
(b) $(x \cot x + \log \sin x) + \frac{1}{\sqrt{x - x^2}}$
(c) $(\sin x)^x (x \cot x + \log x) + \frac{1}{\sqrt{x - x^2}}$
(d) $(\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x - x^2}}$

63. If $f(x) = |x|^3$, then the value of $f''(x)$ is [NCERT]

(a) $f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$ (b) $f''(x) = \begin{cases} -6x, & x \geq 0 \\ 6x, & x < 0 \end{cases}$
(c) $f''(x)$ does not exist (d) None of these

64. If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to [NCERT Exemplar]

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$
(c) $\frac{1}{2}$ (d) $2\sqrt{2}$

65. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{6}\right)$ is equal to [NCERT Exemplar]

(a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{2}(1 + \sqrt{3})$
(c) $\frac{1}{2}(\sqrt{3} - 1)$ (d) $\sqrt{3}$

66. For $|x| < 1$, let $y = 1 + x + x^2 + \ldots$ to ∞, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{y}$ (b) $\frac{x^2}{y^2}$
(c) $\frac{x}{y^2}$ (d) $xy^2 + y$

67. Let a function $y = y(x)$ be defined parametrically by $x = 2t - |t|, y = t^2 + t|t|$. Then, $y'(x), x > 0$

(a) 0 (b) $4x$
(c) $2x$ (d) does not exist

68. The differential coefficient of $f(\sin x)$ with respect to x, where $f(x) = \log x$, is

(a) $\tan x$ (b) $\cot x$ (c) $f(\cos x)$ (d) $\frac{1}{x}$

69. If $f'(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) $6 \sin \log(5)$ (b) $5 \sin \log(6)$
(c) $12 \sin \log(5)$ (d) $5 \sin \log(12)$

70. If $f(x) = |x - 2|$ and $g(x) = f\{f(x)\}$, then for $x > 20$, $g'(x)$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

71. The differential coefficient of the function $|x - 1| + |x - 3|$ at the point $x = 2$ is

(a) -2 (b) 0
(c) 2 (d) undefined

72. If $y = \frac{a + bx^{3/2}}{x^{5/4}}$ and $y' = 0$ at $x = 5$, then the ratio $a : b$ is equal to

(a) $\sqrt{5} : 1$ (b) $5 : 2$ (c) $3 : 5$ (d) $1 : 2$

73. The derivative of $f(x) = 3|2 + x|$ at the point $x_0 = -3$, is

(a) 3 (b) -3
(c) 0 (d) does not exist

74. The value of $\frac{d}{dx}(|x - 1| + |x - 5|)$ at $x = 3$ is

(a) -2 (b) 0 (c) 2 (d) 4

75. If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then the value of $f'\left(\frac{\pi}{4}\right)$ is

(a) 1 (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

76. Let $\phi(x)$ be the inverse of the function $f(x)$ and $f'(x) = \frac{1}{1+x^5}$, then $\frac{d}{dx}\phi(x)$ is equal to

(a) $\frac{1}{1+[\phi(x)]^5}$ (b) $\frac{1}{1+[f(x)]^5}$

(c) $1+[\phi(x)]^5$ (d) $1+f(x)$

77. If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then $\frac{d^2u}{ds^2}$ is equal to

(a) 12 (b) 32 (c) 36 (d) 10

78. If $P(x)$ is a polynomial such that $P(x^2+1) = \{P(x)\}^2 + 1$ and $P(0) = 0$, then $P'(0)$ is equal to

(a) -1 (b) 0

(c) 1 (d) None of these

79. If $x = e^t \sin t$, $y = e^t \cos t$, t is a parameter, then $\frac{d^2y}{dx^2}$ at (1, 1) is equal to

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$ (c) 0 (d) $\frac{1}{2}$

80. If $y = f(x)$ and $y\cos x + x\cos y = \pi$, then the value of $f''(0)$ is

(a) π (b) $-\pi$ (c) 0 (d) 2π

81. If the function $f(x)$ is defined by $f(x) = a + bx$ and $f^r = f\,f\,f\ldots$ (repeated r times), then $f^r(x)$ is equal to

(a) $a + b^r x$ (b) $ar + b^r x$

(c) $ar + bx^r$ (d) $a\left(\frac{b^r - 1}{b-1}\right) + b^r x$

82. Given that $\frac{d}{dx}f(x) = f'(x)$. The relationship $f'(a+b) = f'(a) + f'(b)$ is valid, if $f(x)$ is equal to

(a) x (b) x^2

(c) x^3 (d) x^4

83. If $f(x) = x^3 + x^2 + f'(1) + xf''(2) + f'''(3)$, $\forall\, x \in R$ where $f(x)$ is a polynomial of degree 3, then

(a) $f(0) + f(2) = f(1)$

(b) $f(0) + f(3) = 0$

(c) $f(1) + f(3) = f(2)$

(d) All of the above

84. Derivatives of $y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$ with respect to x, is

(a) $\sqrt{\frac{6}{\cos 4x + \cos 2x}}$ (b) $\frac{6}{\sqrt{\cos 4x + \cos 2x}}$

(c) $\frac{\sqrt{6}}{\cos 4x + \cos 2x}$ (d) None of these

85. If $x = \tan\frac{y}{2} - \log\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2}\sin y\,(1 - \sin y + \cos y)$

(b) $\sin y\,(1 - \sin y - \cos y)$

(c) $\frac{1}{2}\sin y\,(1 + \sin y + \cos y)$

(d) None of the above

Round II (Mixed Bag)

Only One Correct Option

1. If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{2t}{t^2+1}$ (b) $\frac{2t}{t^2-1}$

(c) $\frac{2t}{1-t^2}$ (d) None of these

2. If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to

(a) 0

(b) $\frac{1}{\sqrt{x}+1}$

(c) 1

(d) None of these

3. The solution set of $f'(x) > g'(x)$, where $f(x) = \frac{1}{2}(5)^{2x+1}$ and $g(x) = 5^x + 4x\log_e 5$ is

(a) $(1, \infty)$ (b) $(0, 1)$

(c) $(\infty, 0)$ (d) $(0, \infty)$

4. If $y = \frac{a^{\cos^{-1}x}}{1+a^{\cos^{-1}x}}$ and $z = a^{\cos^{-1}x}$, then $\frac{dy}{dz}$ is equal to

(a) $\frac{1}{1+a^{\cos^{-1}x}}$ (b) $-\frac{1}{1+a^{\cos^{-1}x}}$

(c) $\frac{1}{(1+a^{\cos^{-1}x})^2}$ (d) None of these

5. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ is equal to

(a) 4 (b) 2

(c) -2 (d) 0

6. Given, $x = a\cos t\sqrt{\cos 2t}$ and $y = a\sin t\sqrt{\cos 2t}$ $(a > 0)$, then $\left|\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}\right|$ at $\dfrac{\pi}{6}$ is given by

(a) $\dfrac{a}{3}$ (b) $a\sqrt{2}$ (c) $\dfrac{\sqrt{2}}{3a}$ (d) $\dfrac{\sqrt{2}a}{3}$

7. If $y = \log x \cdot e^{(\tan x + x^2)}$, then $\dfrac{dy}{dx}$ is equal to

(a) $e^{(\tan x + x^2)}\left[\dfrac{1}{x} + (\sec^2 x + x)\log x\right]$

(b) $e^{(\tan x + x^2)}\left[\dfrac{1}{x} + (\sec^2 x - x)\log x\right]$

(c) $e^{(\tan x + x^2)}\left[\dfrac{1}{x} + (\sec^2 x + 2x)\log x\right]$

(d) $e^{(\tan x + x^2)}\left[\dfrac{1}{x} + (\sec^2 x - 2x)\log x\right]$

8. If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$, $a>0$ assuming $y > 0$, then $y''(0)$ is equal to

(a) $\dfrac{2}{a}e^{-\pi/2}$ (b) $-\dfrac{2}{a}e^{\pi/2}$

(c) $-\dfrac{2}{a}e^{-\pi/2}$ (d) None of these

9. If $y = \log_{\sin x}(\tan x)$, then $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$ is

(a) $\dfrac{4}{\log 2}$ (b) $-\dfrac{4}{\log 2}$

(c) $\dfrac{1}{\log 2}$ (d) None of these

10. If $y = \ln\left(\dfrac{x}{a+bx}\right)^x$, then $x^3\dfrac{d^2y}{dx^2}$ is equal to

(a) $\left(\dfrac{dy}{dx} + x\right)^2$ (b) $\left(\dfrac{dy}{dx} - y\right)^2$

(c) $\left(x\dfrac{dy}{dx} + y\right)^2$ (d) $\left(x\dfrac{dy}{dx} - y\right)^2$

11. If $y = \left(\dfrac{ax+b}{cx+d}\right)$, then $2\dfrac{dy}{dx}\cdot\dfrac{d^3y}{dx^3}$ is equal to

(a) $\left(\dfrac{d^2y}{dx^2}\right)^2$ (b) $3\dfrac{d^2y}{dx^2}$ (c) $3\left(\dfrac{d^2y}{dx^2}\right)^2$ (d) $3\dfrac{d^2x}{dy^2}$

12. If $\dfrac{d^2x}{dy^2}\left(\dfrac{dy}{dx}\right)^3 + \dfrac{d^2y}{dx^2} = k$, then k is equal to

(a) 0 (b) 1

(c) 2 (d) None of these

13. If $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$, then the value of $\dfrac{dy}{dx}$ at $x = 0$ is

(a) 0 (b) -1

(c) 1 (d) None of these

14. If $y^2 = P(x)$ is a polynomial of degree 3, then $2\dfrac{d}{dx}\left[y^3\dfrac{d^2y}{dx^2}\right]$ is equal to

(a) $P'''(x) + P'x$ (b) $P''(x)\cdot P'''(x)$

(c) $P(x)\cdot P'''(x)$ (d) None of these

15. If $5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ and $y = x\,f(x)$, then $\left(\dfrac{dy}{dx}\right)_{x=1}$ is equal to

(a) 14 (b) 7/8

(c) 1 (d) None of these

16. If $y = |\sin x|^{|x|}$, then the value of $\dfrac{dy}{dx}$ at $x = -\dfrac{\pi}{6}$ is

(a) $\dfrac{2^{\frac{-\pi}{6}}}{6}[6\log 2 - \sqrt{3}\pi]$ (b) $\dfrac{2^{\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$

(c) $\dfrac{2^{-\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$ (d) None of these

17. If $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f''(x)$.

If $F(x) = \left[f\left(\dfrac{x}{2}\right)\right]^2 + \left[g\left(\dfrac{x}{2}\right)\right]^2$ and $F(5) = 5$, then $F(10)$ is

(a) 0 (b) 5 (c) 10 (d) 25

18. If y is a function of x and $\log(x+y) = 2xy$, then the value of $y'(0)$ is equal to

(a) 1 (b) -1 (c) 2 (d) 0

More Than One Correct Option

19. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{2y-1}$ (b) $\dfrac{x}{x+2y}$ (c) $\dfrac{1}{\sqrt{1+4x}}$ (d) $\dfrac{y}{2x+y}$

20. $f(x) = |x^2 - 3|x| + 2|$, then which of the following is/are true?

(a) $f'(x) = 2x - 3$ for $x \in (0, 1) \cup (2, \infty)$

(b) $f'(x) = 2x + 3$ for $x \in (-\infty, -2) \cup (-1, 0)$

(c) $f'(x) = -2x - 3$ for $x \in (-2, -1)$

(d) None of the above

21. If $f(x-y)$, $f(x)f(y)$ and $f(x+y)$ are in AP for all x, y and $f(0) \neq 0$, then

(a) $f(4) = f(-4)$ (b) $f(2) + f(-2) = 0$

(c) $f'(4) + f''(-4) = 0$ (d) $f'(2) = f''(-2)$

Comprehension Based Questions

Passage

Directions (Q. Nos. 22-24) $\dfrac{df(x)}{dg(x)} = \dfrac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)} = \dfrac{f'(x)}{g'(x)}$

22. The differentiation of $\sin^3 x + \cos^3 x$ with respect to $\sin x + \cos x$ is

(a) $\sin x \cos x$ (b) $-3 \sin x \cos x$
(c) $-\sin x \cos x$ (d) $3 \sin^2 x \cos^2 x$

23. The differentiation of $\tan^{-1} \dfrac{3x - x^3}{1 - 3x^2}$ with respect to $\tan^{-1} \dfrac{2x}{1 - x^2}$, where $x \in \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ is

(a) 2 (b) 3 (c) $\dfrac{2}{3}$ (d) $\dfrac{3}{2}$

24. The differentiation of $\tan^{-1} \dfrac{2x}{1 - x^2}$ with respect to $\sin^{-1} \dfrac{2x}{1 + x^2}$, where $x \in (-1, 1)$ is

(a) 1 (b) −1 (c) 2 (d) −2

Assertion and Reason

Directions (Q. Nos. 25-34) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

25. **Statement I** If $u = f(\tan x), v = g(\sec x)$ and $f'(1) = 2$, $g'(\sqrt{2}) = 4$, then $\left(\dfrac{du}{dv}\right)_{x = \pi/4} = \dfrac{1}{\sqrt{2}}$.

Statement II If $u = f(x), v = g(x)$, then the derivative of f with respect to g is $\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$.

26. **Statement I** For $x < 0$, $\dfrac{d}{dx}(\ln |x|) = -\dfrac{1}{x}$.

Statement II For $x < 0, |x| = -x$.

27. Consider, if $u = f(x), v = g(x)$, then the derivative of f with respect to g is $\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$.

Statement I Derivative of $\sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$ with respect to $\cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$ is 1 for $0 < x < 1$.

Statement II $\sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) \neq \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$ for $-1 \le x \le 1$.

28. Consider for $f(x, y) = 0$, $\dfrac{d}{dx} f(x, y) = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$, then

Statement I If $e^{xy} + \log(xy) + \cos(xy) + 5 = 0$, then $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

Statement II $\dfrac{d}{dx}(xy) = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x}$

29. **Statement I** Let $f(x) = x[x]$ and $[\cdot]$ denotes greatest integral function, when x is not an integral, then rule for $f'(x)$ is given by $[x]$.

Statement II $f'(x)$ does not exist for any $x \in$ integer.

30. **Statement I** If $f(x)$ is an odd function, then $f'(x)$ is an even function.

Statement II If $f'(x)$ is an even function, then $f(x)$ is an odd function.

31. **Statement I** Let $f : R \to R$ is a real valued function, $\forall x, y \in R$ such that $|f(x) - f(y)| \le |x - y|^3$, then $f(x)$ is a constant function.

Statement II If derivative of the function wtih respect to x is zero, then function is constant.

32. **Statement I** If differentiable function $f(x)$ satisfies the relation $f(x) + f(x - 2) = 0$, $\forall x \in R$ and if $\left\{\dfrac{d}{dx} f(x)\right\}_{x = a} = b$, then $\left\{\dfrac{d}{dx} f(x)\right\}_{a + 4000} = b$.

Statement II $f(x)$ is a periodic function with period 4.

33. If for some differentiable function $f(\alpha) = 0$ and $f'(\alpha) = 0$.

Statement I Then, sign of $f(x)$ does not change in the neighbourhood of $x = \alpha$.

Statement II α is repeated root of $f(x) = 0$.

34. Consider function $f(x)$ satisfies the relation, $f(x + y^3) = f(x) + f(y^3)$, $\forall x, y \in R$ and differentiable for all x.

Statement I If $f'(2) = a$, then $f'(-2) = a$.

Statement II $f(x)$ is an odd function.

Previous Years' Question

35. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to [JEE Main 2013]

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

36. $\frac{d^2x}{dy^2}$ is equal to [AIEEE 2011]

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

37. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then, $y'(1)$ is equal to [AIEEE 2009]

(a) -1 (b) 1 (c) $\log 2$ (d) $-\log 2$

38. If $x^m y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is [AIEEE 2006]

(a) $\frac{x+y}{xy}$ (b) xy (c) $\frac{x}{y}$ (d) $\frac{y}{x}$

39. If $x = e^{y + e^{y + \dots \infty}}$, $x > 0$, then $\frac{dy}{dx}$ is equal to [AIEEE 2004]

(a) $\frac{x}{1+x}$ (b) $\frac{1}{x}$ (c) $\frac{1-x}{x}$ (d) $\frac{1+x}{x}$

40. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \dots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

[AIEEE 2003]

(a) 2^n (b) 2^{n-1} (c) 0 (d) 1

41. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is [AIEEE 2002]

(a) $\frac{1+x}{1+\log x}$ (b) $\frac{1-\log x}{1+\log x}$

(c) not defined (d) $\frac{\log x}{(1+\log x)^2}$

42. If $\sin y = x \sin(a + y)$, then $\frac{dy}{dx}$ is equal to [AIEEE 2002]

(a) $\frac{\sin(a+y)}{\sin a}$ (b) $\frac{\sin^2(a+y)}{\sin a}$

(c) $\frac{2\sin(a+y)}{\sin a}$ (d) $\frac{\sin^2(a+y)}{\sin y}$

43. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is [AIEEE 2002]

(a) n^2y (b) $-n^2y$

(c) $-y$ (d) $2x^2y$

Answers

Check Point 1

1. (d) **2.** (b) **3.** (b) **4.** (a) **5.** (b) **6.** (c) **7.** (b) **8.** (a) **9.** (d) **10.** (a)

Check Point 2

1. (a) **2.** (c) **3.** (b) **4.** (a) **5.** (d) **6.** (d) **7.** (c) **8.** (d)

Round I

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (d)
11. (c)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (c)	**19.** (a)	**20.** (d)
21. (a)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (d)	**30.** (d)
31. (d)	**32.** (a)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (c)	**37.** (b)	**38.** (c)	**39.** (d)	**40.** (a)
41. (c)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (d)	**46.** (b)	**47.** (b)	**48.** (c)	**49.** (c)	**50.** (c)
51. (d)	**52.** (c)	**53.** (d)	**54.** (d)	**55.** (c)	**56.** (c)	**57.** (d)	**58.** (d)	**59.** (c)	**60.** (b)
61. (d)	**62.** (d)	**63.** (a)	**64.** (a)	**65.** (b)	**66.** (d)	**67.** (d)	**68.** (b)	**69.** (c)	**70.** (c)
71. (b)	**72.** (a)	**73.** (b)	**74.** (b)	**75.** (b)	**76.** (c)	**77.** (d)	**78.** (c)	**79.** (a)	**80.** (a)
81. (d)	**82.** (b)	**83.** (d)	**84.** (a)	**85.** (c)					

Round II

1. (b)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (d)
11. (c)	**12.** (a)	**13.** (c)	**14.** (c)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (a,c,d)	**20.** (a,b,c)
21. (a,c)	**22.** (b)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (d)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (c)
31. (a)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (c)	**37.** (a)	**38.** (d)	**39.** (c)	**40.** (c)
41. (d)	**42.** (b)	**43.** (a)							

the Guidance

Round I

1. Let $y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$

$$= \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$$

$$\Rightarrow \quad y = x + \frac{3}{4}[\log(x-2) - \log(x+2)]$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[x + \frac{3}{4}\{\log(x-2) - \log(x+2)\}\right]$$

$$= 1 + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] = 1 + \frac{3}{(x^2-4)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2-1}{x^2-4}$$

2. Let $y = 2\sqrt{\cot(x^2)}$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}2(\cot x^2)^{1/2} = 2.\frac{1}{2}\{\cot(x^2)\}^{\frac{1}{2}-1}\frac{d}{dx}\cot(x^2)$$

[using chain rule, $\frac{d}{dx}f\{g(x)\} = f'(x)\frac{d}{dx}g(x)$

$$= \frac{1}{\sqrt{\cot(x^2)}}[-\operatorname{cosec}^2 x^2]\frac{d}{dx}(x^2) \quad \text{(using chain rule)}$$

$$= -\frac{\operatorname{cosec}^2(x^2)\,2x}{\sqrt{\cot(x^2)}} = \frac{-2x\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

3. Write $\sqrt{e^{\sqrt{x}}}$ as $(e^{\sqrt{x}})^{1/2}$ and then firstly use the rule $\frac{d}{dx}(x^n) = nx^{n-1}$

Let $y = (e^{\sqrt{x}})^{1/2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}(e^{\sqrt{x}})^{\frac{1}{2}-1}\frac{d}{dx}e^{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}(e^{\sqrt{x}})^{-\frac{1}{2}}.e^{\sqrt{x}}.\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\frac{e^{\sqrt{x}}}{\sqrt{e^{\sqrt{x}}}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$$

4. Given, $x = 4t$, $y = \frac{4}{t}$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = 4 \quad \text{and} \quad \frac{dy}{dt} = 4(-1)t^{-2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4t^{-2}}{4} = -\frac{1}{t^2} \qquad \left(\because \frac{dy}{dx} = \frac{dy/dt}{dx/dt}\right)$$

5. Given, $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta + \theta\sin\theta) = a\left\{\frac{d}{d\theta}(\cos\theta) + \frac{d}{d\theta}(\theta\sin\theta)\right\}$$

$$= a\{-\sin\theta + (\theta\cos\theta + \sin\theta\cdot 1)\} = a\theta\cos\theta$$

[using product rule in $\frac{d}{d\theta}(\theta\sin\theta)$]

and $\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta - \theta\cos\theta)$

$$= a\left\{\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right\}$$

$$= a[\cos\theta - \{\theta(-\sin\theta) + \cos\theta.1\}] = a\theta\sin\theta$$

[using product rule in $\frac{d}{d\theta}(\theta\cos\theta)$]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

6. Let $y = \sqrt{\tan\sqrt{x}}$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}(\sec^2\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

7. We have, $x = a\sec^3\theta$ and $y = a\tan^3\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) = 3a\sec^3\theta\tan\theta$$

and $\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta$

Thus, $$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$$

Hence, $$\left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{3}}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

8. Let $y=\log[\log(\log x^5)]$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\log(\log x^5)}\cdot\frac{d}{dx}[\log(\log x^5)]$$

$$=\frac{1}{\log(\log x^5)}\cdot\frac{1}{(\log x^5)}\cdot\frac{d}{dx}(\log x^5)$$

$$=\frac{1}{\log(\log x^5)}\cdot\frac{1}{(\log x^5)}\cdot\frac{1}{x^5}\cdot\frac{d}{dx}(x^5)$$

$$=\frac{5x^4}{x^5\log x^5\cdot\log(\log x^5)}$$

$$=\frac{5}{x\cdot 5\log x\cdot\log(\log x^5)}$$

$$=\frac{1}{x\log x\cdot\log(\log x^5)}$$

9. Given that,

$$x=t+\frac{1}{t}\quad\text{and}\quad y=t-\frac{1}{t}$$

We have,

$$\left(t+\frac{1}{t}\right)^2-\left(t-\frac{1}{t}\right)^2=4$$

$$\Rightarrow\qquad x^2-y^2=4$$

Now, differentiating w.r.t. x, we get

$$2x-2y\frac{dy}{dx}=0$$

$$\Rightarrow\qquad \frac{dy}{dx}=\frac{x}{y}=\frac{t^2+1}{t^2-1}$$

10. Given that,

$$x=\frac{1+\log t}{t^2}\quad\text{and}\quad y=\frac{3+2\log t}{t}$$

On differentiating w.r.t. x, we get

$$\frac{dx}{dt}=\frac{t^2\cdot\frac{1}{t}-2t(1+\log t)}{t^4}$$

and $$\frac{dy}{dt}=\frac{t\cdot\frac{2}{t}-(3+2\log t)}{t^2}$$

$$\Rightarrow\qquad \frac{dx}{dt}=\frac{t-2t-2t\log t}{t^4}$$

and $$\frac{dy}{dt}=\frac{2-3-2\log t}{t^2}$$

$$\Rightarrow\qquad \frac{dx}{dt}=\frac{-t-2t\log t}{t^4}$$

and $$\frac{dy}{dt}=\frac{-1-2\log t}{t^2}$$

$$\Rightarrow\qquad \frac{dx}{dt}=-\frac{(1+2\log t)}{t^3}$$

and $$\frac{dy}{dt}=-\frac{(1+2\log t)}{t^2}$$

$$\therefore\qquad \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$=-\frac{(1+2\log t)}{t^2}\times\frac{t^3}{(1+2\log t)}$$

$$\Rightarrow\qquad \frac{dy}{dx}=-t$$

11. Let $y=\dfrac{\tan^2 2x-\tan^2 x}{1-\tan^2 2x\tan^2 x}$

$$=\frac{(\tan 2x-\tan x)}{(1+\tan 2x\tan x)}\cdot\frac{(\tan 2x+\tan x)}{(1-\tan 2x\tan x)}$$

$$=\tan(2x-x)\tan(2x+x)=\tan x\tan 3x$$

On differentiating w.r.t. x, we get

$$\frac{d}{dx}(y\cdot\cot 3x)=\frac{d}{dx}(\tan x\tan 3x\cot 3x)$$

$$=\frac{d}{dx}(\tan x)=\sec^2 x$$

12. $\because y=\log^n x$

On differentiating w.r.t. x, we get

$$x\log x\log^2 x\log^3 x\ldots\log^{n-1}x\log^n x\frac{dy}{dx}$$

$$=\frac{x\log x\log^2 x\log^3 x\ldots\log^{n-1}x\log^n x\cdot 1}{x\log x\log^2 x\log^3 x\ldots\log^{n-1}x}=\log^n x$$

13. We have, $$f(x)=\sqrt{1+\cos^2(x^2)}\qquad\ldots\text{(i)}$$

On differentiating Eq. (i) w.r.t. x, we get

$$f'(x)=\frac{-2\sin x^2\cos x^2}{\sqrt{1+\cos^2 x^2}}(x)$$

$$\Rightarrow\qquad f'(x)=\frac{-\sin 2x^2}{\sqrt{1+\cos^2 x^2}}(x)\qquad\ldots\text{(ii)}$$

Put $x=\dfrac{\sqrt{\pi}}{2}$ in Eq. (ii), we get

$$f'\left(\frac{\sqrt{\pi}}{2}\right)=-\frac{\sqrt{\pi}}{2}\cdot\frac{\sin 2\left(\frac{\pi}{4}\right)}{\sqrt{1+\frac{1}{2}}}=-\frac{\sqrt{\pi}}{2}\cdot\frac{\sin\frac{\pi}{2}}{\sqrt{\frac{3}{2}}}=-\sqrt{\frac{\pi}{6}}$$

14. $f(x)=(\log_{\cot x}\tan x)(\log_{\tan x}\cot x)^{-1}+\tan^{-1}\dfrac{4x}{4-x^2}$

$$=\frac{\log\tan x}{\log\cot x}\cdot\frac{\log\tan x}{\log\cot x}+\tan^{-1}\left(\frac{4x}{4-x^2}\right)$$

$$=\frac{(\log\tan x)^2}{(-\log\tan x)^2}+\tan^{-1}\left(\frac{4x}{4-x^2}\right)=1+\tan^{-1}\left(\frac{4x}{4-x^2}\right)$$

$$\therefore \quad f'(x) = \frac{1}{1+\left(\frac{4x}{4-x^2}\right)^2} \cdot \frac{(4-x^2)\,4 - 4x(-2x)}{(4-x^2)^2}$$

$$= \frac{16 - 4x^2 + 8x^2}{(4-x)^2 + 16x^2} = \frac{4(4+x^2)}{(4-x^2)^2 + (4x)^2}$$

Hence, $\quad f'(2) = \frac{4(4+4)}{0+(8)^2} = \frac{32}{64} = \frac{1}{2}$

15. Given,

$$f(x) = \log_x(\log_e x) = \frac{\log_e \log_e x}{\log_e x}$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\log_e x \cdot \frac{1}{\log_e x} \cdot \frac{1}{x} - \log_e \log_e x \cdot \frac{1}{x}}{(\log_e x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{1 - \log_e \log_e x}{x(\log_e x)^2}$$

$$\Rightarrow \quad f'(e) = \frac{1 - \log_e \log_e e}{e(\log_e e)^2} = \frac{1 - \log_e 1}{e} = \frac{1}{e}$$

16. $f(x) = e^x$ and $g(x) = \sin^{-1} x$

and $\quad h(x) = f[g(x)]$

$\Rightarrow \quad h(x) = f(\sin^{-1} x) = e^{\sin^{-1} x}$

$\therefore \quad h(x) = e^{\sin^{-1} x}$

$$\Rightarrow \quad h'(x) = e^{\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{h'(x)}{h(x)} = \frac{1}{\sqrt{1-x^2}}$$

17. Given, $f(x) = \sin x, g(x) = x^2$

and $\quad h(x) = \log_e x$

Also, $\quad F(x) = (hogof)\,(x)$

$\therefore \quad (hogof)(x) = (hog)\,(\sin x)$

$\Rightarrow \quad = h(\sin x^2)$

$\Rightarrow \quad F(x) = 2 \log \sin x$

On differentiating, we get

$$F'(x) = 2 \cot x$$

Again, differentiating, we get

$$F''(x) = -2 \operatorname{cosec}^2 x$$

18. Given, $xy + y^2 = \tan x + y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow \quad \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\left[\text{using product rule}, \frac{d}{dx}(u \cdot v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$\Rightarrow \quad x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

19. Given, $\quad \sin^2 x + \cos^2 y = 1$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(1)$$

$$2 \sin x \cos x + 2 \cos y\left(-\sin y \frac{dy}{dx}\right) = 0$$

$$\left[\text{using chain rule}, \frac{d}{dx} f\{g(x)\} = f'(x)\frac{d}{dx}g(x)\right]$$

$$\Rightarrow \quad -2 \sin y \cos y \frac{dy}{dx} = -2 \sin x \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin 2x}{-\sin 2y} = \frac{\sin 2x}{\sin 2y} \quad (\because \sin 2x = 2 \sin x \cdot \cos x)$$

20. Given, $\quad x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x} \quad \ldots(i)$$

On squaring both sides of Eq.(i), we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2 - y^2 + x^2y - y^2x = 0$$

$$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0$$

$$\Rightarrow \quad (x-y)\{x + y + xy\} = 0$$

$$\Rightarrow \quad x - y = 0 \text{ or } x + y + xy = 0$$

$$\Rightarrow \quad y = x \text{ or } y \cdot (1+x) = -x$$

$$\Rightarrow \quad y = x \text{ or } y = \frac{-x}{1+x}$$

But $y = x$ does not satisfy the given equation.

So, we consider only $= -\frac{x}{1+x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{-x}{1+x}\right)$$

$$= -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= -\frac{(\ +x) \cdot 1 - x(0+1)}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

21. Given, $\quad (x-a)^2 + (y-b)^2 = c^2 \quad \ldots(i)$

On differentiating w.r.t. x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x-a}{y-b} \quad \ldots(ii)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{(y-b)\cdot 1-(x-a)\dfrac{dy}{dx}}{(y-b)^2} \quad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}\right]$$

$$= -\frac{(y-b)\cdot 1-(x-a)\left(-\dfrac{x-a}{y-b}\right)}{(y-b)^2} \quad \text{[using Eq. (ii)]}$$

$$= -\frac{(y-b)^2+(x-a)^2}{(y-b)^3} = -\frac{c^2}{(y-b)^3}$$

Now, $$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left[1+\left(\dfrac{x-a}{y-b}\right)^2\right]^{3/2}}{-\dfrac{c^2}{(y-b)^3}}$$

$$= \frac{\{(y-b)^2+(x-a)^2\}^{3/2}}{\{(y-b)^2\}^{3/2}\dfrac{(-c^2)}{(y-b)^3}}$$

$$= \frac{(c^2)^{3/2}}{-c^2} = \frac{c^3}{-c^2} = -c$$

which is independent of a and b.

22. Given, $\cos y = x\cos(a+y) \Rightarrow x = \dfrac{y}{\cos(a+y)}$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\left\{\frac{\cos y}{\cos(a+y)}\right\}$$

$$= \frac{\cos(a+y)(-\sin y)-\cos y\{-\sin(a+y)\cdot 1\}}{\cos^2(a+y)}$$

$$\left[\text{using quotient rule, } \frac{d}{dy}\left(\frac{u}{v}\right) = \frac{v\dfrac{d}{dy}u-u\dfrac{d}{dy}v}{v^2}\right]$$

$$= \frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$[\because \sin(A-B) = \sin A\cos B-\cos A\sin B]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{\cos^2(a+y)}{\sin a}$$

23. Here, $x\sin(a+y)+\sin a\cos(a+y) = 0$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{d}{dx}[x\sin(a+y)]+\frac{d}{dx}[\sin a\cos(a+y)] = 0$$

$$\Rightarrow \left[x\frac{d}{dx}\{\sin(a+y)\}+\sin(a+y)\frac{d}{dx}(x)\right] + \sin a\frac{d}{dx}\cos(a+y) = 0$$

(using product rule and chain rule)

$$\Rightarrow \left[x\cos(a+y)\frac{d}{dx}(a+y)+\sin(a+y)\right] + \sin a\left[-\sin(a+y)\frac{d}{dx}(a+y)\right] = 0$$

$$\Rightarrow \left[x\cos(a+y)\left(0+\frac{dy}{dx}\right)+\sin(a+y)\right] - \sin a\sin(a+y)\left(0+\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x\cos(a+y)\frac{dy}{dx}+\sin(a+y)-\sin a\sin(a+y)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}[x\cos(a+y)-\sin a\sin(a+y)] = -\sin(a+y)$$

$$\left[\text{from Eq. (i), putting } x = -\sin a\frac{\cos(a+y)}{\sin(a+y)}\right]$$

$$\Rightarrow \frac{dy}{dx}\left[-\sin a\frac{\cos^2(a+y)}{\sin(a+y)}-\sin a\sin(a+y)\right] = -\sin(a+y)$$

$$\Rightarrow -\frac{dy}{dx}\left[\frac{\sin a\cos^2(a+y)+\sin a\sin^2(a+y)}{\sin(a+y)}\right] = -\sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \sin(a+y)\left[\frac{\sin(a+y)}{\sin a\{\cos^2(a+y)+\sin^2(a+y)\}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} \quad (\because \sin^2\theta+\cos^2\theta = 1)$$

24. Given that, $y = \sqrt{\sin x+y}$

On squaring on both sides, we get $y^2 = \sin x+y$

Now, differentiating, *w.r.t.* x, we gt

$$2y\cdot\frac{dy}{dx} = \cos x+\frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

25. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$

Putting $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{x}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$(\because 1+\tan^2\theta = \sec^2\theta)$$

$$\Rightarrow u = \tan^{-1}\left[\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\left(\begin{array}{l} 1-\cos\theta = 2\sin^2\dfrac{\theta}{2} \\ \sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\end{array}\right)$$

$$= \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$$

Thus, we have $u = \frac{1}{2}\tan^{-1} x$ and $v = \tan^{-1} x$

On differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2} \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2(1+x^2)} \times (1+x^2) = \frac{1}{2}$$

26. Since, it is difficult to differentiate by apply product rule to a product of a function.

Take log on both sides of $y = \cos x \cos 2x \cos 3x$ and use the rule $\log(m \cdot n) = \log m + \log n$ and then differentiate.

Let $\quad y = \cos x \cos 2x \cos 3x$

Taking log on both side, we get

$$\log y = \log(\cos x \cos 2x \cos 3x)$$

$$(\because \log m \times n \times l = \log m + \log n + \log l)$$

or $\quad \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\log y = \frac{d}{dx}\log(\cos x) + \frac{d}{dx}\log(\cos 2x) + \frac{d}{dx}\log(\cos 3x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}\left\{-\sin 2x \frac{d}{dx}(2x)\right\} + \frac{1}{\cos 3x}\left\{-\sin 3x \cdot \frac{d}{dx}(3x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cos x \cos 2x \cos 3x + \{\tan x + 2x \tan 2x + 3x \tan 3x\}$$

27. Given, $\quad y^x = x^y$

Taking log on both sides, we get

$$\log y^x = \log x^y$$

$$\Rightarrow \quad x \log y = y \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x \log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \quad x\left(\frac{1}{y}\right)\frac{dy}{dx} + (\log y) = y\frac{1}{x} + (\log x)\frac{dy}{dx}$$

(using product rule)

$$\Rightarrow \quad \frac{x}{y}\frac{dy}{dx} - (\log x)\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \quad \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \quad \left(\frac{x - y\log x}{y}\right)\frac{dy}{dx} = \frac{y - x\log y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

28. Given, $xy = e^{(x-y)}$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(xy) = \frac{d}{dx}(e^{x-y})$$

$$\Rightarrow \quad x\frac{dy}{dx} + y \cdot 1 = e^{x-y}\frac{d}{dx}(x-y)$$

(using product rule in LHS and chain rule in RHS)

$$\Rightarrow \quad x\frac{dy}{dx} + y = e^{x-y}\left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow \quad x\frac{dy}{dx} + e^{x-y}\frac{dy}{dx} = e^{x-y} - y$$

$$\Rightarrow \quad (x + e^{x-y})\frac{dy}{dx} = e^{x-y} - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{x-y} - y}{x + e^{x-y}} = \frac{xy - y}{x + xy} \quad (\because e^{x-y} = xy \text{ is given})$$

$$= \frac{y(x-1)}{x(1+y)}$$

29. Take logarithm on both sides of the equation and use the rule $\log(m \cdot n) = \log m + \log n$ and then differentiate the equation.

Given, $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking log on both sides, we get

$$\log\{f(x)\} = \log\{(1+x)(1+x^2)(1+x^4)(1+x^8)\}$$

$$\Rightarrow \quad \log\{f(x)\} = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\log\{f(x)\} = \frac{d}{dx}\log(1+x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$$

$$\frac{1}{f(x)}f'(x) = \frac{1}{1+x}(1+0) + \frac{1}{1+x^2}(0+2x) + \frac{1}{1+x^4}(0+4x^3) + \frac{1}{1+x^8}(0+8x^7)$$

$$\Rightarrow \quad f'(x) = f(x)\left\{\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right\}$$

$$= (1+x)(1+x^2)(1+x^4)(1+x^8)\left\{\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right\}$$

$$\therefore \quad f'(1) = (1+1)(1+1)(1+1)(1+1)\left\{\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right\}$$

$$= 2^4\left\{\frac{1}{2} + 1 + 2 + 4\right\} = 16\left(\frac{15}{2}\right) = 120$$

30. Let $y = (\log x)^{\log x}$

Taking log on both sides, we get

$$\log y = \log[(\log x)^{\log x}]$$

$$\Rightarrow \quad \log y = \log x \log(\log x) \quad (\because \log m^n = n \log m)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = (\log x)\frac{d}{dx}\log(\log x) + \log\log(x)\frac{d}{dx}\log(x)$$

$$= (\log x)\frac{1}{\log x}\frac{1}{x} + \log(\log x)\frac{1}{x} = \frac{1}{x}\{1 + \log(\log x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\{1 + \log(\log x)\} = \frac{(\log x)^{\log x}}{x}[1 + \log(\log x)]$$

$$= (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

31. Here, $y^x = e^{(y-x)}$

Taking log on both sides, $\log y^x = \log e^{y-x}$

$(\because \log a^b = b\log a \text{ and } \log e = 1)$

$$x\log y = (y - x)\log e, \quad x\log y = y - x \qquad \ldots(i)$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}(x\log y) = \frac{d}{dx}(y - x) \qquad \text{(using product rule)}$$

$$\Rightarrow \quad \left[x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)\right] = \frac{dy}{dx} - 1$$

$$\Rightarrow \quad x\left(\frac{1}{y}\right)\frac{dy}{dx} + \log y\,(1) = \frac{dy}{dx} - 1$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y} - 1\right) = -1 - \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{y}{(1 + \log y)y} - 1\right] = -(1 + \log y)$$

$$\left[\because \text{ from Eq. (i), } x = \frac{y}{(1 + \log y)}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1 - 1 - \log y}{(1 + \log y)}\right] = -(1 + \log y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(1 + \log y)^2}{-\log y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$$

32. Given that, $x = e^{x/y}$

Taking log on both sides, we get

$$\log x = \frac{x}{y}\cdot\log e = \frac{x}{y}\cdot 1$$

$$\Rightarrow \quad x = y\log x$$

Now, differentiating w.r.t. x, we get

$$\Rightarrow \quad 1 = y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x - y}{x\log x}$$

33. Given that,

$$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$$

$$\Rightarrow \quad y = (\cos x)^y$$

Taking log on both sides,

$$\log y = y\log\cos x$$

Now, differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{-y\tan x + \log\cos x\cdot\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad (1 - y\log\cos x)\frac{dy}{dx} = -y^2\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2\tan x}{(y\log\cos x - 1)}$$

34. Given that,

$$x^m\cdot y^n = (x + y)^{m+n}$$

Taking log on both sides, we get

$$\log(x^m\cdot y^n) = \log(x + y)^{(m+n)}$$

$$\Rightarrow \quad \log x^m + \log y^n = (m + n)\log(x + y)$$

$$\Rightarrow \quad m\log x + n\log y = (m + n)\log(x + y)$$

Now, differentiating w.r.t. x on both sides, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m + n)\cdot\frac{1}{(x + y)}\cdot\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left\{\frac{m}{n} - \frac{m + n}{x + y}\right\} = \left\{\frac{m + n}{x + y} - \frac{n}{y}\right\}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{(my - nx)}{x(x + y)} = \frac{(my - nx)}{y(x + y)}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\cdot\frac{1}{y} - \frac{1}{x}\right)\cdot\left(\frac{my - nx}{x + y}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \qquad \ldots(i) \quad \left(\because \frac{my - nx}{x + y} \neq 0\right)$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = 0$$

Again, differentiating w.r.t. x, we get

$$x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 - \frac{dy}{dx} = 0$$

$$\Rightarrow \quad x\cdot\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 0 \qquad (\because x \neq 0)$$

35. Given that, $x = \exp\left\{\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right\}$

Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right)$$

$$\Rightarrow \quad \frac{y - x^2}{x^2} = \tan(\log x)$$

$$\Rightarrow \quad y = x^2\tan(\log x) + x^2$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2x\tan(\log x) + x^2\frac{\sec^2(\log x)}{x} + 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x \tan(\log x) + x \sec^2(\log x) + 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x[1 + \tan(\log x)] + x \sec^2(\log x)$$

36. $f(x) = \cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}$

Taking log on both sides, we get

$$\log f(x) = \log \cos\frac{x}{2} + \log \cos\frac{x}{2^2} + \ldots \log \cos\frac{x}{2^n}$$

On differentiating w.r.t. x on both sides, we get

$$\frac{f'(x)}{f(x)} = -\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}\right)$$

$$\Rightarrow \quad \frac{-f'(x)}{f(x)} = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$$

37. Substitute $x = \tan\theta \Rightarrow \tan^{-1} x = \theta$

$$\therefore \quad y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$\Rightarrow \quad y = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\theta \Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$$

On differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - 2\frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

38. Substitute $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow \quad y = \cos^{-1}(\sin 2\theta) \qquad \left(\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow \quad y = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right]$$

$$\left[\because \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\theta \Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x \qquad (\because \theta = \tan^{-1} x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2}{1+x^2}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

39. Use the identities $\sin^2 x + \cos^2 x = 1$, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and the identities $(a+b)^2 = a^2 + b^2 + 2ab$ and $(a-b)^2 = a^2 + b^2 - 2ab$ to simplify inside the bracket.

Let $\quad y = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$$

$$\left[\begin{array}{l} \because 1 + \sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}.\cos\frac{x}{2} = \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 \\ \text{and } 1 - \sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 \end{array}\right]$$

$$= \cot^{-1}\left[\frac{2\cos(x/2)}{2\sin(x/2)}\right] = \cot^{-1}\left[\cot\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}$$

Note If $\frac{\pi}{2} < x < \pi$, then $\sqrt{1-\sin x} = \sin\frac{x}{2} - \cos\frac{x}{2}$. Take logarithm on both sides of the given equation $y = (\log x)^{\log x}$ and then differentiate w.r.t. x.

40. Given, $\quad y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$

On putting $\sin^{-1} x = \theta$

$$\Rightarrow \quad x = \sin\theta$$

$$\Rightarrow \quad y = \theta + \sin^{-1}\sqrt{1-\sin^2\theta}$$

$$\Rightarrow \quad y = \theta + \sin^{-1}(\cos\theta) \qquad (\because 1 - \sin^2\theta = \cos^2\theta)$$

$$\Rightarrow \quad y = \theta + \sin^{-1}\sin\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \quad y = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2} \qquad \left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0$$

41. Use the formula $\cos a \cos b + \sin a \sin b = \cos(a-b)$ and $\cos^{-1}(\cos x) = x$ to simplify $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$ and then differentiate w.r.t. x.

Let $\quad y = \cos^{-1}\left\{\frac{\cos x + \cos x}{\sqrt{2}}\right\}$

$$y = \cos^{-1}\left\{\cos x\left(\frac{1}{\sqrt{2}}\right) + \sin x\left(\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \cos^{-1}\left\{\cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin x\left(\frac{\pi}{4}\right)\right\}$$

$$\left(\because \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right)$$

$$y = \cos^{-1}\left[\cos\left(x - \frac{\pi}{4}\right)\right] \quad \ldots(i)$$

$$[\because \cos(A-B) = \cos A \cos B + \sin A \sin B]$$

Here, $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$$\Rightarrow \left(-\frac{\pi}{4} - \frac{\pi}{4}\right) < \left(x - \frac{\pi}{4}\right) < \left(\frac{\pi}{4} - \frac{\pi}{4}\right) \Rightarrow -\frac{\pi}{2} < \left(x - \frac{\pi}{4}\right) < 0$$

So, from Eq. (i),

$$y = -\left(x - \frac{\pi}{4}\right)$$

$$[\text{since, } \cos^{-1}(\cos\theta) = -\theta, \text{ if } \theta \in [-\pi, 0]]$$

$$y = -x + \frac{\pi}{4}$$

On differentiating it w.r.t. x, $\frac{dy}{dx} = -1$

42. Let $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

or $y = \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$$

$$\left[\because \tan^{-1}(A+B) = \tan^{-1}\left(\frac{A+B}{1+AB}\right)\right]$$

$$\Rightarrow y = \tan^{-1}\left(\frac{a}{b}\right) - x \qquad \left[\because -\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$

On differentiating w.r. t. x, we get

$$\frac{dy}{dx} = 0 - 1 = -1$$

43. Here, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin A$, $y = \sin B$

$$\therefore \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \cos A + \cos B = a(\sin A - \sin B)$$

$$[\because (1 - \sin^2\theta) = \cos^2\theta]$$

$$a = \frac{\cos A + \cos B}{\sin A - \sin B}$$

$$\Rightarrow \alpha = \frac{2\cos\frac{A+B}{2} \times \cos\frac{A-B}{2}}{2\cos\frac{A+B}{2} \times \sin\frac{A-B}{2}}$$

$$\left[\begin{array}{l} \because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \\ \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{array}\right]$$

$$\Rightarrow a = \cot\left(\frac{A-B}{2}\right) \Rightarrow \cot^{-1} a = \frac{A-B}{2}$$

$$\Rightarrow 2\cot^{-1} a = A - B$$

$$\Rightarrow 2\cot^{-1} a = \sin^{-1} x - \sin^{-1} y \qquad (\because x = \sin A, y = \sin B)$$

On differentiating w.r.t. x, we get

$$\frac{d}{dx}(2\cot^{-1} a) = \frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\sin^{-1} y)$$

$$\Rightarrow 0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

44. Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Let $x = \tan^2\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$

$$y = \sin^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) = \sin^{-1}\left(\frac{1}{\sec\theta}\right)$$

$$y = \sin^{-1}(\cos\theta) + \sin^{-1}\{\sin(\pi/2 - \theta)\}$$

$$y = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \tan^{-1}\sqrt{x}$$

Now, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{1+x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{x}(1+x)}$$

45. We have, $y = \tan^{-1}(\sec x - \tan x)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right)$$

$$= \frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right) = -\frac{1}{2}$$

46. $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} x = \frac{\pi}{2} - \sin^{-1} y$$

$$\Rightarrow \sin^{-1} x = \cos^{-1} y$$

$$\Rightarrow y = \sqrt{1-x^2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{y}$$

47. Let $y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$\Rightarrow \quad y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$\Rightarrow \quad y = \sin^2\cot^{-1}\left\{\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right\}$$

$$= \sin^2\cot^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow \quad y = \sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = 2\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = -\frac{\sin(\pi-\theta)}{2} = -\frac{\sin\theta}{2} = -\frac{1}{2}\sqrt{1-x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{-1}{2}\sqrt{1-x^2}\,\frac{d}{dx}(\cos^{-1}x)$$

$$= -\frac{\sqrt{1-x^2}}{2}\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{1}{2}$$

48. Let $y = \frac{d}{dx}[\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})]$

Put $x = \sin\alpha$ and $\sqrt{x} = \sin\beta$

$$\therefore \quad y = \frac{d}{dx}[\sin^{-1}(\sin\alpha\sqrt{1-\sin^2\beta} - \sin\beta\sqrt{1-\sin^2\alpha})]$$

$$= \frac{d}{dx}[\sin^{-1}\sin(\alpha-\beta)] = \frac{d}{dx}(\alpha-\beta)$$

$$= \frac{d}{dx}[\sin^{-1}x - \sin^{-1}\sqrt{x}]$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x(1-x)}}$$

49. $\because y = \tan^{-1}\left[\frac{\log\left(\frac{e}{x^2}\right)}{\log ex^2}\right] + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$$= \tan^{-1}\left(\frac{1-\log x^2}{1+\log x^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$$

$$\therefore \quad y = \tan^{-1}(1) + \tan^{-1}(3)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0$$

50. $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \ldots +$ upto n terms

$$= \tan^{-1}\frac{(x+1)-x}{1+x(x+1)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)} + \ldots + \text{upto } n \text{ terms}$$

$$= [\tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \ldots + \tan^{-1}(x+n) - \tan^{-1}\{x+(n-1)\}]$$

$$= \tan^{-1}(x+n) - \tan^{-1}x$$

$$\therefore \quad y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

$$\Rightarrow \quad y'(0) = \frac{1}{1+n^2} - 1 = \frac{n^2}{1+n^2}$$

51. $f(x) = \begin{vmatrix} x^3 & x^2 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$

$$\Rightarrow f(x) = x^3(-6p^3 - 4p^2) - x^2(p^3 - 4p) + 3x^2(p^2 + 6p)$$

$$\Rightarrow f(x) = -6p^3x^3 - 4p^2x^3 - x^2p^3 + 4px^2 + 3p^2x^2 + 18px^2$$

On differentiating w.r.t. x, we get

$$\frac{d}{dx}f(x) = -18p^3x^2 - 12p^2x^2 - 2xp^3 + 8px + 6p^2x + 36px$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2}{dx^2}f(x) = -36p^3x - 24p^2x - 2p^3 + 8p + 6p^2 + 36p$$

Again, differentiating w.r.t. x, we get

$$\frac{d^3}{dx^3}f(x) = -36p^3 - 24p^2 = \text{constant}$$

52. Given, $\quad y = 3\cos(\log x) + 4\sin(\log x) \quad \ldots(i)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = y_1 = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}\log x$$

$$= -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x} \qquad \left(\because \frac{dy}{dx} = y_1\right)$$

On multiplying by x, we get

$$xy_1 = -3\sin(\log x) + 4\cos(\log x) \quad \ldots(ii)$$

Again, differentiating w.r.t. x, we get

$$xy_2 + y_1\cdot 1 = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}\log x$$

$$= -3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}$$

On multiplying throughout by x, we get

$$x^2y_2 + xy_1 = -[3\cos(\log x) + 4\sin(\log x)] \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -y$$

53. Given, $y = 500\, e^{7x} + 600\, e^{-7x}$ ...(i)

On differentiating twicely w.r.t. x, we get

$$\frac{dy}{dx} = 500\, e^{7x} \frac{d}{dx}(7x) + 600\, e^{-7x} \frac{d}{dx}(-7x)$$

$$= 500\, e^{7x} \cdot 7 + 600\, e^{-7x} \cdot (-7)$$

and $$\frac{d^2y}{dx^2} = (7 \times 500)\, e^{7x} \cdot 7 - (7 \times 600)\, e^{-7x}(-7)$$

$$= 49\, \{500\, e^{7x} + 600\, e^{-7x}\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 49\, y$$

54. Given, $e^y (x + 1) = 1$ ...(i)

$$\Rightarrow \quad e^y = \frac{1}{x+1}$$

On differentiating both sides w.r.t. x, we get

$$e^y \frac{dy}{dx} = -\frac{1}{(x+1)^2}$$

$$\Rightarrow \quad \left(\frac{1}{x+1}\right)\frac{dy}{dx} = -\frac{1}{(x+1)^2} \qquad \left(\because e^y = \frac{1}{x+1}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x+1}{(x+1)^2} = -\frac{1}{x+1} \qquad \text{...(ii)}$$

On differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} \quad \text{or} \quad \frac{d^2y}{dx^2} = \left(-\frac{1}{x+1}\right)^2 = \left(\frac{dy}{dx}\right)^2$$

55. Given, $y = (\tan^{-1} x)^2$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2 \tan^{-1} x \frac{d}{dx}(\tan^{-1} x)$$

$$= 2\,(\tan^{-1} x) \frac{1}{1+x^2}$$

or $$(1 + x^2)\, y_1 = 2 \tan^{-1} x \qquad \left(\because \frac{dy}{dx} = y_1\right)$$

Again, differentiating w.r.t. x, we get

$$(1 + x^2) \frac{dy_1}{dx} + y_1 \frac{d}{dx}(1 + x^2) = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2) y_2 + y_1 (0 + 2x) = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2)^2 y_2 + 2x(1 + x^2) y_1 = 2 \qquad \left[\because \frac{d}{dx}(y_1) = y_2\right]$$

56. Given, $y = e^{a \cos^{-1} x}$ ...(i)

On differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = e^{a \cos^{-1} x} \frac{d}{dx} a \cos^{-1} x = e^{a \cos^{-1} x} \cdot \frac{-a}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \qquad (\because e^{a \cos^{-1} x} = y)$$

$$\Rightarrow \quad \sqrt{1-x^2}\, \frac{dy}{dx} = -ay \qquad \text{...(ii)}$$

Again, differentiating w.r.t. x, we get

$$\sqrt{1-x^2}\, \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -a \frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1-x^2}\, \frac{d^2y}{dx^2} + \left(\frac{-x}{\sqrt{1-x^2}}\right)\frac{dy}{dx} = \frac{-a(-ay)}{\sqrt{1-x^2}}$$

[using Eq.(ii)]

On multiplying throughout by $\sqrt{1-x^2}$, we get

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

57. Given that,

$$x = \sin t \quad \text{and} \quad y = \sin pt$$

$$\Rightarrow \quad \frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = p \cdot \cos pt$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{p \cos pt}{\cos t} \qquad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{1}{P}\frac{d^2y}{dx^2} = \left\{\frac{\cos t \cdot (-p \sin pt) - \cos pt(-\sin t)}{\cos^2 t}\right\} \cdot \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{1}{P}\frac{d^2y}{dx^2} = \frac{\cos pt \cdot \sin t - p \sin pt \cdot \cos t}{\cos^2 t} \cdot \frac{1}{\cos t}$$

$$\Rightarrow \quad \cos^2 t \frac{dy}{dt} = \frac{p \cos pt \cdot \sin t - p^2 \sin pt \cdot \cos t}{\cos t}$$

$$\Rightarrow \quad (1 - \sin^2 t) \frac{d^2y}{dx^2} = (1 - x^2) \frac{d^2y}{dx^2}$$

$$= \frac{p \cos pt \cdot \sin t - p^2 \sin pt \cdot \cos t}{\cos t} \qquad \text{...(ii)}$$

$$-x \frac{dy}{dx} = \frac{-p \sin t \cdot \cos pt}{\cos t} \qquad \text{...(iii)}$$

$$p^2 y = p^2 \sin pt \qquad \text{...(iv)}$$

On adding Eqs. (ii), (iii) and (iv), we get

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + p^2 y = \frac{p \cos pt \cdot \sin t - p^2 \sin pt \cdot \cos t}{\cos t} - \frac{p \sin t \cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos t}\{\cos pt \cdot \sin t - p^2 \sin pt \cos t - p \sin t \cdot \cos p + p^2 \sin pt \cdot \cos t\}$$

$$= \frac{1}{\cos t} \times 0 = 0$$

58. $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

$$\Rightarrow \quad y = e^{-x}$$

On differentiating w.r.t. x, we get $\frac{dy}{dx} = -e^{-x}$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -e^{-x}(-1) = e^{-x} = y$$

59. We know that by Newton's Leibnitz formula

If $$I = \int_u^v f(t)\,dx,$$

then $$\frac{dI}{dx} = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

where, u and v are functions of x.

$$\therefore \quad \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{1+9y^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{9y}{\sqrt{1+9y^2}} \cdot \frac{dy}{dx}$$

$$= \frac{9y}{\sqrt{1+9y^2}}\sqrt{1+9y^2} = 9y$$

60. $f(1) = 5,\ f'(x) = nx^{n-1}$ so $f'(1) = n$

$f''(1) = n(n-1), \ldots f^n(1) = 1 \cdot 2 \ldots \cdot n$

Thus, $$f(1) + \frac{f'(1)}{1!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 5 + \frac{n}{1} + \frac{n(n-1)}{2!} + \ldots + \frac{n!}{n!}$$

$$= (1+1)^n + 4 = 2^n + 4$$

61. $f(x) = x^{2n},\ f'(x) = 2n \cdot x^{2n-1};\ f''(x) = 2n(2n-1)\,x^{2n-2} = \ldots$

and so on

$$f^r(x) = \frac{2n!}{(2n-r)!}x^{2n-r}$$

So, $$f^n(x) = \frac{2n!}{n!} \cdot x^n$$

62. $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let $u = (\sin x)^x,\ v = \sin^{-1}\sqrt{x}$

$$\therefore \quad y = u + v$$

On differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $$u = (\sin x)^x$$

Taking log on both sides, $\log u = \log(\sin x)^x$

$$\Rightarrow \quad \log u = x\log(\sin x)$$

On differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}(x)$$

(using product rule)

$$= \frac{x}{\sin x}\cos x + \log(\sin x)$$

$$\frac{du}{dx} = u[x\cot x + \log\sin x] = (\sin x)^x\,[x\cot x + \log\sin x]$$

Again, $$v = \sin^{-1}\sqrt{x}$$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}x^{1/2} = \frac{1}{\sqrt{1-x}}\frac{1}{2}x^{-1/2}$$

(using chain rule)

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}}$$

On putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in Eq. (i), we get

$$\frac{dy}{dx} = (\sin x)^x\,[x\cot x + \log\sin x] + \frac{1}{2\sqrt{x-x^2}}$$

63. Here, $$f(x) = |x|^3$$

When $x \geq 0$, $$f(x) = |x|^3 = x^3$$

On differentiating both sides w.r.t.x, we get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^3) \Rightarrow f'(x) = 3x^2$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(3x^2) \Rightarrow f''(x) = 6x$$

when $x < 0$,

$$f(x) = |x|^3 = -x^3$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(-x^3) \Rightarrow f'(x) = -3x^2$$

Again, differentiating both sides w.r.t.x, we get

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(-3x^2) - 6x$$

$$\frac{d}{dx}f''(x) = -6x$$

Hence, $$f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$$

64. When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$ so that $|\cos x| = -\cos$,

i.e., $f(x) = -\cos x,\ f'(x) = \sin x$

Hence, $$f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

65. When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so that $\cos x - \sin x > 0$ *i.e.*,

$$f(x) = \cos x - \sin x$$

$$f'(x) = -\sin x - \cos x$$

Hence, $$f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1+\sqrt{3})$$

66. $\because y = 1 + x + x^2 + \ldots \infty$

$$\therefore \quad y = \frac{1}{1-x} = (1-x)^{-1}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$\therefore$ $$\frac{dy}{dx} - y = \frac{1}{(1-x)^2} - \frac{1}{(1-x)} = \frac{1-1+x}{(1-x)^2} = \frac{x}{(1-x)^2}$$

$\Rightarrow$ $$\frac{dy}{dx} - y = xy^2$$

$\Rightarrow$ $$\frac{dy}{dx} = xy^2 + y$$

67. $\because$ $x = 2t - |t| = \begin{cases} t, & t \geq 0 \\ 3t, & t < 0 \end{cases}$,

$\therefore$ $$t = \begin{cases} x, & x \geq 0 \\ x/3, & x < 0 \end{cases}$$

$\therefore$ $$y = t^2 + t|t| = \begin{cases} 2t^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$= \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Hence, $$y'(x) = \begin{cases} 4x, & x > 0 \\ 0, & x < 0 \end{cases}$$

$\therefore$ We can't find $\frac{dx}{dt}$ as the derivative does not exist at $t = 0$.

68. Let $y = f(\sin x) \Rightarrow y = \log(\sin x)$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sin x}\cos x = \cot x$$

69. $$\frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right)\frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$$

$$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]\left[\frac{(3-2x)(2)-(2x+3)(-2)}{(3-2x)^2}\right]$$

$$= \frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{x=1} = \frac{12}{(3-2)^2}\sin\log(5) = 12\sin\log 5$$

70. $f\{f(x)\} = f(|x-2|)$

$= ||x-2|-2| = x-4$ $\quad (\because x > 20)$

$\Rightarrow$ $g(x) = x - 4$

$\therefore$ $g'(x) = 1$

71. Let $f(x) = |x-1| + |x-3|$

$$f(x) = \begin{cases} -(x-1)-(x-3), & x < 1 \\ (x-1)-(x-3), & 1 \leq x < 3 \\ (x-1)+(x-3), & x \geq 3 \end{cases}$$

$$f(x) = \begin{cases} 4-2x, & x < 1 \\ 2, & 1 \leq x < 3 \\ 2x-4, & x \geq 3 \end{cases}$$

At $x = 2$,

$$f(x) = 2 \Rightarrow f'(x) = 0$$

72. $\because y = \frac{a + bx^{3/2}}{x^{5/4}}$

On differentiating w.r.t. x, we get

$$y' = \frac{\frac{3}{2}bx^{7/4} - \frac{5}{4}(a + bx^{3/2})x^{1/4}}{(x^{5/4})^2}$$

$\because$ $y' = 0$ at $x = 5$

$\therefore$ $\frac{3}{2}bx^{7/4} - \frac{5}{4}(a + bx^{3/2})x^{1/4} = 0$, at $x = 5$

$\Rightarrow$ $6bx^{3/2} - 5(a + bx^{3/2}) = 0$, at $x = 5$

$\Rightarrow$ $bx^{3/2} = 5a$, at $x = 5$

$\Rightarrow$ $b(5)^{3/2} = 5a$

$\Rightarrow$ $$\frac{a}{b} = \frac{5^{3/2}}{5}$$

$\Rightarrow$ $a : b = \sqrt{5} : 1$

73. Given, $f(x) = 3|2 + x|$

$$f(x) = \begin{cases} 3(2+x), & x \geq -2 \\ -3(2+x), & x \leq -2 \end{cases}$$

On differentiating w.r.t. x, we get

$$f'(x) = \begin{cases} 3, & x \geq 2 \\ -3, & x \leq -2 \end{cases}$$

at $x = -3$, $f'(3) = -3$

74. Let $f(x) = |x-1| + |x-5|$

$$f(x) = \begin{cases} -(x-1)-(x-5), & x < 1 \\ (x-1)-(x-5), & 1 \leq x \leq 5 \\ x-1+x-5, & x > 5 \end{cases}$$

$$f(x) = \begin{cases} 6-2x, & x < 1 \\ 4, & x \leq 5 \\ 2x-6, & x > 5 \end{cases}$$

$\because$ $x = 3 \in (1, 5)$

$\therefore$ For $x = 3$, $f(x) = 4 \Rightarrow f'(x) = 0$

75. We know that, $\cos A \cos 2A \cos 2^2 A \ldots \cos 2^{n-1} A$

$$= \frac{\sin(2^n A)}{2^n \sin A}$$

$\therefore$ $\cos x \cos 2x \cos 4x \cos 8x \cos 16x = \frac{\sin 32x}{32 \sin x}$

$\Rightarrow$ $$f(x) = \frac{1}{32} \cdot \frac{\sin 32x}{\sin x}$$

$\therefore$ $$f'(x) = \frac{1}{32} \times \frac{\sin x(32\cos 32x) - \sin 32x \cos x}{\sin^2 x}$$

$\Rightarrow$ $$f'\left(\frac{\pi}{4}\right) = \frac{1}{\sin\frac{\pi}{4}} = \sqrt{2}$$

76. We have, $\phi(x) = f^{-1}(x) \Rightarrow x = f[\phi(x)]$

On differentiating both sides w.r.t. x, we get

$$1 = f'[\phi(x)] \cdot \phi'(x)$$

$$\Rightarrow \quad \phi'(x) = \frac{1}{f'[\phi(x)]} \quad \text{...(i)}$$

Since, $f'(x) = \dfrac{1}{1+x^5}$ (given)

$$\Rightarrow \quad f'[\phi(x)] = \frac{1}{1+[\phi(x)]^5}$$

From Eq. (i),

$$\phi'(x) = \frac{1}{f'[\phi(x)]} = 1 + [\phi(x)]^5$$

77. Given, $u = x^2 + y^2,\ x = s + 3t, y = 2s - t$

Now, $\dfrac{dx}{ds} = 1,\ \dfrac{dy}{ds} = 2$ ...(i)

$$\frac{d^2x}{ds^2} = 0,\ \frac{d^2y}{ds^2} = 0 \quad \text{...(ii)}$$

Now, $u = x^2 + y^2$

$$\frac{du}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds}$$

$$\frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right)^2 + 2x\frac{d^2x}{ds^2} + 2\left(\frac{dy}{ds}\right)^2 + 2y\left(\frac{d^2y}{ds^2}\right)$$

$$\Rightarrow \quad \frac{d^2u}{ds^2} = 2(1)^2 + 2x(0) + 2(2)^2 + 2y(0) = 2 + 8 = 10$$

78. Polynomial $P(x)$, satisfying the given relation can be taken as x

i.e., $P(x) = x$

$\therefore \quad P'(x) = 1 \Rightarrow P'(0) = 1$

79. Given that, $x = e^t \sin t,\ \ y = e^t \cos t$ ...(i)

At point $(1, 1), 1 = e^t \sin t, 1 = e^t \cos t$

$$\tan t = 1 \Rightarrow t = \frac{\pi}{4}$$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dt} = e^t(\cos t - \sin t)$$

and $\dfrac{dx}{dt} = e^t(\sin t + \cos t)$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\cos t - \sin t}{\cos t + \sin t}\right)\frac{dt}{dx}$$

$$= \left[\frac{\{(\cos t + \sin t)(-\sin t - \cos t) - (\cos t - \sin t)(-\sin t + \cos t)\}}{(\cos t + \sin t)^2}\right]\frac{dt}{dx}$$

$$= \frac{-2}{(\cos t + \sin t)^2} \cdot \frac{1}{e^t(\sin t + \cos t)}$$

$$= \frac{-2}{(e^t \cos t + e^t \sin t)} \cdot \frac{1}{(\cos t + \sin t)^2}$$

$$= \frac{-2}{x+y} \cdot \frac{1}{(\cos t + \sin t)^2} \quad \text{[from Eq. (i)]}$$

At $t = \dfrac{\pi}{4}, x = 1, y = 1$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-2}{1+1} \cdot \frac{1}{\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)^2} = \frac{-1}{\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]} = -\frac{1}{2}$$

80. Given, $y \cos x + x \cos y = \pi$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}\cos x + y(-\sin x) + x(-\sin y)\frac{dy}{dx} + \cos y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \sin x - \cos y}{\cos x - x \sin y}$$

Again, differentiating both sides *w.r.t.* x, we get

$$\frac{d^2y}{dx^2} = \frac{\left[\begin{array}{l}(\cos x - x\sin y)\left(y\cos x + \sin x\frac{dy}{dx} + \sin y\frac{dy}{dx}\right) \\ -(y\sin x - \cos y)\left(-\sin x - \sin y - x\cos y\frac{dy}{dx}\right)\end{array}\right]}{(\cos x - x\sin y)^2}$$

At $x = 0$,

$$f''(0) = \frac{1(y + \sin y) - (-1)(-\sin y)}{(1-0)^2} = y$$

As $y \cos x + x \cos y = \pi$

$\therefore$ At $x = 0 \Rightarrow y = \pi$

Hence, $f''(0) = \pi$

81. $\because \quad f(x) = a + bx$

$$f\{f(x)\} = a + b(a + bx)$$
$$= ab + a + b^2x = a(1 + b) + b^2x$$

$$f[f\{f(x)\}] = f\{a(1 + b) + b^2x\}$$
$$= a + b\{a(1 + b) + b^2x\}$$
$$= a(1 + b + b^2) + b^3x$$

$$\therefore \quad f^r(x) = a(1 + b + b^2 + \ldots + b^{r-1}) + b^r x$$
$$= a\left(\frac{b^r - 1}{b - 1}\right) + b^r x$$

82. Let $f(x) = x^2$

On differentiating w.r.t. x, we get

$$f'(x) = 2x$$

Given that, $f'(a + b) = f'(a) + f'(b)$

$\Rightarrow \quad 2(a + b) = 2a + 2b$

$\Rightarrow \quad 2a + 2b = 2a + 2b$

83. Given, $f(x)$ is a polynomial of degree 3.

$\therefore \quad f(x) = x^3 + ax^2 + bx + c$

But $f(x) = x^3 + x^2 + f'(1) + xf''(2) + f'''(3)$

$\therefore \quad a = f'(1), \quad b = f''(2), \quad c = f'''(3)$

Now, $\quad f'(x) = 3x^2 + 2ax + b$

$f''(x) = 6x + 2a$

$f'''(x) = 6$

$\therefore \quad a = f'(1) = 3 + 2a + b$

$\Rightarrow \quad -a - b = 3 \quad$ ...(i)

$b = 12 + 2a$

$\Rightarrow \quad -2a + b = 12 \quad$ ...(ii)

and $\quad c = 6 \quad$ ...(iii)

On solving Eqs. (i), (ii) and (iii), we get

$a = -5, \quad b = 2 \quad$ and $\quad c = 6$

$\therefore \quad f(x) = x^3 - 5x^2 + 2x + 6$

Thus, $f(0) = 6, \quad f(1) = 4, \quad f(2) = -2, \quad f(3) = -6$

84. $y = \cos^{-1}\sqrt{\dfrac{\cos 3x}{\cos^3 x}} \Rightarrow \cos y = \sqrt{\dfrac{\cos 3y}{\cos^3 x}}$

$$\Rightarrow \quad \cos^2 y = \frac{4\cos^3 x - 3\cos x}{\cos^3 x}$$

$$= 4 - 3\sec^2 x = 1 - 3\tan^2 x \quad \text{...(i)}$$

$\Rightarrow \quad \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$

On differentiating w.r.t. x both sides, we get

$$\cos y \cdot \frac{dy}{dx} = \sqrt{3} \cdot \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{3}\sec^2 x}{\cos y} = \sqrt{\frac{3}{\cos^2 y \cos^4 x}}$$

$$= \sqrt{\frac{3}{(1 - 3\tan^2 x)\cos^4 x}} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{3}{\cos^2 x(\cos^2 x - 3\sin^2 x)}}$$

$$= \sqrt{\frac{3}{\cos^2 x(4\cos^2 x - 3)}}$$

$$= \sqrt{\frac{3}{\cos x(4\cos^3 x - 3\cos x)}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cos 3x}} = \sqrt{\frac{6}{\cos 4x + \cos 2x}}$$

85. $$x = \tan\frac{y}{2} - \ln\left[\frac{\left(1 + \tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right] \quad \text{...(i)}$$

Let $\quad \tan\dfrac{y}{2} = t \Rightarrow \dfrac{y}{2} = \tan^{-1} t$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2}{1 + t^2} = 1 + \cos y \quad \text{...(ii)}$$

From Eq. (i),

$$x = t - \ln\left[\frac{(1+t)^2}{t}\right]$$

$$\frac{dx}{dt} = 1 - \frac{t}{(1+t)^2}\left(-\frac{1}{t^2} + 1\right) = \frac{t^2 + 1}{t^2 + t}$$

$$\frac{dx}{dt} = \frac{1 + \tan^2\frac{t}{2}}{\tan^2\frac{t}{2} + \tan\frac{t}{2}} = \frac{2}{1 - \cos y + \sin y} \quad \text{...(iii)}$$

On dividing Eq. (ii) by Eq. (iii), we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 + \cos y)(1 - \cos y + \sin y)}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\sin y\,(1 + \sin y + \cos y)$$

Round II

1. $x = \dfrac{2t}{1 + t^2}, y = \dfrac{1 - t^2}{1 + t^2}$

Put $t = \tan\theta$

$$\therefore \quad x = \frac{2\tan\theta}{1 + \tan^2\theta} = \sin 2\theta$$

$$y = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{2\cos 2\theta} = -\tan 2\theta$$

$$= \frac{-2\tan\theta}{1 - \tan^2\theta} = \frac{-2t}{1 - t^2} = \frac{2t}{t^2 - 1}$$

2. $$y = \sec^{-1}\left[\frac{\sqrt{x} + 1}{\sqrt{x} - 1}\right] + \sin^{-1}\left[\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right]$$

$$= \cos^{-1}\left[\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right] + \sin^{-1}\left[\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 \qquad \left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$$

3. Since, $f'(x) > g'(x)$

$$\Rightarrow \quad \left(\frac{1}{2}\right)5^{2x+1}\log 5 \times 2 > 5^x \log_e 5 + 4\log_e 5$$

$$\Rightarrow \quad 5^{2x} \cdot 5 > 5^x + 4$$

$\Rightarrow \quad 5\cdot 5^{2x} - 5^x - 4 > 0$

$\Rightarrow \quad (5^x - 1)\ (5\cdot 5^x + 4) > 0$

$\therefore \quad 5^x > 1 \Rightarrow x > 0$

4. $y = \dfrac{a^{\cos^{-1}x}}{1 + a^{\cos^{-1}x}}$, $z = a^{\cos^{-1}x} \Rightarrow y = \dfrac{z}{1+z}$

$$\Rightarrow \quad \frac{dy}{dz} = \frac{(1+z)1 - z(1)}{(1+z)^2} = \frac{1}{(1+z)^2} = \frac{1}{(1 + a^{\cos^{-1}x})^2}$$

5. Since, $f(x)$ is odd.

$\therefore \quad f(-x) = -f(x)$

$\Rightarrow \quad f'(-x)(-1) = -f'(x)$

$\Rightarrow \quad f'(-x) = f'(x)$

$\therefore \quad f'(-3) = f'(3) = -2$

6. We have,

$$\frac{dx}{dt} = a\left[-\sin t\sqrt{\cos 2t} - \frac{\cos t\cdot \sin 2t}{\sqrt{\cos 2t}}\right] = \frac{-a\sin 3t}{\sqrt{\cos 2t}}$$

and $\dfrac{dy}{dt} = a\left[\cos t\sqrt{\cos 2t} - \dfrac{\sin t\cdot \sin 2t}{\sqrt{\cos 2t}}\right] = \dfrac{a\cos 3t}{\sqrt{\cos 2t}}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\cot 3t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\operatorname{cosec}^2 3t\cdot\frac{dt}{dx}$$

$$= \frac{-3\operatorname{cosec}^2 3t\cdot\sqrt{\cos 2t}}{a\sin 3t}$$

$$= -\left(\frac{3}{a}\right)\operatorname{cosec}^3 3t\cdot\sqrt{\cos 2t}$$

$$\therefore \quad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}$$

$$= (1 + \cot^2 3t)^{3/2} \Big/ \left(\frac{-3}{a}\right)\operatorname{cosec}^3 3t\sqrt{\cos 2t}$$

$$\Rightarrow \quad \left|\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}\right| \text{ at } t = \frac{\pi}{6} \text{ is } \frac{a}{3\sqrt{\cos\frac{\pi}{3}}} = \frac{\sqrt{2}a}{3}$$

7. Given, $y = \log x\cdot e^{(\tan x + x^2)}$

$$\therefore \quad \frac{dy}{dx} = e^{(\tan x + x^2)}\cdot\frac{1}{x} + \log x\cdot e^{(\tan x + x^2)}(\sec^2 x + 2x)$$

$$= e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$$

8. When $x = 0$, $y > 0 \Rightarrow y = ae^{\pi/2}$

Taking log on both sides of the given equation, we get

$$\frac{1}{2}\log(x^2 + y^2) = \log a + \tan^{-1}\left(\frac{y}{x}\right)$$

On differentiating both the sides w.r.t. x, we get

$$\frac{1}{2}\times\frac{2x + 2yy'}{x^2 + y^2} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\times\frac{xy' - y}{x^2}$$

$$\Rightarrow \quad x + yy' = xy' - y \qquad \ldots(i)$$

On differentiating again w.r.t. x, we get

$1 + (y')^2 + yy'' = xy'' + y' - y'$

$$\Rightarrow \quad 1 + (y')^2 = (x - y)y'' \quad \Rightarrow \quad y'' = \frac{1 + (y')^2}{x - y}$$

when $x = 0$, we get from Eq. (i), $y' = -1$

$$\Rightarrow \quad y''(0) = \frac{2}{-ae^{\pi/2}} = \frac{-2}{a}e^{-\pi/2}$$

9. We have, $y = \log_{\sin x}(\tan x) = \dfrac{\log\tan x}{\log\sin x}$

$$\therefore \quad \frac{dy}{dx} = \frac{\log\sin x\cdot\frac{1}{\tan x}\cdot\sec^2 x - \log\tan x\cdot\frac{1}{\sin x}\cos x}{(\log\sin x)^2}$$

At $x = \dfrac{\pi}{4}$, $\left(\dfrac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \dfrac{\log\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})^2}{\left(\log\frac{1}{\sqrt{2}}\right)^2}$

$$= \frac{(\sqrt{2})^2}{\log\left(\frac{1}{\sqrt{2}}\right)} = -\frac{2\times 2}{\log 2} = -\frac{4}{\log 2}$$

10. $\because \quad y = x\ln\left(\dfrac{x}{a + bx}\right) = x(\ln x - \ln(a + bx))$

or $\quad \left(\dfrac{y}{x}\right) = \ln x - \ln(a + bx)$

On differentiating both sides w.r.t. x, we get

$$\left(\frac{x\frac{dy}{dx} - y\cdot 1}{x^2}\right) = \frac{1}{x} - \frac{b}{a + bx} = \frac{a}{x(a + bx)} \qquad \ldots(i)$$

or $\quad \left(x\dfrac{dy}{dx} - y\right) = \left(\dfrac{ax}{a + bx}\right)$

On taking log on both sides, we get

$$\ln\left(x\frac{dy}{dx} - y\right) = \ln(ax) - \ln(a + bx)$$

On differentiating both sides w.r.t. x, we get

$$\frac{x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx}}{\left(x\frac{dy}{dx} - y\right)} = \frac{1}{x} - \frac{b}{a + bx} = \frac{a}{x(a + bx)}$$

$$= \frac{\left(x\frac{dy}{dx} - y\right)}{x^2} \qquad \text{[from Eq. (i)]}$$

or $\quad x^3\dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$

11. $\because \quad y = \left(\dfrac{ax + b}{cx + d}\right)$ or $cxy + dy = ax + b$

On differentiating both sides w.r.t. x, we get

$$c\left\{x\frac{dy}{dx} + y\cdot 1\right\} + d\frac{dy}{dx} = a$$

or $\quad x\dfrac{dy}{dx} + y + \left(\dfrac{d}{c}\right)\dfrac{dy}{dx} = \left(\dfrac{a}{c}\right)$

Again, differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}+\left(\frac{d}{c}\right)\frac{d^2y}{dx^2}=0$$

or
$$x+\frac{2\frac{dy}{dx}}{\left(\frac{d^2y}{dx^2}\right)}+\frac{d}{c}=0$$

Again, differentiating both sides w.r.t. x, we get

$$1+\frac{\left(\frac{d^2y}{dx^2}\cdot 2\frac{d^2y}{dx^2}-2\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}\right)}{\left(\frac{d^2y}{dx^2}\right)}+0=0$$

$$\Rightarrow \quad 2\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}=3\left(\frac{d^2y}{dx^2}\right)^2$$

12. $\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1}$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\left(\frac{dx}{dy}\right)^{-2}\left\{\frac{d}{dx}\left(\frac{dx}{dy}\right)\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\left(\frac{dx}{dy}\right)^{-2}\left\{\frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{dx}\right\}$$

$$=-\left(\frac{dy}{dx}\right)^2\left\{\frac{d^2x}{dy^2}\cdot\frac{dy}{dx}\right\}=-\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2}=0$$

13. Since, $y=(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$

$$\Rightarrow \quad (1-x)\,y=(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$$

$$=(1-x^4)(1+x^4)\ldots(1+x^{2^n})$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$=(1-x^{2^n})(1+x^{2^n})=1-x^{2^{n+1}}$$

$$\therefore \quad y=\frac{1-x^{2^{n+1}}}{(1-x)}$$

$$\frac{dy}{dx}=\frac{(1-x)(-2^{n+1})\cdot x^{2^{n+1}-1}-(1-x^{2^{n+1}})(-1)}{(1-x)^2}$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=0}=\frac{(1-0)(-2^{n+1}\cdot 0)-(1-0)(-1)}{1}=1$$

14. $y^2=P(x) \quad \Rightarrow \quad 2yy'=P'(x)$ ...(i)

$$\Rightarrow \quad (2y)\,y''+y'(2y')=P''(x)$$

$$\Rightarrow \quad 2yy''=P''(x)-2(y')^2$$

$$\Rightarrow \quad 2y^3y''=y^2P''(x)-2(yy')^2$$

$$=y^2P''(x)-2\frac{\{P'(x)\}^2}{4}$$

[from Eq. (i)]

$$\Rightarrow \quad 2y^3\cdot y''=P(x)P''(x)-\frac{1}{2}\{P'(x)\}^2$$

$$\therefore \quad \frac{d}{dx}(2y^3\cdot y'')=P(x)P'''(x)+P''(x)P'(x)-P'(x)P''(x)$$

$$=P(x)\cdot P'''(x)$$

$$\Rightarrow \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)=P(x)P'''(x)$$

15. $5f(x)+3f\left(\frac{1}{x}\right)=x+2$ ...(i)

On replacing x by $\frac{1}{x}$, we get

$$5f\left(\frac{1}{x}\right)+3f(x)=\frac{1}{x}+2 \quad \text{...(ii)}$$

On multiplying Eq. (i) by 5 and Eq. (ii) by 3 and then on subtracting, we get

$$\therefore \quad 16f(x)=5x-\frac{3}{x}+4$$

$$\Rightarrow \quad xf(x)=\frac{5x^2-3+4x}{16}=y$$

$$\therefore \quad \frac{dy}{dx}=\frac{10x+4}{16}$$

$$\left[\frac{dy}{dx}\right]_{x=1}=\frac{10+4}{16}=\frac{7}{8}$$

16. Given, $y=|\sin x|^{|x|}$

In the neighbourhood of $-\frac{\pi}{6}$ $|x|$ and $|\sin x|$ both are negative

i.e., $\quad y=(-\sin x)^{(-x)}$

Taking log on both sides, we get

$$\log y=(-x)\cdot\log(-\sin x)$$

On differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx}=(-x)\left(\frac{1}{-\sin x}\right)\cdot(-\cos x)+\log(-\sin x)\cdot(-1)$$

$$=-x\cot x-\log(-\sin x)$$

$$=-[x\cot x+\log(-\sin x)]$$

$$\Rightarrow \quad \frac{dy}{dx}=-y\,[x\cot x+\log(-\sin x)]$$

$$=\frac{(2)^{-\frac{\pi}{6}}[6\log 2-\sqrt{3}\pi]}{6}$$

17. Given, $\frac{d}{dx}\{f'(x)\}=-f(x)$

$$\Rightarrow \quad g'(x)=-f(x) \quad [\because g(x)=f'(x) \text{ given}]$$

Also, given $\quad F(x)=\left\{f\left(\frac{x}{2}\right)\right\}^2+\left\{g\left(\frac{x}{2}\right)\right\}^2$

$$\Rightarrow \quad F'(x)=2\left\{f\left(\frac{x}{2}\right)\right\}\cdot f'\left(\frac{x}{2}\right)\cdot\frac{1}{2}+2\left\{g\left(\frac{x}{2}\right)\right\}\cdot g'\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$=0$$

Hence, $f(x)$ is constant. Therefore, $F(10)=5$

18. At $x = 0$, $\log(0 + y) = 0$

$\Rightarrow y = 1$

Given, $\log(x + y) = 2xy$

$$\Rightarrow \quad \frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = 2x\frac{dy}{dx} + 2y \cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y(x+y)-1}{1-2(x+y)x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

19. $y^2 = x + y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y-1}$

Also, $y = \dfrac{x}{y} + 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x+y}$$

Also, $y^2 - y - x = 0$

$$\Rightarrow \quad y = \frac{1 \pm \sqrt{1+4x}}{2}$$

$$\Rightarrow \quad y = \frac{1+\sqrt{1+4x}}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1+4x}} \qquad (\text{as } y > 0)$$

20. $f(x) = |x^2 - 3|x| + 2|$

$$= \begin{cases} |x^2 - 3x + 2|, & x \ge 0 \\ |x^2 + 3x + 2|, & x < 0 \end{cases}$$

$$= \begin{cases} x^2 - 3x + 2, & x^2 - 3x + 2 \ge 0, & x \ge 0 \\ -x^2 + 3x - 2, & x^2 - 3x + 2 < 0, & x \ge 0 \\ x^2 + 3x + 2, & x^2 + 3x + 2 \ge 0, & x < 0 \\ -x^2 - 3x - 2, & x^2 + 3x + 2 < 0, & x < 0 \end{cases}$$

$$= \begin{cases} x^2 - 3x + 2, & x \in [0,1] \cup (2, \infty) \\ -x^2 + 3x - 2, & x \in (1, 2) \\ x^2 + 3x + 2, & x \in (-\infty, -2] \cup [-1, 0) \\ -x^2 - 3x + 2, & x \in (-2, -1) \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 2x - 3, & x \in (0,1) \cup (2, \infty) \\ -2x + 3, & x \in (1, 2) \\ 2x + 3, & x \in (-\infty, -2) \cup (-1, 0) \\ -2x - 3, & x \in (-2, -1) \end{cases}$$

21. $f(x - y)$, $f(x)$, $f(y)$ and $f(x + y)$ are in AP.

$\Rightarrow \quad f(x + y) + f(x - y) + 2f(x)f(y)$ for all x, y

Putting $x = 0, y = 0$ in Eq. (i), we get

$$f(0) + f(0) = 2f(0)\, f(0)$$

$$\Rightarrow \quad f(0) = 1 \qquad [\because f(0) \ne 0]$$

Putting $x = 0, y = x$, we get

$$f(x) + f(-x) = 2f(0)\, f(x)$$

$$\Rightarrow \quad f(x) = f(-x) \qquad \ldots(\text{i})$$

$$\Rightarrow \quad f(4) = f(-4), f(3) = f(-3)$$

On differentiating Eq. (i) w.r.t. x, we get

$$f'(x) + f'(-x) = 0$$

$$\Rightarrow \quad f'(4) + f'(-4) = 0 \text{ and } f'(2) = f'(-2)$$

22. $u = \sin^3 x + \cos^3 x$

$$\Rightarrow \quad \frac{du}{dx} = 3 \sin x \cos x (\sin x - \cos x) \qquad \ldots(\text{i})$$

and $v = \sin x + \cos x$

$$\frac{dv}{dx} = \cos x - \sin x \qquad \ldots(\text{ii})$$

On dividing Eq. (i) by Eq. (ii), we get

$$\Rightarrow \quad \frac{du}{dv} = -3 \sin x \cos x$$

23. $4 = \tan^{-1} \dfrac{3x - x^3}{1 - 3x^2}$.

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$u = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta$$

$u = 3\tan^{-1} x$

$$\frac{du}{dx} = \frac{3}{1+x^2} \qquad \ldots(\text{i})$$

and $v = \tan^{-1}\dfrac{2x}{1-x^2}$ (put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$)

$$v = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$v = 2\tan^{-1} x$

$$\frac{dv}{dx} = \frac{2}{1+x^2} \qquad \ldots(\text{ii})$$

On dividing Eq. (i) by Eq. (ii), we get

$$\Rightarrow \quad \frac{du}{dv} = \frac{3}{2}$$

24. $u = \tan^{-1}\dfrac{2x}{1-x^2}$ (put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$)

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$$u = 2\tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{2}{1+x^2} \qquad \ldots(\text{i})$$

and $v = \sin^{-1}\dfrac{2x}{1+x^2}$ (put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$)

$$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$v = 2\tan^{-1} x$

$$\frac{dv}{dx} = \frac{2}{1+x^2} \qquad \ldots(\text{ii})$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{du}{dv} = 1$$

25. Given, $u = f(\tan x) \Rightarrow \dfrac{du}{dx} = f'(\tan x)\sec^2 x$

and $\quad v = g(\sec x)$

$\Rightarrow \quad \dfrac{dv}{dx} = g'(\sec x)\sec x \tan x$

$\therefore \quad \dfrac{du}{dv} = \dfrac{(du/dx)}{(dv/dx)} = \dfrac{f'(\tan x)}{g'(\sec x)} \cdot \dfrac{1}{\sin x}$

$\therefore \quad \left(\dfrac{du}{dv}\right)_{x=\pi/4} = \dfrac{f'(1)}{g'(2)} \cdot \sqrt{2}$

$= \dfrac{2}{4} \cdot \sqrt{2} = \dfrac{1}{\sqrt{2}}$

26. For $x < 0$, $\dfrac{d}{dx}(\ln|x|) = \dfrac{d}{dx}[\ln(-x)] = \dfrac{1}{(-x)}(-1) = \dfrac{1}{x}$

27. For $0 < x < 1$,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let $\quad u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$\therefore \quad \dfrac{du}{dv} = 1 \qquad (\because u = v)$

28. $\because e^{xy} + \log(xy) + \cos(xy) + 5 = 0$

Then, $e^{xy}\dfrac{d}{dx}(xy) + \dfrac{1}{(xy)}\dfrac{d}{dx}(xy) - \sin(xy)\dfrac{d}{dx}(xy) = 0$

$\Rightarrow \quad \dfrac{d}{dx}(xy)\left\{e^{xy} + \dfrac{1}{xy} - \sin(xy)\right\} = 0$

$\because \quad e^{xy} + \dfrac{1}{xy} - \sin(xy) \neq 0$

$\therefore \quad \dfrac{d}{dx}(xy) = 0$

$\Rightarrow \quad x\dfrac{dy}{dx} + y \cdot 1 = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

29. $f(x) = x[x] = \begin{cases} -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ \ldots & \ldots \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 2, & 2 < x < 3 \\ \ldots & \ldots \end{cases} \quad \Rightarrow \quad f'(x) = [x]$

30. Statement I is always true, but statement II is not always true, as if $f'(x) = \cos x$, then $f(x)$ can be $\sin x$ which is odd function, but if $f(x) = -\sin x + 2$, then $f(x)$ is neither odd nor even.

31. Since, $|f(x) - f(y)| \le |x - y|^3$, where $x \neq y$

$\therefore \quad \left|\dfrac{f(x) - f(y)}{x - y}\right| \le |x - y|^2$

Taking lim as $y \to x$, we get

$$\lim_{y \to x}\left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{y \to x}|x - y|^2$$

$\Rightarrow \quad \left|\lim_{y \to x}\dfrac{f(x) - f(y)}{x - y}\right| \le \left|\lim_{y \to x}(x - y)^2\right|$

$\Rightarrow \quad |f'(x)| \le 0$

$\Rightarrow \quad |f'(x)| = 0 \qquad (\because |f'(x)| \ge 0)$

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad f(x) = C$ (constant)

32. $\quad f(x) + f(x - 2) = 0 \qquad \ldots(i)$

Replace x by $x - 2 \Rightarrow f(x - 2) + f(x - 4) = 0 \qquad \ldots(ii)$

From Eqs. (i) and (ii),

$$f(x) - f(x - 4) = 0$$

Replace x by

$$x + 4 \Rightarrow f(x + 4) = f(x)$$

$\Rightarrow \quad f(x) = f(x + 4) = f(x + 8) = \ldots = f(x + 4000)$

$\Rightarrow \quad f'(x) = f'(x + 4000)$

Hence, both the statements are true and statement II is correct explanation of statement I.

Hence, $f(x)$ is periodic with period 4.

33. Statement II is true as $f(\alpha) = 0$ and $f'(\alpha) = 0$, then definitely α is repeated root of $f(x) = 0$.

But from data, we are not sure how many times a root repeats.

Also, $f(x) = (x - \alpha)^n \times g(x)$, which changes sign at $x = \alpha$, when n is odd and does not if n is even. Hence, statement I is false.

34. Given, $f(x + y^3) = f(x) + f(y^3), \quad \forall\, x, y \in R$

Put $x = y = 0$, we get $f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0$

Now, put $y = -x^{1/3}$, we get $f(0) = f(x) + f(-x)$

$\Rightarrow \quad f(x) + f(-x) = 0$

$\Rightarrow \quad f(x)$ is an odd function

$\Rightarrow \quad f'(x)$ is an even function

$\Rightarrow \quad f(-2) = a$

35. Given, $y = \sec(\tan^{-1} x)$

Let $\quad \tan^{-1} x = \theta$

$\Rightarrow \quad x = \tan\theta$

$\therefore \quad y = \sec\theta = \sqrt{1 + x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

At $x = 1$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

36. Since, $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-2} \frac{d^2y}{dx^2} \cdot \frac{dx}{dy} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

37. $x^{2x} - 2x^x \cot y - 1 = 0$ ...(i)

Now, $x = 1,\ 1 - 2\cot y - 1 = 0$

$\Rightarrow \quad \cot y = 0 \quad \Rightarrow \quad y = \frac{\pi}{2}$

On differentiating Eq. (i) w.r.t. x, we get

$$2x^{2x}(1 + \log x) - 2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx} + \cot y\, x^x (1 + \log x)\right] = 0$$

At $\left(1, \frac{\pi}{2}\right)$,

$$2(1 + \log 1) - 2\left\{1(-1)\left(\frac{dy}{dx}\right)_{\left(1, \frac{\pi}{2}\right)} + 0\right\} = 0$$

$$\Rightarrow \quad 2 + 2\left(\frac{dy}{dx}\right)_{\left(1, \frac{\pi}{2}\right)} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\left(1, \frac{\pi}{2}\right)} = -1$$

38. Taking log on the given equation, we get $m \log x + n \log y = (m + n) \log (x + y)$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{my + ny - nx - ny}{y(x+y)}\right\} = \frac{mx + my - mx - nx}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

39. $x = e^{y + e^{y + \dots \infty}}$

$\therefore \quad x = e^{y + x}$

Taking log on both sides, we get

$$\log x = (y + x)$$

On differentiating w.r.t. x, we get

$$\frac{1}{x} = \frac{dy}{dx} + 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1 - x}{x}$$

40. Since, $f(x) = x^n \quad \Rightarrow \quad f(1) = 1$

$$f'(x) = nx^{n-1} \quad \Rightarrow \quad f'(1) = n$$

$$f''(x) = n(n-1)\, x^{n-2} \quad \Rightarrow \quad f''(1) = n(n-1)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$f^n(x) = n(n-1)(n-2)\dots 2 \cdot 1$$

$$\Rightarrow \quad f^n(1) = n(n-1)(n-2)\dots 2 \cdot 1$$

We have,

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \dots + \frac{(-1)^n f^n(1)}{n!}$$

$$= 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \dots + \frac{(-1)^n n(n-1)(n-2)\dots 2 \cdot 1}{n!}$$

$$= (1 - 1)^n = 0$$

41. Since, $x^y = e^{x - y}$

Taking log on both sides, we get

$$y \log x = (x - y) \log_e e$$

$$\Rightarrow \quad y = \frac{x}{1 + \log x}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + \log x) - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

42. Given, $x = \frac{\sin y}{\sin (a + y)}$

On differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\cos y \sin (a + y) - \sin y \cos (a + y)}{\sin^2 (a + y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2 (a + y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2 (a + y)}{\sin a}$$

43. $\frac{d}{dx}(y) = n(x + \sqrt{1 + x^2})^{n-1}\left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$

$$\Rightarrow \quad (\sqrt{1 + x^2})\frac{dy}{dx} = n(x + \sqrt{1 + x^2})^n$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}(\sqrt{1 + x^2}) + \frac{dy}{dx}\left(\frac{x}{\sqrt{1 + x^2}}\right) = n^2(x + \sqrt{1 + x^2})^{n-1}\left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$\Rightarrow \quad \frac{\frac{d^2y}{dx^2}(1 + x^2) + \frac{dy}{dx} \cdot x}{\sqrt{1 + x^2}} = \frac{n^2(x + \sqrt{1 + x^2})^n}{\sqrt{1 + x^2}}$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2(x + \sqrt{1 + x^2})^n$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y$$

28 Application of Derivatives

JEE Main MILESTONE

- Errors and its Approximations
- Tangents and Normals
- Angle of Intersection of Two Curves
- Decreasing Function
- Monotonic Function
- Mean Value Theorem

28.1 Errors and its Approximations

Let $y = f(x)$. Then, $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$

$\therefore \quad \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) + \in$, where $\in \to 0$ when $\delta x \to 0$

$\Rightarrow \quad f(x+\delta x) - f(x) = f'(x) \cdot \delta x + \in \cdot \delta x$

$\Rightarrow \quad f(x+\delta x) - f(x) = f'(x) \cdot \delta x$ (approximately)

$\Rightarrow \quad \delta y = f'(x) \cdot \delta x$ approximately $\quad [\because f(x+\delta x) - f(x) = \delta y]$

Thus, if δx is an error in x, then the corresponding error in y is δy. These small values δx and δy are called diffferentials.

(i) **Absolute Error** δx is called an absolute error in x.

(ii) **Relative Error** $\frac{\delta x}{x}$ is called the relative error.

(iii) **Percentage Error** $\left(\frac{\delta x}{x} \times 100\right)$ is called the percentage error.

In this chapter, we will study applications of the derivative in various disciplines. For instance we will learn how the derivative can be used (i) *to determine rate of change of quantities,* (ii) *to find the equations of tangent and normal to a curve at a point* (iii) *to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing.*

***Sample* Problem 1** *If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is* [**NCERT**]

(*a*) 47.66 (*b*) 57.66 (*c*) 67.66 (*d*) 77.66

Interpret (*d*) Consider $f(x) = 3x^2 + 15x + 5 \Rightarrow f'(x) = 6x + 15$

Let $x = 3$ and $\Delta x = 0.02$

Also, $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$.

$\Rightarrow \quad f(x + \Delta x) \simeq (3x^2 + 15x + 5) + (6x + 15)\Delta x$

$\Rightarrow \quad f(3.02) \simeq 3 \times 3^2 + 15 \times 3 + 5 + (6 \times 3 + 15)(0.02)$

(as $x = 3$, $\Delta x = 0.02$)

$= 27 + 45 + 5 + (18 + 15)(0.02)$

$= 77 + 33(0.02) = 77 + 0.66 \Rightarrow f(3.02) \simeq 77.66$

Hence, the approximate value of $f(3.02)$ is 77.66.

***Sample* Problem 2** *The approximate change in the volume of a cube of side x m caused by increasing the side by 3%.* **[NCERT]**

(a) $0.06x^3\ m^3$
(b) $0.6x^3\ m^3$
(c) $0.09x^3\ m^3$
(d) $0.9x^3\ m^3$

Interpret (c) We know that the volume V of a cube of side x is given by $V = x^3$

$\Rightarrow \quad \dfrac{dV}{dx} = 3x^2$

Let Δx be change in side = 3% of $x = 0.03x$

Now, change in volume, $\Delta V = \left(\dfrac{dV}{dx}\right)\Delta x = (3x^2)\Delta x = (3x^2)(0.03x)$

(as Δx = 3% of x is $0.03x$)

$= 0.09x^3\ m^3$

Hence, the approximate change in the volume of the cube is $0.09x^3\ m^3$.

Hot Spot 1

Derivative as The Rate of Change

Derivative as the rate of change is an important topic for JEE (Main) examination. Generally, questions are seen from this topic. The level of question is from moderate to typical. But a small practice can help you in solving such problems very quickly. While solving problems based on derivative as the rate of change, always keep in mind the formulae of area, surface and volume of 2D and 3D figures.

While we solving such types of problem, we follow the steps given below.

Step I First, we read the given question very carefully and observe that which formula of 2D or 3D figure use here.

Step II Now, differentiate the given formula according to given condition.

Step III Now, put the given values in step II.

Step IV Finally, get the desired rate of change.

If a variable quantity y is some function of time t *i.e.*, $y = f(t)$, then small change in time Δt have a corresponding change Δy in y. Thus, average rate of change $= \dfrac{\Delta y}{\Delta t}$.

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to t at the instant t.

i.e., $$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

Note The differential coefficient of y with respect to x *i.e.*, $\dfrac{dy}{dx}$ is nothing but the rate of increase of y relative to x.

Interpretation of $\dfrac{dy}{dx}$ as a rate measurer Let x be a variable which changes with time t. Since, x changes with time, x is a function of t. Let the value of x at time $t = x$.

and the value of x at time $t + \delta t = x + \delta x$

Change in x in time $\delta t = x + \delta x - x = \delta x$

$\therefore$ Average rate of change of x in time $\delta t = \dfrac{\delta x}{\delta t}$.

As $\delta t \to 0$, $\dfrac{\delta x}{\delta t}$ gives the rate of change of x at time t.

$\therefore$ Rate of change of x at time $t = \lim\limits_{\delta \to 0} \dfrac{\delta x}{\delta t} = \dfrac{dx}{dt}$.

Thus, $\dfrac{dx}{dt}$ = rate of change of x at time t.

Similarly, if y is a function of t, then $\dfrac{dy}{dt}$

= rate of change of y at time t.

Now, let $y = f(x)$ be a function of x and x is a function of t, then y is also a function of t.

Let δt be the change in the value of t and corresponding change in the values of x and y be δx and δy, respectively. When $\delta t \to 0$, $\delta x \to 0$ and $\delta y \to 0$

Now, $$\frac{\delta y}{\delta x} = \frac{\dfrac{\delta y}{\delta t}}{\dfrac{\delta x}{\delta t}},$$

$$\therefore \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta x} = \lim_{\delta t \to 0} \frac{\dfrac{\delta y}{\delta t}}{\dfrac{\delta x}{\delta t}}$$

$$\because \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta t \to 0} \dfrac{\delta y}{\delta t}}{\lim\limits_{\delta t \to 0} \dfrac{\delta y}{\delta t}} \qquad (\because \text{ when } \delta t \to 0, \delta x \to 0)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{Rate of change of } y}{\text{Rate of change } x}$$

Thus, differential coefficient of y with respect to x is equal to the ratio of the rate of change of y and rate of change of x.

Note

- Marginal cost is the rate of change of total cost with respect to the output.
- Marginal revenue is the rate of change of total revenue with the respect to the number unit sold.

***Sample* Problem 3** *A balloon which always remains spherical on inflation, is being inflated by pumping in 900 cu cm of gas per second. The rate at which the radius of the balloon increases when the radius is 15 cm, is* [**NCERT**]

(a) $\frac{2}{\pi}$ *cm/s* (b) $\frac{1}{\pi}$ *cm/s*

(c) $\frac{2}{\pi^2}$ *cm/s* (d) $\frac{1}{2\pi}$ *cm/s*

Interpret (*b*) At any instant of time t let the radius of the balloon be r and its volume be V, then

Volume of balloon $V = (4/3)\,\pi\, r^3$

The balloon is being inflated at 900 cu cm/s *i.e.*, the rate of change of volume with respect to time is 900 cm^3/s.

On differentiating w.r.t. t, we get

Rate of change of volume $\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right)$

Given, $r = 15$ cm

$$\Rightarrow \quad 900 = \left(\frac{4}{3}\pi\right)\left\{3(15)^2\frac{dr}{dt}\right\}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{3\times(15)^2}\times\frac{3}{4\pi}$$

Rate of change of radius r,

$$\frac{dr}{dt} = \frac{900}{4\pi\times(15)^2} = \frac{225}{\pi\times 225} = \frac{1}{\pi}\text{ cm/s}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm, is $\frac{1}{\pi}$ cm/s.

***Sample* Problem 4** *A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?* [**NCERT**]

(a) $-\frac{8}{3}$ *m/s* (b) $\frac{8}{3}$ *m/s*

(c) $\frac{4}{3}$ *m/s* (d) $-\frac{2}{3}$ *m/s*

Interpret (*a*) The ladder when leaning against the wall form a right angled triangle with length of ladder = hypotenuse of the triangle and base and altitude equal to the distance of foot from wall and distance of top from ground. Use the Pythagoras theorem, to make an equation and then differentiate it.

Let $AB = 5$ m be the ladder and y be the height of the wall at which the ladder touches. Also, let the foot of the ladder be at B whose distance from the wall is x.

Given that the bottom of ladder is pulled along the ground at 2 cm/s, so $\frac{dx}{dt} = 2$ m/s.

As we know that ΔABC is right angled, so by Pythagoras theorem, we have

$$x^2 + y^2 = 5^2 \quad \ldots(i)$$

when $x = 4$, then

$$y^2 = 5^2 - 4^2 \Rightarrow y = \sqrt{25-16}$$

$$\Rightarrow \quad y = 3 \text{ m}$$

On differentiating Eq. (i) w.r.t. time t on both sides, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 4\times 2 + 3\times\frac{dy}{dt} = 0 \qquad (\because x = 4 \text{ and } \frac{dx}{dt} = 2)$$

$\Rightarrow$ The rate of fall of height on the wall $\frac{dy}{dt} = -\frac{8}{3}$ m/s

(negative sign shows that height of ladder on the wall is decreasing at the rate of $\frac{8}{3}$ m/s.)

Motion in a Straight Line

If x and v denotes the displacement and velocity of a particle at any instant t, then velocity is given by

$$v = \frac{dx}{dt}$$

and
$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d^2x}{dt^2}$$

where, a is acceleration of particle. If the sign of acceleration is opposite to that of velocity, then the acceleration is called retardation which means decrease in magnitude of the velocity.

***Sample* Problem 5** *If $s = \frac{1}{2}t^3 - 6t$, then acceleration at the time when the velocity vanishes, is*

(a) 3 *units/s*^{2} (b) 6 *units/s*^{2}
(c) 2 *units/s*^{2} (d) *None of these*

Interpret (b) Given, $s = \frac{1}{2}t^3 - 6t$

$\therefore$ $$v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right) \quad \text{...(i)}$$

and $$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t \quad \text{...(ii)}$$

when $v = 0$

$\Rightarrow$ $$\frac{3t^2}{2} - 6 = 0$$

$\Rightarrow$ $$t^2 = 4 \Rightarrow t = 2$$

$\therefore$ Acceleration when velocity vanishes
$$= 3 \times 2 = 6 \text{ units/s}^2$$

28.2 Tangents and Normals

Tangent (Definition) The tangent to a curve at a point P on it is defined as the limiting position of the secant PQ as the point Q approaches the point P, provided such a limiting position exists.

The dynamic approach to tangency. The tangent to the curve at P is line through P whose slope is the limit of the secant as $Q \to P$ from either side, such the points P (fixed) and Q (moving) coincide.

Slope of Tangent

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be the point on it.

Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of tangent to the curve $y = f(x)$ at a point P.

i.e., $$\left(\frac{dy}{dx}\right)_P = \tan\theta$$
$$= \text{Slope of tangent at } P$$

where, θ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of x-axis as shown in the figure.

Note If tangent is parallel to x-axis, then

$$\theta = 0°$$

$\Rightarrow$ $$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

If tangent is perpendicular to x-axis (or parallel to y-axis), then

$$\theta = 90° \Rightarrow \tan\theta = \infty \text{ or } \cot\theta = 0$$

$\Rightarrow$ $$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

***Sample* Problem 6** *The slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$ is* **[NCERT]**

(a) $\frac{1}{64}$ (b) $-\frac{1}{64}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{4}$

Interpret (b), Given curve is $y = \frac{x-1}{x-2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x-2)}{(x-2)^2} \quad \text{(using quotient rule)}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{(x-2)\times 1 - (x-1)\times 1}{(x-2)^2} = \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

$\therefore$ Slope of tangent at $x = 10$ is

$$\left(\frac{dy}{dx}\right)_{(x=10)} = \frac{-1}{(10-2)^2} = -\frac{1}{8^2} = -\frac{1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $-\frac{1}{64}$.

***Sample* Problem 7** *The points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis, are* **[NCERT]**

(a) $(3, -20)$ *and* $(-1, 12)$ (b) $(3, 20)$ *and* $(1, 12)$
(c) $(1, -10)$ *and* $(2, 6)$ (d) *None of these*

Interpret (*a*) Tangent to the curve is parallel to the *x*-axis when slope of the tangent is 0.

The equation of the curve is $y = x^3 - 3x^2 - 9x + 7$ ...(i)

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to *x*-axis , then slope of the tangent is zero or we can say $\frac{dy}{dx} = 0$.

$$\Rightarrow \quad 3x^2 - 6x - 9 = 0 \Rightarrow 3(x^2 - 2x - 3) = 0$$

$$\Rightarrow \quad (x-3)(x+1) = 0 \Rightarrow x = 3, -1$$

When $x = 3$, then from Eq.(i), we get

$$y = 3^3 - (3)\cdot(3)^2 - 9\cdot 3 + 7$$
$$= 27 - 27 - 27 + 7 = -20$$

When $x = -1$, then from Eq. (i), we get

$$y = (-1)^3 - 3(-1)^2 - 9(-1) + 7$$
$$= -1 - 3 + 9 + 7 = 12$$

Hence, the points at which the tangent is parallel to *x*-axis are $(3, -20)$ and $(-1, 12)$.

Hot Spot 2

Tangents Related Problems

These types of problems are generally asked in AIEEE (JEE Main), examination, directly or indirectly and a small practice can help you in solving such problems very quickly as the level of question is easy to average.

While solving problems based on tangents of any curve, always keep in mind the slope of tangents.

Let $P(x_1, y_1)$ be any point lying on the curve.

Let m is the slope of tangent to $y = f(x)$ at P, then

$$\Rightarrow \quad m = \left(\frac{dy}{dx}\right)_{P(x_1, y_1)}$$

$\therefore$ The equation of the tangent line through point $P(x_1, y_1)$ and slope m is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad (y - y_1) = \left\{\left(\frac{dy}{dx}\right)_{P(x_1, y_1)}\right\}(x - x_1)$$

Geometrical Interpretation of $\frac{dy}{dx}$

Since, $\frac{dy}{dx}$ represents the slope of the tangent (m) to the given curve $y = f(x)$ at any point $P(x_1, y_1)$ on the curve.

Also, from the principles of coordinate geometry, slope of any line (m) is $\tan\theta$, where θ is the angle made by the line with the positive direction of *x*-axis or in counter clockwise direction.

So, geometrically, $\frac{dy}{dx} = \tan\theta$

***Sample* Problem 8** *The equation of the tangent to the curve* $y = \sqrt{3x-2}$ *which is parallel to the line* $4x - 2y + 5 = 0$ *is* [**NCERT**]

(*a*) $24x - 48y = 23$
(*b*) $48y - 24x = 23$
(c) $48x - 24y = 23$
(d) *None of the above*

Interpret (*c*), Firstly, find $\frac{dy}{dx}$ of given curve and put $\frac{dy}{dx}$ of curve is equal to the slope of the line. Solving them and get the values of x_1 and y_1. Then, use the equation $y - y_1 = \frac{dy}{dx}(x - x_1)$ to find the required equation of tangent.

The equation of the given curve is $y = \sqrt{3x-2}$.

The slope of the tangent at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{1}{2}(3x-2)^{\frac{1}{2}-1}\cdot 3 = \frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x - 2y + 5 = 0$.

$$\Rightarrow \quad y = 2x + \frac{5}{2} \quad \text{(which is of the form } y = mx + c)$$

$\therefore$ Slope of this line is 2.

Now, the tangent to the curve is parallel to the line $4x - 2y + 5 = 0$ *i.e.*, the slope of the tangent is equal to the slope of the line.

$$\Rightarrow \quad \frac{3}{2\sqrt{3x-2}} = 2 \Rightarrow 3 = 4\sqrt{3x-2}$$

$$\Rightarrow \quad 3x - 2 = \left(\frac{3}{4}\right)^2 \Rightarrow 3x = 2 + \frac{9}{16} \Rightarrow x = \frac{41}{48}$$

Now, putting $x = \frac{41}{48}$, in $y = \sqrt{3x-2}$, we get

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41-32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ Equation of the tangent passing through the point $\left(\frac{41}{48}, \frac{3}{4}\right)$ having slope 2 is given by

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right) \Rightarrow y - \frac{3}{4} = 2x - \frac{41}{24}$$

$$\Rightarrow \quad 48x - 24y = 23$$

Hence, the equation of the required tangent is $48x - 24y = 23$.

Normal (Definition) The normal to the curve at any point P on it is the straight line which passes through P and is perpendicular to the tangent to the curve at P.

Slope of Normal

We know that normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passing through P.

$\therefore$ Slope of the normal at $P = -\dfrac{1}{\text{Slope of the tangent at } P}$

$\Rightarrow$ Slope of normal at $P(x_1, y_1) = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$

$\Rightarrow$ Slope of normal at $P(x_1, y_1) = -\left(\dfrac{dx}{dy}\right)_{(x_1, y_1)}$

Note If normal is parallel to x-axis.

$\Rightarrow \quad -\left(\dfrac{dx}{dy}\right)_{(x_1, y_1)} = 0$ or $\left(\dfrac{dx}{dy}\right)_{(x_1, y_1)} = 0$

If normal is perpendicular to x-axis (or parallel to y-axis).

$\Rightarrow \quad -\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 0$

***Sample* Problem 9** *The slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \dfrac{\pi}{2}$, is* [NCERT]

(a) $\dfrac{a}{2b}$ (b) $\dfrac{2a}{b}$

(c) $\dfrac{a}{b}$ (d) $-\dfrac{a}{2b}$

Interpret (*d*) It is given that $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

On differentiating x and y both w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[1 - a\sin\theta] = -a\cos\theta \quad \text{and}$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[b\cos^2\theta] = 2b\cos\theta(-\sin\theta) = -2b\cos\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

$\therefore$ Slope of normal at the point $\theta = \dfrac{\pi}{2}$, $-\dfrac{dx}{dy} = \dfrac{-1}{dy/dx}$

$$= \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(\theta = \pi/2)}} = \frac{-1}{\dfrac{2b}{a}\sin\left(\dfrac{\pi}{2}\right)} = \frac{-a}{2b}$$

Hot Spot 3

Normals Related Problems

These types of problems are generally asked in JEE (Main) examination, directly or indirectly and a small practice can help you in solving such problems very quickly as the level of question is easy to average.

While solving problems based on Normals of any curve, always keep in mind the slope of normals.

Equation of normal to the curve $y = f(x)$ at the point $P(x_1, y_1)$ on the curve having a slope m is

$$(y - y_1) = m(x - x_1)$$

or
$$(y - y_1) = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{P(x_1, y_1)}}(x - x_1)$$

***Sample* Problem 10** *The equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$, is* [NCERT]

(a) $2x + 3my - 3am^3 - 2am^2 = 0$

(*b*) $2x + 3my - 3am^4 - 2am^2 = 0$

(c) $2x + 3m^2y - 3am^3 - 2am^3 = 0$

(*d*) *None of the above*

Interpret (*b*) The equation of the given curve is

$$ay^2 = x^3 \quad \ldots(i)$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$a(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of the tangent to the given curve at (am^2, am^3) is

$$\left(\frac{dy}{dx}\right)_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3}{2}m$$

$\therefore$ Slope of normal at $(am^2, am^3) = -\dfrac{1}{\dfrac{3}{2}m} = -\dfrac{2}{3m}$

Hence, equation of normal at (am^2, am^3) is given by

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

$$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow \quad 2x + 3my - 3am^4 - 2am^2 = 0$$

Parametric Form

If the equation(s) of the curve $y = f(x)$ are represented in the form of the parameter t, such that $x = f_x(t)$ and $y = f_y(t)$

(i) **Slope of Tangent**

$$m = \left(\frac{dy}{dx}\right) \Rightarrow m = \frac{dy}{dt} \cdot \frac{dt}{dx} \Rightarrow m = \frac{f_y'}{f_x'}$$

(ii) **Slope of Normal**

$$M = -\frac{1}{m} = -\frac{f_x'}{f_y'}$$

(iii) **Equation of Tangent**

$$y - f_y(t) = \frac{f_y'}{f_x'}[x - f_x(t)]$$

(iv) **Equation of Normal**

$$y - f_y(t) = -\frac{f_x'}{f_y'}[x - f_x(t)]$$

28.3 Angle of Intersection of Two Curves

The angle of intersection of two curves is the angle subtended between the tangents at their point of intersection.

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$, respectively.

Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 at their point of intersection.

Let θ be the angle between the two tangents PT_1 and PT_2 and θ_1 and θ_2 are the angles made by tangents with the positive direction of x-axis in anti-clockwise sense.

Then, $$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{C_1}$$

and $$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{C_2}$$

From the figure it follows,

$$\theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}\right| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Angle of intersection of these curves is defined as acute angle between the tangents.

***Sample* Problem 11** *The acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersections is*

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

(c) $\frac{\pi}{6}$ (d) $\tan^{-1}\left(\frac{4}{7}\right)$

Interpret (b) The points of intersection are $(\pm\sqrt{2}, 1)$.

Since, the curves are symmetrical about y-axis, the angle of intersection at $(-\sqrt{2}, 1)$

= The angle of intersection at $(\sqrt{2}, 1)$

At $(\sqrt{2}, 1)$, $m = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$.

$$\therefore \quad \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

Orthogonal Curves

If the angle of intersection of two curves is right angle, then two curves are said to be orthogonal curves.

If the curves are orthogonal, then $\theta = \frac{\pi}{2}$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

$$\Rightarrow \quad m_1 m_2 = -1$$

If two curves touches each other, then $m_1 = m_2$.

***Sample* Problem 12** *The curves $x = y^2$ and $xy = k$ cut at right angle (orthogonally), if $8k^2$ is equal to* [NCERT]

(a) 1 (b) 3

(c) $\frac{1}{2}$ (d) *None of these*

Interpret (b) When the curve cut at right angle, their tangents at the point of intersection are also perpendicular *i.e.,* the product of their slopes is equal to –1.

The equation of the given curves are $x = y^2$ ...(i)

and $xy = k$ ...(ii)

The two curves meet where $\frac{k}{y} = y^2$

[eliminating x between Eqs. (i) and (ii)]

$\Rightarrow$ $y^3 = k \Rightarrow y = k^{1/3}$

On substituting this value of y in Eq. (i), we get $x = (k^{1/3})^2 = k^{2/3}$

$\therefore$ Eq. (i) and Eq. (ii) intersect at the point $(k^{2/3}, k^{1/3})$.

On differentiating Eq. (i), w.r.t. x, we get

$$1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$\therefore$ Slope of the tangent to the first curve Eq. (i) at $(k^{2/3}, k^{1/3})$

$$= \frac{1}{2k^{1/3}} \quad \text{...(iii)}$$

From Eq. (ii), $y = \frac{k}{x} \Rightarrow \frac{dy}{dx} = -\frac{k}{x^2}$

$\therefore$ Slope of the tangent to the second curve Eq. (ii) at $(k^{2/3}, k^{1/3})$

$$= -\frac{k}{(k^{2/3})^2} = -\frac{1}{k^{1/3}} \quad \text{...(iv)}$$

We know that two curves intersect at right angles, if the tangents to the curves at the point of intersection *i.e.,* at $(k^{2/3}, k^{1/3})$ are perpendicular to each other.

This implies that we should have the product of the tangents = –1

$$\Rightarrow \quad \left(\frac{1}{2k^{1/3}}\right)\left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow \quad 1 = 2k^{2/3} \Rightarrow 1^3 = (2k^{2/3})^3 \Rightarrow 1 = 8k^2$$

Hence, the given two curves cut at right angles, if $8k^2 = 1$

Length of Tangent, Normal, Subtangent, Subnormal

Let $y = f(x)$ be the equation of the given curve and $P(x, y)$ be any general point on it. Through P, draw a tangent so that it intersects the x-axis at point T. Also, PN is the normal line to the same curve at the same point P (see figure). Now, PT and PN be the length of tangent and normal respectively and the projections of PT and PN, *i.e.,* ST and SN are the lengths of sub-tangent and sub-normal, respectively.

(i) Length of Tangent (PT)

In right angled ΔPTS, right angled at S, we have

$$\sin\theta = \frac{|PS|}{|PT|}$$

$$\Rightarrow \quad |PT| = |PS| \operatorname{cosec}\theta$$

$$\Rightarrow \quad |PT| = |PS|\sqrt{1 + \cot^2\theta}$$

$$\Rightarrow \quad |PT| = y\sqrt{1 + \frac{1}{y'^2}}$$

$$\therefore \quad \text{Length of tangent} = \frac{y\sqrt{1 + y'^2}}{y'}$$

(ii) Length of Normal (PN)

In right angled ΔPSN right angled at S, we have

$$\cos\theta = \frac{|PS|}{|PN|} \Rightarrow |PN| = |PS| \sec\theta$$

$$\Rightarrow \quad |PN| = |PS|\sqrt{1 + \tan^2\theta}$$

$$\therefore \quad \text{Length of normal} = y\sqrt{1 + y'^2}$$

(iii) Length of Subtangent (ST)

In right angled ΔPST, right angled at S, we have

$$\tan\theta = \frac{|PS|}{|ST|} \Rightarrow |ST| = \frac{y}{y'}$$

$$\therefore \quad \text{Length of subtangent} = \frac{y}{y'}$$

(iv) Length of Subnormal (SN)

In right angled ΔPSN, right angled at S, we have

$$\cot\theta = \frac{|PS|}{|SN|}$$

$$\Rightarrow \quad |SN| = y\tan\theta \Rightarrow |SN| = yy'$$

$\therefore$ Length of subnormal = yy'

Condition for a Given Line to Touch the Given Curve

Let the line be a tangent to the given curve at (x_1, y_1), then write the equation of the tangent as

$$y - y_1 = \left(\frac{dy}{dx}\right)(x - x_1)$$

On comparing the equation with the given equation, we get

$$ax + by + c = 0$$

Eliminating, x, y, we will get desired values for which the line touches the curve.

***Sample* Problem 13** *In the curve $x^{m+n} = a^{m-n} y^{2n}$, the mth power of the subtangent varies as the kth power of subnormal, then k is*

(a) m (b) n (c) $1/n$ (d) $1/m$

Interpret (b) Given, $x^{m+n} = a^{m-n} y^{2n}$

$\Rightarrow$ $(m+n)\log x = (m-n)\log a + 2n \log y$

On differentiating w.r.t. x, we get

$$\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{(m+n)}{2n}\left(\frac{y}{x}\right)$$

Now, $$\frac{(\text{subtangent})^m}{(\text{subnormal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$$

$$= \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n} y^{2n}} = \left(\frac{2n}{m+n}\right)^{m+n} \times a^{m-n}$$

= Constant (independent of x and y)

$\Rightarrow$ $(\text{Subtangent})^m \propto (\text{Subnormal})^n$

***Sample* Problem 14** *The equation of family of curves for which the length of normal at any point P is equal to the distance of P from origin, is*

(a) $x^2 = -y + C$ (b) $y^2 = \pm x^2 + C$

(c) $x = \pm y + C$ (d) $2x^2 = \pm y^2 + C$

Interpret (b), Let $P(x, y)$ be the point on the curve.

OP = Radius vector = $\sqrt{x^2 + y^2}$

PN = Length of normal

Now, $$\tan\phi = -\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$PN = \frac{y}{\sin\phi}$$

It is given $OP = PN$

$\Rightarrow$ $$\sqrt{x^2 + y^2} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$\Rightarrow$ $$x^2 + y^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$\Rightarrow$ $$x^2 = y^2\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$$

$\Rightarrow$ $y\,dy = \pm x\,dx$ integrating both sides,

$y^2 = \pm x^2 + C$ is the required family of curves.

Check Point **1**

1. If the line joining the points (0, 3) and (5, −2) is a tangent to the curve $y = \frac{c}{x+1}$, then the value of c is
   (a) 1 (b) −2
   (c) 4 (d) None of these

2. The equation of tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at (a, b) is
   (a) $\frac{x}{a} + \frac{y}{b} = 2$ (b) $\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$
   (c) $\frac{x}{b} - \frac{y}{b} = 2$ (d) $ax + by = 2$

3. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then
   (a) $a > 0, b < 0$ (b) $a < 0, b > 0$
   (c) Both (a) and (b) (d) None of these

4. The equation of normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$, is
   (a) $x + y = 1$ (b) $x - y = 1$
   (c) $x + y = -1$ (d) $x - y = -1$

5. If the subnormal at any point on $y = a^{1-n}x^n$ is of constant length, then the value of n is
   (a) 1 (b) 0.5 (c) 2 (d) −2

6. If the area of the triangle, included between the axes and any tangent to the curve $xy^n = a^{n+1}$ is constant, then the value of n is
   (a) −1 (b) −2 (c) 1 (d) 2

7. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally, then
   (a) $a^2 + b^2 = l^2 + m^2$ (b) $a^2 - b^2 = l^2 - m^2$
   (c) $a^2 - b^2 = l^2 + m^2$ (d) $a^2 + b^2 = l^2 - m^2$

8. The point on the curve $\sqrt{x} + \sqrt{y} = 2a^2$, where the tangent is equally inclined to the axes, is
   (a) (a^4, a^4) (b) $(0, 4a^4)$
   (c) $(4a^4, 0)$ (d) None of these

9. Two cyclists start from the junction of two perpendicular roads, their velocities being 3v m/min and 4v m/min. The rate at which the two cyclists are separating, is
   (a) $\frac{7}{2}$v m/min (b) 5 v m/min
   (c) v m/min (d) None of these

10. If there is an error of k% in measuring the edge of a cube, then the per cent error in estimating its volume is
   (a) k (b) 3k
   (c) k/3 (d) None of these

Hot Spot 4

Increasing Function

Increasing function is the most important topic from JEE (Main) examination point of view. Generally, questions are seen from this topic. The level question is from moderate to typical.

These functions are of two types
(i) Strictly increasing function
(ii) Non-decreasing function

(i) Strictly Increasing Function

A function $f(x)$ is known as strictly increasing function in its domain, if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Therefore, for the smaller input, we have smaller output and for higher value of input we have higher output.

Graphically it can be expressed as, shown in the adjoing figure.

Here, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, $f(x)$ is strictly increasing function.

In the graph, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

As $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, $f(x) < f(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{\text{positive}}{\text{positive}}$$

i.e., $f'(x) > 0$

Thus, $f(x)$ will be strictly increasing, if $f'(x) > 0, \forall\ x \in$ domain.

Classification of Strictly Increasing Function

Concave up
when $f'(x) > 0$
and $f''(x) > 0, \forall x \in$ domain

Neither concave up nor concave down
when $f'(x) > 0$
and $f''(x) = 0, \forall x \in$ domain

Concave down
when $f'(x) > 0$
and $f''(x) < 0, \forall x \in$ domain

(ii) Non-decreasing Function

A function $f(x)$ is said to be non-decreasing, if $x_1 < x_2$

$\Rightarrow$ $f(x_1) \le f(x_2)$

As shown in adjoint figure.

For AB and CD portion, $x_1 < x_2$

$\Rightarrow$ $f(x_1) < f(x_2)$

and for BC, $x_1 < x_2$

$\Rightarrow$ $f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-decreasing function (or increasing function),

if $x_1 < x_2$

$\Rightarrow$ $f(x_1) \le f(x_2)$

Obviously, for this $f'(x) \ge 0$, where equality holds for horizontal path of the graph *i.e.,* in the interval of BC.

In these type of question ask the interval in which a given function is increasing. While solving such types of problems, generally we follow the steps given below.

Step I First, we write the given function in standard form.

Step II Now, differentiate the given with respect to given independent variable.

Step III Now, use the appropriate condition of increasing function and Simplify for number line.

Step IV Get the range of given function from number line.

Step V Finally, we get the required interval in which the given function is increasing.

Sample Problem 15 *The function* $y = \dfrac{\sin\theta}{2 + \cos\theta} - \theta$ *is an increasing on* θ *in interval* [NCERT]

(a) $\left[0, \dfrac{\pi}{3}\right]$ (b) $\left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$

(c) $\left[0, \dfrac{\pi}{2}\right]$ (d) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$

Interpret (c) We have, $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\frac{4\sin\theta}{2+\cos\theta} - \theta\right]$$

$$= \frac{(2+\cos\theta)\dfrac{d}{d\theta}(4\sin\theta) - 4\sin\theta\dfrac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{4\cos\theta(2+\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

In interval $\left[0, \dfrac{\pi}{2}\right]$, we have $\cos\theta \geq 0$. Also, $4 > \cos\theta$

$\Rightarrow$ $(4 - \cos\theta) > 0$.

$\therefore$ $\cos\theta(4-\cos\theta) \geq 0$ and also $(2+\cos\theta)^2 > 0$

$\Rightarrow$ $\dfrac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$

$\Rightarrow$ $\dfrac{dy}{d\theta} \geq 0$

Hence, given function is increasing in interval $\left[0, \dfrac{\pi}{2}\right]$.

Note Since, it is a continuous function,
(i) for increasing function, $f' \geq 0$. (ii) for strictly increasing function, $f' > 0$

***Sample* Problem 16** *Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \dfrac{1}{x}$ is* **[NCERT]**

(a) *strictly increasing on I*
(b) *strictly decreasing on I*
(c) *neither increasing nor decreasing on I*
(d) *None of the above*

Interpret (c), Given, $f(x) = x + \dfrac{1}{x}$

$\Rightarrow$ $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2}$ (differentiate w.r.t. x)

Since, intervals are disjoint from $(-1, 1)$. Therefore,
If $x < -1$, then $f'(x) > 0$. If $x > 1$, then $f'(x) > 0$
$\therefore$ $f(x)$ is strictly increasing on interval I disjoint from $(-1, 1)$.

28.4 Decreasing Function

These functions are also of two types
(i) Strictly decreasing function
(ii) Non-increasing function

(i) Strictly Decreasing Function

A function $f(x)$ is known as strictly decreasing function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Therefore, for the smaller input we have higher output and for higher value of input we have smaller output.

Graphically it can be expressed as shown in the adjoint figure.

Here, $x_1 < x_2$

$\Rightarrow f(x_1) > f(x_2)$ thus, $f(x)$ is strictly decreasing.

In graph, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

As $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Thus, $f(x+h) < f(x) \Rightarrow f'(x) = \lim\limits_{h \to 0} \dfrac{-\text{ve}}{+\text{ve}}$

i.e., $f'(x) < 0$

Thus, $f(x)$ will be strictly decreasing, if $f'(x) < 0 \ \forall x \in$ domain

Classification of Strictly Decreasing Function

Concave up
when $f'(x) < 0$
and $f''(x) > 0$,
$\forall x \in$ domain

Neither concave up nor concave down
when $f'(x) < 0$
and $f''(x) = 0, \forall x \in$ domain

Concave down
when $f'(x) < 0$
and $f''(x) < 0, \forall x \in$ domain

(ii) Non-Increasing Function

A function $f(x)$ is said to be non-increasing, if for $x_1 < x_2$

$\Rightarrow \quad f(x_1) \geq f(x_2)$

As shown in adjoint figure

For AB and CD portion, $x_1 < x_2$

$\Rightarrow \quad f(x_1) > f(x_2)$

and for BC, $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-increasing function (or decreasing function), if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

Obviously, for this $f'(x) \leq 0$, where equality holds for horizontal path of the graph *i.e*, in the interval of BC.

***Sample* Problem 17** *On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?* **[NCERT]**

(a) (0, 1) (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) *None of these*

Interpret (*d*), Given, $f(x) = x^{100} + \sin x - 1$

$\Rightarrow \quad f'(x) = 100x^{99} + \cos x$

(a) In interval (0, 1), $\cos x > 0$ and $100x^{99} > 0$

$\therefore \quad f'(x) > 0$

$\left[\because \text{In} \left(0, \frac{\pi}{2}\right), \cos x > 0 \text{ i.e.}, (0, 1.57), \quad \cos x > 0\right.$

$\left.\therefore (0, 1), \quad \cos x > 0\right]$

Thus, function f is strictly increasing in interval (0, 1).

(b) In interval $\left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$ and $100x^{99} > 0$

Also, $100x^{99} > \cos x$

$\therefore \quad f'(x) > 0$ in $\left(\frac{\pi}{2}, \pi\right)$

Thus, function f is strictly increasing in interval $\left(\frac{\pi}{2}, \pi\right)$.

(c) In interval $\left(0, \frac{\pi}{2}\right)$, $\cos x > 0$ and $100x^{99} > 0$.

$\therefore \quad 100x^{99} + \cos x > 0$

$\Rightarrow \quad f'(x) > 0$ in $\left(0, \frac{\pi}{2}\right)$

Thus, f is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$.

Hence, function f is not strictly decreasing in the given intervals.

The correct answer is (d).

28.5 Monotonic Function

A function f is said to be monotonic or monotone in an interval I. If it is either increasing or decreasing in the interval I. $f(x) = \ln x$, $f(x) = 2^x$, $f(x) = -2x + 3$ are monotonic functions. $f(x) = x^2$ is monotonic in $(-\infty, 0)$ or $(0, \infty)$ but is not monotonic in R.

Properties of Monotonic Functions

(i) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ $[f'(c) < 0]$ for each $c \in (a, b)$, then $f(x)$ is monotonically (or strictly) decreasing function on $[a, b]$.

(ii) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ $[f'(c) > 0]$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.

(iii) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing function on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.

(v) If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

Operations of Monotonic Functions

I : Increasing, D : Decreasing, λ : Neither increasing nor decreasing				
$f(x)$	I	D	I	D
$g(x)$	I	I	D	D
$-f(x)$	D	I	D	I
$-g(x)$	D	D	I	I
$f(x) + g(x)$	I	I or D or λ	I or D or λ	D
$f(x) - g(x)$	I or D or λ	D	I	I or D or λ
$f(x) \cdot g(x)$	I	I or D or λ	I or D or λ	D
$\frac{f(x)}{g(x)}$	I or D or λ	D	I	I or D or λ
$\frac{1}{f(x)}$	D	I	D	I
$\frac{1}{g(x)}$	D	D	I	I
$(fog)x$	I	D	D	I

Use of monotonicity for proving inequalities Comparision of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of $h(x) = f(x) - g(x)$,

If $f(a) = g(a)$ and $f'(x) \geq g'(x) \forall x \geq a \Rightarrow f(x) \geq g(a) \forall x \geq a$.

***Sample* Problem 18** *Which of the following statements is/are true?*

(a) $\log(1+x) > x - \frac{x^2}{2} \forall\ x \in (0, \infty)$

(b) $\log(1+x) < x - \frac{x^2}{2} \forall\ x \in (0, \infty)$

(c) $\sin x < x < \tan x \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $\sin x > x > \tan x \forall\ x \in \left(0, \frac{\pi}{2}\right)$

Interpret (a) Consider the function $f(x) = \ln(1+x) - x + \frac{x^2}{2}$, $x \in (0, \infty)$

Then, $$f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0 \quad \forall\ x \in (0, \infty)$$

$\Rightarrow$ $f(x)$ increases in $(0, \infty)$ $\Rightarrow$ $f(x) > f(0^+) = 0$

i.e., $$\ln(1+x) > x - \frac{x^2}{2}$$

which is the desired result.

Concavity, Convexity and Point of Inflexion

A function $f(x)$ is concave in (a, b), if tangent drawn at every point $[x_0\{f(x_0)\}]$, for $x_0 \in (a, b)$ lie below the curve, $f(x)$ is convex in (a, b) if tangent drawn at each point $\{x_0, f(x_0)\}$, $x_0 \in (a, b)$ lie above the curve.

A point $\{c, f(c)\}$ of the graphy $y = f(x)$ is said to be a point of inflection of the graph, If $f(x)$ is concave in $(c - \delta, c)$ and convex in $(c, c + \delta)$ (or *vice versa*), for some $\delta \in R^+$.

Results

(i) If $f''(x) > 0$, $\forall\ x \in (a, b)$, then the curve $y = f(x)$ is concave in (a, b).

(ii) If $f''(x) < 0, \forall\ x \in (a, b)$, then the curve $y = f(x)$ is convex in (a, b).

(iii) If f is continuous at $x = c$ and $f''(x)$ has opposite signs on either sides of c, then the point $\{c, f(c)\}$ is a point of inflexion of the curve.

(iv) If $f''(c)$ and $f'''(c) \neq 0$, then the point $\{c, f(c)\}$ is a point of inflexion.

28.6 Mean Value Theorem

In calculus, the **mean value theorem** states, roughly, that gives a section of a smooth curve, there is a point on that section at which the derivative (slope) of the curve is equal to the 'average' derivative of the section.

This theorem can be understood concretely by applying in motion : if a car travels 100 miles in one hour, so that its average speed during that time was 100 mph, then at some of its instantaneous speed must have been exactly 100 mph.

Rolle's Theorem

If a function $f(x)$

(i) continuous in the closed interval $[a, b]$, *i.e.,* continuous at each point in the interval $[a, b]$

(ii) differentiable in an open interval (a, b) *i.e.,* differentiable at each point in the open interval (a, b)

(iii) $f(a) = f(b)$

Then, there will be atleast one point c in the interval (a, b) such that $f'(c) = 0$.

Geometrical Meaning of Rolle's Theorem

In the graph of a function $y = f(x)$ be continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between point A and B is unique, *i.e.,* tangent at each point between A and B exists and ordinates, *i.e.,* y-coordinates of points A and B are equal, then, there will be atleast one point P on the curve between A and B at which tangent will be parallel to x-axis.

Fig. (i)

Fig. (ii)

In fig. (i) there is only one such point P, where tangent is parallel to x-axis, but in Fig. (ii) there are more than one such point, where tangents are parallel to x-axis.

Note Converse of Rolle's theorem is not true, *i.e.,* if a funtion $f(x)$ is such that $f'(c) = 0$ for atleast one c in the open interval (a, b), then it is not necessary that

(i) $f(x)$ is continuous in $[a, b]$ (ii) $f(x)$ is differentiable in (a, b)

(iii) $f(a) = f(b)$

Application of Rolle's Theorem for Isolating the Real Roots of an Equation $f(x) = 0$

Suppose a and b are two real numbers such that

(a) $f(x)$ and its derivatve $f'(x)$ are continuous for $a \le x \le b$.

(b) $f(a)$ and $f(b)$ have opposite signs.

(c) $f'(x)$ is different from zero for all values of x between a and b.

Then, there is one and only real root of the equation $f(x) = 0$ between a and b.

***Sample* Problem 19** *What are the values of c for which Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2] is verified ?*

(a) $c = \pm 1$ (b) $c = 1 \pm \frac{1}{\sqrt{3}}$

(c) $c = \pm 2$ (d) None of these

Interpret (b) Here, we observe that

(a) $f(x)$ is a polynomial, so it is continuous in the interval [0, 2].

(b) $f'(x) = 3x^2 - 6x + 2$ exists for all $x \in (0, 2)$.

So, $f(x)$ is differentiable for all $x \in (0, 2)$ and

(c) $f(0) = 0$, $f(2) = 2^3 - 3(2)^2 + 2(2) = 0$

$\therefore$ $f(0) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied.

So, there must exist $c \in (0, 2)$ such that $f'(c) = 0$

$\Rightarrow$ $f'(c) = 3c^2 - 6c + 2 = 0$

$\Rightarrow$ $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$

Hot Spot 5 Lagrange's Mean Value Theorem

Lagrange's Mean Value Theorem is an important topic for JEE Main examination and a small practice can help you in solving such problems very quickly as the level of questions is easy to average.
These types of question generally ask the value of c in the given interval for the given function.

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, *i.e.,* continuous at each point in the interval $[a, b]$.

(ii) differentiable in an open interval (a, b), *i.e.,* differentiable at each point in the interval (a, b).

Then, there will be atleast one point c, where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Another Form of Lagrange's Mean Value Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a + h]$

(ii) differentiable in the open interval $(a, a + h)$

Then, there exists atleast one value θ, $0 < \theta < 1$ such that

$$f(a + h) = f(a) + hf'(a + \theta h)$$

Geometrical Meaning of Lagrange's Mean Value Theorem

Let $A\{a, f(a)\}$ and $B\{b, f(b)\}$ be two points on the curve $y = f(x)$.

Then, $OL = a$, $OM = b$, $Al = f(a)$, $BM = f(b)$

Now, slope of chord AB

$$\tan\theta = \frac{BK}{AK} = \frac{f(b) - f(a)}{b - a} \quad \text{...(i)}$$

By Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c) = \text{Slope of tangent at point } P\{c, f(c)\}$$

From Eq. (i), $\tan\theta$ = Slope of tangent at P

Slope of chord AB = Slope of tangent at P

Hence, chord AB || tangent PT

Thus, geometrical meaning of mean value theorem is as follows

In the graph of a curve $y = f(x)$ be continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between A and B exists, *i.e.,* tangent is unique, then there will be atleast one point P on the curve between A and B, where tangent will be parallel to chord AB. In Fig. (i) there is only one such point P where tangent is parallel to chord AB but in Fig. (ii) there are more than one such points where tangents are parallel to cord AB.

Fig. (i)

Fig. (ii)

***Sample* Problem 20** *The value of c from the Lagrange's mean value theorem for which $f(x) = \sqrt{25 - x^2}$ in [1, 5], is*

(a) 5 (b) 1
(c) $\sqrt{15}$ (d) None of these

Interpret (c) It is clear that $f(x)$ has a definite and unique value of each $x \in [1, 5]$.

Thus, every point in the interval [1, 5], the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous in the interval [1, 5].

Also, $f'(x) = \dfrac{-x}{\sqrt{25 - x^2}}$, which clearly exists for all x in an open interval (1, 5).

Hence, $f(x)$ is differentiable in (1, 5).

So, there must be a value $c \in (1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{0 - \sqrt{24}}{4} = \frac{-\sqrt{6}}{2}$$

But $$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

$\therefore$ $$\frac{-c}{\sqrt{25 - c^2}} = \frac{-\sqrt{6}}{2}$$

$\Rightarrow$ $$4c^2 = 6(25 - c^2)$$

$\Rightarrow$ $$c = \pm\sqrt{15}$$

Clearly, $c = \sqrt{15} \in (1, 5)$ such that Lagrange's theorem is satisfied.

Check Point **2**

1. The value of a for which the function $(a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically through out for all real x, are
   (a) $a < -2$ (b) $a > -2$
   (c) $-3 < a < 0$ (d) $-\infty < a \le -3$
2. The interval in which the function $f(x) = xe^{2-x}$ increases, is
   (a) $(-\infty, 1)$ (b) $(2, \infty)$
   (c) $(0, 2)$ (d) None of these
3. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a monotonically decreasing function of x in the largest possible interval $\left(-2, -\frac{2}{3}\right)$, then
   (a) $\lambda = 4$ (b) $\lambda = 2$
   (c) $\lambda = -1$ (d) λ has no real value
4. Let $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$, $0 \le x \le 1$, then
   (a) g(x) increases on $\left[\frac{1}{2}, 1\right]$ (b) g(x) decreases on $\left[\frac{1}{2}, 1\right]$
   (c) g(x) decreases on $\left[0, \frac{1}{2}\right]$ (d) g(x) increases on $\left[0, \frac{1}{3}\right]$
5. If $f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4\theta + \sin^4\theta)\, d\theta$ is a decreasing function of x for all $x \in R$ and $b \in R$, b being independent of x, then
   (a) $a \in (0, \sqrt{6})$ (b) $a \in (-\sqrt{6}, \sqrt{6})$
   (c) $a \in (-\sqrt{6}, 0)$ (d) None of these
6. Let f(x) be differentiable $\forall x$. If $f(1) = -2$ and $f'(x) \ge 2\ \forall x \in [1, 6]$, then
   (a) $f(6) < 8$ (b) $f(6) \ge 8$
   (c) $f(6) \ge 5$ (d) $f(6) \le 5$
7. From mean value theorem, $f(b) - f(a) = (b - a)f'(a)$; $a \le x_1 < b$, if $f(x) = \frac{1}{x}$, then x_1 is equal to
   (a) $\sqrt{ab}$ (b) $\frac{a + b}{2}$
   (c) $\frac{2ab}{a + b}$ (d) $\frac{b - a}{b + a}$

WORKED OUT
Examples

Example 1 *Using differentials, the approximate value of* (17/8′) *is* **[NCERT]**

(*a*) 0.632 (*b*) 0.677
(*c*) 0.683 (*d*) 0.66

Solution (*b*) Use the relation $f(x+\Delta x) \simeq f(x) + \Delta x\, f'(x)$, to determine the approximate value.

Consider $f(x) = x^{1/4} \Rightarrow \quad f'(x) = \frac{1}{4}x^{-3/4}$

Let $\quad x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$,

Now, $\quad f'(x+\Delta x) \simeq f(x) + \Delta x\, f'(x)$

$$\Rightarrow \quad (x+\Delta x)^{1/4} \simeq x^{1/4} + \frac{1}{4x^{3/4}} \times \Delta x$$

$$\Rightarrow \quad \left(\frac{16}{81}+\frac{1}{81}\right)^{1/4} \simeq \left(\frac{16}{81}\right)^{1/4} + \frac{\frac{1}{81}}{4\left(\frac{16}{81}\right)^{3/4}}$$

$$\Rightarrow \quad \left(\frac{17}{81}\right)^{1/4} \simeq \frac{2}{3} + \frac{1}{81\times 4\times\left(\frac{2}{3}\right)^3} = \frac{2}{3}+\frac{1}{96} = \frac{65}{96}$$

Hence, $\quad \left(\frac{17}{81}\right)^{1/4} \simeq 0.677$

Example 2 *The perpendicular distance of the normal at any point θ to the curve* $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$, *from the origin, is* **[NCERT]**

(*a*) tan θ (*b*) cot θ
(*c*) constant (*d*) *None of these*

Solution (*a*) Firstly, we find the equation of normal to the curve at any point θ and then find the perpendicular distance (*d*) from origin by using the relation

$$d = \frac{|\text{constant}|}{\sqrt{(\text{coefficient to } x)^2 + (\text{coefficient of } y)^2}}$$

The given curve is $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\sin\theta + a[\theta\cos\theta + \sin\theta]$$
$$= -a\sin\theta + a\theta\cos\theta + a\sin\theta = a\theta\cos\theta$$

and $\quad \frac{dy}{d\theta} = a\cos\theta - a[\theta(-\sin\theta) + \cos\theta]$

$$= a\cos\theta + a\theta\sin\theta - a\cos\theta = a\theta\sin\theta$$

$\therefore$ Slope of the tangent at θ, $\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Slope of the normal at $\theta = -\frac{1}{\frac{dy}{dx}} = -\frac{1}{\tan\theta} = -\cot\theta$

The equation of the normal at a given point (x, y) is given by

$$y - [a\sin\theta - a\theta\cos\theta] = -\cot\theta[x - (a\cos\theta + a\theta\sin\theta)]$$

$$\Rightarrow y - [a\sin\theta - a\theta\cos\theta] = -\frac{\cos\theta}{\sin\theta}[x - (a\cos\theta + a\theta\sin\theta)]$$

$$\Rightarrow \quad y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = a(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = a$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$= \frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$= \frac{|-a|}{\sqrt{1}} = |-a| \qquad (\because \cos^2\theta + \sin^2\theta = 1)$$

which is independent of θ. Hence, the perpendicular distance of the normal from the origin is constant.

Example 3 *Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, the rate at which they are being separated.* **[NCERT Exemplar]**

(*a*) $\sqrt{2-\sqrt{2}}\cdot V$ (*b*) $\sqrt{2+\sqrt{2}}\cdot V$
(*c*) $\sqrt{\sqrt{2}-1}\cdot V$ (*d*) $\sqrt{2+\sqrt{2}}\cdot V$

Solution (*a*) Let *L* and *M* be the positions of two men *A* and *B* at any time *t*.

Let $\quad OL = x$ and $LM = y$

Then, $\quad OM = x$

Given, $\frac{dx}{dt} = v$ and we have to find $\frac{dy}{dt}$

From ΔLOM, $\cos 45° = \dfrac{OL^2 + OM^2 - LM^2}{2 \cdot OL \cdot OM}$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}} = \dfrac{x^2 + x^2 - y^2}{2 \cdot x \cdot x} = \dfrac{2x^2 - y^2}{2x^2}$

$\Rightarrow \quad \sqrt{2}x^2 = 2x^2 - y^2 \Rightarrow (2 - \sqrt{2})\, x^2 = y^2$

$\therefore \quad y = \sqrt{2 - \sqrt{2}}\, x$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = \sqrt{2 - \sqrt{2}}\, \frac{dx}{dt} = \sqrt{2 - \sqrt{2}}\, v \qquad \left(\because \frac{dx}{dt} = v\right)$$

Hence, they are being separated from each other at the rate $\sqrt{2 - \sqrt{2}}\, v$.

Example 4 *The point on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes is*

(a) $\left(4, \pm \dfrac{8}{3}\right)$ (b) $\left(4, \dfrac{-8}{3}\right)$

(c) $\left(4, \pm \dfrac{3}{8}\right)$ (d) $\left(\pm 4, \dfrac{3}{8}\right)$

Solution (a) Given curve is $\quad 9y^2 = x^3 \quad$ ...(i)

On differentiating, we get

$$18y \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{6y} \qquad \text{...(ii)}$$

Let (α, β) be a point on Eq. (i) at which normal makes equal intercepts on the axes, then

$$9\beta^2 = \alpha^3 \qquad \text{...(iii)}$$

From Eq.(ii), slope of the normal at (α, β)

$$= \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(\alpha, \beta)}} = \frac{-1}{\alpha^2/(6\beta)} = \frac{-6\beta}{\alpha^2} \qquad \text{...(iv)}$$

Since, normal of the curve makes equal intercepts with the axes, so slope of normal

$$= \tan 45° \text{ or } \tan 135° = \pm 1 \qquad \text{...(v)}$$

$\therefore$ From Eq. (iv), we get

$$\frac{-6\beta}{\alpha^2} = \pm 1 \Rightarrow \beta = \mp \frac{\alpha^2}{6}$$

On putting the value of β in Eq. (iii), we get

$$9\left(\mp \frac{\alpha^2}{6}\right)^2 = \alpha^3$$

$\Rightarrow \quad \alpha^4 = 4\alpha^3$

$\Rightarrow \quad \alpha^3 (\alpha - 4) = 0$

$\Rightarrow \quad \alpha = 0 \text{ or } \alpha = 4$

When $\alpha = 0, \beta = 0$, then normal passes through (0, 0) it mean that they do not intercepts.

Taking $\alpha = 4$, we get $\beta = \mp \dfrac{4^2}{6} = \mp \dfrac{8}{3}$. Hence, the correct option is (a).

Example 5 *The interval in which the function $f(x) = x^{1/x}$ is increasing, is*

(a) $(-\infty, e)$ (b) (e, ∞)

(c) $(-\infty, \infty)$ (d) *None of these*

Solution (a) Given, $f(x) = x^{1/x}$

$\Rightarrow \quad f'(x) = \dfrac{1}{x^2}(1 - \log x)\, x^{1/x}$

$f'(x) > 0$, if $1 - \log x > 0$

$\Rightarrow \quad \log x < 1$

$\Rightarrow \quad x < e$

$\therefore$ $f(x)$ is increasing in the interval $(-\infty, e)$.

Example 6 *At which point the tangent to the curve $x^2 + y^2 = 25$ is parallel to the line $3x - 4y = 7$?*

(a) $(3, 4), (-3, -4)$ (b) $(3, -4), (-3, 4)$

(c) $(4, 3), (-4, -3)$ (d) $(-4, 3), (4, -3)$

Solution (b) We have, $\quad x^2 + y^2 = 25 \quad$ ...(i)

$\Rightarrow \quad 2x + 2y \dfrac{dy}{dx} = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-x}{y}$

Now, slope of the line $3x - 4y = 7$ is $m = \dfrac{3}{4}$.

Since, tangent is parallel to given line.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{3}{4}$

$\Rightarrow \quad -\dfrac{x}{y} = \dfrac{3}{4}$

$\Rightarrow \quad y = -\dfrac{4}{3}x \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$x^2 + \frac{16}{9}x^2 = 25$$

$\Rightarrow \quad x = \pm 3$

If $x = 3, y = -4$ [from Eq. (ii)]

If $x = -3, y = 4$ [from Eq. (ii)]

$\therefore$ Points of contact are $(3, -4)$ and $(-3, 4)$.

Example 7 *The equation of the tangent to the curve $y = \sqrt{9 - 2x^2}$ at the point where the ordinate and the abscissa are equal, is*

(a) $2x + y + 3\sqrt{3} = 0$ (b) $2x + y + \sqrt{3} = 0$

(c) $2x + y - \sqrt{3} = 0$ (d) *None of these*

Solution (a) Given, $x_1 = y_1$

$\therefore \quad x_1 = \sqrt{9 - 2x_1^2}$

$\Rightarrow \quad x_1^2 = 9 - 2x_1^2$

$\Rightarrow \quad x_1 = \pm \sqrt{3}.$

Since, $y_1 > 0$, therefore the point is $(\sqrt{3}, \sqrt{3})$.

Now, $$y = \sqrt{9 - 2x^2}$$

On differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = -4x \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(\sqrt{3}, \sqrt{3})} = -2$$

So, the required equation of tangent is

$$(y - \sqrt{3}) = -2(x - \sqrt{3})$$

$$\Rightarrow \quad 2x + y - 3\sqrt{3} = 0$$

Example 8 *The angle of intersection of the curves* $y = 2\sin^2 x$ *and* $y = \cos 2x$ *at* $x = \frac{\pi}{6}$ *is*

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (d) *None of these*

Solution (b) Given, $y = 2\sin^2 x$ ...(i)

and $y = \cos 2x$ ...(ii)

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 4\sin x \cos x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x = \frac{\pi}{6}} = \sqrt{3} = m_1 \quad \text{(say)}$$

On differentiating Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} = -2\sin 2x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x = \frac{\pi}{6}} = -\sqrt{3} = m_2 \quad \text{(say)}$$

Let θ be the angle between them,

$$\therefore \quad \theta = \pm\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$= \pm\tan^{-1}\left(\frac{\sqrt{3} + \sqrt{3}}{1 - 3}\right)$$

$$= \pm\tan^{-1}(-\sqrt{3}) = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

Example 9 *If* $a_0, a_1, a_2, \ldots, a_n$ *are real numbers such that* $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0$, $(n \neq 1)$, *then the equation* $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ *has a real root lying between*

(a) 0, 1 (b) 1, 2

(c) 1, 3 (d) *None of these*

Solution (a) Consider the polynomial

$$f(x) = a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \ldots + a_n\frac{x^{n+1}}{n+1} \quad \ldots\text{(i)}$$

We have, $f(0) = 0$

and $$f(1) = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0$$

Therefore, 0 and 1 are roots of $f(x)$.

Hence, by algebraic interpretation of Rolle's theorem

$$f'(x) = 0$$

i.e., $$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

must have a root lying between 0 and 1.

Example 10 *The interval in which the function* $y = x^3 + 5x^2 - 1$ *is decreasing is*

(a) $\left(0, \frac{10}{3}\right)$ (b) $(0, 10)$

(c) $\left(\frac{-10}{3}, 0\right)$ (d) *None of these*

Solution (c) Given, $y = x^3 + 5x^2 - 1$

$$\frac{dy}{dx} = 3x^2 + 10x = x(3x + 10)$$

For function to be decreasing, $\frac{dy}{dx} < 0$

$$\therefore \quad x(3x + 10) < 0 \Rightarrow \frac{-10}{3} < x < 0$$

Example 11 *Let* $f(x)$ *and* $g(x)$ *be two differentiable functions in R and* $f(2) = 8$, $g(2) = 0$, $f(4) = 10$ *and* $g(4) = 8$, *then*

(a) $g'(x) > 4f'(x)\ \forall x \in (2, 4)$

(b) $3g'(x) = 4f'(x)$ *for atleast one* $x \in (2, 4)$

(c) $g(x) > f(x)\ \forall x \in (2, 4)$

(d) $g'(x) = 4f'(x)$ *for atleast one* $x \in (2, 4)$

Solution (d) Consider $h(x) = g(x) - 4f(x)$, in $[2, 4]$

Also, $h(2) = g(2) - 4f(2) = -32, h(4) = -32$

$\Rightarrow h'(x) = 0$ for at least one $x \in (2, 4)$ using Rolle's theorem.

Example 12 *A point on the curve* $y = (x - 3)^2$, *where the tangent is parallel to the chord joining the points* (3, 0) *and* (4, 1) *is* **[NCERT]**

(a) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{5}{2}, \frac{1}{4}\right)$

(c) $\left(\frac{5}{4}, \frac{1}{2}\right)$ (d) $\left(\frac{7}{2}, \frac{1}{4}\right)$

Solution (d) Here, $f(x) = y = (x - 3)^2$

We know that polynomial function is always continuous and differentiable.

Here, y is a polynomial function, so it is continuous and differentiable.

So, Lagrange's mean value theorem is applicable.

Thus, there exist a point c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 2(c-3) = \frac{f(4)-f(3)}{4-3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1-0}{1}$$

$$[\because f(x) = (x-3)^2 \Rightarrow f'(x) = 2(x-3)]$$

$$\Rightarrow \quad 2c = 7 \Rightarrow c = \frac{7}{2}$$

$$\Rightarrow \quad y = \left(\frac{7}{2} - 3\right)^2 \Rightarrow y = \frac{1}{4}$$

So, required point on the curve is $(c, y) = \left(\frac{7}{2}, \frac{1}{4}\right)$.

Example 13 *If a, b, c and d are real numbers such that* $\frac{3a+2b}{c+d} + \frac{3}{2} = 0$, *then the equation* $ax^3 + bx^2 + cx + d = 0$ *has*

(a) *atleast one root in* $[-2, 0]$
(b) *atleast one root in* $[0, 2]$
(c) *atleast two roots in* $[-2, 2]$
(d) *No root in* $[-2, 2]$

Solution (b) Let $f'(x) = ax^3 + bx^2 + cx + d$

$$f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + e$$

Given $6a + 4b + 3c + 3d = 0$, $f(+2) = e = f(0)$ use Rollis theorem.

Example 14 *If* $f(x)$ *is continuous in* $[a, b]$ *and differentiable in* (a, b) *then there exist atleast one* $c \in (a, b)$ *such that* $f'(c)$ *is equal to*

(a) $\frac{f(b)-f(a)}{b-a} \cdot c$
(b) $\frac{f(b)-f(a)}{b^2-a^2} \cdot c^2$
(c) $\frac{f(b)-f(a)}{b^3-a^3} \cdot 3c^2$
(d) $\frac{f(b)-f(a)}{b-a}$

Solution (a) We have to prove

$$(b^3 - a^3)f'(c) - \{f(b) - f(a)\}(3c^2) = 0$$

Let us assume a function

$$F(x) = (^2 - a^3)f(x) - \{f(b) - f(a)\} \cdot x^3$$

which will be continuous in $[a, b]$ and differenttiable in (a, b) as $f(x)$ and x^3 both are continuous

Also, $\quad F(a) = b^3 f(a) - a^3 f(b) = F(b)$

So, according to Rolle's theorem, there exists atleast one $c \in (a, b)$ such that, $F'(c) = 0$, which proves the required result.

Example 15 *The curve* $y = ax^3 + bx^2 + cx + 5$ *touches the x-axis at* $P(-2, 0)$ *and cuts the y-axis at a point Q, where its gradient is 3. Then,*

(a) $a = -\frac{1}{2}, b = -\frac{3}{4}$ *and* $c = 3$
(b) $a = \frac{1}{2}, b = -\frac{3}{4}$ *and* $c = -3$
(c) $a = \frac{1}{2}, b = -\frac{1}{4}$ *and* $c = 3$
(d) *None of the above*

Solution (a) Since, we have the curve $y = ax^3 + bx^3 + cx + 5$ touches x-axis at $P(-2, 0)$, then x-axis is the tangent at $(-2, 0)$. The curve meets y-axis in $(0, 5)$.

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{0,5} = 0 + 0 + c = 3 \qquad \text{(given)}$$

$$\Rightarrow \quad c = 3 \qquad \ldots\text{(i)}$$

and $\quad \left(\frac{dy}{dx}\right)_{(-2,0)} = 0 \Rightarrow 12a - 4b + c = 0$

$$\Rightarrow \quad 12a - 4b + 3 = 0 \qquad \text{[from Eq. (1)]} \ldots\text{(ii)}$$

and $(-2, 0)$ lies on the curve, then

$$0 = -8a + 4b - 2c + 5$$

$$\Rightarrow \quad 0 = -8a + 4b - 1 \qquad (\because c = 3)$$

$$\Rightarrow \quad 8a - 4b + 1 = 0 \qquad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we get $a = -\frac{1}{2}, b = -\frac{3}{4}$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Derivatives at the Rate of change

1. A balloon, which always remains spherical, has a variable diameter $3/2(2x+1)$. The rate of change of its volume with respect to x, is [NCERT]

(a) $\frac{27\pi}{8}(2x+1)^2$ (b) $\frac{8\pi}{27}(2x+1)^2$

(c) $\frac{27\pi}{8}(x+2)^2$ (d) $\frac{8\pi}{27}(x+2)^2$

2. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the base. How fast height of the sand cone increasing when the height is 4 cm? [NCERT]

(a) $\frac{\pi}{48}$ cm/s (b) $\frac{1}{48\pi}$ cm/s

(c) $\frac{48}{\pi}$ cm/s (d) 48π cm/s

3. The total cost $C(x)$ in rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

The marginal cost when $x = 17$ units are produced is [NCERT]

(a) ₹ 19.196 (b) ₹ 20.225 (c) ₹ 20.967 (d) ₹ 21.297

4. The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. The marginal revenue when $x = 7$ is [NCERT]

(a) ₹ 197 (b) ₹ 199 (c) ₹ 205 (d) ₹ 208

5. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base? [NCERT]

(a) $3b$ cm^2/s (b) $\sqrt{2}b$ cm^2/s

(c) $\frac{b}{\sqrt{2}}$ cm^2/s (d) $\sqrt{3}b$ cm^2/s

6. A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m. [NCERT Exemplar]

(a) 8 m/s (b) 12 m/s (c) 16 m/s (d) 19 m/s

7. An angle $\theta, 0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine, is [NCERT Exemplar]

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

8. The rate of change of the surface area of a sphere of radius r, when the radius is increasing at the rate of 2 cm/s is proportional to

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) r (d) r^2

9. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 10 cm, is

(a) $\sqrt{3}$ cm^2/s (b) 10 cm^2/s

(c) $10\sqrt{3}$ cm^2/s (d) $\frac{10}{\sqrt{3}}$ cm^2/s

10. If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing, is

(a) a constant
(b) proportional to the radius
(c) inversely proportional to the radius
(d) inversely proportional to the surface area

11. Moving along the x-axis there are two points with $x = 10 + 6t, x = 3 + t^2$. The speed with which they are reaching from each other at the time of encounter is (x is in centimetre and t is in seconds)

(a) 16 cm/s (b) 20 cm/s

(c) 8 cm/s (d) 12 cm/s

12. Gas is being pumped into a spherical balloon at the rate of $30\,\text{ft}^3/\text{min}$. Then, the rate at which the radius increases when it reaches the value 15 ft, is

(a) $\frac{1}{30\pi}$ ft/min (b) $\frac{1}{15\pi}$ ft/min

(c) $\frac{1}{20}$ ft/min (d) $\frac{1}{15}$ ft/min

13. An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, its y-coordinate is decreasing at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is (in unit per second)

(a) 2 (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$

14. The position of a point in time 't' is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time 't' is

(a) $b - c$ (b) $b + c$

(c) $2b - 2c$ (d) $2\sqrt{b^2 + c^2}$

15. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm²/s in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, the rate of decrease of the slant height of water, is **[NCERT Exemplar]**

(a) $\frac{\sqrt{2}}{4\pi}$ cm/s (b) $\frac{1}{4\pi}$ cm/s

(c) $\frac{1}{\pi\sqrt{2}}$ cm/s (d) None of these

Errors and its Approximations

16. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then the approximate error in calculating its volume is **[NCERT]**

(a) $3.12\,\pi\ \text{m}^3$ (b) $3.92\,\pi\ \text{m}^3$

(c) $3.56\,\pi\ \text{m}^3$ (d) $4.01\,\pi\ \text{m}^3$

17. The approximate value of $f(5.001)$, where

$$f(x) = x^3 - 7x^2 + 15, \text{ is}$$ **[NCERT]**

(a) -34.995 (b) -33.995 (c) -33.335 (d) -35.993

Tangent and Normals

18. For the curve $y = 4x^3 - 2x^5$, the points at which the tangent passes through the origin are **[NCERT]**

(a) (0, 0), (1, 2) and (−1, −2)

(b) (0, 0), (2, 4) and (−1, −3)

(c) (0, 0), (2, 3) and (−3, −1)

(d) None of the above

19. The point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to $y = 3x + 9$, will be

(a) (2, 1) (b) (1, 2)

(c) (3, 9) (d) (−2, 1)

20. The length of subtangent to the curve $x^2y^2 = a^4$ at the point $(-a, a)$ is

(a) $3a$ (b) $2a$

(c) a (d) $4a$

21. For the curve $xy = c^2$, the subnormal at any point varies

(a) x^2 (b) x^3

(c) y^2 (d) y^3

22. The abscissa of the points, where the tangent to the curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to x-axis, are

(a) $x = 0$ and 0 (b) $x = 1$ and -1

(c) $x = 1$ and -3 (d) $x = -1$ and 3

23. The slope of the tangent to the curve $x = 3t^2 + 1$, $y = t^3 - 1$, at $x = 1$ is

(a) 0 (b) $\frac{1}{2}$

(c) ∞ (d) -2

24. If tangent to the curve $x = at^2$, $y = 2at$ is perpendicular to x-axis, then its point of contact is

(a) (a, a) (b) $(0, a)$

(c) (0, 0) (d) $(a, 0)$

25. The equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the x-axis, is

(a) $x + 5y = 2$ (b) $x - 5y = 2$

(c) $5x - y = 2$ (d) $5x + y - 2 = 0$

26. The point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at which the normal is parallel to the x-axis, is

(a) (0, 0) (b) $(0, a)$

(c) $(a, 0)$ (d) (a, a)

27. Coordinates of a point on the curve $y = x\log x$ at which the normal is parallel to the line $2x - 2y = 3$, are

(a) (0, 0) (b) (e, e)

(c) $(e^2, 2e^2)$ (d) $(e^{-2}, -2e^{-2})$

28. The length of the normal at point 't' of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is

(a) $a \sin t$

(b) $2a \sin^3(t/2) \sec(t/2)$

(c) $2a \sin(t/2) \tan(t/2)$

(d) $2a \sin(t/2)$

29. The tangent drawn at the point (0, 1) on the curve $y = e^{2x}$, meets x-axis at the point

(a) $\left(\frac{1}{2}, 0\right)$ (b) $\left(-\frac{1}{2}, 0\right)$

(c) (2, 0) (d) (0, 0)

30. The tangent to the curve $y = 2x^2 - x + 1$ at a point P is parallel to $y = 3x + 4$, then the coordinates of P are
(a) (2, 1) (b) (1, 2) (c) (−1, 2) (d) (2, −1)

31. The product of the lengths of subtangent and subnormal at any point of a curve is
(a) square of the abscissa (b) square of the ordinate
(c) constant (d) None of these

32. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{2}$ with the positive x-axis, then $f'(3)$ is equal to
(a) −1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

33. Tangent of the angle at which the curves $y = a^x$ and $y = b^x$ $(a \neq b > 0)$ intersect, is given by
(a) $\frac{\log ab}{1 + \log ab}$ (b) $\frac{\log \frac{a}{b}}{1 + (\log a)(\log b)}$
(c) $\frac{\log ab}{1 + (\log a)(\log b)}$ (d) None of these

34. The equation of tangent to the curve $y = be^{-x/a}$ at the point where it crosses y-axis, is **[NCERT Exemplar]**
(a) $ax + by = 1$ (b) $ax - by = 1$
(c) $\frac{x}{a} - \frac{y}{b} = 1$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

35. At what point on the curve $x^3 - 8a^2y = 0$, the slope of the normal is $-\frac{2}{3}$?
(a) (a, a) (b) $(2a, -a)$
(c) $(2a, a)$ (d) None of these

36. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point
(a) (1, 0) (b) at no point
(c) (0, 1) (d) (0, 0)

37. The point (s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical (parallel to y-axis), is (are)
(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$
(c) (0, 0) (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

38. If $y = 4x - 5$ is tangent to the curve $y^2 = px^3 + q$ at $(2, 3)$, then (p, q) is
(a) (2, 7) (b) −2, 7)
(c) (−2, −7) (d) (2, −7)

39. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is
(a) −1 (b) 3 (c) −3 (d) 1

40. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_2, y_2), then
(a) $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$ (b) $\frac{x_2}{y_1} + \frac{x_1}{y_2} = -1$
(c) $\frac{x_1}{x_2} + \frac{y_1}{y_2} = -1$ (d) $\frac{x_2}{x_1} + \frac{y_2}{y_1} = 1$

Angle of Intersection of two Curves

41. The angle between the curves $y = \sin x$ and $y = \cos x$ is
(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$
(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

42. The condition for the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $xy = c^2$ to intersect orthogonally, is **[NCERT Exemplar]**
(a) $a^2 + b^2 = 0$ (b) $a^2 - b^2 = 0$
(c) $a = b$ (d) None of these

43. $y = \log(1 + x) - \frac{2x}{2 + x}$, $x > -1$, is an increasing function of x throughout in, **[NCERT]**
(a) $x > -1$ (b) $x > 1$
(c) $x < 0$ (d) $x > 0$

Increasing Function

44. The least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2) is **[NCERT]**
(a) −2 (b) −1
(c) 0 (d) 2

45. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in [a, b]$. Then, f is an increasing function on **[NCERT]**
(a) (a, b) (b) $(a, b]$
(c) $[a, b]$ (d) $[a, b)$

46. The function x^x is increasing, when
(a) $x > \frac{1}{e}$ (b) $x < \frac{1}{e}$
(c) $x < 0$ (d) for all real x

47. If $g(x) = \min(x, x^2)$, where x is real number, then
(a) $g(x)$ is an increasing function
(b) $g(x)$ is a decreasing function
(c) $f(x)$ is a constant function
(d) $g(x)$ is a continuous function except at $x = 0$

48. The function $f(x) = \frac{x}{1 + |x|}$ is
(a) strictly increasing
(b) strictly decreasing
(c) neither increasing nor decreasing
(d) not differential at $x = 0$

49. The function $f(x) = x + \cos x$ is
(a) always increasing
(b) always decreasing
(c) increasing for certain range of x
(d) None of the above

50. $2x^3 - 6x + 5$ is an increasing function, if
(a) $0 < x < 1$ (b) $-1 < x < 1$
(c) $x < -1$ or $x > 1$ (d) $-1 < x < -\frac{1}{2}$

51. The length of the longest interval, in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$ (d) π

52. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
(a) increasing on $\left[-\frac{1}{2}, 1\right]$ (b) decreasing on R
(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

53. Consider the following statements S and R

S : Both $\sin x$ and $\cos x$ are decreasing function in $\left(\frac{\pi}{2}, \pi\right)$.

R : If a differentiable function decreases in (a, b), then its derivative also decreases in (a, b).

Which of the following is true ?
(a) Both S and R are wrong.
(b) Both S and R are correct but R is not the correct explanation for S.
(c) S is correct and R is the correct explanation for S.
(d) S is correct, R is wrong.

54. $f(x) = \cos\frac{\pi}{x}$ increases in
(a) $\left(\frac{1}{2n+1}, \frac{1}{2n}\right)$ (b) $\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)$
(c) $(0, 2n)$ (d) R

55. The function $f(x) = \sin^4 x + \cos^4 x$ increases, if
(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$
(c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

56. The function 'g' defined by $g(x) = f(x^2 - 2x + 8) + f(14 + 2x - x^2)$, where $f(x)$ is twice differentiable function, $f''(x) \geq 0$ for all real numbers x. The function $g(x)$ is increasing in the interval
(a) $[-1, 1] \cup [2, \infty)$ (b) $(\infty, -1] \cup [1, 3]$
(c) $[-1, 1] \cup [3, \infty)$ (d) $(\infty, -2] \cup [1, \infty)$

Decreasing Function

57. The function $f(x) = \log(\cos x)$ is strictly decreasing and strictly increasing in **[NCERT]**
(a) $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ (d) $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{3\pi}{2}\right)$

58. The function which is neither decreasing nor increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, is
(a) $\operatorname{cosec} x$ (b) $\tan x$
(c) x^2 (d) $|x-1|$

59. The value of x for which the polynomial $2x^3 - 9x^2 + 12x + 4$ is a decreasing function of x, is
(a) $-1 < x < 1$ (b) $0 < x < 2$
(c) $x > 3$ (d) $1 < x < 2$

60. If $f(x) = \frac{1}{x+1} - \log(1 + x)$, $x > 0$, then f is
(a) an increasing function
(b) a decreasing function
(c) both increasing and decreasing function
(d) None of the above

61. If $f(x) = \sin x - \cos x$, the interval in which function is decreasing in $0 \leq x \leq 2\pi$, is
(a) $\left[\frac{5\pi}{6}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(c) $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ (d) None of these

62. The function $f(x) = x^{1/x}$ is
(a) increasing in $(1, \infty)$
(b) decreasing in $(1, \infty)$
(c) increasing in $(1, e)$ and decreasing in (e, ∞)
(d) decreasing in $(1, e)$ and increasing in (e, ∞)

63. Among $\sin(\tan 2)$, $\sin\left(\tan\frac{5}{2}\right)$, $\sin(\tan 3)$ and $\sin(\tan 4)$, which one is smallest?
(a) $\sin(\tan 2)$ (b) $\sin\left(\tan\frac{5}{2}\right)$
(c) $\sin(\tan 3)$ (d) $\sin(\tan 4)$

64. Which of the following is biggest?
(a) $\cos 1$ (b) $\cos 2$
(c) $\cos 3$ (d) $\cos 4$

65. $g(x) = 2f\left|\frac{x^2}{2}\right| + f(6 - x^2)$ for all $x \in R$. It is given that $f''(x) > 0$ for all $x \in R$, then $g(x)$ decreases for
(a) $(-\infty, \sqrt{3}) \cup (0, \sqrt{3})$ (b) R
(c) $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ (d) $(-\infty, -2] \cup [0, 2]$

66. The function $f(x) = \tan x - 4x$ is strictly decreasing on **[NCERT Exemplar]**

(a) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (b) $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

(c) $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Monotonic Function

67. Function $f(x) = \frac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ is monotonic increasing, if

(a) $\lambda > 1$ (b) $\lambda < 1$ (c) $\lambda < 4$ (d) $\lambda > 4$

Mean Value Theorem

68. If the function $f(x) = x^3 - 6x^2 + ax + b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right) = 0$, then

(a) $a = -11$ (b) $a = -6$ (c) $a = 6$ (d) $a = 11$

69. $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2] and $f(x)$ is continuous in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

70. If $f(x)$ satisfies the conditions for Rolle's theorem in [3, 5], then $\int_3^5 f(x)\,dx$ is equal to

(a) 2 (b) -1

(c) 0 (d) $-\frac{4}{3}$

71. In the mean value theorem $\frac{f(b) - f(a)}{b - a} = f'(c)$, if $a = 0, b = \frac{1}{2}$ and $f(x) = x(x-1)(x-2)$, then value of c is

(a) $1 - \frac{\sqrt{15}}{6}$ (b) $1 + \sqrt{15}$ (c) $1 - \frac{\sqrt{21}}{6}$ (d) $1 + \sqrt{21}$

72. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in [0, 1], is

(a) -2 (b) -1

(c) 0 (d) 1/2

73. Rolle's theorem holds for the function $x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $\frac{4}{3}$, the values of b and c are

(a) $b = 8, c = -5$ (b) $b = -5, c = 8$

(c) $b = 5, c = -8$ (d) $b = -5, c = -8$

74. Suppose the cubic equation $x^3 - px + q = 0$ has three distinct real roots, where $p > 0$ and $q > 0$. Then, which one of the following holds?

(a) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(b) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(c) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(d) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

75. If $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has atleast one real root in

(a) (0, 1) (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (d) $(-1, 1)$

Round II (Mixed Bag)

Only One Correct Option

1. Line joining the points (0, 3) and (5, – 2) is a tangent to the curve $y = \frac{ax}{1+x}$, then

(a) $a = 1 \pm \sqrt{3}$

(b) $a = \phi$

(c) $a = -1 \pm \sqrt{3}$

(d) $a = -2 \pm 2\sqrt{3}$

2. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is a strictly decreasing function

(d) $f(x)$ is unbounded

3. $f(x) = \int_0^x |\log_2 [\log_3\{\log_4 (\cos t + a)\}]|\,dt$. If $f(x)$ is increasing for all real values of x, then

(a) $a \in (-1, 1)$ (b) $a \in (1, 5)$

(c) $a \in (1, \infty)$ (d) $a \in (5, \infty)$

4. The parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect orthogonally at point $P(x_1, y_1)$, where $x_1, y_1 \neq 0$, then

(a) $b = a^2$ (b) $b = a^3$

(c) $b^3 = a^2$ (d) None of these

5. $y = f(x)$ is a parabola, having its axis parallel to y-axis. If the line $y = x$ touches this parabola at $x = 1$, then

(a) $f''(1) + f'(0) = 1$ (b) $f''(0) - f'(1) = 1$

(c) $f''(1) - f'(0) = 1$ (d) $f''(0) + f'(1) = 1$

6. If $f'(x) > 0$ and $f''(x) > 0$, $\forall x \in R$, then for any two real numbers x_1 and x_2, $(x_1 \neq x_2)$

(a) $f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$

(b) $f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$

(c) $f'\left(\frac{x_1 + x_2}{2}\right) > \frac{f'(x_1) + f'(x_2)}{2}$

(d) $f'\left(\frac{x_1 + x_2}{2}\right) < \frac{f'(x_1) + f'(x_2)}{2}$

7. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$ (c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

8. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, – 1) is

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$

(c) –6 (d) None of these

9. The values of a for which the function $(a + 2)x^3 - 3ax^2 + 9ax - 1 = 0$ decreases monotonically throughout for all real x, are

(a) $a < -2$ (b) $a > -2$

(c) $-3 < a < 0$ (d) $-\infty < a \leq -3$

10. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions

(b) both $f(x)$ and $g(x)$ are decreasing

(c) $f(x)$ is an increasing function

(d) $g(x)$ is an increasing function

11. The function $\frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ is decreasing, if

(a) $ad - bc > 0$ (b) $ad - bc < 0$

(c) $ab - cd > 0$ (d) $0 \leq x \leq -2$

12. The function $f(x) = x + \cos x$ is

(a) always increasing

(b) always decreasing

(c) increasing for certain range of x

(d) None of the above

13. The function $f(x) = x(x + 3)e^{-(1/2)x}$ satisfies all the conditions of Rolle's theorem in [–3, 0]. The value of c is

(a) 0 (b) –1

(c) –2 (d) –3

14. Let $f(x)$ satisfy all the conditions of mean value theorem in [0, 2]. If $f(0) = 0$ and $|f'(x)| \leq \frac{1}{2}$ for all x, in [0, 2], then

(a) $f(x) < 2$

(b) $|f(x)| \leq 1$

(c) $f(x) = 2x$

(d) $f(x) = 3$ for atleast one x in [0, 2]

15. The function $f(x) = x^3 - 6x^2 + ax + b$ satisfy the conditions of Rolle's theorem in [1, 3]. The values of a and b are

(a) 11, – 6 (b) –6, 11

(c) –11, 6 (d) 6, –11

16. The function $f(x) = (x - 3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{7}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{1}{4}\right)$

(c) (1, 4) (d) (4, 1)

17. In the mean value theorem, $f(b) - f(a) = (b - a) f'(c)$, if $a = 4, b = 9$ and $f(x) = \sqrt{x}$, then the value of c is

(a) 8.00 (b) 5.25 (c) 4.00 (d) 6.25

18. Let $a + b = 4$, $a < 2$ and $g(x)$ be a monotonically increasing function of x. Then,

$$f(x) = \int_0^a g(x)\, dx + \int_0^b g(x)\, dx$$

(a) increases with increase in $(b - a)$

(b) decreases with increase in $(b - a)$

(c) increases with decreases in $(b - a)$

(d) None of the above

19. The sum of intercepts on coordinate axes made by tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is

(a) a (b) $2a$

(c) $2\sqrt{a}$ (d) None of these

20. The function which is neither decreasing nor increasing in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is

(a) cosec x (b) tan x

(c) x^2 (d) $|x - 1|$

21. The function $f(x) = \cos x - 2px$ is monotonically decreasing for

(a) $p < \frac{1}{2}$ (b) $p > \frac{1}{2}$

(c) $p < 2$ (d) $p > 2$

22. The abscissa of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangents can be obtained by mean value theorem for the interval [–2, 2], are

(a) $\pm \frac{2}{\sqrt{3}}$ (b) $+\sqrt{3}$

(c) $\pm \frac{\sqrt{3}}{2}$ (d) 0

23. The equation of the tangent to the curve $y = 1 - e^{x/2}$ at the point of intersection with the y-axis is
(a) $x + 2y = 0$ (b) $2x + y = 0$
(c) $x - y = 2$ (d) None of these

24. The chord joining the points, where $x = p$ and $x = q$ on the curve $y = ax^2 + bx + c$ is parallel to the tangent at the point on the curve whose abscissa is
(a) $\frac{p+q}{2}$ (b) $\frac{p-q}{2}$
(c) $\frac{pq}{2}$ (d) None of these

25. If $f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4\theta + \sin^4\theta)\, d\theta$ is decreasing function of x for all $x \in R$ and $b \in R$, b being independent of x, then
(a) $a \in (0, \sqrt{6})$ (b) $a \in (-\sqrt{6}, \sqrt{6})$
(c) $a \in (-\sqrt{6}, 0)$ (d) None of these

26. The values of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x, is given by
(a) $a < 1$ (b) $a \geq 1$
(c) $a \leq \sqrt{2}$ (d) $a < \sqrt{2}$

27. In [0, 1], Lagrange's mean value theorem is not applicable to
(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$
(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$
(c) $f(x) = x|x|$
(d) $f(x) = |x|$

28. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$
(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

29. The tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at
(a) (6, 7) (b) (–6, 7)
(c) (6, –7) (d) (–6, –7)

30. If m be the slope of a tangent to the curve $e^y = 1 + x^2$, then
(a) $|m| > 1$ (b) $m < 1$
(c) $|m| < 1$ (d) $|m| \leq 1$

31. If the line joining the points (0, 3) and (5, – 2) is a tangent to the curve $y = \frac{c}{x+1}$, then the value of c is
(a) 1 (b) –2
(c) 4 (d) None of these

32. The total number of parallel tangents of $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$ are
(a) 2 (b) 0 (c) 1 (d) infinite

33. The angle of intersection of curves, $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where [·] denotes greatest integral function is
(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}(2)$ (d) None of these

34. The area of the triangle formed by the coordinate axes and a tangent to the curve $xy = a^2$ at the point (x_1, y_1) on it is
(a) $\frac{a^2 x_1}{y_1}$ (b) $\frac{a^2 x_2}{x_1}$ (c) $2a^2$ (d) $4a^2$

35. The equation of tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at (a, b) is
(a) $\frac{x}{a} + \frac{y}{b} = 2$ (b) $\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$
(c) $\frac{x}{b} - \frac{y}{a} = 2$ (d) $ax + by = 2$

36. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then
(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

37. The interval in which $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^3 x - 3$ decreases is $x \in [0, \pi]$
(a) $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ (b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$
(c) $\left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ (d) None of these

38. Function $f(x) = 2x^2 - \log|x|$, $x \neq 0$ monotonically increases in
(a) $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$
(c) $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$ (d) None of these

39. Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall x_1 > x_2$, then solution set of
$$f\{g(\alpha^2 - 2\alpha)\} > f\{g(3\alpha - 4)\} \text{ is}$$
(a) R (b) ϕ (c) (1, 4) (d) $R - [1, 4]$

40. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is
(a) increasing on $(0, \infty)$
(b) decreasing in $(0, \infty)$
(c) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$
(d) decreasing on $(0, \pi/e)$, increasing on $(\pi/e, \infty)$

41. Assuming the petrol burnt (per hour) in driving a motorboat varies as the cube of its velocity, then the most economical speed when going against a current of c miles per hour

(a) $\frac{c}{2}$ mph (b) $\frac{2c}{3}$ mph

(c) $\frac{3c}{2}$ mph (d) $2c$ mph

42. Let x be the length of one of the equal sides of an isosceles triangle and let θ be the angle between them. If x is increasing at the rate $\left(\frac{1}{12}\right)$ mph and θ is increasing at the rate of $\frac{\pi}{180}$ radius/h, then the rate in m^2/h at which the area of the triangle is increasing when $x = 12$ m and $\theta = \frac{\pi}{4}$

(a) $2^{1/2}\left(1+\frac{2\pi}{5}\right)$ (b) $\frac{73}{2}\cdot 2^{1/2}$

(c) $\frac{3^{1/2}}{2}+\frac{\pi}{5}$ (d) $2^{1/2}\left(\frac{1}{2}+\frac{\pi}{5}\right)$

More Than One Correct Option

43. For $F(x)=\int_0^x 2|t|\,dt$, the tangent lines which are parallel to the bisector of the first coordinate angle is

(a) $y = x-\frac{1}{4}$ (b) $y = x+\frac{1}{4}$

(c) $y = x-\frac{3}{2}$ (d) $y = x+\frac{3}{2}$

44. Let $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$ for every real number x, then

(a) h is increasing whenever f is increasing

(b) h is increasing whenever f is decreasing

(c) h is decreasing whenever f is decreasing

(d) h is decreasing whenever f is increasing

45. Let $f(x) = 2\sin^3 x - 3\sin^2 x + 12\sin x + 5,\ 0 \le x \le \frac{\pi}{2}$.

Then, $f(x)$ is

(a) decreasing in $\left[0, \frac{\pi}{2}\right]$

(b) increasing in $\left[0, \frac{\pi}{2}\right]$

(c) increasing in $\left[0, \frac{\pi}{4}\right]$ and decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

(d) None of the above

46. If a, b, c be non-zero real numbers such that $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$ $= \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$, then the equation $ax^2+bx+c=0$ has

(a) roots of opposite sign

(b) root in (0, 1)

(c) root in (1, 2)

(d) one root in $[-\infty, 1]$ and other root in $[2, \infty]$

Comprehension Based Questions

Passage

Directions (Q. Nos. 47-51) *A cubic $f(x) = ax^3+bx^2+cx+d$ vanishes at $x=-2$ and has relative minimum/maximum at $x=-1$ and $x=\frac{1}{3}$ and if $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$.*

On the basis of above information, answer the following questions.

47. The value of c is

(a) -2 (b) -1

(c) 0 (d) 2

48. The value of d is

(a) 5 (b) 2

(c) 0 (d) -4

49. The function $f(x)$ is

(a) x^3+x^2+x-2 (b) x^3-x^2+x-2

(c) x^3-x^2-x+2 (d) x^3+x^2-x+2

50. $f(x)$ decreases in the interval

(a) $\left(-\frac{1}{3}, 1\right)$ (b) $\left(-\frac{1}{3}, -1\right)$

(c) $\left(-1, \frac{1}{3}\right)$ (d) $\left(1, \frac{3}{2}\right)$

51. The nature of roots of $f(x) = 3$ is

(a) one root is real and other two are distinct

(b) all roots real and distinct

(c) all roots are real; two of them are equal

(d) None of the above

Assertion and Reason

Directions (Q. Nos. 52-60) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) and (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

52. **Statement I** If Rolle's theorem be applied in $f(x)$, then Lagrange Mean Value Theorem (LMVT) is also applied in $f(x)$.

Statement II Both Rolle's theorem and LMVT cannot be applied in $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

53. Consider the ordinate of a point describing the circle $x^2 + y^2 = 25$ decreases at the rate of 1.5 cm/s.

Statement I Then the rate of change of the abscissa of the point when ordinate equals to 4 cm in 2 cm/s.

Statement II $x\,dx + y\,dy = 0$

54. If $g(x)$ is a differentiable function $g(1) \neq 0$, $g(-1) \neq 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2 - 1}{g(x)}$ in $[-1, 1]$, then

Statement I $g(x)$ has atleast one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$. Rolle's theorem is applicable for $x \in (a, b)$.

55. **Statement I** Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

56. **Statement I** Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then, f is one one.

Statement II $f(x)$ is decreasing function.

57. **Statement I** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists atleast one $c \in (a, b)$, then $\frac{f(b) - f(a)}{b^3 - a^3} = \frac{f'(c)}{3c^2}$.

Statement II $f'(c) = \frac{f(b) - f(a)}{b - a}, c \in (a, b)$

58. **Statement I** If $f(x) = x(x+3)e^{-x/2}$, then Rolle's theorem applies for $f(x)$ in $[-3, 0]$.

Statement II Lagrange mean value theorem is applied in $f(x) = x(x+3)e^{-x/2}$ in any interval.

59. **Statement I** A tangent parallel to x-axis can be drawn for $f(x) = (x-1)(x-2)(x-3)$ in the interval $[1, 3]$.

Statement II A horizontal tangent can be drawn in Rolle's theorem.

60. Consider the equation $x^3 - 3x + k = 0, k \in R$.

Statement I There is no value of k for which the given equation has two distinct roots in $(0, 1)$

Statement II Between two consecutive roots of $f'(x) = 0$, $\{f(x)$ is a polynomial$\}$, $f(x) = 0$ must have one root.

Previous Years' Questions

61. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$ **[JEE Main 2013]**

(a) lies between 1 and 2 (b) lies between 2 and 3
(c) lies between –1 and 0 (d) does not exist

62. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to **[JEE Main 2013]**

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

63. A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is **[AIEEE 2012]**

(a) $\frac{9}{7}$ m/min (b) $\frac{7}{9}$ m/min
(c) $\frac{2}{9}$ m/min (d) $\frac{9}{2}$ m/min

64. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[AIEEE 2010]**

(a) $y = 0$ (b) $y = 1$
(c) $y = 2$ (d) $y = 3$

65. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have ? **[AIEEE 2008]**

(a) 5 (b) 7
(c) 1 (d) 3

66. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is **[AIEEE 2007]**

(a) $2\log_3 e$ (b) $\frac{1}{2}\log_e 3$
(c) $\log_3 e$ (d) $\log_e 3$

67. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[AIEEE 2007]**

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(b) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$
(d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Answers

Check Point 1

1. (c)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (b)
11. (c)									

Check Point 2

1. (d)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (a)

Round I

1. (a)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (d)
11. (c)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (b)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (c)
21. (d)	**22.** (d)	**23.** (a)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (b)
31. (b)	**32.** (d)	**33.** (b)	**34.** (d)	**35.** (c)	**36.** (a)	**37.** (d)	**38.** (d)	**39.** (c)	**40.** (a)
41. (a)	**42.** (b)	**43.** (a)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (a)	**48.** (a)	**49.** (a)	**50.** (c)
51. (a)	**52.** (a)	**53.** (d)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (d)	**60.** (b)
61. (d)	**62.** (c)	**63.** (c)	**64.** (a)	**65.** (d)	**66.** (c)	**67.** (d)	**68.** (d)	**69.** (b)	**70.** (d)
71. (c)	**72.** (d)	**73.** (b)	**74.** (c)	**75.** (a)					

Round II

1. (b)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (b)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (c)
11. (b)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (d)	**30.** (d)
31. (c)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (b)
41. (c)	**42.** (d)	**43.** (a,b)	**44.** (a,c)	**45.** (b)	**46.** (b,c)	**47.** (b)	**48.** (b)	**49.** (d)	**50.** (c)
51. (c)	**52.** (b)	**53.** (a)	**54.** (c)	**55.** (a)	**56.** (c)	**57.** (b)	**58.** (b)	**59.** (b)	**60.** (c)
61. (d)	**62.** (a)	**63.** (c)	**64.** (d)	**65.** (c)	**66.** (a)	**67.** (b)			

the **Guidance**

Round I

1. Use the relation, volume of sphere $= \frac{4}{3}\pi$ (radius)^3 $= \frac{4}{3}\pi\left(\frac{\text{Diameter}}{2}\right)^3$. Differentiating w.r.t. x and get the required result.

Given, the diameter of the balloon $= \frac{3}{2}(2x+1)$

$\therefore$ Radius of the balloon

$$= \frac{\text{Diameter}}{2} = \frac{1}{2}\left[\frac{3}{2}(2x+1)\right] = \frac{3}{4}(2x+1)$$

For the volume V, the balloon is given by

$$V = \frac{4}{3}\pi\,(\text{radius})^3 = \frac{4}{3}\pi\left[\frac{3}{4}(2x+1)\right]^3 = \frac{9\pi}{16}(2x+1)^3$$

For the rate of change of volume, differentiate w.r.t. x, we get

$$\frac{dV}{dx} = \frac{9\pi}{16} \times 3(2x+1)^2 \times 2 = \frac{27\pi}{8}(2x+1)^2$$

Thus, rate of change of volume is $\frac{27\pi}{8}(2x+1)^2$.

2. Let r be the radius, h be the height and V be the volume of sand cone at any time t.

It is given that $\frac{dV}{dt} = 12$ cm^3/s and $h = \frac{1}{6}r \Rightarrow r = 6h$

$$\therefore \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6h)^2 h = 12\pi h^3$$

On differentiating w.r.t. t, we get

$$\frac{dV}{dt} = (12\pi)\left(3h^2\frac{dh}{dt}\right) = 36\pi h^2\frac{dh}{dt}$$

$$\Rightarrow \quad 12 = 36\pi(4)^2\frac{dh}{dt}$$

$$\left(\because h = 4 \text{ cm and } \frac{dV}{dt} = 12 \text{ cm}^3/s\right)$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi \times 16} = \frac{1}{48\pi} \text{ cm/s}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

3. Marginal cost is the rate of change of total cost with respect to output.

$\therefore$ Marginal cost (MC)

$$= \frac{dC}{dt} = 0.007(3x^2) - 0.003(2x) + 15$$

$= 0.021x^2 - 0.006x + 15$

When $x = 17$, MC $= 0.021(17)^2 - 0.006(17) + 15$

$= 0.021(289) - 0.006(17) + 15 = 6.069 - 0.102 + 15$

$= 20.967$

Hence, when 17 units are produced, the marginal cost is ₹ 20.967.

4. Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal revenue (MR) $= \frac{dR}{dx} = \frac{d}{dx}(13x^2 + 26x + 15)$

$= 13 \times 2x + 26 = 26x + 26$

When $x = 7$, MR $= 26(7) + 26 = 182 + 26 = 208$

Hence, the required marginal revenue is ₹ 208.

5. Let ΔABC be isosceles triangle, where BC is the base of fixed length b.

Let the length of two equal sides of ΔABC be x.

Draw $AD \perp BC$ in figure.

Now, in ΔADC, by applying the Pythagoras theorem, we have

$$AD = \sqrt{x^2 - \left(\frac{b}{2}\right)^2} = \sqrt{x^2 - \frac{b^2}{4}}$$

$\therefore$ Area of triangle $(A) = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$$

The rate of change of the area A w.r.t. time t is given by

$$\frac{dA}{dt} = \frac{1}{2} b \times \frac{1}{2} \frac{2x}{\sqrt{x^2 - \frac{b^2}{4}}} \times \frac{dx}{dt} = \frac{xb}{\sqrt{4x^2 - b^2}} \frac{dx}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm/s.

$\therefore \frac{dx}{dt} = -3$ cm/s (negative sign use for decreasing)

$$\therefore \quad \frac{dA}{dt} = \frac{-3xb}{\sqrt{4x^2 - b^2}} \text{ cm}^2/\text{s}$$

When $x = b$, we have $\frac{dA}{dt} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}\, b \text{ cm}^2/\text{s}$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\, b \text{ cm}^2/\text{s}$.

Note If the rate of change is increasing, we take positive sign and if the rate of change is decreasing, we take negative sign.

6. Let AB be the position of boy who is flying the kite and C be the position of the kite at any time t.

Let $BD = x$ and $AC = y$, then $AE = x$

Given, $AB = 1.5$ m, $CD = 151.5$ m

$\therefore$ $CE = 150$ m

Given, $\frac{dx}{dt} = 10$ m/s

Here, we have to find $\frac{dy}{dt}$ when $y = 250$ m

Now, from ΔCAE, $y^2 = x^2 + 150^2$

On differentiating, we get

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} = \frac{x}{y} \cdot 10 \quad \text{...(i)}$$

In ΔACE, $x = \sqrt{250^2 - 150^2}$ $(\because y = 250)$

$= 200$ m

$\therefore$ From Eq. (i), we get $\frac{dy}{dt} = \frac{200}{250} \times 10 = 8$ m/s

7. Given, $2\frac{d}{dt}(\sin\theta) = \frac{d\theta}{dt}$

$$\Rightarrow \quad 2 \times \cos\theta \frac{d\theta}{dt} = \frac{d\theta}{dt}$$

$$\Rightarrow \quad 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

8. Surface area of sphere, $S = 4\pi r^2$ and $\frac{dr}{dt} = 2$

$$\therefore \quad \frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r \times 2 = 16\pi r \Rightarrow \frac{dS}{dt} \propto r$$

9. If x is the side of an equilateral triangle and A is its area, then

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4} 2x\frac{dx}{dt}$$

Here, $x = 10$ cm and $\frac{dx}{dt} = 2$ cm/s

$\therefore \quad A = \frac{\sqrt{3}}{4} 2(10)2 = 10\sqrt{3}\ \text{cm}^2/\text{s}$

10. Given that, $\frac{dV}{dt} = k$ (say)

$\because \quad V = \frac{4}{3}\pi R^3$

$\Rightarrow \quad \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$

$\Rightarrow \quad \frac{dR}{dt} = \frac{k}{4\pi R^2}$

Rate of increasing radius is inversely proportional to its surface area.

11. They will encounter, if

$$10 + 6t = 3 + t^2$$

$\Rightarrow \quad t^2 - 6t - 7 = 0 \quad \Rightarrow \quad t = 7$

At $t = 7$ s, moving in a first point

$$v_1 = \frac{d}{dt}(10 + 6t) = 6 \text{ cm/s}$$

At $t = 7$ s, moving in a second point

$$v_2 = \frac{d}{dt}(3 + t^2)$$

$$= 2t = 2 \times 7 = 14 \text{ cm/s}$$

$\therefore$ Resultant velocity $= v_2 - v_1 = 14 - 6 = 8$ cm/s

12. Given that, $\frac{dv}{dt} = 30$ ft^3/min and $r = 15$ ft

Volume of spherical balloon

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$\Rightarrow \quad 30 = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{30}{4 \times \pi \times 15 \times 15} = \frac{1}{30\pi}$ ft/min

13. The equation of given circle is

$$x^2 + y^2 = 1$$

On differentiating w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

But we have, $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dt} = -3$, then

$$\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \Rightarrow \frac{dx}{dt} = 3\sqrt{3}$$

14. Given point is $x = a + bt - ct^2$

Acceleration in x direction $= \frac{d^2x}{dt^2} = -2c$

and acceleration in y direction $= \frac{d^2y}{dt^2} = 2b$

$\therefore$ Resultant acceleration

$$= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

$$= \sqrt{(-2c)^2 + (2b)^2} = 2\sqrt{b^2 + c^2}$$

15. If S represents the surface area, then

$\frac{dS}{dt} = 2$ cm^2/s

$$S = \pi r l = \pi l \cdot \sin\frac{\pi}{4} l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\frac{dS}{dt} = \frac{2\pi}{\sqrt{2}} l \cdot \frac{dl}{dt} = \sqrt{2}\pi l \cdot \frac{dl}{dt}$

when $l = 4$ cm, $\frac{dl}{dt} = \frac{l}{\sqrt{2}\pi \cdot 4} \cdot 2 = \frac{1}{2\sqrt{2}\pi} = \frac{\sqrt{2}}{4\pi}$ cm/s

16. Use the relation between volume V and radius r *i.e.*, $V = \frac{4}{3}\pi r^3$, then differentiate it w.r.t. r and use $\Delta V = \left(\frac{dV}{dr}\right)\Delta r$, to find the approximate change in volume V.

Let r be the radius of the sphere and Δr be the error in measuring radius.

Then, $r = 7$ m and $\Delta r = 0.02$ m

Now, volume of a sphere is given by $V = \frac{4}{3}\pi r^3$

On differentiate w.r.t. r, we get $\frac{dV}{dr} = \left(\frac{4}{3}\pi\right)(3r^2) = 4\pi r^2$

$\therefore \quad \Delta V = \left(\frac{dV}{dr}\right)\Delta r = (4\pi r^2)\Delta r$

$$= 4\pi \times 7^2 \times 0.02 = 3.92\ \pi \text{ m}^3$$

Hence, the approximate error in calculating the volume is $3.92\ \pi$ m^3.

17. Firstly, break the number 5.001 as $x = 5$ and $\Delta x = 0.001$ and use the relation $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$.

Consider $\quad f(x) = x^3 - 7x^2 + 15$

$\Rightarrow \quad f'(x) = 3x^2 - 14x$

Let $\quad x = 5$

and $\quad \Delta x = 0.001$

Also, $\quad f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$

Therefore,

$$f(x + \Delta x) \simeq (x^3 - 7x^2 + 15) + \Delta x(3x^2 - 14x)$$

$\Rightarrow \quad f(5.001) \simeq (5^3 - 7 \times 5^2 + 15) + (3 \times 5^2 - 14 \times 5)(0.001)$

(as $x = 5$, $\Delta x = 0.001$)

$= 125 - 175 + 15 + (75 - 70)(0.001)$

$= -35 + (5)(0.001)$

$= -35 + 0.005 = -34.995$

18. Find the slope of the tangent to the curve and write the general equation for tangent. Since, tangent passes through the origin, so (0, 0) will satisfy it.

The equation of the given curve is $y = 4x^3 - 2x^5$

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by

$$Y - y = (12x^2 - 10x^4)(X - x) \quad \ldots(i)$$

When the tangent passes through the origin (0, 0), then $X = Y = 0$

Therefore, Eq.(i) reduced to

$$-y = (12x^2 - 10x^4)(-x) \Rightarrow y = 12x^3 - 10x^5$$

Also, we have $y = 4x^3 - 2x^5$

$\therefore \quad 12x^3 - 10x^5 = 4x^3 - 2x^5$

$\Rightarrow \quad 8x^5 - 8x^3 = 0$

$\Rightarrow \quad x^5 - x^3 = 0$

$\Rightarrow \quad x^3(x^2 - 1) = 0$

$\Rightarrow \quad x = 0, \pm 1$

When $x = 0$, $y = 4(0)^3 - 2(0)^5 = 0$. When $x = 1$,

$$y = 4(1)^3 - 2(1)^5 = 2.$$

When $x = -1$, $y = 4(-1)^3 - 2(-1)^5 = -2$

Hence, the required points are (0, 0), (1, 2) and (–1, –2).

19. Given curve is $y = 2x^2 - x + 1$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 4x - 1$$

Since, this is parallel to the given curve $y = 3x + 9$.

$\therefore$ These slopes are equal.

$\Rightarrow \quad 4x - 1 = 3 \Rightarrow x = 1$

At $x = 1$, $y = 2(1)^2 - 1 + 1 \Rightarrow y = 2$

Thus, the point is (1, 2).

20. Equation of the curve is $x^2y^2 = a^4$.

On differentiating w.r.t. x, we get

$$x^2 2y\frac{dy}{dx} + y^2 2x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(-a, a)} = -\left(\frac{a}{-a}\right) = 1$$

Therefore, length of subtangent at the point $(-a, a)$

$$= \frac{y}{\left(\frac{dy}{dx}\right)} = \frac{a}{1} = a$$

21. Given curve is $xy = c^2$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

Length of subnormal $= y\dfrac{dy}{dx}$

$$= \frac{y \times (-c)^2}{x^2} = \frac{-yc^2}{\left(\frac{c^2}{y}\right)^2} \quad \left[\text{from Eq. (i)}, x = \frac{c^2}{y}\right]$$

$$= \frac{-yc^2y^2}{c^4} = \frac{-y^3}{c^2}$$

$\therefore$ Subnormal varies as y^3.

22. $\because \quad y = x^3 - 3x^2 - 9x + 5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Since, tangent is parallel to x-axis.

$\therefore \quad \dfrac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow \quad (x + 1)(x - 3) = 0 \Rightarrow x = -1, 3$

23. Given curve is $x = 3t^2 + 1, y = t^3 - 1$

For $x = 1$, $\quad 3t^2 + 1 = 1 \Rightarrow t = 0$

$\therefore \quad \dfrac{dx}{dt} = 6t, \quad \dfrac{dy}{dx} = 3t^2$

Now, $\quad \dfrac{dy}{dx} = \left(\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \dfrac{3t^2}{6t} = \dfrac{t}{2}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(t=0)} = \dfrac{0}{2} = 0$

24. We have, $x = at^2 \Rightarrow \dfrac{dx}{dt} = 2at$

and $y = 2at \Rightarrow \dfrac{dy}{dt} = 2a$

$\therefore$ Slope of tangent $= \dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

$\Rightarrow \quad \dfrac{1}{t} = \infty$

$\Rightarrow \quad t = 0 \Rightarrow$ Point of contact is (0, 0).

25. The given curve is $(1+x^2)\,y = 2-x$ ...(i)

It meets x-axis, where $y=0 \Rightarrow 0 = 2-x \Rightarrow x=2$

So, Eq. (i) meets x-axis at the point (2, 0).

Also, from Eq. (i), $y = \dfrac{2-x}{1+x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+x^2)(-1)-(2-x)(2x)}{(1+x^2)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2-4x-1}{(1+x^2)^2}$$

$\therefore$ Slope of tangent at (2, 0) $= \dfrac{2^2-4(2)-1}{(1+2^2)^2}$

$$= \frac{4-8-1}{(1+4)^2} = -\frac{5}{25} = -\frac{1}{5}$$

$\therefore$ Equation of tangent at (2, 0) with slope $-\dfrac{1}{5}$ is

$$y-0 = -\frac{1}{5}(x-2)$$

$$\Rightarrow \quad 5y = -x+2$$

$$\Rightarrow \quad x+5y = 2$$

26. The equation of given curve is $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

$$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

The normal is parallel to x-axis, if

$$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \quad \Rightarrow \quad x_1 = 0$$

$\therefore$ From equation of curve, $y_1 = a$

$\therefore$ Required point is $(0, a)$.

27. Given curve is $y = x\log x$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1+\log x$$

The slope of the normal $= -\dfrac{1}{(dy/dx)} = \dfrac{-1}{1+\log x}$

The slope of the given line $2x-2y=3$ is 1.

Since, these lines are parallel.

$$\therefore \quad \frac{-1}{1+\log x} = 1$$

$$\Rightarrow \quad \log x = -2$$

$$\Rightarrow \quad x = e^{-2}$$

and $\quad y = -2\,e^{-2}$

$\therefore$ Coordinates of the point are $(e^{-2}, -2\,e^{-2})$.

28. Given curve is

$$x = a(t+\sin t),\ y = a(1-\cos t)$$

$$\Rightarrow \quad \frac{dx}{dt} = a(1+\cos t),\ \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\sin t)}{a(1+\cos t)}$$

$$= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

Length of the normal $= y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$$= a(1-\cos t)\sqrt{1+\tan^2\left(\frac{t}{2}\right)}$$

$$= a(1-\cos t)\sec\left(\frac{t}{2}\right)$$

$$= 2\,a\sin^2\left(\frac{t}{2}\right)\sec\left(\frac{t}{2}\right)$$

$$= 2\,a\sin\left(\frac{t}{2}\right)\tan\left(\frac{t}{2}\right)$$

29. Given curve is $y = e^{2x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\,e^{2x} \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 2\,e^0 = 2$$

Equation of tangent at (0, 1) with slope 2 is

$$y-1 = 2(x-0) \quad \Rightarrow \quad y = 2x+1$$

This tangent meets x-axis.

$$\therefore \quad y = 0$$

$$\Rightarrow \quad 0 = 2x+1 \quad \Rightarrow \quad x = -\frac{1}{2}$$

$\therefore$ Coordinates of the point on x-axis is $\left(-\dfrac{1}{2}, 0\right)$.

30. Given curve is $y = 2x^2-x+1$.

Let the coordinate of P are (h, k).

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 4x-1$$

At the point (h, k), the slope $= \left(\dfrac{dy}{dx}\right)_{(h,k)} = 4h-1$

Since, the tangents is parallel to the given line $y = 3x+4$.

$$\Rightarrow \quad 4h-1 = 3 \quad \Rightarrow \quad h=1, k=2$$

$\therefore$ Coordinates of point P are (1, 2).

31. Length of subtangent $= y\dfrac{dx}{dy}$

and length of subnormal $= y\dfrac{dy}{dx}$

$$\therefore \quad \text{Product} = y^2$$

$\Rightarrow$ Required product is the square of the ordinate.

32. Slope of the curve at an angle $\theta = \frac{3\pi}{4}$ is

$$\frac{dy}{dx} = \tan\frac{3\pi}{4} = -1$$

$$\text{Slope of the normal} = \frac{-1}{dy/dx}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(3,4)} = 1 \Rightarrow \phi'(3) = 1$$

33. The equations of given curves are

$$y = a^x \quad \text{...(i)}$$

and $$y = b^x \quad \text{...(ii)}$$

From Eq. (i), $m_1 = \frac{dy}{dx} = a^x \log a$

and from Eq. (ii), $m_2 = \frac{dy}{dx} = b^x \log b$

From Eqs. (i) and (ii), we get

$$a^x = b^x \Rightarrow x = 0$$

Let α be the angle at which the two curves intersect.

$$\therefore \quad \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{a^x \log a - b^x \log b}{1 + a^x b^x (\log a)(\log b)}$$

$$= \frac{\log\frac{a}{b}}{1 + (\log a)(\log b)}$$

34. Given equation of curve is $y = be^{-x/a}$ ...(i)

Since, the curve crosses y-axis *i.e.*, $x = 0$.

$$\Rightarrow \quad y = be^{-0} \Rightarrow y = b$$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

At point $(0, b)$, $\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a} e^{-0/a} = \frac{-b}{a}$

$\therefore$ Required equation of tangent is

$$y - b = \frac{-b}{a}(x - 0)$$

$$\Rightarrow \quad \frac{y}{b} - 1 = -\frac{x}{a}$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

35. Given curve is $x^3 - 8a^2 y = 0$.

On differentiating w.r.t. x, we get

$$3x^2 - 8a^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{8a^2}$$

$$\therefore \text{ Slope of the normal} = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{\frac{3x^2}{8a^2}} = -\frac{8a^2}{3x^2}$$

Given, $\frac{-8a^2}{3x^2} = \frac{-2}{3} \Rightarrow x^2 = 4a^2$

$$\Rightarrow \quad x = \pm 2a$$

At $x = \pm 2a, y = \pm a$

$$(x, y) = (2a, a)$$

36. Given equation is

$$y - e^{xy} + x = 0$$

$$\Rightarrow \quad e^{xy} = x + y$$

On taking log on both sides, we get

$$\log(x + y) = xy$$

On differentiating w.r.t. x, we get

$$\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y+x} - x\right) = y - \frac{1}{y+x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(y+x) - 1}{1 - x(y+x)}$$

Since, the curve has a vertical tangent.

$$\therefore \quad \frac{dy}{dx} = \infty \Rightarrow 1 - x(x+y) = 0$$

which is satisfied by the point (1, 0).

37. Given curve is $y^3 + 3x^2 = 12y$

On differentiating w.r.t. y, we get

$$3y^2 \frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(3y^2 - 12) + 6x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$$

Since, tangent is parallel to y-axis.

$$\frac{dx}{dy} = 0 \Rightarrow 12 - 3y^2 = 0$$

$$\Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2$$

Then, $$x = \pm\frac{4}{\sqrt{3}}$$

At $y = -2$, x cannot be real.

$\therefore$ The required point is $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

38. Curve is $y^2 = px^3 + q$

$$\therefore \quad 2y\frac{dy}{dx} = 3px^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3p \cdot 4}{2 \cdot 3}$$

$\Rightarrow \quad 4 = 2p$

$\Rightarrow \quad p = 2$

Also, curve is passing through (2, 3).

$\therefore \quad 9 = 8p + q$

$\Rightarrow \quad q = -7$

$\therefore$ (p, q) is $(2, -7)$.

39. Given curve is $y = f(x) = x^2 + bx - b$.

On differentiating, we get $\frac{dy}{dx} = 2x + b$

The equation of the tangent at (1, 1) is

$$y - 1 = \left(\frac{dy}{dx}\right)_{(1,1)} (x - 1)$$

$\Rightarrow \quad y - 1 = (b + 2)(x - 1)$

$\Rightarrow \quad (2 + b)x - y = 1 + b$

$\Rightarrow \quad \dfrac{x}{\frac{(1+b)}{(2+b)}} - \dfrac{y}{1+b} = 1$

So, $\quad OA = \dfrac{1+b}{2+b}$ and $OB = -(1+b)$

Now, $\quad$ area of $\Delta AOB = \frac{1}{2} \cdot \frac{1+b}{2+b} \cdot [-(1+b)] = 2$ $\quad$ (given)

$\Rightarrow \quad 4(2 + b) + (1 + b)^2 = 0$

$\Rightarrow \quad 8 + 4b + 1 + b^2 + 2b = 0$

$\Rightarrow \quad b^2 + 6b + 9 = 0$

$\Rightarrow \quad (b + 3)^2 = 0 \quad \Rightarrow \quad b = -3$

40. $\frac{dy}{dx} = -\frac{x_1^2}{y_1^2}$

$$\text{tangent } y - y_1 = -\frac{x_1^2}{y_1^2}(x - x_1) \quad \ldots\text{(i)}$$

$$x_2^3 + y_2^3 = a^3 \quad \ldots\text{(ii)}$$

$$x_1^3 + y_1^3 = a^3 \quad \ldots\text{(iii)}$$

On putting $y = y_2$ and $x = x_2$ and solving with Eqs. (ii) and (iii), we get result

41. Equation of given curves are

$$y = \sin x \quad \ldots\text{(i)}$$

and $\quad y = \cos x \quad \ldots\text{(ii)}$

On solving Eqs. (i) and (ii), we get

$$x = \frac{\pi}{4}$$

$\therefore$ Point of intersection of curves is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

For $y = \sin x$, $\frac{dy}{dx} = \cos x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\pi/4} = \frac{1}{\sqrt{2}} = m_1$ $\quad$ (say)

For $y = \cos x \quad \Rightarrow \quad \frac{dy}{dx} = -\sin x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\pi/4} = -\frac{1}{\sqrt{2}} = m_2$ $\quad$ (say)

$\therefore \quad \tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} = \dfrac{\frac{2}{\sqrt{2}}}{\frac{1}{2}}$

$\Rightarrow \quad \tan\theta = 2\sqrt{2} \quad \Rightarrow \quad \theta = \tan^{-1}(2\sqrt{2})$

42. Let the curves intersect at (x_1, y_1). Therefore,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$

$\Rightarrow$ Slope of tangent at the point of intersection

$$(m_1) = \frac{b^2 x_1}{a^2 y_1}$$

Again, $xy = c^2 \quad \Rightarrow \quad x\frac{dy}{dx} + y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow \quad m_2 = \frac{-y_1}{x_1}$

For orthogonality, $m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{b^2}{a^2} = 1$ or $a^2 - b^2 = 0$

43. y is an increasing function throughout its domain, if $y' > 0$ throughout the domain.

Given, $\quad y = \log(1 + x) - \frac{2x}{(2 + x)}$.

On differentiating, we get $\frac{dy}{dx} = \frac{d}{dx}\left[\log(1 + x) - \frac{2x}{2 + x}\right]$

$$= \frac{1}{1+x} - \frac{(2+x)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(2+x)}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4 + 2x - 2x}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2} = \frac{4 + x^2 + 4x - 4 - 4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2}$$

When, $x \in (-1, \infty)$, then $\frac{x^2}{(2+x)^2} > 0$ and $(1+x) > 0$

$\therefore \quad y' > 0$ when $x > -1$

Hence, y is an increasing function throughout $(x > -1)$ its domain.

44. Firstly, we find $f'(x)$, then for $1 < x < 2$, put $f'(x) > 0$ and find the required least value of a.

Given, $f(x) = x^2 + ax + 1 \Rightarrow f'(x) = 2x + a$

In interval $(1, 2), 1 < x < 2 \Rightarrow 2 < 2x < 4$

$\Rightarrow \quad (2 + a) < (2x + a) < (4 + a)$

Since, $f(x)$ is strictly increasing function, then $(2 + a) > 0$ [for this $f'(x) > 0$ and $(2x + a) > (2 + a)$]

$\therefore \quad (2 + a) > 0 \Rightarrow a > -2$

Hence, the least value of $a = -2$

45. Given, $f'(x) > 0$ on $[a, b]$

$\therefore$ f is a differentiable function on $[a, b]$

Also, every differentiable function is continuous, therefore f is continuous on $[a, b]$.

Let $x_1, x_2 \in [a, b]$ and $x_2 > x_1$, then by Lagrange mean value theorem, there exist $c \in [a, b]$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\Rightarrow \quad f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$

$\Rightarrow \quad f(x_2) - f(x_1) > 0$ as $x_2 > x_1$

and $\quad f'(x) > 0 \Rightarrow f(x_2) > f(x_1)$

$\therefore$ For $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Hence, f is an increasing function on (a, b).

46. Let $y = x^x$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = x^x(1 + \log x)$$

For increasing function,

$$\frac{dy}{dx} > 0$$

$\Rightarrow \quad x^x(1 + \log x) > 0 \Rightarrow 1 + \log x > 0$

$\Rightarrow \quad \log_e x > \log_e \frac{1}{e} \Rightarrow x > \frac{1}{e}$

Function is increasing when $x > \frac{1}{e}$.

47. $\because \; g(x) = \min(x, x^2)$

$\therefore$ $g(x)$ is an increasing function.

48. Given, $\quad f(x) = \frac{x}{1 + |x|}$

$$\therefore \quad f'(x) = \frac{(1 + |x|) \cdot 1 - x \cdot \frac{|x|}{x}}{(1 + |x|)^2}$$

$$= \frac{1}{(1 + |x|)^2} > 0, \forall x \in R$$

$\Rightarrow$ $f(x)$ is strictly increasing.

49. $\because \; f(x) = x + \cos x$

On differentiating, we get $f'(x) = 1 - \sin x$

$f'(x) > 0$ for all values of x

($\because \sin x$ is lying between -1 to $+1$)

$\therefore$ $f(x)$ is always increasing.

50. Let $f(x) = 2x^3 - 6x + 5$

On differentiating w.r.t. x, we get

$$f'(x) = 6x^2 - 6$$

Since, it is increasing function.

$\Rightarrow \quad 6x^2 - 6 > 0$

$\Rightarrow \quad (x - 1)(x + 1) > 0$

$\Rightarrow \quad x > 1$ or $x < -1$

51. Let $f(x) = 3 \sin x - 4 \sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad -\frac{\pi}{2} \le 3x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{6} \le x \le \frac{\pi}{6}$

Thus, length of interval $= \left|\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{\pi}{3}$

52. $\because f(x) = xe^{x(1-x)}$

On differentiating w.r.t. x, we get

$$f'(x) = e^{x(1-x)} + x \cdot e^{x(1-x)} \cdot (1 - 2x)$$

$$= e^{x(1-x)} \{1 + x(1 - 2x)\}$$

$$= e^{x(1-x)} \cdot (-2x^2 + x + 1)$$

It is clear that $e^{x(1-x)} > 0$ for all x.

Now, by sign rule for $-2x^2 + x + 1$

$f'(x) > 0$, if $x \in \left[-\frac{1}{2}, 1\right]$.

So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

53. We know that, $\sin x$ and $\cos x$ decrease in $\frac{\pi}{2} < x < \pi$, so the statement S is correct.

The statement R is incorrect which is clear from the graph that $f(x)$ is differentiable in (a, b).

Also, $\quad a < x_1 < x_2 < b$

But $\quad f'(x_1) = \tan \phi_1 < \tan \phi_2 = f'(x_2)$

$\Rightarrow$ Derivate is increasing.

54. $$\sin\frac{\pi}{x}+\left(\frac{\pi}{x^2}\right)\geq 0$$

$$2n\pi<\frac{\pi}{x}<(2n+1)\,\pi$$

$$\frac{1}{2n+1}<x<\frac{1}{2n}$$

55. $f(x)=(\sin^2 x+\cos^2 x)-2\sin^2 x\cos^2 x$

$$=1-\frac{1}{2}\sin^2 2x$$

Now, evaluate $f'(x)$ and decide monotonicity.

56. $f''(x)\geq 0 \Rightarrow f'(x)$ is increasing function and

$$g'(x)=(2x-2)\,[f'(x^2-2x+8)-f'(14+2x-x^2)]$$

For $g'(x)>0$

(i) $x>1$ and $x^2-2x+8>14+2x-x^2$

or

(ii) $x<1$ and $x^2-2x+8<14+2x-x^2$

57. Given, $f(x)=\log(\cos x)$

$\Rightarrow \quad f'(x)=\dfrac{1}{\cos x}.(-\sin x)=-\tan x$ (differentiate w.r.t. x)

In interval $\left(0,\dfrac{\pi}{2}\right)$, $\tan x>0$ ($\because \tan x$ is in Ist quadrant)

$\Rightarrow \quad -\tan x<0$ ($\because \tan x$ is in Ist quadrant)

$\therefore \quad f'(x)<0$ in $\left(0,\dfrac{\pi}{2}\right)$

Hence, f is strictly decreasing in $\left(0,\dfrac{\pi}{2}\right)$.

Also, in interval $\left(\dfrac{\pi}{2},\pi\right)$, $\tan x<0 \Rightarrow -\tan x>0$

($\because \tan x$ is in IInd quadrant)

$\therefore \quad f'(x)>0$ in $\left(\dfrac{\pi}{2},\pi\right)$

Hence, f is strictly increasing in $\left(\dfrac{\pi}{2},\pi\right)$.

58. The graph of cosec x is opposite in interval $\left(\dfrac{\pi}{2},\dfrac{3\pi}{2}\right)$.

59. Let $\quad f(x)=2x^3-9x^2+12x+4$

$\Rightarrow \quad f'(x)=6x^2-18x+12$

$f'(x)<0$ for function to be decreasing

$\Rightarrow \quad 6(x^2-3x+2)<0$

$\Rightarrow \quad (x^2-2x-x+2)<0$

$\Rightarrow \quad (x-2)(x-1)<0$

$\Rightarrow \quad 1<x<2$

60. Given curve is $f(x)=\dfrac{1}{x+1}-\log(1+x)$

On differentiating w.r.t. x, we get

$$f'(x)=-\frac{1}{(x+1)^2}-\frac{1}{1+x}$$

$$\Rightarrow \quad f'(x)=-\left[\frac{1}{x+1}+\frac{1}{(x+1)^2}\right]$$

$\Rightarrow \quad f'(x)=-$ve, when $x>0$

$\therefore$ $f(x)$ is a decreasing function.

61. $\because \quad f(x)=\sin x-\cos x$

On differentiating w.r.t. x, we get

$f'(x)=\cos x+\sin x$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$=\sqrt{2}\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$

$$=\sqrt{2}\left[\cos\left(x-\frac{\pi}{4}\right)\right]$$

For decreasing, $f'(x)<0$

$\dfrac{\pi}{2}<\left(x-\dfrac{\pi}{4}\right)<\dfrac{3\pi}{2}$ (within $0\leq x\leq 2\pi$)

$$\Rightarrow \quad \frac{\pi}{2}+\frac{\pi}{4}<\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right)<\frac{3\pi}{2}+\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{3\pi}{4}<x<\frac{7\pi}{4}$$

62. Let $y=x^{1/x}$

On taking log on both sides, we get

$$\log y=\frac{1}{x}\log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2}-\frac{\log x}{x^2}=\frac{1-\log x}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=x^{1/x}\left(\frac{1-\log x}{x^2}\right)$$

Now, $x^{1/x}>0$ for all positive values of x

and $\quad \dfrac{1-\log x}{x^2}>0$ in $(1, e)$

and $\quad \dfrac{1-\log x}{x^2}<0$ in (e, ∞)

$\therefore$ $f(x)$ is increasing in $(1, e)$ and decreasing in (e, ∞).

63. $2, \frac{5}{2}, 3 \in \left(\frac{\pi}{2}, \pi\right)$, in which sine function decreases and tangent function increases.

Hence, the function $\sin(\tan x)$ is a decreasing function in $\left(\frac{\pi}{2}, \pi\right)$. Thus, $\sin(\tan 3)$ is smallest among first three numbers. Now, $\sin(\tan 3) < 0$ and $\sin(\tan 4) > 0$. Thus, $\sin(\tan 3)$ is the smallest.

64. Among $\cos 1, \cos 2, \cos 3$; $\cos 1$ is biggest as cosine function decreases in $[0, \pi]$. As $\cos 4 < 0$, $\cos 1$ is the biggest.

65. $x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\} = 0$

$\Rightarrow \quad x = 0$

or $\quad f'\left(\frac{x^2}{2}\right) = f'(6 - x^2)$

$- \quad + \quad - \quad +$
$-2 \quad 0 \quad +2$

or $\quad \frac{x^2}{2} = 6 - x^2$

or $\quad 3x^2 = 12$

$\Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

66. $f(x) = \tan x - 4x \quad \Rightarrow \quad f'(x) = \sec^2 x - 4$

When $\frac{-\pi}{3} < x < \frac{\pi}{3}, 1 < \sec x < 2$

Therefore, $1 < \sec^2 x < 4 \quad \Rightarrow \quad -3 < (\sec^2 x - 4) < 0$

Thus, for $\frac{-\pi}{3} < x < \frac{\pi}{3}, f'(x) < 0$

Hence, f is strictly decreasing on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

67. $\because \quad f(x) = \frac{\lambda \sin x + 6 \cos x}{2 \sin x + 3 \cos x} \quad \ldots(i)$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\left[\begin{array}{l}(2 \sin x + 3 \cos x)(\lambda \cos x - 6 \sin x) \\ -(\lambda \sin x + 6 \cos x)(2 \cos x - 3 \sin x)\end{array}\right]}{(2 \sin x + 3 \cos x)^2}$$

The function is monotonic increasing, if $f'(x) > 0$

$\Rightarrow \quad 3\lambda(\sin^2 x + \cos^2 x) - 12(\sin^2 x + \cos^2 x) > 0$

$\Rightarrow \quad 3\lambda - 12 > 0 \qquad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \quad \lambda > 4$

68. $\because f(x) = x^3 - 6x^2 + ax + b$

On differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 12x + a$$

By the definition of Rolle's theorem

$$f'(c) = 0 \quad \Rightarrow \quad f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$$

$$\Rightarrow \quad 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow \quad 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow \quad 12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0$$

$$\Rightarrow \quad a = 11$$

69. $\int_1^2 f'(x)\, dx = [f(x)]_1^2 = f(2) - f(1) = 0$

[$\because f(x)$ satisfies the conditions of Rolle's theorem]

$\therefore \quad f(2) = f(1)$

70. Since, $f(x)$ satisfies all the conditions of Rolle's theorem in $[3, 5]$.

Let $\quad f(x) = (x-3)(x-5) = x^2 - 8x + 15$

Now, $\quad \int_3^5 f(x)\, dx = \int_3^5 (x^2 - 8x + 15)\, dx$

$$= \left[\frac{x^3}{3} - \frac{8x^2}{2} + 15x\right]_3^5$$

$$= \left(\frac{125}{3} - 100 + 75\right) - (9 - 36 + 45)$$

$$= \frac{50}{3} - 18 = -\frac{4}{3}$$

71. From mean value theorem $f'(c) = \frac{f(b) - f(a)}{b - a}$

Given, $\quad a = 0 \quad \Rightarrow \quad f(a) = 0$

and $\quad b = \frac{1}{2} \quad \Rightarrow \quad f(b) = \frac{3}{8}$

Now, $\quad f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$

$\therefore \quad f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$

$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$

$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2$

By definition of mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0} = \frac{3}{4}$$

$$\Rightarrow \quad 3c^2 - 6c + \frac{5}{4} = 0$$

This is a quadratic equation in c.

$$\therefore \quad c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}$$

Since, c lies between $\left[0, \frac{1}{2}\right]$.

$$\therefore \quad c = 1 - \frac{\sqrt{21}}{6} \qquad \left(\text{neglecting } c = 1 + \frac{\sqrt{21}}{6}\right)$$

72. To satisfy Rolle's theorem it should be continuous in [0, 1] *i.e.*, RHL = $f(0)$

$$\Rightarrow \quad \lim_{x\to 0^+} \frac{\log x}{x^{-\alpha}} = 0$$

$$\Rightarrow \quad \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\alpha\, x^{-\alpha-1}} = 0 \qquad \text{(using L'-Hospital's rule)}$$

$$\Rightarrow \quad \lim_{x\to 0^+} -\frac{1}{\alpha\, x^{-\alpha}} = 0$$

$$\Rightarrow \quad \lim_{x\to 0^+} -\frac{x^{\alpha}}{\alpha} = 0$$

which shows $\alpha > 0$ otherwise it would be discontinuous also.

When $\alpha > 0$, $f(x)$ is differentiable in (0, 1) and $f(1) = f(0) = 0$.

Clearly, $\alpha > 0$, thus $\alpha = \frac{1}{2}$ is the possible answer.

73. $f(1) = f(2)$

$$\Rightarrow \quad 1 + b + c = 8 + 4b + 2c$$

$$f'(4/3) = 0 \quad \Rightarrow \quad 3 \cdot \frac{16}{9} + 2b\,\frac{4}{3} + c = 0$$

On solving both, we get $b = -5, c = 8$.

74. $f'(x) = 3x^2 - p$

$$= 3\left(x - \sqrt{\frac{p}{3}}\right)\left(x + \sqrt{\frac{p}{3}}\right)$$

75. Let $f'(x) = ax^2 + bx + c$

$$\Rightarrow \quad f(x) = \frac{2ax^3 + 3bx^2 + 6cx + d}{6}$$

$$\Rightarrow \quad f(0) = f(1) = \frac{d}{6}$$

Round II

1. Equation of line joining the points (0, 3) and (5, −2) is $y = 3 - x$. If this line is tangent to $y = \frac{ax}{(x+1)}$, then $(3 - x)(x + 1) = ax$ should have equal roots. Thus, $(a - 2)^2 + 12 = 0 \Rightarrow$ no value of $a \Rightarrow a \in \phi$.

2. We have, $f(x) = x^3 + bx^2 + cx + d$

$$\Rightarrow \quad f'(x) = 3x^2 + 2bx + c$$

Let D_1 be the discriminant of $f'(x) = 3x^2 + 2bx + c$.

Then, $\quad D_1 = 4b^2 - 12c = 4(b^2 - c) - 8c < 0$

$(\because b^2 < c \text{ and } c > 0)$

$$\Rightarrow \quad f'(x) > 0 \text{ for all } x \in (-\infty, \infty)$$

$f(x)$ is strictly increasing function on $(-\infty, \infty)$.

3. $f'(x) = |\log_2[\log_3\{\log_4(\cos x + a)\}]|$

Clearly, $f(x)$ is increasing for all values of x, if

$$\log_2[\log_3\{\log_4(\cos x + a)\}]$$

is defined for all values of x.

$$\Rightarrow \quad \log_3[\log_4(\cos x + a)] > 0, \forall x \in R$$

$$\Rightarrow \quad \log_4(\cos x + a) > 1, \forall x \in R$$

$$\Rightarrow \quad \cos x + a > 4, \forall x \in R$$

$$\Rightarrow \quad a > 5$$

4. On solving, $y^2 = 4ax$ and $x^2 = 4by$, we get $x = 0$ or $x^3 = 64 - ab^2$. Slope of the curves at the common points are $\frac{2a}{y}$ and $\frac{x}{2b}$, respectively. If these parabola intersect orthogonally, then $\frac{2a}{y} \cdot \frac{x}{2b} = -1$

$$\Rightarrow \quad ax + by = 0$$

$$\Rightarrow \quad ax + \frac{x^2}{4} = 0$$

$$\Rightarrow \quad x = -4a$$

$$\Rightarrow \quad -x^3 = 64a^3 \qquad (\because x \neq 0)$$

$$\Rightarrow \quad 64ab^2 + 64a^3 = 0$$

$$\Rightarrow \quad a^2 + b^2 = 0$$

which is not possible.

5. Let $y = f(x) = ax^2 + bx + c$, we have

$$f(1) = 1 \quad \Rightarrow \quad a + b + c = 1.$$

Also, $ax^2 + bx + c = x$ should have $x - 1$ as it's repeated root.

$$\Rightarrow \quad ax^2 + (b - 1)x + c = a(x - 1)^2$$

$$\Rightarrow \quad 1 - b = 2a,\ a = c$$

We have, $\quad f'(x) = 2ax + b,\ f''(x) = 2a$

$$\Rightarrow \quad f''(1) = 2a,\ f'(0) = b$$

$$\Rightarrow \quad f''(1) + f'(0) = 1$$

6. Let $A \equiv \{x_1, f(x_1)\}$ and $B \equiv \{x_2, f(x_2)\}$ be any two points on the graph of $y = f(x)$. Chord AB will be lie completely above the graph of $y = f(x)$. Hence, $\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right)$.

7. $g'(x) = f'(\sin x)\cdot\cos x - f'(\cos x)\cdot\sin x$

$\Rightarrow \quad g''(x) = -f'(\cos x)\cdot\sin x + \cos^2 x$

$f''(\sin x) + f''(\cos x)\cdot\sin^2 x - f'(\cos x)\cdot\cos x > 0,$

$\forall\, x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$. Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow \quad g'(x) > 0,\ \forall\, x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0,\ \forall\, x \in \left(0, \frac{\pi}{4}\right)$

Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$.

8. $t = 2$, for the point $(2, -1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3} = \frac{6}{7}, \text{ for } t = 2$$

9. Let $f(x) = (a+2)\,x^3 - 3\,ax^2 + 9\,ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow \quad 3(a+2)\,x^2 - 6\,ax + 9\,a \le 0$ for all $x \in R$

$\Rightarrow \quad (a+2)\,x^2 - 2\,ax + 3\,a \le 0$ for all $x \in R$

$\Rightarrow \quad a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow \quad a < -2,\ -8\,a^2 - 24\,a \le 0$

$\Rightarrow \quad a < -2$ and $a\,(a+3) \ge 0$

$\Rightarrow \quad a < -2,\ a \le -3$ or $a \ge 0$

$\Rightarrow \quad a \le -3$

$\Rightarrow \quad -\infty < a \le -3$

10. Now, $\quad f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x}$

$$= \frac{\cos x\,(\tan x - x)}{\sin^2 x}$$

$\therefore \quad f'(x) > 0$ for $0 < x \le 1$

$\therefore \quad f(x)$ is an increasing function.

Now, $\quad g'(x) = \dfrac{\tan x - x\sec^2 x}{\tan^2 x}$

$$= \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$$

Again, now $\dfrac{d}{dx}(\sin 2x - 2x) = 2\cos 2x - 2$

$$= 2(\cos 2x - 1) < 0$$

$\Rightarrow \quad \sin 2x - 2x$ is decreasing.

$\Rightarrow \quad \sin 2x - 2x < 0$

$\therefore \quad g'(x) < 0$

$\therefore \quad g(x)$ is decreasing.

11. Let $y = \dfrac{a\sin x + b\cos x}{c\sin x + d\cos x}$

The function will be decreasing, when $\dfrac{dy}{dx} < 0$.

$$\frac{\left[\begin{array}{l}(c\sin x + d\cos x)(a\cos x - b\sin x)\\ \qquad -(a\sin x + b\cos x)(c\cos x - d\sin x)\end{array}\right]}{(c\sin x + d\cos x)^2} < 0$$

$\Rightarrow \quad ac\sin x\cos x - bc\sin^2 x + ad\cos^2 x$

$\quad - bd\sin x\cos x - ac\sin x\cos x + ad\sin^2 x$

$\quad - bc\cos^2 x + bd\sin x\cos x < 0$

$\Rightarrow \quad ad(\sin^2 x + \cos^2 x) - bc(\sin^2 x + \cos^2 x) < 0$

$\Rightarrow \quad (ad - bc) < 0$

12. Given, $\quad f(x) = x + \cos x$

$\Rightarrow \quad f'(x) = 1 - \sin x$

$\Rightarrow \quad f'(x) > 0$ for all values of x

$\therefore \quad f(x)$ is always increasing.

13. To determine c in Rolle's theorem, $f'(c) = 0$.

Here, $\quad f'(x) = (x^2 + 3x)\,e^{-(1/2)x}\left(-\frac{1}{2}\right)$

$\quad + (2x+3)\,e^{-(1/2)x}$

$$= e^{-\left(\frac{1}{2}\right)x}\left\{-\frac{1}{2}(x^2 + 3x) + 2x + 3\right\}$$

$$= -\frac{1}{2}\,e^{-(x/2)}(x^2 - x - 6)$$

$\because \quad f'(c) = 0$

$\Rightarrow \quad c^2 - c - 6 = 0$

$\Rightarrow \quad c = 3, -2$

But $\quad c = 3 \notin [-3, 0]$

14. Since, $\quad \dfrac{f(2) - f(0)}{2 - 0} = f'(x)$

$\Rightarrow \quad \dfrac{f(2) - 0}{2} = f'(x) \Rightarrow \dfrac{df(x)}{dx} = \dfrac{f(2)}{2}$

$\Rightarrow \quad f(x) = \dfrac{f(2)}{2}\,x + c$

$\because \quad f(0) = 0 \Rightarrow c = 0$

$\therefore \quad f(x) = \dfrac{f(2)}{2}\,x \qquad \ldots(i)$

Also, $\quad |f'(x)| \le \dfrac{1}{2}$

$\Rightarrow \quad \left|\dfrac{f(2)}{2}\right| \le \dfrac{1}{2} \qquad \ldots(ii)$

From Eq. (i),

$$|f(x)| = \left|\frac{f(2)}{2}\,x\right| = \left|\frac{f(2)}{2}\right|\,|x| \le \frac{1}{2}\,|x|$$

In interval $[0, 2]$, for maximum $x\,(x = 2)$

$$|f(x)| \le \frac{1}{2}\cdot 2$$

$\Rightarrow \quad |f(x)| \le 1$

15. Since, $f(1) = f(3)$

$\Rightarrow \quad a + b - 5 = 3a + b - 27$

$\Rightarrow \quad a = 11$, which given in option (a) only.

16. Let the point be (x_1, y_1). Therefore, $y_1 = (x_1 - 3)^2$ ...(i)

Now, slope at the tangent at (x_1, y_1) is $2(x_1 - 3)$, but it is equal to 1.

Therefore, $2(x_1 - 3) = 1$

$$\Rightarrow \quad x_1 = \frac{7}{2}$$

$$\therefore \quad y_1 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$$

Hence, the point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

17. $f(x) = \sqrt{x}$

$\therefore f(a) = \sqrt{4} = 2$

$f(b) = \sqrt{9} = 3, \; f'(x) = \frac{1}{2\sqrt{x}}$

Also, $$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3 - 2}{9 - 4} = \frac{1}{5}$$

$$\therefore \quad \frac{1}{2\sqrt{c}} = \frac{1}{5}$$

$$\Rightarrow \quad c = \frac{25}{4} = 6.25$$

18. $a + b = 4 \Rightarrow b = 4 - a$ and $b - a = 4 - 2a = t$ (say)

Now,

$$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx = \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx = I(a)$$

$$\Rightarrow \quad \frac{dI(a)}{da} = g(a) - g(4 - a)$$

As $a < 2$ and $g(x)$ is increasing.

$\Rightarrow \quad 4 - a > a$

$\Rightarrow \quad g(a) - g(4 - a) < 0$

$$\Rightarrow \quad \frac{dI(a)}{da} < 0$$

Now, $$\frac{dI(a)}{d(a)} = \frac{dI(a)}{dt}\frac{dt}{da} = -2 \cdot \frac{dI(a)}{dt}$$

$$\Rightarrow \quad \frac{dI(a)}{dt} > 0$$

Thus, $I(a)$ is an increasing function of t. Hence, the given expression increasing with $(b - a)$.

19. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ ... (i)

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Hence, tangent at (x, y) is $Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$.

$\Rightarrow \quad Y\sqrt{x} - y\sqrt{x} = -X\sqrt{y} + x\sqrt{y}$

$\Rightarrow \quad X\sqrt{y} + Y\sqrt{x} = x\sqrt{y} + y\sqrt{x} = \sqrt{x}\sqrt{y}(\sqrt{x} + \sqrt{y})$

$\Rightarrow \quad X\sqrt{y} + Y\sqrt{x} = \sqrt{axy}$ [from Eq. (i)]

$$\Rightarrow \quad \frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1$$

Clearly, its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of intercepts

$$= \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a} \cdot \sqrt{a} = a$$

20. The graph of cosec x is opposite in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

21. $f(x)$ will be monotonically decreasing, if $f(x) < 0$.

$\Rightarrow \quad f'(x) = -\sin x - 2p < 0$

$\Rightarrow \quad \frac{1}{2}\sin x + p > 0$

$\Rightarrow \quad p > \frac{1}{2} \quad (\because -1 \le \sin x \le 1)$

22. Given that, equation of curve $y = x^3 = f(x)$

So, $f(2) = 8$ and $f(-2) = -8$

Now, $f'(x) = 3x^2$

$$\Rightarrow \quad f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$$

$$\Rightarrow \quad \frac{8 - (-8)}{4} = 3x^2$$

$$\therefore \quad x = \pm\frac{2}{\sqrt{3}}$$

23. For y-axis, $x = 0$

$\therefore \quad y = 1 - e^0 = 1 - 1 = 0$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{1}{2}e^{x/2} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = -\frac{1}{2}$$

$\therefore$ Equation of tangent is

$$y - 0 = -\frac{1}{2}(x - 0)$$

$\Rightarrow \quad x + 2y = 0$

24. For $x = p$, $y = ap^2 + bp + c$

And for $x = q$, $y = aq^2 + bq + c$

$$\text{Slope} = \frac{aq^2 + bq + c - ap^2 - bp - c}{q - p}$$

$$= a(q + p) + b$$

$$\frac{dy}{dx} = 2ax + b = a(q + p) + b$$

(according to the question)

$$\therefore \quad x = \frac{q + p}{2}$$

25. $f'(x) = (ab - b^2 - 2) + \cos^4 x + \sin^4 x < 0$

$$= ab - b^2 - 2 + (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x < 0$$

$$\Rightarrow \quad ab - b^2 - 1 < \left(\frac{1}{2}\right)\sin^2 2x < \frac{1}{2}$$

$$\Rightarrow \quad 2ab - 2b^2 - 2 < 1$$

$$\Rightarrow \quad 2b^2 - 2ab + 3 > 0$$

$$\therefore \quad (-2a)^2 - 4 \cdot 2 \cdot 3 < 0$$

$$\Rightarrow \quad a^2 < 6$$

$$\Rightarrow \quad -\sqrt{6} < a < \sqrt{6}$$

26. Given, $f'(x) < 0, \forall x \in R$

$$\Rightarrow \sqrt{3}\cos x + \sin x - 2a < 0, \forall x \in R$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x < a, \forall x \in R$$

$$\Rightarrow \quad \sin\left(x + \frac{\pi}{3}\right) < a, \forall x \in R$$

$$\Rightarrow \quad a \geq 1 \qquad \left[\because \sin\left(x + \frac{\pi}{3}\right) \leq 1\right]$$

27. There is only one function in option (a), whose critical point $\frac{1}{2} \in (0, 1)$ but in other parts critical point $0 \notin (0, 1)$. Then, we can say that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1).

Now, for $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$

Here, $\quad Lf'\left(\frac{1}{2}\right) = -1$

and $\quad Rf'\left(\frac{1}{2}\right) = 2\left(\frac{1}{2} - \frac{1}{2}\right)(-1) = 0$

$$\therefore \quad Lf'\left(\frac{1}{2}\right) \neq Rf'\left(\frac{1}{2}\right)$$

$\Rightarrow$ f is non-differentiable at $x = \frac{1}{2} \in (0, 1)$.

$\therefore$ Lagrange mean value theorem is not applicable to $f(x)$ in [0, 1].

28. The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$$y - e^c = e^c(x - c) \qquad \ldots(i)$$

Equation of the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ is

$$y - e^{c-1} = e^c \cdot \frac{(e - e^{-1})}{2} \cdot [x - (c - 1)]$$

$$\Rightarrow [x - (c - 1)][2 - (e - e^{-1})] = 2e^{-1} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 2$$

$$\Rightarrow \quad x < c$$

29. The tangent to the parabola $x^2 = y - 6$ at (1, 7) is

$$x(1) = \frac{1}{2}(y + 7) - 6$$

$$\Rightarrow \quad y = 2x + 5$$

which is also a tangent to the given circle.

i.e., $x^2 + (2x + 5)^2 + 16x + 12(2x + 5) + c = 0$

$\Rightarrow$ $(5x^2 + 60x + 85 + c = 0)$ must have equal roots.

Let the roots be $\alpha = \beta$.

$$\therefore \quad \alpha + \beta = -\frac{60}{5} \Rightarrow \alpha = -6$$

$$\therefore \quad x = -6 \text{ and } y = 2x + 5 = -7$$

30. On differentiating w.r.t. x, $e^y \cdot \frac{dy}{dx} = 2x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{1 + x^2} \qquad (\because e^y = 1 + x^2)$$

$$\Rightarrow \quad m = \frac{2x}{1 + x^2}$$

or $\quad |m| = \frac{2|x|}{1 + |x|^2}$

But $\quad 1 + |x|^2 - 2|x| = (1 - |x|)^2 \geq 0$

$$\Rightarrow \quad 1 + |x|^2 \geq 2|x| \quad \therefore \quad |m| \leq 1$$

31. The equation of the line is $y - 3 = \frac{3 + 2}{0 - 5}(x - 0)$

i.e., $\quad x + y - 3 = 0$

$$y = \frac{c}{x + 1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x + 1)^2}$$

Let the line touches the curve at (α, β).

$$\Rightarrow \alpha + \beta - 3 = 0, \left[\frac{dy}{dx}\right]_{\alpha, \beta} = \frac{-c}{(\alpha + 1)^2} = -1 \text{ and } \beta = \frac{c}{\alpha + 1}$$

$$\Rightarrow \frac{c}{(c/\beta)^2} = 1 \text{ or } \beta^2 = c \text{ or } (3 - \alpha)^2 = c = (\alpha + 1)^2$$

$$\Rightarrow \quad 3 - \alpha = \pm(\alpha + 1) \text{ or } 3 - \alpha = \alpha + 1$$

$$\Rightarrow \quad \alpha = 1$$

So, $c = (1 + 1)^2 = 4$

32. Here, $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$

$\Rightarrow$ $f_1'(x_1) = 2x_1 - 1$ and $f_2'(x_1) = 3x_2^2 - 2x_2 - 2$

Let tangents drawn to the curves $y = f_1(x)$ and $y = f_2(x)$ is $\{x_1, f_1(x_1)\}$ and $\{x_2, f_2(x_2)\}$ be parallel.

$\Rightarrow$ $2x_1 - 1 = 3x_2^2 - 2x_2 - 2$ or $2x_1 = (3x_2^2 - 2x_2 - 1)$

which is possible for infinite numbers of ordered pairs.

$\Rightarrow$ Infinite solutions.

33. We know that,

$$1 \le |\sin x| + |\cos x| \le \sqrt{2}$$

$$\Rightarrow \quad y = [|\sin x| + |\cos x|] = 1$$

Let P and Q be the points of intersection of given curves.

Clearly, the given curves meet at points where $y = 1$, so we get

$$x^2 + 1 = 5 \Rightarrow x = \pm 2$$

Now, $P(2, 1)$ and $Q(-2, 1)$

On differentiating $x^2 + y^2 = 5$ w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(2,1)} = -2 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,1)} = 2$$

Clearly, the slope of line $y = 1$ is zero and the slope of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

34. $y = \dfrac{a^2}{x} \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{a^2}{x^2}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a^2}{x_1^2}$$

$\Rightarrow$ Tangent at (x_1, y_1) to the curve $xy = a^2$ is $y - y_1 = -\dfrac{a^2}{x_1^2}$

$$(x - x_1) \quad \text{or} \quad a^2x + x_1^2y = x_1(x_1y_1 + a^2)$$

But $\quad x_1(x_1y_1 + a^2) = x_1(a^2 + a^2) = 2a^2x_1 \qquad (\because x_1y_1 = a^2)$

$\Rightarrow$ Tangent is $a^2x^2 + x_1^2y = 2a^2x_1$

This meets the x-axis, where $y = 0$

$\therefore \quad a^2x = 2a^2x_1 \quad \therefore \quad x = 2x_1$

$\therefore$ Point on the x-axis is $(2x_1, 0)$.

Again tangent meets the y-axis, where $x = 0$

$$\therefore \quad x_1^2 = 2a^2x_1 \quad \therefore \quad y = \frac{2a^2}{x_1}$$

$\therefore$ Point on the y-axis is $\left(0, \dfrac{2a^2}{x_1}\right)$.

Required area $= \dfrac{1}{2}(2x_1)\left(\dfrac{2a^2}{x_1}\right) = 2a^2$

35. $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$

$$\Rightarrow \quad n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b}{a}\frac{(x/a)^{n-1}}{(y/b)^{n-1}}$$

At (a, b), $\quad \dfrac{dy}{dx} = -\dfrac{b}{a} \cdot \dfrac{(a/a)^{n-1}}{(b/b)^{n-1}} = -\dfrac{b}{a}$

$\therefore$ Tangent at (a, b) is $y - b = -\dfrac{b}{a}(x - a)$

$$\Rightarrow \quad ay - ab = -bx + ab$$

$$\Rightarrow \quad bx + ay = 2ab$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$$

36. On differentiating the given curve $xy = 1$ w.r.t. x, we get

$$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Slope of normal at point (x_1, y_1) is $\dfrac{x_1}{y_1}$.

The given equation $ax + by + c = 0$ has slope $\dfrac{-a}{b}$.

According to the question, $\dfrac{x_1}{y_1} = \dfrac{-a}{b}$

Now, (x_1, y_1) lies on the curve $xy = 1$, which lies in first or third quadrant.

$\Rightarrow$ $x_1, y_1 > 0$ or $x_1, y_1 < 0$ $\Rightarrow$ $\dfrac{-a}{b} > 0$ $\Rightarrow$ a and b have opposite sign.

37. Given, $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$

$$\Rightarrow \quad f'(x) = 12\cos^3 x(-\sin x) + 30\cos^2 x(-\sin x)$$

$$+ 12\cos x(-\sin x)$$

$$\Rightarrow \quad f'(x) = -3\sin 2x(2\cos^2 x + 5\cos x + 2)$$

$$\Rightarrow \quad f'(x) = -3\sin 2x(2\cos x + 1)(\cos x + 2)$$

When $f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = 0, \dfrac{\pi}{2}, \pi$

$$\Rightarrow \quad 2\cos x + 1 = 0 \Rightarrow x = \frac{2\pi}{3}$$

as $\quad \cos x + 2 \ne 0$

Using sign scheme for $f'(x)$ in $[0, \pi]$ is shown in figure.

So, $f(x)$ decreases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ and increases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

38. The given function $y = 2x^2 - \log|x|, x \neq 0$

On differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 4x - \frac{1}{x} = \frac{4}{x}\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) : x \neq 0 \qquad \text{...(i)}$$

$$\underset{-1/2}{\bullet}\ \ - \quad + \quad \underset{0}{\bullet}\ \ - \quad \underset{1/2}{\bullet}\ \ +$$

Sign scheme of $f'(x)$

Hence, y is increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

and y is decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

39. Obviously, f is increasing and g is decreasing in (x_1, x_2)

Hence, $f\{g(\alpha^2 - 2\alpha)\} > f\{g(3\alpha - 4)\}$ as f is increasing

$\Rightarrow \quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\Rightarrow \quad \alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing

$\Rightarrow \quad \alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow \quad (\alpha - 1)(\alpha - 4) < 0 \quad \Rightarrow \quad \alpha \in (1, 4)$

40. We have, $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$$\Rightarrow \quad f'(x) = \frac{\left(\frac{1}{\pi + x}\right)\ln(e + x) - \frac{1}{(e + x)}\ln(\pi + x)}{[\ln(e + x)]^2}$$

$$= \frac{(e + x)\ln(e + x) - (\pi + x)\ln(\pi + x)}{(e + x)(\pi + x)\{\ln(e + x)\}^2} < 0 \text{ as } (0, \infty)$$

Since, $1 < e < \pi$

$\therefore$ $f(x)$ decreases on $(0, \infty)$

41. Let the speed of the motorboat be v mph.

$\Rightarrow$ Velocity of the boat relative to the current $= (v - c)$ mph

If s miles is the distance covered, then the time taken to cover this distance is $t = \frac{s}{(v - c)}$ hours.

Since, the petrol burnt $= kv^3$ per hour

where, k is a constant.

$\Rightarrow$ z = Total amount of petrol burnt for a distance of s miles $= kv^3 \cdot \frac{s}{(v - c)}$

$$\Rightarrow \quad \frac{dz}{dv} = \frac{2ksv^2(v - 3c/2)}{(v - c)^2}$$

For max or min of z, $\frac{dz}{dv} = 0 \quad \Rightarrow \quad v = \frac{3c}{2}$

If v is little less of little greater than $\frac{3c}{2}$, then the sign of $\frac{dz}{dv}$ changes from –ve to +ve. Hence, z is minimum when $v = \frac{3c}{2}$ mph.

Since, minima is the only extreme value, z is least at $v = \frac{3c}{2}$

i.e., the most economical speed is $\frac{3c}{2}$ mph.

42. $A = \frac{1}{2}x^2 \sin\theta \quad \Rightarrow \quad 2A = x^2 \sin\theta$

$$2\frac{dA}{dt} = x^2 \cos\theta \frac{d\theta}{dt} + \sin\theta\, 2x\frac{dx}{dt}$$

$$2\frac{dA}{dt} = (144)\left(\frac{1}{\sqrt{2}}\right)\frac{\pi}{180} + \frac{1}{\sqrt{2}} \cdot 2 \cdot 12 \cdot \frac{1}{12}$$

$$= \frac{12\pi}{15\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$\frac{dA}{dt} = \frac{2\pi}{5\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}\pi}{5} + \frac{\sqrt{2}}{2} = \sqrt{2}\left(\frac{\pi}{5} + \frac{1}{2}\right)$$

43. $\because \quad F(x) = \int_0^x 2|t|\, dt$

$\therefore \quad F'(x) = 2|x| = 1$

$\Rightarrow \quad x = \pm\frac{1}{2}$

$\Rightarrow \quad F\left(\frac{1}{2}\right) = \int_0^{1/2} 2|t|\, dt = \{t^2\}_0^{1/2} = \frac{1}{4}$

and $\quad F\left(-\frac{1}{2}\right) = \int_0^{-1/2} 2|t|\, dt$

$$= -2\int_0^{-1/2} t\,dt = -\frac{1}{4}$$

$\therefore$ Equation of tangent at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and at $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ are

$$y - \frac{1}{4} = 1 \cdot \left(x - \frac{1}{2}\right)$$

$\Rightarrow \quad y = x - \frac{1}{4}$

and $\quad y + \frac{1}{4} = 1\left(x + \frac{1}{2}\right)$

$\Rightarrow \quad y = x + \frac{1}{4}$

44. Given,

$$h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$$

$\therefore \quad h'(x) = f'(x) - 2f(x)f'(x) + 3\{f(x)\}^2 f'(x)$

$$= f'(x)[1 - 2f(x) + 3\{f(x)\}^2]$$

Discriminant of the quadratic equation

$3\{f(x)\}^2 - 2f(x) + 1 = 0$ is

$$(-2)^2 - 4 \cdot 3 \cdot 1 = -8 < 0$$

$\Rightarrow \quad 3\{f(x)\}^2 - 2f(x) + 1 > 0, \forall f(x) \in R \qquad (\because 3 > 0)$

$\Rightarrow$ Sign of $h'(x)$ will be same as that of sign of $f'(x)$.

$\therefore$ $h(x)$ is increasing whenever $f(x)$ is increasing and $h(x)$ is decreasing whenever $f(x)$ is decreasing.

45. $f'(x) = 6\sin^2 x\cos x - 6\sin x\cos x + 12\cos x$

$$= 6\cos x\{\sin^2 x - \sin x + 2\}$$

$$= 6\cos x\left\{\left(\sin x - \frac{1}{2}\right)^2 + \frac{7}{4}\right\}$$

$\therefore$ $\ln\left[0, \frac{\pi}{2}\right]$, $f'(x) \geq 0$, so $f(x)$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

46. Since, $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

$\Rightarrow$ Roots of $ax^2+bx+c=0$ should lie between $(0, 1)$

and $\int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

$\Rightarrow$ Roots of $ax^2+bx+c=0$ should lie between $(1, 2)$

Solutions (Q. Nos. 47-49)

$\because$ $f'(-1) = f'\left(\frac{1}{3}\right) = 0$

$\therefore$ $f'(x) = a(x+1)\left(x - \frac{1}{3}\right)$

$$= a'\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$$

On integrating w.r.t. x, we get

$$f(x) = \frac{a'}{3}(x^3 + x^2 - x) + \lambda \quad \text{...(i)}$$

where, λ is constant of integration.

and $f(-2) = 0$

$\Rightarrow$ $\frac{a'}{3}(-8+4+2) + \lambda = 0$

$\therefore$ $\lambda = \frac{2a'}{3}$

From Eq. (i),

$$f(x) = \frac{a'}{3}(x^3 + x^2 - x + 2)$$

Also, $\int_{-1}^1 f(x)\,dx = \frac{14}{3}$

$\Rightarrow$ $\frac{a'}{3}\int_{-1}^1 (x^3 + x^2 - x + 2)\,dx = \frac{14}{3}$

$\Rightarrow$ $0 + \frac{2a'}{3}\int_0^1 (x^2+2)\,dx = \frac{14}{3}$

$\Rightarrow$ $\frac{2a'}{3}\left\{\frac{1}{3} + 2\right\} = \frac{14}{3}$

$\Rightarrow$ $\frac{14a'}{9} = \frac{14}{3}$

$\Rightarrow$ $a' = 3$

Then, $f(x) = x^3 + x^2 - x + 2$

On comparing with $f(x) = ax^3 + bx^2 + cx + d$

$\therefore$ $a = 1,\ b = 1,\ c = -1,\ d = 2$

50. $f'(x) = 3(x+1)\left(x - \frac{1}{3}\right) < 0$

$\therefore$ $-1 < x < \frac{1}{3} \Rightarrow x \in \left(-1, \frac{1}{3}\right)$

51. Given, $f(x) = 3$

$\Rightarrow$ $x^3 + x^2 - x + 2 = 3$

$\Rightarrow$ $x^3 + x^2 - x - 1 = 0$

$\Rightarrow$ $x^2(x+1) - (x+1) = 0$

$\Rightarrow$ $(x^2 - 1)(x+1) = 0$

$\therefore$ $x = -1, -1, 1$

52. For Rolle's theorem and LMVT, $f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b).

Hence, Statement I is true.

Since, $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is non-differentiable at $x = 0$.

Hence, statement II is also true.

53. Given, $x^2 + y^2 = 25$

$\Rightarrow$ $2x\,dx + 2y\,dy = 0$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow$ $\frac{dy/dt}{dx/dt} = -\frac{x}{y}$

$\Rightarrow$ $-\frac{1.5}{dx/dt} = -\frac{3}{4}$

$\Rightarrow$ $\frac{dx}{dt} = \frac{1.5 \times 4}{3} = 2$ cm/s

54. **Statement I** As $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies $f(x)$ is either discontinuous or $f'(x)$ does not exist at atleast one point in $(-1, 1)$.

$g(x) = 0$ at atleast one value of x in $(-1, 1)$.

Statement II is false.

55. Common normal is $y = x$.

On solving $x + y = 2 \Rightarrow (1, 1)$

and $x^2 + y^2 = 16 \Rightarrow (2\sqrt{2}, 2\sqrt{2})$

The distance between AB is $(4-\sqrt{2})$.

But as curve are not smooth, check at slope points. The coordinates in Ist quadrant are (2, 0) and (4, 0) and here distance = 2.

$\therefore$ $4-\sqrt{2}$ is not shortest.

56. $f'(x) = 3x^2 + 2x + 3 + \cos x$

$$= 3\left\{x^2 + \frac{2x}{3} + 1\right\} + \cos x$$

$$= 3\left\{\left(x+\frac{1}{3}\right)^2 - \frac{1}{9} + 1\right\} + \cos x$$

$$= 3\left(x+\frac{1}{3}\right)^2 + \frac{8}{3} + \cos x > 0$$

$$\left[\because 3\left(x+\frac{1}{3}\right)^2 \geq 0, -1 \leq \cos x \leq 1\right]$$

$\therefore$ $f(x)$ is an increasing function.

$\Rightarrow$ $f(x)$ is one one.

57. Let $h(x) = f(x) - f(a) + \lambda(x^3 - a^3)$

where, λ is selected in such a way

$$h(b) = f(b) - f(a) + \lambda(b^3 - a^3) = 0 \quad \text{...(i)}$$

but $h(a) = 0$

Hence, $h(x)$ satisfies all conditions of Rolle's theorem.

$\because$ $c \in (a, b)$

$\therefore$ $h'(c) = 0$

$\Rightarrow$ $f'(c) - 0 + 3\lambda c^2 = 0$

$\Rightarrow$ $\lambda = -\dfrac{f'(c)}{3c^2}$

From Eq. (i), $\lambda = -\dfrac{f(b)-f(a)}{(b^3-a^3)}$

$\therefore$ $\dfrac{f(b)-f(a)}{b^3-a^3} = \dfrac{f'(c)}{3c^2}$

58. $x(x+3)$ and $e^{-x/2}$ are continuous and differentiable everywhere, so $x(x+3)\,e^{-x/2}$ is continuous and differentiable.

and $f(-3) = f(0) = 0$

and $f'(x) = (x^2+3x)\cdot e^{-x/2}$

$$= \left(-\frac{1}{2}\right) + e^{-x/2}\cdot(2x+3)$$

$$= -\frac{1}{2}e^{-x/2}(x^2+3x-4x-6)$$

$$= -\frac{1}{2}e^{-x/2}(x^2-x-6)$$

$$= -\frac{1}{2}e^{-x/2}(x-3)(x+2)$$

$\therefore$ $f'(x) = 0 \Rightarrow x = 3, -2$

$3 \notin [-3, 0]$

$\therefore$ $x = -2 \in [-3, 0]$

$\therefore$ Rolle's theorem is verified.

LMVT is also applied.

$\because$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

i.e., Rolle's theorem is a special case of LMVT.

Since, $f(a) = f(b) \Rightarrow f'(c) = 0$

59. $f(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$\therefore$ $f'(x) = 3x^2 - 12x + 11$

Put $f'(x) = 0$

$\Rightarrow$ $3x^2 - 12x + 11 = 0$

$\Rightarrow$ $x = \dfrac{12 \pm \sqrt{144-132}}{6}$

$= 2 \pm \dfrac{1}{\sqrt{3}}$

$\therefore$ $1 \leq 2 \pm \dfrac{1}{\sqrt{3}} \leq 3$

60. Say $f(x) = x^3 - 3x = x(x-\sqrt{3})(x+\sqrt{3})$

Now, $f(x) = -k$

will have roots when $y = -k$ and $f(x) = y$ cuts

$f'(x) = 0$ at $x = \pm 1$

$f(x)$ is decreasing so $y = -k$ will cut at only one place moreover, $f(x)$ is one-one between two consecutive extrema.

61. Let $f(x) = 2x^3 + 3x + k$

On differentiating w.r.t. x, we get

$$f'(x) = 6x^2 + 3 > 0, \forall x \in R$$

$\Rightarrow$ $f(x)$ is strictly increasing function.

$\Rightarrow$ $f(x) = 0$ has only one real root, so two roots are not possible.

62. Given, $y = \int_0^x |t|\,dt$

$\therefore$ $\dfrac{dy}{dx} = |x|\cdot 1 - 0 = |x|$ (by Leibnitz rule)

$\because$ Tangent to the curve $y = \int_0^x (t)\,dt$, $x \in R$ are parallel to the line $y = 2x$

$\therefore$ Slope of both are equal

$\Rightarrow$ $x = \pm 2$

$\therefore$ Points, $y = \int_0^{\pm 2} |t|\,dt = \pm 2$

$\therefore$ Equation of tangent is

$$y-2=2(x-2) \quad \text{and} \quad y+2=2(x+2)$$

For x-intercept put $y=0$, we get

$$0-2=2(x-2) \quad \text{and} \quad 0+2=2(x+2)$$

$$\Rightarrow \qquad x=\pm 1$$

63. Since, the balloon is spherical in shape, hence the volume of the balloon is $V=\frac{4}{3}\pi r^3$.

On differentiating both the sides w.r.t. t, we get

$$\frac{dV}{dt}=\frac{4}{3}\pi\left(3r^2\times\frac{dr}{dt}\right)$$

$$\Rightarrow \qquad \frac{dr}{dt}=\frac{dV/dt}{4\pi r^2} \qquad \ldots\text{(i)}$$

Now, to find $\frac{dr}{dt}$ at the rate $t=49$ min, we require $\frac{dV}{dt}$ the radius (r) at that stage

$$\frac{dV}{dt}=-72\,\pi\,\text{m}^3/\text{min}$$

Also, amount of volume lost in 49 min $=72\,\pi\times 49\,\text{m}^3$

$\therefore$ Final volume at the end of 49 min

$$=(4500\,\pi-3528\pi)\,\text{m}^3$$

$$=972\,\pi\,\text{m}^3$$

If r is the radius at the end of 49 min, then

$$\frac{4}{3}\pi r^3=972\,\pi$$

$$\Rightarrow \qquad r^3=729$$

$$\Rightarrow \qquad r=9$$

Radius of the balloon at the end of 49 min $=9$ m

Hence, from Eq. (i), we get

$$\frac{dr}{dt}=\frac{dV/dt}{4\pi r^2}$$

$$\Rightarrow \qquad \left(\frac{dr}{dt}\right)_{t=49}=\frac{\left(\frac{dV}{dt}\right)_{t=49}}{4\pi\,(r^2)_{t=49}}$$

$$\Rightarrow \qquad \left(\frac{dr}{dt}\right)_{t=49}=\frac{72\,\pi}{4\pi(9^2)}=\frac{2}{9}\,\text{m/min}$$

64. We have, $y=x+\frac{4}{x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=1-\frac{8}{x^3}$$

Since, the tangent is parallel to x-axis, therefore

$$\frac{dy}{dx}=0$$

$$\Rightarrow \qquad x^3=8$$

$$\Rightarrow \qquad x=2 \text{ and } y=3$$

65. Let $f(x)=x^7+14x^5+16x^3+30x-560$

$$f'(x)=7x^6+70x^4+48x^2+30>0,\ \forall x\in R$$

$\therefore$ $f(x)$ is increasing.

$\therefore$ $f(x)=0$ has only one solution.

66. Using mean value theorem,

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \qquad \frac{1}{c}=\frac{\log_e 3-\log_e 1}{3-1}$$

$$\Rightarrow \qquad c=\frac{2}{\log_e 3}$$

$$=2\log_3 e$$

67. $f'(x)=\frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)$

$$=\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

For $f(x)$ to be increasing,

$$\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)>0$$

$$\Rightarrow \qquad -\frac{\pi}{2}<x+\frac{\pi}{4}<\frac{\pi}{2}$$

$$\Rightarrow \qquad \frac{-3\pi}{4}<x<\frac{\pi}{4}$$

29 Maxima and Minima

JEE Main MILESTONE

- Maxima and Minima/Extremum
- Critical Points
- Test for Local Maximum/Minimum
- Points of Inflection
- Some Cases of Extremum of Discontinuous Functions
- Application of Maxima and Minima to Problems

29.1 Maxima and Minima/Extremum

Concept of Local Maxima and Local Minima

Let $y = f(x)$ be a function defined at $x = a$ and also in the vicinity of the point $x = a$. Then, $f(x)$ is said to have a local maximum at $x = a$, if the value of the function at $x = a$ is greater than the value of the function at the neighbouring points of $x = a$. Mathematically, $f(a) > f(a - h)$ and $f(a) > f(a + h)$, where $h > 0$. (very small quantity).

Similarly, $f(x)$ is said to have a local minimum at $x = a$, if the value of the function at $x = a$ is less than the value of the function at the neighbouring points of $x = a$.

Mathematically, $f(a) < f(a - h)$ and $f(a) < f(a + h)$, where $h > 0$.

A local maximum or a local minimum is also called a local extremum.

Pierrede format was one of the first Mathematicians to propose a general technique for finding maxima and minima.

More generally, the maximum and minimum of a set are the greatest and least element in the set. Unbounded infinite sets such as the set of real numbers have no minimum and maximum.

29.2 Critical Points

Critical points is an important topic for JEE (Main) examination and a small practice can help you in solving such problems very quickly as the level of question is easy to average.

It is a collection of points for which,

(i) $f(x)$ does not exists

(ii) $f'(x)$ does not exists or

(iii) $f'(x) = 0$

All the values of x obtained from above conditions are said to be critical points.

It should be noted that critical points are the interior points of an interval.

Sample Problem 1 *The number of the critical points for* $f(x) = (x-2)^{2/3}(2x+1)$ *is*

(a) 0 (b) 1 (c) 2 (d) 3

Interpret (c) Given, $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3} \cdot 2$$

$$\Rightarrow \quad f'(x) = 2\left[\frac{(2x+1)}{3(x-2)^{1/3}} + \frac{(x-2)^{2/3}}{1}\right]$$

Clearly, $f'(x)$ is not defined at $x = 2$, so $x = 2$ is a critical point.

Another critical point is given by,

$$f'(x) = 0$$

$$\Rightarrow \quad 2\left[\frac{(2x+1)+3(x-2)}{3(x-2)^{1/3}}\right] = 0$$

$$\Rightarrow \quad 5x - 5 = 0$$

$$\Rightarrow \quad x = 1$$

Hence, $x = 1$ and $x = 2$ are two critical points of $f(x)$.

29.3 Test for Local Maximum/Minimum

We have two cases to consider

(a) **Test for local maximum/minimum at $x = a$, if $f(x)$ is differentiable at $x = a$**

Test for local maximum/minimum is the most important topic of this chapter generally question seen from this topic. The level of question is from moderate to typical.

If $f(x)$ is differentiable at $x = a$ and if it is a critical point of the function (*i.e.*, $f'(a) = 0$), then we have the following three tests to decide whether $f(x)$ has a local maximum or local minimum or neither at $x = a$.

(i) First Derivative Test

If $f'(a) = 0$ and $f'(x)$ changes its sign while passing through the point $x = a$, then

(a) $f(x)$ would have a local maximum at $x = a$, if $f'(a-0) > 0$ and $f'(a+0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.

(b) $f(x)$ would have local minimum at $x = a$, if $f'(a-0) < 0$ and $f'(a+0) > 0$. It means that $f'(x)$ should change its sign from negative to positive.

(c) If $f'(x)$ does not change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor minimum at $x = a$

$\therefore$ at $x = -1$, we have local maximum $\Rightarrow f(x) = -2$ and at $x = 1$, we have local minimum $\Rightarrow f_{\min}(x) = 2$

Sample Problem 2 *Let* $f(x) = x + \frac{1}{x}$, $x \neq 0$, *then at which* $f(x)$ *assumes maximum and minimum are respectively*

(a) $-1, 1$ (b) $1, -1$

(c) $0, 1$ (d) *None of these*

Interpret (a) Here, $f'(x) = 1 - \frac{1}{x^2}$

$$\Rightarrow \quad f'(x) = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

Sign scheme for $f'(x)$

$+ \quad -1 \quad - \quad 1 \quad +$

Using number line rule, we have maximum at $x = -1$ and minimum at $x = 1$.

$\therefore$ At $x = -1$, we have local maximum $\Rightarrow f_{\max}(x) = -2$ and at $x = 1$, we have local minimum $\Rightarrow f_{\min}(x) = 2$.

(ii) Second Derivative Test

First we find the roots of $f'(x) = 0$. Suppose $x = a$ is one of the roots of $f'(x) = 0$.

Now, find $f''(x)$ at $x = a$.

(a) If $f''(a)$ = negative, then $f(x)$ is maximum at $x = a$.

(b) If $f''(a)$ = positive, then $f(x)$ is minimum at $x = a$.

(c) If $f''(a)$ = zero, then we find $f'''(x)$ at $x = a$.

If $f'''(a) \neq 0$, then $f(x)$ has neither maximum nor minimum (inflexion point) at $x = a$.

But, if $f'''(a) = 0$, then find $f^{iv}(a)$.

If $f^{iv}(a)$ = positive, then $f(x)$ is minimum at $x = a$.

If $f^{iv}(a)$ = negative, then $f(x)$ is maximum at $x = a$.

and so on, process is repeated till point is discussed.

Sample Problem 3 *Locate the position and nature of any turning points of the function* $y = x^3 - 3x + 2$ *is*

(a) *Maxima at* (1, 0) (b) *Minima at* (1, 0)

(c) *Maxima at* (−1, 0) (d) *None of these*

Interpret (b) We need to find where the turning points are and whether we have maximum or minimum points. First of all, we carry out the differentiation and set $\frac{dy}{dx}$ equal to zero. This will enable us to look for any stationary points. Including any turning points.

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

At stationary points, $\frac{dy}{dx} = 0$ and so

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0 \quad \text{(factorising)}$$

$$3(x-1)(x+1) = 0$$

(factorising the difference of two squares)

It follows that either $x - 1 = 0$

or $x + 1 = 0$ and so either $x = 1$ or $x = -1$

We have found the x-coordinates of the points on the graph, where $\frac{dy}{dx} = 0$, that is the stationary points. We need the y-coordinates which are found by substituting the x values in the original function $y = x^3 - 3x + 2$.

When $x = 1$, $y = 1^3 - 3(1) + 2 = 0$

When $x = -1$, $y = (-1)^3 - 3(-1) + 2 = 4$

To summarise, we have located two stationary points and these occur at (1,0) and (−1, 4).

Next, we need to determine whether, we have maximum or minimum points, or possibly points such as C in which are neither maxima nor minima.

We have seen that the first derivative $\frac{dy}{dx} = 3x^2 - 3$. Differentiating this we can find the second derivative.

$$\frac{d^2y}{dx^2} = 6x$$

We now take each point in turn and use our test.

When $x = 1$, $\frac{d^2y}{dx^2} = 6x = 6(1) = 6.$

We are not really interested in this value. What is important is its sign. Because it is positive. We know, we are dealing with a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = 6x = 6(-1) = -6$

Again, what is important its sign. Because it is negative we have a maximum point.

nth Derivative Test

It is nothing but the general version of the second derivative test, it says that if, $f'(a) = f''(a) = f'''(a) = \ldots f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanishes and $(n+1)$th order derivative does not vanish at $x = a$, then $f(x)$ would have a local maximum or local minimum at $x = a$, if n is odd natural number and that $x = a$ would be a point of local maxima, if $f^{n+1}(a) < 0$ and would be a point of local minima, if $f^{n+1}(a) > 0$.

However if n is even, then f has neither a maxima nor a minima at $x = a$. It is clear that the last two tests are basically the Mathematical representation of the first derivative test. But that should not diminish the importance of these tests. Because at that times it becomes very difficult to decide whether $f'(x)$ changes it's sign or not while passing through point $x = a$ and the remaining tests may come handy in these kind of situations.

Note It must be remembered that this method is not applicable to those critical points, where $f'(x)$ remains undefined.

$\therefore$ at $x = -1$, we have local maximum $\Rightarrow f_{max}(x) = -2$ and at $x = 1$, we hae local minimum $\Rightarrow f_{min}(x) = 2$.

(b) **Test for local maximum/ minimum at $x = a$, if $f(x)$ is not differentiable at $x = a$.**

***Case* I** When $f(x)$ is continuous at $x = a$ and $f'(a-h)$ and $f'(a+h)$ exists and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$, if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h) > 0$ and $f'(a+h) < 0$, then $x = a$ will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a+h) > 0$, then $x = a$ will be a point of local minimum.

***Case* II** When $f(x)$ is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the following about the existence of local maxima/minima from the basic definition of local maxima/ minima.

***Case* III** If $f(x)$ is not continuous at $x = a$ and $f'(a-h)$ and/ or $f'(a+h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

Remark It is advisable to draw the graph of the function in the vicinity of the point $x = a$, because the graph would give us the clear picture about the existence of local maxima/minima at $x = a$.

Sample Problem 4 *Let*

$$f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \geq 0 \end{cases}, \text{ then at } x = 0 \ f(x) \text{ is}$$

(a) *local minimum*
(b) *local maximum*
(c) *Neither maximum nor minimum*
(d) *None of the above*

Interpret (b) Clearly, $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$ as $f(0) = f(0-0) = f(0+0) = 0$

$$f'-(0) = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-h^3 + h^2 - 10h - 0}{-h} = 10$$

But $f'_+(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{-h} = \lim_{h \to 0} \frac{-3\sin h}{h} = -3$

Since, $f'_-(0) > 0$ and $f'_+(0) < 0$, $x = 0$ is the point of local maximum.

Hot Spot 1

Concept of Global (Absolute) Maximum and Minimum

Concept of Absolute maximum/minimum is also an important topic for JEE (Main) examination and a small practice can help you in solving such problems very quickly as the level of questions is easy to average. While solving the problems based on concept of absolute maximum/minimum, always keep in mind.

(a) Critical Points

(b) Second derivative test

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maximum and minimum in $[a, b]$ would occur at critical point of $f(x)$ within $[a, b]$ or at the end points of the interval.

Global (Absolute) Maxima or Minima in $[a, b]$

Step I Find out all the critical points of $f(x)$ in (a, b). Let $c_1, c_2, \ldots, c_n$ be the different critical points.

Step II Find the value of the function at these critical points and also at the end points of the domain. Let the values are $f(c_1), f(c_2), \ldots, f(c_n)$.

Step III Find $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Now, M_1 is the maximum value of $f(x)$ in $[a, b]$, so M_1 is absolute maximum and M_2 is the minimum value of $f(x)$ in $[a, b]$, so M_2 is absolute minimum.

Let $y = f(x)$ be the function defined on $[a, b]$ in the graph, then

(i) $f(x)$ has local maximum values at $x = a_1, a_3, a_5, a_7$

(ii) $f(x)$ has local minimum values at

$$x = a_2, a_4, a_6, a_8$$

(iii) The absolute maximum value of the function is $f(a_7)$ and absolute minimum value is $f(a)$.

Note

- Between two local maximum values, there is a local minimum value and *vice-versa*.
- A local minimum value may be greater than a local maximum value. In the above graph, local minimum at a_6 is greater than local maximum at a_1.

Absolute Maxima or Minima in (a, b)

To find the absolute maxima and minima in (a, b) step I and step II are same. Now,

Step III Find $M_1 = \max\{f(c_1), f(c_2), \ldots, f(c_n)\}$

and $M_2 = \min\{f(c_1), f(c_2), \ldots, f(c_n)\}$

Now, if $\lim\limits_{\substack{x \to a^+ \\ \text{or } x \to b^-}} f(x) > M_1$ or $\lim\limits_{\substack{x \to a^+ \\ \text{or } x \to b^-}} f(x) < M_2$, then

$f(x)$ would not have absolute maximum or absolute minimum in (a, b)

and if $\lim\limits_{x \to a^+ \text{ and } x \to b^-} f(x) < M_1$

and $\lim\limits_{x \to a^+ \text{ and } x \to b^-} f(x) > M_2$,

then M_1 and M_2 would respectively be the absolute maximum and absolute minimum of $f(x)$ in (a, b).

Sample Problem 5 *Let $f(x) = 2x^3 - 9x^2 + 12x + 6$, then absolute maxima of $f(x)$ in $[0, 2]$ and $(1, 3)$ are respectively*

(a) 0, 2 (b) 1, 2

(c) 2, 2 (d) *None of these*

Interpret (b) $f(x) = 2x^3 - 9x^2 + 12x + 6$

$$f'(x) = 6x^2 - 18x + 12$$

$$= 6(x - 1)(x - 2)$$

In $[0, 2]$, Critical point of $f(x)$ in $[0, 2]$ is $x = 1$.

$\therefore$ $f(0) = 6, f(1) = 11, f(2) = 10$

Thus, $x = 0$ is the point of absolute minimum and $x = 1$ is the point of absolute maximum of $f(x)$ in $[0, 2]$.

In $(1, 3)$, Critical point of $f(x)$ in $(1, 3)$ is $x = 2$.

$\therefore$ $f(2) = 10, \lim\limits_{x \to 1^+} f(x) = 11$

and $\lim\limits_{x \to 3^-} f(x) = 15$

Thus, $x = 2$ is the point of absolute minimum in $(1, 3)$ and absolute maximum in $(1, 3)$ does not exist.

29.4 Points of Inflection

Consider function $f(x) = x^3$. At $x = 0$. $f'(x) = 0$. Also, $f''(x) = 0$ at $x = 0$. Such point is called point of inflection, where 2nd derivatives is zero. Consider another function $f(x) = \sin x$, $f''(x) = -\sin x$. Now, $f''(x) = 0$ when $x = n\pi$, then these points are called points of inflection.

At point of inflection

(i) It is not necessary that Ist derivative is zero.

(ii) 2nd derivative must be zero or 2nd derivative changes sign in the neighbourhood of point of inflection.

(iii) Graph of curve changes its concavity.

(iv) If $f''(x) > 0$ graph is concave towards negative y-axis and if $f''(x) < 0$, graph is concave towards positive y-axis.

Note

- For a continous function maximum and minimum value occurs alternately.
- If a function is discontinuous at a point $x = a$, it may have maximum value although it decreases on the left and increases on the right side of $x = a$.

29.5 Some Cases of Extremum of Discontinuous Functions

Clearly, $x = a$ is point of maxima as $f(a) > f(a - h)$ and $f(a) > f(a + h)$

Clearly, $x = a$ is not a point of extremum as $f(a) > f(a + h)$ and $f(a) < f(a - h)$

Clearly, $x = a$ is a point of minima as $f(a) < f(a - h)$ and $f(a) < f(a + h)$.

Clearly, $x = a$ is not a point of extremum as $f(a) < f(a - h)$ and $f(a) > f(a + h)$.

29.6 Application of Maxima and Minima to Problems (Mensuration and Geometry)

This is the most important topic of this chapter. Generally, questions are seen from this topic. The level of question is from moderate to typical because this topic include application of mensuration or geometry before proceeding the second derivative test.

For solving this type of problem, we follows these steps.

Step I First, we read the given problem very carefully for an objective function.

Step II If objective function is of two parameter, then we converts it in terms of one parameter with the help of other given condition in the problem.

Step III Now, we proceed the second derivative test for maxima/minima and get the desired result.

Useful formulae of mensuration to remember

(i) Volume of cubiod $= lbh$

(ii) Surface area of cuboid $= 2(lb + bh + hl)$

(iii) Volume of cube $= a^3$

(iv) Surface area of cube $= 6a^2$

(v) Volume of cone $= \frac{1}{3}\pi r^2 h$

(vi) Curved surface area of cone $= \pi rl$ (l = slant height)

(vii) Curved surface of a cylinder $= 2\pi rh$

(viii) Total surface of a cylinder $= 2\pi rh + 2\pi r^2$

(ix) Volume of a sphere $= \frac{4}{3}\pi r^3$

(x) Surface area of sphere $= 4\pi r^2$

(xi) Area of a circular sector $= \frac{1}{2}r^2\theta$ (where, θ is in radians)

(xii) Volume of a prism = Area of the base × Height

(xiii) Lateral surface of a prism = Perimeter of a base × Height

(xiv) Total surface of a prism = Lateral surface + 2 Area of the base

Note The lateral surface of a prism are all rectangle.

(xv) Volume of a pyramid $= \frac{1}{3}$ Area of the base × Height

(xvi) Curved surface of a pyramid $= \frac{1}{2}$ Perimeter of the base × Slant height

Note The slant surfaces of a pyramid are triangles

Sample Problem 6 *The height of the cylinder of maximum volume that can be inscribed in a sphere of radius R and the volume of the largest cylinder inscribed in a sphere of radius R are*

(a) $\frac{2R}{\sqrt{3}}, \frac{4\pi R^3}{3\sqrt{3}}$ (b) $\frac{2R}{3}, \frac{4\pi R^3}{3}$

(c) $\frac{2R}{3\sqrt{3}}, \frac{4\pi R^3}{\sqrt{3}}$ (d) *None of these*

Interpret (a) Let r be the radius and h the height of the inscribed cylinder $ABCD$. Let V be its volume.

Then, $V = \pi r^2 h$ ...(i)

Clearly, $AC = 2R$

Aslo, $AC^2 = AB^2 + BC^2$

$\Rightarrow$ $(2R)^2 = (2r)^2 + h^2$

$\Rightarrow$ $r^2 = \frac{1}{4}(4R^2 - h^2)$ ...(ii)

Using Eqs. (i) and (ii), we get

$$V = \frac{\pi h}{4}(4R^2 - h^2)$$

$\Rightarrow$ $$\frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4}\pi h^2\right)$$

and $$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$$

For a maxima or minima, we have $(dV/dh) = 0$

Now, $$\frac{dV}{dh} = 0 \Rightarrow \pi R^2 - \frac{3}{4}\pi h^2 = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$$

$$\left[\frac{d^2V}{dh^2}\right]_{h=(2R/\sqrt{3})} = -\frac{3}{2}\pi \times \frac{2R}{\sqrt{3}} = \pi R\sqrt{3} < 0$$

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$

Hence, the height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$.

Largest volume of the cylinder $= \pi \times \frac{1}{4}\left[4R^2 - \frac{4R^2}{3}\right] \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$

Sample Problem 7 *An open box is to be made out of a piece of paper of cardboard measuring (24 cm × 24 cm) by cutting off equal squares from the corners and turning up the sides, then the height of the box when it has maximum volume is*

(a) 2 *cm* (b) 4 *cm*

(c) 6 *cm* (d) 8 *cm*

Interpret (b) Let the length of the side of each square cut off from the corners be x cm.

Then, height of the box = x cm.

$\therefore$ $V = (24 - 2x)^2 \times (x) = 4x^3 - 96x^2 + 576x$

$\Rightarrow$ $$\frac{dV}{dx} = 12(x^2 - 16x + 48)$$

and $$\frac{d^2V}{dx^2} = 24(x - 8)$$

Now, $\frac{dV}{dx} = 0 \Rightarrow x^2 - 16x + 48 = 0$ *i.e.*, $(x - 12)(x - 4) = 0$

$\Rightarrow$ $x = 4$ $(\because x \neq 12)$

$$\left[\frac{d^2V}{dx^2}\right]_{x=4} = -96 < 0$$

$\therefore$ V is maximum at $x = 4$.

Hence, the volume of the box is maximum when its height is 4 cm.

WORKED OUT
Examples

Example 1 *In which of the following functions do not have maxima or minima?* **[NCERT]**

(a) $f(x) = e^x$ (b) $g(x) = \log x$

(c) $h(x) = x^3 + x^2 + x + 1$ (d) *All of these*

Solution (d)

(a) Given function is $f(x) = e^x \Rightarrow f'(x) = e^x$

Now, if $f'(x) = 0$, then $e^x = 0$. But, the exponential function can never assume 0 for any value of x.

Therefore, there does not exist any $x \in R$ such that $f'(x) = 0$

Hence, function f does not have maxima or minima.

(b) Given function is $g(x) = \log x \Rightarrow g'(x) = \frac{1}{x}$

Since, $\log x$ is defined for a positive number x, then $g'(x) > 0$ for any x. Therefore, there does not exist any $x \in R$ such that $g'(x) = 0$

Hence, function g does not have maxima or minima.

(c) Given function is $h(x) = x^3 + x^2 + x + 1$

$$\Rightarrow \quad h'(x) = 3x^2 + 2x + 1$$

Now, put $h'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$

$$\Rightarrow \quad x = \frac{-2 \pm \sqrt{4 - 4 \times 3 \times 1}}{6} = \frac{-2 \pm \sqrt{-8}}{6}$$

$$\left(\text{using } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$\Rightarrow \quad x = \frac{-2 \pm 2\sqrt{2}\, i}{6} = \frac{2(-1 \pm \sqrt{2}\, i)}{6} = \frac{-1 \pm \sqrt{2}\, i}{3} \notin R$$

Therefore, there does not exist any $x \in R$ such that $h'(x) = 0$

Hence, function h does not have maxima or minima.

Example 2 *A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?* **[NCERT]**

(a) 2 *cm* (b) 3 *cm* (c) 5 *cm* (d) 8 *cm*

Solution (c) Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45 - 2x$ and the breadth is $24 - 2x$.

Let V be the corresponding volume of the box, then

$$V = x(24 - 2x)(45 - 2x)$$

$$\Rightarrow \quad V = x(4x^2 - 138x + 1080) = 4x^3 - 138x^2 + 1080x$$

On differentiating twice w.r.t. x, we get

$$\frac{dV}{dx} = 12x^2 - 276x + 1080$$

and $$\frac{d^2V}{dx^2} = 24x - 276$$

For maxima put $\frac{dV}{dx} = 0$

$$\Rightarrow \quad 12x^2 - 276x + 1080 = 0$$

$$\Rightarrow \quad x^2 - 23x + 90 = 0$$

$$\Rightarrow \quad (x - 18)(x - 5) = 0 \Rightarrow x = 5, 18$$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

At $x = 5$, $$\left(\frac{d^2V}{dx^2}\right)_{x=5} = 24 \times 5 - 276$$

$$= 120 - 276 = -156 < 0$$

$\therefore$ By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

Example 3 *Rectangles are inscribed in a circle of radius r. The dimensions of the rectangle which has the maximum area, are* **[NCERT]**

(a) r, r (b) $2r, 2r$

(c) $\sqrt{2}r, \sqrt{2}r$ (d) *None of these*

Solution (c) Let $ABCD$ be the rectangle inscribed in a circle of radius r. Let $AB = x$ and $BC = y$.

Then, $x^2 + y^2 = 4r^2$

$$\Rightarrow \quad y = \sqrt{4r^2 - x^2} \quad \ldots(i)$$

Area of rectangle, $A = xy = x\sqrt{4r^2 - x^2}$

Let $\quad u = A^2 = x^2(4r^2 - x^2) \Rightarrow \dfrac{du}{dx} = 8r^2x - 4x^3$

Put $\dfrac{du}{dx} = 0$, for maxima or minima

$$4x(2r^2 - x^2) = 0 \Rightarrow x = \sqrt{2}r$$

Also, $$\frac{d^2u}{dx^2} = 8r^2 - 12x^2$$

$\therefore$ $$\left(\frac{d^2u}{dx^2}\right)_{x=\sqrt{2}r} = 8r^2 - 24r^2 < 0$$

$\therefore u$ and A are maximum at $x = \sqrt{2}r$.

From Eq. (i), $\quad y = \sqrt{2}r = x$

$\therefore$ Dimensions of the rectangle are $\sqrt{2}r$ and $\sqrt{2}r$.

Example 4 *Of all the closed cylindrical cans (right circular), which enclosed a given volume 100 cubic centimeters, the dimensions of the minimum surface area is* [**NCERT**]

(a) $\left(\dfrac{50}{\pi}\right)^{1/3}$ *cm* (b) $2\left(\dfrac{50}{\pi}\right)^{1/3}$ *cm*

(c) $2\left(\dfrac{25}{\pi}\right)^{1/3}$ *cm* (d) *None of these*

Solution (b) Hence, we have two independent variable r and h. Since, here surface area is minimum so find the value of h in terms of r by using $V = \pi r^2 h$ and put in surface area for eliminating h. Then, apply second derivative test.

Let r cm be the radius of base and h cm be the height of the cylindrical can. Let its volume be V and S its total surface area. Then, $V = 100$ cm^3

$\Rightarrow \quad \pi r^2 h = 100$ (given)

$\Rightarrow \quad h = \dfrac{100}{\pi r^2}$ ...(i)

Also, $\quad S = 2\pi r^2 + 2\pi rh$ ...(ii)

In Eq. (ii), there are two independent variables r and h. We eliminate h, by putting the value of h in Eq. (ii) from Eq. (i), we get

$$S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right) = 2\pi r^2 + \frac{200}{r} \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2} \quad \text{...(iv)}$$

Now, for maxima or minima put $\dfrac{dS}{dr} = 0$

$\Rightarrow \quad 4\pi r = \dfrac{200}{r^2}$

$\Rightarrow \quad r^3 = \dfrac{200}{4\pi} \Rightarrow r = \left(\dfrac{50}{\pi}\right)^{1/3}$

On differentiating Eq. (iv) w.r.t. r, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

At $r = \left(\dfrac{50}{\pi}\right)^{1/3}$,

$$\frac{d^2S}{dr^2} = \frac{400}{\left(\frac{50}{\pi}\right)} + 4\pi = \frac{400}{50} \times \pi + 4\pi = 8\pi + 4\pi = 12\pi > 0$$

$\therefore$ By second derivative test, the surface area is minimum when the radius of the cylinder is $\left(\dfrac{50}{\pi}\right)^{1/3}$. Hence, S is minimum when $r = \left(\dfrac{50}{\pi}\right)^{1/3}$.

On putting value of r is Eq. (i), we get

$$h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}} = 2\left(\frac{50}{\pi}\right)\left(\frac{50}{\pi}\right)^{-2/3} = 2\left(\frac{50}{\pi}\right)^{1/3} \text{ cm}$$

Example 5 *The maximum and minimum value of* $3x^4 - 8x^3 + 12x^2 - 48x + 25$ *on the interval* [0, 3] *are* [**NCERT**]

(a) 25 *and* –39 (b) 16 *and* –12

(c) 10 *and* –8 (d) 5 *and* –3

Solution (a) Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$\Rightarrow \quad f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x^3 - 2x^2 + 2x - 4)$

$= 12\{x^2(x-2) + 2(x-2)\} = 12(x-2)(x^2+2)$

For maxima or minima put $f'(x) = 0$

$\Rightarrow \quad 12(x-2)(x^2+2) = 0 \Rightarrow$ if $x - 2 = 0 \Rightarrow x = 2 \in [0, 3]$

and if $\quad x^2 + 2 = 0 \Rightarrow x^2 = -2 \Rightarrow x = \sqrt{-2}$

Hence, the only real root is $x = 2$ which is considered as critical point.

Now, we evaluate the value of f at critical point $x = 2$ and at the end points of the interval [0, 3].

At $x = 2$, $\quad f(2) = 3 \times 2^4 - 8 \times 2^3 + 12 \times 2^2 - 48 \times 2 + 25$

$= 48 - 64 + 48 - 96 + 25 = -39$

At $x = 0$, $\quad f(0) = 0 - 0 + 0 - 0 + 25 = 25$

At $x = 3$, $\quad f(3) = 3 \times 3^4 - 8 \times 3^3 + 12 \times 3^2 - 48 \times 3 + 25$

$= 243 - 216 + 108 - 144 + 25 = 16$

Hence, we can conclude that the absolute maximum value of f is 25 at $x = 0$ and the absolute minimum value of f is –39 at $x = 2$.

Example 6 *For all real values of* x, *the minimum value of* $\dfrac{1 - x + x^2}{1 + x + x^2}$ *is* [**NCERT**]

(a) *zero* (b) 1 (c) 3 (d) $\dfrac{1}{3}$

Solution (c) Let $y = \dfrac{1 - x + x^2}{1 + x + x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + x + x^2)(-1 + 2x) - (1 - x + x^2)(1 + 2x)}{(1 + x + x^2)^2}$$

$$= \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

$$= \frac{-2+2x^2-2x^2+2x^2}{(1+x+x^2)^2} = \frac{2x^2-2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

Put $\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Now, $\frac{d^2y}{dx^2} = \frac{2\,[(1+x+x^2)^2\,(2x) - (x^2-1)\,(2)\,(1+x+x^2)\,(1+2x)]}{(1+x+x^2)^4}$

$$= \frac{4\,(1+x+x^2)\,[\,(1+x+x^2)\,x - (x^2-1)\,(1+2x)]}{(1+x+x^2)^4}$$

$$= \frac{4\,[x+x^2+x^3-x^2-2x^3+1+2x]}{(1+x+x^2)^3} = \frac{4(1+3x-x^3)}{(1+x+x^2)^3}$$

At $x = 1$, $\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{4[1+3(1)-1^3]}{(1+1+1^2)^3} = \frac{4(3)}{3^3} = \frac{4}{9} > 0$

At $x = -1$, $\left(\frac{d^2y}{dx^2}\right)_{x=-1} = \frac{4[1+3(-1)-(-1)^3]}{[1+(-1)+(-1)^2]^3}$

$$= \frac{4(1-3+1)}{(1-1+1)^3} = 4(-1) = -4 < 0$$

So, y is minimum at $x = -1$.

Now, the minimum value of y is

$$y = \frac{1+1+1}{1-1+1} = 3$$

Example 7 *The volume of the largest cone that can be inscribed in a sphere of radius R is* **[NCERT]**

(a) $\frac{3}{8}$ *of the volume of the sphere*

(b) $\frac{8}{27}$ *of the volume of the sphere*

(c) $\frac{2}{7}$ *of the volume of the sphere*

(d) *None of the above*

Solution (b) Let $OC = x$, $CQ = r$

Now, $OA = R$ (given)

Height of the cone $= h = x + R$

$\therefore$ Volume of the cone $= V = \frac{1}{3}\pi r^2 h$ ...(i)

Also, from right angled ΔOCQ

$OC^2 + CQ^2 = OQ^2 \Rightarrow x^2 + r^2 = R^2 \Rightarrow r^2 = R^2 - x^2$ ...(ii)

From Eqs. (i) and (ii), we have

$$V = \frac{1}{3}\pi(R^2 - x^2)\,(x + R) \quad \text{...(iii)}$$

$(\because h = x + R)$

On differentiating Eq. (iii) w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{3}\pi\,[(R^2 - x^2) - 2x\,(x + R)]$$

$$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}\,[R^2 - x^2 - 2x^2 - 2xR]$$

$$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}\,(R^2 - 2xR - 3x^2)$$

$$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}\,(R - 3x)\,(R + x) \quad \text{...(iv)}$$

For maxima, put $\frac{dV}{dx} = 0$

$$\Rightarrow \frac{\pi}{3}\,(R - 3x)\,(R + x) = 0$$

$$\Rightarrow x = \frac{R}{3} \text{ or } x = -R \Rightarrow x = \frac{R}{3}$$

($\because x$ cannot be negative)

On differentiating Eq. (iv) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}\,[(-3)\,(R + x) + (R - 3x)]$$

$$= \frac{\pi}{3}\,(-2R - 6x) = -\frac{\pi}{3}\,(2R + 6x)$$

At $x = \frac{R}{3}$, $\frac{d^2V}{dx^2} = \frac{-\pi}{3}\left(2R + \frac{6R}{3}\right) = -\frac{4\pi}{3}R < 0$

$\therefore$ V has a local maxima at $x = R/3$.

Now, substituting the value of x in Eq. (iii), we get

$$V = \frac{\pi}{3}\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$$

$$= \frac{\pi}{3}\cdot\frac{8R^2}{9}\cdot\frac{4R}{3} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$\Rightarrow V = \frac{8}{27} \times \text{Volume of sphere}$$

Example 8 *The minimum value of* $\frac{x}{\log x}$ *is*

(a) e (b) $\frac{1}{e}$

(c) e^2 (d) e^3

Solution (a) Let $f(x) = \frac{x}{\log x}$

$$f'(x) = \frac{\log x - 1}{(\log x)^2}$$

Pur $f'(x) = 0$, for maxima or minima, we get

$$\log x - 1 = 0$$

$$\Rightarrow x = e$$

Now, $\frac{f''(x) = (\log x)^2 \cdot \frac{1}{x} - (\log x - 1)\cdot\frac{2\log x}{x}}{(\log x)^4}$

$$\Rightarrow f''(e) = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e} > 0$$

$\therefore$ Function is minimum at $x = e$.

Hence, minimum value of $f(x)$ at $x = e$ is $f(e) = e$.

Example 9 *The sum of two numbers is 3, the maximum value of the product of the first and the square of second is* **[NCERT]**

(a) 4 (b) 1 (c) 3 (d) 0

Solution (a) Let two numbers are x and $3 - x$.

$\therefore$ Product $P = x(3-x)^2$

$\Rightarrow \quad \dfrac{dP}{dx} = -2x(3-x) + (3-x)^2$

$\Rightarrow \quad \dfrac{dP}{dx} = (3-x)(3-3x)$ and $\dfrac{d^2P}{dx^2} = 6x - 12$

For maxima or minima, put $\dfrac{dP}{dx} = 0$

$\Rightarrow \quad (3-x)(3-3x) = 0$

$\Rightarrow \quad x = 3, 1$

At $x = 3$,

$$\frac{d^2P}{dx^2} = 18 - 12 = 6 > 0 \qquad \text{(minima)}$$

At $x = 1$, $\dfrac{d^2P}{dx^2} = -6 < 0$

$\therefore$ P is maximum at $x = 1$.

Maximum value of $P = 1(3-1)^2 = 4$

Example 10 *A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle, then the minimum length of the hypotenuse is* **[NCERT]**

(a) $\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^{\frac{3}{2}}$ (b) $\left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)^{\frac{3}{2}}$

(c) $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$ (d) d$\left(a^{\frac{3}{2}} + b^{\frac{3}{2}}\right)^{\frac{2}{3}}$

Solution (c) With the help of trigonometry, write the length of the hypotenuse in terms of a and b. Follow second derivative test to prove that required result.

Let P be a point on the hypotenuse AC of right angled ΔABC.

Such that $PL \perp AB = a$ and $PM \perp BC = b$

Let $\angle APL = \angle ACB = \theta$ (say)

$AP = a \sec\theta$,

$PC = b \operatorname{cosec}\theta$

Let l be the length of the hypotenuse, then

$l = AP + PC$

$\Rightarrow \quad l = a\sec\theta + b\operatorname{cosec}\theta,\ 0 < \theta < \dfrac{\pi}{2}$

On differentiating w.r.t. θ, we get

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$$

For maxima or minima put $\dfrac{dl}{d\theta} = 0$

$\Rightarrow \quad a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$

$\Rightarrow \quad \dfrac{a\sin\theta}{\cos^2\theta} = \dfrac{b\cos\theta}{\sin^2\theta} \Rightarrow \dfrac{\sin^3\theta}{\cos^3\theta} = \dfrac{b}{a}$

$\Rightarrow \quad \tan^3\theta = \dfrac{b}{a} \Rightarrow \tan\theta = \left(\dfrac{b}{a}\right)^{1/3}$

Now, $\dfrac{d^2l}{d\theta^2} = a(\sec\theta \times \sec^2\theta + \tan\theta \times \sec\theta\tan\theta)$

$$- b[\operatorname{cosec}\theta(-\operatorname{cosec}^2\theta) + \cot\theta(-\operatorname{cosec}\theta\cot\theta)]$$

$$= a\sec\theta(\sec^2\theta + \tan^2\theta) + b\operatorname{cosec}\theta(\operatorname{cosec}^2\theta + \cot^2\theta)$$

Since, $0 < \theta < \dfrac{\pi}{2}$, so, trigonometric ratios are positive.

Also, $a > 0$ and $b > 0$.

$\therefore \quad \dfrac{d^2l}{d\theta^2}$ is positive.

$\Rightarrow \quad l$ is least when $\tan\theta = \left(\dfrac{b}{a}\right)^{1/3}$

$\therefore$ Least value of $l = a\sec\theta + b\operatorname{cosec}\theta$

$$= a\frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$$

$$= \sqrt{a^{2/3} + b^{2/3}}\,(a^{2/3} + b^{2/3}) = (a^{2/3} + b^{2/3})^{3/2}$$

$$\left(\because \text{ in } \Delta EFG \ \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and } \operatorname{cosec}\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}\right)$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Critical Points and Test for Local Maximum/Minimum

1. The critical points of the function

$$f(x) = 2\sin^2\left(\frac{x}{6}\right) + \sin\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)$$

whose coordinates satisfy the inequality $x^2 - 10 < -19.5x$, is

(a) -6π (b) 6π
(c) $\frac{9\pi}{2}$ (d) -4π

2. The points of extrema of $f(x) = \int_0^x \frac{\sin t}{t}\,dt$ in the domain $x > 0$ are

(a) $(2n+1)\frac{\pi}{2}, n = 1, 2, \ldots$ (b) $(4n+1)\frac{\pi}{2}, n = 1, 2, \ldots$
(c) $(2n+1)\frac{\pi}{4}, n = 1, 2, \ldots$ (d) $n\pi,\ n = 1, 2, \ldots$

3. The number of critical points of

$$f(x) = |x|(x-1)(x-2)(x-3) \text{ is}$$

(a) 1 (b) 2
(c) 3 (d) 4

4. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value? [NCERT]

(a) $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ (b) $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$
(c) $x = \frac{-\pi}{4}$ and $x = \frac{-3\pi}{4}$ (d) None of these

5. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a. [NCERT]

(a) 30 (b) 40
(c) 75 (d) 120

6. The maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$ is [NCERT]

(a) 2π and 0 (b) π and $\frac{1}{2}$
(c) $\frac{\pi}{2}$ and -1 (d) None of these

7. The maximum value of $[x(x-1)+1]^{1/3}, 0 \le x \le 1$ is [NCERT]

(a) $\left(\frac{1}{3}\right)^{1/\beta}$ (b) $\frac{1}{2}$
(c) 1 (d) zero

8. The function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has [NCERT Exemplar]

(a) one local maxima
(b) one local minima
(c) one local maxima and two local minima
(d) neither maxima nor minima

9. All the points of local maxima and local minima of the function $f(x) = \frac{-3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$ are [NCERT Exemplar]

(a) $x = 0, -5$ and $x = -3$ (b) $x = 1, 3$ and $x = -2$
(c) $x = 0, -3$ and $x = -2$ (d) None of these

10. The value of 'a' for which the function $f(x) = a\sin x + \frac{1}{3}\sin 3x$ has an extremum at $x = \frac{\pi}{3}$ is

(a) 1 (b) -1 (c) 2 (d) 0

11. If $f(x) = x + 1, \forall x \in R$ and $g(x) = e^x, \forall x \in [-2, 0]$, then the maximum value of $f(|x|) - g(x)$ is

(a) $3 + \frac{1}{e}$ (b) $3 + \frac{1}{e^2}$
(c) $-3 - \frac{1}{e^2}$ (d) $3 - \frac{1}{e^2}$

12. On the interval [0,1], the function $x^{25}(1-x)^{75}$ takes its maximum value at the point

(a) 0 (b) 1/4
(c) 1/2 (d) 1/3

13. The function $f(x) = x^{-x}$, $(x > 0)$ attains a maximum value at x which is
(a) 2 (b) 3
(c) $\frac{1}{e}$ (d) 1

14. The function $y = a(1 - \cos x)$ is maximum when x is equal to
(a) π (b) $\pi/2$
(c) $-\pi/2$ (d) $-\pi/6$

15. The function $f(x) = x + \sin x$ has
(a) a minimum but no maximum
(b) a maximum but no minimum
(c) neither maximum nor minimum
(d) both maximum and minimum

16. If a differential function $f(x)$ has a relative minimum at $x = 0$, then the function $\phi(x) = f(x) + ax + b$ has a relative minimum at $x = 0$ for
(a) all a and all b (b) all b, if $a = 0$
(c) all $b > 0$ (d) all $a > 0$

17. The denominator of a fraction is greater than 16 of the square of numerator, then least value of fraction is
(a) $-1/4$ (b) $-1/8$
(c) $1/12$ (d) $1/16$

18. The function $f(x) = ax + \frac{b}{x}$, b, $x > 0$ takes the least value at x equal to
(a) b (b) $\sqrt{a}$
(c) $\sqrt{b}$ (d) $\sqrt{\frac{b}{a}}$

19. In $(-4, 4)$ the function $f(x) = \int_{-10}^{x} (t^4 - 4)\, e^{-4t}\, dt$ has
(a) no extrema (b) one extremum
(c) two extrema (d) four extrema

20. The function $f(x) = a \cos x + b \tan x + x$ has extreme values at $x = 0$ and $x = \frac{\pi}{6}$, then
(a) $a = -\frac{2}{3}, b = -1$ (b) $a = \frac{2}{3}, b = -1$
(c) $a = -\frac{2}{3}, b = 1$ (d) $a = \frac{2}{3}, b = 1$

21. The absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$ **[NCERT Exemplar]**
(a) 2.25 and 2 (b) 1.25 and 1
(c) 1.75 and 1.5 (d) None of these

22. The maximum value of $f(x) = \frac{x}{4 + x + x^2}$ on $[-1, 1]$ is
(a) $-\frac{1}{4}$ (b) $-\frac{1}{3}$
(c) $\frac{1}{6}$ (d) $\frac{1}{5}$

23. In interval $[1, e]$, the greatest value of $x^2 \log x$ is
(a) e^2 (b) $\frac{1}{e} \log \frac{1}{\sqrt{e}}$
(c) $e^2 \log \sqrt{e}$ (d) None of these

24. The function $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$, where $p > 0, q > 0, r > 0$
(a) $p \neq q$ (b) $r \neq q$
(c) $r \neq p$ (d) $p = q = r$

Point of Inflection and Application of Maxima and Minima

25. The point of inflexion for the curve $y = x^{5/2}$ is
(a) (1, 1) (b) (0, 0)
(c) (1, 0) (d) (0, 1)

26. The two positive numbers x and y such that their sum is 35 and the product is $x^2 y^5$ is maximum, are **[NCERT]**
(a) 15 and 20 (b) 10 and 25
(c) 5 and 30 (d) None of these

27. The two positive numbers whose sum is 16 and the sum of whose cubes is minimum, are **[NCERT]**
(a) 4 and 12 (b) 6 and 10
(c) 8 and 8 (d) None of these

28. A square piece of tin of side 18 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps of the box. What should be the side of the square to be cut-off so that the volume of the box is maximum possible? **[NCERT]**
(a) 3 cm (b) 4 cm (c) 5 cm (d) 9 cm

29. The closed right circular cylinder of given surface and maximum volume is such that its height is equal to **[NCERT]**
(a) the radius of the base
(b) the diameter of the base
(c) the twice of diameter of the base
(d) None of the above

30. The right circular cone of least curved surface area and given volume has an altitude equal to **[NCERT]**
(a) two times the radius of the base.
(b) $\sqrt{3}$ times the radius of the base.
(c) $\sqrt{2}$ times the radius of the base.
(d) None of the above

31. The semi-vertical angle of the cone of the maximum volume and of given slant height is **[NCERT]**
(a) $\tan^{-1}\sqrt{3}$
(b) $\tan^{-1}\sqrt{2}$
(c) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$
(d) None of the above

32. The semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$. [NCERT]

(a) $\sin^{-1}\left(\frac{1}{3}\right)$ (b) $\sin^{-1}\left(\frac{1}{2}\right)$

(c) $\sin^{-1}(\sqrt{3})$ (d) None of these

33. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is [NCERT]

(a) $(2\sqrt{2}, 4)$ (b) $(2\sqrt{2}, 0)$ (c) (0, 0) (d) (2, 2)

34. The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis. [NCERT]

(a) $\frac{3}{4}ab$ sq unit (b) $\frac{3}{4}\sqrt{3}\,ab$ sq unit

(c) $\frac{\sqrt{3}}{4}ab$ sq unit (d) None of these

35. The sum of the perimeter of a circle and square is k, where k is some constant, then the sum of their areas is least when the side of square is [NCERT]

(a) equal to the radius of the circle

(b) double the radius of the circle

(c) triple the radius of the circle

(d) None of the above

36. A window is in the form of a rectangle surmounted by a semi-circle opening. The perimeter of the window is 10 m. The dimensions of the window to admit maximum light through the whole opening is [NCERT Exemplar]

(a) length = breadth = $\frac{1}{\pi+4}$

(b) length = $\frac{20}{\pi+4}$ and breadth = $\frac{10}{\pi+4}$

(c) length = $\frac{2}{\pi+4}$ and breadth = $\frac{1}{\pi+4}$

(d) None of the above

37. The altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is [NCERT Exemplar]

(a) $\frac{r}{2}$ (b) $\frac{r}{3}$ (c) $\frac{3r}{4}$ (d) $\frac{4r}{3}$

38. The height of the cylinder of maximum volume that can be inscribed in a sphere of radius R and the maximum volume respectively [NCERT Exemplar]

(a) $\frac{2R}{\sqrt{3}}$ and $\frac{4\pi R^3}{3\sqrt{3}}$ (b) $\frac{R}{\sqrt{3}}$ and $\frac{\pi R^3}{3\sqrt{3}}$

(c) $\frac{4R}{\sqrt{3}}$ and $\frac{2\pi R^3}{3\sqrt{3}}$ (d) None of these

39. The height of the cylinder of greatest volume which can be inscribed in a circular cone of height h and having semi-vertical angle α and the greatest volume of cylinder are respectively [NCERT Exemplar]

(a) $\frac{2}{3}$ that of the cone, $\frac{2}{27}\pi h^3 \tan\alpha$

(b) $\frac{4}{3}$ that of the cone, $\frac{4}{27}\pi h^2 \tan^3\alpha$

(c) $\frac{1}{3}$ that of the cone, $\frac{4}{27}\pi h^3 \tan^2\alpha$

(d) $\frac{5}{3}$ that of the cone, $\frac{4}{27}\pi h \tan\alpha$

40. If the sum of the length of the hypotenuse and a side of a right angled triangle is given. Then, the area of the triangle is maximum when the angle between them is [NCERT Exemplar]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

41. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12 (c) 16 (d) 32

42. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle, is

(a) π (b) $\pi/3$

(c) $\pi/4$ (d) $\pi/2$

43. If $ab = 2a + 3b$, $a > 0$, $b > 0$, then the minimum value of ab is

(a) 12 (b) 24

(c) $\frac{1}{4}$ (d) None of these

44. The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $a - b$ (b) $a + b$

(c) $2a + b$ (d) None of these

45. The perimeter of a sector is p. The area of the sector is maximum, when its radius is

(a) $\sqrt{p}$ (b) $\frac{1}{\sqrt{p}}$

(c) $\frac{p}{2}$ (d) $\frac{p}{4}$

46. If $xy = c^2$, then minimum value of $ax + by$ is

(a) $c\sqrt{ab}$ (b) $2c\sqrt{ab}$

(c) $-c\sqrt{ab}$ (d) $-2c\sqrt{ab}$

47. If $a^2x^4 + b^2y^4 = c^6$, then maximum value of xy is

(a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^3}{ab}$

(c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c^3}{2ab}$

48. A cone of maximum volume is inscribed in the given sphere, then ratio of the height of the cone to diameter of the sphere is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

49. A given right circular cone has volume p and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then, $p:q$ is

(a) 9 : 4 (b) 8 : 3
(c) 7 : 2 (d) None of these

50. The set of all values of the parameter a for which the points of minimum of the function $y = 1 + a^2x - x^3$ satisfy the inequality $\frac{x^2 + x + 2}{x^2 + 5x + 6} \le 0$, is

(a) an empty set
(b) $(-3\sqrt{3}, -2\sqrt{3})$
(c) $(2\sqrt{3}, 3\sqrt{3})$
(d) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

51. The tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \le x \le 5$ has maximum slope at x which is equal to

(a) 2 (b) 3
(c) 4 (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. $f(x) = x^2 - 4|x|$ and
$g(x) = \begin{cases} \min\{f(t) : -6 \le t \le x\}, x \in [-6, 0] \\ \max\{f(t) : 0 < t \le x\}, x \in (0, 6] \end{cases}$, then $g(x)$ has

(a) exactly one point of local minima
(b) exactly one point of local maxima
(c) no point to local maxima but exactly one point of local minima
(d) neither a point of local maxima nor minima

2. The function $f(x) = \frac{x^2 - 2}{x^2 - 4}$ has

(a) no point of local minima
(b) no point of local maxima
(c) exactly one point of local minima
(d) exactly one point of local maxima

3. If $f(x) = \int_0^x (t^2 - 1)\cos t\, dt, x \in (0, 2\pi)$. Then, $f(x)$ attains local maximum value at

(a) $x = \frac{\pi}{2}$ (b) $x = 1$
(c) $x = \frac{3\pi}{2}$ (d) None of these

4. $f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \le x < 3 \\ x - 18, & x \ge 3 \end{cases}$

Complete the set of values of a such that $f(x)$ has a local maxima at $x = 3$, is

(a) $[-1, 2]$ (b) $(-\infty, 1) \cup (2, \infty)$
(c) $[1, 2]$ (d) $(-\infty, -1) \cup (2, \infty)$

5. The function $f(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a local maxima at $(2, -1)$, then

(a) $b = 1, a = 0$ (b) $a = 1, b = 0$
(c) $b = -1, a = 0$ (d) $a = -1, b = 0$

6. If $f(x) = \frac{x}{1 + x\tan x}, x \in \left(0, \frac{\pi}{2}\right)$, then

(a) $f(x)$ has exactly one point of minima
(b) $f(x)$ has exactly one point of maxima
(c) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
(d) $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

7. A straight line is drawn through the point $P(3, 4)$ meeting the positive direction of coordinate axes at the points A and B. If O is the origin, then minimum area of ΔOAB is equal to

(a) 12 sq units (b) 6 sq units
(c) 24 sq units (d) 48 sq units

8. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segment BC, CA and AB, respectively. Then, maximum area of such a parallelogram is

(a) $\frac{1}{2}$(area of ΔABC) (b) $\frac{1}{4}$(area of ΔABC)
(c) $\frac{1}{6}$(area of ΔABC) (d) $\frac{1}{8}$(area of ΔABC)

9. Let $y = f(x)$ be a parametrically defined expression such that $x = 3t^2 - 18t + 7$ and $y = 2t^3 - 15t^2 + 24t + 10$, $\forall, x \in [0, 6]$. Then, the minimum and maximum values of $y = f(x)$ are

(a) 36, 3 (b) 46, 6
(c) 40, −6 (d) 46, −6

10. Let $f(x)=\begin{cases}|x^2-2|, & -1\le x<\sqrt{3}\\ \dfrac{x}{\sqrt{3}}, & \sqrt{3}\le x<2\sqrt{3}\\ 3-x, & 2\sqrt{3}\le x\le 4\end{cases}$, then the points, where $f(x)$ takes maximum and minimum values, are

(a) 1, 4 (b) 0, 4
(c) 2, 4 (d) None of these

11. The value of a so that the sum of the squares of the roots of the equation $x^2-(a-2)x-a+1=0$ assume the least value is

(a) 2 (b) 1
(c) 3 (d) 0

12. A minimum value of $\int_0^x te^{-t^2}\,dt$ is

(a) 1 (b) 2 (c) 3 (d) 0

13. If $y=a\log x+bx^2+x$ has its extremum value at $x=1$ and $x=2$, then (a, b) is equal to

(a) $\left(1,\frac{1}{2}\right)$ (b) $\left(\frac{1}{2},2\right)$
(c) $\left(2,\frac{-1}{2}\right)$ (d) $\left(\frac{-2}{3},\frac{-1}{6}\right)$

14. If $P=(1,1)$, $Q=(3,2)$ and R is a point on x-axis, then the value of $PR+RQ$ will be minimm at

(a) $\left(\frac{5}{3},0\right)$ (b) $\left(\frac{1}{3},0\right)$
(c) (3, 0) (d) (1, 0)

15. If the function $f(x)=2x^3-9ax^2+12a^2x+1$, where $a>0$ attains its maximum and minimum at p and q respectively such that $p^2=q$, then a is equal to

(a) 3 (b) 1
(c) 2 (d) $\frac{1}{2}$

16. Let $f(x)=1+2x^2+2^2x^4+\ldots+2^{10}x^{20}$, then $f(x)$ has

(a) more than one minimum
(b) exactly one minimum
(c) atleast one maximum
(d) None of the above

17. The point in the interval $[0,2\pi]$, where $f(x)=e^x\sin x$ has maximum slope is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) None of these

18. Let $f(x)=\begin{cases}|x^3+x^2+3x+\sin x|\left(3+\sin\frac{1}{x}\right), & x\ne 0\\ 0, & x=0\end{cases}$,

then number of points [where, $f(x)$ attains its minimum value] is

(a) 1 (b) 2
(c) 3 (d) infinite many

19. If $f(x)=a\log_e|x|+bx^2+x$ has extremum at $x=1$ and $x=3$, then

(a) $a=-3/4, b=-1/8$ (b) $a=3/4, b=-1/8$
(c) $a=-3/4, b=1/8$ (d) None of these

20. The total number of local maxima and local minima of the function $f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{2/3}, & -1<x<2\end{cases}$ is

(a) 0 (b) 1
(c) 2 (d) 3

21. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that minimum $f(x)>g(x)$, then the relation between b and c is

(a) $0<c<b\sqrt{2}$
(b) $|c|<|b|\sqrt{2}$
(c) $|c|>|b|\sqrt{2}$
(d) no real values of b and c

22. The curved surface of the cone inscribed in a given sphere is maximum, if

(a) $h=\frac{4R}{3}$ (b) $h=\frac{R}{3}$
(c) $h=\frac{2R}{3}$ (d) None of these

23. The minimum intercepts made by the axes on the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

(a) 25 (b) 7
(c) 1 (d) None of these

24. Maximum value of $\left(\frac{1}{x}\right)^x$ is [NCERT Exemplar]

(a) $(e)^e$ (b) $(e)^{1/e}$
(c) $(e)^{-e}$ (d) $\left(\frac{1}{e}\right)^e$

25. The largest term in the sequence $a_n=\frac{n^2}{n^3+200}$ is given by

(a) $\frac{529}{49}$ (b) $\frac{8}{89}$
(c) $\frac{49}{543}$ (d) None of these

26. If $A>0, B>0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A\cdot\tan B$ is [NCERT Exemplar]

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$ (c) 3 (d) $\sqrt{3}$

27. All possible values of the parameter a so that the function $f(x)=x^3-3(7-a)x^2-3(9-a^2)x+2$ has a negative point of local minimum are

(a) all real values (b) no real values
(c) $(0,\infty)$ (d) $(-\infty,0)$

28. $f(x)=\begin{cases}\cos\dfrac{\pi x}{2}, x>0\\ x+a, x\le 0\end{cases}$,

then $x=0$ will be a point of local maxima for $f(x)$, if

(a) $a\in(-1,1)$ (b) $a\in(0,1)$
(c) $a\le 0$ (d) $a\ge 1$

29. Let (h, k) be a fixed point, where $h>0, k>0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Which of the following is the minimum area of the ΔOPQ, O being the origin?

(a) hk (b) $2hk$
(c) $\frac{1}{2}hk$ (d) None of these

30. The coordinates of a point of the parabola $y=x^2+7x+2$ which is closest to the straight line $y=3x-3$ is

(a) $(-2,8)$ (b) $(-2,-8)$
(c) $(2,-8)$ (d) None of these

More Than One Correct Option

31. Let the parabolas $y=x^2+ax+b$ and $y=x(c-x)$ touch each other at a point (1,0). Then,

(a) $a=-3$ (b) $b=1$ (c) $c=2$ (d) $b+c=3$

32. Let $f(x)=\begin{cases}x^3+x^2-10x & -1\le x<0\\ \sin x & 0\le x<\dfrac{\pi}{2}\\ 1+\cos x & \dfrac{\pi}{2}\le x\le\pi\end{cases}$,

then $f(x)$ has

(a) local maxima at $x=\frac{\pi}{2}$ (b) local minima at $x=\frac{\pi}{2}$
(c) absolute maxima at $x=0$ (d) absolute maxima at $x=-1$

33. Let $f(x)=(x^2-1)^n(x^2+x+1)$, then $f(x)$ has local extremum at $x=1$, when

(a) $n=2$ (b) $n=3$
(c) $n=4$ (d) $n=6$

Comprehension Based Questions

Passage

Directions (Q. Nos. 34 and 35) *A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass.*

Suppose that y is the length and x is the breadth of the rectangular portion and P is the perimeter.

On the basis of above information, answer the following questions.

34. The ratio of the sides $y:x$ of the rectangle so that the window transmit the maximum light is

(a) $3:2$ (b) $6:6+\pi$
(c) $6+\pi:6$ (d) $1:2$

35. If μ is the amount of light per square metre for the coloured glass and L is the total light transmitted, then $\left(\dfrac{dL}{dy}\right)_{y=1}$ is equal to

(a) $\frac{\mu}{2}\left[3P-12-\frac{5\pi}{2}\right]$ (b) $\mu\left[2P-6-\frac{3\pi}{2}\right]$
(c) $\mu\left[P-2-\frac{\pi}{2}\right]$ (d) $2\mu\left[3P-\frac{\pi}{2}\right]$

Assertion and Reason

Directions (Q. Nos. 36-43) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

36. If $f'(x)=(x-1)^3(x-2)^8$, then

Statement I $f(x)$ has neither maximum nor minimum at $x=2$.

Statement II $f'(x)$ changes sign from negative to positive at $x=2$.

37. Consider the cubic expression $y=x^3+ax^2+bx+c$

Statement I $a^2<3b$, then function y have no any critical points.

Statement II Either y is increasing function or decreasing function for all $x\in R$.

38. Consider the function $f(x)=-x^2+4x+1+\sin^{-1}\left(\dfrac{x}{2}\right)$

Statement I Minimum value of interval $[-1,1]$ is $\left(-4,-\dfrac{\pi}{6}\right)$.

Statement II Minimum value of $f(x)$ in interval $[-1,1]$ is $\min\{f(-1), f(1)\}$.

39. **Statement I** The minimum distance of the fixed point $(0, y_0)$, where $0\le y_0\le\frac{1}{2}$, from the curve $y=x^2$ is y_0.

Statement II Maxima and minima of a function is always a root of the equation $f'(x)=0$.

40. **Statement I** The minimum value of the expression $x^2 + 2bx + c$ is $c - b^2$.

Statement II The first order derivative of the expression at $x = -b$ is zero.

41. **Statement I** The local maximum of the function $x \cos x$ will occur between $\pi / 4$ and $\pi / 3$.

Statement II The function $x \tan x$ is increasing in $\left(0, \frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

42. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement I f has local maximum at $x = -1$ and at $x = 2$.

Statement II $a = 1/2$ and $b = -1/4$.

43. Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Statement I $x = 0$ is point of minima of f.

Statement II $f'(0) = 0$

Previous Years' Questions

44. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a ΔOPQ, where, O is the origin, if the area of the ΔOPQ is least, then the slope of the line PQ is **[AIEEE 2012]**

(a) $-1/4$ (b) -4
(c) -2 (d) $-1/2$

45. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then, f has **[AIEEE 2011]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local maximum at 2π
(d) local maximum at π and 2π

46. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is **[AIEEE 2009]**

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

47. Let $f: R \longrightarrow R$ be defined by

$$f(x) = \begin{cases} k - 2x, \text{if } x \leq -1 \\ 2x + 3, \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k, is **[AIEEE 2009]**

(a) 1 (b) 0
(c) $-\frac{1}{2}$ (d) -1

48. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$ **[AIEEE 2009]**

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

49. Suppose the cubic $x^3 - px + q$ has three distinct real roots, where $p > 0$ and $q > 0$. Then, which one of the following holds? **[AIEEE 2008]**

(a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$.
(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$.
(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$.
(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$.

50. If x is real, the maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is **[AIEEE 2007]**

(a) 41 (b) 1
(c) $\frac{17}{7}$ (d) $\frac{1}{4}$

Answers

Round I

1. (b)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (c)
11. (d)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (a)
21. (b)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (c)
31. (b)	**32.** (a)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (d)	**38.** (a)	**39.** (c)	**40.** (c)
41. (b)	**42.** (d)	**43.** (b)	**44.** (b)	**45.** (d)	**46.** (b)	**47.** (c)	**48.** (a)	**49.** (a)	**50.** (c)
51. (d)									

Round II

1. (d)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (b)
11. (b)	**12.** (d)	**13.** (d)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (a)
21. (c)	**22.** (a)	**23.** (a)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (d)	**29.** (b)	**30.** (b)
31. (a,d)	**32.** (a,c,d)	**33.** (a,c,d)	**34.** (b)	**35.** (a)	**36.** (c)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (a)
41. (c)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (c)	**46.** (a)	**47.** (d)	**48.** (b)	**49.** (a)	**50.** (a)

the **Guidance**

Round I

1. $\because f'(x) = 2\left(\frac{1}{3}\right)\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) + \left(\frac{1}{3}\right)\cos\frac{x}{3} - \left(\frac{1}{3}\right)$

$$= \left(\frac{1}{3}\right)\left[2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) - 2\sin^2\left(\frac{x}{6}\right)\right]$$

$$= \left(\frac{2}{3}\right)\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) - 2\sin\left(\frac{x}{6}\right)$$

Put $\quad f'(x) = 0 \Leftrightarrow \sin\left(\frac{x}{6}\right) = 0$

$\Rightarrow \quad \tan\left(\frac{x}{6}\right) = 1$

$\Rightarrow \quad \frac{x}{6} = k\pi,\ k \in I$ or $\frac{x}{6} = n\pi + \frac{\pi}{4}, n \in I$

$$x^2 - 10 < -19.5x$$

$\Leftrightarrow \quad (x + 9.75)^2 < 105.0625$

$\Leftrightarrow \quad (x - 0.5)(x + 20) < 0$

$\Leftrightarrow \quad -20 < x < 0.5$

So, the critical points satisfying the last inequality will be $0, 6\pi, -\frac{9\pi}{2}$.

2. Given curve is $f(x) = \int_0^x \frac{\sin t}{t}\,dt$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\sin x}{x}$$

For point of extrema, put $f'(x) = 0$

$\Rightarrow \quad \frac{\sin x}{x} = 0 \Rightarrow \sin x = 0$

$\Rightarrow \quad x = n\pi, \quad n = 1, 2, 3, \ldots$

3. We have, $f(x) = |x|(x-1)(x-2)(x-3)$

$$= \begin{cases} x(x-1)(x-2)(x-3), x \geq 0 \\ -x(x-1)(x-2)(x-3), x < 0 \end{cases}$$

It is clear from the figure that, there are four critical points *i.e.*, $0, \alpha, \beta, \gamma$.

4. Let $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x$

For maxima or minima put $f'(x) = 0$

$\Rightarrow \quad 2\cos 2x = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Thus, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

At $x = 0$, $\quad f(0) = \sin(2 \times 0) = 0$

At $x = 2\pi$, $\quad f(2\pi) = \sin(2 \times 2\pi) = 0$

At $x = \frac{\pi}{4}$, $\quad f\left(\frac{\pi}{4}\right) = \sin\left(2 \times \frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$

At $x = \frac{3\pi}{4}$,

$$f\left(\frac{3\pi}{4}\right) = \sin\left(2 \times \frac{3\pi}{4}\right) = \sin\frac{3\pi}{2}$$

$$= \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

At $x = \frac{5\pi}{4}$,

$$f\left(\frac{5\pi}{4}\right) = \sin\left(2 \times \frac{5\pi}{4}\right) = \sin\frac{5\pi}{2} = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

At $x = \frac{7\pi}{4}$,

$$f\left(\frac{7\pi}{4}\right) = \sin\left(2 \times \frac{7\pi}{4}\right) = \sin\frac{7\pi}{2}$$

$$= \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$$

Hence, we can conclude that absolute maximum value of f on $[0, 2\pi]$ is 1 occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

5. Let $f(x) = x^4 - 62x^2 + ax + 9 \Rightarrow f'(x) = 4x^3 - 124x + a$

It is given that function f attains its maximum value on the interval $[0, 2]$ at $x = 1$.

$\therefore \quad f'(1) = 0$

$\Rightarrow \quad 4 \times 1^3 - 124 \times 1 + a = 0$

$\Rightarrow \quad 4 - 124 + a = 0 \Rightarrow a = 120$

Hence, the value of a is 120.

6. Let $f(x) = x + \sin 2x, f'(x) = 1 + 2\cos 2x$

For maxima or minima put $f'(x) = 0$

$\Rightarrow \quad 1 + 2\cos 2x = 0$

$\Rightarrow \quad \cos 2x = -\frac{1}{2}$

$\Rightarrow \quad \cos 2x = -\cos\frac{\pi}{3} = \cos\frac{2\pi}{3}$

$\Rightarrow \quad \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right), \cos\left(\pi + \frac{\pi}{3}\right),$

$\cos\left(3\pi - \frac{\pi}{3}\right), \cos\left(3\pi + \frac{\pi}{3}\right)$

($\because$ we know that $\cos x$ is negative in second and third quadrant)

Then, $2x = 2n \pm \frac{2\pi}{3}, n \in Z \Rightarrow 2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$

Then, we evaluate the value of f at critical points

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

and at the end points of the interval $[0, 2\pi]$.

At $x = 0$, $\quad f(0) = 0 + \sin 0 = 0$

At $x = \frac{\pi}{3}$, $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

At $x = \frac{2\pi}{3}$, $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

At $x = \frac{4\pi}{3}$,

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \sin\frac{2\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

At $x = \frac{5\pi}{3}$, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \sin\frac{2\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$

At $x = 2\pi$, $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

Thus, maximum value is 2π at $x = 2\pi$ and minimum value is 0 at $x = 0$.

7. Let $f(x) = [x(x-1) + 1]^{1/3}$, $0 \le x \le 1 = (x^2 - x + 1)^{1/3}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{3}(x^2 - x + 1)^{\frac{1}{3}-1}(2x - 1) = \frac{1(2x-1)}{3(x^2 - x + 1)^{2/3}}$$

Now, put $\quad f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \in [0, 1]$.

So, $x = \frac{1}{2}$ is a critical point.

Now, we evaluate the value of f at critical point $x = \frac{1}{2}$ and at the end points of the interval [0, 1].

At $x = 0$, $\quad f(0) = (0 - 0 + 1)^{1/3} = 1$

At $x = 1$, $\quad f(1) = (1 - 1 + 1)^{1/3} = 1$

At $x = \frac{1}{2}$, $\quad f\left(\frac{1}{2}\right) = \left(\frac{1}{4} - \frac{1}{2} + 1\right)^{1/3} = \left(\frac{3}{4}\right)^{1/3}$

$\therefore$ Maximum value of $f(x)$ is 1 at $x = 0, 1$. Hence, (c) is the correct option.

8. $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9) = 3(2x - 3)^2$

$f'(x) = 0 \Rightarrow x = \frac{3}{2}$ (critical point)

Since, $f'(x) > 0$ for all $x < \frac{3}{2}$ and for all $x > \frac{3}{2}$

Hence, $x = \frac{3}{2}$ is a point of inflexion *i.e.*, neither a point of maxima nor a point of minima.

$x = \frac{3}{2}$ is the only critical point and f has neither maxima nor minima.

9. $f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15) = -3x(x + 5)(x + 3)$

$f'(x) = 0 \Rightarrow x = -5, x = -3, x = 0$

$f''(x) = -9x^2 - 48x - 45 = -3(3x^2 + 16x + 15)$

$f''(0) = -45 < 0$. Therefore, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Therefore, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Therefore, $x = -5$ is point of local maxima

10. $f'\left(\frac{\pi}{3}\right)=0 \Rightarrow \frac{a}{2}-1=0 \Rightarrow a=2$

11. $f(|x|)=|x|+1=\begin{cases}-x+1; x>0\\ x+1; x>0\end{cases}, g(x)=\{e^x, -2 \le x \le 0$

$f(|x|)-g(x)=\{-x+1-e^x, -2 \le x \le 0$

Say $\quad H(x)=x+1-e^x, -2 \le x \le 0$

$H'(x)=-1-e^x<0$ for all $x \in [-2,0)$. So, $H(x)$ is maximum at $x=-2$.

So, its maximum value at $x=-2$,

$H(x)=-(-2)+1-e^{-2}=3-\frac{1}{e^2}$

12. $f'(x)=25x^{24}(1-x)^{75}-75x^{25}(1-x)^{74}$

$=25x^{24}(1-x)^{74}[(1-x)-3]$

$=25x^{24}(1-x)^{74}(1-4x)$; $f'(x)$ changes sign about $x=\frac{1}{4}$ only

13. Given, $\quad f(x)=x^{-x}$

$\Rightarrow \quad \log f(x)=-x\log x$

On differentiating w.r.t. x, we get

$$\frac{1}{f(x)}\cdot f'(x)=-\log x-1$$

$\Rightarrow \quad f'(x)=-f(x)(1+\log x)$

Put $\quad f'(x)=0$

$\Rightarrow \quad \log x=-1 \Rightarrow x=e^{-1}$

$\therefore \quad f''(x)=-f'(x)(1+\log x)-f(x)\frac{1}{x}$

$$=f(x)(1+\log x)^2-\frac{f(x)}{x}$$

At $x=\frac{1}{e}$,

$$f''(x)=-ef\left(\frac{1}{e}\right)<0, \text{ maxima}$$

Hence, at $x=\frac{1}{e}$, $f(x)$ is maximum.

14. $\because y=a(1-\cos x)$

On differentiating w.r.t. x, we get

$$y'=a\sin x \quad \ldots(i)$$

Put $y'=0$ for maxima or minima,

$$\sin x=0$$

$\Rightarrow \quad x=0, \pi$

Again, differentiating w.r.t. x of Eq. (i), we get

$y''=a\cos x \Rightarrow y''(0)=a$ and $y''(\pi)=-a$

Hence, y is maximum when $x=\pi$.

15. Given function is $f(x)=x+\sin x$

On differentiating w.r.t. x, we get

$$f'(x)=1+\cos x$$

For maxima or minima put $f'(x)=0$

$\Rightarrow \quad 1+\cos x=0 \Rightarrow \cos x=-1 \Rightarrow x=\pi$

Again, differentiating w.r.t. x we get

$f''(x)=-\sin x$, at $x=\pi$, $f''(\pi)=0$

Again, differentiating w.r.t. x, we get

$$f'''(x)=-\cos x,$$
$$f'''(\pi)=1$$

At $x=\pi$, $f(x)$ is minimum.

16. $\phi'(x)=f'(x)+a$

$\because \quad \phi'(0)=0$

$\Rightarrow \quad f'(0)+a=0$

$\Rightarrow \quad a=0 \quad [\because f'(0)=0]$

Also, $\quad \phi'(0)>0 \quad [\because f''(0)>0]$

$\Rightarrow \phi(x)$ has relative minimum at $x=0$ for all b, if $a=0$

17. Let the number be x, then $f(x)=\frac{x}{x^2+16}$

On differentiating w.r.t. x, we get

$$f'(x)=\frac{(x^2+16)\cdot 1-x(2x)}{(x^2+16)^2}$$

$$=\frac{x^2+16-2x^2}{(x^2+16)^2}=\frac{16-x^2}{(x^2+16)^2} \quad \ldots(i)$$

Put $f'(x)=0$ for maxima or minima

$f'(x)=0 \Rightarrow 16-x^2=0 \Rightarrow x=4, -4$

Again, on differentiating w.r.t. x, we get

$$f''(x)=\frac{(x^2+16)^2(-2x)-(16-x^2)2(x^2+16)2x}{(x^2+16)^4}$$

At $x=4$, $f''(x)<0$

$\therefore f(x)$ is maximum at $x=4$.

and at $x=-4$, $f''(x)>0$, $f(x)$ is minimum.

$\therefore$ Least value of $f(x)=\frac{-4}{16+16}=-\frac{1}{8}$

18. Given, $\quad f(x)=ax+\frac{b}{x}$

On differentiating w.r.t. x, we get

$f'(x)=a-\frac{b}{x^2}$

For maxima or minima, put $f'(x)=0$

$\Rightarrow \quad x=\sqrt{\frac{b}{a}}$

Again, differentiating w.r.t. x, we get

$$f''(x)=\frac{2b}{x^3}$$

At $x=\sqrt{\frac{b}{a}}$, $f''(x)=$ positive

$\Rightarrow f(x)$ is minimum at $x=\sqrt{\frac{b}{a}}$.

$\therefore f(x)$ has the least value at $x=\sqrt{\frac{b}{a}}$.

19. Given, $f(x) = \int_{-10}^{x} (t^4 - 4)\, e^{-4t}\, dt$

On differentiating w.r.t x, we get

$$f'(x) = (x^4 - 4)\, e^{-4x}$$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow \quad x = \pm\sqrt{2}, \pm\sqrt{2}$

Again, on differentiating w.r.t. x, we get

$$f''(x) = -4(x^4 - 4)\, e^{-4x} + 4x^3 e^{-4x}$$

At $x = \sqrt{2}$ and $x = -\sqrt{2}$, the given function has two extreme values.

20. $f'(x) = -a \sin x + b \sec^2 x + 1$

Now, $\quad f'(0) = 0$ and $f'\left(\frac{\pi}{6}\right) = 0$

$\Rightarrow \quad b + 1 = 0$ and $-\frac{a}{2} + \frac{4b}{3} + 1 = 0$

$\Rightarrow \quad b = -1,\ a = -\frac{2}{3}$

21. Given, $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$

Now, $f'(x) = 2\cos x(-\sin x) + \cos x = -2\sin x \cos x + \cos x$

For maximum or minimum put $f'(x) = 0$

$\Rightarrow \quad -2\sin x \cos x + \cos x = 0$

$\Rightarrow \quad \cos x(-2\sin x + 1) = 0 \Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}$

For absolute maximum and absolute minimum, we have to evaluate

$f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{\pi}{2}\right), f(\pi)$

At $x = 0$, $\quad f(0) = \cos^2 0 + \sin 0 = 1^2 + 0 = 1$

At $x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right) + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4} = 1.25$

At $x = \frac{\pi}{2}$, $\quad f\left(\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{2} = 0^2 + 1 = 1$

At $x = \pi$, $\quad f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$

Hence, the absolute maximum value of f is 1.25 occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2}$ and π.

Note If close interval is given, to determine global maximum (minimum), check the value at all critical points as well as end points of a given interval.

22. $\because \quad f(x) = \frac{x}{4 + x + x^2}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{4 + x + x^2 - x(1 + 2x)}{(4 + x + x^2)^2}$$

For maximum, put $f'(x) = 0 \Rightarrow \frac{4 - x^2}{(4 + x + x^2)^2} = 0$

$\Rightarrow \quad x = 2, -2$

Both the values of x are not in the interval $[-1, 1]$.

$\therefore \quad f(-1) = \frac{-1}{4 - 1 + 1} = \frac{-1}{4}$

$f(1) = \frac{1}{4 + 1 + 1} = \frac{1}{6}$ (maximum)

23. Given, $\quad f(x) = x^2 \log x$

On differentiating w.r.t. x, we get

$$f'(x) = (2\log x + 1)\, x$$

For a maximum, put $f'(x) = 0$

$\Rightarrow \quad (2\log x + 1)\, x = 0 \quad \Rightarrow \quad x = e^{-1/2}, 0$

$\because \quad 0 < e^{-1/2} < 1$

None of these critical points lies in the interval $[1, e]$.

So, we only compute the value of $f(x)$ at the end points 1 and e.

We have, $\quad f(1) = 0, \quad f(e) = e^2$

Hence, greatest value of $f(x) = e^2$

24. We have, $f(x) = |px - q| + r|x|,\ x \in (-\infty, \infty)$

$$= \begin{cases} -px + q - rx, & x \le 0 \\ -px + q + rx, & 0 < x < q/p \\ px - q + rx, & q/p < x \end{cases}$$

Thus, f has infinite points of minimum, if $r = p$.

In case, $p \ne r$, then $x = 0$ is point of minimum, if $r > p$ and $x = \frac{q}{p}$ is point of minimum, if $r < p$.

25. Given, $\quad y = x^{5/2}$

$\therefore \quad \frac{dy}{dx} = \frac{5}{2}x^{3/2}, \quad \frac{d^2y}{dx^2} = \frac{15}{4}x^{1/2}$

At $x = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3}$ is not defined,

when $x = 0, y = 0$

$\therefore (0, 0)$ is a point of inflection.

26. Let the numbers be x and y and $P = x^2y^5$, then $x + y = 35$

$\Rightarrow \quad x = 35 - y$

$\therefore \quad P = (35 - y)^2 y^5$

On differentiating twice w.r.t. y, we get

$$\frac{dP}{dy} = (35 - y)^2\, 5y^4 + y^5\, 2(35 - y)(-1)$$

$$= y^4 (35 - y)[5(35 - y) - 2y]$$

$$= y^4 (35 - y)(175 - 5y - 2y)$$

$$= y^4 (35 - y)(175 - 7y) = (35y^4 - y^5)(175 - 7y)$$

and $$\frac{d^2P}{dy^2} = (35y^4 - y^5)(-7) + (175 - 7y)(4 \times 35 \times y^3 - 5y^4)$$

$$= -7y^4 (35 - y) + 7(25 - y) \times 5y^3 (28 - y)$$

$$= -7y^4 (35 - y) + 35y^3 (25 - y)(28 - y)$$

For maxima put $\frac{dP}{dy} = 0$

$\Rightarrow \quad y^4 (35 - y)(175 - 7y) = 0 \Rightarrow y = 0, 35 - y = 0,$

$175 - 7y = 0 \Rightarrow y = 0, y = 25, y = 35$

When $y = 0$, $x = 35 - 0 = 35$ and the product x^2y^5 will be 0.

When $y = 35$ and $x = 35 - 35 = 0$. This will make the product x^2y^5 equal to 0.

$\therefore$ $y = 0$ and $y = 35$ cannot be the possible value of y.

When $y = 25$,

$$\left(\frac{d^2P}{dy^2}\right)_{y=25} = -7 \times (25)^4 \times (35 - 25) + 35 \times (25)^3 \times (25 - 25)(28 - 25)$$

$$= -7 \times 390625 \times 10 + 35 \times 15625 \times 0 \times 3$$

$$= -27343750 + 0 = -27343750 < 0$$

$\therefore$ By second derivative test, P will be the maximum when $y = 25$ and $x = 35 - 25 = 10$. Hence, the required numbers are 10 and 25.

Hence, the required numbers are 10 and 25.

27. Let one number is x. Then, the other number will be $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by S.

Then,

$$S = x^3 + (16 - x)^3$$

On differentiating w.r.t. x, we get

$$\frac{dS}{dx} = 3x^2 + 3(16 - x)^2(-1) = 3x^2 - 3(16 - x)^2$$

$$\Rightarrow \quad \frac{d^2S}{dx^2} = 6x + 6(16 - x) = 96$$

For minima put $\frac{dS}{dx} = 0$

$\Rightarrow \quad 3x^2 - 3(16 - x)^2 = 0$

$\Rightarrow \quad x^2 - (256 + x^2 - 32x) = 0$

$\Rightarrow \quad 32x = 256 \Rightarrow x = 8$

At $x = 8$, $$\left(\frac{d^2S}{dx^2}\right)_{x=8} = 96 > 0$$

$\therefore$ By second derivative test, $x = 8$ is the point of local minima of S.

Thus, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$.

Hence, the required numbers are 8 and 8.

28. First of all, draw the figure of square piece. Establish a relation between volume V and the side of the box formed, differentiate it. Put $\frac{dV}{dx} = 0$. Find the critical points and apply the second derivative test to find the required side of the square.

Let the side of the square to be cut-off be x cm $(0 < x < 9)$. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Let V the volume of the open box formed by folding up the flaps, then $V = x(18 - 2x)(18 - 2x)$

$= 4x(9 - x)^2 = 4x(81 + x^2 - 18x)$

$= 4(x^3 - 18x^2 + 81x)$

On differentiating twice w.r.t. x, we get

$$\frac{dV}{dx} = 4(3x^2 - 36x + 81) = 12(x^2 - 12x + 27)$$

and $$\frac{d^2V}{dx^2} = 12(2x - 12) = 24(x - 6)$$

For maxima, put $\frac{dV}{dx} = 0 \Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow \quad x^2 - 12x + 27 = 0$

$\Rightarrow \quad (x - 3)(x - 9) = 0 \Rightarrow x = 3, 9$

But $x = 9$ is not possible.

$\because \quad 2x = 2 \times 9 = 18$

which is equal to the side of square piece.

At $x = 3$, $$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3 - 6) = -72 < 0$$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima.

Hence, if we cut-off the side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

29. Hence, we have two independent variable r and h, so we eliminate one variable. For this, find the value of h in terms of r and S and put in volume, then use the second derivative test. Let r and h be the radius and height of the cylinder respectively.

Then, the surface area S of the cylinder is

$$S = 2\pi r^2 + 2\pi rh \quad \text{(given)}$$

$$\Rightarrow \quad 2\pi rh = S - 2\pi r^2 \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

Also, $$V = \pi r^2 h \quad \text{...(ii)}$$

On putting the value of h from Eq. (i) in Eq. (ii), we get

$$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{Sr}{2} - \pi r^3 \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$$

For maxima or minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S}{2} - 3\pi r^2 = 0 \quad \Rightarrow \quad S = 6\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi} \quad \text{...(iv)}$$

Now, $$\frac{d^2V}{dr^2} = -6\pi r$$

At $r^2 = \frac{S}{6\pi}$, $$\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0$$

By second derivative test, the volume is maximum when $r^2 = \frac{S}{6\pi}$.

when $$r^2 = \frac{S}{6\pi} \text{ or } S = 6\pi r^2$$

Then, $$h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r$$

Hence, the volume is maximum when the height is twice the radius *i. e.*, when the height is equal to the diameter.

30. Let r be the radius of the base, h be the height, V be the volume and S be the curved surface area of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad 3V = \pi r^2 h$$

$$\Rightarrow \quad 9V^2 = \pi^2 r^4 h^2 \Rightarrow h^2 = \frac{9V^2}{\pi^2 r^4} \quad \text{...(i)}$$

and $$S = \pi r l \quad \Rightarrow \quad S = \pi r\sqrt{r^2 + h^2} \quad (\because l = \sqrt{h^2 + r^2})$$

$$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2) = \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4} + r^2\right) \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad S^2 = \frac{9V^2}{r^2} + \pi^2 r^4 \quad \text{...(ii)}$$

when S is least, S^2 is also least.

Now, $$\frac{d}{dr}(S^2) = -\frac{18V^2}{r^3} + 4\pi^2 r^3 \quad \text{...(iii)}$$

For minima, put $$\frac{d}{dr}(S^2) = 0$$

$$\Rightarrow \quad -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow \quad 18V^2 = 4\pi^2 r^6$$

$$\Rightarrow \quad 9V^2 = 2\pi^2 r^6 \quad \text{...(iv)}$$

Again, differentiating Eq. (iii) w.r.t. r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

At $$9V^2 = 2\pi^2 r^6, \quad \frac{d^2}{dr^2}(S^2) = \frac{54}{r^4}\left(\frac{2\pi^2 r^6}{9}\right) + 12\pi^2 r^2$$

$$= \frac{12\pi^2 r^6}{r^4} + 12\pi^2 r^2 = 24\pi^2 r^2 > 0$$

Hence, S^2 and therefore S is minimum when $9V^2 = 2\pi^2 r^6$

On putting $9V^2 = 2\pi^2 r^6$ in Eq. (i), we get

$2\pi^2 r^6 = \pi^2 r^4 h^2$

$$\Rightarrow \quad 2r^2 = h^2 \Rightarrow h = \sqrt{2}r$$

Hence, altitude of right circular cone is $\sqrt{2}$ times the radius of the base.

31. Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$.

Let r, h and l be the radius, height and the slant height of the cone respectively.

The slant height of the cone is given *i. e.*, consider as constant.

Now, $$r = l\sin\theta$$

and $$h = l\cos\theta$$

Let V be the volume of the cone; $V = \frac{\pi}{3}r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi(l^2\sin^2\theta)(l\cos\theta) = \frac{1}{3}\pi l^3\sin^2\theta\cos\theta$$

On differentiating w.r.t. θ, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$$

$$= \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta)$$

For maxima, put $$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \quad \sin^3\theta = 2\sin\theta\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta = 2 \quad \Rightarrow \quad \tan\theta = \sqrt{2}$$

$$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$

Then, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$$= -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ By second derivative test, the volume V is maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

32. With usual notation, given that total surface area $S = \pi r l + \pi r^2$

$\Rightarrow \quad S = \pi r\sqrt{r^2 + h^2} + \pi r^2 \quad \left(\because l = \sqrt{r^2 + h^2}\right)$

$\Rightarrow \quad \dfrac{S}{\pi r} - r = \sqrt{r^2 + h^2}$

$\Rightarrow \quad \dfrac{S^2}{\pi^2 r^2} - \dfrac{2S}{\pi} = h^2$

$\Rightarrow \quad h = \sqrt{\dfrac{S^2}{\pi^2 r^2} - \dfrac{2S}{\pi}}, \quad \left(\because \dfrac{S^2}{\pi^2 r^2} > \dfrac{2S}{\pi}\right) \quad \ldots(i)$

and volume $\quad V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2 \sqrt{\dfrac{S^2}{\pi^2 r^2} - \dfrac{2S}{\pi}}$

$\Rightarrow \quad V = \dfrac{r}{3}\sqrt{S^2 - 2S\pi r^2}, r^2 < \dfrac{S}{2\pi}$ *i.e.*, $0 < r < \sqrt{\dfrac{S}{2\pi}}$

Since, V is maximum, then V^2 is maximum.

Now,

$V^2 = \dfrac{S^2 r^2}{9} - \dfrac{2S\pi r^4}{9}, 0 < r < \sqrt{\dfrac{S}{2\pi}}$

$\therefore \quad \dfrac{d}{dr}(V^2) = \dfrac{2rS^2}{9} - \dfrac{8S\pi r^3}{9}$

and $\dfrac{d^2}{dr^2}(V^2) = \dfrac{2S^2}{9} - \dfrac{24S\pi r^2}{9}$

For maxima, put $\dfrac{dV}{dr} = 0$

$\Rightarrow \quad \dfrac{2rS^2}{9} - \dfrac{8S\pi r^3}{9} = 0 \Rightarrow r^2 = \dfrac{S}{4\pi}$

$\Rightarrow \quad r = \sqrt{\dfrac{S}{4\pi}}$

Here, $\quad \dfrac{d^2(V^2)}{dr^2} < 0$ for $r = \sqrt{\dfrac{S}{4\pi}}$

So, V^2 and hence V is maximum, when $r = \sqrt{\dfrac{S}{4\pi}}$

From Eq. (i), $h = \sqrt{\dfrac{S^2}{\pi^2 r^2} - \dfrac{2S}{\pi}} = \sqrt{\dfrac{S^2(4\pi)}{\pi^2 S} - \dfrac{2S}{\pi}} = \sqrt{\dfrac{2S}{\pi}}$

If θ is the semi-vertical angle of the cone when the volume is maximum, then

in right ΔAOC,

$$\sin\theta = \frac{r}{\sqrt{r^2 + h^2}} = \frac{\sqrt{\frac{S}{4\pi}}}{\sqrt{\frac{S}{4\pi} + \frac{2S}{\pi}}} = \frac{1}{\sqrt{1+8}} \text{ i.e., } \theta = \sin^{-1}\left(\frac{1}{3}\right)$$

33. Firstly, consider any point (x, y) on the curve and use the formulas of distance between two points and adjust them in one variable and simplify it.

Let d be the distance of the point (x, y) on $x^2 = 2y$ from the point (0, 5), then

$$d = \sqrt{(x-0)^2 + (y-5)^2} = \sqrt{x^2 + (y-5)^2} \quad \ldots(i)$$

$$= \sqrt{2y + (y-5)^2} \quad \text{(putting } x^2 = 2y\text{)}$$

$$= \sqrt{y^2 - 8y + 25} = \sqrt{y^2 - 8y + 4^2 + 9} = \sqrt{(y-4)^2 + 9}$$

d is least when $(y-4)^2 = 0$ *i.e.*, when $y = 4$

When $y = 4$, then $x^2 = 2 \times 4 \Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

$\therefore$ The points $(2\sqrt{2}, 4)$ and $(-2\sqrt{2}, 4)$ on the given curve are nearest to the point (0, 5). So (a) is the correct option.

34. Let the equation of an ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then any point P on the ellipse is $(a\cos\theta, b\sin\theta)$.

From P, draw $PM \perp OX$ and produce it to meet the ellipse at Q, then APQ is an isosceles triangle, let S be its area, then

$$S = 2 \times \frac{1}{2} \times AM \times MP = (OA - OM) \times MP$$

$$= (a - a\cos\theta) \cdot b\sin\theta$$

$$\Rightarrow \quad S = ab(\sin\theta - \sin\theta\cos\theta) = ab\left(\sin\theta - \frac{1}{2}\sin 2\theta\right)$$

On differentiating w.r.t. θ, we get

$\dfrac{dS}{d\theta} = ab(\cos\theta - \cos 2\theta)$

Again, differentiating w.r.t. θ, we get

$$\frac{d^2S}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$$

For maxima or minima, put $\dfrac{dS}{d\theta} = 0$

$\Rightarrow \quad \cos\theta = \cos 2\theta$

$\Rightarrow \quad 2\theta = 2\pi - \theta$

$\Rightarrow \quad \theta = \dfrac{2\pi}{3}$

At $\theta = \dfrac{2\pi}{3}$, $\left(\dfrac{d^2S}{d\theta^2}\right)_{\theta = \frac{2\pi}{3}} = ab\left[-\sin\dfrac{2\pi}{3} + 2\sin\left(2 \times \dfrac{2\pi}{3}\right)\right]$

$$= ab\left[-\sin\left(\pi - \frac{\pi}{3}\right) + 2\sin\left(\pi + \frac{\pi}{3}\right)\right]$$

$$= ab\left(-\sin\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right) \quad \left[\begin{array}{l} \because \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \\ \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\pi \end{array}\right]$$

$$= ab\left(-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}ab}{2} < 0$$

$\therefore$ S is maximum, when $\theta = \dfrac{2\pi}{3}$

and maximum value of

$$S = ab\left(\sin\frac{2\pi}{3} - \frac{1}{2}\cdot 2\sin\frac{2\pi}{3}\cos\frac{2\pi}{3}\right)$$

$(\because \sin 2\theta = 2\sin\theta\cos\theta)$

$$= ab\left[\sin\left(\pi - \frac{\pi}{3}\right) - \sin\left(\pi - \frac{\pi}{3}\right)\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= ab\left[\sin\frac{\pi}{3} - \sin\frac{\pi}{3}\times\left(-\cos\frac{\pi}{3}\right)\right]$$

$$= ab\left(\sin\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{3}\right) = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\times\frac{1}{2}\right)$$

$$= ab\left(\frac{2\sqrt{3}+\sqrt{3}}{4}\right) = \frac{3\sqrt{3}}{4}\,ab \text{ sq unit}$$

Thus, maximum area of isosceles triangle is $\frac{3\sqrt{3}}{4}\,ab$ sq unit.

35. Let x be the side of square and r be the radius of circle.
Perimeter of a circle $= 2\pi r$ and perimeter of a square $= 4x$

Given, $\quad 2\pi r + 4x = k \;\Rightarrow\; x = \frac{k - 2\pi r}{4}$ ...(i)

$$\therefore \quad A = x^2 + \pi r^2 = \left[\frac{k-2\pi r}{4}\right]^2 + \pi r^2$$

$$= \left(\frac{1}{16}\right)(k^2 - 4k\pi r + 4\pi^2 r^2) + \pi r^2$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = \left(\frac{1}{16}\right)(-4k\pi + 8\pi^2 r) + 2\pi r$$

Again, differentiating w.r. t. r, we get

$$\frac{d^2A}{dr^2} = \frac{1}{16}[0 + 8\pi^2] + 2\pi = 2\pi + \frac{\pi^2}{2} > 0$$

For maximum or minimum, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 2\pi r - \frac{4k\pi}{16} + \frac{8\pi^2 r}{16} = 0 \Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4}$$

$$\Rightarrow \quad r = \frac{\left(\frac{k\pi}{4}\right)}{2\pi + \frac{\pi^2}{2}} = \frac{k}{8+2\pi} \quad \text{...(ii)}$$

Now, $\quad \left(\frac{d^2A}{dr^2}\right)_{r=\frac{k}{8+2\pi}} = \text{positive}$

$\therefore A$ is least, when $r = \frac{k}{8+2\pi}$ and put this value in Eq. (i), we get

$$x = \frac{k-2\pi r}{4} = \frac{1}{4}\left(k - 2\pi\times\frac{k}{8+2\pi}\right) = \frac{1}{4}\left[\frac{8k + 2\pi k - 2\pi k}{8+2\pi}\right]$$

$$= \frac{2k}{8+2\pi} = 2\left(\frac{k}{8+2\pi}\right) = 2r \quad \text{[using Eq. (ii)]}$$

Hence, S is least when side of the square is double the radius of the circle.

36. Let radius of semi-circle $= r$

$\therefore$ One side of rectangle $= 2r$
Let other side $= x$
$\therefore P =$ Perimeter $= 10$ (given)

$$\Rightarrow \quad 2x + 2r + \frac{1}{2}(2\pi r) = 10$$

$$\Rightarrow \quad 2x = 10 - r(\pi + 2) \quad \text{...(i)}$$

Let A be area of the figure, then

$$A = \text{Area of semi-circle} + \text{Area of rectangle} = \frac{1}{2}\pi r^2 + 2rx$$

$$\Rightarrow \quad A = \frac{1}{2}(\pi r^2) + r[10 - r(\pi + 2)] \quad \text{[using Eq. (i)]}$$

$$= \frac{1}{2}(\pi r^2) + 10r - r^2\pi - 2r^2 = 10r - \frac{\pi r^2}{2} - 2r^2$$

On differentiating twice w.r.t. r, we get

$$\frac{dA}{dr} = 10 - \pi r - 4r \quad \text{...(ii)}$$

and $\quad \frac{d^2A}{dr^2} = -\pi - 4 \quad$ ...(iii)

For maxima or minima, put $\frac{dA}{dr} = 0 \;\Rightarrow\; 10 - \pi r - 4r = 0$

$$\Rightarrow \quad 10 = (4 + \pi)r \;\Rightarrow\; r = \frac{10}{4+\pi}$$

On putting $r = \frac{10}{4+\pi}$ in Eq. (iii), we get $\frac{d^2A}{dr^2} =$ negative

Thus A has local maximum when $r = \frac{10}{4+\pi}$ ...(iv)

$\therefore$ Radius of semi-circle $= \frac{10}{4+\pi}$

and one side of rectangle $= 2r = \frac{2\times 10}{4+\pi} = \frac{20}{4+\pi}$

and other side of rectangle *i.e.*, x from Eq. (i) is given by

$$x = \frac{1}{2}[10 - r(\pi + 2)] = \frac{1}{2}\left[10 - \left(\frac{10}{\pi+4}\right)(\pi + 2)\right]$$

[from Eq. (iv)]

$$= \frac{10\pi + 40 - 10\pi - 20}{2(\pi+4)} = \frac{20}{2(\pi+4)} = \frac{10}{\pi+4}$$

Light is maximum when area is maximum.

So, dimensions of the window are length $= 2r = \frac{20}{\pi+4}$,

breadth $= x = \frac{10}{\pi+4}$.

37. Let R be the radius and h be the height of cone.

$\therefore \quad OA = h - r$

In ΔOAB, $r^2 = R^2 + (h-r)^2 \Rightarrow r^2 = R^2 + h^2 + r^2 - 2rh$

$$\Rightarrow \quad R^2 = 2rh - h^2$$

The volume V of the cone is given by $V = \frac{1}{3}\pi R^2 h$

$$= \frac{1}{3}\pi h(2rh - h^2) = \frac{1}{3}\pi(2rh^2 - h^3)$$

On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2)$$

For maximum or minimum, put $\frac{dV}{dh} = 0$

$\Rightarrow \quad 4rh = 3h^2 \Rightarrow 4r = 3h$

$\therefore \quad h = \frac{4r}{3} \quad (h \neq 0)$

Now, $\quad \frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$

At $h = \frac{4r}{3}$, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r) = \frac{-4r\pi}{3} < 0$$

$\Rightarrow V$ is maximum when $h = \frac{4r}{3}$. Hence, volume of the cone is maximum when $h = \frac{4r}{3}$, which is the altitude of cone.

38. Draw the diagram of a cylinder of height h and radius r inscribed in a sphere of radius R. Now, express volume V in terms of R and h and then apply second derivative test to prove the required results.

Radius of the sphere $= R$

Let h be the height and x be the diameter of the base of the inscribed cylinder. Then,

$$h^2 + x^2 = (2R)^2 \Rightarrow h^2 + x^2 = 4R^2 \quad \text{...(i)}$$

Volume of the cylinder $= \pi\,(\text{radius})^2 \times$ height

$$\Rightarrow V = \pi\left(\frac{x}{2}\right)^2 \cdot h = \frac{1}{4}\pi x^2 h \Rightarrow V = \frac{1}{4}\pi h(4R^2 - h^2) \quad \text{...(ii)}$$

[from Eq. (i), $x^2 = 4R^2 - h^2$]

$$\Rightarrow \quad V = \pi R^2 h - \frac{1}{4}\pi h^3$$

On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2 = \pi\left(R^2 - \frac{3}{4}h^2\right)$$

Put $\quad \frac{dV}{dh} = 0 \Rightarrow R^2 = \frac{3}{4}h^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$

Also, $\quad \frac{d^2V}{dh^2} = -\frac{3}{4} \times 2\pi h$

At $\quad h = \frac{2R}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -\frac{3}{4} \times 2\pi\left(\frac{2R}{\sqrt{3}}\right) = -\sqrt{3}\pi R = -\text{ve}$

$\Rightarrow V$ is maximum at $h = \frac{2R}{\sqrt{3}}$

Maximum volume at $h = \frac{2R}{\sqrt{3}}$ is

$$V = \frac{1}{4}\pi\left(\frac{2R}{\sqrt{3}}\right)\left(4R^2 - \frac{4R^2}{3}\right) \quad \text{[using Eq. (ii)]}$$

$$= \frac{\pi R}{2\sqrt{3}}\left(\frac{8R^2}{3}\right) = \frac{4\pi R^3}{3\sqrt{3}} \text{ sq unit}$$

Thus, volume of the cylinder is maximum when $h = \frac{2R}{\sqrt{3}}$

39. Let VAB be the cone of height h, semi-vertical angle α and let x be the radius of the base of the cylinder $A'B'DC$ which is inscribed in the cone VAB. Then, OO' is the height of the cylinder $= VO - VO' = h - x\cot\alpha$

Volume of the cylinder,

$$V = \pi x^2(h - x\cot\alpha) \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = 2\pi xh - 3\pi x^2\cot\alpha$$

For maxima or minima, put $\frac{dV}{dx} = 0$

$\Rightarrow \quad 2\pi xh - 3\pi x^2\cot\alpha = 0$

$\Rightarrow \quad x = \frac{2h}{3}\tan\alpha \quad (\because x \neq 0)$

Now, $\quad \frac{d^2V}{dx^2} = 2\pi h - 6\pi x\cot\alpha$

At $x = \frac{2h}{3}\tan\alpha$, $\frac{d^2V}{dx^2} = \pi(2h - 4h) = -2\pi h < 0$

$\Rightarrow V$ is maximum, when $x = \frac{2h}{3}\tan\alpha$.

Now, $\quad OO' = h - x\cot\alpha = h - \frac{2h}{3} = \frac{h}{3}$

$\therefore$ The maximum volume of the cylinder is

$$V = \pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(h - \frac{2h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\alpha$$

40. Given, $\quad AC + BC = \text{constant} = k \quad \text{...(i)}$

Let $\quad \angle ACB = \theta$

and $AC = x$, then $BC = x\cos\theta$ and $AB = x\sin\theta$

Let y be the area of ΔABC.

Then,

$$y = \frac{1}{2}BC \cdot AB = \frac{1}{2}x\cos\theta \cdot x\sin\theta$$

$$= \frac{1}{2}x^2\sin\theta\cos\theta \quad \text{...(ii)}$$

From Eq. (i), $x + x\cos\theta = k \Rightarrow x = \frac{k}{1 + \cos\theta} \quad \text{...(iii)}$

On putting the value of x in Eq. (ii), we get

$$y = \frac{k^2}{2} \cdot \frac{\sin\theta\cos\theta}{(1 + \cos\theta)^2} \quad \text{...(iv)}$$

On differentiating w.r.t. θ, we get

$$\frac{dy}{dx} = \frac{k^2}{2} \frac{(1+\cos\theta)^2 (\cos^2\theta - \sin^2\theta) - \sin\theta\cos\theta \cdot 2(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{(1+\cos\theta)[(1+\cos\theta)(\cos^2\theta - \sin^2\theta) + 2\sin^2\theta\cos\theta]}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{\cos^2\theta - \sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta + 2\cos\theta\sin^2\theta}{(1+\cos\theta)^3}$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta - 1 + \cos^3\theta + \cos\theta\sin^2\theta)$$

$$(\because \sin^2\theta = 1 - \cos^2\theta)$$

$$= \frac{k^2}{2(1+\cos\theta)^3}[2\cos^2\theta - 1 + \cos\theta(\cos^2\theta + \sin^2\theta)]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1) \quad \ldots(v)$$

$$(\because \cos^2\theta + \sin^2\theta = 1)$$

Since, $0 < \theta < \frac{\pi}{2}$

$\therefore$ $\frac{k^2}{2(1+\cos\theta)^3} > 0$

Therefore, sign scheme for $\frac{dy}{dx}$ will be same as that of

$$2\cos^2\theta + \cos\theta - 1.$$

Now, $2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow$ $(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$ $\quad (\because \cos\theta \neq -1)$

$\Rightarrow$ $\theta = \frac{\pi}{3}$ $\quad \left(\because 0 < \theta < \frac{\pi}{2}\right)$

Since, sign scheme for $\frac{dy}{dx}$ *i.e.*, for $(2\cos^2\theta + \cos\theta - 1)$ is

Thus, y has maximum value when $\theta = \frac{\pi}{3}$.

41. Let $f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve $f'(x) = -3x^2 + 6x + 9$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

On differentiating w.r.t. x, we get

$$g'(x) = -6x + 6$$

For maxima or minima put $g'(x) = 0 \Rightarrow x = 1$

Now, $g''(x) = -6 < 0$ and hence, at $x = 1$, $g(x)$ (slope) will have maximum value.

$\therefore$ $[g(1)]_{max} = -3 \times 1 + 6(1) + 9 = 12$

42. Let $PQ = a$ and $PR = b$, then $\Delta = \frac{1}{2}ab\sin\theta$

$\because$ $-1 \le \sin\theta \le 1$

$\therefore$ Area is maximum when $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

43. Given,

$ab = 2a + 3b \Rightarrow (a-3)b = 2a \Rightarrow b = \frac{2a}{a-3}$

Now, let $z = ab = \frac{2a^2}{a-3}$

On differentiating w.r.t. x, we get

$$\frac{dz}{da} = \frac{2[(a-3)2a - a^2]}{(a-3)^2} = \frac{2[a^2 - 6a]}{(a-3)^2}$$

For a minimum, put $\frac{dz}{da} = 0$

$\Rightarrow$ $a^2 - 6a = 0 \Rightarrow a = 0, 6$

At $a = 6$, $\frac{d^2z}{da^2} = \text{positive}$

When $a = 6$, $b = 4$

$\therefore$ $(ab)_{min} = 6 \times 4 = 24$

44. Let radius vector is r.

$\therefore$ $r^2 = x^2 + y^2$

$\Rightarrow$ $r^2 = \frac{a^2y^2}{y^2 - b^2} + y^2$ $\quad \left(\because \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1\right)$

For minimum value of r,

$$\frac{d(r^2)}{dy} = 0 \Rightarrow \frac{-2yb^2a^2}{(y^2 - b^2)^2} + 2y = 0$$

$\Rightarrow$ $y^2 = b(a+b)$

$\therefore$ $x^2 = a(a+b)$

$\Rightarrow$ $r^2 = (a+b)^2$

$\Rightarrow$ $r = a + b$

45. $\because$ Perimeter of a sector $= p$

Let AOB be the sector with radius r.

If angle of the sector be θ radians, then area of sector,

$$A = \frac{1}{2}r^2\theta \quad \ldots(i)$$

and length of arc, $s = r\theta \Rightarrow \theta = \frac{s}{r}$

$\therefore$ Perimeter of the sector

$$p = r + s + r = 2r + s \quad \ldots(ii)$$

On substituting $\theta = \frac{s}{r}$ in Eq. (i), we get

$$A = \left(\frac{1}{2}r^2\right)\left(\frac{s}{r}\right) = \frac{1}{2}rs$$

$\Rightarrow$ $s = \frac{2A}{r}$

Now, on substituting the value of s in Eq. (ii), we get

$$p = 2r + \left(\frac{2A}{r}\right) \Rightarrow 2A = pr - 2r^2$$

On differentiating w.r.t. r, we get

$$2\frac{dA}{dr} = p - 4r$$

For the maximum area, put

$$\frac{dA}{dr} = 0 \Rightarrow p - 4r = 0 \Rightarrow r = \frac{p}{4}$$

46. Given curve is

$$xy = c^2 \Rightarrow y = \frac{c^2}{x}$$

Let $$f(x) = ax + by = ax + \frac{bc^2}{x}$$

On differentiating w.r.t. x, we get

$$f'(x) = a - \frac{bc^2}{x^2}$$

For a maxima or minima, put $f'(x) = 0$

$$\Rightarrow \quad ax^2 - bc^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{bc^2}{a} \Rightarrow x = \pm c\sqrt{\frac{b}{a}}$$

Again, on differentiating w.r.t. x, we get

$$f''(x) = \frac{2bc^2}{x^3}$$

At $x = c\sqrt{\frac{b}{a}}$, $\quad f''(x) > 0$

$\therefore f(x)$ is minimum at $x = c\sqrt{\frac{b}{a}}$.

The minimum value at $x = c\sqrt{\frac{b}{a}}$ is

$$\therefore \quad f\left(c\sqrt{\frac{b}{a}}\right) = a \cdot c\sqrt{\frac{b}{a}} + \frac{bc^2}{c} \cdot \sqrt{\frac{a}{b}}$$

$$= \frac{abc + abc}{\sqrt{ab}} = \frac{2abc}{\sqrt{ab}} = 2c\sqrt{ab}$$

47. Given, $\quad a^2x^4 + b^2y^4 = c^6$

$$\Rightarrow \quad y = \left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$$

and let $$f(x) = xy = x\left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$$

$$\Rightarrow \quad f(x) = \left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$$

For maxima or minima, put $f'(x) = 0$

$$\Rightarrow \quad \frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$$

$$\Rightarrow \quad \frac{4x^3}{b^2}(c^6 - 2a^2x^4) = 0$$

$$\Rightarrow \quad x^4 = \frac{c^6}{2a^2} \Rightarrow \pm\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$$

At $x = \frac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum.

$$\therefore \quad f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4}$$

$$= \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4} = \frac{c^3}{\sqrt{2ab}}$$

48. Let the diameter of the sphere is $AE = 2r$.

Let the radius of cone is x and height is y.

$$\therefore \quad AD = y$$

Since, $\quad BD^2 = AD \cdot DE$

$$\Rightarrow \quad x^2 = y(2r - y) \quad \ldots(i)$$

Volume of cone,

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi y(2r - y)y = \frac{1}{3}\pi(2ry^2 - y^3)$$

On differentiating w.r.t. y, we get

$$\frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2)$$

For maxima and minima, put $\frac{dV}{dy} = 0$

$$\Rightarrow \quad \frac{1}{3}\pi(4ry - 3y^2) = 0$$

$$\Rightarrow \quad y(4r - 3y) = 0$$

$$\Rightarrow \quad y = \frac{4}{3}r, 0$$

Again, on differentiating w.r.t. y, we get

$$\frac{d^2V}{dy^2} = \frac{1}{3}\pi(4r - 6y)$$

At $y = \frac{4}{3}r$, $\quad \frac{d^2V}{dy^2} = \frac{1}{3}\pi(4r - 8r) =$ negative

$\therefore$ Volume of cone is maximum at $y = \frac{4}{3}r$.

Now, required ratio $= \frac{\text{Height of cone}}{\text{Diameter of sphere}} = \frac{y}{2r} = \frac{\frac{4r}{3}}{2r} = \frac{2}{3}$

49. Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height.

$$\therefore \quad h = QL = OL - OQ$$

$$= H - x\cot\alpha$$

V = Volume of the cylinder

$= \pi x^2(H - x\cot\alpha)$

Also, $p = \frac{1}{3}\pi (H\tan\alpha)^2 H$ ...(i)

$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha)$

$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow x = 0,$

$x = \frac{2}{3}H\tan\alpha$

$\Rightarrow \left[\frac{d^2V}{dx^2}\right]_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$

$\therefore V$ is maximum when $x = \frac{2}{3}H\tan\alpha$

and $q = V_{max} = \pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H = \frac{4}{9}p$ [from Eq. (i)]

Hence, $p : q = 9 : 4$

50. $\frac{dy}{dx} = a^2 - 3x^2 = 0 \Leftrightarrow x = \pm\frac{a}{\sqrt{3}}$

Since, $\frac{d^2y}{dx^2} = -6x$, so y is minimum for $x = -\frac{a}{\sqrt{3}}$.

Since, $x^2 + x + 2 > 0$ for all x, so for $\frac{x^2+x+2}{x^2+5x+6} \le 0$,

we must have $x^2 + 5x + 6 < 0$. If $x = -\frac{a}{\sqrt{3}}$, we have

$\frac{a^2}{3} - \frac{5a}{\sqrt{3}} + 6 < 0$

$\Rightarrow \quad a^2 - 5\sqrt{3}\,a + 18 < 0$

$\Rightarrow \quad (a - 2\sqrt{3})(a - 3\sqrt{3}) < 0$

$\Rightarrow \quad a \in (2\sqrt{3}, 3\sqrt{3})$

51. $\because \quad y = x^3 - 6x^2 + 9x + 4$

Now, $\frac{dy}{dx} = 3x^2 - 12x + 9$

Let $u = \frac{dy}{dx} = 3x^2 - 12x + 9$

Now, $\frac{du}{dx} = 6x - 12$

Put $\frac{du}{dx} = 0$ for maximum or minimum

$\therefore \quad 6x - 12 = 0$

$\Rightarrow \quad x = 2$

Now, at $x = 0$, $u = 9$

At $x = 2$, $u = -3$

and at $x = 5$, $u = 24$

Thus, the maximum of $u(x)$, $0 \le x \le 5$ is $u(5)$.

Hence, $x = 5$

Round II

1. Bold line represents the graph of $y = g(x)$, clearly $g(x)$ has neither a point of local maxima nor a point of local minima.

2. For $y = \frac{x^2-2}{x^2-4} \Rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2-4)^2}$

$\Rightarrow \quad \frac{dy}{dx} > 0$ for $x < 0$

and $\frac{dy}{dx} < 0$ for $x > 0$

Thus, $x = 0$ is the point of local maxima for y. Now, $(y)_{x=0} = \frac{1}{2}$ (positive). Thus, $x = 0$ is also the point of local maximum for $y = \left|\frac{x^2-2}{x^2-4}\right|$.

3. $f'(x) = (x^2 - 1)\cos x$

Sign scheme of $f'(x)$, clearly $x = \frac{\pi}{2}$ is the point of local maxima.

4. Clearly, $f(x)$ in increasing just before $x = 3$ and decreasing after $x = 3$. For $x = 3$ to be the point of local maxima.

$f(3) \ge f(3 - 0)$

$\Rightarrow -15 \ge 12 - 27 + \log(a^2 - 3a + 3)$

$\Rightarrow \quad 0 < a^2 - 3a + 3 \le 1$

$\Rightarrow \quad 1 \le a \le 2$

5. Clearly, $f(2) = -1$

$\Rightarrow \quad -1 = \frac{2a + b}{(2-1)(2-4)}$

$\Rightarrow \quad 2a + b = 2$

Now, $f'(x) = \frac{4a + 5b - 2bx - ax^2}{(x-1)^2(x-4)^2}$, $f'(2) = 0$

$\Rightarrow \quad b = 0$

$\Rightarrow \qquad a = 1$

$\Rightarrow \qquad f'(x) = -\dfrac{(x-2)(x+2)}{(x-1)^2(x-4)^2}$

Clearly, for $x > 2$, $f'(x) < 0$ and for $x < 2$, $f'(x) > 0$. Thus, $x = 2$ is indeed the point of local maxima for $y = f(x)$.

6. $f'(x) = \dfrac{1 - x^2 \sec^2 x}{(1 + x\tan x)^2}$

$= \dfrac{\sec^2 x(\cos x + x)(\cos x - x)}{(1 + x\tan x)^2}$

Clearly, $f'(x_0) = 0$

and $f'(x) > 0, \forall\, x \in (0, x_0)$,

$f'(x) < 0, \forall\, x \in \left(x_0, \dfrac{\pi}{2}\right)$

Thus, $x = x_0$ is the only point of local maxima for $y = f(x)$.

7. Let the equation of drawn line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where $a > 3$, $b > 4$, as the line passes through $(3, 4)$ and meets the positive direction of coordinate axes.

We have, $\dfrac{3}{a} + \dfrac{4}{b} = 1 \Rightarrow b = \dfrac{4a}{(a-3)}$

Now, area of ΔAOB,

$$\Delta = \frac{1}{2}ab = \frac{2a^2}{(a-3)}$$

$$\frac{d\Delta}{da} = \frac{2a(a-6)}{(a-3)^2}$$

Clearly, $a = 6$ is the point of minima for Δ. Thus,

$$\Delta_{\min} = \frac{2 \times 36}{3} = 24 \text{ sq units}$$

8. $AF \,||\, DE$ and $AE \,||\, FD$

Now, in ΔABC and ΔEDC,

$\angle DEC = \angle BAC$, $\angle ACB$ is common.

$\Rightarrow \qquad \Delta ABC \cong \Delta EDC$

Now, $\dfrac{b-y}{b} = \dfrac{x}{c}$

$\Rightarrow \qquad x = \dfrac{c}{b}(b - y)$

Now, $\qquad S$ = Area of parallelogram

$AFDE = 2$ (Area of ΔAEF)

$\Rightarrow \qquad S = 2\left(\dfrac{1}{2} \times y \sin A\right) = \dfrac{c}{b}(b - y)\sin A$

$$\frac{dS}{dy} = \left(\frac{c}{b}\sin A\right)(b - 2y)$$

Sign scheme of $\dfrac{dx}{dy}$,

Hence, S is maximum when $y = \dfrac{b}{2}$.

$\therefore \qquad S_{\max} = \dfrac{c}{b}\left(\dfrac{b}{2}\right) \times \dfrac{b}{2}\sin A$

$= \dfrac{1}{2}\left(\dfrac{1}{2}bc\sin A\right)$

$= \dfrac{1}{2}$ (Area of ΔABC)

9. We have, $\qquad \dfrac{dy}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4)$

and $\qquad \dfrac{dx}{dt} = 6t - 18 = 6(t-3)$

Thus, $\qquad \dfrac{dy}{dx} = \dfrac{(t-1)(t-4)}{(t-3)}$

which indicates that $t = 1, 3$ and 4 are the critical points of $y = f(x)$.

Now, $\qquad \dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx}$

$$= \frac{t^2 - 6t + 11}{(t-3)^2} \times \frac{1}{6(t-3)}$$

At $(t = 1)$, $\dfrac{d^2y}{dx^2} < 0 \Rightarrow t = 1$ is a point of local maxima.

At $(t = 4)$, $\dfrac{d^2y}{dx^2} > 0 \Rightarrow t = 4$ is a point of local minima.

At $(t = 3)$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are not defined and change its sign.

$\dfrac{d^2y}{dx^2}$ is unknown in the vicinity of $t = 3$, thus $t = 3$ is a point of neither maxima nor minima.

Finally, maximum and minimum values of expression $y = f(x)$ are 46 and –6, respectively.

10. $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \dfrac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$

From the above graph,

Maximum occurs at $x = 0$ and minimum at $x = 4$.

11. Let α and β be the roots of the equation

$$x^2 - (a-2)\,x - a + 1 = 0.$$

Then, $\alpha + \beta = a - 2,\ \alpha\beta = -a + 1$

Let $z = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= (a-2)^2 + 2(a-1) = a^2 - 2a + 2$$

$\Rightarrow \quad \dfrac{dz}{da} = 2a - 2 \quad \Rightarrow a = 1$

Put $\dfrac{dz}{da} = 0$, then

$\therefore \quad \dfrac{d^2z}{da^2} = 2 > 0$

So, z has minima at $a = 1$.

So, $\alpha^2 + \beta^2$ has least value for $a = 1$. This is because, we have only one stationary value at which we have minima. Hence, $a = 1$.

12. Let $f(x) = \int_0^x te^{-t^2}dt$

$\Rightarrow \quad f'(x) = xe^{-x^2} = 0$

Put $f'(x) = 0 \Rightarrow x = 0$

Now, $f''(x) = e^{-x^2}(1 - 2x^2)$

$f''(0) = 1 > 0$

$\therefore$ Minimum value $f(0) = 0$

13. $\dfrac{dy}{dx} = \dfrac{a}{x} + 2bx + 1$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow \quad a = -2b - 1$

and $\quad \left(\dfrac{dy}{dx}\right)_{x=2} = \dfrac{a}{2} + 4b + 1 = 0$

$\Rightarrow \quad \dfrac{-2b-1}{2} + 4b + 1 = 0$

$\Rightarrow \quad -b + 4b + \dfrac{1}{2} = 0$

$\Rightarrow \quad 3b = \dfrac{-1}{2}$

$\Rightarrow \quad b = \dfrac{-1}{6}$ and $a = \dfrac{1}{3} - 1 = \dfrac{-2}{3}$

14. Let the coordinate of $R(x, 0)$.

Now, $PR + RQ = \sqrt{(x-1)^2 + (0-1)^2} + \sqrt{(x-3)^2 + (0-2)^2}$

$$= \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 6x + 13}$$

For minimum value of $PR + RQ$,

$$\frac{d}{dx}(PR + RQ) = 0$$

$\Rightarrow \quad \dfrac{d}{dx}\sqrt{(x^2 - 2x + 2)} + \dfrac{d}{dx}\sqrt{(x^2 - 6x + 13)} = 0$

$\Rightarrow \quad \dfrac{(x-1)}{\sqrt{x^2 - 2x + 2}} = -\dfrac{(x-3)}{\sqrt{x^2 - 6x + 13}}$

On squaring both sides, we get

$$\frac{(x-1)^2}{(x^2 - 2x + 2)} = \frac{(x-3)^2}{x^2 - 6x + 13}$$

$\Rightarrow \quad 3x^2 - 2x - 5 = 0$

$\Rightarrow \quad (3x - 5)(x + 1) = 0$

$\Rightarrow \quad x = \dfrac{5}{3}, -1$

Also, $1 < x < 3$

$\therefore \quad R = (5/3, 0)$

15. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$$f'(x) = 6x^2 - 18ax + 12a^2$$

$$f''(x) = 12x - 18a$$

For maximum and minimum,

$$6x^2 - 18ax + 12a^2 = 0$$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$ at $x = a$ maximum and at $x = 2a$ minimum

$\because \quad p^2 = q$

$\therefore \quad a^2 = 2a$

$\Rightarrow \quad a = 2$ or $a = 0$

But $a > 0$, therefore $a = 2$

16. Given, $f(x) = 1 + 2x^2 + 2^2x^4 + 2^3x^6 + \ldots + 2^{10}x^{20}$

$$f'(x) = x(4 + 4\cdot 2^2x^2 + \ldots + 20\cdot 2^{10}x^{18})$$

Put $f(x) = 0 \Rightarrow x = 0$ only

Also, $f''(0) > 0$

17. (Slope) $\quad f'(x) = e^x \cos x + \sin x e^x$

$$= e^x\sqrt{2}\sin(x + \pi/4)$$

$$f''(x) = \sqrt{2}e^x\{\sin(x + \pi/4) + \cos(x + \pi/4)\}$$

$$= 2e^x \cdot \sin(x + \pi/2)$$

For maximum slope, put $f''(x) = 0$

$\Rightarrow \quad \sin(x + \pi/2) = 0$

$\Rightarrow \quad \cos x = 0$

$\therefore \quad x = \pi/2, 3\pi/2$

$$f'''(x) = 2e^x \cos(x + \pi/2)$$

$$f'''(\pi/2) = 2e^x \cdot \cos\pi = -\text{ve}$$

Maximum slope is at $x = \pi/2$.

18. $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\dfrac{1}{x}\right)\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $\quad g(x) = x^3 + x^2 + 3x + \sin x$

$$g'(x) = 3x^2 + 2x + 3 + \cos x$$

$$= 3\left(x^2 + \frac{2x}{3} + 1\right) + \cos x$$

$$= 3\left\{\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right\}+\cos x>0$$

and $$2<3+\sin\left(\frac{1}{x}\right)<4$$

Hence, minimum value of $f(x)$ is 0 at $x=0$.

Hence, number of points $=1$

19. For $x>0$ or $x<0$

$$f'(x)=\frac{a}{x}+2bx+1$$

$\because$ $$f'(1)=0\Rightarrow a+2b+1=0 \quad \text{...(i)}$$

and $$f'(3)=0\Rightarrow \frac{a}{3}+6b+1=0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$a=-3/4,\ b=-1/8$$

20. $$f'(x)=\begin{cases}3(2+x)^2, & -3<x\le -1\\ \frac{2}{3}x^{-1/3} & -1<x<2\end{cases}$$

Clearly, $f'(x)$ changes its sign at $x=-1$ from positive to negative and so $f(x)$ has local maxima at $x=-1$.

Also, $f'(0)$ does not exist but $f'(0^-)<0$ and $f'(0^+)<0$. It can only be inferred that $f(x)$ has a possibility of a minimum at $x=0$.

Hence, it has one local maxima at $x=-1$ and one local minima at $x=0$.

21. Minimum of $f(x)=-\dfrac{D}{4a}=\dfrac{-(4b^2-8c^2)}{4}=2c^2-b^2$

and maximum of $g(x)=-\dfrac{(4c^2+4b^2)}{4(-1)}=b^2+c^2$

Since, $\min f(x)>\max g(x)$

$\Rightarrow$ $2c^2-b^2>b^2+c^2$

$\Rightarrow$ $c^2>2b^2$

$\Rightarrow$ $|c|>\sqrt{2}\,|b|$

22. Let S be the curved surface area of a cone.

$\therefore$ $$S=\pi rl$$
$$=\pi(\sqrt{2Rh-h^2})(\sqrt{h^2+r^2})$$
$$=(\pi\sqrt{2Rh-h^2})(\sqrt{2Rh})$$

Let $S^2=P$

$\therefore$ $P=\pi^2\,2R(2Rh^2-h^3)$

$\because S$ is maximum, if P is maximum, then

$$\frac{dP}{dh}=2\pi^2R(4Rh-3h^2)=0$$

$\therefore$ $$h=0,\frac{4R}{3}$$

Again, differentiating $\dfrac{dP}{dh}$, we get

$$\frac{d^2P}{dh^2}=2\pi^2R(4R-6h)$$
$$\frac{d^2P}{dh^2}<0 \text{ at } h=\frac{4R}{3}$$

23. Any tangent to the ellipse is $\dfrac{x}{4}\cos t+\dfrac{y}{3}\sin t=1$, where the point of contact is $(4\cos t, 3\sin t)$

or $$\frac{x}{4\sec t}+\frac{y}{3\operatorname{cosec} t}=1$$

It means the axes $Q(4\sec t, 0)$ and $R(0, 3\operatorname{cosec} t)$.

$\therefore$ The distance of the line segment QR is

$$QR^2=D=16\sec^2 t+9\operatorname{cosec}^2 t$$

So, the minimum value of D is $(4+3)^2$ or $QR=7$.

24. $f(x)=\left(\dfrac{1}{x}\right)^x\Rightarrow f'(x)=\left(\dfrac{1}{x}\right)^x\left(\log\dfrac{1}{x}-1\right)$

$$f'(x)=0\Rightarrow\log\frac{1}{x}=1=\log_e\Rightarrow\frac{1}{x}=e\Rightarrow x=\frac{1}{e}$$

Also, for $x<1/e$, $f'(x)$ is positive and for $x>1/e$, $f'(x)$ is negative.

Hence, $x=1/e$ is point of maxima.

Therefore, maximum value of function is $e^{1/e}$.

25. Consider the function $f(x)=\dfrac{x^2}{(x^3+200)}$

$$f'(x)=x\frac{(400-x^3)}{(x^3+200)^2}=0$$

when $x=(400)^{1/3}$, $\quad(\because x\neq 0)$

$x=(400)^{1/3}-h\Rightarrow f'(x)>0$

$x=(400)^{1/3}+h\Rightarrow f'(x)<0$

$\therefore f(x)$ has maxima at $x=(400)^{1/3}$

Since, $7<(400)^{1/3}<8$, either a_7 or a_8 is the greatest term of the sequence.

$\because$ $a_7=\dfrac{49}{543}$ and $a_8=\dfrac{8}{89}$ and $\dfrac{49}{543}>\dfrac{8}{89}$

$\Rightarrow$ $a_7=\dfrac{49}{543}$ is the greatest term.

26. Given, $A+B=60^\circ\Rightarrow B=60^\circ-A$

$\Rightarrow$ $$\tan B=\tan(60^\circ-A)=\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$$

Now, $z=\tan A\tan B$

or $$z=\frac{t(\sqrt{3}-t)}{1+\sqrt{3}t}=\frac{\sqrt{3}t-t^2}{1+\sqrt{3}t}$$

where, $t=\tan A$

$$\frac{dz}{dt}=-\frac{(t+\sqrt{3})(\sqrt{3}t-1)}{(1+\sqrt{3}t)^2}=0$$

$\Rightarrow \quad t = 1/\sqrt{3}$

$\Rightarrow \quad t = \tan A = \tan 30°$

The other value is rejected as both A and B are positive acute angles.

If $t < \frac{1}{\sqrt{3}}, \frac{dz}{dt}$ = positive and if $t > \frac{1}{\sqrt{3}}, \frac{dz}{dt}$ = negative

Hence, max when $t = \frac{1}{\sqrt{3}}$ and max value $= \frac{1}{3}$.

27. $f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$

$$f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2)$$

For real root $D \geq 0$,

$\Rightarrow \quad 49 + a^2 - 14a + 9 - a^2 \geq 0$

$\Rightarrow \quad a \leq \frac{58}{14}$

For local minimum $f''(x) = 6x - 6(7-a) > 0 \Rightarrow 7 - x$ has x must be negative

$\Rightarrow \quad 7 - a < 0 \Rightarrow a > 7$

Thus constradictory, *i.e.*, for real roots $a \leq \frac{58}{14}$ and for negative point of local minimum $a > 7$.

No possible values of a.

28. Clearly $f(x)$ is increasing before $x = 0$ and starts decreasing after $x = 0$, $f(0) = a$.

For $x = 0$ to be the point of local maxima

$$f(0) \geq \lim_{x \to 0^+} f(x)$$

$\Rightarrow \quad a \geq 1$

29. Let the line in intercepts form be $\frac{x}{a} + \frac{y}{b} = 1$.

It passes through $(h,k) \Rightarrow \frac{h}{a} + \frac{k}{b} = 1$

$\Rightarrow \quad \frac{k}{b} = 1 - \frac{h}{a} = \frac{a-h}{a}$

$\Rightarrow \quad b = \frac{ak}{a-h}$

$$\Delta = \frac{1}{2}ab = \frac{1}{2}a \cdot \frac{ak}{a-h} = \frac{1}{2}\frac{h}{\frac{a-h}{a^2}} \quad \text{...(i)}$$

Δ is min when $y = \frac{a-h}{a^2} = \frac{1}{a} - \frac{h}{a^2}$ is max

$\Rightarrow \quad \frac{dy}{da} = -\frac{1}{a^2} + \frac{2h}{a^3} = 0 \Rightarrow a = 2h \quad$ ...(ii)

$$\frac{d^2y}{da^2} = \frac{2}{a^3} - \frac{6h}{a^4} = \frac{2}{a^3} - \frac{3}{a^3} \quad \text{[by Eq. (ii)]}$$

$\Rightarrow \quad \frac{d^2y}{da^2} = -\frac{1}{a^3}$ = negative

$\therefore$ Maximum

Now, put $a = 2h$ in Eq. (i), $\Delta = \frac{1}{2} \cdot 4h^2 \cdot \frac{k}{h} = 2hk$

30. Let (x,y) be one the parabola $y = x^2 + 7x + 2$

Its distance from the line $y = 3x - 3$ or $3x - y - 3 = 0$ is

$$D = \left|\frac{3x - y - 3}{\sqrt{10}}\right| = \left|\frac{3x - (x^2 + 7x + 2) - 3}{\sqrt{10}}\right|$$

$$= \left|\frac{-x^2 - 4x - 5}{\sqrt{10}}\right|$$

$$D = \left|\frac{x^2 + 4x + 5}{\sqrt{10}}\right| = \frac{(x+2)^2 + 1}{\sqrt{10}} = \frac{(x+2)^2 + 1}{\sqrt{10}}$$

as $\frac{\text{Numerator}}{\text{Denominator}}$ is positive

$\frac{dD}{dx} = \frac{2(x+2)}{\sqrt{10}} = 0 \Rightarrow x = -2$ and hence $y = -8$,

i.e., Point is $(-2, -8)$

$\frac{d^2D}{dx^2} = \frac{2}{\sqrt{10}} =$ Positive and hence min at $(-2, -8)$.

31. $(1,0)$ is on both the curves. So, $0 = 1 + a + b$ and $0 = c - 1$

For the first parabola, $\frac{dy}{dx} = 2x + a$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,0)} = 2 + a$

For the second parabola, $\frac{dy}{dx} = c - 2x$

$\therefore \quad \left[\frac{dy}{dx}\right]_{(1,0)} = c - 2$

$\therefore \quad 2 + a = c - 2$ and

$0 = c - 1 \Rightarrow c = 1, a = -3$

$\therefore \quad 0 = 1 + (-3) + b$ or $b = 2$

32. $f'(x) = \begin{cases} 3x^2 + 2x - 10 & , -1 < x < 0 \\ \cos x & , 0 < x < \frac{\pi}{2} \\ -\sin x & , \frac{\pi}{2} < x < \pi \end{cases}$

$f(x)$ is continuous everywhere decide sign of $f'(x)$ about 0 and $\frac{\pi}{2}$.

To find absolute maxima or minima compare

$$f(-1), f(0), f\left(\frac{\pi}{2}\right), f(\pi)$$

33. $\because \quad f'(x) = (x^2 - 1)^n\{(x^2 - 1)(2x + 1) + 2nx(x^2 + x + 1)\}$

$$= (x+1)^{n+1}(x-1)^{n+1}(2x+1) + 2nx(x+1)^n$$

$$(x-1)^n(x^2 + x + 1)$$

If power even, then neither max nor min.

$\therefore \quad n = 2, 4, 6 \quad (\because n + 1 = \text{odd})$

34. Let y be the length and x be the breadth of the rectangular portion. Total perimeter of the window is

$$2x + 2y + \left(\frac{1}{2}\right)\pi y = P \qquad \text{(say)}$$

Let amount of light per square metre for the coloured glass be μ. If L is the total light transmitted, then

$L = 3\mu \times$ Area of rectangular portion

$+ \mu \times$ Area of semi-circular portion

$$= 3\mu xy + \frac{1}{8}\mu\,\pi y^2$$

$$= \mu\left[\frac{3}{2}y\left\{P - \left(2 + \frac{\pi}{2}\right)y\right\} + \frac{1}{8}\pi y^2\right]$$

$$\Rightarrow \quad \frac{dL}{dy} = \frac{\mu}{2}\left[3P - 6\left(2 + \frac{\pi}{2}\right)y + \frac{\pi}{2}y\right]$$

Put $\quad \dfrac{dL}{dy} = 0$

$$\Rightarrow \quad y = \frac{3P}{12 + \frac{5\pi}{2}}$$

$$\frac{d^2L}{dy^2} = \frac{\mu}{2}\left(-12 - \frac{5\pi}{2}\right) < 0$$

$$\therefore \quad \frac{y}{x} = \frac{2 \cdot \frac{3P}{12 + 5\pi/2}}{P - \left(2 + \frac{\pi}{2}\right) \cdot \frac{3P}{12 + \frac{5\pi}{2}}}$$

$$= \frac{6}{(12 + 5\pi/2) - 3(2 + \pi/2)} = \frac{6}{6 + \pi}$$

35. $\left(\dfrac{dL}{dy}\right)_{y=1} = \dfrac{\mu}{2}\left(3P - 12 - \dfrac{5\pi}{2}\right)$

36. It is clear from figure, $f'(x)$ has no sign change at $x = 2$.

Hence, $f(x)$ is neither maximum nor minimum at $x = 2$.

37. If y have no critical points for all $x \in R$, then

$$\frac{dy}{dx} > 0$$

or $\quad \dfrac{dy}{dx} < 0, \forall\, x \in R$

But $\quad \dfrac{dy}{dx} = 3x^2 + 2ax + b > 0$

$$\Rightarrow \quad 3x^2 + 2ax + b > 0$$

$$\Rightarrow \quad D < 0$$

$$\Rightarrow \quad 4a^2 - 4 \cdot 3 \cdot b < 0$$

$$\Rightarrow \quad a^2 < 3b$$

38. $\because \quad f(x) = -x^2 + 4x + 1 + \sin^{-1}\left(\dfrac{x}{2}\right)$

$$\Rightarrow \quad f'(x) = -2x + 4 + \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$$

$$= 2(2 - x) + \frac{1}{\sqrt{4 - x^2}} > 0, \forall\, x \in [-1, 1]$$

Therefore, $f(x)$ increasing in interval $[-1, 1]$.

$\therefore$ Minimum value of $f(x)$

$$f(-1) = -4 - \frac{\pi}{6}$$

39. Let the point on the parabola be (t, t^2). Let d be the distance between (t, t^2) and $(0, y_0)$, then

$$d^2 = t^2 + (t^2 - y_0)^2 = t^4 + (1 - 2y_0)t^2 + y_0^2$$

$$= z^2 + (1 - 2y_0)z + y_0^2,\ z \geq 0$$

Its vertex is at $x = y_0 - \dfrac{1}{2} < 0$.

$\therefore$ The minimum value of d^2 is at $z = 0$ *i.e.*, $t^2 = 0$

$$\therefore \quad d = y_0$$

Hence, statement I is true. Statement II is false because extremum can occur at a point, where $f'(x)$ does not exist.

40. Minimum value of $ax^2 + bx + c$ is $-\dfrac{b^2 - 4ac}{4a}$ at $x = -b/2a$.

41. $f'(x) = \cos x - x\sin x$, equal to zero when $\tan x = x$. By sketching the graph we can easily interprete the assertion.

42. $f(x) = \log|x| + bx^2 + ax$

$$\Rightarrow \quad f'(x) = \frac{1}{x} + 2bx + a$$

$$\Rightarrow \quad f''(x) = \frac{-1}{x^2} + 2b$$

Since, f has extrema at $x = -1$ and $x = 2$.

Hence,

$$f'(-1) = 0 = f'(2)$$

$$f'(-1) = 0 \Rightarrow a - 2b = 1 \qquad \text{...(i)}$$

and $\quad f'(2) = 0 \Rightarrow a + 4b = \dfrac{-1}{2} \qquad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a = \frac{1}{2} \text{ and } b = \frac{-1}{4}$$

$$\Rightarrow \quad f''(x) = \frac{-1}{x^2} + \frac{-1}{2} = -\left(\frac{x^2 + 2}{2x^2}\right)$$

$$\Rightarrow \quad f''(-1) < 0$$

and $\quad f''(2) < 0$

Thus, f has local maxima at both $x = -1$ and $x = 2$.

Hence, obviously statement I is correct.

Also, while solving for statement I, we found the values of a and b which justify that statement II is also correct.

However, statement II does not explain statement I in any way.

43. $f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

As $\dfrac{\tan x}{x} > 1, \forall\, x \neq 0$

$\therefore$ $f(0 + h) > f(0)$ and $f(0 - h) > f(0)$

At $x = 0$, $f(x)$ attains minima.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{\tan h}{h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\tan h - h}{h^2} \quad \text{(using L' Hospital's rule)}$$

$$= \lim_{h \to 0} \frac{\sec^2 h - 1}{2h} \quad (\because \tan^2\theta = \sec^2\theta - 1)$$

$$= \lim_{h \to 0} \frac{\tan^2 h}{2h^2} \cdot h = \frac{1}{2} \cdot 0 = 0$$

$\therefore$ Therefore, statement II is true.

Hence, both statements are true but statement II is not the correct explanation of statement I.

44. Let m be the slope of the line PQ, then the equation of PQ is

$$y - 2 = m(x - 1)$$

Now, PQ meets x-axis at $P\left(1 - \dfrac{2}{m}, 0\right)$ and y-axis at $Q(0, 2 - m)$.

$\therefore$ $OP = 1 - \dfrac{2}{m}$

and $OQ = 2 - m$

Also, area of $\Delta OPQ = \dfrac{1}{2}(OP)(OQ) = \dfrac{1}{2}\left|\left(1 - \dfrac{2}{m}\right)(2 - m)\right|$

$$= \frac{1}{2}\left|2 - m - \frac{4}{m} + 2\right|$$

$$= \frac{1}{2}\left|4 - \left(m + \frac{4}{m}\right)\right|$$

Let $f(m) = 4 - \left(m + \dfrac{4}{m}\right)$

$\Rightarrow$ $f'(m) = -1 + \dfrac{4}{m^2}$

Now, $f'(m) = 0 \Rightarrow m^2 = 4$

$\Rightarrow$ $m = \pm 2 \Rightarrow f(2) = 0$

and $f(-2) = 8$

Since, the area cannot be zero, hence the required value of m is -2.

45. Here, $f(x) = \int_0^x \sqrt{t}\, \sin t\, dt$, where $x \in \left(0, \dfrac{5}{2}\right)$

$$f'(x) = \{\sqrt{x}\, \sin x - 0\} \quad \text{...(i)}$$

(using Newton-Leibnitz formula)

$\therefore$ $f'(x) = \sqrt{x}\, \sin x = 0$

$\Rightarrow$ $\sin x = 0 \;\therefore\; x = \pi, 2\pi$

$$f''(x) = \sqrt{x}\, \cos x + \frac{1}{2\sqrt{x}} \sin x$$

$$f''(x) = -\sqrt{\pi} < 0$$

$\therefore$ Local maximum at $x = \pi$

$$f''(2\pi) = \sqrt{x} > 0$$

Hence, local minimum at x is 2π.

46. To find the shortest distance between $y - x = 1$ and $x = y^2$ is along the common normal.

$\therefore$ Tangent at P is parallel to

$$y = x + 1 \quad \text{...(i)}$$

$\therefore$ Slope of tangent at $P(t^2, t)$.

$\Rightarrow$ $\dfrac{dy}{dx} = \left(\dfrac{1}{2y}\right)_{(t^2, t)} = \dfrac{1}{2t}$ ...(ii)

$\Rightarrow$ $\dfrac{1}{2t} = 1$ [since, Eqs. (i) and (ii) are parallel]

$\Rightarrow$ $t = \dfrac{1}{2} \therefore P\left(\dfrac{1}{4}, \dfrac{1}{2}\right)$

Shortest distance $= |PQ| = \dfrac{\left|\dfrac{1}{4} - \dfrac{1}{2} + 1\right|}{\sqrt{1 + 1}} = \dfrac{3}{4\sqrt{2}}$

$\therefore$ Shortest distance $= \dfrac{3\sqrt{2}}{8}$

47.

$\because$ $k + 2 \leq 1$

$\therefore$ $k \leq -1$

48. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow \quad P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$\because$ $x = 0$ is a solution for $P'(x) = 0$, $\Rightarrow c = 0$

$\therefore \quad P(x) = x^4 + ax^3 + bx^2 + d \quad \ldots(i)$

Also, we have $\quad P(-1) < P(1)$

$\Rightarrow \quad 1 - a + b + d < 1 + a + b + d$

$\Rightarrow \quad a > 0$

Since, $P'(x) = 0$ only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1$, 0 and 1 only.

Also, we have $\quad P(-1) < P(1)$

$\therefore$ Maximum of $P(x) = \max\{P(0), P(1)\}$

and minimum of $P(x) = \min\{P(-1), P(0)\}$

In the interval $[0, 1]$,

$$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$$

$\because$ $P'(x)$ has only one root $x = 0$, then

$4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore \quad (3a)^2 - 32b < 0 \Rightarrow \dfrac{3a^2}{32} < b$

$\therefore \quad b > 0$

Thus, we have $a > 0$ and $b > 0$.

$\therefore \quad P'(x) = 4x^3 + 4ax^2 + 2bx > 0, \forall\, x \in (0, 1)$

Hence, $P(x)$ is increasing in $[0, 1]$.

$\therefore$ Maximum of $P(x) = P(1)$

Similarly, $P(x)$ is decreasing in $[-1, 0]$.

Therefore, minimum $P(x)$ does not occur at $x = -1$.

49. Let $\quad f(x) = x^3 - px + q$

Then, $\quad f'(x) = 3x^2 - p$

Put $\quad f'(x) = 0 \Rightarrow x = \sqrt{\dfrac{p}{3}}, -\sqrt{\dfrac{p}{3}}$

Now, $\quad f''(x) = 6x$

$\therefore$ At $\quad x = \sqrt{\dfrac{p}{3}}, f''(x) = 6\sqrt{\dfrac{p}{3}} > 0 \quad$ (minima)

and at $\quad x = -\sqrt{\dfrac{p}{3}}, f''(x) < 0, \quad$ (maxima)

50. Let $\quad f(x) = 1 + \dfrac{10}{3\left(x^2 + 3x + \dfrac{7}{3}\right)}$

$$= 1 + \frac{10}{3\left[\left(x + \dfrac{3}{2}\right)^2 + \dfrac{1}{12}\right]}$$

So, the maximum value of $f(x)$ at $x = -\dfrac{3}{2}$ is

$$f\left(-\frac{3}{2}\right) = 1 + \frac{10}{3\left(\dfrac{1}{12}\right)} = 1 + 40 = 41$$

30 Indefinite Integration

JEE Main MILESTONE

- Integral as an Anti-Derivative
- Fundamental Integration Formulae
- Rules of Integration
- Integration of Different Types of Functions
- Substitution for Some Irrational Functions
- Integration by Partial Fractions
- Integration using Trigonometric Identities
- Some Special Integrals
- Reduction Formulae

30.1 Integral as an Anti-Derivative

Integration is called the inverse process of differentiation. This helps to find the function whose differential coefficient is known.

Thus, a function $f(x)$ is called a **primitive** or an **anti-derivative** of a function $f(x)$, if

$$F'(x) = f(x)$$

Since, the differential coefficient of a constant is zero.

$$\therefore \quad \frac{d}{dx}[F(x) + c] = \frac{d}{dx}F(x) + 0 = f(x)$$

Hence, $$\int f(x)\,dx = F(x) + C$$

where, $\int$ is the notation of integration, $f(x)$ is called the **integrand** and x is called the variable of integration. c is the constant of integration and can take any constant value. This shows that $F(x)$ and $F(x) + c$ are both integrals of the same function $f(x)$. Thus, for different value of c, we obtain different integral of $f(x)$. This implies that the integral of $f(x)$ is not definite. Due to this $F(x)$ is called the indefinite integral of $f(x)$.

*In calculus, an **anti-derivative, primitive integral** or **indefinite integral** of a function f is a differentiable function F whose derivative is equal to f, i.e., $F' = f$. The process of solving for anti-derivatives is **anti-differentiation (or indefinite integration)** and its opposite operation is differentiation, which is the process of finding a derivative. Anti-derivatives are related to definite integrals through the fundamental theorem of calculus .*

30.2 Fundamental Integration Formulae

(i) Algebraic Formulae

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(b) $\int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C, \; n \neq -1$

(c) $\int \frac{1}{x}\,dx = \log|x| + C$

(d) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}(\log|ax+b|) + C$

(e) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$, when $x < a$

(f) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$, when $x > a$

(g) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

(h) $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\frac{x}{a} + C$

(i) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(j) $\int \frac{-1}{a^2+x^2}\,dx = \frac{1}{a}\cot^{-1}\frac{x}{a} + C$

(k) $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(l) $\int \frac{-1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

(m) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x + \sqrt{x^2-a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(n) $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log|x + \sqrt{x^2+a^2}| + C$

$= \sinh^{-1}\left(\frac{x}{a}\right) + C$

(o) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

(p) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2}$

$- \frac{1}{2}a^2\log|x + \sqrt{x^2-a^2}| + C$

(q) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\log|x$

$+ \sqrt{x^2+a^2}| + C$

***Sample* Problem 1** $\int \frac{dx}{\sqrt{1+4x^2}}$ *is equal to* [NCERT]

(a) $\frac{1}{2}\log|x + \sqrt{2x^2+1}| + C$

(b) $\frac{1}{2}\log|2x + \sqrt{4x^2+1}| + C$

(c) $\log|2x + \sqrt{4x^2+1}| + C$

(d) *None of the above*

Interpret (b) $\int \frac{dx}{\sqrt{1+4x^2}} = \int \frac{dx}{\sqrt{4\left\{\left(\frac{1}{2}\right)^2 + x^2\right\}}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2+\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\left|\log\left|x + \sqrt{x^2+\left(\frac{1}{2}\right)^2}\right|\right| + c$

$\left(\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx + \log|x + \sqrt{x^2+a^2}|\right)$

$= \frac{1}{2}\log\left|x + \frac{\sqrt{4x^2+1}}{2}\right| + C$

$= \frac{1}{2}\log|2x + \sqrt{4x^2+1}| - \frac{1}{2}\log 2 + C$

$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$

$= \frac{1}{2}\log|2x + \sqrt{4x^2+1}| + C$

($\because \frac{1}{2}\log 2$ is constant and constant + constant = C)

***Sample* Problem 2** $\int \frac{dx}{\sqrt{9-25x^2}}$ *is equal to* [NCERT]

(a) $\sin^{-1}\left(\frac{5x}{3}\right) + C$ (b) $\frac{3}{2}\sin^{-1}\left(\frac{5x}{3}\right) + C$

(c) $\frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$ (d) *None of these*

Interpret (c) $\int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{25\left[\left(\frac{3}{5}\right)^2 - x^2\right]}}$

$= \frac{1}{5}\int \frac{dx}{\sqrt{\left[\left(\frac{3}{5}\right)^2 - x^2\right]}}$

$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C$ $\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right]$

$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

(ii) Trigonometric Formulae

(a) $\int \sin x\,dx = -\cos x + C$

(b) $\int \cos x = \sin x + C$

(c) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(d) $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(e) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(f) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(g) $\int \sec^2 x\,dx = \tan x + C$

(h) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(i) $\int \sec x \cdot \tan x\, dx = \sec x + C$

(j) $\int \text{cosec}\, x \cdot \cot x\, dx = -\text{cosec}\, x + C$

***Sample* Problem 3** $\int \frac{dx}{\sin^2 x \cdot \cos^2 x}$ *is equal to* **[NCERT]**

(*a*) $\tan x + \cot x + C$
(*b*) $\tan x - \cot x + C$
(*c*) $\tan x \cdot \cot x + C$
(*d*) $\tan x - \cot 2x + C$

Interpret (b) $\int \frac{dx}{\sin^2 x \cdot \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x}$$

$$= \int \sec^2 x\, dx + \int \text{cosec}^2 x\, dx = \tan x - \cot x + C$$

***Sample* Problem 4** $\int \sin^3 x \cos^3 x\, dx$ *is equal to*

(*a*) $\frac{1}{32}\left[-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$

(*b*) $\frac{1}{16}\left[-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$

(*c*) $\frac{1}{64}\left[-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$

(*d*) *None of the above*

Interpret (a) $\int \sin^3 x \cos^3 x\, dx = \frac{1}{8}\int (2 \sin x \cos x)^3\, dx$

$$= \frac{1}{8}\int \sin^3 2x\, dx = \frac{1}{8}\int \frac{3 \sin 2x - \sin 6x}{4} dx$$

$$= \frac{1}{32}\int (3 \sin 2x - \sin 6x)\, dx$$

$$= \frac{1}{32}\left[-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$$

(iii) Exponential Formulae

(a) $\int e^x dx = e^x + C$

(b) $\int e^{(ax+b)}\, dx = \frac{1}{a} \cdot e^{(ax+b)} + C$

(c) $\int a^x\, dx = \frac{a^x}{\log_e a} + C$

(d) $\int a^{(bx+c)} dx = \frac{1}{b} \cdot \frac{a^{(bx+c)}}{\log_e a} + C$

***Sample* Problem 5** $\int \frac{2^x + 3^x}{5^x} dx$ *is equal to*

(*a*) $\frac{2^x}{\log 2} + \frac{3^x}{\log 3} - \frac{5^x}{\log 5} + C$

(*b*) $\frac{2/5}{x \log_e (2/5)} + \frac{3/5}{x \log_e (3/5)} + C$

(*c*) $\frac{(2/5)^x}{\log_e (2/5)} + \frac{(3/5)^x}{\log_e (3/5)} + C$

(*d*) *None of the above*

Interpret (c) $\int \frac{2^x + 3^x}{5^x} dx = \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right) dx$

$$= \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right] dx$$

$$= \frac{(2/5)^x}{\log_e 2/5} + \frac{(3/5)^x}{\log_e 3/5} + C$$

(iv) Logarithmic Formulae

(a) $\int \log x\, dx = x \log x - x + C$

(b) $\int \log (ax + b)\, dx = x \log (ax + b) - x + \frac{b}{a} \log (ax + b) + C$

30.3 Rules of Integration

(i) $\int \{f(x) \pm g(x)\} dx = \int f(x) dx \pm \int g(x) dx$

(ii) $\int k f(x) dx = k \int f(x) dx$, where k is a constant

(iii) $\frac{d}{dx}\int [f(x) dx] = f(x)$

(iv) $\int f(x)\, dx = g(x) + C$

$$\Rightarrow \quad \int f(ax + b)\, dx = \frac{g(ax + b)}{a} + C$$

Hot Spot 1

Integration by Substitution

Integration by substitution is the most important topic for JEE (Main) examination. Mostly question are seen from this topic. The level of question is from moderate to typical.
Integration of certain functions cannot be obtained directly, if they are not in one of the standard forms, but they may be reduced to standard forms by proper substitution.

If $g(x)$ is a continuous differentiable function, then to evaluate integrals of the form

$$I = \int f[g(x)] \cdot g'(x)\, dx$$

We substitute

$$g(x) = t \quad \text{and} \quad g'(x)\, dx = dt$$

The substitution reduces the integral to $\int f(t)\, dt$. After evaluating this integral we substitute back the value of t.

Lets now review the five steps for integration by substitution.

Step I Choose a new variable t.

Step II Determine the value dx

Step III Make the substitution

Step IV Integrate resulting integral.

Step V Return to the initial variable x

Some Substitution

Function	Substitution
$f(\sqrt{a^2 - x^2})$	$x = a \sin\theta$ or $x = a\cos\theta$
$f(\sqrt{x^2 - a^2})$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$f(x^2 + a^2), f(\sqrt{x^2 + a^2})$	$x = a\tan\theta$ or $x = a\cot\theta$
$f\left(\sqrt{\frac{a-x}{a+x}}\right), f\left(\sqrt{\frac{a+x}{a-x}}\right)$	$x = a\cos 2\theta$
$f\left(\sqrt{\frac{x-a}{b-x}}\right), f\{\sqrt{(x-a)(x-b)}\}$	$x = a\cos^2\theta + b\sin^2\theta$
$\int (x \pm \sqrt{(x^2 \pm a^2)})^n$	expression inside the bracket $= t$
$\int \left[\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}\right], (n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

Sample **Problem 6** $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ *is equal to* **[NCERT]**

(a) $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x+x^2} + C$

(b) $2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$

(c) $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$

(d) *None of the above*

Interpret (c)

Step I Choose a substitution function t, the substitution function is $\sqrt{x} = \cos t$

$$\Rightarrow \qquad t = \cos^{-1}\sqrt{x}$$

Step II Determine the value of dx

$$x = \cos^2 t \Rightarrow dx = 2\cos t\,(-\sin t)\, dt$$

Step III Make the substitution

$$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx = \int \sqrt{\frac{1-\cos t}{1+\cos t}} \cdot [2\cos t(-\sin t)]\, dt$$

$$\Rightarrow \qquad I = -2\int \sqrt{\frac{2\sin^2 t/2}{2\cos^2 t/2}} \cdot 2\sin\frac{t}{2} \cdot \cos\frac{t}{2} \cdot \cos t\, dt$$

$$\left(\because 1 - \cos x = 2\sin^2\frac{x}{2}, 1 + \cos x = 2\cos^2\frac{x}{2} \text{ and } \sin 2t = 2\sin t\cos t\right)$$

$$= -4\int \sin^2\frac{t}{2} \cdot \cos t\, dt = -4\int \frac{1-\cos t}{2} \cdot \cos t\, dt$$

$$= -2\int (\cos t - \cos^2 t)\, dt$$

$$= -2\int \left(\cos t - \frac{1+\cos 2t}{2}\right) dt$$

Step IV Integrate resulting integral

$$= -2\sin t + t + \frac{1}{2}\sin 2t + C$$

$$= -2\sin t + t + \frac{1}{2} \times 2\sin t\cos t + C$$

$$= -2\sqrt{1-\cos^2 t} + t + \frac{1}{2}[2 + \sqrt{1-\cos^2 t} \cdot \cos t] + C$$

Step V Return to the initial variable x

$$= 2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x} \cdot \sqrt{1-x} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

The solution is

$$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

***Sample* Problem 7** $\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$ *is equal to*

(a) $\left(\frac{x+3}{x-4}\right)^{\frac{1}{16}}+C$ (b) $\frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{\frac{1}{16}}+C$

(c) $\frac{16}{5}\left(\frac{x-4}{x+3}\right)^{\frac{1}{16}}+C$ (d) *None of these*

Interpret (b) $I=\int \frac{dx}{(x+3)^{\frac{15}{16}}(x-4)^{\frac{17}{16}}}=\int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{\frac{15}{16}}(x-4)^2}$

Put $\frac{x+3}{x-4}=t \Rightarrow \left[\frac{(x-4)-(x-3)}{(x-4)^2}\right]dx=dt \Rightarrow \frac{dx}{(x-4)^2}=\frac{dt}{-7}$

So, $I=\frac{-1}{7}\int \frac{dt}{t^{\frac{15}{16}}}=\frac{-1}{7}\int t^{\frac{-15}{16}}dt$

$=\frac{-16}{7}t^{\frac{1}{16}}+C=\frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16}+C$

***Sample* Problem 8** $\int \frac{dx}{(x+\sqrt{x^2-4})^{5/3}}$ *is equal to*

(a) $\frac{3}{4}t^{-8/3}(1-t^2)+C$ (b) $\frac{3}{4}t^{\frac{8}{3}}(1-t^2)+C$

(c) $\frac{4}{3}t^{\frac{-8}{3}}(1-t^2)+C$ (d) *None of these*

where, $t=(x+\sqrt{x^2-4})$

Interpret (a) $I=\int \frac{dx}{(x+\sqrt{x^2-4})^{5/3}}$

Put $x+\sqrt{x^2-4}=t \Rightarrow \left(1+\frac{x}{\sqrt{x^2-4}}\right)dx=dt$

$\therefore$ $x+\sqrt{x^2-4}=t \Rightarrow \sqrt{x^4-4}=t-x$

$\Rightarrow$ $x=\frac{t^2+4}{2t} \Rightarrow \sqrt{x^2-4}=\frac{t^2-4}{2t}$

So, $I=\int\left(\frac{t^2-4}{2t^2}\right)\frac{1}{t^{5/3}}dt=\frac{1}{2}\int t^{-5/3}dt-2\int t^{-11/3}dt$

$=\frac{1}{2}\frac{t^{-2/3}}{-2/3}-2\frac{t^{-8/3}}{-8/3}+C=\frac{3}{4}t^{-8/3}[1-t^2]+C,$

where, $t=\left(x+\sqrt{x^2-4}\right)$

***Sample* Problem 9** $\int \frac{dx}{x(\log x)^m}$ *is equal to* [**NCERT**]

(a) $(\log x)^{1-m}+C$ (b) $\frac{1}{m}(\log x)^m+C$

(c) $\frac{(\log x)^{1-m}}{1-m}+C$ (d) *None of these*

Interpret (c) $\int \frac{dx}{x(\log x)^m}$

Let $\log x=t \Rightarrow \frac{1}{x}=\frac{dt}{dx} \Rightarrow dx=xdt$

$\therefore$ $\int \frac{dx}{x(\log x)^m}=\int \frac{x}{x(t)^m}dt=\int t^{-m}dt=\frac{t^{-m+1}}{-m+1}+C$

$=\frac{(\log x)^{1-m}}{1-m}+C$

***Sample* Problem 10** $\int \frac{e^{2x}-1}{e^{2x}+1}dx$ *is equal to* [**NCERT**]

(a) $\frac{e^x+e^{-x}}{e^x-x^{-x}}+C$ (b) $\log \frac{e^x+e^{-x}}{e^x-x^{-x}}+C$

(c) $\log|e^x+e^{-x}|+C$ (d) $\log|e^x-e^{-x}|+C$

Interpret (c) $\int \frac{e^{2x}-1}{e^{2x}+1}dx=\int \frac{(e^x.e^x-1)}{(e^x\cdot e^x+1)}dx$

$=\int \frac{e^x\left(e^x-\frac{1}{e^x}\right)}{e^x\left(e^x+\frac{1}{e^x}\right)}dx=\int \frac{(e^x-e^{-x})}{(e^x+e^{-x})}dx$

Let $(e^x+e^{-x})=t \Rightarrow e^x-e^{-x}=\frac{dt}{dx} \Rightarrow dx=\frac{dt}{e^x-e^{-x}}$

$\int \frac{e^{2x}-1}{e^{2x}+1}dx=\int \frac{(e^x-e^{-x})}{t}.\frac{dt}{(e^x-e^{-x})}=\int \frac{dt}{t}=\log|t|+C$

$=\log|e^x+e^{-x}|+C$

Check Point **1**

1. $\int \frac{\sec^4 x}{\sqrt{\tan x}}dx$ is equal to

(a) $2\tan^{1/2}x+\frac{2}{5}\tan^{5/2}x+C$ (b) $\tan^{1/2}x+\frac{1}{5}\tan^{5/2}x+C$

(c) $2\tan x+\frac{2}{5}\tan^{5/2}x+C$ (d) None of these

2. If $I_r(x)$ means $\underbrace{\log\log\log\ldots\log}_{r\text{ times}}$ x, the log being repeated r times, then $\int \frac{dx}{xI_1(x)I_2(x)\ldots I_r(x)}$ is equal to

(a) $I_{r+1}(x)+C$ (b) $\frac{I_{r+1}(x)}{r+1}+C$

(c) $I_r(x)+C$ (d) None of these

3. $\int \frac{\cos x+x\sin x}{x(x+\cos x)}dx$ is equal to

(a) $\log\left(\frac{1}{x+\cos x}\right)+C$ (b) $\log\left(\frac{x}{x+\sin x}\right)+C$

(c) $\log\left(\frac{x}{x+\cos x}\right)+C$ (d) $\log\left(\frac{x}{x-\cos x}\right)+C$

4. If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = A \cos 4x + B$, then

(a) $A = -\frac{1}{2}$ (b) $A = -\frac{1}{8}$

(c) $A = -\frac{1}{4}$ (d) None of these

5. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ is equal to

(a) $\frac{1}{2}\sin 2x + C$ (b) $-\frac{1}{2}\sin 2x + C$

(c) $-\frac{1}{2}\sin x + C$ (d) $-\sin^2 x + C$

6. $\int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}} dx$ is equal to

(a) $\frac{2}{3}\sin^{-1}(\cos^{3/2} x) + C$

(b) $\frac{3}{2}\sin^{-1}(\cos^{3/2} x) + C$

(c) $\frac{2}{3}\cos^{-1}(\cos^{3/2} x) + C$

(d) None of the above

30.4 Integration of Different Types of Functions

(i) Integration of type $\int \sin^m x \cos^n x\, dx$

(a) If m is an odd integer, put $\cos x = t$

(b) If n is an odd integer, put $\sin x = t$

(c) If $m + n$ is negative even integer, then put $\tan x = t$ or $\cot x = t$

***Sample* Problem 11** $\int \sin^{-7/5} x \cos^{-3/5} dx$ *is equal to*

(a) $\frac{-2}{5}(\tan x)^{2/5} + C$ (b) $-\frac{5}{2}(\cot x)^{2/5} + C$

(c) $\frac{-5}{2}(\tan x)^{2/5} + C$ (d) *None of these*

Interpret (b) Here, $p = -\frac{7}{5}, q = -\frac{3}{5}$

$$\frac{p + q - 2}{2} = -2$$

$$I = \int \sin^{-7/5} x \cos^{-3/5} x dx = \int \frac{\cos^{-3/5} x}{\sin^{-3/5} x \sin^2 x} dx$$

$$= \int (\cot x)^{-3/5} \operatorname{cosec}^2 x dx$$

Put $\cot x = t \Rightarrow \operatorname{cosec}^2 x dx = -dt$

So, $I = -\int t^{-3/5} dt = -\frac{5}{2}(\cot x)^{2/5} + C$

***Sample* Problem 12** $\int \sin^3 x \cos^3 x\, dx$ *is equal to* **[NCERT]**

(a) $\frac{\sin^6 x}{6} - \frac{\cos^4 x}{4} + C$ (b) $\frac{\sin^6 x}{3} - \frac{\cos^4 x}{2} + C$

(c) $\frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$ (d) *None of these*

Interpret (c) $\int \sin^3 x \cdot \cos^3 x dx$

Let $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow dx = -\frac{dt}{\sin x}$

$$\therefore \int \sin^3 x \cdot \cos^3 x dx = \int \sin^{3.} x \cdot t^3 \frac{dt}{-\sin x}$$

$$= -\int \sin^2 x \cdot t^3 dt = -\int t^3 (1 - \cos^2 x) dt$$

$$= -\int t^3 (1 - t^2) dt = -\int (t^3 - t^5) dt$$

$$= -\left[\frac{t^4}{4} - \frac{t^6}{6}\right] + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

(ii) If the integrals are of the form $\int \frac{dx}{ax^2 + bx + c}$, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ and $\int \sqrt{ax^2 + bx + c}\, dx$, then express $ax^2 + bx + c$ as the sum or difference of two squares *i.e.*, in the form of perfect square and then apply the standard results.

***Sample* Problem 13** $\int \frac{dx}{9x^2 + 6x + 5}$ *is equal to* **[NCERT]**

(a) $\tan^{-1}\left(\frac{3x+1}{2}\right) + C$ (b) $6\tan^{-1}\left(\frac{3x+1}{2}\right) + C$

(c) $\cot^{-1}\left(\frac{3x+1}{2}\right) + C$ (d) $\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$

Interpret (d) $\int \frac{dx}{9x^2 + 6x + 5} = \frac{1}{9}\int \frac{dx}{x^3 + \frac{2}{3}x + \frac{5}{9}}$

$$= \frac{1}{9}\int \frac{dx}{x^2 + 2 \cdot \frac{1}{3} \cdot x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \left(\frac{1}{3}\right)^2}$$

$$= \frac{1}{9}\int \frac{dx}{x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \frac{1}{9}}$$

$$= \frac{1}{9}\int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \frac{1}{9}\left(\frac{1}{2/3}\right)\tan^{-1}\left(\frac{x + \frac{1}{3}}{2/3}\right) + C \quad \left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

***Sample* Problem 14** $\int \frac{dx}{\sqrt{x^2+2x+2}}$ *is equal to* [**NCERT**]

(a) $\log|(x+1)+\sqrt{x^2+2x+2}|+C$

(b) $\log|x+\sqrt{(x^2+2x+2)}|+C$

(c) $\log|x+1-\sqrt{x^2+2x+2}|+C$

(d) *None of the above*

Interpret (a) Let $I=\int \frac{dx}{\sqrt{x^2+2x+2}}=\int \frac{dx}{\sqrt{(x^2+2x+1)+1}}$

$$=\int \frac{1}{\sqrt{(x+1)^2+(1)^2}}dx \quad (\text{let } x+1=t)$$

$\Rightarrow$ $dx=dt \therefore I=\int \frac{dt}{\sqrt{t^2+1}}=\log|t+\sqrt{t^2+1}|+C$

$$\left(\because \int \frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|\right)$$

$$=\log|(x+1)+\sqrt{(x+1)^2+1}|+C$$

$$=\log|(x+1)+\sqrt{x^2+2x+2}|+C$$

***Sample* Problem 15** $\int \sqrt{x^2+4x+1}dx$ *is equal to* [**NCERT**]

(a) $(x+2)\sqrt{x^2+4x+1}-\frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}|+C$

(b) $\frac{(x+2)}{2}\sqrt{x^2+4x+1}+\frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}|+C$

(c) $\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}|+C$

(d) *None of the above*

Interpret (c) Let

$$I=\int \sqrt{x^2+4x+1}\,dx=\int \sqrt{x^2+4x+1-2^2+2^2}dx$$

$$=\int \sqrt{(x+2)^2-(\sqrt{3})^2}dx$$

$$\left(\because \int \sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right)$$

$$\Rightarrow \quad I=\frac{x+2}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}|+C$$

(iii) If the integrals are of the form $\int \frac{px+q}{ax^2+bx+c}dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ and $\int (px+q)\sqrt{ax^2+bx+c}\,dx$.

Then, put $px+q=A$ (differential coefficient of $ax^2+bx+c)+B$

Find A and B by comparing the coefficients of x and constant term on both the sides of equation. Then,

$$\int \frac{px+q}{ax^2+bx+c}dx$$

$$=A\int \frac{ax+b}{ax^2+bx+c}dx+B\int \frac{1}{ax^2+bx+c}dx$$

Now, we can integrates it easily. Similarly, other two cases.

***Sample* Problem 16** $\int \frac{5x-2}{1+2x+3x^2}dx$ *is equal to* [**NCERT**]

(a) $\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$

(b) $\frac{5}{3}\log|1+2x+3x^2|-\frac{11}{3}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$

(c) $\frac{1}{3}\log|1+2x+3x^2|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$

(d) *None of the above*

Interpret (a) $\int \frac{5x-2}{1+2x+3x^2}dx$

Let $\quad 5x-2=A\frac{d}{dx}(1+2x+3x^2)+B$

$\Rightarrow \quad 5x-2=A(2+6x)+B \Rightarrow 5x-2=6Ax+(2A+B)$

On equating the coefficient of x and constant on both sides, we get

$$5=6A\Rightarrow A=\frac{5}{6} \text{ and } 2A+B=-2\Rightarrow B=\frac{-11}{3}$$

$$\therefore \quad I=\int \frac{5x-2}{1+2x+3x^2}dx=\int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int \frac{dx}{1+2x+3x^2}$$

Let $\quad I_1=\int \frac{2+6x}{1+2x+3x^2}$

and $\quad I_2=\int \frac{dx}{1+2x+3x^2}$

$\therefore \quad I=\frac{5}{6}I_1-\frac{11}{3}I_2 \quad$ ...(i)

Now, $\quad I_1=\int \frac{2+6x}{1+2x+3x^2}dx$

Let $\quad 1-2x+3x^2=t$

$\Rightarrow \quad (2+6x)\,dx=dt$

$\Rightarrow \quad I_1=\int \frac{dt}{t}=\log|t|+C_1$

$\therefore \quad I_1=\log|1+2x+3x^2|+C_1 \quad$ ...(ii)

Also, $\quad I_2=\int \frac{dx}{1+2x+3x^2}$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$. Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$\therefore I_2 = \frac{1}{3}\int \frac{dx}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]} = \frac{1}{3}\left[\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right)\right] + C_2$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C_2$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C_2 \quad \text{...(iii)}$$

On substituting the values of I_1 and I_2 from Eq. (ii) and Eq. (i), we get

$$I = \frac{5}{6}[\log|1+2x+3x^2|] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

$$\left(\because \frac{5}{6}C_1 = \frac{11}{3}C_2 = C\right)$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

***Sample* Problem 17** $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$ *is equal to* **[NCERT]**

(a) $\sqrt{x^2+2x+3} + \log|x+1+\sqrt{x^2+2x+3}| + C$

(b) $\sqrt{x^2+2x+3} - \log|x+1+\sqrt{x^2+2x+3}| + C$

(c) $\sqrt{x^2+2x+3} + \log|x+1-\sqrt{x^2+2x+3}| + C$

(d) *None of the above*

Interpret (a) Let $x+2 = A\frac{d}{dx}(x^2+2x+3) + B$

$\Rightarrow \quad x+2 = A(2x+2) + B$

$\Rightarrow \quad x+2 = 2Ax + (2A+B)$

On equating the coefficient of x and constant term on both sides, we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

and $\quad 2A + B = 2 \Rightarrow 2\times\frac{1}{2} + B = 2$

$\Rightarrow \quad B = 2 - 1 = 1$

$\Rightarrow \quad x+2 = \frac{1}{2}(2x+2) + 1$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}}$$

Let $\quad I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}}$

Then, $\quad \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad$ ...(i)

Now, $\quad I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $\quad x^2+2x+3 = t \Rightarrow (2x+2)dx = dt$

$$I_2 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = \frac{t^{(-1/2)+1}}{1/2+1} + C = 2\sqrt{t} + C_1$$

$$= 2\sqrt{x^2+2x+3} + C_1$$

and $\quad I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}} = \int \frac{dx}{\sqrt{x^2+2x+(1)^2+3-(1)^2}}$

$$= \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

Let $x+1 = t \Rightarrow dx = dt$

$$I_2 = \int \frac{dt}{\sqrt{t^2+(\sqrt{2})^2}} = \log|t+\sqrt{t^2+2}| + C_2$$

$$\left(\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|\right)$$

$$= \log|x+1+\sqrt{(x+1)^2+2}| + C_2$$

$$= \log|x+1+\sqrt{x^2+2x+3}| + C_2$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}[2\sqrt{x^2+2x+3}]$$

$$+ \log|x+1+\sqrt{x^2+2x+3}|$$

$$\left(\because \frac{1}{2}C_1 + C_2 = C\right)$$

$$= \sqrt{x^2+2x+3} + \log|x+1+\sqrt{x^2+2x+3}| + C$$

(iv) $\int \frac{(ax^2+bx+c)}{(px^2+qx+r)}dx$, $\int \frac{(ax^2+bx+c)}{\sqrt{(px^2+qx+r)}}dx$,

$\int \frac{(ax^2+bx+c)}{\sqrt{(px^2+qx+r)}}dx$

Substitute, $ax^2+bx+c = \lambda(px^2+qx+r)$

$$+ \mu\left\{\frac{d}{dx}(px^2+qx+r)\right\} + \gamma$$

Find λ, μ and γ. These integrations reduce to integration of three independent functions.

***Sample* Problem 18** $\int \frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}dx$ *is equal to*

(a) $\left(\frac{x+\frac{1}{2}}{2}\right) + \sqrt{x^2+x+1} + \frac{3}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$

$$+ 3\sqrt{x^2+x+1} + \frac{9}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + C$$

(b) $\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1} + \frac{1}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$

$$+ \sqrt{x^2+x+1} + \frac{1}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + C$$

(c) $\left(x-\frac{1}{2}\right)+\sqrt{x^2+x+1}+\frac{1}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$

$$-\sqrt{x^2+x+1}+\frac{1}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C$$

(d) *None of the above*

Interpret (a) Let $x^2+4x+7=A(x^2+x+1)+B(2x+1)+C$

Comparing the coefficients of x^2, x and constant term, we get

$$A=1, A+2B=4, A+B+C=7 \Rightarrow A=1, B=\frac{3}{2}, C=\frac{9}{2}$$

So, $I=\int\sqrt{x^2+x+1}\,dx+\frac{3}{2}\int\frac{(2x+1)}{\sqrt{x^2+x+1}}dx+\frac{9}{2}\int\frac{dx}{\sqrt{x^2+x+1}}$

Now, $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$

$$I=\left(\frac{x+\frac{1}{2}}{2}\right)+\sqrt{x^2+x+1}+\frac{3}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$$

$$+3\sqrt{x^2+x+1}+\frac{9}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C$$

***Sample* Problem 19** $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$ *is equal to*

(a) $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$

(c) $-\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$ (d) *None of these*

Interpret (a) Put $x=\frac{1}{t}\Rightarrow dx=\frac{-1}{t^2}dt$

$$\therefore\quad I=\int\frac{-dt}{t^2\left(1+\frac{1}{t^2}\right)\sqrt{1-\left(\frac{1}{t}\right)^2}}=-\int\frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Again, put $t^2-1=y^2\Rightarrow t\,dt=y\,dy$

$$\therefore\quad I=-\int\frac{y\,dy}{(y^2+2)y}$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$$

***Sample* Problem 20** $\int\frac{dx}{x\sqrt{ax-x^2}}$ *is equal to* [**NCERT**]

(a) $-\frac{1}{a}\sqrt{\frac{x-a}{x}}+C$ (b) $\frac{-2}{a}\sqrt{\frac{a-x}{x}}+C$

(c) $\frac{2}{a}\sqrt{\frac{a-x}{x}}+C$ (d) *None of these*

Interpret (b) Let $I=\int\frac{dx}{x\sqrt{ax-x^2}}$, put $x=\frac{a}{t}\Rightarrow dx=-\frac{a\,dt}{t^2}$

$$\therefore\quad I=\int\frac{1}{\frac{a}{t}\sqrt{a\left(\frac{a}{t}\right)-\frac{a^2}{t^2}}}\cdot\left(\frac{-a}{t^2}\right)dt=\int\frac{-a}{a\cdot a\sqrt{t-1}}\cdot dt$$

$$=-\frac{1}{a}\int(t-1)^{-1/2}\cdot dt=\frac{-1}{a}\cdot\frac{(t-1)^{(-1/2)+1}}{-(1/2)+1}+C$$

$$=-\frac{2}{a}\sqrt{t-1}+C=\frac{-2}{a}\sqrt{\frac{a}{x}-1}+C\quad\left(\because x=\frac{a}{t}\Rightarrow t=\frac{a}{x}\right)$$

$$=\frac{-2}{a}\sqrt{\frac{a-x}{x}}+C$$

(v) If integral is of the form $\int\frac{x^2+1}{x^4+kx^2+1}dx$, $\int\frac{x^2-1}{x^4+kx^2+1}dx$, $\int\frac{dx}{x^4+kx^2+1}$ where k is any constant, then divide numerator (Nr) and denominator (Dr) by x^2 and put $x\mp\frac{1}{x}=t$.

(vi) If the integral is of the form $\int\frac{dx}{(ax+b)^m(cx+d)^n}$ (where, m and n are positive integers, then put $\frac{ax+b}{cx+d}=t$.

(vii) If the integrals are of the form $\int\frac{dx}{a\cos^2x+b}$, $\int\frac{dx}{a+b\sin^2x}$, $\int\frac{dx}{a\sin^2x+b\cos^2x}$, $\int\frac{dx}{(a\sin x+b\cos x)^2}$ and $\int\frac{dx}{a^2\sin x+2b\sin x\cos x+c^2\cos^2x}$, then multiply the numerator and denominator by $\sec^2x$ and then, put $\tan x=t$. (use $\sec^2x=1+\tan^2x$)

(viii) If the integrals are of the form

$$\int\frac{1}{a\cos x+b\sin x}dx, \int\frac{dx}{a+b\cos x},$$

$$\int\frac{dx}{a\sin x+b\cos x+c}\text{ and }\int\frac{dx}{a+b\sin x},$$

then convert sines and cosines into their respective tangents of half the angles and then put $\tan\frac{x}{2}=t$.

(ix) If the integrals are of the form

$$\int\frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+n}dx, \int\frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx,$$

$$\int\frac{p\cos x}{a\cos x+b\sin x}dx\text{ and }\int\frac{q\sin x}{a\cos x+b\sin x}dx,$$

then express the numerator as

$$\text{Nr}\equiv A(\text{Dr})+B\frac{\text{d}}{\text{dx}}(\text{Dr})+C$$

(the value of c is zero in last three forms) and compare the coefficients of $\sin x$ and $\cos x$ and then, proceed.

(x) If integral is of the form $\int\frac{ae^x+be^{-x}}{ce^x+de^{-x}}dx$, then

put $ae^x+be^{-x}=A(ce^x+de^{-x})+B(ce^x-de^{-x})$

Now, find A and B by comparing the coefficients of e^x and e^{-x}.

Now, $$\int \frac{ae^x + be^{-x}}{ce^x + de^{-x}}\,dx = A\int dx + B\int \frac{f'(x)}{f(x)}\,dx,$$

where $f(x) = ce^x + de^{-x}$

(xi) $\int x^m (a + bx^n)^p dx$ $(p \neq 0)$. Here we have the following cases.

(a) If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

(b) If p is a negative integer and m and n are rational number, put $x = t^k$, when k is the LCM of denominator of m and n.

(c) If $\frac{m+1}{n}$ is an integer and p is rational number, put $(a + bx^n) = t^k$, when k is the denominator of p.

(d) If $\frac{m+1}{n}$ is an integer, put $\frac{a+bx^n}{x^n} = t^k$, where k is the denominator of p.

***Sample* Problem 21** *Evaluate* $\int x^{\frac{-1}{3}} (1+x^{1/3})^{1/4} dx$.

(a) $\frac{-4}{15}(1+x^{1/3})^{5/4}(4+9x^{1/2}) + C$

(b) $\frac{1}{15}(1+x^{1/3})^{5/4}(4+9x^{1/3}) + C$

(c) $\frac{2}{15}(1+x^{1/3})^{1/4}(4+9x^{1/3}) + C$

(d) *None of the above*

Interpret (a) Here, $m = \frac{-1}{3}, n = \frac{1}{3}, p = \frac{1}{4}$

$$\frac{m+1}{n} = 2, \text{ which is an integer}$$

So, $$(1+x^{1/3}) = t^4 \Rightarrow \frac{dx}{3x^{2/3}} = 4t^3 dt$$

$$I = 12\int (t^4 - 1)t^4 dt$$

$$= -\frac{4}{15}(1+x^{1/3})^{5/4}[4+9x^{1/3}] + C$$

***Sample* Problem 22** *Evaluate* $\int \frac{(5\sin x + 6)}{\sin x + 2\cos x + 3}\,dx$

(a) $\tan^{-1}\left(\frac{1+\tan\frac{1}{2}}{2}\right) + C$ (b) $\cot^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right) + C$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right) + C$ (d) *None of these*

Interpret (a) Let

$$5\sin x + 6 = A(\sin x + 2\cos x + 3) + B(\cos x - 2\sin x) + C$$

On equating the coefficients of sin$x$, cos$x$ and constant term, we get

$$\left.\begin{aligned} A - 2B &= 5 \\ 2A + B &= 0 \\ 3A + C &= 6 \end{aligned}\right\} \Rightarrow A = 1, B = -2, C = 3$$

$$I = \int dx - 2\int \frac{(\cos x - 2\sin x)\,dx}{\sin x + 2\cos x + 3} + 3\int \frac{dx}{\sin x + \cos x + 3}x$$

$$- 2\log|\sin x + 2\cos x + 3| + 3\ell_1$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2\,dt$

So, $$I_1 = \int \frac{2dt}{t^2 + 2t + 5} = \int \frac{2dt}{(t+1)^2 + 4} = \tan^{-1}\left(\frac{t+1}{2}\right) + C$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right) + C$$

***Sample* Problem 23** *Evaluate* $\int x^{\frac{-2}{3}}(1+x^{2/3})^{-1}$.

(a) $\tan^{-1}(x^{1/3}) + C$ (b) $\tan^{-1}(x^{2/3}) + C$

(c) $3\tan^{-1}(x^{1/3}) + C$ (d) *None of these*

Interpret (c) Here, $p = -1$, is a negative interger and m and n are rational numbers.

Put $x = t^3$

$\Rightarrow$ $dx = 3t^2 dt$

So, $$I = \int t^{-2}(1+t^2)^{-1}\,3t^2\,dt$$

$$= \int \frac{3\,dt}{1+t^2} = 3\tan^{-1}(x^{1/3}) + C$$

***Sample* Problem 24** $\int \frac{dx}{2 + \sin x + \cos x}$ *is equal to*

[NCERT Exemplar]

(a) $\sqrt{2}\tan^{-1}\left(\frac{\tan x/2 + 1}{\sqrt{2}}\right) + C$ (b) $\tan^{-1}\left(\frac{\tan x/2 + 1}{\sqrt{2}}\right) + C$

(c) $\sqrt{2}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{2}}\right) + C$ (d) *None of these*

Interpret (a) Let $I = \int \frac{dx}{2 + \sin x + \cos x}$

$$\Rightarrow \quad I = \int \frac{dx}{2 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\sec^2\frac{x}{2}dx}{2 + 2\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$I = \int \frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore \quad I = \int \frac{2\,dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$$

$$= 2\int \frac{dt}{(t+1)^2+(\sqrt{2})^2} = 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right)+C$$

$$\Rightarrow \quad I = \sqrt{2}\tan^{-1}\left(\frac{\tan x/2+1}{\sqrt{2}}\right)+C$$

***Sample* Problem 25** $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$ *is equal to*

(a) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{x+1}\right)+C$ (b) $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x}{\sqrt{3(x+1)}}\right]+C$

(c) $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x}{(x+1)^2}\right]+C$ (d) *None of these*

Interpret (b) Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Put $x+1 = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore I = \int \frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}\cdot(2t)\,dt$$

$$= 2\int \frac{t^2+1}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}\,dt = 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{3})^2}\,dt$$

$$= 2\int \frac{du}{u^2+(\sqrt{3})^2} \qquad \left(\text{where, } u = t-\frac{1}{t} \Rightarrow du = 1+\frac{1}{t^2}\,dt\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C$$

$$\therefore \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}\,t}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x}{\sqrt{3(x+1)}}\right]+C$$

30.5 Substitution for Some Irrational Functions

(i) $\int \frac{dx}{(x-\alpha)(x-\beta)}, \int \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\,dx$

$\int \sqrt{(x-\alpha)(\beta-x)}\,dx$, put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(ii) $\int \frac{dx}{(px+q)\sqrt{ax+b}}$, put $ax+b = t^2$

(iii) $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$, put $px+q = t^2$

(iv) $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, put $px+q = \frac{1}{t}$

(v) $\int \frac{dx}{(px^2+r)\sqrt{(ax^2+c)}}$ at first $x = \frac{1}{t}$ and then $a+ct^2 = z^2$

(vi) $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, where $r \geq 2$ and $r \in I$

Put $x-k = \frac{1}{t}$

(vii) $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}\,dx$

Substitute, $ax^2+bx+c = A_1(dx+e)(2fx+g) + B_1(dx+e)+C_1$

(viii) If the integral is of the form

$$\int \frac{dx}{(ax^2+bx+c)\sqrt{lx^2+mx+n}},$$

then put $\frac{ax^2+bx+c}{lx^2+mx+n} = t^2$

***Sample* Problem 26** $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+8}}$ *is equal to*

(a) $\log(x+2+\sqrt{x^2+4x+8}) + \frac{1}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(b) $4\log(x+2+\sqrt{x^2+4x+8}) + \frac{1}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(c) $\log(x+2+\sqrt{x^2+4x+8}) + \log\left|(x+2)+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(d) *None of the above*

Interpret (b) Let $4x+7 = A(x+2)+B \Rightarrow A = 4, B = -1$

So, $$I = 4\int \frac{dx}{\sqrt{x^2+4x+8}} - \int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$$

$$= 4\log(x+2+\sqrt{x^2+4x+8}) + \frac{1}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$$

***Sample* Problem 27** $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$ *is equal to*

(a) $2\sqrt{x^2+4x+8} - \log|(x+2)+\sqrt{x^2+4x+8}| - \frac{5}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+c$

(b) $\sqrt{x^2+4x+8} + \log|(x+2)+\sqrt{x^2+4x+8}| + 5\log\left|(x+2)+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(c) $\sqrt{x^2+4x+8} + \log|(x+2)+\sqrt{x^2+4x+8}| + 5\log\left|(x+2)+\sqrt{(x+2)^2+4}\right|+C$

(d) *None of the above*

Interpret (*a*) Put

$$2x^2 + 7x + 11 = A(x+2)(2x+4) + B(x+2) + C$$

On comparing the coefficients of x^2, x and constant term, we get

$$A = 1, 7 = 8A + B, C + 2B + 8A = 11$$

$$\Rightarrow \qquad B = -1, C = 5$$

So, $I = \int \frac{2x+4}{\sqrt{x^2+4x+8}} - \int \frac{dx}{\sqrt{x^2+4x+8}} + 5\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$$= 2\sqrt{x^2+4x+8} - \log\left|(x+2) + \sqrt{x^2+4x+8}\right|$$

$$-\frac{5}{2}\log\left|\frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}}\right| + C$$

***Sample* Problem 28** $\int \frac{x\,dx}{(2x^2+3)\sqrt{x^2-1}}$ *is equal to*

(a) $\tan^{-1}\left(\sqrt{\frac{2}{5}}\sqrt{x^2-1}\right) + C$ (*b*) $10\tan^{-1}\left(\sqrt{\frac{2}{5}}\cdot\sqrt{x^2-1}\right) + C$

(c) $\frac{1}{10}\tan^{-1}\left(\sqrt{\frac{2}{5}}\cdot\sqrt{x^2-1}\right) + C$ (*d*) *None of these*

Interpret (c) Put $x^2 - 1 = t^2 \Rightarrow x\,dx = t\,dt$

So, $I = \int \frac{t\,dt}{(2t^2+5)t}$

$$= \int \frac{dt}{2t^2+5} = \frac{1}{2}\int \frac{dt}{t^2 + \frac{5}{2}} = \frac{1}{10}\tan^{-1}\left(\sqrt{\frac{2}{5}}\sqrt{x^2-1}\right) + C$$

Hot Spot 2

Integration by Parts

Integration by parts is an important topic for JEE Main examination and a small practice can help you in solving such problems very quickly as the level of question is easy to average.

If u and v are the differentiable functions of x, then

$$\int u \cdot v\,dx = u\int v\,dx - \int\left[\left\{\frac{d}{dx}(u)\right\}\left(\int v\,dx\right)\right]dx$$

i.e., The integral of the product of two functions = (First functions) × (Integral of the second function)–Integral of {(differentiation of first function) × (integral of second function)}.

How to Choose Ist and IInd Function

If two functions are of different types take the function as Ist which comes first in the word **ILATE**, where *I* stands for inverse circular function, *L* stands for logarithmic function, *A* stands for algebraic function, *T* stands for trigonometric and *E* for exponential function.

Note

- If the integral contains a single logarithmic or single inverse trigonometric function take unity as the second function.
- If the integrals of both the functions are known, the function which is easy to integrate is taken as the second function.
- In certain cases, integration by parts will lead to a simple equation involving the integral. Solve the equation and determine the integral.

***Sample* Problem 29** $\int x^2 \log x\,dx$ *is equal to* [**NCERT**]

(a) $\frac{x^3}{3}\log x - \frac{x^3}{9} + C$ (*b*) $\frac{x^2}{2}\log x - \frac{x^3}{9} + C$

(c) $\frac{x}{3}\log x - \frac{x^3}{3} + C$ (*d*) *None of these*

Interpret (*a*) On taking $\log x$ as first function and x^2 as second function and integrating by parts, we get

($\because$ log function comes before algebraic function in ILATE)

$$I = \int x^2 \log x\,dx = \log x \int x^2 dx - \int\left[\frac{d}{dx}(\log x)\int x^2 dx\right]dx$$

$$= \frac{x^3}{3}\log x - \int\left[\frac{1}{x}\cdot\frac{x^3}{3}\right]dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

***Sample* Problem 30** $\int x \sin^{-1} x\,dx$ *is equal to* [**NCERT Exemplar**]

(*a*) $\frac{\sin^{-1}x}{2}(2x^2-1) + \frac{1}{4}x\sqrt{1-x^2} + C$

(*b*) $\frac{\sin^{-1}x}{4}(2x^2-1) + \frac{1}{4}x\sqrt{1-x^2} + C$

(c) $\frac{\sin^{-1}x}{4}(2x^2-1) + \frac{1}{2}x\sqrt{1-x^2} + C$

(*d*) *None of the above*

Interpret (*b*) Let $I = \int x\sin^{-1}x\,dx$

On taking $\sin^{-1}x$ as first function and x as second function and integrating by parts, we get

($\because$ inverse functions comes before, algebraic functions in ILATE)

$$I = \sin^{-1}x\int x\,dx - \int\left[\frac{d}{dx}(\sin^{-1}x)\int x\,dx\right]dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \left[\frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \right] dx$$

$$= \frac{x^2}{2} \cdot \sin^{-1} x + \int \left[\frac{1-1-x^2}{\sqrt{1-x^2}} \cdot \frac{1}{2} \right] dx$$

(add and subtract 1 in numerator of second term)

$$= \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx$$

$$\Rightarrow \quad I = \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] + C$$

$$\Rightarrow \quad I = \sin^{-1} x \left[\frac{x^2}{2} - \frac{1}{4} \right] + \frac{1}{4} x\sqrt{1-x^2} + C$$

$$\Rightarrow \quad I = \frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{4} x\sqrt{1-x^2} + C$$

***Sample* Problem 31** $I = \int 1 \cdot \tan^{-1} x \, dx$ *is equal to* [NCERT]

(*a*) $x \tan^{-1} x + \log |1 + x^2| + C$ (*b*) $x \tan^{-1} x - \log |1 + x^2| + C$

(*c*) $\frac{x}{2} \tan^{-1} x - \log |1 + x^2| + C$ (*d*) $x \tan^{-1} x - \frac{1}{2} \log |1 + x^2| + C$

Interpret (*d*) There is no standard integration for $\tan^{-1} x$.

So, take $\tan^{-1} x$ as first function and 1 as second function and integrating by parts.

Let $\quad I = \int 1 \cdot \tan^{-1} x \, dx$

$$\Rightarrow \quad I = \tan^{-1} x \int 1 \, dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int 1 \, dx \right] dx$$

$$= x \tan^{-1} x - \int \frac{1}{1+x^2} \cdot x \, dx$$

Let $\quad 1 + x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

$$\therefore \quad I = x \tan^{-1} x - \int \frac{x}{t} \cdot \frac{dt}{2x} = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{t} dt$$

$$= x \tan^{-1} x - \frac{1}{2} \log |t| + C = x \tan^{-1} x - \frac{1}{2} \log |1 + x^2| + C$$

30.6 Integration by Partial Fractions

If a function is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials of degree m and n respectively.

If $m < n$, this fraction is proper fraction and if $m \geq n$ this fraction is improper fraction to convert it into a proper fraction divide $f(x)$ by $Q(x)$, so that $\frac{P(x)}{Q(x)}$ is expressed in the form

$$\frac{P(x)}{Q(x)} = M(x) + \frac{P_1(x)}{Q(x)} \left(\text{where, } \frac{P_1(x)}{Q(x)} \text{ is proper fraction} \right)$$

where, $M(x)$ and $P_1(x)$ are polynomials such that the degree of $P_1(x)$ is less than that of $Q(x)$.

Any proper rational function $\frac{P(x)}{Q(x)}$ can be expressed as the sum of rational functions, each having a simple factor of $Q(x)$. Each such fraction is called a partial fraction and the process of obtained them is called the resolutions or decomposition of $\frac{P(x)}{Q(x)}$ into partial fractions and it depends upon the nature of the factors of $Q(x)$.

***Case* I** If $Q(x)$ has non-repeating linear factors, then

$$\frac{P_1(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_{n+1}}{a_nx + b_n}$$

$A_1, A_2, \ldots, A_n$ are independent of x, *i.e.*, constants. To find A_1 put $x = -\frac{b_1}{a_1}$ in the LHS except in the factor $(a_1x + b_1)$, we get A_1. Similarly, find $A_2, A_3, \ldots, A_n$. Now, integrate

$$\int \frac{P_1(x)}{Q(x)} dx = \int \left(\frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_n}{a_nx + b_n} \right) dx$$

***Case* II** If $Q(x)$ has repeating linear factors, then

$$\frac{P_1(x)}{Q(x)} = \frac{P_1(x)}{(a_1x + b_1)(a_2x + b_2)^n}$$

$$= \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_{n+1}}{(a_nx + b_n)^n}$$

Now, find the constants $A_1, A_2, \ldots, A_{n-1}$ and then integrate.

***Case* III** When some of the factors of denominator $Q(x)$ are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, assume partial fraction of the type $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants.

$$\frac{P_1(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{Ax + B}{ax^2 + bx + c}$$

Now, find the constants A_1, A and B and then integrate.

***Case* IV** When some of the factors of $Q(x)$ are quadratic and repeating, then

$$\frac{P_1(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3x + B_1}{ax^2 + bx + c} + \frac{A_4x + B_2}{(ax^2 + bx + c)^2}$$

Now, find the constants A_1, A_2, A_3, B_1, A_4 and B_2 and then integrate.

***Sample* Problem 32** $\int \frac{dx}{(x+2)(x^2+1)}$ *is equal to* [NCERT Exemplar]

(a) $\frac{1}{5}\log|x+2| - \frac{1}{10}\log(x^2+1) + \frac{2}{5}\tan^{-1}x + C$

(b) $\log|x+2| + \frac{1}{10}\log(x^2+1) + \frac{1}{5}\tan^{-1}x + C$

(c) $5\log|x+2| + 10\log(x^2+1) + \tan^{-1}x + C$

(d) *None of the above*

Interpret (a) Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{(x^2+1)}$

$\Rightarrow \quad 1 = A(x^2+1) + (Bx+C)(x+2)$

Put $x = -2$, we get $A = \frac{1}{5}$

Now, comparing the coefficients of x^2 and constant term, we get

$0 = A + B$

and $1 = A + 2C$

$\Rightarrow \quad B = \frac{1}{5}, C = \frac{2}{5}$

So, $I = \frac{1}{5}\int \frac{dx}{x+2} - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{2}{5}\int \frac{dx}{x^2-1}$

$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log(x^2+1) + \frac{2}{5}\tan^{-1}x + C$

***Sample* Problem 33** $\int \frac{3x-1}{(x-1)(x-2)(x-3)}dx$ *is equal to* [NCERT]

(a) $\log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

(b) $\log|x-1| - \log|x-2| + 4\log|x-3| + C$

(c) $5\log|x-1| - \log|x-2| + 4\log|x-3| + C$

(d) *None of the above*

Interpret (a) Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$\Rightarrow \frac{3x-1}{(x-1)(x-2)(x-3)}$

$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$\Rightarrow 3x-1 = A[x^2-5x+6] + B[x^2-4x+3] + C[x^2-3x+2]$

$\Rightarrow 3x-1 = x^2(A+B+C) + x(-5A-4B-3C) + (6A+3B+2C)$

On equating the coefficients of x^2, x and constant term on both sides, we get

$A + B + C = 0$ ...(i)

$-5A - 4B - 3C = 3$ ...(ii)

and $6A + 3B + 2C = -1$ ...(iii)

From Eq. (i), we get $A = -(B+C)$

On putting the value of A in Eqs. (ii) and (iii), we get

$-5\{-(B+C)\} - 4B - 3C = 3$

$\Rightarrow \quad 5B + 5C - 4B - 3C = 3$

$\Rightarrow \quad B + 2C = 3$ ...(iv)

and $6\{-(B+C)\} + 3B + 2C = -1$

$\Rightarrow \quad -6B - 6C + 3B + 2C = -1$

$\Rightarrow \quad -3B - 4C = -1$ ...(v)

On solving Eqs. (iv) and (v), we get $C = 4$

On putting the value of C in Eq. (iv), we get

$B + 2 \times 4 = 3$

$\Rightarrow \quad B = -5$

On putting the value of B and C in Eq. (i), we get

$A + (-5) + 4 = 0 \Rightarrow A = 1$

$\therefore \quad A = 1, B = -5, C = 4$

Now, $\int \frac{3x-1}{(x-1)(x-2)(x-3)}dx = \int\left[\frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}\right]dx$

$= \int \frac{1}{(x-1)}dx + \int \frac{(-5)}{(x-2)}dx + \int \frac{4}{(x-3)}dx$

$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C \quad \left(\because \frac{1}{x}dx = \log x\right)$

***Sample* Problem 34** $\int \frac{1-x^2}{x(1-2x)}dx$ *is equal to* [NCERT]

(a) $x + \log x - \frac{3}{4}\log|x-1| + C$

(b) $\frac{x}{2} + \log x + \frac{1}{4}\log|x-1| + C$

(c) $\frac{x}{2} + \log x - \frac{3}{4}\log|2x-1| + C$

(d) *None of the above*

Interpret (c) In rational algebraic functions, if the degree of numerator is greater than or equal to the degree of denominator, then always divide the numerator by denominator and use the result.

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Thus, we can convert the given integrand into simple form and then integrate easily.

Let $I = \int \frac{1-x^2}{x(1-2x)}dx$. Here, degree of numerator is equal to degree of denominator, so divide the numerator by denominator.

Thus, $\frac{1-x^2}{x(1-2x)} = \frac{x^2-1}{2x^2-x} = \frac{1}{2} + \frac{\frac{1}{2}x-1}{2x^2-x}$

$\therefore \quad I = \int \frac{1}{2}dx + \int \frac{\frac{1}{2}x-1}{(2x^2-x)}dx$

$\Rightarrow \quad I = I_1 + I_2$ ...(i)

where, $I_1 = \int \frac{1}{2}dx$

and $I_2 = \int \frac{\frac{1}{2}x-1}{(2x^2-x)}dx$

Now, $I_1 = \int \frac{1}{2}dx = \frac{1}{2}x + C_1$

and $$I_2 = \int \frac{\frac{1}{2}x - 1}{2x^2 - x}\,dx$$

Let $$\frac{\frac{1}{2}x - 1}{(2x^2 - x)} = \frac{A}{x} + \frac{B}{(2x-1)} \Rightarrow \frac{\frac{1}{2}x - 1}{(2x^2 - x)} = \frac{A(2x-1) + Bx}{x(2x-1)}$$

$$\Rightarrow \quad \frac{1}{2}x - 1 = 2Ax - A + Bx$$

$$\Rightarrow \quad \frac{1}{2}x - 1 = x(2A + B) - A$$

On comparing the coefficient of x and constant term on both sides, we get

$$2A + B = \frac{1}{2} \quad \text{...(ii)}$$

and $$-A = -1 \Rightarrow A = 1 \quad \text{...(iii)}$$

From Eq. (iii), $2 \times 1 + B = \frac{1}{2}$

$$\Rightarrow \quad B = \frac{1}{2} - 2 = \frac{-3}{2}$$

$$\therefore \quad I_2 = \int \left[\frac{1}{x} - \frac{3}{2(2x-1)}\right] dx$$

$$= \int \frac{1}{x}\,dx - \frac{3}{2}\int \frac{1}{2x-1}\,dx$$

$$\Rightarrow \quad I_2 = \log x - \frac{3}{2}\,\frac{\log|2x-1|}{2} + C_2$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = \frac{1}{2}x + \log x - \frac{3}{4}\log|2x-1| + C \qquad (C_1 + C_2 = C)$$

Check Point **2**

1. $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}\,dx$ is equal to

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$ (b) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + C$

(c) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$ (d) $\frac{1}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + C$

2. $\int \frac{x\tan^{-1}x}{\sqrt{(1+x^2)^3}}\,dx$ is equal to

(a) $\frac{x - \tan^{-1}x}{\sqrt{1-x^2}} + C$ (b) $\frac{x - \tan^{-1}x}{1-x^2} + C$

(c) $\frac{x + \sqrt{1-x^2}}{\sqrt{1+x^2}} + C$ (d) $\frac{x + \tan^{-1}x}{\sqrt{1-x^2}} + C$

3. $\int \sin(\log x)\,dx$ is equal to

(a) $\frac{x}{2}\{\sin(\log x) + \cos(\log x)\} + C$

(b) $\frac{x}{2}\{\cos(\log x) - \sin(\log x)\} + C$

(c) $\frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$

(d) None of the above

4. If $\int \frac{1}{(\sin x + 4)(\sin x - 1)}\,dx = \frac{A}{\tan\frac{x}{2} - 1} + B\tan^{-1}[f(x)] + C$, then

(a) $A = \frac{1}{5}, B = -\frac{2}{5\sqrt{5}}, f(x) = \frac{4\tan x + 3}{\sqrt{15}}$

(b) $A = -\frac{1}{5}, B = \frac{1}{\sqrt{15}}, f(x) = \frac{4\tan\left(\frac{x}{2}\right) + 1}{\sqrt{15}}$

(c) $A = \frac{2}{5}, B = -\frac{2}{5}, f(x) = \frac{4\tan x + 1}{5}$

(d) $A = \frac{2}{5}, B = -\frac{2}{5\sqrt{15}}, f(x) = \frac{4\tan\frac{x}{2} + 1}{\sqrt{15}}$

5. If $\int \log(\sqrt{1-x} + \sqrt{1+x})\,dx = x\,f(x) + Ax + B\sin^{-1}x + C$, then

(a) $f(x) = \log(\sqrt{1-x} + \sqrt{1+x})$

(b) $A = -\frac{1}{2}$

(c) $B = \frac{1}{2}$

(d) All of the above

6. $\int \frac{2x - 1}{(x+1)(x^2+2)}$ is equal to

(a) $-\ln|x+1| + \frac{1}{2}\ln|x^2+1| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(b) $\ln|x-1| + \frac{1}{2}\ln|x^2+1| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(c) $\ln|x+1| + \frac{1}{2}\ln|x^2-1| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(d) None of the above

30.7 Integration using Trigonometric Identities

To evaluate integrals of the form $\int \sin mx \cdot \cos nx\, dx$, $\int \sin mx \cdot \sin nx\, dx$, $\int \cos mx \cdot \cos nx\, dx$ and $\int \cos mx \cdot \sin nx\, dx$, we use the following trigonometrical identities

(i) $2 \sin A \cdot \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \cdot \sin B = \sin (A + B) - \sin (A - B)$

(iii) $2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \cdot \sin B = \cos (A - B) - \cos (A + B)$

(v) $2 \sin A \cdot \cos A = \sin 2A$

(vi) $\cos^2 A = \left(\frac{1 + \cos 2A}{2}\right)$ (vii) $\sin^2 A = \left(\frac{1 - \cos 2A}{2}\right)$

(viii) $\cos^2 A - \sin^2 A = \cos 2A$ (ix) $\sin^2 A + \cos^2 A = 1$

***Sample* Problem 35** $\int \cos 4x \cos 7x\, dx$ *is equal to*

(a) $\sin 3x + \sin 11x + C$ (b) $\frac{\sin 3x}{3} + \frac{\sin 11x}{11} + C$

(c) $\frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$ (d) *None of these*

Interpret (c) Let $I = \int \cos 4x \cos 7x\, dx$

Here, $\cos 4x \cos 7x = \frac{1}{2}(\cos 3x + \cos 11x)$

$$\therefore \quad I = \int \cos 4x \cos 7x\, dx = \frac{1}{2}\int (\cos 3x + \cos 11x)\, dx$$

$$= \frac{1}{2}\int \cos 3x\, dx + \frac{1}{2}\int \cos 11x\, dx = \frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$$

30.8 Some Special Integrals

(i) $\int e^{ax+b}[af(x) + f'(x)]\, dx = e^{ax+b} f(x) + C$

***Sample* Problem 36** $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx$ *is equal to*

[NCERT Exemplar]

(a) $e^x \sec \frac{x}{2} + C$ (b) $\frac{e^x}{1 + \cos x} + C$

(c) $e^x \tan \frac{x}{2} + C$ (d) *None of these*

Interpret (c) $\int e^x \left(\frac{1}{1 + \cos x} + \frac{\sin x}{1 + \cos x}\right) dx$

$$= \int e^x \left(\frac{1}{2\cos^2 \frac{x}{2}} + \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}}\right) dx$$

$(\because \cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2 \sin x \cos x)$

$$= \int e^x \left(\frac{\sec^2 \frac{x}{2}}{2} + \tan \frac{x}{2}\right) dx = \int e^x \left(\tan \frac{x}{2} + \frac{1}{2}\sec^2 \frac{x}{2}\right) dx$$

Let $f(x) = \tan \frac{x}{2} \Rightarrow f'(x) = \frac{\sec^2 \frac{x}{2}}{2}$

$$\therefore \int e^x \left(\tan \frac{x}{2} + \frac{\sec^2 \frac{x}{2}}{2}\right) dx = e^x \tan \frac{x}{2} + C$$

$[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x)]$

(ii) (a) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}$

$$(a \sin bx - b \cos bx) + C = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + C$$

(b) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}$

$$(a \cos bx + b \sin bx) + C = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left(bx - \tan^{-1}\frac{b}{a}\right) + C$$

***Sample* Problem 37** $\int e^{2x} \sin x\, dx$, *is equal to* **[NCERT]**

(a) $\frac{e^x}{5}(\sin x - \cos x) + C$ (b) $\frac{e^{2x}}{5}(2 \sin x - \cos x) + C$

(c) $\frac{e^{2x}}{5}(2 \cos x - \sin x) + C$ (d) $\frac{e^x}{5}(2 \cos x - \sin x) + C$

Interpret (b) Let $I = \int e^{2x} \sin x\, dx$

On taking $\sin x$ as first function and e^{2x} as second function and integrating by parts, we get

$$\int e^{2x} \sin x\, dx = \sin x \cdot \int e^{2x}\, dx - \int \left[\frac{d}{dx}(\sin x) \cdot \int e^{2x}\, dx\right] dx$$

$$= \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2}\, dx$$

$$= \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x} \cos x\, dx$$

$$\Rightarrow \quad \int e^{2x} \sin x\, dx = \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} I_1 \quad \text{...(i)}$$

where, $I_1 = \int e^{2x} \cos x\, dx$

On taking $\cos x$ as first function and e^{2x} as second function and integrating by parts, we get

$$I_1 = \cos x \cdot \int e^{2x}\, dx - \int \left[\frac{d}{dx}(\cos x) \cdot \int e^{2x}\, dx\right] dx$$

$$= \cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \cdot \frac{e^{2x}}{2}\, dx + C_1$$

$$= \frac{e^{2x} \cos x}{2} + \frac{1}{2}\int e^{2x} \sin x\, dx + C_1$$

On putting value of I_1 in Eq. (i), we get

$$\int e^{2x} \sin x\, dx = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\frac{e^{2x} \cos x}{2} + \frac{1}{2}\int e^{2x} \sin x\, dx + C_1\right]$$

$$\Rightarrow \int e^{2x} \sin x\, dx = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4}\int e^{2x} \sin x\, dx - \frac{1}{2} C_1$$

$$\Rightarrow \int e^{2x} \sin x\, dx + \frac{1}{4}\int e^{2x} \sin x\, dx = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{2} C_1$$

$$\Rightarrow \frac{5}{4}\int e^{2x} \sin x\, dx = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{2} C_1$$

$$\Rightarrow \int e^{2x} \sin x\, dx = \frac{4}{5}\left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{2} C_1\right]$$

$$= \frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x - \frac{2}{5} C_1$$

$$= \frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x + C$$

where, $C = -\frac{2}{5} C_1 = \frac{e^{2x}}{5}(2 \sin x - \cos x) + C$

30.9 Reduction Formulae

(i) $I_n = \int \tan^n x dx = \int \tan^2 x \cdot \tan^{n-2} dx$

$$= \int (\sec^2 x - 1) \cdot \tan^2 x\, dx$$

$$\Rightarrow I_n = \int \sec^2 x \cdot \tan^{n-2} x\, dx - I_{n-2}$$

$$\Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, n \geq 2$$

(ii) $I_n = \int \cot^n x\, dx = \int \cot^2 x \cdot \cot^{n-2} x\, dx$

$$= \int (\text{cosec}^2 x - 1) \cot^{n-2} x\, dx$$

$$I_n = \int \text{cosec}^2 x \cdot \cot^{n-2} x \cdot dx - I_{n-2}$$

$$\Rightarrow I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}, n \geq 2$$

(iii) $I_n = \int \sec^n x dx = \int \sec^2 x \cdot \sec^{n-1} x\, dx$

$$\Rightarrow I_n = \tan x \cdot \sec^{n-2} x - \int (\tan x)(n-2) \sec^{n-2} x \sec x \cdot \tan x\, dx$$

$$\Rightarrow I_n = \tan x \cdot \sec^{n-2} x - (n-2)\int (\sec^2 x - 1) \cdot \sec^{n-2} x\, dx$$

$$\Rightarrow (n-1) I_n = \tan x \cdot \sec^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{\tan x \cdot \sec^{n-2} x}{n-1} + \frac{(n-2)}{(n-1)} \cdot I_{n-2}$$

(iv) $I_n = \int \text{cosec}^n x\, dx = \int \text{cosec}^2 x \cdot \text{cosec}^{n-2} x\, dx$

$$\Rightarrow I_n = -\cot x \cdot \text{cosec}^{(n-2)} x + \int (\cot x)(n-2)(-\text{cosec}^{n-3} x \cdot \text{cosec} x \cdot \cot) x\, dx$$

$$\Rightarrow I_n = \cot x \cdot \text{cosec}^{n-2} x - (n-2)\int \cot^2 \cdot x \text{cosec}^{n-2} x\, dx$$

$$\Rightarrow I_n = -\cot x \cdot \text{cosec}^{n-2} x - (n-2) \int (\text{cosec}^2 (x-1) \cdot \text{cosec}^{n-2} x\, dx$$

$$\Rightarrow (n-1) I_n = -\cot x \cdot \text{cosec}^{n-2} x + (n-2) \cdot I_{n-1}$$

$$I_n = \frac{\cot x \cdot \text{cosec}^{n-2} x}{-(n-1)} + \frac{(n-2)}{(n-1)} \cdot I_{n-2}$$

(v) $\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

(vi) $\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

(vii) $\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

(viii) $\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} \cdot e^{ax} dx$

(ix) $\int (\log x)_n dx = x(\log x)^n - n\int (\log x)^{n-1} dx$

***Sample* Problem 38** *If* $I_n = \int \tan^n x\, dx$, *then* $(n-1)(I_n + I_{n-2})$ *is equal to*

(a) $\tan^n x$ (b) $\cot^n x$

(c) $\tan^{n-1} x$ (d) $\cot^{n-1} x$

Interpret (c) Here, $I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx$

$$= \int \tan^{n-2} (\sec^2 x - 1)\, dx = \int \tan^{n-2} x dx - \int \tan^{n-2} x dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - I_{n-2}$$

$$I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence, $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$

WORKED OUT Examples

Example 1 $\int \frac{1}{x^{1/2}+x^{1/3}}dx$ *is equal to* [NCERT]

(a) $x^{1/2}-3x^{1/3}-6x^{1/6}+6log|x^{1/6}+1|+C$

(b) $2x^{1/2}+3x^{1/3}-x^{1/6}-6log|x^{1/6}+1|+C$

(c) $2x^{1/2}-3x^{1/3}+6x^{1/6}-6log|x^{1/6}+1|+C$

(d) *None of the above*

Solution (c) Let $I=\int \frac{1}{x^{1/2}+x^{1/3}}dx$

Put $x^{1/6}=t \Rightarrow x=t^6$

$\Rightarrow$ $dx=6t^5dt$

$\therefore$ $I=\int \frac{6t^5}{(t^6)^{1/2}+(t^6)^{1/3}}dt=6\int \frac{t^5}{t^3+t^2}dt$

$$=6\int \frac{t^5}{t^2(t+1)}dt=6\int \frac{t^3}{(t+1)}dt=6\int\left(\frac{t^3+1-1}{t+1}\right)dt$$

$$=6\int\left[\frac{(t+1)(t^2-t+1)}{t+1}-\frac{1}{t+1}\right]dt=6\int\left\{t^2-t+1+\frac{-1}{t+1}\right\}dt$$

$$=6\left\{\frac{t^3}{3}-\frac{t^2}{2}+t-\log|t+1|\right\}+C$$

$$=6\left\{\frac{(x^{1/6})^3}{3}-\frac{(x^{1/6})^2}{2}+(x^{1/6})-\log|x^{1/6}+1|\right\}+C \qquad (\because t=x^{1/6})$$

$$=2x^{1/2}-3x^{1/3}+6x^{1/6}-6\log|x^{1/6}+1|+C$$

Example 2 $\int \frac{\sin x}{\sin(x-a)}dx$ *is equal to* [NCERT Exemplar]

(a) $x+\sin a\cos a\, log|\sin(x-a)|+C$

(b) $\cos ax+\sin a\, log|\sin(x-a)|+C$

(c) $\sin ax+\cos a\, log|\sin(x-a)|+C0$

(d) *None of the above*

Solution (b) $\int \frac{\sin x}{\sin(x-a)}dx=\int \frac{\sin\{(x-a)+a\}}{\sin(x-a)}dx$

$$=\int \frac{\sin(x-a)\cos a+\cos(x-a)\sin a}{\sin(x-a)}dx$$

$[\because \sin(A+B)=\sin A\cos B+\cos A\sin B]$

$$=\int \frac{\sin(x-a)\cos a}{\sin(x-a)}dx+\int \frac{\cos(x-a)\sin a}{\sin(x-a)}dx$$

$$=\cos a\int 1\,dx+\sin a\int \cot(x-a)\,dx$$

$$=(\cos a)\,x+\sin a\cdot\log|\sin(x-a)|+C$$

Example 3 $\int (\sin^{-1}x)^2dx$ *is equal to* [NCERT]

(a) $(\sin^{-1}x)^2+2\sqrt{1-x^2}-2x+C$

(b) $x(\sin^{-1}x)^2+2\sqrt{1-x^2}-2x+C$

(c) $x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x+C$

(d) $x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$

Solution (d) Let $I=\int(\sin^{-1}x)^2dx$

Put $\sin^{-1}x=\theta$

$\Rightarrow$ $x=\sin\theta$

$\Rightarrow$ $dx=\cos\theta\, d\theta$

$\therefore$ $I=\int(\sin^{-1}x)^2dx=\int\theta^2\cos\theta\, d\theta$

On taking θ^2 as first function and $\cos\theta$ as second function and integrating by parts, we get

$$=\theta^2\int\cos\theta\, d\theta-\int\left[\frac{d}{d\theta}(\theta^2)\int\cos\theta\, d\theta\right]d\theta$$

$$=\theta^2\sin\theta-\int 2\theta\sin\theta\, d\theta$$

Again, integrating by parts, we get

$$I=\theta^2\sin\theta-2[\theta(-\cos\theta)-\int 1(-\cos\theta)\,d\theta]+C$$

$$=\theta^2\sin\theta+2\theta\cos\theta-2\int\cos\theta\, d\theta+C$$

$$=\theta^2\sin\theta+2\theta\cos\theta-2\sin\theta+C$$

$$=(\sin^{-1}x)^2x+2\sin^{-1}x\sqrt{1-\sin^2\theta}-2x+C$$

(put $\theta=\sin^{-1}x$ and $\sin\theta=x$)

$$=x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Example 4 $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$ *is equal to* [NCERT]

(a) $log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right|+C$

(b) $log\left|\left(x+\frac{a-b}{2}\right)+\sqrt{(x+a)(x+b)}\right|+C$

(c) $log\left|\left(x+\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right|+C$

(d) *None of the above*

Solution (*a*) Let $I = \int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx = \int \frac{1}{\sqrt{x^2-(a+b)x+ab}}\,dx$

$$= \int \frac{1}{\sqrt{\left[x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2+ab\right]}}\,dx$$

$$= \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2+ab-\left(\frac{a+b}{2}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2+ab-\left(\frac{a^2+b^2+2ab}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2+\frac{4ab-a^2-b^2-2ab}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2+\frac{2ab-a^2-b^2}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\left(\frac{a^2+b^2-2ab}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}}\,dx$$

Let $x-\frac{a+b}{2} = t \Rightarrow dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}\,dt$$

$$= \log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right| + C$$

$$\left(\because \int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|\right)$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}\right| + C$$

$$\left[\because t = x-\frac{(a+b)}{2}\right]$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right| + C$$

Example 5 $\int \frac{1}{\sin x \cos^3 x}\,dx$ *is equal to* **[NCERT]**

(*a*) $\log\,|\tan x| + \tan^2 x + C$ (*b*) $\frac{1}{2}\log|\tan x| + \tan^2 x + C$

(*c*) $\log|\tan x| + \frac{1}{2}\tan^2 x + C$ (*d*) *None of these*

Solution (*c*) Here, we can convert the given integral in the terms of $\tan x$ and $\sec^2 x$. So, put $\tan x = t$ to convert integral in t and then integrate easily.

Let $$I = \int \frac{1}{\sin x \cos^3 x}\,dx = \int \frac{\cos x}{\sin x}\sec^4 x\,dx$$

$$= \int \frac{\sec^2 x \sec^2 x}{\tan x}\,dx = \int \frac{(1+\tan^2 x)\sec^2 x}{\tan x}\,dx$$

$(\because \sec^2 x = 1+\tan^2 x)$

Putting $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{\sec^2 x}$

$$\therefore \quad I = \int\left[\frac{1+t^2}{t}\right]\sec^2 x\,\frac{dt}{\sec^2 x} = \int \frac{1+t^2}{t}\,dt = \int\left[\frac{1}{t}+t\right]dt$$

$$= \log|t| + \frac{t^2}{2} + C$$

$$\Rightarrow \quad \log|\tan x| + \frac{\tan^2 x}{2} + C$$

Example 6 $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$ *is equal to* **[NCERT Exemplar]**

(*a*) $\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C$

(*b*) $\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C$

(*c*) $\frac{1}{\sin(b-a)}\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C$

(*d*) *None of the above*

Solution (*c*) $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\cos(x-a)\cos(x-b)}\,dx$$

On multiplying by $\sin(b-a)$ in numerator and denominator both and

$\sin(b-a) = \sin[(x-a)-(x-b)]$.

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\,dx$$

$[\because \sin(C-D) = \sin C\cos D - \cos C\sin D]$

$$= \frac{1}{\sin(b-a)}\int\left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\right]dx$$

$$= \frac{1}{\sin(b-a)}\int\{\tan(x-a)-\tan(x-b)\}\,dx$$

$$= \frac{1}{\sin(b-a)}\cdot\{-\log|\cos(x-a)| + \log|\cos(x-b)|\} + C$$

$(\because \int \tan x\,dx = -\log|\cos x|)$

$$= \frac{1}{\sin(b-a)}\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C$$

Example 7 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$ *is equal to* [NCERT Exemplar]

(a) $\left[\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\right]+C$

(b) $a\left[\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\right]+C$

(c) $a\left[\tan^{-1}\sqrt{\frac{x}{a}}\cdot\frac{(a+x)}{a}\right]+C$

(d) $a\left[\tan^{-1}\sqrt{\frac{x}{a}}\cdot\frac{(a+x)}{a}-\sqrt{\frac{x}{a}}\right]+C$

Solution (d) Let $I=\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Put $x=a\tan^2\theta \Rightarrow dx=2a\tan\theta\sec^2\theta\, d\theta$

Now, $I=\int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\cdot 2a\tan\theta\sec^2\theta\, d\theta$

$=\int \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\cdot 2a\tan\theta\sec^2\theta\, d\theta$ $(\because 1+\tan^2\theta=\sec^2\theta)$

$=\int \sin^{-1}(\sin\theta)\, 2a\tan\theta\sec^2\theta\, d\theta$

$=2a\int \underset{\mathrm{I}}{\theta}\,(\underset{\mathrm{II}}{\tan\theta\sec^2\theta})\, d\theta$ $[\because \sin^{-1}(\sin\theta)=\theta]$

$=2a\left[\theta\int(\tan\theta\sec^2\theta)\, d\theta-\int\left\{\frac{d}{d\theta}(\theta)\int \tan\theta\sec^2\theta\, d\theta\right\}d\theta\right]$

$=2a\left[\theta\frac{\sec^2\theta}{2}-\int 1\frac{\sec^2\theta}{2}d\theta\right]+C$

$\left(\because 1+\tan^2\theta=\sec^2\theta \text{ and } \sec^2\theta=1+\frac{x}{a}\right)$

$\left(\int \tan\theta\sec^2\theta=\int z\, dz=\frac{z^2}{2}=\frac{\sec^2\theta}{2}, \text{ where } z=\sec\theta\right)$

$=a[\theta\sec^2\theta-\tan\theta]+C=a\left[\tan^{-1}\sqrt{\frac{x}{a}}\cdot\frac{a+x}{a}-\sqrt{\frac{x}{a}}\right]+C$

Example 8 $\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$ *is equal to* [NCERT]

(a) $-\left(\frac{\sin 2x}{2}+\sin x\right)+C$ (b) $\left(\frac{\sin 2x}{2}+\cos x\right)+C$

(c) $\left(\frac{\sin 2x}{2}+\sin x\right)+C$ (d) *None of these*

Solution (a) $\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx=\int \frac{\sin 3x(\cos 5x+\cos 4x)}{\sin 3x-2\cos 3x\sin 3x}dx$

(multiplying numerator and denominator by $\sin 3x$)

$=\int \frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}\cdot 2\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x-\sin 6x}$

$\left(\because \sin 3x=2\sin\frac{3x}{2}\cos\frac{3x}{2} \text{ and } \cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right)$

$=4\int \frac{\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}dx$

$\left[\because \sin C-\sin D=2\sin\frac{C-D}{2}\cos\frac{C+D}{2} \text{ and } \sin(-\theta)=-\sin\theta\right]$

$=-\int 2\cos\frac{3x}{2}\cos\frac{x}{2}dx=-\int(\cos 2x+\cos x)\, dx$

$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$

$=-\left(\frac{\sin 2x}{2}+\sin x\right)+C$

Example 9 $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ *is equal to* [NCERT]

(a) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\cdot\left[\log\left(\frac{x^2+1}{x^2}\right)-\frac{2}{3}\right]+C$

(b) $-\left(1+\frac{1}{x^2}\right)^{1/2}\cdot\left[\log\left(\frac{x^2+1}{x^2}\right)-\frac{2}{3}\right]+C$

(c) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{1/2}\cdot\left[\log\left(\frac{x^2+1}{x^2}\right)-\frac{1}{3}\right]+C$

(d) *None of the above*

Solution (a) Let $I=\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$

$=\int \frac{\sqrt{x^2+1}}{x^4}[\log(x^2+1)-\log x^2]dx$

$=\int \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(1+\frac{1}{x^2}\right)\right]dx=\int \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]dx$

Put $\frac{x^2+1}{x^2}=t\Rightarrow 1+\frac{1}{x^2}=t \Rightarrow -\frac{2}{x^3}dx=dt\Rightarrow dx=\frac{-x^3}{2}dt$

$\therefore$ $I=\int\sqrt{t}\log t\cdot\frac{1}{x^3}\left(-\frac{x^3dt}{2}\right)$

$=-\frac{1}{2}\int(t)^{1/2}\log t\, dt$

On integrating by parts, we get

$I=-\frac{1}{2}\left[\log t\cdot\int t^{1/2}dt-\int\left\{\left(\frac{d}{dt}\log t\right)\int t^{1/2}dt\right\}dt\right]$

$=-\frac{1}{2}\left[\log t\cdot\frac{t^{3/2}}{\frac{3}{2}}-\int\frac{1}{t}\cdot\frac{t^{3/2}}{\frac{3}{2}}dt\right]=-\frac{1}{2}\left[\frac{2}{3}\log t\cdot t^{3/2}-\frac{2}{3}\int t^{1/2}dt\right]$

$=-\frac{1}{2}\times\frac{2}{3}[\log t\cdot t^{3/2}-\int t^{1/2}dt]$

$=-\frac{1}{3}\left[\log t\cdot t^{3/2}-\frac{t^{3/2}}{\frac{3}{2}}\right]+C=-\frac{1}{3}t^{3/2}\left[\log t-\frac{2}{3}\right]+C$

$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(\frac{x^2+1}{x^2}\right)-\frac{2}{3}\right]+C$

Example 10 $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$ *is equal to* [**NCERT**]

(a) $\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$ (b) $\frac{-1}{\sin\alpha}\sqrt{\frac{\sin x}{\sin(x+\alpha)}} + C$

(c) $\frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$ (d) *None of these*

Solution (c) Let $I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$

$$= \int \frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha + \cos x \sin\alpha)}} dx$$

$[\because \sin(A+B) = \sin A\cos B + \cos A \sin B]$

$$= \int \frac{1}{\sqrt{\sin^4 x(\cos\alpha + \cot x \sin\alpha)}} dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

Put $\cos\alpha + \cot x \sin\alpha = t$

$\Rightarrow$ $-\operatorname{cosec}^2 x \sin\alpha\, dx = dt$

$\Rightarrow$ $dx = \frac{dt}{-\operatorname{cosec}^2 x \sin\alpha}$

$\therefore$ $I = \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}} = \frac{-1}{\sin\alpha}[2\sqrt{t}] + C$

$$= \frac{-1}{\sin\alpha}[2\sqrt{\cos\alpha + \cot x \sin\alpha}] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

$[\because \sin(A+B) = \sin A\cos B + \cos A \sin B]$

Example 11 $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$ *is equal to* [**NCERT**]

(a) $\frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] - x + C$

(b) $\frac{\pi}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] + C$

(c) $\frac{\pi}{2}[\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] - x + C$

(d) *None of the above*

Solution (a) Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$\Rightarrow$ $\cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$

$\therefore$ $I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$

$$= \int \frac{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} dx = \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int 1\, dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x$$

$\Rightarrow$ $I = \frac{4}{\pi} I_1 - x + C$ ...(i)

where, $I_1 = \int \sin^{-1}\sqrt{x}\, dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt$

$$I_1 = \int \sin^{-1} t\, 2t\, dt = 2\int \underset{\text{I}}{\sin^{-1} t} \cdot \underset{\text{II}}{t}\, dt$$

$$= 2\left[\sin^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt\right]$$

[using integration by part I, $\int uv dx = u\int v dx - \int\left(\frac{d}{dx}u \int v\, dx\right)dx$]

$$= t^2 \sin^{-1} t - \int \frac{t^2}{\sqrt{1-t^2}} dt = t^2 \sin^{-1} t - \int \frac{-(1-t^2)+1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \int \sqrt{1-t^2}\, dt - \int \frac{1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1} t - \sin^{-1} t$$

$$= \left(t^2 - \frac{1}{2}\right)\sin^{-1} t + \frac{1}{2} t\sqrt{1-t^2}$$

$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}]$$

$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}]$$

On putting the value of I_1 in Eq.(i), we get

$$\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$$

$$= \frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] - x + C$$

Example 12 $\int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x} dx$ *is equal to* [**NCERT**]

(a) $\sqrt{\tan x} + C$ (b) $\sqrt{\cot x} + C$

(c) $2\sqrt{\tan x} + C$ (d) $2\sqrt{\cot x} + C$

Solution (c) $\int \frac{\sqrt{\tan x}}{\sin x\cos x} dx = \int \frac{\sqrt{\tan x}}{\sin x\cos x \times \frac{\cos x}{\cos x}} dx$

$\left(\text{multiply by } 1 = \frac{\cos x}{\cos x} \text{ in denominator}\right)$

$$= \int \frac{\sqrt{\tan x}}{\tan x \cdot \cos^2 x} dx = \int \frac{\sqrt{\tan x}}{\tan x} \cdot \sec^2 x \, dx$$

Let $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow$ $dx = \frac{dt}{\sec^2 x}$

$\therefore$ $\int \frac{\sqrt{\tan x}}{\tan x} \cdot \sec^2 x \, dx = \int \frac{\sqrt{t}}{t} \cdot \sec^2 x \frac{dt}{\sec^2 x}$

$$= \int \frac{1}{\sqrt{t}} dt = \int t^{-1/2} dt = \frac{t^{-1/2+1}}{\left(-\frac{1}{2}+1\right)} + C$$

$$= 2\sqrt{t} + C = 2\sqrt{\tan x} + C$$

Example 13 $\int \frac{1}{(1+\cot x)} dx$ *is equal to* [**NCERT**]

(*a*) $\frac{1}{2}[x + \log|\sin x - \cos x|] + C$

(*b*) $\frac{1}{2}[x - \log|\sin x + \cos x|] + C$

(*c*) $\frac{1}{2}\left[\frac{x}{2} - \log|\sin x + \cos x|\right] + C$

(*d*) *None of the above*

Solution (*b*) Here, the given integrand cannot integrate directly. So, use the trigonometric ratios to convert it into simplest form and then apply substitution method to integrate it.

Let $I = \int \frac{1}{(1+\cot x)} dx = \int \frac{1}{1 + \frac{\cos x}{\sin x}} dx$

$$= \int \frac{1}{\frac{\sin x + \cos x}{\sin x}} dx = \int \frac{\sin x}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\int \frac{2(\sin x)}{\sin x + \cos x} dx = \frac{1}{2}\int \frac{\sin x + \sin x + \cos x - \cos x}{\sin x + \cos x} dx$$

(add and subtract $\cos x$ in numerator)

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\left[\int \frac{(\sin x + \cos x)}{(\sin x + \cos x)} dx + \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)} dx\right]$$

$$= \frac{1}{2}\left[\int 1 dx + \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)} dx\right]$$

Let $\sin x + \cos x = t$

$\Rightarrow$ $\cos x - \sin x = \frac{dt}{dx}$

$\Rightarrow$ $dx = \frac{dt}{-[\sin x - \cos x]}$

$\therefore$ $I = \frac{1}{2}\left[\int 1 dx + \int \frac{(\sin x - \cos x)}{t} \frac{dt}{-[\sin x - \cos x]}\right]$

$$= \frac{1}{2}\left[\int 1 dx - \int \frac{1}{t} dt\right]$$

$$= \frac{1}{2}[x - \log|t|] + C$$

$$= \frac{1}{2}[x - \log|\sin x + \cos x|] + C$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Integral as an Anti-derivative

1. If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then, $f(x)$ is [NCERT]

(a) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (b) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(c) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (d) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

2. If $\int f(x)\, dx = f(x) + c$, then $\int \{f(x)\}^2\, dx$ is

(a) $\{f(x)\}^2 + K$ (b) $\frac{1}{3}\{f(x)\}^3 + K$

(c) $f(x) + K$ (d) $\frac{1}{2}\{f(x)\}^2 + K$

3. If the primitive of $\frac{1}{f(x)}$ is equal to $\log\{f(x)\}^2 + C$, then $f(x)$ is

(a) $x + d$ (b) $\frac{x}{2} + d$ (c) $\frac{x^2}{2} + d$ (d) $x^2 + d$

Fundamental Integration

4. $\int x\sqrt{x+2}\, dx$ is equal to [NCERT]

(a) $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

(b) $\frac{1}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

(c) $\frac{2}{5}(x+2)^{5/2} - \frac{2}{3}(x+2)^{3/2} + C$

(d) None of the above

5. $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\, dx$ is equal to [NCERT Exemplar]

(a) $\frac{x}{2} + C$ (b) $\frac{x^2}{2} + C$

(c) $\frac{x^3}{3} + C$ (d) $\frac{x^4}{2} + C$

6. $\int (1 - \cos x)\operatorname{cosec}^2 x\, dx$ is equal to

(a) $\tan x + C$ (b) $\tan\frac{x}{2} + C$

(c) $\frac{1}{2}\tan\frac{x}{2} + C$ (d) None of these

Rules of Integration

7. $\int \frac{1}{x - \sqrt{x}}\, dx$ is equal to [NCERT]

(a) $2\log|\sqrt{x}+1| + C$ (b) $\log|\sqrt{x}+1| + C$

(c) $2\log|\sqrt{x}-1| + C$ (d) $2\log|x+1| + C$

8. $\int (x^3 - 1)^{1/3} \cdot x^5\, dx$ is equal to [NCERT]

(a) $\frac{1}{7}(x^3-1)^{3/7} + \frac{1}{4}(x^3-1)^{3/4} + C$

(b) $\frac{1}{7}(x^3-1)^{7/3} + \frac{1}{4}(x^3-1)^{4/3} + C$

(c) $\frac{3}{7}(x^3-1)^{7/3} + \frac{3}{4}(x^3-1)^{4/3} + C$

(d) None of the above

9. $\int \frac{dx}{x(\log x)^m}$ is equal to [NCERT Exemplar]

(a) $\frac{(\log x)^m}{m} + C$ (b) $\frac{(\log x)^{m-1}}{m-1} + C$

(c) $\frac{(\log x)^{1-m}}{1-m} + C$ (d) $\frac{(\log x)^{1-m}}{m} + C$

10. $\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$ is equal to

(a) $-e^{\tan^{-1}x} + C$ (b) $e^{\tan^{-1}x} + C$

(c) $\tan^{-1} x + C$ (d) $-\tan^{-1} x + C$

11. $\int \frac{e^{2x} - 1}{e^{2x} + 1}\, dx$ is equal to [NCERT]

(a) $\log|e^x + e^{-x}| + C$ (b) $\log|e^x - e^{-x}| + C$

(c) $\log|e^{2x} + 1| + C$ (d) $\log|1 - e^{2x}| + C$

12. $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$ is equal to [NCERT Exemplar]

(a) $\log|e^{2x} - e^{-2x}| + C$ (b) $\frac{1}{2}\log|e^{2x} + e^{-2x}| + C$

(c) $\log|e^{2x} + e^{-2x}| + C$ (d) None of these

13. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$ is equal to [NCERT]

(a) $\frac{1}{2}(\sin^{-1} x) + C$ (b) $(\sin^{-1} x)^2 + C$

(c) $\frac{1}{2}(\sin^{-1} x)^2 + C$ (d) $\frac{-1}{2}\sin^{-1} x + C$

14. $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$ is equal to [NCERT]

(a) $\frac{1}{2}\log|3\cos x + 2\sin x| + C$

(b) $\frac{1}{2}\log|3\sin x + 2\cos x| + C$

(c) $\log|3\cos x + 2\sin x| + C$

(d) None of the above

15. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$ is equal to [NCERT]

(a) $2\cos\sqrt{x} + C$ (b) $2\sin\sqrt{x} + C$

(c) $\frac{1}{2}\sin\sqrt{x} + C$ (d) None of these

16. $\int \cot x \cdot \log\sin x \cdot dx$ is equal to [NCERT]

(a) $2\cot x + C$ (b) $2\log\sin x + C$

(c) $2(\cot x)^2 + C$ (d) $\frac{1}{2}(\log\sin x)^2 + C$

17. $\int \frac{dx}{1 - \tan x}$ is equal to [NCERT]

(a) $\frac{1}{2}[x - \log|\cos x - \sin x|] + C$

(b) $2[x + \log|\cos x - \sin x|] + C$

(c) $x - \log|\cos x + \sin x| + C$

(d) None of the above

18. $\int \frac{x^3 \sin(\tan^{-1} x^4)}{(1+x^8)} dx$ is equal to [NCERT]

(a) $\frac{1}{4}\sin(\cot^{-1} x^4) + C$ (b) $\frac{-1}{4}\cos(\tan^{-1} x^4) + C$

(c) $\frac{1}{4}\cos(\cot^{-1} x^4) + C$ (d) None of these

19. $\int \left(\frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\right) dx$ is equal to [NCERT]

(a) $10^x + x^{10} + C$

(b) $10^x + x^{10} + C$

(c) $(10^x - x^{10})^{-1} + C$

(d) $\log|10^x + x^{10}| + C$

20. $\int \sin^3(2x+1)\, dx$ is equal to [NCERT]

(a) $-\frac{1}{2}\cos(2x+1) + \frac{\cos^3(2x+1)}{6} + C$

(b) $-\cos(2x+1) + \frac{1}{3}\cos^3(2x+1) + C$

(c) $\frac{1}{2}\cos(2x+1) - \frac{1}{3}\cos^3(2x+1) + C$

(d) None of the above

21. $\int \frac{\cos x - \sin x}{1 + \sin 2x} dx$ is equal to [NCERT]

(a) $\sin x + \cos x + C$ (b) $-(\sin x + \cos x) + C$

(c) $-\frac{1}{(\sin x + \cos x)} + C$ (d) None of these

22. $\int \tan^3 2x \cdot \sec 2x\, dx$ is equal to [NCERT]

(a) $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$ (b) $\frac{1}{2}\sec^3 2x - \sec 2x + C$

(c) $\frac{1}{6}\sec^2 2x - \frac{1}{2}\sec 2x + C$ (d) None of these

23. $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ is equal to [NCERT]

(a) $-\cot(e^x x) + C$

(b) $\tan(xe^x) + C$

(c) $\tan(e^x) + C$

(d) $\cot(e^x) + C$

24. $\int \frac{3x}{1+2x^4} dx$ is equal to [NCERT]

(a) $\frac{3}{2}\tan^{-1}(\sqrt{2}x^2) + C$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{3}{2}x^2\right) + C$

(c) $\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{x^2}{\sqrt{2}}\right) + C$ (d) $\frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C$

25. $\int \frac{\sec^2 x}{\tan^2 x + 4} dx$ is equal to [NCERT]

(a) $\log|\tan x + \sqrt{\tan^2 x + 4}| + C$

(b) $\frac{1}{2}\log|\tan x + \sqrt{\tan^2 x + 4}| + c$

(c) $-\log|\tan x + \sqrt{\tan^2 x + 4}| + C$

(d) None of the above

Integration by Substitution

26. $\int \frac{dx}{e^x - 1}$ is equal to [NCERT]

(a) $-\log\left|\frac{e^x - 1}{e^x}\right| + C$ (b) $\log\left|\frac{e^x}{e^x - 1}\right| + C$

(c) $\log\left|\frac{e^x - 1}{e^{-x}}\right| + C$ (d) $\log\left|\frac{e^x - 1}{e^x}\right| + C$

27. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ is equal to [**NCERT**]

(a) $(1+x^{-4})^{1/4}+C$ (b) $(1+x^4)^{1/4}+C$

(c) $-(1+x^4)^{1/4}+C$ (d) $-(1+x^{-4})^{1/4}+C$

28. $\int f'(ax+b)[f(ax+b)]^n\,dx$ is equal to [**NCERT**]

(a) $\frac{[f(ax+b)]^{n+1}}{n+1}+C$ (b) $\frac{1}{a}\frac{[f(ax+b)]^n}{n}+C$

(c) $\frac{1}{a}\frac{[f(ax+b)]^{n+1}}{n+1}+C$ (d) $\frac{1}{(n+1)}\frac{[f(ax+b)]^{n+1}}{n}+C$

29. $\int \frac{1}{e^x+e^{-x}}\,dx$ is equal to [**NCERT**]

(a) $\tan^{-1} e^x+C$ (b) $\tan^{-1} e^{-x}+C$

(c) $\log(e^x-e^{-x})+C$ (d) $\log(e^x+e^{-x})+C$

30. $\int \frac{\sqrt{x+1}}{x^4}\,dx$ is equal to [**NCERT**]

(a) $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$ (b) $\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

(c) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$ (d) $\frac{3}{2}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

31. $\int \frac{dx}{e^x+e^{-x}+2}$ is equal to

(a) $\frac{1}{e^x+1}+C$ (b) $-\frac{1}{e^x+1}+C$

(c) $\frac{1}{1+e^{-x}}+C$ (d) None of these

32. $\int \sqrt{\frac{x}{1-x^3}}\,dx$ is equal to [**NCERT Exemplar**]

(a) $\frac{2}{3}\sin^{-1}(x^{2/3})+C$ (b) $\frac{3}{2}\sin^{-1}(x^{3/2})+C$

(c) $\frac{3}{2}\sin^{-1}(x^{2/3})+C$ (d) $\frac{2}{3}\sin^{-1}(x^{3/2})+C$

33. $\int \frac{x^2}{(x\sin x+\cos x)^2}$ is equal to [**NCERT Exemplar**]

(a) $\frac{x\cos x}{x\sin x+\cos x}+\tan x+C$ (b) $-\frac{x\sec x}{x\sin x+\cos x}+\tan x+C$

(c) $-\frac{x\cos x}{x\sin x+\cos x}+\cot x+C$ (d) None of these

34. If $\int \frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = gof(x)+\text{constant}$, then

(a) $f(x)=\log(x+\sqrt{x^2+1})$

(b) $f(x)=\log(x+\sqrt{x^2+1})$ and $g(x)=x^2$

(c) $f(x)=\log(x+\sqrt{x^2+1})$ and $g(x)=\frac{x^2}{2}$

(d) $f(x)=\frac{x^2}{2}$ and $g(x)=\log(x+\sqrt{x^2+1})$

35. $\int \frac{dx}{(a^2+x^2)^{3/2}}$ is equal to

(a) $\frac{x}{(a^2+x^2)^{1/2}}+C$

(b) $\frac{x}{a^2(a^2+x^2)^{1/2}}+C$

(c) $\frac{1}{a^2(a^2+x^2)^{1/2}}+c$

(d) None of the above

36. $\int \frac{3^x}{\sqrt{9^x-1}}\,dx$ is equal to

(a) $\frac{1}{\log 3}\log|3^x+\sqrt{9^x-1}|+C$

(b) $\frac{1}{\log 3}\log|9^x+\sqrt{9^x-1}|+C$

(c) $\frac{1}{\log 9}\log|3^x+\sqrt{9^x-1}|+C$

(d) $\frac{1}{\log 9}\log|3^x-\sqrt{9^x-1}|+C$

37. $\int \frac{dx}{x(\log x)(\log\log x)\ldots\underbrace{(\log\log\ldots x)}_{8\text{ times}}}$ is equal to

(a) $\underbrace{(\log\log\ldots x)}_{8\text{ times}}+C$ (b) $\underbrace{(\log\log\ldots x)}_{7\text{ times}}+C$

(c) $\underbrace{(\log\log\ldots x)}_{9\text{ times}}+C$ (d) None of these

38. $\int x^2(ax+b)^{-2}\,dx$ is equal to

(a) $\frac{2}{a^2}\left\{x-\frac{b}{a}\log(ax+b)\right\}+C$

(b) $\frac{2}{a^2}\left\{x-\frac{b}{a}\log(ax+b)\right\}-\frac{x^2}{a(ax+b)}+C$

(c) $\frac{2}{a^2}\left\{x+\frac{b}{a}\log(ax+b)\right\}+\frac{x^2}{a(ax+b)}+C$

(d) $\frac{2}{a^2}\left\{x+\frac{b}{a}\log(ax+b)\right\}-\frac{x^2}{a(ax+b)}+C$

39. $\int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}\,dx$ is equal to

(a) $\tan^{-1}\left(x+\frac{1}{x}\right)+C$ (b) $\cot^{-1}\left(x+\frac{1}{x}\right)+C$

(c) $\log\left(x+\frac{1}{x}\right)+C$ (d) $\log\left[\tan^{-1}\left(x+\frac{1}{x}\right)\right]+C$

40. $\int \frac{mx^{m+2n-1}-nx^{n-1}}{x^{2m+2n}+2x^{m+n}+1}\,dx$ is equal to

(a) $\frac{x^m}{x^{m+n}+1}+C$ (b) $\frac{x^n}{x^{m+n}+1}+C$

(c) $\frac{x^{m+n}-1}{x^{m+n}+1}+C$ (d) $-\frac{x^n}{x^{m+n}+1}+C$

41. If the primitive of $\sin^{-3/2} x \sin^{-1/2}(x+\theta)$ is $-2\operatorname{cosec}\theta\sqrt{f(x)}+C$, then

(a) $f(x)=\dfrac{\sin x}{\sin(x+\theta)}$ (b) $f(x)=\tan(x+\theta)$

(c) $f(x)=\dfrac{\sin(x+\theta)}{\sin x}$ (d) $f(x)=\dfrac{\tan(x+\theta)}{\sin x}$

Integration of Different Types of Functions

42. $\int \dfrac{dx}{\sqrt{8+3x-x^2}}$ is equal to [NCERT Exemplar]

(a) $\dfrac{2}{3}\sin^{-1}\left(\dfrac{2x-1}{\sqrt{41}}\right)+C$ (b) $\dfrac{3}{2}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$

(c) $\dfrac{1}{\sqrt{41}}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$ (d) $\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$

43. $\int \dfrac{1}{\sqrt{9x-4x^2}}\,dx$ is equal to [NCERT]

(a) $\dfrac{1}{9}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (b) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$

(c) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (d) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

44. $\int \sqrt{x^2+4x+6}\,dx$ is equal to [NCERT]

(a) $\dfrac{x+2}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

(b) $(x+2)\sqrt{x^2+4x+6}+\dfrac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

(c) $(x+2)\sqrt{x^2+4x+6}-\dfrac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

(d) None of the above

45. $\int \sqrt{1+3x-x^2}\,dx$ is equal to [NCERT]

(a) $\dfrac{2x-3}{2}\sqrt{1+3x-x^2}+\dfrac{13}{4}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{13}}\right)+C$

(b) $\dfrac{2x-3}{4}\sqrt{1+3x-x^2}-\dfrac{13}{8}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{13}}\right)+C$

(c) $\dfrac{2x-3}{4}\sqrt{1+3x-x^2}+\dfrac{13}{8}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{13}}\right)+C$

(d) None of the above

46. $\int \sqrt{x^2+3x}\,dx$ is equal to [NCERT]

(a) $\dfrac{2x+3}{2}\sqrt{x^2+3x}-\dfrac{9}{4}\log\left|\left(\dfrac{2x+3}{2}\right)+\sqrt{x^2+3x}\right|+C$

(b) $\dfrac{2x+3}{4}\sqrt{x^2+3x}-\dfrac{9}{4}\log\left|\left(\dfrac{2x+3}{2}\right)+\sqrt{x^2+3x}\right|+C$

(c) $\left(x+\dfrac{3}{2}\right)\sqrt{x^2+3x}-\dfrac{9}{8}\log\left|\left(x+\dfrac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$

(d) None of the above

47. $\int \dfrac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$ is equal to [NCERT Exemplar]

(a) $\dfrac{5}{13}x+\dfrac{12}{13}\ln|3\cos x+2\sin x|+c$

(b) $-\dfrac{5}{13}x-\dfrac{12}{13}\ln|3\cos x+2\sin x|+C$

(c) $-\dfrac{5}{13}x+\dfrac{12}{13}\ln|3\cos x+2\sin x|+C$

(d) $-\dfrac{5}{13}x+\dfrac{12}{13}\ln|2\cos x+3\sin x|+C$

48. $\int \dfrac{x^2-1}{x^4+x^2+1}\,dx$ is equal to

(a) $\log(x^4+x^2+1)+C$ (b) $\log\dfrac{x^2-x+1}{x^2+x+1}+C$

(c) $\dfrac{1}{2}\log\dfrac{x^2-x+1}{x^2+x+1}+C$ (d) $\dfrac{1}{2}\log\dfrac{x^2+x+1}{x^2-x+1}+C$

49. If $\int \dfrac{2e^x+3e^{-x}}{3e^x+4e^{-x}}\,dx = Ax+B\log(3e^{2x}+4)+C$, then

(a) $A=-\dfrac{3}{4}, B=\dfrac{1}{24}$ (b) $A=\dfrac{3}{4}, B=-\dfrac{1}{24}$

(c) $A=\dfrac{1}{4}, B=\dfrac{1}{24}$ (d) $A=-\dfrac{3}{4}, B=\dfrac{1}{4}$

50. The value of $\int \dfrac{dx}{\sqrt{x}+\sqrt[3]{x}}$ is

(a) $3\sqrt{x}+3(\sqrt[3]{x})-6\sqrt[6]{x}+6\log(\sqrt[6]{x}+1)+C$

(b) $2\sqrt{x}+6(\sqrt[6]{x})-6\log(\sqrt[6]{x}+1)+C$

(c) $2\sqrt{x}-3(\sqrt[3]{x})+6(\sqrt[6]{x})-6\log(\sqrt[6]{x}+1)+C$

(d) None of the above

51. $\int \dfrac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ is equal to

(a) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{a\tan x}{b}\right)+C$ (b) $\dfrac{a}{b}\tan^{-1}\left(\dfrac{a\tan x}{b}\right)+C$

(c) $\dfrac{b}{a}\tan^{-1}\left(\dfrac{b\tan x}{a}\right)+C$ (d) None of these

52. $\int \dfrac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ is equal to

(a) $\sqrt{\tan x}+\dfrac{\tan^{5/2}x}{5}+C$ (b) $\sqrt{\tan x}+\dfrac{2}{5}\tan^{5/2}x+C$

(c) $2\sqrt{\tan x}+\dfrac{2}{5}\tan^{5/2}x+C$ (d) None of these

53. If $\int \dfrac{4e^x+6e^{-x}}{9e^x-4e^{-x}} = Ax+B\log_e(9e^{2x}-4)+C$, then the values of A, B and C are [NCERT Exemplar]

(a) $\dfrac{-3}{2}, \dfrac{35}{36}$ and C is a constant

(b) $-\dfrac{1}{2}, \dfrac{3}{2}$ and $C=0$

(c) $\dfrac{-3}{2}, \dfrac{33}{2}$ and $C=1$

(d) None of the above

Substitution for Some Irrational Functions

54. $\int \frac{dx}{(x^2+5)(\sqrt{2x^2-3})} dx$ is equal to [NCERT Exemplar]

(a) $\log\left|\frac{\sqrt{2-3/x^2}-\sqrt{13/5}}{\sqrt{2-3/x^2}+\sqrt{13/5}}\right|+C$

(b) $\frac{1}{5}\log\left|\frac{\sqrt{2-3/x^2}-\sqrt{13/5}}{\sqrt{2-3/x^2}+\sqrt{13/5}}\right|+C$

(c) $\log\left|\frac{\sqrt{2x^2-3}-\sqrt{13}}{\sqrt{2x^2-3}+\sqrt{13}}\right|+C$

(d) None of the above

55. $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$ is equal to [NCERT Exemplar]

(a) $-\frac{1}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(b) $-\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(c) $\frac{1}{2}\log\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$

(d) None of the above

Integration by Parts

56. $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}} dx$ is equal to [NCERT Exemplar]

(a) $x-\sqrt{1-x^2}\cos^{-1}x+C$ (b) $\sqrt{1-x^2}-x\cos^{-1}x+C$

(c) $-\sqrt{1-x^2}\cos^{-1}x-x+C$ (d) None of these

57. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ is equal to [NCERT]

(a) $x\tan^{-1}x+\log|1+x^2|+C$ (b) $2x\tan^{-1}x-\log|1+x^2|+C$

(c) $2\tan^{-1}x-\frac{1}{2}\log|1+x^2|+C$ (d) None of these

58. $\int \cos(\log_e x)\,dx$ is equal to

(a) $\frac{1}{2}x[\cos(\log_e x)+\sin(\log_e x)]$

(b) $x[\cos(\log_e x)+\sin(\log_e x)]$

(c) $\frac{1}{2}x[\cos(\log_e x)-\sin(\log_e x)]$

(d) $x[\cos(\log_e x)-\sin(\log_e x)]$

59. $\int |x|\,dx$ is equal to

(a) $\frac{1}{2}x^2+C$ (b) $-\frac{x^2}{2}+C$

(c) $x|x|+C$ (d) $\frac{1}{2}x|x|+C$

Integration by Partial Fractions

60. $\int \frac{x}{(x^2+1)(x-1)} dx$ is equal to [NCERT]

(a) $\frac{1}{2}\log|x-1|-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$

(b) $\frac{1}{4}\log|x-1|-\frac{1}{2}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$

(c) $\frac{1}{4}\log|x-1|+\frac{1}{2}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$

(d) None of the above

61. $\int \frac{3x+5}{x^3-x^2-x+1} dx$ is equal to [NCERT]

(a) $\log\left|\frac{x-1}{x+1}\right|-\frac{1}{x-1}+C$ (b) $\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|-\frac{4}{x-1}+C$

(c) $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+C$ (d) None of these

62. $\int \frac{x^3+x+1}{x^2-1} dx$ is equal to [NCERT]

(a) $\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{1}{2}\log(x-1)+C$

(b) $\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log(x-1)+C$

(c) $x^2+\frac{1}{2}\log|x+1|+\frac{3}{2}\log(x-1)+C$

(d) None of the above

63. $\int \frac{dx}{x(x^n+1)}$ is equal to

(a) $\log\left|\frac{x^n+1}{x^n}\right|+C$ (b) $-\log\left|\frac{x^n}{x^n+1}\right|+C$

(c) $\log\left|\frac{x^n}{x^n+1}\right|+C$ (d) $n\log\left|\frac{x^n}{x^n+1}\right|+C$

64. $\int \frac{dx}{x(x^4-1)}$ is equal to [NCERT]

(a) $\frac{1}{2}\log\left|\frac{x^4}{x^4-1}\right|+C$ (b) $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$

(c) $\frac{1}{4}\log\left|\frac{x^4}{x^4-1}\right|+C$ (d) None of these

65. $\int \frac{1}{\sin x-\sin 2x} dx$ is equal to [NCERT Exemplar]

(a) $-\frac{1}{2}\log|1-\cos x|+C$

(b) $\frac{2}{3}\log|1-2\cos x|+C$

(c) $\frac{-1}{6}(1+\cos x)+C$

(d) None of the above

66. $\int \frac{e^x}{(1+e^x)(2+e^x)}\,dx$ is equal to [NCERT Exemplar]

(a) $\log\left|\frac{2+e^x}{1+e^x}\right|+C$ (b) $\log\left|\frac{1+e^x}{1-e^x}\right|+C$

(c) $\frac{1}{2}\log\left|\frac{2+e^x}{1+e^x}\right|+C$ (d) $\log\left|\frac{1+e^x}{2+e^x}\right|+C$

67. If $\int \frac{2x^2+3}{(x^2-1)(x^2+4)}\,dx = a\log\left(\frac{x-1}{x+1}\right)+b\tan^{-1}\frac{x}{2}+C$, then the values of a and b are respectively

(a) 1/2, 1/2 (b) 1, 1

(c) 1/2, 1 (d) None of these

68. $\int \frac{\cos x\,dx}{\sin^2 x(\sin x+\cos x)}$ is equal to [NCERT Exemplar]

(a) $\log\left|\frac{1+\tan x}{\tan x}\right|-\cot x+C$

(b) $\log\left|\frac{\tan x}{1+\tan x}\right|-\tan x+C$

(c) $\log\left|\frac{\tan x}{1+\tan x}\right|+\cot x+C$

(d) None of the above

69. $\int \frac{dx}{(\sin x-2\cos x)(2\sin x+\cos x)}$ is equal to

(a) $\log_e \sqrt[5]{\frac{\tan x-2}{2\tan x+1}}+C$

(b) $\frac{1}{4}\log\left(\frac{\tan x-2}{\tan x+2}\right)+C$

(c) $-\frac{1}{4}\log\left(\frac{2\sin x+\cos x}{\sin x+2\cos x}\right)+C$

(d) None of the above

Integration using Trigonometric Identities

70. $\int \cos 2x\cdot\cos 4x\cdot\cos 6x\,dx$ is equal to [NCERT]

(a) $\frac{\sin 12x}{12}+\frac{\sin 8x}{4}+\frac{\sin 4x}{2}+C$

(b) $\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{4}+\frac{\sin 4x}{2}\right]+C$

(c) $\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+\frac{\sin 4x}{4}\right]+C$

(d) None of the above

71. $\int \cos^4 2x\,dx$ is equal to [NCERT]

(a) $\frac{1}{8}[3x+\sin 8x+\sin 4x]+C$

(b) $\frac{1}{8}[3x+\frac{\sin 8x}{8}+\sin 4x]+C$

(c) $\frac{1}{8}[x+\sin 4x+\sin 8x]+C$

(d) None of the above

72. $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}\,dx$ is equal to [NCERT Exemplar]

(a) $2\sin x+2x\cos\alpha+C$ (b) $\sin x+x\cos\alpha+C$

(c) $2\sin x+x\cos x+C$ (d) None of these

73. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cdot\cos^2 x}\,dx$ is equal to [NCERT]

(a) $-\frac{\sin 2x}{2}+C$ (b) $-\frac{\cos 2x}{2}+C$

(c) $-\frac{1}{4}\sin 2x+C$ (d) None of these

74. If $\int \sqrt{2}\sqrt{1+\sin x}\,dx = -4\cos(ax+b)+C$, then the value of a, b are respectively

(a) $\frac{1}{2},\frac{\pi}{4}$ (b) $1,\frac{\pi}{2}$

(c) 1, 1 (d) None of these

75. $\int \frac{dx}{\sin x-\cos x+\sqrt{2}}$ is equal to [NCERT Exemplar]

(a) $-\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$ (b) $\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

(c) $\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$ (d) $-\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

76. If $\int \frac{\cos 4x+1}{\cot x-\tan x}\,dx = A\cos 4x+B$, then

(a) $A=-\frac{1}{2}$ (b) $A=-\frac{1}{8}$

(c) $A=-\frac{1}{4}$ (d) None of these

77. The value of $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+\theta)+\sin\theta}}$ is

(a) $\sqrt{(\tan x+\tan\theta)\sec\theta}+C$

(b) $\sqrt{2(\tan x+\tan\theta)\sec\theta}+C$

(c) $\sqrt{2(\sin x+\tan\theta)\sec\theta}+C$

(d) None of the above

Some Special Integrals

78. $\int \frac{xe^x}{(1+x)^2}\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{e^x}{1+x}+C$ (b) $\frac{-1}{1+x}+C$

(c) $\frac{-e^x}{(1+x)}+C$ (d) $\frac{e^x}{(1+x)^2}+C$

79. $\int \frac{x-3}{(x-1)^3}\cdot e^x\cdot dx$ is equal to [NCERT]

(a) $\frac{e^x}{(x-1)}+c$ (b) $\frac{-e^x}{(x-1)^2}+C$

(c) $\frac{2e^x}{(1-x)^2}+C$ (d) $\frac{e^x}{(x-1)^2}+C$

80. $\int e^x \sec x(1+\tan x)\,dx$ is equal to [NCERT Exemplar]

(a) $e^x \cos x + C$ (b) $e^x \sec x + C$

(c) $e^x \sin x + C$ (d) $e^x \tan x + C$

81. $\int \frac{\cos x - 1}{\sin x + 1} e^x\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{e^x \cos x}{1+\sin x} + C$ (b) $C - \frac{e^x \sin x}{1+\sin x}$

(c) $C - \frac{e^x}{1+\sin x}$ (d) $C - \frac{e^x \cos x}{1+\sin x}$

82. $\int \left(\frac{x+2}{x+4}\right)^2 e^x\,dx$ is equal to [NCERT Exemplar]

(a) $e^x\left(\frac{x}{x+4}\right) + C$ (b) $e^x\left(\frac{x+2}{x+4}\right) + C$

(c) $e^x\left(\frac{x-2}{x+4}\right) + C$ (d) $\left(\frac{2xe^x}{x+4}\right) + C$

83. $\int (1 + x - x^{-1})\, e^{x+x^{-1}}\,dx$ is equal to

(a) $(x+1)e^{x+x^{-1}} + C$ (b) $(x-1)e^{x+x^{-1}} + C$

(c) $-xe^{x+x^{-1}} + C$ (d) $xe^{x+x^{-1}} + C$

Miscellaneous

84. If $I_n = \int (\log x)^n\,dx$, then $I_n + nI_{n-1}$ is equal to

(a) $x(\log x)^n$ (b) $(x\log x)^n$

(c) $(\log x)^{n-1}$ (d) $n(\log x)^n$

85. $\int \sin^{-1}(\cos x)\,dx$ is equal to [NCERT]

(a) $\frac{\pi}{2} + \frac{x^2}{2} + c$ (b) $\frac{2}{\pi} - \frac{x^2}{2} + C$

(c) $\frac{\pi}{2}x - \frac{x^2}{2} + C$ (d) $\frac{2x}{\pi} + x^2 + C$

86. $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$ is equal to [NCERT Exemplar]

(a) $(\sin^{-1} x)\frac{x}{\sqrt{1-x^2}} + \log(\sqrt{1-x^2}) + C$

(b) $\frac{\sin^{-1} x}{\sqrt{1-x^2}} + \log(\sqrt{1-x^2}) + C$

(c) $\frac{x\sin^{-1} x}{\sqrt{1-x^2}} - \log(\sqrt{1-x^2}) + C$

(d) None of the above

87. $\int \tan^{-1}\sqrt{x}\,dx$ is equal to [NCERT Exemplar]

(a) $x\tan^{-1}\sqrt{x} - \sqrt{x} + \frac{1}{2}\tan^{-1}\sqrt{x} + C$

(b) $x\tan^{-1}\sqrt{x} + C$

(c) $-\sqrt{x} + x\tan^{-1}\sqrt{x} + C$

(d) None of the above

88. $\int \sqrt{x}\, e^{\sqrt{x}}\,dx$ is equal to

(a) $e^{\sqrt{x}} + C$ (b) $2xe^{\sqrt{x}} + C$

(c) $(2x - 4\sqrt{x} + 4)e^{\sqrt{x}} + C$ (d) None of these

89. $\int \cos\left[2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]dx$ is equal to

(a) $\frac{1}{2}x^2 + C$ (b) $\frac{1}{2}\sin\left[2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right] + C$

(c) $-\frac{1}{2}x^2 + C$ (d) $\frac{1}{2}x + C$

90. If $f(x) = \cos x - \cos^2 x + \cos^3 x - \ldots \infty$, then $\int f(x)\,dx$ is equal to

(a) $\tan\frac{x}{2} + C$ (b) $x - \tan\frac{x}{2} + C$

(c) $x - \frac{1}{2}\tan\frac{x}{2} + C$ (d) $\frac{x - \tan\frac{x}{2}}{2} + C$

91. If $f(x) = \frac{x+2}{2x+3}$. Then, $\int \left\{\frac{f(x)}{x^2}\right\}^{1/2} dx$ is equal to $\frac{1}{\sqrt{2}}\, g\left\{\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right\} - \sqrt{\frac{2}{3}}\, h\left\{\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right\} + C$, where

(a) $g(x) = \tan^{-1} x, h(x) = \log|x|$

(b) $g(x) = \log|x|, h(x) = \tan^{-1} x$

(c) $g(x) = h(x) = \tan^{-1} x$

(d) $g(x) = \log|x|, h(x) = \log|x|$

92. $\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$ is equal to

(a) $\sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

(b) $\sqrt{2}\sin^{-1}(\sin x + \cos x) + C$

(c) $\sqrt{2}\tan^{-1}(\sin x - \cos x) + C$

(d) None of the above

93. $\int \cos^{-1}\left(\frac{1}{x}\right)dx$ is equal to

(a) $x\sec^{-1} x + \cosh^{-1} x + C$ (b) $x\sec^{-1} x - \cosh^{-1} x + C$

(c) $x\sec^{-1} x - \sin^{-1} x + C$ (d) None of these

94. $\int x\{f(x^2)\, g''(x^2) - f''(x^2) g(x^2)\}\,dx$ is equal to

(a) $f(x^2)\, g'(x^2) - g(x^2) f'(x^2) + C$

(b) $\frac{1}{2}\{f(x^2)\, g(x^2) f'(x^2)\} + C$

(c) $\frac{1}{2}\{f(x^2)\, g'(x^2) - g(x^2) f'(x^2)\} + C$

(d) None of the above

95. Let $\int e^x\{f(x) - f'(x)\}\,dx = \phi(x)$. Then, $\int e^x f(x)\,dx$ is equal to

(a) $\phi(x) + e^x f(x)$ (b) $\phi(x) - e^x f(x)$

(c) $\frac{1}{2}[\phi(x) + e^x f(x)]$ (d) $\frac{1}{2}[\phi(x) + e^x f'(x)]$

96. If $\int f(x)\,dx = F(x)$, then $\int x^3 f(x^2)\,dx$ is equal to

(a) $\frac{1}{2}\{x^2F(x^2) - \int F(x^2)\,dx^2\}$ (b) $\frac{1}{2}\{x^2F(x^2) - \int F(x^2)\,dx\}$

(c) $\frac{1}{2}\left\{x^2F(x) - \frac{1}{2}\int F(x^2)\,dx\right\}$ (d) None of these

97. The anti-derivative of $f(x) = \log(\log x) + (\log x)^{-2}$ whose graph passes through (e, e), is

(a) $x[\log(\log x) + (\log x)^{-1}]$

(b) $x[-\log(\log x) + (\log x)^{-1}] + e$

(c) $x[\log(\log x) - (\log x)^{-1}] + 2e$

(d) None of the above

98. If $f(0) = f'(0) = 0$ and $f''(x) = \tan^2 x$, then $f(x)$ is

(a) $\log\sec x - \frac{x^2}{2}$ (b) $\log\cos x + \frac{x^2}{2}$

(c) $\log\sec x + \frac{x^2}{2}$ (d) None of these

99. $\int \sqrt{\tan x}\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{x}{2} + C$ (b) $x + \frac{\pi}{2} + C$

(c) $\frac{x}{2} + \frac{\pi}{4} + C$ (d) $\frac{x}{2} + \frac{\pi}{2} + C$

100. If $0 < a < 1$, then $\int \frac{dx}{1 - 2a\cos x + a^2}$ is equal to [NCERT Exemplar]

(a) $\frac{1}{1-a^2}\tan^{-1}\left(t \cdot \frac{1-a}{1+a}\right) + C$

(b) $\frac{2}{1-a^2}\tan^{-1}\left(t \cdot \frac{1+a}{1-a}\right) + C$

(c) $\frac{1}{1+a^2}\tan^{-1}(t) + C$

(d) None of the above

Round II (Mixed Bag)

Only One Correct Option

1. $\int \frac{x\cos x + 1}{\sqrt{2x^3 e^{\sin x} + x^2}}\,dx$

(a) $\log\left|\frac{\sqrt{2xe^{\sin x}+1} - 1}{\sqrt{2xe^{\sin x}+1} + 1}\right| + C$ (b) $\log\left|\frac{\sqrt{2xe^{\sin x}-1} + 1}{\sqrt{2xe^{\sin x}+1} + 1}\right| + C$

(c) $\log\left|\frac{\sqrt{2xe^{\sin x}+1} + 1}{\sqrt{2xe^{\sin x}-1} + 1}\right| + C$ (d) $\log\left|\frac{\sqrt{2xe^{\sin x}+1} + 1}{\sqrt{2xe^{\sin x}-1} - 1}\right| + C$

2. $\int e^x\left[\frac{1}{\sqrt{1+x^2}} + \frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right]dx$

(a) $e^x\left[\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{(1+x^2)^3}}\right] + C$

(b) $e^x\left[\frac{1}{\sqrt{1+x^2}} - \frac{x}{\sqrt{(1+x^2)^3}}\right] + C$

(c) $e^x\left[\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{(1+x^2)^5}}\right] + C$

(d) $e^x\left[\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{(1+x^2)}}\right] + C$

3. $\int \frac{(1+x)\sin x}{(x^2+2x)\cos^2 x - (1+x)\sin 2x}\,dx$

(a) $\frac{1}{2}\log_e\left|\frac{\sin x - (x+1)\cos x - 1}{\sin x - (x+1)\cos x + 1}\right| + C$

(b) $\frac{1}{2}\tan^{-1}\{\sin x - (x+1)\cos x\} + C$

(c) $\frac{1}{2}\sin^{-1}\{\sin x - (x+1)\cos x\} + C$

(d) $\frac{1}{2}\sin^{-1}(\cos x + \sin x) + C$

4. $\int \frac{\operatorname{cosec}^2 x - 2005}{\cos^{2005} x}\,dx$

(a) $\frac{\cot x}{(\cos x)^{2005}} + C$ (b) $\frac{\tan x}{(\cos x)^{2005}} + C$

(c) $-\frac{\tan x}{(\cos x)^{2005}} + C$ (d) $\frac{-\cot x}{(\cos x)^{2005}} + C$

5. $\int \frac{(x - \sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\frac{x}{2}}{(x - \sin x)} + 3x\right\}dx$ is equal to

(a) $x^{3/2}(x - \sin x)^{3/2} + C$

(b) $2x^{3/2}(x - \sin x)^{3/2} + C$

(c) $2x^{3/2}(x + \sin x)^{3/2} + C$

(d) $2x^{3/2}(x - \sin x)^{3/2} - C$

6. If $\int \frac{(\sqrt{x})^5\,dx}{(\sqrt{x})^7 + x^6} = \lambda\log\left(\frac{x^a}{x^a + 1}\right) + C$, then $a + \lambda$

(a) 2 (b) >2 (c) <2 (d) >3

7. If $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2 - a^2)}\log f(x) + K$, then $f(x)$ is equal to

(a) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x + C}$ (b) $\frac{1}{a^2\sin^2 x - b^2\cos^2 x + C}$

(c) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x + C}$ (d) $\frac{1}{a^2\cos^2 x - b^2\sin^2 x + C}$

8. The value of $\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$ is equal to

(a) $e^x(\sqrt{1-x^2})+C$ (b) $e^x\frac{\sqrt{1+x^{2n}}}{1+x^{2n}}+C$

(c) $\frac{e^x\sqrt{1-x^n}}{1-x^{2n}}+C$ (d) $\frac{e^x\sqrt{1-x^{2n}}}{1-x^n}+C$

9. If $f(x)=\lim_{n\to\infty} n^2\{x^{1/n}-x^{1/(n+1)}\}$, $x>0$, then $\int x f(x)\,dx$ is equal to

(a) $x^2/2$ (b) 0

(c) $x^2\log x-\frac{1}{2}x^2+c$ (d) None of these

10. If $f(x)=\lim_{n\to\infty}[2x+4x^3+\ldots+2nx^{2n-1}]$ $(0<x<1)$, then $\int f(x)\,dx$ is equal to

(a) $-\sqrt{1-x^2}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{x^2-1}$ (d) $\frac{1}{1-x^2}$

11. $\int (x+1)^{3/2}[\sin^{-1}(\log x)+\cos^{-1}(\log x)]\,dx$ is equal to

(a) $\frac{2\pi}{5}(x+1)^{5/2}+C$ (b) $\frac{\pi}{5}(x+1)^{5/2}+C$

(c) does not exist (d) None of these

12. $\int e^{2ax}\frac{1-\cos 2ax}{1+\sin 2ax}dx$ is equal to

(a) $-\frac{1}{a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$

(b) $-\frac{1}{2a}e^{2ax}\cot\left(\frac{\pi}{4}+ax\right)+C$

(c) $-\frac{1}{2a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$

(d) $-\frac{1}{a}e^{2ax}\operatorname{cosec}\left(\frac{\pi}{4}+ax\right)+C$

13. Let $x^2\neq n\pi-1$, $n\in N$. The value of $\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$ is equal to

(a) $\log\left|\frac{1}{2}\sec(x^2+1)\right|+C$

(b) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$

(c) $\frac{1}{2}\log|\sec(x^2+1)|+C$

(d) None of the above

14. Given, $f(x)=\begin{vmatrix}0 & x^2-\sin x & \cos x-2\\ \sin x-x^2 & 0 & 1-2x\\ 2-\cos x & 2x-1 & 0\end{vmatrix}$, then $\int f(x)\,dx$ is equal to

(a) $\frac{x^3}{3}-x^2\sin x+\sin 2x+C$

(b) $\frac{x^3}{3}-x^2\sin x-\cos 2x+C$

(c) $\frac{x^3}{3}-x^2\cos x-\cos 2x+C$

(d) None of the above

15. Let $g(x)$ be a differentiable function satisfying $\frac{d}{dx}\{g(x)\}=g(x)$ and $g(0)=1$, then $\int g(x)\left(\frac{2-\sin 2x}{1-\cos 2x}\right)dx$ is equal to

(a) $g(x)\cot x+C$ (b) $-g(x)\cot x+C$

(c) $\frac{g(x)}{1-\cos 2x}+C$ (d) None of these

16. Let $f(x)$ be a polynomial satisfying $f(0)=2$, $f'(0)=3$ and $f''(x)=f(x)$. Then, $f(4)$ is equal to

(a) $5(e^8+1)$ (b) $\frac{5e^8-1}{2e^4}$

(c) $\frac{2e^4}{5(e^8-1)}$ (d) $\frac{2e^4}{5(e^8+1)}$

17. Let $f(x)=\int\frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)=0$, then the value of $f(1)$ be

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log(1+\sqrt{2})+\frac{\pi}{2}$ (d) None of these

18. If an anti-derivative of $f(x)$ is e^x and that of $g(x)$ is $\cos x$, then $\int f(x)\cos x\,dx+\int g(x)e^x dx$ is equal to

(a) $f(x)\cdot g(x)+C$ (b) $f(x)+g(x)+C$

(c) $e^x\cos x+C$ (d) $f(x)-g(x)+C$

19. $\int\cos 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ is equal to

(a) $(\cos\theta-\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$

(b) $(\cos\theta+\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$

(c) $\frac{(\cos\theta-\sin\theta)^2}{2}\log\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)+C$

(d) $\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta+C$

20. If $\int\frac{f(x)}{\log\sin x}dx=\log\log\sin x$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$

(c) $\log\sin x$ (d) $\cot x$

21. $\int \frac{-1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$ is equal to

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$ (b) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4}+C$

(c) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$ (d) $\frac{1}{3}\left(\frac{x+2}{x-1}\right)^{1/4}+C$

22. $\int |x|\log|x|\,dx$ is equal to $(x \neq 0)$

(a) $\frac{x^2}{2}\log|x|-\frac{x^2}{4}+C$ (b) $\frac{1}{2}x|x|\log x+\frac{1}{4}x|x|+C$

(c) $-\frac{x^2}{2}\log|x|+\frac{x^2}{4}+C$ (d) $\frac{1}{2}x|x|\log|x|-\frac{1}{4}x|x|+C$

23. Let $f(x)$ be a function such that, $f(0)=f'(0)=0$, $f''(x)=\sec^4 x+4$, then the function is

(a) $\log|(\sin x)|+\frac{1}{3}\tan^3 x+x$

(b) $\frac{2}{3}\log|(\sec x)|+\frac{1}{6}\tan^2 x+2x^2$

(c) $\log|(\cos x)|+\frac{1}{6}\cos^2 x+\frac{x^2}{5}$

(d) None of the above

24. $\int(\sin 2x-\cos 2x)\,dx=\frac{1}{\sqrt{2}}\sin(2x-a)+b$, then

(a) $a=\frac{5\pi}{4}, b\in R$ (b) $a=-\frac{5\pi}{4}, b\in R$

(c) $a=\frac{\pi}{4}, b\in R$ (d) None of these

25. $\int x^x(1+\log|x|)\,dx$ is equal to

(a) $x^x\log|x|+C$ (b) $e^{x^x}+C$

(c) x^x+C (d) None of these

More Than One Correct Option

26. $\int\frac{dx}{x^2(x^4+1)^{3/4}}=A\left(\frac{x^4+1}{x^4}\right)^B+C$

(a) $A=-1$ (b) $B=\frac{1}{4}$

(c) $A=\frac{1}{2}$ (d) $B=\frac{1}{2}$

27. If $\int\frac{(2x-1)dx}{x^4-2x^3+x+1}=A\tan^{-1}\left[\frac{f(x)}{\sqrt{3}}\right]+C$, then

(a) $A=\frac{2}{\sqrt{3}}$ (b) $A=2$

(c) $f(x)=2x^2-2x-1$ (d) $f(x)=2x^2+2x+1$

28. If $\int\frac{2x^2\sec^2 x}{(\tan x-x\sec^2 x)^2}\,dx=\frac{f(x)}{\tan x(\tan x-x\sec^2 x)}+g(x)+x+C$, then

(a) $f(x)=x$ (b) $f(x)=x^2$

(c) $g(x)=\cot x$ (d) $g(x)=\tan x$

29. If $\int\frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}\,dx=\frac{A}{2}\sqrt[3]{x^2}+B\tan^{-1}\sqrt[6]{x}+C$, then

(a) $A+B=9$ (b) $A=3$

(c) $B=6$ (d) $B-A=3$

Comprehension Based Questions

Passage

Directions (Q. Nos. 30 and 31) *If the integrand is a rational function of x and fractional powers of a linear fractional function of the form* $\frac{ax+b}{cx+d}$, *then rationalization of the integral is affected by the substitution* $\frac{ax+b}{cx+d}=t^m$, *where m is LCM of fractional powers of* $\frac{ax+b}{cx+d}$.

30. If $I=\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}=A\sqrt[4]{\frac{x-1}{x+2}}+C$, then A is equal to

(a) 1/3 (b) 2/3 (c) 3/4 (d) 4/3

31. If $I=\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=k\sqrt[3]{\frac{1+x}{1-x}}+C$, then k is equal to

(a) 2/3 (b) 3/2 (c) 1/3 (d) 1/2

Directions (Q. Nos. 32-42) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

32. **Statement I** $\int\frac{(2-2x)}{\sqrt{(4+2x-x^2)}}\,dx=2\sqrt{(4+2x-x^2)}+\sin^{-1}\left(\frac{x-1}{\sqrt{5}}\right)+C$

Statement II $\int\frac{dx}{\sqrt{a^2-x^2}}=\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)$

33. If $I_n=\int\tan^n x\,dx$, then

Statement I $5(I_6+I_4)=\tan^5 x$

Statement II $(n-1)(I_n+I_{n-2})=\tan^{n-1}x$

34. **Statement I** $\int\frac{d(x^2+1)}{\sqrt{\lambda^2+2}}$ is equal to $2\sqrt{x^2+2}+C$

Statement II $\int\frac{x^{a/2}}{\sqrt{1+x^{11}}}\,dx$ is $\frac{2}{11}\log|x+\sqrt{1+x^{11}}|+C$

35. If $a > 0$ and $b^2 - 4ac < 0$, then

Statement I The value of the integral $\int \frac{dx}{ax^2 + bx + c}$ will be of the type of $\mu \tan^{-1} \frac{x + A}{B} + C$, where A, B, C and μ are constants.

Statement II $ax^2 + bx + c$ can be written as sum of two squares.

36. **Statement I** $\int \tan 5x \tan 3x \tan 2x \, dx$

$$= \frac{\log|\sec 5x|}{5} - \frac{\log|\sec 3x|}{3} - \frac{\log|\sec 2x|}{2} + C$$

Statement II

$$\tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$$

37. If $y(x - y)^2 = x$, then

Statement I $\int \frac{dx}{(x - 3y)} = \frac{1}{2} \log(x - y)^2 - 1$

Statement II $\int \frac{dx}{(x - 3y)} = \log(x - 3y) + C$

38. **Statement I** If $\int \frac{1}{f(x)} dx = 2\log|f(x)| + c$, then $f(x) = \frac{x}{2}$

Statement II When $f(x) = \frac{x}{2}$, then

$$\int \frac{1}{f(x)} dx = \int \frac{2}{x} dx = 2 \log|x| + C$$

39. **Statement I** The function $F(x) = \int \sin^2 x \, dx$ satisfies

$$F(x + \pi) = F(x), \ \forall x \in R$$

Statement II $\sin^2(x + \pi) = \sin^2 x$

40. If $I_n = \int \cot^n x \, dx$, then

Statement I $5(I_6 + I_4) = -\cot^5 x$

Statement II $I_n = -\frac{\cot^{n-1}}{n} - I_{n-2}$, where $n \geq 2$

41. **Statement I** $\int \frac{xe^x}{(x + 1)^2} dx = \frac{e^x}{x + 1} + C$

Statement II $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

42. **Statement I** $\int \frac{dx}{e^x + e^{-x} + 2} = -\frac{1}{e^x + 1} + C$

Statement II $\int \frac{d\{f(x)\}}{\{f(x)\}^2} = -\frac{1}{f(x)} + C$

Previous Years' Questions

43. If $\int f(x) \, dx = \psi(x)$, then $\int x^5 f(x^3) \, dx$ is equal to **[JEE Main 2013]**

(a) $\frac{1}{3}[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + C$

(b) $\frac{1}{3} x^3 \psi(x^3) - 3 \int x^3 \psi(x^3) dx + C$

(c) $\frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + C$

(d) $\frac{1}{3}[x^3 \psi(x^3) - \int x^3 \psi(x^3) dx] + C$

44. If the integral

$$\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln|\sin x - 2\cos x| + K,$$

then a is equal to **[AIEEE 2012]**

(a) -1 (b) -2 (c) 1 (d) 2

45. The value of $\sqrt{2} \int \frac{\sin x \, dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is **[AIEEE 2008]**

(a) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$

(b) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$

(c) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

(d) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

46. $\int \frac{dx}{\cos x + \sqrt{3} \sin x}$ is equal to **[AIEEE 2007]**

(a) $\frac{1}{2} \log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

(b) $\frac{1}{2} \log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

(c) $\log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

(d) $\log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

47. $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is equal to **[AIEEE 2005]**

(a) $\frac{xe^x}{1 + x^2} + C$ (b) $\frac{x}{(\log x)^2 + 1} + C$

(c) $\frac{\log x}{(\log x)^2 + C}$ (d) $\frac{x}{x^2 + 1} + C$

Answers

Check Point 1

1. (a)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (c)

Check Point 2

1. (a)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (a)

Round I

1. (a)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (a)
21. (c)	**22.** (a)	**23.** (b)	**24.** (d)	**25.** (a)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (a)	**30.** (c)
31. (b)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (b)	**39.** (d)	**40.** (d)
41. (c)	**42.** (d)	**43.** (b)	**44.** (a)	**45.** (c)	**46.** (b)	**47.** (c)	**48.** (c)	**49.** (b)	**50.** (c)
51. (a)	**52.** (a)	**53.** (a)	**54.** (b)	**55.** (a)	**56.** (c)	**57.** (b)	**58.** (a)	**59.** (d)	**60.** (a)
61. (c)	**62.** (b)	**63.** (c)	**64.** (b)	**65.** (d)	**66.** (d)	**67.** (a)	**68.** (a)	**69.** (a)	**70.** (c)
71. (b)	**72.** (a)	**73.** (a)	**74.** (a)	**75.** (c)	**76.** (b)	**77.** (b)	**78.** (a)	**79.** (d)	**80.** (a)
81. (a)	**82.** (a)	**83.** (d)	**84.** (a)	**85.** (c)	**86.** (a)	**87.** (d)	**88.** (c)	**89.** (c)	**90.** (b)
91. (d)	**92.** (a)	**93.** (b)	**94.** (c)	**95.** (c)	**96.** (a)	**97.** (c)	**98.** (d)	**99.** (c)	**100.** (b)

Round II

1. (a)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (d)	**9.** (d)	**10.** (d)
11. (c)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (c)	**19.** (d)	**20.** (d)
21. (a)	**22.** (d)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (a,b)	**27.** (a,c)	**28.** (a,c)	**29.** (a,b,c,d)	**30.** (d)
31. (c)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (c)	**38.** (a)	**39.** (a)	**40.** (c)
41. (a)	**42.** (a)	**43.** (c)	**44.** (d)	**45.** (d)	**46.** (a)	**47.** (b)			

the Guidance

Round I

1. It is an anti-derivative of $f(x)$, so we integrate both sides and use the given condition to find the value of constant of integration.

Given, $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$

$\Rightarrow$ Anti-derivative of $\left(4x^3 - \frac{3}{x^4}\right) = f(x)$

$\therefore$ $f(x) = \int \left(4x^3 - \frac{3}{x^4}\right) dx$

$= 4\int x^3 dx - 3\int x^{-4} dx$

$\Rightarrow$ $f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C = x^4 + \frac{1}{x^3} + C$

Also, $f(x) = 0$

$\therefore$ $f(2) = (2)^4 + \frac{1}{(2)^3} + C$

$\Rightarrow$ $16 + \frac{1}{8} + C = 0$

$\Rightarrow$ $C = -\left(16 + \frac{1}{8}\right)$

$\Rightarrow$ $C = -\frac{129}{8}$

$\therefore$ $f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$

2. Given, $\int f(x)\, dx = f(x) + C$

$\Rightarrow$ $f(x) = f'(x)$

$\therefore$ $\int \{f(x)^2\}\, dx = \int f(x) \cdot f'(x)\, dx = \frac{1}{2}\{f(x)\}^2 + C$

3. Given that, $\int \frac{1}{f(x)} dx = \log\{f(x)\}^2 + C$

$\Rightarrow$ $\frac{1}{f(x)} = \frac{1}{\{f(x)\}^2} 2f(x) f'(x) \Rightarrow f'(x) = \frac{1}{2}$

$\Rightarrow$ $f(x) = \frac{x}{2} + D$

(where, D is a integration constant)

4. $\int x\sqrt{x+2}\,dx = \int (x+2-2)\sqrt{x+2}\,dx$

$= \int (x+2)\sqrt{x+2}\,dx - 2\int \sqrt{x+2}\,dx$

$= \int (x+2)^{3/2}dx - 2\int (x+2)^{1/2}dx$

$= \dfrac{(x+2)^{(3/2)+1}}{(3/2)+1} - 2\dfrac{(x+2)^{(1/2)+1}}{(1/2)+1} + C \qquad \left(\because \int x^n\,dx = \dfrac{x^{n+1}}{n+1}\right)$

$= \dfrac{2}{5}(x+2)^{5/2} - \dfrac{2\times 2}{3}(x+2)^{3/2} + C$

$= \dfrac{2}{5}(x+2)^{5/2} - \dfrac{4}{3}(x+2)^{3/2} + C$

Alternate Method

Let $x+2=t$ or $x=(t-2) \Rightarrow 1 = \dfrac{dt}{dx} \Rightarrow dx = dt$

Then, $\int x\sqrt{x+2}\,dx = \int (t-2)\sqrt{t}\,dt = \int (t^{3/2} - 2t^{1/2})\,dt$

$= \dfrac{t^{(3/2)+1}}{(3/2)+1} - 2\dfrac{t^{(1/2)+1}}{(1/2)+1} + C$

$= \dfrac{2}{5}t^{5/2} - \dfrac{4}{3}t^{3/2} + C$

$= \dfrac{2}{5}(x+2)^{5/2} - \dfrac{4}{3}(x+2)^{3/2} + C$

5. $\int \dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx = \int \dfrac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}dx \quad (\because \log m^n = n\log m)$

$= \int \dfrac{x^6 - x^5}{x^4 - x^3}dx = \int \dfrac{x^5(x-1)}{x^3(x-1)}dx = \int x^2\,dt = \dfrac{x^3}{3} + C$

6 $\int (1-\cos x)\,\text{cosec}^2 x\,dx$

$= \int \text{cosec}^2 x\,dx - \int \text{cosec}^2 x\cos x\,dx$

$\cot x - \int \text{cosec}\,x\cot x\,dx$

$= -\cot x + \text{cosec}\,x + C = \dfrac{1-\cos x}{\sin x} + C$

$= \dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} + C = \tan\dfrac{x}{2} + C$

7. $\int \dfrac{1}{x-\sqrt{x}}dx = \int \dfrac{1}{\sqrt{x}(\sqrt{x}-1)}dx$

Let $\sqrt{x}-1 = t \Rightarrow \dfrac{1}{2\sqrt{x}} = \dfrac{dt}{dx} \Rightarrow dx = 2\sqrt{x}\,dt$

$\therefore \int \dfrac{1}{\sqrt{x}(\sqrt{x}-1)}dx = \int \dfrac{1}{\sqrt{x}\,t}2\sqrt{x}\,dt$

$= \int \dfrac{2}{t}dt = 2\cdot\log|t| + C = 2\log|\sqrt{x}-1| + C$

8. $\int (x^3-1)^{1/3}x^5dx = \int (x^3-1)^{1/3}x^3\cdot x^2dx$

Let $x^3 - 1 = t \Rightarrow x^3 = t+1$

Differentiating w.r.t. x, we get $3x^2 = \dfrac{dt}{dx} \Rightarrow dx = \dfrac{dt}{3x^2}$

$\therefore \int (x^3-1)^{1/3}x^3\cdot x^2dx = \int t^{1/3}(t+1)x^2\dfrac{dt}{3x^2} = \dfrac{1}{3}\int (t^{4/3} + t^{1/3})\,dt$

$= \dfrac{1}{3}\left[\dfrac{t^{7/3}}{\frac{7}{3}} + \dfrac{t^{4/3}}{\frac{4}{3}}\right] + C = \dfrac{1}{3}\left[\dfrac{3}{7}t^{7/3} + \dfrac{3}{4}t^{4/3}\right] + C$

$= \dfrac{1}{7}t^{7/3} + \dfrac{1}{4}t^{4/3} + C$

$= \dfrac{1}{7}(x^3-1)^{7/3} + \dfrac{1}{4}(x^3-1)^{4/3} + C$

9. Here, given integrand is not in any standard form. So, we convert it by substitution method in a standard form and then integrate it.

$$\int \frac{1}{x(\log x)^m}dx$$

Let $\log x = t \Rightarrow \dfrac{1}{x} = \dfrac{dt}{dx} \Rightarrow dx = x\,dt$

$\therefore \int \dfrac{1}{x(\log x)^m}dx = \int \dfrac{1}{x(t)^m}x\,dt = \int t^{-m}dt$

$= \dfrac{t^{-m+1}}{-m+1} + C = \dfrac{(\log x)^{1-m}}{1-m} + C$

10. Here, the given integrand is of the form $e^{f(x)}\cdot f'(x)$, so we use substitution method to integrate it.

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$$

Let $\tan^{-1}x = t \Rightarrow \dfrac{1}{1+x^2} = \dfrac{dt}{dx}$

$\Rightarrow dx = (1+x^2)\,dt$

$\therefore \int \dfrac{e^{\tan^{-1}x}}{1+x^2}dx = \int \dfrac{e^t}{1+x^2}(1+x^2)\,dt$

$= \int e^t dt = e^t + C = e^{\tan^{-1}x} + C$

11. $\int \dfrac{e^{2x}-1}{e^{2x}+1}dx = \int \dfrac{(e^x.e^x-1)}{(e^x.e^x+1)}dx = \int \dfrac{e^x\left(e^x - \frac{1}{e^x}\right)}{e^x\left(e^x + \frac{1}{e^x}\right)}dx$

$= \int \dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}dx$

Let $e^x + e^{-x} = t \Rightarrow e^x - e^{-x} = \dfrac{dt}{dx} \Rightarrow dx = \dfrac{dt}{e^x - e^{-x}}$

$\int \dfrac{e^{2x}-1}{e^{2x}+1}dx = \int \dfrac{(e^x - e^{-x})}{t}\dfrac{dt}{e^x - e^{-x}}$

$= \int \dfrac{1}{t}dt = \log|t| + C = \log|e^x + e^{-x}| + C$

12. $\int \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}dx$

Let $e^{2x} + e^{-2x} = t$

$\Rightarrow 2e^{2x} - 2e^{-2x} = \dfrac{dt}{dx}$

$$\Rightarrow \quad dx = \frac{dt}{2(e^{2x} - e^{-2x})}$$

$$\therefore \quad \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x} - e^{-2x}}{t} \frac{dt}{2[e^{2x} - e^{-2x}]}$$

$$= \frac{1}{2} \int \frac{1}{t} dt = \frac{1}{2} \log|t| + C$$

$$= \frac{1}{2} \log|e^{2x} + e^{-2x}| + C$$

13. Let $I = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Let $\sin^{-1} x = t$

$$\Rightarrow \quad \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow dx = \sqrt{1-x^2} dt$$

$$\therefore \quad I = \int \frac{t}{\sqrt{1-x^2}} \sqrt{1-x^2} dt$$

$$= \int t\, dt = \frac{t^2}{2} + C = \frac{(\sin^{-1} x)^2}{2} + C$$

14. $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \int \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)} dx$

Let $3\cos x + 2\sin x = t$

$$\Rightarrow \quad -3\sin x + 2\cos x = \frac{dt}{dx}$$

$$\Rightarrow \quad dx = \frac{dt}{2\cos x - 3\sin x}$$

$$\therefore \quad \int \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)} dx$$

$$= \int \frac{2\cos x - 3\sin x}{2(t)} \frac{dt}{2\cos x - 3\sin x}$$

$$= \frac{1}{2} \int \frac{1}{t} dt = \frac{1}{2} \log|t| + C$$

$$= \frac{1}{2} \log|3\cos x + 2\sin x| + C$$

15. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

$$\Rightarrow \quad dx = 2\sqrt{x}\, dt$$

$$\therefore \quad \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int \frac{\cos t}{\sqrt{x}} 2\sqrt{x}\, dt = 2\int \cos t\, dt$$

$$= 2\sin t + C = 2\sin\sqrt{x} + C$$

16. $\int \cot x \log \sin x\, dx$

Let $\log \sin x = t \quad \Rightarrow \quad \frac{1}{\sin x} \times \cos x = \frac{dt}{dx}$

$$\Rightarrow \quad \cot x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{\cot x}$$

$$\therefore \quad \int \cot x \log \sin x\, dx = \int \cot x \cdot t \frac{dt}{\cot x}$$

$$= \int t\, dt = \frac{t^2}{2} + C = \frac{(\log \sin x)^2}{2} + C$$

17. Here, the given integrand cannot integrate directly. So, replace $\tan x$ by $\frac{\sin x}{\cos x}$ and then simplify it to convert in the form where we can use substitution method for integrating.

Let $I = \int \frac{1}{1 - \tan x} dx$

$$= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx = \int \frac{1}{\frac{\cos x - \sin x}{\cos x}} dx$$

$$= \frac{1}{2} \int \frac{\cos x + \cos x + \sin x - \sin x}{(\cos x - \sin x)} dx$$

(add and subtract $\cos x$ in numerator)

$$= \frac{1}{2} \int \frac{2(\cos x)\, dx}{(\cos x - \sin x)}$$

$$= \frac{1}{2} \left[\int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx + \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \right]$$

$$= \frac{1}{2} \left[\int 1 dx + \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \right]$$

Let $\cos x - \sin x = t \Rightarrow -\sin x - \cos x = \frac{dt}{dx}$

$$\Rightarrow \quad -[\sin x + \cos x] = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{-[\sin x + \cos x]}$$

$$\therefore \quad I = \frac{1}{2} \left[\int 1 dx + \int \frac{\cos x + \sin x}{t} \frac{dt}{-[\sin x + \cos x]} \right]$$

$$= \frac{1}{2} \left[\int 1 dx - \int \frac{1}{t} dt \right]$$

$$= \frac{1}{2} [x - \log|t|] + C = \frac{1}{2} [x - \log|\cos x - \sin x|] + C$$

18. Let $I = \int \frac{x^3 \sin(\tan^{-1} x^4)}{(1 + x^8)} dx$

Let $\tan^{-1} x^4 = t \quad \Rightarrow \quad \frac{1}{1 + x^8} . 4x^3 = \frac{dt}{dx}$

$$\Rightarrow \quad dx = \frac{(1 + x^8)}{4x^3} dt$$

$$\therefore \quad I = \int \frac{x^3 \sin t}{(1 + x^8)} . \frac{1 + x^8 dt}{4x^3}$$

$$= \frac{1}{4} \int \sin t\, dt = -\frac{1}{4} \cos t + C$$

$$= -\frac{1}{4} \cos(\tan^{-1} x^4) + C$$

19. $\int \left(\frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} \right) dx$

Let $x^{10} + 10^x = t \Rightarrow (10x^9 + 10^x \log_e 10)\, dx = dt$

$$\Rightarrow \quad dx = \frac{dt}{10x^9 + 10^x \log_e 10}$$

$$\therefore \int \left(\frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\right) dx = \int \frac{10x^9 + 10^x \log_e 10}{t} \times \frac{dt}{10x^9 + 10^x \log_e 10}$$

$$= \int \frac{dt}{t} = \log|t| + C = \log|10^x + x^{10}| + C$$

20. $\int \sin^3(2x+1)\,dx = \int \sin^2(2x+1)\cdot\sin(2x+1)\,dx$

$= \int [1 - \cos^2(2x+1)]\sin(2x+1)\,dx \qquad (\because \sin^2 x = 1 - \cos^2 x)$

Let $\cos(2x+1) = t \Rightarrow -2\sin(2x+1)\,dx = dt$

$\Rightarrow \sin(2x+1)\,dx = -\frac{dt}{2}$

$$\therefore \int \sin^3(2x+1)\,dx = \frac{-1}{2}\int (1-t^2)\,dt = \frac{-1}{2}\left(t - \frac{t^3}{3}\right) + C$$

$$= \frac{-1}{2}\left[\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right] + C$$

$$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$

21. Let $I = \int \frac{\cos x - \sin x}{1 + \sin 2x}\,dx$

$$= \int \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2\sin x\cos x}\,dx$$

$(\because \sin^2 x + \cos^2 x = 1;\ \sin 2x = 2\sin x\cos x)$

$$= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\,dx$$

Let $\cos x + \sin x = t \Rightarrow -\sin x + \cos x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{(\cos x - \sin x)}$

$$\therefore I = \int \frac{\cos x - \sin x}{t^2}\cdot\frac{dt}{(\cos x - \sin x)} = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt$$

$$= \frac{t^{-2+1}}{-2+1} + C = \frac{-1}{\cos x + \sin x} + C$$

22. $\int \tan^3 2x \sec 2x\,dx$

Let $\sec 2x = t \Rightarrow 2\sec 2x \tan 2x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{2\sec 2x\cdot\tan 2x}$

$$\therefore \int \tan^3 2x \sec 2x\,dx = \int \tan^3 2x \sec 2x \frac{dt}{2\sec 2x\cdot\tan 2x}$$

$$= \frac{1}{2}\int \tan^2 2x\,dt = \frac{1}{2}\int [\sec^2 2x - 1]\,dt \qquad (\because \tan^2 x = \sec^2 x - 1)$$

$$= \frac{1}{2}\int [(t^2 - 1)dt] = \frac{1}{2}\left[\frac{t^3}{3} - t\right] + C$$

$$= \frac{1}{2}\left[\frac{\sec^3 2x}{3} - \sec 2x\right] + C = \frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$$

23. $\int \frac{e^x(1+x)}{\cos^2(e^x x)}\,dx$

Let $xe^x = t \Rightarrow (xe^x + e^x) = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{e^x(x+1)}$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}\,dx = \int \frac{e^x(1+x)}{\cos^2 t} \times \frac{dt}{e^x(1+x)}$$

$$= \int \frac{1}{\cos^2 t}\,dt = \int \sec^2 t\,dt$$

$$= \tan t + C = \tan(xe^x) + C$$

24. $\int \frac{3x}{1+2x^4}\,dx = \frac{3}{2}\int \frac{x\,dx}{\frac{1}{2} + x^4} = \frac{3}{2}\int \frac{x\,dx}{\frac{1}{2} + (x^2)^2}$

Let $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

$$\therefore \frac{3}{2}\int \frac{x\,dx}{\frac{1}{2} + (x^2)^2} = \frac{3}{2}\int \frac{x}{\frac{1}{2} + (t)^2}\frac{dt}{2x}$$

$$= \frac{3}{4}\int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{3}{4} \times \frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right) + C$$

$$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C \ (\because t = x^2)$$

25. The given integrand cannot integrate directly, so put $\tan x = t$ to convert into standard form and then integrate it.

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\,dx$$

Let $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{\sec^2 x}$

$$\therefore \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\,dx = \int \frac{\sec^2 x}{\sqrt{t^2 + 4}}\frac{dt}{\sec^2 x} = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log|t + \sqrt{t^2 + 4}| + C$$

$$\left(\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}|\right)$$

$$= \log|\tan x + \sqrt{\tan^2 x + 4}| + C \qquad (\because t = \tan x)$$

26. Let $I = \int \frac{1}{e^x - 1}\,dx$

On multiplying numerator and denominator by e^{-x}, we get

$$I = \int \frac{e^{-x}}{1 - e^{-x}}\,dx$$

Put $1 - e^{-x} = t \Rightarrow -e^{-x}(-1) = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{e^{-x}} \Rightarrow e^{-x}dx = dt$

$$\therefore I = \int \frac{dt}{t} = \log|t| + C = \log|1 - e^{-x}| + C = \log\left|\frac{e^x - 1}{e^x}\right| + C$$

27. Let $I=\int \frac{1}{x^2(x^4+1)^{3/4}}dx=\int \frac{1}{x^2\left\{x^3\left(1+\frac{1}{x^4}\right)\right\}^{3/4}}dx$

$$=\int \frac{1}{x^2\cdot x^3\left(1+\frac{1}{x^4}\right)^{3/4}}dx=\int \frac{1}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}dx$$

Put $\quad 1+\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt$

$\Rightarrow \quad \frac{1}{x^5}dx=-\frac{dt}{4}$

$\therefore \quad I=\frac{1}{-4}\int \frac{1}{t^{3/4}}dt=-\frac{1}{4}\left[\frac{t^{1/4}}{1/4}\right]+C=-(1+x^{-4})^{1/4}+C$

28. Let $I=\int f'(ax+b)\,[f(ax+b)]^n dx$

Put $\quad f(ax+b)=t$

$\Rightarrow \quad f'(ax+b)\,a\,dx=dt$

$\Rightarrow \quad dx=\frac{dt}{af'(ax+b)}$

$\therefore \quad I=\int f'(ax+b)t^n\frac{dt}{f'(ax+b)a}$

$$=\frac{1}{a}\int t^n\,dt=\frac{1}{a}\left(\frac{t^{n+1}}{n+1}\right)+C=\frac{1}{a}\cdot\frac{[f(ax+b)]^{n+1}}{n+1}+C$$

29. $I=\int \frac{1}{e^x+e^{-x}}dx=\int \frac{1}{e^x+\frac{1}{e^x}}dx=\int \frac{e^x}{(e^x)^2+1}dx$

Put $e^x=t\Rightarrow e^x dx=dt \qquad \left(\because \int \frac{dx}{1+x^2}=\tan^{-1}x\right)$

$\therefore \quad I=\int \frac{e^x}{t^2+1}\frac{dt}{e^x}=\tan^{-1}(t)+C=\tan^{-1}e^x+C$

30. Let $I=\int \frac{\sqrt{x^2+1}}{x^4}dx$

Put $\quad x=\tan\theta\Rightarrow dx=\sec^2\theta\,d\theta$

$\therefore \quad I=\int \frac{\sqrt{\tan^2\theta+1}}{\tan^4\theta}\sec^2\theta\,d\theta=\int \frac{\sqrt{\sec^2\theta}}{\tan^4\theta}\sec^2\theta\,d\theta$

$$=\int \frac{\sec^3\theta}{\tan^4\theta}d\theta=\int \frac{1}{\cos^3\theta}\frac{\cos^4\theta}{\sin^4\theta}d\theta=\int \frac{\cos\theta\,d\theta}{\sin^4\theta}$$

$=\int \frac{dt}{t^4} \qquad$ (put $\sin\theta=t\Rightarrow\cos\theta\,d\theta=dt$)

$=\int t^{-4}dt=\frac{t^{-3}}{-3}+C=-\frac{1}{3t^3}+C$

$$=-\frac{1}{3\sin^3\theta}+C=-\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3+C$$

$\left(\text{from figure } \sin\theta=\frac{x}{\sqrt{1+x^2}}\right)$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$$

31. $I=\int \frac{dx}{e^x+e^{-x}+2}=\int \frac{e^x\,dx}{e^{2x}+2e^x+1}$

$$=\int \frac{e^x}{(e^x+1)^2}dx=-\frac{1}{e^x+1}+C$$

32. $I=\int \sqrt{\frac{x}{1-x^3}}dx=\int \frac{\sqrt{x}}{\sqrt{1-x^3}}dx=\int \frac{\sqrt{x}}{\sqrt{1-(x^{3/2})^2}}dx$

Put $\quad x^{3/2}=t \Rightarrow \sqrt{x}\,dx=\frac{2}{3}dt$

So, $\quad I=\frac{2}{3}\int \frac{dt}{\sqrt{1-t^2}}=\frac{2}{3}\sin^{-1}(x^{3/2})+C$

33. $I=\int \frac{x^2}{(x\sin x+\cos x)^2}=\int x\cdot\sec x\left[\frac{x\cos x}{(x\sin x+\cos x)^2}\right]dx$

$$=\frac{-x\sec x}{x\sin x+\cos x}+\tan x+C$$

34. On putting $\log(x+\sqrt{1+x^2})=t$

$\Rightarrow \quad \frac{1}{x+\sqrt{1+x^2}}\times\left(1+\frac{x}{\sqrt{1+x^2}}\right)dx=dt$

$\Rightarrow \quad \frac{dx}{\sqrt{1+x^2}}=dt$

$\therefore \quad \int \frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx=\int t\,dt=\frac{1}{2}t^2+C$

$$=\frac{1}{2}\log(x+\sqrt{x^2+1})^2+C$$

Thus, $\quad f(x)=\log(x+\sqrt{x^2+1})$ and $g(x)=\frac{x^2}{2}$

35. Let $I=\int \frac{dx}{(a^2+x^2)^{3/2}}$

Put $x=a\tan\theta\Rightarrow dx=a\sec^2\theta\,d\theta$

$\therefore \quad I=\int \frac{a\sec^2\theta}{(a^2+a^2\tan^2\theta)^{3/2}}d\theta$

$$=\int \frac{a\sec^2\theta}{a^3(\sec^2\theta)^{3/2}}d\theta$$

$\Rightarrow \quad I=\frac{1}{a^2}\int \frac{d\theta}{\sec\theta}=\frac{1}{a^2}\int \cos\theta\,d\theta$

$$=\frac{1}{a^2}\sin\theta+C$$

$\Rightarrow \quad =\frac{x}{a^2(x^2+a^2)^{1/2}}+C$

36. Let $I=\int \frac{3^x}{\sqrt{9^x-1}}dx=\int \frac{3^x dx}{\sqrt{(3^x)^2-1}}$

Put $3^x=t \Rightarrow 3^x \log 3\, dx=dt$

$\Rightarrow \quad 3^x\, dx=\frac{dt}{\log 3}$

$\therefore \quad I=\frac{1}{\log 3}\int \frac{dt}{\sqrt{t^2-1}}$

$=\frac{1}{\log 3}\log[t+\sqrt{t^2-1}]+C$

$=\frac{1}{\log 3}\log[3^x+\sqrt{9^x-1}]+C$

37. Let $I=\int \frac{dx}{x(\log x)(\log\log x)\ldots\underbrace{(\log\log\ldots x)}_{8\text{ times}}}$

Put $\underbrace{(\log\log\ldots x)}_{8\text{ times}}=t$

$\Rightarrow \quad I=\frac{1}{\underbrace{(\log\log\ldots x)}_{7\text{ times}}\ldots(\log\log x)(\log x)x}dx=dt$

$\therefore \quad I=\int \frac{dt}{t}=\log t+C=\underbrace{(\log\log\log\ldots x)}_{9\text{ times}}+C$

38. $I=\int \frac{x^2}{(ax+b)^2}dx$

Put $ax+b=t$

$\Rightarrow \quad dx=\frac{1}{a}dt$

and $x=\left(\frac{t-b}{a}\right)$

$\therefore \quad I=\frac{1}{a^3}\int \frac{(t-b)^2}{t^2}dt=\frac{1}{a^3}\int\left(1+\frac{b^2}{t^2}-\frac{2b}{t}\right)dt$

$=\frac{1}{a^3}\left(t-\frac{b^2}{t}-2b\log t\right)+C$

$=\frac{1}{a^3}\left[ax+b-\frac{b^2}{ax+b}-2b\log(ax+b)\right]+C$

$=\frac{2}{a^2}\left[x-\frac{b}{a}\log(ax+b)\right]-\frac{x^2}{a(ax+b)}+C$

39. Let $I=\int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

$=\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}+3\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

Put $x+\frac{1}{x}=t$

$\Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx=dt$ and $\left(x+\frac{1}{x}\right)^2=t^2$

$\Rightarrow \quad x^2+\frac{1}{x^2}=t^2-2$

$\therefore \quad I=\int \frac{1}{(t^2+1)\tan^{-1}t}dt=\log(\tan^{-1}t)+C$

$=\log\left[\tan^{-1}\left(x+\frac{1}{x}\right)\right]+C$

40. Let $I=\int \frac{mx^{m+2n-1}-nx^{n-1}}{x^{2m+2n}+2x^{m+n}+1}dx$

$=\int \frac{mx^{m+2n-1}-nx^{n-1}}{(1+x^{m+n})^2}dx$

$\therefore \quad I=\int \frac{mx^{m-1}-\frac{n}{x^{n+1}}}{\left(x^m+\frac{1}{x^n}\right)^2}dx$

Let $x^m+\frac{1}{x^n}=t$

$\Rightarrow \quad mx^{m-1}-\frac{n}{x^{n+1}}dx=dt$

$\therefore \quad I=\int \frac{1}{t^2}dt=-\frac{1}{t}+C$

$=\frac{-1}{\left(x^m+\frac{1}{x^n}\right)}+C=\frac{-x^n}{x^{m+n}+1}+C$

41. The primitive of the given function is

$\int \frac{dx}{\sqrt{\sin^3 x\sin(x+\theta)}}=\int \frac{dx}{\sqrt{\sin^3 x(\sin x\cos\theta+\cos x\sin\theta)}}$

$=\int \frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\cos\theta+\cot x\sin\theta}}$

Put $\cot x=t \Rightarrow -\operatorname{cosec}^2 x\,dx=dt$

$=-\frac{1}{\sqrt{\sin\theta}}\int \frac{dt}{\sqrt{\cot\theta+1}}$

$=-\frac{2}{\sqrt{\sin\theta}}(\cot\theta+t)^{1/2}+C$

$=-\frac{2}{\sin\theta}(\cos\theta+t\sin\theta)^{1/2}+C$

$=\frac{-2\operatorname{cosec}\theta(\sin x\cos\theta+\sin\theta\cos x)^{1/2}}{\sqrt{\sin x}}+C$

$=-2\operatorname{cosec}\theta\left(\frac{\sin(\theta+x)}{\sin x}\right)^{1/2}+C$

42. Let $I=\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{8-\left[x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]}}dx$

$=\int \frac{1}{\sqrt{8-\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right]}}dx$

$=\int \frac{1}{\sqrt{8+\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$

Let $x-\frac{3}{2}=t \Rightarrow dx=dt$

$$\therefore \quad I=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}\,dt=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)\right]=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C \qquad \left(\because t=x-\frac{3}{2}\right)$$

43. $\int \frac{1}{\sqrt{9x-4x^2}}\,dx=\frac{1}{\sqrt{4}}\int \frac{1}{\sqrt{\frac{9}{4}x-x^2}}\,dx$

$$=\frac{1}{2}\int \frac{1}{\sqrt{-\left[x^2-\frac{9}{4}x+\left(\frac{9}{8}\right)^2\right]+\left(\frac{9}{8}\right)^2}}\,dx$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}\,dx$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)+C \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

44. Here, we can convert the integrand into some standard integrand of the form $\sqrt{x^2 \pm a^2}$ and then integrate.

Let $\quad I=\int \sqrt{x^2+4x+6}\,dx$

$$=\int \sqrt{x^2+4x+2^2+6-4}\,dx$$

$$=\int \sqrt{(x+2)^2+(\sqrt{2})^2}\,dx$$

$$\left(\because \int \sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right)$$

$$=\frac{x+2}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log|(x+2)+\sqrt{x^2+4x+6}|+C$$

$$\Rightarrow I=\frac{x+2}{2}\sqrt{x^2+4x+6}+\log|(x+2)+\sqrt{x^2+4x+6}|+C$$

45. Let $I=\int \sqrt{1+3x-x^2}\,dx=\int \sqrt{-(x^2-3x-1)}\,dx$

$$=\int \sqrt{-\left[x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-1\right]}\,dx$$

$$=\int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-1\right]}\,dx=\int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2-\frac{9+4}{4}\right]}\,dx$$

$$=\int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2-\frac{13}{4}\right]}\,dx=\int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

$$\therefore \quad I=\frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)$$

$$\left(\because \int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C\right)$$

46. Let $\quad I=\int \sqrt{x^2+3x}\,dx=\int \sqrt{x^2+3x-\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}\,dx$

$$=\int \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I=\frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}-\frac{9}{4\times 2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

$$\left(\because \int \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right)$$

$$\Rightarrow \quad I=\frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(\frac{2x+3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

47. Let $I=\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

Let $3\sin x+2\cos x=A\frac{d}{dx}(3\cos x+2\sin x)$
$\qquad +B(3\cos x+2\sin x)$

$\Rightarrow 3\sin x+2\cos x=A(-3\sin x+2\cos x)$
$\qquad +B(3\cos x+2\sin x)$

On comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$-3A+2B=3$ and $2A+3B=2$

$$\Rightarrow \quad B=\frac{12}{13} \text{ and } A=-\frac{5}{13}$$

$$\therefore \quad I=\int \frac{B(-3\sin x+2\cos x)+A(3\cos x+2\sin x)}{3\cos x+2\sin x}\,dx$$

$$\Rightarrow \quad I=A\int 1\,dx+B\int \frac{-3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$$

$$=Ax+B\int \frac{dt}{t}, \text{ where } t=3\cos x+2\sin x$$

$$=Ax+B\ln|t|+C$$

$$=\frac{-5}{13}x+\frac{12}{13}\ln|3\cos x+2\sin x|+C$$

48. $\int \frac{x^2-1}{x^4+x^2+1}\,dx$

$$\Rightarrow \quad I=\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}\,dx=\int \frac{1-\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2-1}\,dx$$

Put $\quad x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt$

$$\therefore \quad I=\int \frac{dt}{t^2-1^2}=\frac{1}{2\times 1}\log\frac{t-1}{t+1}+C$$

$$=\frac{1}{2}\log\frac{x^2+1-x}{x^2+1+x}+C$$

49. $\int \frac{2\,e^x+3\,e^{-x}}{3\,e^x+4\,e^{-x}}\,dx=\int \frac{2\,e^{2x}+3}{3\,e^{2x}+4}\,dx$

Let $\quad 2\,e^{2x}+3=A(3\,e^{2x}+4)+B(6\,e^{2x})$

$\Rightarrow \quad 2\,e^{2x}+3=(3A+6B)\,e^{-2x}+4A$

On comparing both sides, we get

$$2=3A+6B \qquad \ldots(i)$$

and $\quad 3=4A \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$2=\frac{9}{4}+6B$$

$$\Rightarrow \quad 6B=2-\frac{9}{4}=-\frac{1}{4}$$

$$\Rightarrow \quad B=-\frac{1}{24}$$

$$\therefore \quad \int \frac{2e^{2x}+3}{3e^{2x}+4}dx=\int \frac{3\,(3e^{2x}+4)}{4\,(3e^{2x}+4)}dx-\frac{1}{24}\int \frac{6e^{2x}}{3e^{2x}+4}dx$$

$$=\frac{3}{4}x-\frac{1}{24}\log(3e^{2x}+4)+C$$

50. Let $\quad I=\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Now, let $\quad x=t^6 \Rightarrow dx=6t^5\,dt$

$$\Rightarrow \quad I=\int \frac{6\,t^5}{t^3+t^2}dt=6\int \frac{t^3}{t+1}dt$$

$$=6\int\left(t^2-t+1-\frac{1}{t+1}\right)dt$$

$$=2\,t^3-3\,t^2+6\,t-6\log(t+1)+C$$

$$=2\sqrt{x}-3\,(\sqrt[3]{x})+6(\sqrt[6]{x})-6\log(\sqrt[6]{x}+1)+C$$

51. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}=\int \frac{\sec^2 x\,dx}{a^2\tan^2 x+b^2}=\frac{1}{a^2}\int \frac{dt}{t^2+\left(\frac{b}{a}\right)^2}$

where, $\quad t=\tan x$

$$=\frac{1}{a^2}\cdot\frac{a}{b}\tan^{-1}\frac{t}{b/a}+C$$

$$=\frac{1}{ab}\tan^{-1}\frac{a(\tan x)}{b}+C$$

52. Let $I=\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$$=\int \frac{dx}{\cos^3 x\sqrt{4\sin x\cos x}}=\frac{1}{2}\int \frac{dx}{\cos^{7/2}x\sin^{1/2}x}$$

$$=\frac{1}{2}\int \frac{\sec^4 x}{\sqrt{\tan x}}dx=\frac{1}{2}\int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}}dx$$

Put $\quad \tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$\therefore \quad I=\frac{1}{2}\int \frac{1+t^2}{\sqrt{t}}dt$$

$$=\frac{1}{2}\int t^{-1/2}dt+\frac{1}{2}\int t^{3/2}dt=t^{1/2}+\frac{t^{5/2}}{5}+C$$

$$=\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x+C$$

53. Let $\quad I=\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$

Using the rule

$$\text{Nr}=\lambda\,\text{Dr}+\mu\,(\text{dc of Dr})$$

$$\Rightarrow \quad 4e^x+6e^{-x}=\lambda(9e^x-4e^{-x})+\mu\,(9e^x+4e^{-x})$$

$$\Rightarrow \quad 4e^x+6e^{-x}=9(\lambda+\mu)e^x+4(\mu-\lambda)e^{-x}$$

$$\therefore \quad \lambda+\mu=\frac{4}{9} \text{ and } \lambda-\mu=-\frac{6}{4}$$

On solving, we have $\lambda=-\frac{19}{36}, \mu=\frac{35}{36}$

$$\therefore \quad I=\int \frac{\lambda\,\text{Dr}+\mu(\text{derivative of Dr})}{\text{Dr}}dx$$

$$\Rightarrow \quad I=\lambda\int dx+\mu\int \frac{(\text{derivative of Dr})}{\text{Dr}}dx$$

$$\Rightarrow \quad I=\lambda x+\mu\log|\text{Dr}|+C$$

$$\Rightarrow \quad I=-\frac{19}{36}x+\frac{35}{36}\log(9e^x-4e^{-x})+C$$

$$\Rightarrow \quad I=-\frac{19}{36}x+\frac{35}{36}\log(9e^{2x}-4)-\frac{35}{36}\log e^x+C$$

$$\Rightarrow \quad I=\left(-\frac{19}{36}-\frac{35}{36}\right)x+\frac{35}{36}\log(9e^{2x}-4)+C$$

$$I=-\frac{54}{36}x+\frac{35}{36}\log(9e^{2x}-4)+C$$

$$I=-\frac{3}{2}x+\frac{35}{36}\log(9e^{2x}-4)+C$$

$\therefore \quad A=-\frac{3}{2}, B=\frac{35}{36}$ and C is a constant

54. Put $\quad x=\frac{1}{t} \Rightarrow dx=-\frac{dt}{t^2}$

So, $\quad I=\int \frac{-dt}{t^2\left(\frac{1}{t^2}+5\right)\sqrt{\frac{2}{t^2}-3}}=\int \frac{-t\,dt}{(1+5t^2)\sqrt{2-3t^2}}$

Put $\quad 2-3t^2=y^2 \Rightarrow -t\,dt=\frac{y\,dy}{3}$

So, $\quad I=-\frac{1}{3}\int \frac{y\,dy}{\left(\frac{13-5y^2}{3}\right)y}=\frac{1}{5}\ln\frac{y-\sqrt{13/5}}{y+\sqrt{13/5}}+C$

$$=\frac{1}{5}\log\left|\frac{\sqrt{2-3t^2}-\sqrt{13/5}}{\sqrt{2-3t^2}+\sqrt{13/5}}\right|+C$$

$$=\frac{1}{5}\log\left|\frac{\sqrt{2-3/x^2}-\sqrt{13/5}}{\sqrt{2-3/x^2}+\sqrt{13/5}}\right|+C$$

55. Put $x+2=\frac{1}{t} \Rightarrow dx=\frac{-dt}{t^2}$

Now, $x^2+4x+8=(x+2)^2+4$

So, $I=\int \frac{\frac{-dt}{t^2}}{\frac{1}{t}\sqrt{\frac{1}{t^2}+4}}=-\int \frac{dt}{\sqrt{1+4t^2}}$

$$=-\frac{1}{2}\int \frac{dt}{\sqrt{t^2+\frac{1}{4}}}=-\frac{1}{2}\ln\left|t+\sqrt{t^2+\frac{1}{4}}\right|+C$$

$$=-\frac{1}{2}\ln\left|\frac{1}{x+2}t+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+C$$

56. Let $I=\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx \Rightarrow I=\int \cos^{-1}x\cdot\frac{x}{\sqrt{1-x^2}}dx$

Consider $\cos^{-1}x$ as first function and $\frac{x}{\sqrt{1-x^2}}$ as second function and integrating by parts, we get

$$I=\cos^{-1}x\int \frac{x}{\sqrt{1-x^2}}dx-\int\left[\frac{d}{dx}(\cos^{-1}x)\int \frac{x}{\sqrt{1-x^2}}dx\right]dx$$

Let $1-x^2=t^2 \Rightarrow -2x=2t\frac{dt}{dx} \Rightarrow dx=-\frac{t}{x}dt$

$$\therefore \quad I=\cos^{-1}x\int \frac{x}{t}\left(\frac{-t}{x}\right)dt-\int\left[\frac{d}{dx}(\cos^{-1}x)\int \frac{x}{t}\left(\frac{-t}{x}\right)dt\right]dx$$

$$=\cos^{-1}x(-t)-\int\left[\frac{(-1)}{\sqrt{1-x^2}}(-t)\,dt\right]dx$$

$$=-\sqrt{1-x^2}\cdot\cos^{-1}x-\int\left[\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx$$

$(\because 1-x^2=t^2 \Rightarrow t=\sqrt{1-x^2})$

$$=-\sqrt{1-x^2}\cdot\cos^{-1}x-\int 1\,dx$$

$$=-\sqrt{1-x^2}\cdot\cos^{-1}x-x+C$$

57. Let $I=\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

On putting $x=\tan\theta$, we get $I=\int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)dx$

$$=\int \sin^{-1}(\sin 2\theta)\,dx \qquad \left(\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=2\int \theta\,dx=2\int \tan^{-1}x\,dx \quad (\because x=\tan\theta \Rightarrow \theta=\tan^{-1}x)$$

$$=2\int \underset{\text{II}}{1}\cdot\underset{\text{I}}{\tan^{-1}x}\,dx$$

$$=2\tan^{-1}x\int 1\,dx-2\int\left[\frac{d}{dx}\tan^{-1}x\int 1\,dx\right]dx$$

(using integration by parts)

$$\Rightarrow \quad I=2\tan^{-1}x\cdot x-2\int\left[\frac{1}{1+x^2}\cdot x\right]dx$$

Put $1+x^2=t \Rightarrow 2x=\frac{dt}{dx} \Rightarrow dx=\frac{dt}{2x}$

$$\therefore \quad I=2x\tan^{-1}x-2\int\left[\frac{x}{t}\cdot\frac{dt}{2x}\right]$$

$$=2x\tan^{-1}x-\int \frac{1}{t}dt=2x\tan^{-1}x-\log|t|+C$$

$$\Rightarrow \quad I=2x\tan^{-1}x-\log|1+x^2|+C$$

58. Let $I=\int \cos(\log_e x)\,dx=\int \cos(\log_e x)\cdot 1dx$

$$=\cos(\log_e x)\,x-\int \frac{-\sin(\log_e x)}{x}\cdot x\,dx$$

$$=x\cos(\log_e x)+\int \sin(\log_e x)\,dx$$

$$=x\cos(\log_e x)+\int \sin(\log_e x)\cdot 1dx$$

$$=x\cos(\log_e x)+\sin(\log_e x)x-\int \frac{\cos(\log_e x)}{x}x\,dx$$

$$=x\cos(\log_e x)+x\sin(\log_e x)-I$$

$$2I=x[\cos(\log_e x)+\sin(\log_e x)]$$

$$I=\frac{x}{2}[\cos(\log_e x)+\sin(\log_e x)]$$

59. Let $I=\int |x|\cdot 1dx$

$$=|x|x-\int \frac{|x|}{x}x\,dx=x|x|-\int |x|\,dx$$

$$\Rightarrow \quad 2I=x|x|$$

$$\Rightarrow \quad I=\frac{x|x|}{2}+C$$

60. $\int \frac{x}{(x^2+1)(x-1)}dx$

First, we resolve the given integrand into partial fractions.

Let $\frac{x}{(x^2+1)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$ ...(i)

$\Rightarrow \quad x=A(x^2+1)+(Bx+C)(x-1)$ ...(ii)

On substituting $x=1$ and 0 in Eq. (ii), we get

$$1=A(2) \text{ and } 0=A-C$$

$$\Rightarrow \quad A=\frac{1}{2} \text{ and } C=A=\frac{1}{2}$$

On equating the coefficient of x^2 on the both sides in Eq. (ii), we get

$$0=A+B \Rightarrow B=-A=-\frac{1}{2}$$

$$\therefore \quad \int \frac{x}{(x^2+1)(x-1)}dx=\int\left\{\frac{\frac{1}{2}}{x-1}+\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}\right\}dx$$

$$=\frac{1}{2}\int \frac{1}{x-1}dx+\frac{1}{2}\int \frac{(-x+1)}{x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}I_1 + \frac{1}{2}\tan^{-1}x + C_1 \quad \text{...(iii)}$$

where, $I_1 = \int \frac{2x}{x^2+1}dx$

Let $(x^2+1) = t \Rightarrow 2x\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| + C_2 = \log|x^2+1| + C_2$

On putting these values in Eq. (iii), we get

$$\int \frac{x}{(x^2+1)(x-1)}dx = \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2 + 1| + \frac{1}{2}\tan^{-1}x + C \; (C = C_1 + C_2)$$

61. Here, the denominator of the given integral can be factorise, so we convert it into the product of linear factors and then apply partial fractions to convert the given integrand in simple form and then integrate.

$$\int \frac{3x+5}{x^3-x^2-x+1}dx = \int \frac{3x+5}{(x^2-1)(x-1)}dx$$

$$= \int \frac{3x+5}{(x-1)(x+1)(x-1)}dx = \int \frac{3x+5}{(x-1)^2(x+1)}dx$$

Let $\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$\Rightarrow \quad \frac{3x+5}{(x-1)^2(x+1)} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$

$\Rightarrow \quad 3x+5 = A(x^2-1) + Bx + B + C(x^2+1-2x)$

$\Rightarrow \quad 3x+5 = x^2(A+C) + x(B-2C) + (-A+B+C)$

On comparing the coefficients of x^2, x and constant term on both sides, we get

$A + C = 0 \Rightarrow A = -C$ ...(i)

$B - 2C = 3$ ...(ii)

and $-A + B + C = 5$ ...(iii)

On putting the value of A from Eq. (i) in Eq. (iii), we get

$-(-C) + B + C = 5$

$\Rightarrow \quad B + 2C = 5$ ...(iv)

On adding Eq. (ii) and Eq. (iv), we get $2B = 8 \Rightarrow B = 4$

On putting the value of B in Eq. (ii), we get $4 - 2C = 3$

$\Rightarrow \quad 1 = 2C \Rightarrow C = \frac{1}{2}$

Also, $A = -C = -\frac{1}{2}$

$$\therefore \int \frac{3x+5}{(x-1)^2(x+1)}dx = \int \left\{\frac{-\frac{1}{2}}{x-1} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}\right\}dx$$

$$= -\frac{1}{2}\int \frac{1}{x-1}dx + 4\int \frac{1}{(x-1)^2}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= -\frac{1}{2}\log|x-1| + 4\frac{(x-1)^{-2+1}}{(-2+1)} + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C \quad \left[\because \log b - \log a = \log\left(\frac{b}{a}\right)\right]$$

62. In rational algebraic functions, if the degree of numerator is greater than or equal to the degree of denominator, then always divide the numerator by denominator and use the result.

$$\frac{\text{Numerator}}{\text{Benominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Thus, we can convert the given integrand into the simple form and then integrate easily.

$$\int \frac{x^3+x+1}{x^2-1}dx = \int x\,dx + \int \frac{2x+1}{x^2-1}dx$$

$$\left(\because \frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}\right)$$

$$= \int x\,dx + \int \frac{2x+1}{(x+1)(x-1)}dx$$

Let $\frac{2x+1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$\Rightarrow \quad \frac{2x+1}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$

$\Rightarrow \quad 2x+1 = x(A+B) - A + B$

On comparing the coefficients of x and constant term on both sides, we get

$A + B = 2$

and $-A + B = 1$

On adding above equations, we get $2B = 3 \Rightarrow B = \frac{3}{2}$

and then $A = \frac{1}{2}$

$$\therefore \int \frac{x^2+x+1}{x^2-1}dx = \int x\,dx + \int \frac{A}{(x+1)}dx + \int \frac{B}{(x-1)}dx$$

$$= \int x\,dx + \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{3}{2}\int \frac{1}{(x-1)}dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

63. Here, partial fractions of given integrand cannot find easily, so we convert it into easy form for partial fractions by multiply x^{n-1} in both numerator and denominator and then put $x^n = t$. Now, apply partial fractions method and then integrate.

Let $I = \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$

Put $x^n = t \Rightarrow nx^{n-1}dx = dt \Rightarrow x^{n-1}dx = \frac{1}{n}dt$

$\therefore \quad I = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$ ...(i)

Now, $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$

$\Rightarrow$ $1 = A(1+t) + Bt$ ...(ii)

On comparing, we get

$A + B = 0,\ A = 1$

$\Rightarrow$ $B = -1$

$\therefore$ $I = \int \left[\frac{1}{t} - \frac{1}{(t+1)}\right] dt$

$= \log|t| - \log|t+1| + C$

$= \log\left|\frac{t}{t+1}\right| + C$

$= \log\left|\frac{x^n}{x^n+1}\right| + C$

64. Here, partial fractions of given integrand cannot find easily, so we multiply both numerator and denominator by x^3 and then put $x^4 = t$ and then use partial fractions to convert it into simple form . After integration put $t = x^4$.

Let

$$I = \int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx = \frac{1}{4}\int \frac{4x^3}{x^4(x^4-1)} dx$$

Put $x^4 = t \Rightarrow 4x^3 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{4x^3}$

$\therefore$ $I = \frac{1}{4}\int \frac{4x^3}{t(t-1)} \frac{dt}{4x^3} = \frac{1}{4}\int \frac{dt}{t(t-1)}$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)} \Rightarrow 1 = A(t-1) + Bt$ ...(i)

On substituting $t = 0$ and 1 in Eq. (i), we get

$A = -1$ and $B = 1$

$\therefore$ $\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$\therefore$ $I = \frac{1}{4}\int \left(-\frac{1}{t} + \frac{1}{t-1}\right) dt = \frac{1}{4}\{-\log|t| + \log|t-1|\} + C$

$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C = \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$ (put $t = x^4$)

65. Let $I = \int \frac{1}{\sin x - \sin 2x} dx$

$= \int \frac{1}{\sin x - 2\sin x\cos x} dx$

$= \int \frac{1}{\sin x(1-2\cos x)} dx$

$= \int \frac{\sin x}{\sin^2 x(1-2\cos x)} dx$

$= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)} dx$

(put $\cos x = t \Rightarrow -\sin x\, dx = dt$)

$\therefore$ $I = \int \frac{-dt}{(1-t^2)(1-2t)} = \int \frac{-1}{(1-t)(1+t)(1-2t)}$ ...(i)

Here, in Eq. (i), we have linear and non-repeated factors, thus we use partial fractions for,

$$\frac{-1}{(1-t)(1+t)(1-2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1-2t)}$$

or $-1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$ ...(ii)

On putting $(t+1) = 0$ or $t = -1$, we get

$-1 = B(2)(1+2) \Rightarrow B = -\frac{1}{6}$

On putting $(1-t) = 0$ or $t = 1$, we get

$-1 = A(2)(-1) \Rightarrow A = \frac{1}{2}$

On putting $(1-2t) = 0$ or $t = \frac{1}{2}$, we get

$-1 = C\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right) \Rightarrow C = -\frac{4}{3}$

$\therefore$ $\frac{-1}{(1-t)(1+t)(1-2t)} = \frac{1}{2(1-t)} - \frac{1}{6(1+t)} - \frac{4}{3(1-2t)}$

So, Eq. (i) reduces to,

$I = \frac{1}{2}\int \frac{1}{1-t} dt - \frac{1}{6}\int \frac{1}{1+t} dt - \frac{4}{3}\int \frac{1}{1-2t} dt$

$= -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| - \frac{4}{3}\times\left(-\frac{1}{2}\right)\log|1-2t| + C$

$= \left[-\frac{1}{2}\log|1-\cos x| - \frac{1}{6}|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C\right]$

66. Let $I = \int \frac{e^x}{(1+e^x)(2+e^x)} dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $I = \int \frac{e^x}{(1+t)(2+t)} \frac{dt}{e^x} = \int \frac{1}{(1+t)(2+t)} dt$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$

$\Rightarrow$ $1 = A(2+t) + B(1+t) = (2A+B) + t(A+B)$

On equating the coefficients of t and constant term on both sides, we get $A + B = 0$ and $2A + B = 1$

On solving both equations, we get $A = 1$ and $B = -1$

$\therefore$ $I = \int \frac{1}{(1+t)} dt - \int \frac{1}{(2+t)} dt$

$= \log|1+t| - \log|2+t| + C$

$= \log\left|\frac{1+t}{2+t}\right| + C = \log\left|\frac{1+e^x}{2+e^x}\right| + C$

67. $\int \frac{2x^2+3}{(x^2-1)(x^2+4)} dx = \int \frac{1}{x^2-1} dx + \int \frac{1}{x^2+4} dx$

$= \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

But $\int \frac{2x^2+3}{(x^2-1)(x^2+4)} dx = \left[a\log\left(\frac{x-1}{x+1}\right) + b\tan^{-1}\left(\frac{x}{2}\right) + C\right]$

$\therefore$ $a = \frac{1}{2}$ and $b = \frac{1}{2}$

68. Here, $R(\sin x, \cos x) = \dfrac{\cos x\, dx}{\sin^2 x(\sin x + \cos x)}$

Here, $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

So, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int \frac{\cos x \sec^2 x\, dx}{\sec^2 x \sin^2 x(\sin x + \cos x)} = \int \frac{dt}{t^2(1+t)}$$

Let $\dfrac{1}{t^2(1+t)} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{(1+t)}$ or $1 = At(1+t) + B(1+t) + Ct^2$

Put $t = 0$, we get $B = 1$, put $t = -1$, we get $C = 1$

On compare the coefficients of t^2, we get $0 = A + C$

$\Rightarrow$ $A = -1$

$$\therefore \int \left[\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{(1+t)}\right] dt$$

$$= -\log|t| + \left(\frac{-1}{t}\right) + \log|1+t| + C$$

$$= -\frac{1}{t} + \log\left|\frac{1+t}{t}\right| + C$$

$$= -\frac{1}{\tan x} + \log\left|\frac{1+\tan x}{\tan x}\right| + C$$

$$= -\cot x + \log\left|\frac{1+\tan x}{\tan x}\right| + C$$

69. Let $I = \displaystyle\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

$$= \int \frac{\sec^2 x\, dx}{(\tan x - 2)(2\tan x + 1)}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore I = \int \frac{dt}{(t-2)(2t+1)}$$

$$= \frac{1}{5}\int \frac{1}{t-2} dt - \frac{2}{5}\int \frac{1}{2t+1} dt$$

$$= \frac{1}{5}\log(t-2) - \frac{1}{5}\log(2t+1) + C$$

$$= \log \sqrt[5]{\frac{\tan x - 2}{2\tan x + 1}} + C$$

70. There is no standard integral formula for given integrand. So, we simplify the function by using the formula, $2\cos A \cos B = [\cos(A+B) + \cos(A-B)]$ and then integrate it.

$$\int \cos 2x \cos 4x \cos 6x\, dx = \int \cos 2x \left[\frac{1}{2}\{\cos(4x+6x) + \cos(4x-6x)\}\right] dx$$

$[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)]$

$$= \frac{1}{2}\int [\cos 2x \cos 10x + \cos 2x \cos(-2x)]\, dx$$

$$= \frac{1}{2}\int [\cos 2x \cos 10x + \cos^2 2x]\, dx \qquad (\because \cos(-\theta) = \cos\theta)$$

$$= \frac{1}{2}\int \left[\frac{1}{2}\{\cos(2x+10x) + \cos(2x-10x)\} + \left(\frac{1+\cos 4x}{2}\right)\right] dx$$

$(\therefore \cos 2\theta = 2\cos^2\theta - 1)$

$$= \frac{1}{4}\int [\cos 12x + \cos 8x + 1 + \cos 4x]\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4}\right] + C$$

71. $\displaystyle\int \cos^4 2x\, dx = \int (\cos^2 2x)^2\, dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx$

$$= \frac{1}{4}\int (1+\cos 4x)^2\, dx = \frac{1}{4}\int (1 + \cos^2 4x + 2\cos 4x)\, dx$$

$$= \frac{1}{4}[\int 1\, dx + \int \cos^2 4x\, dx + 2\int \cos 4x\, dx]$$

$$= \frac{1}{4}\left[\int 1\, dx + \int \frac{1+\cos 8x}{2} dx + 2\int \cos 4x\, dx\right]$$

$\left(\because \cos^2 x = \dfrac{1+\cos 2x}{2}\right)$

$$= \frac{1}{4}\left[\int 1\, dx + \frac{1}{2}\{\int 1\, dx + \int \cos 8x\, dx\} + 2\int \cos 4x\, dx\right]$$

$$= \frac{1}{4}\left[x + \frac{1}{2}\left\{x + \frac{\sin 8x}{8}\right\} + \frac{2\sin 4x}{4}\right] + C$$

$$= \frac{1}{4}x + \frac{x}{8} + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C = \left[\frac{3x}{8} + \frac{\sin 8x}{64} + \frac{\sin 4x}{8}\right] + C$$

72. $\displaystyle\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{(\cos x - \cos\alpha)} dx \qquad (\because \cos 2x = 2\cos^2 x - 1)$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{(\cos x - \cos\alpha)} dx = \int \frac{2(\cos^2 x - \cos^2\alpha)}{(\cos x - \cos\alpha)} dx$$

$$= 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)} dx$$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$$= 2[\int \cos x\, dx + \cos\alpha \int 1\, dx]$$

$$= 2[\sin x + \cos\alpha \cdot x] + C = 2\sin x + 2x\cos\alpha + C$$

73. $\displaystyle\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x} dx = \int \frac{\{(\sin^4 x)^2 - (\cos^4 x)^2\}}{1 - 2\sin^2 x\cos^2 x} dx$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x\cos^2 x} dx$$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$$= \int \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)[(\sin^2 x)^2 + (\cos^2 x)^2]}{1 - 2\sin^2 x\cos^2 x} dx$$

$$= \int \frac{(\sin^2 x - \cos^2 x)1\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x\}}{1 - 2\sin^2 x\cos^2 x} dx$$

$[\because \sin^2 x + \cos^2 x = 1$ and $a^2 + b^2 = (a+b)^2 - 2ab]$

$$= \int \frac{-\cos 2x\{1-2\sin^2 x\cdot\cos^2 x\}}{1-2\sin^2 x\cos^2 x}dx \quad (\because \cos 2x = \cos^2 x - \sin^2 x)$$

$$= -\int \cos 2x\, dx = -\frac{\sin 2x}{2} + C$$

74. Let $I = \int \sqrt{2}\sqrt{1+\sin x}\,dx = \sqrt{2}\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx$

$$= 2\int \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx = -4\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+C$$

But $I = -4\cos(ax+b)+C$ (given)

On comparing, we get

$$a = \frac{1}{2}, \quad b = \frac{\pi}{4}$$

75. $\int \frac{dx}{\sin x - \cos x + \sqrt{2}}$

$$= \int \frac{dx}{\sin x\frac{\sqrt{2}}{\sqrt{2}} - \cos x\frac{\sqrt{2}}{\sqrt{2}} + \sqrt{2}}$$

$$= \int \frac{dx}{\sqrt{2}(\sin x\sin\frac{\pi}{4} - \cos x\cos\frac{\pi}{4} + 1)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{1-\cos\left(x+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{1-\cos 2\left(\frac{x}{2}+\frac{\pi}{8}\right)} = \frac{1}{\sqrt{2}}\int \frac{dx}{2\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}$$

$$= \frac{1}{2\sqrt{2}}\int \operatorname{cosec}^2\left(\frac{x}{2}+\frac{\pi}{8}\right)dx$$

$$= -\frac{1}{2\sqrt{2}}\cdot\frac{-\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\frac{1}{2}} + C = \frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$$

76. Given, $\int \frac{\cos 4x+1}{\cot x - \tan x}dx = A\cos 4x + B$

Let $I = \int \frac{\cos 4x+1}{\cot x - \tan x}dx$

$$= \int \frac{2\cos^2 2x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}dx = \int \frac{2\cos^2 2x}{\frac{\cos 2x}{\sin x\cos x}}dx$$

$$= \int \sin 2x\cos 2x\,dx$$

$$= \frac{1}{2}\int 2\sin 2x\cos 2x\,dx$$

$$= \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\left[-\frac{\cos 4x}{4}\right]+B$$

$$= -\frac{1}{8}\cos 4x + B$$

But $I = A\cos 4x + B$(given)

$\therefore \quad A = -\frac{1}{8}$

77. $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+\theta)+\sin\theta}}$

$$= \int \frac{\sec x\,dx}{\sqrt{2\sin(x+\theta)\cos x}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^{3/2} x\,dx}{\sqrt{\sin x\cos\theta + \sin\theta\cos x}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2 x\,dx}{\sqrt{\tan x\cos\theta + \sin\theta}}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t\cos\theta + \sin\theta}}$$

$$= \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{t\cos\theta+\sin\theta}}{\cos\theta} + C$$

$$= \sqrt{2(\tan x\sec\theta + \tan\theta\sec\theta)} + C$$

78. Let $I = \int \frac{xe^x}{(1+x)^2}dx = \int e^x\frac{(x+1)-1}{(1+x)^2}dx$

$$= \int e^x\left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]dx$$

Let $f(x) = \frac{1}{1+x} \Rightarrow f'(x) = -\frac{1}{(1+x)^2}$

We know that $\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x)$

$$\Rightarrow \quad I = \int e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\}dx = \frac{e^x}{1+x} + C$$

79. Let $I = \int \frac{x-3}{(x-1)^3}e^x\,dx = \int \frac{x-1-2}{(x-1)^3}e^x\,dx$

$$= \int e^x\left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3}\right]dx$$

$$= \int e^x\left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right]dx$$

Let $f(x) = \frac{1}{(x-1)^2}$

$$\Rightarrow \quad f'(x) = \frac{-2}{(x-1)^3}$$

$$\therefore \quad I = \frac{e^x}{(x-1)^2} + C \qquad [\because \int e^x\{f(x)+f'(x)\}\,dx = e^x f(x)]$$

80. Let $I = \int e^x \sec x\,(1+\tan x)\,dx$

$$\Rightarrow I = \int e^x\sec x\,dx + \int e^x\sec x\tan x\,dx \qquad \text{...(i)}$$

Now, $\int e^x\sec x\,dx = \sec x\int e^x\,dx - \int\left[\frac{d}{dx}\sec x\int e^x\,dx\right]dx$

$$= e^x\sec x - \int \sec x\tan x\,e^x\,dx \qquad \text{...(ii)}$$

On putting the value from Eq. (ii) in Eq. (i), we get

$$I = e^x\sec x - \int e^x\sec x\tan x\,dx + \int \sec x\tan x\,e^x\,dx + C$$

$$\Rightarrow \quad I = e^x\sec x + C$$

81. $\int \frac{\cos x-1}{\sin x+1} e^x\, dx$

$$= \int \frac{\cos x}{1+\sin x} e^x\, dx - \int \frac{1}{1+\sin x} e^x\, dx$$

$$= \frac{(\cos x)\, e^x}{1+\sin x} - \int \frac{-(1+\sin x)\sin x - \cos^2 x}{(1+\sin x)^2} e^x\, dx - \int \frac{e^x}{\sin x+1} dx + C$$

$$= \frac{e^x \cos x}{1+\sin x} + \int \frac{1}{1+\sin x} e^x dx - \int \frac{e^x}{1+\sin x} dx + C$$

$$= \frac{e^x \cos x}{1+\sin x} + C$$

82. $\int \left(\frac{x+2}{x+4}\right)^2 e^x\, dx = \int e^x \left[\frac{x^2+4x+4}{(x+4)^2}\right] dx$

$$= \int e^x \left[\frac{x(x+4)}{(x+4)^2} + \frac{4}{(x+4)^2}\right] dx$$

$$= \int e^x \left[\frac{x}{x+4} + \frac{4}{(x+4)^2}\right] dx$$

$$= \int \frac{e^x x}{(x+4)} dx + \int \frac{4e^x}{(x+4)^2} dx$$

$$= e^x\left(\frac{x}{x+4}\right) - \int \frac{4e^x}{(x+4)^2} dx + \int \frac{4e^x}{(x+4)^2} dx = \frac{x\, e^x}{(x+4)} + C$$

83. $\int (1+x-x^{-1}) e^{x+x^{-1}} dx$

$$= \int [x e^{x+x^{-1}}\left(1-\frac{1}{x^2}\right) + e^{x+x^{-1}}] dx$$

$$= x e^{x+x^{-1}} - \int e^{x+x^{-1}} dx + \int e^{x+x^{-1}} dx = x e^{x+x^{-1}} + C$$

84. $I_n = \int (\log x)^n\, dx$ ...(i)

$\therefore$ $I_{n-1} = \int (\log x)^{n-1} dx$ ...(ii)

Now, $I_n = \int (\log x)^n \cdot 1 dx$

$$= (\log x)^n x - n\int (\log x)^{n-1} \frac{1}{x} x\, dx$$

$$= x(\log x)^n - n\int (\log x)^{n-1} dx$$

$\Rightarrow$ $I_n = x(\log x)^n - n I_{n-1}$

$\therefore$ $I_n + n\, I_{n-1} = x(\log x)^n$

85. $\int \sin^{-1}(\cos x)\, dx = \int \left\{\frac{\pi}{2} - \cos^{-1}(\cos x)\right\} dx$

$$\left(\because \sin^{-1} t + \cos^{-1} t = \frac{\pi}{2} \text{ for } |t| \le 1\right)$$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2} x - \frac{x^2}{2} + C$$

86. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t$

$\Rightarrow$ $\frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow dx = \sqrt{1-x^2}\, dt$

$\therefore$ $I = \int \frac{t}{(1-\sin^2 t)\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dt$

$$= \int \frac{t}{(1-\sin^2 t)} dt = \int \underset{\text{I}}{t}\ \underset{\text{II}}{\sec^2} dt$$

$(\because 1-\sin^2 t = \cos^2 t)$

$$= t\int \sec^2 t\, dt - \int \left[\frac{d}{dt} t \int \sec^2 t\, dt\right] dt \text{ (integration by parts)}$$

$$= t\, \tan t - \int \tan t\, dt = t \tan t + \log \cos t + C$$

$$= (\sin^{-1} x) \frac{x}{\sqrt{1-x^2}} + \log(\sqrt{1-x^2}) + C$$

(from figure $\tan t = \frac{x}{\sqrt{1-x^2}}$ and $\cos t = \sqrt{1-x^2}$)

87. $I = \int 1 \cdot \tan^{-1}\sqrt{x}\, dx$

$$= \tan^{-1}\sqrt{x} \cdot x - \int \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \cdot x\, dx$$

$$= x \tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x} dx$$

Let $I_1 = \int \frac{\sqrt{x}}{x+1} dx$

Assume that $\sqrt{x} = t$

$\Rightarrow$ $\frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow$ $dx = 2t\, dt$

$\therefore$ $I_1 = \int \frac{t \cdot 2t}{1+t^2} dt = 2\int \left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right) dt$

$$= 2t - 2\tan^{-1} t$$

$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x}$$

$\therefore$ $I = x\tan^{-1}\sqrt{x} - \frac{1}{2}(2\sqrt{x} - 2\tan^{-1}\sqrt{x}) + C$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + (\tan^{-1}\sqrt{x}) + C$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

88. Let $I = \int \sqrt{x}\, e^{\sqrt{x}} dx$

Put $\sqrt{x} = t$

$\Rightarrow$ $\frac{1}{2\sqrt{x}} dx = dt$

$\therefore$ $I = 2\int t^2 e^t\, dt$

$$= 2\{t^2 e^t - 2\int t\, e^t\, dt\}$$

$$= 2\{t^2 e^t - 2(te^t - e^t)\} + C$$

$$= 2\{t^2 e^t - 2te^t + 2e^t\} + C$$

$$= (2x - 4\sqrt{x} + 4)\, e^{\sqrt{x}} + C$$

89. $\int \cos\left[2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]dx = \int \cos[\cos^{-1}(-x)]\,dx$

$$= \int (-x)\,dx = -\frac{x^2}{2} + C$$

90. $\because f(x) = \cos x - \cos^2 x + \cos^3 x - \ldots \infty$

$$= \frac{\cos x}{1+\cos x}$$

$$\therefore \quad \int f(x)\,dx = \int \frac{1+\cos x}{1+\cos x}dx - \int \frac{1}{1+\cos x}dx$$

$$= x - \frac{1}{2}\int \sec^2\frac{x}{2}dx$$

$$= x - \frac{1}{2}\tan\frac{x}{2}\cdot 2 + c = x - \tan\frac{x}{2} + c$$

91. On putting $y^2 = f(x) = \frac{x+2}{2x+3}$, we have

$$x = \frac{3y^2-2}{1-2y^2} \quad \text{and} \quad dx = -\frac{2y}{(1-2y^2)^2}dy$$

$$\therefore \quad \int \{f(x)\}^{1/2}\frac{dx}{x} = -\int y\,\frac{2y}{(1-2y^2)^2}\cdot\frac{1-2y^2}{3y^2-2}dy$$

$$= 2\int \frac{y^2}{(2y^2-1)(3y^2-2)}dy$$

$$= -2\int\left[\frac{1}{2y^2-1} - \frac{2}{3y^2-2}\right]dy$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}y}{1-\sqrt{2}y}\right| - \sqrt{\frac{2}{3}}\log\left|\frac{\sqrt{3}y+\sqrt{2}}{\sqrt{3}y-\sqrt{2}}\right| + C$$

Thus, $g(x) = \log|x|$ and $h(x) = \log|x|$

92. $\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$= \int \left(\frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}\right)dx$$

$$= \int \frac{\sqrt{2}(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$$

Put $\sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)\,dx = dt$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1} t + C$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

93. $\int \cos^{-1}\left(\frac{1}{x}\right)dx = \int \sec^{-1}x\cdot 1dx$

$$= \sec^{-1}x\int dx - \int\left[\frac{d}{dx}\sec^{-1}x\int dx\right]dx$$

$$= x\sec^{-1}x - \int \frac{1}{x\sqrt{x^2-1}}x\,dx$$

$$= x\sec^{-1}x - \int \frac{1}{\sqrt{x^2-1}}dx$$

$$= x\sec^{-1}x - \cosh^{-1}x + C$$

94. Let $I = \int x\{f(x^2)g''(x^2) - f''(x^2)g(x^2)\}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\left[\int \{f(t)g''(t)\}\,dt - \int f''(t^2)\,g(t^2)\,dt\right]$$

$$= \frac{1}{2}\left[f(t)g'(t) - \int f'(t)g(t)\,dt - g(t)f'(t) + \int g'(t)f'(t)\,dt\right] + C$$

$$= \frac{1}{2}[f(t)g'(t) - g(t)f'(t)] + C$$

$$= \frac{1}{2}[f(x^2)g'(x^2) - g(x^2)f'(x^2)] + C$$

95. Given, $\int e^x\{f(x) - f'(x)\}\,dx = \phi(x)$

$\Rightarrow \quad e^x\{f(x) - f'(x)\} = \phi'(x)$

Let $\quad I = \int e^x f(x)\,dx$

$$= f(x)e^x - \int f'(x)e^x\,dx$$

$$= f(x)e^x - \left[\int e^x f(x)\,dx - \int \phi'(x)\,dx\right]$$

$\Rightarrow \quad 2I = f(x)e^x + \phi(x)$

$\therefore \quad I = \frac{1}{2}[e^x f(x) + \phi(x)]$

96. We have, $\quad \int f(x)\,dx = F(x) \quad \ldots$(i)

Let $\quad I = \int x^3 f(x^2)\,dx$

Put $\quad x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int t\,f(t)\,dt$$

$$= \frac{1}{2}\left\{t\int f(t)\,dt - \int\left\{\int f(t)\,dt\right\}dt\right\}$$

$$= \frac{1}{2}\left\{t\,F(t) - \int F(t)\,dt\right\} \quad \text{[from Eq. (i)]}$$

$$= \frac{1}{2}\left\{x^2F(x^2) - \int F(x^2)\,dx^2\right\}$$

97. An anti-derivative of $f(x)$

$$= F(x) = \int [\log(\log x) + (\log x)^{-2}]dx + C$$

$$= x\log(\log x) - \int \frac{x}{x\log x}dx + \int (\log x)^{-2}dx + C$$

(integrating by parts the first integral)

$$= x\log(\log x) - [x(\log x)^{-1} + \int (\log x)^{-2}dx] + \int (\log x)^{-2}dx + C$$

[again, integrating by parts $(\log x)^{-1}$]

$$= x\log(\log x) - x(\log x)^{-1} + C$$

On putting $x = e$, we have, $e = 0 - e + C$, so $C = 2e$

Thus, $\quad F(x) = x[\log(\log x) - (\log x)^{-1}] + 2e$

98. Given, $f''(x) = \tan^2 x = (1 - \sec^2 x)$

On integrating both sides, we get

$$f'(x) = x - \tan x + C_1$$

$$f'(0) = 0 - 0 + C_1 \Rightarrow C_1 = 0$$

$$\Rightarrow \quad f'(x) = x - \tan x$$

Again, integrating both sides, we get

$$f(x) = \frac{x^2}{2} - \log \sec x + C_2$$

$$\Rightarrow \quad f(0) = 0 - \log 1 + C_2 \Rightarrow C_2 = 0$$

$$\therefore \quad f(x) = \frac{x^2}{2} - \log \sec x$$

99. $\int \sqrt{\tan x}\, dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x = 2t \dfrac{dt}{dx}$

$$= \int \sqrt{t^2}\, \frac{2t\, dt}{\sec^2 x} = 2\int \frac{t^2}{(1+\tan^2 x)} dt$$

$$= 2\int \frac{t^2}{(1+t^4)} dt = 2\int \frac{t^2}{(t^4+1+2t^2-2t^2)} dt$$

$$= 2\int \frac{t^2}{(t^2+1)^2 - (\sqrt{2}t)^2} dt$$

$$= 2\int \frac{t^2}{(t^2-\sqrt{2}t+1)(t^2+\sqrt{2}t+1)} dt$$

By partial fraction, $\dfrac{t^2}{(t^2-\sqrt{2}t+1)(t^2+\sqrt{2}t+1)}$

$$= \frac{At+B}{(t^2-\sqrt{2}t+1)} + \frac{Ct+D}{(t^2+\sqrt{2}t+1)} \quad \ldots\text{(i)}$$

$$\Rightarrow \quad t^2 = (At+B)(t^2-\sqrt{2}t+1) + (Ct+D)(t^2-\sqrt{2}t+1)$$

$$\Rightarrow \quad t^2 = (At^3 + Bt^2 + \sqrt{2}\,At^2 + \sqrt{2}\,Bt + At + B) + (Ct^3 + Dt^2 - \sqrt{2}Ct^2 - D\sqrt{2}t + Ct + D)$$

On comparing,

$A + C = 0,\ B + \sqrt{2}\,A - \sqrt{2}\,C + D = 1,$

$\sqrt{2}B + A - D\sqrt{2} + C = 0$ and $B + D = 0$

$$\Rightarrow \quad A - C = \frac{1}{\sqrt{2}},\ B - D = 0$$

$$\therefore \quad A = \frac{1}{2\sqrt{2}},\ C = -\frac{1}{2\sqrt{2}} \text{ and } B = D = 0$$

From Eq. (i),

$$\therefore \quad \frac{1}{2\sqrt{2}}\int \frac{dt}{t^2-\sqrt{2}t+1} - \frac{1}{2\sqrt{2}}\int \frac{dt}{t^2+\sqrt{2}\,t+1}$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dt}{\left\{t^2-\sqrt{2}t+\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\right\}} - \frac{1}{2\sqrt{2}}\int \frac{dt}{\left\{t^2+\sqrt{2}\,t+\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\right\}}$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dt}{\left(t-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{2\sqrt{2}}\int \frac{dt}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2\sqrt{2}} \times \sqrt{2}\tan^{-1}\left\{\frac{t-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right\} - \frac{\sqrt{2}}{2\sqrt{2}}\tan^{-1}\left(\frac{t+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(\sqrt{2}t-1) - \frac{1}{2}\tan^{-1}(\sqrt{2}t+1) + C$$

$$= \frac{1}{2}\tan^{-1}(\sqrt{2\tan x}-1) - \frac{1}{2}\tan^{-1}(\sqrt{2\tan x}+1) + C$$

$$= \frac{1}{2}\{\tan^{-1}(\sqrt{2\tan x}-1) - \tan^{-1}(\sqrt{2\tan x}+1)\} + C$$

$$= \frac{1}{2}\tan^{-1}\left\{\frac{\sqrt{2\tan x}-1-\sqrt{2\tan x}-1}{1+(\sqrt{2\tan x}-1)(\sqrt{2\tan x}+1)}\right\} + C$$

$$= \frac{1}{2}\tan^{-1}\left\{\frac{-2}{1+2\tan x-1}\right\} + C$$

$$= \frac{1}{2}\tan^{-1}(-\cot x) + C = \frac{1}{2}\tan^{-1}\left\{\tan\left(\frac{\pi}{2}+x\right)\right\} + C$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+x\right) + C = \frac{\pi}{4} + \frac{x}{2} + C$$

100. Let $\quad I = \int \dfrac{dx}{(1+a^2) - 2a\cos x}$

$$\Rightarrow \quad I = \int \frac{dx}{(1+a^2) - 2a\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$\Rightarrow \quad I = \int \frac{\sec^2\frac{x}{2}\, dx}{(1+a^2) - \sec^2\frac{x}{2} - 2a\left(1-\tan^2\frac{x}{2}\right)}$$

$$I = \int \frac{\sec^2\frac{x}{2}\, dx}{(1+a^2)\left(1+\tan^2\frac{x}{2}\right) - 2a\left(1+\tan^2\frac{x}{2}\right)}$$

Put $t = \tan\dfrac{x}{2}$

$$\Rightarrow \quad 2dt = \left(\sec^2\frac{x}{2}\right)dx$$

$$\Rightarrow \quad I = \int \frac{2dt}{(1+a^2)(1+t^2) - 2a(1-t^2)}$$

$$\Rightarrow \quad I = \int \frac{2dt}{(1-a)^2 + (1+a)^2 t^2}$$

$$\Rightarrow \quad I = \frac{2}{(1+a)^2}\int \frac{dt}{t^2 + \left(\frac{1-a}{1-a}\right)^2}$$

$$\therefore \quad I = \frac{2}{1-a^2}\tan^{-1}\left\{t\left(\frac{1+a}{1-a}\right)\right\} + C \text{ where, } t = \tan\frac{x}{2}$$

Round II

1. $I=\int \frac{(x\cos x+1)\, e^{\sin x}dx}{e^{\sin x}x\sqrt{2xe^{\sin x}+1}}$

Put $\quad 2x\,e^{\sin x}+1=t^2$

$\Rightarrow \quad 2[e^{\sin x}+e^{\sin x}\cos x\, x]\,dx=2t\,dt$

$\Rightarrow \quad e^{\sin x}(1+x\cos x)\,dx=t\,dt$

$\therefore \quad I=\int \frac{t\,dt}{\frac{(t^2-1)}{2}xt}$

$I=2\int \frac{dt}{t^2-1}$

$=\log\left|\frac{t-1}{t+1}\right|+C,$

where $\quad t^2=2xe^{\sin x}+1$

2. $I=\int e^x\left[\frac{1}{\sqrt{1+x^2}}-\frac{x}{\sqrt{(1+x^2)^3}}+\frac{x}{\sqrt{(1+x^2)^3}}+\frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right]dx$

$I=e^x\frac{1}{\sqrt{1+x^2}}+e^x\frac{x}{\sqrt{(1+x^2)^3}}+C$

$I=e^x\left[\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^3}}\right]+C$

$[\because \text{ using } \int e^x[f(x)+f'(x)]dx=e^xf(x)+C]$

3. $I=\int \frac{(1+x)\sin x\,dx}{(x+1)^2\cos^2 x-\cos^2 x-2(1+x)\sin x\cos x}$

$I=\frac{(1+x)\sin x\,dx}{(x+1)^2\cos^2 x+\sin^2 x-2(1+x)\sin x\cos x-1}$

$I=\frac{(1+x)\sin x}{(\sin x-(x+1)\cos x)^2-1}dx$

Put $\quad \sin x-(x+1)\cos x=t$

$\Rightarrow \quad \sin x(x+1)dx=dt$

$\therefore \quad I=\int \frac{dt}{t^2-1}$

$=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C$

$=\frac{1}{2}\log_e\left|\frac{\sin x-(x+1)\cos x-1}{\sin x-(x+1)\cos x+1}\right|+C$

4. $I=\int(\cos x)^{-2005}\operatorname{cosec}^2 x\,dx-2005\int\frac{dx}{\cos^{2005}x}$

$I=\int(\cos x)^{-2005}(-\cot x)\;-\int(-2005)(\cos x)^{-2006}$

$(-\sin x)(-\cot x)\,dx-2005\int\frac{dx}{\cos^{2005}x}$

$I=-\frac{\cot x}{(\cos x)^{2005}}+C$

5. $3\int \underset{\text{I}}{x^{3/2}}\,(\underset{\text{II}}{\sqrt{x-\sin x}\cdot 2\sin^2 x/2})\,dx+3\int\sqrt{x}(x-\sin x)^{3/2}dx$

Apply integration by parts in first integration

$=3\left[x^{3/2}\int\sqrt{x-\sin x}\cdot 2\sin^2 x/2\,dx\right.$

$\left.-\int\left\{\left(\frac{d}{dx}x^{3/2}\right)\int\sqrt{x-\sin x}\cdot 2\sin^2\frac{x}{2}dx\right\}dx\right]$

$+3\int\sqrt{x}\,(x-\sin x)^{3/2}dx$

$\left[\text{put } t=x-\sin x\; dt=(1-\cos x)\,dx=2\sin^2\frac{x}{2}dx\right]$

$=3\left[x^{3/2}\int\sqrt{t}\,dt-\int\left\{\frac{3}{2}\sqrt{x}\int\sqrt{t}\,dt\right\}dx\right]+3\int\sqrt{x}(x-\sin x)^{3/2}dx$

$=3\left[x^{3/2}\cdot\frac{2}{3}t^{3/2}-\int\left(\frac{3}{2}\sqrt{x}\cdot\frac{2}{3}t^{3/2}\right)dx\right]+3\int\sqrt{x}(x-\sin x)^{3/2}dx$

$=3\left[\frac{2}{3}x^{3/2}\cdot(x-\sin x)^{3/2}-\int\sqrt{x}\cdot(x-\sin x)^{3/2}dx\right]$

$+3\int\sqrt{x}(x-\sin x)^{3/2}dx$

$=2x^{3/2}(x-\sin x)^{3/2}-3\int\sqrt{x}(x-\sin x)^{3/2}dx$

$+3\int\sqrt{x}(x-\sin x)^{3/2}dx$

$=2x^{3/2}(x-\sin x)^{3/2}+C$

6. $I=\int\frac{dx}{(\sqrt{x})^2+(\sqrt{x})^7}$

$I=\int\frac{dx}{(\sqrt{x})^7\left(\frac{1}{(\sqrt{x})^5}+1\right)}$

Put $\quad 1+\frac{1}{(\sqrt{x})^5}=t\Rightarrow -\frac{5}{2}\cdot\frac{1}{(\sqrt{x})^7}dx=dt$

$I=-\frac{2}{5}\int\frac{dt}{t}=-\frac{2}{5}\log t+C$

$=-\frac{2}{5}\log\left[1+\frac{1}{(\sqrt{x})^5}\right]+C$

$=\frac{-2}{5}\log\left(\frac{x^{5/2}+1}{x^{5/2}}\right)+C$

$\Rightarrow \quad \lambda=\frac{-2}{5}$ and $a=\frac{5}{2}$

$\therefore \quad a+\lambda=2\cdot 1>2$

7. $f(x)\sin x\cos x=\frac{1}{2(b^2-a^2)}\frac{f'(x)}{f(x)}$

$\int\frac{f'(x)dx}{(f(x))^2}=\int 2(b^2-a^2)\sin x\cos x\,dx$

$\Rightarrow\frac{-1}{f(x)}=(b^2-a^2)=\frac{1}{a^2\sin^2 x-b^2\sin^2 x}$

$=\frac{1}{a^2\sin^2 x-b^2(1-\cos^2 x)}=\frac{1}{a^2\sin^2 x+b^2\cos^2 x-b^2}$

8. Let $I=\int \frac{e^x(1+nx^{n-1}-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}}dx$

$$=\int e^x\left\{\frac{\sqrt{1-x^{2n}}}{1-x^n}+\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right\}dx$$

Let $f(x)=\frac{\sqrt{1-x^{2n}}}{(1-x^n)}$

$$\Rightarrow\ f'(x)=\frac{\frac{(1-x^n)(-2nx^{2n-1})}{2\sqrt{1-x^{2n}}}+\sqrt{1-x^{2n}}(nx^{n-1})}{(1-x^n)^2}$$

$$\Rightarrow\ f'(x)=\frac{(1-x^{2n})(nx^{n-1})-n(1-x^n)x^{2n-1}}{(1-x^n)^2\sqrt{1-x^{2n}}}$$

$$=\frac{n(1-x^n)x^{n-1}\{(1+x^n)-x^n\}}{(1-x^n)^2\sqrt{1-x^{2n}}}=\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$$

$$\therefore\quad I=\frac{e^x\sqrt{1-x^{2n}}}{(1-x^n)}+C$$

9. $f(x)=\lim_{n\to\infty} n^2x^{1/(n+1)}\left[x^{\frac{1}{n}-\frac{1}{n+1}}-1\right]$

$$=\lim_{n\to\infty}\frac{x^{1/(n+1)}\left(x^{\frac{1}{n(n+1)}}-1\right)}{\frac{1}{n(n+1)}\times\frac{n(n+1)}{n^2}}\log x=\log x$$

Hence, $\int xf(x)\,dx=\int x\log x\,dx=\frac{x^2}{2}\log x-\frac{1}{4}x^2+C$

10. Let $g_n(x)=1+x^2+x^4+\ldots+x^{2n}=\frac{x^{2n+2}-1}{x^2-1}$

$$\therefore\quad h_n(x)=g_n'(x)=\frac{2x(nx^{2n+2}-(n+1)x^{2n}+1)}{(x^2-1)^2}$$

Now, $f(x)=\lim_{n\to\infty}h_n(x)=\frac{2x}{(x^2-1)^2}$

As $0<x<1$

Thus, $\int f(x)\,dx=\int\frac{2x}{(x^2-1)^2}dx$

$$=-\frac{1}{x^2-1}=\frac{1}{1-x^2}$$

11. $\because$ $\sin^{-1}x$ and $\cos^{-1}x$ are defined for $[-1,1]$.

For $x>e$, $\log x>1$

$\Rightarrow$ $\sin^{-1}(\log x)+\cos^{-1}(\log x)$ is not defined.

Hence, the given integral does not exist.

12. Let $I=\int e^{2ax}\frac{1-\cos 2ax}{1+\sin 2ax}dx$

$$\Rightarrow\quad I=\frac{1}{a}\int e^{2t}\frac{1-\cos 2t}{1+\sin 2t}dt\ ,\text{where } ax=t$$

$$\Rightarrow\quad I=\frac{1}{a}\int e^{2t}\frac{1-2\sin\left(\frac{\pi}{4}+t\right)\cdot\cos\left(\frac{\pi}{4}+t\right)}{2\sin^2\left(\frac{\pi}{4}+t\right)}dt$$

$$\Rightarrow\quad I=\frac{1}{a}\int e^{2t}\left\{\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}+t\right)-\cot\left(\frac{\pi}{4}+t\right)\right\}dt$$

$$\Rightarrow\quad I=\frac{1}{2a}\int \underset{\mathrm{I}}{e^{2t}}\ \underset{\mathrm{II}}{\operatorname{cosec}^2}\left(\frac{\pi}{4}+t\right)dt-\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt$$

$$\Rightarrow\quad I=-\frac{1}{2a}e^{2t}\cot\left(\frac{\pi}{4}+t\right)+\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt$$

$$-\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt+C$$

$$\Rightarrow\quad I=-\frac{1}{2a}e^{2t}\cot\left(\frac{\pi}{4}+t\right)+C$$

$$I=\frac{-1}{2a}e^{2an}\cot\left(\frac{\pi}{4}+ax\right)+C$$

13. We have, $\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$

$$=\int x\sqrt{\frac{2\sin(x^2+1)-2\sin(x^2+1)\cdot\cos(x^2+1)}{2\sin(x^2+1)+2\sin(x^2+1)\cdot\cos(x^2+1)}}dx$$

$$=\int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx$$

$$=\int x\tan\left(\frac{x^2+1}{2}\right)dx$$

$$=\int\tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)=\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$$

14. We have, $f(x)=\begin{vmatrix}0 & x^2-\sin x & \cos x-2\\ \sin x-x^2 & 0 & 1-2x\\ 2-\cos x & 2x-1 & 0\end{vmatrix}$

$$\Rightarrow\quad f(x)=\begin{vmatrix}0 & \sin x-x^2 & 2-\cos x\\ x^2-\sin x & 0 & 2x-1\\ \cos x-2 & 1-2x & 0\end{vmatrix}$$

(interchanging rows and columns)

$$\Rightarrow\quad f(x)=(-1)^3\begin{vmatrix}0 & x^2-\sin x & \cos x-2\\ \sin x-x^2 & 0 & 1-2x\\ 2-\cos x & 2x-1 & 0\end{vmatrix}$$

[taking (-1) common from each column]

$\Rightarrow$ $f(x)=-f(x)\Rightarrow f(x)=0$

$\Rightarrow$ $\int f(x)\,dx=0$

15. We have, $\frac{d}{dx}\{g(x)\}=g(x)$

$\Rightarrow$ $g'(x)=g(x)$

$\Rightarrow$ $\int\frac{g'(x)}{g(x)}=\int 1dx$

$\Rightarrow \quad \log_e \{g(x)\} = x + \log C_1$

$\Rightarrow \quad g(x) = C_1 e^x$

Now, $\quad g(0) = 1 \Rightarrow C_1 = 1 \quad \therefore \quad g(x) = e^x$

So, $\quad \int g(x) \left(\frac{2 - \sin x}{1 - \cos x}\right) dx = \int e^x (\text{cosec}^2 x - \cot x)\, dx$

$= -e^x \cot x + C = -g(x) \cot x + C$

16. We have, $f''(x) = f(x) \Rightarrow 2f'(x) f''(x) = 2f(x) f'(x)$

$\Rightarrow \quad \frac{d}{dx}\{f'(x)\}^2 = \frac{d}{dx}\{f(x)\}^2$

$\Rightarrow \quad \{f'(x)\}^2 = \{f(x)\}^2 + C \quad \ldots(i)$

Now, $\quad f(0) = 2$ and $f'(0) = 3$

Therefore, from Eq. (i), we get

$\{f'(0)\}^2 = \{f(0)\}^2 + C$

$\Rightarrow \quad 9 = 4 + C \Rightarrow C = 5$

$\Rightarrow \quad \{f'(x)\}^2 = \{f(x)\}^2 + 5$

$\Rightarrow \quad f'(x) = \sqrt{5 + \{f(x)\}^2}$

$\Rightarrow \quad \int \frac{d\{f(x)\}}{\sqrt{(5)^2 + \{f(x)\}^2}} = \int 1 dx$

$\Rightarrow \quad \log[f(x) + \sqrt{5 + \{f(x)\}^2}] = x + C_1$

$\therefore \quad f(0) = 2 \Rightarrow \log\{2 + 3\} = C_1$

$\Rightarrow \quad C_1 = \log 5$

$\therefore \quad \log\{f(x) + \sqrt{5 + \{f(x)\}^2}\} = x + \log 5$

$\Rightarrow \quad \log\left\{\frac{f(x) + \sqrt{5 + \{f(x)\}^2}}{5}\right\} = x$

$\Rightarrow \quad f(x) + \sqrt{5 + \{f(x)\}^2} = 5e^x$

$\Rightarrow \quad \sqrt{5 + \{f(x)\}^2} - f(x) = e^{-x}$

$\Rightarrow \quad 2f(x) = 5e^x - e^{-x} \Rightarrow f(x) = \frac{1}{2}(5e^x - e^{-x})$

$\Rightarrow \quad f(4) = \frac{1}{2}(5e^4 - e^{-4}) \Rightarrow f(4) = \frac{5e^8 - 1}{2e^4}$

17. $f(x) = \int \frac{x^2 dx}{(1 + x^2)(1 + \sqrt{1 + x^2})}$

Put $\quad x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta = (1 + x^2)\, d\theta$

$\therefore \quad f(x) = \int \frac{\tan^2\theta \sec^2\theta}{\sec^2\theta(1 + \sec\theta)} d\theta$

$= \int \frac{1 - \cos^2\theta}{\cos\theta(1 + \cos\theta)} d\theta$

$= \int \sec\theta\, d\theta - \int d\theta$

$= \log(\sec\theta + \tan\theta) - \theta + C$

$= \log(x + \sqrt{1 + x^2}) - \tan^{-1} x + C$

$\Rightarrow \quad f(0) = \log(0 + \sqrt{1 + 0}) - \tan^{-1}(0) + C$

and $\quad f(1) = \log(1 + \sqrt{2}) - \frac{\pi}{4}$

18. $\int f(x) \cos x\, dx + \int g(x)\, e^x dx$

$= \frac{e^x}{2}(\cos x + \sin x) - \frac{e^x}{2}(\sin x - \cos x) + C$

$= \frac{e^x}{2}(2\cos x) + C = e^x \cos x + C$

19. Since, $\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) = \log\tan\left(\frac{\pi}{4} + \theta\right)$

and $\quad \int \sec\theta\, d\theta = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\Rightarrow \quad \int \sec 2\theta\, d\theta = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \theta\right)$

$2\sec 2\theta = \frac{d}{d\theta}\log\tan\left(\frac{\pi}{4} + \theta\right)$

$\therefore \quad I = \frac{1}{2}\sin 2\theta \log\tan\left(\frac{\pi}{4} + \theta\right) - \int \tan 2\theta\, d\theta$

$= \frac{1}{2}\sin 2\theta \log\tan\left(\frac{\pi}{4} + \theta\right) - \frac{1}{2}\log\sec 2\theta + C$

20. $\int \frac{f(x)}{\log\sin x} dx = \log\log\sin x$

On differentiating both sides w.r.t. x, we get

$\frac{f(x)}{\log\sin x} = \frac{\cot x}{\log\sin x} \Rightarrow f(x) = \cot x$

21. $\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}} dx$

$= \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2} dx$

$= \frac{1}{3}\int \frac{1}{t^{3/4}} dt \quad \left(\text{put } \frac{x-1}{x+2} = t \Rightarrow \frac{3}{(x+2)^2} dx = dt\right)$

$= \frac{1}{3}\left(\frac{t^{1/4}}{\frac{1}{4}}\right) + C = \frac{4}{3}t^{1/4} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

22. ***Case I*** If $x > 0$, then $|x| = x$

$\therefore \quad \int |x| \log|x|\, dx = \int x \log x\, dx$

$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{x^2}{2} \cdot \log x - \frac{x^2}{4} + C = -\frac{x^2}{2}\log|x| - \frac{x^2}{4} + C$

Case II If $x < 0$, then $|x| = -x$

$\therefore \quad \int |x| \log|x|\, dx = -\int x \log(-x)\, dx = -\left\{\log(-x) \cdot \frac{x^2}{2} - \frac{x^2}{4}\right\} + C$

$= -\frac{x^2}{2}\log|x| + \frac{x^2}{2} + C$

On combining both cases, we get

$\int |x| \log|x|\, dx = \frac{1}{2}x|x|\log|x| - \frac{1}{4}x|x| + C$

23. Since, $f''(x) = \sec^4 x + 4$

$f''(x) = (1+\tan^2 x)\sec^2 x + 4$

$\Rightarrow \quad f'(x) = \tan x + \dfrac{\tan^3 x}{3} + 4x + C$

$\Rightarrow \quad f'(0) = 0$

$\Rightarrow \quad 0 = C$

Then, $f'(x) = \tan x + \dfrac{\tan^3 x}{3} + 4x$

$= \tan x + \dfrac{1}{3}\tan x(\sec^2 x - 1) + 4x$

$\Rightarrow \quad f'(x) = \dfrac{2}{3}\tan x + \dfrac{1}{3}\tan x \sec^2 x + 4x$

$\therefore \quad f(x) = \dfrac{2}{3}\log|\sec x| + \dfrac{\tan^2 x}{6} + 2x^2 + d$

But $f(0) = 0$

$\Rightarrow \quad d = 0$

Then, $f(x) = \dfrac{2}{3}\log|\sec x| + \dfrac{1}{6}\tan^2 x + 2x^2$

24. $\int (\sin 2x - \cos 2x)\,dx = \dfrac{1}{\sqrt{2}}\sin(2x - a) + b$

$\Rightarrow \quad \sqrt{2}\int\left(\dfrac{1}{\sqrt{2}}\sin 2x - \dfrac{1}{\sqrt{2}}\cos 2x\right)dx = \dfrac{1}{\sqrt{2}}\sin(2x-a) + b$

$\Rightarrow \quad -\sqrt{2}\int\left(\dfrac{1}{2}\cos 2x - \dfrac{1}{\sqrt{2}}\sin 2x\right)dx = \dfrac{1}{\sqrt{2}}\sin(2x-a) + b$

$\Rightarrow \quad -\sqrt{2}\int \cos\left(2x + \dfrac{\pi}{4}\right)dx = \dfrac{1}{\sqrt{2}}\sin(2x-a) + b$

$\Rightarrow \quad -\dfrac{\sqrt{2}}{2}\sin\left(2x + \dfrac{\pi}{4}\right) + C = \dfrac{1}{\sqrt{2}}\sin(2x-a) + b$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\sin\left(2x + \dfrac{5\pi}{4}\right) + C = \dfrac{1}{\sqrt{2}}\sin(2x-a) + b$

$\therefore \quad a = -\dfrac{5\pi}{4}, \quad b \in R$

25. Put $x^x = t$

$\Rightarrow \quad x^x(1 + \log_e|x|)\,dx = dt$

$\int x^x(1 + \log_e|x|)\,dx = t + C = x^x + C$

26. $I = \int \dfrac{dx}{x^2(x^4+1)^{3/4}} = \int \dfrac{dx}{x^2 \cdot x^3\left(1 + \dfrac{1}{x^4}\right)^{3/4}}$,

Put $1 + x^{-4} = t$

$= \dfrac{-4}{x^5}dx = dt = -\dfrac{1}{4}\int \dfrac{dt}{t^{3/4}}$

$= \dfrac{1}{4}\int t^{-3/4}\,dt = -\dfrac{1}{4}\cdot\dfrac{t^{1/4}}{1/4} + C$

$= -\left(1 + \dfrac{1}{x^4}\right)^{1/4} + C = -\left(\dfrac{1+x^4}{x^4}\right)^{1/4} + C$

Hence, $A = -1, B = \dfrac{1}{4}$

27. $I = \int \dfrac{(2x-1)dx}{(x^2-x)^2 - (x^2-x) + 1}$

Put $x^2 - x = t$

$I = \int \dfrac{dt}{t^2 - t + 1} = \int \dfrac{dt}{\left(t - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}}$

$I = \int \dfrac{dt}{\left(t - \dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2}$

$I = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x^2 - 2x - 1}{\sqrt{3}}\right) + C$

28. $I = \int \dfrac{x}{\tan x}\dfrac{2x\sec^2 x \tan x}{(\tan x - x\sec^2 x)^2}dx$

$I = \dfrac{x}{\tan x}\dfrac{1}{(\tan x - x\sec^2 x)} - \int \dfrac{\tan x - \sec^2 x}{\tan^2 x}\dfrac{1}{\tan x - x\sec^2 x}dx$

$I = \dfrac{x}{\tan x(\tan x - x\sec^2 x)} - \int \cot^2 x \cdot dx$

$I = \dfrac{x}{\tan x(\tan x - x\sec^2 x)} + \cot x + x + C$

29. Put $x = t^6 \Rightarrow dx = 6t^5\,dt$

$\therefore \quad I = \int \dfrac{(t^6 + t^4 + t)\,6t^5}{t^6(1+t^2)}dt$

$I = 6\int \dfrac{(t^5 + t^3 + 1)}{(t^2+1)}dt$

$I = 6\int t^3 dt + 6\int \dfrac{dt}{1+t^2}$

$I = \dfrac{3}{2}t^4 + 6\tan^{-1} t + C$

$I = \dfrac{3}{2}\sqrt[3]{x^2} + 6\tan^{-1}\sqrt[6]{x} + C$

30. Given, $I = \int \dfrac{dx}{(x-1)^2\sqrt[4]{\left(\dfrac{x+2}{x-1}\right)^5}}$

Put $\dfrac{x+2}{x-1} = t \Rightarrow \dfrac{-3}{(x-1)^2}dx = dt$

$\therefore \quad I = -\dfrac{1}{3}\int \dfrac{dt}{x^{5/4}} = \dfrac{4}{3}\left[\dfrac{1}{t^{-1/4}}\right] + C = \dfrac{4}{3}\left[\sqrt[4]{\dfrac{x-1}{x+2}}\right] + C$

$\therefore \quad A = \dfrac{4}{3}$

31. $I = \int \dfrac{dx}{(1-x)^2\sqrt[3]{\left(\dfrac{x+1}{1-x}\right)^2}}$

Put $\dfrac{1+x}{1-x} = t \Rightarrow \dfrac{2}{(1-x)^2}dx = dt$

$\therefore \quad I = \dfrac{1}{2}\int \dfrac{dt}{t^{2/3}} = \dfrac{3}{2}[t^{1/3}] + C = \dfrac{3}{2}\left[\sqrt{\dfrac{1+x}{1-x}} + C\right]$

$\therefore \quad k = \dfrac{3}{2}$

45. Let $I = \int \frac{(2-2x)}{\sqrt{(4+2x-x^2)}}\,dx + \int \frac{dx}{\sqrt{(4+2x-x^2)}}$

$$= 2\sqrt{4+2x-x^2} + \int \frac{dx}{\sqrt{5-(x-1)^2}}$$

$$= 2\sqrt{4+2x-x^2} + \sin^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C$$

46. Given, $I_n = \int \tan^n x\,dx = \int \tan^{n-2}x \cdot \tan^2 x\,dx$

$$= \int \tan^{n-2}x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x \cdot \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \int \tan^{n-2} x \cdot \sec^2 x\,dx - I_{n-2}$$

$\Rightarrow$ $$I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence, $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$

Put $n = 6, 5\ (I_6 + I_4) = \tan^5 x$

47. $I = \int \frac{dt}{\sqrt{1+t}}$, where $x^2 + 1 = t$

$$= 2\sqrt{1+t} + C = 2\sqrt{x^2+2} + C$$

$$I = \int \frac{x^{a/2}}{\sqrt{1+x^{11}}}\,dx$$

Put $x^{11/2} = t$, then

$$t = \frac{2}{11}\log(x^{11/2} + \sqrt{1+x^{11}}) + C$$

48. If $a > 0$ and $b^2 - 4ac < 0$, then

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$\Rightarrow$ $$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a\left(x+\frac{b}{2a}\right)^2 + k^2}$$

where, $k^2 = \frac{4ac - b^2}{4a} > 0$

which will have an answer of the type

$$\frac{1}{a}\cdot\frac{1}{\frac{K}{\sqrt{a}}}\tan^{-1}\frac{x+\frac{b}{2a}}{\frac{K}{\sqrt{a}}} + C$$

or $$\mu \tan^{-1}\frac{x+A}{B} + C$$

49. $\because$ $5x = 3x + 2x$

$\Rightarrow$ $\tan 5x = \frac{\tan 3x + \tan 2x}{1 - \tan 3x \tan 2x}$

$\therefore$ $\tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$

50. Let $P = \int \frac{dx}{(x-3y)} = \frac{1}{2}\log\{(x-y)^2 - 1\}$

$\because$ $P = \int \frac{dx}{(x-3y)} \Rightarrow \frac{dP}{dx} = \frac{1}{(x-3y)}$ ...(i)

Also, $P = \frac{1}{2}\log\{(x-y)^2 - 1\}$

$\therefore$ $$\frac{dP}{dx} = \frac{2(x-y)\left(1-\frac{dy}{dx}\right)}{2\{(x-y)^2-1\}} = \frac{(x-y)\left(1-\frac{dy}{dx}\right)}{(x-y)^2-1} \quad \text{...(ii)}$$

Given, $y(x-y)^2 = x$

$\Rightarrow$ $\log y + 2\log(x-y) = \log x$

$\Rightarrow$ $$\frac{1}{y}\frac{dy}{dx} + \frac{2}{(x-y)}\left(1-\frac{dy}{dx}\right) = \frac{1}{x}$$

$\Rightarrow$ $$\frac{dy}{dx}\left(\frac{1}{y} - \frac{2}{x-y}\right) = \frac{1}{x} - \frac{2}{x-y} = \frac{x-y-2y}{x(x-y)}$$

$\Rightarrow$ $$\frac{dy}{dx}\left(\frac{x-3y}{y(x-y)}\right) = -\frac{(x+y)}{x(x-y)}$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{y(x+y)}{x(x-3y)}$$

Now, from Eq. (ii),

$$\frac{dP}{dx} = \frac{(x-y)\left\{1+\frac{y(x+y)}{x(x-3y)}\right\}}{(x-y)^2-1} = \frac{(x-y)\left\{\frac{x^2-2xy+y^2}{x(x-3y)}\right\}}{\left(\frac{x}{y}-1\right)}$$

$$= \frac{y(x-y)^2}{x(x-3y)} = \frac{1}{x-3y} \quad \text{...(iii)}$$

$\therefore$ It is true from Eq. (i),

$\therefore$ $$\int \frac{dx}{x-3y} = \frac{1}{2}\log\{(x-y)^2 - 1\}$$

$\because y$ is variable.

$\therefore$ $$\int \frac{dx}{x-2y} \neq \log(x-3y)$$

51. $\because$ $\int \frac{1}{f(x)}dx = 2\log|f(x)| + C$

On differentiating both sides w.r.t. x, then

$$\frac{1}{f(x)} = \frac{2}{f(x)}f'(x) \text{ or } f'(x) = \frac{1}{2}$$

$\therefore$ $$f(x) = \frac{x}{2} + C$$

If $f(0) = 0$, then $f(x) = \frac{x}{2}$

52. $F(x+\pi) = \int \sin^2(x+\pi)\,dx$

$= \int \sin^2 x\,dx$ $\quad [\because \sin^2(\pi + x) = \sin^2 x]$

$= F(x)$

53. Given, $I_n = \int \cot^n x\,dx = \int \cot^{n-2} x(\operatorname{cosec}^2 x - 1)\,dx$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\,dx - I_{n-2} = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

Put $n = 6, 5\,(I_6 + I_4) = -\cot^5 x$

54. Statement II is true.

Now, $\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{(x+1-1)e^x}{(x+1)^2}\,dx$

$\int e^x \left\{\frac{1}{x+1} - \frac{1}{(x+1)^2}\right\} dx = \frac{e^x}{x+1} + C$ (by using statement II)

55. Statement II is true.

Now, $\int \frac{dx}{e^x + e^{-x} + 2} = \int \frac{e^x\,dx}{(e^x+1)^2} = \int \frac{d(e^x+1)}{(e^x+1)^2}$

$= -\frac{1}{e^x+1} + C$ (by using statement II)

56. Given, $\int f(x)\,dx = \psi(x)$

Let $I = \int x^5 f(x^3)\,dx$

Put $x^3 = t \Rightarrow x^2 dx = \frac{dt}{3}$ ...(i)

$\therefore$ $I = \frac{1}{3}\int t\, f(t)\,dt$

$= \frac{1}{3}\left[t \cdot \int f(t)\,dt - \int \left\{\frac{d}{dt}(t) \int f(t)\,dt\right\} dt\right]$ (by parts)

$= \frac{1}{3}[t\,\psi(t) - \int \psi(t)\,dt]$

$= \frac{1}{3}[x^3\,\psi(x^3) - 3\int x^2\,\psi(x^3)dx] + C$ [$\because$ from Eq. (i)]

$= \frac{1}{3}x^3\,\psi(x^3) - \int x^2\,\psi(x^3)\,dx + C$

57. Given, $\int \frac{5\tan x}{\tan x - 2}\,dx = x + a\ln|\sin x - 2\cos x| + k$ ...(i)

Now, let us assume that

$$I = \int \frac{5\tan x}{\tan x - 2}\,dx$$

On multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5\sin x}{\sin x - 2\cos x}\,dx$$

This special integration requires special substitution of type

$$\text{Nr} = A\,(\text{Dr}) + B\left(\frac{d\text{Dr}}{dx}\right)$$

$\Rightarrow$ Let $5\sin x = A\,(\sin x - 2\cos x) + B\,(\cos x + 2\sin x)$

$\Rightarrow$ $0\cos x + 5\sin x = (A + 2B)\sin x + (B - 2A)\cos x$

On comparing the coefficients of $\sin x$ and $\cos x$, we get

$$A + 2B = 5 \text{ and } B - 2A = 0$$

On solving the above 2 equations in A and B, we get

$$A = 1 \text{ and } B = 2$$

$\Rightarrow$ $5\sin x = (\sin x - 2\cos x) + 2\,(\cos x + 2\sin x)$

$\Rightarrow$ $I = \int \frac{5\sin x}{\sin x - 2\cos x}\,dx$

$= \int \frac{(\sin x - 2\cos x) + 2\,(\cos x + 2\sin x)}{(\sin x - 2\cos x)}\,dx$

$\Rightarrow$ $I = \int \frac{\sin x - 2\cos x}{\sin x - 2\cos x}\,dx + 2\int \frac{(\cos x + 2\sin x)}{(\sin x - 2\cos x)}\,dx$

$\Rightarrow$ $I = \int 1\,dx + 2\int \frac{d\,(\sin x - 2\cos x)}{(\sin x - 2\cos x)}$

$\Rightarrow$ $I = x + 2\log|(\sin x - 2\cos x)| + k$ ...(ii)

where k is the constant of integration.

Now, by comparing the value of I in Eqs. (i) and (ii), we get $a = 2$

58. Let $I = \sqrt{2}\int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)}\,dx$

Put $x - \frac{\pi}{4} = t \Rightarrow dx = dt$

$\therefore$ $I = \sqrt{2}\int \frac{\sin\left(\frac{\pi}{4} + t\right)dt}{\sin t} = \sqrt{2}\int\left[\frac{1}{\sqrt{2}}\cot t + \frac{1}{\sqrt{2}}\right]dt$

$= \log|\sin t| + t + C = x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

59. Let $I = \int \frac{dx}{2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)} = \frac{1}{2}\int \sec\left(x - \frac{\pi}{3}\right)dx$

$= \frac{1}{2}\log\tan\left(\frac{x}{2} - \frac{\pi}{6} + \frac{\pi}{4}\right) + C = \frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

60. Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

[$\because \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + C$]

$\therefore \int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt = \frac{e^t}{1+t^2} + C = \frac{x}{1+(\log x)^2} + C$

31 Definite Integration

JEE Main MILESTONE

- Definition
- Geometrical Interpretation of a Definite Integral
- Fundamental Theorem of Calculus
- Properties of Definite Integration
- Definite Integral as a Limit of Sum
- Reduction Formulae in Definite Integrals

31.1 Definition

Every definite integral has a unique value.

Let $\phi(x)$ be the anti-derivative of a function $f(x)$ defined on $[a, b]$ *i.e.*, $\frac{d}{dx}[\phi(x)] = f(x)$.

Definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$. *i.e.*,

$$\int_a^b f(x)dx = \phi(b) - \phi(a).$$

This is also called **Newton-Leibnitz** formula.

The numbers a and b are called the limits of integration 'a' is called **lower limit** and 'b' is **upper limit**. The interval $[a, b]$ is called the interval of integration. The interval $[a, b]$ is also known as **range of integration**.

Anti-derivatives are related to definite integrals through the fundamental theorem of calculus : the definite integral of a function over an interval is equal to the difference between the values of an anti-derivate evaluated at the end points of the intervals. The discrete equivalent of the notion of anti-derivative is anti-difference.

31.2 Geometrical Interpretation of a Definite Integral

Geometrically it represents an algebraic sum of the areas of regions bounded by graph of the function $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$. The areas above x-axis are taken as positive and the areas below x-axis are taken as negative.

$\therefore$ $$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$$

where A_1, A_2, A_3, A_4 and A_5 are the areas of the shaded region.

Note

- In a definite integral, there is no need to keep the constant of integration.
- The indefinite integral $\int f(x)\,dx$ is a function of x, whereas definite integral $\int_a^b f(x)\,dx$ is a number.
- Given $\int f(x)\,dx$ we can find $\int_a^b f(x)\,dx$, but given $\int_a^b f(x)\,dx$ we cannot find $\int f(x)dx$.

Sample **Problem 1** $\int_{-1}^{4}(2x-3)\,dx$ *is equal to*

(a) $\frac{3}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) 0

Interpret (d) $y = 2x - 3$ is a straight line, which lie below the x-axis in $\left[-1, \frac{3}{2}\right)$ and above in $\left(\frac{3}{2}, 4\right]$.

Now, area of $\Delta ABC = \frac{1}{2}\times\frac{5}{2}\times 5 = \frac{25}{4}$

Area of $\Delta CDE = \frac{1}{2}\times\frac{5}{2}\times 5 = \frac{25}{4}$

So, $\int_{-1}^{4}(2x-3)\,dx = -\frac{25}{4} + \frac{25}{4} = 0$

31.3 Fundamental Theorem of Calculus

The fundamental theorem of calculus is a theorem that links the concept of the derivative of a function with the concept of the integral.

(i) First Fundamental Theorem of Calculus

Let f be a continuous real valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$ by

$$F(x) = \int_a^x f(t)\,dt$$

Then, F is continuous on $[a, b]$, differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in (a, b).

(ii) Second Fundamental Theorem of Calculus/Newton-Leibnitz Axiom

Let f and g be real valued functions defined on a closed interval $[a, b]$. Such that the derivative of g is f. *i.e.*, f and g are functions such that for all x in $[a, b]$.

$$f(x) = g'(x)$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

31.4 Properties of Definite Integration

(i) If $\int f(x)dx = F(x) + C$, then $\int_a^b f(x)dx = F(b) - F(a)$

Sample **Problem 2** $\int_0^{2/3}\frac{1}{4+9x^2}\,dx$ *is equal to* **[NCERT]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$

(c) $\frac{\pi}{24}$ (d) $\frac{\pi}{4}$

Interpret (c) $\int_0^{2/3}\frac{1}{4+9x^2}\,dx = \frac{1}{9}\int_0^{2/3}\frac{1}{\frac{4}{9}+x^2}\,dx$

$$= \frac{1}{9}\int_0^{2/3}\frac{1}{\left(\frac{2}{3}\right)^2 + x^2}\,dx = \frac{1}{9}\cdot\frac{1}{2/3}\left[\tan^{-1}\left(\frac{x}{2/3}\right)\right]_0^{2/3}$$

$$\left(\because \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{6}\left[\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{2/3}$$

$$= \frac{1}{6}\left[\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \tan^{-1}0\right]$$

$$= \frac{1}{6}(\tan^{-1}1 - 0) = \frac{1}{6}\cdot\frac{\pi}{4} = \frac{\pi}{24}$$

(ii) $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$

Here, x is a dummy variable, it can be replaced by any other variable t, u, ...

i.e., The value of a definite integral does not change with change of variable of integration provided the limits of integration remains the same.

Sample Problem 3 $\int_2^3 \frac{2x^2}{x^4+3x^2+1}dx$ *is equal to*

(a) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{7}{54}\right)+\tan^{-1}\left(\frac{5}{56}\right)$

(b) $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{54}\right)+\tan^{-1}\left(\frac{5}{26}\right)$

(c) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{7\sqrt{5}}{54}\right)+\tan^{-1}\left(\frac{5}{56}\right)$

(d) *None of the above*

Interpret (c) Let $I=\int_2^3\left(\frac{2x^2}{x^4+3x^2+1}\right)dx$

$$=\int_2^3\frac{(x^2+1)}{x^4+3x^2+1}dx+\int_2^3\frac{(x^2-1)\,dx}{x^4+3x^2+1}$$

$$=\int_2^3\frac{[1+(1/x^2)]\,dx}{[x-(1/x)]^2+5}+\int_2^3\frac{[1-(1/x^2)]\,dx}{[x+(1/x)]^2+1}$$

In 1st put $x-\frac{1}{x}=t$, in 2nd put $x+\frac{1}{x}=y$

$$I=\int_{3/2}^{8/3}\frac{dt}{t^2+5}+\int_{5/2}^{10/3}\frac{dy}{y^2+1}$$

$$=\frac{1}{\sqrt{5}}\left[\tan^{-1}\left(\frac{8}{3\sqrt{5}}\right)-\tan^{-1}\left(\frac{3}{2\sqrt{5}}\right)\right]+\tan^{-1}\left(\frac{10}{3}\right)-\tan^{-1}\left(\frac{5}{2}\right)$$

$$=\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{7\sqrt{5}}{54}\right)+\tan^{-1}\left(\frac{5}{56}\right)$$

(iii) $\int_a^b f[g(x)]g'(x)dx=\int_{g(a)}^{g(b)}f(t)dt$

Sample Problem 4 *The value of the integral* $\int_{1/3}^1\frac{(x-x^3)^{\frac{1}{3}}}{x^4}dx$ *is* [NCERT]

(a) 6 (b) 0 (c) 3 (d) 4

Interpret (a) Let $I=\int_{1/3}^1\frac{(x-x^3)^{\frac{1}{3}}}{x^4}dx=\int_{1/3}^1\frac{(x^3)^{1/3}\left(\frac{x}{x^3}-\frac{x^3}{x^3}\right)^{1/3}}{x^4}dx$

[on multiplying and dividing the numerator by $(x^3)^{1/3}$]

$$=\int_{1/3}^1\frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}dx$$

Put $\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3dx}=dt\Rightarrow\frac{dx}{x^3}=\frac{dt}{(-2)}$

For limit when $x=1\Rightarrow t=1$ and when $x=\frac{1}{3}\Rightarrow t=3^2=9$

$\left(\because t=\frac{1}{x^2}\right)$

$$\therefore\quad I=\int_9^1(t-1)^{1/3}\frac{dt}{(-2)}=-\frac{1}{2}\left[\frac{(t-1)^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_9^1$$

$$=-\frac{1}{1}\times\frac{3}{4}\left[(t-1)^{\frac{4}{3}}\right]_9^1=-\frac{3}{8}\left[(1-1)^{\frac{4}{3}}-(9-1)^{\frac{4}{3}}\right]$$

$$=-\frac{3}{8}\left[0-(2^3)^{\frac{4}{3}}\right]=-\frac{3}{8}\times(-16)=6$$

(iv) Interchanging the limits of the definite integral does not change the absolute value but change the sign of the integral.

$$\int_a^b f(x)dx=-\int_b^a f(x)dx$$

Sample Problem 5 $\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx$ *is equal to* [NCERT]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) π

Interpret (b) Let $I=\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x=t\Rightarrow-\sin x dx=dt$

$\Rightarrow\quad dx=-\frac{dt}{\sin x}$

For limit when $x=0\Rightarrow t=\cos 0=1$ $(\because t=\cos x)$

and when $x=\frac{\pi}{2}\Rightarrow t=\cos\frac{\pi}{2}=0$

$$\therefore\quad I=\int_1^0\frac{\sin x}{1+t^2}\cdot\left(\frac{dt}{-\sin x}\right)$$

$$=-\int_1^0\frac{1}{1+t^2}dt=\int_0^1\frac{1}{1+t^2}dt$$

$$=\left[\frac{1}{1}\tan^{-1}\left(\frac{t}{1}\right)\right]_1^0\qquad\left(\because\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$=-\left(0-\frac{\pi}{4}\right)=\frac{\pi}{4}$$

(v) $\int_a^b f(x)\,dx=\int_a^c f(x)dx+\int_c^b f(x)dx,\qquad(a<c<b)$

Sample Problem 6 $\int_{-1}^1 f(x)\,dx$, *where* $f(x)=\begin{cases}1-2x, & x\le 0\\ 1+2x, & x\ge 0\end{cases}$ *is equal to*

(a) 1 (b) 2 (c) 3 (d) 4

Interpret (d) $\int_{-1}^1 f(x)\,dx=\int_{-1}^0 f(x)\,dx+\int_0^1 f(x)\,dx$

$$=\int_{-1}^0(1-2x)\,dx+\int_0^1(1+2x)\,dx=[x-x^2]_{-1}^0+[x+x^2]_0^1=4$$

(vi) $\int_a^b f(x)dx=\int_a^{c_1}f(x)dx+\int_{c_1}^{c_2}f(x)dx$

$$+\int_{c_2}^{c_3}f(x)dx+\ldots+\int_{c_n}^b f(x)dx$$

where, $a<c_1<c_2<c_3<\ldots c_n<b$

Sample Problem 7 $\int_{-5}^{5}|x+2|\,dx$ *is equal to* [NCERT]

(a) 13 (b) 19
(c) 21 (d) 29

Interpret (d) Here, the given integrand is in the form of absolute function and we define the absolute function as $|x| = x,\ x \geq 0$ or $|x| = -x, x < 0$ by using it, we convert the given integrand in simple form and then integrate it.

Let $$I = \int_{-5}^{5} |x+2|\,dx$$

It can be seen that $(x+2) \leq 0$ on $[-5, -2]$ and $(x+2) \geq 0$ on $[-2, 5]$.

$$\therefore \quad I = \int_{-5}^{-2} -(x+2)\,dx + \int_{-2}^{5}(x+2)\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$\Rightarrow \quad I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4 = 29$$

Note In an absolute integral function, please careful while breaking the limit.

Sample Problem 8 $\int_2^8 (x-5)\,dx$ *is equal to* [NCERT]

(a) 9 (b) 3
(c) 6 (d) 15

Interpret (a) Let $I = \int_2^8 |x-5|\,dx$

It can be seen that $(x-5) \leq 0$ on $[2, 5]$ and $(x-5) \geq 0$ on $[5, 8]$.

$$\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\therefore \quad I = \int_2^5 \{-(x-5)\}\,dx + \int_5^8 (x-5)\,dx$$

$$= \left[5x - \frac{x^2}{2}\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= \left(25 - \frac{25}{2}\right) - \left(10 - \frac{4}{2}\right) + \left(\frac{64}{2} - 40\right) - \left(\frac{25}{2} - 25\right)$$

$$= \frac{25}{2} - 8 - 8 + \frac{25}{2}$$

$$= 25 - 16 = 9$$

(vii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

In particular, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx$$

$$= \int_a^b \frac{f(a+b-x)}{f(x) + f(a+b-x)}\,dx$$

$$\Rightarrow \quad 2I = \int_a^b \frac{f(x) + f(a+b-x)}{f(x) + f(a+b-x)}\,dx$$

$$\Rightarrow \quad 2I = \int_a^b dx = (b-a)$$

$$\therefore \quad I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx = \frac{b-a}{2} \quad \text{...(i)}$$

Eq. (i) is a special case of 4th property and is used as standard result.

(a) $\int_0^{b-c} f(x+c)\,dx = \int_c^b f(x)\,dx$

(b) $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$

Sample Problem 9 $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ *is equal to* [NCERT]

(a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) *None of these*

Interpret (b) Firstly, reduce the integrand into simplest form by using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, add them and integrate.

Let $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{...(i)}$$

Then, $$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{...(ii)}$$

$$\left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 \quad \Rightarrow \quad I = \frac{\pi}{4}$$

Sample Problem 10 $\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}dx$ *is equal to*

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 0

Interpret (c) $\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$ ...(i)

$$I=\int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{9-(-x)}}dx$$

$$I=\int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_2^7\left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}+\frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\right)dx=\int_2^7 dx=|x|_2^7=5$$

So, $I=\frac{5}{2}$

(viii) $\int_{-a}^a f(x)dx=\begin{cases}2\int_0^a f(x)dx, & \text{if } f(-x)=f(x) \text{ i.e., } f(x) \text{ is even}\\ 0, & \text{if } f(-x)=-f(x) \text{ i.e., } f(x) \text{ is odd}\end{cases}$

Sample Problem 11 *The value of*

$$\int_{-\pi/2}^{\pi/2}(x^3+x\cos x+\tan^5 x+1)\,dx \text{ is}$$

[NCERT]

(a) 0 (b) 2 (c) π (d) 1

Interpret (c) Let $I=\int_{-\pi/2}^{\pi/2}(x^3+x\cos x+\tan^5 x+1)\,dx$

$$\Rightarrow I=\int_{-\pi/2}^{\pi/2}x^3dx+\int_{-\pi/2}^{\pi/2}x\cos x\,dx+\int_{-\pi/2}^{\pi/2}\tan^5 x\,dx+\int_{-\pi/2}^{\pi/2}1dx$$

We know that, $\int_{-a}^a f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even}\\ 0, & \text{if } f(x) \text{ is odd}\end{cases}$

$\therefore$ $I=0+0+0+2\int_0^{\pi/2}1\,dx$

[$\because x^3$, $x\cos x$ and $\tan^5(x)$ are odd functions]

$\therefore$ $I=2[x]_0^{\pi/2}=\frac{2\pi}{2}=\pi$

(ix) $\int_{-a}^a f(x)dx=\int_0^a[f(x)+f(-x)]\,dx$

Sample Problem 12 $\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{(1+e^x)(1+x^2)}$ *is equal to*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) π

Interpret (a) $I=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{(1+e^x)(1+x^2)}$

Here, $f(x)=\frac{1}{(1+e^x)(1+x^2)}$

$$\Rightarrow f(-x)=\frac{1}{(1+e^{-x})\{1+(-x)^2\}}=\frac{e^x}{(1+e^x)(1+x^2)}$$

[using property $\int_{-a}^a f(x)\,dx=\int_0^a\{f(x)+f(-x)\}\,dx$]

$$\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{(1+e^x)(1+x^2)}=\int_0^{\sqrt{3}}\frac{1}{(1+e^x)(1+x^2)}+\frac{e^x}{(1+e^x)(1+x^2)}dx$$

So, $I=\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\tan^{-1}x|_0^{\sqrt{3}}=\frac{\pi}{3}$

(x) $\int_0^{2a}f(x)dx=\int_0^a[f(x)+f(2a-x)]\,dx$

Sample Problem 13 $\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$ *is equal to*

(a) $\frac{\pi}{2\sqrt{2}}$ (b) $\frac{\pi^2}{2\sqrt{2}}$

(c) $\frac{\pi}{\sqrt{2}}$ (d) $\frac{\pi^2}{\sqrt{2}}$

Interpret (b) $I=\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$ ... (i)

$$I=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2(\pi-x)}$$

$$=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2 x} \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi\,dx}{1+\cos^2 x}\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{1+\cos^2 x}$$

$$=\frac{\pi}{2}\left[\int_0^{\pi}\frac{dx}{1+\cos^2 x}+\int_0^{\pi/2}\frac{dx}{1+\cos^2(\pi-x)}\right]$$

$$=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{2+\tan^2 x}$$

Put $\tan x=t$; $I=\pi\int_0^{\infty}\frac{dt}{t^2+2}=\frac{\pi}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^{\infty}=\frac{\pi^2}{2\sqrt{2}}$

(xi) $\int_0^{2a}f(x)\,dx=\begin{cases}2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x)\\ 0, & \text{if } f(2a-x)=-f(x)\end{cases}$

Sample Problem 14 $\int_0^{\pi}\frac{dx}{1+2\sin^2 x}$ *is equal to*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3\sqrt{3}}$ (c) $\frac{\pi}{\sqrt{3}}$ (d) 0

Interpret (c) $\int_0^{\pi}\frac{dx}{1+2\sin^2 x}=2\int_0^{\pi/2}\frac{dx}{1+2\sin^2 x}$

$$\left[\int_0^{2a}f(x)\,dx=2\int_0^a f(x)\,dx, \text{if } f(2a-x)=f(x)\right]$$

$$=2\int_0^{\pi/2}\frac{\sec^2 x\,dx}{\sec^2 x+2\tan^2 x}=2\int_0^{\pi/2}\frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

Note that in the beginning we can not divide numerator and denominator by $\cos^2 x$, as $\cos x=0$ at $x=\frac{\pi}{2}$

$$= 2\int_0^{\infty} \frac{dt}{1+3t^2} \ (\tan x = t)$$

$$= 2\frac{1}{\sqrt{3}}[\tan^{-1} t\sqrt{3}]_0^{\infty} = \frac{2}{\sqrt{3}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

(xii) $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)x + a]dx$

Sample **Problem 15** $\int_{-5}^{-4} e^{(x-4)^2}dx$ *is equal to*

(a) $3\int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$ (b) $\int_{2/3}^{1/3} e^{9(x-3/2)^2}dx$

(c) $\int_{1/3}^{2/3} e^{9(x-3/2)^2}dx$ (d) *None of these*

Interpret (d) Let $I = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$

$$= 3\left[\left(\frac{2}{3}-\frac{1}{3}\right)\right]\int_0^1 e^{9\left\{\left(\frac{2}{3}-\frac{1}{3}\right)x+\frac{1}{3}-\frac{2}{3}\right\}^2}dx = \int_0^1 e^{9\left(\frac{x}{3}-\frac{1}{3}\right)^2}dx$$

$$= \int_0^1 e^{(x-1)^2}dx$$

Also, $\int_{-5}^{-4} e^{(x+4)^2}dx = \int_0^1 e^{(x-1)^2}dx$

(xiii) $\int_a^b f(x)\,dx = \begin{cases} 0, \text{if } f(a+x) = -f(b-x) \\ 2\int_0^{\frac{a+b}{2}} f(x)dx, \text{if } f(a+x) = f(b-x) \end{cases}$

(xiv) **Periodic Properties of Definite Integral** If $f(x)$ is a periodic function with period T, then

(a) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx,\ n \in I$

Sample **Problem 16** $\int_0^{4\pi} |\sin x|\,dx$ *is equal to*

(a) 0 (b) 2 (c) 4 (d) 8

Interpret (d) We know that, $|\sin x|$ is a periodic function of π. Hence,

$$\int_0^{4\pi} |\sin x|dx = 4\int_0^{\pi} |\sin x|dx = 4\int_0^{\pi} \sin x\,dx = 4[-\cos x]_0^{\pi} = 8$$

(b) $\int_{mt}^{nt} f(x)dx = (n-m)\int_0^T f(x)dx,\ m, n \in I$

Sample **Problem 17** $\int_{-3/2}^{10} \{2x\}\,dx$, *where* $\{.\}$ *denotes the fractional part of x, is equal to*

(a) $\frac{1}{2}$ (b) $\frac{13}{2}$ (c) $\frac{23}{2}$ (d) $\frac{23}{4}$

Interpret (d) $f(x) = \{2x\}$ is a periodic function with period $\frac{1}{2}$.

Let $I = \int_{-3/2}^{10} \{2x\}\,dx = \int_{-3(1/2)}^{20(1/2)} \{2x\}\,dx$

$$= 23\int_0^{1/2} 2x\,dx \quad \left(\text{as } \{2x\} = 2x - \{2x\} \text{ and when } \in \left[0, \frac{1}{2}\right],\ [2x] = 0\right)$$

$$= 23\,[x^2]_0^{1/2} = \frac{23}{4}$$

(c) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)\,dx,\ n \in I,\ a \in R$

Sample **Problem 18** $\int_0^{n\pi+w} |\sin x|\,dx$, *where* $n \in N$ *and* $0 \le w < \pi$, *is equal to*

(a) $(2n+1) + \sin w$ (b) $2n + \cos w$

(c) $(2n+1) - \cos w$ (d) *None of these*

Interpret (c) $I = \int_0^{n\pi+w} |\sin x|\,dx$

$$= \int_0^w |\sin x|\,dx + \int_w^{n\pi+w} |\sin x|\,dx = I_1 + I_2$$

$$I_1 = \int_0^w |\sin x|\,dx = \int_0^w \sin x\,dx$$

(as $0 \le w < \pi$ and $\sin x \ge 0$, when $n \in [0, \pi]$)

$$= -[\cos x]_0^w = -\cos w + 1 = 1 - \cos w$$

$$I_2 = \int_w^{n\pi+w} |\sin x|\,dx = n\int_0^{\pi} |(\sin x)|\,dx$$

$$= n\int_0^{\pi} \sin x\,dx = n[-\cos x]_0^{\pi} = 2n$$

So, $I = 1 - \cos w + 2n = (2n+1) - \cos w$

(d) $\int_{nt}^{a+nT} f(x)dx = \int_0^a f(x)dx,\ n \in I, a \in R$

(e) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx,\ n \in I, a, b \in R$

Sample **Problem 19** $\int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)\,dx$, *is equal to*

(a) $\sqrt{3}$ (b) $1 + \sqrt{3}$

(c) $\sqrt{3} - 1$ (d) 1

Interpret (c) $f(x) = \sin x + \cos x$ is periodic with period 2π.

Let $I = \int_{10\pi+\pi/6}^{10\pi+\pi/3} (\sin x + \cos x)\,dx$

$$= \int_{\pi/6}^{\pi/3} (\sin x + \cos x)\,dx$$

$$= |\sin x - \cos x|_{\pi/6}^{\pi/3}$$

$$= \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right] - \left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right] = (\sqrt{3} - 1)$$

(f) If $f(x)$ is an odd periodic function defined on the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$, where T is period of $f(x)$, then $g(x) = \int_{\alpha}^{x} f(t)\,dt$ is periodic with period T, *i.e.*, $g(x+T) = g(x)$.

(g) Let $f(x)$ be an odd periodic function with period T, then

$f(x) = \int_{\alpha}^{x} f(t)\,dt$ is an even function with period T.

(h) Let $f(x)$ be an even periodic function with period T, then

$f(x) = \int_{\alpha}^{x} f(t)\,dt$ is an odd function with period T.

(xv) **Some special properties**

(a) If $f(t)$ is an odd function, then $g(x) = \int_a^x f(t)\,dt$ is an even function.

(b) $f(-t) = f(t)$ *i.e.,* $f(t)$ is an even function, then $g(x) = \int_0^x f(t)\,dt$

(c) • $\int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

• $\int_a^{\pi-a} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)\,dx$

(d) • $\int_{-a}^a f(x)^2\,dx = 2\int_0^a f(x)^2 dx,\ f(-x) = f(x)$

i.e., $f(x)$ is an even function.

• $\int_{-a}^a x\,f(x)^2\,dx = 0,\ f(-x) = -f(x)$

i.e., $f(x)$ is an odd function.

(e) $\int_0^{n\pi} f(\cos^2 x)\,dx = n\int_0^\pi f(\cos^2 x)\,dx,\ n \in I^+$

(xvi) **Leibnitz theorem**

Let $\phi(x)$ and $\psi(x)$ be two functions defined on $[\alpha, \beta]$ and differentiable at a point $x \in]\alpha, \beta[$ and $f(x, t)$ is continuous function.

(a) **For implicit function**

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt = \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt + \left\{\frac{d}{dx}[\psi(x)]f[x, \psi(x)] - \frac{d}{dx}[\phi(x)]\right\} f[x, \phi(x)]\right]$$

where, $f(x, t)$ is an implicit function.

(b) **For explicit function**

If function $\phi(x)$ and $\psi(x)$ are defined on $[\alpha,\beta]$ and differentiable at a point $]\alpha,\beta[$ and $f(t)$ is continuous on $[\phi(\alpha), \phi(\beta)]$, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = \left\{\frac{d}{dt}\psi(x)\right\} f[\psi(x)] - \left[\frac{d}{dx}[\phi(x)]\right]\{f\,\phi(x)\}$$

(c) Let a function $f(x,\alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$, then for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x,\alpha)\,dx$, then $\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\cdot dx$

Sample **Problem 20** $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right)$ *is equal to*

(a) $\frac{1}{\log x}$ (b) $\frac{x^2}{\log x}$

(c) $\frac{x^2 - x}{\log x}$ (d) *None of these*

Interpret (c) $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}.dt\right)$

$$= \frac{1}{\log x^3}\cdot\frac{d}{dx}(x)^3 - \frac{1}{\log x^2}\cdot\frac{d}{dx}(x^2)$$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right) = \frac{1}{\log x}.(x^2 - x)$$

(xvii) **Approximation in definite integration**

(a) If $f(x) \ge g(x)$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ (where, $b > (a)$

(b) If $f(x) \le g(x)$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ (where, $b < a$)

(c) If $f(x) \ge 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

(d) If f is continuous on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

(e) If f is continuous on $[a, b]$ and $|f(x)| \le k$.

$\forall x \in [a, b]$, then $\left|\int_a^b f(x)\,dx\right| \le k(b-a)$.

(f) If $f_1(x), f(x)$ and $f_2(x)$ are continuous function in $[a, b]$. Such that, $f_1(x) \le f(x) \le f_2(x), \forall x \in [a, b]$, then

$$\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)\,dx$$

(g) $\left|\int_a^b f(x)g(x)\,dx\right| \le \sqrt{\left\{\int_a^b f^2(x)\,dx\right\}\left\{\int_a^b g^2(x)\,dx\right\}}$

where, $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$.

(h) If m and M are global minima and global maxima of $f(x)$ in $[a, b]$ *i.e.,* $m \le f(x) \le M$ for $a \le x \le b$, then $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$

Sample **Problem 21** *Which of the following is correct?*

(a) $\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$ (b) $\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$

(c) $\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} = 0$ (d) *None of these*

Interpret (a) $0 \le x \le 1 \Rightarrow 0 \le x^3 \le x^2 \le 1$

$$\Rightarrow \quad -x^2 \le -x^3 \le 0 \Rightarrow 4 - x^2 - x^2 \le 4 - x^2 - x^3 \le 4 - x^2$$

$$\Rightarrow \quad \frac{1}{\sqrt{4 - x^2}} \le \frac{1}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{4 - 2x^2}}$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4 - x^2}} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}}$$

$$\Rightarrow \quad \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$$

Sample **Problem 22** *Maximum value of* $\int_1^3 \sqrt{3+x^3}\,dx$ *is*

(a) $\sqrt{30}$
(b) $2\sqrt{30}$
(c) 2
(d) *None of the above*

Interpret (b) Since, the function $f(x)=\sqrt{3+x^3}$ increases monotonically in the interval [1, 3].

$\therefore \quad M = \text{Maximum value of } \sqrt{3+x^3} = \sqrt{3+3^3} = \sqrt{30}$

$m = \text{Minimum value of } \sqrt{3+x^3} = \sqrt{3+1^3} = 2$

Now, $\quad b-a=2$

Hence, $\quad 2\cdot 2 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

$\Rightarrow \quad 4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

(xviii) **Definite integral of piecewise continuous functions**

Suppose we have to evaluate $\int_a^b f(x)dx$, but either $f(x)$ is not continuous at $x = c_1, c_2, \ldots c_n$ or it is not defined at these points. In both cases we have to break the limit at $c_1, c_2, \ldots, c_n$.

Sample **Problem 23** $\int_1^2 [x^3-1]\,dx$, *where* [·] *denotes the greatest integer function, is equal to*

(a) 6 (b) 12 (c) 15 (d) 14

Interpret (d) $1 \le x \le 2 \Rightarrow 1 \le x^3 \le 8 \Rightarrow 0 \le x^3 - 1 \le 7$

So, $\quad I = \int_1^2 [x^3-1]\,dx = \int_1^{2^{1/3}} [x^3-1]\,dx + \int_{2^{1/3}}^{3^{1/3}} [x^3-1]\,dx + \ldots + \int_{7^{1/3}}^{2} [x^3-1]\,dx$

Now, if $x \in \left(1, 2^{\frac{1}{3}}\right)$, then $x^3 \in [1,2)$ or $[x^3-1]=0$ and so on

Therefore, $I = \int_1^{2^{1/3}} 0\,dx + \int_{2^{1/3}}^{3^{1/3}} 1\cdot dx + \ldots \int_{7^{1/3}}^{2} 6\,dx$

$= [3^{1/3} - 2^{1/3}] + 2[4^{1/3} - 3^{1/3}] + 3[5^{1/3} - 4^{1/3}] + 4[6^{1/3} - 4^{1/3}] + 5[6^{1/3} - 5^{1/3}] + 6[2^{1/3} - 7^{1/3}]$

$= 12 - [7^{1/3} + 6^{1/3} + 5^{1/3} + 4^{1/3} + 3^{1/3} + 2^{1/3}]$

(xix) **Mean value of a function over an interval**

Let $f(x)$ be a continuous function defined on $[a, b]$, then there exists a point $c \in (a, b)$, such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

$\therefore f(c) = \dfrac{1}{b-a}\int_a^b f(x)\,dx$, is called the mean value of $f(x)$ over $[a, b]$.

Hot Spot 1 PROPERTIES of Definite Integration

Properties of definite integration is the most important topic of this chapter for JEE Main Examination. Mostly question are asked from this topic. The level of question is from moderate to typical. While solving problems based on properties of definite integration, always keep in mind some selective properties.

(i) $\int_a^b f(x)\,dx = \int_a^b f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx,$

(where $a < c_1 < c_2 < c_3 \ldots < b$)

(ii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(iii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x), \text{ i.e., } f(x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x), \text{ i.e., } f(x) \text{ is odd} \end{cases}$

(iv) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

(v) Periodic properties of definite integral.

(vi) Approximation in definite integration.

(vii) Definite integral of piecewise continuous functions.

Generally we use the following steps to solve such types of problems.

Step I We write the given integral and observe it very carefully.

Step II Now, apply the appropriate property of definite integral mention above to reduce the given integral in simplest form.

Step III Now, use the formulae of indefinite integral (if need) and solve the integration.

Step IV After integration, we apply the upper and lower limit (if need according to property).

Step V Finally get the desired result of given integration.

Check Point 1

1. If $f(x) = ae^{2x} + be^{x} + cx$ satisfies the conditions $f(0) = -1$; $f'(\log 2) = 31$, $\int_0^{\log 4} \{f(x) - cx\}\, dx = \frac{39}{2}$, then
   (a) $a = 5$ (b) $b = 6$
   (c) $c = 2$ (d) $a = 3$

2. The value of $\int_0^{2\pi} |\cos x - \sin x|\, dx$ is
   (a) $\frac{4}{\sqrt{2}}$ (b) $2\sqrt{2}$
   (c) $\frac{2}{\sqrt{2}}$ (d) $4\sqrt{2}$

3. The greatest value of $f(x) = \int_{-1/2}^{x} |t|\, dt$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is
   (a) $\frac{3}{8}$ (b) $\frac{1}{4}$
   (c) $-\frac{3}{8}$ (d) $-\frac{1}{2}$

4. The value of the integral $\int_0^{\pi} \frac{\sin 2kx}{\sin x}\, dx$, where $x \in I$, is
   (a) $\frac{\pi}{2}$ (b) π
   (c) 0 (d) 1

5. If $f(a + b - x) = f(x)$, then $\int_a^b x f(x)\, dx$ is equal to
   (a) $\frac{a+b}{2} \int_a^b f(b - x)\, dx$
   (b) $\frac{a+b}{2} \int_a^b f(x)\, dx$
   (c) $\frac{b-a}{2} \int_a^b f(x)\, dx$
   (d) None of the above

6. $\int_0^{10\pi} (|\sin x| + |\cos x|)\, dx$ is equal to
   (a) 20 (b) 40
   (c) 10 (d) 30

Hot Spot 2

Definite Integration by Substitution

Definite integration by substitution is an important topic for JEE Main examination and a small practice can help you in solving such problems very quickly as the level of questions is easy to average.
While solving problems based on definite integration by substitution, always keep in mind.

(i) Formulae of indefinite integration.
(ii) Properties of definite integration.

When we substitute, we are changing the **variable**, so we cannot use the same upper and lower limits. We can either do the problem as an indefinite integral first, then use upper and lower limits later.
Or do the problem throughout using the new variable and the new upper and lower limits.
Show the correct variable for the upper and lower limit during the substitution phase.

Sample Problem 24 *The value of the integral* $\int_0^r xy\, dx$, *where* $y = \sqrt{r^2 - x^2}$, *is*

(a) $\frac{r^3}{6}$ (b) $\frac{r^3}{3}$ (c) $\frac{r^3}{2}$ (d) $\frac{r^4}{5}$

Interpret (b) Let $I = \int_0^r xy\, dx = \int_0^r x\sqrt{r^2 - x^2}\, dx$

Put $r^2 - x^2 = t^2$

when $x \to 0, t \to r^2$ and when $x \to r, t \to 0$

$$\Rightarrow \quad x\, dx = \left(-\frac{1}{2}\right) dt$$

$$\therefore \quad I = \int_{r^2}^{0} \left(-\frac{1}{2}\right) \sqrt{t}\, dt$$

$$= \frac{1}{2} \int_0^{r^2} \sqrt{t}\, dt$$

$$= \left[\frac{t^{3/2}}{3}\right]_0^{r^2} = \frac{1}{3} r^3$$

31.5 Definite Integral as a Limit of Sum

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n parts. The point of division on x-axis are

$a, a+h, a+2h, \ldots, a+(n-1)h, a+nh,$

where $h = \dfrac{b-a}{n}$.

Let S_n denotes the area of these n rectangles. Then,

$S_n = hf(a) + hf(a+h)$
$\quad + hf(a+2h) + \ldots + hf[a+(n-1)h]$

Clearly, S_n is area very close to the area of the region bounded by curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$.

Hence, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{r=0}^{n-1} hf(a+rh)$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\left(\frac{b-a}{n}\right) f\left\{a + \frac{(b-a)r}{n}\right\}$$

Note

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha + (n-1)\beta]$
$$= \frac{\sin(\alpha + \alpha + (n_1)\beta \sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$
- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$
$$= \cos\frac{\left(\frac{\text{1st angle + Last angle)}}{2}\right)\cdot\sin\frac{n\beta}{2}}{\sin\left(\frac{\beta}{2}\right)}$$

Method to Express the Infinite Series as Definite Integral

(i) Express the given series in the form $\sum \dfrac{1}{n} f\left(\dfrac{r}{n}\right)$.

(ii) Then, the limit is its sum when $n \to \infty$, *i.e.*, $\displaystyle\lim_{n\to\infty}\sum \frac{1}{n} f\left(\frac{r}{n}\right)$.

(iii) Replace $\dfrac{r}{n}$ by x and $\dfrac{1}{n}$ by (dx) and $\displaystyle\lim_{n\to\infty}\sum$ by the sign of $\int$.

(iv) The lower and the upper limit of integration are limiting values of $\dfrac{r}{n}$ for the first and the last term of r, respectively.

Some Particular Cases

(i) $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n} f\left(\frac{r}{n}\right)$

or $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right)$

$$= \int_0^1 f(x)\,dx$$

(ii) $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{p_n}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \displaystyle\lim_{n\to\infty}\frac{r}{n} = 0$ $\quad(\because r = 1)$

$\beta = \displaystyle\lim_{n\to\infty}\frac{r}{n} = p$ $\quad(\because r = pn)$

Check Point 2

1. $\displaystyle\int_0^{\pi/2}\cos^5\left(\frac{x}{2}\right)\sin x\,dx$ is equal to
   (a) $\frac{2}{7}\left(1-\frac{1}{8\sqrt{2}}\right)$ (b) $-\frac{4}{7}\left(1-\frac{1}{8\sqrt{2}}\right)$
   (c) $\frac{4}{7}\left(1-\frac{1}{8\sqrt{2}}\right)$ (d) None of these

2. $\displaystyle\int_0^a \frac{dx}{a+\sqrt{a^2-x^2}}$ is equal to
   (a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}-1$ (c) $1-\frac{\pi}{2}$ (d) $\frac{\pi}{3}+2$

3. $\displaystyle\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right]$ is equal to
   (a) 1 (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

4. $\displaystyle\lim_{n\to\infty}\frac{\pi}{n}\left\{\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right\}$ is equal to
   (a) 0 (b) π
   (c) 2 (d) None of these

5. $\displaystyle\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ is equal to
   (a) $\frac{e^{\pi/2}}{2e^2}$ (b) $2e^2e^{\pi/2}$
   (c) $\frac{2}{e^2}\cdot e^{\pi/2}$ (d) None of these

6. $\displaystyle\int_{-\pi/2}^{\pi/2}\sin^2 x\cos^2 x(\sin x+\cos x)\,dx$ is equal to
   (a) $\frac{4}{15}$ (b) $\frac{2}{15}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

31.6 Reduction Formulae in Definite Integrals

(i) $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$= [-\sin^{n-1} x \cdot \cos x]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{(n-2)} x \cdot \cos^2 x\,dx$

$= (n-1)\int_0^{\pi/2} \sin^{n-2} x \cdot (1-\sin^2 x)\,dx$

$= (n-1)\int_0^{\pi/2} \sin^{n-2} x \cdot dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx$

$\Rightarrow \quad I_n + (n-1)I_n = (n-1)I_{n-2}$

$\Rightarrow \quad I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Note • $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

• $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots I_0$ or I_1

According to n even or odd,

$I_0 = \frac{\pi}{2}, I_1 = 1$

So, $I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\left(\frac{1}{2}\right)\cdot\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\left(\frac{2}{3}\right)\cdot 1, & \text{if } n \text{ is odd} \end{cases}$

(ii) $I_n = \int_0^{\pi/4} (\tan x)^{(n-2)} \cdot \tan^2 x\,dx$

$= \int_0^{\pi/4} (\tan x)^{n-2} (\sec^2 x - 1)\,dx$

$= \int_0^{\pi/4} (\tan x)^{(n-2)} \cdot \sec^2 x\,dx - \int_0^{\pi/4} (\tan x)^{n-2}\,dx$

$= \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$I_n = \frac{1}{n-1} - I_{n-2}$

$\therefore \quad I_n + I_{n-2} = \frac{1}{n-1}$

(iii) $I_{mn} = \int_0^{\pi/2} \sin^{m-1} x\,(\sin x \cos^n x)\,dx$

$= \left[-\frac{\sin^{m-1} x \cdot \cos^{n+1} x}{n+1}\right]_0^{\pi/2}$

$+ \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} \cdot (m-1) \cdot \sin^{m-2} x \cos x\,dx$

$= \frac{(m-1)}{(n+1)} \int_0^{\pi/2} \sin^{m-2} x \cdot \cos^n x \cdot \cos^2 x\,dx$

$= \frac{(m-1)}{(n+1)} \int_0^{\pi/2} (\sin^{m-2} x \cdot \cos^n x - \sin^m x \cdot \cos^n x)\,dx$

$= \frac{(m-1)}{(n+1)} I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$

$\Rightarrow \quad \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{m+1}\right) I_{m-2,n}$

$\Rightarrow \quad I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

Note According to m is even or odd.

$I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right)\ldots I_{0,n}$ or $I_{1,n}$

$I_{0,n} = \int_0^{\pi/2} \cos^n x\,dx$

and $\quad I_{1,n} = \int_0^{\pi/2} \sin x \cdot \cos^n x\,dx = \frac{1}{n+1}$

WORKED OUT Examples

Example 1 $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ *is equal to* **[NCERT]**

(a) $\frac{\pi}{2} - \log 2$ (b) $-\frac{1}{2}\log 2$

(c) $\log 2$ (d) $-\log 2$

Solution (a) In angle of $\sin^{-1}$ we put $x = \tan\theta$ and use the identity $\sin^{-1}(\sin 2\theta) = 2\theta$ and change the limit also by using it and then integrate under proper limits.

Let $$I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Put $\tan\theta = x \Rightarrow \sec^2\theta\, d\theta = dx$ or $\theta = \tan^{-1} x$

For limit when $x = 0 \Rightarrow \theta = 0$ and when $x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$(\because \theta = \tan^{-1} x)$

$$\therefore \quad I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta \qquad \left(\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= 2\int_0^{\pi/4} \underset{\mathrm{I}}{\theta}\ \underset{\mathrm{II}}{\sec^2\theta}\, d\theta$$

Now, applying rule of integration by parts taking θ as the first function and $\sec^2\theta$ as second function, we get

$$I = 2\left[\theta\int \sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\right\}\int \sec^2\theta\, d\theta\right]_0^{\pi/4}$$

$$= 2[\theta\tan\theta - \int 1\tan\theta\, d\theta]_0^{\pi/4} \qquad (\because \int \sec^2\theta d\theta = \tan\theta)$$

$$= 2[\theta\tan\theta + \log|\cos\theta|]_0^{\pi/4} \quad (\because \int \tan\theta\, d\theta = -\log\cos\theta)$$

$$= 2\left\{\left(\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right|\right) - (|0 + \log|\cos 0|)\right\}$$

$$= \frac{\pi}{2} + 2\log\frac{1}{\sqrt{2}} - 0 \qquad (\because \log 1 = 0)$$

$$= \frac{\pi}{2} + 2\log 2^{-1/2} = \frac{\pi}{2} - 2\times\frac{1}{2}\log 2 \quad (\because \log m^n = n\log m)$$

$$= \frac{\pi}{2} - \log 2$$

Example 2 $\int_0^{\pi/4} \log(1+\tan x)\, dx$ *is equal to* **[NCERT]**

(a) $\frac{1}{2}\log 2$ (b) $\frac{\pi}{2}\log 2$

(c) $\frac{\pi}{4}\log 2$ (d) $\frac{\pi}{8}\log 2$

Solution (d) Let $I = \int_0^{\pi/4} \log(1+\tan x)\, dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4} - x\right)\right]dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$$

$$\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}, \text{ here } A = \pi/4, B = x\right]$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1+\tan x)\}\, dx \qquad \text{...(ii)}$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/4} \log 2\, dx$$

$$= \log 2\int_0^{\pi/4} 1\, dx = \log 2\,[x]_0^{\pi/4} = (\log 2)\left(\frac{\pi}{4} - 0\right)$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2 \quad \Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Example 3 $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\, dx$ *is equal to* **[NCERT]**

(a) 0 (b) $-\log 2$

(c) $-\frac{\pi}{2}\log 2$ (d) *None of these*

Solution (c) Let $I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\, dx$

$$= \int_0^{\pi/2} (\log\sin^2 x - \log\sin 2x)\, dx \qquad (\because m\log n = \log n^m)$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right) dx \qquad \left(\because \log m - \log n = \log \frac{m}{n}\right)$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2 \sin x \cos x}\right) dx \qquad (\because \sin 2x = 2 \sin x \cos x)$$

$$= \int_0^{\pi/2} \log\left(\frac{\tan x}{2}\right) dx = \int_0^{\pi/2} \log(\tan x) - \log 2 \, dx$$

$$= \int_0^{\pi/2} \log(\tan x) \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$\Rightarrow \qquad I = I_1 - (\log 2)\,[x]_0^{\pi/2} = I_1 - \left(\frac{\pi}{2} - 0\right)\log 2 \qquad \ldots(i)$$

where, $\quad I_1 = \int_0^{\pi/2} \log(\tan x) \, dx \qquad \ldots(ii)$

$$\Rightarrow \qquad I_1 = \int_0^{\pi/2} \log\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \qquad I_1 = \int_0^{\pi/2} \log(\cot x) \, dx \qquad \ldots(iii)$$

On adding Eqs. (ii) and (iii), we get

$$2I_1 = \int_0^{\pi/2} \{\log(\tan x) + \log(\cot x)\} \, dx$$

$$= \int_0^{\pi/2} \log(\tan x \cot x) \, dx \qquad [\because \log m + \log n = \log(mn)]$$

$$= \int_0^{\pi/2} \log 1 \, dx = 0 \quad \Rightarrow \quad I_1 = 0 \qquad \left(\because \tan x = \frac{1}{\cot x}\right)$$

On putting the value of I_1 in Eq. (i), we get

$$I = 0 - \frac{\pi}{2}\log 2 = -\frac{\pi}{2}\log 2$$

Example 4 $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$ *is equal to* [NCERT]

(a) $2 \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

(c) $2 \sin^{-1}\left(\frac{1+\sqrt{3}}{2}\right)$ (d) *None of these*

Solution (a) $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

We know that, $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x$

$$\Rightarrow \qquad (\sin x - \cos x)^2 = 1 - \sin 2x$$

$$(\because \sin^2 x + \cos^2 x = 1 \text{ and } 2 \sin x \cos x = \sin 2x)$$

$$\Rightarrow \qquad \sin 2x = 1 - (\sin x - \cos x)^2$$

Put $\qquad \sin x - \cos x = t \qquad \ldots(i)$

$$\Rightarrow \qquad (\cos x + \sin x)\,dx = dt \quad \Rightarrow \quad dx = \frac{dt}{\cos x + \sin x}$$

For limit when $\quad x = \frac{\pi}{6} \Rightarrow t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2}$

when $\quad x = \frac{\pi}{3}, t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2}$

$$\therefore \qquad I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{\sin x + \cos x}{\sqrt{1-t^2}} \frac{dt}{\cos x + \sin x} = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{1-t^2}} dt = [\sin^{-1} t]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$= \left[\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left\{-\left(\frac{\sqrt{3}-1}{2}\right)\right\}\right]$$

$$= \left[\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right] \qquad [\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$$

$$= 2 \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Example 5 $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ *is equal to* [NCERT]

(a) $(\pi + 2)$ (b) $-\frac{\pi}{2}(\pi - 2)$

(c) $\frac{\pi}{2}(\pi - 2)$ (d) 0

Solution (c) Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \qquad \ldots(i)$

$$\Rightarrow \qquad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \qquad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \qquad \ldots(ii)$$

$$[\because \tan(\pi - x) = -\tan x \text{ and } \sec(\pi - x) = -\sec x]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx = \pi \int_0^{\pi} \frac{\sin x(1 - \sin x)}{1 - \sin^2 x} dx$$

$$(\because \sin^2 x + \cos^2 x = 1)$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx = \pi \int_0^{\pi} \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right) dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x)\,dx$$

$$= \pi \int_0^{\pi} \{\sec x \tan x - (\sec^2 x - 1)\}\,dx \qquad (\because 1 + \tan^2 x = \sec^2 x)$$

$$= \pi\,[\sec x - \tan x + x]_0^{\pi}$$

$$= \pi\,[\sec \pi - \tan \pi + \pi - (\sec 0 - \tan 0 + 0)]$$

$$= \pi\,(-1 + \pi - 1)$$

$$\therefore \qquad I = \frac{\pi}{2}(\pi - 2)$$

Example 6 $\int_1^4 \{|x-1|+|x-2|+|x-3|\}dx$ *is equal to* [**NCERT**]

(a) $\frac{1}{2}$ (b) $\frac{7}{2}$

(c) $\frac{19}{2}$ (d) $\frac{21}{2}$

Solution (c) Firstly, define the absolute function as

$$|x-1| = \begin{cases} x-1, x \geq 1 \\ -(x-1), x < 1 \end{cases}, |x-2| = \begin{cases} x-2, x \geq 2 \\ -(x-2), x < 2 \end{cases},$$

$$|x-3| = \begin{cases} x-3, x \geq 3 \\ -(x-3), x < 3 \end{cases}$$

And then it is easy to solve the integration.

$$\int_1^4 \{|x-1|+|x-2|+|x-3|\}\, dx$$

$$= \int_1^2 (|x-1|+|x-2|+|x-3|)\, dx + \int_2^3 \{|x-1|+|x-2|+|x-3|\}\, dx + \int_3^4 \{|x-1|+|x-2|+|x-3|\}\, dx$$

$$= \int_1^2 \{x-1-(x-2)-(x-3)\}\, dx + \int_2^3 \{x-1+x-2-(x-3)\}\, dx + \int_3^4 (x-1+x-2+x-3)\, dx$$

$$= \int_1^2 (-x+4)\, dx + \int_2^3 x\, dx + \int_3^4 (3x-6)\, dx$$

$$= \left[-\frac{x^2}{2}+4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2}-6x\right]_3^4$$

$$= \left(\frac{-2^2}{2}+8\right) - \left(\frac{-1}{2}+4\right) + \frac{1}{2}(3^2-2^2) + \left(\frac{3}{2}\times 4^2 - 6\times 4\right) - \left(\frac{3}{2}\times 3^2 - 6\times 3\right)$$

$$= 6 - \frac{7}{2} + \frac{5}{2} + (24-24) - \left(-\frac{9}{2}\right) = \frac{12-7+5}{2} + 0 + \frac{9}{2} = \frac{19}{2}$$

Example 7 *The value of* $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ *is* [**NCERT**]

(a) 1 (b) 0

(c) –1 (d) $\frac{\pi}{4}$

Solution (b) Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$= \int_0^1 \tan^{-1}\left\{\frac{x+(x-1)}{1-x(x-1)}\right\}dx = \int_0^1 \{\tan^{-1}x + \tan^{-1}(x-1)\}\, dx$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$\Rightarrow \quad I = \int_0^1 \{\tan^{-1}x - \tan^{-1}(1-x)\}\, dx \quad \ldots(\text{i})$$

Also, $$I = \int_0^1 \{\tan^{-1}(1-x) - \tan^{-1}[1-(1-x)]\}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]\, dx \quad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get

$$2I = 0 \Rightarrow I = 0$$

Example 8 $\int_0^\pi x \log \sin x\, dx$ *is equal to* [**NCERT Exemplar**]

(a) $-\frac{\pi}{2}\log\frac{1}{2}$ (b) $\frac{1}{2}\log\frac{1}{2}$

(c) $\frac{\pi^2}{2}\log\frac{1}{2}$ (d) $\frac{\pi}{2}\log 2$

Solution (c) Let $I = \int_0^\pi x \log \sin x\, dx$ ...(i)

$$= \int_0^\pi (\pi - x)\log\sin(\pi - x)\, dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^\pi \pi \log\sin(x)\, dx - \int_0^\pi x\log\sin x\, dx \quad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get $[\because \sin(\pi - x) = \sin x]$

$$2I = \pi\int_0^\pi \log\sin x\, dx$$

Since, $f(\pi - x) = f(x) \quad \therefore \quad 2I = 2\pi\int_0^{\pi/2}\log\sin x\, dx$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log\sin x\, dx \quad \ldots(\text{iii})$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx = \pi\int_0^{\pi/2}\log\cos x\, dx \quad \ldots(\text{iv})$$

Now, adding Eqs. (iii) and (iv), we get

$$2I = \pi\int_0^{\pi/2}\log\sin x\, dx + \pi\int_0^{\pi/2}\log\cos x\, dx$$

$$= \pi\int_0^{\pi/2}(\log\sin x + \log\cos x)\, dx \quad (\because \log m + \log n = \log mn)$$

$$= \pi\int_0^{\pi/2}\log\sin x\cdot\cos x\, dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log\left(\frac{2\sin x\cdot\cos x}{2}\right)dx = \pi\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}(\log\sin 2x)\, dx - \pi\int_0^{\pi/2}\log 2\, dx \quad \ldots(\text{v})$$

$$\left(\because \log\frac{m}{n} = \log m - \log n\right)$$

Now, let $I_1 = \int_0^{\pi/2}\log\sin 2x\, dx$

Put $2x = t \Rightarrow 2dx = dt$

when $x = 0 \Rightarrow t = 0$ and when $x = \frac{\pi}{2}, t = \pi$

$$\therefore \quad I_1 = \int_0^\pi \log\sin t\,\frac{dt}{2} = \frac{1}{2}\int_0^\pi \log\sin t\, dt$$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin t \, dt$$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \int_0^{\pi/2} \log \sin x \, dx = \frac{I}{\pi}$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right] \quad \text{[from Eq. (iii)]}$$

On putting this value in Eq. (v), we get

$$2I = I - \pi \int_0^{\pi/2} \log 2 \, dx$$

$$\Rightarrow \quad I = -\frac{\pi^2}{2}\log 2 = \frac{\pi^2}{2}\log\frac{1}{2} \quad \left(\because -\log m = \log m^{-1} = \log\frac{1}{m}\right)$$

Example 9 *The value of* $\lim_{x\to\infty} \dfrac{\left(\int_0^x e^x\, dx\right)^2}{\int_0^x e^{2x^2}\, dx}$ *is*

(a) 0 (b) 1 (c) 2 (d) ∞

Solution (a) $\lim_{x\to\infty} \dfrac{\left(\int_0^x e^x\, dx\right)^2}{\int_0^x e^{2x^2}\, dx} = \lim_{x\to\infty} \dfrac{(e^x-1)^2}{\int_0^x e^{2x^2}\, dx}$

$$= \lim_{x\to\infty} \frac{2(e^x-1)e^x}{e^{2x^2}} \quad \text{(by L' Hospital's rule)}$$

$$= 2\lim_{x\to\infty} \frac{e^x-1}{e^{2x^2-x}}$$

$$= 2\lim_{x\to\infty} \frac{e^x}{e^{2x^2-x}(4x-1)} \quad \text{(by L' Hospital's rule)}$$

$$= 2\lim_{x\to\infty} \frac{1}{e^{2x^2-2x}(4x-1)} = \frac{1}{\infty} = 0$$

Example 10 *The value of integral* $\int_0^{\pi/2} \dfrac{\sin^2 x}{\sin x + \cos x}\, dx$ *is equal to*

(a) $\sqrt{2}(\log \sqrt{2})$ (b) $\sqrt{2}(\sqrt{2}+1)$

(c) $\log(\sqrt{2}+1)$ (d) *None of these*

Solution (d) Let $I = \int_0^{\pi/2} \dfrac{\sin^2 x}{\sin x + \cos x}\, dx$ ...(i)

Now, $I = \int_0^{\pi/2} \dfrac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\, dx$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\, dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos\left(x-\frac{\pi}{4}\right)}\, dx = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sec\left(x-\frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x-\frac{\pi}{4}\right) + \tan\left(x-\frac{\pi}{4}\right)\right\}\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}[\log(\sqrt{2}+1) - \log(\sqrt{2}-1)]$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2 = \sqrt{2}\log(\sqrt{2}+1)$$

Example 11 $\int_0^2 [x^2]\, dx$ *is equal to*

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$

(c) $\sqrt{2}-1$ (d) $-\sqrt{2}-\sqrt{3}+5$

Solution (d) $\int_0^2 [x^2]\, dx = \int_0^1 [x^2]\, dx + \int_1^{\sqrt{2}} [x^2]\, dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\, dx + \int_{\sqrt{3}}^2 [x^2]\, dx$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{\sqrt{3}} dx + \int_{\sqrt{3}}^2 3\, dx$$

$$= 0 + 1(\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5 - \sqrt{3} - \sqrt{2}$$

Example 12 *If* $\int_{\log 2}^x \dfrac{1}{\sqrt{e^x-1}}\, dx = \dfrac{\pi}{6}$, *then the value of x is*

(a) *log* 2 (b) *log* 3

(c) *log* 4 (d) *None of these*

Solution (c) Let $I = \int_{\log 2}^x \dfrac{1}{\sqrt{e^x-1}}\, dx$

Put $e^x - 1 = t^2 \Rightarrow e^x dx = 2t\, dt$

$$\therefore \quad I = 2\int_1^{\sqrt{e^x-1}} \frac{1}{1+t^2}\, dt = 2\,[\tan^{-1}\sqrt{e^x-1} - \tan^{-1} 1]$$

$$= 2\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{2}$$

But $\int_{\log 2}^x \dfrac{1}{\sqrt{e^x-1}}\, dx = \dfrac{\pi}{6}$

$$\Rightarrow \quad 2\tan^{-1}\sqrt{e^x-1} = \frac{2\pi}{3} \Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow \quad e^x - 1 = 3 \Rightarrow e^x = 4 \Rightarrow x = \log 4$$

Example 13 $\lim_{n\to\infty} \dfrac{(n!)^{1/n}}{n}$ *is equal to*

(a) e (b) e^2 (c) e^3 (d) e^{-1}

Solution (d) Let $A = \lim_{n\to\infty} \dfrac{(n!)^{1/n}}{n}$

On taking log on both sides, we get

$$\log A = \lim_{n\to\infty} \log\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\ldots\cdot\frac{n}{n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log x\, dx = [x\log x - x]_0^1$$

$$\Rightarrow \quad \log A = -1 \Rightarrow A = e^{-1}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Fundamental Theorem of Calculus

1. $\int_0^{\pi/2} \cos^2 x \, dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

2. $\int_0^1 \left(xe^x + \sin \frac{\pi \cdot x}{4} \right) dx$ is equal to [NCERT]

(a) $1 - \frac{\pi}{4} + \frac{2\sqrt{2}}{\pi}$ (b) $1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

(c) $1 + \frac{\pi}{4} + \frac{\pi}{2\sqrt{2}}$ (d) None of these

3. $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$ is equal to [NCERT]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{3}$

4. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} \, dx$ is equal to [NCERT]

(a) $\frac{e^2}{2}\left(\frac{e^2}{2} - 1\right)$ (b) $\frac{e}{2}\left(\frac{e}{2} - 1\right)$

(c) $\frac{e}{2}\left(\frac{e^2}{2} - 1\right)$ (d) $\frac{e^2}{2}\left(\frac{e}{2} + 1\right)$

5. $\int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$ is equal to [NCERT]

(a) e^{π} (b) $e^{2\pi}$

(c) $e^{\pi/2}$ (d) 0

6. $\int_0^1 x (\tan^{-1} x)^2 \, dx$ is equal to [NCERT Exemplar]

(a) $\frac{\pi^2 + 4}{16} + \log \sqrt{2}$ (b) $\frac{\pi^2 - 4\pi}{16} + \log \sqrt{2}$

(c) $\frac{\pi^2 + 4}{16} - \log \sqrt{2}$ (d) None of these

7. If $\int_0^1 \frac{e^t}{1+t} dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2} dt$ is equal to [NCERT Exemplar]

(a) $a - 1 + \frac{e}{2}$ (b) $a + 1 - \frac{e}{2}$

(c) $a - 1 - \frac{e}{2}$ (d) $a + 1 + \frac{e}{2}$

Properties of Definite Integration

8. $\int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx + \int_{2\pi}^{\pi/4} (\cos x - \sin x) \, dx$ is equal to

(a) $\sqrt{2} - 2$ (b) $2\sqrt{2} - 2$

(c) $3\sqrt{2} - 2$ (d) $4\sqrt{2} - 2$

9. If $f(\pi) = 2$ and $\int_0^{\pi} [f(x) + f''(x)] \sin x \, dx = 5$, then $f(0)$ is equal to, {it is given that $f(x)$ is continuous in $[0, \pi]$}

(a) 7 (b) 3

(c) 5 (d) 1

10. If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is [NCERT]

(a) $\sin x + x \sin x$

(b) $x \sin x$

(c) $x \cos x$

(d) $\sin x + x \cos x$

11. $\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) None of these

12. $\int_0^{\pi} \frac{x}{1 + \sin x} dx$ is equal to [NCERT]

(a) π (b) $\frac{\pi}{2}$

(c) 2π (d) $\frac{\pi}{8}$

13. $\int_0^{2\pi} \cos^5 x\, dx$ is equal to [NCERT]

(a) $\frac{7}{25}$ (b) $\frac{3}{7}$ (c) $\frac{1}{6}$ (d) 0

14. $\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

15. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

16. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{a}{2}$

17. $\int_0^4 |x - 1|\, dx$ is equal to [NCERT]

(a) $\frac{5}{2}$ (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ (d) 5

18. If f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$, then $\int_0^a f(x)\, g(x)\, dx$ is equal to [NCERT]

(a) $\int_0^a f(x)\, dx$ (b) $2\int_0^a f(x)\, dx$

(c) $2\int_0^a f(x)\, g(x)\, dx$ (d) None of these

19. The value of $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ is [NCERT]

(a) 2 (b) $\frac{3}{4}$ (c) 0 (d) -2

20. $\int_{-1}^1 x^{17} \cos^4 x\, dx$ is equal to [NCERT]

(a) $\frac{17}{4}$ (b) $\frac{13}{2}$

(c) 0 (d) -1

21. If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to [NCERT]

(a) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$ (b) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\, dx$ (d) $\frac{a+b}{2}\int_a^b f(x)\, dx$

22. $\int_{-a}^a \sqrt{\frac{a-x}{a+x}}\, dx$ is equal to

(a) π (b) a (c) $\frac{a\pi}{2}$ (d) $a\pi$

23. $\int_{-1}^1 x^3 e^{x^4}\, dx$ is equal to

(a) 0 (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 3

24. Given function $f(x) = \begin{cases} x^2, \text{ for } 0 \le x < 1 \\ \sqrt{x}, \text{ for } 1 \le x \le 2 \end{cases}$, then $\int_0^2 f(x)\, dx$ is equal to

(a) $(4\sqrt{2} - 1)$ (b) $\frac{1}{3}(4\sqrt{2} - 1)$

(c) $\frac{1}{3}(\sqrt{2} - 1)$ (d) None of these

25. $\int_{-2}^3 |x^2 - 1|\, dx$ is equal to

(a) 3 (b) $\frac{1}{3}$ (c) $\frac{17}{3}$ (d) $\frac{28}{3}$

26. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ is equal to [NCERT Exemplar]

(a) $\frac{1}{\sqrt{2}}\log(\sqrt{2} + 1)$ (b) $\frac{1}{2}\log(\sqrt{2} + 1)$

(c) $-\log(\sqrt{2} + 1)$ (d) None of these

27. $\int_{-1}^2 f(x)\, dx$, where $f(x) = |x+1| + |x| + |x-1|$ is equal to [NCERT Exemplar]

(a) $\frac{7}{2}$ (b) $\frac{9}{2}$ (c) $\frac{13}{2}$ (d) $\frac{19}{2}$

28. If f and g are continuous functions in [0,1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x) \cdot g(x)\, dx$ is equal to [NCERT Exemplar]

(a) $\frac{a}{2}$ (b) $\frac{a}{2}\int_0^a f(x)\, dx$

(c) $\int_0^a f(x)\, dx$ (d) $a\int_0^a f(x)\, dx$

29. If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$, then a is equal to [NCERT Exemplar]

(a) 3 (b) 6 (c) 9 (d) 1

30. $\int_{-1}^1 \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$ is equal to [NCERT Exemplar]

(a) $\log 2$ (b) $2\log 2$

(c) $\frac{1}{2}\log 2$ (d) $4\log 2$

31. $\int_{-2}^2 |x \cos \pi x|\, dx$ is equal to [NCERT Exemplar]

(a) $\frac{8}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{2}{\pi}$ (d) $\frac{1}{\pi}$

32. $\int_0^{\pi} \sqrt{\frac{\cos 2x + 1}{2}}\, dx$ is equal to

(a) 0 (b) 2

(c) -2 (d) None of these

33. $\int_0^{2\pi} (\sin x + |\sin x|)\, dx$ is equal to

(a) 0 (b) 4 (c) 8 (d) 1

34. $\int_0^{\pi/2} \sin 2x \log \tan x\, dx$ is equal to

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) 2π

35. The value of the integral $\int_{1/n}^{(an-1)/n} \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}\, dx$ is

(a) $\frac{a}{2}$ (b) $\frac{na+2}{2n}$

(c) $\frac{na-2}{2n}$ (d) None of these

36. $\int_{-1}^{1} \log(x+\sqrt{x^2+1})\, dx$ is equal to

(a) 0 (b) log 2

(c) $\log \frac{1}{2}$ (d) None of these

37. If $\int_0^a f(2a-x)\, dx = \mu$ and $\int_0^a f(x)\, dx = \lambda$, then $\int_0^{2a} f(x)\, dx$ is equal to

(a) $2\lambda - \mu$ (b) $\lambda + \mu$

(c) $\mu - \lambda$ (d) $\lambda - 2\mu$

38. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x\, dx$ has the value

(a) π (b) 1

(c) 0 (d) None of these

39. $\int_0^{\pi} e^{\sin^2 x} \cos^3 x\, dx$ is equal to

(a) -1 (b) 0

(c) 1 (d) π

40. The value of $\int_{-1}^{1} [x\{1+\sin \pi x\}+1]\, dx$ is ([·] denotes the greatest integer)

(a) 2 (b) 0

(c) 1 (d) None of these

41. If $f(x) = \begin{cases} |x|, & -1 \le x \le 1 \\ |x-2|, & 1 < x \le 3 \end{cases}$, then $\int_{-1}^{3} f(x)\, dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 4

42. Suppose that $f(x) \ge 0$ for all $x \in [0,1]$ and f is continuous in [0, 1] and $\int_0^1 f(x)\, dx = 0$, then $\forall\, x \in [0,1]$, f is

(a) entirely increasing (b) entirely decreasing

(c) constant (d) None of these

43. Let $f:(0,\infty) \to R$ and $F(x) = \int_0^x f(x)\, dt$.

If $F(x^2) = x^2(1+x)$, then $f(4)$ is equal to

(a) 5/4 (b) 7

(c) 4 (d) 2

44. Suppose f is such that $f(-x) = -f(x)$ for every real x and $\int_0^1 f(x)\, dx = 5$, then $\int_{-1}^{0} f(t)\, dt$ is equal to

(a) 10 (b) 5

(c) 0 (d) -5

45. $\int_{-2}^{2} |[x]|\, dx$ is equal to

(a) 1 (b) 2

(c) 3 (d) 4

46. If $f:R \to R$ and $g:R \to R$ are one to one, real valued functions, then the value of the integral

$\int_{-\pi}^{\pi} [f(x)+f(-x)][g(x)-g(-x)]\, dx$ is

(a) 0 (b) π

(c) 1 (d) None of these

47. If $P = \int_0^{3\pi} f(\cos^2 x)\, dx$ and $Q = \int_0^{\pi} f(\cos^2 x)\, dx$, then

(a) $P - Q = 0$ (b) $P - 2Q = 0$

(c) $P - 3Q = 0$ (d) $P - 5Q = 0$

48. Let $I_1 = \int_a^{\pi-a} x f(\sin x)\, dx$, $I_2 = \int_a^{\pi-a} f(\sin x)\, dx$, then I_2 is equal to

(a) $\frac{\pi}{2} I_1$ (b) πI_1

(c) $\frac{2}{\pi} I_1$ (d) $2 I_1$

49. $\int_{-1/7}^{20/7} e^{5\{x\}} dx$ is equal to (where {·} denotes fractional part of x)

(a) $\frac{3}{5}(e^5)$ (b) $\frac{3}{5}(e^5-1)$

(c) $\frac{3}{5}$ (d) None of these

50. If $f(x) = \begin{cases} e^{\cos x} \sin x, & |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)\, dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

51. $\int_{-1}^{1} [x \sin \pi x]\, dx$ is equal to

(a) 2 (b) -2 (c) 1 (d) 0

52. If $f(x)$ is continuous for all real values of x, then $\sum_{r=1}^{10} \int_0^1 f(r-1+x)\, dx$ is equal to

(a) $\int_0^{10} f(x)\, dx$ (b) $\int_0^1 f(x)\, dx$

(c) $10 \int_0^1 f(x)\, dx$ (d) $9\int_0^1 f(x)\, dx$

53. $\int_{-1}^{1} \max\{x-[x], -x-[-x]\}\, dx$ is equal to

(a) 3 (b) $\frac{3}{2}$

(c) $\frac{3}{4}$ (d) 2

54. If $I_1 = \int_0^1 \frac{1}{|x|} dx$ and $I_2 = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$, then

(a) $I_1 = I_2$ (b) $I_1 < I_2$
(c) $I_1 > I_2$ (d) $\frac{I_1}{I_2} = 2$

55. $I_1 = \int_0^{\pi/2} \sin x\, dx, I_2 = \int_0^{\pi/2} \sin^3 x\, dx$, then

(a) $I_1 > I_2$ (b) $I_1 < I_2$
(c) $I_1 = I_2$ (d) $I_2 = 0$

56. If f is a positive function and

$$I_1 = \int_{1-k}^{k} x f[x(1-x)]\, dx,$$

$$I_2 = \int_{1-k}^{k} f[x(1-x)]\, dx,$$

where $2k - 1 > 0$, then $\frac{I_1}{I_2}$ is

(a) 2 (b) k
(c) $\frac{1}{2}$ (d) 1

57. $f(x) = \min\{x+2, 1, 2-x\}$, then $\int_{-2}^{2} f(x)\, dx$ is equal to

(a) 1 (b) 2
(c) 3 (d) 0

58. If $\phi(x) = \cos x - \int_0^x (x-t)\ \phi(t)\, dt$, then $\phi''(x) + \phi(x)$ is equal to

(a) $-\cos x$ (b) 0
(c) $\int_0^x (x-t)\, \phi(t)\, dt$ (d) $-\int_0^{-x} (x-t)\, \phi(t)\, dt$

59. $\int_{-1/7}^{20/7} \sin(x - [x])\, dx$ (where [·] denotes greatest integer) is

(a) 0 (b) 3 (cos 1 – 1)
(c) 3 (1 – cos 1) (d) None of these

60. Consider the following statements

I. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\ dx = \frac{3}{4}$

II. $\int_0^4 (|x-1| + |x-3|)\, dx = 10$

Which of these is/are correct ?

(a) Only I (b) Only II
(c) Both I and II (d) None of these

61. The value of $\int_0^1 (1 + e^{-x^2})\, dx$ is

(a) -1 (b) 2
(c) $1 + e^{-1}$ (d) None of these

62. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then, $\int_{-1}^{1} f(x)\, dx$ is equal to

(a) 1 (b) 2
(c) 0 (d) $-\frac{1}{2}$

Definite Integration by Substitution

63. $\int_0^1 x e^{x^2}\ dx$ is equal to [NCERT]

(a) $\frac{1}{2}(e+1)$ (b) $e-1$ (c) $\frac{1}{2}(e-1)$ (d) $\frac{e^2}{2}$

64. $\int_0^2 \frac{6x+3}{x^2+4} dx$ is equal to [NCERT]

(a) $3\log 2 + \frac{\pi}{8}$ (b) $2\log 3 + \frac{\pi}{8}$
(c) $3\log 2 + \frac{3\pi}{8}$ (d) None of these

65. $\int_0^{\pi/2} \sqrt{\sin\phi} \cdot \cos^5 \phi d\phi$ is equal to [NCERT]

(a) $\frac{64}{231}$ (b) $\frac{35}{231}$ (c) $\frac{64}{221}$ (d) $\frac{35}{251}$

66. $\int_0^{\pi/4} \frac{\sin x \cdot \cos x}{\cos^4 x + \sin^4 x} dx$ is equal to [NCERT]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}$ (d) π

67. $\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$ is equal to [NCERT]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{8}$

68. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$ is equal to [NCERT]

(a) $\frac{1}{2}\log 3$ (b) $2\log 9$
(c) $\frac{1}{30}\log 3$ (d) $\frac{1}{40}\log 9$

69. $\int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x)\, dx$ is equal to [NCERT]

(a) $1 + \frac{\pi}{2}$ (b) $\frac{\pi}{2} - 1$ (c) $\frac{\pi}{2}$ (d) 1

70. $\int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$ is equal to

(a) $\frac{1}{3}\log\left\{\frac{\sqrt{17}+1}{\sqrt{37}+1}\right\}$ (b) $\frac{3}{2}\log\left\{\frac{\sqrt{17}+1}{\sqrt{37}+1}\right\}$
(c) $\log\log\left\{\frac{3}{2}\cdot\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right\}$ (d) None of these

71. $\int_0^1 \frac{dx}{e^x + e^{-x}} dx$ is equal to [NCERT Exemplar]

(a) $\pi/4$ (b) $\tan^{-1} e - \frac{\pi}{4}$
(c) $\tan^{-1} e$ (d) $\frac{\pi}{4}\tan^{-1} e$

72. $\int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x} dx$ is equal to [NCERT Exemplar]

(a) 0 (b) $\frac{1}{m}$

(c) $\frac{2}{m}$ (d) $\frac{1}{2m}$

73. $\int_0^{\frac{1}{2}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ is equal to [NCERT Exemplar]

(a) $\frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{2}{3}}$

(b) $\tan^{-1} \sqrt{\frac{3}{2}}$

(c) 0

(d) $\frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{3}{2}}$

74. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx$ is equal to [NCERT Exemplar]

(a) $\frac{3}{64}$ (b) $-\frac{3}{128}$

(c) $\frac{-3}{64}$ (d) $-\frac{1}{128}$

75. $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) 0

76. If $\int_0^1 \frac{e^t\, dt}{t+1} = a$, then $\int_{b-1}^{b} \frac{e^{-t}\, dt}{t-b-1}$ is equal to

(a) ae^{-b} (b) $-ae^{-b}$

(c) $-be^{-a}$ (d) ae^{b}

Definite Integration as a Limit of Sum

77. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right)$ is equal to

(a) log 2 (b) log 3

(c) log 5 (d) log 6

78. $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n+1}+\frac{2}{n+2}+\ldots+\frac{3n}{4n}\right)$ is equal to

(a) log 4

(b) $-$log 4

(c) 1 $-$ log 4

(d) None of the above

Reduction Formulae Indefinite Integrals

79. If $I(m,n) = \int_0^1 t^m (1+t)^n\, dt$, then the expression for $I(m,n)$ in terms of $I(m+1, n-1)$ is

(a) $\frac{2^n}{m+1} - \frac{n}{m+1} I(m+1, n-1)$

(b) $\frac{n}{m+1} I(m+1, n-1)$

(c) $\frac{2^n}{m+1} + \frac{n}{m+1} I(m+1, n-1)$

(d) $\frac{m}{n+1} I(m+1, n-1)$

Miscellaneous

80. $\int_0^{\pi} \log(1+\cos x)\, dx$ is equal to [NCERT]

(a) 0 (b) $\frac{\pi}{2}\log 2$

(c) $-\pi \log 2$ (d) $2\pi \log 2$

81. $\int_0^{\infty} e^{-x} \cdot x^3 dx$ is equal to

(a) 0 (b) 2

(c) 6 (d) 9

82. The value of $\int_0^1 \log \sin\left(\frac{\pi x}{2}\right) dx$ is equal to

(a) log 2 (b) $-$log 2

(c) log 3 (d) 0

83. $\int_{a+c}^{b+c} f(x)\, dx$ is equal to [NCERT Exemplar]

(a) $\int_a^b f(x-c)\, dx$ (b) $\int_a^b f(x+c)\, dx$

(c) $\int_a^b f(x)\, dx$ (d) $\int_{a-c}^{b-c} f(x)\, dx$

84. The value of $\int_{-1}^{3}\left[\tan^{-1}\left(\frac{x}{x^2+1}\right)+\tan^{-1}\left(\frac{x^2+1}{x}\right)\right] dx$ is

(a) 2π (b) π

(c) $\pi/2$ (d) $\pi/4$

85. $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\, dx$ is equal to

(a) $\frac{3\pi}{64}$ (b) $\frac{3\pi}{572}$ (c) $\frac{3\pi}{256}$ (d) $\frac{3\pi}{128}$

86. $\int_0^1 \tan^{-1}\left(\frac{1}{x^2-x+1}\right) dx$ is equal to

(a) log 2 (b) $-$log 2

(c) $\frac{\pi}{2}$ + log 2 (d) $\frac{\pi}{2}$ $-$ log 2

Round II (Mixed Bag)

Only One Correct Option

1. $100\int_0^1 \{x\}\,dx$, where $\{x\}$ denotes the fraction part of x.

(a) 100 (b) 25 (c) 75 (d) 50

2. $\int_0^{41\pi/2} \sin x\,dx$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

3. If $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$, then

(a) $I \ge \frac{\pi}{6}$ (b) $I \ge \frac{\pi}{4}$ (c) $I \ge \frac{\pi}{5}$ (d) $I \ge \frac{\pi}{2}$

4. $\int_0^x [\sin t]\,dt$, where $x \in \{2n\pi, (2n+1)\pi\}, n \in N$ and $[\cdot]$ denotes the greatest integer function, is equal to

(a) $-n\pi$ (b) $-(n+1)\pi$ (c) $-2n\pi$ (d) $-(2n+1)\pi$

5. The value of $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$ is

(a) $\frac{\pi}{2}$ (b) 1

(c) $\frac{\pi}{4}$ (d) None of these

6. Equation of tangent to $y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^2}}$ at $x = 1$ is

(a) $y = x\sqrt{2} - 1$ (b) $y\sqrt{2} = x - 1$

(c) $y = x\sqrt{2} + 1$ (d) $y\sqrt{2} = x + 1$

7. Find the value of $\int_0^9 [\sqrt{x} + 2]\,dx$, where $[\cdot]$ is the greatest integer function

(a) 31 (b) 23 (c) 22 (d) 27

8. The value of $\int_0^{\sqrt{2}} [x^2]\,dx$, where $[\cdot]$ is the greatest integer function

(a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$

(c) $\sqrt{2} - 1$ (d) $\sqrt{2} - 2$

9. Value of $\int_1^5 \{\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2(x-1)}\}\,dx$ is

(a) $\frac{8}{3}$ (b) $\frac{16}{3}$ (c) $\frac{32}{3}$ (d) $\frac{34}{3}$

10. $I = \int_{-2}^1 \left(\tan^{-1} x + \cot^{-1}\frac{1}{x}\right)dx$ is equal to

(a) $\frac{5\pi}{2} + 4\tan^{-1} 2 - \ln\frac{5}{2}$ (b) $\frac{5\pi}{2} - 4\tan^{-1} 2 + \ln\frac{5}{2}$

(c) $\frac{5\pi}{2} - 3\tan^{-1} 2 - \ln\frac{5}{2}$ (d) $\frac{5\pi}{2} - 3\tan^{-1} 2 + \ln\frac{5}{2}$

11. $I_1 = \int_0^{\pi/4} 2^{\sin x}\,dx$ and $I_2 = \int_0^{\pi/4} 2^{\cos x}\,dx$, then

(a) $I_1 = I_2$ (b) $I_1 > I_2$

(c) $I_1 < I_2$ (d) cannot say

12. Let $g(x) = \int_0^x f(t)\,dt$, where f is such that $1/2 \le f(t) \le 1$ for $t \in [0,1]$ and $0 \le f(t) \le 1/2$ for $t \in [1,2]$. Then, $g(2)$ satisfies the inequality.

(a) $1/2 \le g(2) \le 3/2$ (b) $0 \le g(2) < 2$

(c) $3/2 \le g(2) \le 5/2$ (d) $2 < g(2) < 4$

13. The expression $\frac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx}$, where $[x]$ and $\{x\}$ are integral and fractional part of x and $n \in N$, is equal to

(a) $\frac{1}{n-1}$ (b) $\frac{1}{n}$

(c) n (d) $n - 1$

14. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx$ is equal to

(a) $\sin\theta + \operatorname{cosec}\theta$ (b) $\sin^2\theta$

(c) $\operatorname{cosec}^2\theta$ (d) None of these

15. If $f(x) = \frac{x-1}{x+1}$, $f^2(x) = f(fx), \ldots, f^{k+1}(x) = f\{f^k(x)\}$, $k = 1, 2, 3, \ldots$ and $g(x) = f^{1998}(x)$, then $\int_{1/e}^1 g(x)\,dx$ is equal to

(a) 0 (b) 1 (c) -1 (d) e

16. $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x)\,dx = \frac{n^2}{2}$, $\forall\, n \in I$, then $\int_{-3}^5 f(|x|)\,dx$ is equal to

(a) $\frac{19}{2}$ (b) $\frac{35}{2}$

(c) $\frac{17}{2}$ (d) None of these

17. $\int_1^4 (\{x\})^{[x]}\,dx$, where $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer function and fractional part respectively is equal to

(a) $\frac{13}{12}$ (b) $\frac{1}{2}$

(c) $\frac{15}{12}$ (d) None of these

18. $I_{1(n)} = \int_0^{\pi/2} \frac{\sin(2n-1)}{\sin x}\,dx$, $I_{2(n)} = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$, $n \in N$, then

(a) $I_{2(n+1)} - I_{2(n)} = I_{1(n)}$ (b) $I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$

(c) $I_{2(n+1)} + I_{1(n)} = I_{2(n)}$ (d) $I_{2(n+1)} + I_{1(n+1)} = I_{2(n)}$

19. $\int_0^{3\pi/2} \sin\left[\frac{2x}{\pi}\right] dx$, where $[\cdot]$ denotes the greatest integer function is equal to

(a) $\frac{\pi}{2}(\sin 1 + \cos 1)$ (b) $\frac{\pi}{2}(\sin 1 - \sin 2)$

(c) $\frac{\pi}{2}(\sin 1 - \cos 1)$ (d) $\frac{\pi}{2}(\sin 1 + \sin 2)$

20. $\int_{-1}^{1} \{x^2 + x - 3\}\, dx$, where $\{x\}$ denotes the fractional part of x, is equal to

(a) $\frac{1}{3}(1 + 3\sqrt{5})$ (b) $\frac{1}{6}(1 + 3\sqrt{5})$

(c) $\frac{1}{3}(3\sqrt{5} - 1)$ (d) $\frac{1}{6}(3\sqrt{5} - 1)$

21. If $I = \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}}\, dx$, then

(a) $0 < I < 1$ (b) $I > \frac{\pi}{2\sqrt{2}}$

(c) $I < \sqrt{2}\,\pi$ (d) $I < 2\pi$

22. $\int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}]\, dx$ is equal to

(a) $\sqrt{2}\,\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{\sqrt{2}}$ (d) 2π

23. If $g(1) = g(2)$, then $\int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\}\, g'(x)\, dx$ is equal to

(a) 1 (b) 2

(c) 0 (d) None of these

24. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x}\, dx$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these

25. Let a, b and c be non-zero real numbers such that $\int_0^3 (3ax^2 + 2bx + c)\, dx = \int_1^3 (3ax^2 + 2bx + c)\, dx$, then

(a) $a + b + c = 3$ (b) $a + b + c = 1$

(c) $a + b + c = 0$ (d) $a + b + c = 2$

26. If $f(x)$ is continuous function, then

(a) $\int_{-2}^{2} f(x)\, dx = \int_0^2 [f(x) - f(-x)]\, dx$

(b) $\int_{-3}^{5} 2f(x)\, dx = \int_6^{10} f(x-1)\, dx$

(c) $\int_{-3}^{5} f(x)\, dx = \int_{-4}^{4} f(x-1)\, dx$

(d) $\int_{-3}^{5} f(x)\, dx = \int_{-2}^{6} f(x-1)\, dx$

27. $\lim_{n \to \infty} \left[\frac{n!}{n^n}\right]^{1/n}$ is equal to

(a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

28. The points of intersection of $f_1(x) = \int_2^x (2t - 5)\, dt$ and $f_2(x) = \int_0^x 2t\, dt$, are

(a) $\left(\frac{6}{5}, \frac{36}{25}\right)$ (b) $\left(\frac{2}{3}, \frac{4}{9}\right)$ (c) $\left(\frac{1}{3}, \frac{1}{9}\right)$ (d) $\left(\frac{1}{5}, \frac{1}{25}\right)$

29. The value of integral $\sum_{k=1}^{n} \int_0^1 f(k - 1 + x)\, dx$ is

(a) $\int_0^1 (x)\, dx$ (b) $\int_0^2 f(x)\, dx$

(c) $\int_0^n f(x)\, dx$ (d) $n \int_0^1 f(x)\, dx$

30. Let $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{3}{x} e^{\sin x^3}\, dx = f(k) - f(1)$, then the possible value of k, is

(a) 15 (b) 16 (c) 63 (d) 64

31. The value of the definite integral $\int_{a+2\pi}^{a+5\pi/2} \{\sin^{-1}(\cos x) + \cos^{-1}(\sin x)\}\, dx$ is equal to

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{2}$ (d) π^2

32. Let f be integrable over $[0, a]$ for any real a. If we define

$$I_1 = \int_0^{\pi/2} \cos\theta\, f(\sin\theta + \cos^2\theta)\, d\theta$$

and $$I_2 = \int_0^{\pi/2} \sin 2\theta\, f(\sin\theta + \cos^2\theta)\, d\theta, \text{ then}$$

(a) $I_1 = I_2$ (b) $I_1 = -I_2$ (c) $I_1 = 2I_2$ (d) $I_1 = -2I_2$

33. If $f(x) = \int_0^x (\sin^4 t + \cos^4 t)\, dt$, then $f(x + \pi)$ will be equal to

(a) $f(x) + f\left(\frac{\pi}{2}\right)$ (b) $f(x) + f(\pi)$ or $f(x) + 2f\left(\frac{\pi}{2}\right)$

(c) $f(x) - f(\pi)$ (d) $f(x) - 2f\left(\frac{\pi}{2}\right)$

34. If $I_k = \int_{-2k\pi}^{2k\pi} |\sin x|[\sin x]\, dx, \forall\, k \in N$, where $[\cdot]$ denotes the greatest integer function, then $\sum_{k=1}^{10} I_k$ is equal to

(a) –110 (b) –440 (c) –330 (d) –220

35. The value of $\int_0^{\frac{3\pi}{2}} \frac{|\tan^{-1}\tan x| - |\sin^{-1}\sin x|}{|\tan^{-1}\tan x| + |\sin^{-1}\sin x|}\, dx$ is equal to

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{3\pi}{2}$ (d) None of these

36. The value of $\int_{-1}^{15} \text{sgn}(\{x\})\, dx$, where $\{\cdot\}$ denotes the fractional part function, is

(a) 8 (b) 16 (c) 24 (d) 0

37. If $\int_{\sin x}^{1} t^2 f(t)\,dt = 1 - \sin x, \forall\, x \in \left(0, \frac{\pi}{2}\right)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) 3 (b) $\sqrt{3}$
(c) 1/3 (d) None of these

More Than One Correct Option

38. If $I_1 = \int_x^1 \frac{dt}{1+t^2}$ and $I_2 = \int_1^{1/x} \frac{dt}{1+t^2}, x > 0$, then

(a) $I_1 = I_2$ (b) $I_1 > I_2$
(c) $I_2 > I_1$ (d) $I_2 = \frac{\pi}{4} - \tan^{-1} x$

39. Let $A = \int_{e^{-1}}^{\tan x} \frac{t\,dt}{t^2+1}$ and $B = \int_{e^{-1}}^{\cot x} \frac{dt}{1(t^2+1)}$

(a) at $x = \frac{\pi}{4}, A + B = -1$
(b) $A + B = 1$ for all x in $\left(0, \frac{\pi}{2}\right)$
(c) $A + B = 1$ for all x in $\left(0, \frac{\pi}{4}\right)$ and 2 for all x in $\left(\frac{\pi}{4}, \pi\right)$
(d) $A = B$ for all x

40. If f is odd and is periodic with period 2, if $g(x) = \int^x f(t)\,dt$, then

(a) $g(x)$ is an odd function
(b) $g(x)$ is an even function
(c) $g(x)$ as periodic with period 2
(d) $g(x)$ is not periodic

41. The value of α which satisfy $0 \le \alpha \le 2\pi$ and $\int_{\pi/2}^{\alpha} \sin x\,dx = \sin 2\alpha$ are

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

42. The integral $\int_0^{\pi} x\, f(\sin x)\,dx$ is equal to

(a) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ (b) $\frac{\pi}{4}\int_0^{\pi} f(\sin x)\,dx$
(c) $\pi\int_0^{\pi/2} f(\sin x)\,dx$ (d) $\pi\int_0^{\pi/2} f(\cos x)\,dx$

Comprehension Based Questions

Passage

Directions (Q. Nos. 43-45) *By using properties of definite integral* $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

43. If $I_1 = \int_{1-k}^{k} x \sin\{x(1-x)\}\,dx$ and $I_2 = \int_{1-k}^{k} \sin\{x(1-x)\}\,dx$, then

(a) $I_1 = 2I_2$ (b) $2I_1 = I_2$
(c) $I_1 = I_2$ (d) None of these

44. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} xf\{x(3-x)\}\,dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\{x(3-x)\}\,dx$ where f is a continuous function and z is any real number, then I_1 / I_2 is equal to

(a) $\frac{3}{2}$ (b) 1/2 (c) 1 (d) 5

45. If f and g are continuous functions on $[0, p]$ satisfying $f(x) + f(p-x) = g(x) + g(p-x) = 1$, then $\int_0^p [f(x) + g(x)]\,dx$ is equal to

(a) p (b) $2p$ (c) $p/2$ (d) $3p/2$

Assertion and Reason

Directions (Q. Nos. 46-58) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

46. **Statement I** $\int_0^{4x} [|\sin x| + |\cos x|]\,dx$, (where [·] denotes greatest integer function) equals to 8π.

Statement II If $f(x) = |\sin x| + |\cos x|$, then $1 \le f(x) \le \sqrt{2}$.

47. **Statement I** If satisfies $(x+y) = f(x) + f(y), \forall\, x, y \in R$, then $\int_{-5}^{5} f(x)\,dx = 0$

Statement II If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

48. **Statement I** If y is a function of x such that $y(x-y)^2 = x$, then $\int \frac{dx}{x-3y} = \frac{1}{2}[\log(x-y)^2 - 1]$

Statement II $\int \frac{dx}{x-3y} = \log(x-3y) + C$

49. **Statement I** $f(x) = \int_1^x \frac{\log t\,dt}{1+t+t^2} (x > 0)$, then $f(x) = -f\left(\frac{1}{x}\right)$

Statement II If $f(x) = \int_1^x \frac{\log t\,dt}{t+1}$, then

$$f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2$$

50. **Statement I** If $\int_0^1 e^{\sin x}\,dx = \lambda$, then $\int_0^{200} e^{\sin x}\,dx = 200\lambda$

Statement II $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx, n \in I$ and $f(a+x) = f(x)$

51. **Statement I** $\int_0^6 \{x+5\}^2\,dx = 41$,

where {·} denotes the fractional part function.

Statement II $\{x+5\}$ is a periodic function.

52. **Statement I** $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2$

Statement II $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $c \in (a, b)$

53. **Statement I** $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx = 0$, where [·] denotes the greatest integer function.

Statement II $2\sin x$ is a decreasing function in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

54. **Statement I** A polynomial of least degree that has a maximum equal to 6 at $x = 1$ minimum equal to 2 at $x = 3$ is $x^3 - 6x^2 + 9x + 2$.

Statement II The polynomial is everywhere differentiable and the points of extremum can only be roots of derivative.

55. If $n > 1$, then

Statement I $\int_0^\infty \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Statement II $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

56. **Statement I** The value of the integral $\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx\,(n \in N)$ is π.

Statement II $\int_0^\pi \sin mx\,dx = 0\,(m \in N)$

57. **Statement I** $\int_0^{\pi/2} (\sin^6 x + \cos^6 x)\,dx$ lie in the interval $\left(\frac{\pi}{8}, \frac{\pi}{2}\right)$.

Statement II $\sin^6 x + \cos^6 x$ is periodic with period $\pi/2$.

58. **Statement I** $\int_0^1 e^{-x^2}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\,dx$

Statement II $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx,\ \forall\ f(x) \ge g(x)$

Previous Years' Questions

59. If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ is equal to [AIEEE 2012]

(a) $\frac{g(x)}{g(\pi)}$ (b) $g(x) + g(\pi)$

(c) $g(x) - g(\pi)$ (d) $g(x) \cdot g(\pi)$

60. Let [·] denotes the greatest integer function, then the value of $\int_0^{1.5} x[x^2]\,dx$ is [AIEEE 2011]

(a) $\frac{5}{4}$ (b) 0 (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

61. Let $p(x)$ be a function defined on R such that $\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$, $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then, $\int_0^1 p(x)\,dx$ is equal to [AIEEE 2010]

(a) $\sqrt{41}$ (b) 21 (c) 41 (d) 42

62. $\int_0^\pi [\cot x]\,dx$, [·] denotes the greatest integer function, is equal to [AIEEE 2009]

(a) $\frac{\pi}{2}$ (b) 1

(c) -1 (d) $-\frac{\pi}{2}$

63. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$. Then, which one of the following is true? [AIEEE 2008]

(a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$

(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

64. Let $f(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^x \frac{\log t}{1+t}\,dt$.

Then, $f(e)$ is equal to [AIEEE 2007]

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) 2

65. The value of $\int_1^a [x]\,f'(x)\,dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x is [AIEEE 2006]

(a) $[a]\,f(a) - \{f(1) + f(2) + \ldots + f([a])\}$

(b) $[a]\,f([a]) - \{f(1) + f(2) + \ldots + f(a)\}$

(c) $a\,f([a]) - \{f(1) + f(2) + \ldots + f(a)\}$

(d) $a\,f(a) - \{f(1) + f(2) + \ldots + f([a])\}$

66. $\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)]\,dx$ is equal to [AIEEE 2006]

(a) $\left(\frac{\pi^4}{32}\right) + \frac{\pi}{2}$ (b) $\frac{\pi}{2}$

(c) $\left(\frac{\pi}{4}\right) - 1$ (d) $\frac{\pi^4}{32}$

67. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$, $a > 0$, is [AIEEE 2005]

(a) 2π (b) π/a (c) $\pi/2$ (d) $a\pi$

Answers

Check Point 1

1. (a) **2.** (d) **3.** (b) **4.** (c) **5.** (b) **6.** (b)

Check Point 2

1. (c) **2.** (b) **3.** (c) **4.** (c) **5.** (c) **6.** (`a)

Round I

1. (c)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (b)
11. (c)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (a)	**16.** (d)	**17.** (d)	**18.** (b)	**19.** (c)	**20.** (c)
21. (d)	**22.** (d)	**23.** (a)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (b)
31. (a)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (a)
41. (c)	**42.** (c)	**43.** (c)	**44.** (d)	**45.** (d)	**46.** (a)	**47.** (c)	**48.** (c)	**49.** (b)	**50.** (c)
51. (d)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (a)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (c)	**60.** (b)
61. (d)	**62.** (a)	**63.** (c)	**64.** (c)	**65.** (a)	**66.** (c)	**67.** (a)	**68.** (d)	**69.** (b)	**70.** (c)
71. (b)	**72.** (d)	**73.** (a)	**74.** (b)	**75.** (b)	**76.** (b)	**77.** (d)	**78.** (d)	**79.** (a)	**80.** (c)
81. (c)	**82.** (b)	**83.** (b)	**84.** (a)	**85.** (c)	**86.** (d)				

Round II

1. (d)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (b)
11. (c)	**12.** (a)	**13.** (d)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (b)
21. (b)	**22.** (c)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (d)
31. (b)	**32.** (a)	**33.** (b)	**34.** (d)	**35.** (d)	**36.** (b)	**37.** (a)	**38.** (a,d)	**39.** (a,b)	**40.** (b,c)
41. (a,b,c,d)	**42.** (a,c,d)	**43.** (b)	**44.** (a)	**45.** (a)	**46.** (d)	**47.** (a)	**48.** (c)	**49.** (d)	**50.** (d)
51. (d)	**52.** (b)	**53.** (d)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (b)	**58.** (b)	**59.** (b)	**60.** (d)
61. (b)	**62.** (d)	**63.** (c)	**64.** (a)	**65.** (a)	**66.** (b)	**67.** (c)			

the Guidance

Round I

1. There is no standard integration of $\cos^2 x$, therefore we convert it in standard form by using the relation $\frac{\cos 2x + 1}{2}$ and then integrate and put the limits.

$$\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \left(\frac{\cos 2x + 1}{2}\right) dx \quad (\because \cos 2x = 2\cos^2 x - 1)$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx + \frac{1}{2}\int_0^{\pi/2} 1\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2} + \frac{1}{2}[x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\sin \pi}{2} - \frac{\sin 0}{2}\right) + \left(\frac{\pi}{2} - 0\right)\right]$$

$$= \frac{\pi}{4} + 0 - 0 = \frac{\pi}{4}$$

2. $$\int_0^1 \left(xe^x + \sin\frac{\pi \cdot x}{4}\right) dx = \int_0^1 xe^x\, dx + \int_0^1 \sin\left(\frac{\pi \cdot x}{4}\right) dx$$

$$= [xe^x - \int 1 \cdot e^x\, dx]_0^1 + \left[\frac{-\cos\frac{\pi \cdot x}{4}}{\frac{\pi}{4}}\right]_0^1$$

First integral solved by using the integration by parts and for second $\left(\int \sin x\, dx = -\frac{\cos ax}{a}\right)$

$$= [xe^x - e^x]_0^1 - \frac{4}{\pi}\left[\cos\left(\frac{\pi \cdot x}{4}\right)\right]_0^1$$

$$= (e - e) - (0 - e^0) - \frac{4}{\pi}\left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= 0 - (-1) - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right) = 1 - \frac{4}{\pi}\left(\frac{1 - \sqrt{2}}{\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 1 + \frac{2}{\pi}(2 - \sqrt{2}) = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

3. $$\int_{-1}^1 \frac{1}{x^2 + 2x + 5} dx = \int_{-1}^1 \frac{1}{(x^2 + 2x + 1) + 4} dx$$

$$= \int_{-1}^1 \frac{1}{(x+1)^2 + 2^2} dx$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x+1}{2}\right]_{-1}^1 \qquad \left(\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{0}{2}\right)\right] = \frac{1}{2}[\tan^{-1} 1] = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$$

4. Let $I=\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\int_1^2\frac{1}{x}e^{2x}dx-\int_1^2\frac{1}{2x^2}e^{2x}dx$

Evaluate the first integral by rule of integration by parts taking $\frac{1}{x}$ as the first function, we get

$$I=\left[\frac{1}{x}\cdot\frac{e^{2x}}{2}\right]_1^2+\int_1^2\frac{1}{x^2}\cdot\frac{e^{2x}}{2}dx-\int_1^2\frac{1}{2x^2}e^{2x}dx$$

$$=\left[\frac{1}{x}\frac{e^{2x}}{2}\right]_1^2=\frac{e^4}{4}-\frac{e^2}{2}=\frac{e^2}{2}\left(\frac{e^2}{2}-1\right)$$

5. Let $I=\int_{\pi/2}^{\pi}e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$=\int_{\pi/2}^{\pi}e^x\left\{\frac{1-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}\right\}dx$$

$$\left(\because \sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\text{ and }1-\cos x=2\sin^2\frac{x}{2}\right)$$

$$=\int_{\pi/2}^{\pi}e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)dx$$

$$=\int_{\pi/2}^{\pi}e^x\left(-\cot\frac{x}{2}+\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx$$

$$\left[\text{here, }\frac{d}{dx}\left(-\cot\frac{x}{2}\right)=\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\text{ it is the form of }\int e^x\{f(x)+f'(x)\}dx=e^xf(x)\right]$$

$$\therefore\quad I=\left[e^x\left(-\cot\frac{x}{2}\right)\right]_{\pi/2}^{\pi}=-e^{\pi}\cot\left(\frac{\pi}{2}\right)-\left[-e^{\pi/2}\cot\left(\frac{\pi}{4}\right)\right]$$

$$=-e^{\pi}\cdot 0+e^{\pi/2}\cdot 1=e^{\pi/2}$$

6. $I=\int_0^1 x(\tan^{-1}x)^2dx$

On integrating by parts, we have

$$I=\frac{x^2}{2}[(\tan^{-1}x)^2]_0^1-\frac{1}{2}\int_0^1 x^2\cdot 2\frac{\tan^{-1}x}{1+x^2}dx$$

$$=\frac{\pi^2}{32}-\int_0^1\frac{x^2}{1+x^2}\cdot\tan^{-1}x\,dx$$

$$=\frac{\pi^2}{32}-I_1,\text{ where }I_1=\int_0^1\frac{x^2}{1+x^2}\tan^{-1}x\,dx$$

Now, $I_1=\int_0^1\frac{x^2+1-1}{1+x^2}\tan^{-1}x\,dx$

$$=\int_0^1\tan^{-1}x\,dx-\int_0^1\frac{1}{1+x^2}\tan^{-1}x\,dx$$

$$=I_2-\frac{1}{2}\{(\tan^{-1}x^2)\}_0^1=I_2-\frac{\pi^2}{32}$$

Here, $I_2=\int_0^1\tan^{-1}x\,dx=[x\tan^{-1}x]_0^1-\int_0^1\frac{x}{1+x^2}dx$

$$=\frac{\pi}{4}-\frac{1}{2}[\log|1+x^2|]_0^1=\frac{\pi}{4}-\frac{1}{2}\log 2$$

Thus, $I_1=\frac{\pi}{4}-\frac{1}{2}\log 2-\frac{\pi^2}{32}$

Therefore,

$$I_1=\frac{\pi^2}{32}-\frac{\pi}{4}+\frac{1}{2}\log 2+\frac{\pi^2}{32}=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$$

$$=\frac{\pi^2-4\pi}{16}+\log\sqrt{2}$$

7. Since, $I=\int_0^1\frac{e^t}{1+t}dt=\left[\frac{1}{1+t}e^t\right]_0^1+\int_0^1\frac{e^t}{(1+t)^2}dt=a$ (given)

Therefore, $\int_0^1\frac{e^t}{(1+t)^2}=a-\frac{e}{2}+1$

8. $\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)\,dx$

$$+\int_{2\pi}^{\pi/4}(\cos x-\sin x)\,dx$$

$$=\{[\sin x+\cos x]_0^{\pi/4}-[\sin x+\cos x]_{\pi/4}^{5\pi/4}+[\sin x+\cos x]_{2\pi}^{\pi/4}\}$$

$$=\left[\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-(\sin 0+\cos 0)\right]$$

$$-\left[\sin\frac{5\pi}{4}+\cos\frac{5\pi}{4}-\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)\right]$$

$$+\left[\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-(\sin 2\pi+\cos 2\pi)\right]$$

$$=\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right]-\left[-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]$$

$$+\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right]$$

$$=[\sqrt{2}-1]-[-\sqrt{2}-\sqrt{2}]+[\sqrt{2}-1]$$

$$=[\sqrt{2}-1+2\sqrt{2}+\sqrt{2}-1]=4\sqrt{2}-2$$

9. $\int_0^{\pi}f(x)\sin x\,dx+\int_0^{\pi}f''(x)\sin x\,dx=5$

$$\Rightarrow [f(x)(-\cos x)]_0^{\pi}+\int_0^{\pi}f'(x)\cos x\,dx+\int_0^{\pi}f''(x)\sin x\,dx=5$$

$$\Rightarrow\quad [-f(x)\cos x]_0^{\pi}+[f'(x)\sin x]_0^{\pi}$$

$$-\int_0^{\pi}f''(x)\sin x\,dx+\int_0^{\pi}f''(x)\sin x\,dx=5$$

$$\Rightarrow\quad f(\pi)+f(0)=5\Rightarrow f(0)=5-f(\pi)=3$$

10. Firstly, we integrate the given function and then differentiate it to get the result.

Given, $f(x)=\int_0^x t\sin t\,dt$

On integrating by parts, we get

$$f(x)=t\int_0^x\sin t\,dt-\int_0^x\left[\left(\frac{d}{dt}t\right)\int\sin t\,dt\right]dt$$

$$=[t(-\cos t)]_0^x-\int_0^x(-\cos t)\,dt=[-t\cos t+\sin t]_0^x$$

$\Rightarrow f(x)=-x\cos x+\sin x$

Now, differentiate w.r.t. x, we get

$$f'(x)=-[\{x(-\sin x)\}+\cos x]+\cos x$$

$$=x\sin x-\cos x+\cos x=x\sin x$$

Using the product rule, $\frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}$

11. Let $I=\int_0^{\pi/2}\frac{\sin^{3/2}x}{\sin^{3/2}x+\cos^{3/2}x}dx$ ...(i)

Then, $I=\int_0^{\pi/2}\frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)}dx$

$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\Rightarrow$ $I=\int_0^{\pi/2}\frac{\cos^{3/2}x}{\cos^{3/2}x+\sin^{3/2}x}dx$ ...(ii)

$\left[\because \sin\left(\frac{\pi}{2}-x\right)=\cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right)=\sin x\right]$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{\sin^{3/2}x+\cos^{3/2}x}{\sin^{3/2}x+\cos^{3/2}x}dx$$

$$=\int_0^{\pi/2}1\,dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0 \Rightarrow I=\frac{\pi}{4}$$

12. Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx$ ...(i)

Then, $I=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow$ $I=\int_0^{\pi}\frac{\pi-x}{1+\sin x}dx$ ...(ii)

$[\because \sin(\pi-x)=\sin x]$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi}{(1+\sin x)}dx=\pi\int_0^{\pi}\frac{1}{(1+\sin x)}dx$$

$$=\pi\int_0^{\pi}\frac{1-\sin x}{(1+\sin x)(1-\sin x)}dx$$

[multiply numerator and denominator by $(1-\sin x)$]

$\Rightarrow$ $2I=\pi\int_0^{\pi}\frac{1-\sin x}{1-\sin^2 x}dx=\pi\int_0^{\pi}\frac{1}{\cos^2 x}dx-\pi\int_0^{\pi}\frac{\sin x}{\cos^2 x}dx$

$(\because \sin^2 x+\cos^2 x=1)$

$\Rightarrow$ $2I=\pi\int_0^{\pi}\sec^2 x\,dx-\pi\int_0^{\pi}\sec x\cdot\tan x\,dx$

$\Rightarrow$ $2I=\pi\,[\tan x-\sec x]_0^{\pi}$

$\Rightarrow$ $2I=\pi\,[\tan\pi-\sec\pi-(\tan 0-\sec 0)]$

$\Rightarrow$ $2I=\pi\,[0+1-0+1] \Rightarrow 2I=2\pi \Rightarrow I=\pi$

13. $\int_0^{2\pi}\cos^5 x\,dx=2\int_0^{\pi}\cos^5 x\,dx$

$\left[\because \int_0^{2a} f(x)=2\int_0^a f(x)\,dx, \text{ where } f(2a-x)=f(x); \text{ hence } 2a=2x\right.$

$\left.\therefore \cos^5(2x-x)=\cos^5 x\right]$

$=2\times 0=0$ $\left[\because \int_0^{2a} f(x)=0, \text{ if } f(2a-x)=-f(x); \text{ here } 2a=\pi \;\therefore \cos^5(\pi-x)=-\cos^5 x\right]$

14. Let $I=\int_0^{\pi/2}\frac{\cos^5 x}{\sin^5 x+\cos^5 x}dx$ ...(i)

$\Rightarrow$ $I=\int_0^{\pi/2}\frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx$

$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$=\int_0^{\pi/2}\frac{\sin^5 x}{\cos^5 x+\sin^5 x}dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{\cos^5 x+\sin^5 x}{\cos^5 x+\sin^5 x}dx=\int_0^{\pi/2}1dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0$$

$\Rightarrow$ $I=\frac{\pi}{4}$

15. Let $I=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ ...(i)

$\Rightarrow$ $I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$

$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$=\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\cos x\sin x}dx$ ...(ii)

$\left[\because \sin\left(\frac{\pi}{2}-x\right)=\cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right)=\sin x\right]$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{0}{1+\sin x\cos x}dx=0 \Rightarrow I=0$$

16. Here, we reduce the integrand into simplest form, by using the property $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ and then add them and integrate.

Let $I=\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ ...(i)

$I=\int_0^a\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}dx$

$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$=\int_0^a\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$\Rightarrow$ $2I=\int_0^a\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \Rightarrow 2I=\int_0^a 1\,dx$

$=[x]_0^a=a-0=a \Rightarrow I=\frac{a}{2}$

17. Let $I = \int_0^4 |x-1|\,dx$

It can be seen that, $(x-1) \le 0$ when $0 \le x \le 1$ and $(x-1) \ge 0$ when $1 \le x \le 4$

$$\therefore \quad I = \int_0^1 |x-1|\,dx + \int_0^4 |x-1|\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$= \int_0^1 (1-x)\,dx + \int_1^4 (x-1)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

$$= \left(1 - \frac{1}{2}\right) - 0 + \left(\frac{4^2}{2} - 4\right) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} + 4 + \frac{1}{2} = 5$$

18. Let $I = \int_0^a f(x)\,g(x)\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^a f(a-x)\,g(a-x)\,dx \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^a f(x)\,\{4 - g(x)\}\,dx \quad \text{...(ii)}$$

$[\because f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$ (given)$]$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^a 4\,f(x)\,dx \;\Rightarrow\; I = 2\int_0^a f(x)\,dx$$

19. Let $I = \int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right)dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\left[\frac{4 + 3\sin(\pi/2 - x)}{4 + 3\cos(\pi/2 - x)}\right]dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right)dx \quad \text{...(ii)}$$

$$\left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2}\left[\log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) + \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right)\right]dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x} \times \frac{4 + 3\cos x}{4 + 3\sin x}\right)dx$$

$(\because \log m + \log n = \log mn)$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log 1\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 0\,dx \quad (\because \log 1 = 0)$$

$$\Rightarrow \quad I = 0$$

20. Let $f(x) = x^{17}\cos^4 x$

$$\Rightarrow \quad f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$$

Therefore, $f(x)$ is an odd function.

We known that, if $f(x)$ is an odd function, then $\int_{-a}^a f(x)\,dx = 0$

$$\therefore \quad \int_{-1}^1 x^{17}\cos^4 x\,dx = 0$$

Note If $f(x)$ is odd, then $\int_{-a}^a f(x)\,dx = 0$

If $f(x)$ is even, then $\int_{-a}^a f(x)\,dx = 2\int_a^a f(x)dx$

and if $f(x)$ is neither odd nor even, then we do not apply the any rule.

21. Let $I = \int_a^b x\,f(x)\,dx$ ...(i)

Then, by a property of definite integrals

$$I = \int_a^b (a + b - x)\,f(a + b - x)\,dx = \int_a^b (a + b - x)\,f(x)\,dx \quad \text{...(ii)}$$

$[\because$ given $f(a + b - x) = f(x)]$

On adding Eqs. (i) and (ii), we get

$$2I = \int_a^b (a + b)\,f(x)\,dx \;\Rightarrow\; I = \frac{a + b}{2}\int_a^b f(x)\,dx$$

Hence, the option (d) is correct.

22. $I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}}\,dx = \int_{-a}^a \frac{a-x}{\sqrt{a^2 - x^2}}\,dx = a\int_{-a}^a \frac{dx}{\sqrt{a^2 - x^2}} - \int_{-a}^a \frac{x\,dx}{\sqrt{a^2 - x^2}}$

$$= a \cdot 2\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} - 0 \qquad \left(\because \frac{x}{\sqrt{a^2 - x^2}} \text{ is an odd function}\right)$$

$$= 2a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$\Rightarrow \quad 2a[\sin^{-1}(1) - \sin^{-1}(0)] = 2a\left[\frac{\pi}{2} - 0\right] = \pi a$$

23. Let $f(x) = x^3 e^{x^4}$, then $f(-x) = (-x)^3 \cdot e^{-x^4} = -x^3 e^{x^4} = -f(x)$

Hence, $f(x)$ is an odd function.

$$\therefore \quad \int_{-1}^1 f(x)\,dx = 0$$

or $$\int_{-1}^1 x^3 e^{x^4}\,dx = 0$$

24. Here, $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$$\therefore \quad \int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 \sqrt{x}\,dx$$

$$\Rightarrow \quad \int_0^2 f(x)\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2}{3}x\sqrt{x}\right]_1^2 = \left[\frac{1}{3} - 0\right] + \frac{2}{3}[2\sqrt{2} - 1]$$

$$= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} = \frac{4\sqrt{2}}{3} - \frac{1}{3}$$

$$\therefore \quad \int_0^2 f(x)\,dx = \frac{1}{3}(4\sqrt{2} - 1)$$

25. $\int_{-2}^3 \left|x^2 - 1\right|dx = \int_{-2}^{-1}\left|x^2 - 1\right|dx + \int_{-1}^1 \left|x^2 - 1\right|dx + \int_1^3 \left|x^2 - 1\right|dx$

(here, modulus function will change at the points, when $x^2 - 1 = 0$ *i.e.*, at $x = \pm 1$)

So, $$I = \int_{-2}^{-1}(x^2 - 1)\,dx + \int_{-1}^1 (1 - x^2)\,dx + \int_1^3 (x^2 - 1)\,dx$$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 + \left[\frac{x^3}{3} - x\right]_1^3$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + 6 + \frac{2}{3} = \frac{28}{3}$$

26. We have, $I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$

$$=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \text{(by } P_4\text{)}$$

$$I=\int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx$$

Thus, we get $2I=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx=\frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right\}\right]_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right]$$

$$=\frac{1}{\sqrt{2}}[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)]=\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$=\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{1}\right\}=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

Hence, $I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

27. We can redefine f as $f(x)=\begin{cases}2-x, & \text{if } -1<x\le 0\\ x+2, & \text{if } 0<x\le 1\\ 3x, & \text{if } 1<x\le 2\end{cases}$

Therefore, $\int_{-1}^{2}f(x)\,dx=\int_{-1}^{0}(2-x)\,dx+\int_0^1(x+2)dx+\int_1^2 3x\,dx$

(by P_2)

$$=\left[2x-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}+2x\right]_0^1+\left[\frac{3x^2}{2}\right]_1^2$$

$$=0-\left(-2-\frac{1}{2}\right)+\left(\frac{1}{2}+2\right)+3\left(\frac{4}{2}-\frac{1}{2}\right)=\frac{5}{2}+\frac{5}{2}+\frac{9}{2}=\frac{19}{2}$$

28. Since, $I=\int_0^a f(x)\cdot g(x)\,dx$

$$=\int_0^a f(a-x)\,g(a-x)\,dx=\int_0^a f(x)\{a-g(x)\}\,dx$$

$$=a\int_0^a f(x)\,dx-\int_0^a f(x)\cdot g(x)\,dx=a\int_0^a f(x)\,dx-I$$

or $I=\frac{a}{2}\int_0^a f(x)\,dx$

29. Since, $x=\int_0^y\frac{dt}{\sqrt{1+9t^2}}\Rightarrow\frac{dx}{dy}=\frac{1}{\sqrt{1+9y^2}}$

which gives $\frac{d^2y}{dx^2}=\frac{18y}{2\sqrt{1+9y^2}}\frac{dy}{dx}=9y$

30. Since, $I=\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx$

$$I=\int_{-1}^{1}\frac{x^3}{x^2+2|x|+1}+\int_{-1}^{1}\frac{|x|+1}{x^2+2|x|+1}dx=0+2\int_0^1\frac{|x|+1}{(|x|+1)^2}dx$$

(odd function + even function)

$$=2\int_0^1\frac{x+1}{(x+1)^2}dx=2\int_0^1\frac{1}{x+1}dx=2\,[\log|x+1|]_0^1=2\log 2$$

31. Since, $I=\int_{-2}^{2}|x\cos\pi x|\,dx=2\int_0^2|x\cos\pi x|\,dx$

$$=2\left\{\int_0^{\frac{1}{2}}|x\cos\pi x|\,dx+\int_{\frac{1}{2}}^{\frac{3}{2}}|x\cos\pi x|\,dx+\int_{\frac{3}{2}}^{2}|x\cos\pi x|\,dx\right\}=\frac{8}{\pi}$$

32. $\int_0^{\pi}\sqrt{\frac{\cos 2x+1}{2}}dx=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}|\cos x|\,dx$

$$=\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{\pi}\cos x\,dx=2$$

33. $\int_0^{2\pi}(\sin x+|\sin x|)\,dx$

$$=\int_0^{\pi}(\sin x+\sin x)\,dx+\int_{\pi}^{2\pi}(\sin x-\sin x)\,dx$$

$$=\int_0^{\pi}2\sin x\,dx+\int_{\pi}^{2\pi}0\,dx=2\,[-\cos x]_0^{\pi}+0$$

$$=-2(\cos\pi-\cos 0)=-2(-1-1)=4$$

34. Let $I=\int_0^{\pi/2}\sin 2x\log\tan x\,dx$ ...(i)

$$I=\int_0^{\pi/2}\sin 2\left(\frac{\pi}{2}-x\right)\log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\because\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$I=\int_0^{\pi/2}\sin 2x\log\cot x\,dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\sin 2x\log(\tan x\cot x)\,dx$$

$$=\int_0^{\pi/2}\sin 2x\log 1\,dx=\int_0^{\pi/2}0\,dx$$

$\Rightarrow$ $I=0$

35. $I=\int_{1/n}^{(an-1)/n}\frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}dx$

$$=\int_{1/n}^{a-(1/n)}\frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad \text{...(i)}$$

$$=\int_{1/n}^{a-(1/n)}\frac{\sqrt{\frac{1}{n}+a-\frac{1}{n}-x}\,dx}{\sqrt{a-\left(\frac{1}{n}+a-\frac{1}{n}-x\right)}+\sqrt{\left(\frac{1}{n}+a-\frac{1}{n}-x\right)}}$$

$$\Rightarrow \quad I=\int_{\frac{1}{n}}^{a-\frac{1}{n}}\frac{\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{1/n}^{a-(1/n)} 1\,dx = [x]_{\frac{1}{n}}^{a-\frac{1}{n}}$$

$$\Rightarrow \quad 2I = a - \frac{1}{n} - \frac{1}{n} = \frac{na-2}{n} \Rightarrow I = \frac{na-2}{2n}$$

36. Let $f(x) = \log(x + \sqrt{1+x^2})$ and replacing x by $-x$, we get

$$f(-x) = \log(\sqrt{1+x^2} - x)$$

$$= \log(\sqrt{1+x^2} - x)\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}$$

$$= \log\frac{[(1+x^2)-x^2]}{(\sqrt{1+x^2}+x)}$$

$$= \log 1 - \log(\sqrt{1+x^2}+x)$$

$$= -\log(\sqrt{1+x^2}+x)$$

$\Rightarrow \quad f(-x) = -f(x)$

Hence, $f(x)$ is an odd function.

$$\therefore \quad \int_{-1}^{1} \log(x+\sqrt{1+x^2})\,dx = 0$$

37. We have, $\int_0^a f(2a-x)\,dx = \mu$

and $\int_0^a f(x)\,dx = \lambda$

Now, using properties of definite integral

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\Rightarrow \quad \int_0^{2a} f(x)\,dx = \lambda + \mu$$

38. Given, $\int_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\,dx$

Let $f(x) = e^{\cos^2 x}\cdot\cos^3(2n+1)x$

Then, $f(\pi - x) = e^{\cos^2(\pi-x)}\cdot\cos^3[(2n+1)\pi - (2n+1)x]$

$$= -e^{\cos^2 x}\cdot\cos^3(2n+1)x$$

$\Rightarrow \quad f(\pi - x) = -f(x)$

Then, $f(x)$ is an odd function.

$$\therefore \quad \int_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\,dx = 0$$

39. Let $I = \int_0^{\pi} e^{\sin^2 x}\cos^3 x\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} e^{\sin^2(\pi-x)}\cos^3(\pi-x)\,dx$$

$$\Rightarrow \quad I = -\int_0^{\pi} e^{\sin^2 x}\cos^3 x\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = 0 \quad \Rightarrow \quad I = 0$$

40. $\int_{-1}^{1}[x\{1+\sin\pi x\}+1]\,dx$

$$= \int_{-1}^{0}[x\{1+\sin\pi x\}+1]\,dx + \int_0^1[x\{1+\sin\pi x\}+1]\,dx$$

Now, $-1 < x < 0 \quad \Rightarrow \quad [1+\sin\pi x] = 0$

$$0 < x < 1 \quad \Rightarrow \quad [1+\sin\pi x] = 1$$

$$\Rightarrow \quad [x\{1+\sin\pi x\}+1] = 1$$

$$\therefore \quad \int_{-1}^{1}[x(1+\sin\pi x)+1]\,dx = 2$$

41. $\int_{-1}^{1}|x|\,dx + \int_1^3|x-2|\,dx$

$$= 2\int_0^1 x\,dx + \int_1^2(2-x)\,dx + \int_2^3(x-2)\,dx$$

$$= 2\times\frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3 = 1 + \frac{1}{2} + \frac{1}{2} = 2$$

42. Let $I = \int_0^1 f(x)\,dx = 0$ and $f(x) \geq 0 \Rightarrow f(x) = 0$

43. Given, $F(x^2) = x^2(1+x)$

$$\Rightarrow \quad \int_0^{x^2} f(t)\,dt = x^2(1+x)$$

On differentiating both sides w.r.t. x, we get

$$2xf(x^2) = 2x + 3x^2$$

$$\Rightarrow \quad f(x^2) = 1 + \frac{3x}{2}$$

$$\Rightarrow \quad f(2^2) = 1 + \frac{3}{2}(2) = 4$$

$$\Rightarrow \quad f(4) = 4$$

44. Given, $f(-x) = -f(x), \forall$ values of real x.

We know that,

$$\int_{-a}^{a} f(x)\,dx = 0 = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad [\because f(-x) = -f(x)]$$

$$\Rightarrow \quad \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx = 0$$

$$\Rightarrow \quad \int_{-1}^{0} f(x)\,dx = -5 \qquad \left[\because \int_0^1 f(x)\,dx = 5\right]$$

$$\Rightarrow \quad \int_{-1}^{0} f(t)\,dt = -5$$

45. $\int_{-2}^{2}|[x]|\,dx$

$$= \int_{-2}^{-1}|[x]|\,dx + \int_{-1}^{0}|[x]|\,dx + \int_0^1|[x]|\,dx + \int_1^2|[x]|\,dx$$

$$= \int_{-2}^{-1} 2\,dx + \int_{-1}^{0} 1\,dx + \int_0^1 0\,dx + \int_1^2 1\,dx$$

$$= 2[x]_{-2}^{-1} + [x]_{-1}^{0} + 0 + [x]_1^2$$

$$= 2(-1+2) + (0+1) + (2-1) = 2+1+1 = 4$$

46. Let $\phi(x) = [f(x) + f(-x)][g(x) - g(-x)]$,

then $\phi(-x) = [f(-x) + f(x)][g(-x) - g(x)]$

$$\phi(-x) = -\phi(x)$$

$\Rightarrow$ It is an odd function.

$$\therefore \quad \int_{-\pi}^{\pi}\phi(x)\,dx = 0$$

$$\Rightarrow \quad \int_{-\pi}^{\pi}[f(x) + f(-x)]\,[g(x) - g(-x)]\,dx = 0$$

47. Given, $P = \int_0^{3\pi} f(\cos^2 x)\ dx$ ...(i)

and $Q = \int_0^{\pi} f(\cos^2 x)\ dx$ ...(ii)

From Eq. (i),

$$P = 3\int_0^{\pi} f(\cos^2 x)\, dx$$

$\Rightarrow$ $P = 3Q \Rightarrow P - 3Q = 0$

48. Given,

$$I_1 = \int_a^{\pi-a} x f(\sin x)\ dx$$

and $I_2 = \int_a^{\pi-a} f(\sin x)\ dx$

Now, $I_1 = \int_a^{\pi-a} x f(\sin x)\ dx$

$$= \int_a^{\pi-a} (\pi - x)\, f[\sin(\pi - x)]\, dx$$

$$= \int_a^{\pi-a} (\pi - x)\, f(\sin x)\ dx$$

$$= \int_a^{\pi-a} \pi\, f(\sin x)\, dx - I_1$$

$\Rightarrow$ $2I_1 = \pi I_2$

$\Rightarrow$ $I_2 = \dfrac{2}{\pi} I_1$

49. $\int_{-1/7}^{20/7} e^{5\{x\}}\, dx = \int_{-1/7}^{3-\frac{1}{7}} e^{5\{x\}} dx$

$$= 3\int_0^1 e^{5\{x\}}\, dx = 3\int_0^1 e^{5x} dx \quad (\{x\} = x \text{ as } 0 \le x < 1)$$

$$= 3\left[\frac{e^{5x}}{5}\right]_0^1 = 3\left(\frac{e^5 - 1}{5}\right)$$

50. $\int_{-2}^{3} f(x)\, dx = \int_{-2}^{2} f(x)\, dx + \int_2^3 f(x)\, dx = \int_{-2}^{2} e^{\cos x} \sin x\, dx + \int_2^3 2\, dx$

Function of Ist integral is an odd function, therefore the value of Ist integral is zero.

$\therefore$ $\int_{-2}^{3} f(x)\, dx = 0 + 2\,[x]_2^3 = 0 + 2\,(3-2) = 2$

51. $\int_{-1}^{1} [x \sin \pi x]\, dx = 2\int_0^1 [x \sin \pi x]\, dx$ ...(i)

(as $[x \sin \pi x]$ is an even function)

Now, $0 \le x \le 1$

$\Rightarrow$ $0 \le \pi x \le \pi$

$\Rightarrow$ $0 \le \sin \pi x \le 1$ ...(ii)

$\Rightarrow$ $0 \le x \sin \pi x < 1$

$\Rightarrow$ $[x \sin \pi x] = 0$

From Eqs. (i) and (ii), we get

$$\int_0^1 [x \sin \pi x]\, dx = 0$$

52. $I = \sum_{r=1}^{10} \int_0^1 f(r - 1 + x)\ dx$

$$I_t = \int_0^1 f(t - 1 + x)\ dx$$

Put $t - 1 + x = y \Rightarrow dx = dy$

$$I_t = \int_{t-1}^{t} f(y)\, dy \Rightarrow I_t = \int_{t-1}^{t} f(x)\, dx$$

$\therefore$ $I_1 = \int_0^1 f(x)\, dx,\ I_2 = \int_1^2 f(x)\, dx,$

$$I_3 = \int_2^3 f(x)\, dx, \ldots,\quad I_{10} = \int_9^{10} f(x)\, dx$$

So, $I = I_1 + I_2 + \ldots + I_{10} = \int_0^{10} f(x)\, dx$

53. $\int_{-1}^{1} \max(\{x\}, \{-x\})\, dx$

$$= \int_{-1}^{-0.5} \{-x\} dx + \int_{-0.5}^{0} \{x\}\ dx + \int_0^{0.5} \{-x\}\ dx + \int_{0.5}^{1} \{x\}\ dx$$

$$= \int_{-1}^{-0.5} (-x)\ dx + \int_{-0.5}^{0} \{x - [x]\}\ dx + \int_0^{0.5} (-x - [-x])\ dx + \int_{0.5}^{1} (x - [x])\ dx$$

$$= \left[-\frac{x^2}{2}\right]_{-1}^{-0.5} + \left[\frac{x^2}{2} + x\right]_{-0.5}^{0} + \left[x - \frac{x^2}{2}\right]_0^{0.5} + \left[\frac{x^2}{2}\right]_{0.5}^{1}$$

$$= -\frac{0.25}{2} + \frac{1}{2} + \left(-\frac{0.25}{2} + 0.5\right) + 0.5 - \frac{0.25}{2} + \frac{1}{2} - \frac{0.25}{2}$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

54. $|x| < \sqrt{1 + x^2} \Rightarrow \dfrac{1}{|x|} > \dfrac{1}{\sqrt{1 + x^2}}$

$\therefore$ $\int_0^1 \dfrac{1}{|x|} dx > \int_0^1 \dfrac{1}{\sqrt{1 + x^2}} dx \Rightarrow I_1 > I_2$

55. $\sin x > \sin^3 x$

$\Rightarrow$ $\int_0^{\pi/2} \sin x\, dx > \int_0^{\pi/2} \sin^3 x\, dx$

$\Rightarrow$ $I_1 > I_2$

56. $I_1 = \int_{1-k}^{k} (1 - x) f[x(1 - x)]\, dx$

$\Rightarrow$ $2I_1 = I_2 \Rightarrow \dfrac{I_1}{I_2} = \dfrac{1}{2}$

57. $\int_{-2}^{2} f(x)\, dx = \int_{-2}^{-1} (x + 2)\, dx + \int_{-1}^{1}\ dx + \int_1^2 (2 - x)\, dx$

$$= \frac{1}{2} + 2 + \frac{1}{2} = 3$$

58. $\phi(x) = \cos x - \int_0^x (x - t)\, \phi(t)\, dt$

$$= \cos x - x\int_0^x \phi(t)\, dt + \int_0^x t\, \phi(t)\, dt$$

$\Rightarrow \quad \phi'(x) = -\sin x - \int_0^x \phi(t)\,dt - x\phi(x) + x\phi(x)$

$\Rightarrow \quad \phi''(x) = -\cos x - \phi(x)$

$\Rightarrow \quad \phi(x) + \phi''(x) = -\cos x$

59. $I = \int_{-1/7}^{20/7} \sin(x-[x])\,dx = \int_{-1/7}^{3-\frac{1}{7}} \sin(\{x\})\,dx$

$= 3\int_0^1 \sin x\,dx = 3\,[-\cos x]_0^1 = 3\,(1-\cos 1)$

60. **Statement I** $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx$

$= \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\,\sin x\,dx$

$= 2\int_0^{\pi/2} \sqrt{\cos x}\,\sin x\,dx = -\frac{4}{3}[\cos^{3/2} x]_0^{\pi/2} = \frac{4}{3}$

Statement II $I = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx + \int_0^3 |x-3|\,dx + \int_3^4 |x-3|\,dx$

$= \int_0^1 -(x-1)\,dx + \int_1^4 (x-1)\,dx + \int_0^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx$

$= 10$

61. If $f(x)$ is a continuous function defined on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

where, M and m are maximum and minimum values respectively of $f(x)$ in $[a, b]$.

Here, $f(x) = 1 + e^{-x^2}$ is continuous on [0, 1].

Now, $\quad 0 < x < 1 \Rightarrow x^2 < x$

$\Rightarrow \quad e^{x^2} < e^x \Rightarrow e^{-x^2} > e^{-x}$

Again, $\quad 0 < x < 1 \Rightarrow x^2 > 0$

$\Rightarrow \quad e^{x^2} > e^0 \Rightarrow e^{-x^2} > 1$

$\therefore \quad e^{-x} < e^{-x^2} < 1$ for all $x \in [0, 1]$

$\Rightarrow \quad 1 + e^{-x} < 1 + e^{-x^2} < 2$ for all $x \in [0, 1]$

$\Rightarrow \quad \int_0^1 (1+e^{-x})\,dx < \int_0^1 (1+e^{-x^2})\,dx < \int_0^1 2\,dx$

$\Rightarrow \quad 2 - \frac{1}{e} < \int_0^1 (1+e^{-x^2})\,dx < 2$

62. $\int_{-1}^1 f(x)\,dx = \int_{-1}^1 (x-[x])\,dx = \int_{-1}^1 x\,dx - \int_{-1}^1 [x]\,dx$

$= 0 - \int_{-1}^1 [x]\,dx \qquad (\because x \text{ is an odd function})$

But $\quad [x] = \begin{cases} -1, & \text{if } -1 \le x < 0 \\ 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

$\therefore \quad \int_{-1}^1 [x]\,dx = \int_{-1}^0 [x]\,dx + \int_0^1 [x]\,dx$

$= \int_{-1}^0 (-1)\,dx + \int_0^1 0\,dx = -[x]_{-1}^0 + 0 = 1$

Thus, $\int_{-1}^1 f(x)\,dx = 1$

63. Let $I = \int_0^1 x e^{x^2}\,dx$

Put $\quad x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

Limits when $x = 0 \Rightarrow t = 0$ and when $x = 1 \Rightarrow t = 1$

$\therefore \quad I = \int_0^1 x e^t \frac{dt}{2x} = \frac{1}{2}\int_0^1 e^t\,dt$

$= \frac{1}{2}[e^t]_0^1 = \frac{1}{2}[e^1 - e^0] = \frac{1}{2}[e-1]$

64. Let $\quad I = \int_0^2 \frac{6x+3}{x^2+4}\,dx = \int_0^2 \frac{6x}{x^2+4}\,dx + \int_0^2 \frac{3}{x^2+4}\,dx$

Put $\quad x^2 + 4 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

Lower limit, when $x = 0, t = 0 + 4 = 4$

Upper limit, when $x = 2, t = 4 + 4 = 8$

$\therefore \quad I = \int_4^8 \frac{6x}{t}\frac{dt}{2x} + \int_0^2 \frac{3}{x^2+4}\,dx$

$= 3\int_4^8 \frac{1}{t}\,dt + 3\int_0^2 \frac{1}{x^2+2^2}\,dx$

$= 3\,[\log t]_4^8 + \frac{3}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \qquad \left(\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$

$= 3\,[\log(8) - \log(4)] + \frac{3}{2}\left[\tan^{-1}\frac{2}{2}\right]$

$= 3\log\left(\frac{8}{4}\right) + \frac{3}{2} \times \frac{\pi}{4} \qquad \left(\because \log b - \log a = \log\frac{b}{a}\right)$

$= 3\log 2 + \frac{3\pi}{8}$

65. $\int_0^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi = \int_0^{\pi/2} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\,d\phi$

$(\because \cos^2\theta + \sin^2\theta = 1)$

Put $\quad \sin\phi = t \Rightarrow \cos\phi = \frac{dt}{d\phi} \Rightarrow d\phi = \frac{dt}{\cos\phi}$

For limit when $\phi = 0 \Rightarrow t = 1$ and when $\phi = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore \quad I = \int_0^1 \sqrt{t}(1-t^2)^2 dt = \int_0^1 \sqrt{t}(t^4 + 1 - 2t^2)\,dt$

$= \int_0^1 (t^{9/2} + t^{1/2} - 2t^{5/2})\,dt$

$= \left[\frac{t^{\frac{9}{2}+1}}{\left(\frac{9}{2}+1\right)} + \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2\left(\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right)\right]_0^1$

$= \left[\frac{2}{11} + \frac{2}{3} - \frac{4}{7}\right] - 0 = \frac{42+154-132}{11\times 3\times 7} = \frac{64}{231}$

66. Here, the given integrand cannot integrate directly, so we put $\sin^2 x = t$ and then convert it into the standard form $\frac{1}{x^2+a^2}$ and then integrate under the proper limits.

Let $I = \int_0^{\pi/4} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}\,dx$

$= \int_0^{\pi/4} \frac{\sin x\cos x}{(1-\sin^2 x)^2 + (\sin^2 x)^2}\,dx \quad (\because \cos^2 x = 1 - \sin^2 x)$

Put $\sin^2 x = t \Rightarrow 2 \sin x \cos x\, dx = dt$

For limit when $x = 0 \Rightarrow t = 0$ and when $x = \frac{\pi}{4} \Rightarrow t = \frac{1}{2}$

$$\therefore \quad I = \int_0^{1/2} \frac{\sin x \cos x}{(1-t)^2 + t^2} \frac{dt}{2 \sin x \cos x}$$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{(1-t)^2 + t^2} dt = \frac{1}{2}\int_0^{1/2} \frac{1}{2t^2 - 2t + 1} dt$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{1/2} \frac{1}{t^2 - t + \frac{1}{2}} dt = \frac{1}{4}\int_0^{1/2} \frac{1}{t^2 - t + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \frac{1}{2}} dt$$

$$= \frac{1}{4}\int_0^{1/2} \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dt = \frac{1}{4}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^{1/2}$$

$$= \frac{1}{2}[\tan^{-1} 0 - \tan^{-1}(-1)] = \frac{1}{2}\cdot\frac{\pi}{4} \qquad [\because \tan(-\theta) = -\tan\theta]$$

$$= \frac{\pi}{8}$$

67. Let $I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x} dx \qquad (\because \sin^2 x = 1 - \cos^2 x)$$

$$= -\frac{1}{3}\int_0^{\pi/2} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(1 - \frac{4}{4 - 3\cos^2 x}\right) dx$$

$$= -\frac{1}{3}[x]_0^{\pi/2} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{4\sec^2 x - 3} dx$$

(divide numerator and denominator in second integral by $\cos^2 x$)

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{4(1 + \tan^2 x) - 3} dx \quad (\because 1 + \tan^2 x = \sec^2 x)$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

when $x = 0 \Rightarrow t = 0$ and when $x = \left(\frac{\pi}{2}\right) \Rightarrow t = \infty$

$$\therefore \quad I = -\frac{\pi}{6} + \frac{4}{3}\int_0^{\infty} \frac{dt}{4(1 + t^2) - 3} = -\frac{\pi}{6} + \frac{4}{3}\int_0^{\infty} \frac{dt}{4t^2 + 1}$$

$$= -\frac{\pi}{6} + \frac{4}{3}\times\frac{1}{4}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = -\frac{\pi}{6} + \frac{1}{3}\cdot 2\left[\tan^{-1}\frac{t}{\frac{1}{2}}\right]_0^{\infty}$$

$$\left(\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= -\frac{\pi}{6} + \frac{2}{3}[\tan^{-1} 2t]_0^{\infty} = -\frac{\pi}{6} + \frac{2}{3}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2} - 0\right] = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$$

68. $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

We know that,

$(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x$

$\Rightarrow \quad (\sin x - \cos x)^2 = 1 - \sin 2x$

$(\because \sin^2 x + \cos^2 x = 1$ and $2\sin x\cos x = \sin 2x)$

$\Rightarrow \quad \sin 2x = 1 - (\sin x - \cos x)^2$

Put $\sin x - \cos x = t \Rightarrow \cos x + \sin x = \frac{dt}{dx}$ ...(i)

$\Rightarrow \quad dx = \frac{dt}{\cos x + \sin x}$

when $x = 0, t = 0 - \cos 0 = -1$, $x = \frac{\pi}{4}, t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$\therefore I = \int_{-1}^{0} \frac{\sin x + \cos x}{9 + 16(1 - t^2)} \frac{dt}{\cos x + \sin x} = \int_{-1}^{0} \frac{1}{9 + 16(1 - t^2)} dt$$

$$= \int_{-1}^{0} \frac{1}{9 + 16 - 16t^2} dt$$

$$= \int_{-1}^{0} \frac{1}{25 - 16t^2} dt = \frac{1}{16}\int_{-1}^{0} \frac{1}{\left(\frac{5}{4}\right)^2 - t^2} dt$$

$$= \frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\left[\log\left|\frac{\frac{5}{4} + t}{\frac{5}{4} - t}\right|\right]_{-1}^{0}$$

$$\left(\because \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right|\right)$$

$$= \frac{1}{40}\left[\log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^{0} = \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$$

$$= \frac{1}{40}[\log 1 - (\log 1 - \log 9)] = \frac{1}{40}\log 9$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

69. Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$

$$= \int_0^{\pi/2} 2\sin x\cos x \tan^{-1}(\sin x)\, dx \quad (\because \sin 2x = 2\sin x\cos x)$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt \Rightarrow dx = \frac{dt}{\cos x}$

when $x = 0 \Rightarrow t = 0$ and when $x = \frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} = 1$

$$\therefore \quad I = \int_0^1 2t\cos x \tan^{-1} t \frac{dt}{\cos x} = 2\int_0^1 \underset{\text{II}}{t}\cdot\underset{\text{I}}{\tan^{-1} t}\, dt$$

$$= 2\left(\left[(\tan^{-1} t)\frac{t^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1 + t^2}\cdot\frac{t^2}{2} dt\right)$$

(using integration by parts)

$$= 2\left[\frac{\tan^{-1} 1}{2} - 0\right] - \int_0^1 \frac{t^2}{1 + t^2} dt$$

$$= 2\left(\frac{\pi}{8}\right) - \int_0^1 \frac{(1+t^2)-1}{1+t^2}\,dt$$

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt = \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$

$$= \frac{\pi}{4} - [1 - \tan^{-1} 1 - (0 - 0)] = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

70. $I = \int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$

So, $$I = \int_{1/2}^{1/3} \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\frac{4}{t^2}+1}} = -\int_{1/2}^{1/3} \frac{dt}{\sqrt{4+t^2}}$$

$$= [\ln(t + \sqrt{4+t^2})]_{1/3}^{1/2} = \ln\left\{\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right\}$$

71. $\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x}{e^{2x}+1}\,dx$

Let $t = e^x$

$\Rightarrow$ $dt = e^x\,dx$

$$= \int_1^e \frac{dt}{t^2+1} = [\tan^{-1} t]_1^e$$

$$= \tan^{-1} e - \tan^{-1}(1) = \tan^{-1} e - \frac{\pi}{4}$$

72. $\int_0^{\pi/2} \frac{\tan x\,dx}{1 + m^2 \tan^2 x}$

Let $t = m \tan x$

$dt = m \sec^2 x\,dx$

$dx = \frac{\cos^2 x}{m}\,dt$

$$= \int_0^\infty \frac{\tan x}{1+t^2} \cdot \frac{\cos^2 x}{m}\,dt = \frac{1}{2m}\int_0^\infty \frac{\sin 2x}{1+t^2}\,dt$$

$$= \frac{1}{2m}\int_0^\infty \frac{1}{1+t^2} \times \frac{2\tan x}{1+\tan^2 x}\,dt = \frac{1}{2m}\int_0^\infty \frac{2t}{(1+t^2)^2}\,dt$$

Put $z = 1 + t^2$

$dz = 2t\,dt$

$$= \frac{1}{2m}\int_1^\infty \frac{dz}{z^2} = \frac{1}{2m}\left[-\frac{1}{z}\right]_1^\infty = \frac{1}{2m}[-0+1] = \frac{1}{2m}$$

73. $\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$$= \int_0^{\pi/6} \frac{\cos\theta\,d\theta}{(1+\sin^2\theta)\cos\theta} = \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\,d\theta$$

$$\int_0^{\pi/6} \frac{\sec^2\theta}{1 + (\sqrt{2}\tan\theta)^2}\,d\theta$$

Put $t = \sqrt{2}\tan\theta$

$dt = \sqrt{2}\sec^2\theta\,d\theta$

$$= \frac{1}{\sqrt{2}}\int_0^{\sqrt{\frac{2}{3}}} \frac{dt}{1+t^2} = \frac{1}{\sqrt{2}}[\tan^{-1} t]_0^{\sqrt{2/3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - \tan^{-1} 0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$$

74. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\sqrt{2}\cos\frac{x}{2}}{2^{5/2}\sin^5\frac{x}{2}}\,dx = \frac{1}{4}\int_{\pi/3}^{\pi/2} \frac{\cos\frac{x}{2}}{\sin^5\frac{x}{2}}\,dx$

Let $t = \sin\frac{x}{2}$

$dt = \frac{1}{2}\cos\frac{x}{2}\,dx$

$$= \frac{1}{2}\int_{1/2}^{1/\sqrt{2}} \frac{dt}{t^5} = \frac{1}{2}\left[\frac{-1}{4t^4}\right]_{1/2}^{1/\sqrt{2}} = \frac{-1}{8}\left[\frac{1}{4} - \frac{1}{16}\right] = \frac{-3}{128}$$

75. Let $I = \int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)dx$

Put $x = \cos 2\theta$

Then, $\sin\left[2\tan^{-1}\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right] = \sin[2\tan^{-1}(\cot\theta)]$

$$= \sin\left[2\tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\}\right]$$

$$= \sin\left[2\left(\frac{\pi}{2} - \theta\right)\right] = \sin(\pi - 2\theta) = \sin 2\theta$$

$$= \sqrt{1 - \cos^2 2\theta} = \sqrt{1 - x^2}$$

Now, $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)dx = \int_0^1 \sqrt{1-x^2}\,dx$

$$= \left[\frac{1}{2}x \cdot \sqrt{1-x^2}\right]_0^1 + \frac{1}{2}[\sin^{-1} x]_0^1$$

$$= \frac{1}{2}[1\sqrt{1-1} - 0] + \frac{1}{2}[\sin^{-1}(1) - 0]$$

$$= \frac{1}{2}[0] + \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

76. $\int_{b-1}^{b} \frac{e^{-t}}{t-b-1}\,dt = \int_{-1}^{0} \frac{e^{-(t+b)}}{(t+b)-b-1}\,dt$

$$= e^{-b}\int_{-1}^{0} \frac{e^{-t}}{t-1}\,dt = e^{-b}\int_0^1 \frac{e^{-(t-1)}}{t-2}\,dt$$

Put $t - 1 = -s$

$\Rightarrow$ $dt = -ds$

$$= -e^{-b}\int_1^0 \frac{e^s}{-(s+1)}\,ds$$

$$= e^{-b}\int_1^0 \frac{e^s}{s+1}\,ds$$

$$= -e^{-b}\int_0^1 \frac{e^t}{t+1}\,dt = -ae^{-b}$$

77. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right)$

$$=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+5n}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{5n}\left(\frac{1}{n+r}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\left(\frac{1}{1+\frac{r}{n}}\right)$$

$\because$ Lower limit of $r=1$

$\therefore$ Lower limit of integration $=\lim_{n\to\infty}\frac{1}{n}=0$

$\because$ Upper limit of $r=5n$

$\therefore$ Upper limit of integration $=\lim_{n\to\infty}\frac{5n}{n}=5$

From Eq. (i), $\int_0^5\frac{1}{1+x}dx=[\ln(1+x)]_0^5=\ln 6-\ln 1=\ln 6$

78. $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n+1}+\frac{2}{n+2}+\ldots+\frac{3n}{4n}\right)$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{\frac{1}{n}}{1+\frac{1}{n}}+\frac{\frac{2}{n}}{1+\frac{2}{n}}+\ldots+\frac{\frac{3n}{n}}{1+\frac{3n}{n}}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n}\left(\frac{\frac{r}{n}}{1+\frac{r}{n}}\right)=\int_0^3\frac{x}{1+x}dx$$

$$=\int_0^3\frac{x+1-1}{(1+x)}dx=\int_0^3 dx-\int_0^3\frac{1}{1+x}dx$$

$$=[x-\log(x+1)]_0^3=3-\log 4$$

79. We have, $I(m,n)=I=\int_0^1 t^m(1+t)^n dt$

$$\Rightarrow\quad I(m,n)=\left[(1+t)^n\cdot\frac{t^{m+1}}{m+1}\right]_0^1-\frac{n}{m+1}\int_0^1(1+t)^{n-1}t^{m+1}dt$$

$$\Rightarrow\quad I(m,n)=\frac{2^n}{m+1}-\frac{n}{m+1}I(m+1,n-1)$$

80. Let $\quad I=\int_0^\pi\log(1+\cos x)\,dx \qquad\ldots(i)$

$$\Rightarrow\quad I=\int_0^\pi\log\{1+\cos(\pi-x)\}\,dx$$

$$\left[\because\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^\pi\log(1-\cos x)\,dx \qquad\ldots(ii)$$

$$[\because\cos(\pi-x)=-\cos x]$$

$$=\int_0^\pi\log\left\{2\sin^2\left(\frac{x}{2}\right)\right\}dx \qquad \left(\because 1-\cos x=2\sin^2\frac{x}{2}\right)$$

$$=\int_0^\pi\left\{\log 2+2\log\left(\sin\frac{x}{2}\right)\right\}dx$$

$$[\because\log(mn^2)=\log m+2\log n]$$

$$=\int_0^\pi\log 2\,dx+2\int_0^\pi\log\left(\sin\frac{x}{2}\right)dx$$

In the second integral, put $\frac{x}{2}=t\quad\Rightarrow\quad dx=2dt$

and limits when $x=0, t=0$ and when $x=\pi, t=\pi/2$

$$\therefore\qquad I=\log 2\,[x]_0^\pi+2\int_0^{\pi/2}\log(\sin t)\,2dt$$

$$=(\log 2)(\pi-0)+4\left(-\frac{\pi}{2}\log 2\right)$$

$$\left(\because\int_0^{\pi/2}\log\sin x\,dx=-\frac{\pi}{2}\log 2\right)$$

$$=-\pi\log 2$$

81. By definition of Gamma function,

$$\int_0^\infty e^{-x}x^3\,dx=\int_0^\infty e^{-x}x^{4-1}\,dx=\Gamma 4=3!$$

$$\therefore\qquad\int_0^\infty e^{-x}x^3\,dx=6$$

82. Let $\quad I=\int_0^1\log\sin\left(\frac{\pi x}{2}\right)dx$

Put $\quad\frac{\pi x}{2}=t\quad\Rightarrow\quad dx=\frac{2}{\pi}dt$

$$\therefore\qquad I=\frac{2}{\pi}\int_0^{\pi/2}\log\sin t\,dt=\frac{2}{\pi}I_1 \qquad\ldots(i)$$

where, $\quad I_1=\int_0^{\pi/2}\log\sin t\,dt=\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-t\right)dt$

$$=\int_0^{\pi/2}\log\cos t\,dt$$

$$\therefore\qquad 2I_1=\int_0^{\pi/2}(\log\sin t+\log\cos t)\,dt$$

$$=\int_0^{\pi/2}\log(\sin t\cos t)\,dt=\int_0^{\pi/2}\log\left(\frac{\sin 2t}{2}\right)dt$$

$$=\int_0^{\pi/2}(\log\sin 2t-\log 2)\,dt$$

$$=\frac{1}{2}\int_0^\pi\log\sin z\,dz-\log 2\int_0^{\pi/2}dt$$

(put $2x=z$ in first integral, $\therefore 2dx=dz$)

$$=\frac{1}{2}\cdot 2\int_0^{\pi/2}\log\sin z\,dz-\frac{\pi}{2}\log 2$$

$$\Rightarrow\quad 2I_1=I_1-\frac{\pi}{2}\log 2\ \Rightarrow\ I_1=-\frac{\pi}{2}\log 2$$

$\therefore$ From Eq. (i), $I=\frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right)=-\log 2$

83. Since, by putting $x=t+c$, we get $I=\int_a^b f(c+t)\,dt=\int_a^b f(x+c)\,dx$

84. $\int_{-1}^3\left[\tan^{-1}\left(\frac{x}{x^2+1}\right)+\tan^{-1}\left(\frac{x^2+1}{x}\right)\right]dx$

$$=\int_{-1}^3\left[\tan^{-1}\left(\frac{x}{x^2+1}\right)+\cot^{-1}\left(\frac{x}{x^2+1}\right)\right]dx \quad\left(\because\tan^{-1}x=\cot^{-1}\frac{1}{x}\right)$$

$$=\int_{-1}^3\left(\frac{\pi}{2}\right)dx=\left[\frac{\pi x}{2}\right]_{-1}^3=\frac{\pi}{2}[3+1]=2\pi$$

$$\left(\because\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right)$$

85. $I=\int_{-\pi/2}^{\pi/2}\sin^4 x\cos^6 x\,dx$

This is an even function.

$\therefore \qquad I=2\int_0^{\pi/2}\sin^4 x\cos^6 x\,dx$

Applying Gamma function

$$\Rightarrow \qquad I=\frac{2\,\Gamma\frac{4+1}{2}\,\Gamma\frac{6+1}{2}}{2\,\Gamma 6}=\frac{2\,\Gamma\frac{5}{2}\,\Gamma\frac{7}{2}}{2\,\Gamma 6}$$

$$=\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{5\cdot4\cdot3\cdot2\cdot1}=\frac{3\pi}{2^8}=\frac{3\pi}{256}$$

86. $\int_0^1\tan^{-1}\left(\frac{1}{x^2-x+1}\right)dx$

$$=\int_0^1\tan^{-1}\left\{\frac{x-(x-1)}{1+x(x-1)}\right\}$$

$$=\int_0^1\tan^{-1}x\,dx-\int_0^1\tan^{-1}(x-1)\,dx$$

$$=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(1-1+x)\,dx$$

$$=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}x\,dx$$

$$=2\int_0^1\tan^{-1}x\,dx$$

$$=2\left[x\tan^{-1}x-\int\frac{x}{1+x^2}dx\right]_0^1$$

$$=2\left[x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\right]_0^1$$

$$=2\left[\left\{1\tan^{-1}1-\frac{1}{2}\log(2)\right\}-\left\{0-\frac{1}{2}\log 1\right\}\right]$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2-0\right]=\frac{\pi}{2}-\log 2$$

Round II

1. $100\int_0^1\{x\}\,dx=100\int_0^1\{x\}\,dx$

$$\frac{100}{2}[x^2]_0^1=50$$

2. $\sin x$ is periodic function with period 2π

$$I=\int_0^{41\pi/2}\sin x\,dx=\int_0^{\pi/2}\sin x\,dx+\int_{\pi/2}^{\pi/2+20\pi}\sin x\,dx$$

$$I=0+\int_0^{\pi/2}\sin x\,dx=1$$

3. $1+x^4<1+x^2,\ 0\le x\le 1$

$$\sqrt{1+x^4}<1+x^4<1+x^2$$

$$\frac{1}{\sqrt{1+x^4}}>\frac{1}{1+x^2}$$

$$\int_0^1\frac{dx}{\sqrt{1+x^4}}\ge\int_0^1\frac{dx}{1+x^2}\Rightarrow I\ge\frac{\pi}{4}$$

4. $I=\int_0^x[\sin t]\,dt$

$$=\int_0^{2x\pi}[\sin t]\,dt+\int_{2n\pi}^{x}[\sin t]\,dt$$

$$=n\int_0^{\pi}[\sin t]\,dt+\int_{2n\pi}^{x}[\sin t]\,dt$$

$$=n(-\pi)+0=-n\pi$$

5. $f(x)=\int_0^{\sin^2 x}\sin^{-1}\sqrt{t}\,dt+\int_0^{\cos^2 x}\cos^{-1}\sqrt{t}\,dt$

$$f'(x)=\sin(\sqrt{\sin^2 x})\,2\sin x\cos x+\cos^{-1}\sqrt{\cos^2 x}(-2\cos x\sin x)$$

$$=\sin 2x[\sin^{-1}(\sin x)-\cos^{-1}(\cos x)]=\sin 2x(x-x)=0$$

$f'(x)=\text{constant}$

Find $f(x)$ for any value of x

$$\int_0^{1/2}\frac{\pi}{2}dx=\frac{\pi}{2}\cdot\frac{1}{2}=\frac{\pi}{4}$$

6. When $x=1, y=0$, slope $=\frac{1}{\sqrt{1+x^6}}(3x^2)-\frac{1}{\sqrt{1+x^4}}(2x)=\frac{1}{\sqrt{2}}$

$\therefore \qquad \sqrt{2}y=x-1$

7. $\int_0^9[\sqrt{x}+2]\,dx$

$$=\int_0^1[\sqrt{x}+2]\,dx+\int_1^4[\sqrt{x}+2]\,dx+\int_4^9[\sqrt{x}+2]\,dx$$

$$=\int_0^1 2dx+\int_1^4(1+2)\,dx+\int_4^9(2+2)\,dx$$

$$=2+9+20=31$$

8. $\int_0^1[x^2]dx+\int_1^{\sqrt{2}}[x^2]dx$

$$=\int_0^1 0\,dx+\int_1^{\sqrt{2}}1\,dx=(\sqrt{2}-1)$$

9. $\sqrt{x+(2\sqrt{x-1})}=\left|\sqrt{x-1}+1\right|$

and $\qquad =\sqrt{x-2\sqrt{x-1}}=\left|\sqrt{x-1}-1\right|$

$$I=\int_1^2 2dx+\int_2^5 2\sqrt{x-1}\,dx$$

$$=2\cdot1+2\cdot\left[(x-1)^{3/2}\cdot\frac{2}{3}\right]_2^5=2+\frac{28}{3}=\frac{34}{3}$$

10. $I=\int_{-2}^1\left(\tan^{-1}x+\cot^{-1}\frac{1}{x}\right)dx$

$$=\int_{-2}^0(\tan^{-1}x+\pi+\tan^{-1}x)\,dx+\int_0^1(\tan^{-1}x+\tan^{-1}x)\,dx$$

$$=\frac{5\pi}{2}-4\tan^{-1}2+\log\frac{5}{2}$$

11. $\cos x>\sin x, \forall x\in(0,\pi/4)$

$$\Rightarrow \qquad \int_0^{\pi/4}2^{\cos x}dx>\int_0^{\pi/4}2^{\sin x}\,dx\Rightarrow I_2>I_1$$

12. Use property of $f_1(x) \le f(x) \le f_2(x), \forall x \in [a, b]$

Then, $\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)\,dx$

$$= \frac{1}{2} \le f(t) \le 1, \forall t \in [0,1]$$

$$= \frac{1}{2}(1-0) \le \int_0^1 f(t)dt \le 1(1-0)$$

$$= \frac{1}{2} \le \int_0^1 f(t)dt \le 1$$

Similarly, $0 \le f(t) \le \frac{1}{2}\ t \in [1,2]$

$$(2-1)\,0 \le 0 \int_1^2 f(t)dt \le (2-1)\cdot\frac{1}{2}$$

$$0 \le \int_1^2 f(t)dt \le \frac{1}{2}$$

$$g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$$

$\Rightarrow$ $$0 + \frac{1}{2} \le g(2) \le 1 + \frac{1}{2}$$

$$\frac{1}{2} \le g(2) \le \frac{3}{2}$$

13. We have, $\int_0^n [x]dx$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \ldots + \int_{n-1}^n (n-1)\,dx$$

$$= 1(2-1) + 2(3-2) + 3(4-3) + \ldots + (n-1)\{n-(n-1)\}$$

$$= 1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

and $$\int_0^n \{x\}\,dx = \int_0^n (x - [x])\,dx = \frac{n}{2}$$

$\therefore$ $$\frac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx} = n-1$$

14. We have, $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$

$\Rightarrow$ $$f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$$

$$I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx = \int_{\sin\theta}^{\operatorname{cosec}\theta} \left\{-\frac{1}{x^2} f\left(\frac{1}{x}\right)\right\} dx$$

$$= \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\,dt, \text{ where } t = \frac{1}{x}$$

$\Rightarrow$ $$I = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(t)\,dt = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

15. We have, $f(x) = \frac{x-1}{x+1}$

$\Rightarrow$ $$f^2(x) = f\{f(x)\} = f\left(\frac{x-1}{x+1}\right)$$

$$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\frac{1}{x}$$

$\Rightarrow$ $$f^4(x) = f^2[f^2(x)] = f^2\left(-\frac{1}{x}\right) = \frac{-1}{-\frac{1}{x}} = x$$

$\therefore$ $$g(x) = f^{1998}(x) = f^2 o f^{1996}(x)$$

$\Rightarrow$ $$g(x) = f^2[f^{1996}(x)]$$

$\Rightarrow$ $$g(x) = f^2(x)$$

$$[\because f^{1996}(x) = \{\underbrace{f^4 o f^4 o f^4 o \ldots f^4}_{499 \text{ times}})\}\ (x) = x]$$

$\Rightarrow$ $$g(x) = -\frac{1}{x}$$

$\therefore$ $$\int_{1/e}^1 g(x)\,dx = \int_{1/e}^1 \left(-\frac{1}{x}\right)dx = -[\log_e x]_{1/e}^1$$

$\Rightarrow$ $$\int_{1/e}^1 g(x)\,dx = -\left[\log_e 1 - \log_e \frac{1}{e}\right]$$

$$= -[0 + 1] = -1$$

16. $$\int_{-3}^5 f(|x|)\,dx = \int_{-3}^3 f(|x|)\,dx + \int_3^5 f(|x|)\,dx$$

$$= 2\int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx\right] + \left\{\int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx\right\}$$

$$= 2\left(0 + \frac{1}{2} + \frac{2^2}{2}\right) + \left(\frac{9}{2} + \frac{16}{2}\right) = \frac{35}{2}$$

17. $$\int_1^4 (\{x\})^{[x]}dx = \int_1^2 (\{x\})^{[x]}dx + \int_2^3 (\{x\})^{[x]}dx + \int_3^4 (\{x\})^{[x]}dx$$

$$= \int_1^2 (x-1)\,dx + \int_2^3 (x-2)^2\,dx + \int_3^4 (x-3)^3 dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-3)^4}{4}\right]_3^4 = \frac{13}{12}$$

18. $$I_{2(n)} - I_{2(n-1)} = \int_0^{\pi/2} \frac{\{\sin^2 nx - \sin^2(n-1)\,x\}}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)\,x \cdot \sin x}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)\,x}{\sin x}\,dx = I_{1(n)}$$

$\Rightarrow$ $$I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$$

19. $$\int_0^{3\pi/2} \sin\left[\frac{2x}{\pi}\right]dx = \int_0^{\pi/2} \sin\left[\frac{2x}{\pi}\right]dx + \int_{\pi/2}^{\pi} \sin\left[\frac{2x}{\pi}\right]dx + \int_{\pi}^{3\pi/2} \sin\left[\frac{2x}{\pi}\right]dx$$

$$= 0 + \sin 1 \int_{\pi/2}^{\pi} dx + \sin 2 \int_{\pi}^{3\pi/2} dx$$

$$= \frac{\pi}{2}(\sin 1 + \sin 2)$$

20. $\int_{-1}^{1} \{x^2 + x - 3\}\, dx = \int_{-1}^{1} \{x^2 + x\}\, dx$

$$= \int_{-1}^{1} (x^2 + x - [x^2 + x])\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} - \int_{-1}^{0} [x^2 + x]\, dx$$

$$- \int_{0}^{\frac{\sqrt{5}-1}{2}} [x^2 + x]\, dx - \int_{(\sqrt{5}-1)/2}^{1} [x^2 + x]\, dx$$

$$= \frac{2}{3} + 1 - 0 - 1\left(1 - \frac{\sqrt{5}-1}{2}\right) = \frac{2}{3} + \frac{\sqrt{5}-1}{2} = \frac{1}{6}(1 + 3\sqrt{5})$$

21. Since, $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow 1 \le 1 + \sin^3 x \le 2$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} \le \frac{1}{\sqrt{1 + \sin^3 x}} \le 1$$

$$\Rightarrow \quad \int_0^{\pi/2} \frac{1}{\sqrt{2}}\, dx \le \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}} \le \int_0^{\pi/2} dx$$

$$\Rightarrow \quad \frac{\pi}{2\sqrt{2}} \le I \le \frac{\pi}{2}$$

22. Let $I = \int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}]\, dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$$

$$= \sqrt{2} \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$$\therefore \quad I = \sqrt{2} \int_{-1}^{0} \frac{dt}{\sqrt{1 - t^2}} \quad \Rightarrow I = \sqrt{2}\, [\sin^{-1} t]_{-1}^{0}$$

$$= \sqrt{2}\, [0 - (-\pi/2)] = \frac{\pi}{\sqrt{2}}$$

23. Let $I = \int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\}\, \{g'(x)\}\, dx$

Let $\quad f\{g(x)\} = z$

$\Rightarrow \quad f'\{g(x)\}\, g'(x)\, dx = dz$

when $x = 1$, $z = f\{g(1)\}$

when $x = 2$, $z = f\{g(2)\}$

$$\therefore \quad I = \int_{f\{g(1)\}}^{f\{g(2)\}} \frac{1}{z}\, dz = [\log z]_{f\{g(1)\}}^{f\{g(2)\}}$$

$$\Rightarrow \quad I = \log f\{g(2)\} - \log f\{g(1)\} = 0 \quad [\because g(2) = g(1)]$$

24. Let $\quad I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x}\, dx$

$$= \int_{-\pi/2}^{0} \frac{\cos x}{1 + e^x}\, dx + \int_0^{\pi/2} \frac{\cos x}{1 + e^x}\, dx \quad \ldots(i)$$

On putting $x = -t$ in Ist integral, we get

$$\int_{-\pi/2}^{0} \frac{\cos x}{1 + e^x}\, dx = \int_0^{\pi/2} \frac{e^x \cos x}{1 + e^x}\, dx$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{e^x \cos x}{1 + e^x}\, dx + \int_0^{\pi/2} \frac{\cos x}{1 + e^x}\, dx$$

$$= \int_0^{\pi/2} \frac{(1 + e^x) \cos x}{(1 + e^x)}\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2} = 1$$

25. $\int_0^3 (3ax^2 + 2bx + c)\, dx = \int_1^3 (3ax^2 + 2bx + c)\, dx$

$$\Rightarrow \quad \int_0^1 (3ax^2 + 2bx + c)\, dx + \int_1^3 (3ax^2 + 2bx + c)\, dx$$

$$= \int_1^3 (3ax^2 + 2bx + c)\, dx$$

$$\Rightarrow \quad \int_0^1 (3ax^2 + 2bx + c)\, dx = 0$$

$$\Rightarrow \quad \left[\frac{3ax^3}{3} + \frac{2bx^2}{2} + cx\right]_0^1 = 0$$

$$\Rightarrow \quad a + b + c = 0$$

26. Since, f is continuous function.

Let $x = t - 1$

$\therefore$ $dx = dt$. When x tends to -3 and 5, then t tends to -2, 6.

Therefore, $\quad \int_{-3}^{5} f(x)\, dx = \int_{-2}^{6} f(t - 1)\, dt = \int_{-2}^{6} f(x - 1)\, dx$

27. Let $\quad P = \lim_{n \to \infty} \left[\frac{n!}{n^n}\right]^{1/n} = \lim_{n \to \infty} \left[\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \frac{n}{n}\right]^{1/n}$

$$\therefore \quad \log P = \lim_{n \to \infty} \frac{1}{n}\left[\log \frac{1}{n} + \log \frac{2}{n} + \ldots + \log \frac{n}{n}\right]$$

$$\Rightarrow \quad \log P = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log \frac{r}{n}$$

$$\Rightarrow \quad \log P = \int_0^1 \log x\, dx = [x \log x - x]_0^1 = -1$$

$$\Rightarrow \quad P = \frac{1}{e}$$

28. Let $f_1(x) = y_1 = \int_2^x (2t - 5)\, dt$ and $f_2(x) = y_2 = \int_0^x 2t\, dt$

Now, point of intersection means those point at which

$y_1 = y_2 = y \Rightarrow y_1 = x^2 - 5x + 6$ and $y_2 = x^2$

On solving, we get $x^2 = x^2 - 5x + 6$

$$\Rightarrow \quad x = \frac{6}{5} \text{ and } y = x^2 = \frac{36}{25}$$

Thus, point of intersection is $\left(\frac{6}{5}, \frac{36}{25}\right)$.

29. Let $\quad I = \int_0^1 f(k - 1 + x)\, dx$

$$\Rightarrow \quad I = \int_{k-1}^{k} f(t)\, dt, \text{ where } t = k - 1 + x$$

$$\Rightarrow \quad I = \int_{k-1}^{k} f(x)\, dx$$

$$\therefore \sum_{k=1}^{n} \int_{k-1}^{k} f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx + \ldots + \int_{n-1}^{n} f(x)\, dx$$

$$= \int_0^n f(x)\, dx$$

30. $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$

$\Rightarrow \quad \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$

$\therefore \quad \log P = \lim_{n \to \infty} \frac{1}{n}\left[\log \frac{1}{n} + \log \frac{2}{n} + \ldots + \log \frac{n}{n}\right]$

$\Rightarrow \quad \log P = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log \frac{r}{n}$

$\Rightarrow \quad \log P = \int_0^1 \log x dx = [x \log x - x]_0^1 = -1$

$\Rightarrow \quad P = \frac{1}{e}$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$f(t) = \int_1^{64} \frac{e^{\sin t}}{t} dt = \int_1^{64} f(t)\, dt = f(64) - f(1)$

On comparing, we get $k = 64$

31. $\because$ $\sin^{-1}(\cos x) + \cos^{-1}(\sin x)$ is periodic with period 2π, then

$\int_{a+2\pi}^{a+5\pi/2} \{\sin^{-1}(\cos x) + \cos^{-1}(\sin x)\} dx$

$= \int_0^{\pi/2} \{\sin^{-1}(\cos x) + \cos^{-1}(\sin x)\} dx$

$= \int_0^{\pi/2} \left[\sin^{-1}\left\{\cos\left(\frac{\pi}{2} - x\right)\right\} + \cos^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}\right] dx$

$= \int_0^{\pi/2} (\sin^{-1} \sin x + \cos^{-1} \cos x)\, dx$

$= 2\int_0^{\pi/2} x dx = 2\left\{\frac{x^2}{2}\right\}_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$

32. $I_1 - I_2 = \int_0^{\pi/2} (\cos\theta - \sin 2\theta) f(\sin\theta + \cos^2\theta)\, d\theta$

Put $\quad \sin\theta + \cos^2\theta = t$

$\Rightarrow \quad (\cos\theta - \sin 2\theta)\, d\theta = dt$

Then, $\quad I_1 - I_2 = \int_1^1 f(t)\, dt = 0$

$\therefore \quad I_1 = I_2$

33. $\because$ $\sin^4 t + \cos^4 t$ is periodic with period $\frac{\pi}{2}$.

Now, $\quad f(x + \pi) = \int_0^{\pi + x} (\sin^4 t + \cos^4 t) dt$

$= \int_0^x (\sin^4 t + \cos^4 t)\, dt + \int_x^{x+\pi} (\sin^4 t + \cos^4 t)\, dt$

$= f(x) + \int_0^{\pi} (\cos^4 t + \sin^4 t)\, dt = f(x) + f(\pi)$

or

$= f(x) + 2\int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt = f(x) + 2f\left(\frac{\pi}{2}\right)$

34. $\because$ $I_k = \int_{-2k\pi}^{0} |\sin x| [\sin x] dx + \int_0^{2k\pi} |\sin x| [\sin x] dx$

$= -\int_{2k\pi}^{0} |\sin x| [-\sin x] dx + \int_0^{2k\pi} |\sin x| [\sin x] dx$

$= \int_0^{2k\pi} |\sin x| ([\sin x] + [-\sin x])\, dx$

$= 2k\int_0^{\pi} |\sin x| ([\sin x] + [-\sin x])\, dx = 2k\int_0^{\pi} \sin x (0 - 1)\, dx$

$= -2k[-\cos x]_0^{\pi}$ $\quad \left[\begin{array}{l} \because \sin x > 0,\ x \in (0, \pi) \\ \therefore -\sin x < 0,\ x \in (0, \pi) \end{array}\right]$

$= -4k$

$\therefore \quad \sum_{k=1}^{10} I_k = -4 \sum_{k=1}^{10} k = -4 \cdot \frac{10 \cdot 11}{2} = -220$

35. $\because$ Integrand is discontinuous at $\frac{\pi}{2}$, then

$\int_0^{\pi/2} 0 \cdot dx + \int_{\pi/2}^{3\pi/2} 0 \cdot dx = 0$

$\because \quad 0 < x < \frac{\pi}{2}, |\tan^{-1} \tan x| = |\sin^{-1} \sin x|$

and $\quad \frac{\pi}{2} < x < \frac{3\pi}{2}, |\tan^{-1} \tan x| = |\sin^{-1} \sin x|$

36. $\int_{-1}^{15} \text{sgn}(\{x\})\, dx = \int_0^{16} \text{sgn}(\{x - 1\})\, dx$ (by property)

$= \int_0^{16} \text{sgn}(\{x\})\, dx$

$= 16\int_0^1 \text{sgn}(\{x\})\, dx = 16\int_0^1 \text{sgn}(x)\, dx$

$= 16\int_0^1 1 \cdot dx = 16$

37. $\frac{d}{dx}\int_{\sin x}^{1} t^2 f(t) = \frac{d}{dx}(1 - \sin x)$

$\Rightarrow [1^2 f(1) \cdot (0)] - (\sin^2 x) \cdot f(\sin x) \cdot \cos x = -\cos x$

$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x}$

$\Rightarrow \quad f\left(\frac{1}{\sqrt{3}}\right) = (\sqrt{3})^2 = 3 \quad \left(\text{put } \sin x = \frac{1}{\sqrt{3}}\right)$

38. $I_2 = \int_1^{1/x} \frac{dt}{1 + t^2}$

Put $\quad t = 1/y$

$= -\int_1^x \frac{dy}{1 + y^2} = \int_x^1 \frac{dt}{1 + t^2} = I_1$

So, choice (a) is true and (b) and (c) are ruled out. On integrating

$\int_1^{1/x} \frac{dt}{1 + t^2} = \tan^{-1}\left(\frac{1}{x}\right) - \frac{\pi}{4} = \cot^{-1}(x) - \frac{\pi}{4} = \frac{\pi}{2} - \tan^{-1}(x) - \frac{\pi}{4}$

Hence, choice (d) is true.

39. Let $A + B = f(x) = \int_{e^{-1}}^{\tan x} \frac{t dt}{t^2 + 1} + \int_{e^{-1}}^{\cot x} \frac{dt}{t(1 + t^2)}$

$\Rightarrow \quad f(x) = \frac{\tan x}{\tan^2 x + 1} \cdot \sec^2 x + \frac{1}{\cot x (1 + \cot^2 x)} (-\text{cosec}^2 x) = 0$

$\Rightarrow f(x)$ is constant $\Rightarrow f(x) = f\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad f(x) = \int_{e^{-1}}^{1} \frac{t dt}{t^2 + 1} + \int_{e^{-1}}^{1} \frac{dt}{t(t^2 + 1)} = \int_{e^{-1}}^{1} \frac{dt}{t} = 1$

$\Rightarrow \quad f(x) = 1$ for all x in $\left(0, \frac{\pi}{2}\right)$

40. $f(-x) = -f(x)$

$$f(x+2) = f(x)$$

To test whether $g(x)$ is odd or even we compute $g(-x)$ indeed

$g(-x) = \int_0^{-x} f(t)dt$

On putting $t = -z, dt = -dz$

$$\int_0^x f(-z)\,(-dz) = -\int_0^x f(-z)\,dz = \int_0^x f(t)\,dt = g(x)$$

41. $\int_{\pi/2}^{\alpha} \sin x dx = \sin 2\alpha$

$$\sin 2\alpha + \cos\alpha = 0$$

$$\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

42. Let $\quad I = \int_0^{\pi} x\, f(\sin x)\, dx \quad$ ...(i)

$I = \int_0^{\pi} (\pi - x)\, f(\sin x)\, dx \quad$ (by property)...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^{\pi} f(\sin x)\, dx$$

$\therefore \quad I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx \quad$ [option (a)]

$= \frac{\pi}{2} \cdot 2 \int_0^{\pi/2} f(\sin x)\, dx$

$= \pi \int_0^{\pi/2} f(\sin x)\, dx \quad$ [option (c)] (by property)

$= \pi \int_0^{\pi/2} f(\cos x)\, dx \quad$ [option (d)] (by property)

43. $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x) f[(3-x)x]\, dx$

$$I_1 = 3\int_{\sec^2 z}^{2-\tan^2 z} f[(3-x)x]\, dx - \int_{\sec^2 z}^{2-\tan^2 z} x[(3-x)x]\, dx$$

$$I_1 = 3I_2 - I_1 \Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

44. $\int_0^p [f(x) + g(x)]\, dx$

$$= \int_0^p [f(p-x) + g(p-x)]\, dx$$

$$= \int_0^p [1 - f(x) + 1 - g(x)]\, dx$$

$$= \int_0^p 2dx - \int_0^p [f(x) + g(x)]\, dx$$

$\Rightarrow \quad 2\int_0^p [f(x) + g(x)]\, dx = 2\int_0^p dx$

$$\int_0^p [f(x) + g(x)]\, dx = p$$

45. Let

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \quad \text{...(i)}$$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}} \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1 + \sqrt{\tan x}} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$\Rightarrow \quad 2I = [x]_{\pi/6}^{\pi/3}$

$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12}$

Statement I is false.

But $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$ is a true statement by property of definite integrals.

46. $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

$\Rightarrow \quad [|\sin x| + |\cos x|] = 1$

$$I = \int_0^{4\pi} [|\sin x| + |\cos x|]\, dx = \int_0^{4\pi} 1 \cdot dx = 4\pi$$

47. Option (a) is correct as $f(x) = kx$

$$f(x+y) = f(x) + f(y), \forall x, y$$

$\Rightarrow \quad f(x) = kx$

$$\int_{-5}^{5} f(x)\, dx = \int_{-5}^{5} kx\, dx = 0$$

48. The statement II is false, since in writing $\int \frac{dx}{x-3y} = \log(x-3y) + C$, we are assuming that y is a constant. We will now prove the assertion. From the given relation

$(x-y)^2 = \frac{x}{y}$ and $2\log(x-y) = \log x - \log y$

Also, $\frac{dy}{dx} = \left(-\frac{y}{x}\right) \cdot \frac{x+y}{x-3y}$. To prove the integral relation it is sufficient to show that $\frac{d}{dx}\text{RHS} = \frac{1}{x-3y}$

Now, $\frac{1}{2}\log\left(\frac{x}{y} - 1\right) \quad \left[\because (x-y)^2 = \frac{x}{y}\right] = \frac{1}{2}[\log(x-y) - \log y]$

$$= \frac{1}{2}\left[\frac{\log x - \log y}{2} - \log y\right] = \frac{1}{4}[\log x - 3\log y]$$

$\Rightarrow \quad \frac{d}{dx}\text{RHS} = \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\frac{dy}{dx}\right]$

$$= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\left(-\frac{y}{x}\right)\frac{x+y}{x-3y}\right] = \frac{1}{x-3y}$$

49. $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log t\, dt}{1 + t + t^2}$

On putting $t = \frac{1}{z}, dt = -\frac{1}{z^2}dz$

$$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log 1/z}{1 + \frac{1}{z} + \frac{1}{z^2}}\left(-\frac{dz}{z^2}\right)$$

$$= \int_1^x \frac{\log z\, dz}{z^2 + z + 1} = \int_1^x \frac{\log t\, dt}{1 + t + t^2} = f(x)$$

$\Rightarrow$ Assertion (A) is false.

The reason (R) is true which can be proved in the same style in which assertion A has been disproved.

50. $\because$ Period of $e^{\sin x}$ is 2π.

$\therefore \quad \int_0^{200} e^{\sin x} dx \neq 200\lambda$

51. $\because \int_0^6 \{x+5\}^2 dx = \int_0^5 \{x+6\}^2 dx$ (by property)

$= \int_0^5 \{x\}^2 dx = 5\int_0^1 \{x\}^2 dx$

($\because \{\cdot\}$ is periodic with period 1)

$= 5\int_0^1 x^2 dx = \frac{5}{3}$

52. $\because |\sin x|$ is an even function.

$\therefore \quad \int_{-\pi/2}^{\pi/2} |\sin x|\, dx = 2\int_0^{\pi/2} |\sin x|\, dx = 2\int_0^{\pi/2} \sin x\, dx$

$= -2 (\cos x)_0^{\pi/2} = -2 (0 - 1) = 2$

53. $\because \int_{\pi/2}^{3\pi/2} [2 \sin x]\, dx = \int_{\pi/2}^{5\pi/6} [2 \sin x]\, dx + \int_{5\pi/6}^{\pi} [2 \sin x]\, dx$

$+ \int_{\pi}^{7\pi/6} [2 \sin x]\, dx + \int_{7\pi/6}^{3\pi/2} [2 \sin x]\, dx$

$= \int_{\pi/2}^{5\pi/6} 1 \cdot dx + 0 - \int_{\pi}^{7\pi/6} 1 \cdot dx - 2\int_{7\pi/6}^{3\pi/2} 1 \cdot dx$

$= \frac{\pi}{3} - \frac{\pi}{6} - \frac{2\pi}{3} = -\frac{\pi}{2}$

$\left[\because 2 \sin x \text{ is decreasing function in } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right]$

54. Let $p'(x) = a(x-1)(x-3)$

$\Rightarrow \quad p(x) = \int_1^x a(x^2 - 4x + 3)\, dx + C$

$\Rightarrow \quad p(x) = a\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^x + 60$ $[\because p(1) = 6]$

$\Rightarrow \quad p(x) = a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$

Since, $p(3) = 2$

then $a = 3$

$\therefore \quad p(x) = x^3 - 6x^2 + 9x + 2$

Statement II is also true and it is a correct explanation for Statement I.

55. In LHS, put $x^n = \tan^2 \theta$

$\Rightarrow \quad nx^{n-1}\, dx = 2 \tan \theta \sec^2 \theta\, d\theta$

$\therefore \quad \int_0^{\infty} \frac{dx}{1 + x^n} = \frac{2}{n}\int_0^{\pi/2} \tan^{1-2+2/n} \theta\, d\theta$

$= \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n)-1} \theta\, d\theta$

In RHS, put $x^n = \sin^2 \theta$

$\Rightarrow \quad nx^{n-1}\, dx = 2 \sin \theta \cos \theta\, d\theta$

$\therefore \quad \int_0^1 \frac{dx}{(1 - x^n)^{1/n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{1}{\cos^{2/n} \theta} \sin^{\frac{2}{n}-1} \theta \cos \theta\, d\theta$

$= \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n)-1} \theta\, d\theta$

56. $\because \sin \frac{x}{2} [1 + 2(\cos x + \cos 2x + \cos 3x + \ldots + \cos nx)]$

$= \sin\left(n + \frac{1}{2}\right) x$

$\therefore \quad \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right) x}{\sin \frac{x}{2}} dx = \int_0^{\pi} dx$

$+ 2\left[\int_0^{\pi} \cos dx + \int_0^{\pi} \cos 2x\, dx + \ldots + \int_0^{\pi} \cos nx\, dx\right]$

$= \pi + 2(0 + 0 + \ldots + 0) = \pi$

$\Rightarrow$ Statement I is true.

$\because \quad \int_0^{\pi} \sin mx\, dx = -\frac{1}{m} [\cos mx]_0^{\pi}$

$= -\frac{1}{m} (\cos m\pi - 1)$

$= -\frac{1}{m} [(-1)^m - 1]$

$\neq 0$ when m is odd.

57. $\because \sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)$

$= 1 - 3 \sin^2 x \cos^2 x$

$= 1 - \frac{3}{4} \sin^2 2x$ $\left(\because \text{period } \frac{\pi}{2}\right)$

$\therefore$ Least and greatest value of $\sin^6 x + \cos^6 x$ are $\frac{1}{4}$ and 1.

Hence, $\left(\frac{\pi}{2} - 0\right) \times \frac{1}{4} < \int_0^{\pi/2} (\sin^6 x + \cos^6 x)\, dx$

$< \left(\frac{\pi}{2} - 0\right) \times 1$

$\Rightarrow \quad \frac{\pi}{8} < \int_0^{\pi/2} (\sin^6 x + \cos^6 x)\, dx < \frac{\pi}{2}$

58. For $0 < x < 1$, then

$x > x^2$

$\Rightarrow \quad -x < -x^2$

$\Rightarrow \quad e^{-x} < e^{-x^2}$

$\Rightarrow \quad \int_0^1 e^{-x} \cos^2 x\, dx < \int_0^1 e^{-x^2} \cos^2 x\, dx$

If $f(x) \geq g(x)$, then

$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$

59. Given, integral $g(x) = \int_0^x \cos 4t\, dt$

$g(x) = \int_0^x \cos 4t\, dt$

$\Rightarrow \quad g(x + \pi) = \int_0^{x+\pi} \cos 4t\, dt$

$= \int_0^x \cos 4t\, dt + \int_x^{x+\pi} \cos 4t\, dt$

$$= g(x) + I_1 \text{ (say)}$$

$$I_1 = \int_x^{x+\pi} \cos 4t\, dt$$

$$= \int_0^{\pi} \cos 4t\, dt \text{ (definite integral property)}$$

$$= g(\pi)$$

$\Rightarrow \quad g(x+\pi) = g(x) + g(\pi)$

But wait the value of I_1 is zero.

$$\Rightarrow \quad I_1 = \left[\frac{\sin 4t}{4}\right]_0^{\pi} = \left(\frac{\sin 4\pi}{4} - \frac{\sin 0}{4}\right) = 0$$

$\Rightarrow \quad g(x+\pi) = g(x) - g(\pi)$

60. Here, $\int_0^{1.5} x[x^2]\, dx$

$$I = \int_0^1 x \cdot 0\, dx + \int_1^{\sqrt{2}} x \cdot 1\, dx + \int_{\sqrt{2}}^{1.5} x \cdot 2\, dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5}$$

$$= \frac{1}{2}\{2-1\} + \{(1.5)^2 - 2\}$$

$$= \frac{1}{2} + 2.25 - 2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

61. We have, $p'(x) = p'(1-x), \ \forall \ x \in [0, 1], \ p(0) = 1$

$$p(1) = 41$$

$\Rightarrow \quad p(x) = -p(1-x) + C$

$\Rightarrow \quad 1 = -41 + C$

$\Rightarrow \quad C = 42$

$\therefore \quad p(x) + p(1-x) = 42$

Now, $\quad I = \int_0^1 p(x)\, dx = \int_0^1 p(1-x)\, dx$

$\Rightarrow \quad 2I = \int_0^1 \{p(x) + p(1-x)\}\, dX$

$$= \int_0^1 42\, dx = 42$$

$\Rightarrow \quad I = 21$

62. Let $\quad I = \int_0^{\pi} [\cot x]\, dx \quad \ldots(i)$

$\Rightarrow \quad I = \int_0^{\pi} [\cot(\pi - x)]\, dx = \int_0^{\pi} [-\cot x]\, dx \quad \ldots(ii)$

On adding Eqs. (i) and (ii),

$$2I = \int_0^{\pi} [\cot x]\, dx + \int_0^{\pi} [-\cot x]\, dx$$

$$= \int_0^{\pi} (-1)\, dx$$

$(\because [x] + [-x] = -1, \text{ if } x \notin z = 0, \text{ if } x \in z)$

$$= [-x]_0^{\pi} = -\pi$$

$$\therefore \quad I = -\frac{\pi}{2}$$

63. Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\, dx < \int_0^1 \frac{x}{\sqrt{x}}\, dx,$

because in $x \in (0, 1), x > \sin x$

$$I < \int_0^1 \sqrt{x}\, dx = \frac{2}{3}[x^{3/2}]_0^1$$

$\Rightarrow \quad I < \frac{2}{3}$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\, dx < \int_0^1 x^{\frac{-1}{2}}\, dx = 2$

$$J < 2$$

64. Since, $f(e) = f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{1+t}\, dt + \int_1^{1/e} \frac{\log t}{1+t}\, dt$

$$= \int_1^e \frac{\log t}{1+t}\, dt + \int_1^e \frac{\log t}{t(1+t)}\, dt$$

$$= \int_1^e \left(\frac{\log t}{t}\right) dt = \left[\frac{(\log t)^2}{2}\right]_1^e$$

$$= \frac{1}{2}[(\log e)^2 - (\log 1)^2] = \frac{1}{2}$$

65. $\therefore \int_1^a [x] f'(x)\, dx = [f(x)]_1^2 + 2[f(x)]_2^3 + \ldots + [a][f(x)]_{[a]}^a$

$$= f(2) - f(1) + 2f(3) - 2f(2) + \ldots + [a] f(a) - [a] f([a])$$

$$= [a] f(a) - \{f(1) + f(2) + \ldots + f([a])\}$$

66. Let $I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2 x]\, dx$

$$\Rightarrow \quad I = \int_{-3\pi/2}^{-\pi/2} \left[\left(-\frac{\pi}{2} - \frac{3\pi}{2} - x + \pi\right)^3 + \cos^2\left(-\frac{\pi}{2} - \frac{3\pi}{2} - x\right)\right] dx$$

$$\Rightarrow \quad I = \int_{-3\pi/2}^{-\pi/2} [-(x+\pi)^3 + \cos^2 x]\, dx$$

$$\Rightarrow \quad 2I = \int_{-3\pi/2}^{-\pi/2} 2\cos^2 x\, dx$$

$$= \int_{-3\pi/2}^{-\pi/2} (1 + \cos 2x)\, dx = \left[x + \frac{\sin 2x}{2}\right]_{-3\pi/2}^{-\pi/2}$$

$$= \left[-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} - \left\{-\frac{3\pi}{2} + \frac{\sin(-3\pi)}{2}\right\}\right] = \pi$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

67. Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x}\, dx, a > 0$

Put $x = -x$, we get $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^{-x}}\, dx$

$$\Rightarrow \quad 2I = \int_{-\pi}^{\pi} \frac{(1 + a^x)\cos^2 x}{(1 + a^x)}\, dx$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2x)\, dx$$

$$\Rightarrow \quad 2I = \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right]_{-\pi}^{\pi}$$

$$\Rightarrow \quad I = \frac{1}{4}(\pi + \pi) = \frac{\pi}{2}$$

32 Area Bounded by Curves

JEE Main MILESTONE

- Curve Sketching
- Sign Convention for Finding the Areas Using Integration
- Variable Area, Greatest and Least Values

Continuous curve which is bounded by under the certain condition, the space which is occupied, is called area bounded by curves.

32.1 Curve Sketching

For the evaluation of area of bounded regions it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

One of the important applications of integration is to find the area bounded by a curve. Often such an area a can have a physical significance like the work done by a motor or the distance travelled by a vehicle.

(i) Symmetry

(a) **Symmetry about x-axis** If all powers of y in the equation of the given curve are even, then it is symmetric about x-axis, *i.e.*, the shape of the curve above x-axis is exactly identical to its shape below x-axis.
e.g., $y^2 = 4ax$ is symmetric about x-axis.

(b) **Symmetry about y-axis** If all powers of x in the equation of the given curve are even, then it is symmetric about y-axis.
e.g., $x^2 = 4ay$ is symmetric about y-axis.

(c) **Symmetry in opposite quadrants** If by putting $-x$ for x and $-y$ for y, the equation of curve remains same, then it is symmetric in opposite quadrants.
e.g., $xy = c^2$, $x^2 + y^2 = a^2$ are symmetric in opposite quadrants.

(d) **Symmetric about the line $y = x$** If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of $45°$ with positive direction of x-axis.

(ii) Origin and Tangents at the Origin

See whether the curve passes through origin or not. If the point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such a case to find the equations of the tangents at the origin, equate the lowest degree term to zero.

e.g., $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$.

So, $x = 0$, *i.e.*, y-axis is tangent at the origin to $y^2 = 4ax$.

(iii) Points of Intersection of the Curve with the Coordinate Axes

By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the x-axis. Similarly, by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the y-axis.

e.g., To find the points where the curve $xy^2 = 4a^2(2a - x)$ meets x-axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or $x = 2a$. So, the curve $xy^2 = 4a^2(2a - x)$, meets x-axis at $(2a, 0)$. This curve does not intersect y-axis, because by putting $x = 0$ in the equation of the given curve get an absurd result.

(iv) Asymptotes

The straight line AB is called the asymptote of curve $y = f(x)$, if the distance MK from M a point on the curve $y = f(x)$ to the straight line AB tends to zero as M recedes infinity. In other words, the straight line AB meets the curve $y = f(x)$ at infinity (K is a point on AB).

Thus,

(a) If $f(x) \to \pm\infty$ for $x \to a$, then the straight line $x = a$ is the asymptote of the curve $y = f(x)$.

(b) If in the right hand member of the equation of the curve $y = f(x)$ it is possible to single out a linear part so that the remaining part tends to zero as $x \to \pm\infty$, *i.e.*, if $y = f(x) = Kx + b + g(x)$ and $g(x) \to 0$ for $x \to \pm\infty$, then the straight line $y = Kx + b$ is the asymptote of the curve.

(c) If there exist finite limits $\lim_{x \to \pm\infty} \frac{f(x)}{x} = K$ and $\lim_{x \to \pm\infty} [f(x) - Kx] = b$, then the straight line $y = Kx + b$ is the asymptote of the curve.

While constructing the graphs of functions, it is expedient to follow the procedure given below

- Find the domain of definition of the function.
- Determine the odd-even nature of the function.
- Find the period of the function if its periodic.
- Find the asymptotes of the function.
- Check the behaviour of the function for $x \to 0 \pm$
- Find the value of x, if possible for which $f(x) \to 0$.

The interval of increase and decrease of the function in its range. Hence, determine the greatest and the least values of the function, if any.

(v) Regions Where the Curve does not Exist

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

e.g., The values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of y-axis. Similarly, the curve $a^2y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

(vi) Special Points

Find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to x-axis.

Find the points at which $\frac{dx}{dy} = 0$. At these points the tangents to the curve is parallel to y-axis.

(vii) Sign of $\frac{dy}{dx}$ and Points of Maxima and Minima

Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing, find the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing. Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the points so obtained, to find the points of maxima and minima.

Keeping the above facts in mind and plotting some points on the curve one can easily have a rough sketch of the curve.

Following examples will clear the procedure.

Sample Problem 1 *The graph of the function* $f(x) = \frac{x^2 - 1}{x^2 + 1}$ *is*

(a)

(b)

(c)

(d)

Interpret (c) Here, $f(x) = \dfrac{x^2-1}{x^2+1} = y$ (say)

Symmetry On putting $x = -x$, the graph is unaltered.
$\therefore$ Graph is symmetrical about y-axis.
Passes through origin Since, (0, 0) not satisfies the given curve.
$\therefore$ Graph is not passing through origin.
Intersection with coordinate axes Put $x = 0$, then $y = -1$ and put

$$y = 0, \text{ then } x = \pm 1$$

$\therefore$ Graph cuts the y-axis at $(0, -1)$ and the x-axis at $(1, 0)$ and $(-1, 0)$.

Asymptotes $$y = \frac{x^2-1}{x^2+1}$$

Here, no asymptotes is parallel to y-axis.
(since, $x^2 + 1 \neq 0$ for any value of x)

Now, $$x^2 = \frac{y+1}{1-y}$$

$$1 - y = 0 \text{ at } y = 1.$$

So, $y = 1$ is an asymptote parallel to x-axis.
Hence, required graph is

Note • If function is periodic and we can find it's period, then plot curve for the interval equal to one period and repeat it.
• If $f(a)f(b) < 0$, curve intersects x-axis atleast once. Similarly, if $f(a)f(b) > 0$ curve intersects x-axis even number of times.

***Sample* Problem 2** *The graph of the function* $f(x) = x + \dfrac{1}{x}$ *is*

(a)

(b)

(c)

(d) *None of these*

Interpret (a) The function is defined for all x except for $x = 0$.
It is an odd function for $x \neq 0$.
It is not a periodic function.
For $x \to 0 +$ $f(x) \to +\infty$, for $x \to 0 -$ $f(x) \to -\infty$
For $x \to -\infty$ $f(x) \to -\infty$, for $x \to \infty$ $f(x) \to \infty$

$$\therefore \quad \lim_{x \to \pm\infty} \{f(x) - x\} = 0$$

$\therefore$ The straight lines $x = 0$ and $y = x$ are the asymptotes of the graph of the given function.

Now, consider $f(x_2) - f(x_1)$ (for $x_2 > x_1$)

$$= (x_2 - x_1) + \frac{1}{x_2} - \frac{1}{x_1}$$

$$= (x_2 - x_1)\left[1 - \frac{1}{x_1 x_2}\right] < 0 \text{ for } x_1 x_2 \in (0, 1]$$

and it is > 0 for $x_1 x_2 \in [1, \infty)$.
Thus, $f(x)$ increases for $x \in [1, \infty)$ and decreases for $x \in (0, 1]$.
Thus, the least value of the function is at $x = 1$ which is $f(1) = 2$. Thus, its graph can be drawn as

***Sample* Problem 3** *The graph of the function* $f(x) = \dfrac{1}{1 + e^{1/x}}$ *is*

(a)

(b)

(c)

(d) *None of the above*

Interpret (c) The function is defined for all x except for $x = 0$. It is neither even nor an odd function. It is not a periodic function.
For $x \to 0 +$ $f(x) \to 0$ for $x \to 0 -$ $f(x) \to 1$
For $x \to \infty$ $f(x) \to \dfrac{1}{2}$ for $x \to -\infty$ $f(x) \to \dfrac{1}{2}$

$$\therefore \quad \lim_{x \to \pm\infty} f(x) = \frac{1}{2}$$

$\therefore$ The straight line $y = \dfrac{1}{2}$ is asymptote of the graph of the given function.

As x increases from $(0, \infty)$, $\frac{1}{x}$ decreases from $(0, \infty)$ and $e^{1/x}$ decreases from $(0, \infty)$. Thus, $(1 + e^{1/x})$ decreases from $(2, \infty)$.

$\therefore$ $f(x)$ in creases from $\left(0, \frac{1}{2}\right)$ for $x \in (0, \infty)$.

Similarly, f(x) increases from $\left(\frac{1}{2}, 1\right)$ for $x \to (-\infty, 0)$.

i.e., $f(x)$ is an increasing function except for $x = 0$. Thus, its graph can be drawn as shown in figure.

32.2 Sign Convention for Finding the Areas Using Integration

We can tactfully apply the concept of definite integration to find the area enclosed between the curves. But then, we must be very careful, while applying the discussed sign convention we will discuss the three cases.

***Case* I** If in the expression $\int_a^b f(x)dx$, if $b > a$ and $f(x) > 0$ for all $a \le x \le b$, then this integration will give the area enclosed between the curve $f(x)$, x-axis and the line $x = a$ and $x = b$ which is positive. No need of any modification.

***Case* II** If in the expression $\int_a^b f(x)dx$, if $b > a$ and $f(x) < 0$ for all $a \le x \le b$, then this integration will calculate to be negative. But the numerical or the absolute value is to be taken to mean the area enclosed between the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$.

***Case* III** If in the expression $\int_a^b f(x)dx$, where $b > a$ but $f(x)$ changes its sign a numbers of times in the interval $a \le x \le b$, then we must divide the region $[a, b]$ in such a way that we clearly get the points lying between $[a, b]$, where $f(x)$ changes its sign. For the region, where $f(x) > 0$ we just integrate to get the area in that region and then add the absolute value of the integration calculated in the region, where $f(x) < 0$ to get the desired area between the curve $y = f(x)$, x-axis and the line $x = a$ and $x = b$.

Hence, if $f(x)$ is as shown in figure, the area enclosed by $y = f(x)$; x-axis and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^c f(x)dx + \left|\int_c^d f(x)dx\right| + \int_d^e f(x)dx + \left|\int_e^f f(x)dx\right| + \int_f^b f(x)dx$$

***Sample* Problem 4** *The area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$ is* [**NCERT**]

(a) 1 *sq unit* (b) 2 *sq units* (c) 4 *sq units* (d) 8 *sq units*

Interpret (c) The graph of $y = \sin x$ can be drawn as

Required area = Area of $OABO$ + Area $BCDB$

$$= \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx$$

$$= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx$$

$(\because \sin x \ge 0$ for $x \in [0, \pi]$ and $\sin x \le 0$ for $x \in [\pi, 2\pi])$

$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$$

$$= -\cos\pi + \cos 0 + \cos 2\pi - \cos\pi$$

$$= -(-1) + 1 + 1 - (-1)$$

$$= 4 \text{ sq units}$$

Hot Spot 1

Area Between a Curve and Axis

Area between a curve and axis is the most important topic for JEE Main examination point of view and a small practice can help you in solving such problems very quickly as the level of questions is easy to average while solving problems based on area between a curve and axis, always keep in mind the concept of curve sketching.

(i) The area bounded by the curve $y = F(x)$ above the X-axis and between the lines $x = a$, $x = b$ is given by $\int_a^b y\,dx = \int_a^b F(x)dx$

(ii) If the curve between the lines $x = a$, $x = b$ lies below the X-axis, then the required area is given by

$$\left|\int_a^b (-y)dx\right| = \left|-\int_a^b y\,dx\right| = \left|-\int_a^b F(x)dx\right|$$

(iii) The area bounded by the curve $x = F(y)$ right to the Y-axis and the lines $y = c$, $y = d$ is given by

$$\int_c^d x\,dy = \int_c^d F(y)dy$$

(iv) If the curve between the lines $y = c$, $y = d$ left to the Y-axis, then the area is given by

$$\left|\int_c^d (-x)dy\right| = \left|-\int_c^d x\,dy\right| = \left|-\int_c^d F(y)dy\right|$$

When ever we solve this type of question. Generally, we follow the steps given below.

(i) First we sketch the given curve.

(ii) Now, we find the intersection of curve with axis and given line.

(iii) We select the bounded region in the figure and take interval between bounded region.

(iv) Now, we apply the appropriate formula to calculate the area of bounded region.

Sample Problem 5 *The area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis is* **[NCERT]**

(a) $\frac{14}{3}$ *sq units* (b) $\frac{7}{3}$ *sq units*

(c) $\frac{11}{3}$ *sq units* (d) *None of these*

Interpret (a) The area enclosed by the curve $y = f(x)$, the X-axis and the ordinates $x = a$, $x = b$ is given by $A = \int_a^b |y|\,dx$.

The area of the region bounded by the curve, $y^2 = x$, the lines $x = 1$ and $x = 4$ and the X-axis is shown in the figure.

Here, $a = 1$, $b = 4$

Since, the given curve $y^2 = x$ is parabola which is symmetrical about X-axis ($\because$ it contains even power of Y) and passes through the origin.

$\therefore$ Required area (shaded region)

$$= \int_a^b |y|\,dx = \int_1^4 |y|\,dx = \int_1^4 \sqrt{x}\,dx \quad (\because y^2 = x, \therefore |y| = \sqrt{x})$$

$$= \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4 = \frac{2}{3}[4^{3/2} - 1^{3/2}] \qquad \left(\because \int x^n = \frac{x^{n+1}}{n+1}\right)$$

$$= \frac{2}{3}[(2^2)^{3/2} - 1] = \frac{2}{3}[8 - 1] = \frac{14}{3} \text{ sq units}$$

Note We can solve these questions without drawing the curve.

Sample Problem 6 *The area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the Y-axis in the first quadrant is* **[NCERT]**

(a) $(4 - \sqrt{2})$ *sq units* (b) $(4 + \sqrt{2})$ *sq units*

(c) $\frac{8}{3}(4 - \sqrt{2})$ *sq units* (d) *None of these*

Interpret (c) To determine the required area, integrate x w.r.t. y and take $y = 2$ as lower limit and $y = 4$ as upper limit.

The given curve $x^2 = 4y$ is a parabola, which is symmetrical about Y-axis.

($\because$ it contains even power of x) only and passes through the origin.

The area of the region bounded by the curve $x^2 - 4y$, $y = 2$ and $y = 4$ and the Y-axis is shown in the figure.

Required area (shaded region)

$$= \int_{y=a}^{y=b} |x|\,dy$$

(here, $|x| = \sqrt{4y}$ and $a = 2$, $b = 4$) $= \int_2^4 |x|\,dy$

(considering the elementary strip on Y-axis) $= \int_2^4 2\sqrt{y}\,dy$

$(\because x^2 = 4y, \therefore |x| = 2\sqrt{y})$

$$= 2\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_2^4 = \frac{4}{3}[4^{3/2} - 2^{3/2}] = \frac{4}{3}[8 - 2\sqrt{2}] = \frac{8}{3}[4 - \sqrt{2}] \text{ sq units}$$

Hot Spot 2

Area Between the Given Curves

Area between the given curves is the most important topic of this chapter from JEE Main examination point of view. Generally, questions are seen from this topic. The level of question is from moderate to typical. But a small practice can help you in solving such problems very quickly.

While solving problems based on area between the given curves, always keep in mind.

(i) Concept of curve sketching (ii) Area between a curve and axis.

Area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $y = b$.

Let the curves $y = f(x)$ and $y = g(x)$ be represented by AB and CD, respectively. We assume that the two curves do not intersect each other in the interval $[a, b]$.

Thus, the shaded area

= Area of curvilinear trapezoid $APQB$

– Area of curvilinear trapezoid $CPQD$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$

Now, consider the case when $f(x)$ and $g(x)$ intersect each other in the interval $[a, b]$.

First of all we should find the intersection point of $y = f(x)$ and $y = g(x)$. For that we solve $f(x) = g(x)$. Let the root is $x = c$.

(we consider only one intersection point to illustrate the phenomenon).

Thus, the required (shaded) area

$$= \int_a^c \{f(x) - g(x)\}\,dx + \int_c^b \{g(x) - f(x)\}\,dx$$

If confusion arises in such case evaluate $\int_a^b |f(x) - g(x)|\,dx$ which gives the required area.

Note Area between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is always given by $\int_a^b \{f(x) - g(x)\}\,dx$ provided $f(x) > g(x)$ in $[a, b]$; the position of the graph is immaterial. As shown in figures.

Whenever, we solve such types of problems, generally we follow the steps given below.

(i) First, we sketch the given curves. (if necessary)

(ii) Now, we find the intersection points of the given curves by solving them.

(iii) Now, we select the bounded region in the figure of curves and take the interval between bounded region.

(iv) Now, we apply the appropriate formula to calculate the area of bounded region.

***Sample* Problem 7** *The area of the region in the first quadrant enclosed by x-axis and $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is* **[NCERT]**

(a) 0 *sq unit* (b) $\frac{3\pi}{4}$ *sq units* (c) $\frac{\pi}{3}$ *sq units* (d) $\frac{\pi}{2}$ *sq units*

Interpret (c) Firstly, find the intersection point of $x = \sqrt{3}$ and $x^2 + y^2 = 4$ in the first quadrant and then draw a rough diagram to indicate the required area. Now, divide the area into two parts one lying under the line joining point of intersection to (0, 0) and other lying under the circle and then integrate separately to find required area.

Given equation of circle is $x^2 + y^2 = 4$ and $x = \sqrt{3}y$ or $y = \frac{1}{\sqrt{3}}x$ represents a line through the origin.

The line $y = \frac{1}{\sqrt{3}}x$ intersect the circle, so it will satisfy the equation of circle

$$\therefore \quad x^2 + \left(\frac{1}{\sqrt{3}}x\right)^2 = 4 \Rightarrow \frac{4}{3}x^2 = 4$$

$$\Rightarrow \quad x^2 = \frac{4 \times 3}{4} = 3$$

$$\Rightarrow \quad x = \pm\sqrt{3}$$

When $x = \sqrt{3}$, then $y = \frac{1}{\sqrt{3}}\sqrt{3} = 1$

(for first quadrant we take $x = \sqrt{3}$ and neglect $x = -\sqrt{3}$)

$\therefore$ The line and the circle meet at the point $(\sqrt{3}, 1)$.

Required area (shaded region in first quadrant)

$= $ (Area under the line $y = \frac{1}{\sqrt{3}}x$ from $x = 0$ to $x = \sqrt{3}$)

+ (Area under the circle from $x = \sqrt{3}$ to $x = 2$)

$$= \int_0^{\sqrt{3}} \frac{1}{\sqrt{3}} x\,dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\,dx \qquad (\because x^2 + y^2 = 4 \Rightarrow y = \sqrt{4 - x^2})$$

$$= \frac{1}{\sqrt{3}} \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{2\sqrt{3}}[(\sqrt{3})^2 - 0^2] + \left[0 + 2\sin^{-1}(1) - \frac{\sqrt{3}}{2}\sqrt{4 - 3} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + 2\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{3}\right)$$

$$= \pi - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq units}$$

***Sample* Problem 8** *The area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$ is* **[NCERT]**

(a) $\frac{1}{6} + \frac{1}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ *sq unit*

(b) $\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ *sq unit*

(c) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\sin^{-1}\left(2 \cdot \sqrt{\frac{2}{3}}\right)$ *sq unit*

(d) *None of the above*

Draw a proper figure and find which area is required.

Find the points of intersection and then integrate with proper limits.

Interpret (b) Given, circle is $4x^2 + 4y^2 = 9$ and the given parabola is $x^2 = 4y$. The two curves meet where

$$4(4y) + 4y^2 = 9 \Rightarrow 4y^2 + 16y - 9 = 0$$

$$\Rightarrow y = \frac{-16 \pm \sqrt{256 + 144}}{2 \times 4} = \frac{-16 \pm \sqrt{400}}{8} = \frac{-16 \pm 20}{8} = \frac{1}{2}, -\frac{9}{2}$$

But $y > 0$, therefore the two curves meet when $y = \frac{1}{2}$.

i.e., when $x^2 = 4 \times \frac{1}{2} = 2$, *i.e.*, when $x = \pm\sqrt{2}$

$\therefore$ Required area (shown in shaded region)

$$= 2\int_0^{\sqrt{2}} (y_2 - y_1)\,dx = 2\left\{\int_0^{\sqrt{2}} \sqrt{\frac{9 - 4x^2}{4}}\,dx - \int_0^{\sqrt{2}} \frac{x^2}{4}\,dx\right\}$$

$$= 2\int_0^{\sqrt{2}} \sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx - \frac{2}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right)\right]_0^{\sqrt{2}} - \frac{1}{6}[(\sqrt{2})^3 - 0]$$

$$= \left[x\sqrt{\frac{9}{4} - x^2} + \frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\sqrt{2}} - \frac{2\sqrt{2}}{6}$$

$$= \left[\sqrt{2}\sqrt{\frac{9}{4} - 2} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] - \{0 - 0\} - \frac{2\sqrt{2}}{6}$$

$$= \frac{\sqrt{2}}{2} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{6} = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ sq unit}$$

***Sample* Problem 9** *The area bounded by the curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$ is* **[NCERT]**

(a) $\left(\frac{\pi}{3} - \frac{1}{\sqrt{3}}\right)$ *sq unit* (b) $\frac{1}{2}\left(\frac{\pi}{\sqrt{3}} - \frac{1}{2}\right)$ *sq unit*

(c) $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ *sq unit* (d) *None of these*

Interpret (c) Given, curve $(x - 1)^2 + y^2 = 1$ ...(i)

$\therefore$ $y = \sqrt{1 - (x - 1)^2}$

which represents a circle with centre (1, 0) and radius 1 and curve

$x^2 + y^2 = 1$ ...(ii)

$\therefore$ $y = \sqrt{1 - x^2}$

which represents a circle with centre (0, 0) and radius 1.

Both the curves are circle and meet where $(x - 1)^2 = x^2$ *i.e.*, where $2x = 1$ or $x = \frac{1}{2}$.

Required area (shown in shaded region)

$$= 2\left[\int_0^{1/2} y_1\,dx + \int_{1/2}^{1} y_2\,dx\right]$$

$$= 2\left\{\int_0^{1/2} \sqrt{1 - (x - 1)^2}\,dx + \int_{1/2}^{1} \sqrt{1 - x^2}\,dx\right\}$$

$$= 2\left[\frac{x - 1}{2}\sqrt{1 - (x - 1)^2} + \frac{1}{2}\sin^{-1}\frac{x - 1}{1}\right]_0^{1/2} + 2\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$$

$$= 2\left[\frac{\frac{1}{2}-1}{2}\sqrt{1-\frac{1}{4}} + \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right) - \left(\frac{-1}{2}\right)0 - \frac{1}{2}\sin^{-1}(-1)\right]$$

$$+ 2\left[0 + \frac{1}{2}\sin^{-1}(1) - \frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[-\frac{1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{6} + 0 + \frac{1}{2}\cdot\frac{\pi}{2}\right] + \frac{\pi}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq unit}$$

32.3 Variable Area, Greatest and Least Values

An Important Concept

If $y = f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at $x = a, x = b, y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum, when $c = \frac{a+b}{2}$.

Proof $A = \int_a^c \{f(c) - f(x)\}\,dx + \int_c^b \{f(x) - f(c)\}\,dx$

$$= f(c)(c-a) - \int_a^c \{f(x)\}\,dx + \int_c^b \{f(x)\}\,dx - f(c)(b-c)$$

$$= \{(c-a) - (b-c)\}\,f(c) + \int_c^b \{f(x)\}\,dx - \int_a^c \{f(x)\}\,dx$$

$$A = [2c - (a+b)]\,f(c) + \int_c^b \{f(x)\}\,dx - \int_a^c f(x)\}\,dx$$

Differentiate w.r.t. c, we get

$$\frac{dA}{dc} = [2c - (a+b)]\,f'(c) + 2f(c) + 0 - f(c) - [f(c) - 0]$$

For maximum and minimum $\frac{dA}{dc} = 0$

$$f'(c)\,[2c - (a+b)] = 0 \quad [\text{as } f'(c) \neq 0]$$

Hence, $c = \frac{a+b}{2}$

Also, $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$

and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$

Hence, A is minimum when $c = \frac{a+b}{2}$

***Sample* Problem 10** *If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$; $x = 2$ and the x-axis is minimum, then the value of a is*

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{5}{3}$ (d) *None of these*

Interpret (b) $f'(x) = x^2 - 2x = x(x-2) = 0$

[note that $f(x)$ is monotonic in (0, 2)]

Thus, for the minimum and $f(x)$ must cross the x-axis are $\frac{0+2}{2} = 1$.

Hence, $f(1) = \frac{1}{3} - 1 + a = 0 \Rightarrow a = \frac{2}{3}$

***Sample* Problem 11** *The value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least is*

(a) 1

(b) -1

(c) 0

(d) *None of the above*

Interpret (b) $f(x) = x^3 + 3x^2 + x + a$

$$f'(x) = 3x^2 + 6x + 1 = 0$$

$$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$$

Hence, $f(x)$ cuts the x-axis at $\frac{1}{2}\left[\left(-1 + \frac{\sqrt{6}}{3}\right) + \left(-1 - \frac{\sqrt{6}}{3}\right)\right] = -1$

$$f(-1) = -1 + 3 - 1 + a = 0$$

$$\Rightarrow \quad a = -1$$

Area of Important Curves

Curves	Point of intersection	Area of shaded region
(i) $f(x, y): ax^2 \le y \le mx$ $\therefore\ y = ax^2,\ y = mx$	$B(0, 0),\ A\left(\frac{m}{a}, \frac{m^2}{a}\right)$	Area $= \frac{1}{6}\frac{m^3}{a^2}$ sq unit
(ii) $f(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, $\frac{x}{a} + \frac{y}{b} \ge 1$ $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}$	$A\ (a, 0),\ B(0, b)$	Area $= ab\frac{(\pi - 2)}{4}$ sq unit
(iii) Area of parabola $y^2 = 4ax$ and its latusrectum $x = a$	$A\ (a, 2a),\ B(a, -2a)$	Area $= \frac{8}{3}a^2$ sq units
(iv) $f(x, y): y^2 = 4ax$ and $y = \lvert mx \rvert$	$O(0, 0),\ A\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$	Area $= \frac{8a^2}{3m^3}$ sq unit
(v) $f(x, y): x^2 + y^2 \le 2ax$ and $y^2 \ge ax$	$O(0, 0),\ A(a, a),\ B(a, -a)$	(i) For $x \ge 0,\ y \ge 0$ (area of Ist quadrant) Area $= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$ sq unit (ii) For $x \ge 0$ (for Ist and IVth quadrant) Area $= 2a^2\left(\frac{\pi}{4} - \frac{2}{3}\right) = a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$
(vi) $f(x, y): x^2 = 4ay$ $y^2 = 4bx$	$O(0, 0)$, $A(4a^{2/3}b^{1/3}, 4a^{1/3}b^{2/3})$	Area $= \frac{16}{3}(ab)$ sq units
(vii) Area bounded by $y = c^2x^2$, x-axis and the lines $y = a, y = b$	$O(0, 0)$, $A\left(\frac{\sqrt{c^2}}{a}, a\right)$, $B\left(\frac{\sqrt{c^2}}{b}, b\right)$	Area $= \frac{2(b^{3/2} - a^{3/2})}{3c}$ sq unit

Curves	Point of intersection	Area of shaded region	
(viii) $y = k\cos 3x, \forall\ 0 \le x \le \frac{\pi}{6}$	As $0 \le x \le \frac{\pi}{6}$ $\therefore 0 \le 3x \le \frac{\pi}{2}$	Area $= \frac{k}{3}$ sq unit	
(ix) Area bounded by $y^2 = 4a(x + a)$ $y^2 = 4b(b - x)$	$A(b - a, 2\sqrt{ab})$ $B(b - a, -2\sqrt{ab})$	Area $= \frac{8}{3}\sqrt{ab}\,(a + b)$ sq units	
(x) Common area bounded by the ellipses $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{1}{a^2b^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2b^2}$, $0 < a < b$	$x = y = \frac{1}{\sqrt{a^2 + b^2}}$ $x = y = -\frac{1}{\sqrt{a^2 + b^2}}$	Area = Area of $PQRS$ $= 4 \times$ Area of $OLQM$ $= \frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ sq unit	
(xi) If $\alpha, \beta > 0$, $\alpha > \beta$, the area between the hyperbola $xy = p^2$, the x-axis and the ordinates $x = \alpha$, $x = \beta$		Area $= p^2 \log\left(\frac{\alpha}{\beta}\right)$	

WORKED OUT
Examples

Example 1 *The area of the region bounded by the curves $y = 2^x$, $y = 2x - x^2$ and $x = 2$, is*

(a) $\dfrac{3}{\log 2}$ *sq unit* (b) $\dfrac{3}{\log 2} - \dfrac{4}{9}$ *sq unit*

(c) $\dfrac{3}{2} - \dfrac{\log 2}{9}$ (d) *None of these*

Solution (d) Required area

$$= \int_0^2 [2^x - (2x - x^2)]\, dx$$

$$= \int_0^2 (2^x - 2x + x^2)\, dx$$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2}$$

$$= \frac{3}{\log 2} - \frac{4}{3} \text{ sq unit}$$

Example 2 *The area of one curvilinear triangle formed by the curves $y = \sin x$, $y = \cos x$ and x-axis, is*

(a) 2 *sq units* (b) $(2 + \sqrt{2})$ *sq units*

(c) $(2 - \sqrt{2})$ *sq units* (d) *None of these*

Solution (c) Required area

$$= \int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$$

$$= -[\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$

$$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = (2 - \sqrt{2}) \text{ sq units}$$

Example 3 *The whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes is*

(a) $4\pi a^2$ *sq units* (b) $4a^2$ *sq units*

(c) $2a^2$ *sq units* (d) *None of these*

Solution (b) Given curve is $x^2y^2 = a^2(y^2 - x^2)$

which is symmetrical about both the axes. Its asymptotes are $x = \pm a$.

$\therefore$ Required area, $A = 4\int_0^a y\, dx$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\, dx$$

Put $x = a \sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$\therefore \quad A = 4a^2 \int_0^{\pi/2} \frac{\sin\theta}{a\sqrt{1 - \sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/2} \sin\theta\, d\theta = 4a^2 [-\cos\theta]_0^{\pi/2}$$

$$= 4a^2 \text{ sq units}$$

Example 4 *The area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptotes is*

(a) a^2 *sq units* (b) πa^2 *sq units*

(c) $4\pi a$ *sq units* (d) *None of these*

Solution (d) The given curve is symmetrical about x-axis and meet it at $(2a, 0)$. The line $x = 0$ *i.e.*, y-axis is asymptote.

$\therefore$ Required area, $A = 2\int_0^{2a} y\, dx$

$$= 2\int_0^{2a} 2a\sqrt{\frac{2a - x}{x}}\, dx$$

Put $x = 2a\sin^2\theta \Rightarrow dx = 4a\sin\theta\cos\theta\, d\theta$

$$\therefore \quad A = 16a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta = 4\pi a^2 \text{ sq units}$$

Example 5 *If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is least, then its least area*

(a) $\dfrac{64}{3}$ *sq units* (b) $\dfrac{32}{3}$ *sq units*

(c) $\dfrac{25}{3}$ *sq units* (d) *None of these*

Solution (*b*) x_1 and x_2 are the roots of the equation

$$x^2 + 2x - 3 = kx + 1$$

$$x^2 + (2 - k)x - 4 = 0$$

$$x_1 + x_2 = k - 2$$

$$x_1x_2 = -4$$

$$A = \int_{x_1}^{x_2} [(kx + 1) - (x^2 + 2x - 3)]\, dx$$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{x_1}^{x_2}$$

$$= \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}(x_2^3 - x_1^3) + 4(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\{(x_2 + x_1)^2 - x_1x_2\} + 4\right]$$

$$= \sqrt{(x_2 + x_1)^2 - 4x_1x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\{(k-2)^2 + 4\} + 4\right]$$

$$= \frac{\sqrt{(k-2)^2 + 16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{16}{3}\right]$$

$$= \frac{[(k-2)^2 + 16]^{3/2}}{6}$$

$\Rightarrow A$ is minimum, if $k = 2$

Hence, $A_{\min} = \dfrac{32}{3}$

Example 6 *The area of the region bounded by the x-axis and the curves defined by* $y = \tan x, -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$ *and*

$$y = \cot x, \frac{\pi}{6} \le x \le \frac{3\pi}{2} \text{ is}$$

(a) $\log \dfrac{3}{2}$ *sq units* (b) $\log 2$ *sq units*

(c) $\log 3$ *sq units* (d) *None of these*

Solution (*b*) Given curves are

$$y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$$

$$y = \cot x, \frac{\pi}{6} \le x \le \frac{3\pi}{2}$$

$\therefore$ Required area = Area $OABO$ + Area $BACB$

$$= \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$$

$$= [\log \sec x]_0^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/2}$$

$$= \log\sqrt{2} - \log 1 + \log 1 - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - \log\frac{1}{\sqrt{2}} = \log\sqrt{2} - (\log 1 - \log\sqrt{2}) = 2\log\sqrt{2}$$

$$= \log(\sqrt{2})^2 = \log 2 \text{ sq units}$$

Example 7 *The area bounded by* $y = x|\sin x|$ *and x-axis between* $x = 0$, $x = 2\pi$ *is*

(a) 2π *sq units* (b) 3π *sq units*

(c) π *sq units* (d) 4π *sq units*

Solution (*d*) $y = \begin{cases} x\sin x, \text{ if } \sin x \ge 0 \text{ i.e., } 0 \le x \le \pi \\ -x\sin x, \text{ if } \sin x < 0 \text{ i.e, } \pi < x \le 2\pi \end{cases}$

$\therefore$ Required area $= \int_0^{\pi} x\sin x\, dx + \int_{\pi}^{2\pi} (-x\sin x)\, dx$

$$= [x(-\cos x)]_0^{\pi} - \int_0^{\pi}(-\cos x)\, dx - [\{x - \cos x\}]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}(-\cos x)\, dx$$

$$= [\pi + \sin x]_0^{\pi} + [(2\pi + \pi) - \sin x]_{\pi}^{2\pi} = 4\pi \text{ sq units}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Area between a Curve and Axis

1. The area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the X-axis in the first quadrant is [NCERT]

(a) 16 sq units (b) $4\sqrt{2}$ sq units
(c) $4(4 - \sqrt{2})$ sq units (d) $4(4 + \sqrt{2})$ sq units

2. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, the value of a is [NCERT]

(a) $(2)^{2/3}$ (b) $\sqrt{2}$ (c) $(4)^{4/3}$ (d) $(4)^{2/3}$

3. The area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$ is [NCERT]

(a) $2\sqrt{3}$ sq units (b) $8\sqrt{3}$ sq units
(c) $4\sqrt{3}$ sq units (d) $3\sqrt{3}$ sq units

4. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is [NCERT]

(a) π units (b) $\frac{\pi}{2}$ units
(c) $\frac{\pi}{3}$ units (d) $\frac{\pi}{4}$ units

5. Sketch the graph of $y = |x + 3|$ and the value of $\int_{-6}^{0} |x + 3|\,dx$ is [NCERT]

(a) 9 sq units (b) $\frac{9}{2}$ sq units
(c) 3 sq units (d) 11 sq units

6. The area bounded by the curve $y = x|x|$, X-axis and the coordinates $x = -1$ and $x = 1$ is given by [NCERT]

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

7. Using integration, the area of the region bounded by the line $2y = 5x + 7$, X-axis and the lines $x = 2$ and $x = 8$ is [NCERT Exemplar]

(a) 96 sq units (b) 72 sq units
(c) 84 sq units (d) None of these

8. Draw a rough sketch of the curve $y = \sqrt{x - 1}$ in the interval $[1, 5]$. The area under the curve and between the lines $x = 1$ and $x = 5$ is [NCERT Exemplar]

(a) $\frac{4}{3}$ sq units (b) $\frac{8}{3}$ sq units
(c) $\frac{16}{3}$ sq units (d) None of these

9. The area of the region bounded by the curve $ay^2 = x^3$, the Y-axis and the lines $y = a$ and $y = 2a$ is [NCERT Exemplar]

(a) $\frac{3}{5}a^2(2 \cdot 2^{2/3} - 1)$ sq unit
(b) $\frac{2}{5}a(2^{2/3} - 1)$ sq unit
(c) $\frac{3}{5}a^2(2^{2/3} + 1)$ sq unit
(d) None of the above

10. The area bounded by the curve $x = 2 - y - y^2$ and Y-axis is

(a) $\frac{3}{2}$ sq units (b) $\frac{5}{2}$ sq units
(c) $\frac{9}{2}$ sq units (d) None of these

11. The area bounded by the curve $|x| + y = 1$ and axis of x is

(a) 1 sq unit (b) 2 sq units
(c) 8 sq units (d) None of these

12. The area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, X-axis and the lines $x = 3$, $x = 4$, is

(a) 3 sq units (b) $3 + 16 \log 2$ sq units
(c) $16 \log 2$ sq units (d) None of these

13. The area included between the curves $y = \frac{1}{x^2 + 1}$ and X-axis is

(a) $\frac{\pi}{2}$ sq units (b) π sq units
(c) 2π sq units (d) None of these

14. The area bounded by $x = 1, x = 2, xy = 1$ and X-axis is
(a) (log 2) sq unit (b) 2 sq units
(c) 1 sq unit (d) None of these

15. The area between the curve $y = 4 + 3x - x^2$ and X-axis is
(a) 125/6 sq units (b) 125/3 sq units
(c) 125/2 sq units (d) None of these

16. The area bounded by the curves $f(x) = ce^x (c > 0)$, the X-axis and the two ordinates $x = p$ and $x = q$, is proportional to
(a) $f(p)f(q)$ (b) $|f(p) - f(q)|$
(c) $f(p) + f(q)$ (d) $\sqrt{f(p)f(q)}$

17. If the area above X-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\frac{3}{\log 2}$, then the value of k is
(a) 1/2 (b) 1 (c) −1 (d) 2

18. The area bounded by $y = \sin^{-1} x, x = \frac{1}{\sqrt{2}}$ and X-axis is
(a) $\left(\frac{1}{\sqrt{2}} + 1\right)$ sq unit (b) $\left(1 - \frac{1}{\sqrt{2}}\right)$ sq unit
(c) $\frac{\pi}{4\sqrt{2}}$ sq unit (d) $\left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$ sq unit

19. The area bounded by $y = \tan^{-1} x, x = 1$ and X-axis is
(a) $\left(\frac{\pi}{4} + \log\sqrt{2}\right)$ sq unit
(b) $\left(\frac{\pi}{4} - \log\sqrt{2}\right)$ sq unit
(c) $\left(\frac{\pi}{4} - \log\sqrt{2} + 1\right)$ sq unit
(d) None of the above

20. The area bounded by the curve $y = \sin^2 x$ and lines $x = \frac{\pi}{2}, x = \pi$ and X-axis is
(a) $\frac{\pi}{2}$ sq unit (b) $\frac{\pi}{4}$ sq unit
(c) $\frac{\pi}{8}$ sq unit (d) None of these

21. The area bounded by the curve $y = \sec^2 x$, $y = 0$ and $|x| = \frac{\pi}{3}$ is
(a) $\sqrt{3}$ sq units (b) $\sqrt{2}$ sq units
(c) $2\sqrt{3}$ sq units (d) None of these

22. Area enclosed between the curve $y^2(2a - x) = x^3$ and line $x = 2a$ above X-axis is
(a) πa^2 sq units (b) $\frac{3\pi a^2}{2}$ sq units
(c) $2\pi a^2$ sq units (d) $3\pi a^2$ sq units

23. The area bounded by $y = e^{-x}$, X-axis and $x \geq 0$ is
(a) 1 (b) 2
(c) $\frac{1}{e}$ (d) e

24. The area bounded by $y = x^3 - 4x$ and X-axis is
(a) 5 (b) 9 (c) 8 (d) 12

25. The area bounded by the curve $y = \log_e x$, the X-axis and the straight line $x = e$ is
(a) e sq unit (b) 1 sq unit
(c) $1 - \frac{1}{e}$ sq unit (d) $1 + \frac{1}{e}$ sq unit

26. The ratio of the areas between the curves $y = \cos 2x$ and $y = \cos x$ and X-axis from $x = 0$ to $x = \frac{\pi}{3}$ is
(a) 1 : 2 (b) 2 : 1
(c) $\sqrt{3}$: 1 (d) None of these

Area between the Given Curves

27. The area of the smaller part of the circle $x^2 + y^2 = a^2$ cut-off by the line $x = \frac{a}{\sqrt{2}}$ is **[NCERT]**
(a) $\frac{a}{2}\left(\frac{\pi}{2} + 1\right)$ sq units (b) $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ sq units
(c) $a\left(\frac{\pi}{2} - 1\right)$ sq units (d) None of these

28. The area of the region bounded by the parabola $y = x^2$ and $y = |x|$ is **[NCERT]**
(a) $\frac{1}{2}$ sq unit (b) $\frac{1}{3}$ sq unit
(c) $\frac{2}{3}$ sq unit (d) None of these

29. The area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$ is **[NCERT]**
(a) $\frac{9}{8}$ sq units (b) $\frac{3}{8}$ sq unit
(c) $\frac{1}{8}$ sq unit (d) None of these

30. The area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$ is **[NCERT]**
(a) $\frac{1}{2}$ sq unit (b) $\frac{5}{2}$ sq units
(c) $\frac{7}{2}$ sq units (d) $\frac{21}{2}$ sq units

31. Using integration the area of region bounded by the triangle whose vertices are (−1, 0), (1, 3) and (3, 2) is **[NCERT]**
(a) 4 sq units (b) 6 sq units
(c) $\frac{5}{2}$ sq units (d) None of these

Round II (Mixed Bag)

Only One Correct Option

1. The larger of the area bounded by $y = \cos x$, $y = x + 1$ and $y = 0$ is

(a) $\frac{1}{2}$ sq unit (b) $\frac{3}{2}$ sq units
(c) 1 sq unit (d) 2 sq units

2. The area of the closed figure bounded by $y = 3 - |3 - x|$ and $y = \frac{6}{|x+1|}$ is

(a) $\frac{13}{2} + 6\log 2$ sq units (b) $\frac{9}{2} + 6\log 2$ sq units
(c) $\frac{13}{2} - 6\log 2$ sq units (d) $\frac{9}{2} - 6\log 2$ sq units

3. The region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$, then the area of this region, is

(a) $\frac{4}{3}$ sq units (b) $\frac{5}{3}$ sq units
(c) $\frac{3}{2}$ sq units (d) None of these

4. The area of the region $R = \{(x, y) : |x| \le |y|$ and $x^2 + y^2 \le 1\}$ is

(a) $\frac{3\pi}{8}$ sq units (b) $\frac{5\pi}{8}$ sq units
(c) $\frac{\pi}{2}$ sq units (d) $\frac{\pi}{8}$ sq unit

5. If $y = f(x)$ makes positive intercepts of 2 and 1 unit on x and Y-coordinates axes and encloses an area of $\frac{3}{4}$ sq unit with the axes, then $\int_0^2 xf'(x)dx$ is

(a) $\frac{3}{2}$ sq units (b) 1 sq unit
(c) $\frac{5}{4}$ sq units (d) $-\frac{3}{4}$ sq unit

6. The parabola $y^2 = 2x$ divides the circle $x^2 + y^2 = 8$ in two parts. Then, the ratio of the areas of these parts is

(a) $\frac{3\pi - 2}{10\pi + 2}$ (b) $\frac{3\pi + 2}{9\pi - 2}$
(c) $\frac{6\pi - 3}{11\pi - 5}$ (d) $\frac{2\pi - 9}{9\pi + 2}$

7. The area bounded by the curve $xy^2 = 4(2 - x)$ and Y-axis is

(a) 2π sq units
(b) 4π sq units
(c) 12π sq units
(d) 6π sq units

8. A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point $P(x, y)$ on the curve,lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinates area in $m : n$, then the value of $f(x)$ is equal to

(a) Cx^{m+n} (b) Cx^{m-n}
(c) $x^{m/n}$ (d) $Cx^{m/n}$

9. If area bounded by parabola $y^2 = x$ and straight line $2y = x$ is

(a) $\frac{4}{3}$ sq units (b) 1 sq unit
(c) $\frac{2}{3}$ sq unit (d) $\frac{1}{3}$ sq unit

10. The area of bounded region by the curve $y = \log_e x$ and $y = (\log_e x)^2$ is

(a) $3 - e$ (b) $e - 3$
(c) $\frac{1}{2}(3 - e)$ (d) $\frac{1}{2}(e - 3)$

11. The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$ is

(a) $\frac{\pi^3 - 8}{4}$ (b) $\frac{\pi^3}{4}$
(c) $\frac{\pi^3 - 16}{4}$ (d) $\frac{\pi^3 - 8}{2}$

12. The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axes is

(a) 9 sq units (b) 27/4 sq units
(c) 1 sq unit (d) 18 sq units

13. The area bounded by the curve $y = \log x$, $y = \log|x|$, $y = |\log x|$ and $y = |\log|x||$ is

(a) 4 sq units (b) 6 sq units
(c) 10 sq units (d) None of these

14. The sine and cosine curves intersects infinitely many times giving bounded regions of equal areas. The area of one of such region is

(a) $\sqrt{2}$ sq units (b) $2\sqrt{2}$ sq units
(c) $3\sqrt{2}$ sq units (d) $4\sqrt{2}$ sq units

15. If a curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area bounded by the curve, line $x = 4$ and X-axis is 8 sq units, then

(a) $a = 3, b = -1$
(b) $a = 3, b = 1$
(c) $a = -3, b = 1$
(d) $a = -3, b = -1$

16. Let y be the function which passes through (1, 2) having slope $(2x + 1)$. The area bounded between the curve and X-axis is

(a) 6 sq units (b) $\frac{5}{6}$ sq unit

(c) $\frac{1}{6}$ sq unit (d) None of these

17. The area between the curve $y = 2x^4 - x^2$, the X-axis and the ordinates of two minima of the curve is

(a) $\frac{7}{120}$ sq unit (b) $\frac{9}{120}$ sq unit

(c) $\frac{11}{120}$ sq unit (d) $\frac{13}{120}$ sq unit

18. The area of the region bounded by $1 - y^2 = |x|$ and $|x| + |y| = 1$ is

(a) 1/3 sq unit (b) 2/3 sq unit

(c) 4/3 sq unit (d) 1 sq unit

19. Area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ and x-axis lying between the ordinates $x = 0$ and $x = 3$ is equal to

(a) $\frac{9}{4}$ sq units (b) $\frac{11}{4}$ sq units

(c) $\frac{13}{4}$ sq units (d) $\frac{15}{4}$ sq units

20. If the ordinate $x = a$ divides the area bounded by X-axis part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates $x = 2$, $x = 4$ into two equal parts, then a is equal to

(a) $\sqrt{2}$ sq units (b) $2\sqrt{2}$ sq units

(c) $3\sqrt{2}$ sq units (d) None of these

21. Area bounded by the curve $y = x \sin x$ and X-axis between $x = 0$ and $x = 2\pi$ is

(a) 2π sq units (b) 3π sq units

(c) 4π sq units (d) 5π sq units

22. Let $f(x) = \min\{x + 1, \sqrt{(1 - x)}\}$, then area bounded by $f(x)$ and X-axis is

(a) $\frac{1}{6}$ sq unit (b) $\frac{5}{6}$ sq unit

(c) $\frac{7}{6}$ sq units (d) $\frac{11}{6}$ sq units

23. The area bounded by the graph $y = |[x - 3]|$, the X-axis and the lines $x = -2$ and $x = 3$ is ([·] denotes the greatest integer function)

(a) 7 sq units (b) 15 sq units

(c) 21 sq units (d) 28 sq units

24. The value of c for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight lines $x = 1$ and $x = c$ and the x-axis is equal to $\frac{16}{3}$ is

(a) 2 (b) $\sqrt{8 - \sqrt{17}}$

(c) 3 (d) -1

25. The slope of the tangent to a curve $y = f(x)$ at $\{x, f(x)\}$ is $2x + 1$. If the curve passes through the point (1, 2), then the area of the region bounded by the curve, the X-axis and the line $x = 1$ is

(a) $\frac{5}{6}$ sq unit (b) $\frac{6}{5}$ sq units

(c) $\frac{1}{6}$ sq unit (d) 6 sq units

26. The area of the region bounded by the curve $a^4y^2 = (2a - x)x^5$ is to that of the circle whose radius is a, is given by the ratio

(a) 4 : 5 (b) 5 : 8 (c) 2 : 3 (d) 3 : 2

27. The area bounded by $y = xe^{|x|}$ and lines $|x| = 1$, $y = 0$ is

(a) 4 sq units (b) 6 sq units

(c) 1 sq unit (d) 2 sq units

28. The area enclosed by the curves $|y + x| \le 1$, $|y - x| \le 1$ and $2x^2 + 2y^2 = 1$ is

(a) $\left(2 + \frac{\pi}{2}\right)$ sq units (b) $\left(2 - \frac{\pi}{2}\right)$ sq unit

(c) $\left(3 + \frac{\pi}{4}\right)$ sq units (d) $\left(3 - \frac{\pi}{4}\right)$ sq unit

29. The area bounded by the curve $[x] + [y] = 4$ in first quadrant is (where [·] denotes the greatest integer function)

(a) 3 sq units (b) 4 sq units

(c) 5 sq units (d) 6 sq units

30. The area enclosed by the curve $4 \le x^2 + y^2 \le 2(|x| + |y|)$ is

(a) 4 sq units (b) 6 sq units

(c) 8 sq units (d) 10 sq units

31. The area of the curve enclosed by the curve $|x + y| + |x - y| \le 4$, $|x| \le 1$, $y \ge \sqrt{x^2 - 2x + 1}$ is

(a) 1 sq unit (b) 4 sq units

(c) 2 sq units (d) 6 sq units

32. Area of the region bounded by the curve $y = 25^x + 1.6$ and curve $y = b \cdot 5^x + 4$ whose tangent at the point $x = 1$, makes an angle $\tan^{-1}(40 \log 5)$ with the X-axis is

(a) $2\log_5\left(\frac{e^4}{27}\right)$ (b) $4\log_5\left(\frac{e^4}{27}\right)$

(c) $3\log_5\left(\frac{e^4}{27}\right)$ (d) None of these

33. The area of the region bounded by the curves $y = ex \log x$ and $y = \frac{\log x}{ex}$ is

(a) $\frac{e^2 - 5}{4e}$ (b) $\frac{e^2 + 1}{2e}$

(c) $\frac{e^2}{2}$ (d) None of these

34. The area enclosed by circle $x^2 + y^2 = 4$, parabola $y = x^2 + x + 1$, the curve $y = \left[\sin^2 \frac{x}{4} + \cos \frac{x}{4}\right]$ and X-axis (where $[\cdot]$ is the greatest integer function) is

(a) $\left(\frac{2\pi}{3} - \frac{1}{\sqrt{3}} + \frac{1}{3}\right)$ sq unit (b) $\left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right)$ sq unit

(c) $\left(\frac{\pi}{3} + \frac{1}{\sqrt{3}} - \frac{1}{6}\right)$ sq unit (d) None of these

35. The area of the region satisfying and $\max\{|x|, |y|\} \le 2$ is

(a) $(14 + 2\log 2)$ sq units (b) $2\log 3$ sq units

(c) $\left(\frac{1}{4} + \log 3\right)$ sq unit (d) None of these

36. The area bounded by the curve $y = \int_{1/8}^{\sin^2 x} (\sin^{-1}\sqrt{t})\, dt + \int_{1/8}^{\cos^2 x} (\cos^{-1}\sqrt{t})\, dt$ $(0 \le x \le \pi/2)$ and the curve satisfying the differential equation $y(x + y^3)dx = x(y^3 - x)dy$ passing through, $(4, -2)$

(a) $\frac{1}{2}\left(\frac{3\pi}{16}\right)^2$ sq unit (b) $\frac{3}{8}\left(\frac{3\pi}{16}\right)^4$ sq unit

(c) $\frac{1}{8}\left(\frac{3\pi}{16}\right)^4$ sq unit (d) None of these

More Than One Correct Option

37. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded the lines $x = 4$, $y = 4$ and the coordinates axes. If S_1, S_2 and S_3 are respectively the areas of these parts numbered from top to bottom, then

(a) $\frac{S_3}{S_2} = \frac{1}{2}$ (b) $\frac{S_1}{S_2} = 1$

(c) $\frac{S_1}{S_2} = \frac{1}{2}$ (d) $\frac{S_2}{S_3} = 1$

38. The line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$ and $x = \frac{3}{2}$ and the curve $y = -x^2 + 4x + 1$. Then, the value m is equal to

(a) $\frac{13}{6}$ (b) > 2

(c) < 1 (d) 2

39. Area bounded by the line $y = x$, curve $y = f(x)$, $\{f(x) > x, \forall\, x > 1\}$ and the lines $x = 1$, $x = t$ is $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$ for all $t > 1$. Then, $f(x)$ is equal to

(a) $1 + x + \frac{x}{\sqrt{1 + x^2}}$ (b) $> 1 + x$

(c) $\frac{x}{\sqrt{1 + x^2}}$ (d) $1 + x$

40. Area bounded by curves $y = \frac{x^2}{4a}$ and $y = \frac{8ab}{x^2 + 4a^2}$ is

(a) $\frac{1}{3}(6\pi - 4)$, if $a = 1$ (b) $\frac{1}{3}(4\pi + 3)$, if $a = 1$

(c) $\frac{4}{3}(6\pi - 4)$, if $a = 2$ (d) $\frac{1}{3}(2\pi + 3)$, if $a = 1$

Assertion and Reason

Directions (Q. Nos. 41-44) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

41. **Statement I** Area enclosed by the curve $|x| + |y| = 2$ is 8 units.

Statement II $|x| + |y| = 2$ represents a square of side length $\sqrt{8}$ units.

42. **Statement I** The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is least, if $k = 0$.

Statement II The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is $\sqrt{k^2 + 20}$.

43. **Statement I** The area of the ellipse $2x^2 + 3y^2 = 6$ will be more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

Statement II The length of the semi-major axis of ellipse $2x^2 + 3y^2 = 6$ is more than the radius of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

44. Consider the curve $f(x) = y = e^{x^3}$

Statement I Area enclosed by the curve $f(x)$ between the lines $x = a$, $x = b$ and x-axis is $\int_a^b e^{x^3}\, dx$.

Statement II $f(x)$ is an increasing function.

Previous Years' Questions

45. The area (in square units) bounded by the curves $y=\sqrt{x}, 2y-x+3=0$, x-axis and lying in the first quadrant is **[JEE Main 2013]**

(a) 9 (b) 36

(c) 18 (d) $\frac{27}{4}$

46. The area bounded between the parabolas $x^2=\frac{y}{4}$ and $x^2=9y$ and the straight line $y=2$ is **[AIEEE 2012]**

(a) $20\sqrt{2}$ (b) $\frac{10\sqrt{2}}{3}$

(c) $\frac{20\sqrt{2}}{3}$ (d) $10\sqrt{2}$

47. The area bounded by the curves $y^2=4x$ and $x^2=4y$ is **[AIEEE 2011]**

(a) 0 (b) $\frac{32}{3}$

(c) $\frac{16}{3}$ (d) $\frac{8}{3}$

48. The area of the region enclosed by the curves $y=x$, $x=e$, $y=\frac{1}{x}$ and the positive X-axis is **[AIEEE 2011]**

(a) 1 sq unit (b) $\frac{3}{2}$ sq unit

(c) $\frac{5}{2}$ sq unit (d) $\frac{1}{2}$ sq unit

Answers

Round I

1. (c)	**2.** (d)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (c)
11. (a)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (b)
21. (c)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (b)	**29.** (a)	**30.** (d)
31. (a)									

Round II

1. (b)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (a)
11. (a)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (b)
21. (c)	**22.** (c)	**23.** (b)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (c)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (a)	**36.** (c)	**37.** (b,d)	**38.** (a,b)	**39.** (a,b)	**40.** (a,c)
41. (a)	**42.** (c)	**43.** (b)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (c)	**48.** (b)		

the Guidance

Round I

1. Since, the given curve $y^2 = 9x$ is a parabola which is symmetrical about X-axis ($\because$ the power to y is even) and passes through the origin).

The area of the region bounded by the curve, $y^2 = 9x$, $x = 2$ and $x = 4$ and the X-axis is the area shown in the figure.

Required area (shaded region)

$= \int_2^4 |y|\, dx = \int_2^4 3\sqrt{x}\, dx$ $\quad (\because y^2 = 9x \Rightarrow |y| = 3\sqrt{x})$

$= 3\left[\dfrac{x^{3/2}}{3/2}\right]_2^4 = \dfrac{3\times 2}{3}[4^{3/2} - 2^{3/2}] = 2\,[4\sqrt{4} - 2\sqrt{2}] = 2[8 - 2\sqrt{2}]$

$= 4[4 - \sqrt{2}]$ sq units

2. Given curve $x = y^2$ is a parabola symmetrical about X-axis and passing through the origin.

The line $x = a$, divides the area bounded by the parabola and $x = 4$ into two equal parts.

Area of OAD = Area of $ABCD$

$\therefore$ Area of OED = Area of $EFCD$

$\Rightarrow$ Area of $OED = \int_0^a y\, dx$ and area of $EFCD = \int_0^4 \sqrt{x}\, dx$

$(\because y^2 = x \Rightarrow |y| = \sqrt{x})$

$\Rightarrow \quad \int_0^a \sqrt{x}\, dx = \int_a^4 \sqrt{x}\, dx \Rightarrow \left[\dfrac{x^{3/2}}{3/2}\right]_0^a = \left[\dfrac{x^{3/2}}{3/2}\right]_a^4$

$\Rightarrow \quad \dfrac{2}{3}[a^{3/2} - 0] = \dfrac{2}{3}[4^{3/2} - a^{3/2}]$

$\Rightarrow \quad a^{3/2} = 4^{3/2} - a^{3/2} \Rightarrow 2a^{3/2} = 8$

$\Rightarrow \quad a^{3/2} = 4 \Rightarrow a = (4)^{2/3}$

Therefore, the value of a is $(4)^{2/3}$.

3. The given curve is $y^2 = 4x$, which represents a right hand parabola with vertex at (0, 0) and axis along X-axis and the equation of line is $x = 3$.

Required area

$= 2 \times$ (Area of shaded region in the first quadrant only)

$= 2\int_0^3 |y|\, dx = 2\int_0^3 2\sqrt{x}\, dx$ $\quad (\because y^2 = 4x \Rightarrow y = 2\sqrt{x})$

$= 4\left[\dfrac{x^{3/2}}{3/2}\right]_0^3 = \dfrac{8}{3}[3^{3/2} - 0] = \dfrac{8}{3}(3\sqrt{3}) = 8\sqrt{3}$ sq units

Therefore, the required area is $8\sqrt{3}$ sq units.

4. The area bounded by the circle and the lines $x = 0$ and $x = 2$, in the first quadrant is represented in the figure by shaded region.

Required area $= \int_0^2 |y|\, dx = \int_0^2 \sqrt{4 - x^2}\, dx$

$= \left[\dfrac{x}{2}\sqrt{4 - x^2} + \dfrac{4}{2}\sin^{-1}\left(\dfrac{x}{2}\right)\right]_0^2$

$= 0 + 2\sin^{-1}(1) - 0 = 2 \times \dfrac{\pi}{2} = \pi$ sq units

5. $y = |x + 3| = \begin{cases} -(x+3) \text{ for } x < -3 \\ x + 3 \text{ for } x \geq -3 \end{cases}$

When $x < -3$, $y = -x - 3$

x	-4	-5	-6
y	1	2	3

When $x \geq -3$

x	-1	-2	-3
y	2	1	0

Draw these points on the graph paper and we get the required figure.

$\therefore$ Required area

$= \text{Area of region } ABC + \text{Area of region } OAD$

$= \int_{-6}^{-3} |x+3|\, dx + \int_{-3}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} (-x-3)\, dx + \int_{-3}^{0} (x+3)\, dx$

$= \left[\frac{-x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$

$= \left[\left\{\frac{-(-3)^2}{2} - 3\times(-3)\right\} - \left\{\frac{-(-6)^2}{2} - 3\times(-6)\right\}\right]$

$+ \left[0 - \left\{\frac{(-3)^2}{2} + 3\times(-3)\right\}\right]$

$= \left[\left(\frac{-9}{2} + 9\right) - (-18+18)\right] + \left[\frac{9}{2}\right] = \frac{9}{2} = \frac{9}{2} = 9 \text{ sq units}$

6. Given, $y = x|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

Required area

$= 2$ [Area under the curve $y = x^2$, between $x = 0$, $x = 1$]

($\because$ the curve is symmetrical in opposite quadrant.)

$= 2\int_0^1 x|x| = 2\int_0^1 x^2\, dx$

$= 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}(1^3 - 0^3)$

$= \frac{2}{3}$ sq unit

7. Given equations of the curves are

$2y = 5x + 7$ ...(i)

$x = 2$ ...(ii)

and $x = 8$ ...(iii)

Eq. (i) represents the line passing through the point $\left(0, \frac{7}{2}\right)$ and $\left(-\frac{7}{5}, 0\right)$

Eq. (ii) represents the line parallel to Y-axis and passes through the point $(2, 0)$.

Eq. (iii) represents the line parallel to Y-axis passes through the point (8, 0).

Also, from Eq. (i), $y = \frac{5x+7}{2}$

$\therefore$ Required area (shown in shaded region)

$= \int_2^8 y\, dx = \int_2^8 \frac{5x+7}{2}\, dx$

$= \frac{1}{2}\left[\frac{5x^2}{2} + 7x\right]_2^8 = \frac{1}{2}\left[\left\{\frac{5(64)}{2} + 56\right\} - \left\{\frac{5(4)}{2} + 14\right\}\right]$

$= \frac{1}{2}[(160+56) - (10+14)] = \frac{1}{2}(216 - 24) = \frac{192}{2}$

$= 96$ sq units

8. Given equations of the curves are

$y = \sqrt{x-1}$ ...(i)

$x = 1$ ...(ii)

and $x = 5$ ...(iii)

Eq. (i) represents the upper portion of the parabola, whose vertex is (1, 0) and axis is X-axis.

Eq. (ii) represents the line parallel to Y-axis and passes through the point (1, 0).

Eq. (iii) represents the line parallel to Y-axis and passes through the point (5, 0).

$\therefore$ Required area (shown in shaded region)

$= \int_1^5 y\, dx = \int_1^5 \sqrt{x-1}\, dx = \left[\frac{2(x-1)^{3/2}}{3}\right]_1^5$

$= \frac{2}{3}[(5-1)^{3/2} - (1-1)^{3/2}] = \frac{2}{3}[(4)^{3/2}] = \frac{16}{3}$ sq units

9. We have,

Area $BMNC = \int_a^{2a} x\, dy\ a$

$= \int_a^{2a} a^{1/3}\, y^{2/3}\, dy$

$= \frac{3a^{1/3}}{5}\left|y^{5/3}\right|_a^{2a}$

$= \frac{3a^{\frac{1}{3}}}{5}\left|(2a)^{\frac{5}{3}} - a^{\frac{5}{3}}\right|$

$= \frac{3}{5}a^{\frac{1}{3}}a^{\frac{5}{3}}\left|(2)^{\frac{5}{3}} - 1\right| = \frac{3}{5}a^2\left|2\cdot 2^{\frac{2}{3}} - 1\right|$ sq unit

10. The required area $= \int_{-2}^{1} x\, dy$

$= \int_{-2}^{1} (2 - y - y^2)\, dy$

$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1} = \frac{9}{2}$ sq units

11. Given curve is $|x| + y = 1$

$\therefore$ Curve is $x + y = 1$, when $x \ge 0$

and $-x + y = 1$, when $x \le 0$

The graph of the curve is as given in the figure.

$\therefore$ Required area

$= \text{Area } CAOC + \text{Area } OABO$

$= \int_{-1}^{0} y\, dx + \int_0^1 y\, dx$

$= \int_{-1}^{0} (x+1)\, dx + \int_0^1 (1-x)\, dx$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \left[0 - \left(\frac{1}{2} - 1\right)\right] + \left[\left(1 - \frac{1}{2}\right) - 0\right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq unit}$$

12. The equation of the curve is

$$xy - 3x - 2y - 10 = 0 \Rightarrow y = \frac{3x + 10}{x - 2}$$

$\therefore$ Required area $= \int_3^4 \frac{3x + 10}{x - 2}\, dx = \int_3^4 \left\{3 + \frac{16}{(x - 2)}\right\} dx$

$$= [3x + 16 \log(x - 2)]_3^4$$

$$= 12 + 16 \log 2 - 9 - \log 1$$

$$= 3 + 16 \log 2 \text{ sq units}$$

13. Required area $= \int_{-\infty}^{\infty} \frac{1}{x^2 + 1}\, dx$

$$= 2\,[\tan^{-1} x]_0^{\infty} = \pi \text{ sq units}$$

14. Required area $= \int_1^2 \frac{1}{x}\, dx$

$$= [\log |x|]_1^2 = \log 2 \text{ sq unit}$$

15. Equation of curve are $y = 0$ ...(i)

and $y = 4 + 3x - x^2$ ...(ii)

On solving Eqs. (i) and (ii), we get $x = -1, 4$

$\therefore$ Curve does not intersect x-axis between $x = -1$ and $x = 4$.

$\therefore$ Required area $= \int_{-1}^{4} (4 + 3x - x^2)\, dx$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left[16 + 24 - \frac{64}{3} + 4 - \frac{3}{2} - \frac{1}{3}\right]$$

$$= 44 - \frac{65}{3} - \frac{3}{2}$$

$$= \frac{264 - 130 - 9}{6} = \frac{125}{6} \text{ sq units}$$

16. Required area $= \int_q^p ce^x\, dx$

$$= [ce^x]_q^p = c[e^p - e^q]$$

$$= f(p) - f(q)$$

17. Area bounded by curves $y = 2^{kx}$ and $x = 0$ and $x = 2$ is given by

$$A = \int_0^2 2^{kx}\, dx = \left[\frac{2^{kx}}{k \log 2}\right]_0^2 = \left[\frac{2^{2k} - 1}{k \log 2}\right]$$

But $A = \frac{3}{\log 2}$

$\therefore$ $\frac{2^{2k} - 1}{k \log 2} = \frac{3}{\log 2} \Rightarrow 2^{2k} - 1 = 3k$

This relation is satisfied by only option (b).

18. Required area

= Area of rectangle $OABC$

– Area of curve $OABO$

$$= \frac{\pi}{4\sqrt{2}} - \int_0^{\pi/4} \sin y\, dy$$

$$= \frac{\pi}{4\sqrt{2}} + [\cos y]_0^{\pi/4}$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \left\{\frac{1}{\sqrt{2}} - 1\right\}\right] \text{ sq units}$$

19. Required area

= Area of rectangle $OABC$ – Area of curve $OABO$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \tan y\, dy = \frac{\pi}{4} + [\log \cos y]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \log \cos \frac{\pi}{4} - \log \cos(0)$$

$$= \frac{\pi}{4} + \log 1 - \log \sqrt{2} - \log 1 = \left(\frac{\pi}{4} - \log \sqrt{2}\right) \text{ sq unit}$$

20. Required area, $A = \int_{\pi/2}^{\pi} \sin^2 x\, dx$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} (1 - \cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi} = \frac{\pi}{4} \text{ sq units}$$

21. Required area $= \int_{-\pi/3}^{\pi/3} \sec^2 x\, dx$

$= [\tan x]_{-\pi/3}^{\pi/3} = 2\sqrt{3}$ sq units

22. Given equation of curve is

$$y^2(2a - x) = x^3$$

which is symmetrical about x-axis and passes through origin.

Also, $\dfrac{x^3}{2a - x} < 0$

for $x > 2a$ or $x < 0$

So, curve does not lie in $x > 2a$ and $x < 0$, therefore curve lies wholly on $0 \le x \le 2a$

$\therefore$ Required area $= \int_0^{2a} \dfrac{x^{3/2}}{\sqrt{2a - x}}\, dx$

Put $x = 2\, a \sin^2 \theta$

$\Rightarrow$ $dx = 2a \cdot 2 \sin\theta \cos\theta\, d\theta$

$\therefore$ Required area $= \int_0^{\pi/2} 8a^2 \sin^4 \theta\, d\theta = 8a^2 \left[\dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}\right]$

(using gamma function)

$= \dfrac{3\pi a^2}{2}$ sq units

23. $A = \int_0^{\infty} e^{-x}\, dx = 1$

24. $y = x(x^2 - 2^2)$

$\Rightarrow$ $y = x(x - 2)(x + 2)$

$\therefore$ Required area $= \int_{-2}^{0} (x^3 - 4x)\, dx + \int_0^2 |x^3 - 4x|\, dx$

$= \left[\dfrac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\dfrac{x^4}{4} - 2x^2\right]_0^2 = 8$ sq units

25. Required area

$= \int_1^e \log_e x \cdot dx$

$= [x \log_e x - x]_1^e$

$= [e \log_e e - e - \log 1 + 1]$

$= [e - e + 1] = 1$ sq unit

26. $\dfrac{\int_0^{\pi/3} \cos x\, dx}{\int_0^{\pi/3} \cos 2x\, dx} = \dfrac{[\sin x]_0^{\pi/3}}{\left[\dfrac{\sin 2x}{2}\right]_0^{\pi/3}} = \dfrac{2}{1}$

27. Given line, $x = \dfrac{a}{\sqrt{2}}$ ...(i)

and circle, $x^2 + y^2 = a^2$ ...(ii)

Since, given line cuts the circle. So, put $x = \dfrac{a}{\sqrt{2}}$ in Eq. (ii), we get

$$\left(\frac{a}{\sqrt{2}}\right)^2 + y^2 = a^2$$

$\Rightarrow$ $y^2 = \dfrac{a^2}{1} - \dfrac{a^2}{2} = \dfrac{a^2}{2}$

$\Rightarrow$ $|y| = \dfrac{a}{\sqrt{2}} \Rightarrow y = \pm \dfrac{a}{\sqrt{2}}$

When $y = \dfrac{a}{\sqrt{2}}$, $x^2 = a^2 - \left(\dfrac{a^2}{2}\right)$

$\Rightarrow$ $x^2 = \dfrac{a^2}{2} \Rightarrow x = \pm \dfrac{a}{\sqrt{2}}$

$\therefore$ Intersection point in first quadrant is $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$.

Required area

= 2 (Area of shaded region in first quadrant only).

$= 2\int_{a/\sqrt{2}}^{a} |y|\, dx = 2\int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2}\, dx$

$= 2\left[\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}$

$\left[\because \int \sqrt{a^2 - x^2}\, dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)\right]$

$= 2\left[0 + \dfrac{a^2}{2}\sin^{-1}(1) - \dfrac{a}{2\sqrt{2}}\sqrt{a^2 - \dfrac{a^2}{2}} - \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)\right]$

$= 2\left[\dfrac{a^2}{2}\left(\dfrac{\pi}{2}\right) - \dfrac{a}{2\sqrt{2}} \cdot \dfrac{a}{\sqrt{2}} - \dfrac{a^2}{2}\left(\dfrac{\pi}{4}\right)\right]$

$= 2\left[\dfrac{a^2\pi}{4} - \dfrac{\pi a^2}{8} - \dfrac{a^2}{4}\right] = \left(\dfrac{a^2\pi}{4} - \dfrac{a^2}{2}\right) = \dfrac{a^2}{2}\left(\dfrac{\pi}{2} - 1\right)$ sq unit

Therefore, the area of smaller part of the circle, $x^2 + y^2 = a^2$, cut-off by the line $x = \dfrac{a}{\sqrt{2}}$ is $\dfrac{a^2}{2}\left(\dfrac{\pi}{2} - 1\right)$ sq units.

28. Firstly, draw the curve $y = x^2$ and line

$$y = |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

The common shaded region is the required area.

Given parabola $y = x^2$ which is symmetrical about Y-axis and passes through (0, 0) and the curve $y = |x|$

On putting $x = -x$, we get $y = |-x| = |x|$

$\therefore$ Curve $y = |x|$ is symmetrical about y-axis and passes through origin.

The area bounded by the parabola, $y = x^2$ and the line $y = |x|$ or $y = \pm x$ can be represented in the figure.

The point of intersection of parabola, $x^2 = y$ and line, $y = x$ in first quadrant is $A(1, 1)$.

The given area is symmetrical about Y-axis.

$\therefore$ Area $OACO$ = Area $ODBO$

Required area

$= 2$ (Area of shaded region in the first quadrant only)

$$= 2\int_0^1 (y_2 - y_1)\, dx = 2\int_0^1 (x - x^2)\, dx$$

($\because$ the curve $y = |x|$ lies above the curve $= x^2$ in $[0, 1]$ so, we take $y_2 = x$ and $y_1 = x^2$)

$$= 2\left[\int_0^1 x\, dx - \int_0^1 x^2\, dx\right] = 2\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right)$$

$$= 2\left\{\left(\frac{1}{2} - 0\right) - \left(\frac{1}{3} - 0\right)\right\} = \frac{1}{3} \text{ sq unit}$$

Therefore, required area is $\frac{1}{3}$ sq unit.

29. Find the points of intersection of line $x = 4y - 2$ and $x^2 = 4y$. Find the proper limits, the required area is given by area under the parabola with respect to X-axis subtracted from area under the line with respect to X-axis under the proper limits.

Given curve is $x^2 = 4y$ ...(i)

which represents an upward parabola with vertex (0, 0) and axis along Y-axis and the equation of straight line

$x = 4y - 2$ ...(ii)

For intersection point put the value of $4y$ from Eq. (i) in Eq. (ii), we get

$$x^2 = x + 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x - 2)(x + 1) = 0$$

$$\Rightarrow \quad x = 2, -1$$

When $x = 2$, then from Eq. (ii), we get

$$4y = 2 + 2$$

$$\Rightarrow \quad y = 1$$

When $x = -1$, then from Eq. (ii) we get

$$4y = 2 - 1 = 1 \Rightarrow y = \frac{1}{4}$$

$\therefore$ The line meets the parabola at the points $B\left(-1, \frac{1}{4}\right)$ and $A(2, 1)$.

Required area

= (Area under the line $x = 4y - 2$)

– (Area under the parabola $x^2 = 4y$)

$$= \int_{-1}^{2}\left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4}\, dx$$

[from Eq. (ii), $y = \frac{x+2}{4}$ and from Eq. (i), $y = \frac{x^2}{4}$]

$$= \frac{1}{4}\left\{\left[\frac{x^2}{2} + 2x\right]_{-1}^{2}\right. - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left\{\frac{2^2}{2} + 2 \times 2 - \left(\frac{1}{2} - 2\right)\right\}$$

$$- \frac{1}{12}[2^3 - (-1)^3]$$

$$= \frac{1}{4}\left(6 + \frac{3}{2}\right) - \frac{1}{12} \times 9 = \frac{15}{8} - \frac{3}{4} = \frac{9}{8} \text{ sq units}$$

Therefore, required area is $\frac{9}{8}$ sq units.

30. Given curve $y = x^2 + 2$ represents a parabola and it is symmetrical about Y-axis having vertex (0, 2).

The given region bounded by $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$, is represented by the shaded area.

The point of intersection of the curve $y = x^2 + 2$ and the line $x = 3$ is (3, 11).

Required area (shown in shaded region)

= Area $OABDO$ – Area $OCDO$

= [Area under $y = x^2 + 2$ between $x = 0$, $x = 3$]

– [Area under $y = x$ between $x = 0$, $x = 3$]

$$= \int_0^3 (x^2 + 2)\, dx - \int_0^3 x\, dx = \left[\frac{x^3}{3} + 2x\right]_0^3$$

$$- \left[\frac{x^2}{2}\right]_0^3 = \left[\frac{3^3}{3} + 6 - 0\right] - \left[\frac{3^2}{2} - 0\right]$$

$$= 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq units}$$

31. Firstly, draw the triangle on the axis and find the equations of three lines joining these three points. Now, find the area by integrating under proper limits.

Let the given points are $A = (-1, 0)$, $B(1, 3)$ and $C(3, 2)$. Equation of the line joining points (x_1, y_1) and (x_2, y_2) is

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Equation of line AB is

$$y - 0 = \frac{3-0}{1+1}(x+1) \Rightarrow y = \frac{3}{2}(x+1)$$

Equation of line BC is $y - 3 = \frac{2-3}{3-1}(x-1)$

$$\Rightarrow \quad y = -\frac{1}{2}x + \frac{1}{2} + 3$$

$$\Rightarrow \quad y = -\frac{1}{2}x + \frac{7}{2}$$

Equation of the line AC is

$$y - 0 = \frac{2-0}{3+1}(x+1) \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$$

Area of ΔABC

$$= \text{Area } AMB + \text{Area } BMNC - \text{Area } ANC$$

$$= \int_{-1}^{1} \frac{3}{2}(x+1)\,dx + \int_{1}^{3}\left(-\frac{1}{2}x + \frac{7}{2}\right)dx - \int_{-1}^{3}\left(\frac{1}{2}x + \frac{1}{2}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1} + \left[-\frac{x^2}{4} + \frac{7}{2}x\right]_{1}^{3} - \left[\frac{x^2}{4} + \frac{1}{2}x\right]_{-1}^{3}$$

$$= \frac{3}{2}\left[\frac{1}{2} + 1 - \frac{1}{2} - (-1)\right] + \left[-\frac{9}{4} + \frac{21}{2} + \frac{1}{4} - \frac{7}{2}\right] - \left[\frac{9}{4} + \frac{3}{2} - \frac{1}{4} + \frac{1}{2}\right]$$

$$= \frac{3}{2}[2] + [7-2] - [2+2] = 3 + 5 - 4 = 4 \text{ sq units}$$

Round II

1. $y = \cos x$ and $y = x + 1$ meet at the point $(0, 1)$ $y = x + 1$ passes through the points $(-1, 0)$ and $[0, 1]$ $y = \cos x$ meets X-axis at $\left(\frac{-p}{2}, 0\right)$ and $\left(\frac{p}{2}, 0\right)$.

$\therefore$ Required area $= \int_{-1}^{0}(x+1)dx + \int_{0}^{\pi/2}\cos x\,dx$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + [\sin x]_0^{\pi/2}$$

$$= 0 - \left(\frac{1}{2} - 1\right) + 1 = \frac{3}{2} \text{ sq units}$$

2. First consider $y = 3 - |3 - x|$

For $x < 3$, $y = 3 - (3 - x) = x$

For $x \geq 3$, $y = 3 + 3 - x = 6 - x$

Again, $\quad y = \frac{6}{|x+1|}$

For $x < -1$, $y = -\frac{6}{x+1} \Rightarrow (x+1)y = -6$

For $x > -1$, $y = -\frac{6}{x+1} \Rightarrow (x+1)y = -6$

$\therefore$ Required area

$$= \text{Area of shaded portion}$$

$$= \frac{1}{2} \times 6 \times 3 - \left(\int_{2}^{5}\frac{6}{x+1}dx + \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 1 \times 1\right)$$

$$= 9 - 6\log 2 - 2 - \frac{1}{2} = \frac{13}{2} - 6\log 2$$

3. Since, the inverse of a logarithmic function is an exponential function and *vice-versa* and these two curves are on the opposite sides of the line $y = x$. Thus, $y = 2^x$ and $y = \log x$ do not intersect. Their graphs are shown in figure. The shaded region in figure shows the area bounded by the given curves. Let us slice this region into vertical strips as shown in figure. For the approximating rectangle shown in figure, we have

$$\text{length} = (y_1 - y_2), \text{width} = \Delta x$$

$$\text{area} = y_1 - y_2)dx$$

As the approximating rectangle can move horizontally between $x = 1/2$ and 2.

$\therefore$ Required area

$$= \int_{1/2}^{2}(y_1 - y_2)dx$$

$$= \int_{1/2}^{2}(2^x - \log x)dx \quad \left[\begin{array}{l} \because P(x_1, y_1) \text{ and } Q(x_2, y_2) \text{ lie on} \\ y = 2^x \text{ and } y = \log x, \text{respectively.} \\ \therefore y_1 = 2^x \text{ and } y_2 = \log x \end{array}\right]$$

$$= \left[\frac{2^x}{\log 2} - x\log x + x\right]_{1/2}^{2}$$

$$= \left\{\frac{4}{\log 2} - 2\log 2 + 2\right\} - \left\{\frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log 2 + \frac{1}{2}\right\}$$

$$= \frac{(4-\sqrt{2})}{\log 2} - \frac{5}{2}\log 2 = \frac{3}{2} \text{ sq units}$$

4. Required area = Area of the shaded region

$$= 4 \text{ (Area of the shaded region in first quadrant)}$$

$$= 4\int_0^{1/\sqrt{2}} (y_1 - y_2)dx$$

$$= 4\int_0^{1/\sqrt{2}} (\sqrt{1-x^2} - x)\, dx$$

$$= 4\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \frac{x^2}{2}\right]_0^{1/\sqrt{2}}$$

$$= 4\left[\frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{4}\right] = \frac{\pi}{2} \text{ sq units}$$

5. Clearly, $y = f(x)$ passes through $(2, 0)$ and $(0, 1)$.

$\therefore$ $0 = f(2)$ and $1 = f(0)$

Also, $\int_0^2 f(x)\, dx = \frac{3}{4}$ (given)

Now, $\int_0^2 xf'(x)\, dx = [xf(x)]_0^2 - \int_0^2 f(x)\, dx$

$\Rightarrow$ $\int_0^2 xf'(x)\, dx = [2f(2) - 0f(0)] - \frac{3}{4}$

$\Rightarrow$ $\int_0^2 xf'(x)\, dx = 2 \times 0 - 0 \times 1 - \frac{3}{4} = -\frac{3}{4}$

6. Let the area of the smaller part of circle be A_1 and that of the bigger part be A_2. We have to find $\frac{A_1}{A_2}$.

The point B is a point of intersection (lying in the first quadrant) of the given parabola and the circle, whose coordinates can be obtained by solving the two equations $y^2 = 2x$ and $x^2 + y^2 = 8$.

$\Rightarrow$ $x^2 + 2x = 8$

$\Rightarrow$ $(x - 2)(x + 4) = 0$

$\Rightarrow$ $x = 2, -4$

$x = -4$ is not possible as both the points of intersection have the same positive x-coordinate.

Thus, $C \equiv (2, 0)$

Now, $A_1 = 2[\text{Area } (OBCO) + \text{Area } (CBAC)]$

$$= 2\left[\int_0^2 y_1 dx + \int_2^{2\sqrt{2}} y_2\, dx\right],$$

where y_1 and y_2 are respectively the values of y from the equations of the parabola and that of the circle.

or $A_1 = 2\left[\int_0^2 \sqrt{2x}\, dx + \int_2^{2\sqrt{2}} \sqrt{8 - x^2} dx\right]$

$$\Rightarrow A_1 = 2\left[\sqrt{2} \cdot \frac{2}{3} \cdot x^{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{8-x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_2^{2\sqrt{2}}$$

$$= \frac{16}{3} + 2\left[2\pi + \left(2 + 4 \times \frac{\pi}{4}\right)\right]$$

$$= \left(\frac{4}{3} + 2\pi\right) \text{ sq units}$$

Area of the circle $= \pi(2\sqrt{2})^2 = 8\pi$ sq units

Hence, $A_2 = 8\pi - A_1 = 6\pi - \frac{4}{3}$

Then, the required ratio

$$\frac{A_1}{A_2} = \frac{\frac{4}{3} + 2\pi}{6\pi - \frac{4}{3}} = \frac{2 + 3\pi}{9\pi - 2}$$

7. In the equation of curve $xy^2 = 4(2 - x)$, the degree of y is even. Therefore, the curve is symmetrical about Y-axis and lies in $0 < x \le 2$.

The bounded area by the curve is $2\int_0^2 y\, dx$

$$= 2\int_0^2 2\sqrt{\frac{2-x}{x}}\, dx = 4\int_0^2 \sqrt{\frac{2-x}{x}}\, dx$$

Put $x = 2\sin^2\theta$

$\Rightarrow$ $dx = 4\sin\theta \cdot \cos\theta d\theta$

$\therefore$ $A = 4\int_0^{\pi/2} \sqrt{\frac{2 - 2\sin^2\theta}{2\sin^2\theta}} \cdot 4\sin\theta \cdot \cos\theta d\theta$

$$= 8\int_0^{\pi/2} 2\cos^2\theta d\theta$$

$$= 8\int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= 8\left[\theta + \sin\frac{2\theta}{2}\right]_0^{\frac{\pi}{2}} = 8\left[\frac{\pi}{2} + 0 - 0\right] = 4\pi$$

8. Area $(OAPB) = xy$

Area $(OAPO) = \int_0^x f(t)dt$

Therefore, area

$(OBPO) = xy - \int_0^x f(t)dt$

According to the given condition,

$$\frac{xy - \int_0^x f(t)dt}{\int_0^x f(t)dt} = \frac{m}{n}$$

$\Rightarrow$ $$nxy = (m+n)\int_0^x t(t)dt$$

On differentiating w.r.t. x, we get

$$n\left(x\frac{dy}{dx} + y\right) = [m + nf(x)] = (m+n)y \qquad [\because y = f(x)]$$

$\Rightarrow$ $$\frac{m}{n}\cdot\frac{dx}{x} = \frac{dy}{y}$$

$\Rightarrow$ $\frac{m}{n}\cdot \ln x = \ln y - \ln C$, where C is a constant

$\Rightarrow$ $$y = Cx^{m/n} \Rightarrow f(x) = Cx^{m/n}$$

9. Given curves are $y^2 = x$ and $2y = x$

$\Rightarrow$ $$y^2 + 2y \Rightarrow y = 0, 2$$

$\therefore$ $$A = \left|\int_0^2 (y^2 - 2y)dy\right| = \left|\left[\frac{y^3}{3} - y^2\right]_0^2\right| = \frac{4}{3}\text{ sq units}$$

10. Area $A = \int_1^e [\log x - (\log x)^2]dx$

$$A = \int_1^e \log x\,dx - \int_1^e (\log x)^2 dx$$

$$= [x\log x - x]_1^e - [x(\log x)^2 - 2(x\log x - x)]_1^0$$

$$= [e - e - (-1)] - [e(1)^2 - 2e + 2e - (2)] = 1 - (e-2) = 3 - e$$

11. Area of the circle in 1st quadrant is

$$\frac{\pi(\pi^2)}{4} = \frac{\pi^3}{4}$$

Also, area of curve $y = \sin x$ and X-axis is 2 sq units.

$\therefore$ Required area, $A = \frac{\pi^3}{4} - 2 = \frac{\pi^3 - 8}{4}$

12. $A = \int_{1-e}^0 \log_e(x+e)dx$

$= \int_1^e \log t\,dt$ (put $x + e = t$)

$= [t\log t - t]_1^e = 1$ sq unit

13. $\log x$ is defined for $x > 0$

$|\log x| \geq 0$ and $|\log|x|| \geq 0$

$\therefore A = 4\int_0^1 |\log x|\,dx = -4\int_0^1 \log x\,dx$

$= -4[x\log x - x]_0^1$

$= -4(-1) = 4$ sq units

14.

Intersection of curves are

$$y = \sin x, y = \cos x \text{ are } \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

Since, $\sin x \geq \cos x$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx$$

$$= -[\cos x + \sin x]_{\pi/4}^{5\pi/4}$$

$$= 2\sqrt{2} \text{ sq units}$$

15. Since, $y = a\sqrt{x} + bx$ passes through (1, 2).

$\therefore$ $2 = a + b$ ...(i)

Area bounded by this curve and line $x = 4$ and X-axis is 8 sq units.

Then, $$\int_0^4 (a\sqrt{x} + bx)\,dx = 8$$

$\Rightarrow$ $$\frac{2a}{3}[x^{3/2}]_0^4 + \frac{b}{2}[x^2]_0^4 = 8$$

$\Rightarrow$ $$\frac{2a}{3}\cdot 8 + 8b = 8$$

$\Rightarrow$ $2a + 3b = 3$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$$a = 3$$

and $$b = -1$$

16. $\frac{dy}{dx} = 2x + 1$

$\Rightarrow$ $$y = x^2 + x + c$$

$\Rightarrow$ $y = x^2 + x$ $\quad$ ($c = 0$, put $x = 1, y = 2$)

$\therefore$ $$\left(x + \frac{1}{2}\right)^2 = y + \frac{1}{4}$$

which is equation of parabola having vertex $V\left(-\frac{1}{2}, -\frac{1}{4}\right)$.

$$A = \left|\int_{-1}^0 (x^2 + x)dx\right|$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^0 = \frac{1}{6}\text{ sq unit}$$

17. $\because y = 2x^4 - x^2 \therefore \frac{dy}{dx} = 8x^3 - 2x$

For maxima or minima, put $\frac{dy}{dx} = 0$, we get

$$x = -\frac{1}{2}, 0, \frac{1}{2}$$

Then, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}} > 0, \left(\frac{d^2y}{dx^2}\right)_{x=0} < 0$

$\therefore$ Required area $= \left|\int_{-1/2}^{1/2}(2x^4 - x^2)dx\right| = \frac{7}{120}$ sq unit

18. Since, $|x| + |y| = 1$

$$\Rightarrow \begin{cases} x+y=1, & x>0, \ y>0 \\ x-y=1, & x>0, \ y<0 \\ -x+y=1, & x<0, \ y>0 \\ -x-y=1, & x<0, \ y<0 \end{cases}$$

and $1 - y^2 = |x|$

$$\Rightarrow \begin{cases} 1-y^2 = x, & x \ge 0 \\ 1-y^2 = -x, & x < 0 \end{cases}$$

$\therefore$ Required area

$$= \left|2\int_0^1 \sqrt{(1-x)}\,dx\right| + \left|2\int_{-1}^0 \sqrt{(x+1)}\,dx\right| - 4\left(\frac{1}{2}\cdot 1\cdot 1\right)$$

$$= \frac{2}{3} \text{ sq unit}$$

19. Required area

$$= \left|\int_0^1 (x-1)(x-2)(x-3)\,dx\right| + \int_1^2 (x-1)(x-2)(x-3)\,dx + \left|\int_2^3 (x-1)(x-2)(x-3)dx\right|$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4} \text{ sq units}$$

20. Area $= \int_2^4 \left(1 + \frac{8}{x^2}\right)dx$

Since, the ordinate $x = a$ divides area into two equal parts, therefore

$$\int_2^a \left(1 + \frac{8}{x^2}\right)dx = \frac{1}{2}\int_2^4 \left(1 + \frac{8}{x^2}\right)dx$$

$$\Rightarrow \left[x - \frac{8}{x}\right]_2^a = \frac{1}{2}\left[x - \frac{8}{x}\right]_2^4$$

$$\Rightarrow \left(a - \frac{8}{a}\right) - (2-4) = \frac{1}{2}[(4-2) - (2-4)]$$

$$\Rightarrow a - \frac{8}{a} + 2 = 2$$

$$\Rightarrow a = \sqrt{8} = 2\sqrt{2} \text{ sq units}$$

21. Required area

$$= \int_0^\pi x\sin x\,dx + \left|\int_\pi^{2\pi} x\sin x\,dx\right|$$

$$= \{-x\cos x + \sin x\}_0^\pi + |\{-x\cos x + \sin x\}_\pi^{2\pi}|$$

$$= (\pi + 0) - (0 + 0) + |(-2\pi + 0) - (\pi + 0)|$$

$$= 4\pi + 3\pi = 4\pi \text{ sq units}$$

22. $f(x) = \min\{x+1, \sqrt{(1-x)}\} = \begin{cases} x+1, & -1 \le x < 0 \\ \sqrt{1-x}, & 0 < x \le 1 \end{cases}$

$\therefore$ Required area $= \left|\int_{-1}^0 (x+1)\,dx\right| + \left|\int_0^1 \sqrt{(1-x)}\,dx\right|$

$= 7/6$ sq units

23. Required area

$$= \int_{-2}^3 |[x-3]|\,dx$$

$$= \int_{-2}^{-1} |[x-3]|\,dx + \int_1^0 |[x-3]|\,dx + \int_0^1 |[x-3]|\,dx + \int_1^2 |[x-3]|\,dx + \int_2^3 |[x-3]\,dx$$

$$= \int_{-2}^{-1} 5\cdot dx + \int_{-1}^0 4\cdot dx + \int_0^{-1} 3\cdot dx + \int_1^2 2\cdot dx + \int_2^3 1\cdot dx$$

$$= 5(1) + 4(1) + 3(1) + 2(1) + 1(1) = 15 \text{ sq units}$$

24. For $c < 1$, $\int_c^1 (8x^2 - x^5)dx = \frac{16}{3}$

$$\Rightarrow \frac{8}{3} - \frac{1}{6} - \frac{8c^3}{3} + \frac{c^6}{6} = \frac{16}{3}$$

$$\Rightarrow c^3\left[-\frac{8}{3} + \frac{c^3}{6}\right] = \frac{16}{3} - \frac{8}{3} + \frac{12}{6} = \frac{17}{6}$$

$\Rightarrow c = -1$ satisfy the above equation.

For $c \ge 1$, none of the values of c satisfy the required condition that

$$\int_1^c (8x^2 - x^5)dx = \frac{16}{3}$$

25. We have, $\frac{dy}{dx} = 2x + 1$

$\Rightarrow$ $y = x^2 + x + c$, it passes through $(1, 2)$

$\therefore$ $c = 0$

Then, $y = x^2 + x$

$\therefore$ Required area $= \int_0^1 (x^2 + x)\,dx = \frac{5}{6}$ sq units

26. Given curve $a^4y^2 = (2a - x)x^5$

Cut-off X-axis, when $y = 0$

$$0 = (2a - x)x^5 \quad \therefore x = 0, 2a$$

Hence, the area bounded by the curve

$a^4y^2 = (2a - x)x^5$ is

$$A_1 = \int_0^{2a} \frac{\sqrt{(2a-x)}\,x^{5/2}}{a^2}\,dx$$

Put $x = 2a\sin^2\theta$

$\therefore \quad dx = 4a\sin\theta\cos\theta\, d\theta$

$$\therefore \quad A_1 = \int_0^{\pi/2} \frac{\sqrt{2a}\,\cos\theta\,(2a)^{5/2}\sin^5\theta\; 4a\sin\theta\cos\theta}{a^2}\, d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^6\theta\cos^2\theta\, d\theta$$

$$= 32a^2\cdot\frac{(5\cdot3\cdot1)(1)}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} \qquad \text{(by walli's formula)}$$

$$= \frac{5\pi a^2}{8}$$

Area of circle, $A_2 = \pi a^2 \therefore \; \dfrac{A_1}{A_2} = \dfrac{5}{8} \Rightarrow A_1 : A_2 = 5:8$

27. Since, $|x| = 1$

$\therefore \quad x = \pm 1$

$$y = xe^{|-x|} = \begin{cases} xe^{-x}, & -1 < x < 0 \\ xe^{x}, & 0 \le x < 1 \end{cases}$$

$\therefore$ Required area $= \left|\int_{-1}^{0} xe^{-x}\,dx\right| + \left|\int_0^1 xe^x\,dx\right|$

$= |\{-xe^{-x} - e^{-x}\}_{-1}^{0}| + |\{xe^x - e^x\}_0^1| = 2$ sq units

28. Area of the square $ABCD = 2$ sq units

Area of the circle $= \pi \times \dfrac{1}{2} = \dfrac{\pi}{2}$ sq units

Required area $= \left(2 - \dfrac{\pi}{2}\right)$ sq unit

29. 5 sq units.

30. Required area $= 4\left[\dfrac{(\pi\sqrt{2})^2}{2} - (\pi - 2)\right] = 8$ sq units

31. Required area $= \dfrac{1}{2}\times 2\times 2 = 2$ sq units

32. For $x = 1$, $y = b\cdot5^x + 4 = 5b + 4$ and $\dfrac{dy}{dx} = b\cdot5^x\log 5$

$\Rightarrow \quad 5b\log 5 = 40\log 5$

$\Rightarrow \quad b = 8$

The two curves intersect at points where

$$8\cdot5^x + 4 = 25^x + 16$$

$\Rightarrow \quad 5^{2x} - 8\cdot5 + 12 = 0 \Rightarrow x\log_5 2,\, x\log_5 6$

Hence, the area of the given region

$$= \int_{\log_5 2}^{\log_5 6}\{8\cdot5^x + 4 - (25x^x + 16)\}\,dx$$

$$= \int_{\log_5 2}^{\log_5 6}(8\cdot5^x - 25^x - 12)dx$$

$$= \left[\frac{8\cdot5^x}{\log_e 5} - 12x - \frac{25^x}{\log_e 25}\right]_{\log_5 2}^{\log_5 6}$$

$$= \frac{-5^{\log_5 36}}{\log_e 5} + \frac{8\cdot5^x}{\log_e 5} - 12(\log_5 6 - \log_5 2) + \frac{5^{\log_5 4}}{\log_e 25} - \frac{8\cdot5^{\log_5 2}}{\log_e 5}$$

$$= \frac{36}{2\log_e 5} + \frac{48}{\log_e 5} - 12[\log_5 3] + \frac{4}{2(\log_e 5)} - \frac{16}{(\log_e 5)}$$

$$= \frac{16}{\log_e 5} - 12\log_5 3 = 4\log_5 e^4 - 4\log_5 27 = 4\log_5\left[\frac{e^4}{27}\right]$$

33. Both the curves are defined for $x > 0$. Both are positive when $x > 1$ and negative when $0 < x < 1$.

We know that, $\lim\limits_{x\to0^+}\log x \to -\infty$

Hence, $\quad \lim\limits_{x\to0^-}\dfrac{\log x}{ex} \to -\infty.$

Thus, y-axis is asymptote of second curve.

and $\lim\limits_{x\to0^+} ex\log x\{(0)(-\infty)$ form$\}$

$$= \lim_{x\to0^+}\frac{e\log x}{1/x}\left(-\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to0^+}\frac{e(1/x)}{(-1/x^2)} = 0 \text{(using L' Hospital's rule)}$$

Thus, the first curve starts from (0, 0) but does not include (0, 0).

Now, the given curves intersect, where

$$ex\log x = \frac{\log x}{ex}$$

i.e., $\quad (e^2x^2 - 1)\log x = 0$

i.e., $\quad x = 1, \dfrac{1}{e} \qquad$ (since $x > 0$)

Hence, using above results figure could be plotted as

$\therefore$ The required area $= \int_{1/e}^{1}\left(\frac{\log x}{ex} - ex\log x\right)dx$

$$= \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1} - e\left[\frac{x^2}{4}(2\log x - 1)\right]_{1/e}^{1}$$

$$= \frac{e^2 - 5}{4e}$$

34. $\because y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$

$\because \quad 1 < \sin^2\frac{x}{4} + \cos\frac{x}{4}$

For $x \in (-2, 2)$

$\therefore \quad y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right] = 1$

Now, we have to find out the area enclosed by the circle $x^2 + y^2 = 4$, parabola $\left(y - \frac{3}{4}\right) = \left(x + \frac{1}{2}\right)^2$, line $y = 1$ and X-axis. Required area is shaded area in the figure.

Hence required area,

$$= \sqrt{3} \times 1 + (\sqrt{3} - 1) \times 1 + \int_{-1}^{0}(x^2 + x + 1)dx + 2\int_{\sqrt{3}}^{2}(\sqrt{4 - x^2})dx$$

$$= (2\sqrt{3} - 1) + \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} + 2\left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= 2\sqrt{3} - 1) + \left[0 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)\right] + 2\left[(0 + \pi) - \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)\right]$$

$$= (2\sqrt{3} - 1) + \frac{5}{6} + \frac{2\pi}{3} - \sqrt{3} = \left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right) \text{ sq units}$$

35. Consider the curve

$$|y| + \frac{1}{2} = e^{-|x|} \Rightarrow |y| = e^{-|x|} - \frac{1}{2}$$

This curve is symmetrical about both the X and Y-axes.

Also, for $x = 0$ gives $|y| = \frac{1}{2}$ and $|y| = 0$ for $x = \ln 2$

Also, $x > \ln 2. |y| < 0$ is not possible.

Also, max $\{|x|, |y|\} \le 2$ is the interior of the square with vertices $(2, 2), (2, -2), (-2, 2)$ and $(-2, -2)$.

The region $|y| + \frac{1}{2} \ge e^{-|x|}$ and max $\{|x|, |y|\} \le 2$ is given above (as shaded portion).

$\therefore$ The required area $= 4(4 - \text{ar } OABO)$

$$\text{Area } OABO = \int_0^{\ln 2}\left(e^{-x} - \frac{1}{2}\right)dx = \frac{1}{2}(1 - \ln 2)$$

$\therefore$ Required area $= 4\left(\frac{7}{2} + \frac{\ln 2}{2}\right) = (14 + 2\ln 2)$ sq units

36. $y(x + y^3)dx = x(y^3 - x)dy$

$\Rightarrow \quad xydx + y^2dx = xy^3dy - x^2dy$

$\Rightarrow \quad xd(xy) = x^2y^3\left(\frac{1}{x}dy - \frac{y}{x^2}dx\right)$

$\Rightarrow \quad \frac{d(xy)}{(xy)^2} = \frac{y}{x}d\left(\frac{y}{x}\right)$

$\Rightarrow \quad -\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + c$

At $\quad x = 4, y = -2$

Hence, $\quad \frac{1}{8} = \frac{1}{2}\left(-\frac{1}{2}\right)^2 + c$

$\Rightarrow \quad c = 0$

Hence, $\quad y^3 + 2x = 0$

So, $\quad f(x) = (-2x)^{1/3}$

The second equation given is

$$y = \int_{1/8}^{\sin^2 x}\sin^{-1}\sqrt{t}dt + \int_{1/8}^{\cos^2 x}\cos^{-1}\sqrt{t}dt$$

$\Rightarrow \quad y' = x \cdot 2\sin x\cos x + x \cdot 2\cos x(-\sin x) = 0$

Hence, y is constant.

Put $\sin x = \cos x = \frac{1}{\sqrt{2}}$

Hence, $\quad y = \int_{1/8}^{1/2}(\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})dt$

$$= \int_{1/8}^{1/2}\left(\frac{\pi}{2}\right)dt = \frac{\pi}{2} \cdot \frac{3}{8} = \frac{3\pi}{16},$$

So, $\quad g(x) = \frac{3\pi}{16}$

So, we must find the area between $y = f(x), y = \frac{3\pi}{16}$

At $\quad y = \frac{3\pi}{16}; x = -\frac{1}{2}\left(\frac{3\pi}{16}\right)^3 = P$ say)

Hence, required area $= \int_P^0\left(\frac{3\pi}{16} + (2x)^{1/3}\right)dx$

$$= \left(\frac{3\pi}{16}x + 2^{1/3} \cdot \frac{x^{4/3}}{4/3}\right)_P^0 = \frac{1}{8}\left(\frac{3\pi}{16}\right)^4$$

37. Area $S_1 = \frac{1}{4}\int_0^4 y^2 - dy = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3}$ sq units

Area $S_3 = \frac{1}{4}\int_0^4 x^2 \cdot dx = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3}$ sq units

Area of square $(ABCD) = 4 \times 16$ sq units

$\Rightarrow \quad S_2 = 16 - S_1 - S_3 = 16 - \frac{16}{3} = \frac{16}{3}$ sq units

$\Rightarrow \quad \frac{S_1}{S_2} = \frac{S_2}{S_3} = 1$

38. By the given condition,

Area $OLMP = 2$ area of ΔOQL

$= \int_0^{3/2} (1 + 4x - x^2)\, dx = 2\left(\frac{1}{2} \cdot \frac{3}{2} \cdot QL\right)$

$= \left[x + 2x^2 - \frac{x^3}{3}\right]_0^{3/2} = 2\left(\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3m}{2}\right)$

$\Rightarrow \quad \frac{3}{2} + \frac{9}{2} - \frac{24}{24} = 2\left(\frac{9}{8}m\right)$

$\Rightarrow \quad \frac{39}{8} = \frac{9}{4}m \Rightarrow m = \frac{39}{9 \times 2} = \frac{13}{6}$

$\Rightarrow \quad m = \frac{39}{9 \times 2} = \frac{13}{6}$

39. The area bounded by $y = f(x)$ and $y = x$ between the lines $x = 1$ and $x = t$ is

$$\int_1^t (f(x) - x)\, dx$$

But it is equal to $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$

$\therefore \quad \int_1^t \{f(x) - x\} dx = (t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$

On differentiating both sides w.r.t. t, we get

$$f(x) - t = 1 + \frac{t}{\sqrt{1 + t^2}} \Rightarrow f(t) = 1 + t + \frac{t}{\sqrt{1 + t^2}}$$

$\Rightarrow \quad f(x) = 1 + x + \frac{x}{\sqrt{1 + x^2}}$

40. The curve $y(x^2 + 4a^2) = 8a^3$ is symmetrical about Y-axis and cut it at $A(0, 2a)$.

Tangent at A is parallel to X-axis and X-axis is asymptote. This curve meets $x^2 = 4ay$, where $\frac{x^2}{4a} = \frac{8a^3}{x^2 + 4a^2}$

$\Rightarrow \quad x^4 + 4a^2x^2 - 32a^4 = 0$

$\Rightarrow \quad (x^2 - 4a^2)(x^2 + 8a^2) = 0 \Rightarrow x = \pm 2a$

$\therefore$ Required area $= 2\left[\int_0^{2a} \frac{8a^3}{x^2 + 4a^2} dx - \int_0^{2a} \frac{x^2}{4a} dx\right]$

$= 16a^3 \cdot \frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a} - \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$

$= 8a^2 \cdot \frac{\pi}{4} - \frac{1}{6a} \cdot 8a^3$

$= 2\pi a^2 - \frac{4}{3}a^2 = \frac{a^2}{3}(6\pi - 4)$

41. $AB = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\therefore$ Required area $= (\sqrt{8})^2 = 8$

42. Let the lie $y = kx + 2$ cuts $y = x^2 - 3$ at $x = \alpha$ and $x = \beta$ (where α and β are the roots of the equation $x^2 - kx - 5 = 0$, $\alpha < \beta$)

Then, area bounded by these two curves

$f(x) = \int_\alpha^\beta (y_1 - y_2)\, dx$

$= \int_\alpha^\beta (kx + 2) - (x^2 - 3)\, dx = \left[\frac{kx^2}{2} + 2x - \frac{x^3}{3} + 3x\right]_\alpha^\beta$

$= \left(\frac{k\beta^2}{2} + 2\beta - \frac{\beta^3}{3} + 3\beta\right) = \left(\frac{k\alpha^2}{2} + 2\alpha - \frac{\alpha^3}{3} + 3\alpha\right)$

$= \frac{k}{2}(\beta^2 - \alpha^2) + 2(\beta - \alpha) - \frac{1}{3}(\beta^3 - \alpha^3) + 3(\beta - \alpha)$

$= (\beta - \alpha)\left[\frac{k}{2}(\beta + \alpha) + 2 - \frac{1}{3}(\beta^2 + \alpha^2 + \alpha\beta) + 3\right]$

Now, as α, β are the roots of the equation

$$x^2 - kx - 5 = 0,\ \alpha + \beta = -k;\ \alpha\beta = -5$$

where, $\quad |\beta - \alpha| = \sqrt{k^2 + 20}$

Thus, $f(k)$ can be easily combuted, we will get

$$f(k) = \frac{(k^2 + 20)^{3/2}}{6}$$

This show that a statement II is false but $f(k)$ is least when $dk = 0$.

$\Rightarrow$ A statement I is true.

43. The ellipse is $\frac{x^2}{3} + \frac{y^2}{2} = 1$ and the circle is $(x - 1)^2 + (y + 2)^2 = 1$

$\Rightarrow$ Area of ellipse $= \pi\sqrt{3}\sqrt{2} = \sqrt{6}\pi$ and area of circle $= \pi.1^2 = \pi$

$\Rightarrow$ A statement I is true

The statement II is true in this particular example in general may not be true.

44. Since, $y = e^{x^3}$ $\therefore \dfrac{dy}{dx} = e^{x^3} \cdot 3x^2 > 0$

Hence, y is an increasing function and area enclosed by the curve $y = e^{x^3}$ between the lines $x = a$, $x = b$ and X-axis is $\int_a^b e^{x^3} dx$

45. Given curves are

$$y = \sqrt{x} \qquad \text{...(i)}$$

and $$2y - x + 3 = 0 \qquad \text{...(ii)}$$

On solving Eqs.(i) and (ii), we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$$\Rightarrow \quad (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow \quad (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$$

$$\Rightarrow \quad \sqrt{x} = 3 \qquad (\because \sqrt{x} = -1 \text{ is not possible})$$

$$\therefore \quad y = 3$$

$\therefore$ Required area $= \int_0^3 (x_2 - x_1)\, dy = \int_0^3 \{(2y + 3) - y^2\}\, dy$

$$= \left[y^2 + 3y - \frac{y^3}{3} \right]_0^3 = 9 + 9 - 9 = 9$$

46.

$\therefore$ Required area $= 2\int_0^2 \left(3\sqrt{y} - \dfrac{\sqrt{y}}{2}\right) dy = 2\int_0^2 \left(\dfrac{5}{2}\sqrt{y}\right) dy$

$$= 5\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_{y=0}^{y=2} = \frac{10}{3}(2^{3/2} - 0) = \frac{20\sqrt{2}}{3}$$

47. For the point of intersection for $y^2 = 4x$ and $x^2 = 4y$

$$\Rightarrow \quad \left(\frac{x^2}{4}\right)^2 = 4x$$

$$\Rightarrow \quad x^4 = 4^3 x \Rightarrow x = 0, 4$$

$\therefore$ Area bounded between curves

$$= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4 = \frac{4}{3} \cdot (4)^{3/2} - \frac{(4)^3}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

48. Given, $y = x$, $x = e$ and $y = \dfrac{1}{x}$, $x \geq 0$

Since, $y = x$ and $x \geq 0 \Rightarrow y \geq 0$

$\therefore$ Area to be calculated in Ist quadrant shown in figure.

$\therefore$ Area = Area of ΔODA + Area of $DABCD$

$$= \frac{1}{2}(1 \times 1) + \int_1^e \frac{1}{x}\, dx = \frac{1}{2} + [\log|x|]_1^e$$

$$= \frac{1}{2} + [\log|x|]_1^e = \frac{1}{2} + 1 = \frac{3}{2} \text{ sq units}$$

33 Differential Equations

JEE Main MILESTONE

- Differential Equation
- Order and Degree of a Differential Equation
- Application of First Order Differential Equation
- Orthogonal Trajectory

33.1 Differential Equation

An equation involving independent variable, dependent variable, derivatives of dependent variable with respect to independent variable and constant, is called a differential equation.

e.g., $x\dfrac{dy}{dx} + xy\dfrac{d^2y}{dx^2} + 4 = 0, \dfrac{dy}{dx} + y\cos x = \sin x$ are differential equations.

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential Equations play a prominent role in Engineering, Physics, Economics and other disciplines.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and or time (expressed as derivatives) is known or postulated.

Ordinary and Partial Differential Equations

(i) Ordinary

If the unknown function depends on only one argument (independent variable), then the differential equation is called an ordinary differential equation.

e.g.,

(i) $\dfrac{dy}{dx} = x\ln x$ (ii) $dy = \cos x\, dx$

(iii) $y = x\dfrac{dy}{dx} + a$ (iv) $\left[1 + \dfrac{dy}{dx}\right]^4 = b\left(\dfrac{d^2y}{dx^2}\right)^2$

(ii) Partial

A differential equation is said to be partial, if there are two or more independent variable occurring in it.

e.g., (i) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = kz$ (ii) $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$

33.2 Order and Degree of a Differential Equation

The order of highest differential coefficiant (or highest order derivative) appearing in a differential equation is the order of differential equation, *e.g.*, Differential equation $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + xy\frac{d^2y}{dx^2} + 4 = 0$ is of order. 3. The differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is of order 2.

The highest exponent of the highest derivative is called degree of a differential equation provided exponent of each derivative and the unknown variable appearing in the differential equation is a non-negative integer, *e.g.*,

(i) $\frac{d^2y}{dx^2} + \sqrt{\frac{dy}{dx}} + x = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{d^2y}{dx^2} + x\right)^2$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) + 2x\frac{d^2y}{dx^2} + x^2 = 0$$

Here, degree is 2.

(ii) $\left(\frac{d^3y}{dx^3}\right)^{2/3} + x + y = 0$

$$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)^2 = (-x - y)^3$$

So, degree is 2.

***Sample* Problem 1** *The degree of the differential equation* $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ *is* **[NCERT]**

(*a*) 3
(*b*) 2
(*c*) 1
(*d*) *None of the above*

Interpret (*d*) The given differential equation is not a polynomial equation in $\frac{dy}{dx}$. Therefore, its degree is not defined.

***Sample* Problem 2** *The order of the differential equation* **[NCERT]**

$$2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is}$$

(*a*) 2 (*b*) 1
(*c*) *zero* (*d*) 3

Interpret (*a*) The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

Linear and Non-linear Differential Equation

A differential equation is said to be linear, if the dependent variable say y and all its derivatives occur in the first degree, otherwise it is non-linear.

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n y = 0$$

where, $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are either constants or functions of independent variable x.

If a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Else it is non-linear.

Important

A differential equation will be non-linear differential equation, if
(a) its degree is more than one.
(b) any of the differential coefficient has exponent more than one.
(c) exponent of dependent variable is more than unity.
(d) products containing dependent variable and its differential coefficients are present.

Hot Spot 1

Formation of a Differential Equation

Formation of a differential equation is an important topic for JEE Main examination point of views and a small practice can help you in solving such problems very quickly as the level of questions is easy to average.

In these types of problem we have given any curve with one or more parameters to find the differential equation of given curve, we have to eliminate the given parameters in the given curve by differentiating the curve. After that we get the desired differential equation of given curve.

If an equation in independent, dependent variables involving some arbitrary constants is given, then a differential equation is obtained as follows

(i) Differentiate the given equation with respect to the independent variable (say x) as many times as the number of arbitrary constants in it.

(ii) Eliminate the arbitrary constants.

(iii) The obtained equation is the required differential equation *i.e.*, if we have an equation $f(x, y, c_1, c_2, \dots, c_n) = 0$ containing n arbitrary constants $c_1, c_2, \dots, c_n$, then by differentiating this n times, we shall get n-equations.

Now, among these n equations and the given equation, in all $(n + 1)$ equations. If the n arbitrary constants $c_1, c_2, \dots, c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

***Sample* Problem 3** *The differential equation of the family of curves $y = ae^{3x} + be^{-2x}$, where a and b are arbitrary constants, is*

[NCERT]

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$ (b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 6y = 0$

(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ (d) *None of these*

Interpret (c) Given family is $y = ae^{3x} + be^{-2x}$. ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = a\frac{d}{dx}(e^{3x}) + b\frac{d}{dx}(e^{-2x})$$

$$\Rightarrow \quad \frac{dy}{dx} = ae^{3x}\frac{d}{dx}(3x) + be^{-2x}\frac{d}{dx}(-2x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3ae^{3x} - 2be^{-2x} \quad \text{...(ii)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 3a\,e^{3x}\frac{d}{dx}(3x) - 2be^{-2x}\frac{d}{dx}(-2x) = 9ae^{3x} + 4be^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3(3ae^{3x}) + 4be^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\left(\frac{dy}{dx} + 2be^{-2x}\right) + 4be^{-2x}$$

$$\left(\because \frac{dy}{dx} + 2be^{-2x} = 3ae^{3x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 6be^{-2x} + 4be^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 10\,be^{-2x} \quad \text{...(iii)}$$

On multiplying Eq. (i) by 3 and then subtracting from Eq. (ii), we get

$$\frac{dy}{dx} - 3y = -2be^{-2x} - 3be^{-2x}$$

$$\Rightarrow \quad 3y - \frac{dy}{dx} = 5be^{-2x}$$

$$\Rightarrow \quad b = \frac{3y - \frac{dy}{dx}}{5e^{-2x}} \quad \text{...(iv)}$$

On putting the value of b in Eq. (iii), we get

$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 10\,\frac{3y - \frac{dy}{dx}}{5e^{-2x}}\cdot e^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 2\left(3y - \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 6y - 2\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + 6y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

which is the required differential equation.

Hot Spot 2

Solution of a Differential Equation

Solution of a differential equation is the most important topic in this chapter and JEE Main examination point of view. General question are seen from this topic. The leveled of question is from moderate to typical.

In these types of problem generally asked the general solution and particular solution of the given differential equation at some particular conditions. While solving problems based on solution of a differential equation, always keep in mind the various method to solve.

A function $y = f(x)$ is called a solution (or primitive) of a differential equation, if when substituted in the equation, it reduces the equation to an identity.

The process of finding all the solutions of a differential equation is called integrating the differential equation.

General Solution A solution of a differential equation, containing independent arbitrary constants equal in number to the order of differential equation is called general solution or (complete primitive).

Particular Solution A solution obtained by giving particular values of the arbitrary constants in the general solution is called particular solution of differential equation or particular integral.

The differential equation of first order. While we solving the given differential equation, we just follow the steps below.

(i) First, we apply the suitable method, to get solution of given differential equation, of first order differential equation and get the general solution.

(ii) Now, apply the given condition in general solution, to get the value of constant.

(iii) Now, put the value of constant further in general solution and get the desired particular solution.

Note The general solution of a differential equation of the nth order must contain n and only n independent arbitrary constants.

Sample **Problem 4** *A particular solution of the differential equation* $(x+1)\dfrac{dy}{dx} = 2e^{-y} - 1$, *given that* $y = 0$ *when* $x = 0$ *is* **[NCERT]**

(a) $y = \log\left|\dfrac{x+1}{x-1}\right|, (x \neq 1)$ (b) $y = \log\left|\dfrac{2x+1}{x+1}\right|, (x \neq -1)$

(c) $y = \log\left|\dfrac{x+1}{2x-1}\right|, \left(x \neq \dfrac{1}{2}\right)$ (d) None of these

Interpret (b) Given, differential equation is

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1 \quad \ldots(i)$$

On separating the variables

$$\Rightarrow \quad \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1} \Rightarrow \frac{e^y\, dy}{2 - e^y} = \frac{dx}{x+1}$$

On integrating both sides, we get

$$\int \frac{e^y\, dy}{2 - e^y} = \int \frac{dx}{x+1} \quad \ldots(ii)$$

Put $\quad 2 = e^y = t \Rightarrow -e^y = \dfrac{dt}{dy} \Rightarrow e^y\, dy = -dt$

Then, Eq. (ii) becomes

$$\int \frac{-dt}{t} = \int \frac{dx}{x+1} \Rightarrow -\log|t| = \log|x+1| + \log C$$

$$\Rightarrow \quad -\log|2 - e^y| = \log|C(x+1)|$$

$$\Rightarrow \quad \frac{1}{2 - e^y} = C(x+1) \qquad \left(\because -k\log x = \log x^{-1} = \frac{1}{x}\right)$$

$$\Rightarrow \quad 2 - e^y = \frac{1}{C(x+1)} \quad \ldots(iii)$$

Which is the required general solution.

Now, at $x = 0$ and $y = 0$, $2 - 1 = \dfrac{1}{C} \Rightarrow C = 1$

On putting the value of C in Eq. (iii), we get

$$2 - e^y = \frac{1}{(x+1)} \Rightarrow e^y = 2 - \frac{1}{x+1}$$

$$\Rightarrow \quad e^y = \frac{2x+2-1}{x+1} \Rightarrow e^y = \frac{2x+1}{x+1}$$

$$\Rightarrow \quad y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1) \qquad (\because \text{if } \log_e x = m \Rightarrow e^m = x)$$

Hot Spot 3

Differential Equation of First Order and First Degree

Differential equation of first order and first degree is an important topic for JEE Main examination. Generally, question are seen from this topic. The level of question is from moderate to typical while solving problems based on this concept always keep in mind the following methods.

(i) Variable separable method
(ii) differential equation reducible to variable separable method.
(iii) Homogeneous differential equations.
(iv) Differential equation reducible to homogeneous form
(v) Linear differential equation.
(vi) Differential equation reducible to linear form.
(vii) Bernaulli's equation
(viii) Exact differential equation.
(ix) General form of variable separation.

All differential equations of first order cannot be always solved. They can be solved by the following methods.

To solve such types of problems we follow the steps given below.

(i) We write the given differential equation in standard form.
(ii) Now, we use the suitable methods given above.
(iii) At last, we get the general solution of given differential equation.

The most general differential equation of first order and first degree is

$$\frac{dy}{dx} = f(x, y) \text{ or } \frac{f(x, y)}{g(x, y)}$$

(i) Variables Separable Method

Equations in which the variables are separable are those equations which can be expressed that the coefficient of dx is only a function of x and that of dy is only a function of y.

Thus, the general form of such an equation is $f(x)\,dx + g(y)\,dy = 0$. The solution of this equation is obtained by integrating $f(x)$ and $g(y)$ with respect to x and y respectively *i.e.*, solution is given by

$$\int f(x)\,dx = \int g(y)\,dy + c$$

Sample **Problem 5** *The solution of the differential equation* $\sec^2 x \cdot \tan y\,dx + \sec^2 y \cdot \tan x\,dy = 0$ *is* [NCERT]

(a) $\tan x \cdot \cot y = C$ (b) $\cot x \cdot \tan y = C$
(c) $\tan x \cdot \tan y = C$ (d) $\sin x \cdot \cos y = C$

Interpret (c) Given, $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

On separating the variables, we get

$$\Rightarrow \quad \sec^2 x \tan y\,dx = -\sec^2 y \tan x\,dy$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

On integrating both sides, we get

$$\int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy$$

Let $\tan x = v \Rightarrow \sec^2 x = \frac{du}{dy} \Rightarrow dx = \frac{du}{\sec^2 y}$

and $\tan y = v = \sec^2 y = \frac{dv}{dy} = dy = \frac{dv}{\sec^2 y}$

$$\int \frac{\sec^2 x}{u}\,\frac{du}{\sec^2 x} = -\int \frac{\sec^2 y}{v}\,\frac{dv}{\sec^2 y}$$

$$\Rightarrow \quad \int \frac{du}{u} = -\int \frac{dv}{v}$$

$$\Rightarrow \quad \log|u| = -\log|v| + \log|C|$$

$$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + \log|C|$$

$$\Rightarrow \quad \log|\tan x \tan y| = \log|C| \qquad (\because \log m + \log n = \log mn)$$

$$\Rightarrow \quad \tan x \cdot \tan y = C \qquad (\because \log m = \log n \Rightarrow m = n)$$

which is the required general solution.

(ii) Differential Equations Reducible to Variable Separable Method

Sometimes differential equation of the first order cannot be solved directly by variable separable method. But by some substitution, we can reduce it to a differential equation with separable variable. Let the differential equation is of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

can be reduced to variable separable form by the substitution $ax + by + c = z$

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \quad \left(\frac{dz}{dx} - a\right)\frac{1}{b} = f(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = a + bf(z)$$

Now, apply variable separable method.

***Sample* Problem 6** *The solution of the differential equation* $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ *is*

(a) $\log\left|\tan\frac{x+y}{2}+1\right| = x + C$ (b) $\log\left|\cot\frac{x+y}{2}+1\right| = x + C$

(c) $\log\left|\tan\frac{x-y}{2}-1\right| = C$ (d) *None of these*

Interpret (a) We have, $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ ...(i)

Let $z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

From Eq. (i),

$$\frac{dz}{dx} - 1 = \sin z + \cos z$$

$$\Rightarrow \quad \frac{dz}{\sin z + \cos z + 1} = dx \quad \text{(variables are separated)}$$

On integrating both sides, we get

$$\int \frac{dz}{\sin z + \cos z + 1} = \int 1 \cdot dx + C$$

$$\Rightarrow \quad \int \frac{dz}{\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}} + \frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}} + 1} = x + C$$

$$\Rightarrow \quad \int \frac{\sec^2\frac{z}{2}\,dz}{2\tan\frac{z}{2} + 2} = x + C$$

$$\Rightarrow \quad \int \frac{dt}{t+1} = x + C, \text{ where } t = \tan\frac{z}{2}$$

$$\Rightarrow \quad \log|t+1| = x + C$$

$$\Rightarrow \quad \log\left|\tan\frac{x+y}{2} + 1\right| = x + C$$

This is the required general solution.

(iii) Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equation.

For solving such equations, put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

These substitutions transform the given equation into an equation of the form

$$v + x\frac{dv}{dx} = f(v)$$

i.e.,

$$x\frac{dv}{dx} = f(v) - v$$

The variables are now separable. Separating the variables and integrating, we get

$\int \frac{dv}{f(v) - v} = \log x + C$, where C is an arbitrary constant. Now, replacing r by $\left(\frac{y}{x}\right)$ a factor integration, we get the required solution.

***Sample* Problem 7** *The solution of the differential equation* $(x - y)\,dy - (x + y)\,dx = 0$ *is* **[NCERT]**

(a) $\tan^{-1}\frac{x}{y} + \frac{1}{2}\log(x^2 + y^2) = C$

(b) $\tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2 + y^2) = C$

(c) $\cot^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\log(x^2 + y^2) = C$

(d) *None of the above*

Interpret (b) Given, $(x - y)\,dy - (x + y)\,dx = 0$

$$\Rightarrow \quad (x - y)\,dy = (x + y)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{...(i)}$$

Thus, the given differential equation is homogeneous.

So, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

On putting values of $\frac{dy}{dx}$ and y in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx} \Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v}{1-v} - v \Rightarrow x\frac{dv}{dx} = \frac{1 + v - v + v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x} \Rightarrow \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dx = \log|x| + C$$

Let $1 + v^2 = t \Rightarrow 2v = \frac{dt}{dv} \Rightarrow dv = \frac{dt}{2v}$

$$\therefore \quad \tan^{-1} v - \int \frac{v}{t} \times \frac{dt}{2v} = \log|x| + C$$

$$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\log|t| = \log|x| + C$$

$$\Rightarrow \quad 2\tan^{-1} v - [\log(1 + v^2) + 2\log(x)] = 2C \quad (\text{put } t = 1 + v^2)$$

$$\Rightarrow \quad 2\tan^{-1} v - \log[(1 + v^2)\,x^2] = 2C$$

$$\Rightarrow \quad 2\tan^{-1}\frac{y}{x} - \log\left[\left(\frac{x^2 + y^2}{x^2}\right)x^2\right] = 2C \quad \left(\text{put } v = \frac{y}{x}\right)$$

$$\Rightarrow \quad 2\tan^{-1}\frac{y}{x} - \log(x^2 + y^2) = 2C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2 + y^2) = C$$

This is the required solution of the given differential equation.

(iv) Differential Equations Reducible to Homogeneous Form

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1 x + b_1 y + c_1}, \text{ where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

can be reduced to homogeneous form by taking new variable x and y such that $x = X + h$ and $y = Y + k$, where h and k are constants to be so chosen as to make the given equation homogeneous. With the above substitutions, we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$. Hence, the given equation becomes

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + C}{a_1 (X + h) + b_1 (Y + k) + C_1}$$

$$= \frac{aX + bY + (ah + bk + C)}{a_1 X + b_1 Y + (a_1 h + b_1 k + C_1)}$$

Now, choose h and k such that

$$ah + bk + C = 0 \text{ and } a_1 h + b_1 k + C_1 = 0$$

Then, the differential equation becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a_1 X + b_1 Y}, \text{ which is homogeneous .}$$

Now, this equation can be solved as in homogeneous equations by substituting $Y = vX$. Finally by replacing X by $(x - h)$ and Y by $(x - k)$ we shall get the solution in original variables x and y.

If however $\frac{a}{a_1} = \frac{b}{b_1} = m$(say), then the differential equation becomes of the form $\frac{dy}{dx} = \frac{m(a_1 x + b_1 y) + C}{a_1 x + b_1 y + C_1}$

To solve such a differential equation put $v = a_1 x + b_1 y$, get rid of y and then the transformed equation will be such that the variables are separable.

***Sample* Problem 8** *The solution of the differential equation* $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$ *is*

(a) $\log [(y + 3)^2 + (x + 2)^2] + 2 \tan^{-1} \{(y + 3)/(x + 2)\} = C$

(b) $\log [(y + 3)^2 + (x + 2)^2] + \tan^{-1}\{(y + 3)/(x + 2)\} = C$

(c) $\log [(y + 3)^2 - (x + 2)^2] + 2 \cot^{-1} \{(y + 3)/(x + 2)\} = C$

(d) *None of the above*

Interpret (a) Here, $\frac{a}{a_1} \neq \frac{b}{b_1}$ *i.e.*, the coefficients of x and y in the numerator and denominator of the expression for $\left(\frac{dy}{dx}\right)$ are not proportional. Such equations can be reduced to homogeneous form by taking new variables X and Y such that $x = X + h$ and $y = Y + k$, where h and k are constants to be taken at our choice. With these substitutions the given equation reduce to

$$\frac{dY}{dX} = \frac{Y + k - X - h + 1}{Y + k + X + h + 5} = \frac{Y - X + (k - h + 1)}{Y + X + (k + h + 5)} \quad \ldots\text{(i)}$$

Choose h and k such that

$k - h + 1 = 0$ and $k + h + 5 = 0$. Solving these, we get $k = -3, h = -2$, with these values of h and k the Eq. (i) becomes

$$\frac{dY}{dX} = \frac{Y - X}{Y + X} \quad \ldots\text{(ii)}$$

Now, Eq. (ii) is a homogeneous equation, so putting $Y = vk$, we have

$$v + x\frac{dv}{dx} = \frac{v - 1}{v + 1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = \frac{-1 - v^2}{v + 1}$$

$$\Rightarrow \quad \frac{v + 1}{v^2 + 1} dv = \frac{-dx}{x}$$

or $$\frac{v}{v^2 + 1} dv + \frac{dv}{v^2 + 1} = \frac{-dx}{x}$$

$$\Rightarrow \quad \frac{2v}{v^2 + 1} dv + \frac{2}{v^2 + 1} dv = \frac{-2}{x} dx$$

On integrating, we have

$$\log(v^2 + 1) + 2 \tan^{-1} v = -2 \log x + C$$

$$\Rightarrow \quad \log\{(v^2 + 1) x^2\} = -2 \tan^{-1} v + C$$

$$\Rightarrow \quad \log \{y^2 + x^2\} = -2 \tan^{-1}\left(\frac{y}{x}\right) + C \qquad \left(\because v = \frac{y}{x}\right)$$

$$\Rightarrow \log [(x + 2)^2 + (y + 3)^2] + 2 \tan^{-1} \{(y + 3)/(x + 2)\} = C$$

$$(\because Y = y - k = y + 3, X = x - h = x + 2)$$

which is the required solution.

***Sample* Problem 9** *Solution of the differential equation* $\frac{dy}{dx} = \frac{x + y + 3}{2x + 2y + 1}$ *is*

(a) $x + C = \frac{1}{3}(x + y) - \frac{5}{4} \log (3x + 3y + 4)$

(b) $C = \frac{1}{3}(x + y) - \frac{5}{9} \log(3x + 3y + 4)$

(c) $x + C = \frac{2}{3}(x + y) + \frac{4}{9} \log(3x + 3y + 4)$

(d) $x + C = \frac{2}{3}(x + y) - \frac{5}{9} \log (3x + 3y + 4)$

Interpret (d) Here, $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{2}$ *i.e.*, the coefficients of x and y in the Nr and Dr of the expression for $\frac{dy}{dx}$ are proportional. Proper substitution in this case, therefore, will be to put v for $x + y$. Let $x + y = v$. Then, $1 + \frac{dy}{dx} = \frac{dv}{dx}$ with these substitutions the given equation reduces to

$$\frac{dv}{dx} - 1 = \frac{v + 3}{2v + 1} \text{ or } \frac{dv}{dx} = \frac{v + 3}{2v + 1} + 1 = \frac{3v + 4}{2v + 1}$$

or $$dx = \frac{2v+1}{3v+4}\,dv = \left[\frac{2}{3} - \frac{\frac{5}{3}}{3v+4}\right] dv$$

$\therefore$ On integrating, $x + C = \frac{2}{3}v - \frac{5}{3}\cdot\frac{1}{3}\log(3v+4)$

$$\Rightarrow \quad x + C = \frac{2}{3}v - \frac{5}{9}\log(3v+4)$$

or $$x + C = \frac{2}{3}(x+y) - \left(\frac{5}{9}\right)\log(3x+3y+4), \quad (\because v = x+y)$$

which is the required solution.

Check Point 1

1. The differential equation whose solution is a circle centred at (h, k) and radius a is (where, a is a constant)
   (a) $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = a^2\frac{d^2y}{dx^2}$ (b) $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$
   (b) $\left\{1+\left(\frac{dy}{dx}\right)\right\}^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$ (d) None of these
2. The order of the differential equation of all circles of radius r, having centre on y-axis and passing through the origin, is
   (a) 1 (b) 2
   (c) 3 (d) 4
3. The solution of the differential equation $\frac{dy}{dx} - x\tan(y-x) = 1$ is
   (a) $\sin(x-y) = ce^{x^2/2}$
   (b) $\sin(x-y) = ce^{x}$
   (c) $\sin(y-x) = ce^{x^2/2}$
   (d) $\sin(x+y) = ce^{x^2}$
4. The solution of differential equation $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1+x+x^2)$ is
   (a) $\log(1+y^2) = \tan^{-1}x + C$ (b) $\log\frac{\sqrt{1+y^2}}{x} = \tan^{-1}x + C$
   (c) $\frac{1}{2}\log\frac{\sqrt{1+y^2}}{x} = \tan^{-1}x + C$ (d) None of these
5. The solution of differential equation $(x-y)^2\frac{dy}{dx} = a^2$ is
   (a) $\log\left(\frac{x+y-a}{x+y+a}\right) = 2y + C$ (b) $a\log\left(\frac{x-y-a}{x-y+a}\right) = 2y + C$
   (c) $a\log\left(\frac{x+y+a}{x+y-a}\right) = y + C$ (d) $a\log\left(\frac{x-y+a}{x-y-a}\right) = y + C$
6. The solution of $(x+y+1)\,dy = dx$ is
   (a) $x + y + 2 = Ce^y$ (b) $x + y + 4 = C\log y$
   (c) $\log(x+y+2) = Cy$ (d) $\log(x+y+2) = C - y$

(v) Linear Differential Equation

A differential equation is said to be linear, if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.

The most general form of a linear equation of the first order is

$$\frac{dy}{dx} + Py = Q$$

where, P and Q are any function of x.

To solve such equations find integrating factor

i.e., $$\text{IF} = e^{\int P\,dx}$$

On multiplying the form by $e^{\int P\,dx}$ on both sides, we get

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Q\cdot e^{\int P\,dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\cdot e^{\int P\,dx} + y\cdot\frac{d}{dx}e^{\int P\,dx} = Qe^{\int P\,dx}$$

$$\left[\text{since, } \frac{d}{dx}\cdot(e^{\int P\,dx}) = P\cdot e^{\int P\,dx}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(y\cdot e^{\int P\,dx}) = Q\cdot e^{\int P\,dx}$$

$$\Rightarrow \quad \int\frac{d}{dx}\cdot(y\cdot e^{\int P\,dx})\,dx = \int Q\cdot e^{\int P\,dx}\,dx$$

$$\Rightarrow \quad y\cdot e^{\int P\,dx} = \int Q\cdot e^{\int P\,dx} + C$$

which is the required solution of the given differential equation.

Note If linear equation of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y. Then, $\text{IF} = e^{\int P\,dy}$ and solution is given by

$$x\cdot\text{IF} = \int Q\cdot\text{IF}\,dy + c$$

***Sample* Problem 10** *The solution of*

$$x\frac{dy}{dx} + y - x + xy\cot x = 0, (x \neq 0) \text{ is}$$

[NCERT]

(a) $y = x - \cot x + \frac{C}{\sin x}$ (b) $y = \frac{1}{x} - \cot x - \frac{xC}{x\sin x}$

(c) $y = \frac{1}{x} - \cot x + \frac{C}{x\sin x}$ (d) $y = \frac{1}{x} + \cot x - \frac{C}{\sin x}$

Interpret (c) Given, $x\frac{dy}{dx} + y - x + xy\cot x = 0$

$$\Rightarrow \quad x\frac{dy}{dx} + y(1 + x\cot x) = x$$

On dividing by x both sides, we get

$$\frac{dy}{dx}+\frac{y(1+x\cot x)}{x}=1$$

On comparing with the form $\frac{dy}{dx}+Py=Q$, we get

$$\therefore \quad P=\frac{1+x\cot x}{x} \text{ and } Q=1$$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log x+\log|\sin x|}=e^{\log(x\sin x)}$$

$$\Rightarrow \quad \text{IF}=x\sin x \qquad (\because e^{\log ax}=ax) \quad \ldots(i)$$

The general solution of the differential equation is given by

$$y\cdot\text{IF}=\int Q\times\text{IF}\,dx+C$$

$$\Rightarrow \quad y(x\sin x)=\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin x}\,dx+C$$

$$\Rightarrow y(x\sin x)=-x\cos x+\int\cos x\,dx+C \quad \text{(integration by parts)}$$

$$\Rightarrow \quad y(x\sin x)=-x\cos x+\sin x+C$$

$$\Rightarrow \quad y=\frac{-x\cos x}{x\sin x}+\frac{\sin x}{x\sin x}+\frac{C}{x\sin x}$$

$$\Rightarrow \quad y=\frac{1}{x}-\cot x+\frac{C}{x\sin x}$$

***Sample* Problem 11** *The solution of* $(x+y)\frac{dy}{dx}=1$ *is* [NCERT]

(a) $x+y+1=Ce^{y}$ (b) $x+y+1=Ce^{-y}$

(c) $x+y-1=Ce^{y}$ (d) $x+y-1=Ce^{-y}$

Interpret (a) Given equation can be converted into linear equation by dividing $\frac{dy}{dx}$ and then solve it.

$$(x+y)\frac{dy}{dx}=1$$

$$\Rightarrow \quad \frac{dx}{dy}=(x+y)\Rightarrow\frac{dx}{dy}-x=y$$

On comparing with the form $\frac{dx}{dy}+Px=Q$, we get

$$P=-1, Q=y$$

$$\therefore \quad \text{IF}=e^{\int P\,dy}\ e^{-\int 1\,dy}\Rightarrow \text{IF}=e^{-y} \quad \ldots(i)$$

The general solution of the given differential equation is given by the relation,

$$x\cdot\text{IF}=\int Q\times\text{IF}\,dy+C$$

$$\Rightarrow \quad x\cdot e^{-y}=\int e^{-y}\cdot y\,dy+C$$

$$\Rightarrow \quad xe^{-y}=y\int e^{-y}\,dy-\int\left[\frac{d}{dy}y\int e^{-y}\,dy\right]dy+C$$

(integration by parts)

$$\Rightarrow \quad xe^{-y}=-y(e^{-y})+\int e^{-y}dy+C$$

$$\Rightarrow \quad xe^{-y}=-ye^{-y}-e^{-y}+C\Rightarrow x=-y-1+e^{y}C$$

$$\Rightarrow \quad x+y+1=Ce^{y}$$

(vi) Differential Equation Reducible to Linear Form

An equation of the form

$$f'(y)\frac{dy}{dx}+Pf(y)=Q \quad \ldots(i)$$

where P and Q are constants or function of x alone, can be reduced to linear form by putting

$$f(y)=v\Rightarrow f'(y)\frac{dy}{dx}=\frac{dv}{dx}$$

Then, Eq. (i) becomes $\frac{dv}{dx}+Pv=Q$

which is linear in v and x.

***Sample* Problem 12** *The solution of*

$$\{xy^3(1+\cos x)-y\}\,dx+x\,dy=0 \text{ is}$$

(a) $\frac{x^3}{3}-x^2\sin x+x\cos x-2\sin x+C$

(b) $\frac{x^3}{3}+x^2\sin x+2x\cos x-2\sin x+C$

(c) $\frac{x^3}{3}+2x^2\sin x-x\cos x+\sin x+C$

(d) *None of the above*

Interpret (b) The given equation can be written as

$$\frac{dy}{dx}+y^3(1+\cos x)-\frac{y}{x}=0$$

i.e., $$\frac{1}{y^3}\frac{dy}{dx}-\frac{1}{y^2x}=-(1+\cos x)$$

Using the transformation

$$\frac{-1}{y^2}=u, \text{ we get } \frac{2}{y^3}dy=du$$

The above equation reduces to $\frac{1}{2}\frac{du}{dx}+\frac{u}{x}=-(1+\cos x)$

whose $$\text{IF}=e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$$

Hence, the solution of the given differential equation is given by

$$ux^2=-2\int x^2(1+\cos x)\,dx$$

i.e., $$\frac{x^2}{2y^2}=\int x^2\,dx+\int x^2\cos x\,dx=\frac{x^3}{3}+x^2\sin x-\int 2x\sin x\,dx$$

$$=\frac{x^3}{3}+x^2\sin x+2x\cos x-2\int\cos x+C$$

$$=\frac{x^3}{3}+x^2\sin x+2x\cos x-2\sin x+C$$

(vii) A Special Case Bernoulli's Equation

An equation of the form $\frac{dy}{dx}+Py=Qy^n$

where P and Q are functions of x only and $n\neq 0, 1$ is known as Bernoulli's differential equation.

It is easy to reduce the equation into linear form as below.

On dividing both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $$y^{1-n} = z \Rightarrow (1-n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ Given equation becomes $\dfrac{dz}{dx} + (1-n)\,Pz = (1-n)\,Q$

which is a linear differential equation in z.

Here, $$\text{IF} = e^{\int (1-n)P\,dx}$$

Required solution is

$$z(\text{IF}) = \int \{(1-n)\cdot Q\cdot e^{\int (1-n)P\,dx}\,dx$$

Note If equation is of the form $\dfrac{dx}{dy} + Px = Qx^n$, where $n \neq 0, 1$ and P and Q are functions of y only. Then, divide by x^n and put $x^{1-n} = z \Rightarrow (1-n)\,x^{-n}\dfrac{dx}{dy} = \dfrac{dz}{dx}$.

***Sample* Problem 13** *The solution of* $x\left(\dfrac{dy}{dx}\right) + y = y^2 \log x$ *is*

(a) $C = y\,(1 + \log x) + xy$ (b) $1 = y\,(1 + \log x) + Cxy$

(c) $1 = y\,(1 + \log x) - Cxy$ (d) $C = y\,(1 + \log x) - xy$

Interpret (b) The given equation can be written as $\dfrac{dy}{dx} + \dfrac{1}{x}\cdot y = \dfrac{y^2}{x}\log x$, (dividing throughout by x).

or $$\frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{\log x}{x}. \quad \text{...(i)}$$

(dividing throughout by y^2)

Now, put $\dfrac{1}{y} = v$, so that $\left(-\dfrac{1}{y^2}\right)\dfrac{dy}{dx} = \left(\dfrac{dv}{dx}\right)$

With these substitutions Eq. (i), becomes

$$-\frac{dv}{dx} + \frac{1}{x}\cdot v = \frac{\log x}{x}$$

or $$\frac{dv}{dx} - \frac{1}{x}\cdot v = -\frac{\log x}{x}$$

This is linear with v as the dependent variable.

Here, $P = -\dfrac{1}{x}$ and $Q = -(\log x)/x$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$$

Hence, the solution is

$$v\left(\frac{1}{x}\right) = \int \{-(\log x)/x\}\cdot\left(\frac{1}{x}\right)dx + C$$

$$\Rightarrow \quad \frac{v}{x} = -\int\left(\frac{1}{x^2}\right)\log x\,dx + C$$

$$= -\left[\left(-\frac{1}{x}\right)\log x - \int\left(\frac{1}{x}\right)\left(-\frac{1}{x}\right)dx\right] + C$$

(integrating by parts $\dfrac{1}{x^2}$ as the second function)

$$= \left(\frac{1}{x}\right)\log x - \int\left(\frac{1}{x^2}\right)dx + C = \left(\frac{1}{x}\right)\log x + \frac{1}{x} + C$$

or $$\frac{1}{(xy)} = \left(\frac{1}{x}\right)(1 + \log x) + C \text{ or } 1 = y\,(1 + \log x) + Cxy$$

(viii) Exact Differential Equations

A differential equation is said to be exact, if it can be obtained from its primitive (solution) directly by differentiation, without involving any subsequent process of multiplication, elimination, etc. Thus, the differential equation of the form $M\,dx + N dy = 0$, where M and N are the some functions of x and y is exact, if it can be obtained directly by differentiating an equation of the form $u = c$, where u is some function of x and y and c is an arbitrary constant.

Note The necessary and sufficient condition for the ordinary differential equation $Mdx + N\,dy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working rule for solving an exact differential equation

(i) Integrate M with respect to x treating y as constant.

(ii) Integrate with respect to y only those terms of N which do not contain x.

(iii) Equate the sum of these two integrals [found (i) and (ii)] to an arbitrary constant. Thus, get the required solution.

***Sample* Problem 14** *Solution of the differential equation* $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$ *is*

(a) $x^3 + y^3 - 3axy = 3C$ (b) $x^3 + y^3 + 3axy = C$

(c) $x^3 + y^3 - axy = C$ (d) $x^3 + y^3 - 3xy = 3C$

Interpret (a) Here, $M = x^2 - ay$ and $N = -(ax - y^2)$

we have $\dfrac{\partial M}{\partial y} = -a$ and $\dfrac{\partial N}{\partial x} = -a$ *i.e.*, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence, the given equation is exact.

Now, $\int M\,dx$ (regarding y as a constant)

$$= \int (x^2 - ay)\,dx = \frac{1}{3}x^3 - ayx \quad \text{...(i)}$$

and $\int N\,dy$ (taking in N only those terms which do not contain x)

$$= \int y^2\,dy = \frac{1}{3}y^3 \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii) equal to C (an arbitrary constant)

$$\Rightarrow \quad \frac{1}{3}x^3 - ayx + \frac{1}{3}y^3 = c$$

$$\Rightarrow \quad x^3 + y^3 - 3axy = 3c$$

(ix) General Form of Variable Separation

If we can write the differential equation in the form $f\{f_1(x, y)\}d\{f_1(x, y)\} + \phi\{f_2(x, y)\}\, d\{f_2(x, y)\} + \ldots + = 0$, then each term can be easily integrated separately. For this use the following results

(a) $d(x \pm y) = dx \pm dy$

(b) $d(xy) = xdy + y\,dx$

(c) $d\left(\frac{x}{y}\right) = \frac{y\,dx - xdy}{y^2}$

(d) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(e) $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2dy}{y^2}$

(f) $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2dx}{x^2}$

(g) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2\,dx - 2x^2y\,dy}{y^4}$

(h) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$

(i) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$

(j) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(k) $d\,[\log(xy)] = \frac{x\,dy + y\,dx}{xy}$

(l) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

(m) $d\left[\frac{1}{2}\log(x^2 + y^2)\right] = \frac{x\,dx + y\,dy}{x^2 + y^2}$

(n) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

(o) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$

(p) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

(q) $d\left(\frac{e^y}{x}\right) = \frac{xe^y\,dy - e^y\,dx}{x^2}$

(r) $d(x^my^n) = x^{m-1}y^{n-1}\,(my\,dx + nx\,dy)$

(s) $d\,(\sqrt{x^2 + y^2}) = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$

(t) $d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

(u) $\frac{d\,[f(x, y)^{1-n}}{1 - n} = \frac{f'(x, y)}{\{f(x, y)\}^n}$

(v) $d\,[\log(x + y)] = \frac{dx + dy}{x + y}$

(w) Sometimes, transformations to the polar coordinates facilitates separation of variables. It is convenient to remember the followings differentials

If $x = r\cos\theta$, $y = r\sin\theta$

- $xdx + y\,dy = rdr$
- $xdy + ydx = r^2\,d\theta$
- $dx^2 + dy^2 = dr^2 + r^2\,d\theta$

Sample Problem 15 *The solution of the differential equation* $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\cos^2(x^2 + y^2)}{y^3}$ *is*

(a) $\tan(x^2 + y^2) = \frac{x^2}{y^2} + C$

(b) $\cot(x^2 + y^2) = \frac{y^2}{x^2} + C$

(c) $\cot(x^2 + y^2) = \frac{x^2 + y^2}{y^2} + c$

(d) *None of the above*

Interpret (a) The given differential equation can be written as

$$\frac{x\,dx + y\,dy}{(y\,dx - xdy)/y^2} = y^2\,\frac{x}{y^3}\,\{\cos^2(x^2 + y^2)\}$$

$$\Rightarrow \quad \sec^2(x^2 + y^2)\,\frac{1}{2}\,d(x^2 + y^2) = \frac{x}{y}\,d\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{1}{2}\tan(x^2 + y^2) = \frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{C}{2}$$

$$\therefore \quad \tan(x^2 + y^2) = \frac{x^2}{y^2} + C$$

33.3 Applications of First Order Differential Equation

(i) **Growth and Decay Problems** Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that $\frac{dN}{dt}$, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $\frac{dN}{dt} = kN$

or $$\frac{dN}{dt} - kN = 0 \qquad \text{...(a)}$$

Where, k is the constant of proportionality. We are assuming that $N(t)$ is a differentiable, hence continuous function of time.

***Sample* Problem 16** *The population of a village increases at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in year 1999 and 25000 in the year 2004, what will be the population of the village in 2009?* **[NCERT]**

(a) 31250 (b) 27520 (c) 30320 (d) 33000

Interpret (a) Let P be the population at time t, then $\frac{dy}{dt} \propto y$

$\Rightarrow \quad \frac{dy}{dt} = ky$, where k is constant

$\Rightarrow \quad \frac{dy}{y} = k\,dt$

On integrating both sides, we get $\log y = kt + C$ ...(i)

In the year 1999, $t = 0, y = 20000$

$\therefore$ From Eq. (i), $\log 20000 = k(0) + C$

$\Rightarrow \quad \log 20000 = C$ ...(ii)

In the year 2004, $t = 5, y = 25000$, so from Eq. (i),

$\log 25000 = k\,5 + C$

$\Rightarrow \quad \log 25000 = 5k + \log 20000$ [using Eq. (ii)]

$\Rightarrow \quad 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow \quad k = \frac{1}{5}\log\frac{5}{4}$

For year 2009, $t = 10$ yr

Now, substituting the values of t, k and C in Eq. (i), we get

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \quad \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right] \quad (\because \log m + n = \log mn)$$

$$y = 20000 \times \frac{5}{4} \times \frac{5}{4}$$

$$\Rightarrow \quad y = 31250$$

Hence, the population of the village in 2009 will be 31250.

***Sample* Problem 17** *A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10% of its original mass, find* (i) *and expression for the mass of the material remaining at any time t,* (ii) *the mass of the material after four hours and* (iii) *the time at which the material has decayed to one half of its initial mass.*

(a) $50\,e^{-\frac{1}{2}\log\cdot 9t}$, $50e^{-2\log\cdot 9}$ and $\frac{\log\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\log\cdot 9\right)}\,h$

(b) $50\,e^{2\log\cdot 9t}$, $25\,e^{-\frac{1}{2}\log\cdot 9}$ and $\frac{\log 2}{\left(\frac{1}{2}\log\cdot 9\right)}\,h$

(c) $25\,e^{-\log\cdot 9t}$, $50e^{2\log\cdot 9}$ and $\left(\frac{\log 2}{-2\log\cdot 9}\right)h$

(d) *None of the above*

Interpret (a)

(i) Let N denote the amount of material present at time t. Then, from equation (a) $\frac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is

$$N = Ce^{kt} \quad \text{...(i)}$$

At $t = 0$, we are given that $n = 50$. Therefore, from Eq. (i), $50 = ce^{k(0)}$ or $c = 50$. Thus,

$$N = 50\,e^{kt} \quad \text{...(ii)}$$

At $t = 2$, 10 per cent of the original mass of 50 mg or 5 mg, has decayed. Hence, at $t = 2$, $N = 50 - 5 = 45$. Substituting these values into Eq. (ii) and solving for k, we have

$$45 = 50e^{2k} \text{ or } k = \frac{1}{2}\ln\frac{45}{50}$$

On substituting this value into Eq. (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{-1/2\ln\cdot 9\,t} \quad \text{... (iii)}$$

where t is measured in hours.

(ii) We require N at $t = 4$. Substituting $t = 4$ into Eq. (iii) and then solving for N, we find $N = 50\,e^{-2\ln\cdot 9}$

(iii) We require t when $N = \frac{50}{2} = 25$. Substituting $N = 25$ into Eq. (i) and solving for t, we find

$$25 = 50e^{-1/2\ln\cdot 9t} \Rightarrow t = \ln\frac{1}{2}/(-1/2\ln\cdot 9)\text{ h}$$

(ii) **Temperature Problems** Newton's law of cooling which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then, the time rate of change temperature of the body is $\frac{dT}{dt}$ and Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -k(T - T_m)$$

or as $\frac{dT}{dt} + kT = kT_m$ ...(a)

where, k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make $\frac{dT}{dt}$ negative in a cooling process, when T is greater than T_m and positive in a heating process, when T is less than T_m.

***Sample* Problem 18** *A metal bar at a temperature of 100°F is placed in a room at a constant temperature of 0°F. If after 20 min the temperature of the bar is 50°F, find* (i) *the time it will take the bar to reach a temperature of 25°F and* (ii) *the temperature of the bar after 10 min.*

(a) 39.6 *min and* $100\, e^{\frac{1}{20}\log\frac{1}{2}\cdot 10}$ °F

(b) 36.9 *min and* $100\, e^{\frac{1}{2}\log\frac{1}{2}\cdot 10}$ °F

(c) 38.9 *min and* $100\, e^{-\frac{1}{2}\log\frac{1}{2}\cdot 10}$ °F

(d) *None of the above*

Interpret (a) Use equation (a) with $T_m = 0$; the medium, here is the room which is being held at a constant temperature of 0°F. Thus, we have

$$\frac{dT}{dt} + kT = 0$$

whose solution is $\quad T = ce^{-kt} \quad$ ...(i)

Since, $T = 100$ at $t = 0$ (the temperature of the bar is initially 100°F), it follows Eq. (i) and $100 = ce^{-k(0)}$ or $100 = c$. Substituting this value into Eq. (i), we obtain

$$t = 100\, e^{-kt} \quad \text{...(ii)}$$

At $t = 20$, we are given that $t = 50$, hence, from Eq. (ii),

$$50 = 100\, e^{-20k} \text{ from which } k = -\frac{1}{20}\ln\frac{50}{100}$$

On substituting this value into Eq. (ii), we obtain the temperature of the bar at any time t as

$$T = 100\, e^{\frac{1}{2}\ln\frac{1}{2}t} \quad \text{...(iii)}$$

(i) We require t when $T = 25$. Substituting $T = 25$ into Eq. (iii), we have

$$25 = 100\, e^{\frac{1}{20}\ln\frac{1}{2}t}$$

Solving, we find that $t = 39.6$ min.

(ii) We require T when $t = 10$. Substituting $t = 10$ into Eq. (iii) and then solving for T, we find that

$$T = 100\, e^{\frac{1}{20}\ln\frac{1}{2}\cdot 10} \text{ °F}$$

It should be noted that since Newton's law is valid only for small temperature differences, the above calculations represent only a first approximation to the physical situation.

(iii) **Geometrical Applications** Let $P(x, y)$ be any point on the curve $y = f(x)$. Let the tangent and normal at $P(x, y)$ to the curve meets x-axis and y-axis at T and N, respectively.

Now, draw perpendicular from P on x-axis.

$\therefore \quad PM = y$

It tangent at P makes angle ψ with positive direction of x-axis, then $\frac{dy}{dx} = \tan\psi$

(a) **Length of Sub-tangent** TM is defined as sub-tangent. In ΔPTM

$$TM = |y \cot\psi| = \left|\frac{y}{\tan\psi}\right| = \left|y\frac{dx}{dy}\right|$$

$\therefore$ Length of sub-tangent $= \left|y\frac{dx}{dy}\right|$

(b) **Length of Sub-normal** MN is defined as sub-normal. In ΔPMN

$$MN = |y \cot(90° - \psi)| = |y \tan\psi| = \left|y\frac{dy}{dx}\right|$$

$\therefore$ Length of sub-normal $= \left|y\frac{dy}{dx}\right|$

(c) **Length of Tangent** PT is defined as length of tangent. In ΔPMT,

$$PT = |y \operatorname{cosec}\psi| = |y\sqrt{(1 + \cot^2\psi)}|$$

$\therefore$ Length of tangent $= \left|y\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\right|$

(d) **Length of Normal** PN is defined as length of normal. In ΔPMN

$$PN = |y \operatorname{cosec}(90° - \psi)| = |y \sec\psi|$$

$$= |y\sqrt{(1 + \tan^2\psi)}| = \left|y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\right|$$

$\therefore \quad \left|y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\right|$

(e) **Intercepts Made by the Tangent on the Coordinate Axes** The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{...(i)}$$

Putting $Y = 0$ in Eq. (i), we get

$$X = x - y\frac{dx}{dy}$$

Hence, the tangent of intercept OT that the tangent cuts off from the x-axis is $x - y\frac{dx}{dy}$.

Again, the tangent meets y-axis then putting $X = 0$ in Eq. (i), we get $Y = y - x\frac{dy}{dx}$

Hence, the length of intercept OL that the tangent cuts off from the y-axis is

$$y - x\frac{dy}{dx}.$$

***Sample* Problem 19** *At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). The equation of the curve given that it passes through (– 2, 1) is* [**NCERT**]

(a) $y + 3 = x^2$ (b) $(y + 3) = x^2 + 4$

(c) $y - 3 = (x + 4)^2$ (d) $y + 3 = (x + 4)^2$

Interpret (d) First, we find the slope of the line segment which passes through the points (x, y) and $(-4, -3)$ and then apply the given condition.

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope of the line segment joining the points $(x_2, y_2) \to (x, y)$ and $(x_1, y_1) \to (-4, -3)$

$$= \frac{y-(-3)}{x-(-4)} = \frac{y+3}{x+4} \qquad \left(\because \text{slope of tangent} = \frac{y_2 - y_1}{x_2 - x_1}\right)$$

According to question (slope of tangent is twice the slope of the line), we must have, $\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$

Now, separating the variables, we get $\frac{dy}{y+3} = \left(\frac{2}{x+4}\right)dx$

On integrating both sides, we get $\int \frac{dy}{y+3} = \int \left(\frac{2}{x+4}\right)dx$

$$\Rightarrow \quad \log|y+3| = 2\log|x+4| + \log|C|$$

$$\Rightarrow \quad \log|y+3| = \log|x+4|^2 + \log|C|$$

$$\Rightarrow \quad \log\frac{|y+3|}{|x+4|^2} = \log|C| \qquad \left(\because \log m - \log n = \log\frac{m}{n}\right)$$

$$\Rightarrow \quad \frac{|y+3|}{|x+4|^2} = C \qquad \ldots(i)$$

The curve passes through the point $(-2, 1)$, therefore

$$\Rightarrow \quad \frac{|1+3|}{|-2+4|^2} = C$$

$$\Rightarrow \quad C = 1$$

On substituting $C = 1$ in Eq. (i), we get

$$\frac{|y+3|}{|x+4|^2} = 1 \Rightarrow y + 3 = (x+4)^2$$

which is the required equation of curve.

***Sample* Problem 20** *The curve for which the length of normal is equal to the radius vector is*

(a) $y^2\left(\frac{dy}{dx}\right)^2 = x^2$ (b) $y\left(\frac{dy}{dx}\right)^2 = x$

(c) $x^2\left(\frac{dy}{dx}\right) = y^2$ (d) None of these

Interpret (a) Here, radius vector $= OP$ and length of normal $= PN$

According to question, $PN = OP$

$$\Rightarrow \quad y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{(x^2+y^2)}$$

$$\Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 = x^2$$

***Sample* Problem 21** *The curve for which the intercept cut off by any tangent on y-axis is proportional to the square of the ordinate of the point of tangency is*

(a) $\frac{C_1}{x} - \frac{C_2}{y} = 1$ (b) $\frac{C_1}{x} + \frac{C_2}{y} = 1$

(c) $\frac{C_1}{x} + \frac{C_2}{y} = \frac{1}{xy}$ (d) *None of these*

Interpret (b) Let $P(x, y)$ be any point on the curve.

$\therefore$ Length of intercept on y-axis $= y - x\frac{dy}{dx}$ (according to the question)

Length of intercept on y-axis $\propto y^2$

$$\Rightarrow \quad y - x\frac{dy}{dx} = ky^2$$

where k is constant of proportionality

$$\text{or} \quad \frac{y}{x} - \frac{dy}{dx} = \frac{ky^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -\frac{ky^2}{x}$$

$$\text{Put} \quad y^{-1} = v$$

$$\therefore \quad -y^{-2}\frac{dy}{dx} = \frac{dv}{dx},$$

$$\text{then} \quad \frac{dv}{dx} - \frac{v}{x} = -\frac{k}{x}$$

$$\text{or} \quad \frac{dv}{dx} + \frac{v}{x} = \frac{k}{x}$$

which is linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\therefore \quad \text{The solution is } v \cdot x = \int \frac{k}{x}x\,dx + C$$

$$\Rightarrow \quad vx = kx + C \Rightarrow \frac{x}{y} = kx + C \qquad \left(\because v = \frac{1}{y}\right)$$

$$\Rightarrow \quad \frac{C_1}{x} + \frac{C_2}{y} = 1 \quad \left(\text{where } -\frac{C}{k} = C_1 \text{ and } \frac{1}{k} = C_2\right)$$

33.4 Orthogonal Trajectory

A curve which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. To find the orthogonal trajectory.

(i) Let $f(x, y, c) = 0$ ...(i)

be the equation of given family of curves, where c is a parameter (or arbitrary constant).

(ii) Find the differential equation of Eq. (i) and then substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in the differential equation. Thus, we will get the differential equation for the orthogonal trajectories.

(iii) Now, solve this differential equation to obtain general solution which will give us the required orthogonal trajectories.

Note If the orthogonal trajectories from the same family of curves as the given family of curves, then the given system of curves is called self orthogonal.

***Sample* Problem 22** *The orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$, where a is a parameter is*

(a) $x^2 + y^2 = c^2$ (b) $x^2 + y^2 = cx$

(c) $x^2 + y^2 = cx + 1$ (d) *None of these*

Interpret (b) Here, $x^2 + y^2 - ay = 0$; differentiating w.r.t. x,

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} - \frac{x^2 + y^2}{y}\frac{dy}{dx} = 0$$

$$\left(\therefore x^2 + y^2 - ay = 0 \Rightarrow a = \frac{x^2 + y^2}{y}\right)$$

or $$2x + \frac{y^2 - x^2}{y}\frac{dy}{dx} = 0$$

This is the differential equation of the circles. The equation of orthogonal trajectories is

$$2x + \frac{y^2 - x^2}{y}\cdot\left(-\frac{dx}{dy}\right) = 0 \left(\text{putting } \frac{-dx}{dy} \text{ in place of } \frac{dy}{dx}\right)$$

or $$2xy\,dy + (x^2 - y^2)\,dx = 0$$

It is a homogeneous equation

Put $y = vx$; then $\frac{dy}{dx} = v + x\frac{dy}{dx}$

$\because$ $$2x\cdot vx\cdot\left(v + x\frac{dv}{dx}\right) + x^2 - v^2x^2 = 0$$

or $$2v\left(v + x\frac{dv}{dx}\right) + 1 - v^2 = 0$$

$\therefore$ $$1 + v^2 + 2vx\frac{dv}{dx} = 0$$

or $$\frac{dx}{x} + \frac{2v}{1 + v^2}dv = 0$$

On integrating, we get

$$\log x + \log(1 + v^2) = \log C$$

$\therefore$ $$x(1 + v^2) = C$$

or $$x\left(1 + \frac{y^2}{x^2}\right) = C, \text{ i.e., } x^2 + y^2 = Cx$$

Check Point 2

1. The equation of the curve passing through the origin and satisfying the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ is

(a) $(1 + x^2)y = x^3 + C$ (b) $2(1 + x^2)y = 3x^3 + C$

(c) $3(1 + x^2)y = 4x^3 + C$ (d) None of these

2. The family of curves represented by $\frac{dy}{dx} = \frac{x^2 + x + 1}{y^2 + y + 1}$ and the family represented by $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$

(a) touch each other (b) are orthogonal

(c) are one and the same (d) None of these

3. The solution of the differential equation

$$2y\sin x\left(\frac{dy}{dx}\right) = 2\sin x\cos x - y^2\cos x,\ x = \frac{\pi}{2},\ y = 1 \text{ is}$$

(a) $y^2 = \sin x$ (b) $y = \sin^2 x$

(c) $y^2 = \cos x + 1$ (d) None of these

4. The solution of the differential equation $x\left(\frac{dy}{dx}\right) + y = y^2\log x$ is

(a) $\frac{1}{y} = \log x + 1 + Cx$ (b) $y = \log x + 1 + Cx$

(c) $\frac{1}{y} = \log x - 1 + \frac{C}{x}$ (d) None of these

WORKED OUT Examples

Example 1 *Form the differential equation having* $y = (\sin^{-1} x)^2 + A\cos^{-1} x + B$, *where A and B are arbitrary constant, then its general solution is* **[NCERT Exemplar]**

(a) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$

(b) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2 = 0$

(c) $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 2 = 0$

(d) $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 2` = 0$

Solution (a) Given, $y = (\sin^{-1} x)^2 + A\cos^{-1} x + B$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2(\sin^{-1} x)\times 1}{\sqrt{1-x^2}} + A\left[\frac{-1}{\sqrt{1-x^2}}\right] + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} - \frac{A}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin^{-1} x - A}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1} x - A$$

Again, differentiating w.r.t. x, we get

$$\sqrt{1-x^2}\,\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(\sqrt{1-x^2}) = 2\frac{d}{dx}\sin^{-1} x - \frac{d}{dx}(A)$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \frac{2\times 1}{\sqrt{1-x^2}} - 0$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx}}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

This is the required differential equation of the given general solution.

Example 2 *The solution of differential equation* $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$ *is* **[NCERT Exemplar]**

(a) $\dfrac{1}{x}\log\left(\dfrac{x}{y}\right) = C$ (b) $\dfrac{1}{y}\log\left(\dfrac{x}{y}\right) = C$

(c) $\log\left(\dfrac{y}{x}\right) = Cx$ (d) $\log\left(\dfrac{y}{x}\right) = Cy$

Solution (c) Given, $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

$$\Rightarrow \quad x\frac{dy}{dx} = y\left[\log\left(\frac{y}{x}\right) + 1\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right) + 1\right] \quad \text{...(i)}$$

This is a homogencous equation. To simplify it, put

$$y = vx \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

On substituting the value of y and $\dfrac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left[\log\left(\frac{vx}{x}\right) + 1\right]$$

$$\Rightarrow \quad x\frac{dv}{dx} = v[\log v + 1] - v$$

$$x\frac{dv}{dx} = v[\log v + 1 - 1]$$

$$\Rightarrow \quad \frac{1}{v\log v}dv = \frac{1}{x}dt$$

On integrating both sides, we get

$$\int\frac{1}{v\log v}dv = \int\frac{1}{x}dx \quad \text{...(ii)}$$

Putting $\log v = t \Rightarrow \dfrac{1}{v} = \dfrac{dt}{dv} \Rightarrow dv = v\,dt$

Then, Eq. (ii) becomes

$$\int\frac{1}{vt}v\,dt = \int\frac{1}{x}dx$$

$$\Rightarrow \quad \int\frac{1}{t}dt = \int\frac{1}{x} \Rightarrow \log(t) = \log(x) + \log C$$

$$\Rightarrow \quad \log(\log v) - \log x = \log C \quad (\because t = \log v)$$

$$\Rightarrow \quad \log\left(\frac{\log v}{x}\right) = \log C \Rightarrow \frac{\log v}{x} = C$$

$$\Rightarrow \quad \frac{\log\left(\frac{y}{x}\right)}{x} = C$$

$$\Rightarrow \quad \log\left(\frac{y}{x}\right) = xC \qquad (\because y = vx)$$

which is the required solution.

Example 3 *The solution of the equation* $(x-h)^2 + (y-k)^2 = a^2$, *where h and k are parameters, is*

(a) $\left[1+\left(\frac{d^2y}{dx^2}\right)\right]^3 = a^2\left(\frac{dy}{dx}\right)^2$ (b) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(c) $\left[1+\left(\frac{dy}{dx}\right)^3\right]^2 = a^2\left(\frac{d^2y}{dx^2}\right)$ (d) *None of these*

Solution (*b*) Three relations are necessary to eliminate two constants. Thus, besides the given relation, we require two more and they will be obtained by differentiating the given relation twice successively.

Thus, we have

$$(x-h) + (y-k)\frac{dy}{dx} = 0 \qquad \ldots(i)$$

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we obtain $y - k = -\dfrac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$

and $$x - h = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

On substituting these values in the given relation, we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Example 4 *The solution of differential equation* $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$ *is*

(a) $\frac{1}{2}\sqrt{1+x^2} + \log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) = -\sqrt{1+y^2} + C$

(b) $-\frac{1}{2}\sqrt{1+x^2} + \log\left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right) = \sqrt{1+y^2} + C$

(c) $-\left[\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)\right] = \sqrt{1+y^2} + C$

(d) *None of the above*

Solution (*c*) $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

$$\Rightarrow \quad -\frac{\sqrt{1+x^2}}{x}dx = \frac{y\,dy}{\sqrt{1+y^2}}$$

On integrating, we get

$$-\int\frac{\sqrt{1+x^2}}{x}dx = \int\frac{y\,dy}{\sqrt{1+y^2}}$$

$$\Rightarrow \quad -\left[\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)\right] = \sqrt{1+y^2} + C$$

This is the solution to the given differential equation.

Example 5 *The solution of the differential equation* $(x+y)^2\frac{dy}{dx} = a^2$ *is*

(a) $y = a\tan^{-1}(x+y) + C$ (b) $y = a\tan^{-1}\left(\frac{x+y}{a}\right) - C$

(c) $y = \tan^{-1}\left(\frac{x+y}{a}\right) + C$ (d) *None of these*

Solution (*b*) $$(x+y)^2\frac{dy}{dx} = a^2 \qquad \ldots(i)$$

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \left(\frac{dt}{dx} - 1\right)$

$\therefore$ Eq. (i) reduces to,

$$t^2\left\{\frac{dt}{dx} - 1\right\} = a^2 \Rightarrow t^2\frac{dt}{dx} = a^2 + t^2,$$

On separating the variable and then on integrating, we get

$$\int dx = \int\frac{t^2}{a^2+t^2}dt = \int\left(1 - \frac{a^2}{a^2+t^2}\right)dt$$

$$\Rightarrow \quad x = t - a\tan^{-1}\left(\frac{t}{a}\right) + C$$

$$\Rightarrow \quad x = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) + C$$

$$\Rightarrow \quad y = a\tan^{-1}\left(\frac{x+y}{a}\right) - C$$

which is the required general solution.

Example 6 *The general solution of the differential equation* $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ *is* **[NCERT]**

(a) $(x - y - 1) = A(1 + x + y - 2xy)$

(b) $(x - y + 1) = A(1 - x + y - 2xy)$

(c) $(x + y + 1) = A(1 + x + y + 2xy)$

(d) $(x + y + 1) = A(1 - x - y - 2xy)$, where A is a parameter.

Solution (*d*) Given, differential equation is

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0 \Rightarrow \frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$$

On integrating both sides, we get

$$\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C$$

$$\Rightarrow \quad \int \frac{dy}{y^2+y+1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} + \int \frac{dx}{x^2+x+1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} = C$$

$$\Rightarrow \quad \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(1-\frac{1}{4}\right)} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(1-\frac{1}{4}\right)} = C$$

$$\Rightarrow \quad \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = C$$

$$\left(\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = \frac{\sqrt{3}C}{2} = k \quad \text{(say)}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right] = k$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{2y+1+2x+1}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right] = k$$

$$\Rightarrow \quad \frac{2\sqrt{3}\,(x+y+1)}{3-(4xy+2x+2y+1)} = \tan k$$

$$\Rightarrow \quad \frac{2\sqrt{3}\,(x+y+1)}{2(1-x-y-2xy)} = \tan k$$

$$\Rightarrow \quad x+y+1 = \frac{1}{\sqrt{3}}\tan k\,(1-x-y-2xy)$$

Let, $A = \frac{1}{\sqrt{3}}\tan k$ is an arbitrary constant.

$$\Rightarrow \quad x \times y + 1 = A(1-x-y-2xy)$$

Hence, the given result is proved.

Example 7 *The solution of the differential equation* $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ *is*

(*a*) $x = \tan^{-1}y + 1 + Ce^{-\tan^{-1}y}$ (*b*) $y = \tan^{-1}y + 1 + Ce^{-\tan^{-1}y}$

(*c*) $x = \tan^{-1}y + Ce^{-\tan^{-1}y}$ (*d*) *None of these*

Solution (*d*) The given differential equation is

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

This equation can be rewritten as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Here, $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1}y}{1+y^2}$

$$\therefore \quad \text{IF} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Therefore, required solution is

$$xe^{\tan^{-1}y} = \int e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}\,dy + C$$

$$\Rightarrow \quad xe^{\tan^{-1}y} = (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + C$$

$$\Rightarrow \quad x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$$

Example 8 *The solution of* $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$ *is* **[NCERT]**

(*a*) $y\log x = \frac{1}{x}(1+\log x) + C$ (*b*) $y\log x = \frac{2}{x}(1+\log x) + C$

(*c*) $y\log x = 2x(1+\log x) + C$ (*d*) $y\log x = \frac{-2}{x}(1+\log x) + C$

Solution (*d*) Given, $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

On dividing by $x\log x$ on both sides, we get $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x\log x} \text{ and } Q = \frac{2}{x^2}$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int \frac{1\,dx}{x\log x}}$$

Let $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow dx = x\,dt$

$$\therefore \quad \text{IF} = e^{\int \frac{1}{t}dt} = e^{\log|t|} = t = \log x \quad (\because t = \log x)$$

The general solution of the given differential equation is given by,

$$y\cdot\text{IF} = \int Q\,\text{IF}\,dx + C \Rightarrow y\cdot\log x = \int \frac{2}{x^2}\log x\,dx + C$$

$$\Rightarrow \quad y\log x = 2\int\left(\underset{\text{I}}{\log x}\cdot\underset{\text{II}}{\frac{1}{x^2}}\right)dx + C$$

$$= 2\left[\log x\int\frac{1}{x^2}dx - \int\left(\frac{d}{dx}(\log x)\int\frac{1}{x^2}dx\right)dx\right] + C$$

(integration by part)

$$= 2\left[\log x\left(-\frac{1}{x}\right) - \int\left\{\frac{1}{x}\cdot\left(-\frac{1}{x}\right)\right\}dx\right] + C$$

$$= 2\left[-\frac{\log x}{x} + \int\frac{1}{x^2}dx\right] + C = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right] + C$$

$$\Rightarrow \quad y\log x = -\frac{2}{x}(1+\log x) + C$$

Example 9 *The solution of differential equation* $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}\cdot x\,dy$ *is* **[NCERT]**

(a) $\cos\left(\frac{y}{x}\right)=A\,xy$ (b) $\cos\left(\frac{x}{y}\right)=A\,xy$

(c) $xy\cos\left(\frac{x}{y}\right)=A$ (d) $xy\cos\left(\frac{y}{x}\right)=A$

Solution (d) Given, $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx$

$$=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y\left\{x\cos\frac{y}{x}+y\sin\frac{y}{x}\right\}}{x\left\{y\sin\frac{y}{x}-x\cos\frac{y}{x}\right\}} \quad \ldots(\text{i})$$

Thus, the given differential equation is homogeneous.

So, put $y=vx$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore \quad v+x\frac{dv}{dx}=\frac{vx\{x\cos v+vx\sin v\}}{x\{vx\sin v-x\cos v\}}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{v\{\cos v+v\sin v\}}{v\sin v-\cos v}$$

$$x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \quad \left(\frac{v\sin v-\cos v}{v\cos v}\right)dv=\frac{2}{x}dx$$

$$\Rightarrow \quad \left(\tan v-\frac{1}{v}\right)dv=\frac{2}{x}dx$$

On integrating both sides, we get $\int\left(\tan v-\frac{1}{v}\right)dv=\int\frac{2dx}{x}$

$$\Rightarrow \quad \int\tan v\,dv-\int\frac{1}{v}dv=2\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\log|\cos v|-\log|v|=2\log|x|+C$$

$$\Rightarrow \quad \log|v\cos v|+2\log|x|=-C \quad (\because \log m+\log n=\log mn)$$

$$\Rightarrow \quad \log[(v\cos v)x^2]=-C$$

$$\Rightarrow \quad (v\cos v)x^2=e^{-C} \quad (\because \log_e x=m\Rightarrow e^m=x)$$

$$\Rightarrow \quad x^2v\cos v=A \text{ (where, } A=e^{-C})$$

$$\Rightarrow \quad x^2\frac{y}{x}\cos\frac{y}{x}=A \quad \left(\because y=\frac{v}{x}\right)$$

$$\Rightarrow \quad xy\cos\frac{y}{x}=A$$

This is the required solution of the given differential equation.

Example 10 *The solution of the differential equation* $\frac{dy}{dx}=e^{x-y}(e^x-e^y)$ *is*

(a) $e^y=(e^x+1)+Ce^{-e^x}$ (b) $e^y=(e^x-1)+C$

(c) $e^y=(e^x-1)+Ce^{-e^x}$ (d) *None of these*

Solution (c) $\frac{dy}{dx}=\frac{e^x}{e^y}(e^x-e^y)$

$$\Rightarrow \quad e^y\frac{dy}{dx}+e^x\cdot e^y=e^x\cdot e^x$$

Let $\quad e^y=t\Rightarrow e^y\frac{dy}{dx}\frac{y}{x}=\frac{dt}{dx}$

Then, given equation reduces to

$$\frac{dt}{dx}+e^xt=e^{2x}$$

Here, $\quad P=e^x$ and $Q=e^{2x}$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int e^x dx}=e^{e^x}$$

Required solution is

$$t\cdot e^{e^x}=\int e^{2x}\cdot e^{e^x}\,dx+C$$

$$\Rightarrow \quad e^y=(e^x-1)+Ce^{-e^x}$$

Example 11 *What is the solution of the equation* $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}$ *tan y sin y ?*

(a) $2x=(1-2Cx^2)\sin y$ (b) $x=(1-2Cx^2)\sin y$

(c) $2x=(1+2Cx^2)\sin y$ (d) *None of these*

Solution (a) The given differential equation can be rewritten as

$$\cot y\,\text{cosec}\,y\frac{dy}{dx}+\frac{1}{x}\text{cosec}\,y=\frac{1}{x^2}$$

On putting $v=-\text{cosec}\,y$ and $\frac{dv}{dx}=\text{cosec}\,y\cot y\frac{dy}{dx}$, we get

$$\frac{dv}{dx}-\frac{1}{x}v=\frac{1}{x^2}$$

Here, $\quad P=-\frac{1}{x}$ and $Q=\frac{1}{x^2}$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int-\frac{1}{x}dx}=\frac{1}{x}$$

Required solution is

$$v\frac{1}{x}=\int\frac{1}{x}\cdot\frac{1}{x^2}dx+C=\frac{-1}{2x^2}+C$$

$$\Rightarrow \quad 2x=(1-2Cx^2)\sin y$$

Example 12 *The solution of the differential equation* $\frac{dy}{dx}=\frac{yf'(x)-y^2}{f(x)}$ *is*

(a) $f(x)=y+C$ (b) $f(x)=y(x+C)$

(c) $f(x)=x+C$ (d) *None of these*

Solution (*b*) The given equation is

$$\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$$

$$\Rightarrow \quad yf'(x)\,dx - f(x)\,dy = y^2\,dx$$

$$\Rightarrow \quad \frac{yf'(x)\,dx - f(x)\,dy}{y^2} = dx$$

$$\Rightarrow \quad d\left[\frac{f(x)}{y}\right] = dx$$

On integration, we get

$$\frac{f(x)}{y} = x + C$$

$$\Rightarrow \quad f(x) = y(x + C)$$

Example 13 *The equation of the curve passing through the point* $\left(1, \frac{\pi}{4}\right)$ *and having slope of tangent at any point* (x, y) *as* $\frac{y}{x} - \cos^2\frac{y}{x}$, *is*

(*a*) $x = e^{1 - \tan\left(\frac{y}{x}\right)}$ (*b*) $x = e^{1 + \tan\left(\frac{y}{x}\right)}$

(*c*) $x = 1 - \tan\left(\frac{y}{x}\right)$ (*d*) *None of these*

Solution (*a*) Given, $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$

$$\Rightarrow \quad \frac{x\,dy - y\,dx}{x} = -\left(\cos^2\frac{y}{x}\right)dx$$

$$\Rightarrow \quad \sec^2\frac{y}{x}\left(\frac{x\,dy - y\,dx}{x^2}\right) = -\frac{dx}{x}$$

$$\Rightarrow \quad \sec^2\frac{y}{x}\,d\left(\frac{y}{x}\right) = -\frac{dx}{x}$$

$$\Rightarrow \quad \tan\frac{y}{x} = -\log x + C$$

At $\quad x = 1, y = \frac{\pi}{4}, C = 1$

$$\therefore \quad \tan\frac{y}{x} = 1 - \log x$$

$$\Rightarrow \quad \log x = 1 - \tan\left(\frac{y}{x}\right)$$

$$\therefore \quad x = e^{1 - \tan\left(\frac{y}{x}\right)}$$

Example 14 *The differential equation of the family of circles touching the y-axis at origin is* $\left(y' = \frac{dy}{dx}\right)$ **[NCERT]**

(*a*) $2xyy' + x^2 = y^2$ (*b*) $2xy' + x^2 = y^2$

(*c*) $xyy' + 2x^2 = y^2$ (*d*) *None of these*

Solution (*a*) The equation of family of circle touching the *x*-axis at origin is given by $(x - a)^2 + y^2 = a^2$, where *a* is radius of circle.

Differentiate this equation once because one arbitrary constant is present in the equation.

The centre of the circle touching the *y*-axis at origin lies on the *x*-axis.

Let $(a, 0)$ be the centre of the circle and its radius is *a*.

Now, the equation of the family of circle with centre $(a, 0)$ and radius (a) is

$$(x - a)^2 + y^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 = 2ax \qquad \ldots(\text{i})$$

On differentiating Eq. (i) w.r.t *x*, we get

$$2x + 2yy' = 2a \Rightarrow x + yy' = a$$

Now, substituting the value of *a* in Eq. (i), we get

$$x^2 + y^2 = 2(x + yy')x$$

$$\Rightarrow \quad x^2 + y^2 = 2x^2 + 2xyy'$$

$$\Rightarrow \quad 2xyy' + x^2 = y^2$$

which is the required differential equation.

Example 15 *The differential equation of the family of ellipse having foci on y-axis and centre at origin is* $\left(y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$ **[NCERT]**

(*a*) $xy'' + x(y')^2 - xy' = 0$ (*b*) $xyy'' + 2x(y')^2 - yy' = 0$

(*c*) $xyy'' + x(y')^2 - yy' = 0$ (*d*) $xyy'' - x(y')^2 + yy' = 0$

Solution (*a*) An ellipse having foci on *y*-axis and centre at origin represented by

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Differentiate this equation twicely because two arbitrary constants are present in this equation.

Equation of family of ellipse is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots(\text{i})$$

(since, foci is on *y*-axis, so we draw a vertical ellipse)

On differentiating Eq. (i) w.r.t. *x*, we get

$$\frac{d}{dx}\left(\frac{x^2}{b^2}\right) + \frac{d}{dx}\left(\frac{y^2}{a^2}\right) = \frac{d}{dx}(1) \qquad \ldots(\text{ii})$$

$$\Rightarrow \quad \frac{1}{b^2}2x + \frac{1}{a^2}2yy' = 0 \Rightarrow \frac{yy'}{x} = -\frac{a^2}{b^2}$$

Again, differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{x\frac{d}{dx}(yy') - yy'\frac{d}{dx}(x)}{x^2} = 0$$

(using quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$)

$$\Rightarrow \quad \frac{x[yy' + (y')^2] - yy'\cdot 1}{x^2} = 0$$

Using product rule $\frac{d}{dx}(u\cdot v) = \left(u\frac{d}{dx}v + v\frac{d}{dx}u\right)$

$$\Rightarrow \quad x(y')^2 + xyy'' - yy' = 0$$

$$\Rightarrow \quad xyy'' + x(y')^2 - yy' = 0$$

which is the required differential equation.

Example 16 *The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 s it is 6 units. Then, the radius of balloon after t seconds is* [**NCERT**]

(a) $(63t + 27)^{1/3}$

(b) $(27t + 63)^{1/3}$

(c) $(27t + 9)^{1/3}$

(d) $(9t + 63)^{1/3}$

Solution (a) Let the rate of change of the volume of the balloon be k (where, k is constant)

$$\frac{d}{dt}(\text{volume}) = \text{constant}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left(\because \text{volume of sphere} = \frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right) = k$$

On separating the variables, we get $4\pi r^2\, dr = k\, dt$ ...(i)

On integrating both sides, we get $4\pi\int r^2 dr = k\int dt$

$$\Rightarrow \quad 4\pi\frac{r^3}{3} = kt + C \Rightarrow 4\pi r^3 = 3(kt + C) \qquad \text{...(ii)}$$

Now, at $t = 0$, $r = 3$ initially;

$$\therefore \quad 4\pi(3)^3 = 3(k\times 0 + C)$$

$$\Rightarrow \quad 108\pi = 3C \Rightarrow C = 36\pi$$

Also, when $t = 3$, $r = 6$, then from Eq. (ii),

$$4\pi(6)^3 = 3(k\times 3 + C) \Rightarrow 864\pi = 3(3k + 36\pi)$$

$$\Rightarrow \quad 3k = 288\pi - 36\pi = 252\pi \Rightarrow k = 84\pi$$

On substituting the values of k and C in Eq. (ii), we get

$$4\pi r^3 = 3[84\pi t + 36\pi] \Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$

$$\Rightarrow \quad r^3 = 63t + 27 \Rightarrow r = (6t + 27)^{1/3}$$

which is the required radius of the balloon at time t.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Order and Degree of a Differential Equation

1. Order and degree of the differential equation $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ are [NCERT]

(a) order = 4, degree = 1
(b) order = 3, degree= 1
(c) order = 4, degree = 0
(d) order = 4, degree = not defined

2. Order and degree of the differential equation $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ are [NCERT]

(a) order = 2, degree = 1 (b) order = 2, degree = 4
(c) order = 1, degree = 4 (d) order = 1, degree= 1

3. Order and degree of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \sin y = 0$ are [NCERT]

(a) order = 1, degree = 1
(b) order = 1, degree = not defined
(c) order = 2, degree = 1
(d) order = 2, degree = not defined

4. The number of arbitrary constants in the general solution of a differential equation of fourth order is [NCERT]

(a) zero (b) 2 (c) 3 (d) 4

5. The number of arbitrary constants in the particular solution of a differential equation of third order is [NCERT]

(a) 3 (b) 2 (c) 1 (d) zero

6. Family $y = Ax + A^3$ of curves is represented by the differential equation of degree {NCERT Exemplar]

(a) 1 (b) 2
(c) 3 (d) 4

7. The order of the differential equation whose general solution is given by $y = c_1e^{2x+c_2} + c_3e^x + c_4\sin(x + c_5)$ is

(a) 5 (b) 4 (c) 3 (d) 2

8. The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4e^{x+c_5}$, where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants is

(a) 5 (b) 6 (c) 3 (d) 2

9. The degree of the differential equation to of all tangent lines to the parabola $y^2 = 4ax$ is

(a) 1 (b) 2 (c) 3 (d) 4

Formation of a Differential Equation

10. The differential equation of the curve $y^2 = a(b^2 - x^2)$ representing the given family of curves, where a and b are constants, is [NCERT]

(a) $x[(y'')^2 + yy']$ (b) $x[(y')^2 + yy'']$
(c) $[(y'')^2 + yy']$ (d) None of these

11. The differential equation of the curve $y = e^x(a\cos x + b\sin x)$ representing the given family of curves, where a and b are constants, is [NCERT]

(a) $y'' - 2y' + 2y = 0$ (b) $y'' + 2y' - 2y = 0$
(c) $y'' + 2y' + 2y = 0$ (d) None of these

12. The differential equation of the family of parabolas having vertex at origin and axis along positive y-axis is [NCERT]

(a) $xy' - y = 0$ (b) $xy' + 2y = 0$
(c) $xy' - 2y = 0$ (d) $xy' + y = 0$

13. The differential equation of the family of hyperbolas having foci on x-axis and centre at origin is [NCERT]

(a) $xyy'' + x(y')^2 - yy' = 0$
(b) $xy'' + x(y')^2 - 2yy' = 0$
(c) $xy'' + 2x(y')^2 - yy' = 0$
(d) None of the above

14. Which of the following differential equation has $y = c_1e^x + c_2e^{-x}$ as the general solution? [NCERT]

(a) $\frac{d^2y}{dx^2} + y = 0$ (b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{d^2y}{dx^2} + 1 = 0$ (d) $\frac{d^2y}{dx^2} - 1 = 0$

15. The differential equation of the family of circles in the first quadrant which touch the coordinate axes is [NCERT]

(a) $(x - y)^2[1 + (y')] = [x + yy']^2$

(b) $(x - y)^2[1 + (y')^2] = (x + yy')^2$

(c) $(x + y)^2[1 + y')^2] = (x + yy')^2$

(d) None of the above

16. The differential equation of all circles which pass through origin and whose centres lie on y-axis is [NCERT Exemplar]

(a) $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ (b) $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$

(c) $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ (d) None of these

17. The differential equation by eliminating A and B in $Ax^2 + By^2 = 1$ is [NCERT Exemplar]

(a) $x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

(b) $x\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

(c) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(d) None of the above

18. The differential equation of system of concentric circles with centre (1,2) is [NCERT Exemplar]

(a) $\frac{dy}{dx} = \frac{x-1}{x-2}$ (b) $\frac{dy}{dx} = \frac{x-1}{2-x}$

(c) $\frac{dy}{dx} = \frac{1-x}{2-y}$ (d) $\frac{dy}{dx} = \frac{1-x}{y-2}$

19. The differential equation of the family of curves $x^2 + y^2 - 2ay = 0$, where a is arbitrary constant, is [NCERT Exemplar]

(a) $(x^2 - y^2)\frac{dy}{dx} = 2xy$ (b) $2(x^2 + y^2)\frac{dy}{dx} = xy$

(c) $2(x^2 - y^2)\frac{dy}{dx} = xy$ (d) $(x^2 + y^2)\frac{dy}{dx} = 2xy$

20. The differential equation of the family of curves $y^2 = 4a(x + a)$ is [NCERT Exemplar]

(a) $y^2 = 4\frac{dy}{dx}\left(x + \frac{dy}{dx}\right)$ (b) $2y\frac{dy}{dx} = 4a$

(c) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ (d) $2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 - y = 0$

21. The differential equation of the rectangular hyperbola whose axes are the asymptotes of the hyperbola, is

(a) $y\frac{dy}{dx} = x$ (b) $x\frac{dy}{dx} = -y$

(c) $x\frac{dy}{dx} = y$ (d) $x\,dy + y\,dx = C$

22. The differential equation of the family of parabolas with focus at the origin and the x-axis as axis, is

(a) $y\left(\frac{dy}{dx}\right)^2 + 4x\frac{dy}{dx} = 4y$

(b) $-y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$

(c) $y\left(\frac{dy}{dx}\right)^2 + y = 2xy\frac{dy}{dx}$

(d) $y\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y = 0$

23. The differential equation of all circles passing through the origin and having their centres on the x-axis is

(a) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (b) $y^2 = x^2 - 2xy\frac{dy}{dx}$

(c) $x^2 = y^2 + xy\frac{dy}{dx}$ (d) $x^2 = y^2 - xy\frac{dy}{dx}$

24. The differential equation that represents all parabolas each of which has a latusrectum $4a$ and whose axes are parallel to x-axis, is

(a) $a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ (b) $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

(c) $2a\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3$ (d) $a\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 = 0$

25. The equation of the curve through the point (1, 1) and whose slope is $\frac{2ay}{x(y-a)}$, is

(a) $y^a \cdot x^{2a} = e^{y-1}$ (b) $y^a \cdot x^{2a} = e^y$

(c) $y^{2a} \cdot x^a = e^{y-1}$ (d) $y^a \cdot x^a = e^y$

Solution of a Differential Equation

26. The particular solution of the differential equation $\cos\left(\frac{dy}{dx}\right) = a, (a \in R)$, satisfying the condition, $y = 2$ when $x = 0$ is [NCERT]

(a) $\cos\left(\frac{y+2}{x}\right) = a$ (b) $\cos\left(\frac{x+2}{y}\right) = a$

(c) $\cos\left(\frac{y-2}{x}\right) = a$ (d) $\cos\left(\frac{x-2}{y}\right) = a$

27. The equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$ is [NCERT]

(a) $2y - 1 = e^x(\sin x - \cos x)$

(b) $2y - 1 = e^{-x}(\cos x - \sin x)$

(c) $2y + 1 = e^{-x}(\cos x - \sin x)$

(d) None of the above

28. The particular solution of the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$, satisfying the condition $y = \frac{\pi}{4}$, when $x = 1$, is [NCERT]

(a) $\tan\left(\frac{y}{x}\right) = \log(ex)$ (b) $\tan\left(\frac{x}{y}\right) = \log(ex)$

(c) $\cot\left(\frac{y}{x}\right) = \log\left(\frac{e}{x}\right)$ (d) $\cot\left(\frac{y}{x}\right) = \log(ex)$

29. The particular solution of the differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, satisfying the condition, $y = 2$ when $x = 1$, is [NCERT]

(a) $y = \frac{x}{1 - \log|x|}$ (b) $y = \frac{x}{1 + \log|x|}$

(c) $y = \frac{2x}{1 - \log|x|}$ (d) None of these

30. Particular solution of the differential equation $\frac{dy}{dx} - 3y\cot x = \sin 2x$, where $y = 2$ and $x = \frac{\pi}{2}$ is [NCERT]

(a) $y = 4\sin^3 x - \sin^2 x$ (b) $y = 4\sin^3 x - 2\sin^2 x$

(c) $y = 4\sin^3 x + \sin^2 x$ (d) None of these

31. The equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos y\, dx + \cos x \sin y\, dy = 0$, is [NCERT]

(a) $y = \frac{\cos x}{\sqrt{2}}$ (b) $y = \frac{\sin x}{2}$

(c) $y = \frac{\sec x}{\sqrt{2}}$ (d) $y = \frac{\operatorname{cosec} x}{\sqrt{2}}$

32. The particular solution of the differential equation $(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$ given that $y = 1$ when $x = 0$, is [NCERT]

(a) $\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$

(b) $\tan^{-1} x + \tan^{-1} e^y = \frac{\pi}{2}$

(c) $\tan^{-1} x + \tan^{-1} e^y = \frac{\pi}{4}$

(d) $\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{3}$

33. A particular solution of the differential equation $(x - y)(dx + dy) = dx - dy$ given that $y = -1$, when $x = 0$, is [NCERT]

(a) $\log|x - y| = x + y + 1$ (b) $\log|x - y| = x - y - 1$

(c) $\log|x - y| = x + y - 1$ (d) None of these

34. Solution of the differential equation $x^2\frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right)$, where $x \neq 0$ and $x = 1$, $y = \frac{\pi}{2}$, is [NCERT Exemplar]

(a) $\tan\left(\frac{y}{x}\right) = -\frac{1}{2x^2} - \frac{3}{2}$

(b) $\tan\left(\frac{2y}{x}\right) = -\frac{1}{x^2} + \frac{3}{2}$

(c) $\tan\left(\frac{y}{2x}\right) = -\frac{1}{x^2} - \frac{3}{2}$

(d) $\tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$

35. If $y(x)$ is a solution of $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then the value of $y\left(\frac{\pi}{2}\right)$ is [NCERT Exemplar]

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) 1 (d) 0

36. If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then the value of $y(1)$ is [NCERT Exemplar]

(a) 1 (b) -1

(c) $-\frac{1}{2}$ (d) 0

37. The function $f(\theta) = \frac{d}{d\theta}\int_0^\theta \frac{dx}{1 - \cos\theta\cos x}$ satisfies the differential equation

(a) $\frac{df}{d\theta} + 2f(\theta) = 0$ (b) $\frac{df}{d\theta} - 2f(\theta) = 0$

(c) $\frac{df}{d\theta} - 2f(\theta) = \tan\theta$ (d) $\frac{df}{d\theta} + 2f(\theta)\cot\theta = 0$

38. The differential equation $\frac{d^2y}{dx^2} = 2$ represents

(a) a parabola whose axis is parallel to x-axis

(b) a parabola whose axis is parallel to y-axis

(c) a circle

(d) None of the above

39. The solution of the equation $x^2\frac{d^2y}{dx^2} = \log x$ when $x = 1$, $y = 0$ and $\frac{dy}{dx} = -1$ is

(a) $y = \frac{1}{2}(\log x)^2 + \log x$ (b) $y = \frac{1}{2}(\log x)^2 - \log x$

(c) $y = -\frac{1}{2}(\log x)^2 + \log x$ (d) $y = -\frac{1}{2}(\log x)^2 - \log x$

40. A continuously differential function $\phi(x)$ in $(0, \pi)$ satisfying $y' = 1 + y^2$, $y(0) = 0 = y(\pi)$, is

(a) $\tan x$ (b) $x(x - \pi)$

(c) $(x - \pi)(1 - e^x)$ (d) Not possible

41. If $\phi(x) = \int \{\phi(x)\}^{-2} dx$ and $\phi(1) = 0$, then $\phi(x)$ is equal to

(a) $\{2(x-1)\}^{1/4}$ (b) $\{5(x-2)\}^{1/5}$

(c) $\{3(x-1)\}^{1/3}$ (d) $\{2(x-1)\}^{1/2}$

42. Solution of the differential equation $\frac{dx}{dy} - \frac{x \log x}{1 + \log x} = \frac{e^y}{1 + \log x}$, if $y(1) = 0$, is

(a) $x^x = e^{ye^y}$ (b) $e^y = x^{e^y}$

(c) $x = ye^y$ (d) $y = e^{x^y}$

43. The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$, satisfying $y(1) = 0$, is given by a

(a) hyperbola (b) circle

(c) ellipse (d) parabola

Differential Equation of First Order and First Degree

44. The general solution of differential equation $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$ is [NCERT]

(a) $y = \log|e^x - e^{-x}| + C$ (b) $y = \log\left|\frac{e^x - e^{-x}}{e^x + e^{-x}}\right| + C$

(c) $y = \log|e^x + e^{-x}| + C$ (d) None of these

45. The general solution of differential equation $e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$ is [NCERT]

(a) $(Ce^{x-1}) = \tan y$

(b) $(e^x + 1) = C \tan y$

(c) $(Ce^{x-1}) = \cot y$

(d) None of the above

46. Solution of the differential equation $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$ is [NCERT]

(a) $x\left(1 + \cos\frac{y}{x}\right) = c\sin\left(\frac{y}{x}\right)$

(b) $x\left(1 - \cos\frac{y}{x}\right) = c\sin\left(\frac{x}{y}\right)$

(c) $x\left(1 - \cos\frac{y}{x}\right) = c\sin\left(\frac{y}{x}\right)$

(d) $\left(1 - \cos\frac{y}{x}\right) = cx\sin\left(\frac{y}{x}\right)$

47. Solution of the differential equation $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$ is [NCERT]

(a) $\log\left(\frac{y}{x}\right) + 1 = Cy$ (b) $\log\left(\frac{y}{x}\right) - 1 = Cy$

(c) $\log\left(\frac{y}{x}\right) - 1 = Cx$ (d) None of these

48. Solution of the differential equation $\frac{dy}{dx} + y\sec x = \tan x\left(0 \le x < \frac{\pi}{2}\right)$ is [NCERT]

(a) $y(\sec x - \tan x) = (\sec x + \tan x) - x + C$

(b) $y(\sec x + \tan x) = (\sec x - \tan x) - x + C$

(c) $y(\sec x + \tan x) = (\sec x + \tan x) - x + C$

(d) None of the above

49. Solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$ is [NCERT]

(a) $\frac{x^2}{16}(4\log x - 1) + Cx^{-2}$ (b) $\frac{x^2}{16}(2\log x - 1) + Cx^2$

(c) $\frac{x^2}{4}(4\log x - 1) + Cx^2$ (d) None of these

50. Solution of the differential equation $(1 + x^2)dy + 2xy\,dx = \cot x\,dx$ is [NCERT]

(a) $y = \log|\sin x|(1 + x^2) + C(1 + x^2)^{-1}$

(b) $y = \log|\sin x|(1 + x^2)^{-1} + C(1 + x^2)$

(c) $y = \log|\sin x|(1 + x^2) + C(1 + x)$

(d) $y = \log x|\sin x|(1 + x^2)^{-1} + C(1 + x^2)^{-1}$

51. Solution of the differential equation $ye^{x/y}dx = (xe^{x/y} + y^2)\,dy (y \ne 0)$, is [NCERT]

(a) $e^{x/y} = x + C$ (b) $e^{x/y} + x = C$

(c) $e^{y/x} = x + C$ (d) $e^{x/y} = y + C$

52. Solution of the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1\ (x \ne 0)$ [NCERT]

(a) $ye^{2x} = 2\sqrt{x} + C$ (b) $ye^{2x} = \frac{1}{2}\sqrt{x} + C$

(c) $ye^{2\sqrt{x}} = 2\sqrt{x} + C$ (d) $ye^{\frac{1}{2}\sqrt{x}} = 2\sqrt{x} + C$

53. The general solution of the differential equation $e^x dy + (ye^x + 2x)dx = 0$ is [NCERT]

(a) $xe^y + x^2 = C$ (b) $xe^y + y^2 = C$

(c) $ye^x + x^2 = C$ (d) $ye^y + x^2 = C$

54. Solution of the differential equation $xdy - ydx = \sqrt{x^2 + y^2}$ is [NCERT Exemplar]

(a) $y + \sqrt{x^2 + y^2} = Cx$ (b) $y + \sqrt{x^2 + y^2} = Cx^2$

(c) $y + \sqrt{x^2 + y^2} = C$ (d) None of these

55. Solution of the differential equation $(x^2 - 1)\dfrac{dy}{dx} + 2xy = \dfrac{1}{x^2 - 1}$ is [NCERT Exemplar]

(a) $y(x^2 - 1) = \log\left|\dfrac{x-1}{x+1}\right| + C$

(b) $y(x^2 - 1) = \dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right| + C$

(c) $y(x^2 + 1) = \log\left|\dfrac{x+1}{x-1}\right| + C$

(d) None of the above

56. Solution of the differential equation $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$ is [NCERT Exemplar]

(a) $y \cdot x^2 = x(2\sin x - x\cos x) + \dfrac{x^3}{9}(3\log x - 1) + 2\cos x + C$

(b) $y \cdot x^2 = x(\sin x - 2x\cos x) + \dfrac{x^3}{3}(3\log x - 1) + \cos x + C$

(c) $y \cdot x^2 = x(\sin x - x\cos x) + \dfrac{x^3}{3}(\log x - 3) + 2\cos x + C$

(d) None of the above

57. The general solution of

$(1 + \tan y)(dx - dy) + 2x\,dy = 0$, is [NCERT Exemplar]

(a) $x = \dfrac{1}{1 + \tan y} + \dfrac{C \cdot e^{-y}}{1 + \tan y}$

(b) $x = \dfrac{1}{1 + \tan y} + \dfrac{C \cdot e^{y}}{1 + \tan y}$

(c) $x = \dfrac{1}{1 + \tan y} + \dfrac{C \cdot e^{-y}}{(1 + \tan y)\sec y}$

(d) None of the above

58. Solution of the differential equation $(1 + y^2)\tan^{-1} x\,dx + 2y(1 + x^2)dy = 0$ is [NCERT Exemplar]

(a) $\dfrac{\tan^{-1} x}{2} + \log(1 + y^2) = C$

(b) $\dfrac{(\tan^{-1} x)^2}{2} + \log(1 + y^2) = C$

(c) $2(\tan^{-1} x)^2 - \log(1 + y^2) = C$

(d) None of the above

59. The differential equation $y\dfrac{dy}{dx} + x = c$ represents family of [NCERT Exemplar]

(a) hyperbolas (b) parabolas

(c) ellipses (d) circles

60. The general solution of $\dfrac{dy}{dx} = 2x\,e^{x^2 - y}$ is [NCERT Exemplar]

(a) $e^{x^2 - y} = C$ (b) $e^{-y} + e^{x^2} = C$

(c) $e^{y} = e^{x^2} + C$ (d) $e^{x^2 + y} = C$

61. The solution of the equation $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$ is

(a) $y\sin y = x^2\log x + C$ (b) $y = x^2 + \log x + C$

(c) $y\sin y = x^2 + C$ (d) None of these

62. The solution of $\dfrac{dy}{dx} + xy = xy^2$ is

(a) $y = 1 + ce^{x^2/2}$ (b) $\dfrac{1}{y} = 1 + ce^{x^2/2}$

(c) $\dfrac{1}{y} = 1 + ce^{x/2}$ (d) None of these

63. The solution of differential equation $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is

(a) $e^{y} = \dfrac{x^3}{3} + C$ (b) $e^{x} = \dfrac{x^3}{3} + e^{y} + C$

(c) $e^{y} = \dfrac{x^3}{3} + e^{x} + C$ (d) None of these

64. The solution of $x\,dy - y\,dx + x^2e^x dx = 0$ is

(a) $\dfrac{y}{x} + e^x = C$ (b) $\dfrac{x}{y} + e^x = C$

(c) $x + e^y = C$ (d) $y + e^x = C$

65. The general solution of the differential equation $\dfrac{dy}{dx} + \sin\left(\dfrac{x+y}{2}\right) = \sin\left(\dfrac{x-y}{2}\right)$ is

(a) $\log\tan\left(\dfrac{y}{2}\right) = C - 2\sin x$

(b) $\log\tan\left(\dfrac{y}{4}\right) = C - 2\sin\left(\dfrac{x}{2}\right)$

(c) $\log\tan\left(\dfrac{y}{2} + \dfrac{\pi}{4}\right) = C - 2\sin x$

(d) $\log\tan\left(\dfrac{y}{4} + \dfrac{\pi}{4}\right) = C - 2\sin\left(\dfrac{x}{2}\right)$

66. The solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{\phi\left(\dfrac{y}{x}\right)}{\phi'\left(\dfrac{y}{x}\right)}$ is

(a) $\phi\left(\dfrac{y}{x}\right) = kx$ (b) $x\phi\left(\dfrac{y}{x}\right) = k$

(c) $\phi\left(\dfrac{y}{x}\right) = ky$ (d) $y\,\phi\left(\dfrac{y}{x}\right) = k$

67. The solution of differential equation $\dfrac{dt}{dx} = \dfrac{t\left[\dfrac{d}{dx}\{g(x)\}\right] - t^2}{g(x)}$ is

(a) $t = \dfrac{g(x) + C}{x}$ (b) $t = \dfrac{g(x)}{x} + C$

(c) $t = \dfrac{g(x)}{x + C}$ (d) $t = g(x) + x + C$

68. The solution of differential equation $t = 1 + (ty)\frac{dy}{dt} + \frac{(ty)^2}{2!}\left(\frac{dy}{dt}\right)^2 + \ldots \infty$ is

(a) $y = \pm\sqrt{(\log t)^2 + C}$ (b) $ty = t^y + C$

(c) $y = \log t + C$ (d) $y = (\log t)^2 + C$

69. The solution of $\left(y - \frac{xdy}{dx}\right) = 3\left(1 - x^2\frac{dy}{dx}\right)$ is

(a) $(y + 3)(1 - 3x) = Cx$ (b) $(y - 3) = 3xy + Cx$

(c) $(y + 3)(1 + 3x) = Cx$ (d) $y + 3 = 3xy - Cx$

70. The solution of $ydx - xdy = xydx$ is

(a) $y = Cxe^{-x}$ (b) $2y = Cxe^{-x}$

(c) $y = 3Cxe^{-x}$ (d) $y^2 = Cxe^{-x}$

71. The function $f(x)$ satisfying the equation, $f^2(x) + 4f'(x).f(x) + [f'(x)]^2 = 0$

(a) $f(x) = C \cdot e^{(-2 \pm \sqrt{3})x}$ (b) $f(x) = -C \cdot e^{(2+\sqrt{3})x}$

(c) $f(x) = C \cdot e^{(\sqrt{3}-2)x}$ (d) $f(x) = C \cdot e^{(2-\sqrt{3})x}$

where C is an arbitrary constant.

72. The solution of $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$ is

(a) $\sqrt{1+y^2} = Cxe^{\tan^{-1}x}$ (b) $\sqrt{1-y^2} = Cxe^{\tan^{-1}x}$

(c) $\sqrt{1+y^2} = Ce^{\tan^{-1}x}$ (d) $\sqrt{1+y^2} = Cx$

73. The solution of $x\frac{dy}{dx} = y + 2\sqrt{y^2 - x^2}$ is

(a) $\frac{1}{2}\ln\left(\frac{y + \sqrt{y^2 - x^2}}{x}\right) = \ln Cx$

(b) $\frac{1}{2}\ln\left(\frac{y - \sqrt{y^2 - x^2}}{x}\right) = \ln Cx$

(c) $\frac{1}{2}\ln(y - \sqrt{y^2 - x^2}) = \ln Cx$

(d) $\ln(y - \sqrt{y^2 - x^2}) = \ln Cx$

74. The real value of m for which the substitution $y = u^m$ will transform the differential equation $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is

(a) $m = 0$ (b) $m = 1$

(c) $m = \frac{3}{2}$ (d) $m = \frac{2}{3}$

75. Solution of $x\frac{dy}{dx} + y = x^2y^4$ is

(a) $x^2y^2(3 + Cx) = 1$

(b) $x^2y^3(3 + Cx) = 1$

(c) $x^3y^3(3 + Cx) = 1$

(d) $x^2y^3(3 - Cx) = 1$

Round II (Mixed Bag)

Only One Correct Option

1. The degree of the differential equation satisfying the relation $\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda(x\sqrt{1+y^2} - y\sqrt{1+x^2})$ is

(a) 1 (b) 2

(c) 3 (d) None of these

2. The orthogonal trajectories of the family of curves $a^{n-1}y = x^n$ are given by (a is the arbitrary constant)

(a) $x^n + n^2y$ = constant (b) $ny^2 + x^2$ = constant

(c) $n^2x + y^n$ = constant (d) $n^2x - y^n$ = constant

3. A curve passes through the point (0,1) and the gradient at (x, y) on it is $y(xy - 1)$. The equation of the curve is

(a) $y(x - 1) = 1$ (b) $y(x + 1) = 1$

(c) $x(y + 1) = 1$ (d) $x(y - 1) = 1$

4. A tangent to the curve $y = f(x)$ cuts the line $y = x$ at a point which is at a distance of 1 unit from y-axis. The equation of the curve is

(a) $\frac{x-1}{y-1} = C$ (b) $\frac{x}{y} = C$

(c) $xy = C$ (d) None of these

5. The integrating factor of the differentiable equation $(xy - 1)\frac{dy}{dx} + y^2 = 0$ is

(a) $\frac{1}{x}$ (b) $\frac{1}{y}$

(c) $\frac{1}{xy}$ (d) xy

6. An inverted conical tank of 2 m radius and 4 m height is initially full of water has an outlet at bottom. The outlet is opened at some instant. The rate of flow through the outlet at any time t is $6\,h^{3/2}$, where h is height of water level above the outlet at time t. Then, the time it takes to empty the tank, is

(a) $\frac{2\pi}{9}$ unit (b) $\frac{\pi}{9}$ unit

(c) $\frac{2\pi}{8}$ unit (d) None of these

7. The equation of a curve passing through (2,7/2) and having gradient $1 - \frac{1}{x^2}$ at (x, y) is

(a) $y = x^2 + x + 1$ (b) $xy = x^2 + x + 1$
(c) $xy = x + 1$ (d) None of these

8. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If is its graph passes through the point (2,1) and at the point the tangent to the graph is $y = 3x - 5$, then the function

(a) $(x + 1)^3$ (b) $(x - 1)^3$
(c) $(x + 1)^3$ (d) $(x - 1)^2$

9. The differential equation of the family of curves for which the length of the normal is equal to a constant k, is given by

(a) $y^2\left(\frac{dy}{dx}\right)^2 = k^2 - y^2$ (b) $\left[y\frac{dy}{dx}\right]^2 = k^2 - y^2$
(c) $y\frac{dy}{dx} = k^2 - y^2$ (d) $\left[y\frac{dy}{dx}\right]^2 = k^2 + y^2$

10. The solution of the differential equation $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$ is equal to

(a) $2\cos(xy) + x^{-2} = C$ (b) $2\cos(xy) + y^{-2} = C$
(c) $2\sin(xy) + x^{-2} = C$ (d) $2\sin(xy) + y^{-2} = C$

11. The solution of the differential equation $\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0$ is

(a) $\ln\left|\frac{x}{y}\right| + \frac{xy}{(x-y)} = C$ (b) $\ln|xy| + \frac{xy}{(x-y)} = C$
(c) $\frac{xy}{(x-y)} = Ce^{x/y}$ (d) $\frac{xy}{(x-y)} = Ce^{xy}$

(where, c is a arbitrary constant)

12. The equation of the curve in which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact is

(a) $y = \frac{kx^3}{3} + Cx$ (b) $y - \frac{kx^2}{2} + C$
(c) $y = -\frac{kx^3}{2} + Cx$ (d) $y = \frac{kx^3}{3} + \frac{Cx^2}{2}$

(k is constant of proportionality)
(where, C is arbitrary constant)

13. The equation of the curve for which the square of the ordinate is twice the rectangle contained by the abscissa and the intercept of the normal on x-axis and passing through (2,1) is

(a) $x^2 + y^2 - x = 0$ (b) $4x^2 + 2y^2 - 9y = 0$
(c) $2x^2 + 4y^2 - 9x = 0$ (d) $4x^2 + 2y^2 - 9x = 0$

14. A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at point P is $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate at that point, then the equation of the curve is

(a) $y = e^{ax} - 1$ (b) $y = e^{ax} + 1$
(c) $y = e^{ax} + a$ (d) $y = e^{a(x-1)}$

15. The differential equation representing all lines at a distance p from the origin is

(a) $(x^2 + y^2)\frac{dy}{dx} = 2y\left\{x - p\left(\frac{dy}{dx}\right)^2\right\}$
(b) $\left(x\frac{dy}{dx} - y\right)^2 - p^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = 0$
(c) $\left(x\frac{dy}{dx} - y\right)^2 - \left(p\frac{dy}{dx} + x\frac{dy}{dx}\right) = 0$
(d) $(x - y) - \left(\frac{dy}{dx} - \frac{dx}{dy}\right) = 0$

16. Solution of the differential equation $x^2\frac{dy}{dx}.\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$, where $y \to 1$ as $x \to \infty$, is

(a) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (b) $y = \frac{x+1}{x\sin\frac{1}{x}}$
(c) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (d) $y = \frac{x+1}{x\cos\frac{1}{x}}$

17. A tangent and a normal to curve at any point P meet the x and y axes at A, B and C, D, respectively. If the centre of circle through O, C, P and B lies on the line $y = x$ (O is the origin), then the differential equation of all such curves is

(a) $\frac{dy}{dx} = \frac{y - x}{y + x}$ (b) $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$
(c) $\frac{dy}{dx} = \frac{x - y}{xy}$ (d) $\frac{dy}{dx} = \frac{xy}{y + x}$

18. Let $y = f(x)$ be a curve passing through (e, e^e), which satisfy the differential equation $(2ny + xy\log_e x)dx - x\log_e x\, dy = 0, \quad x > 0, y > 0$. If $g(x) = \lim_{n\to\infty} f(x)$, then $\int_{1/e}^{e} g(x)dx$ equal to

(a) e (b) 1
(c) 0 (d) 2

19. A curve is such that the portion of the x-axis cut off between the origin and tangent at a point is twice the abscissa and which passes through the point (1, 2). The equation of the curve is

(a) $xy = 1$ (b) $xy = 2$
(c) $xy = 3$ (d) $xy = 0$

20. A particle of mass m is moving in straight line is acted on by an attractive force $\frac{mk^2a^2}{x^2}$ for $x \geq a$ and $\frac{2mk^2x}{a}$ for $x < a$. If the particle starts from rest at the point $x = 2a$ then it will reach the point $x = 0$ with a speed

(a) $k\sqrt{a}$ (b) $k\sqrt{2a}$ (c) $k\sqrt{3a}$ (d) $\frac{1}{k}\sqrt{a}$

More Than One Correct Option

21. The solution of $\left(\frac{x\,dx + y\,dy}{x\,dy - y\,dx}\right) = \sqrt{\left(\frac{a^2 - x^2 - y^2}{x^2 + y^2}\right)}$ is

(a) $\sqrt{(x^2 + y^2)} = a\sin\{(\tan^{-1} y/x) + \text{constant}\}$

(b) $\sqrt{(x^2 + y^2)} = a\cos\{(\tan^{-1} y/x) + \text{constant}\}$

(c) $\sqrt{(x^2 + y^2)} = a\{\tan(\sin^{-1} y/x) + \text{constant}\}$

(d) $y = x\tan\{\text{constant} + \sin^{-1}\frac{1}{a}\sqrt{(x^2 + y^2)}\}$

22. The solution of $\frac{dy}{dx} + x = xe^{(n-1)y}$ is

(a) $\frac{1}{n-1}\log\left(\frac{e^{(n-1)y} - 1}{e^{(n-1)y}}\right) = \frac{x^2}{2} + C$

(b) $e^{(n-1)y} = Ce^{(n-1)y + (n-1)x^2/2} + 1$

(c) $\log\left(\frac{e^{(n-1)y} - 1}{(n-1)e^{(n-1)y}}\right) = x^2 + C$

(d) $e^{(n-1)y} = Ce^{(n-1)x^2/2 + x} + 1$

23. The solution of the differential equation $(y^2dx - 2xydy) = x^3y^3dy + x^2y^4dx$ is

(a) $\log\left(\frac{x}{y^2}\right) = \frac{(xy)^2}{2} + C$ (b) $\frac{x}{Cy^2} = e^{\frac{(xy)^2}{2}}$

(c) $\left(\frac{x}{y^2}\right) = 2\log(xy) + C$ (d) $\frac{x}{y^2} = e^{(2xy)} + C$

24. If $x\frac{dy}{dx} + y = x \cdot \frac{f(xy)}{f'(xy)}$, then $f(xy)$ is equal to

(a) $ke^{\frac{x^2}{2}}$ (b) $ke^{\frac{y^2}{2}}$

(c) $ke^{c + \frac{x^2}{2}}$ (d) $ke^{\frac{xy}{2}}$

25. The first integral of $\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) - x^2y\frac{dy}{dx} = xy^2$ will be

(a) $\frac{dy}{dx} = \sqrt{C - x^2y^2}$ (b) $\frac{dy}{dx} = \sqrt{C + x^2y^2}$

(c) $\frac{dy}{dx} = -\sqrt{C + x^2y^2}$ (d) $\frac{dy}{dx} = -\sqrt{C - x^2y^2}$

Comprehension Based Questions

Passage

Directions (Q. Nos. 26-29) *Let $y = f(x)$ and $y = g(x)$ be the pair of curves such that*

(i) *the tangents at point with equal abscissae intersect on y-axis.*

(ii) *the normal drawn at points with equal abscissae intersect on x-axis and*

(iii) *one curve passes through* (1, 1) *and the other one passes through* (2, 3), *then*

26. The curve $f(x)$ is given by

(a) $\frac{2}{x} - x$ (b) $2x^2 - \frac{1}{x}$

(c) $\frac{2}{x^2} - x$ (d) $\frac{2}{x} - x^2$

27. The curve $g(x)$ is given by

(a) $x - \frac{1}{x}$ (b) $x + \frac{2}{x}$

(c) $x^2 - \frac{1}{x^2}$ (d) $x^2 + \frac{1}{x^2}$

28. The number of positive integral solutions for $f(x) = g(x)$, are

(a) 4 (b) 5

(c) 6 (d) None of these

29. The value of $\int_1^2 \{g(x) - f(x)\}\,dx$, is

(a) 2 (b) 3

(c) 4 (d) 5

Assertion and Reason

Directions (Q. Nos. 30-37) *Each of these question contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these question also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

30. Let us consider a linear equation $\frac{dy}{dx} + Py = Q$

Statement I The value of $y(2)$, if y satisfies $x^2\frac{dy}{dx} + xy = \sin x$, $y(1) = 2$ is $\frac{1}{2}\int_1^2 \frac{\sin t}{t}\,dt$.

Statement II The integrating factor of given equation is $e^{\int P dx}$.

31. **Statement I** The elimination of four arbitrary constants in $y = (c_1 + c_2 + c_3 e^{c_4})x$ results into a differential equation of the first order $x\dfrac{dy}{dx} = y$

Statement II Elimination of n arbitrary constants requires in general a differential equation of the nth order.

32. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2 - 1}dy - y\sqrt{y^2 - 1}dx = 0$ satisfy $y(2) = \dfrac{2}{\sqrt{3}}$

Statement I $y(x) = \sec\left(\sec^{-1} x - \dfrac{\pi}{6}\right)$

Statement II $y(x)$ is given by $\dfrac{1}{y} = \dfrac{2\sqrt{3}}{x} - \sqrt{1 - \dfrac{1}{x^2}}$

33. Consider the differential equation $(x^2 + 1) \cdot \dfrac{d^2y}{dx^2} = 2x \cdot \dfrac{dy}{dx}$

Statement I For any member of this family $y \to \infty$ as $x \to \infty$

Statement II Any solution of this differential equation is a polynomial of odd degree with positive coefficient of maximum power.

34. **Statement I** The differential equation of non-vertical lines in a plane is $\dfrac{d^2x}{dy^2} = 0$

Statement II The general equation of all non-vertical lines in a planes is $ax + by = 1$, where $b \neq 0$

35. **Statement I** The general solution of $\dfrac{dy}{dx} + y = 1$ is $ye^x = e^x + C$

Statement II The number of arbitrary constants is in the general solution of the differential equation is equal to the order of differential equation.

36. **Statement I** The curve $y = Cx^2, C$ be any arbitrary constant, intersects the curves $x^2 + 2y^2 = 2C$ right angles.

Statement II As above, curves trace orthogonal trajectories.

37. **Statement I** The differential equation $y^3dy + (x + y)^2 dx = 0$ becomes homogeneous, if we put $y^2 = t$

Statement II All differential equation of first order and first degree become homogeneous if we put $y = tx$

Previous Years' Questions

38. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\dfrac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is [JEE Main 2013]

(a) 2500 (b) 3000 (c) 3500 (d) 4500

39. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\dfrac{dp(t)}{dt} = 0.5(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is [AIEEE 2012]

(a) $2\log 18$ (b) $\log 9$ (c) $\dfrac{1}{2}\log 18$ (d) $\log 18$

40. If $\dfrac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to [AIEEE 2011]

(a) 5 (b) 13 (c) -2 (d) 7

41. Let l be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\dfrac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then, the scrap value $V(T)$ of the equipment is [AIEEE 2011]

(a) $l - \dfrac{kT^2}{2}$ (b) $l - \dfrac{k(T - l)^2}{2}$

(c) e^{-kT} (d) $T^2 - \dfrac{1}{k}$

42. Consider the differential equation $y^2dx + \left(x - \dfrac{1}{y}\right)dy = 0$. If $y(1) = 1$, then x is given by [AIEEE 2011]

(a) $1 - \dfrac{1}{y} + \dfrac{e^{1/y}}{e}$ (b) $4 - \dfrac{2}{y} - \dfrac{e^{1/y}}{e}$

(c) $3 - \dfrac{1}{y} - \dfrac{e^{1/y}}{e}$ (d) $1 + \dfrac{1}{y} - \dfrac{e^{1/y}}{e}$

Answers

Check Point 1

1. (b) **2.** (a) **3.** (c) **4.** (b) **5.** (b) **6.** (a)

Check Point 2

1. (c) **2.** (b) **3.** (a) **4.** (a)

Round I

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (c)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (b)
11. (a)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (d)
21. (b)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (d)	**29.** (c)	**30.** (b)
31. (c)	**32.** (a)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (d)	**38.** (b)	**39.** (d)	**40.** (d)
41. (c)	**42.** (a)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (a)	**50.** (d)
51. (d)	**52.** (c)	**53.** (c)	**54.** (b)	**55.** (b)	**56.** (a)	**57.** (c)	**58.** (b)	**59.** (d)	**60.** (c)
61. (a)	**62.** (b)	**63.** (c)	**64.** (a)	**65.** (b)	**66.** (a)	**67.** (c)	**68.** (a)	**69.** (b)	**70.** (a)
71. (a)	**72.** (a)	**73.** (a)	**74.** (c)	**75.** (b)					

Round II

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (a)	**12.** (c)	**13.** (d)	**14.** (d)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (c)
21. (a,b)	**22.** (a,b)	**23.** (a,b)	**24.** (a,c)	**25.** (b,c)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (b)	**30.** (d)
31. (a)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (a)	**40.** (d)
41. (a)	**42.** (d)								

the Guidance

Round I

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0 \Rightarrow y''' + \sin(y''') = 0$

The highest order derivative which occurs in the given differential equation is y'''', therefore its order is 4.

As the given differential equation is not a polynomial equation in $\frac{dy}{dx}$ (*i.e.*, y'), therefore its degree is not defined.

2. The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, the index of its highest power is one, hence the given differential equation is of order 2 and degree 1.

3. The highest order derivative, present in the differential equation is y''. Therefore, its order is two.

This is a polynomial equation in y'' and y' and the highest power raised to y'' is one. Hence, its degree is one.

4. We know that the number of constants in the general solution of a differential equation of order n is equal to its order. Therefore, the number of constants in the general equation of fourth order differential equation is four.

5. In the particular solution of a differential equation of third order, there is no arbitrary constant because in the particular solution of any differential equation, we removes all the arbitrary constant by substituting some particular values.

6. Given curve, $y = Ax + A^3$ ...(i)

On differentiating w.r.t. x, we get

$\Rightarrow$ $\frac{dy}{dx} = A$ ...(ii)

From Eqs. (i) and (iii), we get

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 \qquad (\because \text{degree} = 3)$$

7. Given equation is

$$y = C_1e^{2x+c_2} + C_3e^x + C_4\sin(x + C_5)$$
$$= C_1e^{c_2}e^{2x} + C_3e^x + C_4(\sin x\cos C_5 + \cos x\sin C_5)$$
$$= Ae^{2x} + C_3e^x + B\sin x + D\cos x$$

Here, $A = C_1e^{c_2}, B = C_4\cos c_5, D = C_4\sin c_5$

Since, equation consists of four arbitrary constants.

So, the order of differential equation is 4.

8. $y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x + c_5}$

$y_1 = -(C_1 + C_2)\sin(x + C_3) - C_4 e^{x + c_5}$

$y_2 = -(C_1 + C_2)\cos(x + C_3) - C_4 e^{x + c_5} = -y - 2C_4 e^{x + c_5}$

$$y_3 = -y_1 - 2C_4 e^{x + c_5}$$

$$y_3 = -y_1 + y_2 + y$$

$\therefore$ Differential equation is

$$y_3 - y_2 + y_1 - y = 0$$

which is of order 3.

9. The equation of any tangent to the parabola

$y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$, where m is any arbitrary constant.

On differentiating w.rt. x, we get $\dfrac{dy}{dx} = m$

On substituting the value of m in Eq. (i), we get

$$y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}} \Rightarrow x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + a = 0$$

which is a differential equation of degree 2.

10. Given, family is $y^2 = a(b^2 - x^2)$

On differentiating w.r.t. x, we get

$$2yy' = a(0 - 2x) \Rightarrow 2yy' = a(-2x) \Rightarrow yy' = -ax \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$yy'' + (y')^2 = -a$ (using product rule of differentiation) ...(ii)

Now, for eliminating a, put the value of a from Eq. (ii) in Eq. (i), we get

$$yy' = [yy'' + (y')^2]\,x$$

$$\Rightarrow \quad yy' = x[(y')^2 + yy'']$$

which is the required differential equation.

11. Firstly rearrange the equation by taking e^x to LHS from RHS and then differentiate and eliminate the arbitrary constants.

Given, $\quad y = e^x(a\cos x + b\sin x) \quad$ ...(i)

On dividing both sides by

$$e^x,\ e^{-x}y = (a\cos x + b\sin x) \quad \text{...(ii)}$$

On differentiating both sides w.r.t. x, we get

$$e^{-x}y' + ye^{-x}(-1) = -a\sin x + b\cos x$$

(using product rule, $\dfrac{d}{dx}(u \cdot v) = u\dfrac{d}{dx}v + v\dfrac{d}{dx}u$)

Again, differentiating both sides w.r.t x, we get

$$e^{-x}\frac{d}{dx}(y') + y'\frac{d}{dx}(e^{-x}) - \left[y\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}y\right]$$

$$= -a\cos x - b\sin x$$

$$\Rightarrow \quad e^{-x}y'' + y'e^{-x}(-1) - [ye^{-x}(-1) + e^{-x}y']$$

$$= -a\cos x - b\sin x$$

$$\Rightarrow \quad e^{-x}y'' - 2y'e^{-x} + ye^{-x} = -(ye^{-x}) \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad e^{-x}[y'' - 2y' + 2y] = 0$$

$$\Rightarrow \quad y'' - 2y' + 2y = 0 \quad \text{(dividing by } e^{-x}\text{)}$$

which is the required differential equation.

12. A parabola having vertex at origin and axis along positive Y-axis represented by $x^2 = 4ay$. Differentiate this equation once because one arbitrary constant is present in this equation.

The equation of the family of parabola having the vertex at origin and the axis along the positive y-axis is

$$x^2 = 4ay \quad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. x, we get

$$2x = 4ay' \quad \text{...(ii)}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{2x}{x^2} = \frac{4ay'}{4ay} \Rightarrow \frac{2}{x} = \frac{y'}{y}$$

$$\Rightarrow \quad xy' = 2y \Rightarrow xy' - 2y = 0$$

which is the required differential equation.

13. A hyperbola having foci on x-axis and centre at origin is represented by $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and differentiate twicely to get the required differential equation.

The equation of the family of hyperbolas with the centre at origin and foci along the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{x} = \frac{a^2}{b^2}$$

Again, differentiating w.r.t. x, we get

$$\frac{x\frac{d}{dx}(yy') - yy' \cdot \frac{d}{dx}(x)}{x^2} = 0$$

(using quotient rule of differentiation)

$$\Rightarrow \quad \frac{x[yy'' + (y')^2] - yy' \cdot 1}{x^2} = 0$$

(using product rule of differentiation)

$$\Rightarrow \quad x(y')^2 + xyy'' - yy' = 0$$

$$\Rightarrow \quad xyy'' + x(y')^2 - yy' = 0$$

which is the required differential equation.

14. Given, general solution is $y = C_1e^x + C_2e^{-x}$ ...(i)

On differentiating twice w.r.t x, we get

$$y' = C_1e^x + C_2e^{-x}(-1)$$

Again, differentiating w.r.t. x, we get

$$y'' = C_1e^x - C_2e^{-x}(-1)$$

$$\Rightarrow \quad y'' = C_1e^x + C_2e^{-x}$$

$\Rightarrow \quad y'' = y$ [from Eq. (i)]

$\Rightarrow \quad y'' - y = 0$

which is the required differential equation of the given general solution.

Replacing y' by $\frac{dy}{dx}$ and y'' by $\frac{d^2y}{dx^2}$.

15. Let the radius of circle be a, then centre of the circle is (a, a).

Hence, the equation of the circle is

$$(x-a)^2 + (y-a)^2 = a^2 \quad \ldots(i)$$

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

On differentiating Eq. (i) w.r.t. x, we get

$$2x + 2yy' - 2a - 2ay' = 0$$

$$\Rightarrow \quad x + yy' - a(1 + y') = 0$$

$$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}$$

On substituting this value of a in Eq. (i), we get

$$\left(x - \frac{x + yy'}{1 + y'}\right)^2 + \left(y - \frac{x + yy'}{1 + y'}\right)^2 = \left(\frac{x + yy'}{1 + y'}\right)^2$$

$$\Rightarrow \quad (xy' - yy')^2 + (y - x)^2 = (x + yy')^2$$

$$\Rightarrow \quad (x - y)^2(y')^2 + (x - y)^2 = (x + yy')^2$$

$$\Rightarrow \quad (x - y)^2[1 + (y')^2] = (x + yy')^2$$

which is the required differential equation.

16. The equation of family of circles which pass through the origin and whose centres lie on y-axis is,

$$(x - 0)^2 + (y - k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 + k^2 - 2yk = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2yk = 0 \quad \ldots(i)$$

where k is parameter.

On differentiate w.r.t.x, we get

$$2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx}k = 0$$

$$\Rightarrow \quad x + (y - k)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + \left\{y - \frac{x^2 + y^2}{2y}\right\}\frac{dy}{dx} = 0 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x + \frac{1}{2y}(y^2 - x^2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2xy = (x^2 - y^2)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

which is the required equation.

17. Given that, $\quad Ax^2 + By^2 = 1 \quad \ldots(i)$

On differentiate w.r.t. x, we get

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow \quad Ax + By\frac{dy}{dx} = 0 \quad \ldots(ii)$$

Again, differentiate w.r.t. x, we get

$$\Rightarrow \quad A + B\left\{y\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx}\right\} = 0 \quad \ldots(iii)$$

On multiplying Eq. (iii) by x and subtracting Eq. (ii), we get

$$B\left\{xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2\right\} - By\frac{dy}{dx} = 0$$

$$B\left\{xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}\right\} = 0 \quad (\because B \neq 0)$$

$$\therefore \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required solution.

18. The equation of family of circles (concentric) with centre $(1, 2)$ is

$$(x - 1)^2 + (y - 2) = r^2 \quad \ldots(i)$$

(where, r is parameter)

On differentiate w.r.t. x, we get

$$2(x - 1) + 2(y - 2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - x}{y - 2}$$

which is the required solution.

19. Given family of curves, $\quad x^2 + y^2 - 2ay = 0 \quad \ldots(i)$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + (y - a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + \left\{y - \frac{x^2 + y^2}{2y}\right\}\frac{dy}{dx} = 0 \quad \left[\text{from Eq. (i), } a = \frac{x^2 + y^2}{2y}\right]$$

$$\Rightarrow \quad x + \frac{(y^2 - x^2)}{2y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} = 2xy$$

20. Given family of curves, $\quad y^2 = 4a(x + a) \quad \ldots(i)$

where, $(a \rightarrow \text{parameter})$

On differentiate w.r. t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx} \quad \text{[from Eq. (i)]}$$

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \quad y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 - y = 0$$

21. The differential equation of the rectangular hyperbola $xy = c^2$ is

$$y + x\frac{dy}{dx} = 0 \Rightarrow x\frac{dy}{dx} = -y$$

22. Equation of family of parabolas with focus at (0, 0) and x-axis as axis is

$$y^2 = 4a(x + a) \quad \ldots(i)$$

On differentiating Eq. (i), we get

$2yy_1 = 4a$, putting the value of a in Eq. (i),

$$\Rightarrow \quad y^2 = 2yy_1\left(x + \frac{yy_1}{2}\right)$$

$$\Rightarrow \quad y = 2xy_1 + yy_1^2$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$$

23. The equation of circle passing through the origin and having their centres on the x-axis is

$$x^2 + y^2 + 2gx = 0 \quad \ldots(i)$$

where g is an arbitrary constant.

On differentiating w.r.t. x, we get

$$x + y\frac{dy}{dx} + g = 0 \Rightarrow g = -\left(x + y\frac{dy}{dx}\right)$$

On putting this value of g in Eq. (i1), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0, \Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

24. Equation of the family of such parabola is

$$(y - k)^2 = 4a(x - h) \quad \ldots(i)$$

where h and k are arbitrary constants.

On differentiating w.r.t, x we get

$$(y - k)\frac{dy}{dx} = 2a \quad \ldots(ii)$$

On differentiating again,

$$(y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \ldots(iii)$$

On putting value of $(y - k)$ from Eq. (ii) in Eq. (iii),

we get $2a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + 0$,

which is the required differential equation.

25. We have, slope $\dfrac{dy}{dx} = \dfrac{2ay}{x(y - a)}$

$$\Rightarrow \quad \frac{y - a}{y}dy = \frac{2a}{x}dx$$

On integrating both sides, we get

$$a\log|y| - y = -2a\log|x| + \log c$$

$$\Rightarrow \quad y^2 \cdot x^{2a} = ce^y$$

This passes through (1, 1), therefore $1 = ce \Rightarrow c = \dfrac{1}{e}$

So, the equation of the curve is $y \cdot x^{2a} = e^{y-1}$

26. Given, $\cos\left(\dfrac{dy}{dx}\right) = a \Rightarrow \dfrac{dy}{dx} = \cos^{-1} a$

On separating the variables, we get $dy = (\cos^{-1} a)\, dx$

On integrating both sides, we get $\int dy = \cos^{-1} a\int dx$

$$\Rightarrow \quad y = \cos^{-1} a(x) + C \quad \ldots(i)$$

On putting $y = 2$ and $x = 0$, we get $2 = \cos^{-1}(a) \times 0 + C$

$$\Rightarrow \quad C = 2$$

On substituting $C = 2$ in Eq. (i), we get

$$y = x\cos^{-1} a + 2 \Rightarrow \frac{y - 2}{x} = \cos^{-1} a \Rightarrow \cos\left(\frac{y - 2}{x}\right) = a$$

which is the required particular solution.

27. The given differential equation of the curve is $y' = e^x \sin x$ or $\dfrac{dy}{dx} = e^x \sin x$.

On separating the variables, we get $dy = e^x \sin x\, dx$

On integrating both sides, we get

$$\int dy = \int e^x \sin x\, dx \quad \ldots(i)$$

Let $I = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\, dx \quad \ldots(ii)$

$$\Rightarrow \quad I = \sin x \int e^x dx - \int\left(\frac{d}{dx}(\sin x)\int e^x\, dx\right)dx$$

(using integration by parts)

$$= \sin x \int e^x\, dx - \int \underset{\text{I}}{\operatorname{cosec}} \cdot \underset{\text{II}}{e^x} dx$$

$$\Rightarrow \quad = \sin x \cdot e^x - \left[\cos x \cdot \int e^x\, dx - \int\left(\frac{d}{dx}(\cos x) \cdot \int e^x\, dx\right)dx\right]$$

(again, integration by parts)

$$\Rightarrow \quad I = \sin x \cdot e^x - [\cos x \cdot e^x - \int(-\sin x) \cdot e^x\, dx]$$

$$= \sin x \cdot e^x - e^x \cdot \cos x - \int e^x \cdot \sin x \cdot dx$$

$$\Rightarrow \quad I = \sin x \cdot e^x - \cos x \cdot e^x - I \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad 2I = e^x(\sin x - \cos x) \Rightarrow I = \frac{e^x(\sin x - \cos x)}{2}$$

On substituting this value in Eq. (i), we get

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \quad \text{[from Eq. (iii)]}$$

Now, the curve passes through the point (0, 0) or we can say $x = 0, y = 0$

$$0 = \frac{e(\sin 0 - \cos 0)}{2} + C \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 0 = \frac{1(0 - 1)}{2} + C \Rightarrow C = \frac{1}{2}$$

On substituting $C = \dfrac{1}{2}$ in Eq. (iii), we get

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2} \Rightarrow 2y = e^x(\sin x - \cos x) + 1$$

$$\Rightarrow \quad 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the curve is

$$2y - 1 = e^x(\sin x - \cos x)$$

28. Given, $\sin^2\left(\frac{y}{x}\right) - \frac{y}{x} + \frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin^{-2}\left(\frac{y}{x}\right)$ ...(i)

Thus, the given differential equation is homogeneous.

So, put $\frac{y}{x} = v$

i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, Eq. (i) becomes

$$v + x\frac{dv}{dx} = v - \sin^2 v \Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

$\Rightarrow \quad \operatorname{cosec}^2 v\, dv = -\frac{1}{x}dx$

On integrating both sides, we get

$$\int \operatorname{cosec}^2 v\, dv = -\int \frac{dx}{x} \Rightarrow -\cot v = -\log|x| + C$$

$\Rightarrow \quad \log|x| - \cot v = C \Rightarrow \log|x| - \cot\left(\frac{y}{x}\right) = C \left(\text{put } v = \frac{y}{x}\right)$ ...(ii)

when, $x = 1$, then $y = \frac{\pi}{4}$, therefore,

$$\log|1| - \cot\frac{\pi}{4} = C \Rightarrow C = 0 - 1 = -1$$

On putting the value of C in Eq. (ii), we get

$$\log|x| - \cot\left(\frac{y}{x}\right) = -1 \Rightarrow \log|x| - \cot\left(\frac{y}{x}\right) = -\log e$$

$(\because 1 = \log e)$

$\Rightarrow \quad \cot\left(\frac{y}{x}\right) = \log|ex| \quad (\because \log m + \log n = \log mn)$

This is the required solution of the given differential equation.

29. Given, $2x^2\frac{dy}{dx} = 2xy + y^2 \Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$ ...(i)

Thus, the given differential equation is homogeneous.

So, put $\frac{y}{x} = v$ *i.e.*, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, Eq. (i) becomes, $v + x\frac{dv}{dx} = \frac{2x^2 v + x^2 v^2}{2x^2}$

$\Rightarrow \quad v + x\frac{dy}{dx} = \frac{x^2(2v + v^2)}{2x^2} \Rightarrow v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{2}v^2 \Rightarrow \frac{2}{v^2}dv = \frac{1}{x}dx$

On integrating both sides, we get

$$2\int v^2\, dv = \int \frac{dx}{x} \Rightarrow 2\left(\frac{v^{-2+1}}{-2+1}\right) = \log|x| + C$$

$\Rightarrow \quad 2\left(\frac{v^{-1}}{-1}\right) = \log|x| + C \Rightarrow -\frac{2}{v} = \log|x| + C$

$\Rightarrow \quad -\frac{2x}{y} = \log|x| + C \quad \left(\because v = \frac{y}{x}\right)$ ...(ii)

when, $x = 1$, then $y = 2$, therefore, $\frac{-2 \times 1}{2} = \log 1 + C$

$\Rightarrow \quad -1 = 0 + C$

On putting the value of C in Eq. (ii), we get

$$-\frac{2x}{y} = \log|x| - 1 \Rightarrow y = \frac{2x}{1 - \log|x|}$$

30. Given, $\frac{dy}{dx} - 3y\cot x = \sin 2x$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = -3\cot x, Q = \sin 2x$$

$$\text{IF} = e^{-3\int \cot x\, dx} = e^{-3\log|\sin x|} = e^{\log|\sin x|^{-3}}$$

$\Rightarrow \quad \text{IF} = \frac{1}{\sin^3 x}$ ...(i)

The general solution of the given differential equation is given by

$$y \times \text{IF} = \int Q \times \text{IF}\, dx + C$$

$\Rightarrow \quad y \times \frac{1}{\sin^3 x} = \int \frac{1}{\sin^3 x}\sin 2x\, dx$

$\Rightarrow \quad y \times \frac{1}{\sin^3 x} = 2\int \frac{\sin x \cos x}{\sin^3 x}dx + C$

$\Rightarrow \quad \frac{1}{\sin^3 x} \times y = 2\int \frac{\cos x}{\sin^2 x}dx + C$

$\Rightarrow \quad \frac{y}{\sin^3 x} = 2\int \cot x \operatorname{cosec} x\, dx + C$

$\Rightarrow \quad \frac{y}{\sin^3 x} = -2\operatorname{cosec} x + C$

$\Rightarrow \quad y = -2\left(\frac{1}{\sin x} \times \sin^3 x\right) + C\sin^3 x$

$\Rightarrow \quad y = -2\sin^2 x + + C\sin^3 x$ ...(ii)

Given, $y = 2$ and $x = \frac{\pi}{2}$, then from Eq. (ii), we get

$$2 = -2\sin^2\left(\frac{\pi}{2}\right) + C\sin^3\left(\frac{\pi}{2}\right)$$

$\Rightarrow \quad 2 = -2 + C \Rightarrow C = 4$

On putting the value of C in Eq. (ii), we get

$$y = -2\sin^2 x + 4\sin^3 x$$

$\Rightarrow \quad y = 4\sin^3 x - 2\sin^2 x$

31. The differential equation of the given curve is

$$\sin x \cos y\, dx + \cos x \sin y\, dy = 0$$

$\Rightarrow \quad \frac{\sin x}{\cos x}dx + \frac{\sin y}{\cos y}dy = 0$

$\Rightarrow \quad \tan x\, dx + \tan y\, dy = 0$

On integrating both sides, we get

$$\int \tan x\, dx + \int \tan y\, dy = \log C$$

$\Rightarrow \quad \log(\sec x) + \log(\sec y) = \log C$

$\sec x \sec y = C$ ...(i)

The curve passes through the point $\left(0, \frac{\pi}{4}\right)$, therefore put

$x = 0, y = \frac{\pi}{4}$, we get $\sec\theta \sec\frac{\pi}{4} = C \Rightarrow C = \sqrt{2}$

On putting the value of C in Eq. (i), we get $\sec x \cdot \sec y = \sqrt{2}$

$$\Rightarrow \quad \sec x \cdot \frac{1}{\cos y} = \sqrt{2} \Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$.

32. Given, differential equation is $(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$

Separating the variables, we get $\frac{dy}{1+y^2} + \frac{e^x dx}{1+e^{2x}} = C$

On integrating both sides, we get $\int \frac{dy}{1+y^2} + \int \frac{e^x\, dx}{1+e^{2x}} = C$

Put $t = e^x \Rightarrow e^x dx = dt \Rightarrow \tan^{-1} y + \int \frac{dt}{1+t^2} = C$

$$\Rightarrow \quad \tan^{-1} y + \tan^{-1} t = C \Rightarrow \tan^{-1} y + \tan^{-1} e^x = C \qquad \ldots(i)$$

Now, put $x = 0$ and $y = 1$

$$\therefore \quad \tan^{-1} 1 + \tan^{-1} e^0 = C \Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C \Rightarrow C = \frac{\pi}{2}$$

On putting the value of C in Eq. (i), we get

$$\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$$

which is the required particular solution.

33. Given, differential equation is $(x - y)(dx + dy) = dx - dy$

$$\Rightarrow \quad dx + dy = \frac{dx - dy}{x - y}$$

On integrating both sides, we get

$$\int (dx + dy) = \int \frac{dx - dy}{x - y} + C$$

Let $\quad x - y = t \Rightarrow dx - dy = dt$

$$\therefore \int dx + dy - C = \int \frac{dx - dy}{x - y} = \int \frac{dt}{t} = \log t = \log|x - y|$$

$$\Rightarrow \quad x + y = \log|x - y| + C \qquad \ldots(i)$$

It is given that when $x = 0, y = -1$

$$\therefore \quad 0 + (-1) = \log(0 + 1) + C \Rightarrow C = -1$$

On substituting this value in Eq. (i), we get the required particular solution as $x + y = \log|x - y| - 1 \Rightarrow \log|x - y| = x + y + 1$.

34. Given equation can be written as

$$x^2 \frac{dy}{dx} - xy = 2\cos^2\left(\frac{y}{2x}\right), x \neq 0$$

$$\Rightarrow \quad \frac{x^2 \frac{dy}{dx} - xy}{2\cos^2\left(\frac{y}{2x}\right)} = 1 \Rightarrow \frac{\sec^2\left(\frac{y}{2x}\right)}{2}\left[x^2 \frac{dy}{dx} - xy\right] = 1$$

On dividing both sides by x^3, we get

$$\frac{\sec^2\left(\frac{y}{2x}\right)}{2}\left[\frac{x\frac{dy}{dx} - y}{x^2}\right] = \frac{1}{x^3} \Rightarrow \frac{d}{dx}\left[\tan\left(\frac{y}{2x}\right)\right] = \frac{1}{x^3}$$

On integrating both sides, we get

$$\tan\left(\frac{y}{2x}\right) = \frac{-1}{2x^2} + k$$

On substituting $x = 1, y = \frac{\pi}{2}$, we get

$$k = \frac{3}{2}, \text{ therefore, } \tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$$

is the required solution.

35. Given that, $\quad \left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$

$$\Rightarrow \quad \frac{dy}{dx} = -\cos x\left(\frac{1 + y}{2 + \sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{\cos x}{2 + \sin x} \cdot y = -\frac{\cos x}{2 + \sin x}$$

which is a linear differential equation.

$$\text{IF} = e^{\int \frac{\cos x}{2 + \sin x} dx} = e^{\log(2 + \sin x)} = 2 + \sin x$$

Complete solution,

$$y \cdot (\text{IF}) = \int \frac{-\cos x}{2 + \sin x} \cdot (\text{IF}) dx + C$$

$$y(2 + \sin x) = \int \frac{-\cos x}{2 + \sin x} \cdot (2 + \sin x) dx + C$$

$$y(2 + \sin x) = -\int \cos x\, dx + C$$

$$\Rightarrow \quad y(2 + \sin x) = -\sin x + C \qquad \ldots(i)$$

Now, $x = 0$ when $y = 1$

$$\Rightarrow \quad 1 \cdot (2 + \sin 0) = -\sin 0 + C$$

$$\Rightarrow \quad 2 = C \Rightarrow C = 2$$

$$\therefore \quad y(2 + \sin x) = -\sin x + 2$$

$$\Rightarrow \quad y = \frac{2 - \sin x}{2 + \sin x} \qquad \ldots(ii)$$

$$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{2 - \sin\frac{\pi}{2}}{2 + \sin\frac{\pi}{2}} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$$

36. Given that, $\quad (1 + t)\frac{dy}{dt} - ty = 1$

$$\Rightarrow \quad \frac{dy}{dt} - \frac{t}{1+t} y = \frac{1}{1+t}$$

which is a linear differential equation.

$$\therefore \quad \text{I F} = e^{-\int \frac{t}{1+t} dt} = e^{-\int \left(1 - \frac{1}{1+t}\right) dt}$$

$$= e^{-\{t - \log(1+t)\}}$$

$$= e^{\log(1+t) - t} = (1 + t) \cdot e^{-t}$$

Complete solution,

$$y \cdot (\text{I.F}) = \int \frac{1}{(1+t)} \cdot (\text{IF}) dt + C$$

$$\Rightarrow \quad y \cdot (1+t)e^{-t} = \int \frac{1}{(1+t)} \cdot (1+t) \cdot e^{-t} \cdot dt + C$$

$$\Rightarrow \quad y(1+t)e^{-t} = \int e^{-t} dt + C$$

$$\Rightarrow \quad y(1+t)e^{-t} = -e^{-t} + C \quad \ldots(\text{i})$$

$$\therefore \quad y = \frac{-1}{(1+t)} + \frac{Ce^t}{1+t}$$

Now, $y(0) = \frac{-1}{(1+0)} + \frac{C \cdot e^0}{(1+0)} = -1$

$\Rightarrow \quad -1 + C = -1 \Rightarrow C = 0$

From Eq. (i), $y(1+t)e^{-t} = -e^{-t} \Rightarrow y = \frac{-1}{1+t}$

$$\therefore \quad y(1) = \frac{-1}{2}$$

37. Given that,

$$f(\theta) = \frac{d}{d\theta}\int_0^\theta \frac{dx}{1-\cos\theta \cdot \cos x} = \left[\frac{1}{1-\cos\theta\cdot\cos x}\right]_0^\theta$$

$$= \frac{1}{1-\cos\theta\cdot\cos\theta} \times 1 - \frac{1}{1-\cos\theta\cdot\cos 0} \times 0$$

$$= \frac{1}{1-\cos^2\theta} - 0 = \frac{1}{\sin^2\theta} = \text{cosec}^2\,\theta$$

Now, $\frac{df}{d\theta} = 2\,\text{cosec}\,\theta \cdot (-\text{cosec}\,\theta \cdot \cot\theta)$

$$= -2\,\text{cosec}^2\,\theta \cdot \cot\theta = -2f(\theta) \cdot \cot\theta$$

$$\Rightarrow \quad \frac{df}{d\theta} + 2f(\theta) \cdot \cot\theta = 0$$

38. Given differential equation is

$$\frac{d^2y}{dx^2} = 2 \Rightarrow \frac{dy}{dx} = 2x + a \Rightarrow y = x^2 + ax + b$$

$\therefore$ It represents a parabola whose axis is parallel to y-axis.

39. Given equation is $\frac{d^2y}{dx^2} = \frac{\log x}{x^2}$

On integrating both sides, we get

$$\int \frac{d^2y}{dx^2} dx = \int \frac{\log x}{x^2} dx$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\log x}{x} + \int \frac{1}{x^2} dx + C$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\log x}{x} - \frac{1}{x} + C$$

At $x = 1, y = 0$ and $\frac{dy}{dx} = -1 \Rightarrow C = 0 \therefore \frac{dy}{dx} = -\frac{(\log x + 1)}{x}$

Again, on integrating both sides, we get

$$\int \frac{dy}{dx} dx = -\int \frac{\log x + 1}{x} dx + C_1$$

$$y = -\frac{1}{2}(\log x)^2 - \log x + C_1$$

At $x = 1, y = 0$

$\Rightarrow \quad C_1 = 0$

$$\therefore \quad y = -\frac{1}{2}(\log x)^2 - \log x$$

40. Given that, $\frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{dy}{1+y^2} = dx$

On integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\Rightarrow \quad \tan^{-1} y = x + C$$

At $x = 0, y = 0$, then $C = 0$

At $x = \pi, y = 0$, then $\tan^{-1} 0 = \pi + C \Rightarrow C = -\pi$

$\therefore \quad \tan^{-1} y = x \Rightarrow y = \tan x = \phi(x)$

Therefore, solution becomes $y = \tan x$.

But $\tan x$ is not continuous function in $(0, \pi)$.

So, $\phi(x)$ is not possible in $(0, \pi)$.

41. $\phi(x) = \int \frac{1}{\{\phi(x)\}^2}$, $\phi'(x) \frac{1}{\{\phi(x)\}^2}$, $\phi'(x) \cdot \{\phi(x)\}^2 = 1$

On integrating both the sides, $\frac{\{\phi(x)\}^3}{3} = x + C$

42. $(1 + \log x)\frac{dx}{dy} - x\log x = e^y$

Put $x\log x = t \Rightarrow (1 + \log x)dx = dt$

$$\therefore \quad \frac{dt}{dy} - t = e^y$$

$$\Rightarrow \quad te^{-y} = \int e^{-y}e^y dy + C$$

$$\Rightarrow \quad t = Ce^y + ye^y$$

$$\Rightarrow \quad x\log x = (C + y)e^y$$

Since, $y(1) = 0$ *i.e*, $C = 0$

$$\therefore \quad ye^y = x\log x$$

$$\Rightarrow \quad x^x = e^{ye^y}$$

43. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

$$\Rightarrow \quad 2xydy = (x^2 + 1)dx + y^2dx$$

$$\Rightarrow \quad \frac{xd(y^2) - y^2dx}{x^2} = \left(\frac{x^2+1}{x^2}\right)dx$$

$$\Rightarrow \quad \int d(y^2/x) = \int \left(1 + \frac{1}{x^2}\right)dx$$

$$\Rightarrow \quad \frac{y^2}{x} = x - \frac{1}{x} + C \Rightarrow y^2 = (x^2 - 1 + Cx)$$

$x = 1, y = 0, \therefore 0 = 1 - 1 + x \Rightarrow C = 0$

$y^2 = x^2 - 1 \Rightarrow x^2 - y^2 = 1$

44. Given $(e^x + e^{-x})\,dy - (e^x - e^{-x})\,dx = 0$

$\Rightarrow \quad (e^x + e^{-x})\,dy = (e^x - e^{-x})\,dx$

On separating the variables, we get $dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$

On integrating, we get $\int dy = \int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx$

Let $\quad e^x + e^{-x} = t \Rightarrow e^x - e^{-x} = \frac{dt}{dx}$

$\Rightarrow \quad dx = \frac{dt}{e^x - e^{-x}}$

$\therefore \quad \int dy = \int \frac{e^x - e^{-x}}{t} \frac{dt}{e^x - e^{-x}} = \int \frac{1}{t} dt$

$\Rightarrow \quad y = \log|t| + C \Rightarrow y = \log|e^x + e^{-x}| + C$

which is the required general solution.

45. $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

On separating the variables, we get

$$e^x \tan y\, dx = -(1 - e^x) \sec^2 y\, dy$$

$\Rightarrow \quad \frac{e^x}{(e^x - 1)} dx = \frac{\sec^2 y}{\tan y} dy$

On integrating, we get $\int \frac{e^x}{(e^x - 1)} dx = \int \frac{\sec^2 y}{\tan y} dy$

Let $e^x - 1 = t \Rightarrow e^x = \frac{dt}{dx} = dx = \frac{dt}{e^x}$ and $\tan y = v$

$\Rightarrow \quad \sec^2 y = \frac{dv}{dy} = dy = \frac{dv}{\sec^2 y}$

$\therefore \quad \int \frac{e^x}{t} \frac{dt}{e^x} = \int \frac{\sec^2 y}{v} \frac{dv}{\sec^2 y}$

$\Rightarrow \quad \log|t| = \log|v| - \log|C|$

$\Rightarrow \quad \log|e^x - 1| = \log|\tan y| = \log|C|$

$\Rightarrow \quad \log|C(e^x - 1)| = \log|\tan y|$

$(\because \log m + \log n = \log mn)$

$\Rightarrow \quad C(e^x - 1) = \tan y \quad (\because \log m = \log n \Rightarrow m = n)$

which is the required general solution.

46. Given, $\quad \frac{dy}{dx} = \frac{y}{x} - \sin\frac{y}{x} \quad \ldots(i)$

Thus, the given differential equation is homogeneous.

So put, $\quad \frac{y}{x} = v \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, Eq. (i) becomes $\; v + x\frac{dv}{dx} = v - \sin v$

$\Rightarrow \quad \operatorname{cosec} v\, dv = -\frac{1}{x} dx$

On integrating both sides, we get

$$\int \operatorname{cosec} v\, dv = -\int \frac{dx}{x}$$

$\Rightarrow \quad \log|\operatorname{cosec} v - \cot v| = -\log|x| + A$

$(\because \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x|)$

$\Rightarrow \quad \log|(\operatorname{cosec} v - \cot v)| + \log|x| = A$

$\Rightarrow \quad \log|(\operatorname{cosec} v - \cot v)\, x| = A$

$\Rightarrow \quad |x(\operatorname{cosec} v - \cot v)| = e^A$

$\Rightarrow \quad x\left(\frac{1}{\sin v} - \frac{\cos v}{\sin v}\right) = e^A$

$\Rightarrow \quad x\left(\frac{1 - \cos v}{\sin v}\right) = e^A \Rightarrow x(1 - \cos v) = e^A \sin v$

$\Rightarrow \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C \sin\left(\frac{y}{x}\right) \left(\text{put } C = e^A \text{ and } v = \frac{y}{x}\right)$

This is the required solution of given differential equation.

47. Given, $\quad y\, dx + x \log\left(\frac{y}{x}\right) dy - 2x\, dy = 0$

$\Rightarrow \quad \frac{y}{x} + \log\left(\frac{y}{x}\right)\frac{dy}{dx} - 2\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x}}{2 - \log\frac{y}{x}} \quad \ldots(i)$

Thus, the given differential equation is homogeneous.

So, put $\quad \frac{y}{x} = v$ *i.e.*, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, Eq. (i) becomes

$$v + x\frac{dv}{dx} = \frac{v}{2 - \log v} \Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\log v - v}{2 - \log v} \Rightarrow \frac{2 - \log v}{v\log v - v} dv = \frac{1}{x} dx$

$\Rightarrow \quad \frac{1 - (\log v - 1)}{v(\log v - 1)} dv = \frac{1}{x} dx$

$\Rightarrow \quad \left(\frac{1}{v(\log v - 1)} - \frac{(\log v - 1)}{v(\log v - 1)}\right) dv = \frac{1}{x} dx$

On integrating both sides, we get

$$\int \left(\frac{1}{v(\log v - 1)} - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$\Rightarrow \quad \int \frac{1}{v(\log v - 1)} dv - \int \frac{1}{v} dv = \int \frac{dx}{x}$

Let $\log v - 1 = t \Rightarrow \frac{1}{v} dv = dt$

$\therefore \quad \int \frac{dt}{t} - \int \frac{1}{v} dv = \int \frac{dx}{x}$

$\Rightarrow \quad \log|t| - \log|v| = \log|x| + \log C$

$\Rightarrow \quad \log|\log v - 1| - \log|v| = \log|x| + \log C$

$(\because t = \log v - 1)$

$\Rightarrow \quad \log|\log v - 1| - \log|v| - \log|x| = \log C$

$\Rightarrow \quad \log\left|\frac{\log v - 1}{vx}\right| = \log C \quad \left(\because \log\frac{m}{n} = \log m - \log n\right)$

$\Rightarrow \quad \left|\frac{\log v - 1}{vx}\right| = C \Rightarrow \frac{\log v - 1}{vx} = C$

$\Rightarrow \quad \frac{\log\left(\frac{y}{x} - 1\right)}{y} = C \quad (\because v = y/x)$

$\Rightarrow \quad \log\left(\frac{y}{x}\right) - 1 = Cy$

This is the required solution of the given differential equation.

48. The given differential equation is $\frac{dy}{dx} + y \sec x = \tan x$.

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get $P = \sec x$ and $Q = \tan x$

$\therefore \quad IF = e^{\int P\,dx} = e^{\int \sec x\,dx}$

$\Rightarrow \quad IF = e^{\log|\sec x + \tan x|} = \sec x + \tan x \quad \ldots(i)$

The general solution of the given differential equation is given by

$$y \cdot IF = \int Q \times IF\,dx + C$$

$\Rightarrow \quad y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$

$\Rightarrow \quad y(\sec x + \tan x) = \int \tan x \sec x\,dx + \int \tan^2 x\,dx + C$

$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \int \sec^2 x\,dx - \int 1\,dx + C$

$(\because \tan^2 x = \sec^2 x - 1)$

$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \tan x - x + C$

49. Given, $\quad x\frac{dy}{dx} + 2y = x^2 \log x$

On dividing by x on both sides, we get $\frac{dy}{dx} + 2\frac{y}{x} = x \log x$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2}{x} \text{ and } Q = x \log x$$

$\therefore \quad IF = e^{\int P\,dx} = e^{2\int \frac{1}{x}dx} \Rightarrow IF = e^{2\log|x|} = x^2$

The general solution of the given differential equation is given by

$$y \cdot IF = \int Q \times IF\,dx + C \Rightarrow x^2 y = \int x^2 \cdot x \log x\,dx + C$$

$\Rightarrow \quad x^2 y = \int \underset{I}{x^3} \underset{II}{\log x}\,dx + C$

$\Rightarrow \quad x^2 y = \log x \cdot \int x^3 - \int\left[\frac{d}{dx}\log x \int x^3\,dx\right]dx + C$

$\left[\text{integration by parts } \int \underset{I}{u} \cdot \underset{II}{v}\,dx = u\int v\,dx - \int\left(\frac{d}{dx}(u)\int v\,dx\right)dx\right]$

$\Rightarrow \quad x^2 y = \log x \cdot \frac{x^4}{4} - \int\left[\frac{d}{dx}\log x \int x^3 dx\right]dx + C$

$\Rightarrow \quad x^2 y = \log x \cdot \frac{x^4}{4} - \int\left[\frac{1}{x} \times \frac{x^4}{4}\right]dx + C$

$\Rightarrow \quad x^2 y = \log x \cdot \frac{x^4}{4} - \int \frac{x^3}{4}\,dx + C$

$\Rightarrow \quad x^2 y = \frac{x^4}{4}\log x - \frac{x^4}{16} + C$

$\Rightarrow \quad y = \frac{x^2}{16}(4\log x - 1) + Cx^{-2}$

50. Given, $\quad (1 + x^2)\,dy + 2xy\,dx = \cot x\,dx$

$\Rightarrow \quad (1 + x^2)\,dy = dx(\cot x - 2xy) \Rightarrow \frac{dy}{dx} = \frac{\cot x - 2xy}{1 + x^2}$

$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{\cot x}{1 + x^2}$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$\therefore \quad P = \frac{2x}{1 + x^2}$ and $Q = \frac{\cot x}{1 + x^2}$

$\therefore \quad IF = e^{\int P\,dx} = e^{\int \frac{2x}{1 + x^2}dx}$

Let $\quad 1 + x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

$\therefore \quad IF = e^{\int \frac{2x}{t} \times \frac{dt}{2x}} \Rightarrow IF = e^{\log|t|} = t = 1 + x^2 \quad \ldots(i)$

The general solution of the given differential equation is given by

$$y \cdot IF = \int Q \times IF\,dx + C$$

$\Rightarrow \quad (1 + x^2)y = \int\left((1 + x^2)\frac{\cot x}{(1 + x^2)}\right)dx + C$

$\Rightarrow \quad (1 + x^2)y = \log|\sin x| + C$

$\Rightarrow \quad y = \log x|\sin x|(1 + x^2)^{-1} + C(1 + x^2)^{-1}$

51. Firstly,check the homogeneous nature of the differential equation, if it is homogeneous, then put $x = vy$ and

$$\frac{dx}{dy} = v + y\frac{dv}{dy}.$$

Given, differential equation is $ye^y dx = (xe^{x/y} + y^2)dy$

$\Rightarrow \quad e^{x/y}\frac{dx}{dy} = \frac{x}{y}e^{x/y} + y \Rightarrow e^{x/y}\frac{dx}{dy} - \frac{x}{y}e^{x/y} = y \quad \ldots(i)$

On putting $\quad \frac{x}{y} = v \Rightarrow x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dx}$

Then, Eq. (i) becomes,

$$e^v\left(v + y\frac{dv}{dy}\right) - ve^v = y \Rightarrow e^v y\frac{dv}{dy} = y$$

$\Rightarrow \quad e^v\frac{dv}{dy} = 1 \Rightarrow e^v dv = dy$

On integrating both sides, we get

$$\int e^v dv = \int 1\,dy \Rightarrow e^v = y + C \Rightarrow e^{x/y} = y + C$$

52. The given differential equation is $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \quad \ldots(i)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \Rightarrow \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \quad \ldots(ii)$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{\sqrt{x}}, Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \therefore IF = e^{\int \frac{1}{\sqrt{x}}dx} \Rightarrow IF = e^{2x^{1/2}} = e^{2\sqrt{x}}$$

The general solution of the given differential equation is given by

$$y \cdot IF = \int Q \times IF\,dx + C \Rightarrow ye^{2\sqrt{x}} = \int e^{2\sqrt{x}} \times \frac{e^{-2\sqrt{x}}}{\sqrt{x}}dx + C$$

$\Rightarrow \quad ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}dx + C \Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + C$

53. Given, differential equation is $e^x dy + (ye^x + 2x)dx = 0$

$$\Rightarrow \quad e^x \frac{dy}{dx} + ye^x + 2x = 0 \Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

On comparing, $P = 1$ and $Q = -2xe^{-x}$ $\therefore$ $= e^{\int 1 dx} = e^x$

The general solution of the given differential equation is given by

$$y \cdot \text{IF} = \int (-2xe^{-x} \times e^x)dx + C \Rightarrow ye^x = -\int 2x dx + C \Rightarrow ye^x$$
$$= -x^2 + C \Rightarrow ye^x + x^2 = C$$

54. Given equation can be written as $xdy = (\sqrt{x^2 + y^2} + y)dx$, *i.e*,

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x} \quad \text{...(i)}$$

Clearly, RHS of Eq. (i) is a homogeneous function of degree zero. Therefore, the given equation is a homogeneous differential equation. Substituting $y = vx$, we get from Eq. (i)

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2x^2} + vx}{x} \; i.e, \; v + x\frac{dv}{dx} = \sqrt{1 + v^2} + v$$

$$x\frac{dv}{dx} = \sqrt{1 + v^2} \Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x} \quad \text{...(ii)}$$

On integrating both sides of Eq. (ii), we get

$$\log(v + \sqrt{1 + v^2}) = \log x + \log C \Rightarrow v + \sqrt{1 + v^2} = Cx$$

$$\Rightarrow \quad \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx \Rightarrow y + \sqrt{x^2 + y^2} = Cx^2$$

55. The given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1} \Rightarrow \frac{dy}{dx} + \frac{2x}{x^2 - 1}y = \frac{1}{(x^2 - 1)^2} \quad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $\quad P = \dfrac{2x}{x^2 - 1}$ and $Q = \dfrac{1}{(x^2 - 1)^2}$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \frac{2x}{x^2 - 1} dx} = e^{\log(x^2 - 1)} = (x^2 - 1)$$

The general solution of the given differential equation is

$$y \cdot \text{IF} = \int Q \times \text{IF} \, dx + C \Rightarrow y(x^2 - 1) = \int \frac{1}{x^2 - 1} dx + C$$

$$\Rightarrow \quad y(x^2 - 1) = \frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right| + C$$

which is the required solution.

56. Given that, $\quad y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$

$$\Rightarrow \quad y + x\frac{dy}{dx} + y = x\sin x + x\log x$$

$$\Rightarrow \quad \frac{2y}{x} + \frac{dy}{dx} = \sin x + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x} \cdot y = (\sin x + \log x)$$

which is linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$$

Complete solution,

$$y\,(\text{IF}) = \int (\sin x + \log x)\,(\text{IF})dx + C$$

$$y \cdot x^2 = \int x^2(\sin x + \log x)\,dx + C$$

$$\Rightarrow \quad y \cdot x^2 = \int x^2 \sin x dx + \int x^2 \log x dx + C$$

$$\Rightarrow \quad y \cdot x^2 = -x^2 \cos x + 2\int x \cos x dx + \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx + C$$

$$\Rightarrow \quad y \cdot x^2 = -x^2 \cos x + 2x\sin x - 2\int \sin x dx + \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

$$\Rightarrow \quad y \cdot x^2 = -x^2 \cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

$$\Rightarrow \quad yx^2 = x(2\sin x - x\cos x) + \frac{x^3}{9}(3\log x - 1) + 2\cos x + C$$

57. Given that, $\quad (1 + \tan y)(dx - dy) + 2xdy = 0$

$$\Rightarrow \quad (1 + \tan y)dx + (2x - 1 - \tan y)dy = 0$$

$$\Rightarrow \quad (1 + \tan y)dx = (1 + \tan y - 2x)dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-2}{1 + \tan y} \cdot x + 1$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{2}{1 + \tan y} \cdot x = 1$$

which is a linear differential equation.

$$\text{IF} = e^{\int \frac{2}{1 + \tan y} dy} = e^{\int \frac{2dt}{(1 + t)(1 + t^2)}} \quad \text{...(i)}$$

Let $\quad t = \tan y$

$$dt = \sec^2 y dy$$

$$\frac{dt}{1 + t^2} = dy$$

$$\frac{1}{(1 + t)(1 + t^2)} = \frac{A}{1 + t} + \frac{Bt + C}{1 + t^2}$$

$$1 = A(1 + t^2) + (1 + t)(Bt + C)$$

$$1 = A + At^2 + Bt + C + Bt^2 + Ct$$

$$1 = (A + B)t^2 + (B + C)t + (A + C)$$

On comparing,

$$A + B = 0, B + C = 0 \text{ and } A + C = 1$$

$$\Rightarrow \quad A - C = 0 \text{ and } A + C = 1$$

$$\Rightarrow \quad 2A = 1 \Rightarrow \left(A = \frac{1}{2} \text{ and } C = \frac{1}{2}\right) \Rightarrow \left(B = \frac{-1}{2}\right)$$

$$\therefore \quad \frac{1}{(1 + t)(1 + t^2)} = \frac{1}{2(1 + t)} + \frac{1}{2}\frac{(-t + 1)}{(1 + t^2)}$$

$$\Rightarrow \int \frac{1}{(1 + t)(1 + t^2)} dt = \frac{1}{2}\int \frac{dt}{1 + t} - \frac{1}{4}\int \frac{2t}{1 + t^2} dt + \frac{1}{2}\int \frac{dt}{1 + t^2}$$

$$= \frac{1}{2}\log|1+t| + \frac{1}{4}\log|1+t^2| + \frac{1}{2}\tan^{-1}t$$

$$= \frac{1}{2}\log(1+t)\sqrt{1+t^2} + \frac{1}{2}\tan^{-1}t$$

Now, from Eq. (i),

$$= e^{2\left\{\frac{1}{2}\log(1+t)\sqrt{1+t^2} + \frac{1}{2}\tan^{-1}t\right\}} = e^{\log(1+t)\sqrt{1+t^2} + \tan^{-1}t}$$

$$= (1+t)\sqrt{1+t^2}\cdot e^{\tan^{-1}t} = (1+\tan y)\sec y e^y$$

$\therefore$ Complete solution,

$$x\cdot(1+\tan y)\sec y e^y = \int \sec y e^y dy + \int \tan y \sec y e^y dy + C$$

$$= \sec y e^y - \int \sec y\cdot\tan y e^y dy + \int \sec y \tan y e^y dy + C$$

$$\therefore \quad x(1+\tan y)\sec y\cdot e^y d = \sec y\cdot e^y + C$$

$$\Rightarrow \quad x = \frac{1}{(1+\tan y)} + \frac{C\cdot e^{-y}}{(1+\tan y)\cdot\sec y}$$

58. Given, $(1+y^2)\tan^{-1}x\,dx + 2y(1+x^2)dy = 0$

$$\Rightarrow \quad (1+y^2)\tan^{-1}x\,dx = -2y(1+x^2)dy = 0$$

$$\Rightarrow \quad \frac{\tan^{-1}x}{1+x^2}dx + \frac{2y}{1+y^2}dy = 0$$

On integrating both sides, we get

$$\int \frac{\tan^{-1}x}{1+x^2}dx + \int \frac{2y}{1+y^2}dy = 0 \quad \ldots(i)$$

Putting $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2} = \frac{dt}{dx} \Rightarrow dx = (1+x^2)dt$

Put $1+y^2 = x \Rightarrow 2y = \frac{dz}{dy} \Rightarrow dy = \frac{dz}{2y}$

Then, Eq. (i) becomes $\int \frac{t}{1+x^2}(1+x^2)\,dt + \int \frac{2y}{z}\frac{dz}{2y} = 0$

$$\Rightarrow \quad \int t\,dt + \int \frac{1}{z}dz = 0 \Rightarrow \frac{t^2}{2} + \log|z| = C$$

$$\Rightarrow \quad \frac{(\tan^{-1}x)^2}{2} + \log|1+y^2| = C \ (\because t = \tan^{-1}x \text{ and } z = 1+y^2)$$

which is the required solution.

59. Given that, $y\frac{dy}{dx} + x = C$

$$\Rightarrow \quad y\,dy = (C-x)dx$$

On integrating, we get

$$\Rightarrow \quad \int y\,dy = \int c(-x)dx$$

$$\Rightarrow \quad \frac{y^2}{2} = Cx - \frac{x^2}{2} + d$$

$$\Rightarrow \quad x^2 + y^2 - 2Cx - 2d = 0$$

which represent a family of circles having centre at $(2C, 0)$ and radius $= \sqrt{4C^2 + 2d}$.

60. Given that, $\frac{dy}{dx} = 2x\cdot e^{x^2-y}$

$$\Rightarrow \quad \int e^y dy = \int 2xe^{x^2}dx \ (\text{put } x^2 = t,\ 2xdx = dt)$$

$$\Rightarrow \quad \int e^y dy = \int e^t dt$$

$$\Rightarrow \quad C^y = e^t + C$$

$$\Rightarrow \quad e^y = e^{x^2} + C$$

61. Given equation is $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

$$\Rightarrow \quad (\sin y + y\cos y)dy = x(2\log x + 1)dx$$

On integration both sides, we get

$$\int \sin y\,dy + y\sin y - \int \sin y\,dy$$

$$= x^2\log x - \int x^2\cdot\frac{1}{x}dx + \int x\,dx + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C$$

62. On dividing by y^2 on both sides of given equation, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{y}x = x$$

Let $\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

Then, given equation reduces to

$$\frac{dz}{dx} - xz = -x$$

$$\therefore \quad \text{IF} = e^{\int -x\,dx} = e^{-x^2/2}$$

The solution is

$$ze^{-x^2/2} = \int(-x)\cdot e^{-x^2/2}dx = e^{-x^2/2} + C$$

i.e., $\frac{1}{y} = 1 + Ce^{x^2/2}$

63. Here, $\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y} \Rightarrow e^y dy = (x^2 + e^x)dx$

This is variable separable form.

$\therefore$ On integrating both sides, we get

$$\int e^y\,dy = \int(x^2 + e^x)dx \Rightarrow e^y = \frac{x^3}{3} + e^x + C$$

which is the general solution of the given differential equation, where C is an arbitrary constant.

64. Given equation is $xdy - y\,dx + x^2e^x dx = 0$

$$\Rightarrow \quad \frac{xdy - y\,dx}{x^2} + e^x dx = 0$$

$$\Rightarrow \quad d\left(\frac{y}{x}\right) + d(e^x) = 0$$

$$\Rightarrow \quad \frac{y}{x} + e^x = C$$

65. Given equation

$$\frac{dy}{dx} + \sin\left(\frac{x+y}{2}\right) = \sin\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \sin\left(\frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -2\sin\left(\frac{y}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\Rightarrow \quad \operatorname{cosec}\left(\frac{y}{2}\right)dy = -2\cos\left(\frac{x}{2}\right)dx$$

On integrating both sides, we get

$$\int \operatorname{cosec}\left(\frac{y}{2}\right)dy = -\int 2\cos\left(\frac{x}{2}\right)dx + C$$

$$\Rightarrow \quad \frac{\log\left(\tan\frac{y}{4}\right)}{\frac{1}{2}} = -\frac{2\sin\left(\frac{x}{2}\right)}{\frac{1}{2}} + C$$

$$\Rightarrow \quad \log\left(\tan\frac{y}{4}\right) = C - 2\sin\left(\frac{x}{2}\right)$$

66. Given equation is, $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ ...(i)

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, Eq. (i) becomes

$$v + x\frac{dv}{dx} = v + \frac{\phi(v)}{\phi'(v)}$$

$$\Rightarrow \quad \frac{\phi'(v)}{\phi(v)}dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{\phi'(v)}{\phi(v)}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad \log\phi(v) = \log x + \log k$$

$$\Rightarrow \quad \log\phi(v) = \log xk$$

$$\Rightarrow \quad \phi(v) = kx \Rightarrow \phi\left(\frac{y}{x}\right) = kx \qquad \left(\because v = \frac{x}{y}\right)$$

67. $$\frac{dt}{dx} - t\frac{g'(x)}{g(x)} = -\frac{t^2}{g(x)}$$

$$\Rightarrow \quad -\frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\frac{g'(x)}{g(x)} = \frac{1}{g(x)} \qquad \text{...(i)}$$

Let $z = \frac{1}{t} \Rightarrow -\frac{1}{t^2}\frac{dt}{dx} = \frac{dz}{dx}$

$\therefore$ From Eq. (i),

$$\frac{dz}{dx} + \frac{g'(x)}{g(x)}z = \frac{1}{g(x)}$$

On comparing with $\frac{dz}{dx} + Pz = Q$, we get

$$P = \frac{g'(x)}{g(x)}, Q = \frac{1}{g(x)}$$

$$\therefore \quad \text{IF} = e^{\int \frac{g'(x)}{g(x)}dx} = e^{\log[g(x)]} = g(x)$$

Thus, complete solution is

$$z \cdot g(x) = \int g(x) \cdot \frac{1}{g(x)}dx + C$$

$$\Rightarrow \quad \frac{1}{t}g(x) = x + C \Rightarrow \frac{g(x)}{x + C} = t$$

68. The given equation is

$$t = 1 + (ty)\left(\frac{dy}{dt}\right) + \frac{(ty)^2}{2!}\left(\frac{dy}{dt}\right)^2 + \ldots\infty$$

$$\Rightarrow \quad t = e^{ty\left(\frac{dy}{dt}\right)}$$

$$\Rightarrow \quad \log t = ty\frac{dy}{dt} \Rightarrow y\,dy = \frac{\log t}{t}dt$$

On integrating both sides, we get

$$\frac{y^2}{2} = \frac{(\log t)^2}{2} + k$$

$$\Rightarrow \quad y = \pm\sqrt{(\log t)^2 + 2k}$$

$$\Rightarrow \quad y = \pm\sqrt{(\log t)^2 + C}$$

69. $$\frac{ydx - xdy}{dx} = 3\frac{(dx - x^2dy)}{dx}$$

$$\Rightarrow \quad \frac{ydx - xdy}{x^2} = 3\left(\frac{dx}{x^2} - dy\right)$$

$$\Rightarrow \quad \int -d\left(\frac{y}{x}\right) = \int 3\left(\frac{dx}{x^2} - dy\right)$$

$$\Rightarrow \quad -\frac{y}{x} = -\frac{3}{x} - 3y - c$$

$$\Rightarrow \quad \frac{y}{x} = \frac{3}{x} + 3y + c$$

$$\Rightarrow \quad y = 3 + 3xy + cx$$

$$\Rightarrow \quad y - 3 = 3xy + cx$$

70. $$y\,dx - x\,dy = xy\,dx$$

$$\Rightarrow \quad \frac{y\,dx - x\,dy}{y^2} = \frac{x}{y}dx$$

$$\Rightarrow \quad \int \frac{d\left(\frac{x}{y}\right)}{\frac{x}{y}} = \int dx$$

$$\Rightarrow \quad \log_e\left(\frac{x}{y}\right) + \log_e C = x \Rightarrow \log_e\left(\frac{Cx}{y}\right) = x$$

$$\Rightarrow \quad y = Cx \cdot e^{-x}$$

71. $\{f(x)\}^2 + 4f(x) \cdot f'(x) + \{f'(x)\}^2 = 0$ solving for $f'(x)$, we get

$$f'(x) = \frac{-4f(x) \pm \sqrt{16\{f(x)\}^2 - 4\{f(x)\}^2}}{2} = -2f(x) \pm \sqrt{3}f(x)$$

$$\therefore \quad \int \frac{f'(x)}{f(x)} = \int -2 \pm \sqrt{3} \Rightarrow \log f(x) = (-2 \pm \sqrt{3})x + C$$

$$\therefore \quad f(x) = C \cdot e^{(-2 \pm \sqrt{3})x}$$

72. $$xy\frac{dy}{dx} = (1 + y^2)\left(1 + \frac{x}{1 + x^2}\right)$$

$$\Rightarrow \quad \frac{y}{1 + y^2}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{1 + x^2}$$

$$\frac{1}{2}\ln(1 + x^2) = \ln x + \tan^{-1}x + \ln c$$

$$\Rightarrow \quad \sqrt{1 + y^2} = cxe^{\tan^{-1}x}$$

73. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

we get $xv + x^2\frac{dv}{dx} = vx + 2x\sqrt{v^2 - 1}$

$\Rightarrow \dfrac{dv}{2\sqrt{v^2-1}} = \dfrac{dx}{x}$, integrating, we get

$$\Rightarrow \quad \frac{1}{2}\ln(v+\sqrt{v^2-1}) = \ln Cx$$

$$\Rightarrow \quad \frac{1}{2}\ln\left(\frac{y+\sqrt{y^2-x^2}}{x}\right) = \ln Cx$$

74. **Hint** $\quad y = u^m \Rightarrow \dfrac{dy}{dx} = mu^{m-1}\dfrac{du}{dx}$

Hence, $\quad 2x^4 \cdot u^m \cdot mu^{m-1} \cdot \dfrac{du}{dx} = u^{4m} = 4x^6$

$$\frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4 u^{2m-1}}$$

$$\Rightarrow \quad 4m = 6 \Rightarrow m = \frac{3}{2}$$

75. $\quad \dfrac{dy}{dx} + \dfrac{1}{x}y = xy^4 \Rightarrow y^{-4}\dfrac{dy}{dx} + \dfrac{1}{x}y^{-3} = x$

$$\Rightarrow \quad \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx} \qquad (\because z = y^{-3})$$

$$\Rightarrow \quad -\frac{1}{3}\frac{dz}{dx} + \frac{1}{x}z = x \Rightarrow \frac{dz}{dx} - \frac{3}{x}z = -3x$$

Solution the above equation is linear differential equation which can be solved.

Round II

1. On putting $x = \tan A$ and $y = \tan B$ in the given relation, we get

$$\cos A + \cos B = \lambda(\sin A - \sin B)$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{1}{\lambda}$$

$$\Rightarrow \quad \tan^{-1}x - \tan^{-1}y = 2\tan^{-1}\left(\frac{1}{\lambda}\right)$$

On differentiating w.r.t. x, we get

$$\frac{1}{1+x^2} - \frac{1}{1+y^2}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

Clearly, it is a differential equation of degree 1.

2. The equation of the given family of curves is

$$a^{n-1}y = x^n \qquad \ldots(\text{i})$$

$$\Rightarrow \quad (n-1)\log a + \log y = n\log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x} \qquad \ldots(\text{ii})$$

This is the differential equation of the family of curves given in Eq. (i).

The differential equation of the orthogonal trajectories of Eq. (i) is obtained by replacing $\dfrac{dy}{dx}$ by $\left(-\dfrac{dx}{dy}\right)$ in Eq. (ii).

$$\therefore \quad \frac{1}{y}\times\left(\frac{-dx}{dy}\right) = \frac{n}{x}$$

$$\Rightarrow \quad x\,dx + ny\,dy = 0$$

On integrating, we get

$$\frac{x^2}{2} + n\frac{y^2}{2} = C$$

$$\Rightarrow \quad x^2 + ny^2 = 2C = \text{Constant}$$

3. We have, $\quad \dfrac{dy}{dx} = y(xy-1)$

$$\Rightarrow \quad dy = xy^2dx - y\,dx$$

$$\Rightarrow \quad y\,dx + dy = xy^2dx$$

$$\Rightarrow \quad \frac{y\,dx + dy}{y^2} = x\,dx$$

$$\Rightarrow \quad \frac{ye^{-x}dx + e^{-x}dy}{y^2} = xe^{-x}dx$$

$$\Rightarrow \quad -d\left(\frac{e^{-x}}{y}\right) = xe^{-x}dx$$

On integrating, we get

$$-\frac{e^{-x}}{y} = -xe^{-x} - e^{-x} - C$$

$$\Rightarrow \quad \frac{1}{y} = x + 1 + Ce^x \qquad \ldots(\text{i})$$

Since, it passes through $(0, 1)$. Therefore,

$$1 = 1 + C \Rightarrow C = 0$$

On putting $C = 0$ in Eq. (i), we get

$$\frac{1}{y} = x + 1 \Rightarrow y(x+1) = 1$$

4. The equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts the line $y = x$ at the point

$$\left(\frac{y - x\frac{dy}{dx}}{1-\frac{dy}{dx}}, \frac{y - x\frac{dy}{dx}}{1-\frac{dy}{dx}}\right)$$

$$\frac{y - x\frac{dy}{dx}}{1-\frac{dy}{dx}} = 1$$

$$\Rightarrow \quad y - x\frac{dy}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dx}{x-1} = \frac{dy}{y-1}$$

$$\Rightarrow \quad (x-1) = C(y-1)$$

5. The given equation can be written as

$$xy\,dy + y^2dx - dy = 0$$

$$\Rightarrow \quad xdy + y\,dx - \frac{1}{y}dy = 0$$

$$\Rightarrow \quad \int d(xy) = \int \frac{dy}{y}$$

$$\Rightarrow \quad xy - \log y = C$$

$\Rightarrow \frac{1}{y}$ is integrating factor.

6. Given that, $\frac{dV}{dt} = -6h^{3/2}$ ...(i)

At time t the volume of water in the tank is

$$V = \frac{1}{3}\pi r^2 h$$

Since, ΔOAB and $\Delta OA'B'$ are similar.

$$\frac{OA'}{A'B'} = \frac{OA}{AB}$$

$$\Rightarrow \quad \frac{h}{r} = \frac{4}{2} \Rightarrow r = \frac{h}{2}$$

$$V = \frac{\pi}{3} \times \frac{h^3}{4}$$

$$\Rightarrow \quad h^3 = \frac{12V}{\pi}$$

From Eq. (i), we get

$$\frac{dV}{dt} = \frac{-6\sqrt{12}}{\sqrt{\pi}}\sqrt{V}$$

$$\Rightarrow \quad \int_{V_0}^{0} \frac{dV}{\sqrt{V}} = \frac{-6\sqrt{12}}{\sqrt{\pi}} \int_0^t dt$$

$$\Rightarrow \quad (2\sqrt{V})_{V_0}^{0} = \frac{-6\sqrt{12}}{\sqrt{\pi}} \cdot t$$

$$\Rightarrow \quad t = \frac{2\sqrt{V_0}\sqrt{\pi}}{6\sqrt{12}}$$

$$\Rightarrow \quad t = \frac{2\sqrt{\pi}}{6\sqrt{12}}\sqrt{\frac{16}{3}\cdot\pi} \qquad \left(\because V_0 = \frac{\pi}{3} \times \frac{2^3}{4}\right)$$

$$\Rightarrow \quad t = \frac{2\pi}{9} \text{ unit}$$

7. We have, $\frac{dy}{dx} = 1 - \frac{1}{x^2} \Rightarrow y = x + \frac{1}{x} + C$

This passes through $\left(2, \frac{7}{2}\right)$.

$$\therefore \quad \frac{7}{2} = 2 + \frac{1}{2} + C \Rightarrow C = 1$$

Thus, the equation of the curve is

$$y = x + \frac{1}{x} + 1 \text{ or } xy = x^2 + x + 1$$

8. Given, $f''(x) = 6(x-1)$

$$\Rightarrow \quad f'(x) = 3(x-1)^2 + C_1 \qquad \text{...(i)}$$

But at point (2, 1) the line $y = 3x - 5$ is tangent to the graph $y = f(x)$.

Hence, $\left(\frac{dy}{dx}\right)_{x=2} = 3$ or $f'(2) = 3$

Then, from Eq. (i) $f'(2) = 3(2-1)^2 + C_1$

$$3 = 3 + C_1$$

$$\Rightarrow \quad C_1 = 0$$

i.e., $f'(x) = 3(x-1)^2$

Given, $f(2) = 1$

$$\therefore \quad f(x) = (x-1)^3 + C_2$$

$$\Rightarrow \quad f(2) = 1 + C_2$$

$$\Rightarrow \quad 1 = 1 + C_2$$

$$\Rightarrow \quad C_2 = 0$$

Hence, $f(x) = (x-1)^3$

9. The length of normal is given by, $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\therefore \quad y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = k \qquad \text{(given)}$$

$$\Rightarrow \quad y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = k^2$$

$$\Rightarrow \quad y^2 + y^2\left(\frac{dy}{dx}\right)^2 = k^2$$

$$\Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 = k^2 - y^2$$

10. $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$

$$\Rightarrow \quad x^4dy + x^3ydx + \operatorname{cosec}(xy)dx = 0$$

$$\Rightarrow \quad x^3(xdy + y\,dx) + \operatorname{cosec}(xy)\,dx = 0$$

$$\Rightarrow \quad x^3d(xy) + \operatorname{cosec}(xy)\,dx = 0$$

$$\Rightarrow \quad \frac{d(xy)}{\operatorname{cosec}(xy)} + \frac{dx}{x^3} = 0$$

On integrating both sides, we get

$$\int \sin(xy)\,d(xy) + \int x^{-3}dx = 0$$

$$\Rightarrow \quad -\cos(xy) + \left(\frac{x^{-2}}{-2}\right) = C_1 \Rightarrow 2\cos(xy) + x^{-2} = C_1$$

$(\because C = -2C_1)$

11. The given equation can be written as

$$\left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{(x^2dy - y^2dx)}{(x-y)^2} = 0$$

$$\Rightarrow \quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)^2} = 0$$

$$\Rightarrow \quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2} = 0$$

On integrating both sides, we get

$$\ln|x| - \ln|y| - \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} = C$$

$$\Rightarrow \quad \ln\left|\frac{x}{y}\right| - \frac{xy}{(y-x)} = C$$

$$\Rightarrow \quad \ln\left|\frac{x}{y}\right| + \frac{xy}{(x-y)} = C$$

12. Equation of tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

For y-axis, $X = 0$.

Then, $$Y = y - x\frac{dy}{dx}$$

Given, $$\left(y - x\frac{dy}{dx}\right) \propto x^3$$

$$\Rightarrow \quad y - x\frac{dy}{dx} = kx^3$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -kx^2$$

$$\text{IF} = e^{\int 1/x\, dx} = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}$$

Then, solution is

$$y\left(\frac{1}{x}\right) = \int \frac{-kx^2}{x} dx$$

$$\Rightarrow \quad \frac{y}{x} = -\frac{kx^2}{2} + C \text{ or } y = -\frac{kx^3}{2} + Cx$$

13. $\therefore$ Equation of normal at (x, y) is

$$Y - y = \frac{dx}{dy}(X - x)$$

Put $$Y = 0$$

Then, $$X = x + y\frac{dy}{dx}$$

Given, $$y^2 = 2xX$$

$$\Rightarrow \quad y^2 = 2x\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - 2x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 2}{2\left(\frac{y}{x}\right)}$$

Put $y = vx$, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then, $$v + x\frac{dv}{dx} = \frac{v^2 - 2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(2 + v^2)}{2v}$$

$$\Rightarrow \quad \frac{2v\, dv}{(2 + v^2)} + \frac{dx}{x} = 0$$

On integrating both sides, we get

$$\ln(2 + v^2) + \ln|x| = \ln C$$

$$\Rightarrow \quad \ln[|x|(2 + v^2)] = \ln C$$

$$\Rightarrow \quad |x|\left(2 + \frac{y^2}{x^2}\right) = C$$

$\because$ It passes through $(2, 1)$, then

$$2\left(2 + \frac{1}{4}\right) = C \Rightarrow C = \frac{9}{2}$$

Then, $$|x|\left(2 + \frac{y^2}{x^2}\right) = \frac{9}{2}$$

$$\Rightarrow \quad 2x^2 + y^2 = \frac{9}{2}|x|$$

$$\Rightarrow \quad 4x^2 + 2y^2 = 9|x|$$

14. $\because$ Equation of normal at $P(1, 1)$ is

$$ay + x = a + 1 \qquad \text{(given)}$$

$\because$ Slope of normal at $(1, 1) = -\frac{1}{a}$

$\therefore$ Slope of tangent at $(1, 1) = a$ ...(i)

Also, given $\frac{dy}{dx} \propto y$

$$\Rightarrow \quad \frac{dy}{dx} = ky$$

$$\left[\frac{dy}{dx}\right]_{(1, 1)} = k = a \qquad \text{[from Eq. (i)]}$$

Then, $$\frac{dy}{dx} = ay$$

$$\Rightarrow \quad \frac{dy}{y} = a\, dx$$

$$\Rightarrow \quad \ln|y| = ax + C$$

$\because$ It is passing through $(1, 1)$, then $c = -a$

$$\Rightarrow \quad \ln|y| = a(x - 1)$$

$$\Rightarrow \quad |y| = e^{a(x-1)}$$

15. [**Hint** All lines at a constant distance p from the origin are tangent to the circle, $x^2 + y^2 = p^2 \Rightarrow$ Equation to the family of such lines are $y = mx \pm p\sqrt{1 + m^2}$. Put $m = \frac{dy}{dx}$ and get the result].

16. $$\frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = -\sec\frac{1}{x} \cdot \frac{1}{x^2}$$

$$\text{IF} = e^{-\int \frac{1}{x^2}\tan\frac{1}{x} dx} = \sec\frac{1}{x}$$

$$\therefore \quad y \cdot \sec\frac{1}{x} = -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2} dx = \tan\frac{1}{x} = C$$

If $y \to -1$, then $x \to \infty \Rightarrow C = -1$

$$\Rightarrow \quad y = \sin\frac{1}{x} - \cos\frac{1}{x}$$

17. Let P be (x, y). C is $\left(x + y\frac{dy}{dx}, 0\right)$ and B is $\left(0, y - x\frac{dy}{dx}\right)$. Centre of the circle through O, C, P and B has its centre at the mid-point of BC. Let it be (α, β), then $2\alpha = x + y\frac{dy}{dx}$ and

$$2\beta = y - x\frac{dy}{dx}$$

Now, (α, β) lies on $y = x$

So, $$y - x\frac{dy}{dx} = x + y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y - x}{x + y}$$

18. $(2ny + xy\log_e x)\,dx = x\log_e x\,dy \Rightarrow \frac{dy}{y} = \left(\frac{2n}{x\log_e x} + 1\right)dx$

$\Rightarrow \quad \log(y) = 2n\log|\log x| + x + C$ and $C = 0$

$\therefore \quad y = e^{x + \log(\log x)^{2n}} = e^{x(\log x)^{2n}} \Rightarrow f(x) = e^x(\log x)^{2n}$

Now, $$g(x) = \lim_{n\to\infty} f(x) = \begin{cases} \to \infty, & \text{if } x < \frac{1}{2} \\ 0, & \text{if } \frac{1}{e} < x < e \\ \to \infty, & \text{if } x > e \end{cases}$$

$\therefore \quad \int_{1/e}^{e} g(x)\,dx = 0$

19. Let $P(x, y)$ be any point on the curve PM the perpendicular to x-axis PT the tangent at P meeting the axis of x at T, as given $OT = 2$, $OM = 2x$. Equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It intersects the axis of x, where $Y = 0$

$\Rightarrow \quad -y = \frac{dy}{dx}(X - x) \Rightarrow X = x - y\frac{dx}{dy} = OT$

Hence, $$x - y\frac{dx}{dy} = 2x \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$$

On integrating, $\log x + \log y = \log C \Rightarrow xy = C$

This passes through $(1, 2)$ $\therefore C = 2$

$\therefore$ The required curve is $xy = 2$

Hence option (b) is the correct answer.

20. For the path BA, the equation of the motion of the particle is

$$mv\frac{dv}{dx} = -\frac{mk^2a^2}{x^2} \Rightarrow \int_0^v v\,dv = -k^2a^2\int_{2a}^{a}\frac{dx}{x^2} \Rightarrow v = k\sqrt{a}$$

For the path AO, the equation of motion of the particle is

$$mv\frac{dv}{dx} = -\frac{2mk^2x}{a} \Rightarrow \int_V^v v\,dx = \frac{2k^2}{a}\int_a^0 x\,dx,$$

where V is velocity at $x = 0$

$\Rightarrow \quad \frac{V^2 - v^2}{2} = k^2a \Rightarrow v^2 = 2k^2a + k^2a = 3k^2a \quad (\because v = k\sqrt{a})$

$\therefore \quad v = k\sqrt{3a}$

21. Taking $x = r\cos\theta$ and $y = r\sin\theta$, so that

$$x^2 + y^2 = r^2 \text{ and } \frac{y}{x} = \tan\theta,$$

we have $\quad x\,dx + y\,dy = r\,dr$

and $\quad x\,dy - y\,dx = x^2\sec^2\theta\,d\theta = r^2d\theta$

The given equation can be transformed into

$$\frac{r\,dr}{r^2d\theta} = \sqrt{\left(\frac{a^2 - r^2}{r^2}\right)}$$

$\Rightarrow \quad \frac{dr}{d\theta} = \sqrt{(a^2 - r^2)}$

$\Rightarrow \quad \frac{dr}{\sqrt{(a^2 - r^2)}} = d\theta$

On integrating both sides, then we get

$$\sin^{-1}\left(\frac{r}{a}\right) = \theta + C$$

$\Rightarrow \quad \sin^{-1}\left\{\frac{\sqrt{(x^2 + y^2)}}{a}\right\} = \tan^{-1}\left(\frac{y}{x}\right) + C \quad \ldots(i)$

$\Rightarrow \quad \sqrt{(x^2 + y^2)} = a\sin\{\tan^{-1}(y/x) + C\}$

$\Rightarrow \quad \sqrt{(x^2 + y^2)} = a\sin\{\tan^{-1}(y/x) + \text{constant}\}$

Also, from Eq. (i),

$$\tan^{-1}\left(\frac{y}{x}\right) = \left\{\sin^{-1}\left[\frac{\sqrt{(x^2 + y^2)}}{a}\right] - C\right\}$$

$\Rightarrow \quad y = x\tan\left\{\sin^{-1}\left[\frac{\sqrt{(x^2 + y^2)}}{a}\right] - C\right\}$

$\Rightarrow \quad = x\tan\left\{\sin^{-1}\left[\frac{\sqrt{(x^2 + y^2)}}{a}\right] + \text{constant}\right\}$

22. Rewriting the given equation, we get $\frac{dy}{dx} = x\{e^{(n-1)y} - 1\}$

$\Rightarrow \quad \frac{dy}{e^{(n-1)y} - 1} - x\,dx$

$\Rightarrow \quad \frac{1}{n-1}\int\frac{(n-1)e^{(n-1)y}}{\{e^{(n-1)y} - 1\}e^{(n-1)y}}dy = \frac{x^4}{2} + C$

$\Rightarrow \quad \frac{1}{n-1}\int\frac{du}{u(u-1)} = \frac{x^2}{2} + C \quad [\text{where } u = e^{(n-1)y}]$

$\Rightarrow \quad \frac{1}{n-1}\log\frac{u-1}{u} = \frac{x^2}{2} + C$

$$\Rightarrow \quad \frac{1}{n-1}\log\left\{\frac{e^{(n-1)y}-1}{e^{(n-1)y}}\right\} = \frac{x^2}{2} + C$$

$$\Rightarrow \quad e^{(n-1)y} = Ce^{(n-1)y+(n-1)x^2/2} + 1$$

23. $$\frac{y^2dx - 2xy\,dy}{y^4} = \frac{x^3y^3dy + x^2y^4dx}{y^4}$$

$$\Rightarrow \quad d\left(\frac{x}{y^2}\right) = \frac{x^2y^3(xdy + ydx)}{y^4}$$

$$\Rightarrow \quad d\left(\frac{x}{y^2}\right) = \frac{x^2}{y}d(xy)$$

$$\Rightarrow \quad d\left(\frac{x}{y^2}\right) = \left(\frac{x}{y^2}\right)\cdot xy\,d(xy)$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y^2}\right)}{(x/y^2)} = xy\,d(xy)$$

$$\Rightarrow \quad \log\left(\frac{x}{y^2}\right) = \frac{(xy)^2}{2} + C$$

24. $\frac{d}{dx}(xy) = x\cdot\frac{f(xy)}{f'(xy)}$

On integration both sides,

$$\int \frac{f'(xy)}{f(xy)}d(xy) = \int x\,dx$$

$$\log f(xy) = \frac{x^2}{2} + C$$

$$f(xy) = e^{\left(\frac{x^2}{2}+c\right)} = e^{x^2/2}\cdot e^c = ke^{x^2/2}$$

25. Rewriting the given differential equation as

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - \left[2x^2y\frac{dy}{dx} + 2xy^2\right] = 0$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 - \frac{d}{dx}(x^2y^2) = 0$$

On integrating $\left(\frac{dy}{dx}\right)^2 - x^2y^2 = C$

$$\frac{dy}{dx} = \pm\sqrt{C + x^2y^2}$$

Solutions (Q. Nos. 26-29)

Let $y = f(x)$ and $y = g(x)$ be the required curves. The equation of the tangents to these two curves at points with equal abscissa x are

$$Y - f(x) = f'(x)(X - x) \text{ and } Y - g(x) = g'(x)(X - x)$$

These two lines intersect on y-axis

$$\therefore \quad Y - f(x) = -xf'(x) \text{ and } Y - g(x) = -xg'(x)$$

$$\Rightarrow \quad f(x) - g(x) = x\{f'(x) - g'(x)\}$$

$$\Rightarrow \quad f(x) - g(x) = x\frac{d}{dx}\{f(x) - g(x)\}$$

On integration,

$$\log\{f(x) - g(x)\} = \log x + \log C$$

$$f(x) - g(x) = Cx \quad \ldots\text{(i)}$$

The equation of the normals at points with equal abscissa x to the two curves are

$$Y - f(x) = -\frac{1}{f'(x)}(X - x)$$

and $$Y - g(x) = -\frac{1}{g'(x)}(X - x)$$

These intersect on x-axis

$$0 - f(x) = \frac{1}{f'(x)}(X - x) \text{ and } 0 - g(x) = \frac{1}{g'(x)}(X - x)$$

$$\Rightarrow \quad X = x + f(x)f'(x) \text{ and } X = x + g(x)\cdot g'(x)$$

$$\Rightarrow \quad f(x)f'(x) = g(x)g'(x)$$

On integration, $\{f(x)\}^2 = \{g(x)\}^2 + C_1$

$$\Rightarrow \quad \{f(x)\}^2 - \{g(x)\}^2 = C_1$$

$$\Rightarrow \quad \{f(x) + g(x)\}Cx = C_1$$

$$\Rightarrow \quad f(x) + g(x) = \frac{C_1}{Cx} \quad \ldots\text{(ii)}$$

On solving Eq. (i) and (ii),

$$f(x) = \frac{1}{2}\left(Cx + \frac{C_1}{x}\right) \quad \ldots\text{(iii)}$$

and $$g(x) = \frac{1}{2}\left(\frac{C_1}{Cx} - Cx\right) \quad \ldots\text{(iv)}$$

Since, Eq. (iii) passes through (1, 1) and Eq. (iv) passes through (2, 3).

$$\therefore \quad 2 = C + \frac{C_1}{C} \text{ and } 6 = \frac{C_1}{2C} - 2C$$

$$\Rightarrow \quad C = -2 \text{ and } C_1 = -8$$

$$\therefore \quad f(x) = -x + \frac{2}{x} \text{ and } g(x) = \frac{2}{x} + x$$

For number of positive integral solutions for $f(x) = g(x)$

$$\therefore \quad -x + \frac{2}{x} = \frac{2}{x} + x$$

$\therefore x = 0$, no solution.

Now, $$\int_1^2[g(x) - f(x)]\,dx = \int_1^2\left[\frac{2}{x} + x + x - \frac{2}{x}\right]dx$$

$$= 2\int_1^2 x\,dx = 2\cdot\left[\frac{x^2}{2}\right]_1^2 = [4 - 1] = 3$$

30. Given, $$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x^2}$$

$$\therefore \quad \text{IF} = e^{\int\frac{1}{x}dx} = x$$

$\therefore$ Solution is $xy = \int_0^x \frac{\sin t}{t}dt + C$

On putting $x = 1, y = 2$, we get

$$C = 2 - \int_0^1 \frac{\sin t}{t}dt$$

$$\Rightarrow \quad y = \frac{1}{x}\int_0^x \frac{\sin t}{t}dt + \frac{C}{x}$$

$$= \frac{1}{x}\int_0^x \frac{\sin t}{t}dt + \frac{2}{x} - \frac{1}{x}\int_0^1 \frac{\sin t}{t}dt$$

$$= \frac{2}{x} + \frac{1}{x}\int_1^x \frac{\sin t}{t}\,dt$$

$$\Rightarrow \quad y(2) = 1 + \frac{1}{2}\int_1^2 \frac{\sin t}{t}\,dt$$

31. Let $c_1 + c_2 + c_3 e^{c_4} = A$ (constant)

Then, $y = Ax$

$$\Rightarrow \quad \frac{dy}{dx} = A$$

$$\Rightarrow \quad y = x\frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{dy}{dx} = y$$

32. Given, $\dfrac{dy}{dx} = \dfrac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$

$$\Rightarrow \quad \int \frac{dy}{y\sqrt{y^2-1}} = \int \frac{dx}{x\sqrt{x^2-1}}$$

$$\Rightarrow \quad \sec^{-1} y = \sec^{-1} x + C$$

At $x = 2, y = \dfrac{2}{\sqrt{3}}$

$$\Rightarrow \quad \frac{\pi}{6} = \frac{\pi}{3} + C \Rightarrow C = -\frac{\pi}{6}$$

$$\therefore \quad y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$$

$$= \cos\left(\cos^{-1}\frac{1}{x} - \cos^{-1}\frac{\sqrt{3}}{2}\right)$$

$$= \cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x} + \sqrt{1-\frac{1}{x^2}}\cdot\sqrt{1-\frac{3}{4}}\right)\right]$$

$$\Rightarrow \quad \frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1-\frac{1}{x^2}}$$

33. The given differential equation is

$$\frac{d\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}} = \frac{2x}{x^2+1}\cdot dx$$

$$\Rightarrow \quad \log\left(\frac{dy}{dx}\right) = \log(x^2+1) + \log C, C > 0$$

$$\Rightarrow \quad \frac{dy}{dx} = C(x^2+1) \Rightarrow y = C\left(\frac{x^3}{3} + x\right) + C'$$

$$C' \in R$$

Obviously $y \to \infty$, as $x \to \infty$ as $C > 0$

34. The general equation of all non-vertical lines in a plane is $ax + by = 1$,

where $b \neq 0$

Now, $ax + by = 1$

$$\Rightarrow \quad a + b\frac{dy}{dx} = 0 \qquad \text{(differential w.r.t. } x\text{)}$$

$$\Rightarrow \quad b\frac{d^2y}{dx^2} = 0 \qquad \text{(differential w.r.t. } x\text{)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \qquad (\because b \neq 0)$$

Hence, the differential equation is $\dfrac{d^2y}{dx^2} = 0$

35. $\dfrac{dy}{dx} + y = 1 \Rightarrow \dfrac{dy}{1-y} = dx$

$$\int \frac{dy}{1-y} = \int dx - \log(1-y) = x$$

$$1 - y = e^{-x}, ye^x = e^x + C$$

Order of differential equation is the number of arbitrary constants.

Both are true but R is not correct reason.

36. The equation of the given family of curves is

$$y = Cx^2 \qquad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. 'x', we get

$$\frac{dy}{dx} = 2Cx \qquad \text{...(ii)}$$

On eliminating c from Eqs. (i) and (ii), we get

$$y = \left(\frac{1}{2x}\frac{dy}{dx}\right)x^2 \Rightarrow 2y = x\frac{dy}{dx} \qquad \text{...(iii)}$$

This is the differential equation of the family of curves given in Eq. (i). The differential equation of the orthogonal trajectory in Eq. (i) is obtained by replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in Eq. (iii).

On replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in Eq. (iii), we get

$$2y = -x\frac{dx}{dy} \Rightarrow 2ydy = -xdx$$

On integrating, we get $y^2 = -\dfrac{x^2}{2} + C \Rightarrow x^2 + 2y^2 = 2C$

This is the required family of the orthogonal trajectories.

37. R is false, since $\dfrac{dy}{dx} = \dfrac{x+y^2}{y+x^2}$ cannot be made homogeneous by putting $y = tx$. But if we put $y^2 = t$ in the differential equation in assertion (A), then $2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$.

And differential equation becomes

$$t\cdot\frac{1}{2}dt + (x+t)\,dx = 0$$

or $\dfrac{dx}{dt} - \dfrac{-t}{2(x+t)}$ which is homogeneous.

38. Given, $\dfrac{dP}{dx} = (100 - 12\sqrt{x})$

$$\Rightarrow \quad dP = (100 - 12\sqrt{x})\,dx$$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})\,dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000$

$\Rightarrow \quad C = 2000$

Now, when $x = 25$, then is

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$
$$= 2500 - 8 \times 125 + 2000$$
$$= 4500 - 1000 = 3500$$

39. Given, $\quad p'(t) = \dfrac{dp(t)}{dt} = 0.5p(t) - 450$

$\Rightarrow \quad \dfrac{2dp(t)}{p(t) - 900} = dt$

$\Rightarrow \quad \int \dfrac{2dp(t)}{p(t) - 900} = \int dt$

$\Rightarrow \quad 2\log|p(t) - 900| = t + C$

To find the value of C, let's substitute $t = 0$

$\Rightarrow \quad 2\log|p(0) - 900| = 0 + C$

$\Rightarrow \quad C = 2\log|850 - 900|$

$\Rightarrow \quad C = 2\log 50$

Now,

$$2\log|p(t) - 900| = t + 2\log 50$$

Now, put $p(t) = 0$, we get

$$2\log|0 - 900| = t + 2\log 50$$

$\Rightarrow \quad t = 2\log\left|\dfrac{900}{50}\right| \Rightarrow t = 2\log 18$

40. Here, $\dfrac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$

$\Rightarrow \quad \int \dfrac{dy}{y + 3} = \int dx$

$\Rightarrow \quad \log|y + 3| = x + C,$

But $\quad y(0) = 2$

$\Rightarrow \quad \log|2 + 3| = 0 + C$

$\therefore \quad C = \log_e 5$

$\Rightarrow \quad \log|y + 3| = x + \log_e 5$

$\therefore$ when $\quad x = \log_e 2$

$\Rightarrow \quad \log|y + 3| = \log_e 2 + \log_e 5 = \log_e 10$

$\Rightarrow \quad y + 3 = 10 \Rightarrow y = 7$

41. Given, $\quad \dfrac{d\{V(t)\}}{dt} = -k(T - t)$

$\therefore \quad d\{V(t)\} = -k(T - t)dt \quad$ ...(i)

When $\quad t = 0, V(t) = I$

$\Rightarrow \quad \int_0^T d\{V(t)\} = \int_0^T -k(T - t)dt$

$\Rightarrow \quad V(T) - V(0) = k\left[\dfrac{(t - T)^2}{2}\right]_0^T$

$\Rightarrow \quad V(T) - I = \dfrac{k}{2}[(T - T)^2 - (0 - T)^2]$

$\therefore \quad V(T) = I - \dfrac{k}{2}T^2$

42. Here, $\dfrac{dx}{dy} + \dfrac{1}{y^2} \cdot x = \dfrac{1}{y^3}$ (linear differential equation in x)

$\because \quad \text{IF} = e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$

$\therefore$ Complete solution is,

$$x \cdot e^{-\frac{1}{y}} = \int \frac{1}{y^3} \cdot e^{-\frac{1}{y}} dy$$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \int \dfrac{1}{y} \cdot \dfrac{1}{y^2} \cdot e^{\frac{1}{y}} dy$

Put $\quad -\dfrac{1}{y} = t$

$\Rightarrow \quad \dfrac{1}{y^2}dy = dt \Rightarrow xe^{-\frac{1}{y}} = \int -t \cdot e^t dt$

$\Rightarrow \quad xe^{-\frac{1}{y}} = -\{t \cdot e^t - \int 1 \cdot e^t dt\} + C$

$\Rightarrow \quad xe^{-\frac{1}{y}} = -te^t + e^1 + C$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \dfrac{1}{y} \cdot e^{-\frac{1}{y}} + e^{-\frac{1}{y}} + C \quad [\because y(1) = 1]$

$\Rightarrow \quad e^{-1} = e^{-1} + e^{-1} + C$

$\Rightarrow \quad C = -\dfrac{1}{e}$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \dfrac{1}{y} \cdot e^{-\frac{1}{y}} + e^{-\frac{1}{y}} - \dfrac{1}{e}$

$\Rightarrow \quad x = \dfrac{1}{y} + 1 - \dfrac{1}{e} \cdot e^{\frac{1}{y}}$

34 Vector Algebra

JEE Main MILESTONE

- Scalar and Vector Quantities
- Representation of a Vector
- Types of Vector
- Addition of Vectors
- Multiplication of a Vector by a Scalar
- Component of a Vector
- Linear Combinations
- Orthogonal System of Unit Vectors
- Product of Two Vectors
- Product of Three Vectors
- Reciprocal System of Vectors
- Application of Vectors in Geometry

34.1 Scalar and Vector Quantities

Physical quantities are divided into two categories scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space, also does not follow the rules of vector algebra, are called scalar quantities or briefly scalars. Examples of scalars are mass, volume, density, work, temperature etc., vectors are those which have both magnitude and direction and follow the rules of vector algebra are called vector quantities. Displacement, velocity, acceleration, momentum, weight, force etc., are examples of vector quantities.

In Mathematics, Physics and Engineering, a Euclidean vector is a geometric object that has magnitude (or length) and direction and can be added to other vectors according to vector algebra. A Euclidean Vector is frequently represented by a line segment with a definite direction or graphically as an arrow, connecting an initial point A with a terminal point B and denoted by **AB**. *We frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc., and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.*

34.2 Representation of a Vector

A vector is generally represented by a direction line segment, say **AB**. A is called the initial point and B is called the terminal point. The positive number representing the measure of the length of the line segment denoting the vector is called the modulus or magnitude of the vector and the arrow head indicates its direction. The modulus of vector **AB** is denoted by $|\mathbf{AB}|$.

34.3 Types of Vector

(i) Zero Vector or Null Vectors

A vector of zero magnitude *i.e.*, which has the same initial and terminal points, is called a zero vector. It is denoted by **O**.

(ii) Unit Vector

A vector is unit magnitude in the direction of a vector **a** is called a unit vector along **a** and is denoted by $\hat{\mathbf{a}}$.

Symbolically, $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

(iii) Equal Vector

If magnitude and direction of two vectors are same, then those are called equal vectors.

(iv) Parallel Vector

Two vectors **a** and **b** are said to be parallel if either both have same line of action or support or there exists a scalar λ such that if $\lambda > 0$, then **a** and **b** have same direction or sense. If $\lambda < 0$, then **a** and **b** have opposite direction or sense.

(v) Like and Unlike Vector

Two parallel vectors are said to be like when they have same sense of direction *i.e.*, angle between them is zero. Otherwise vectors are said to be unlike vectors and angle between them is π.

Vectors **a** and **b** are like vectors and vectors **a** and **c**, **b** and **c** are unlike vectors.

(vi) Negative Vector

A vector having the same magnitude as that of a given vector and direction opposite to that given vector is called negative vector.

(vii) Coinitial and Coterminal Vectors

The vectors which have the same initial point are coinitial vectors. Similarly, the vectors which have the same terminal point are called coterminal vectors.

(viii) Free Vector

The vector whose initial point or tail is not fixed is called free or non-localised vector.

(ix) Collinear Vector

Two or more vectors are known as collinear vectors, if they are parallel to a given straight line. The magnitude of collinear vectors can be different.

(x) Coplanar Vector

Vectors are said to be coplanar, if they occur in same or common plane.

(xi) Reciprocal of a Vector

A vector having the same direction as that of a given vector **a** but magnitude equal to the reciprocal of the given vector is known as the reciprocal of **a** and is denoted by $\mathbf{a}^{-1}$.

(xii) Localized and Free Vectors

A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector.

Force acting on a rigid body is a localized vectors as its effect depends on the line of action of the force.

(xiii) Position Vector

Let O be a fixed origin, then the position vector of a point P is the vector **OP**, if **a** and **b** are position vectors of two points A and B, then $\mathbf{AB} = \mathbf{b} - \mathbf{a}$

= Position vector of **b** – Position vector of **a**

34.4 Addition of Vectors

The addition of two vectors **a** and **b** is denoted by $\mathbf{a} + \mathbf{b}$ and it is known as resultant of **a** and **b.**

There are three methods of addition of vectors.

(i) Triangle Law

If two vectors **a** and **b** lie along the two sides of a triangle in consecutive order (as shown in the adjoining figure), then third side represents the sum (resultant) $\mathbf{a} + \mathbf{b}$.

(ii) Parallelogram Law

If two vectors lie along two adjacent sides of a parallelogram (as shown in the adjoining figure), then diagonal of the parallelogram through the common vertex represents their sum.

(iii) Polygon Law

If $(n - 1)$ sides of a polygon represents vector $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$, in consecutive order, then nth side represents their sum (as shown in the adjoining figure).

Properties of Vector Addition

(i) Vector addition is commutative, *i.e.*,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

(ii) Vector addition is associative *i.e.*,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

(iii) $\mathbf{a} + \mathbf{O} = \mathbf{a} = \mathbf{O} + \mathbf{a}$, where $\mathbf{O}$ is additive identify.

(iv) $\mathbf{a} + (-\mathbf{a}) = \mathbf{O} = (-\mathbf{a}) + \mathbf{a}$ where $(-\mathbf{a})$ is additive inverse.

(v) $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$, equality holds when **a** and **b** are like vectors.

(vi) $|\mathbf{a} + \mathbf{b}| \ge ||\mathbf{a}| - |\mathbf{b}||$, equality holds when **a** and **b** are unlike vectors.

Note If **a** and **b** are two vectors, then the subtraction of **b** from **a** is defined as the vector sum **a** and $-$ **b**, *i.e.*, reverse the direction of **b** and add it to **a**

34.5 Multiplication of a Vector by a Scalar

If **a** is a vector and λ is a scalar, then $\lambda\,\mathbf{a}$ is a vector parallel to **a** whose modulus is $|\lambda|$ times that of **a**. This multiplication is called scalar multiplication.

Properties of Multiplication of a Vector by a Scalar

If λ and μ be two scalars, then

(i) $\lambda(\mu\,\mathbf{a}) = \lambda\mu\,\mathbf{a}$

(ii) $(\lambda + \mu)\,\mathbf{a} = \lambda\,\mathbf{a} + \mu\,\mathbf{a}$

(iii) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\,\mathbf{a} + \lambda\,\mathbf{b}$

Sample **Problem 1** *If* **a** *and* **b** *are the vectors determined by two adjacent sides of a regular hexagon, then vector* **EO** *is*

(*a*) $(\mathbf{a} + \mathbf{b})$ (*b*) $(\mathbf{a} - \mathbf{b})$

(*c*) $2\,\mathbf{a}$ (*d*) $2\,\mathbf{b}$

Interpret (*b*) *OABCDE* is a regular hexagon. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{AB} = \mathbf{b}$. Join *OB* and *OC*, we have

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{a} + \mathbf{b}$$

Since, *OC* is parallel to *AB* and double of *AB*.

$\therefore$ $\mathbf{OC} = 2\mathbf{AB} = 2\mathbf{b}$

Now, $\mathbf{BC} = \mathbf{OC} - \mathbf{OB} = 2\mathbf{b} - (\mathbf{a} + \mathbf{b}) = \mathbf{b} - \mathbf{a}$

$\mathbf{CD} = -\mathbf{OA} = -\mathbf{a}$

and $\mathbf{DE} = -\mathbf{AB} = -\mathbf{b}$

Also, $\mathbf{EO} = -\mathbf{BC} = -(\mathbf{b} - \mathbf{a}) = \mathbf{a} - \mathbf{b}$

34.6 Component of a Vector

The process of splitting a vector is called resolution of a vector. In simpler language it would mean, determining the effect of a vector in a particular direction. The parts of the vector obtained after splitting the vectors are known as the components of the vector.

Components of a Vector in Two Dimensional Space

The planar components of a vector lies in the plane of vector. Since, there are two perpendicular axes involved with a plane, a vector is resolved in two components which lie in the same plane as that of vector. Clearly, a vector is composed of components in only two directions

$$\mathbf{a} = a\cos\alpha\,\hat{\mathbf{i}} + a\cos\beta\,\hat{\mathbf{j}}$$

From the figure, depicting a planar coordinate, it is clear that angle β is compliment of angle *a*. If $\alpha = \theta$, then

$$\alpha = \theta \text{ and } \beta = 90^\circ - \theta$$

Planar Vector

The direction of a planar vector with respect to rectangular axes can be

Putting in the expression for the vector.

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$$

Now, magnitude of the vector **a** is $|\mathbf{a}| = \sqrt{\mathbf{a}x^2 + \mathbf{a}y^2}$

$$\mathbf{a} = a\cos\theta\,\hat{\mathbf{i}} + a\cos(90^\circ - \theta)\hat{\mathbf{j}}$$

$$\mathbf{a} = a\cos\theta\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}$$

From graphical representation, the tangent of the angle that vector makes with *x*-axis is

$$\tan\alpha = \tan\theta = \frac{a\sin\theta}{a\cos\theta} = \frac{a_y}{a_x}$$

Similarly, the tangent of the angle that vector makes with *y*-axis is

$$\tan\beta = \tan(90^\circ - \theta) = \cot\theta = \frac{a\cos\theta}{a\sin\theta} = \frac{a_x}{a_y}$$

Components of a Vector in Three Dimensional Space

A scalar component, also known as projection, of a vector **AB** in the positive direction of a straight line *C′C* is defined as

$$AC = |\mathbf{AB}|\cos\theta = AB\cos\theta$$

where θ is the angle that vector **AB** makes with the specified direction $C'C$ as shown in the figure.

It is clear that **scalar component** can either be **positive** or **negative** depending on the value of angle that the vector makes with the referred direction. The angle lies between the range given by $0 \le \theta \le 180°$. This interval means that we should consider the smaller angle between two vectors.

In accordance with the above definition, we resolve a given vector in three components in **three mutually perpendicular directions of rectangular coordinate system.** Note here that we measure angle with respect to parallel lines to the axis. By convention, we denote components by using the non-bold type face of the vector symbol with a suffix representing direction (x or y or z).

$$AB_x = AB\cos\alpha$$
$$AB_y = AB\cos\beta$$
$$AB_z = AB\cos\gamma,$$

where α, β and γ are the angles that vector AB makes with the positive directions of x, y and z directions respectively. As a vector can be resolved in a set of components, the reverse processing of components in a vector is also expected. A vector is composed from vector components in three directions along the axes of rectangular coordinate system by combining three component vectors. The vector component is the vector counterpart of the scalar component, which is obtained by multiplying the scalar component with the unit vector in axial direction. The vector components of **a** vector a are $a_x\hat{\mathbf{i}}$, $a_y\hat{\mathbf{j}}$ and $a_z\hat{\mathbf{k}}$.

A vector **a** is equal to the vector sum of component vectors in three mutually perpendicular directions of rectangular coordinate system.

$$\mathbf{a} = a_x\,\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors in the respective directions.

$$\mathbf{a} = a\cos\alpha\hat{\mathbf{i}} + a\cos\beta\,\hat{\mathbf{j}} + a\cos\gamma\,\hat{\mathbf{k}}$$

Magnitude of the sum of two vectors $(a_x\,\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})$ in x and y direction, which are perpendicular to each other, is

$$a' = \sqrt{(a_x^2 + a_y^2)} = (a_x^2 + a_y^2)^{1/2}$$

We observe here that resultant vector lies in the plane formed by the two component vectors being added. The resultant vector is, therefore perpendicular to third component vector. Thus, the magnitude of the sum of vector $(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})$ and third vector $a_z\hat{\mathbf{k}}$, which are perpendicular to each other, is

$$\Rightarrow \quad a = \sqrt{[[(a_x^2 + a_y^2)^{1/2}]^2 + a_z^2]}$$

$$\Rightarrow \quad a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$$

Note

- It two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal, their resolved parts will also equal *i.e.*, $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.
- The resolved parts of a resultant vector of addition of two vectors are equal to the sum of resolved parts of those vectors.

34.7 Linear Combinations

Given a finite set of vectors **a, b, c**, ... , then the vector $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$ is called a linear combination of **a, b, c**, ... for any $x, y, z \ldots \in R$. We have the following results

(i) If **a** and **b** are non-zero, non-collinear vectors, then $x\mathbf{a} + y\mathbf{b} = x'\mathbf{a} + y'\mathbf{b} \Rightarrow x = x'; y = y'$

(ii) **Fundamental Theorem** Let **a, b** be non-zero, non-collinear vectors. Then, any vector **r** coplanar with **a, b** can be expressed uniquely as a linear combination of **a**, **b** *i.e.*, There exist some unique $x, y \in R$ such that $x\mathbf{a} + y\mathbf{b} = \mathbf{r}$.

(iii) If **a**, **b** and c are non-zero, non-coplanar vectors, then $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c} \Rightarrow x = x', y = y', z = z'$

(iv) **Fundamental Theorem in Space** Let **a**, **b** and **c** be non-zero, non-coplanar vectors in space. Then, any vector **r**, can be uniquely expressed as a linear combination of **a**, **b** and **c** *i.e.*, There exist some unique $x, y, z \in R$ such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{r}$.

(v) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n non-zero vectors and $k_1, k_2, \ldots, k_n$ are n scalars and if the linear combination $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_n\mathbf{x}_n = 0 \Leftrightarrow k_1 = 0, k_2 = 0 \ldots k_n = 0$, then we say that vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are **linearly independent vectors**.

(vi) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are not **linearly independent**, then they are said to be linearly dependent vectors. *i.e.*, if $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_n\mathbf{x}_n = 0$, if there exists atleast one $K_r \neq 0$ $(r = 1, 2, \ldots, n)$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are said to be linearly dependant.

Linearly Dependent

Note If $K_r \neq 0$; $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + k_3\mathbf{x}_3 + \ldots + k_r\mathbf{x}_r + \ldots + k_n\mathbf{x}_n = 0$

$$-K_r\mathbf{x}_r = k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_{r-1}\mathbf{x}_{r-1} + \ldots + k_{r+1}\mathbf{x}_{r+1} + \ldots + k_n\mathbf{x}_n$$

$$-K_r\frac{1}{K_r}\mathbf{x}_r = K_1\frac{1}{K_r}\mathbf{x}_1 + K_2\frac{1}{K_r}\mathbf{x}_2 + \ldots + K_{r-1}\frac{1}{K_r}\mathbf{x}_{r-1} + \ldots + K_n\frac{1}{K_r}\mathbf{x}_n$$

$$\mathbf{x}_r = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_{r-1}\mathbf{x}_{r-1} + c_r\mathbf{x}_{r-1} + \ldots + c_n\mathbf{x}_n$$

i.e., $\mathbf{x}_r$ is expressed as a linear combination of vectors.

$$\mathbf{x}_1, \mathbf{x}_2 \ldots, \mathbf{x}_{r-1}, \mathbf{x}_{r+1} \ldots, \mathbf{x}_n$$

Hence $\mathbf{x}_r$ with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r-1}, \mathbf{x}_{r+1}, \ldots, \mathbf{x}_n$ forms a linearly dependent set of vectors.

Note

- If $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, then $\mathbf{a}$ is expessed as a linear Combination of vectors. $\hat{\mathbf{i}}, \hat{\mathbf{j}}\ \hat{\mathbf{k}}$. Also, $\mathbf{a}, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ form a lineary dependent set of vectors. In general, every set of four vectors is a linearly dependent system.
- $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are linearly independent set of vectors for $k_1\hat{\mathbf{i}} + k_2\hat{\mathbf{j}} + k_3\mathbf{k} = 0 \Rightarrow k_1 = 0 = k_1 = 0 = k_2 = k_3$

Collinearity of Points or Vectors Collinearity of Three Points

Collinearity of the three points is the most importnt topic of this chapter. The level of question is from moderate to typical some times questions ask in JEE Main examination and a small practice can help you in solving such problems very quickly as the level of questions is medium to average.

Three points with position vectors, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are collinear if and only if there exists three scalars x, y, z not all simultaneously such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$ together with $x + y + z = 0$

Three points representing three position vectors can be represent two vectors in the plane.

Let A, B and C are three points represented by three position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Now, these three position vectors can represent two vectors $\mathbf{AB}$ and $\mathbf{AC}$.

Now, the vectors **AB** and **AC** are collinear, if there exists a linear relation between the two, such that $\mathbf{AB} = \lambda\ \mathbf{AC}$.

Sample **Problem 2** *Let O be the point of intersection of diagonals of a parallelogram ABCD. The points M, N, K and P are the mid-points of OA, MB, NC and KD respectively, then N, O and P are*

(a) *collinear* (b) *non-collinear*
(c) *can't say* (d) *None of these*

Interpret (a) Now, $M \equiv \dfrac{\mathbf{a}}{2}$, $N \equiv \dfrac{\frac{\mathbf{a}}{2} + \mathbf{b}}{2} = \dfrac{\mathbf{a} + 2\mathbf{b}}{4}$

$$K \equiv \frac{\frac{\mathbf{a} + 2\mathbf{b}}{4} - \mathbf{a}}{2} = \frac{2\mathbf{b} - 3\mathbf{a}}{8}$$

$$P \equiv \frac{-\mathbf{b} + \frac{2\mathbf{b} - 3\mathbf{a}}{8}}{2} = \frac{-6\mathbf{b} - 3\mathbf{a}}{16}$$

$\Rightarrow$ $\mathbf{OP} = -\dfrac{3}{16}(2\mathbf{b} + \mathbf{a})$

Also, $\mathbf{ON} = \dfrac{1}{4}(\mathbf{a} + 2\mathbf{b}) = -\dfrac{3}{4}(\mathbf{OP})$

Hence, points N, O and P are collinear.

Collinearity of Four Points

Let A, B, C and D be four points represented by four position vectors $\mathbf{a}$, $\mathbf{b}$ $\mathbf{c}$ and $\mathbf{d}$, respectively.

Now, these four position vectors can represent three vectors **AB, AC** and **AD.**

The vectors **AB, AC** and **AD** are collinear, if

$\mathbf{AC} = m\ \mathbf{AB}$ and $\mathbf{AD} = n\ \mathbf{AB}$ and similar other cases for more than four vectors.

Hot Spot 1 Coplanarity of Points or Vectors

Coplanarity of points or vcetors is the most important of JEE Main examination point of view. Generally question are seen from this topic and small practice can help you in solving such problems vary quickly as the level of question is easy to average.

Coplanarity of Three Points

Three points A, B and C represented by position vectors **a**, **b** and **c** respectively represent two vectors AB and AC and from the figure, two vectors are always coplanar *i.e.*, two vectors always from their own plane.

Coplanarity of Four Points

The necessary and sufficient condition that four points with position vectors **a, b, c** and **d** should be coplanar is that there exist four scalars x, y, z, t not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0,\ x + y + z + t = 0$$

The prove that the four points A, B, C and D having position vectors as **a**, **b**, **c** and **d** are coplanar.

Step I Find the vectors AB, AC and AD having the reference point as A.

Step II Express one of these vectors as the linear combination of the other two

$$\text{AB} = \lambda\ \text{AC} + \mu\ \text{AD}$$

Step III Now, compare the coefficients on LHS and RHS in respective manner and thus find the respective value of λ and μ.

Step IV If real values of the scalars λ and μ exist, then the three vectors representing four points are coplanar otherwise not.

Note Three vectors $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ are coplanar, if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Sample Problem 3 *The vectors*

$$5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c},\ 7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c} \text{ and } 3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$$

are (where **a**, **b**, **c** *are three non-coplanar vectors.)*

(a) *collinear*
(b) *coplanar*
(c) *non-coplanar*
(d) *None of the above*

Interpret (*b*) Let $\mathbf{A} = 5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}$, $\mathbf{B} = 7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}$ and $\mathbf{C} = 3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$

A, **B** and **C** are coplanar.

$$\Rightarrow \quad x\mathbf{A} + y\mathbf{B} + z\mathbf{C} = 0$$

must have a real solution for x, y and z other than (0, 0, 0).

Now, $x(5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}) + y(7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}) + z(3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}) = \mathbf{0}$

$$\Rightarrow \quad (5x + 7y + 3z)\,\mathbf{a} + (6x - 8y + 20z)\mathbf{b} + (7x + 9y + 5z)\mathbf{c} = \mathbf{0}$$

$$5x + 7y + 3z = 0$$
$$6x - 8y + 20z = 0$$
$$7x + 9y + 5z = 0$$

(as **a**, **b** and **c** are non-coplanar vectors)

Now, $$D = \begin{vmatrix} 5 & 7 & 3 \\ 6 & -8 & 20 \\ 7 & 9 & 5 \end{vmatrix} = 0$$

So, the three linear simultaneous equation in x, y and z have a non-trivial solution.

Hence, **A**, **B** and **C** are coplanar vectors.

Check Point 1

1. If $\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ are such that $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c}$, then
   (a) $\mu, \frac{\lambda}{2}, \gamma$ are in AP (b) λ, μ, γ are in AP
   (c) λ, μ, γ are in HP (d) μ, λ, γ are in GP
2. If $\mathbf{DA} = \mathbf{a}$, $\mathbf{AB} = \mathbf{b}$ and $\mathbf{CB} = k\mathbf{a}$, where $k > 0$ and X, Y are the mid-points of DB and AC respectively such that $|\mathbf{a}| = 17$ and $|\mathbf{XY}| = 4$, then k is equal to
   (a) $\frac{8}{17}$ (b) $\frac{9}{17}$
   (c) $\frac{5}{17}$ (d) $\frac{4}{17}$
3. The position vectors of three points are $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$, $\mathbf{a} - 2\mathbf{b} + \lambda\mathbf{c}$ and $\mu\mathbf{a} - 5\mathbf{b}$, where **a**, **b** and **c** are non-coplanar vectors. The points are collinear, when
   (a) $\lambda = -2,\ \mu = \frac{9}{4}$ (b) $\lambda = -\frac{9}{4},\ \mu = 2$
   (c) $\lambda = \frac{9}{4},\ \mu = -2$ (d) None of these
4. A point O is the centre of circle circumscribed about a ΔABC. Then, $\mathbf{OA}\sin 2A + \mathbf{OB}\sin 2B + \mathbf{OC}\sin 2C$ is equal to
   (a) $(\mathbf{OA} + \mathbf{OB} + \mathbf{OC})\sin 2A$
   (b) $3\mathbf{OG}$, where G is centroid of ΔABC
   (c) **O**
   (d) None of the above

5. Let ABCD be a parallelogram whose diagonals intersect at P and let O be the origin, then $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$ equal to
   (a) $\mathbf{OP}$ (b) $2\mathbf{OP}$
   (c) $3\mathbf{OP}$ (d) $4\mathbf{OP}$
6. ABCD is a parallelogram, A_1 and B_1 are mid-points of side BC and CD respectively. If $\mathbf{AA}_1 + \mathbf{BB}_1 = \lambda\mathbf{AC}$, then λ is equal to
   (a) $\frac{1}{2}$ (b) 1
   (c) 3/2 (d) 2
7. If ABCD is a rhombus whose diagonals cut at the origin O, then $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$ equals
   (a) $\mathbf{AB} + \mathbf{AC}$
   (b) $\mathbf{O}$
   (c) $2(\mathbf{AB} + \mathbf{BC})$
   (d) $\mathbf{AC} + \mathbf{BD}$
8. If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of A and B respectively and C is a point on AB produced such that AC = 3AB, then position vector of C is
   (a) $3\mathbf{a} - 2\mathbf{b}$ (b) $3\mathbf{b} - 2\mathbf{a}$
   (c) $3\mathbf{b} + 2\mathbf{a}$ (d) $2\mathbf{a} - 3\mathbf{b}$

34.8 Orthogonal System of Unit Vectors

Let OX, OY and OZ be three mutually perpendicular straight lines. Given any point $P(x, y, z)$ in space, we can construct the rectangular parallelopiped of which OP is a diagonal and $OA = x, OB = y, OC = z$.

Here, A, B and C are $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ respectively and L, M and N are $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$ respectively.

Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ denote unit vectors along OX, OY and OZ respectively. We have, $\mathbf{r} = \mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ as $\mathbf{OA} = x\hat{\mathbf{i}}$, $\mathbf{OB} = y\mathbf{j}$ and $\mathbf{OC} = z\mathbf{k}$.

$$\mathbf{ON} = \mathbf{OA} + \mathbf{AN}$$
$$\mathbf{OP} = \mathbf{ON} + \mathbf{NP}$$

So, $$\mathbf{OP} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}\ (\mathbf{NP} = \mathbf{OC}, \mathbf{AN} = \mathbf{OB})$$
$$r = |\mathbf{r}| = |\mathbf{OP}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}}$$
$$= l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$$
$$\Rightarrow \quad \mathbf{r} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$$

34.9 Product of Two Vectors

There are two methods of products of two vectors.

(i) Scalar Product or Dot Product

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors inclined at an angle θ. Then, the scalar product of $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$ and defined as

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta,\ 0 \le \theta \le \pi$$

Since, scalar product of two vectors is a scalar quantity, so it is called scalar product.

Hot Spot 2

Properties of Scalar Product

Properties of scalar product is the most important topic for JEE Main examination point of views. Generally, questions ask in examination from based on properties of scalar product or dot product . The level of question is from moderate to typical and a small practice can help you in solving such problems very quickly as the level of question medium to average.

While solving problems based on properties of scalar product or dot product, always keep in mind the concept of scalar product

(i) The scalar product of two vectors is commutative *i.e.*, $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

(ii) If m and n be any two scalars and **a** and **b** be any two vectors, then $(m\,\mathbf{a})\cdot(n\,\mathbf{b}) = (n\mathbf{a})\cdot(m\,\mathbf{b})$

(iii) $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\pm\cdot\mathbf{c}$ (distributive law)

and $(\mathbf{b}\pm\mathbf{c})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a}\pm\dot{\mathbf{c}}\cdot\mathbf{a}$

(iv) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

(v) If two vectors **a** and **b** are perpendicular to each other, then $\mathbf{a}\cdot\mathbf{b} = 0$

(a) $\mathbf{a}\cdot\mathbf{b} < 0$, iff **a** and **b** are inclined at an obtuse angle.

(b) $\mathbf{a}\cdot\mathbf{b} > 0$, iff **a** and **b** are inclined at an acute angle.

(vi) If $\mathbf{a}\cdot\mathbf{b} = 0$, then either $\mathbf{a} = 0$, $\mathbf{b} = 0$ or $\mathbf{a}\perp\mathbf{b}$

(vii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$,

then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

(viii) $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$ and $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$

(ix) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then

$$\cos\theta = \left[\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right] = \left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right]$$

If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then both vectors are perpendicular to each other and if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$, then both vectors are parallel to each other.

(x) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

(xi) Projection of **b** along $\mathbf{a} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}$

and projection of **a** along $\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$ and perpendicular to $\mathbf{b} = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

If **b** represents a force, then projection of **b** along **a** represents the effective force in the direction of **a**.

Total work done $= \mathbf{F}\cdot\mathbf{d} = F\,d\cos\theta$ where **F** is the force and **d** is the displacement.

(xii) Maximum value of $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$

(xiii) Minimum value of $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$

(xiv) Any vector **a** can be written as,

$$\mathbf{a} = (\mathbf{a}\cdot\hat{\mathbf{i}})\,\hat{\mathbf{i}} + (\mathbf{a}\cdot\hat{\mathbf{j}})\,\hat{\mathbf{j}} + (\mathbf{a}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}$$

(xv) If **r** is a vector making angles α, β and γ with OX, OY and OZ respectively, then

$$\cos\alpha = \mathbf{r}\cdot\hat{\mathbf{i}},\ \cos\beta = \mathbf{r}\cdot\hat{\mathbf{j}},\ = \cos\gamma = \mathbf{r}\cdot\hat{\mathbf{k}}$$

$$\mathbf{r} = |\mathbf{r}|\cos\alpha\,\hat{\mathbf{i}} + |\mathbf{r}|\cos\beta\hat{\mathbf{j}} + |\mathbf{r}|\cos\gamma\hat{\mathbf{k}}$$

If **r** a unit vect or then,

$$\mathbf{r} = (\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\,\hat{\mathbf{j}} + (\cos\gamma)\,\hat{\mathbf{k}}$$

(xvi) If **a**, **b** and **c** are non-coplanar vectors, in space, any vector **r** in space can be written as $\hat{\mathbf{r}} = (\hat{\mathbf{r}}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}} + (\hat{\mathbf{r}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} + (\hat{\mathbf{r}}\cdot\hat{\mathbf{c}})\hat{\mathbf{c}}$, where $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors along **a**, **b** and **c**, respectively.

(xvii) If is a non-zero vector in space and **a**, **b** and **c** are three vectors,

$\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = \mathbf{r}\cdot\mathbf{c} = 0$

$\Rightarrow$ **a**, **b** and **c** are coplanar.

(xviii) If **r** is a non-zero coplanar to two given vectors **a** and **b**, then $\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = 0 \Rightarrow$ **a** and **b** are collinear.

Sample Problem 4 *The angle between the vectors* $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *and* $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *as* **[NCERT]**

(*a*) 0 (*b*) $\frac{\pi}{2}$

(*c*) $\cos^{-1}\left(\frac{5}{7}\right)$ (*d*) $\cos^{-1}\left(\frac{1}{7}\right)$

Interpret (*c*) The angle between the two vectors **a** and **b** is given by $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. Here, we are not given $\mathbf{a}\cdot\mathbf{b}$ (dot product) and magnitude (**a**) and (**b**), so first of all we have to find these required factors.

Let $\mathbf{a} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Magnitude of **a**, $|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

Magnitude of **b**, $|\mathbf{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

Now, $\mathbf{a}\cdot\mathbf{b} = (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$= 1\cdot 3 + (-2)\cdot(-2) + 3\cdot 1 = 3 + 4 + 3 = 10$

(dot product of two vectors is equal to the sum of the products of their corresponding components.)

Let θ be the required angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Sample Problem 5 *The projection of the vector $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ on the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is* [NCERT]

(*a*) 0 (*b*) $\frac{1}{\sqrt{2}}$

(*c*) $\frac{1}{2}$ (*d*) *None of these*

Interpret (*d*) Use formula for projection of $\mathbf{a}$ on $\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Let $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

Projection of $\mathbf{a}$ on $\mathbf{b}$ is given by,

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{1^2 + 1^2}} = \frac{1\times 1 + (-1)\times 1}{\sqrt{2}} = 0$$

Hence, the projection of vector $\mathbf{a}$ on $\mathbf{b}$ is 0.

Sample Problem 6 *If $\mathbf{a}$ is a non-zero vector of magnitude a and λ is a non-zero scalar, then $\lambda\,\mathbf{a}$ is unit vector if* [NCERT]

(*a*) $\lambda = 1$ (*b*) $\lambda = -1$ (*c*) $a = |\lambda|$ (*d*) $a = 1/|\lambda|$

Interpret (*d*) Vector $\lambda\,\mathbf{a}$ is a unit vector, if $|\lambda\,\mathbf{a}| = 1$

Now, $|\lambda\,\mathbf{a}| = 1 \Rightarrow |\lambda||\mathbf{a}| = 1 \Rightarrow |\lambda||\mathbf{a}| = 1 \Rightarrow |\mathbf{a}| = \frac{1}{|\lambda|}$ $[\because \lambda \neq 0]$

$\Rightarrow$ $a = \frac{1}{|\lambda|}$ (given $|\vec{\mathbf{a}}| = a$)

Sample Problem 7 *If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$, $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ is*

(*a*) 13 (*b*) 26

(*c*) −26 (*d*) −13

Interpret (*d*) We know that,

$$(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

$\Rightarrow$ $0 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$\Rightarrow$ $0 = (3)^2 + (1)^2 + (4)^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = -\frac{26}{2} = -13$

Sample Problem 8 *Two forces $\mathbf{f}_1 = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{f}_2 = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ acting on a particle at A move it to B. The work done, if the position vector of A and B are $-2\hat{\mathbf{i}} + 5\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, is*

(*a*) 20 *units* (*b*) 7 *units*

(*c*) 25 *units* (*d*) *None of these*

Interpret (*c*) Let $\mathbf{R}$ be the resultant of two forces $\mathbf{f}_1$ and $\mathbf{f}_2$ and $\mathbf{d}$ be the displacement.

Then, $\mathbf{R} = \mathbf{f}_1 + \mathbf{f}_2 = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$

$= 4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

and $\hat{\mathbf{i}} = (3\hat{\mathbf{j}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (-2\hat{\mathbf{i}} + 5\hat{\mathbf{k}}) = 5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$\therefore$ The total work done

$= \mathbf{R}\cdot\mathbf{d}$

$= (4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}})\cdot(5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

$= 20 - 7 + 12 = 25$ units

II Vector Product or Cross Product

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector and is given by

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{\mathbf{n}}$$

where, θ be the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a perpendicular unit vector to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ form a right handed system.

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Also, $\hat{\mathbf{n}} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$, where, $\hat{\mathbf{n}}$ indicates direction of $\mathbf{a}\times\mathbf{b}$.

Hot Spot 3

Properties of Vector Product

Properties of vector product is the most important topic of this chapter and also most important for JEE Main examination point of view. Generally, questions are seen from this topic. The level of question is from moderate to typical. But a small practice can help you in sloving such problems very quickly as the level of questions is difficult to average. Generally question ask in examination based on mixed properties of vector product . While solving problems based on properties of vector product, always keep in mind the concept of vector product.

(i) Vector product is not commutative, *i.e.*,

$$\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a}$$

But $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$

(ii) If m and n be two scalars and **a** and **b** be two vectors, then

$$(m\,\mathbf{a})\times(n\mathbf{b})=mn(\mathbf{a}\times\mathbf{b})=(n\mathbf{a})\times(m\mathbf{b})$$

(iii) $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=\mathbf{a}\times\mathbf{b}\pm\mathbf{a}\times\mathbf{c}$ (distributive law)

and $(\mathbf{b}\pm\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}\pm\mathbf{c}\times\mathbf{a}$

(iv) If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$, then

$$\mathbf{a}\times\mathbf{b}=[(a_2b_3-a_3b_2)\hat{\mathbf{i}}+(a_3b_1-a_1b_3)\hat{\mathbf{j}}+(a_1b_2-a_2b_1)\hat{\mathbf{k}}]$$

or $$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

(v) $\mathbf{a}\times\mathbf{b}=0$

If $\mathbf{a}=0$ or $\mathbf{b}=0$ or **a** and **b** are two collinear vectors.

or $\mathbf{a}\,||\,\mathbf{b}$, where **a** and **b** are non-null vectors.

(vi) $\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=0$

(vii) $\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}},\ \hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}$

and $\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\ \hat{\mathbf{k}}\times\hat{\mathbf{j}}=-\hat{\mathbf{i}},\ \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}}$

(viii) Area of parallelogram *OACB*

$$=OA\times BM=ab\sin\theta=|\,\mathbf{a}\times\mathbf{b}\,|$$

(ix) (a) Area of $\Delta ABC=\frac{1}{2}|\,\mathbf{AB}\times\mathbf{AC}\,|$

(b) If **a**, **b** and **c** are position vectors of *A*, *B* and *C* respectively, then area of

$$\Delta ABC=\frac{1}{2}|\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\,|$$

(x) **Moment of a Force about a Point** Let *F* be the magnitude of force acting at a point *A* of the rigid body along the line *AB* and $OM=p$ is the perpendicular distance of fixed point *O* from *AB*, then the moment of force **F** about *O* is

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}$$

= Force × Perpendicular distance of the point from the line of action of force

(xi) Area of quadrilateral *OABC* is $\frac{1}{2}|\,\mathbf{OC}\times\mathbf{BA}\,|$, where **OC** and **BA** are diagonals.

(xii) The unit vector perpendicular to the plane of *a* and *b* are $\pm\frac{\mathbf{a}\times\mathbf{b}}{|\,\mathbf{a}\times\mathbf{b}\,|}$ and **a** vector of magnitude λ perpendicular to the plane of (**a** and **b** or **b** and **a**) is $\frac{\lambda\,(\mathbf{a}\times\mathbf{b})}{|\,\mathbf{a}\times\mathbf{b}\,|}$

(xiii) **Rotating body** A rigid body is spinning with angular velocity ω about an axis through θ. The velocity of a point *P* in the body is $\mathbf{v}=\omega\times\mathbf{OP}=\omega\times\mathbf{r}$

(xiv) **Lagranges Identity** For any two vectors *a* and *b*;

$$(\mathbf{a}\times\mathbf{b})^2=|\,\mathbf{a}\,|^2\cdot|\,\mathbf{b}\,|^2-(\mathbf{a}-\mathbf{b})^2=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}.$$

Sample Problem 9 *If* $\mathbf{a}=\hat{\mathbf{i}}-7\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ *and* $\mathbf{b}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, *then* $|\,\mathbf{a}\times\mathbf{b}\,|$ *is equal to* **[NCERT]**

(*a*) $9\hat{\mathbf{i}}+19\hat{\mathbf{j}}+19\hat{\mathbf{k}}$ (*b*) $0\hat{\mathbf{i}}+19\hat{\mathbf{j}}+19\hat{\mathbf{k}}$

(*c*) $19\hat{\mathbf{i}}+9\hat{\mathbf{j}}+19\hat{\mathbf{k}}$ (*d*) *None of these*

Interpret (*b*) Suppose $\mathbf{a}=a_1\hat{\mathbf{i}}+b_1\hat{\mathbf{j}}+c_1\hat{\mathbf{k}}$ and $\mathbf{b}=a_2\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+c_2\hat{\mathbf{k}}$, then use $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$. We can find the value by expanding this determinant.

It is given that $\mathbf{a}=\hat{\mathbf{i}}-7\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\mathbf{b}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

$$\therefore\ \mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{j}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=(-14+14)\hat{\mathbf{i}}-(2-21)\hat{\mathbf{j}}+(-2+21)\hat{\mathbf{k}}=0\hat{\mathbf{j}}+19\hat{\mathbf{j}}+19\hat{\mathbf{k}}$$

Sample Problem 10 *A unit vector perpendiculat to each of the vectors* $\mathbf{a}+\mathbf{b}$ *and* $\mathbf{a}-\mathbf{b}$, *where* $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ *and* $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ *is* [NCERT]

(a) $\frac{1}{3}\hat{\mathbf{i}}-\frac{1}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{k}}$ or $-\frac{1}{3}\hat{\mathbf{i}}+\frac{1}{3}\hat{\mathbf{j}}+\frac{1}{3}\hat{\mathbf{k}}$

(b) $\frac{4}{3}\hat{\mathbf{i}}-\frac{4}{3}\hat{\mathbf{j}}-\frac{4}{3}\hat{\mathbf{k}}$ or $-\frac{4}{3}\hat{\mathbf{i}}+\frac{4}{3}\hat{\mathbf{j}}+\frac{4}{3}\hat{\mathbf{k}}$

(c) $\frac{2}{3}\hat{\mathbf{i}}-\frac{2}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{j}}$ *or* $-\frac{2}{3}\hat{\mathbf{i}}+\frac{2}{3}\hat{\mathbf{j}}+\frac{1}{3}\hat{\mathbf{k}}$

(d) *None of the above*

Interpret (c) Unit vector perpendicular to two vectors **A** and **B** is given by $\hat{\mathbf{n}}=\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$

Then, determine $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, then unit vector perpendicular to both $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-\mathbf{b})$ is given by $\pm\frac{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})}{|(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})|}$

Given that, $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

$$\therefore \quad \mathbf{a}+\mathbf{b}=(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})+(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}})$$
$$=4\hat{\mathbf{i}}+4\hat{\mathbf{j}}+0\hat{\mathbf{k}}$$

and $$\mathbf{a}-\mathbf{b}=(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}})=2\hat{\mathbf{i}}+4\hat{\mathbf{k}}$$

Now, $$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$$
$$=\hat{\mathbf{i}}(16-0)-\hat{\mathbf{j}}(16-0)+\hat{\mathbf{k}}(0-8)$$
$$=16\hat{\mathbf{i}}-16\hat{\mathbf{j}}-8\hat{\mathbf{k}}$$

$$\Rightarrow \quad |(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})|=\sqrt{(16)^2+(-16)^2+(-8)^2}$$
$$=\sqrt{256+256+64}=\sqrt{576}=24$$

$\therefore$ A unit vector, perpendicular to both $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-\mathbf{b})$ is

$$\pm\frac{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})}{|(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})|}=\pm\frac{16\hat{\mathbf{i}}-16\hat{\mathbf{j}}-8\hat{\mathbf{k}}}{24}$$
$$=\pm\frac{8(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-1\hat{\mathbf{k}})}{24}=\pm\frac{1}{3}(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})$$

$\therefore$ Required vector is either $\frac{2}{3}\hat{\mathbf{i}}-\frac{2}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{k}}$ or $-\frac{2}{3}\hat{\mathbf{i}}+\frac{2}{3}\hat{\mathbf{j}}+\frac{1}{3}\hat{\mathbf{k}}$

Sample Problem 11 *Area of a rectangle having vertices* $A\left(-\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$, $B\left(\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$, $C\left(\hat{\mathbf{i}}-\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$ *and* $D\left(-\hat{\mathbf{i}}-\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$ *is* [NCERT]

(a) $\frac{1}{2}$ (b) 1

(c) 2 (d) 4

Interpret (c) To determine the area of rectangle, use the relation that if **a** and **b** are the adjacent sides of a parallelogram, then area of parallelogram $=|\mathbf{a}\times\mathbf{b}|$

The position vectors of vertices A, B, C and D of rectangle $ABCD$ are given.

First , we compute vectors **AB** and **AD**.

Now, $$\mathbf{AB}=\text{PV of } B-\text{PV of } A=\left(\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)-\left(-\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$$
$$=[1-(-1)]\hat{\mathbf{i}}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{\mathbf{j}}+(4-4)\hat{\mathbf{k}}=2\hat{\mathbf{i}}$$

and $$\mathbf{AD}=\text{Position vector of } D-\text{Position vector of } A$$
$$=\left(-\hat{\mathbf{i}}-\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)-\left(-\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+4\hat{\mathbf{k}}\right)$$
$$=[-1-(-1)]\hat{\mathbf{i}}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{\mathbf{j}}+(4-4)\hat{\mathbf{k}}=-\hat{\mathbf{j}}$$

$$\therefore \quad \mathbf{AB}\times\mathbf{AD}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=-2\hat{\mathbf{k}}$$

Area of rectangle $ABCD=|\mathbf{AB}\times\mathbf{AD}|=\sqrt{(-2)^2}=2$ sq units

(now, it is known that the area of a parallelogram whose adjacent sides are **a** and **b** is $|\mathbf{a}\times\mathbf{b}|$)

Hence, the area of the rectangle is $|\mathbf{AB}|\times|\mathbf{AD}|=2$ sq units

Note If **a** and **b** are the adjacent sides of a parallelogram, then area $=|\mathbf{a}\times\mathbf{b}|$. If **a** and **b** are the diagonals of a parallelogram, then area $=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

Sample Problem 12 *Let the vectors* **a** *and* **b** *be such that* $|\mathbf{a}|=3$ *and* $|\mathbf{b}|=\frac{\sqrt{2}}{3}$, *then* $\mathbf{a}\times\mathbf{b}$ *is a unit vector, if the angle between* **a** *and* **b** *is* [NCERT]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Interpret (b) It is given that $|\mathbf{a}|=3$ and $|\mathbf{b}|=\frac{\sqrt{2}}{3}$

Let θ be the angle between **a** and **a**, then $|\mathbf{a}\times\mathbf{b}|=1$

$$\Rightarrow |\mathbf{a}||\mathbf{b}|\sin\theta=1\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1 \;\Rightarrow\; \sin\theta=\frac{1}{\sqrt{2}}\Rightarrow \theta=\frac{\pi}{4}$$

Hence, $\mathbf{a}\times\mathbf{b}$ is a unit vector, if the angle between **a** and **b** is $\frac{\pi}{4}$

Sample Problem 13 *The area of the triangle with vectices* $A(1, 1, 2)$, $B(2, 3, 5)$ *and* $C(1, 5, 5)$ *is* [NCERT]

(a) $\sqrt{61}$ *sq units* (b) $\frac{1}{2}\sqrt{31}$ *sq units*

(c) $\frac{1}{2}\sqrt{61}$ *sq units* (d) $2\sqrt{61}$ *sq units*

Interpret (c) To determine the area of triangle, use formula $\Delta=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$.

The vertices of ΔABC are given as $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$.

First, we find vectors **AB** and **AC**.

Now, $\mathbf{AB} = \text{PV of } B - \text{PV of } A = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$= (2-1)\hat{\mathbf{i}} + (3-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

and $\mathbf{AC} = \text{PV of } C - \text{PV of } A = (\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$= (1-1)\hat{\mathbf{i}} + (5-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\therefore \quad \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(6-12) - \hat{\mathbf{j}}(3-0) + \hat{\mathbf{k}}(4-0) = -6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Comparing with $\mathbf{X} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we get $x = -6, y = -3, z = 4$

$$\therefore \quad |\mathbf{AB} \times \mathbf{AC}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-6)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{36 + 9 + 16} = \sqrt{61}$$

Area of $\Delta ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2} \times \sqrt{61} = \frac{\sqrt{61}}{2}$ sq units

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ sq units.

Sample Problem 14 *The moment about the point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ of a force represented by $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ acting through the point $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is*

(a) $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (b) $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$

(c) $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ (d) $3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

Interpret (d) Here, $\mathbf{r} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$\Rightarrow \quad \mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

and $\mathbf{F} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Then, the required moment is given by

$$\mathbf{r} \times \mathbf{F} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$$

$\therefore$ Moment about given point $= 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

Check Point 2

1. If $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$ and $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then the angle between **a** and **b** is
   (a) $\frac{\pi}{2}$ (b) π
   (c) 0 (d) None of these
2. If θ be the angle between the unit vectors **a** and **b**, then $\cos\frac{\theta}{2}$ is equal to
   (a) $\frac{1}{2}|\mathbf{a} - \mathbf{b}|$ (b) $\frac{1}{2}|\mathbf{a} + \mathbf{b}|$
   (c) $\frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} + \mathbf{b}|}$ (d) $\frac{|\mathbf{a} + \mathbf{b}|}{|\mathbf{a} - \mathbf{b}|}$
3. If $\mathbf{a} \cdot \hat{\mathbf{i}} = \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then **a** is equal to
   (a) $\hat{\mathbf{i}}$ (b) $\hat{\mathbf{k}}$
   (c) $\hat{\mathbf{j}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
4. If the scalar projection of the vector x, $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ on the vector $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ is $\frac{1}{\sqrt{30}}$, then the value of x is equal to
   (a) $-\frac{5}{2}$ (b) 6
   (c) -6 (d) 3
5. $\mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{a}$, $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$, $\mathbf{a} \neq \lambda\mathbf{b}$, **a** is not perpendicular to **b**, then **r** is equal to
   (a) $\mathbf{a} - \mathbf{b}$ (b) $\mathbf{a} + \mathbf{b}$
   (c) $\mathbf{a} \times \mathbf{b} + \mathbf{a}$ (d) $\mathbf{a} \times \hat{\mathbf{b}} + \mathbf{b}$
6. The area of the parallelogram whose diagonals are $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ is
   (a) $4\sqrt{3}$ (b) $5\sqrt{3}$
   (c) 8 (d) 4
7. A constant force $\mathbf{F} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ is applied at the point $(1, -1, 2)$. Then, vector moment of **F** about the point $(2-1, 3)$ is
   (a) $-2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
   (b) $-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
   (c) $2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
   (d) $2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}\hat{\mathbf{i}}$
8. If the vectors $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ are perpendicular, then λ is equal to
   (a) 4 (b) -5
   (c) 2 (d) 1
9. If θ is the angle between the vectors **a** and **b** and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a} \cdot \mathbf{b}|$, then θ is
   (a) 0 (b) 180°
   (c) 135° (d) 45°
10. The vectors $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\lambda\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ are collinear, if λ is equal to
    (a) 3 (b) 4
    (c) 5 (d) 6

34.10 Product of Three Vectors

There are two methods of products of three vectors.

Scalar Triple Product (Box Product)

The scalar triple product of three vectors **a**, **b** and **c** is defined as $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\phi$ where, θ is the angle between **a** and **b** and ϕ is the angle between $\mathbf{a}\times\mathbf{b}$ and **c**. It is also defined as $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$.

Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$,

$\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$

Then,
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{j}}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple Product

(i) The value of scalar triple product does not depend upon the position of dot and cross. *i.e.,*
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

(ii) If **a, b** and **c** are cyclically permuted. The value of scalar product remains same.

The change of cyclic order of vectors in scalar triple product changes the sign of the scalar but not the magnitude. *i.e.,*
$$[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}]$$
and $$[\mathbf{abc}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}]$$

(iii) The scalar triple product of three vectors is zero, if any two of them are equal.

(iv) The scalar triple product of three vectors is zero, if two of them are parallel or collinear.

(v) If three vectors **a, b** and **c** are coplanar, then $[\mathbf{abc}] = 0$

(vi) $[\mathbf{abc}+\mathbf{d}] = [\mathbf{abc}] + [\mathbf{abd}]$

(vii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by **a, b** and **c** *ie,* $V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(viii) If **a, b, c** are non-coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > 0$ for right handed system and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] < 0$ for left handed system.

(ix) $[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}] = 1$

(x) $[k\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = k[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, where k is any scalar.

(xi) Volume of tetrahedron with 0 as origin and the position vectors of A, B and C being **a, b** and **c** respectively, is given by
$$V = \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

(xii) Four points with position vectors **a, b, c** and **d** will be coplanar, if
$$[\mathbf{d}\,\mathbf{b}\,\mathbf{c}] + [\mathbf{d}\,\mathbf{c}\,\mathbf{a}] + [\mathbf{d}\,\mathbf{a}\,\mathbf{b}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$$

(xiii) $[\mathbf{a}\,\mathbf{a}\,\mathbf{b}] = 0$

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$; $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$, then $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general, if $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}$; $\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$ and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\mathbf{l}\,\mathbf{m}\,\mathbf{n}]$;

where **l**, **m** and **n** are non-coplanar vectors.

(xiv) The position vector of the centroid of a tetrahedron if the PV's of its angular vertices are **a, b, c** and **d** are given by $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$.

Note that this is also the point of concurrence of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Remember that $[\mathbf{a}-\mathbf{b}, \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}] = 0$ and $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

Sample Problem 15 *The value of* $\hat{\mathbf{i}}\cdot(\hat{\mathbf{j}}\times\hat{\mathbf{k}}) + \hat{\mathbf{j}}\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{i}}) + \hat{\mathbf{k}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{i}})$ *is* [NCERT]

(*a*) *zero* (*b*) –1 (*c*) 1 (*d*) 3

Interpret (*d*) We have, $\hat{\mathbf{i}}\cdot(\hat{\mathbf{j}}\times\hat{\mathbf{k}}) + \hat{\mathbf{j}}\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{i}}) + \hat{\mathbf{k}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{i}})$

$= \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} + \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} + \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}$ $[\because \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}, \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}, \hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}]$

$= 1 + 1 + 1 = 3$ $[\because \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1]$

Vector Triple Product

The vector triple product of three vactors **a**, **b** and **c** is defined as $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. If atleast one **a**, **b** and **c** is a zero vector or **b** and **c** are collinear vectors or **a** is perpendicular to both **b** and only **c**, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{0}$. In all other cases $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector **a**.

Thus, we can take $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \lambda\mathbf{b} + \mu\mathbf{c}$, for some scalar λ and μ. Since, $\mathbf{a}\perp\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{a}\cdot[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})] = 0$

$\Rightarrow \qquad \lambda(\mathbf{a}\cdot\mathbf{b}) + \mu(\mathbf{ac}) = 0$

$\lambda(\mathbf{a}\cdot\mathbf{c})\,\alpha\mu = -(\mathbf{a}\cdot\mathbf{b})$ for same scalar α.

Hence, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfying the conditions given in the beginning. In particular, if we take, $\mathbf{a} = \mathbf{b} = \mathbf{i}$, $\mathbf{c} = \mathbf{j}$, then $\alpha = 1$.

Hence, $\qquad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

and $\qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

but $\qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

Sample Problem 16 *For any vector* $\mathbf{a}$ $\mathbf{i}\times(\mathbf{a}\times\mathbf{i}) + \mathbf{j}\times(\mathbf{a}\times\mathbf{j}) + \mathbf{k}\times(\mathbf{a}\times\mathbf{k})$ *is equal to*

(*a*) $\mathbf{a}$ (*b*) $2\mathbf{a}$

(*c*) $\frac{1}{2}\mathbf{a}$ (*d*) $4\mathbf{a}$

Interpret (*b*) $[\mathbf{i}\times(\mathbf{a}\times\hat{\mathbf{i}})] + [\mathbf{j}\times(\mathbf{a}\times\mathbf{j})] + [\mathbf{k}\times(\mathbf{a}\times\mathbf{k})]$

$$= [(\hat{\mathbf{i}}\cdot\hat{\mathbf{i}})\mathbf{a} - (\hat{\mathbf{i}}\cdot\mathbf{a})\hat{\mathbf{i}}][(\hat{\mathbf{j}}\cdot\hat{\mathbf{j}})\mathbf{a} - (\hat{\mathbf{j}}\cdot\hat{\mathbf{a}})\hat{\mathbf{j}}] + [(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})\mathbf{a} - (\hat{\mathbf{k}}\cdot\mathbf{a})\hat{\mathbf{k}}]$$

$$= \mathbf{a} - (\hat{\mathbf{i}}\cdot\mathbf{a})\hat{\mathbf{i}} + \mathbf{a} - (\hat{\mathbf{j}}\cdot\mathbf{a})\hat{\mathbf{j}} + \mathbf{a} - (\hat{\mathbf{k}}\cdot\mathbf{a})\hat{\mathbf{k}}$$

$$[\because \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1]$$

$$= 3\mathbf{a} - [\hat{\mathbf{i}}\cdot\mathbf{a})\hat{\mathbf{i}} + (\hat{\mathbf{j}}\cdot\mathbf{a})\hat{\mathbf{j}} + (\hat{\mathbf{k}}\cdot\mathbf{a})\hat{\mathbf{k}}]$$

Let $\quad \mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$. Then,

$$\hat{\mathbf{i}}\mathbf{a} = \hat{\mathbf{i}}(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) = a_2\,\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} + a_2(\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}) + a_3(\hat{\mathbf{i}}\cdot\hat{\mathbf{k}})$$

$$= a_1(1) + a_2(0) = a_1$$

Similarly, $\qquad \hat{\mathbf{j}}\cdot\mathbf{a} = a_2, \hat{\mathbf{k}}\cdot\mathbf{a} = a_3$

$\therefore \qquad \text{LHS} = 3\mathbf{a} - (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) = 3\mathbf{a} - \mathbf{a} = 2\mathbf{a} = \text{RHS}$

34.11 Reciprocal System of Vectors

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be a system of three non-coplanar vectors. Then, $= \mathbf{b}\cdot\mathbf{a}'$ the system of vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ which satisfies $\quad \mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1 \quad$ and $\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{c}' = \mathbf{c}\cdot\mathbf{a}' = \mathbf{c}\cdot\mathbf{b}' = 0$ is called the reciprocal system to the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. In terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are given by $\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}, \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}, \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}$.

Properties of Reciprocal System of Vectors

(i) Scalar product of any vector of one system with the vector of other system is zero *i.e.*,

$$\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{a}' = \mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = 0$$

(ii) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$

(iii) $\hat{\mathbf{i}}' = \hat{\mathbf{i}}, \hat{\mathbf{j}}' = \hat{\mathbf{j}}, \hat{\mathbf{k}}' = \hat{\mathbf{k}}$

(iv) Let $\mathbf{a}'\,\mathbf{b}'\,\mathbf{c}'$ is a reciprocal system $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{r}$ is any vector, then

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a}')\mathbf{a} + (\mathbf{r}\cdot\mathbf{b}')\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\mathbf{c}$$

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a})\mathbf{a}' + (\mathbf{r}\cdot\mathbf{b})\,\mathbf{b}' + (\mathbf{r}\cdot\mathbf{c})\mathbf{c}'$$

(v) If the system $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar and so are the reciprocal system $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$.

(vi) $(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}' + (\mathbf{b}+\mathbf{c})\cdot\mathbf{b}' + (\mathbf{c}+\mathbf{a})\cdot\mathbf{c}' = 3$

34.12 Application of Vectors in Geometry

(i) 'The points A, B and C are collinear' means

(a) Area of ΔABC is zero

(b) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are collinear vectors

(c) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are parallel

(d) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \mathbf{0}$

(e) There exist α, β and γ not all zero such that $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$ and $\alpha+\beta+\gamma = 0$

Otherwise A, B and C are not collinear.

(ii) 'A, B, C and D are coplanar' means.

(a) Volume of tetrahedron $ABCD$ is zero.

(b) $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{a}$ and $\mathbf{d}-\mathbf{a}$ are coplanar.

(c) $[\mathbf{b}-\mathbf{a}, \mathbf{c}-\mathbf{a}, \mathbf{d}-\mathbf{a}] = 0$

(d) There exist α, β, γ and δ not all zero such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d} = 0$ and $\alpha+\beta+\gamma+\delta = 0$

Otherwise A, B, C and D are not coplanar.

(iii) If $\mathbf{a}$ and $\mathbf{b}$ are the position vector of A and B and $\mathbf{r}$ be the position vector of the point P which divides to join of A and B in the ratio $m : n$, then

$$\mathbf{r} = \frac{m\mathbf{b} \pm n\mathbf{a}}{m \pm n}$$

'+' sign takes for internal ratio and

'–' sign takes for external ratio.

(iv) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the PV of three vertices of ΔABC and $\mathbf{r}$ be the PV of the centroid of ΔABC, then

$$\mathbf{r} = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$

(v) **Equation of straight line in vector form**

(a) Vector equation of the straight line passing through origin and parallel to $\mathbf{b}$ is given by $\mathbf{r} = t\mathbf{b}$, where t is scalar.

(b) Vector equation of the straight line passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ is given by $\mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}$ or $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is scalar.

(c) Vector equation of the straight line passing through $\mathbf{a}$ and $\mathbf{b}$ is given by $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = 0$ or $\mathbf{r} = \mathbf{a} + t\,(\mathbf{b}-\mathbf{a})$, where t is scalar.

(d) Equation of straight line passing through the point $\mathbf{a}$ perpendicular to two non-parallel vectors $\mathbf{c}$ and $\mathbf{d}$ is $(\mathbf{r}-\mathbf{a})\times(\mathbf{c}\times\mathbf{d}) = 0$

Sample Problem 17 *The vector equation of a line passing through a point with position vector* $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and paralled to the line joining the points with position vector* $-\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *is*

(a) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (b) $2\mathbf{i} - 2\hat{\mathbf{k}} + \hat{\mathbf{k}}$

(c) $(2 + 2t)\hat{\mathbf{i}} + \hat{\mathbf{k}}(1 + t)$ (d) *None of these*

Interpret (d) Let $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

Then, the equation of the line will be

$$\begin{aligned}\mathbf{r} &= \mathbf{a} + t(\mathbf{c} - \mathbf{b})\\ &= (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + t(2\mathbf{j} + 2\mathbf{k} + i - 4j - \hat{\mathbf{k}})\\ &= (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + t(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})\\ &= (2 + 2t)\hat{\mathbf{i}} + \hat{\mathbf{j}}(-1 - 2t) + \hat{\mathbf{k}}(1 + t)\end{aligned}$$

(vi) **Equation of a plane in vector form**

(a) Vector equation of the plane through origin and parallel to $\mathbf{b}$ and $\mathbf{c}$ is given by $\mathbf{r} = s\mathbf{b} + t\mathbf{c}$, where s and t are scalars.

(b) Vector equation of the plane passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ is given by $[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$, where s and t are scalars.

(c) Vector equation of the plane passing through $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ or

$$\mathbf{r} = (1 - s - t)\,\mathbf{a} + s\mathbf{b} + t\mathbf{c},$$

where s and t are scalars.

(d) Equation of plane which passes through the point $\mathbf{a}$ and perpendicular to $\mathbf{n}$, is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

(vii) **Equation of a plane containing the line of intersection of two planes**

Let two planes be $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = q_2$, then equation $(\mathbf{r} \cdot n\mathbf{a}_1 - q_1) + \lambda(\mathbf{r} \cdot \mathbf{n}_2 - q_2) = 0$, where λ be any scalar quantity, is the equation of plane passing through the intersection line of planes.

(viii) **Equation of a line of intersection of two planes**

Let $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ be two equation of planes, then the equation of a line of intersection of two planes, is $\mathbf{r} = \mathbf{a} + t(\mathbf{n}_1 \times \mathbf{n}_2)$, where t be any scalar.

(ix) **Solving of vector equation**

Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions).

Generally, to solve vector equations, we express the unknown as the linear combination of three non-coplanar vectors as

$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$ as $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are non-coplanar and find x, y, z using given conditions. Sometimes, we can directly solve the given conditions if would be more clear from some examples.

Sample Problem 18 *Solution of the vector equation* $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{r} \cdot \mathbf{c} = 0$ *provided that* $\mathbf{c}$ *is not perpendicular to* $\mathbf{b}$, *is*

(a) $\mathbf{r} = \mathbf{a} - \left(\dfrac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{c}}\right)\mathbf{b}$ (b) $\mathbf{r} = \mathbf{b} - \dfrac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{b} \cdot \mathbf{r})}\mathbf{a}$

(c) $\mathbf{r} = \mathbf{b} - \left(\dfrac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{c}}\right)\mathbf{a}$ (d) *None of these*

Interpret (a) We are given; $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow$ $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$

Hence, $(\mathbf{r} - \mathbf{a})$ and $\mathbf{b}$ are parallel

$\Rightarrow$ $\mathbf{r} - \mathbf{a} = t\,\mathbf{b}$ ...(i)

and we know $\mathbf{r} \cdot \mathbf{c} = 0$,

$\therefore$ Taking dot product of Eq. (i) by $\mathbf{c}$ we get

$$\mathbf{r} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c} = t(\mathbf{b} \cdot \mathbf{c})$$

$$0 - \mathbf{a} \cdot \mathbf{c} = t(\mathbf{b} \cdot \mathbf{c})$$

$\Rightarrow$ $t = -\left(\dfrac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{c}}\right)$ ...(ii)

$\therefore$ From Eqs. (i) and (ii) solution of $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{a} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{c}}\right)\mathbf{b}$$

Tetrahedron

A tetrahedron is a three dimensional figure formed by four triangles.

In figure, $OABC \rightarrow$ tetrahedron

$\Delta ABC \rightarrow$ base

$OAB, OBC, OCA \rightarrow$ faces

OA, OB, OC, AB, BC and $CA \rightarrow$ edges

OA, BC, OB, CA, OC and $AB \rightarrow$ pair of opposite edges.

Properties of tetrahedron

(i) A tetrahedron in which all edges are equal is called a regular tetrahedron.

(ii) If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.

(iii) The sum of the squares of two opposite edges is the same for each pair of opposite edges.

(iv) Any two opposite edges in a regular tetrahedron are perpendicular.

(v) Volume of a tetrahedron $ABCD$ is $\frac{1}{6}\left|[\mathbf{a} - \mathbf{d}, \mathbf{b} - \mathbf{d}, \mathbf{c} - \mathbf{d}]\right|$ where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are position vectors.

(vi) volume of a tetrahedren whose three coterminous edges are in the right handed system are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is given by $\frac{1}{6}[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]$

(vii) Centroid of tetrahedron is $\dfrac{[\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}]}{4}$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are position vectors.

WORKED OUT
Examples

Example 1 *If* $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$, *then a vector* $\mathbf{c}$ *such that* $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ *and* $\mathbf{a} \cdot \mathbf{c} = 3$ *is* **[NCERT Exemplar]**

(*a*) $\frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}$ (*b*) $\frac{2}{3}\hat{\mathbf{i}} + \frac{5}{3}\hat{\mathbf{j}} + \frac{5}{3}\hat{\mathbf{k}}$

(*c*) $\frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$ (*d*) *None of these*

Solution (*a*) Given, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 0\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Let $\mathbf{c} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$

Now, $$\mathbf{a} \times \mathbf{c} = \mathbf{b} \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 0\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\Rightarrow \quad (z - y)\hat{\mathbf{i}} - \hat{\mathbf{j}}(z - x) + \hat{\mathbf{k}}(y - x) = 0\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\Rightarrow \quad z - y = 0 \Rightarrow y = z \quad \text{...(i)}$$

$$-z + x = 1 \Rightarrow x = 1 + z \quad \text{...(ii)}$$

$$y - x = -1 \quad \text{...(iii)}$$

Also, $$\mathbf{a} \cdot \mathbf{c} = 3 \Rightarrow (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 3$$

$$\Rightarrow \quad x + y + z = 3 \quad \text{...(iv)}$$

On putting the values of x and y from Eqs. (i) and (ii) in Eq. (iv), we get

$$(1 + z) + z + z = 3$$

$$\Rightarrow \quad 3z = 2 \Rightarrow z = \frac{2}{3}$$

On putting value of z in Eqs. (i) and (ii), we get $y = \frac{2}{3}$ and $x = \frac{5}{3}$

These values of x and y also satisfy Eq. (iii),

$$x = \frac{5}{3}, y = \frac{2}{3}, z = \frac{2}{3}$$

$\therefore \mathbf{c} = \frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}$, which is the required vector.

Example 2 *Let* $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ *and a unit vector* $\mathbf{c}$ *be coplanar. If* $\mathbf{c}$ *is perpendicular to* $\mathbf{a}$, *then* $\mathbf{c}$ *is equal to* **[NCERT Exemplar]**

(*a*) $(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (*b*) $\pm\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$

(*c*) $\pm\frac{1}{\sqrt{2}}(\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (*d*) *None of these*

Solution (*b*) $$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\,\mathbf{b}$$

$$= 3(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 6(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$$= -9\hat{\mathbf{i}} + 9\hat{\mathbf{k}}$$

$\therefore$ Required unit vector $= \dfrac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})|}$

$$= \pm\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Example 3 *A girl walks 4 km towards West, then she walks 3 km in a direction 30° East of North and stops. Then, the girl's displacement from her initial point of departures is* **[NCERT]**

(*a*) $-\frac{5}{2}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$ (*b*) $\frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{i}}$

(*c*) $-\frac{1}{2}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$ (*d*) *None of these*

Solution (*a*) Let O and B be the initial and final positions of the girls respectively.

Then, the girl's position can be shown as in the figure. Now, we have $\mathbf{OA} = 4\hat{\mathbf{i}}$

$\mathbf{AB} = \hat{\mathbf{i}}\,|\mathbf{AB}|\cos 60° + \hat{\mathbf{j}}\,|\mathbf{AB}|\sin 60°$

($AB\cos 60°$ is component of $\mathbf{AB}$, along X-axis) and $\mathbf{AB}\sin 60°$ is component of $\mathbf{AB}$ = along y-axis.)

$$= \hat{\mathbf{i}}\,3 \times \frac{1}{2} + \hat{\mathbf{j}}\,3 \times \frac{\sqrt{3}}{2} = \frac{3}{2}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$$

By the triangle law of vector addition, we have

$$\mathbf{OB} = \mathbf{AO} + \mathbf{AB} = (-4\hat{\mathbf{i}}) + \left(\frac{3}{2}\hat{\mathbf{j}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}\right)$$

$$= \left(-4 + \frac{3}{2}\right)\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$$

$$= \left(\frac{-8 + 3}{2}\right)\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}} = \frac{-5}{2}\hat{\mathbf{k}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$$

Hence, the girl's displacement from her initial point of departure is $\frac{-5}{2}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$.

Example 4 *If* **A**, **B** *and* **C** *are non-coplanar vectors and* $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$, *then* λ *is equal to* [**NCERT Exemplar**]

(*a*) $\dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]}$ (*b*) $\dfrac{[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]}{[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]}$

(*c*) $\dfrac{[\mathbf{b}\ \mathbf{d}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ (*d*) *None of these*

Solution (*b*) $\therefore \quad \mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$

$$\therefore \quad \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = \lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + 0 + 0$$
$$= \lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$\Rightarrow \quad \lambda = \frac{[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]}{[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]}$$

Example 5 *If* **a**, **b**, **c** *be three vectors such that* $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, *then* $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$ *and its deduction is*

(*a*) *sine rule* (*b*) *cosine rule*

(*c*) *a and b both* (*d*) *None of these*

Solution (*a*) Let **BC**, **CA** and **AB** represent the vectors, a, b, and c respectively.

Then, we have

$$a + b + c = 0,$$
$$\Rightarrow \quad c = -(a + b)$$
$$\Rightarrow \quad b\times c = b\times(-a - b)$$
$$= -b\times a = a\times b$$

Similarly, $\quad c\times a = a\times b$

Hence, $\quad b\times c = c\times a = a\times b$

$$\Rightarrow \quad bc\sin(\pi - A) = ca\sin(\pi - B) = ab\sin(\pi - C)$$
$$\Rightarrow \quad bc\sin A = ca\sin B = ab\sin C$$
$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 6 *The acute angle that the vector* $2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *makes with the plane contained by the two vector as* $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, *is given by*

(*a*) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (*b*) $\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$

(*c*) $\tan^{-1}\sqrt{(2)}$ (*d*) $\cot^{-1}(\sqrt{2})$

Solution (*d*) Let $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then normal vector to the plane is

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 5\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

$\therefore$ Unit vector perpendicular to the plane of **a** and **b** is

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$$

If θ is the required angle between the plane and $\mathbf{c}\,(= 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$, then $\left(\dfrac{\pi}{2} - \theta\right)$ is the angle between normal vector and **c**.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{3}\cdot\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$$
$$= \frac{1}{\sqrt{3}}$$
$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$\therefore \quad \theta = \cot^{-1}(\sqrt{2})$$

Example 7 *If* $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ *and* $\mathbf{c} = \hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ *are linearly dependent vectors and* $|c| = \sqrt{3}$ *then the value of* α *and* β *are respectively.*

(*a*) $\pm 1, 1$ (*b*) $\pm 2, 1$

(*c*) $0, \pm 1$ (*d*) *None of these*

Solution (*a*) $\because$ **a**, **b** and **c** are linearly dependent vectors.

$$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$
$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$
$$\Rightarrow \quad 1(3\beta - 4\alpha) - 1(4\beta - 4) + 1(4\alpha - 3) = 0$$
$$\Rightarrow \quad -\beta + 1 = 0 \Rightarrow \beta = 1$$

Now, $\quad |\mathbf{c}| = \sqrt{3}$

$$\Rightarrow \quad \sqrt{1 + \alpha^2 + \beta^2} = \sqrt{3}$$
$$\Rightarrow \quad 1 + 1 + \alpha^2 = 3$$
$$\Rightarrow \quad \alpha^2 = 1 \Rightarrow \alpha = \pm 1$$

Example 8 *Forces of magnitudes* 5 *and* 3 *units acting in the direction* $6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *and* $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ *respectively act on a particle which is displaced from the point* (2, 2, – 1) *to* (4, 3, 1). *The work done by the force is*

(*a*) 148 *units* (*b*) $\dfrac{148}{7}$ *units*

(*c*) $\dfrac{78}{7}$ *units* (*d*) *None of these*

Solution (*b*) Let **F** be the resultant force and **d** be the displacement vector. Then,

$$\mathbf{F} = 5\,\frac{(6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})}{\sqrt{36 + 4 + 9}} + 3\,\frac{(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{\sqrt{9 + 4 + 36}} = \frac{1}{7}[39\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 33\hat{\mathbf{j}}]$$

and $\quad \mathbf{d} = (4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\therefore$ Total work done $= \mathbf{F}\cdot\mathbf{d}$

$$= \frac{1}{7}[(39\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 33\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})]$$
$$= \frac{1}{7}[78 + 4 + 66] = \frac{148}{7}\text{ units}$$

Example 9

$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ *is equal to*

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

Solution (*a*) Let $\mathbf{b}\times\mathbf{c}=\mathbf{u}$, $\mathbf{c}\times\mathbf{a}=\mathbf{v}$ and $\mathbf{c}\times\mathbf{d}=\mathbf{w}$

$$\begin{aligned}\therefore\ \mathbf{u}\cdot(\mathbf{a}\times\mathbf{d})+\mathbf{v}\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot\mathbf{w}&\\ =(\mathbf{u}\times\mathbf{a})\cdot\mathbf{d}+(\mathbf{v}\times\mathbf{b})\cdot\mathbf{d}+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{w})&\\ =[(\mathbf{b}\times\mathbf{c})\times\mathbf{a}]\cdot\mathbf{d}+[(\mathbf{c}\times\mathbf{a})\times\mathbf{b}]\cdot\mathbf{d}+[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\cdot\mathbf{a}&\\ =[(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}]\cdot\mathbf{d}+[(\mathbf{c}\cdot\mathbf{b})\ \mathbf{a}&\\ -(\mathbf{a}\cdot\mathbf{b})\mathbf{c}]\cdot\mathbf{d}+[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}]\cdot\mathbf{a}&\\ [(\mathbf{a}\cdot\mathbf{b})\ (\mathbf{c}\cdot\mathbf{d})]-[(\mathbf{c}\cdot\mathbf{a})\ (\mathbf{b}\cdot\mathbf{d})]+[(\mathbf{c}\cdot\mathbf{b})\ (\mathbf{a}\cdot\mathbf{d})]-[(\mathbf{a}\cdot\mathbf{b})\ (\mathbf{c}\cdot\mathbf{d})]&\\ +[(\mathbf{a}\cdot\mathbf{c})\ (\mathbf{b}\cdot\mathbf{d})]-[(\mathbf{a}\cdot\mathbf{d})\ (\mathbf{b}\cdot\mathbf{c})]&\\ =0&\end{aligned}$$

Example 10 *The vectors* $\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}-4\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ *and* $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *are* [NCERT Exemplar]

(*a*) *linearly dependent* (*b*) *linearly independent*
(*c*) *can't say* (*d*) *None of these*

Solution (*b*) Let x, y, z be three scalars such that

$$x(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}})+y(2\hat{\mathbf{i}}-4\hat{\mathbf{j}}-4\hat{\mathbf{k}})+z(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$$

$$\Rightarrow\ (x+2y+3z)\ \hat{\mathbf{i}}+(-3x-4y+2z)\hat{\mathbf{j}}+(2x-4y-z)\hat{\mathbf{k}}=0$$

$$\Rightarrow\quad x+2y+3z=0,\ -3x-4y+2z=0$$

and $\quad 2x-4y-z=0$

This is a homogeneous system of equations. The determinant of coefficient matrix is $\begin{vmatrix}1&2&3\\-3&-4&2\\2&-4&-1\end{vmatrix}\neq 0$

So, the system of equations has only the trivial solution given by $x=0=y=z$

Hence, the given vectors are linearly independent.

Example 11 *If* $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{d}$ *and* $\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$, *then* $\mathbf{a}-\mathbf{d}$ ($\mathbf{a}\neq\mathbf{d}$ *and* $\mathbf{b}\neq\mathbf{c}$) *is,*

(*a*) *perpendicular to* $\mathbf{b}-\mathbf{c}$ (*b*) *parallel to* $\mathbf{b}-\mathbf{c}$
(*c*) *parallel to* $\mathbf{c}+\mathbf{b}$ (*d*) *None of these*

Solution (*b*) The show $(\mathbf{a}-\mathbf{d})$ parallel to $(\mathbf{b}-\mathbf{c})$, we must show their cross product is zero.

i.e., $\quad(\mathbf{a}-\mathbf{d})\times(\mathbf{b}-\mathbf{c})=0$

$\Rightarrow\quad \mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{d}\times\mathbf{b}+\mathbf{d}\times\mathbf{c}=0$

Using $\quad\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{d}$

and $\quad\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$

we get $\quad\mathbf{c}\times\mathbf{d}-\mathbf{b}\times\mathbf{d}-\mathbf{d}\times\mathbf{b}+\mathbf{d}\times\mathbf{c}=0$

$\Rightarrow\quad \mathbf{d}\times\mathbf{c}+\mathbf{d}\times\mathbf{b}-\mathbf{d}\times\mathbf{b}+\mathbf{d}\times\mathbf{c}=\mathbf{0}$

(as $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$)

$\Rightarrow\quad 0=0$

$\therefore\quad(\mathbf{a}-\mathbf{d})\times(\mathbf{b}-\mathbf{c})=0$

which shows $(\mathbf{a}-\mathbf{d})$ is parallel to $(\mathbf{b}-\mathbf{c})$, when, $\mathbf{a}\neq\mathbf{d}$ and $\mathbf{b}\neq\mathbf{c}$

Example 12 *The moment of the couple consisting of the force* $\mathbf{F}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *acting through the point* $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $-\mathbf{F}$ *acting through the point* $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *is*

(*a*) $\sqrt{5}$ (*b*) $5\sqrt{5}$
(*c*) $\sqrt{51}$ (*d*) *None of these*

Solution (*b*) Let A and b be the points having position vectors $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ respectively. Then

$$\mathbf{r}=\mathbf{BA}=(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})-(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}})=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

$$\therefore\ \text{Moment of couple}=|\mathbf{r}\times\mathbf{F}|=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-1&2&2\\3&2&-1\end{vmatrix}$$

$$=|-6\hat{\mathbf{i}}+5\hat{\mathbf{j}}-8\hat{\mathbf{k}}|$$

$$=\sqrt{36+25+64}=5\sqrt{5}$$

Example 13 *Let* G_1, G_2 *and* G_3 *be the centroids of the triangular faces* OBC, OCA *and* OAB *of a tetrahedron* $OABC$. *If* V_1 *denote the volume of the tetrahedron* $OABC$ *and* V_2 *that of the parallelopiped with* OG_1, OG_2 *and* OG_3 *as three concurrent edges, then*

(*a*) $4V_1=9V_2$ (*b*) $9V_1=4V_2$ (*c*) $3V_1=2V_2$ (*d*) $3V_2=2V_1$

Solution (*a*) Taking O as the origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. Then, the position vectors G_1, G_2 and G_3 are $\dfrac{\mathbf{a}+\mathbf{c}}{3}$, $\dfrac{\mathbf{c}+\mathbf{a}}{3}$ and $\dfrac{\mathbf{a}+\mathbf{b}}{3}$, respectively.

$\therefore\quad V_1=\dfrac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

and $\quad V_2=[\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3]$

Now, $\quad V_2=[\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3]$

$\Rightarrow\quad V_2=\left[\dfrac{\mathbf{b}+\mathbf{c}}{3}\ \dfrac{\mathbf{c}+\mathbf{a}}{3}\ \dfrac{\mathbf{a}+\mathbf{b}}{3}\right]$

$\Rightarrow\quad V_2=\dfrac{1}{27}[\mathbf{b}+\mathbf{c}+\mathbf{c}+\mathbf{a}+\mathbf{a}+\mathbf{b}]$

$\Rightarrow\quad V_2=\dfrac{2}{27}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\Rightarrow V_2=\dfrac{2}{27}\times 6\,V_1$

$\Rightarrow\quad 9V_2=4V_1$

Example 14 $\mathbf{a}$ *and* $\mathbf{c}$ *are unit collinear vectors and* $|\mathbf{b}|=6$, *then* $\mathbf{b}-3\mathbf{c}=\lambda\,\mathbf{a}$, *if* λ *is*

(*a*) $-9, 3$ (*b*) $9, 3$
(*c*) $3, -3$ (*d*) *None of these*

Solution (*a*) We have, $\quad\mathbf{b}-3\mathbf{c}=\lambda\mathbf{a}$

Taking scalar product with $\mathbf{c}$, we have

$$(\mathbf{b}-3\mathbf{c})\cdot\mathbf{c}=\lambda\,(\mathbf{a}\cdot\mathbf{c})$$

$$\Rightarrow\quad \mathbf{b}\cdot\mathbf{c}-3(\mathbf{c}\cdot\mathbf{c})=\lambda\,(\mathbf{a}\cdot\mathbf{a})$$

$$\Rightarrow\quad \mathbf{b}\cdot\mathbf{c}-3=\lambda$$

$$\Rightarrow\quad \mathbf{b}\cdot\mathbf{c}=3+\lambda \qquad \ldots(\text{i})$$

($\because |\mathbf{a}|=|\mathbf{c}|=1$ and $\mathbf{a}$ and $\mathbf{c}$ are collinear vectors)

Again, $\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}| = |\lambda\,\mathbf{a}|$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}|^2 = \lambda^2 |\mathbf{a}|^2$

$\Rightarrow \quad |\mathbf{b}|^2 + 9|\mathbf{c}|^2 - 6(\mathbf{b}\cdot\mathbf{c}) = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad 36 + 9 - 6(3 + \lambda) = \lambda^2$ [From Eq. (i)]

$\Rightarrow \quad 27 - 6\lambda = \lambda^2$

$\Rightarrow \quad \lambda^2 + 6\lambda - 27 = 0$

$\therefore \quad \lambda = -9, 3$

Example 15 *If* **a**, **b** *and* **c** *are three mutually perpendicular vectors, then the projection of the vector* $l\dfrac{\mathbf{a}}{|\mathbf{a}|} + m\dfrac{\mathbf{b}}{|\mathbf{b}|} + n\dfrac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$ *along the angle bisector of the vector* **a** *and* **b** *is*

(*a*) $\dfrac{l^2 + m^2}{\sqrt{l^2 + m^2 - n^2}}$ (*b*) $\sqrt{l^2 + m^2 + n^2}$

(*c*) $\dfrac{\sqrt{l^2 + m^2}}{\sqrt{l^2 - m^2 + n^2}}$ (*d*) $\dfrac{l + m}{\sqrt{2}}$

Solution (*d*) A vector parallel to the bisector of the angle between the vectors **a** and **b** is $\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{a} + \mathbf{b}$

$\therefore$ Unit vector along the bisector $= \dfrac{\mathbf{a} + \mathbf{b}}{|\mathbf{a} + \mathbf{b}|} = \dfrac{1}{\sqrt{2}}(\mathbf{a} + \mathbf{b})$

$$\left(\begin{array}{l}\because |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} \\ \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 1 + 1 + 0 = 2\end{array}\right)$$

$\therefore$ Required projection

$$= \left\{l\cdot\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{a}|} + n\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}\right\}\cdot\frac{1}{\sqrt{2}}(\mathbf{a} + \mathbf{b})$$

$$= \frac{1}{\sqrt{2}}(l + m)$$

$[\because |\mathbf{a}| = |\mathbf{b}| = 1$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0]$

Example 16 *Let* $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ *and* $\mathbf{w} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. *If* $\hat{\mathbf{n}}$ *is a unit vector such that* $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ *and* $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ *, then* $|\mathbf{w}\cdot\mathbf{n}|$ *is equal to*

(*a*) 3 (*b*) 0

(*c*) 1 (*d*) 2

Solution (*a*) We have, $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ and $\mathbf{v}\cdot\hat{\mathbf{n}} = 0$

$\Rightarrow \quad \hat{\mathbf{n}} \perp \mathbf{u}$ and $\hat{\mathbf{n}} \perp \mathbf{v}$

$\Rightarrow \quad \hat{\mathbf{n}} = \pm\dfrac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$

Now, $\mathbf{u}\times\mathbf{v} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})\times(\hat{\mathbf{j}} - \hat{\mathbf{k}}) = -2\hat{\mathbf{k}}$

$\therefore \quad \hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$

Hence, $|\mathbf{w}\cdot\hat{\mathbf{n}}| = |(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(\pm\hat{\mathbf{k}})| = 3$

Example 17 *Let* **a**, **b** *and* **c** *be non-zero vectors such that no two are collinear and* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \dfrac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$. *If* θ *is the acute angle between the vectors* **b** *and* **c**, *then* $\sin\theta$ *is equal to*

(*a*) $\dfrac{2\sqrt{2}}{3}$ (*b*) $\dfrac{\sqrt{2}}{3}$ (*c*) $\dfrac{2}{3}$ (*d*) $\dfrac{1}{3}$

Solution (*a*) We have, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \dfrac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = \dfrac{1}{3}|\mathbf{b}||\mathbf{b}|\,\mathbf{a}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - \{(\mathbf{b}\cdot\mathbf{c}) + \dfrac{1}{3}|\mathbf{b}||\mathbf{b}|)\}\,\mathbf{a} = 0$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c}) = 0$ and $\mathbf{b}\cdot\mathbf{c} + \dfrac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$

(**a** and **b** are non-collinear)

$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\cos\theta + \dfrac{1}{3}|\mathbf{b}||\mathbf{c}| = 0 \Rightarrow \cos\theta = -\dfrac{1}{3}$

$\therefore \quad \sin\theta = \sqrt{\dfrac{8}{9}} = \dfrac{2\sqrt{2}}{3}$

Example 18 *Resolved part of vector* **a** *along the vector* **b** *is* $\mathbf{a}_1$ *and that perpendicular to* **b** *is* $\mathbf{a}_2$, *then* $\mathbf{a}_1\times\mathbf{a}_2$ *is equal to*

(*a*) $\dfrac{(\mathbf{a}\times\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$ (*b*) $\dfrac{(\mathbf{a}\times\mathbf{b})\,\mathbf{a}}{|\mathbf{a}|^2}$

(*c*) $\dfrac{(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}|^2}$ (*d*) $\dfrac{(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}\times\mathbf{a}|}$

Solution (*c*) $\mathbf{a}_1 = (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b} = \dfrac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$

$\Rightarrow \quad \mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - \dfrac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$

Thus, $\mathbf{a}_1\times\mathbf{a}_2 = \dfrac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}\times\left(\mathbf{a} - \dfrac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}\right) = \dfrac{(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}|^2}$

Example 19 *A unit vector* **d** *is equally inclined at an angle* α *with the vectors* $\mathbf{a} = \cos\theta\cdot\hat{\mathbf{i}} + \sin\theta\cdot\hat{\mathbf{j}}$, $b = -\sin\theta\cdot\hat{\mathbf{i}} + \cos\theta\cdot\hat{\mathbf{j}}$ *and* $\mathbf{c} = \mathbf{k}$. *Then,* α *is equal to*

(*a*) $\cos^{-1}\dfrac{1}{\sqrt{2}}$ (*b*) $\cos^{-1}\dfrac{1}{\sqrt{3}}$

(*c*) $\cos^{-1}\dfrac{1}{3}$ (*d*) $\dfrac{\pi}{2}$

Solution (*b*) Let $\mathbf{d}\cdot\mathbf{a} = \mathbf{d}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \cos\alpha \Rightarrow \mathbf{d}\cdot(\mathbf{a} - \hat{\mathbf{k}}) = 0$

$$\mathbf{d} \text{ is parallel to } (\mathbf{a} - \hat{\mathbf{k}})\times(\mathbf{b} - \hat{\mathbf{k}}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos 0 & \sin\theta & -1 \\ -\sin\theta & \cos\theta & -1\end{vmatrix}$$

$$\therefore \quad \mathbf{d} = \frac{(\cos\theta - \sin\theta)\,\hat{\mathbf{i}} + (\cos\theta + \sin\theta) + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$$

$$\cos\alpha = \mathbf{d}\cdot\hat{\mathbf{k}} = \frac{1}{\sqrt{3}}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Product of Two Vectors

1. If $\mathbf{a}$ and $\mathbf{b}$ are two collinear vectors, then which of the following are incorrect? [NCERT]

(a) $\mathbf{b} = \lambda\mathbf{a}$, for some scalar λ

(b) $\hat{\mathbf{a}} = \pm\hat{\mathbf{b}}$

(c) The respective components of $\mathbf{a}$ and $\mathbf{b}$ are proportional.

(d) Both the vectors $\mathbf{a}$ and $\mathbf{b}$ have same direction but different magnitudes.

2. The angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, having $\mathbf{a}\cdot\mathbf{b} = \sqrt{6}$ is [NCERT]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

3. The projection of the vector $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ on the vector $7\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}}$ is [NCERT]

(a) $\frac{60}{\sqrt{122}}$ (b) $\frac{30}{\sqrt{144}}$

(c) $\frac{60}{\sqrt{144}}$ (d) $\frac{60}{\sqrt{111}}$

4. The magnitude of two vectors $\mathbf{a}$ and $\mathbf{b}$ having the same magnitude and such that the angle between them is 60° and their scalar product is 1/2 are [NCERT]

(a) $|\mathbf{a}| = \frac{1}{2}$, $|\mathbf{b}| = 1$ (b) $|\mathbf{a}| = |\mathbf{b}| = 1$

(c) $|\mathbf{a}| = 1$, $|\mathbf{b}| = \frac{1}{2}$ (d) None of these

5. If $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 8$ and $|\mathbf{a}| = 8|\mathbf{b}|$, then the value of $|\mathbf{a}|$ and $|\mathbf{b}|$ are [NCERT]

(a) $\frac{16}{3}\sqrt{\frac{2}{7}}, \frac{2}{3}\sqrt{\frac{2}{7}}$ (b) $\frac{4}{3}\sqrt{\frac{2}{7}}, \frac{2}{3}\sqrt{\frac{3}{7}}$

(c) $\frac{12}{5}\sqrt{\frac{2}{7}}, \frac{4}{3}\sqrt{\frac{2}{7}}$ (d) None of these

6. If for a unit vector $\mathbf{a}$, $(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}+\mathbf{a}) = 12$, then $|\mathbf{x}|$ is equal to [NCERT]

(a) 4 (b) 2

(c) $\sqrt{13}$ (d) $\sqrt{11}$

7. If $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ such that $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, then the value of λ is [NCERT]

(a) 2 (b) 4

(c) 6 (d) 8

8. For any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ and $|\mathbf{a}|\mathbf{b} - |\mathbf{b}|\mathbf{a}$ are [NCERT]

(a) parallel (b) perpendicular

(c) non-parallel (d) None of these

9. If $\mathbf{a}\cdot\mathbf{a} = 0$ and $\mathbf{a}\cdot\mathbf{b} = 0$, then what can be conclude about the vector $\mathbf{b}$? [NCERT]

(a) Any vector (b) Zero-vector

(c) Unit vector (d) None of these

10. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ is [NCERT]

(a) 0 (b) $-\frac{1}{2}$

(c) $-\frac{3}{2}$ (d) 2

11. If the vertices A, B, C of a ΔABC have position vectors $(1, 2, 3), (-1, 0, 0), (0, 1, 2)$ respectively, then $\angle ABC$ ($\angle ABC$ is the angle between the vectors BA and BC), is equal to [NCERT]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

12. The points $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$ are [NCERT]

(a) collinear (b) coplanar

(c) non-collinear (d) None of these

13. Given that $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = 0$. What can you conclude about the vectors **a** and **b**? [NCERT]

(a) **a** and **b** can be perpendicular and parallel simultaneously
(b) $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$
(c) $|\mathbf{a}| = 0$ and $|\mathbf{b}| = 0$
(d) None of the above

14. A unit vector in XY-plane, making an angle of 30° in anti-clockwise direction with the positive direction of X-axis is [NCERT]

(a) $\frac{1}{2}\hat{\mathbf{i}} + \frac{3}{2}\hat{\mathbf{i}}$ (b) $4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
(c) $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (d) $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$

15. If $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then which of the following statements is correct? [NCERT]

(a) $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$ (b) $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$
(c) $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$ (d) None of these

16. The value of x for which $x(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ is a unit vector [NCERT]

(a) $\pm\frac{1}{\sqrt{3}}$ (b) $\pm\frac{1}{3}$
(c) $\pm\frac{1}{2}$ (d) $\pm\frac{1}{\sqrt{2}}$

17. A vector of magnitude 5 units and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is [NCERT]

(a) $\pm\frac{3}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$ (b) $\pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{1}{2}\hat{\mathbf{j}}$
(c) $\pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$ (d) None of these

18. Find the position vector of point R which divides the line joining two points $P(2a + b)$ and $Q(a - 3b)$ externally in the ratio 1 : 2. Also, show that P is the middle point of the line segment RQ. [NCERT]

(a) $2\mathbf{a} + \mathbf{b}$ (b) $5\mathbf{a} + 3\mathbf{b}$
(c) $3\mathbf{a} + 5\mathbf{b}$ (d) None of these

19. The two adjacent sides of a parallelogram are $2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$. Then, the unit vector parallel to its diagonal. Also, its area are [NCERT]

(a) $\frac{3}{7}\hat{\mathbf{i}} - \frac{6}{7}\hat{\mathbf{j}} + \frac{2}{7}\hat{\mathbf{k}}$ and $11\sqrt{5}$ sq units
(b) $\frac{2}{7}\hat{\mathbf{i}} - \frac{6}{7}\hat{\mathbf{j}} + \frac{3}{7}\hat{\mathbf{k}}$ and $11\sqrt{3}$ sq units
(c) $\frac{6}{7}\hat{\mathbf{i}} - \frac{2}{7}\hat{\mathbf{j}} + \frac{3}{7}\hat{\mathbf{k}}$ and $11\sqrt{7}$ sq units
(d) None of the above

20. Let $\mathbf{a} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. Then, a vector **d** which is perpendicular to both **a** and **b** and $\mathbf{c} \cdot \mathbf{d} = 15$, is [NCERT]

(a) $\frac{5}{3}(\hat{\mathbf{i}} - 32\hat{\mathbf{j}} - 14\hat{\mathbf{k}})$ (b) $\frac{1}{3}(\hat{\mathbf{i}} - 32\hat{\mathbf{j}} - 14\hat{\mathbf{k}})$
(c) $\frac{5}{3}(32\hat{\mathbf{i}} - \hat{\mathbf{j}} - 14\hat{\mathbf{k}})$ (d) None of these

21. The scalar product of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ with a unit vector along the sum of vectors $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is equal to one. The value of λ is [NCERT]

(a) 1 (b) 2 (c) 3 (d) 4

22. If $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$, $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$, then which of the following option is correct? [NCERT]

(a) $\Leftrightarrow \mathbf{a} \parallel \mathbf{b}$ (b) $\Leftrightarrow \mathbf{a} \perp \mathbf{b}$
(c) $\Leftrightarrow$ **a** and **b** are collinear (d) None of these

23. If θ is the angle between two vectors **a** and **b**, then $\mathbf{a} \cdot \mathbf{b} \geq 0$ only when [NCERT]

(a) $0 < \theta < \frac{\pi}{2}$ (b) $0 \leq \theta \leq \frac{\pi}{2}$
(c) $0 \leq \theta < \pi$ (d) $0 \leq \theta \leq \pi$

24. Let **a** and **b** be two unit vectors and θ is the angle between them. Then, $\mathbf{a} + \mathbf{b}$ is a unit vector, if [NCERT]

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{\pi}{3}$
(c) $\theta = \frac{\pi}{2}$ (d) $\theta = \frac{2\pi}{3}$

25. If θ is the angle between any two vectors **a** and **b**, then $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ when θ is equal to [NCERT]

(a) zero (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) π

26. If the points $(-1, -1, 2)$, $(2, m, 5)$ and $(3, 11, 6)$ are collinear, then the value of m is [NCERT Exemplar]

(a) 2 (b) 4 (c) 6 (d) 8

27. A vector **r** of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z-axes, respectively is

(a) $\mathbf{r} = \pm 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (b) $\mathbf{r} = 3\hat{\mathbf{i}} \neq 3\hat{\mathbf{j}}$
(c) $\mathbf{r} = \pm 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (d) None of these

28. If $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, then the value of λ such that **a** is perpendicular to $\lambda\mathbf{b} + \mathbf{c}$ is [NCERT Exemplar]

(a) -1 (b) -2
(c) 1 (d) 2

29. If $|\mathbf{a}| = 8$, $|\mathbf{b}| = 3$ and $|\mathbf{a} \times \mathbf{b}| = 12$, then value of $\mathbf{a} \cdot \mathbf{b}$ is [NCERT Exemplar]

(a) $6\sqrt{3}$ (b) $8\sqrt{3}$
(c) $12\sqrt{3}$ (d) None of these

30. The two vectors $\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is [NCERT Exemplar]

(a) $\frac{\sqrt{34}}{2}$ (b) $\frac{\sqrt{48}}{2}$
(c) $\sqrt{18}$ (d) None of these

31. The projection of vector $\mathbf{a}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ along $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ is [NCERT Exemplar]

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) 2 (d) $\sqrt{6}$

32. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then what is the angle between $\mathbf{a}$ and $\mathbf{b}$ for $\sqrt{3}\,\mathbf{a}-\mathbf{b}$ to be a unit vector ?

(a) 30° (b) 45°
(c) 60° (d) 90°

33. The unit vector perpendicular to the vectors $\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ forming a right handed system is [NCERT Exemplar]

(a) $\hat{k}$ (b) $-\hat{k}$
(c) $\frac{\hat{i}-\hat{j}}{\sqrt{2}}$ (d) $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$

34. If $|\mathbf{a}|=3$ and $-1\le k\le 2$, then $|k\mathbf{a}|$ lies in the interval [NCERT Exemplar]

(a) [0, 6] (b) [−3, 6]
(c) [3, 6] (d) [1, 2]

35. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B, respectively, then the position vector of a point C in BA produced such that $BC = 1.5\,BA$, is [NCERT Exemplar]

(a) $3\mathbf{a}-\mathbf{b}$ (b) $\mathbf{a}-3\mathbf{b}$
(c) $0.5(\mathbf{a}-3\mathbf{b})$ (d) $0.5(3\mathbf{a}-\mathbf{b})$

36. If $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then which of the following is correct ? [NCERT Exemplar]

(a) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$
(b) $\mathbf{a}\times\mathbf{b}\ne\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$
(c) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}\ne\mathbf{c}\times\mathbf{a}$
(d) $\mathbf{a}\times\mathbf{b}\ne\mathbf{b}\times\mathbf{c}\ne\mathbf{c}\times\mathbf{a}$

37. The sine of the angle between the vectors $\mathbf{a}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, is [NCERT Exemplar]

(a) $\sqrt{\frac{2}{7}}$ (b) $\frac{2}{\sqrt{7}}$
(c) $\frac{\sqrt{2}}{7}$ (d) None of these

38. If A, B, C and D are the points with position vectors $\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}-3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, respectively, Then, the projection of $\mathbf{AB}$ along $\mathbf{CD}$ is [NCERT Exemplar]

(a) $\frac{1}{\sqrt{21}}$ (b) $\sqrt{21}$
(c) $\sqrt{\frac{3}{7}}$ (d) $\frac{2}{\sqrt{21}}$

39. Using vectors, the area of the ΔABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$ is [NCERT Exemplar]

(a) $\sqrt{\frac{137}{2}}$ (b) $\sqrt{137}$
(c) $\frac{1}{2}\sqrt{137}$ (d) $\frac{1}{2}\sqrt{278}$

40. Find the value of λ such that the vectors $\mathbf{a}=2\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ are orthogonal [NCERT Exemplar]

(a) 0 (b) 1 (c) $\frac{3}{2}$ (d) $-\frac{5}{2}$

41. The value of λ for which the vectors $3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ are parallel is [NCERT Exemplar]

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) $\frac{5}{2}$ (d) $\frac{2}{5}$

42. For any vector $\mathbf{a}$, the value of $(\mathbf{a}\times\hat{\mathbf{i}})^2+(\mathbf{a}\times\hat{\mathbf{j}})^2+(\mathbf{a}\times\hat{\mathbf{k}})^2$ is equal to [NCERT Exemplar]

(a) a^2 (b) $3a^2$ (c) $4a^2$ (d) $2a^2$

43. If $|\mathbf{a}|=10$, $|\mathbf{b}|=2$ and $\mathbf{a}\cdot\mathbf{b}=12$, then value of $|\mathbf{a}\times\mathbf{b}|$ is [NCERT Exemplar]

(a) 5 (b) 10
(c) 14 (d) 16

44. The vectors $\lambda\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ are coplanar, if [NCERT Exemplar]

(a) $\lambda=-2$ (b) $\lambda=0$ (c) $\lambda=1$ (d) $\lambda=-1$

45. The number of vectors of unit length perpendicular to the vectors $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is [NCERT Exemplar]

(a) one (b) two
(c) three (d) infinite

46. The value of $[\mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}]$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

47. The value of $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})\times(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is equal to

(a) 0 (b) a
(c) b (d) c

48. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors such that $\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$. If $|\mathbf{u}|=3$, $|\mathbf{v}|=4$ and $|\mathbf{w}|=5$, then $\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{u}$ is equal to

(a) 0 (b) 25
(c) −25 (d) 1

49. The points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}, -7\mathbf{b} + 10\mathbf{c}$ are
(a) collinear (b) non-collinear
(c) can't say (d) None of these

50. The area of the parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is [NCERT]
(a) $15\sqrt{2}$ sq units (b) $3\sqrt{2}$ sq units
(c) $\frac{1}{7}\sqrt{2}$ sq units (d) None of these

51. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ the value of $[\mathbf{a} + \mathbf{b}\ \mathbf{b} + \mathbf{c}\ \mathbf{c} + \mathbf{a}]$ is equal to
(a) [abc] (b) 2[abc]
(c) 3[abc] (d) None of these

52. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear vectors and $x\mathbf{a} + y\mathbf{b} = 0$
(a) $x = 0$ but y is not necessarily zero
(b) $y = 0$ but x is not necessarily zero
(c) $x = 0, y = 0$
(d) None of the above

53. Five points given by A, B, C, D and E are in a plane. Three forces **AC, AD** and **AE** act at A and three forces **CB, DB, EB** act at B. Then, their resultant is
(a) 2AC (b) 3AB
(c) 3DB (d) 2BC

54. If $ABCDEF$ is regular hexagon, then $\mathbf{AB} + \mathbf{EB} + \mathbf{FC}$ is equal to
(a) 0 (b) 2AB (c) 3AB (d) 4AB

55. Let $ABCD$ be the parallelogram whose sides AB and AD are represented by the vectors $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ respectively. Then, if $\mathbf{a}$ is a unit vector parallel to **AC**, then $\mathbf{a}$ equal to
(a) $\frac{1}{3}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (b) $\frac{1}{3}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ (d) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

56. If D, E and F are respectively the mid-points of AB, AC and BC in ΔABC, then $\mathbf{BE} + \mathbf{AF}$ is equal to
(a) DC (b) $\frac{1}{2}$BF
(c) 2BF (d) $\frac{3}{2}$BF

57. If three points A, B and C have position vectors $(1, x, 3)$, $(3, 4, 7)$ and $(y, -2, -5)$ respectively and if they are collinear, then (x, y) is equal to
(a) $(2, -3)$ (b) $(-2, 3)$
(c) $(2, 3)$ (d) $(-2, -3)$

58. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors such that angle between them is 60°. Then, $|\mathbf{a} - \mathbf{b}|$ is equal to
(a) $\sqrt{5}$ (b) $\sqrt{3}$
(c) 0 (d) 1

59. If position vector of a point A is $\mathbf{a} + 2\mathbf{b}$ and any point $P(\mathbf{a})$ divides **AB** in the ratio of $2:3$, then position vector of B is
(a) 2a − b (b) b − 2a
(c) a − 3b (d) b

60. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular unit vectors, then $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$ is equal to
(a) $\sqrt{3}$ (b) 3 (c) 1 (d) 0

61. If $|\mathbf{a}| = 3, |\mathbf{b}| = 4$, then a value of λ for which $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{a} - \lambda\mathbf{b}$, is
(a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\frac{3}{2}$ (d) $\frac{4}{3}$

62. $(\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$ is equal to
(a) a (b) 2a (c) 3a (d) 0

63. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{5\pi}{6}$ and the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\frac{-6}{\sqrt{3}}$, then $|\mathbf{a}|$ is equal to
(a) 6 (b) $\frac{\sqrt{3}}{2}$ (c) 12 (d) 4

64. If $\mathbf{a} \cdot \hat{\mathbf{i}} = \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then $\mathbf{a}$ is equal to
(a) $\hat{\mathbf{i}}$ (b) $\hat{\mathbf{k}}$
(c) $\hat{\mathbf{j}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

65. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\mathbf{c} = 7\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$, then the area of the parallelogram having diagonals $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$ is
(a) $4\sqrt{6}$ (b) $\frac{1}{2}\sqrt{21}$
(c) $\frac{\sqrt{6}}{2}$ (d) $\sqrt{6}$

66. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = \sqrt{37}, |\mathbf{b}| = 3, |\mathbf{c}| = 4$, then the angle between $\mathbf{b}$ and $\mathbf{c}$ is
(a) 30° (b) 45° (c) 60° (d) 90°

67. If the angle between $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + a\hat{\mathbf{k}}$ is $\frac{\pi}{3}$, then the value of a is
(a) 0 or 2 (b) − 4 or 0
(c) 0 or − 2 (d) 2 or − 2

68. If A, B, C, D and E are five coplanar points, then $\mathbf{DA} + \mathbf{DB} + \mathbf{DC} + \mathbf{AE} + \mathbf{BE} + \mathbf{CE}$ is equal to
(a) OE (b) 3DE (c) 2DE (d) 4ED

69. Forces acting on a particle have magnitude 5, 3 and 1 unit and act in the direction of the vectors $6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, respectively. They remain constant while the particle is displaced from the points $A(2, -1, -3)$ to $B(5, -1, 1)$. The work done is
(a) 11 units (b) 33 units
(c) 10 units (d) 30 units

70. If $\sum_{i=1}^{n}|\mathbf{a}_i| = \mathbf{0}$, where $|\mathbf{a}_i| = 1, \forall\, i$, then the value of $\sum_{1\le i<j\le n}\sum \mathbf{a}_i \cdot \mathbf{a}_j$ is

(a) n^2 (b) $-n^2$ (c) n (d) $-\frac{n}{2}$

71. A unit vector in xy-plane that makes an angle 45° with the vector $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ and an angle of 60° with the vector $(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$, is

(a) $\hat{i}$ (b) $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

(c) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (d) None of these

Product of Three Vectors

72. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors and let $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ be vector defined by the relations. $\mathbf{p} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$, $\mathbf{q} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$ and $\mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$. Then, the value of the expression $(\mathbf{a}+\mathbf{b})\cdot\mathbf{p} + (\mathbf{b}+\mathbf{c})\cdot\mathbf{q} + (\mathbf{c}+\mathbf{a})\cdot\mathbf{r}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

73. The moment about the point $M(-2, 4, -6)$ of the force represented in magnitude and position $\mathbf{AB}$, where the points A and B have the coordinates $(1, 2, -3)$ and $(3, -4, 2)$ respectively, is

(a) $8\hat{i} - 9\hat{j} - 14\hat{k}$ (b) $2\hat{i} - 6\hat{j} + 5\hat{k}$

(c) $-3\hat{i} + 2\hat{j} - 3\hat{k}$ (d) $-5\hat{i} - 8\hat{j} - 8\hat{k}$

74. If $\mathbf{x}\cdot\mathbf{a} = \mathbf{x}\cdot\mathbf{b} = \mathbf{x}\cdot\mathbf{c} = 0$, where $\mathbf{x}$ is a non-zero vector. Then, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$ is equal to

(a) $[x\ a\ b]^2$ (b) $[x\ b\ c]^2$

(c) $[x\ c\ a]^2$ (d) 0

75. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors inclined at an angle $2\alpha (0 \le \alpha \le \pi)$ each other, then $|\mathbf{a}+\mathbf{b}| < 1$, if

(a) $\alpha = \frac{\pi}{2}$ (b) $\alpha < \frac{\pi}{3}$

(c) $\alpha > \frac{2\pi}{3}$ (d) $\frac{\pi}{3} < \alpha < \frac{2\pi}{3}$

76. If $|\mathbf{a}\times\mathbf{b}| = 4$ and $|\mathbf{a}\cdot\mathbf{b}| = 2$, then $|\mathbf{a}|^2\,|\mathbf{b}|^2$ is equal to

(a) 2 (b) 6 (c) 8 (d) 20

77. $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]$ is equal to

(a) $(a\times a)\cdot(b\times a)$ (b) $a\cdot(b\times a) - b\cdot(a\times b)$

(c) $[a\cdot(a\times b)]a$ (d) $(a\cdot a)\,(b\times a)$

78. The length of longer diagonal of the parallelogram constructed on $5\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$. If it is given that $|\mathbf{a}| = 2\sqrt{2}$, $|\mathbf{b}| = 3$ and angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$, is

(a) 15 (b) $\sqrt{113}$

(c) $\sqrt{593}$ (d) $\sqrt{369}$

79. If $\mathbf{a} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, $\mathbf{a}\cdot\mathbf{b} = 1$ and $\mathbf{a}\times\mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$, then $\mathbf{b}$ is

(a) $\hat{i} - \hat{j} + \hat{k}$ (b) $2\hat{j} - \hat{k}$

(c) $\hat{i}$ (d) $2\hat{i}$

80. A unit vector coplanar with $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and perpendicular to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is

(a) $\left(\frac{\hat{j} - \hat{k}}{\sqrt{2}}\right)$ (b) $\left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\right)$

(c) $\left(\frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}\right)$ (d) $\left(\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}\right)$

81. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{c} = \hat{\mathbf{i}}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, then $\lambda + \mu$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

82. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors such that $\mathbf{a} \ne 0$ and $\mathbf{a}\times\mathbf{b} = 2\mathbf{a}\times\mathbf{c}$, $|\mathbf{a}| = |\mathbf{c}| = 1$, $|\mathbf{b}| = 4$ and $|\mathbf{b}\times\mathbf{c}| = \sqrt{15}$. If $\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, then λ is equal to

(a) 1 (b) -4 (c) 3 (d) -2

83. $[\mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}\ \mathbf{a}\times\mathbf{b}]$ is equal to

(a) $[a\ b\ c]$ (b) $2[a\ b\ c]$

(c) $[a\ b\ c]^2$ (d) $a\times(b\times c)$

84. The unit vector perpendicular to $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and coplanar with $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ is

(a) $\frac{2\hat{i} - 5\hat{j}}{\sqrt{2a}}$ (b) $2\hat{i} + 5\hat{j}$

(c) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (d) $\hat{i} + \hat{j}$

85. If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}}) + \hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}}) + \hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})$, then length of $\mathbf{b}$ is equal to

(a) $\sqrt{12}$ (b) $2\sqrt{12}$

(c) $3\sqrt{14}$ (d) $2\sqrt{14}$

86. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the three vectors mutually perpendicular to each other to form a right handed system and $|\mathbf{a}| = 1$, $|\mathbf{b}| = 3$ and $|\mathbf{c}| = 5$, then $[\mathbf{a} - 2\mathbf{b}\ \mathbf{b} - 3\mathbf{c}\ \mathbf{c} - 4\mathbf{a}]$ is equal to

(a) 0 (b) -24 (c) 3600 (d) -215

87. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit coplanar vectors, then $[2\mathbf{a} - \mathbf{b}\ 2\mathbf{b} - \mathbf{c}\ 2\mathbf{c} - \mathbf{a}]$ is equal to

(a) 1 (b) 0 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

88. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $(\mathbf{a} - \mathbf{b})\cdot(\mathbf{b} + \mathbf{c})\times(\mathbf{c} + \mathbf{a})$ is equal to

(a) $2a\cdot(b\times c)$ (b) $[a\ b\ c]$

(c) $[a\ b\ c]^2$ (d) 0

89. $[\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] + [\hat{\mathbf{k}}\ \hat{\mathbf{j}}\ \hat{\mathbf{i}}] + [\hat{\mathbf{j}}\ \hat{\mathbf{k}}\ \hat{\mathbf{i}}]$ is equal to

(a) 1 (b) 3 (c) -3 (d) -1

90. The value of $[2\hat{\mathbf{i}}\ 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}]$ is equal to

(a) -30 (b) -25

(c) 0 (d) 11

91. What is the value of $(\mathbf{d}+\mathbf{a})\cdot[\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}]$?
(a) $(\mathbf{d}\cdot\mathbf{a})\cdot[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$ (b) $(\mathbf{a}\cdot\mathbf{d})\ [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$
(c) $(\mathbf{b}\cdot\mathbf{d})\ [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$ (d) $[\mathbf{b}\cdot\mathbf{d})\ [\mathbf{a}\ \mathbf{d}\ \mathbf{c}]$

92. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors inclined to x-axis at angles 30° and 120°, then $|\mathbf{a}+\mathbf{b}|$ is equal to
(a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{2}$
(c) $\sqrt{3}$ (d) 2

93. A tetrahedron has vertical at $O(0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then, the angle between the faces OAB and ABC will be
(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{7}{31}\right)$
(c) 30° (d) 90°

94. If $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{b}=2\hat{\mathbf{i}}-\hat{\mathbf{k}}$ are two vectors, then the point of intersection of two lines $\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{a}$ and $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ is
(a) $\hat{\mathrm{i}}+\hat{\mathrm{j}}-\hat{\mathrm{k}}$ (b) $\hat{\mathrm{i}}-\hat{\mathrm{j}}+\hat{\mathrm{k}}$
(c) $3\hat{\mathrm{i}}+\hat{\mathrm{j}}-\hat{\mathrm{k}}$ (d) $3\hat{\mathrm{i}}-\hat{\mathrm{j}}+\hat{\mathrm{k}}$

95. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the pth, qth, rth terms of an HP and $\mathbf{u}=(q-r)\hat{\mathbf{i}}+(r-p)\hat{\mathbf{j}}+(p-q)\hat{\mathbf{k}}$ and $\mathbf{v}=\frac{\hat{\mathbf{i}}}{a}+\frac{\hat{\mathbf{j}}}{b}+\frac{\hat{\mathbf{k}}}{c}$, then
(a) $\mathbf{u}, \mathbf{v}$ are parallel vectors
(b) $\mathbf{u}, \mathbf{v}$ are orthogonal vectors
(c) $\mathbf{u}\cdot\mathbf{v}=1$
(d) $\mathbf{u}\times\mathbf{v}=\hat{\mathrm{i}}+\hat{\mathrm{j}}+\hat{\mathrm{k}}$

96. Let $\mathbf{A}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}, \mathbf{B}=\hat{\mathbf{i}}, \mathbf{C}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$. If $c_2=-1$ and $c_3=1$, then to make three vectors coplanar
(a) $c_1=0$
(b) $c_1=1$
(c) $c_1=2$
(d) no value of c_1 can be found

97. The vectors $\mathbf{a}(x)=\cos x\hat{\mathbf{i}}+(\sin x)\hat{\mathbf{j}}$ and $\mathbf{b}(x)=x\hat{\mathbf{i}}+\sin x\hat{\mathbf{j}}$ are collinear for
(a) unique value of x, $0<x<\frac{\pi}{6}$
(b) unique value of x, $\frac{\pi}{6}<x<\frac{\pi}{3}$
(c) no value of x
(d) infinitely many values of x, $0<x<\frac{\pi}{2}$

98. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are reciprocal vectors, then $(l\mathbf{a}+m\mathbf{b}+n\mathbf{c})\cdot(l\mathbf{p}+m\mathbf{q}+n\mathbf{r})$ is equal to
(a) $l+m+n$ (b) $l^3+m^3+n^3$
(c) $l^2+m^2+n^2$ (d) None of these

99. If $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}, \mathbf{b}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$ is
(a) 60 (b) 68 (c) −60 (d) −74

100. If $\mathbf{A}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}, \mathbf{B}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ and $\mathbf{c}$ from a left handed system, then $\mathbf{c}$ is
(a) $11\hat{\mathrm{i}}-6\hat{\mathrm{j}}-\hat{\mathrm{k}}$ (b) $-11\hat{\mathrm{i}}+6\hat{\mathrm{j}}+\hat{\mathrm{k}}$
(c) $11\hat{\mathrm{i}}-6\hat{\mathrm{j}}+\hat{\mathrm{k}}$ (d) $11\hat{\mathrm{i}}+6\hat{\mathrm{j}}-\hat{\mathrm{k}}$

101. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ iff
(a) $(\mathrm{a}\times\mathrm{b})\times\mathrm{c}=0$ (b) $\mathrm{c}\times\mathrm{a}=\mathrm{b}$
(c) $(\mathrm{a}\times\mathrm{c})\times\mathrm{b}=0$ (d) $\mathrm{a}\cdot(\mathrm{b}\times\mathrm{c})$

102. The sum of two unit vectors is a unit vector. The magnitude of their difference is
(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) 1

103. For non-zero vector $\mathbf{a}$ and $\mathbf{b}$, if $|\mathbf{a}+\mathbf{b}|<|\mathbf{a}-\mathbf{b}|$, then $\mathbf{a}$ and $\mathbf{b}$ are
(a) collinear
(b) perpendicular to each other
(c) inclined at an acute angle
(d) inclined at an obtuse angle

104. If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-5\mathbf{a}+4\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{b}=3$, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is equal to
(a) $5\mathrm{b}-3\mathrm{c}$ (b) $3\mathrm{c}-4\mathrm{b}$
(c) $3\mathrm{b}-5\mathrm{c}$ (d) $4\mathrm{b}-3\mathrm{c}$

105. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/6$, then angle between $2\mathbf{a}$ and $3\mathbf{b}$ is
(a) $\pi/3$ (b) $\pi/2$ (c) $\pi/6$ (d) $\frac{3\pi}{4}$

106. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors such that $\mathbf{a}+2\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$ are perpendicular to each other, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is
(a) 30° (b) 60° (c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) $\cos^{-1}\left(\frac{4}{7}\right)$

107. Let there be two points A and B on the curve $y=x^2$ in the plane OXY satisfying $OA\cdot\hat{\mathbf{i}}=1$ and $OB\cdot\hat{\mathbf{i}}=-2$ then the length of the vector $2OA-3OB$ is
(a) $\sqrt{14}$ (b) $2\sqrt{51}$ (c) $3\sqrt{41}$ (d) $2\sqrt{41}$

108. If $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ and $|\mathbf{a}|=3$, $|\mathbf{b}|=5$ and $|\mathbf{c}|=7$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

109. Given that $(\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}})=8$ and $\mathbf{x}\cdot\hat{\mathbf{a}}=2$, then the angle between $(\mathbf{x}-\hat{\mathbf{a}})$ and $(\mathbf{x}+\hat{\mathbf{a}})$ is
(a) $\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$ (b) $\cos^{-1}\left(\frac{3}{\sqrt{21}}\right)$
(c) $\cos^{-1}\left(\frac{5}{\sqrt{21}}\right)$ (d) $\cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$

110. The vector $(\mathbf{a}+3\mathbf{b})$ is perpendicular to $(7\mathbf{a}-5\mathbf{b})$ and $(\mathbf{a}-4\mathbf{b})$ is perpendicular to $(7\mathbf{a}-2\mathbf{b})$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is
(a) 30° (b) 45° (c) 60° (d) 15°

Round II (Mixed Bag)

Only One Correct Option

1. If **a** and **b** are unit vectors, then the greatest value of $|\mathbf{a}+\mathbf{b}|+|\mathbf{a}-\mathbf{b}|$ is
(a) 2 (b) 4
(c) $2\sqrt{2}$ (d) $\sqrt{2}$

2. If V is the volume of the parallelopiped having three coterminus edges, as **a, b** and **c**, then the volume of the parallelopiped having three coterminus edges as
$\alpha = (\mathbf{a}\cdot\mathbf{a})\mathbf{a}+(\mathbf{a}\cdot\mathbf{b})\mathbf{b}+(\mathbf{a}\cdot\mathbf{c})\mathbf{c}$
$\beta = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}+(\mathbf{b}\cdot\mathbf{b})\mathbf{b}+(\mathbf{b}\cdot\mathbf{c})\mathbf{c}$
$\gamma = (\mathbf{a}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{b}+(\mathbf{c}\cdot\mathbf{c})\mathbf{c}$
(a) V^3 (b) $3V$
(c) V^2 (d) $2V$

3. If **a, b** and **c** are non-coplanar vectors and r is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $\lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar for
(a) no value of λ
(b) all except one value of λ
(c) all except two values of λ
(d) all values of λ

4. Let **u**, **v** and **w** be three vectors such that $|\mathbf{u}|=1, |\mathbf{v}|=2, |\mathbf{w}|=3$. If the projection of **v** along **u** is equal to that of **w** along **u** , **v** and **w** are perpendicular to each other, then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ is equal to
(a) 4 (b) $\sqrt{7}$
(c) $\sqrt{14}$ (d) 2

5. Vectors **a** and **b** are such that $|\mathbf{a}|=1, |\mathbf{b}|=4$ and $\mathbf{a}\cdot\mathbf{b}=2$. If $\mathbf{c}=2\mathbf{a}\times\mathbf{b}-3\mathbf{b}$, then the angle between **b** and **c** is
(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

6. Let **b** and **c** be non-collinear vectors. If **a** is a vector such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=4$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(x^2-2x+6)\,\mathbf{b}+\sin y\cdot\mathbf{c}$; then (x, y) lies on the line
(a) $x+y=0$ (b) $x-y=0$
(c) $x=1$ (d) $y=\pi$

7. The distance of the point $3\hat{\mathbf{i}}+5\hat{\mathbf{k}}$ from the line parallel to $6\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and passing through the point $8\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is
(a) 1 (b) 2
(c) 3 (d) 4

8. Let $|\mathbf{a}|=2\sqrt{2}$, $|\mathbf{b}|=3$ and the angle between **a** and **b** is $\frac{\pi}{4}\cdot$ If a parallelogram is constructed with adjacent sides $2\mathbf{a}-3\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$, then its longer diagonal is of length
(a) 10 (b) 8
(c) $2\sqrt{26}$ (d) 6

9. If $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=1$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=\cos\theta$, then the maximum value of θ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
(c) $\frac{2\pi}{3}$ (d) $\frac{2\pi}{5}$

10. If the positive numbers a, b and c are the pth, qth and rth terms of GP, then the vectors $\log a\cdot\hat{\mathbf{i}}+\log b\cdot\hat{\mathbf{j}}+\log c\cdot\hat{\mathbf{k}}$ and $(q-r)\hat{\mathbf{i}}+(r-p)\hat{\mathbf{j}}+(p-q)\hat{\mathbf{k}}$ are
(a) equal (b) parallel
(c) perpendicular (d) None of these

11. A vector of magnitude $\sqrt{2}$ coplanar with $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and perpendicular to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is
(a) $-\hat{\mathrm{i}}+\hat{\mathrm{k}}$ (b) $\hat{\mathrm{i}}-\hat{\mathrm{k}}$
(c) $\hat{\mathrm{i}}-\hat{\mathrm{j}}$ (d) $\hat{\mathrm{i}}-2\hat{\mathrm{j}}+\hat{\mathrm{k}}$

12. If **a** is a unit vector and projection of **x** along **a** is 2 and $\mathbf{a}\times\mathbf{r}+\mathbf{b}=\mathbf{r}$, then **r** is equal to
(a) $\frac{1}{2}[\mathrm{a}-\mathrm{b}+\mathrm{a}\times\mathrm{b}]$ (b) $\frac{1}{2}[2\mathrm{a}+\mathrm{b}+\mathrm{a}\times\mathrm{b}]$
(c) $\mathrm{a}+\mathrm{a}\times\mathrm{b}$ (d) $\mathrm{a}-\mathrm{a}\times\mathrm{b}$

13. Let **a**, **b** and **c** be three unit vectors such that **a** is perpendicular to the plane of **b** and **c**. If the angle between **b** and **c** is $\frac{\pi}{3}$, then $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|$ is equal to
(a) 1/3 (b) 1/2 (c) 1 (d) 2

14. If $\mathbf{r}=\alpha\,\mathbf{b}\times\mathbf{c}+\beta\mathbf{c}\times\mathbf{a}+\gamma\mathbf{a}\times\mathbf{b}$ and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=2$, then $\alpha+\beta+\gamma$ is equal to
(a) $\mathrm{r}\cdot(\mathrm{b}\times\mathrm{c}+\mathrm{c}\times\mathrm{a}+\mathrm{a}\times\mathrm{b})$
(b) $\frac{1}{2}\mathrm{r}\cdot(\mathrm{a}+\mathrm{b}+\mathrm{c})$
(c) $2r\cdot(\mathrm{a}+\mathrm{b}+\mathrm{c})$
(d) 4

15. If $\mathbf{a}+2\mathbf{b}+3\mathbf{c}=0$, then
$$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=k\mathbf{a}\times\mathbf{b},$$
where k is equal to
(a) 0 (b) 1
(c) 2 (d) 3

16. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$. If $|\mathbf{c}| = 1$ and $(a \times b) \times c = 0$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is equal to

(a) 0 (b) 1
(c) $|\mathbf{a}|^2 |\mathbf{b}|^2$ (d) $|\mathbf{a} \times \mathbf{b}|^2$

More Than One Correct Option

17. If **a, b** and **c** are coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + \mu\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are coplanar for

(a) all values of μ (b) $\lambda = \frac{1}{2}$
(c) $\lambda = 0$ (d) no value of λ

18. If the vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and **a** form a triangle, then **a** may be

(a) $-\hat{\mathrm{i}} - \hat{\mathrm{k}}$ (b) $\hat{\mathrm{i}} - 2\hat{\mathrm{j}} - \hat{\mathrm{k}}$
(c) $2\hat{\mathrm{i}} + \hat{\mathrm{j}} + \hat{\mathrm{k}}$ (d) $\hat{\mathrm{i}} + \hat{\mathrm{k}}$

19. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between **OA** and **OB** (O is the origin of reference)?

(a) (2, 2, 4) (b) (2, 11, 5)
(c) (−3, −3, −6) (d) (1, 1, 2)

20. In a four dimensional space where unit vectors along the axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = \mathbf{0}$, then

(a) $\lambda = 1$ (b) $\mu = -2/3$
(c) $\gamma = 2/3$ (d) $\delta = 1/3$

21. Unit vectors **a** and **b** are perpendicular and unit vector **c** is inclined at an angle θ to both **a** and **b**. If $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$, then

(a) $a = \beta$ (b) $\gamma^2 = 1 - 2\alpha^2$
(c) $\gamma^2 = -\cos 2\theta$ (d) $\beta^2 = \frac{1 + \cos 2\theta}{2}$

22. **a** and **b** are two given vectors. With these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to **a** is

(a) $\frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2}\mathbf{a} - \mathbf{b}$ (b) $\frac{1}{|\mathbf{a}|^2}\{|\mathbf{a}|^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}\}$
(c) $\frac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a}|^2}$ (d) $\frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2}$

23. $a_1, a_2, a_3 \in R - \{0\}$ and $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all $x \in R$, then

(a) vectors $\mathbf{a} = a_1\hat{\mathrm{i}} + a_2\hat{\mathrm{j}} + a_3\hat{\mathrm{k}}$ and $\mathbf{b} = 4\hat{\mathrm{i}} + 2\hat{\mathrm{j}} + \hat{\mathrm{k}}$ are perpendicular to each other
(b) vectors $\mathbf{a} = a_1\hat{\mathrm{i}} + a_2\hat{\mathrm{j}} + a_3\hat{\mathrm{k}}$ and $\mathbf{b} = -\hat{\mathrm{i}} + \hat{\mathrm{j}} + 2\hat{\mathrm{k}}$ are parallel to each other
(c) if vector $\mathbf{a} = a_1\hat{\mathrm{i}} + a_2\hat{\mathrm{j}} + a_3\hat{\mathrm{k}}$ is of length $\sqrt{6}$ units, then one of the ordered tripplet $(a_1, a_2, a_3) = (1, -1, -2)$
(d) if $2a + 3a_2 + 6a_3 = 26$, then $|a_1\hat{\mathrm{i}} + a_2\hat{\mathrm{j}} + a_3\hat{\mathrm{k}}|$ is $2\sqrt{6}$

24. If **a** and **b** are two vectors and angle between them is θ, then

(a) $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$
(b) $|\mathbf{a} \times \mathbf{b}| = (\mathbf{a} \cdot \mathbf{b})$, if $\theta = \pi/4$
(c) $\mathbf{a} \times \mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\hat{\mathbf{n}}$, ($\hat{\mathbf{n}}$ is normal unit vector), if $\theta = \pi/4$
(d) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$

25. Let **a** and **b** be two non-zero perpendicular vectors. A vector **r** satisfying the equation $\mathbf{r} \times \mathbf{b} = \mathbf{a}$ can be

(a) $\mathbf{b} - \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$ (b) $2\mathbf{b} - \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$
(c) $|\mathbf{a}|\mathbf{b} - \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$ (d) $|\mathbf{b}|\mathbf{b} - \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$

26. If vectors **a** and **b** are non-collinear, then $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$ is

(a) a unit vector
(b) in the plane of **a** and **b**
(c) equally inclined to **a** and **b**
(d) perpendicular to $\mathbf{a} \times \mathbf{b}$

Comprehension Based Questions

Passage I

Directions (Q.Nos. 27-29) *Vectors* **x**, **y** *and* **z** *each of magnitude* $\sqrt{2}$ *make angle of* 60° *with each other.* $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a}$, $\mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b}$ *and* $\mathbf{x} \times \mathbf{y} = \mathbf{c}$.

27. Vector **x** is

(a) $\frac{1}{2}[(\mathbf{a} - \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})]$
(b) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} - \mathbf{b})]$
(c) $\frac{1}{2}[-(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})]$
(d) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})]$

28. Vector **y** is

(a) $\frac{1}{2}[(\mathbf{a} + \mathbf{c}) \times \mathbf{b} - \mathbf{b} - \mathbf{a}]$ (b) $\frac{1}{2}[(\mathbf{a} - \mathbf{c}) \times \mathbf{c} + \mathbf{b} + \mathbf{a}]$
(c) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + \mathbf{b} + \mathbf{a}]$ (d) $\frac{1}{2}[(\mathbf{a} - \mathbf{c}) \times \mathbf{a} + \mathbf{b} - \mathbf{a}]$

29. Vector $\mathbf{z}$ is

(a) $\frac{1}{2}[(\mathbf{a}-\mathbf{c})\times\mathbf{c}-\mathbf{b}+\mathbf{a}]$ (b) $\frac{1}{2}[(\mathbf{a}+\mathbf{b})\times\mathbf{c}+\mathbf{b}-\mathbf{a}]$

(c) $\frac{1}{2}[\mathbf{c}\times(\mathbf{a}-\mathbf{b})+\mathbf{b}+\mathbf{a}]$ (d) None of these

Passage II

Directions (Q. Nos. 30 and 31) *The vertices of a ΔABC are $A\equiv(2,0,2)$, $B\equiv(-1,1,1)$ and $C\equiv(1,-2,4)$. The points D and E divide the sides AB and CA in the ratio $1:2$, respectively. Another point F is taken in space such that perpendicular drawn from F on ΔABC, meets the triangle at the point of intersection of the line segment CD and BE, say P. If the distance of F from the plane of the ΔABC is $\sqrt{2}$ units, then*

On the basis of the above information, answer the following question.

30. The position vector of P is

(a) $\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

(c) $2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

31. The volume of the tetrahedron $ABCF$ is

(a) 7 cu units (b) 3/5 cu units

(c) 7/3 cu units (d) 7/5 cu units

Assertion and Reason

Directions (Q. Nos. 32-36) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

32. **Statement I** Vector $\mathbf{c}=-5\hat{\mathbf{i}}+7\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ is along the bisector of angle between $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=-8\hat{\mathbf{i}}+\hat{\mathbf{j}}-4\hat{\mathbf{k}}$.

Statement II $\mathbf{c}$ is equal inclined to $\mathbf{a}$ and $\mathbf{b}$.

33. **Statement I** A component of vector $\mathbf{b}=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ in the direction perpendicular to the direction of the vector $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $\hat{\mathbf{i}}-\hat{\mathbf{j}}$.

Statement II A component of vector in the direction of $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$.

34. **Statement I** Distance of point $D(1,0,-1)$ from the plane of points $A(1,-2,0)$, $B(3,1,2)$ and $C(-1,1,-1)$ is $\frac{8}{\sqrt{129}}$.

Statement II Volume of tetrahedron formed by the points A, B, C and D is $\frac{\sqrt{129}}{2}$.

35. Let $\mathbf{r}$ be a non-zero vector satisfying $\mathbf{r}\cdot\mathbf{a}=\mathbf{r}\cdot\mathbf{b}=\mathbf{r}\cdot\mathbf{c}=0$ for given non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Statement I $[\mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}]=0$

Statement II $[\mathbf{abc}]=0$

36. **Statement I** If $a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$, $b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ and $c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ are three mutually perpendicular unit vectors, then $a_1\hat{\mathbf{i}}+b_1\hat{\mathbf{j}}+c_1\hat{\mathbf{k}}$, $a_2\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+c_2\hat{\mathbf{k}}$ and $a_3\hat{\mathbf{i}}+b_3\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ may be mutually perpendicular unit vectors.

Statement II Value of determinant and its transpose are the same.

Previous Years' Questions

37. If the vector $\mathbf{AB}=3\hat{\mathbf{i}}+4\hat{\mathbf{k}}$ and $\mathbf{AC}=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ are the sides of a ΔABC, then the length of the median through A is **[JEE Main 2013]**

(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{45}$

38. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors. If the vectors $\mathbf{c}=\mathbf{a}+2\mathbf{b}$ and $\mathbf{d}=5\mathbf{a}-4\mathbf{b}$ are perpendicular to each other, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is **[AIEEE 2012]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

39. Let $ABCD$ be a parallelogram such that $\mathbf{AB}=\mathbf{q}$, $\mathbf{AB}=\mathbf{p}$, and $\angle BAD$ be an acute angle. If $\mathbf{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\mathbf{r}$ is given by **[AIEEE 2012]**

(a) $\mathbf{r}=3\mathbf{q}-\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{q})}\mathbf{p}$ (b) $\mathbf{r}=-\mathbf{q}+\left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{q}\cdot\mathbf{p}}\right)\mathbf{p}$

(c) $\mathbf{r}=\mathbf{q}-\left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{q}}\right)\mathbf{p}$ (d) $\mathbf{r}=-3\mathbf{q}+\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{q})}\mathbf{p}$

40. If $\mathbf{a}=\frac{1}{\sqrt{10}}(3\hat{\mathbf{i}}+\hat{\mathbf{k}})$ and $\mathbf{b}=\frac{1}{7}(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}})$, then the values of $(2\mathbf{a}-\mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}+2\mathbf{b})]$ is **[AIEEE 2011]**

(a) -3 (b) 5 (c) 3 (d) -5

41. The vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular $\mathbf{c}$ and $\mathbf{d}$ are two vectors satisfying $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$ and $\mathbf{a}\cdot\mathbf{d}=0$. Then, the vectors $\mathbf{d}$ is equal to **[AIEEE 2011]**

(a) $\mathbf{c}+\left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b}+\left(\frac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c}-\left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b}-\left(\frac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$

42. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is [AIEEE 2011]

(a) $\mathbf{a} + \mathbf{c}$ (b) $\mathbf{a}$
(c) $\mathbf{c}$ (d) 0

43. If the vectors $p\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + q\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + r\hat{\mathbf{k}}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is [AIEEE 2011]

(a) -2 (b) 2 (c) 0 (d) -1

44. Let $a = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 3$, is [AIEEE 2010]

(a) $-\hat{\mathrm{i}} + \hat{\mathrm{j}} - 2\hat{\mathrm{k}}$ (b) $2\hat{\mathrm{i}} - \hat{\mathrm{j}} + 2\hat{\mathrm{k}}$
(c) $\hat{\mathrm{i}} - \hat{\mathrm{j}} - 2\hat{\mathrm{k}}$ (d) $\hat{\mathrm{i}} + \hat{\mathrm{j}} - 2\hat{\mathrm{k}}$

45. If the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\mathbf{k}$ are mutually orthogonal, then (λ, μ) is equal to [AIEEE 2010]

(a) $(-3, 2)$ (b) $(2, -3)$
(c) $(-2, 3)$ (d) $(3, -2)$

Answers

Check Point 1

1. (a) **2.** (b) **3.** (c) **4.** (c) **5.** (d) **6.** (c) **7.** (b) **8.** (b)

Check Point 2

1. (c) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (a)

Round I

1. (d)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (c)
11. (c)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (c)
21. (a)	**22.** (b)	**23.** (b)	**24.** (d)	**25.** (b)	**26.** (d)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (a)
31. (a)	**32.** (a)	**33.** (a)	**34.** (a)	**35.** (d)	**36.** (a)	**37.** (b)	**38.** (b)	**39.** (a)	**40.** (d)
41. (a)	**42.** (c)	**43.** (d)	**44.** (a)	**45.** (b)	**46.** (a)	**47.** (a)	**48.** (c)	**49.** (a)	**50.** (a)
51. (b)	**52.** (c)	**53.** (b)	**54.** (d)	**55.** (d)	**56.** (a)	**57.** (a)	**58.** (d)	**59.** (c)	**60.** (a)
61. (b)	**62.** (a)	**63.** (d)	**64.** (a)	**65.** (a)	**66.** (c)	**67.** (b)	**68.** (b)	**69.** (b)	**70.** (d)
71. (d)	**72.** (d)	**73.** (a)	**74.** (d)	**75.** (d)	**76.** (d)	**77.** (d)	**78.** (c)	**79.** (c)	**80.** (a)
81. (a)	**82.** (b)	**83.** (c)	**84.** (c)	**85.** (d)	**86.** (d)	**87.** (b)	**88.** (c)	**89.** (d)	**90.** (a)
91. (c)	**92.** (b)	**93.** (a)	**94.** (c)	**95.** (b)	**96.** (d)	**97.** (b)	**98.** (c)	**99.** (d)	**100.** (b)
101. (c)	**102.** (b)	**103.** (d)	**104.** (d)	**105.** (c)	**106.** (b)	**107.** (d)	**108.** (c)	**109.** (d)	**110.** (c)

Round II

1. (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (c)	**9.** (c)	**10.** (c)
11. (a)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (d)	**17.** (a,b,c)	**18.** (a,b,d)	**19.** (a,c,d)	**20.** (a,b,d)
21. (a,b,c,d)	**22.** (a,b,c)	**23.** (a,b,c)	**24.** (a,b,c,d)	**25.** (a,b,c,d)	**26.** (b,c,d)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (b)
31. (c)	**32.** (b)	**33.** (c)	**34.** (d)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (b)	**40.** (d)
41. (c)	**42.** (d)	**43.** (a)	**44.** (a)	**45.** (a)					

the Guidance

Round I

1. Considering each options.

(a) $b = \lambda a$ is correct, it is the condition for collinearity of two vectors.

(b) $\hat{\mathbf{a}} = \pm \hat{\mathbf{b}}$ is also correct, it shows that unit vectors in the direction of given vectors are either in the same direction or in opposite direction *i.e.*, they are collinear.

(c) is also correct, if for two vectors the respective, components are proportional, then their magnitude will be different but they will be collinear.

(d) is incorrect, because if the two vectors are collinear then they may have same directions or opposite directions, in both the areas they will be collinear.

2. To determine the angle between **a** and **b**, we can use the formula $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

It is given that $|\mathbf{a}| = \sqrt{3}, |\mathbf{b}| = 2$ and $\mathbf{a}\cdot\mathbf{b} = \sqrt{6}$

Let θ be the required angle, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Hence, the angle between the given vectors **a** and **b** is $\pi/4$.

3. Let $\mathbf{a} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\mathbf{b} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}}$, then projection of **a** is given by

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \left(\frac{(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}})\cdot(7\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}})}{\sqrt{7^2 + (-1)^2 + 8^2}}\right)$$

$$= \left(\frac{1\times 7 + 3\times(-1) + 7\times 8}{(\sqrt{49 + 1 + 64})^2}\right) = \frac{7 - 3 + 56}{\sqrt{114}} = \frac{60}{\sqrt{144}}$$

Hence, the projection of vector $(\hat{\mathbf{i}} + \hat{\mathbf{j}}3 + 7\hat{\mathbf{k}})$ on $(7\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}})$ is $\dfrac{60}{\sqrt{144}}$.

4. Both vectors have same magnitude *i.e.*, $|\mathbf{a}| = |\mathbf{b}|$ and scalar product of vectors, $\mathbf{a}\cdot\mathbf{b} = \dfrac{1}{2}$ (given)

Let θ be the angle between two vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \Rightarrow \cos 60^\circ = \frac{1/2}{|\mathbf{a}|\,|\mathbf{a}|} \qquad [\because |\mathbf{a}| = |\mathbf{b}|]$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2|\mathbf{a}|^2} \Rightarrow |\mathbf{a}|^2 = 1 \Rightarrow |\mathbf{a}| = 1$$

Thus, $\quad |\mathbf{a}| = |\mathbf{b}| = 1$

5. Given, $\quad (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = 8$ and $|\mathbf{a}| = 8|\mathbf{b}|$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} = 8$$

$$\Rightarrow \quad |\mathbf{a}|^2 - |\mathbf{b}|^2 = 8 \qquad (\because \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \text{ and } \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a})$$

$$\Rightarrow \quad (8|\mathbf{b}|)^2 - |\mathbf{b}|^2 = 8 \Rightarrow 63|\mathbf{b}|^2 = 8 \text{(given, } |\mathbf{a}| = 8|\mathbf{b}|)$$

$$\Rightarrow \quad |\mathbf{b}| = \sqrt{\frac{8}{63}} = \frac{2}{3}\sqrt{\frac{2}{7}}$$

Also, $\quad |\mathbf{a}| = 8|\mathbf{b}| = 8\left(\dfrac{2}{3}\sqrt{\dfrac{2}{7}}\right) = \dfrac{16}{3}\sqrt{\dfrac{2}{7}}$.

6. Given, $\quad |\mathbf{a}| = 1$

($\because$ it is a unit vector, magnitude of a unit vector is 1.)

and $\quad (\mathbf{x} - \mathbf{a})\cdot(\mathbf{x} + \mathbf{a}) = 12$

$$\Rightarrow \quad \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{x} - \mathbf{a}\cdot\mathbf{a} = 12$$

($\because \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$)

$$\Rightarrow \quad |\mathbf{x}|^2 - |\mathbf{a}|^2 = 12 \Rightarrow |\mathbf{x}|^2 - 1^2 = 12$$

($\because |\mathbf{a}| = 1$ as is **a** unit vector)

$$\Rightarrow \quad |\mathbf{x}|^2 = 13 \Rightarrow |\mathbf{x}| = \sqrt{13}$$

7. The given vectors are $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Now, $\quad (\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c}$ (given)

$$\Rightarrow \quad (\mathbf{a} + \lambda\mathbf{b})\cdot\mathbf{c} = 0$$

($\because$ scalar product of two perpendicular vectors is zero)

$$\Rightarrow \quad [(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})]\cdot(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 0$$

$$\Rightarrow \quad [(2 - \lambda)\hat{\mathbf{i}} + (2 + 2\lambda)\hat{\mathbf{i}} + (3 + \lambda)\hat{\mathbf{k}}]\cdot(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 0$$

$$\Rightarrow \quad (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$$

$$\Rightarrow \quad 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow 8 - \lambda = 0 \Rightarrow \lambda = 8.$$

Hence, the required value of λ is 8.

8. Let $\quad \mathbf{p} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ and $\mathbf{q} = |\mathbf{a}|\mathbf{b} - |\mathbf{b}|\mathbf{a}$

Thus, $\quad \mathbf{p}\cdot\mathbf{q} = (|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a})\cdot(|\mathbf{a}|\mathbf{b} - |\mathbf{b}|\mathbf{a})$

$$= |\mathbf{a}|^2(\mathbf{b}\cdot\mathbf{b}) - |\mathbf{a}||\mathbf{b}|(\mathbf{b}\cdot\mathbf{a}) + |\mathbf{b}||\mathbf{a}|(\mathbf{a}\cdot\mathbf{b}) - |\mathbf{b}|^2(\mathbf{a}\cdot\mathbf{a})$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}||\mathbf{b}|(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{a}||\mathbf{b}|(\mathbf{a}\cdot\mathbf{b}) - |\mathbf{b}|^2|\mathbf{a}|^2 = 0$$

$$\Rightarrow \quad \mathbf{p} \perp \mathbf{q} \qquad (\because \text{If } \mathbf{c}\cdot\mathbf{d} = 0 \Rightarrow \mathbf{c} \text{ is perpendicular to } \mathbf{d})$$

Hence, $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ and $|\mathbf{a}|\mathbf{b} - |\mathbf{b}|\mathbf{a}$ are perpendicular to each other for any two non-zero vectors **a** and **b**.

9. It is given that $\mathbf{a}\cdot\mathbf{a} = 0$ or $|\mathbf{a}|^2 = 0 \Rightarrow |\mathbf{a}| = 0 \Rightarrow \mathbf{a} = \mathbf{0}$ *i.e.*, **a** is a zero vector because dot becomes its magnitude is zero.

Hence, $\mathbf{a}\cdot\mathbf{b} = 0$ whatever **b** may ($\because \mathbf{a} = 0$). This means that nothing can be concluded about **b** it can be any vector.

10. As **a, b** and **c** are unit vectors, so use $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and expand $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$ to get the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$.

Given, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$

We have $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0$

$\Rightarrow \mathbf{a}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) + \mathbf{b}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) + \mathbf{c}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0$

$\Rightarrow \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c} = 0$

$\Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$(\because \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a})$

$\Rightarrow 1+1+1+2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0 (\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1)$

$\Rightarrow 3 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = -\dfrac{3}{2}$

11. Here, we have to find $\angle ABC$ *i.e.*, angle between **BA** and **BC**, so first of all, we have to calculate both these vectors after that we can find the angle between them by $\cos\theta = \dfrac{(\mathbf{BA})\cdot(\mathbf{BC})}{|\mathbf{BA}|\,|\mathbf{BC}|}$

We are given the points $A(1, 2, 3)$, $B(-1, 0, 0)$ and $C(0, 1, 2)$. Also, it is given that $\angle ABC$ is the angle between the vectors **BA** and **BC**. Here, **BA** = PV of A − PV of B

$= (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (-\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})$

$= [\hat{\mathbf{i}} - (-\hat{\mathbf{i}}) + (2\hat{\mathbf{j}} - 0) + (3\hat{\mathbf{k}} - 0)] = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$|\mathbf{BA}| = \sqrt{(2)^2 + (2)^2 + (3)^2} = \sqrt{4+4+9} = \sqrt{17}$

BC = PV of C − PV of B

$= (0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (-\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})$

$= [0 - (-\hat{\mathbf{i}}) + (1\hat{\mathbf{j}} - 0) + (2\hat{\mathbf{k}} - 0)]$

$= \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$|\mathbf{BC}| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$

Now, $\mathbf{BA}\cdot\mathbf{BC} = (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$= 2\times1 + 2\times1 + 3\times2 = 10$

$\cos\theta = \dfrac{\mathbf{BA}\cdot\mathbf{BC}}{|\mathbf{BA}|\,|\mathbf{BC}|} \Rightarrow \cos(\angle ABC) = \dfrac{10}{\sqrt{17}\sqrt{6}}$

$\Rightarrow \angle ABC = \cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$

12. For collinear, prove that $|\mathbf{AC}| = |\mathbf{AB}| + |\mathbf{BC}|$

The given points are $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$.

$\therefore$ **AB** = PV of B − PV of $A = (2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

$= (2-1)\hat{\mathbf{i}} + (6-2)\hat{\mathbf{j}} + (3-7)\hat{\mathbf{k}} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

Magnitude of

AB, $|\mathbf{AB}| = \sqrt{(1)^2 + (4)^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

BC = PV of C − PV of B

$= (3\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 1\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$= (3-2)\hat{\mathbf{i}} + (10-6)\hat{\mathbf{j}} + (-1-3)\hat{\mathbf{k}} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

Magnitude of

BC, $|\mathbf{BC}| = \sqrt{(1)^2 + (4)^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

AC = PV of C − PV of $A = (3\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 1\hat{\mathbf{k}}) - (1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

$= (3-1)\hat{\mathbf{i}} + (10-2)\hat{\mathbf{j}} + (-1-7)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$

Magnitude of **AC**, $|\mathbf{AC}|$

$= \sqrt{2^2 + 8^2 + (-8)^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$

$= \sqrt{33} + \sqrt{33}$

$\therefore |\mathbf{AC}| = |\mathbf{AB}| + |\mathbf{BC}|$.

Hence, the given points A, B and C are collinear.

13. Given that, $\mathbf{a}\cdot\mathbf{b} = 0$

Then, either $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\mathbf{a} \perp \mathbf{b}$ (in case **a** and **b** are non-zero) and if $\mathbf{a}\times\mathbf{b} = 0$, then either $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\mathbf{a} \parallel \mathbf{b}$ (in case **a** and **b** are non-zero)

But **a** and **b** cannot be perpendicular and parallel simultaneously.

Hence, $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$

14. Let OP make 30° with X-axis, 60° with Y-axis and 90° with Z-axis and it lies in XY-plane.

$\therefore$ Direction cosines of **OP** are $\cos 30°$, $\cos 60°$ and $\cos 90°$

i.e., $\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0$. Thus, **OP** is

$\dfrac{\sqrt{3}}{2}\hat{\mathbf{i}} + \dfrac{1}{2}\hat{\mathbf{j}}$,

$$|\mathbf{OP}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

which is required unit vector in XY-plane.

15. In ΔABC, let **CB** = **a**, **CA** = **b** and **AB** = **c**

(as shown in the following figure)

Now, by the triangle law of addition, we have **a** = **b** + **c**

It is clearly known that $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$ represent the side of ΔABC

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\mathbf{a}| < |\mathbf{b}| + |\mathbf{c}|$

Hence, it is not true that $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$

16. Given, $x(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ is a unit vector, therefore $|x(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})| = 1$

$\Rightarrow |x\hat{\mathbf{i}} + x\hat{\mathbf{j}} + x\hat{\mathbf{k}}| = 1 \Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$

$\Rightarrow \sqrt{3x^2} = 1 \Rightarrow \pm\sqrt{3}x = 1 \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$

Hence, the required values of x are $\pm\dfrac{1}{\sqrt{3}}$

17. Firstly, determine the resultants of vectors **a** and **b**, vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and then find the direction of **c** by finding unit vector in its direction and then multiplying it by 5, we can find the required vector.

Given vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Let **c** be the resultant of **a** and **b**.

$\therefore\ \mathbf{c} = \mathbf{a} + \mathbf{b} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \Rightarrow \mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

Comparing with $\mathbf{X} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore\quad |\mathbf{c}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 1} = \sqrt{9 + 1} = \sqrt{10}$

$\therefore$ Unit vector in the direction of **c**, $\hat{\mathbf{c}} = \dfrac{\mathbf{c}}{|\mathbf{c}|} = \dfrac{3\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{10}}$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors **a** and **b** is

$$\pm 5\hat{\mathbf{c}} = \pm 5\frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$$

18. It is given that $\mathbf{OP} = 2\mathbf{a} + \mathbf{b}$, $\mathbf{OQ} = \mathbf{a} - 3\mathbf{b}$

If a point divides the line joining point P and Q externally in the ratio $m : n$, then position vector of the point is

$$\frac{m(\text{PV of } Q) - n(\text{PV of } P)}{m - n}$$

It is given that point R divides a line segment joining two points P and Q externally in the ratio $1 : 2$. Then, on using the section formula, we get

Position vector of point $R = \dfrac{(\mathbf{a} - 3\mathbf{b}) \times 1 - (2\mathbf{a} + \mathbf{b}) \times 12}{1 - 2}$

$$= \frac{\mathbf{a} - 3\mathbf{b} - 4\mathbf{a} - 2\mathbf{b}}{-1} = \frac{-3\mathbf{a} - 5\mathbf{b}}{-1} = 3\mathbf{a} + 5\mathbf{b}$$

Now, position vector of mid-point of $RQ = \dfrac{\mathbf{OQ} + \mathbf{OR}}{2}$

$$= \frac{(3\mathbf{a} + 5\mathbf{b}) + (\mathbf{a} - 3\mathbf{b})}{2} = \frac{4\mathbf{a} + 2\mathbf{b}}{2} = 2\mathbf{a} + \mathbf{b}$$

Also, the position vector of point $P = 2\mathbf{a} + \mathbf{b}$

which shows that P is mid-point of line segment RQ.

19. Adjacent sides of a parallelogram are given as

$\mathbf{a} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$.

Then, the diagonal of a parallelogram is given by $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

($\because$ from the figure, it is clear that resultant of adjacent sides of a parallelogram is given by the diagonal)

$\therefore\quad \mathbf{v} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}} + \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$$= (2 + 1)\hat{\mathbf{i}} + (-4 - 2)\hat{\mathbf{j}} + (5 - 3)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Comparing with $\mathbf{X} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we get

$$x = 3, y = -6, z = 2$$

$$\therefore\quad |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(3)^2 + (-6)^2 + (2)^2}$$

$$= \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{7} = \frac{3}{7}\hat{\mathbf{i}} - \frac{6}{7}\hat{\mathbf{j}} + \frac{2}{7}\hat{\mathbf{k}}$$

Also, area of parallelogram $ABCD$, $|\mathbf{a} \times \mathbf{b}| = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$

$$= |\hat{\mathbf{i}}(12 + 10) - \hat{\mathbf{j}}(-6 - 5) + \hat{\mathbf{k}}(-4 + 4)| = |22\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 0\hat{\mathbf{k}}|$$

$$= \sqrt{(22)^2 + (11)^2 + 0^2} = \sqrt{(11)^2(2^2 + 1)^2} = 11\sqrt{5} \text{ sq units}$$

Note If $\mathbf{d}_1$ and $\mathbf{d}_2$ are the diagonals of a parallelogram, then area of parallelogram $= \dfrac{1}{2}|\mathbf{d}_1 \times \mathbf{d}_2|$

20. The vector which is perpendicular to both **a** and **b** most be parallel $\mathbf{a} \times \mathbf{b}$.

Now, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = \hat{\mathbf{i}}(28 + 4) - \hat{\mathbf{j}}(7 - 6) + \hat{\mathbf{k}}(-2 - 12)$

$$= 32\hat{\mathbf{i}} - \hat{\mathbf{j}} - 14\hat{\mathbf{k}}$$

Let $\quad \mathbf{d} = \lambda(\mathbf{a} \times \mathbf{b}) = \lambda(32\hat{\mathbf{i}} - \hat{\mathbf{j}} - 14\hat{\mathbf{k}})$

Also, $\quad \mathbf{c} \cdot \mathbf{d} = 15 \Rightarrow (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \cdot \lambda(32\hat{\mathbf{i}} - \hat{\mathbf{j}} - 14\hat{\mathbf{k}}) = 15$

$\Rightarrow\quad 2 \times (32\lambda) + (-1) \times (-\lambda) + 4 \times (-14\lambda) = 15$

$\Rightarrow\quad 64\lambda + \lambda - 56\lambda = 15 \Rightarrow 9\lambda = 15 \Rightarrow \lambda = \dfrac{15}{9} = \dfrac{5}{3}$

$\therefore$ Required vector, $\mathbf{d} = \dfrac{5}{3}(32\hat{\mathbf{i}} - \hat{\mathbf{j}} - 14\hat{\mathbf{k}})$.

21. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Now, $\quad \mathbf{b} + \mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{i}} - 5\hat{\mathbf{k}} + \lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3$

$$= (2 + \lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\therefore\quad |\mathbf{b} + \mathbf{c}| = \sqrt{(2 + \lambda)^2 + (6)^2 + (-2)^2}$$

$$= \sqrt{4 + \lambda^2 + 4\lambda + 36 + 4} = \sqrt{\lambda^2 + 4\lambda + 44}$$

The unit vector along $(\mathbf{b} + \mathbf{c})$, *i.e.* $\dfrac{\mathbf{b} + \mathbf{c}}{|\mathbf{b} + \mathbf{c}|} = \dfrac{(2 + \lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{\lambda^2 + 4\lambda + 44}}$

Scalar product of $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ with this unit vector is 1.

$$\therefore\quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \frac{\mathbf{b} + \mathbf{c}}{|\mathbf{b} + \mathbf{c}|} = 1$$

$$\Rightarrow\quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \frac{(2 + \lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow\quad \frac{1(2 + \lambda) + 1(6) + 1(-2)}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1 \Rightarrow \frac{(2 + \lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow\quad \lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44} \Rightarrow (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$$

$$\Rightarrow\quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1.$$

Hence, the value of λ is 1.

22. To expand $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$, use $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$ and

$$\mathbf{b}\cdot\mathbf{b}=|\mathbf{b}|^2,\ \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$

Given, $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+|\mathbf{b}|^2$

$\Rightarrow$ $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+|\mathbf{b}|^2$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}=|\mathbf{a}|^2+|\mathbf{b}|^2$

$\Rightarrow$ $|\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$

$$2\mathbf{a}\cdot\mathbf{b}=0$$

$[\because \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ (scalar product is commutative)$]$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b}=0$

$\therefore$ $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. (given $\mathbf{a}\neq 0$, $\mathbf{b}\neq 0$)

23. It is given that, $\mathbf{a}\cdot\mathbf{b}\geq 0$

We know that, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta\geq 0\Rightarrow\cos\theta\geq 0$, $[|\mathbf{a}|$ and $|\mathbf{b}|$ are positive$]$

$\Rightarrow$ $0\leq\theta\leq\dfrac{\pi}{2}$

24. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them.

Then, $|\mathbf{a}|=|\mathbf{b}|=1$

Now, $(\mathbf{a}+\mathbf{b})$ is a unit vector, if

$$|\mathbf{a}+\mathbf{b}|=1\Rightarrow(\mathbf{a}+\mathbf{b})^2=1$$

$\Rightarrow$ $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=1\Rightarrow\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}=1$

$\Rightarrow$ $|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}=1$ $\quad(\because \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow 1^2+1^2+2\mathbf{a}\cdot\mathbf{b}=1\Rightarrow\mathbf{a}\cdot\mathbf{b}=-\dfrac{1}{2}\Rightarrow|\mathbf{a}||\mathbf{b}|\cos\theta=-\dfrac{1}{2}$

$\Rightarrow$ $1\times 1\times\cos\theta=-\dfrac{1}{2}$

$\Rightarrow$ $\cos\theta=-\dfrac{1}{2}\Rightarrow\theta=\dfrac{2\pi}{3}$

25. We have, $|\mathbf{a}\cdot\mathbf{b}|=|\mathbf{a}\times\mathbf{b}|\Rightarrow|\mathbf{a}||\mathbf{b}|\cos\theta=|\mathbf{a}||\mathbf{b}|\sin\theta$

$\Rightarrow$ $\cos\theta=\sin\theta$ $\quad(|\mathbf{a}|$ and $|\mathbf{b}|$ are positive$)$

$\Rightarrow$ $\tan\theta=1\Rightarrow\dfrac{\pi}{4}$

26. Let the given points be $A(-1,-1,2)$, $B(2,m,5)$ and $C(3,11,6)$. Then,

$\mathbf{AB}=(2+1)\hat{\mathbf{i}}+(m+1)\hat{\mathbf{j}}+(5-2)\hat{\mathbf{k}}=3\hat{\mathbf{i}}+(m+1)\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

and $\mathbf{AC}=(3+1)\hat{\mathbf{i}}+(11+1)\hat{\mathbf{j}}+(6-2)\hat{\mathbf{k}}=4\hat{\mathbf{i}}+12\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Since, A, B and C are collinear, we have $\mathbf{AB}=\lambda\,\mathbf{AC}$,

i.e., $(3\hat{\mathbf{i}}+(m+1)\hat{\mathbf{j}}+3\hat{\mathbf{k}})=\lambda(4\hat{\mathbf{i}}+12\hat{\mathbf{j}}+4\hat{\mathbf{k}})$

$\Rightarrow$ $3=4\lambda$ and $m+1=12\lambda$

Therefore, $m=8$

27. Here, $m=\cos\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}$ and $n=\cos\dfrac{\pi}{2}=0$

Therefore, $l^2+m^2+n^2=1$ (gives)

$$l^2+\frac{1}{2}+0=1$$

$\Rightarrow$ $l=\pm\dfrac{1}{\sqrt{2}}$

Hence, the required vector $\mathbf{r}=3\sqrt{2}\,(l\hat{\mathbf{i}}+m\hat{\mathbf{j}}+n\hat{\mathbf{k}})$ is given by

$$\mathbf{r}=3\sqrt{2}\left(\pm\frac{1}{\sqrt{2}}\hat{\mathbf{i}}+\frac{1}{\sqrt{2}}\hat{\mathbf{j}}+0\hat{\mathbf{k}}\right)=\mathbf{r}=\pm 3\hat{\mathbf{i}}+3\hat{\mathbf{j}}.$$

28. We have,

$$\lambda\mathbf{b}+\mathbf{c}=\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})+(\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}})$$

$$=(\lambda+1)\hat{\mathbf{i}}+(\lambda+3)\hat{\mathbf{j}}-(2\lambda+1)\hat{\mathbf{k}}$$

Since, $\mathbf{a}\perp(\lambda\mathbf{b}+\mathbf{c})$, $\mathbf{a}\cdot(\lambda\mathbf{b}+\mathbf{c})=0$

$\Rightarrow$ $(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot[(\lambda+1)\hat{\mathbf{i}}+(\lambda+3)\hat{\mathbf{j}}-(2\lambda+1)\hat{\mathbf{k}}]=0$

$\Rightarrow$ $2(\lambda+1)-(\lambda+3)-(2\lambda+1)=0$

$\therefore$ $\lambda=-2$

29. Using the formula $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\cdot|\mathbf{b}||\sin\theta|$, we get

$$\theta=\pm\frac{\pi}{6}$$

Therefore, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\cdot|\mathbf{b}|\cos\theta=8\times 3\times 3\times\dfrac{\sqrt{3}}{2}=12\sqrt{3}$

30. Median $\mathbf{AD}$ is given by

$$|\mathbf{AD}|=\frac{1}{2}|3\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}}|=\frac{\sqrt{34}}{2}$$

31. Projection of a vector $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}=\frac{(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})}{\sqrt{1+4+4}}=\frac{2}{3}$$

32. We have,

$$(\sqrt{3}\mathbf{a}-\mathbf{b})^2=3\mathbf{a}^2+\mathbf{b}^2-2\sqrt{3}\mathbf{a}\cdot\mathbf{b}$$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b}=\dfrac{\sqrt{3}}{2}\Rightarrow\cos\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=30°$

33. Required unit vector is $\dfrac{(\hat{\mathbf{i}}-\hat{\mathbf{j}})\times(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{|(\hat{\mathbf{i}}-\hat{\mathbf{j}})\times(\hat{\mathbf{i}}+\hat{\mathbf{j}})|}=\dfrac{2\hat{\mathbf{k}}}{2}=\hat{\mathbf{k}}$

34. The smallest value of $|k\mathbf{a}|$ will exist at numerically smallest value of k, *i.e.*, at $k=0$, which gives $|k\mathbf{a}|=|k||\mathbf{a}|=0\times 3=0$

The numerically greatest value of k is 2 at which $|k\mathbf{a}|=6$

35. Given that,

$$\mathbf{OA}=\mathbf{a}\text{ and }\mathbf{OB}=\mathbf{b}$$

Let $\mathbf{c}$ be the position vector of c.

Also given, $\mathbf{BC}=1.5\,\mathbf{BA}$

$\Rightarrow$ $\mathbf{OC}-\mathbf{OB}=1.5\,(\mathbf{OA}-\mathbf{OB})$

$\Rightarrow$ $\mathbf{c}-\mathbf{b}=1.5\,(\mathbf{a}-\mathbf{b})$

$\Rightarrow$ $\mathbf{c}=1.5\mathbf{a}-1.5\mathbf{b}+\mathbf{b}$

$\Rightarrow$ $\mathbf{c}=1.5\mathbf{a}-0.5\mathbf{b}$

$\Rightarrow$ $\mathbf{c}=0.5(3\mathbf{a}-\mathbf{b})$

36. $\mathbf{a}+\mathbf{b}+\mathbf{c}=0\Rightarrow\mathbf{a}+\mathbf{b}=-\mathbf{c}$

$\Rightarrow$ $(\mathbf{a}+\mathbf{b})\times\mathbf{b}=-\mathbf{c}\times\mathbf{b}\Rightarrow\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$

$(\because -\mathbf{c}\times\mathbf{b}=\mathbf{b}\times\mathbf{c})$

$\Rightarrow$ $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$ $\quad(\because \mathbf{a}\times\mathbf{a}=0)$ ...(i)

Again, $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0} \Rightarrow \mathbf{a}+\mathbf{c}=-\mathbf{b} \Rightarrow (\mathbf{a}+\mathbf{c})\times\mathbf{a}=-\mathbf{b}\times\mathbf{a}$

$\Rightarrow \quad \mathbf{a}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}=\mathbf{a}\times\mathbf{b} \quad (\because -\mathbf{b}\times\mathbf{a}=\mathbf{a}\times\mathbf{b})$

$\Rightarrow \quad \mathbf{c}\times\mathbf{a}=\mathbf{a}\times\mathbf{b} \quad (\because \mathbf{a}\times\mathbf{a}=0)$ ...(ii)

From, Eqs. (i) and (ii), we get $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

Here, $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ means that **a, b, c** form a triangle, that means **a**, **b**, **c** vectors are on the same plane. We know that, the cross product of two non-linear vectors will geometrically give the area of the parallelogram they makes and the half of that value would give the area of the triangle.

So, $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ refer to same triangle so the modulus value of all three should be equal. Since, **a, b, c** are in the same plane, the direction of there cross products should also be same which make these vectors product equal.

37. Given that, $\mathbf{a}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

and $\mathbf{b}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Let θ be the angle between them.

$$\therefore \quad \cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{(3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}})}{\sqrt{9+1+4}\cdot\sqrt{4+4+16}}$$

$$\Rightarrow \quad \cos=\frac{6+(-2)+8}{\sqrt{14}\,\sqrt{24}}=\frac{12}{4\sqrt{21}}$$

$$\therefore \quad \sin\theta=\sqrt{1-\cos^2\theta}$$

$$=\sqrt{1-\frac{144}{336}}=\sqrt{\frac{192}{336}}$$

$$\Rightarrow \quad \sin\theta=\frac{2}{\sqrt{7}}$$

38. Let $\mathbf{OA}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{OB}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$,

$\mathbf{OC}=2\hat{\mathbf{i}}-3\hat{\mathbf{k}}$ and $\mathbf{OD}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Now, $\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}})-(\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$

$=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\mathbf{CD}=\mathbf{OD}-\mathbf{OC}=(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})-(2\hat{\mathbf{i}}-3\hat{\mathbf{k}})$

$=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\therefore$ Projection of **AB** along **CD** is $=\dfrac{\mathbf{AB}\cdot\mathbf{CD}}{|\mathbf{CD}|}$

$$=\frac{(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}})}{\sqrt{1+4+16}}$$

$$=\frac{1+4+16}{\sqrt{21}}=\frac{21}{\sqrt{21}}=\sqrt{21}$$

39. Let position vectors of **A**, **B** and **C** are; $\mathbf{OA}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $\mathbf{OB}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

and $\mathbf{OC}=4\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}}$

Now, $\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$=\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{AC}=\mathbf{OC}-\mathbf{OA}=(4\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$=3\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

$\therefore$ Area of $\Delta ABC=\dfrac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$$=\frac{1}{2}|(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(3\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}})|$$

$$=\frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{matrix}\right\|$$

$$=\frac{1}{2}|\hat{\mathbf{i}}(12-3)-\hat{\mathbf{j}}(-4-3)+\hat{\mathbf{k}}(3+9)|$$

$$=\frac{1}{2}|9\hat{\mathbf{i}}+7\hat{\mathbf{j}}+12\hat{\mathbf{k}}|$$

$$=\frac{1}{2}\sqrt{81+49+144}=\frac{1}{2}\sqrt{274}=\sqrt{\frac{137}{2}}$$

40. Since, **a** and **b** are orthogonal, then $\mathbf{a}\cdot\mathbf{b}=0$

$\Rightarrow \quad (2\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})=0$

$\Rightarrow \quad 2+2\lambda+3=0$

$\Rightarrow \quad 2\lambda=-5 \Rightarrow \lambda=-\dfrac{5}{2}$

41. Let $\mathbf{a}=3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{b}=2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$

Since, **a** and **b** are parallel, then

$\mathbf{a}=\mu\mathbf{b}$

$\Rightarrow \quad (3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}})=\mu(2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}})$

$\Rightarrow \quad 3\hat{\mathbf{i}}-6\hat{\mathbf{i}}+\hat{\mathbf{k}}=2\mu\hat{\mathbf{i}}-4\mu\hat{\mathbf{j}}+\lambda\mu\hat{\mathbf{k}}$

On comparing, we get

$2\mu=3 \Rightarrow \mu=\dfrac{3}{2}$

and $\lambda\mu=1$

$\Rightarrow \quad \lambda\cdot\dfrac{3}{2}=1 \Rightarrow \lambda=\dfrac{2}{3}$

42. Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$

Now, $(\mathbf{a}\times\hat{\mathbf{i}})=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\times\hat{\mathbf{i}}$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ 1 & 0 & 0\end{vmatrix}=-\hat{\mathbf{j}}(-a_3)+\hat{\mathbf{k}}(-a_2)$$

$=a_3\hat{\mathbf{j}}-a_2\hat{\mathbf{k}}$

Similarly, $(\mathbf{a}\times\hat{\mathbf{j}})=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\times\hat{\mathbf{j}}$

$=a_1\hat{\mathbf{k}}-a_3\hat{\mathbf{i}}$

and $(\mathbf{a}\times\hat{\mathbf{k}})=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\times\hat{\mathbf{k}}$

$=-a_1\hat{\mathbf{j}}+a_2\hat{\mathbf{i}}$

Now, $(\mathbf{a}\times\hat{\mathbf{i}})^2=(\mathbf{a}\times\hat{\mathbf{i}})\cdot(\mathbf{a}\times\hat{\mathbf{i}})$

$=(a_3\hat{\mathbf{j}}-a_2\hat{\mathbf{k}})\cdot(a_3\hat{\mathbf{j}}-a_2\hat{\mathbf{k}})$

$=a_3^2+a_2^2$

$(\mathbf{a}\times\hat{\mathbf{j}})^2=a_1^2+a_3^2$

and $(\mathbf{a} \times \hat{\mathbf{k}})^2 = a_1^2 + a_2^2$

$\therefore (\mathbf{a} \times \hat{\mathbf{i}})^2 + (\mathbf{a} \times \hat{\mathbf{j}})^2 + (\mathbf{a} \times \hat{\mathbf{k}})^2 = a_3^2 + a_2^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$= 2(a_1^2 + a_2^2 + a_3^2)$

$= 2(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \cdot (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})$

$= 2\mathbf{a} \cdot \mathbf{a} = 2\mathbf{a}^2$

43. Given that, $|\mathbf{a}| = 10, |\mathbf{b}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = 12$

$\because \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$

$\Rightarrow \quad 12 = 10 \times 2 \cdot \cos\theta$

$\Rightarrow \quad \cos\theta = \frac{3}{5}$

$\because \quad \mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \cdot \hat{\mathbf{n}}$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\sin\theta| \cdot |\hat{\mathbf{n}}| \qquad (\because |\hat{\mathbf{n}}| = 1|)$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = 10 \cdot 2 \cdot |\sqrt{1 - \cos^2\theta}| \cdot 1$

$= 20 \cdot |\sqrt{1 - 9/25}|$

$= 20 \times \left|\sqrt{\frac{16}{25}}\right| = 20 \times \frac{4}{5}$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = 16$

44. Given that vectors $\lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ are coplanar, then

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda(\lambda^2 - 1) - (\lambda + 2) + 2(-1 - 2\lambda) = 0$

$\Rightarrow \quad \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow \quad \lambda^2(\lambda + 2) - 2\lambda(\lambda + 2) - 2(\lambda + 2) = 0$

$\Rightarrow \quad (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\Rightarrow \quad \lambda = -2$

45. The number of vectors of unit length perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b} = \pm \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

$= \pm \frac{(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (\hat{\mathbf{j}} + \hat{\mathbf{k}})}{|(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (\hat{\mathbf{j}} + \hat{\mathbf{k}})|}$

$= \pm \frac{(2\hat{\mathbf{i}} - 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{i}})}{|-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|}$

$= \pm \frac{(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1 + 4 + 4}} = \pm \frac{(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{9}}$

$= \pm \frac{1}{3}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$\therefore$ Required number of vectors is 2.

46. $[\mathbf{a} - \mathbf{b}\ \mathbf{b} - \mathbf{c}\ \mathbf{c} - \mathbf{a}]$

$= \{(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c})\} \cdot (\mathbf{c} - \mathbf{a})$

$= (\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})$

$= (\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})$

$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

$= [\mathbf{abc}] - [\mathbf{abc}] = 0$

47. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= \mathbf{a} \cdot \{-\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b})$

$= \mathbf{a} \cdot (-\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}) = 0$

48. $|\mathbf{u} + \mathbf{v} + \mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2(\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u})$

$\Rightarrow \quad 0 = 9 + 16 + 25 + 2(\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u})$

$\Rightarrow \quad \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u} = -25$

49. Let the given points be A, B and C.

Let O be the origin.

Then, $\mathbf{OA} = \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$

$\mathbf{OB} = 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$

and $\mathbf{OC} = -7\mathbf{b} + 10\mathbf{c}$

Now, $\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$= (-7\mathbf{b} + 10\mathbf{c}) - (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c})$

$= -\mathbf{a} - 5\mathbf{b} + 7\mathbf{c}$

$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= (2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}) - (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c})$

$= \mathbf{a} + 5\mathbf{b} - 7\mathbf{c}$

$\therefore \quad \mathbf{AB} = -\mathbf{AC} = (-1) \times \mathbf{AC} = \text{scalar} \times \mathbf{AC}$

Hence, the points A, B and C are collinear.

50. To determine the area of parallelogram, use the formula, $|\mathbf{a} \times \mathbf{b}|$. The area of the parallelogram whose adjacent sides are a and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.

Adjacent sides are given as

$\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and

$\mathbf{b} = 2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$$\therefore \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{\mathbf{i}}(-1 + 21) - \hat{\mathbf{j}}(1 - 6) + \hat{\mathbf{k}}(-7 + 2)$$

$= 20\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

Comparing with $\mathbf{X} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we get

$x = 20, y = 5, z = -5$

$\therefore$ Area of the parallelogram $= |\mathbf{a} \times \mathbf{b}|$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(20)^2 + 5^2 + (-5)^2}$

$= \sqrt{450} = \sqrt{225 \times 2}$

$= 15\sqrt{2}$ sq units

51. We have,

$[\mathbf{a} + \mathbf{b}\ \mathbf{b} + \mathbf{c}\ \mathbf{c} + \mathbf{a}] = \{(\mathbf{a} + \mathbf{b}) \times (\mathbf{b} + \mathbf{c})\} \cdot (\mathbf{c} + \mathbf{a})$

$= (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} + \mathbf{a})$

$= (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} + \mathbf{a}) \qquad (\because \mathbf{b} \times \mathbf{b} = 0)$

$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{c} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{c}$

$+ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{a} + (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

$= [\mathbf{abc}] + 0 + 0 + 0 + 0 + [\mathbf{bca}]$

$(\because [\mathbf{acc}] = 0, [\mathbf{bcc}] = 0, [\mathbf{aba}) = 0, [\mathbf{aca}] = 0$

$= [\mathbf{abc}] + [\mathbf{abc}] = 2[\mathbf{abc}]$

52. If **a** and **b** are two non-zero non-collinear vectors and x, y are two scalars such that $x\mathbf{a} + y\mathbf{b} = 0$, then $x = 0, y = 0$. Because otherwise one will be a scalar multiple of the other and hence collinear, which is a contradiction.

53. All points A, B, C, D, E are in a plane.

$\therefore$ Resultant $= (\mathbf{AC} + \mathbf{AD} + \mathbf{AE}) + (\mathbf{CB} + \mathbf{DB} + \mathbf{EB})$

$= (\mathbf{AC} + \mathbf{CB}) + (\mathbf{AD} + \mathbf{DB}) + (\mathbf{AE} + \mathbf{EB})$

$= \mathbf{AB} + \mathbf{AB} + \mathbf{AB} = 3\mathbf{AB}$

54. *ABCDEF* is a regular hexagon. We know from the hexagon that **AD** is parallel to **BC**.

$\Rightarrow$ $\mathbf{AD} = 2\mathbf{BC}$

Similarly, **EB** is parallel to **FA**.

$\Rightarrow$ $\mathbf{EB} = 2\mathbf{FA}$

and **FC** is parallel to **AB**.

$\Rightarrow$ $\mathbf{FC} = 2\mathbf{AB}$

Thus, $\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = 2\mathbf{BC} + 2\mathbf{FA} + 2\mathbf{AB}$

$= 2(\mathbf{FA} + \mathbf{AB} + \mathbf{BC})$

$= 2(\mathbf{FC}) = 2(2\mathbf{AB}) = 4\mathbf{AB}$

55. $\mathbf{R}_1 = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

and $\mathbf{R}_2 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$\therefore \mathbf{R}$ (along $\mathbf{AC}) = \mathbf{R}_1 + \mathbf{R}_2 = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\therefore \hat{\mathbf{a}}$ (unit vector along $\mathbf{AC}) = \dfrac{\mathbf{R}}{|\mathbf{R}|} = \dfrac{3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{9 + 36 + 4}} = \dfrac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

56. Let O be the origin

$\therefore$ $\mathbf{BE} + \mathbf{AF} = \mathbf{OE} - \mathbf{OB} + \mathbf{OF} - \mathbf{OA}$

$= \dfrac{\mathbf{OA} + \mathbf{OC}}{2} - \mathbf{OB} + \dfrac{\mathbf{OB} + \mathbf{OC}}{2} - \mathbf{OA}$

$= \dfrac{\mathbf{OC}}{2} + \dfrac{\mathbf{OC}}{2} + \dfrac{\mathbf{OA}}{2} - \mathbf{OA} + \dfrac{\mathbf{OB}}{2} - \mathbf{OB}$

$= \mathbf{OC} - \dfrac{\mathbf{OA} + \mathbf{OB}}{2} = \mathbf{OC} - \mathbf{OD} = \mathbf{DC}$

57. Given that, $\mathbf{OA} = \hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$,

$\mathbf{OB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

and $\mathbf{OC} = y\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

Since, A, B, C are collinear. Then, $\mathbf{AB} = \lambda\,\mathbf{BC}$

$\Rightarrow$ $2\hat{\mathbf{i}} + (4 - x)\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \lambda[(y - 3)\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 12\hat{\mathbf{k}}]$

On comparing the coefficient of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$2 = (y - 3)\lambda$ ...(i)

$4 - x = -6\lambda$ ...(ii)

and $4 = -12\lambda \Rightarrow \lambda = -\dfrac{1}{3}$ ...(iii)

On putting the value of λ is Eqs. (i) and (ii), we get

$y = -3$ and $x = 2$

58. $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

$\Rightarrow$ $|\mathbf{a} - \mathbf{b}|^2 = 1 + 1 - 2\cos 60° = 2 - 1$

$\Rightarrow$ $|\mathbf{a} - \mathbf{b}| = 1$

59. Let position vector of B is **r**.

$\therefore$ $\mathbf{a} = \dfrac{2\mathbf{r} + 3(\mathbf{a} + 2\mathbf{b})}{2 + 3}$

$\Rightarrow$ $5\mathbf{a} = 2\mathbf{r} + 3\mathbf{a} + 6\mathbf{b}$

$\Rightarrow$ $2\mathbf{r} = 2\mathbf{a} - 6\mathbf{b}$

$\therefore$ $\mathbf{r} = \mathbf{a} - 3\mathbf{b}$

60. Since, **a**, **b** and **c** are mutually perpendicular unit vectors.

$\Rightarrow$ $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$ ...(i)

Now, $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$= 1 + 1 + 1 + 0 = 3$ [from Eq. (i)]

$\Rightarrow$ $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = \sqrt{3}$

61. Given that, $|\mathbf{a}| = 3, |\mathbf{b}| = 4$ and $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{a} - \lambda\mathbf{b}$.

$\therefore$ $(\mathbf{a} + \lambda\mathbf{b})\cdot(\mathbf{a} - \lambda\mathbf{b}) = 0$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b}\lambda + \lambda\mathbf{b}\cdot\mathbf{a} - \lambda^2\mathbf{b}\cdot\mathbf{b} = 0$

$\Rightarrow$ $|\mathbf{a}|^2 - \lambda^2|\mathbf{b}|^2 = 0$

$\Rightarrow$ $\lambda^2 = \dfrac{|\mathbf{a}|^2}{|\mathbf{b}|^2} \Rightarrow \lambda = \dfrac{|\mathbf{a}|}{|\mathbf{b}|} = \dfrac{3}{4}$

62. We know that, any vector **a** can be uniquely expressed in terms of three non-coplanar vectors as $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ multiply in succession by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$x = \mathbf{a}\cdot\hat{\mathbf{i}}, y = \mathbf{a}\cdot\hat{\mathbf{j}}, z = \mathbf{a}\cdot\hat{\mathbf{k}}$

$\therefore$ $(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = \mathbf{a}$

63. $\because$ $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\dfrac{5\pi}{6} = -\dfrac{|\mathbf{a}||\mathbf{b}|\sqrt{3}}{2}$

$\therefore$ $-\dfrac{6}{\sqrt{3}} = -\dfrac{|\mathbf{a}||\mathbf{b}|\sqrt{3}}{2|\mathbf{b}|}$ (given condition)

$\Rightarrow$ $|\mathbf{a}| = \dfrac{6 \times 2}{3} = 4$

64. Let $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore$ $\mathbf{a}\cdot\hat{\mathbf{i}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{i}} = x$

$\mathbf{a}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = x + y$

and $\mathbf{a}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$= x + y + z$

$\because$ Given that, $\mathbf{a}\cdot\hat{\mathbf{i}} = \mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}) = \mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$\Rightarrow \quad x = x + y = x + y + z$

Take $\quad x = x + y \Rightarrow y = 0$

and $\quad x + y = x + y + z \Rightarrow z = 0$

$\Rightarrow x$ has any real values.

Now, take $x = 1 \therefore \mathbf{a} = \hat{\mathbf{i}}$

65. Given that,

$$\mathbf{a} = \hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}},\ \mathbf{b} = \hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}}$$

and $\quad \mathbf{c} = 7\hat{\mathbf{i}}+9\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

Let $\quad \mathbf{A} = \mathbf{a}+\mathbf{b} = (\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}) + (\hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

$$= 2\hat{\mathbf{i}}+4\hat{\mathbf{j}}+6\hat{\mathbf{k}}$$

and $\quad \mathbf{B} = \mathbf{b}+\mathbf{c} = (\hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}}) + (7\hat{\mathbf{i}}+9\hat{\mathbf{j}}+11\hat{\mathbf{k}})$

$$= 8\hat{\mathbf{i}}+12\hat{\mathbf{j}}+16\hat{\mathbf{k}}$$

If $\mathbf{A}$ and $\mathbf{B}$ are diagonals, then area of parallelogram

$$= \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{K}}\\ 2 & 4 & 6\\ 8 & 12 & 16\end{matrix}\right\|$$

$$= \frac{1}{2}|\hat{\mathbf{i}}(64-72) - \hat{\mathbf{j}}(32-48) + \hat{\mathbf{k}}(24-32)|$$

$$= \frac{1}{2}|-8\hat{\mathbf{i}}+16\hat{\mathbf{j}}-8\hat{\mathbf{k}}|$$

$$= |-4\hat{\mathbf{i}}+8\hat{\mathbf{j}}-4\hat{\mathbf{k}}|$$

$$= \sqrt{(-4)^2+(8)^2+(-4)^2}$$

$$= \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$$

66. Given, $\quad \mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = \sqrt{37}, |\mathbf{b}| = 3$ and $|\mathbf{c}| = 4$

Now, $\quad \mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$

$\Rightarrow \quad \mathbf{a} = -(\mathbf{b}+\mathbf{c})$

$\Rightarrow \quad |\mathbf{a}|^2 = |-(\mathbf{b}+\mathbf{c})|^2$

$\Rightarrow \quad |\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2|\mathbf{b}||\mathbf{c}|\cos\theta$

$$= 9 + 16 + 24\cos\theta$$

$\Rightarrow \quad 37 = 25 + 24\cos\theta$

$\Rightarrow \quad 24\cos\theta = 12 \Rightarrow \theta = 60°$

67. $\cos\frac{\pi}{3} = \frac{(\hat{\mathbf{i}}+\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+a\hat{\mathbf{k}})}{\sqrt{2}\sqrt{1+1+a^2}}$

$\Rightarrow \quad \frac{1}{2} = \frac{1+a}{\sqrt{2}\sqrt{2+a^2}} \Rightarrow \frac{1}{4} = \frac{(1+a)^2}{2(2+a^2)}$

$\Rightarrow \quad 2 + a^2 = 2(1 + a^2 + 2a)$

$\Rightarrow \quad a^2 + 4a = 0$

$\Rightarrow \quad a = 0, -4$

68. $\mathbf{DA}+\mathbf{DB}+\mathbf{DC}+\mathbf{AE}+\mathbf{BE}+\mathbf{CE}$

$$= (\mathbf{DA}+\mathbf{AE}) + (\mathbf{DB}+\mathbf{BE}) + (\mathbf{DC}+\mathbf{CE})$$

$$= \mathbf{DE}+\mathbf{DE}+\mathbf{DE} = 3\mathbf{DE}$$

69. $\because \mathbf{F}_1 = \frac{5(6\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{7}, \mathbf{F}_2 = \frac{3(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+6\hat{\mathbf{k}})}{7}$

$$\mathbf{F}_3 = \frac{1(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-6\hat{\mathbf{k}})}{7}$$

and $\quad \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$= \frac{1}{7}(30\hat{\mathbf{i}}+10\hat{\mathbf{j}}+15\hat{\mathbf{k}}+9\hat{\mathbf{i}}-6\hat{\mathbf{j}}+18\hat{\mathbf{k}}+2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-6\hat{\mathbf{k}})$$

$$= \frac{1}{7}(41\hat{\mathbf{i}}+\hat{\mathbf{j}}+27\hat{\mathbf{k}})$$

and $\quad \mathbf{AB} = 5\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}-2\hat{\mathbf{i}}+\hat{\mathbf{k}}+3\hat{\mathbf{k}}$

$$= 3\hat{\mathbf{i}}+4\hat{\mathbf{k}}$$

$\therefore$ Work done $= \frac{1}{7}[41\hat{\mathbf{i}}+\hat{\mathbf{j}}+27\hat{\mathbf{k}}]\cdot[3\hat{\mathbf{i}}+4\hat{\mathbf{k}}]$

$$= \frac{1}{7}[123+108] = 33 \text{ units}$$

70. $\because \sum_{i=1}^{n}\mathbf{a}_i = \mathbf{0}$

$\therefore \quad \left(\sum_{i=1}^{n}\mathbf{a}_i\right)\left(\sum_{i=1}^{n}\mathbf{a}_j\right) = \sum_{i=1}^{n}|\mathbf{a}_i|^2 + 2\sum\sum_{1\le i<j\le n}\mathbf{a}_i\cdot\mathbf{a}_j$

$\Rightarrow \quad 0 = n + 2\sum\sum_{1\le i<j\le n}\mathbf{a}_i\cdot\mathbf{a}_j$

$\therefore \quad \sum\sum_{1\le i<j\le n}\mathbf{a}_i\cdot\mathbf{a}_j = -\frac{n}{2}$

71. Let the unit vector in xy-plane be $\mathbf{a} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$

$\therefore \quad \cos 45° = \frac{(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{\sqrt{x^2+y^2}\sqrt{1^2+1^2}}$

$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{x+y}{\sqrt{2}\sqrt{x^2+y^2}}$

$\Rightarrow \quad 1 = \frac{x+y}{\sqrt{x^2+y^2}}$

$\Rightarrow \quad x + y = \sqrt{x^2+y^2}$

Since, $\mathbf{a}$ is a unit vector.

$\therefore \quad |\mathbf{a}| = \sqrt{x^2+y^2} = 1$

$\Rightarrow \quad x + y = 1 \quad \ldots(i)$

Again, $\quad \cos 60° = \frac{(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-4\hat{\mathbf{j}})}{\sqrt{x^2+y^2}\sqrt{3^2+4^2}}$

$\Rightarrow \quad \frac{1}{2} = \frac{3x-4y}{1\cdot 5} \Rightarrow \frac{5}{2} = 3x - 4y$

$$5 = 6x - 8y \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{13}{14}, y = \frac{1}{14}$$

$\therefore \quad \mathbf{a} = \frac{1}{14}(13\hat{\mathbf{i}}+\hat{\mathbf{j}})$

No value in the given options satisfies the above relations. Thus, option (d) is correct.

72. Given, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ defined by the relations

$$\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]}, \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} \text{ and } \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}$$

$\therefore \quad \mathbf{a} \cdot \mathbf{p} = \dfrac{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} = \dfrac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]} = 1$

and $\quad \mathbf{a} \cdot \mathbf{q} = \mathbf{a} \cdot \dfrac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} = \dfrac{\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})}{[\mathbf{abc}]} = 0$

Similarly, $\quad \mathbf{b} \cdot \mathbf{q} = \mathbf{c} \cdot \mathbf{r} = 1$

and $\quad \mathbf{a} \cdot \mathbf{r} = \mathbf{b} \cdot \mathbf{p} = \mathbf{c} \cdot \mathbf{q} = \mathbf{c} \cdot \mathbf{p} = \mathbf{b} \cdot \mathbf{r} = 0$

$\therefore \quad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$

$= \mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{q} + \mathbf{c} \cdot \mathbf{q} + \mathbf{c} \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{r}$

$= 1 + 1 + 1 = 3$

73. Force $\quad \mathbf{F} = \mathbf{AB} = (3 - 1)\hat{\mathbf{i}} + (-4 - 2)\hat{\mathbf{j}} + (2 + 3)\hat{\mathbf{k}}$

$= 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

Moment of force $\mathbf{F}$ with respect to $M = \mathbf{MA} \times \mathbf{F}$

$\because \quad \mathbf{MA} = (1 + 2)\hat{\mathbf{i}} + (2 - 4)\hat{\mathbf{j}} + (-3 + 6)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Now, $\quad \mathbf{MA} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$

$= \hat{\mathbf{i}}(-10 + 18) + \hat{\mathbf{j}}(6 - 15) + \hat{\mathbf{k}}(-18 + 4)$

$= 8\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 14\hat{\mathbf{k}}$

74. $\because \quad \mathbf{x} \cdot \mathbf{a} = \mathbf{x} \cdot \mathbf{b} = \mathbf{x} \cdot \mathbf{c} = 0$

For non-zero vector $\mathbf{x}$

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

(three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar)

and $\quad [\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}]$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = 0$

75. Given, $\quad |\mathbf{a} + \mathbf{b}| < 1$

$\Rightarrow \quad \sqrt{1 + 1 + 2\cos 2\alpha} < 1$

$\Rightarrow \quad \sqrt{2(1 + \cos 2\alpha)} < 1$

$\Rightarrow \quad \sqrt{4\cos^2 \alpha} < 1$

$\Rightarrow \quad |\cos \alpha| < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{\pi}{3} < \alpha < \dfrac{2\pi}{3} \qquad (\because 0 \le \alpha \le \pi)$

76. Given, $\quad |\mathbf{a} \times \mathbf{b}| = 4$

$\Rightarrow \quad ||\mathbf{a}||\mathbf{b}| \sin\theta \mathbf{n}| = 4$

$\Rightarrow \quad ||\mathbf{a}||\mathbf{b}| \sin\theta| = 4 \qquad (\because |\mathbf{n}| = 1)$ ...(i)

Also, $\quad |\mathbf{a} \cdot \mathbf{b}| = 2$

$\Rightarrow \quad ||\mathbf{a}||\mathbf{b}| \cos\theta| = 2$ ...(ii)

On squaring and then adding Eqs. (i) and (ii), we get

$|\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta + |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta = 4^2 + 2^2$

$\Rightarrow \quad |\mathbf{a}|^2|\mathbf{b}|^2(\sin^2\theta + \cos^2\theta) = 16 + 4$

$\Rightarrow \quad |\mathbf{a}|^2|\mathbf{b}|^2 = 20$

77. $\quad \mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})] = \mathbf{a} \times \{(\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}\}$

(expanding by vector triple product)

$= (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{a}) - (\mathbf{a} \cdot \mathbf{a})(\mathbf{a} \times \mathbf{b})$

$= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \times \mathbf{a}) \qquad (\because \mathbf{a} \times \mathbf{a} = 0)$

78. Given that, $\quad |\mathbf{a}| = 2\sqrt{2}, |\mathbf{b}| = 3$

The longer vectors is $5\mathbf{a} + 2\mathbf{b} + \mathbf{a} - 3\mathbf{b} = 6\mathbf{a} - \mathbf{b}$

Length of one diagonal

$= |6\mathbf{a} - \mathbf{b}|$

$= \sqrt{36\mathbf{a}^2 + \mathbf{b}^2 - 2 \times 6|\mathbf{a}||\mathbf{b}| \cos 45°}$

$= \sqrt{36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \dfrac{1}{\sqrt{2}}}$

$= \sqrt{288 + 9 - 12 \times 6} = \sqrt{225} = 15$

Other diagonal is $4\mathbf{a} + 5\mathbf{b}$.

Its length $= \sqrt{16 \times 8 + 25 \times 9 + 40 \times 6} = \sqrt{593}$

79. Given that, $\quad \mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{a} \cdot \mathbf{b} = 1$

and $\quad \mathbf{a} \times \mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{K}}$

As we know $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$

$\Rightarrow \quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (\hat{\mathbf{j}} - \hat{\mathbf{k}}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\sqrt{3})^2\mathbf{b}$

$\Rightarrow \quad -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} - 3\mathbf{b}$

$\Rightarrow \quad 3\mathbf{b} = 3\hat{\mathbf{i}}$

$\Rightarrow \quad \mathbf{b} = \hat{\mathbf{i}}$

80. Let unit vector is $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$

$\because a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is perpendicular to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Then, $\quad a + b + c = 0$ ...(i)

and $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}, (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ are coplanar.

$\therefore \quad \begin{vmatrix} a & b & c \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad -3a + b + c = 0$ ...(ii)

From Eqs. (i) and (ii), we get $a = 0$ and $c = -b$

$\because a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is a unit vector, then

$a^2 + b^2 + c^2 = 1$

$\Rightarrow \quad 0 + b^2 + b^2 = 1$

$\Rightarrow \quad b = \dfrac{1}{\sqrt{2}}$

$\therefore \quad a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{k}} = \dfrac{\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}$

81. $\mathbf{a}\cdot\mathbf{c} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot\hat{\mathbf{i}} = 1$

and $\mathbf{b}\cdot\mathbf{c} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})\cdot\hat{\mathbf{i}} = 1$

Now, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} = \mu\mathbf{b} + \lambda\mathbf{a}$

$\Rightarrow$ $\mu = \mathbf{c}\cdot\mathbf{a}$ and $\lambda = -\mathbf{c}\cdot\mathbf{b}$

$\Rightarrow$ $\mu = 1$ and $\lambda = -1$

$\therefore$ $\mu + \lambda = 1 - 1 = 0$

82. Given that, $|\mathbf{a}| = |\mathbf{c}| = 1, |\mathbf{b}| = 4$

Let angle between $\mathbf{b}$ and $\mathbf{c}$ is α, then

$$|\mathbf{b}\times\mathbf{c}| = \sqrt{15} \text{ (given)}$$

$\Rightarrow$ $|\mathbf{b}||\mathbf{c}|\sin\alpha = \sqrt{15}$

$\Rightarrow$ $\sin\alpha = \dfrac{\sqrt{15}}{4\times 1} = \dfrac{\sqrt{15}}{4}$

$\therefore$ $\cos\alpha = \sqrt{1-\sin^2\alpha} = \dfrac{1}{4}$

We have, $\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$

On squaring both sides, we get

$$(\mathbf{b} - 2\mathbf{c})^2 = \lambda^2(\mathbf{a})^2$$

$\Rightarrow$ $\mathbf{b}^2 + 4\mathbf{c}^2 - 4\mathbf{b}\cdot\mathbf{c} = \lambda^2\mathbf{a}^2$

$\Rightarrow$ $16 + 4 - 4|\mathbf{b}||\mathbf{c}|\cos\alpha = \lambda^2$

$\Rightarrow$ $16 + 4 - 4\times 4\times 1\times\dfrac{1}{4} = \lambda^2$

$\Rightarrow$ $\lambda^2 = 16 + 4 - 4 = 16$

$\Rightarrow$ $\lambda = \pm 4$

83. we know, $[\mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}\ \mathbf{a}\times\mathbf{b}]$

$$= (\mathbf{b}\times\mathbf{c})[(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b})]$$

$$= (\mathbf{b}\times\mathbf{c})[(\mathbf{c}\times\mathbf{a})\mathbf{b}]\mathbf{a} - [(\mathbf{c}\times\mathbf{a})\mathbf{a})\mathbf{b}]$$

$$= (\mathbf{b}\times\mathbf{c})[(\mathbf{cab})\mathbf{a} - (\mathbf{caa})\mathbf{b}]$$

$$= (\mathbf{b}\times\mathbf{c})\mathbf{a}[\mathbf{abc}] - 0$$

$$= [\mathbf{abc}][\mathbf{abc}] = [\mathbf{abc}]^2$$

84. Let the unit vector $\dfrac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$ is perpendicular to $\hat{\mathbf{i}} - \hat{\mathbf{j}}$, then we get

$$\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}})\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{2}} = \frac{1-1}{\sqrt{2}} = 0$$

$\therefore \dfrac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$ is the required unit vector.

85. We have, $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

and $\mathbf{b} = \hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}}) + \hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}}) + \hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})$

$$= 3\mathbf{a} - \mathbf{a} = 2\mathbf{a} = 2(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$\Rightarrow$ $|\mathbf{b}| = \sqrt{4 + 16 + 36} = \sqrt{56} = 2\sqrt{14}$

86. Given that, $|\mathbf{a}| = 1, |\mathbf{b}| = 3$ and $|\mathbf{c}| = 5$

$\therefore$ $[\mathbf{a} - 2\mathbf{b}\ \mathbf{b} - 3\mathbf{c}\ \mathbf{c} - 4\mathbf{a}]$

$$= (\mathbf{a} - 2\mathbf{b})\cdot\{(\mathbf{b} - 3\mathbf{c})\times(\mathbf{c} - 4\mathbf{a})\}$$

$$= (\mathbf{a} - 2\mathbf{b})\cdot\{\mathbf{b}\times\mathbf{c} - 4\mathbf{b}\times\mathbf{a} + 12\mathbf{c}\times\mathbf{a}\}$$

$$= (\mathbf{a} - 2\mathbf{b})\cdot(\mathbf{a} + 4\mathbf{c} + 12\mathbf{b})$$

$$= \mathbf{a}\cdot\mathbf{a} - 24\mathbf{b}\cdot\mathbf{b} = 1 - 24\times 9$$

$$= 1 - 216 = -215$$

87. $\because$ **a, b** and **c** are coplanar vectors, so $2\mathbf{a} - \mathbf{b}$, $2\mathbf{b} - \mathbf{c}$ and $2\mathbf{c} - \mathbf{a}$ are also coplanar. Thus,

$$[2\mathbf{a} - \mathbf{b}\ 2\mathbf{b} - \mathbf{c}\ 2\mathbf{c} - \mathbf{a}] = 0$$

88. $(\mathbf{a} - \mathbf{b})\cdot(\mathbf{b} + \mathbf{c})\times(\mathbf{c} + \mathbf{a})$

$$= (\mathbf{a} - \mathbf{b})[\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$$

$$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{a}) + \mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) - \mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) - \mathbf{b}\cdot(\mathbf{b}\times\mathbf{a}) - \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$$

$$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) - \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

$$= [\mathbf{abc}] - [\mathbf{abc}] = 0$$

89. $[\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] + [\hat{\mathbf{k}}\ \hat{\mathbf{j}}\ \hat{\mathbf{i}}] + [\hat{\mathbf{j}}\ \hat{\mathbf{k}}\ \hat{\mathbf{i}}] = [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] + [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] - [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}]$

$$= [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] = \hat{\mathbf{i}}\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{j}})$$

$$= \hat{\mathbf{i}}\cdot(-\hat{\mathbf{i}}) = -1$$

90. $[2\hat{\mathbf{i}}\ 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}] = -30[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}]$

$$= -30 \qquad (\because [\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}] = 1)$$

91. $(\mathbf{d} + \mathbf{a})\cdot[\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}]$

$$= (\mathbf{d} + \mathbf{a})\cdot[\mathbf{a}\times\{(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}\}]$$

$$= (\mathbf{b}\cdot\mathbf{d})[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{c})] - (\mathbf{b}\cdot\mathbf{c})[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{d})] + (\mathbf{b}\cdot\mathbf{d})[\mathbf{a}\cdot(\mathbf{a}\times\mathbf{c})] - (\mathbf{b}\cdot\mathbf{c})[\mathbf{a}\cdot(\mathbf{a}\times\mathbf{d})]$$

$$= (\mathbf{b}\cdot\mathbf{d})[\mathbf{d}\ \mathbf{a}\ \mathbf{c}] = (\mathbf{b}\cdot\mathbf{d})[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$$

92. Clearly, angle between $\mathbf{a}$ and $\mathbf{b} = \dfrac{\pi}{2}$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b} = 0$

$\therefore$ $|\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}$

$$= 1 + 1 + 0 = 2$$

$\Rightarrow$ $|\mathbf{a} + \mathbf{b}| = \sqrt{2}$

93. Angle between the faces *OAB* and *ABC* is same as angle between normals of the faces *OAB* and *ABC*.

Vector along the normals of *OAB*

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}} = \mathbf{a} \text{ (let)}$$

Vector along normals of *ABC*

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}} = \mathbf{b} \text{ (let)}$$

$\therefore$ $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{5 + 5 + 9}{\sqrt{35}\sqrt{35}}$

$\Rightarrow$ $\theta = \cos^{-1}\left(\dfrac{19}{35}\right)$

94. Given that, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{k}}$

Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then

$$\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a} \Rightarrow (\mathbf{r} - \mathbf{b}) \times \mathbf{a} = \mathbf{0}$$

Now, $\mathbf{r} - \mathbf{b} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - \hat{\mathbf{k}})$

$$= (x-2)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z+1)\hat{\mathbf{k}}$$

$$\therefore \quad (\mathbf{r} - \mathbf{b}) \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \quad -(z+1)\hat{\mathbf{i}} + (z+1)\hat{\mathbf{j}} + (x-2-y)\hat{\mathbf{k}} = \mathbf{0}$$

On equating the coefficient of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$z = -1, x - y = 2 \quad \text{...(i)}$$

Now, $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \Rightarrow (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

and $\mathbf{r} - \mathbf{a} = (x-1)\hat{\mathbf{i}} + (y-1)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$$\therefore \quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x-1 & y-1 & z \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \quad (-y+1)\hat{\mathbf{i}} - \hat{\mathbf{j}}(-x+1-2z) + (-2y+2)\hat{\mathbf{k}} = \mathbf{0}$$

$$\Rightarrow \quad y = 1, x + 2z = 1 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x = 3, y = 1, z = -1$$

$$\therefore \quad \mathbf{r} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

95. $\because$ **a**, **b** and **c** are pth, qth and rth terms of HP, respectively.

$$\frac{1}{a} = A + (p-1)D, \frac{1}{b} = A + (q-1)D \text{ and } \frac{1}{c} = A + (r-1)D$$

$$\therefore \quad q - r = \frac{c-b}{bcD}, r - p = \frac{a-c}{acD}$$

and $$q - r = \frac{b-a}{abD}$$

$$\Rightarrow \quad \frac{(q-r)}{a} + \frac{(r-p)}{b} + \frac{(p-q)}{c} = 0$$

$$\Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0$$

96. Given that, $\mathbf{A} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{B} = \hat{\mathbf{i}}, \mathbf{C} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$

Since, **A**, **B**, **C** are coplanar.

$$\therefore \quad [\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0$$

Now,

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = -c_3\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$$

$$\therefore \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (-c_3\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}) = 0$$

$\Rightarrow$ No value of c_1 can be found.

97. For collinearity, $\cos x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}} = \lambda(x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}})$

$$\Rightarrow \quad \cos x = x$$

Let $$f(x) = \cos x - x$$

$$\Rightarrow \quad f'(x) = -\sin x - 1 < 0$$

$f(x)$ is decreasing function and for $x \geq \frac{\pi}{3}$, $f(x) < 0$ and for

$$\frac{\pi}{3} < x < \frac{\pi}{6}, f(x) > 0.$$

Hence, unique solution exist.

98. $\because$ **p**, **q** and **r** are reciprocal vectors of **a**, **b**, **c** respectively.

$$\therefore \quad \mathbf{p} \cdot \mathbf{a} = 1, \mathbf{p} \cdot \mathbf{b} = 0 = \mathbf{p} \cdot \mathbf{c} \text{ etc.}$$

$$\therefore \quad (l\mathbf{a} + m\mathbf{b} + n\mathbf{c}) \cdot (l\mathbf{p} + m\mathbf{q} + n\mathbf{r}) = l^2 + m^2 + n^2$$

99. $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$$

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\therefore \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = -40 - 27 - 7 = 74$$

100. Since, **A B C** from a left handed system

$$\therefore \quad [\mathbf{A}, \mathbf{B}, \mathbf{C}] < 0$$

Now, $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix} = 11\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$\therefore \mathbf{C} = -11\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$, so that

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = -121 - 36 - 1 = -158 < 0$$

101. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ iff

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

i.e., iff $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = 0$

i.e., iff $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = 0$

102. Let $\mathbf{C} = \mathbf{a} + \mathbf{b}$ where **a, b** and **c** are all unit vectors

$$\therefore \quad \mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{ab}$$

$$\Rightarrow \quad 2\mathbf{a} \cdot \mathbf{b} = -1$$

Again $|\mathbf{a} - \mathbf{b}|^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b} = 1 + 1 - (-1) = 3$

$$\therefore \quad |\mathbf{a} - \mathbf{b}| = \sqrt{3}$$

103. Since, $|\mathbf{a} + \mathbf{b}| < |\mathbf{a} - \mathbf{b}|$

$$|\mathbf{a} + \mathbf{b}|^2 < |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow \quad \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} < \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b}$$

$$\Rightarrow \quad 4\mathbf{a} \cdot \mathbf{b} < 0 \Rightarrow \mathbf{a} \cdot \mathbf{b} < 0$$

$\Rightarrow \cos\theta < 0 \Rightarrow \theta$ is obtuse.

104. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a} = -5\mathbf{a} + 4\mathbf{b}$

$$\therefore \quad \mathbf{c} \cdot \mathbf{a} = 4, \mathbf{c} \cdot \mathbf{b} = 5$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$4\mathbf{b} - 3\mathbf{c}$$

105. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\dfrac{\sqrt{3}}{2}$

$$(2\mathbf{a})\cdot(3\mathbf{b}) = 6|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \cos\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

106. Given that **a** and **b** are two unit vectors

$$|\mathbf{a}| = 1 \text{and } |\mathbf{b}| = 1$$

Also, given that $(\mathbf{a}+2\mathbf{b}) \perp (5\mathbf{a}-4\mathbf{b})$

$$\Rightarrow \quad (\mathbf{a}+2\mathbf{b})\cdot(5\mathbf{a}-4\mathbf{b}) = 0$$

$$\Rightarrow \quad 6|\mathbf{a}||\mathbf{b}|\cos\theta = 3$$

(where, θ is the angle between **a** and **b**)

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$$

107. Let $\mathbf{OA} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ and $\mathbf{OB} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$, since $1 = \mathbf{OA}.\ \hat{\mathbf{i}} = x_1$ and $-2 = \mathbf{OB}.\ \hat{\mathbf{i}} = x_2$ Moreover,

$$y_1 = x_1^2 \text{and } y_2 = x_2^2 = 4, \text{so}$$

$$\mathbf{OA} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \text{and } \mathbf{OB} = -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}.$$

Hence,

$$|2\mathbf{OA} - 3\mathbf{OB}| = |8\hat{\mathbf{i}} - 10\hat{\mathbf{j}}| = \sqrt{164} = 2\sqrt{41}$$

108. $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0 \Rightarrow \mathbf{a}+\mathbf{b} = -\mathbf{c} \Rightarrow (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{c}|^2$

Thus, $|a|^2 + |b^2| + 2|a||b|\cos\theta = |c|^2$ where, θ is the angle between *a* and *b*.

Therefore, $\cos\theta = \dfrac{49-9-25}{2\cdot3\cdot5} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$

109. $(\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}}) = 8 \Rightarrow \mathbf{x} = 3$

To determine $(\mathbf{x}-\hat{\mathbf{a}})$, we have

$$(\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}}) = 9+1-4 = 6 \text{so that}$$

$$|\mathbf{x}-\hat{\mathbf{a}}| = \sqrt{6} \text{and similarly } |\mathbf{x}+\hat{\mathbf{a}}| = \sqrt{14}$$

Then, $\quad (\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}}) = \sqrt{14}\times\sqrt{6}\cos q$

$$8 = \sqrt{14}\times\sqrt{6}\cos q$$

$$\cos q = \frac{4}{\sqrt{21}}$$

110. Given, $\quad (\mathbf{a}+3\mathbf{b})\cdot(7\mathbf{a}-5\mathbf{b}) = 0$

$$\Rightarrow \quad 7\mathbf{a}^2 - 15\mathbf{b}^2 + 16\mathbf{a}\cdot\mathbf{b} = 0 \quad \text{...(i)}$$

Also, $\quad (\mathbf{a}-4\mathbf{b})\cdot(7\mathbf{a}-2\mathbf{b}) = 0$

$$\Rightarrow \quad 7\mathbf{a}^2 + 8\mathbf{b}^2 - 30\mathbf{a}\cdot\mathbf{b} = 0 \quad \text{...(ii)}$$

On subtracting $-23\mathbf{b}^2 + 46\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} = \dfrac{\mathbf{b}^2}{2}$ putting this in Eq. (i), we get

$$7\mathbf{a}^2 - 7\mathbf{b}^2 = 0 \Rightarrow |\mathbf{a}| = |\mathbf{b}|$$

Thus, $\mathbf{a}\cdot\mathbf{b} = \mathbf{ab}\cos q \Rightarrow \dfrac{\mathbf{b}^2}{2} = \mathbf{b}^2\cos q \Rightarrow \cos q = \dfrac{1}{2} \Rightarrow q = 60°$

Round II

1. Let θ be an angle between unit vectors **a** and **b**. Then, $\mathbf{a}\cdot\mathbf{b} = \cos\theta$

Now, $\quad |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}$

$$= 2 + 2\cos\theta = 4\cos^2\frac{\theta}{2}$$

$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}| = 2\cos\frac{\theta}{2} \quad |\mathbf{a}-\mathbf{b}| = 2\sin\frac{\theta}{2}$$

$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}| + |\mathbf{a}-\mathbf{b}| = 2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \le 2\sqrt{2}$$

2. We have, $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]| = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ. Then,

$$V_1 = |[\alpha\ \beta\ \gamma]|$$

Now, $\quad [\alpha\ \beta\ \gamma] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{abc}]^3$$

$$\therefore \quad V_1 = |[\alpha\ \beta\ \gamma]|$$

$$= |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3| = V^3$$

3. Let $\quad \alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \beta = \lambda\mathbf{b} + 4\mathbf{c}$

and $\quad \gamma = (2\lambda - 1)\,\mathbf{c}$

Then, $\quad [\alpha\ \beta\ \gamma] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda-1) \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$[\alpha\beta\gamma] = \lambda(2\lambda - 1)[abc]$$

$$\Rightarrow \quad [\alpha\ \beta\ \gamma] = 0,$$

If $\quad \lambda = 0, \dfrac{1}{2} \quad (\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \ne 0)$

Hence, α, β and γ are non-coplanar for all value of λ except two values 0 and $\dfrac{1}{2}\cdot$

4. We have, projection of **v** along **u** = projection of **w** along **u**

$$\Rightarrow \quad \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{u}|}$$

$$\Rightarrow \quad \mathbf{v}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u} \quad \text{...(i)}$$

Also, **v** and **w** are perpendicular to each other

$$\therefore \quad \mathbf{v}\cdot\mathbf{w} = 0 \quad \text{...(ii)}$$

Now, $\quad [\mathbf{u}-\mathbf{v}+\mathbf{w}]^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \ |\mathbf{w}|^2$

$$-2(\mathbf{u}\cdot\mathbf{v}) - 2(\mathbf{v}\cdot\mathbf{w}) + 2(\mathbf{u}\cdot\mathbf{w})$$

$\Rightarrow \quad [\mathbf{u}-\mathbf{v}+\mathbf{w}]^2 = 1+4+9$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}| = \sqrt{14}$

5. Now, $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = 16-4 = 12$

and $|\mathbf{c}|^2 = (2\mathbf{a}\times\mathbf{b}-3\mathbf{b})\cdot(2\mathbf{a}\times\mathbf{b}-3\mathbf{b})$

$= 4|\mathbf{a}\times\mathbf{b}|^2 + 9|\mathbf{b}|^2 = 4\cdot 12 + 9\cdot 16$

$= 192$

$\Rightarrow \quad |\mathbf{c}| = 8\sqrt{3}$

$\therefore \quad \mathbf{b}\cdot\mathbf{c} = \mathbf{b}\cdot(2\mathbf{a}\times\mathbf{b}-3\mathbf{b})$

$= -3|\mathbf{b}|^2 = -48$

$\because \quad \cos\theta = \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = -\dfrac{48}{4\cdot 8\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \theta = \dfrac{5\pi}{6}$

6. By expanding $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, we get

$\mathbf{a}\cdot\mathbf{c} = x^2-2x+6,\ \mathbf{a}\cdot\mathbf{b} = -\sin y$

$\because \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = 4$

$\Rightarrow \quad x^2-2x+2 = \sin y$

$\Rightarrow \quad \sin y = x^2-2x = 2+(x-1)^2+1 \geq 1$

But $\sin y \leq 1$

$\therefore$ Both sides are equal only for $x = 1$

7. The distance of the point $\mathbf{c}$ from the line $\mathbf{r} = \mathbf{a}+5\mathbf{b}$ is $\dfrac{|(\mathbf{c}-\mathbf{a})\times\mathbf{b}|}{|\mathbf{b}|}$.

Now, $(\mathbf{c}-\mathbf{a})\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & -3 & 4 \\ 6 & 1 & -2 \end{vmatrix} = 2\hat{\mathbf{i}}+14\hat{\mathbf{j}}+13\hat{\mathbf{k}}$

$\therefore \quad$ Distance $= \dfrac{\sqrt{4+196+169}}{\sqrt{36+1+4}} = \sqrt{9} = 3$

8. Now, $\mathbf{a}\cdot\mathbf{b} = 2\sqrt{2}\cdot 3\cdot\dfrac{1}{\sqrt{2}} = 6$

The diagonals are $2\mathbf{a}-3\mathbf{b}\pm(\mathbf{a}+\mathbf{b})$

$\therefore$ Length of diagonals are

$|3\mathbf{a}-2\mathbf{b}|^2 = 9\cdot 8+4\cdot 9-12\cdot 6 = 36$

and $|\mathbf{a}-4\mathbf{b}|^2 = 8+16\cdot 9-8\cdot 6 = 104$

$\therefore$ The length of the longer diagonal is $\sqrt{104}$ *i.e.*, $2\sqrt{26}$.

9. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$

$= 1-3\cos^2\theta+2\cos^3\theta$

$= (1-\cos\theta)^2(1+2\cos\theta)$

$\Rightarrow \quad 1+2\cos\theta \geq 0$

$\Rightarrow \quad \theta \leq \dfrac{2\pi}{3}$

10. Let first term and common ratio of a GP be α and β. Then,

$a = \alpha\cdot\beta^{p-1},\ b = \alpha\cdot\beta^{q-1},\ c = \alpha\cdot\beta^{r-1}$

$\therefore \quad \log a = \log\alpha + (p-1)\log\beta$, etc.

The dot product of the given two vectors is

$\Sigma\{\log\alpha + (p-1)\log\beta\}(q-r)$

$= (\log\alpha - \log\beta)\,\Sigma(q-r) + \log\beta\,\Sigma p(q-r)$

$= 0$

11. A vector coplanar with $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$= (1+\lambda)\hat{\mathbf{i}}+(1+2\lambda)\hat{\mathbf{j}}+(2+\lambda)\hat{\mathbf{k}}$

It is perpendicular to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$.

$\therefore \quad 1+\lambda+1+2\lambda+2+\lambda = 0$

$\Rightarrow \quad \lambda = -1$

$\therefore$ Required vectors is $-\hat{\mathbf{j}}+\hat{\mathbf{k}}$.

12. Here, $\mathbf{a}\cdot\mathbf{x} = 2$ and $\mathbf{a}\times\mathbf{r}+\mathbf{b} = \mathbf{r}$ ... (i)

Dot product of Eq. (i) with $\mathbf{a}$ to get, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{r} = 2$

Cross product of Eq. (i) with $\mathbf{a}$ to get,

$\mathbf{a}\times(\mathbf{a}\times\mathbf{r})+\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{r} = \mathbf{r}-\mathbf{b}$

$\Rightarrow \quad 2\mathbf{a}-\mathbf{r}+\mathbf{a}\times\mathbf{b}+\mathbf{r}-\mathbf{b}$

$\therefore \quad \mathbf{r} = \dfrac{1}{2}[2\mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}]$

13. Since, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = 0,\ \mathbf{b}\cdot\mathbf{c} = \dfrac{1}{2}$

$\therefore \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{c}| = 1$

Now, $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|^2 = |\mathbf{a}\times\mathbf{b}|^2+|\mathbf{a}\times\mathbf{c}|^2$

$-2(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$

$= 1+1-2\begin{vmatrix} 1 & 0 \\ 0 & 1/2 \end{vmatrix} = 1$

14. We know that,

$\mathbf{r} = \dfrac{(\mathbf{r}\cdot\mathbf{a})\,\mathbf{b}\times\mathbf{c}+(\mathbf{r}\cdot\mathbf{b})\,\mathbf{c}\times\mathbf{a}+(\mathbf{r}\cdot\mathbf{c})\,\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$\therefore \quad \alpha+\beta+\gamma = \dfrac{\mathbf{r}}{2}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$

15. Taking cross product of the given relation with $\mathbf{a}$ and $\mathbf{b}$ in turn, we get

$2\mathbf{a}\times\mathbf{b} = 3\mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b} = 3\mathbf{b}\times\mathbf{c}$

$\therefore \quad \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = \left(1+\dfrac{1}{3}+\dfrac{2}{3}\right)(\mathbf{a}\times\mathbf{b})$

$= 2\mathbf{a}\times\mathbf{b}$

16. If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = 0$, then $\mathbf{c} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

$\therefore \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = |\mathbf{a}\times\mathbf{b}|$

$\Rightarrow \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = |\mathbf{a}\times\mathbf{b}|^2$

17. For coplanar vectors $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

$\Rightarrow \quad (2\lambda - 1)\lambda = 0 \Rightarrow \lambda = 0, \frac{1}{2}$

18. $\mathbf{a} = [\pm(\hat{\mathbf{i}} - \hat{\mathbf{j}}) \pm (\hat{\mathbf{j}} + \hat{\mathbf{k}})] = \pm(\hat{\mathbf{i}} + \hat{\mathbf{k}}), \pm(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

19. $\mathbf{OA} = -4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \mathbf{OB} = 14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

$$\hat{\mathbf{a}} = \frac{-4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{5}; \hat{\mathbf{b}} = \frac{14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}}{15}$$

$$\mathbf{r} = \frac{\lambda}{15}[-12\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}]$$

$$\mathbf{r} = \frac{\lambda}{15}[2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}]$$

$$\mathbf{r} = \frac{2\lambda}{15}[\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}]$$

20. $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = \mathbf{0}$

i.e., $(\lambda - 1)\mathbf{a}_1 + (1 - \lambda + \mu - 2\gamma)\mathbf{a}_2 + (\mu + \gamma + 1)\mathbf{a}_3 + (\gamma + \delta)\mathbf{a}_4 = \mathbf{0}$

Since, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent.

$\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0, \gamma + y + 1 = 0$ and $\gamma + \delta = 0$

i.e., $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e., $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

21. Since, **a, b** and **c** are unit vectors inclined at an angle θ.

$$|\mathbf{a}| = |\mathbf{b}| = 1 \text{and } \cos\theta = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$$

Now, $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$ ...(i)

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{c} = \alpha(\mathbf{a}\cdot\mathbf{a}) + \beta(\mathbf{ab}) + \gamma\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\}$

$\Rightarrow \quad \cos\theta = \alpha|\mathbf{a}|^2 \quad [\because \mathbf{a}\cdot\mathbf{b} = 0, \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0]$

$\Rightarrow \quad \cos\theta = \alpha$

Similarly, by taking det roduct on both sides of Eq. (i), we get $\beta = \cos\theta$

$\therefore \quad \alpha = \beta$

Again, $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$

$\Rightarrow \quad |\mathbf{c}|^2 = |\alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})|^2$

$= \alpha^2|\mathbf{a}|^2 + \beta^2|\mathbf{b}|^2 + \gamma^2|\mathbf{a}\times\mathbf{b}|^2 + 2\alpha\beta(\mathbf{a}\cdot\mathbf{b}) + 2\alpha\gamma\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\} + 2\beta\gamma[\mathbf{b}\cdot\{\mathbf{a}\times\mathbf{b}\}]$

$\Rightarrow \quad 1 = \alpha^2 + \beta^2 + \gamma^2|\mathbf{a}\times\mathbf{b}|^2$

$\Rightarrow \quad 1 = 2\alpha^2 + \gamma^2\{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\pi/2\}$

$\Rightarrow \quad 1 = 2\alpha^2 + \gamma^2\{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\pi/2\}$

$\Rightarrow \quad 1 = 2\alpha^2 + \gamma^2 \Rightarrow 2\alpha^2 = \frac{1-\gamma^2}{2}$

But $\alpha = \beta = \cos\theta$.

$1 = 2\alpha^2 + \gamma^2 \Rightarrow \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$\therefore \quad \beta^2 = \frac{1-\gamma^2}{2} = \frac{1+\cos 2\theta}{2}$

22. We have,

AM = Projection of **b** on **a** $= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

$\therefore \quad \mathbf{AM} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Now, in ΔADM

$$\mathbf{AD} = \mathbf{AM} + \mathbf{MD} \Rightarrow \mathbf{DM} = \mathbf{AM} - \mathbf{AD}$$

$\Rightarrow \quad \mathbf{DM} = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{a}}{|\mathbf{a}|^2} - \mathbf{b}$

Also, $\mathbf{DM} = \frac{1}{|\mathbf{a}|^2}[(\mathbf{a}\cdot\mathbf{b})\mathbf{a} - 1|\mathbf{a}|^2\mathbf{b}]$

$\Rightarrow \quad \mathbf{MD} = \frac{1}{|\mathbf{a}|^2}|\mathbf{a}|^2\mathbf{b} - (\mathbf{ab})\mathbf{a}]$

Now, $\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}|^2} = \frac{1}{|\mathbf{a}|^2}[(\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b}] = \mathbf{DM}$

23. $a_1 + a_2\cos 2x + a_3\sin x = 0, \forall x \in R$

$\Rightarrow \quad (a_1 + a_2) + \sin^2 x, \forall (a_3 - 2a_2) = 0$

$\Rightarrow \quad a_1 + a_2 = 0$ and $a_3 - 2a_2 = 0$

$$\frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda\ (\neq 0)$$

$\Rightarrow \quad a_1 = -\lambda, a_2 = \lambda, a_3 = 2\lambda$

24. $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$\Rightarrow \quad \sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ ...(i)

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \Rightarrow \cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ ...(ii)

From Eqs. (i) and (ii),

$$\sin^2\theta + \cos^2\theta = 1$$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$

If $\theta = \frac{\pi}{4}$, then $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

Therefore, $|\mathbf{a}\times\mathbf{b}| = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}$ and $\mathbf{a}\cdot\mathbf{b} = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}$

$$|\mathbf{a}\times\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}$$

$\Rightarrow \quad \mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}\hat{\mathbf{n}}$

$\Rightarrow \quad \mathbf{a}\times\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\hat{\mathbf{n}}$

25. Since, **a**, **b** and $\mathbf{a}\times\mathbf{b}$ are non-coplanar,

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b}) \therefore \mathbf{r}\times\mathbf{b} = \mathbf{a}$$

$\Rightarrow \quad x\mathbf{a}\times\mathbf{b} + z\{(\mathbf{a}\cdot\mathbf{b})\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\mathbf{a}\} = \mathbf{a}$

$\Rightarrow \quad -(1 + z|\mathbf{b}|^2)\mathbf{a} + x\mathbf{a}\times\mathbf{b} = 0$ (since, $\mathbf{a}\cdot\mathbf{b} = 0$)

$\therefore \quad x = 0$ and $z = -\frac{1}{|\mathbf{b}|^2}$

Thus, $\mathbf{r} = y\mathbf{b} - \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{b}|^2}$, where y is the parameter.

26. Obviously, $\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$ is a vector in the plane of **a** and **b** and hence, perpendicular to $\mathbf{a}\times\mathbf{b}$. It is also equally inclined to **a** and **b** as it is along the angle bisector.

Solutions (Q. Nos. 27-29)

Given that, $|\mathbf{x}|=|\mathbf{y}|=|\mathbf{z}|=\sqrt{2}$ and they are inclined at an angle of 60° with each other.

$\therefore \quad \mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{z}=\mathbf{z}\cdot\mathbf{x}=\sqrt{2}\times\sqrt{2}\cos 60°=1$

$$\mathbf{x}\times(\mathbf{y}\times\mathbf{z})=\mathbf{a}\Rightarrow(\mathbf{x}\cdot\mathbf{z})\,\mathbf{y}-(\mathbf{x}\cdot\mathbf{y})\mathbf{z}=\mathbf{a}$$

$\Rightarrow \quad \mathbf{y}-\mathbf{z}=\mathbf{a}$ ...(i)

Similarly, $\mathbf{y}\times(\mathbf{z}\times\mathbf{x})=\mathbf{b}\Rightarrow\mathbf{z}-\mathbf{x}=\mathbf{b}$ ...(ii)

$\mathbf{y}=\mathbf{a}+\mathbf{z},\ \mathbf{x}=\mathbf{z}-\mathbf{b}$ [from Eqs. (i) and (ii)] ...(iii)

Now, $\mathbf{x}\times\mathbf{y}=\mathbf{c}$

$\Rightarrow \quad (\mathbf{z}-\mathbf{b})\times(\mathbf{z}+\mathbf{a})=\mathbf{c}$

$\Rightarrow \quad \mathbf{z}\times\mathbf{a}-\mathbf{b}\times\mathbf{z}-\mathbf{b}\times\mathbf{a}=\mathbf{c}$

$\Rightarrow \quad \mathbf{z}\times(\mathbf{a}+\mathbf{b})=\mathbf{c}+(\mathbf{b}\times\mathbf{a})$ ...(iv)

$\Rightarrow \quad (\mathbf{a}+\mathbf{b})\times\{\mathbf{z}\times(\mathbf{a}+\mathbf{b})\}$

$=(\mathbf{a}+\mathbf{b})\times\mathbf{c}+(\mathbf{a}+\mathbf{b})\times(\mathbf{b}\times\mathbf{a})$

$\Rightarrow \quad (\mathbf{a}+\mathbf{b})^2\,\mathbf{z}-\{(\mathbf{a}+\mathbf{b})\cdot\mathbf{z}\}\,(\mathbf{a}+\mathbf{b})$

$=(\mathbf{a}+\mathbf{b})\times\mathbf{c}+|\mathbf{a}|^2\,\mathbf{b}-|\mathbf{b}|^2\,\mathbf{a}+(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}-\mathbf{a})$ ...(v)

Now, Eq. (i) $|\mathbf{a}|^2=|\mathbf{y}-\mathbf{z}|^2=2+2-2=2$

Similarly, Eq. (ii) $\Rightarrow|\mathbf{b}|^2=2$

Also, Eqs. (i) and (ii),

$\Rightarrow \quad \mathbf{a}+\mathbf{b}=\mathbf{y}-\mathbf{x}\Rightarrow|\mathbf{a}+\mathbf{b}|^2=2$ ...(vi)

Also, $(\mathbf{a}+\mathbf{b})\cdot\mathbf{z}=(\mathbf{y}-\mathbf{x})\cdot\mathbf{z}=\mathbf{y}\cdot\mathbf{z}-\mathbf{x}\cdot\mathbf{z}$

$=1-1=0$

and $\mathbf{a}\cdot\mathbf{b}=\mathbf{y}\cdot\mathbf{x}-|\mathbf{z}|^2+\mathbf{z}\cdot\mathbf{x}=-1$

Thus, from Eq. (v), we have

$$2\mathbf{z}=(\mathbf{a}+\mathbf{b})\times\mathbf{c}+2(\mathbf{b}-\mathbf{a})-(\mathbf{b}-\mathbf{a})$$

or $\mathbf{z}=(1/2)\,[(\mathbf{a}+\mathbf{b})\times\mathbf{c}+\mathbf{b}-\mathbf{a}]$

$\therefore \quad \mathbf{y}=\mathbf{a}+\mathbf{z}=\left(\frac{1}{2}\right)[(\mathbf{a}+\mathbf{b})\times\mathbf{c}+\mathbf{b}+\mathbf{a}]$ and

$$\mathbf{x}=\mathbf{z}-\mathbf{b}=\left(\frac{1}{2}\right)[(\mathbf{a}+\mathbf{b})\times\mathbf{c}-(\mathbf{a}+\mathbf{b})]$$

30. The vector equations of *CD* and *BE* are

$$\mathbf{r}_1=(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}})+\frac{\lambda}{3}(7\hat{\mathbf{j}}-7\hat{\mathbf{k}}) \quad ...(i)$$

and $$\mathbf{r}_2=(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+\frac{\mu}{3}(7\hat{\mathbf{i}}-7\hat{\mathbf{j}}+7\hat{\mathbf{k}}) \quad ...(ii)$$

The intersection point of **CD** and **BE**.

i.e., $1=-1+\frac{7\mu}{3},\ -2+\frac{7\lambda}{3}=1-\frac{7\mu}{3}$

and $4-\frac{7\lambda}{3}=1+\frac{7\mu}{3}$

$\Rightarrow \quad \mu=\frac{6}{7}$ and $\lambda=\frac{3}{7}$

On substituting the value of λ in Eq. (i), we get

$$\mathbf{r}_1=\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

31. Area of $\Delta ABC=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$=\frac{1}{2}|(-3\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})\times(-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+2\hat{\mathbf{k}})|$

$=\frac{7\sqrt{2}}{2}$ sq units

$\therefore$ Volume of the tetrahedron *ABCF*

$=\frac{1}{3}$(Area of the base) × Height

$=\frac{1}{3}\cdot\frac{7\sqrt{2}}{2}\cdot\sqrt{2}=\frac{7}{3}$ cu units

32. A vector along the bisector is $\frac{\mathbf{a}}{|\mathbf{b}|}+\frac{\mathbf{b}}{|\mathbf{b}|}=\frac{-5\hat{\mathbf{i}}+7\hat{\mathbf{j}}\ \ 2\hat{\mathbf{k}}}{9}$

Hence, $\mathbf{c}=-5\hat{\mathbf{i}}+7\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ is along the bisector. It is obvious that **c** makes an equal angle with **a** and **b**.

However, statement II does not explain statement I, as a vector inclined to given two vectors is not necessarily coplanar.

33. Component of vector $\mathbf{b}=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ in the direction of $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $\frac{\mathbf{a}\cdot\mathbf{ba}}{|\mathbf{a}||\mathbf{a}|}$ or $3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. Then, component in the direction perpendicular to the direction of $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $\mathbf{b}-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$

34. $\mathbf{AD}=2\hat{\mathbf{j}}-\mathbf{k},\ \mathbf{BD}=-2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

and $\mathbf{CD}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}$

Volume of tetrahedron is

$$\frac{1}{6}[\mathbf{AD}\ \mathbf{BD}\ \mathbf{CD}]=\frac{1}{6}\begin{Vmatrix}0 & 2 & -1\\ -2 & -1 & -3\\ 2 & -1 & 0\end{Vmatrix}=\frac{8}{3}$$

Also, the area of the ΔABC is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|=\frac{1}{2}\begin{Vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 3 & 2\\ -2 & 3 & -1\end{Vmatrix}=\frac{1}{2}|-9\hat{\mathbf{i}}-2\hat{\mathbf{j}}+12\hat{\mathbf{k}}|=\frac{\sqrt{129}}{2}$$

Then, $\frac{8}{2} = \frac{1}{3} \times$ (Distance of D from base ABC) $\times$ (Area of ΔABC)

Distance of D from base $ABC = \frac{16}{\sqrt{129}}$

35. $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{c} = 0$ only if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

Hence, statement II is true.

Also, $[\mathbf{a} - \mathbf{b}\ \mathbf{b} - \mathbf{c}\ \mathbf{c} - \mathbf{a}] = 0$ even if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$.

Hence, statement II is not the correct explanation for statement I.

36. Let the three given unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$. Since, they are mutually perpendicular, $\hat{\mathbf{a}} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = 1$

Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

Hence, $a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$, $a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$ and $a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$ may be mutually perpendicular.

37. We know that, the sum of three vectors of a triangle is zero.

$\therefore \quad \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$

$\Rightarrow \quad \mathbf{BC} = \mathbf{AC} - \mathbf{AB}$

$\Rightarrow \quad \mathbf{BM} = \frac{\mathbf{AC} - \mathbf{AB}}{2}$

($\because$ M is a mid-point of BC)

Also, $\mathbf{AB} + \mathbf{BM} + \mathbf{MA} = 0$ (by properties of a triangle)

$\Rightarrow \quad \mathbf{AB} + \frac{\mathbf{AC} - \mathbf{AB}}{2} = \mathbf{AM}$

$\Rightarrow \mathbf{AM} = \frac{\mathbf{AB} + \mathbf{AC}}{2} = \frac{3\hat{\mathbf{i}} + 4\hat{\mathbf{k}} + 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{2} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{AM}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

38. Given that

(i) $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, *i.e.*, $|\mathbf{a}| = |\mathbf{b}| = 1$

(ii) $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{b} = 5\mathbf{a} - 4\mathbf{b}$

(iii) $\mathbf{c}$ and $\mathbf{d}$ are perpendicular to each other, *i.e.*, $\mathbf{c} \cdot \mathbf{d} = 0$

To find angle between $\mathbf{a}$ and $\mathbf{b}$.

Now, $\quad \mathbf{c} \cdot \mathbf{d} = 0$

$\Rightarrow \quad (\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow \quad 5\mathbf{a} \cdot \mathbf{a} - 4\mathbf{a} \cdot \mathbf{b} + 10\mathbf{b} \cdot \mathbf{a} - 8\mathbf{b} \cdot \mathbf{b} = 0$

$\Rightarrow \quad 6\mathbf{a} \cdot \mathbf{b} = 3$

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{2}$

$\therefore$ Angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$.

39. Given,

(i) A parallelogram $ABCD$ such that $\mathbf{AB} = q$ and $\mathbf{AD} = p$.

(ii) The altitude from vertex B to side AD coincides with a vector $\mathbf{r}$.

To find the vector $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Let E be the foot of perpendicular from B to side AD.

AE = Projection of vector $\mathbf{q}$ on

$\mathbf{AE}$ = Vector along AE of length AE

$$= |\mathbf{AE}|\ \mathbf{AE} = \left(\frac{\mathbf{q} \cdot \mathbf{p}}{|\mathbf{p}|}\right)\mathbf{p} = \frac{(\mathbf{q} \cdot \mathbf{p})\ \mathbf{p}}{|\mathbf{p}|^2}$$

Now, applying triangles law in ΔABE, we get

$\Rightarrow \quad \mathbf{AB} + \mathbf{BE} = \mathbf{AE}$

$\Rightarrow \quad q + r = \frac{(\mathbf{q} \cdot \mathbf{p})\ \mathbf{p}}{|\mathbf{p}|^2}$

$\Rightarrow \quad \mathbf{a} = \frac{(\mathbf{q} \cdot \mathbf{p})\ \mathbf{p}}{|\mathbf{p}|^2} - \mathbf{q}$

$\therefore \quad \mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{q} \cdot \mathbf{p}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$

40. $\mathbf{a} = \frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{k}})$ and $\mathbf{b} = \frac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$

$\therefore \quad (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})\}$

$\Rightarrow \quad (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \times 2\mathbf{b}\}$

$\Rightarrow \quad (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \cdot \mathbf{a})\ \mathbf{b} + (\mathbf{b} \cdot \mathbf{a})\mathbf{a} + 2(\mathbf{a} \cdot \mathbf{b})\ \mathbf{b} - 2\ (\mathbf{b} \cdot \mathbf{b})\ \mathbf{a}\}$

$\Rightarrow \quad (2\mathbf{a} - \mathbf{b}) \cdot \{1(\mathbf{b}) - (0)\ \mathbf{a} + 2(0)\ \mathbf{b} - 2(1)\ \mathbf{a}\}$

$\{\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1\}$

$\Rightarrow \quad (2\mathbf{a} - \mathbf{b})\ (\mathbf{b} - 2\mathbf{a})$

$\Rightarrow \quad -(4|\mathbf{a}|^2 + 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2)$

$\Rightarrow \quad \{4 + 0 + 1\} = -5$

41. Given, $\quad \mathbf{a} \cdot \mathbf{b} \neq 0,\ \mathbf{a} \cdot \mathbf{b} = 0 \quad$ ...(i)

and $\quad \mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow \quad \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = 0$

$\therefore \quad \mathbf{b} \,||\, (\mathbf{c} - \mathbf{d})$

$\Rightarrow \quad \mathbf{c} - \mathbf{d} = \lambda\,\mathbf{b}$

$\Rightarrow \quad \mathbf{d} = \mathbf{c} - \lambda\,\mathbf{d} \quad$ ...(ii)

Taking dot product with **a**.

$$\mathbf{a}\cdot\mathbf{d} = \mathbf{a}\cdot\mathbf{c} - \lambda\mathbf{a}\cdot\mathbf{b} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 0 = \mathbf{a}\cdot\mathbf{c} - \lambda(\mathbf{a}\cdot\mathbf{b})$$

$$\therefore \quad \lambda = \frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}} \quad \text{...(iii)}$$

$$\therefore \quad \mathbf{d} = \mathbf{c} - \frac{(\mathbf{a}\cdot\mathbf{c})}{(\mathbf{a}\cdot\mathbf{b})}\mathbf{b}$$

42. As $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$

$$\Rightarrow \quad \mathbf{a} + 3\mathbf{b} = \lambda\mathbf{c} \quad \text{...(i)}$$

Also, $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$.

$$\Rightarrow \quad \mathbf{b} + 2\mathbf{c} = \mu\,\mathbf{a} \quad \text{...(ii)}$$

From Eq. (i),

$$\mathbf{a} + 3\,\mathbf{b} + 6\mathbf{c} = (\lambda + 6)\,\mathbf{c} \quad \text{...(iii)}$$

From Eq. (ii),

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\mu)\,\mathbf{a} \quad \text{...(iv)}$$

From Eqs. (iii) and (v), we get

$$(\lambda + 6)\,\mathbf{c} = (1 + 3\mu)\,\mathbf{a}$$

Since, $\mathbf{a}$ is not collinear with $\mathbf{a}$

$$\Rightarrow \quad \lambda + 6 = 1 + 3\mu = 0$$

$\therefore$ From Eq. (iv),

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = 0$$

43. Given, $\mathbf{a} = p\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + q\hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + r\hat{\mathbf{k}}$ are coplanar and $(p \neq q \neq r \neq 1)$

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

$$\Rightarrow \quad \begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$$\Rightarrow \quad p(q\,r - 1) - 1(r - 1) + 1(1 - q) = 0$$

$$\Rightarrow \quad pqr - p - r + 1 + 1 - q = 0$$

$$\therefore \quad pqr - (p + q + r) = -2$$

44. We have, $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$

$$\Rightarrow \quad \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{c} = 0$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\,\mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$$

$$\Rightarrow \quad 3\mathbf{a} - 2\mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$$

$$\Rightarrow \quad 2\mathbf{b} = 3\mathbf{a} + \mathbf{a} \times \mathbf{c}$$

$$\Rightarrow \quad 2\mathbf{b} = 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}} - 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

$$\therefore \quad \mathbf{b} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

45. Since, the given vectors are mutually orthogonal, therefore

$$\mathbf{a}\cdot\mathbf{b} = 2 - 4 + 2 = 0$$

$$\mathbf{a}\cdot\mathbf{c} = \lambda - 1 + 2\mu = 0 \quad \text{...(i)}$$

$$\mathbf{b}\cdot\mathbf{c} = 2\lambda + 4 + \mu = 0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\mu = 2 \text{ and } \lambda = -3$$

Hence, $(\lambda, \mu) = (-3, 2)$

35 Three Dimensional Geometry

JEE Main MILESTONE

- Direction Cosines and Direction Ratios
- Plane
- Equation of Plane in Different Form
- Angle between Two Planes
- Distance of a Point from a Plane
- Distance between Two Parallel Planes
- Important Facts Related to Plane
- Equation of Plane Bisecting the Angle between Two Planes
- Straight Line
- Equation of Straight Line in Different Form
- Angle between the Two Lines
- Projection of a Line Segment on a Given Line
- Skew Lines
- Line and Plane
- Area of a Triangle

Geometry Arose as the field of knowledge dealing with spatial relationships. Geometry was one of the two fields of pre-modern ***mathematics****, the other being the study of numbers* ***(arithmetic)***

In Mathematics, analytic geometry (Cartesian geometry) describes every point in three dimensional space by means of three coordinates. Three coordinate axes are given, usually each perpendicular to the other two at the origin, the point at which they cross. they are usually labeled x, y and z. Relative to these axes, the position of any point in three dimensional space is given by an ordered triple of real numbers, each number giving the distance of that point from the origin measured along the given axes, which is equal to the distance of that point from the plane determined by the other two axes.

35.1 Direction Cosines and Direction Ratios

Direction Cosines

If α, β and γ are the angles which a directed line segment OP makes with the positive directions of the coordinate axes OX, OY and OZ respectively, then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are known as the direction cosines of OP and are generally denoted by the letters l, m and n respectively, *i.e.*,

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

Properties of Direction Cosines

(i) If OP is a directed line segment with direction cosines l, m and n such that $OP = r$. Then, the coordinates of P are $(lr, mr$ and $nr)$.

(ii) Sum of squares of direction cosines are always unity *i.e.*, $l^2 + m^2 + n^2 = 1$.

(iii) Parallel lines have same direction cosines.

(iv) Direction cosines of a line are always unique.

(v) $0 \le \alpha, \beta, \gamma \le \pi$.

Direction Ratios

Let l, m and n be direction cosines of a line and a, b and c be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$. Then, direction ratios of the line are proportional to a, b and c.

Relation between Direction Cosines and Direction Ratios

If the direction ratios of a line are proportional to a, b and c, then its direction cosines are

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Note

- The direction cosines of a line are defined as the direction cosines of any directed line segment on it.
- Direction ratios of a line are not unique.
- The direction ratios of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ and direction cosines are $\lambda(x_2 - x_1)$, $\lambda(y_2 - y_1)$ and $\lambda(z_2 - z_1)$.
- If $\mathbf{r} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ be a vector having direction cosines l, m and n. Then,

$$l = \frac{a}{|\mathbf{r}|}, \quad m = \frac{b}{|\mathbf{r}|}$$

and $$n = \frac{c}{|\mathbf{r}|}$$

- If direction ratios of $\mathbf{r}$ are a, b, and c, then

$$\mathbf{r} = \frac{|\mathbf{r}|}{\sqrt{a^2 + b^2 + c^2}} (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$$

Sample Problem 1 *The direction cosines of a line which makes equal angles with the coordinate axes are* [NCERT]

(a) $< \frac{1}{3}, \frac{1}{3}, \frac{1}{3} >$ (b) $< \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3} >$

(c) $< \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} >$ (d) $< \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3} >$

Interpret (c) If the direction cosines of a line are l, m and n, then use the relation $l^2 + m^2 + n^2 = 1$ and simplify it.

Let the line makes an angle α with each of the three coordinate axes, then its direction cosines are

$$l = \cos\alpha, m = \cos\alpha, n = \cos\alpha$$

We know that, $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \quad (\because l = m = n = \cos\alpha)$

$\Rightarrow \quad 3\cos^2\alpha = 1$

$\Rightarrow \quad \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \quad \cos\alpha = \pm \frac{1}{\sqrt{3}}$

$\therefore$ Direction cosines of the line are either

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

Sample Problem 2 *If a line has the direction ratios* $-18, 12, -4$ *then, its direction cosines are* [NCERT]

(a) $\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$ (b) $\frac{-9}{13}, \frac{6}{13}, \frac{-2}{13}$

(c) $\frac{9}{13}, \frac{-6}{13}, \frac{-2}{13}$ (d) *None of these*

Interpret (a) If a, b and c are the direction ratios of a line, then direction cosines are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Given, direction ratios are -18, 12, -4.

Here, $a = -18$, $b = 12$ and $c = -4$, then direction cosines of a line are

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)$$

$$= \left(\frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}\right)$$

$$= \left(\frac{-18}{\sqrt{484}}, \frac{12}{\sqrt{484}}, \frac{-4}{\sqrt{484}}\right)$$

$$= \left(\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}\right)$$

$$= \left(-\frac{9}{11}, \frac{6}{11}, \frac{-2}{11}\right)$$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11}$ and $\frac{-2}{11}$.

Angle between Two Lines

If two lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2), then angle θ between them is given by

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

If direction ratios are given (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively, then

Now, if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, lines are perpendicular and if $l_1 = l_2$, $m_1 = m_2$ and $n_1 = n_2$, lines are parallel.

Similarly, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, then lines are perpendicular and if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then lines are parallel.

35.2 Plane

A plane is a surface such that, if any two points are taken on it, the line segment joining them lies completely on the surface.

35.3 Equation of Plane in Different Form

(i) **General Equation of a Plane** The general equation of a plane is $ax + by + cz + d = 0$,

$$a^2 + b^2 + c^2 \neq 0$$

where a, b and c are the DR's of the normal to the plane.

(a) Plane through the origin is $ax + by + cz = 0$

(b) Planes parallel to the coordinate planes (perpendicular to coordinate axes)

$x = k$ parallel to YOZ plane

$y = k$ parallel to ZOX plane

$z = k$ parallel to XOY plane

(c) Planes parallel to coordinate axes.

$by + cz + d = 0$ parallel to x-axis

$ax + cz + d = 0$ parallel to y-axis

$ax + by + d = 0$ parallel to z-axis

(ii) **Equation of a Plane Passing Through a Given Point** The general equation of a plane passing through a given point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are direction ratios of a line perpendicular to plane.

(iii) **Intercept Form of the Equation of a Plane** The equation of a plane whose intercepts are a, b, and c on the x-axis, y-axis and z-axis respectively, is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

In Vector Form If, $\mathbf{r} \cdot \vec{\eta} = q$ be the vector equation of the plane, where η is the normal vector having $(a\ b\ c)$ as direction ratios, then the vector along normal vector is $\vec{\eta} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, such that

$$\mathbf{r} \cdot (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}) = q$$

$$\Rightarrow \quad (\mathbf{r} \cdot \hat{\mathbf{i}})a + (\mathbf{r} \cdot \hat{\mathbf{j}})b + (\mathbf{r} \cdot \hat{\mathbf{k}})c = q$$

Now, intercept along x (put components in $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ zero) $= \left(\frac{q}{r \cdot \hat{\mathbf{i}}}\right)$

Similarly, intercept along y (put components in $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ zero) $= \left(\frac{q}{r \cdot \hat{\mathbf{j}}}\right)$ and

intercept along z (put components in $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ zero $= \left(\frac{q}{r \cdot \hat{\mathbf{k}}}\right)$

Cartesian Form If $ax + by + cz = d$ is any parametric l cartesian equation of the plane, then the intercepts which this plane makes with axes are intercept along x-axis (put $y = z = 0$) $= \frac{d}{a}$

Similarly, intercept along y-axis (put $x = z = 0$) $= \frac{d}{b}$

Intercept along z-axis (put $x = y = 0$) $= \frac{d}{c}$

(iv) **Equation of Plane in Normal Form** The equation of a plane in normal form is $lx + my + nz = p$ where l, m and n are direction cosines of a line normal to plane and p is perpendicular distance of the plane from origin.

The vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ at a distance d from the origin is $\mathbf{r} \cdot \hat{\mathbf{n}} = d$.

Note

- The vector equation of a plane passing through a point having position vector $\mathbf{a}$ and normal to vector $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

(v) **Equation of a Plane Passing Through Three Non-collinear Points** Equation of a plane passing through the three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Let A, B and C be the three given points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and P be a point in the plane with position vector $\mathbf{r}$.

Then, equation of plane is

$$[\mathbf{r} - \mathbf{a}\ \ \mathbf{b} - \mathbf{a}\ \ \mathbf{c} - \mathbf{a}] = 0$$

or $\quad [\mathrm{r\ b\ c}] + [\mathrm{r\ a\ b}] + [\mathrm{r\ c\ a}] = [\mathrm{a\ b\ c}]$

(vi) **Equation of Plane Passing Through a Point and Parallel to Two Vectors** Equation of a plane passing through a point A whose position vector is $\mathbf{a}$ and parallel to two vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Note Equation of a plane which passes through the point (x_1, y_1, z_1) and parallel to lines whose DR's are $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0$$

Sample **Problem 3** *The cartesian equations of the plane which passes through the point* (5, 2, – 4) *and perpendicular to the line with direction ratios* 2, 3, –1 *is*

(*a*) $2x + 2y - z = 20$ (*b*) $2x + y + z = 10$

(*c*) $2x - 3y + z = 10$ (*d*) *None of these*

Interpret (*d*) We have, the position vector of point (5, 2 – 4) say $\mathbf{a} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and the normal vector $\mathbf{n}$ perpendicular to the plane as $\mathbf{n} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Therefore, the vector equation of the plane is given by

$$(\mathbf{r} - \mathbf{a}).\ \mathbf{n} = 0$$

or $$[\mathbf{r} - (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}).(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}] = 0 \quad \text{...(i)}$$

Transforming Eq. (i) into cartesian form, we have

$$[(x-5)\hat{\mathbf{i}} + (y-2)\hat{\mathbf{j}} + (z+4)\hat{\mathbf{k}}]\cdot(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$$

or $$2(x-5) + 3(y-2) - 1(z+4) = 0$$

i.e., $$2x + 3y - z = 20$$

which is the cartesian equation of plane.

Sample **Problem 4** *The equation of the plane with intercepts* 2, 3 *and* 4 *on the x, y and z-axis repectively, is*

(*a*) $2x + 3y + 4z = 12$ (*b*) $6x + 4y + 3z = 12$

(*c*) $x + 3y + 2z = 6$ (*d*) *None of these*

Interpret (*b*) Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

Here, $a = 2, b = 3, c = 4$

substituting the values of *a*, *b* and *c* in Eq. (i), we get the required equation of the plane as $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

$\Rightarrow$ $$6x + 4y + 3z = 12$$

Sample **Problem 5** *The cartesian equation of the plane which is at a distance of* $\frac{6}{\sqrt{29}}$ *form the origin and the normal vector from the origin is* $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ *is*

(*a*) $3x - 2y + 4z = 3$ (*b*) $2x - 3y + 4z = 6$

(*c*) $x - 3y + 2z = 6$ (*d*) *None of these*

Interpret (*b*) Let $n = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. Then,

$$\mathbf{n} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{4+9+16}} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{29}}$$

Hence, the required equation of the plane is

$$\mathbf{r}\cdot\left(\frac{2}{\sqrt{29}}\hat{\mathbf{i}} + \frac{-3}{\sqrt{29}}\hat{\mathbf{j}} + \frac{4}{\sqrt{20}}\hat{\mathbf{k}}\right) = \frac{6}{\sqrt{29}} \quad \text{...(i)}$$

On putting $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ in Eq. (i),

we get $$\frac{2x}{\sqrt{29}} - \frac{3y}{\sqrt{29}} + \frac{4z}{\sqrt{29}} = \frac{6}{\sqrt{29}}$$

$\Rightarrow$ $$2x - 3y + 4z = 6$$

Sample **Problem 6** *The coordinates of the foot of the perpendicular drawn from the origin to the plane* $2x - 3y + 4z - 6 = 0$ *is*

(*a*) $\left(\frac{12}{\sqrt{29}}, \frac{-18}{\sqrt{29}}, \frac{24}{\sqrt{29}}\right)$ (*b*) $\left(\frac{18}{\sqrt{29}}, \frac{-12}{\sqrt{29}}, \frac{24}{\sqrt{29}}\right)$

(*c*) $\left(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29}\right)$ (*d*) $\left(\frac{12}{11}, \frac{-18}{11}, \frac{24}{11}\right)$

Interpret (*c*) Let the coordinates of the foot of the perpendicular *P* from the origin to the plane is (x_1, y_1, z_1) figure, then the direction rations of the line *OP* are x_1, y_1, z_1.

Writing the equation of the plane in the normal form, we have

$$\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

where, $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}$ and $\frac{4}{\sqrt{29}}$ are the direction cosines of the *OP*.

Since, DC's direction ratio of a line are proportional, we have

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$$

i.e., $$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$$

On substituting these in the equation of the plane, we get $k = \frac{6}{\sqrt{29}}$

Hence, the foot of the perpendicular is $\left(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29}\right)$.

Sample **Problem 7** *The equation of the plane passing through the points* (2 , 5, – 3), (–2, – 3, 5) *and* (5, 3, – 3) *is,*

(*a*) $2x + 3y + 4z = 14$ (*b*) $2x + 3y + 4z = 18$

(*c*) $2x - 3y + 4z = 14$ (*d*) $2x + 3y - 4z = 18$

Interpret (*d*) The equation of the plane passing through the points (2 , 5, – 3), (–2, – 3, 5) and (5, 3, – 3) is,

$$\begin{vmatrix} x-2 & y+2 & z-5 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & y+2 & z-5 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(+16) - (y+2)(-24) + (z-5)(8+24) = 0$$

$$\Rightarrow \quad 16x - 32 + 24y + 48 + 32z - 160 = 0$$

$$\Rightarrow \quad 16x + 24y + 32z - 144 = 0$$

$$\Rightarrow \quad 2x + 3y + 4z = 18$$

35.4 Angle between Two Planes

Angle between planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

where, θ is acute angle between the planes.

The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors.

Let θ be the angle between the planes $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$, then

$$\cos\theta = \frac{\mathbf{n}_2 \cdot \mathbf{n}_2}{|\mathbf{n}_2||\mathbf{n}_2|}$$

Now, two planes are perpendicular, if

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \quad \text{or} \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

and parallel, if $\dfrac{\mathbf{n}_1}{\mathbf{n}_2} = \lambda$

or $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Sample **Problem 8** *The angle between the two planes* $3x - 6y + 2z = 7$ *and* $2x + 2y - 2z = 5$ *is*

(*a*) $\dfrac{\pi}{3}$ (*b*) $\dfrac{\pi}{6}$

(*c*) $\cos^{-1}\left(\dfrac{5\sqrt{3}}{21}\right)$ (*d*) $\cos^{-1}\left(\dfrac{5\sqrt{3}}{7}\right)$

Interpret (*c*) Comparing the given equations of the planes with the equations

$$A_1x + B_1y + C_1z + D_1 = 0$$

and $$A_2x + B_2y + C_2z + D_2 = 0,$$

we get $$A_1 = 3,\ B_1 = -6,\ C_1 = 2$$

$$A_2 = 2,\ B_2 = 2,\ C_2 = -2$$

$$\cos\theta = \left| \frac{3 \times 2 + (-6)(2) + 2(-2)}{\sqrt{\left((3^3 + (-6)^2 + (-2)^2\sqrt{2^2 + 2^2 + (-2^2}\right)}} \right|$$

$$= \left| \frac{-10}{7 \times 2\sqrt{3}} \right| = \frac{5}{7\sqrt{3}} = \frac{5\sqrt{3}}{21}$$

Therefore, $\theta = \cos^{-1}\left(\dfrac{5\sqrt{3}}{21}\right)$

35.5 Distance of a Point from a Plane

The distance of a point $P(x_1, y_1, z_1)$ from a plane $ax + by + cz + d = 0$ is given by

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

The distance of a point $P(\mathbf{a})$ from the plane $\mathbf{r}\cdot\mathbf{n} = q$ is given by

$$\left| \frac{q - \mathbf{a}\cdot\hat{\mathbf{n}}}{|\hat{\mathbf{n}}|} \right|$$

35.6 Distance between Two Parallel Planes

The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$d = \left| \frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Sample **Problem 9** *The distance of a point* (2,5,–3) *from the plane* $6x - 3y + 2z = 4$ *or* $\mathbf{r}.(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 4$ *is*

(*a*) $\dfrac{7}{13}$ (*b*) $\dfrac{4}{13}$ (*c*) $\dfrac{13}{7}$ (*d*) $\dfrac{15}{7}$

Interpret (c) **First method** Here, $\mathbf{a} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $N = 6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $d = 4$

Therefore, the distance of the point (2,5,–3) from the given plane is

$$\frac{|(2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}}).(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - 4|}{|6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|} = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

Second method The cartesian equation of the plane is $6x - 3y + 2z = 4$

$\Rightarrow$ The cartesian equation of the plane is $6x - 3y + 2z - 9 = 0$

Therefore, the distance of the point (2, 5, –3) from the given plane is $\left|\frac{6 \times 2 - 3 \times 5 + 2(-3) - 4}{\sqrt{36 + 9 + 4}}\right| = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$

Sample **Problem 10** *The distance between the parallel planes*

$$x + 2y - 2z + 1 = 0 \text{ and } 2x + 4y - 4z + 5 = 0 \text{ is}$$

(a) 1 (b) 1/2
(c) 2 (d) 0

Interpret (b) Distance between $x + 2y - 2z + 1 = 0$

and $x + 2y - 2z + \frac{5}{2} = 0$ is $\left|\frac{\frac{5}{2} - 1}{\sqrt{1 + 4 + 4}}\right| = \frac{1}{2}$

35.7 Important Facts Related to Plane

(i) **Image of a point in a plane** Image (x, y, z) (or reflection) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

(ii) Foot (x, y, z) of perpendicular drawn from a point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

(iii) Any plane parallel to XY plane is z = constant, similarly plane parallel to YZ plane is x = constant and plane parallel to ZX plane is y = constant, $x = 0$, $y = 0$ and $z = 0$ are respectively YZ, ZX and XY planes.

(iv) Any plane parallel to X-axis is of the form $by + cz = d$.

(v) **Position of the points** Points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are on same side of plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same sign. If they are of opposite sign, then the points are on the opposite sides.

(vi) Equation of plane parallel to planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ and equidistance from them is

$$ax + by + cz + \left(\frac{d_1 + d_2}{2}\right) = 0$$

(vii) If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are coplanar, then

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$$

(viii) Plane through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

where, $a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$

(ix) Plane $ax + by + cz + d = 0$ intersecting a line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ divides in the ratio $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.

(a) If this ratio is positive, A and B are on opposite sides of the plane.

(b) If this ratio is negative, A and B are on the same side of the plane.

Sample **Problem 11** *The vector equation of plane which is at a distance of 8 units from the origin and which is normal to the vector* $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *is* $\mathbf{r} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = \lambda$, *where* λ *is equal to*

(a) 0 (b) 24 (c) 42 (d) 8

Interpret (b) Here, $d = 8$, $\mathbf{n} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$$\therefore \quad \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

Hence, the required equation of plane is

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d \Rightarrow \mathbf{r} \cdot \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3} = 8 \Rightarrow \mathbf{r} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 24$$

Hence, $\lambda = 24$

35.8 Equation of Planes Bisecting the Angle between Two Planes

Equation of the planes bisecting the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{...(i)}$$

Bisector of Acute/Obtuse Angle

Write the equation of the given planes such that their constant terms (*i.e.*, d_1, d_2) are positive.

(i) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then origin lies in obtuse angle and hence, positive sign in Eq. (i) gives the bisector of the obtuse angle.

(ii) If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then origin lies in acute angle and hence, positive sign in Eq. (i) gives the bisector of the acute angle.

Hot Spot 1

Family of Planes

Family of planes is the most important topic for JEE (Main) examination point of view. Generally, questions are seen from this topic. The level of question is from moderate to typical.
Generally, question ask from equation of family of curves.

Equation of Family of Planes

(i) Let $P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$
and $P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$
be two planes, then $P_1 + \lambda P_2 = 0$ (where, λ is a parameter) represents family of planes passing through line of intersection of the planes $P_1 = 0$ and $P_2 = 0$. Let $S_1 \equiv \mathbf{r} \cdot \hat{\mathbf{n}}_1 = q_1$ and $S_2 \equiv \mathbf{r} \cdot \hat{\mathbf{n}}_2 = q_2$ be two planes, then $S_3 \equiv S_1 + \lambda S_2$ represents family of planes.

(ii) $ax + by + cz = k$ represents family of planes parallel to the plane $ax + by + cz + d = 0$. (where, k is a parameter).
While we solving such types of problems generally we follow the steps given below.
(i) First, we write the given planes P_1 and P_2 in the following format
$$P_1 + \lambda P_2 = 0 \quad \text{... (i)}$$
(ii) Now, apply the appropriate Condition given in the question in Eq. (i) and get the value of λ.
(iii) Now, put the value of λ in Eq. (i) and get the desired result.

Sample Problem 12 *The equation of plane containing the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1, 1, 1) is*
(a) $20x + 23y + 26z = 0$
(b) $20x + 23y + 26z + 69 = 0$
(c) $20x + 23y + 26z - 69 = 0$
(d) *None of the above*

Interpret (c) The equation of the plane through the line of intersection of the given planes is
$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \quad \text{...(i)}$$
If it passes through (1, 1, 1), then
$$(1 + 1 + 1 - 6) + \lambda(2 + 3 + 4 + 5) = 0$$
$$\Rightarrow \quad \lambda = \frac{3}{14}$$
On putting $\lambda = \frac{3}{14}$ in Eq. (i), we get
$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$
$$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$$

Sample Problem 13 *The equation of plane passing through the intersection of planes $2x - 4y + 3z + 5 = 0$, $x + y + z = 6$ and parallel to straight line having direction cosines (1, – 1, – 1) is*
(a) $x + 7y - 6z - 33 = 0$
(b) $x + 7y - 6z - 23 = 0$
(c) $x + 6y - 7z - 33 = 0$
(d) *None of the above*

Interpret (b) Equation of required plane be
$$(2x - 4y + 3z + 5) + \lambda(x + y - z - 6) = 0$$
i.e., $(2 + \lambda)x + (-4 + \lambda)y + z(3 - \lambda) + (5 - 6\lambda) = 0$
This plane is parallel to a straight line, so $al + bm + cn = 0$
$$1(2 + \lambda) + (-4 + \lambda) + (-1)(3 - \lambda) = 0$$
i.e., $\lambda = -3$
$\therefore$ Equation of required plane is $-x - 7y + 6z + 23 = 0$
i.e., $x + 7y - 6z - 23 = 0$

Check Point 1

1. The equation of plane which passes through the points (2, 3, –4) and (1, –1, 3) and parallel to x-axis, is
(a) $7y - 4z - 5 = 0$ (b) $4y - 7z - 5 = 0$
(c) $4y + 7z + 5 = 0$ (d) $7y + 4z - 5 = 0$

2. From the point P(a, b, c) the normals drawn to planes YZ and ZX are PA and PB respectively, then the equation of plane OAB is
(a) $bcx + acy + abz = 0$ (b) $bcx + acy - abz = 0$
(c) $bcx - acy + abz = 0$ (d) $-bcx + acy + abz = 0$

3. The distance of the plane $2x - 3y + 6z + 14 = 0$ from the origin is
(a) 14 (b) 2
(c) –2 (d) 11

4. The equation of the plane passing through the point (–1, 3, 2) and perpendicular to planes $x + 2y + 3z = 5$ and $3x + 3y + z = 9$, is
(a) $7x - 8y - 3z + 25 = 0$ (b) $7x - 8y + 3z + 25 = 0$
(c) $7x - 8y + 3z + 25 = 0$ (d) $7x + 8y + 3z - 25 = 0$

5. The position vectors of two points P and Q are $3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ respectively. The equation of the plane through Q and perpendicular to PQ is
(a) $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 28$
(b) $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 32$
(c) $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + 28 = 0$
(d) None of the above

6. The equation of the plane through the points (–1, 1, 1), (1, –1, 1) and perpendicular to the plane x + 2y + 2z = 5, is
   (a) 2x – 2y – 3z + 3 = 0
   (b) 2x + 2y + 3z – 3 = 0
   (c) x + y – 3z + 3 = 0
   (d) 2x + 2y – 3z + 3 = 0
7. The cartesian equation of the plane
   $\mathbf{r} = (1+\lambda-\mu)\hat{\mathbf{i}} + (2-\lambda)\hat{\mathbf{j}} + (3-2\lambda+\mu)\hat{\mathbf{k}}$ is
   (a) 2x + y = 5 (b) 2x – y = 5
   (c) 2x + z = 5 (d) 2x – z = 5
8. Ratio in which the xy-plane divides the join of (1, 2, 3) and (4, 2, 1) is
   (a) 3 : 1internally (b) 3 : 1 externally
   (c) 1 : 2 internally (d) 2 : 1 internally
9. If the DC's of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then
   (a) c < 0 (b) $c = \pm\sqrt{3}$ (c) 0 < c < 1.5 (d) c > 2
10. The equation of the plane passing through the points (3, 2, –1), (3, 4, 2) and (7, 0, 6) is 5x + 3y – 2z = λ , where λ is equal to
   (a) 23 (b) 25 (c) 20 (d) –27

35.9 Straight Line

A straight line is the locus of the intersection of ang two planes

OR

Two intersection planes determine a straight line.

35.10 Equation of Straight Line in Different Forms

(i) Equation of a Straight Line Passing Through a Given Point and Parallel to a Given Vector

Equation of straight line passing through a point A with position vector $\mathbf{p}\,(x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$ and parallel to a vector $\mathbf{q}$ $(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$ is

$$\mathbf{r} = \bar{\mathbf{p}} + \lambda\bar{\mathbf{q}}$$

On putting the value of $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$, we get

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}) + \lambda(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$$

$$\therefore \quad x - x_1 = \lambda a, \quad y - y_1 = \lambda b, \quad z - z_1 = \lambda c$$

$$\Rightarrow \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

which is the required equation of line in cartesian form. Here, a, b and c are direction ratios. If l, m, n are direction cosines, then equation of straight line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

This form of straight line is called symmetrical form.

(ii) Equation of Straight Line Passing Through Two Points

The equation of a line passing through two points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

Equation of a straight line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Note

- Intersection points of two planes also form a straight line, which is known as line of intersection but this is an unsymmetrical form of the straight line.
- The parametric equations of a line through (a_1, a_2, a_3) with direction cosines l, m and n are $x = a_1 + lr$, $y = a_2 + mr$ and $z = a_3 + nr$.
- The general coordinates of a point on a line is given by $(x_1 + lr, y_1 + mr, z_1 + nr)$, where r is distance between point (x_1, y_1, z_1) and point whose coordinates is to be written.

Sample **Problem 14** *The vector and the cartesian equations of the line through the points* (5, 2, – 4) *and which is parallel to the vector to* $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$ *are*

(*a*) $(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}})$ *and* $\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$

(*b*) $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}) + \lambda(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *and* $\frac{x+5}{3} = \frac{y+2}{2} = \frac{z-4}{-8}$

(*c*) $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} + 4\hat{\mathbf{k}})$ *and* $\frac{x+5}{5} = \frac{y+2}{2} = \frac{z-4}{-4}$

(*d*) *None of the above*

Interpret (*a*) We have $\mathbf{r}_1 = 5\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$

Therefore, the vector equation of the line is

$$r = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}})$$

Now, **r** is the position vector of any point $p(x, y, z)$ on the time. Therefore,

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - z\hat{\mathbf{k}} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}})$$

$$= (5 + 3\lambda)\,\hat{\mathbf{i}} + (2 + 2\lambda)\,\hat{\mathbf{j}} + (-4 - 8\lambda)\,\hat{\mathbf{k}}$$

Eliminating λ, we get

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8},$$

which is the equation of the line in cartesian form.

Sample Problem 15 *The vector equation for the line passing through the points* $(-1, 0\ 2)$ *and* $(3, 4, 6)$ *is*

(*a*) $\hat{\mathbf{i}} + 2\hat{\mathbf{k}} + \lambda(4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

(*b*) $2\hat{\mathbf{i}} - \hat{\mathbf{k}} + \lambda(4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

(*c*) $-\hat{\mathbf{i}} + 2\hat{\mathbf{k}} + \lambda(-\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(*d*) $-\hat{\mathbf{i}} + 2\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$

Interpret (*a*) Let **a** and **b** be the posision rectors of the point $A(-1, 0, 2)$ and $B(3, 4, 6)$. Then, $\mathbf{a} = -\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

Therefore, $\mathbf{b} - \mathbf{a} = 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

Let **r** be the position vector of any point on the line. Then, the vector equation of the line is $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{k}} + \lambda(4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

Sample Problem 16 *The vector equation of a line passing through a point with position vector* $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and parallel to the line joining the points with position vectors* $-\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *is* $\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda(x\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$, *where x is equal to*

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 4

Interpret (*c*) Let $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\mathbf{c} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Then, the equation of the line will be

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{c} - \mathbf{b}) = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda\{(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (-\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})\}$$

$$= (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Hence, $x = 2$

35.11 Angle between the Two Lines

Let $\mathbf{r} = \mathbf{a}_1 + \lambda\,\mathbf{b}_1$ and $\mathbf{r} = \mathbf{r}_2 + \lambda\mathbf{b}_2$ be the equations of two straight lines. If θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$$

Also, if θ is the angle between

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2},$$

then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of Perpendicularity

The lines are perpendicular, if

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = 0 \text{ or } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Condition of Parallelism

The lines are parallel, if $\mathbf{b}_1 = \lambda\mathbf{b}_2$, for same scalar λ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Sample Problem 17 *The angle between the pair of lines*

$$\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

and $\mathbf{r} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ *is*

(*a*) 0 (*b*) $\pi/2$

(*c*) π (*d*) *None of these*

Interpret (*d*) Given lines are

$$\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

and $\mathbf{r} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$

Let the angle is θ, then

$$\cos\theta = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$$

$$= \frac{(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{\sqrt{1 + 4 + 4}\sqrt{9 + 4 + 36}}$$

$$= \frac{3 + 4 + 12}{\sqrt{9}\sqrt{49}} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

35.12 Projection of a Line Segment on a Given Line

Projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n is $AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$. Here, PQ and AB need to be coplanar.

Hot Spot 2

Coplanar Line

Coplanar lines is the most important topic for JEE (Main) examination point of views. But a small practice can help you in solving such problems very quickly as the level of questions is easy to average. While solving problems based on coplanar lines, always keep in mind the condition of coplanarity.

A line which is in the same plane as another line. Any two intersecting lines must lie in the same plane and therefore will be coplanar.

Condition of Coplanarity of Two Lines

If the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

and equation of plane containing them is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

If the lines $\mathbf{r}=\mathbf{a}_1+\mathbf{b}_1\lambda$ and $\mathbf{r}=\mathbf{a}_2+\lambda\mathbf{b}_2$ are coplanar, then $(\mathbf{a}_2-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$

Sample Problem 18 *The lines* $\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{3}$ *and* $\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$ *are*

(a) *parallel*
(b) *perpendicular*
(c) *coplanar*
(d) *None of the above*

Interpret (c) Here, x_1-3, $y_1=1$, $z_1=5$ $a_1=-3$, b_1-1, $b_1=1$, $c_1=5$, $x_2=-1$, $y_2=2$, $z_2=5$, $a_2=-1$, $b_2=2$, $c_2=5$

Now, consider the determinant

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_1 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=0$$

Therefore, lines are coplanar.

35.13 Skew Lines

Two non-parallel non-intersecting straight lines are called skew lines.

Shortest Distance between Two Lines

Let the straight lines are $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ and d is shortest distance between them.

Then, $d=|(x_1-x_2)l+(y_1-y_2)m+(z_1-z_2)n|$,

where l, m and n are direction cosines of a line perpendicular to lines with (a_1, b_1, c_1) and (a_2, b_2, c_2).

So, $$l\hat{\mathbf{i}}+m\hat{\mathbf{j}}+n\hat{\mathbf{k}}=\frac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$$

where, $$\mathbf{a}=a_1\hat{\mathbf{i}}+b_1\hat{\mathbf{j}}+c_1\hat{\mathbf{k}}$$

and $$\mathbf{b}=a_2\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+c_2\hat{\mathbf{k}}\text{s}$$

If $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\lambda\mathbf{b}_2$ are two skew lines, then the distance between them is $\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$ and these lines intersect, if

$$(\mathbf{b}_1\times\mathbf{b}_2)(\mathbf{a}_2-\mathbf{a}_1)=0$$

i.e., shortest distance = 0

The distance between two skew lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{[(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2]}}$$

Note If $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}$ are two parallel lines, then the distance between them is given by $d=\frac{|(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}|}{|\mathbf{b}|}$.

Sample Problem 19 *The shortest distance between the lines the lines* l_1 *and* l_2 *whose equations are*

$$\mathbf{r}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\lambda(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

and $$\mathbf{r}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}+\mu(3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+2\hat{\mathbf{k}})\ \textit{is}$$

(a) $\frac{10}{\sqrt{59}}$ (b) $\frac{5}{\sqrt{59}}$

(c) $\frac{6}{\sqrt{59}}$ (d) $\frac{9}{\sqrt{59}}$

Interpret (a) **First Method**

On Comparing Eqs. (i) and (ii) with

$\mathbf{r} = \mathbf{a}_1 + \lambda\, \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\, \mathbf{b}_2$ respectively,

we get $\mathbf{a}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{b}_1 = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\mathbf{a}_2 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, and $\mathbf{b}_2 = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Therefore, $\mathbf{a}_2 - \mathbf{a}_1 = \hat{\mathbf{i}} - \hat{\mathbf{k}}$

and $\mathbf{b}_1 \times \mathbf{b}_2 = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

So, $|\mathbf{b}_1 \times \mathbf{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$

Hence, the shortest distance between the given lines by

$$d = \left| \frac{(\mathbf{b}_1 \times \mathbf{b}_1) \cdot (\mathbf{a}_1 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_1|} \right| = \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Second Method

Here,

$$x_1 = 1, y_1 = 1, z_1 = 0$$
$$a_1 = 2, b_1 = -1, c_1 = 1$$
$$x_2 = 2, y_2 = 1, z_2 = -1$$
$$a_2 = 3, b_2 = -5, c_2 = 2$$

Hence, the shortest distance between the given lines in given by

$$d = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\sum (a_1 b_2 - a_2 b_1)^2}} \right| = \frac{10}{\sqrt{59}}$$

Sample **Problem 20** *The distance between two parallel lines l_1 and l_2 whose equations are*

$$\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

and $\mathbf{r} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}} + \mu(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ *are*

(a) $\frac{\sqrt{297}}{7}$ (b) $\frac{\sqrt{293}}{7}$ (c) $\frac{\sqrt{291}}{7}$ (d) $\frac{\sqrt{293}}{5}$

Interpret (b)The two lines are parallel, we have

$\mathbf{a}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, $\mathbf{a}_2 = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$,

and $\mathbf{b} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

and $\mathbf{r} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}} + \mu(2\mathbf{i} + 3\mathbf{j} + 6\hat{\mathbf{k}})$

Therefore, the distance between the lines is given by

$$d = \left| \frac{|\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}|} \right| \left| \frac{\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4 + 9 + 36}} \right|$$

or $$= \frac{|-9\hat{\mathbf{i}} + 14\hat{\mathbf{j}} - 4\hat{\mathbf{k}}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{49}} = \frac{\sqrt{293}}{7}$$

Hot Spot 3

Important Facts Related to Line

Foot of perpendicular from point $A(\alpha, \beta, \gamma)$ to the given line and length and equation of perpendicular is also a most important topic for JEE (Main) examination point of view. Generally, questions are seen from this topic. The level of question is from moderate to typical.

OR

Let L be the foot of perpendicular drawn from $P(\alpha)$ on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

$\therefore$ Position vector of L is $\Rightarrow \mathbf{a} - \left(\frac{(\mathbf{a} - \alpha) \cdot \mathbf{b}}{|\mathbf{a}|} \right) \mathbf{b}$

which is foot of perpendicular.

Sample **Problem 21** *The length of perpendicular from $P(2, -3, 1)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ is*

(a) $\frac{531}{14}$ *units* (b) $\sqrt{\frac{531}{14}}$ *units*

(c) $\frac{15}{14}$ *units* (d) *None of these*

Interpret (b) Given line is

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1} = (r) \quad \text{(say)} \ldots\text{(i)}$$

and coordinates of point P are $(2, -3, 1)$.

Coordinates of any point on Eq. (i), may be taken as,

$$(2r-1, 3r+3, -r-2)$$

Let $\quad Q = (2r-1, 3r+3, -r-2)$.

Direction ratio's of PQ are $(2r-3, 3r+6, -r-3)$

Direction ratio's of AB are $(2, 3, -1)$.

Since, $\quad PQ \perp AB$

$\therefore \quad 2(2r-3) + 3(3r+6) - 1(-r-3) = 0$

$\Rightarrow \quad r = -\dfrac{15}{14}$

$\therefore \quad Q = \left(-\dfrac{22}{7}, -\dfrac{3}{14}, -\dfrac{13}{14}\right)$

$$PQ^2 = \left(2+\frac{22}{7}\right)^2 + \left(-3+\frac{3}{14}\right)^2 + \left(1+\frac{13}{14}\right)^2 = \frac{531}{14}$$

$\Rightarrow \quad PQ = \sqrt{\dfrac{531}{14}}$ units

35.14 Line and Plane

Equation of Plane Through a Given Line

(i) If equation of the line is given in symmetrical form as $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$, then equation of plane is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ where a, b, c are given by $al + bm + cn = 0$

(ii) If equation of line is given in general form as $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$, then the equation of plane passing through this line is

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

(iii) If the plane pass through parallel lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mathbf{b}$, then equation of the required plane is

$$[\mathbf{r}-\mathbf{a} \quad \mathbf{c}-\mathbf{a} \quad \mathbf{b}] = 0$$

Sample **Problem 22** *The equation of the plane passing through the point* $(0,7,-7)$ *and containing the line* $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ *is*

(a) $x + 7y + 7z = 0$ (b) $x + y + z = 7$

(c) $x + y + z = 0$ (d) $x + y - z = 14$

Interpret (c) Any plane through $(0,7,-7)$ is

$$a(x-0) + b(y-7) + c(z+7) = 0 \quad \text{...(i)}$$

If Eq. (i) contains the given line, it must pass through the point $(-1, 3, -2)$ and must be parallel to this line. As Eq. (i) passes through $(-1, 3, -2)$, we get $a(-1-0) + b(3-7) + c(-2+7) = 0$

i.e., $\quad a + 4b - 5c = 0 \quad$ ...(ii)

Also, Eq. (i) is parallel to the given line

$$(-3)a + 2b + 1.c = 0$$

i.e., $\quad -3a + 2b + c = 0 \quad$ ...(iii)

From Eqs. (i) and (iii),

$\Rightarrow \quad \dfrac{a}{14} = \dfrac{b}{14} = \dfrac{c}{14} \Rightarrow a = b = c = k$ (let)

Putting $a = k$, $b = k$ and $c = k$ in Eq. (i), we get

$$kx + k(y-7) + k(z+7) = 0 \text{ or } x + y + z + 0$$

which is the required equation of the plane.

Angle between a Line and a Plane

Let the equation of line be $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and equation of plane is $\mathbf{r}\cdot\mathbf{n} = q$

Let θ be the angle between a line and a plane, then

$$\sin\theta = \frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

Angle between a line $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$

and plane $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\sin\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Note The line $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ lies in the plane $a_2x + b_2y + c_2z + d_2 = 0$, then $a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$ and $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Sample **Problem 23** *The angle between the line* $\dfrac{x+1}{2} = \dfrac{y}{3} = \dfrac{z-3}{6}$ *and the plane* $10x + 2y - 11z = 3$ *is*

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{6}$ (d) $\sin^{-1}\left(\dfrac{8}{21}\right)$

Interpret (d) Let θ be the angle between the line and the plane, we have

$$a = 2, b = 3, c = 6$$

and $\quad a_1 = 10, b_1 = 2, c_1 = -11$

$$\therefore \quad \cos(90-\theta) = \sin\theta = \left|\frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2+b^2+c^2}.\sqrt{a_1^2+b_1^2+c_1^2}}\right|$$

$$= \left|\frac{(2)(10) + (3)(2) + (6)(-11)}{\sqrt{2^2+3^2+6^2}\cdot\sqrt{10^2+2^2+11^2}}\right|$$

$$= \left|\frac{20+6-66}{\sqrt{49}\cdot\sqrt{225}}\right| = \left|\frac{-40}{7\cdot 15}\right|$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{8}{21}\right)$$

Check Point 2

1. The equation of the line passing through the point (1, 2, 3) and perpendicular to $x + 2y - 5z + 9 = 0$ is
   (a) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{-5}$ (b) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+5}{3}$
   (c) $\frac{x+1}{1} = \frac{y+2}{2} = \frac{z+3}{-5}$ (d) $\frac{x+1}{1} = \frac{y+2}{2} = \frac{z-5}{3}$
2. The shortest distance between lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is
   (a) $3\sqrt{30}$ (b) $\sqrt{30}$ (c) $2\sqrt{30}$ (d) $4\sqrt{30}$
3. The length of perpendicular from $(2, -1, 5)$ to the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$ and the coordinates of the foot are
   (a) $\sqrt{14}, (1, 2, -3)$ (b) $\sqrt{14}, (1, -2, 3)$
   (c) $\sqrt{14}, (1, 2, 3)$ (d) None of these
4. The point of intersection of the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z+2}{3}$ and plane $2x + 3y + z = 0$ is
   (a) (0, 1, –2) (b) (1, 2, 3)
   (c) (–1, 9, –25) (d) $\left(-\frac{1}{11}, \frac{9}{11}, -\frac{25}{11}\right)$
5. The equation of perpendicular from (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is
   (a) $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$
   (b) $\frac{x-3}{1} = \frac{y+1}{6} = \frac{z-11}{4}$
   (c) $\frac{x+3}{-1} = \frac{y+1}{-6} = \frac{z-11}{4}$
   (d) None of the above

35.15 Area of a Triangle

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

So, area of ΔABC is given by the relation

$$\Delta^2 = \Delta_x^{\,2} + \Delta_y^{\,2} + \Delta_z^{\,2}$$

In vector form

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of ΔABC, then

$$\text{Area of triangle} = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|$$

$$= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Sample Problem 24 *The area of the triangle whose vertices are $A(1, 2, 3)$, $B(2, -1, 1)$ and $C(1, 2, -4)$ is*

(a) $7\sqrt{10}$ *sq units*

(b) $\frac{1}{2}\sqrt{10}$ *sq units*

(c) $\frac{7}{2}\sqrt{10}$ *sq units*

(d) *None of the above*

Interpret (c) Let A_x, A_y and A_z be the auras of projection of ΔABC on yz, zx and xy planes. Then,

$$A_x = \frac{1}{2} \times \text{Absolute values of} \begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix} = \frac{21}{2}$$

$$A_y = \frac{1}{2} \times \text{Absolute values of} \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -4 & 1 & 1 \end{vmatrix} = \frac{7}{2}$$

and $$A_z = \frac{1}{2} \times \text{Absolute values of} \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\therefore$ $$\text{Area of } \Delta ABC = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$= \sqrt{\frac{441}{4} + \frac{49}{4} + 0} = \frac{7}{2}\sqrt{10} \text{ sq units.}$$

WORKED OUT
Examples

Example 1 *The equation of the plane passing through* (a, b, c) *and parallel to the plane* $\mathbf{r}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=2$ *is,* [**NCERT**]

(*a*) $x+y+z=1$ (*b*) $ax+by+cz=1$

(*c*) $x+y+z=a+b+c$ (*d*) *None of these*

Solution (*c*) Given plane is $\mathbf{r}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=2$ ...(i)

The equation of the plane parallel to plane (i) is

$$\mathbf{r}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=d \quad \text{...(ii)}$$

or $$(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=d$$

Since, it passes through point (a, b, c).

$\therefore$ $$(a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=d$$

$\Rightarrow$ $$a+b+c=d$$

Putting $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ and $d=a+b+c$ in Eq. (ii), we have

$$(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=a+b+c$$

which is required equation of the plane .

Example 2 *The vector equation of the line* $\frac{x-2}{2}=\frac{2y-5}{-3}$, $z=-1$, *is* $\mathbf{r}=\left(2\hat{\mathbf{i}}+\frac{5}{2}\hat{\mathbf{j}}-\hat{\mathbf{k}}\right)+\lambda\left(2\hat{\mathbf{i}}-\frac{3}{2}\hat{\mathbf{j}}+x\hat{\mathbf{k}}\right)$, *where* x *is equal to*

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

Solution (*a*)The given line is

$$\frac{x-2}{2}=\frac{2y-5}{-3}, z=-1$$

$\Rightarrow$ $$\frac{x-2}{2}=\frac{y-\frac{5}{2}}{-\frac{3}{2}}=\frac{z+1}{0}$$

This shows that the given line passes through the point $\left(2, \frac{5}{2}, -1\right)$ and has direction ratios $2, -\frac{3}{2}, 0$. So, its DC's are

$$\frac{2}{\sqrt{2^2+(-3/2)^2}}, \frac{-3/2}{\sqrt{2^2+(-3/2)^2}}, \frac{0}{\sqrt{2^2+(-3/2)^2}}$$

or $$\left(\frac{4}{5}, -\frac{3}{5}, 0\right)$$

Thus, given line passes through the point having position vector $\mathbf{a}=2\hat{\mathbf{i}}+\frac{5}{2}\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and is parallel to the vector

$$\mathbf{b}=2\hat{\mathbf{i}}-\frac{3}{2}\hat{\mathbf{j}}+0\,\hat{\mathbf{k}}$$

So, its vector equation is

$$\mathbf{r}=\left(2\hat{\mathbf{i}}+\frac{5}{2}\hat{\mathbf{j}}-\hat{\mathbf{k}}\right)+\lambda\left(2\hat{\mathbf{i}}-\frac{3}{2}\hat{\mathbf{j}}+0\,\hat{\mathbf{k}}\right)$$

Hence, $x=0$

Example 3 *The coordinates of the point where the line through* (5,1,6) *and* (3,4,1) *crosses the ZX –plane is* [**NCERT**]

(*a*) $\left(\frac{-17}{3}, 0, \frac{23}{3}\right)$ (*b*) $\left(\frac{15}{2}, 0, \frac{21}{2}\right)$

(*c*) $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$ (*d*) $\left(\frac{15}{2}, 0, \frac{-21}{2}\right)$

Solution (*c*) It is known that equation of *ZX*-plane is $y=0$.

The equation of the line passing through the point (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

The equation of the line joining (5, 1, 6) and (3, 4, 1) is

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6} \Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=\lambda \text{ (say)}$$

Any point on this line is $(5-2\lambda, 1+3\lambda, 6-5\lambda)$ ...(i)

Since, the line passes through *ZX*-plane is $(y=0)$

$\therefore$ From Eq. (i),

$$1+3\lambda=0 \Rightarrow \lambda=-1/3 \quad \text{...(ii)}$$

$\therefore$ Required point of intersection of planes (i) and line (ii) is

$$\left(5+\frac{2}{3}, 0, 6+\frac{5}{3}\right)$$

$\Rightarrow$ $$\left(\frac{17}{3}, 0, \frac{23}{3}\right)$$

Example 4 *The distance between the parallel planes* $2x-y+2z+3=0$ *and* $4x-2y+4z+5=0$ *is*

(*a*) $\frac{1}{36}$ *unit* (*b*) $\frac{1}{3}$ *unit*

(*c*) $\frac{1}{6}$ *unit* (*d*) *None of these*

Solution (c) Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$, then

$$2x_1 - y_1 + 2z_1 + 3 = 0 \quad \ldots(i)$$

The length of perpendicular from $P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$

$$= \frac{|4x_1 - 2y_1 + 4z_1 + 5|}{\sqrt{4^2 + (-2)^2 + 4^2}}$$

$$= \frac{|2(2x_1 - y_1 + 2z_1) + 5|}{\sqrt{36}} = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \text{ unit}$$

Example 5 *The coordinates of the point where the line through* (3,–4,–5) *and* (2, – 3,1) *crosses the planes* $2x + y + z = 7$ *is* [**NCERT**]

(a) (1, – 2, 7) (b) (1, – 2 , – 7)

(c) (3, 2, 1) (d) *None of these*

Solution (a)The equation of the line passing through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The equation of the line joining (3, –4, –5) and (2,–3, 1) is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}$$

$\Rightarrow \quad x = 3 - \lambda, y = \lambda - 4, z = 6\lambda - 5$

Therefore, any point on the line is of the form

$$(3 - \lambda, \lambda - 4, 6\lambda - 5).$$

This point lies on the plane, $2x + y + z = 7$

Therefore, $2(3 - \lambda) + (\lambda - 4) + (6\lambda - 5) = 7 \Rightarrow 5\lambda - 3 = 7$

$\Rightarrow \quad \lambda = 2$

Hence, the coordinates of the required point are

$(3 - 2, 2 - 4, 6 \times 2 - 5)$ *i.e.*, (1, – 2, 7)

Example 6 *The equation of the plane passing through the point* (– 1, 3, 2) *and perpendicular to each of the planes* $x + 2y + 3z = 5$ *and* $3x + 3y + z = 0$ *is* [**NCERT**]

(a) $-14x - 5y + 3z + 25 = 0$ (b) $-x - y + z = 0$

(c) $7x - 8y + 3z + 25 = 0$ (d) *None of these*

Solution (c) Two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Any plane through the given point (– 1, 3, 2) is

$$a\{x - (-1)\} + b(y - 3) + c(z - 2) = 0$$

$\Rightarrow \quad a(x + 1) + b(y - 3) + c(z - 2) = 0 \quad \ldots(i)$

This plane (i) is at right angle to given plane $x + 2y + 3z = 5$.

$\therefore \quad a \cdot 1 + b \cdot 2 + c \cdot 3 = 0 \Rightarrow a + 2b + 3c = 0 \quad \ldots(ii)$

Also, plane (i) is at right angle to given plane $3x + 3y + z = 0$

$$a \cdot 3 + b \cdot 3 + c \cdot 1 = 0 \Rightarrow 3a + 3b + c = 0 \quad \ldots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$\frac{a}{2 \times 1 - 3 \times 3} = \frac{b}{3 \times 3 - 1 \times 1} = \frac{c}{1 \times 3 - 2 \times 3} \Rightarrow \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = \lambda$$

$\Rightarrow \quad a = -7\lambda, b = 8\lambda, c = -3\lambda$

Putting the values of a, b, c in Eq. (i), we obtain

$$-7\lambda(x + 1) + 8\lambda(y - 3) - 3\lambda(z - 2) = 0$$

$\Rightarrow \quad (-7x - 7) + (8y - 24) - 3z + 6 = 0$

$\Rightarrow \quad -7x + 8y - 3z - 25 = 0$

$\Rightarrow \quad 7x - 8y + 3z + 25 = 0$

Hence, the equation of the required plane is

$$7x - 8y + 3z + 25 = 0$$

Example 7 *If the points* (1, 1, p) *and* (–3, 0, 1) *be equidistant from the plane* $\mathbf{r} \cdot (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 12\hat{\mathbf{k}}) + 13 = 0$, *then value of p is* [**NCERT**]

(a) $\frac{1}{3}$ (b) $\frac{7}{3}$ (c) $\frac{4}{3}$ (d) 7

Solution (b) The distance of point (1, 1, p) from the plane $\mathbf{r} \cdot (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 12\hat{\mathbf{k}}) + 13 = 0$ or in cartesian form $3x + 4y - 12z + 13 = 0$ is

$$d_1 = \left|\frac{3 \times 1 + 4 \times 1 - 12 \times p + 13}{\sqrt{3^2 + 4^2 + (-12)^2}}\right| = \left|\frac{3 + 4 - 12p + 13}{\sqrt{169}}\right| = \left|\frac{20 - 12p}{13}\right| \quad \ldots(i)$$

The distance of the point (–3, 0, 1) from the plane

$3x + 4y - 12z + 13 = 0$ is

$$d_2 = \left|\frac{3 \times (-3) + 4 \times 0 - 12 \times 1 + 13}{\sqrt{3^2 + 4^2 + (-12)^2}}\right|$$

$$= \left|\frac{-9 + 0 - 12 + 13}{\sqrt{169}}\right| = \left|\frac{-8}{13}\right| = \frac{8}{13}$$

According to the given condition,

$$d_1 = d_2 \Rightarrow \left|\frac{20 - 12p}{13}\right| = \frac{8}{13}$$

$\Rightarrow \quad \frac{20 - 12p}{13} = \pm \frac{8}{13}$

Taking +ve sign, we get

$$\frac{20 - 12p}{13} = \frac{8}{13} \Rightarrow 20 - 12p = 8$$

$\Rightarrow \quad 12p = 12 \Rightarrow p = 1$

Taking –ve sign, we get

$$\frac{20 - 12p}{13} = \frac{-8}{13} \Rightarrow 20 - 12p = -8$$

$\Rightarrow \quad 12p = 28 \Rightarrow p = \frac{28}{12} = \frac{7}{3}$

Example 8 *The equation of the plane passing through the line of intersection of the planes* $\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1$ *and* $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + 4 = 0$ *and parallel to X-axis is* [**NCERT**]

(a) $x + y - 3z + 6 = 0$ (b) $y - 3z + 6 = 0$

(c) $3y - z + 6 = 0$ (d) $y - z + 6 = 0$

Solution (*b*) The intersection equation of planes $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$ is $(\mathbf{r}\cdot\mathbf{n}_1 - d_1) + \lambda(\mathbf{r}\cdot\mathbf{n}_2 - d_2) = 0$.

Given planes are $\mathbf{r}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1$

and $\mathbf{r}\cdot(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + 4 = 0$

The equation of any plane passing through the line of intersection of these planes is

$$[\mathbf{r}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 1] + \lambda[\mathbf{r}\cdot(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + 4] = 0$$

or $\mathbf{r}\cdot[(2\lambda + 1)\hat{\mathbf{i}} + (3\lambda + 1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}] + (4\lambda - 1) = 0$ ... (i)

Its direction ratios are $(2\lambda + 1)$, $(3\lambda + 1)$ and $(1 - \lambda)$.

The required plane is parallel to X-axis. Therefore, its normal is perpendicular to X-axis.

The direction ratios of X-axis are 1, 0 and 0.

$\therefore$ $1(2\lambda + 1) + 0(3\lambda + 1) + 0(1 - \lambda) = 0$

$\Rightarrow$ $2\lambda + 1 = 0$

$\Rightarrow$ $\lambda = -\frac{1}{2}$

Putting the value of λ in Eq. (i), we obtain

$$\mathbf{r}\cdot\left(-\frac{1}{2}\hat{\mathbf{j}} + \frac{3}{2}\hat{\mathbf{k}}\right) + (-3) = 0$$

$\Rightarrow$ $\mathbf{r}\cdot(\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) + 6 = 0$

Therefore, its cartesian equation is $y - 3z + 6 = 0$

(put $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$)

This is the equation of the required plane.

Example 9 *The equation of the plane which contains line of intersection of the planes* $\mathbf{r}\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - 4 = 0$, $\mathbf{r}\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) + 5 = 0$, *and which is perpendicular to the plane* $\mathbf{r}\cdot(5\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) + 8 = 0$ **[NCERT]**

(*a*) $33x - 45y + 50z - 41 = 0$ (*b*) $33x + 45y + 50z + 41 = 0$

(*c*) $33x + 45y - 50z + 41 = 0$ (*d*) $33x + 45y + 50z - 41 = 0$

Solution (*d*) Given planes are

$$\mathbf{r}\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - 4 = 0 \quad \text{...(i)}$$

$$\mathbf{r}\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) + 5 = 0 \quad \text{...(ii)}$$

and $$\mathbf{r}\cdot(5\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) + 8 = 0 \quad \text{...(iii)}$$

Any plane through the line of intersection of Eqs. (i) and (ii) can be written as

$$[\mathbf{r}\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - 4] + \lambda[\mathbf{r}\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})] + 5 = 0$$

$\Rightarrow$ $[\mathbf{r}\cdot(1 + 2\lambda)\hat{\mathbf{i}} + (2 + \lambda)\hat{\mathbf{j}} + (3 - \lambda)\hat{\mathbf{k}}] + (5\lambda - 4) = 0$ ...(iv)

This plane is at right angle to Eq. (iii),

$$5(1 + 2\lambda) + 3(2 + \lambda) - 6(3 - \lambda) = 0$$

$\Rightarrow$ $19\lambda = 7 \Rightarrow \lambda = \frac{7}{19}$

Substituting this value of λ in Eq. (iv), we get the required plane as

$$\mathbf{r}\cdot\left[\left(1 + \frac{14}{19}\right)\hat{\mathbf{i}} + \left(2 + \frac{7}{19}\right)\hat{\mathbf{j}} + \left(3 - \frac{7}{19}\right)\hat{\mathbf{k}}\right] + \frac{35}{19} - 4 = 0$$

$\Rightarrow$ $\mathbf{r}\left[\left(\frac{33}{19}\right)\hat{\mathbf{i}} + \left(\frac{45}{19}\right)\hat{\mathbf{j}} + \left(\frac{50}{19}\right)\hat{\mathbf{k}}\right] - \frac{41}{19} = 0$

$\Rightarrow$ $\mathbf{r}\cdot(33\hat{\mathbf{i}} + 45\hat{\mathbf{j}} + 50\hat{\mathbf{k}}) - 41 = 0$ ...(v)

This is the vector equation of the required plane.

Now, converting the equation into cartesian equation of this plane can be obtained by substituting $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ in Eq. (v), we get

$$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(33\hat{\mathbf{i}} + 45\hat{\mathbf{j}} + 50\hat{\mathbf{k}}) - 41 = 0$$

$\Rightarrow$ $33x + 45y + 50z - 41 = 0$

Example 10 *If a plane has the intercepts a,b,c and is at a distance of p units from the origin, then* **[NCERT]**

(*a*) $a^2 + b^2 + c^2 = p^2$ (*b*) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$

(*c*) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = p^2$ (*d*) $a^2 + b^2 + c^2 = \frac{1}{p^2}$

Solution (*b*)The equation of a plane having intercepts *a, b, c*, with X, Y and Z-axes respectively, is given by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \left(\frac{1}{a}\right)x + \left(\frac{1}{b}\right)y + \left(\frac{1}{c}\right)z - 1 = 0$$

Distance of this plane from the origin (0, 0, 0) is given to be p.

$$\therefore \quad p = \frac{\left|\frac{1}{a}\cdot 0 + \frac{1}{b}\cdot 0 + \frac{1}{c}\cdot 0 - 1\right|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} \Rightarrow \frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

On squaring both sides, we get $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Example 11 *The equation of the plane through the points* $(2, 1, -1)$ *and* $(-1, 3, 4)$ *and perpendicular to the plane* $x - 2y + 4z = 10$ *is* **[NCERT]**

(*a*) $18x + 17y + 4z = 49$ (*b*) $18x - 17y + 4z = 49$

(*c*) $18x - 17y - 4z = 49$ (*d*) *None of these*

Solution (*a*)The equation of any plane through (2, 1, –1) is

$$a(x - 2) + b(y - 1) + c(z + 1) = 0 \quad \text{...(i)}$$

If it passes through (–1, –3, 4),

then $a(-1 - 2) + b(3 - 1) + c(4 + 1) = 0$

$\Rightarrow$ $-3a + 2b + 5c = 0$ ...(ii)

If plane (i) is perpendicular to the plane $x - 2y + 4z = 10$, then

$$a - 2b + 4c = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) by the method of cross-multiplication, we obtain

$$\frac{a}{8 + 10} = \frac{b}{5 + 12} = \frac{c}{6 - 2} \Rightarrow \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \text{ (say)}$$

$\Rightarrow$ $a = 18\lambda$, $b = 17\lambda$ and $c = 4\lambda$

On putting $a = 18\lambda$, $b = 17\lambda$ and $c = 4\lambda$ in Eq. (i), we obtain

$$18\lambda(x - 2) + 17\lambda(y - 1) + 4\lambda(z + 1) = 0$$

$\Rightarrow$ $18x + 17y + 4z = 49$

This is the required equation of the plane.

Start Practice for JEE Main

Round I (Topically Divided Problems)

Direction Cosines and Direction Ratios

1. If a line makes angles 90°, 35°, 45° and with the positive x, y and z-axes respectively, then its direction cosines are [NCERT]

(a) $0, \frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ (b) $0, \frac{1}{\sqrt{2}}$ and $\frac{-1}{2}$

(c) $0, \frac{-1}{2}$ and $\frac{1}{\sqrt{2}}$ (d) $1, \frac{-1}{2}$ and $\frac{1}{\sqrt{2}}$

2. The points (2, 3, 4), (–1, –2, 1) and (5, 8, 7) are [NCERT]

(a) formed an isosceles triangle
(b) formed an equilateral triangle
(c) collinear
(d) None of the above

3. The direction cosines of the side AB of the ΔABC whose vertices are $A(3, 5, -4)$, $B(-1, 1, 2)$ and $C(-5, -5, -2)$ are [NCERT]

(a) $\frac{-4}{\sqrt{17}}, \frac{-4}{\sqrt{17}}, \frac{6}{\sqrt{17}}$ (b) $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$

(c) $\frac{-2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}$ (d) None of these

4. If a line makes angles α, β and γ with the positive directions of the coordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is [NCERT Exemplar]

(a) 0 (b) 1
(c) 2 (d) –1

5. If a line makes an angle of $\frac{\pi}{4}$ with each of y and z axes, then the angle which it makes with x-axis is [NCERT Exemplar]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

6. The points $P(4, 5, 10)$, $Q(2, 3, 4)$ and $R(1, 2, -1)$ are three vertices of a parallelogram $PQRS$. The coordinates of S are [NCERT Exemplar]

(a) (3, 4, 5) (b) (4, 6, 5)
(c) (1, 1, 5) (d) (1, 3, 5)

7. A line passes through the points $(6, -7, -1)$ and $(2, -3, 1)$. The direction cosines of the line so directed that the angle made by it with the positive direction of x-axis is acute, is

(a) $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$ (b) $-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

(c) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ (d) $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

8. A line makes angles α, β, γ and δ with the four diagonals of a cube, then

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(a) 1 (b) 4/3 (c) 3/4 (d) 4/5

9. If a line makes angles α, β and γ with the axes, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to

(a) –2 (b) –1
(c) 1 (d) 2

10. If the projection of a line segment on x, y and z-axes are 3, 1 and $\sqrt{15}$ respectively then length of line segment is

(a) 5 (b) $4 + \sqrt{15}$

(c) $5 + \sqrt{2}$ (d) 6

11. Lines OA and OB are drawn from O with direction cosines proportional to $(1, -2, -1)$ and $(3, -2, 3)$, respectively. The direction ratios of the normal to the pane AOB are

(a) (4, 3, 2)
(b) (4, – 3, – 2)
(c) (–4, 3, – 2)
(d) (4, 3, – 2)

12. O is the origin. OP makes an angle of 45° and 60° with the positive direction of x and y-axes respectively. $OP = 12$ units. Then, the coordinates of P are
(a) (6, 6, 6) (b) (6, 6, – 6)
(c) $(6\sqrt{2}, 6, \pm 6)$ (d) (6, ± 6, 6)

13. If a line lies in the octant $OXYZ$ and it makes equal angles with the axes, then
(a) $l = m = n = \frac{1}{\sqrt{3}}$ (b) $l = m = n = \pm\frac{1}{\sqrt{3}}$
(c) $l = m = n = \frac{-1}{\sqrt{3}}$ (d) $l = m = n = \pm\frac{1}{\sqrt{2}}$

14. If $\left(\frac{1}{2}, \frac{1}{3}, n\right)$ are the direction cosines of a line, then the value of n is
(a) $\frac{\sqrt{23}}{6}$ (b) $\frac{23}{6}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

15. A line makes acute angles of α, β and γ with the coordinate axes such that $\cos\alpha\cos\beta = \cos\beta\cos\gamma = \frac{2}{9}$ and $\cos\gamma\cos\alpha = \frac{4}{9}$, then $\cos\alpha + \cos\beta + \cos\gamma$ is equal to
(a) $\frac{25}{9}$ (b) $\frac{5}{9}$ (c) $\frac{5}{3}$ (d) $\frac{2}{3}$

16. The points (5, 2, 4), (6, – 1, 2) and (8, – 7, k) are collinear, if k is equal to
(a) – 2 (b) 2 (c) 3 (d) – 1

17. If O is the origin and $OP = 3$ with direction ratios – 1, 2, – 2, then coordinates of P are
(a) (1, 2, 2) (b) (– 1, 2, – 2)
(c) (– 3, 6, – 9) (d) (– 1/3, 2/3, – 2/3)

18. The direction cosines l, m and n of two lines are connected by the relations $l + m + n = 0$, $lm = 0$, then the angles between them is
(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/2$ (d) 0

19. The angle between a line whose direction ratios are in the ratio 2 : 2 : 1 and a line joining (3, 1, 4) to (7, 2, 12) is
(a) $\cos^{-1}(2/3)$ (b) $\cos^{-1}(-2/3)$
(c) $\tan^{-1}(2/3)$ (d) None of these

Problems Based on Plane

20. The vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ is [NCERT]
(a) $\frac{3}{\sqrt{70}}x + \frac{5}{\sqrt{70}}y - \frac{6}{\sqrt{70}}z = 7$
(b) $3x + 5y - 6z = 7$
(c) $3\sqrt{70}\,x + 5\sqrt{70}\,y - 6\sqrt{70}z = 7$
(d) None of the above

21. The coordinates of the foot of the perpendicular drawn form the origin to the plane $2x + 3y + 4z - 12 = 0$ is [NCERT]
(a) $\left(\frac{36}{29}, \frac{24}{29}, \frac{48}{29}\right)$ (b) $\left(\frac{24}{29}, \frac{48}{29}, \frac{36}{29}\right)$
(c) $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$ (d) None of these

22. The equation of the plane that passes through the points (1, 1, 0) (1, 2, 1) and (–2, 2, – 1) is [NCERT]
(a) $2x + 3y - 3z = 5$ (b) $2x + 3y + 3z = 5$
(c) $2x - 3y + 3z = 5$ (d) None of these

23. The equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1) is [NCERT]
(a) $7x + 5y + 4z + 8 = 0$ (b) $7x + 5y + 4z - 8 = 0$
(c) $7x - 5y + 4z - 8 = 0$ (d) None of these

24. The equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$ is, [NCERT]
(a) $-x + z + 2 = 0$ (b) $x - z + 2 = 0$
(c) $x - z - 2 = 0$ (d) $-x + z - 2 = 0$

25. The angle between the planes whose vector equation are $\mathbf{r}\cdot(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 5$ and $\mathbf{r}\cdot(3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) = 3$ is, [NCERT]
(a) $\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$ (b) $\cos^{-1}\left(\frac{15}{\sqrt{7}}\right)$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

26. Two planes $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$ are [NCERT]
(a) non-parallel, perpendicular and have an angle $\frac{\pi}{2}$.
(b) parallel, non-perpendicular and have an angle 0.
(c) non-parallel, non-perpendicular and have an angle $\cos^{-1}\left(\frac{2}{5}\right)$
(d) non-parallel, non-perpendicular and have an angle $\cos^{-1}\left(\frac{1}{5}\right)$

27. The distance of the point (3, – 2, 1) from the plane $2x - y + 2z + 3 = 0$ is, [NCERT]
(a) $\frac{3}{13}$ unit (b) $\frac{13}{3}$ units (c) 3 units (d) 2 units

28. The equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ is [NCERT Exemplar]
(a) $23x + 15y + 50z + 173 = 0$
(b) $51x + 15y + 50z + 173 = 0$
(c) $51x + 15y - 50z + 173 = 0$
(d) None of the above

29. The coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) cross the YZ-plane, is [NCERT]

(a) $(0, \frac{-17}{2}, \frac{13}{2})$ (b) $(0, \frac{17}{2}, \frac{-13}{2})$

(c) $(0, \frac{5}{2}, \frac{7}{2})$ (d) $(0, \frac{-5}{2}, \frac{7}{2})$

30. If O be the origin and the coordinates of P be (1, 2, –3), then the equation of the plane passing through P and perpendicular to OP is, [NCERT]

(a) $x + 2y - 3z - 14 = 0$ (b) $x - 2y + 3z - 14 = 0$

(c) $x + 2y + 3z - 14 = 0$ (d) None of these

31. Distance between the two planes $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is [NCERT]

(a) 2 units (b) 4 units

(c) 8 units (d) $\frac{2}{\sqrt{29}}$ units

32. The planes $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are [NCERT]

(a) perpendicular (b) parallel

(c) intersect along Y-axis (d) passes through $\left(0, 0, \frac{5}{4}\right)$

33. The image of the point having position vector $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ is the plane $\mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + 3 = 0$ is, [NCERT Exemplar]

(a) $3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ (b) $-3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

(c) $-3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ (d) None of these

34. If a plane meets the coordinate axes in A, B, C such that the centroid of the triangle is the point $(1, r, r^2)$, then equation of the plane is

(a) $x + ry + r^2z = 3r^2$ (b) $r^2x + ry + z = 3r^2$

(c) $x + ry + r^2z = 3$ (d) $r^2x + ry + z = 3$

35. The points $(0, -1, -1), (-4, 4, 4), (4, 5, 1)$ and $(3, 9, 4)$ are

(a) collinear (b) coplanar

(c) forming a square (d) forming a triangle

36. If (p_1, q_1, r_1) be the image of (p, q, r) in the plane $ax + by + cz + d = 0$, then

(a) $\frac{p_1 - p}{a} = \frac{q_1 - q}{b} = \frac{r_1 - r}{c}$

(b) $a(p + p_1) + b(q + q_1) + c(r + r_1) + 2d = 0$

(c) both (a) and (b)

(d) All of the above

37. The reflection of the point $(2, -1, 3)$ in the plane $3x - 2y - z = 9$ is

(a) $\left(\frac{26}{7}, \frac{15}{7}, \frac{17}{7}\right)$ (b) $\left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$

(c) $\left(\frac{16}{7}, \frac{26}{7}, \frac{-17}{7}\right)$ (d) $\left(\frac{1}{6}, \frac{2}{3}, \frac{3}{4}\right)$

38. The equation of the right bisecting plane of the segment joining the points (a, a, a) and $(-a, -a, -a)$, $a \neq 0$ is

(a) $x + y + z = a$ (b) $x + y + z = 3a$

(c) $x + y + z = 0$ (d) $x + y + z + a = 0$

39. The cartesian equation of the plane perpendicular to the line $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{2}$ and passing through the origin is

(a) $2x - y + 2z - 7 = 0$ (b) $2x + y + 2z = 0$

(c) $2x - y + 2z = 0$ (d) $2x - y - z = 0$

40. The equation of the plane through $P(x_1, y_1, z_1)$ and perpendicular to OP, (O being the origin) is

(a) $xx_1 + yy_1 + zz_1 = x_1 + y_1$

(b) $xx_1 + yy_1 + zz_1 = y_1 + z_1$

(c) $xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$

(d) $xx_1 + yy_1 = z + z_1$

41. The plane $x - y - z = 4$ is rotated through an angle 90° about its line of intersection with the plane $x + y + 2z = 4$. Then, the equation of the plane in its new position is

(a) $x + y + 4z = 20$ (b) $x + 5y + 4z = 20$

(c) $x + y - 4z = 20$ (d) $5x + y + 4z = 20$

42. If **a, b** and **c** are three unit vectors equally inclined to each other at angle θ. Then, angle between **a** and the plane of **b** and **c** is

(a) $\cos^{-1}\left(\frac{\cos\theta}{\cos(\theta/2)}\right)$ (b) $\sin^{-1}\left(\frac{\sin\theta}{\sin(\theta/2)}\right)$

(c) $\sin^{-1}\left(\frac{\cos\theta}{\cos(\theta/2)}\right)$ (d) $\cos^{-1}\left(\frac{\sin\theta}{\sin(\theta/2)}\right)$

43. Let $A(1, 1, 1), B(2, 3, 5)$ and $C(-1, 0, 2)$ be three points, then equation of a plane parallel to the plane ABC which is at a distance 3 is

(a) $2x - 3y + z + 3\sqrt{14} = 0$ (b) $2x - 3y + z - 2\sqrt{14} = 0$

(c) $2x - 3y + z + 2 = 0$ (d) $2x - 3y + z + \sqrt{14} = 0$

44. Let $P(3, 2, 6)$ be a point in space and Q be point on the line $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \mu(-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}})$. Then, the value of μ for which the vector **PQ** is parallel to the plane $x - 4y + 3z = 1$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$

(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

45. Let the vectors, **a, b, c** and **d** be such that $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$. Let P_1 and P_2 be planes determined by the pairs of vectors **a**, **b** and **c, d** respectively. Then, the angle between p_1 and p_2

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

46. A variable plane at a distance of the one unit from the origin cuts the coordinates axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is

(a) 3 (b) 1 (c) 1/3 (d) 9

47. Two systems of rectangular axes have the same origin. If plane cut the intercepts a', b', c' on coordinates axes for Ist system and intercepts a', b', c' on 2nd system, then pick the correct alternatives

(a) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$

(b) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}=0$

(c) $\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}-\frac{1}{c'^2}=0$

(d) $\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}+\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$

48. The line $x+2y-z-3=0, x+3y-z-4=0$ is parallel to

(a) xy-plane (b) yz-plane
(c) zx-plane (d) z-axis

49. There is a point $P(a, a, a)$ on the line passing through the origin and equally inclined with axes. The equation of plane perpendicular to OP and passing through P cuts the intercepts on axes. The sum of whose reciprocals is

(a) a (b) $\frac{3}{2a}$
(c) $\frac{3a}{2}$ (d) $\frac{1}{a}$

50. Under what condition does a straight line $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ is parallel to the xy-plane ?

(a) $l = 0$ (b) $m = 0$
(c) $n = 0$ (d) $l = 0, m = 0$

51. The equation of the plane through the points (1, 2, 3), (– 1, 4, 2) and (3, 1, 1) is

(a) $5x + y + 12z - 23 = 0$ (b) $5x + 6y + 2z - 23 = 0$
(c) $x + 6y + 2z - 13 = 0$ (d) $x + y + z - 13 = 0$

52. If for a plane, the intercepts on the coordinate axes are 8, 4, and 4, then the length of the perpendicular from the origin to the plane is

(a) 8/3 (b) 3/8
(c) 3 (d) 4/3

53. If a plane meets the coordinate axes at A , B and C such that the centroid of the triangle is (1, 2, 4), then the equation of the plane is

(a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$
(c) $x + 2y + 4z = 3$ (d) $4x + 2y + z = 3$

54. If the distance of the point (1, 1, 1) from the origin is half its distance from the plane $x + y + z + k = 0$, then k is equal to

(a) ± 3 (b) ± 6
(c) $-3, 9$ (d) $3, -9$

55. A plane makes intercepts 3 and 4 respectively on z-axis and x-axis. If plane is parallel to y-axis, then its equation is

(a) $3x + 4z = 12$ (b) $3z + 4x = 12$
(c) $3y + 4z = 12$ (d) $3z + 4y = 12$

56. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to x-axis is

(a) $y - 3z + 6 = 0$ (b) $3y - z + 6 = 0$
(c) $y + 3z + 6 = 0$ (d) $3y - 2z + 6 = 0$

57. The equation of the plane which bisects the line joining (2, 3, 4) and (6, 7, 8), is

(a) $x - y - z - 15 = 0$ (b) $x - y + z - 15 = 0$
(c) $x + y + z - 15 = 0$ (d) $x + y + z + 15 = 0$

58. Equation of the plane which passes through the line of intersection of the planes $P = ax + by + cz + d = 0$, $P' = a'x + b'y + c'z + d' = 0$ and parallel to x-axis, is

(a) $Pa - P'a' = 0$ (b) $P/a = P'/a' = 0$
(c) $Pa + P'a' = 0$ (d) $P/a = P'/a'$

59. The points $A(-1, 3, 0)$, $B(2, 2, 1)$ and $C(1, 1, 3)$ determine a plane. The distance from the plane to the point $D(5, 7, 8)$ is

(a) $\sqrt{66}$ (b) $\sqrt{71}$
(c) $\sqrt{73}$ (d) $\sqrt{76}$

60. If M denotes the mid-point of the line joining $A(4\hat{i}+5\hat{j}-10\hat{k})$ and $B(-\hat{i}+2\hat{j}+\hat{k})$, then equation of the plane through M and perpendicular to AB is

(a) $\mathbf{r}\cdot(5\hat{i}+3\hat{j}-11\hat{k})-\frac{135}{2}=0$

(b) $\mathbf{r}\cdot\left(\frac{3}{2}\hat{i}+\frac{7}{2}\hat{j}-\frac{9}{2}\hat{k}\right)+\frac{135}{2}=0$

(c) $\mathbf{r}\cdot(4\hat{i}+5\hat{j}-10\hat{k})+4=0$

(d) $\mathbf{r}\cdot(-\hat{i}+2\hat{j}+\hat{k})+4=0$

61. If the foot of the perpendicular from the origin to a plane is (a, b, c), then equation of the plane is

(a) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(b) $ax + by + cz = 1$

(c) $ax + by + cz = a^2 + b^2 + c^2$

(d) $ax + by + cz = 0$

62. Equation of the plane through three points A, B and C with position vectors $-6\hat{i}+3\hat{j}+2\hat{k}$, $3\hat{i}-2\hat{j}+4\hat{k}$ and $5\hat{i}+7\hat{j}+3\hat{k}$ is

(a) $\mathbf{r}\cdot(\hat{i}-\hat{j}-7\hat{k})+23=0$
(b) $\mathbf{r}\cdot(\hat{i}+\hat{j}+7\hat{k})=23$
(c) $\mathbf{r}\cdot(\hat{i}+\hat{j}-7\hat{k})+23=0$
(d) $\mathbf{r}\cdot(\hat{i}-\hat{j}-7\hat{k})=23$

Problems Based on Straight Line

63. The values of p, so that the lines $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are at [NCERT]

(a) acute angles (b) obtuse angles
(c) right angles (d) None of these

64. The shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ is [NCERT]

(a) $\sqrt{29}$ units (b) 29 units
(c) $\frac{29}{2}$ units (d) $2\sqrt{29}$ units

65. The vector equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$, is [NCERT]

(a) $\mathbf{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$
(b) $\mathbf{r}=(2\hat{i}+3\hat{j}-6\hat{k})+\lambda(\hat{i}+2\hat{j}-4\hat{k})$
(c) $\mathbf{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(3\hat{i}+8\hat{j}-5\hat{k})$
(d) $\mathbf{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(3\hat{i}-16\hat{j}-7\hat{k})$

66. If the coordinates of the points A, B, C and D be $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$ and $(2, 9, 2)$ respectively, then the angle between the lines AB and CD is [NCERT]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) π (d) None of these

67. The vector equation of the straight line passing through $(1, 2, 3)$ and perpendicular to the plane $\mathbf{r}\cdot(\hat{i}+2\hat{j}-5\hat{k})+9=0$, is [NCERT]

(a) $\mathbf{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}+2\hat{j}-5\hat{k})$
(b) $\mathbf{r}=(\hat{i}+2\hat{j}-5\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$
(c) $\mathbf{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-8\hat{k})$
(d) None of the above

68. The intersection point of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ is [NCERT Exemplar]

(a) $(1, 1, 1)$ (b) $(-1, -1, -1)$
(c) $(1, 2, 3)$ (d) $(2, 2, 2)$

69. The distance of the point $(-2, 4, -5)$ from the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, is [NCERT Exemplar]

(a) $\sqrt{\frac{37}{10}}$ (b) $\frac{37}{\sqrt{10}}$
(c) $\frac{\sqrt{37}}{10}$ (d) None of these

70. The distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\mathbf{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane $\mathbf{r}(\hat{i}-\hat{j}+\hat{k})=5$ is [NCERT Exemplar]

(a) 11 (b) 12 (c) $\sqrt{11}$ (d) 13

71. The coordinates of the foot of perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$ is [NCERT Exemplar]

(a) $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$ (b) $\left(\frac{5}{3}, \frac{-2}{3}, \frac{-19}{3}\right)$
(c) $\left(\frac{-5}{3}, \frac{2}{3}, \frac{-19}{3}\right)$ (d) None of these

72. The image of the point $(1, 6, 3)$ in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$, is [NCERT Exemplar]

(a) $(1, 6, 9)$ (b) $(1, 0, 7)$ (c) $(0, 2, 7)$ (d) $(7, 9, 0)$

73. The lines $x=py+q$, $z=ry+s$ and $x=p'y+q'$, $z=r'y+s'$ are perpendicutar, if [NCERT Exemplar]

(a) $pp'+rr'=1$ (b) $pp'-rr'=1$
(c) $pp'+rr'+1=0$ (d) $p'p+r'r=0$

74. A perpendicular is drawn from the point $P(2, 4, -1)$ to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$. The equation of the perpendicular from P to the given line is

(a) $\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$ (b) $\frac{x+2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$
(c) $\frac{x+2}{-6}=\frac{y-4}{3}=\frac{z+1}{2}$ (d) $\frac{x+2}{6}=\frac{y+4}{3}=\frac{z+1}{2}$

75. Lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, if

(a) $k=0$ (b) $k=-1$ (c) $k=2$ (d) $k=3$

76. Two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=z$ intersect at a point, if k is equal to

(a) $\frac{2}{9}$ (b) $\frac{1}{2}$ (c) $\frac{9}{2}$ (d) $\frac{1}{6}$

77. For the lines L_1 $\mathbf{r}=\mathbf{a}+t(\mathbf{b}+\mathbf{c})$ and L_2 $\mathbf{r}=\mathbf{b}+s(\mathbf{c}+\mathbf{a})$, then L_1 and L_2 intersect at

(a) $\mathbf{a}$ (b) $\mathbf{b}$ (c) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ (d) $\mathbf{a}+2\mathbf{b}$

78. The lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x-1}{-2}=\frac{y-2}{-4}=\frac{z-3}{-6}$ are

(a) interecting (b) skew
(c) parallel (d) coincident

79. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

80. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

(a) $\sqrt{30}$ (b) $2\sqrt{30}$ (c) $5\sqrt{30}$ (d) $3\sqrt{30}$

81. The length of the perpendicular drawn from (1, 2, 3) to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(a) 4 (b) 5 (c) 6 (d) 7

Miscellaneous Problems

82. The coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane passing through three points (2, 2, 1), (3, 0, 1) and (4, –1, 0) is, [NCERT Exemplar]

(a) (1, 2, 7) (b) (-1, 2, -7)
(c) (1, –2, 7) (d) None of these

83. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, then the direction cosines of the line perpendicular to both of these are [NCERT]

(a) $m_1 n_2 + m_2 n_1, n_1 l_2 + n_2 l_1, l_1 m_2 + l_2 m_1$
(b) $m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1$
(c) $m_1 m_2 - n_1 n_2, n_1 n_2 - l_1 l_2, l_1 l_2 - m_1 m_2$
(d) $m_1 m_2 + n_1 n_2, n_1 n_2 + l_1 l_2, l_1 l_2 + m_1 m_2$

84. The equation of the line passing through the point (3, 0, 1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$, is [NCERT Exemplar]

(a) $\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$ (b) $\frac{x-3}{1} = \frac{y-0}{-2} = \frac{z-1}{3}$
(c) $\frac{x-3}{3} = \frac{y-0}{1} = \frac{z-1}{-2}$ (d) None of these

85. The ratio in which $2x + 3y + 5z = 1$ divides the line joining the points (1, 0, –3) and (1, –5, 7) is

(a) 2 : 3 (b) 3 : 2 (c) 2 : 1 (d) 1 : 3

86. If $A(0,0,0), B(a,0,0), C(0,b,0)$, and $D(0,0,c)$, are the vertices of a tetrahedron, then the volume of tetrahedron is

(a) abc cu units (b) $\frac{abc}{2}$ cu units
(c) $\frac{abc}{3}$ cu units (d) $\frac{abc}{6}$ cu units

87. The angle between the line $\mathbf{r} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) + \lambda\,(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and the plane $\mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 4$ is,

(a) 0 (b) $\frac{\pi}{2}$
(c) π (d) None of these

88. The x-coordinate of a point on the line joining the points $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then its z-coordinate is [NCERT Exemplar]

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) 0

89. The angle between the lines whose direction cosines are given by $2l - m + 2n = 0$, $lm + mn + nl = 0$, is [NCERT Exemplar]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

90. The angle between the lines whose direction are given by $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is [NCERT Exemplar]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

91. The angle between any two diagonals of a cube is [NCERT Exemplar]

(a) $\sin^{-1}\frac{2}{3}$ (b) $\cos^{-1}\frac{1}{2}$ (c) $\cos^{-1}\frac{1}{\sqrt{3}}$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

92. $A(3, 2, 0)$, $B(5, 3, 2)$ and $C(-9, 6, -3)$ are the vertices of a ΔABC. If the bisector of $\angle BAC$ meets BC at D, then coordinates of D are

(a) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$
(c) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$ (d) $\left(\frac{19}{8}, \frac{57}{16}, -\frac{17}{16}\right)$

93. The coordinates of a point of the line $\frac{x-1}{2} = \frac{y+1}{-3} = z$ at a distance $4\sqrt{14}$ from the point (1, – 1, 0) nearer the origin is

(a) (9, –13, 4)
(b) $(8\sqrt{14} + 1, -12\sqrt{14} - 1, 4\sqrt{14})$
(c) (–7, 11, – 4)
(d) $(-8\sqrt{14} + 1,\ 12\sqrt{14} - 1, -4\sqrt{14})$

94. A line segment has length 63 and direction ratios are 3, – 2, 6. If the line makes an obtuse angle with x-axis, the components of the line vector are

(a) 27, – 18, 54 (b) –27, 18, 54
(c) –27, 18, – 54 (d) 27, – 18, – 54

95. The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes is

(a) 60 (b) 600 (c) 720 (d) 400

Round II (Mixed Bag)

Only One Correct Option

1. The area of the triangle whose vertices are (1, 0, 1), (2, – 1, 3) and (–1, 2, – 1), is
(a) 1 (b) 2
(c) $\sqrt{2}$ (d) None of these

2. A plane is such that the foot of perpendicular drawn from the origin to it is (2, – 1, 1). The distance of (1, 2, 3) from the plane is
(a) 3/2 (b) $\sqrt{3/2}$
(c) 2 (d) None of these

3. A and B are two given points. Let C divides AB internally and D divides AB externally in the same ratio. Then, AC, AB and AD are in
(a) AP (b) GP
(c) HP (d) None of these

4. The angle between a diagonal of a cube and an edge of the cube intersecting the diagonal is
(a) $\cos^{-1}\frac{1}{3}$ (b) $\cos^{-1}\sqrt{\frac{2}{3}}$
(c) $\tan^{-1}\sqrt{2}$ (d) None of these

5. The equation of the plane through (3, 1, – 3) and (1, – 2, 2) and parallel to the line with DR's 1, 1, – 2 is
(a) $x - y + z + 1 = 0$ (b) $x + y - z + 1 = 0$
(c) $x - y - z - 1 = 0$ (d) $x + y + z - 1 = 0$

6. A line makes an angle θ both with x and y-axes. A possible value of θ is in
(a) $\left[0, \frac{\pi}{4}\right]$ (b) $\left[0, \frac{\pi}{2}\right]$ (c) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{3}, \frac{\pi}{6}\right]$

7. A plane passes through the point (1, – 2, 3) and is parallel to the plane $2x - 2y + z = 0$. The distance of the point (–1, 2, 0) from the plane is
(a) 2 (b) 3 (c) 4 (d) 5

8. The image of the point $A(1, 0, 0)$ in the line $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ is
(a) (5, –8, –4) (b) (3, –4, 2)
(c) (5, –4, –8) (d) (3, 4, –2)

9. The projection of the line segment joining (2, 5, 6) and (3, 2, 7) on the line with direction ratios 2, 1, – 2 is
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 1

10. If the points (1, 2, 3) and (2, – 1, 0) lie on the opposite sides of the plane $2x + 3y - 2z = k$, then
(a) $k < 1$ (b) $k > 2$
(c) $k < 1$ or $k > 2$ (d) $1 < k < 2$

11. The plane passing through the point (–2, – 2, 2) and containing the line joining the points (1, – 1, 2) and (1, 1, 1) makes intercepts on the coordinate axes and the sum of whose length is
(a) 3 (b) 6
(c) 12 (d) 20

12. $OABC$ is a tetrahedron such that $OA = OB = OC = k$ and each of the edges OA, OB and OC is inclined at an angle θ with the other two the range of θ is
(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) $\left[0, \frac{2\pi}{3}\right]$

13. $OABC$ is regular tetrahedron of unit edge. Its volume is
(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{6}}$
(c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{6\sqrt{2}}$

14. Find the distance of the plane $x + 2y - z = 2$ from the point (2, – 1, 3) as measured in the direction with DR's (2, 2, 1).
(a) 2 (b) –3
(c) –2 (d) 3

15. Find the planes bisecting the acute angle between the planes $x - y + 2z + 1 = 0$ and $2x + y + z + 2 = 0$.
(a) $x + z - 1 = 0$ (b) $x + z + 1 = 0$
(c) $x - z - 1 = 0$ (d) None of these

16. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive x-axis, then cos α is equal to
(a) 1/2 (b) 1
(c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

17. If the orthocentre and centroid of a triangle are (–3, 5, 1) and (3, 3, – 1) respectively, then its circumcentre is
(a) (6, 2, –2) (b) (1, 2, 0)
(c) (6, 2, 2) (d) (6, –2, 2)

18. A variable plane at a distance of 1 unit from the origin coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then k is equal to
(a) 3 (b) 1
(c) $\frac{1}{3}$ (d) 9

19. If α, β, γ and δ are the angles between a straight line with the diagonals of a cube, then

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{8}{3}$

(c) $\frac{7}{4}$ (d) None of these

20. The direction cosines to two lines at right angles are $(1, 2, 3)$ and $\left(-2, \frac{1}{2}, \frac{1}{3}\right)$, then the direction cosine perpendicular to both the given lines are

(a) $\sqrt{\frac{25}{2198}}, \sqrt{\frac{38}{1099}}, \sqrt{\frac{729}{2198}}$ (b) $\sqrt{\frac{24}{2198}}, \sqrt{\frac{38}{2198}}, \sqrt{\frac{730}{2198}}$

(c) $\frac{1}{3}, -2, \frac{-7}{2}$ (d) None of these

21. The distance of the point $A(-2, 3, 1)$ from the line PQ through $P(-3, 5, 2)$ which make equal angles with the axes is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{\frac{14}{3}}$

(c) $\frac{16}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

22. A equation of the plane passing through the points $(3, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$ is $5x + 3y - 2z = \lambda$, where λ is

(a) 23 (b) 21

(c) 19 (d) 27

23. A variable plane which remains at a constant distance p from the origin cuts the coordinate axes in A, B, C. The locus of the centroid of the tetrahedron $OABC$ is $y^2z^2 + z^2x^2 + x^2y^2 = kx^2y^2z^2$, where k is equal to

(a) $9p^2$ (b) $\frac{9}{p^2}$

(c) $\frac{7}{p^2}$ (d) $\frac{16}{p^2}$

24. The line joining the points $(1, 1, 2)$ and $(3, -2, 1)$ meets the plane $3x + 2y + z = 6$ at the point

(a) $(1, 1, 2)$ (b) $(3, -2, 1)$

(c) $(2, -3, 1)$ (d) $(3, 2, 1)$

25. The point on the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{-2}$ at a distance of 6 from the point $(2, -3, -5)$ is

(a) $(3, -5, -3)$ (b) $(4, -7, -9)$

(c) $(0, 2, -1)$ (d) $(-3, 5, 3)$

26. The plane passing through the point $(5, 1, 2)$ perpendicular to the line $2(x-2) = y - 4 = z - 5$ will meet the line in the point

(a) $(1, 2, 3)$ (b) $(2, 3, 1)$

(c) $(1, 3, 2)$ (d) $(3, 2, 1)$

27. The point equidistant from the point $(a, 0, 0), (0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$ is

(a) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ (b) (a, b, c)

(c) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (d) None of these

28. P, Q, R, and S are four coplanar points on the sides AB, BC, CD, and DA of a skew quadrilateral. The product $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA}$ is equal to

(a) -2 (b) -1 (c) 2 (d) 1

29. The straight lines whose direction cosines l, m and n are given by the equations $al + bm + cn = 0$, $ul^2 + vm^2 + wn^2 = 0$ are perpendicular or parallel according as

(a) $a(v+w) + b(w+u) + c(u+v) = 0$ or $\frac{a}{u} + \frac{b}{v} + \frac{c}{w} = 0$

(b) $a(v-w) + b(w-u) + c(u-v) = 0$ or $\frac{a}{u} + \frac{b}{v} + \frac{c}{w} = 0$

(c) $a^2(v+w) + b^2(w+u) + c^2(u+v) = 0$ or $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$

(d) None of the above

30. If a variable line in two adjacent positions had direction cosines l, m and n and direction cosines for 2nd position. It $l + \delta l, m + \delta m, n + \delta n$, then the small angle so between two positions is given by

[NCERT Exemplar]

(a) $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$ (b) $\delta\theta = \delta l + \delta m + \delta n$

(c) $\delta\theta^2 = \delta l + \delta m + \delta n$ (d) $\delta\theta = \delta l^2 + \delta m^2 + \delta n^2$

31. A triangle, the lengths of whose sides are a, b and c is placed so that the middle points of the sides are on the axes, then the equation to the plane is

(a) $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 1$ (b) $\alpha x + \beta y + \gamma z = 1$

(c) $\alpha^2 x + \beta^2 y + \gamma^2 z = 1$ (d) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

where,

$\alpha^2 = \frac{(b^2 + c^2 - a^2)}{8}, \beta^2 = \frac{(c^2 + a^2 - b^2)}{8}, \gamma^2 = \frac{(a^2 + b^2 - c^2)}{8}$

32. The four planes $my + nz = 0, nz + lx = 0, lx + my = 0$ and $lx + my + nz = p$ form a tetrahedron whose volume is

(a) $\frac{3p^3}{2lmn}$ (b) $\frac{2p^3}{3lmn}$ (c) $\frac{2p^2}{3lmn}$ (d) $\frac{3p^2}{2lmn}$

33. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Through A, B and C planes are drawn parallel to the coordinate planes, then the locus of their point of intersection is

(a) $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ (b) $x^2 + y^2 + z^2 = p^2$

(c) $x^2 + y^2 + z^2 = 2p^2$ (d) $x^{-2} + y^{-2} + z^{-2} = \frac{p^{-2}}{2}$

34. ABC is any triangle and O is any point in the plane of the triangle. AO, BO, CO meet the sides BC, CA, AB in D, E, F respectively, then $\frac{OD}{AD}+\frac{OE}{BE}+\frac{OF}{CF}$ is equal to

(a) 0 (b) –1
(c) 1 (d) 2

35. The angles between the four diagonals of a rectangular parallelopiped whose edges are a, b and c are **[NCERT Exemplar]**

(a) $\cos^{-1}\left(\frac{a^2+b^2+c^2}{a^2\pm b^2\pm c^2}\right)$ (b) $\cos^{-1}\left(\frac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$
(c) $\frac{1}{2}\cos^{-1}\left(\frac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$ (d) None of these

36. The direction cosines l, m, n of two lines which are connected by the relations $l-5m+3n=0$ and $7l^2+5m^2-3n^2=0$ are

(a) $\frac{-1}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$ and $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}$
(b) $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ and $\frac{1}{\sqrt{14}}, \frac{2}{14}, \frac{3}{14}$
(c) $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ and $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$
(d) $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ and $\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

37. A parallelopiped S has base points A, B, C and D and upper face points A', B', C' and D'. This parallelopiped is compressed by upper face $A¢B¢C¢D¢$ to form a new parallelopiped T having upper face points $A¢¢, B¢¢, C¢¢$ and $D¢¢$. Volume of parallelepiped T is 90% of the volume of the parallelpiped S, the locus of $A¢¢$ is

(a) a plane
(b) a straight line
(c) a parallelopiped
(d) None of the above

38. In a $\Delta ABC, D$ divides BC in the ratio 3:2 and E divides CA in the ratio 1:3. The lines AD and BE meet at H and CH meets AB in F, then the ratio in which F divides AB is

(a) 1:2 (b) 2:1
(c) 3:2 (d) 2:3

39. The equation of the plane through the line of intersection of the planes $ax+by+cz+d=0$ and $\alpha x+\beta y+\gamma z+\delta=0$ perpendicular to the xy-plane is

(a) $(a\gamma+c\alpha)x+(b\gamma-c\beta)y+(d\gamma-c\delta)=0$
(b) $(a\gamma-c\alpha)x+(b\gamma-c\beta)y+(d\gamma-c\delta)=0$
(c) $(a\gamma+c\alpha)x-(b\gamma+c\beta)y-(d\gamma-c\delta)=0$
(d) None of the above

40. A line with direction cosines proportional to (2,7,–5) is drawn to intersect the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$, then the coordinate of the points of intersection are

(a) (2,8,–3) and (0,1,2) (b) (–2,–8,3) and (0,–1,2)
(c) (2,–8,3) and (0,1,–2) (d) None of these

More Than One Correct Option

41. The line $\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z-1}{3}$ and the plane $x+2y+z=6$ meet in

(a) no point
(b) the line lies on the plane
(c) infinitely many points
(d) at one point

42. If the direction cosines l, m, n of a line are related by the equations $l+m+n=0, 2mn+2ml-nl=0$, then the ordered triplet (l, m, n) is

(a) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \frac{-2}{\sqrt{6}}\right)$ (b) $\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \frac{2}{\sqrt{6}}\right)$
(c) $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}}\right)$ (d) $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} -\frac{1}{\sqrt{6}}\right)$

43. Consider the point $(1, 3, 4)$ and the plane $2x-y+z+3=0$ is

(a) the foot of the perpendicular form (1, 3, 4) to the plane is (–1, 4, 3)
(b) image of the point in plane is (–3, 5, 2)
(c) image of the point in plane is (–1, 4, 3)
(d) image of the point in plane is (–1, 2, 4)

44. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B and C. The locus of a point equidistant from origin A, B and C must be

(a) $ayz+bzx+cxy=2xyz$ (b) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$
(c) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$ (d) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3$

45. If $OABC$ is a tetrahedron such that $OA^2+BC^2=OB^2+CA^2=OC^2+AB^2$, then

(a) $OA \perp BC$ (b) $OB \perp CA$
(c) $OC \perp AB$ (d) $AB \perp BC$

46. If α, β and γ are the angles which a line makes with the coordinate axes, then

(a) $\sin^2\alpha=\cos^2\beta+\cos^2\gamma$
(b) $\cos^2\alpha=\cos^2\beta+\cos^2\gamma$
(c) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$
(d) $\sin^2\alpha+\sin^2\beta=1+\cos^2\gamma$

Comprehension Based Questions

Passage

Directions (Q. Nos. 47-49) *Two lines whose equations are* $\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{\lambda}$ *and* $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z-2}{3}$ *lie in the same plane, then*

47. The value of $\sin^{-1}\sin\lambda$ is equal to
(a) 3 (b) $\pi-3$
(c) 4 (d) $\pi-4$

48. Point of intersection of the lines lies on
(a) $3x+y+z=20$ (b) $2x+y+z=25$
(c) $3x+2y+z=24$ (d) $x=y=z$

49. Angle between the plane containing both lines and the plane $4x+y+2z=0$ is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{6}$ (d) $\cos^{-1}\frac{2}{\sqrt{186}}$

Assertion and Reason

Directions (Q. Nos. 50-56) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below* :
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Satement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

50. **Statement I** If $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ then equation $\mathbf{r}\times(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}})=3\hat{\mathbf{i}}+\hat{\mathbf{k}}$

Statement II If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ then equation $\mathbf{r}\times(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})=2\hat{\mathbf{i}}-\hat{\mathbf{j}}$ represent a straight line

51. **Statement I** If centroid and circumcentre of a triangle are known its orthocentre can be found.

Statement II Centroid, orthocentre and circumcentre of a triangle are collinear.

52. **Statement I** The shortest distance between the skew lines $\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2}$ and $\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$ is 9.

Statement II Two lines are skew lines if there exists no plane passing through them.

53. **Statement I** The points $A(2,9,12)$, $B(1,8,8)$, $C(-2,11,8)$ and $D(-1,12,12)$ and the vertices of a rhombus.

Statement II $AB=BC=CD=DA$ and $AC\neq BD$.

54. **Statement I** A point on the straight line $2x+3y-4z=5$ and $3x-2y+4z=7$ can be determined by taking $x=k$ and then solving the two equations for y and z, where k is any real number.

Statement II If $c'\neq kc$, then the straight line $ax+by+cz+d=0, kax+kby+c'z+d'=0$, does not intersect the plane $z=\alpha$, where α is any real number.

55. Consider the line is perpendicular to the plane, then DR's plane is proportional to the line.

Statement I The lines $\frac{x-1}{1}=\frac{y}{-1}=\frac{z+1}{1}$ and $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ are coplanar and equation of the plane containing them is $5x+2y-3z-8=0$.

Statement II The line $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ is perpendicular to the plane $3x+6y+9z-8=0$ and parallel to the plane $x+y-z=0$.

56. Consider the lines
$$L_1:\frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2},\quad L_2:\frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}.$$

Statement I The distance of the point (1, 1, 1) from the plane passing through the point (–1, – 2, – 1) and whose normal is perpendicular to both the lines L_1 and L_2 is $\frac{13}{5\sqrt{3}}$.

Statement II The unit vector perpendicular to both the lines L_1 and L_2 is $\frac{-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}}}{5\sqrt{3}}$.

Previous Years' Questions

57. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is [JEE Main 2013]

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$

58. If the lines

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k} \text{ and } \frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$$

are coplanar, then k can have

(a) any value (b) exactly one value
(c) exactly two values (d) exactly three values

59. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to [AIEEE 2012]

(a) -1 (b) $\frac{2}{9}$ (c) $\frac{9}{2}$ (d) 0

60. An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is [AIEEE 2012]

(a) $x - 2y + 2z - 3 = 0$ (b) $x - 2y + 2z + 1 = 0$
(c) $x - 2y + 2z - 1 = 0$ (d) $x - 2y + 2z + 5 = 0$

61. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ is equal to [AIEEE 2011]

(a) $\frac{3}{2}$ (b) $\frac{2}{5}$
(c) $\frac{5}{3}$ (d) $\frac{2}{3}$

62. The length of the perpendicular drawn from the point $(3, -1, 11)$ to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is [AIEEE 2011]

(a) $\sqrt{66}$ (b) $\sqrt{29}$ (c) $\sqrt{33}$ (d) $\sqrt{53}$

63. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along a straight line $x = y = z$ is [AIEEE 2011]

(a) $3\sqrt{5}$ (b) $10\sqrt{3}$
(c) $5\sqrt{3}$ (d) $3\sqrt{10}$e

64. A line AB in three dimensional space makes angles 45° and 120° with the positive x-axis and the positive y-axis, respectively. If AB makes an acute angle θ with the positive z-axis, then θ is equals

(a) 30° (b) 45° (c) 60° (d) 75°

65. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then, (α, β) is equal to [AIEEE 2009]

(a) $(6, -17)$ (b) $(-6, 7)$
(c) $(5, -15)$ (d) $(-5, 15)$

Answers

Check Point 1

1. (d) **2.** (b) **3.** (b) **4.** (b) **5.** (c) **6.** (d) **7.** (c) **8.** (b) **9.** (b) **10.** (a)

Check Point 2

1. (a) **2.** (a) **3.** (c) **4.** (d) **5.** (a)

Round I

1. (a)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (a)
11. (d)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (a)
21. (c)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (a)
31. (d)	**32.** (b)	**33.** (c)	**34.** (b)	**35.** (b)	**36.** (c)	**37.** (b)	**38.** (c)	**39.** (c)	**40.** (c)
41. (d)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (a)	**46.** (d)	**47.** (a)	**48.** (c)	**49.** (d)	**50.** (c)
51. (b)	**52.** (a)	**53.** (b)	**54.** (d)	**55.** (a)	**56.** (a)	**57.** (c)	**58.** (d)	**59.** (a)	**60.** (a)
61. (c)	**62.** (a)	**63.** (c)	**64.** (d)	**65.** (a)	**66.** (c)	**67.** (a)	**68.** (b)	**69.** (a)	**70.** (d)
71. (a)	**72.** (b)	**73.** (c)	**74.** (a)	**75.** (a)	**76.** (c)	**77.** (c)	**78.** (d)	**79.** (c)	**80.** (d)
81. (d)	**82.** (c)	**83.** (b)	**84.** (a)	**85.** (a)	**86.** (d)	**87.** (d)	**88.** (b)	**89.** (d)	**90.** (c)
91. (d)	**92.** (a)	**93.** (c)	**94.** (c)	**95.** (b)					

Round II

1. (c)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (c)	**7.** (d)	**8.** (a)	**9.** (d)	**10.** (d)
11. (c)	**12.** (d)	**13.** (d)	**14.** (d)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (d)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (a)
31. (d)	**32.** (b)	**33.** (a)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (a)
41. (b,c)	**42.** (a,b,c,d)	**43.** (a,b)	**44.** (a,c)	**45.** (a,b,c)	**46.** (a,c,d)	**47.** (d)	**48.** (d)	**49.** (b)	**50.** (c)
51. (c)	**52.** (c)	**53.** (b)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (c)	**59.** (c)	**60.** (a)
61. (d)	**62.** (d)	**63.** (b)	**64.** (c)	**65.** (b)					

the Guidance

Round I

1. Let a line makes angles α, β and γ with the positive x, y and z- axes respectively. Then, direction cosines of a line are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

Let direction cosines of the line be l, m and n with the x, y and z-axes respectively and given that

$\alpha = 90°, \beta = 135°$ and $\gamma = 45°$

Then, $l = \cos\alpha = \cos 90° = 0,$

$m = \cos\beta = \cos 135° = -1/\sqrt{2}$

and $n = \cos\gamma = \cos 45° = 1/\sqrt{2}$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

2. Firstly, we determine the direction ratio of joining two points *i.e., AB, BC, CA,* if they are proportional, then points are collinear.

Let the given points be A, B and C respectively, then $A(2, 3, 4)$, $B(-1, -2, 1)$ and $C(5, 8, 7)$.

It is known that the direction ratios of line joining the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Then, direction ratios of AB are $(-1-2, -2-3, 1-4)$

$\Rightarrow \quad (-3, -5, -3)$

Direction ratios of AC are $(5-2, 8-3, 7-4) \Rightarrow (3, 5, 3)$

As $\frac{-3}{3} = \frac{-5}{5} = \frac{-3}{3}$, therefore the direction ratios of AB and AC are proportional.

Hence, AB and AC are parallel, but these have a point A in common, therefore AB and AC are along the same line *i.e.*, A, B and C are collinear.

3. Let the vertices of the triangle be $A(3, 5, -4)$, $B(-1, 1, 2)$ and $C(-5, -5, -2)$ respectively.

The direction ratios of side AB are $\{-1, -3, 1-5, 2-(-4)\}$

$(-4, -4, 2+4)$, *i.e*, $(-4, -4, 6)$.

[$\because$ if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then DR's of $AB = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$]

Then, magnitude of AB,

$$|AB| = \sqrt{(-4)^2 + (-4)^2 + (6)^2}$$
$$= \sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$$

Therefore, the direction cosines of AB are

$$\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}} \Rightarrow \frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

($\because$ If a, b and c are direction ratios, then direction cosines are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$)

4. Note that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1-\cos^2\alpha) + (1-\cos^2\beta) + (1-\cos^2\gamma)$$
$$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2$$

5. Let it makes angle α with x-axis. Then,

$$\cos^2\alpha + \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = 1$$

which after simplification gives $\alpha = \frac{\pi}{2}$

6. DR's of PQ are $(2, 2, 6)$

i.e., $(1, 1, 3)$

DR's of SR are $(1, 1, 3)$ (since, $PQ \,||\, SR$)

$R \equiv (1, 2, -1)$

By checking options, coordinates of S are $(3, 4, 5)$.

7. Let l, m and n be the DC's of given the line. Then, as it makes an acute angle with x-axis, therefore $l > 0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$, therefore its DR are

$6-2, -7+3, -1-1$ or $4, -4, -2$ or $2, -2, -1$

DC of the given line are $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$.

8. Let the DC's of given line be (l, m, n). Diagonals of cube are OP, AL, BM, CN. The DC's of the diagonals are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$\therefore \quad \cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{-l+m+n}{\sqrt{3}},$$
$$\cos\gamma = \frac{l-m+n}{\sqrt{3}}, \cos\delta = \frac{l+m-n}{\sqrt{3}}$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}(l^2+m^2+n^2) = \frac{4}{3}$$

9. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma - 3 = -1$

10. Let l, m, n be DC's of the given line segment r so projection on x, y and z is are lr, mr and nr, so $l^2r^2 + m^2r^2 + n^2r^2$

$$3^2 + 1^2 + (\sqrt{15})^2 = r^2 \Rightarrow r = 5$$

11. $\mathbf{a}$ parallel to line OA is $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

$\mathbf{b}$ parallel to line OB is $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

$$\therefore \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & -1 \\ 3 & -2 & 3 \end{vmatrix} = -8\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Required DR's are –8, –6, 4 or 4, 3, –2.

12. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore \quad \frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1, \cos^2\gamma = \frac{1}{4}, \cos\gamma = \pm\frac{1}{2}$$

$\therefore x$-coordinate $= OP \times \cos\alpha = 12 \times \frac{1}{\sqrt{2}} = 6\sqrt{2}$

y-coordinate $= OP \times \cos\beta = 12 \times \frac{1}{2} = 6$

z-coordinate $= OP \times \cos\gamma = 12 \times \left(\pm\frac{1}{2}\right) = \pm 6$

13. Since, it is given that line makes equal angle with the coordinate axes.

$\therefore \quad l = m = n$

We know, $\quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad 3l^2 = 1 \Rightarrow l^2 = \frac{1}{3}$

$\Rightarrow \quad l = \frac{1}{\sqrt{3}}$ (neglect –ve sign)

14. If $\left(\frac{1}{2}, \frac{1}{3}, n\right)$ are the DC's of line, then using the relation $l^2 + m^2 + n^2 = 1$, we get

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + n^2 = 1$$

$\Rightarrow \quad n^2 = 1 - \frac{1}{4} - \frac{1}{9}$

$\Rightarrow \quad n^2 = \frac{23}{36}$

$\Rightarrow \quad n = \frac{\sqrt{23}}{6}$

15. $\because \cos\alpha\cos\beta = \cos\beta\cos\gamma = \frac{2}{9}$ and $\cos\gamma\cos\alpha = \frac{4}{9}$,

then $\cos\alpha = \frac{2}{3}$, $\cos\beta = \frac{1}{3}$ and $\cos\gamma = \frac{2}{3}$

$$\therefore \quad \cos\alpha + \cos\beta + \cos\gamma = \frac{2}{3} + \frac{1}{3} + \frac{2}{3} = \frac{5}{3}$$

16. Let the given points are A, B and C respectively.

$\therefore$ Direction ratios of AB and BC are $1, -3, -2$ and $2, -6, K-2$ respectively.

Since, given points are collinear.

$$\therefore \quad \frac{2}{1} = \frac{-6}{-3} = \frac{K-2}{-2}$$

$\Rightarrow \quad K - 2 = -4$

$\Rightarrow \quad K = -2$

17. The equation of straight line passing through origin and direction cosine (l, m, n) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \quad \text{(say)}$$

Coordinates of any point P are (lr, mr, nr).

Here, $\quad l = \frac{-1}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{-1}{3}, m = \frac{2}{3}, n = \frac{-2}{3}$

and $\quad r = 3$ (given)

$\therefore$ Coordinates of P are $(-1, 2, -2)$.

18. Given that, $\quad l + m + n = 0 \quad$ ...(i)

and $\quad lm = 0 \quad$ ...(ii)

$\therefore$ From Eq. (i), $l = -(m + n)$

and from Eq. (ii), $\quad -(m + n)m = 0$

$\Rightarrow \quad -(m^2 + mn) = 0$

$\Rightarrow \quad m^2 + mn = 0$

$\Rightarrow \quad m = 0, \; m + n = 0$

If $\quad m = 0, \; l + m + n = 0 \quad$ [from Eq. (i)]

Then, $\quad \frac{l_1}{-1} = \frac{m_1}{0} = \frac{n_1}{1}$

and if $\quad l + m + n = 0$,

then $\quad \frac{l_2}{0} = \frac{m_2}{-1} = \frac{n_2}{1}$

$\therefore \quad (l_1, m_1, n_1) = (-1, 0, 1)$

and $\quad (l_2, m_2, n_2) = (0, -1, 1)$

$\therefore$ Angle between them is given by

$$\cos\theta = \frac{0 + 0 + 1}{\sqrt{1 + 0 + 1}\sqrt{0 + 1 + 1}} = \frac{1}{2} = \frac{\pi}{3}$$

19. Let $a_1 = 2x, b_1 = 2x, c_1 = x$

and $\quad a_2 = 7 - 3 = 4, \; b_2 = 2 - 1 = 1,$

$c_2 = 12 - 4 = 8$

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{2x \times 4 + 2x \times 1 + x \times 8}{\sqrt{4x^2 + 4x^2 + x^2}\sqrt{16 + 1 + 64}} = \frac{18x}{3x \times 9} = \frac{2}{3}$$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$

20. Firstly, find $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and use equation of plane with position vector $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, where , $\mathbf{n} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and d is the distance of normal from origin.

The normal vector is, $\mathbf{n} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$.

The unit vector normal to the plane is $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$.

$$\therefore \quad \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 6\hat{\mathbf{k}}}{\sqrt{3^2 + 5^2 + (-6)^2}} = \frac{3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 6\hat{\mathbf{k}}}{\sqrt{70}}$$

$$= \frac{3}{\sqrt{70}}\hat{\mathbf{i}} + \frac{5}{\sqrt{70}}\hat{\mathbf{j}} - \frac{6}{\sqrt{70}}\hat{\mathbf{k}}$$

It is known that the equation of the plane with position vector $\mathbf{r} \cdot \hat{\mathbf{n}} = d$

$$\therefore \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \left(\frac{3}{\sqrt{70}}\hat{\mathbf{i}} + \frac{5}{\sqrt{70}}\hat{\mathbf{j}} - \frac{6}{\sqrt{70}}\hat{\mathbf{k}}\right) = 7$$

$$\Rightarrow \quad \frac{3}{\sqrt{70}}x + \frac{5}{\sqrt{70}}y - \frac{6}{\sqrt{70}}z = 7$$

This is the vector equation of the required plane.

21. Firstly, determine the equation of line passing through (x_1, y_1, z_1) and perpendicular to the given plane is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$ (say). Next, we determine any point on the line which lies on the plane and solve it.

Given plane is $2x + 3y + 4z - 12 = 0$ ...(i)

The direction ratios of normal are 2, 3 and 4.

Also, $\sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$

Equation of the line through origin and at right angles to Eq. (i) are $\frac{x - 0}{2} = \frac{y - 0}{3} = \frac{z - 0}{4}$.

Any point on this line is $(2t, 3t, 4t)$. This point lies in the plane (i), if $2(2t) + 3(3t) + 4(4t) - 12 = 0$ *i.e.*, $t = \frac{12}{29}$

$\therefore$ The required foot of perpendicular is

$$\left(2 \times \frac{12}{29}, 3 \times \frac{12}{29}, 4 \times \frac{12}{29}\right) \text{ or } \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right).$$

22. If three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are given, then find $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$.

If above determinant is equal to zero, then points are collinear but if it is not equal to zero, then use the equation $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{vmatrix}$, solve and get the equation.

The given points are $A(1, 1, 0)$, $B(1, 2, 1)$ and $C(-2, 2, -1)$.

Firstly, check the condition of collinear point.

i.e., $$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix}$$

$$= 1(-2 - 2) - 1(-1 + 2) + 0\,(2 + 4) = -5 \neq 0$$

Therefore, a plane will pass through the points A, B and C.

It is known that the equation of the plane through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 1 & y - 1 & z \\ 0 & 1 & 1 \\ -3 & 0 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad -2(x - 1) - 3(y - 1) + 3z = 0$$

$$\Rightarrow \quad -2x + 2 - 3y + 3 + 3z = 0$$

$$\Rightarrow \quad 2x + 3y - 3z - 5 = 0$$

$$\Rightarrow \quad 2x + 3y - 3z = 5$$

This is the cartesian equation of the required plane.

23. The intersection equation of two planes is $P_1 + P_2\lambda = 0$, as it is passes through a point (x_1, y_1, z_1). Simply and get the value of λ and put λ in $P_1 + P_2\lambda = 0$ and solve it.

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$$(3x - y + 2z - 4) + \lambda\,(x + y + z - 2) = 0 \quad \text{...(i)}$$

The plane passes through the point (2, 2, 1). Therefore, this point will satisfy Eq. (i).

$$\therefore \quad (3 \times 2 - 2 + 2 \times 1 - 4) + \lambda\,(2 + 2 + 1 - 2) = 0$$

$$\Rightarrow \quad (6 - 4) + 3\lambda = 0 \Rightarrow 2 + 3\lambda = 0 \Rightarrow \lambda = \frac{-2}{3}$$

On substituting this value of λ in Eq. (i), we obtain the required plane as

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$\Rightarrow \quad 9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$

$$\Rightarrow \quad 7x - 5y + 4z - 8 = 0$$

This is the required equation of the plane.

24. Equation of any plane through the intersection of given planes can be written as

$$x + y + z - 1 + \lambda\,(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow \quad (1 + 2\lambda)\,x + (1 + 3\lambda)\,y + (1 + 4\lambda)\,z - 1 - 5\lambda = 0 \quad \text{...(i)}$$

The direction ratios, a_1, b_1, c_1, of the plane are $(2\lambda + 1)$, $(3\lambda + 1)$ and $(4\lambda + 1)$.

The plane in Eq. (i) is perpendicular to $x - y + z = 0$. Its direction ratios, a_2, b_2, c_2 are 1, –1 and 1.

Since, the planes are perpendicular.

$$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \quad 1(1 + 2\lambda) - 1(1 + 3\lambda) + 1(1 + 4\lambda) = 0$$

$\Rightarrow \quad 1+2\lambda-1-3\lambda+1+4\lambda=0$

$\Rightarrow \quad 3\lambda=-1$

$\Rightarrow \quad \lambda=-\frac{1}{3}$

On substituting this value of λ in Eq. (i), we obtain the required plane as

$$\left(1-\frac{2}{3}\right)x+\left(1-\frac{3}{3}\right)y+\left(1-\frac{4}{3}\right)z-1+\frac{5}{3}=0$$

$\Rightarrow \quad \frac{1}{3}x-\frac{1}{3}z+\frac{2}{3}=0$

$\Rightarrow \quad x-z+2=0$

This is the required equation of the plane.

25. If equation of planes are $\mathbf{r}\cdot\mathbf{n}_1=d_1$ and $\mathbf{r}\cdot\mathbf{n}_2=d_2$, then angle between them is $\cos\theta=\left|\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$.

Given, planes are

$$\mathbf{r}\cdot(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})=5$$

and $\quad \mathbf{r}\cdot(3\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})=3$

It is known that, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to the plane, $\mathbf{r}\cdot\mathbf{n}_1=d_1$ and $\mathbf{r}\cdot\mathbf{n}_2=d_2$, then the angle between them is given by

$$\cos\theta=\left|\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right| \quad \ldots(i)$$

Here, $\quad \mathbf{n}_1=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $\mathbf{n}_2=3\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}}$

$\therefore \quad \mathbf{n}_1\cdot\mathbf{n}_2=(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

$=2\cdot3+2\cdot(-3)+(-3)\cdot5=6-6-15=-15$

$|\mathbf{n}_1|=\sqrt{(2)^2+(2)^2+(-3)^2}=\sqrt{4+4+9}=\sqrt{17}$

$|\mathbf{n}_2|=\sqrt{(3)^2+(-3)^2+(5)^2}=\sqrt{9+9+25}=\sqrt{43}$

On substituting the value of $\mathbf{n}_1\cdot\mathbf{n}_2$, $|\mathbf{n}_1|$ and $|\mathbf{n}_2|$ in Eq. (i), we obtain

$$\cos\theta=\left|\frac{-15}{\sqrt{17}\sqrt{43}}\right|\Rightarrow\cos\theta=\frac{15}{\sqrt{731}}$$

$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$

26. (i) The direction ratios of normal to the plane, $L_1: a_1x+b_1y+c_1z=0$ are a_1, b_1, c_1 and $L_2: a_2x+b_2y+c_2z=0$ are a_2, b_2, c_2.

(ii) L_1 parallel to L_2, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(iii) L_1 perpendicular to L_2, if $a_1a_2+b_1b_2+c_1c_2=0$

The angle between L_1 and L_2 is given by

$$\theta=\cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Given planes are

$$7x+5y+6z+30=0$$

and $\quad 3x-y-10z+4=0$

Here, $\quad a_1=7, b_1=5, c_1=6$ and $a_2=3, b_2=-1, c_2=-10$

$\therefore \quad a_1a_2+b_1b_2+c_1c_2=7\times3+5\times(-1)+6\times(-10)$

$=21-5-60=-44\neq0$

Therefore, the given planes are not perpendicular.

Here, $\quad \frac{a_1}{a_2}=\frac{7}{3}, \frac{b_1}{b_2}=\frac{5}{-1}=-5, \frac{c_1}{c_2}=\frac{6}{-10}=\frac{-3}{5}$

It can be seen that $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$.

Therefore, the given planes are not parallel.

Let θ be the acute angle between the given planes, then

$$\cos\theta=\left|\frac{7\times3+5\times(-1)+6\times(-10)}{\sqrt{7^2+5^2+6^2}\sqrt{3^2+(-1)^2+(-10)^2}}\right|=\left|\frac{21-5-60}{\sqrt{110}\sqrt{110}}\right|$$

$\Rightarrow \cos\theta=\frac{44}{110}=\frac{2}{5} \Rightarrow \theta=\cos^{-1}\left(\frac{2}{5}\right)$

Hence, the angles between the given planes is $\cos^{-1}(2/5)$.

27. The distance between a point $P(x_1, y_1, z_1)$ and a plane $Ax+By+Cz=D$, is given by

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right| \text{ and simplify it.}$$

The given point is (3,–2,1) and the plane is

$$2x-y+2z+3=0.$$

$\therefore \quad$ From Eq. (i), $d=\frac{|2\times3+(-1)\times(-2)+2\times1+3|}{\sqrt{2^2+(-1)^2}+2^2}$

$=\frac{|6+2+2+3|}{\sqrt{4+1}+4}=\frac{13}{\sqrt{9}}=\frac{13}{3}$ units

28. The equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda(2x+y-z+5)=0 \quad \ldots(i)$$

$\Rightarrow \quad x(1+2\lambda)+y(2+\lambda)+z(3-\lambda)-4+5\lambda=0$

This is perpendicular to the plane $5x+3y+6z+8=0$, therefore

$$5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0$$

(using $a_1a_2+b_1b_2+c_1c_2=0$)

$\Rightarrow \quad 7\lambda+29=0 \Rightarrow \lambda=-29/7$

On putting $\lambda=-20/9$ in Eq. (i), we obtain the equation of the required plane as

$$-51x-15y+50z-173=0$$

$\Rightarrow \quad 51x+15y-50z+173=0$

29. The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.

It is known that equation of YZ-plane is $x=0$.

The equation of the line joining (5, 1, 6) and (3, 4, 1) is

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6} \Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=\lambda \text{ (say)}$$

Any point on this line is $(5-2\lambda, 1+3\lambda, 6-5\lambda)$ ...(i)

Since, the line passes through YZ-plane is $(x=0)$

$\therefore$ From Eq. (i), $5-2\lambda=0 \Rightarrow \lambda=\frac{5}{2}$ ...(ii)

$\therefore$ Required point of intersection of plane (i) and line (ii) is

$$\left(5-2\times\frac{5}{2}, 1+3\times\frac{5}{2}, 6-5\times\frac{5}{2}\right)=\left(0, \frac{17}{2}, \frac{-13}{2}\right).$$

30. The equation of the plane passing through the point (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where a, b and c are the direction ratios of normal.

The coordinates of the points O and P are (0, 0, 0) and (1, 2, – 3), respectively.

Therefore, the direction ratios of OP are

$$(1-0)=1, (2-0)=2 \text{ and } (-3-0)=-3$$

Here , the direction ratios of normal are 1, 2 and – 3 and the point P is (1, 2, –3). Then, equation of plane is

$$(x-1)1+(y-2)2+(z+3)(-3)=0$$

$\Rightarrow$ $x-1+2y-4-3z-9=0$

$\Rightarrow$ $x+2y-3z-14=0$

31. Given planes are $2x+3y+4z=4$ ...(i)

and $4x+6y+8z=12$

$\Rightarrow$ $2x+3y+4z=6$ ...(ii)

From the above equations, it can be seen that given planes are parallel.

It is known that the distance between two planes, $ax+by+cz=d_1$ and $ax+by+cz=d_2$ is given by

$$\text{distance, } d=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right| \Rightarrow d=\left|\frac{6-4}{\sqrt{2^2+3^2+4^2}}\right|$$

$$\Rightarrow \quad d=\frac{2}{\sqrt{4+9+16}}=\frac{2}{\sqrt{29}} \text{ unit}$$

32. Here, DR's of both planes are (2, –1, 4) and (5, –2.5, 10). As

$$\frac{2}{5}=\frac{-1}{-2.5}=\frac{4}{10} \Rightarrow \frac{4}{10}=\frac{4}{10}=\frac{4}{10}$$

Therefore, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Therefore, the given planes are parallel.

33. Let the given point be $P(\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ and Q be the image of P in the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})+3=0$ as shown in the figure.

Then, PQ is the normal to the plane. Since, PQ passes through P and is normal to the given plane, so the equation of PQ is given by

$$\mathbf{r}=(\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

Since, Q lies on the line PQ, the position vector of Q can be expressed as $(\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ *i.e.*,

$(1+2\lambda)\hat{\mathbf{i}}+(3-\lambda)\hat{\mathbf{j}}+(4+\lambda)\hat{\mathbf{k}}$

Since, R is the mid-point of PQ, the position vector of R is

$$\frac{[(1+2\lambda)\hat{\mathbf{i}}+(3-\lambda)\hat{\mathbf{j}}+(4+\lambda)\hat{\mathbf{k}}]+[\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}]}{2}$$

i.e., $(\lambda+1)\hat{\mathbf{i}}+\left(3-\frac{\lambda}{2}\right)\hat{\mathbf{j}}+\left(4+\frac{\lambda}{2}\right)\hat{\mathbf{k}}$

Again, since R lies on the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})+3=0$, we have

$$\left\{(\lambda+1)\hat{\mathbf{i}}+\left(3-\frac{\lambda}{2}\right)\hat{\mathbf{j}}+\left(4+\frac{\lambda}{2}\right)\hat{\mathbf{k}}\right\}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})+3=0$$

$\Rightarrow$ $\lambda=-2$

Hence, the position vector of Q is

$(\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})-2(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$, *i.e.*, $-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+2\hat{\mathbf{k}}$.

34. Let an equation of the required plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

This meets the coordinates axes in

$$A(a, 0, 0), B(0, b, 0) \text{ and } C(0, 0, c).$$

So that the coordinates of the centroid of the ΔABC are $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)=(1, r, r^2)$ (given)

$\Rightarrow$ $a=3, b=3r, c=3r^2$

Hence, the required equation of the plane is

$$\frac{x}{3}+\frac{y}{3r}+\frac{z}{3r^2}=1 \text{ or } r^2x+ry+z=3r^2$$

35. $A\equiv(0,-1,-1), B\equiv(-4,4,4)\ C\equiv(4,5,1), D\equiv(3,9,4)$

$\mathbf{AB}\neq\lambda\mathbf{BC}\therefore$ Points are not collinear.

$|\mathbf{AB}|\neq|\mathbf{BC}|\therefore$ points do not form a square.

$$\mathbf{CD}=\frac{3}{4}(\mathbf{AB})+\frac{1}{4}(\mathbf{BC})$$

$\therefore$ **AB**, **BC** and **CD** are coplanar.

$\therefore A, B, C$ and D coplanar.

36. $\because$ Line joining (p_1, q_1, r_1) and (p, q, r) is perpendicular to the plane. DR's of this line is proportional to a, b, c.

$\therefore$ $\frac{p-p_1}{a}=\frac{q-q_1}{b}=\frac{r-r_1}{c}$

Also, mid-point of line satisfies plane

$\therefore$ $a\left(\frac{p+p_1}{2}\right)+b\left(\frac{q+q_1}{2}\right)+c\left(\frac{r+r_1}{2}\right)+d=0$

37. Using image formula

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{z_2-z_1}{c}=-\frac{2(a_1x_1+b_1y_1+z_1+d)}{a^2+b^2+c^2}$$

$$\frac{x_2-2}{3}=\frac{y_2+1}{-2}=\frac{z_2-3}{-1}=\frac{2(6+2-3-9)}{9+4+1}$$

$$x_2=\frac{26}{7}, y_2=\frac{-15}{7}, z_2=\frac{17}{7}$$

38. The DR's of line perpendicular to plane is $(2a, 2a, 2a)$. The plane passes through mid-point of (a, a, a) and $(-a, -a, -a)$.

$\therefore$ Equation of plane is of the form $2ax + 2ay + 2az = 0$ or $x + y + z = 0$.

39. $\because$ Plane is perpendicular to the line.

$\therefore$ Equation of plane is of the form $2x - y + 2z + k = 0$

$\because$ If passes through origin

$$\therefore \quad k = 0$$

$$\therefore \quad 2x - y + 2z = 0$$

40. The equation of plane in normal form is

$$\frac{xx_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} + \frac{yy_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} + \frac{zz_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\therefore \quad xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$$

41. $x - y - z - 4 = 0, x + y + 2z - 4 = 0$

Required plane is of the form

$$(x - y - z - 4) + \lambda(x + y + 2z - 4) = 0$$

Since, this plane is perpendicular to the plane

$$x - y - z - 4 = 0$$

$$\therefore \quad 1 + \lambda + (\lambda - 1)(-1) + (2\lambda - 1)(-1) = 0, \lambda = \frac{3}{2}$$

$\therefore$ Required plane $5x + y + 4z = 20$

42. Let α be angle between **a** and angular bisector of **b** and **c**

$$\cos\alpha = \frac{\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})}{|\mathbf{a}||\mathbf{b}+\mathbf{c}|} = \frac{\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}}{1\cdot\sqrt{1^2 \times 1^2 + 2\times 1\times 1\times\cos\theta}}$$

$$= \frac{2\cos\theta}{\sqrt{2 + 2\cos\theta}} = \frac{\cos\theta}{\cos\theta/2}$$

43. DR's of AB are 1, 2, 4; Dr's of AC are $-2, -1, 1$.

DR's of normal to plane ABC are $2, -3, 1$. Equation of the plane ABC is $2x - 3y + z = 0$

Let the equation of required plane be $2x - 3y + z = k$, then

$$\left|\frac{2\times 1 - 3\times 1 + 1 - k}{\sqrt{4+9+1}}\right| = 3$$

[since distance of point (1, 1, 1) from required plane = 3]

$$\Rightarrow \quad k = \pm 3\sqrt{14}$$

44. $\mathbf{PQ} = \hat{\mathbf{i}}(-2 - 3\mu) + \hat{\mathbf{j}}(\mu - 3) + \hat{\mathbf{k}}(5\mu - 4)$

PQ is parallel to $x - 4y + 3z = 1$

$$\Rightarrow \quad 1(-2 - 3\mu) - 4(\mu - 3) + 3(5\mu - 4) = 0$$

$$\Rightarrow \quad \mu = \frac{1}{4}$$

45. Given that **a**, **b**, **c** and **d** are vectors such that

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = 0 \quad \ldots(i)$$

P_1 is the plane determined by vectors **a** and **b**.

$\therefore$ Normal vector $\mathbf{n}_1 = \mathbf{a}\times\mathbf{b}$

P_2 is the plane determined by vectors **c** and **d**.

$\therefore$ Normal vector $\mathbf{n}_2 = \mathbf{c}\times\mathbf{d}$

$$\mathbf{n}_1 \times \mathbf{n}_2 = 0$$

$$\Rightarrow \quad n_1 \,||\, n_2$$

Therefore, plane are also parallel to each other.

Therefore, angle between planes = 0.

46. Let the equation of variable plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ and it satisfied the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$$

$$\Rightarrow \quad \frac{9}{a^2} + \frac{9}{b^2} + \frac{9}{c^2} = k$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{k}{9} \quad \ldots(i)$$

Also, given that the distance of plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ from (0, 0, 0) is 1 unit.

$$\Rightarrow \quad \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 1 \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1 \quad \ldots(ii)$$

From Eqs.(i) and (ii), we get $\frac{k}{9} = 1$ *i.e.*, $k = 9$

47. In both systems the perpendicular distance of plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ or $\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1$ from origin will be same

$$\Rightarrow \quad \left|\frac{-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right| = \left|\frac{-1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}\right|$$

On squaring on both sides, we get

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$$

48. If l, m, n are the direction ratios of the line, then $l + 2m - n = 0, l + 3m - n = 0$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{0} = \frac{n}{1}$$

Line makes 90° with y-axis or is parallel to xz-plane.

49. $\because OP = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}\,a$

$\therefore$ DC's of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Equation of plane is $\quad x + y + z = 3a$

$$\Rightarrow \quad \frac{x}{3a} + \frac{y}{3a} + \frac{z}{3a} = 1$$

$\therefore$ Intersection on axes are $3a$, $3a$ and $3a$ respectively.

Sum of their reciprocals $= \frac{1}{3a} + \frac{1}{3a} + \frac{1}{3a} = \frac{1}{a}$

50. Since, the given line is parallel to the xy-plane, it means that the normal line is perpendicular to z-axis.

$\therefore$ DR's of z-coordinate is zero.

i.e., $n = 0$

51. Equation of plane through (1, 2, 3) is

$$a(x-1) + b(y-2) + c(z-3) = 0 \quad \text{...(i)}$$

It passes through (–1, 4, 2) and (3, 1, 1), so

$$-2a + 2b - c = 0 \quad \text{...(ii)}$$

and $$2a - b - 2c = 0 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii),

$$\frac{a}{-5} = \frac{b}{-6} = \frac{c}{-2}$$

$\therefore$ Equation of plane is

$$-5x - 6y - 2z + 5 + 12 + 6 = 0$$

$$\Rightarrow \quad 5x + 6y + 2z - 23 = 0$$

52. Equation of plane is $\frac{x}{8} + \frac{y}{4} + \frac{z}{4} = 1$

$$\Rightarrow \quad x + 2y + 2z = 8$$

Length of perpendicular from origin to the plane

$x + 2y + 2z - 8 = 0$ is

$$\left|\frac{-8}{\sqrt{1+4+4}}\right| = \frac{8}{3}$$

53. Let equation of plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$, then $A(\alpha, 0, 0), B(0, \beta, 0)$ and $C(0, 0, \gamma)$ are the points on coordinate axes.

Since, the centroid of a triangle is (1, 2, 4).

Now, $$\frac{\alpha}{3} = 1$$

$$\therefore \quad \alpha = 3, \ \frac{\beta}{3} = 2 \Rightarrow \beta = 6$$

and $$\frac{\gamma}{3} = 4 \Rightarrow \gamma = 12$$

$\therefore$ Equation of plane is

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1$$

$$\Rightarrow \quad 4x + 2y + z = 12$$

54. Distance of a point (1, 1, 1) from $x + y + z + k = 0$ is

$$\left|\frac{1+1+1+k}{\sqrt{3}}\right| = \left|\frac{3+k}{\sqrt{3}}\right|$$

According to the question,

$$\left|\frac{3+k}{\sqrt{3}}\right| = \pm 2\sqrt{3} \Rightarrow k = 3, -9$$

55. Plane intercept on x-axis at $a = 4$

Plane intercept on z-axis at $c = 3$

Required equation is $\frac{x}{4} + \frac{z}{3} = 1$ or $3x + 4z = 12$

56. The equation of any plane through the intersection of the plane $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ is

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0$$

$$\Rightarrow \quad (1 + 2\lambda)x + (1 + 3\lambda)y + (1 - \lambda)z + 4\lambda - 1 = 0$$

Since, the plane parallel to x-axis.

Therefore, DR's of the above plane *i.e.*, the coefficient of x is zero.

$$\therefore \quad 1 + 2\lambda = 0 \Rightarrow \lambda = -\frac{1}{2}$$

Hence, the required equation will be $y - 3z + 6 = 0$

57. $\because$ Mid-point of line joining (2, 3, 4) and (6, 7, 8) is (4, 5, 6). This point satisfied the equation

$$x + y + z - 15 = 0$$

$\therefore$ $x + y + z - 15 = 0$ is the required equation of plane.

58. Given that equation of planes are

$$P \equiv ax + by + cz + d = 0 \quad \text{...(i)}$$

and $$P' \equiv a'x + b'y + c'z + d' = 0 \quad \text{...(ii)}$$

Equation of intersection of planes is

$$P + \lambda P' = 0 \quad \text{...(iii)}$$

$$\Rightarrow ax + by + cz + d + \lambda(a'x + b'y + c'z + d) = 0$$

This line will be parallel to x-axis, if coefficient of $x = 0$.

$$\Rightarrow \quad a + \lambda a' = 0 \Rightarrow \lambda = -\frac{a}{a'}$$

$\therefore$ From Eq. (iii), we get

$$P - \frac{a}{a'}P' = 0 \Rightarrow \frac{P}{a} = \frac{P'}{a'}$$

59. Equation of plane passing through (–1, 3, 0) is

$$A(x+1) + B(y-3) + C(z-0) = 0 \quad \text{...(i)}$$

Also, plane (i) is passing through the points (2, 2, 1) and (1, 1, 3).

$$3A - B + C = 0 \quad \text{...(ii)}$$

and $$2A - 2B + 3C = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$\frac{A}{-3+2} = \frac{B}{2-9} = \frac{C}{-6+2}$$

$$\therefore \quad A : B : C = -1 : -7 : -4$$

$$\Rightarrow \quad A : B : C = 1 : 7 : 4$$

From Eq. (i), $1(x+1) + 7(y-3) + 4(z) = 0$

$$\Rightarrow \quad x + 7y + 4z - 20 = 0$$

$\therefore$ Distance from the plane to the point (5, 7, 8)

$$= \frac{1 \times 5 + 7 \times 7 + 4 \times 8 - 20}{\sqrt{1^2 + 7^2 + 4^2}}$$

$$= \frac{5 + 49 + 32 - 20}{\sqrt{66}}$$

$$= \frac{66}{\sqrt{66}} = \sqrt{66}$$

60. Since, M is the mid-point of $A(4\hat{\mathbf{i}}+5\hat{\mathbf{j}}-10\hat{\mathbf{k}})$ and $B(-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$.

$\therefore$ Coordinates of point M are $\left(\frac{3}{2},\frac{7}{2},-\frac{9}{2}\right)$.

$$\left\{\mathbf{r}-\left(\frac{3}{2}\hat{\mathbf{i}}+\frac{7}{2}\hat{\mathbf{j}}-\frac{9}{2}\hat{\mathbf{k}}\right)\right\}\cdot\mathbf{AB}=0$$

$$\left\{\mathbf{r}-\left(\frac{3}{2}\hat{\mathbf{i}}+\frac{7}{2}\hat{\mathbf{j}}-\frac{9}{2}\hat{\mathbf{k}}\right)\right\}\cdot(-5\hat{\mathbf{i}}-3\hat{\mathbf{j}}+11\hat{\mathbf{k}})=0$$

$$\mathbf{r}\cdot(5\hat{\mathbf{i}}+3\hat{\mathbf{j}}-11\hat{\mathbf{k}})-\frac{135}{2}=0$$

61. Let P be the foot of the perpendicular from the origin on the plane, then direction ratios of OP, the normal to the plane are $a-0, b-0, c-0$ *i.e.*, a, b, c. Also, since it passes through (a, b, c), the equation of the plane is

$$a(x-a)+b(y-b)+c(z-c)=0$$

$$\Rightarrow \quad ax+by+cz=a^2+b^2+c^2$$

62. Equation of the plane passing through three points A, B, C with position vectors a, b, c is

$$\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$$

So, that if a, b, c represents the given vectors, then

$$(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})$$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -6 & 3 & 2\\ 3 & -2 & 4\end{vmatrix}+\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & -2 & 4\\ 5 & 7 & 3\end{vmatrix}+\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 5 & 7 & 3\\ -6 & 3 & 2\end{vmatrix}$$

$$=[\hat{\mathbf{i}}(12+4-6-28+14-9)$$
$$-\hat{\mathbf{j}}(-24-6+9-20+10+18)$$
$$+\hat{\mathbf{k}}(12-9+21+10+15+42)]$$

$$=-13\hat{\mathbf{i}}+13\hat{\mathbf{j}}+91\hat{\mathbf{k}}$$

and $$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\begin{vmatrix}-6 & 3 & 2\\ 3 & -2 & 4\\ 5 & 7 & 3\end{vmatrix}=299$$

So, the required equation of the plane is

$$\mathbf{r}\cdot(-13\hat{\mathbf{i}}-13\hat{\mathbf{j}}+91\hat{\mathbf{k}})=299$$

$$\Rightarrow \quad \mathbf{r}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}}-7\hat{\mathbf{k}})=-23$$

$$\Rightarrow \quad \mathbf{r}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}}-7\hat{\mathbf{k}})+23=0$$

63. Two lines are at right angles, if $a_1a_2+b_1b_2+c_1c_2=0$, where (a_1, b_1, c_1) and (a_2, b_2, c_2) are the DR's of the lines.

Equation of the given lines can be written as standard form

$$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2} \text{ and } \frac{x-1}{-\frac{3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Direction ratios of these lines are respectively $-3, \frac{2p}{7}, 2$ and $\frac{-3p}{7}, 1, -5$.

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0$.

$$\therefore \quad (-3)\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)(1)+(2)(-5)=0$$

$$\Rightarrow \quad \frac{9p}{7}+\frac{2p}{7}-10=0 \Rightarrow \frac{11p}{7}=10$$

$$\Rightarrow \quad p=\frac{70}{11}$$

Thus, the value of $p=\frac{70}{11}$

Note For making a line in standard form be ensure that coefficient of variable should be non-negative and constant.

64. Firstly, convert the given equations in vector form and use the formula $d=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$ determine the shortest distance.

The given lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}.$$

For line Ist DR's $=(7,-6,1)$ and passes through $(-1,-1,-1)$, then equation of given lines (in vector form) are

$$\mathbf{r}_1=-\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}+\lambda(7\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

Similarly, $\mathbf{r}_2=3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+7\hat{\mathbf{k}}+\mu(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

which are of the form $\mathbf{r}_1=\mathbf{a}_1+\lambda\,\mathbf{b}_1$ and $\mathbf{r}_2=\mathbf{a}_2+\mu\,\mathbf{b}_2$

where, $\mathbf{a}_1=-\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}, \mathbf{b}_1=7\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{a}_2=3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+7\hat{\mathbf{k}}, \mathbf{b}_2=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Now, $$\mathbf{a}_2-\mathbf{a}_1=(3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+7\hat{\mathbf{k}})-(-\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}})=4\hat{\mathbf{i}}+6\hat{\mathbf{j}}+8\hat{\mathbf{k}}$$

and $$\mathbf{b}_1\times\mathbf{b}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 7 & -6 & 1\\ 1 & -2 & 1\end{vmatrix}$$

$$=\hat{\mathbf{i}}(-6+2)-\hat{\mathbf{j}}(7-1)+\hat{\mathbf{k}}(-14+6)$$
$$=-4\hat{\mathbf{i}}-6\hat{\mathbf{j}}-8\hat{\mathbf{k}}$$

$$|\mathbf{b}_1\times\mathbf{b}_2|=\sqrt{(-4)^2+(-6)^2+(-8)^2}$$
$$=\sqrt{16+36+64}=\sqrt{116}=2\sqrt{29}$$

$\therefore$ SD between the given lines

$$d=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{b}_2\times\mathbf{b}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|=\frac{|(-4\hat{\mathbf{i}}-6\hat{\mathbf{j}}-8\hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}}+6\hat{\mathbf{j}}+8\hat{\mathbf{k}})|}{2\sqrt{29}}$$

$$=\frac{|(-4)\times4+(-6)\times6+(-8)\times8|}{2\sqrt{29}}=\frac{|-16-36-64|}{2\sqrt{29}}$$

$$=\frac{116}{2\sqrt{29}}=\frac{58}{\sqrt{29}}=2\sqrt{29} \text{ units}$$

Note The two lines should be parallel and non-intercepting, then we can only determined the shortest distance.

65. Any equation passing through the point (x_1, y_1, z_1) with the direction ratio a, b, c is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Any line through (1, 2, –4) can be written as

$$\frac{x-1}{a}=\frac{y-2}{b}=\frac{z+4}{c} \quad \text{...(i)}$$

where, a, b, c are the direction ratios of line (i).

Now, the line (i) be perpendicular to the lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$

and $$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

In above having direction ratios are (3, –16, 7) and (3, 8, –5). respectively. Which is perpendicular with the Eq. (i).

$\therefore$ $$3a-16b+7c=0 \quad \text{...(ii)}$$

and $$3a+8b-5c=0 \quad \text{...(iii)}$$

By cross-multiplication, we have

$$\frac{a}{80-56}=\frac{b}{21+15}=\frac{c}{24+48}$$

$\Rightarrow$ $$\frac{a}{24}=\frac{b}{36}=\frac{c}{72} \Rightarrow \frac{a}{2}=\frac{b}{3}=\frac{c}{6}=\lambda \text{ (say)}$$

$\therefore$ $$a=2\lambda, b=3\lambda \text{ and } c=6\lambda$$

The equation of required line which is passes through the point (1, 2, –4) and parallel to vector $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ is $\mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+6\hat{\mathbf{k}})$.

66. Here, coordinate of the points A, B, C and D be (1, 2, 3), (4, 5, 7), (–4, 3, –6) and (2, 9, 2), respectively.

$\therefore$ Direction ratios of line AB are (4 – 1),(5 – 2), (7 – 3)

$\Rightarrow$ (3, 3, 4)

Direction ratios of line CD are

$$(2+4),(9-3),(2+6) \Rightarrow (6,6,8)$$

Let direction ratios of AB and CD are (a_1, b_1, c_1) and (a_2, b_2, c_2)

$$\frac{a_1}{a_2}=\frac{3}{6}=\frac{1}{2}, \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}, \frac{c_1}{c_2}=\frac{4}{8}, =\frac{1}{2}$$

It can be seen that $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either $0°$ or $180°$.

67. Given plane is $\mathbf{r}\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-5\hat{\mathbf{k}})+9=0$

In cartesian form, this plane can be written as $x+2y-5z+9=0$.

On putting $$\mathbf{r}=x\hat{\mathbf{i}}+\mathbf{y}\hat{\mathbf{j}}+z\hat{\mathbf{k}}$$

Direction ratio of any line perpendicular to the given plane are 1, 2, –5.

If this line passes through (1, 2, 3), then its equation is

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-5}$$

$$\left(\because \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\right)$$

The above equation can be written in vector form

$$\mathbf{r}=(1\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\lambda(1\hat{\mathbf{i}}+2\hat{\mathbf{j}}-5\hat{\mathbf{k}})$$

where, λ being any real number.

68. The equation of first line is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda \text{ (say)}$$

$\Rightarrow$ $$x=2\lambda+1, y=3\lambda+2 \text{ and } z=4\lambda+3$$

So, the coordinates of a general point on this line are $(2\lambda+1, 3\lambda+2, 4\lambda+3)$.

The equation of second line is

$$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}=\mu \text{ (say)}$$

$\Rightarrow$ $$x=5\mu+4, y=2\mu+1, z=\mu$$

So, the coordinates of a general point on this line are $(5\mu+4, 2\mu+1, \mu)$.

If the line intersect, then they have a common point. So, for some values of λ and μ, we must have

$$2\lambda+1=5\mu+4, 3\lambda+2=2\mu+1 \text{ and } 4\lambda+3=\mu$$

$\Rightarrow$ $$2\lambda-5\mu=3, 3\lambda-2\mu=-1, 4\lambda-\mu=-3$$

On solving first two equations of the above equations, we get

$$\lambda=-1 \text{ and } \mu=-1$$

Since, $\lambda=-1$ and $\mu=-1$ satisfy the third equation. So, the given lines intersect.

On putting $\lambda=-1$ in $(2\lambda+1, 3\lambda+2, 4\lambda+3)$, the coordinates of the required point of intersection are (–1, –1, –1).

69. Here, $P(-2, 4, -5)$ is the given point.

Any point Q on the line is given by $(3\lambda-3, 5\lambda+4, 6\lambda-8)$.

$$\mathbf{PQ}=(3\lambda-1)\hat{\mathbf{i}}+5\lambda\hat{\mathbf{j}}+6\lambda-3)\hat{\mathbf{k}}$$

Since, $$\mathbf{PQ}\perp(3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}}),$$

we have $3(3\lambda-1)+5(5\lambda)+6(6\lambda-3)=0$

$$9\lambda+25\lambda+36\lambda=21,$$

i.e., $$\lambda=\frac{3}{10}$$

Thus, $$\mathbf{PQ}=-\frac{1}{10}\hat{\mathbf{i}}+\frac{15}{10}\hat{\mathbf{j}}+\frac{12}{10}\hat{\mathbf{k}}$$

Hence, $$|\mathbf{PQ}|=\frac{1}{10}\sqrt{1+225+144}=\sqrt{\frac{37}{10}}$$

70. We have, $\mathbf{r}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}+\lambda(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}})$ and $\mathbf{r}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$

On solving these two equations, we get

$$[2\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})+\lambda(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}})]\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$$

which given $\lambda=0$.

Therefore, the point of intersection of line and the plane is $(2,-1,2)$ and the other given point is $(-1,-5,-10)$. Hence, the distance between these two points is

$$\sqrt{[2-(-1)]^2+[-1+5]^2+[2-(-10)]^2},\ i.e., 13$$

71. Let L be the foot of perpendicular drawn form the points $A(1,8,4)$ to the line passing through B and C as shown in the figure. The equation of line BC by using formula $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$, the equation of the line BC is

$$\mathbf{r}=(-\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-4\hat{\mathbf{k}})$$

$$x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}=2\lambda\hat{\mathbf{i}}-(2\lambda+1)\hat{\mathbf{j}}+\lambda(3-4\lambda)\hat{\mathbf{k}}$$

On comparing both sides, we get

$$x=2\lambda, y=-(2\lambda+1), z=3-4\lambda \quad \ldots(i)$$

Thus, the coordinate of L are $\{2\lambda,-(2\lambda+1),(3-4\lambda)\}$, so that the direction ratios of the line AL are

$$(1-2\lambda), 8+(2\lambda+1), 4-(3-4\lambda)\ i.e., 1-2\lambda, 2\lambda+9, 1+4\lambda$$

Since, AL is perpendicular to BC, we have

$$(1-2\lambda)(2-0)+(2\lambda+9)(-3+1)+(4\lambda+1)(-1-3)=0$$

$$\lambda=\frac{-5}{6}$$

The required point is obtained by substituting the value of λ in Eq. (i), which is

$$\left(\frac{-5}{2},\frac{2}{3},\frac{19}{3}\right).$$

72. Let $P(1,6,3)$ be the given point and let L be the foot of perpendicular from P to the given line.

The coordinates of a general point on the given line are

$$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{3}=\lambda\ i.e.,$$

$$x=\lambda, y=2\lambda+1, z=3\lambda+2$$

If the coordinates of L are $(\lambda, 2\lambda+1, 3\lambda+2)$, then the direction ratios of PL are $\lambda-1, 2\lambda-5, 3\lambda-1$ but the direction ratios of given line which is perpendicular to PL are $1,2,3$. Therefore, $(\lambda-1)1+(2\lambda-5)2+(3\lambda-1)3=0$, which gives $\lambda=1$. Hence, coordinates of L are $(1,3,5)$.

Let $Q(x_1,y_1,z_1)$ be the image of $P(1,6,3)$ in the given line. Then, L is the mid-point of PQ. Therefore,

$$\frac{x_1+1}{2}=1, \frac{y_1+6}{2}=3, \frac{z_1+3}{2}=5$$

$$\Rightarrow \quad x_1=1, y_1=0, z_1=7$$

Hence, the image of $(1,6,3)$ in the given line is $(1,0,7)$.

73. Given pair of lines

$$x=py+q$$

$$\Rightarrow \quad \frac{x-q}{p}=y \quad \ldots(i)$$

and $$z=ry+s$$

$$\Rightarrow \quad \frac{z-s}{r}=y \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\frac{x-q}{p}=\frac{y-0}{1}=\frac{z-s}{r} \quad \ldots(iii)$$

Also, $$x=p'y+q'$$

$$\Rightarrow \quad \frac{x-q'}{p'}=y \quad \ldots(iv)$$

and $$z=r'y+s'$$

$$\Rightarrow \quad \frac{z-s'}{r'}=y \quad \ldots(v)$$

From Eqs. (iv) and (v),

$$\frac{x-q'}{p'}=\frac{y-0}{1}=\frac{z-s'}{r'} \quad \ldots(vi)$$

Since, line (iii) and (vi) are perpendicular.

$$\therefore \quad (p)(p')+(1)(1)+(r)(r')=0 \quad (\because l_1l_2+m_1m_2+n_1n_2=0)$$

$$pp'+rr'+1=0$$

74. Obtain the foot of the perpendicular $N(-4,1,-3)$.

Find the equations of the line passing through $P(2,4,-1)$ and $N(-4,1,-3)$

$$\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$$

75. The given lines are coplanar, if

$$0=\begin{vmatrix}2-1 & 3-4 & 4-5\\ 1 & 1 & -k\\ k & 2 & 1\end{vmatrix}=\begin{vmatrix}1 & -1 & -1\\ 1 & 1 & -k\\ k & 2 & 1\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 1 & 2 & 1-k\\ k & k+2 & 1+k\end{vmatrix}$$

or if $$2(1+k)-(k+2)(1-k)=0$$

$$k^2+3k=0$$

or if $$k=0,-3$$

76. $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=r$ (say)

$$\Rightarrow \quad x=2r+1, y=3r-1, z=4r+1$$

As the two given lines intersect,

$$\frac{2r+1-3}{1}=\frac{3r-1-k}{2}=\frac{4r+1}{1}$$

For some value of r,

we have $$2r-2=4r+1$$

$$\Rightarrow \quad r=-\frac{3}{2}$$

Also, $$3r-1-k=8r+2$$

$$\Rightarrow \quad k=-5r-3=\frac{15}{2}-3=\frac{9}{2}$$

77. $\mathbf{a} + t(\mathbf{b} + \mathbf{c}) = \mathbf{b} + s(\mathbf{c} + \mathbf{a})$

We easily get $t = 1, s = 1$

Hence, intersection point $\mathbf{a} + \mathbf{b} + \mathbf{c}$

78. Any point on the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is $(r, 2r, 3r)$ which lies in the second line, if $\frac{r-1}{-2} = \frac{2r-2}{-4} = \frac{3r-3}{-6}$

i.e., if $r - 1 = r - 1 = r - 1$ which is true, $\forall$ all real of r, hence two lines are coincident.

79. Let the line make the angle α with the axes, then we have $3\cos^2\alpha = 1$ ($\because$ sum of the square's of DC's $= 1$).

$$\cos\alpha = \frac{1}{\sqrt{3}} \qquad (\because \text{DC's are positive, given})$$

$$\frac{x-2}{1/\sqrt{3}} = \frac{y+1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = r \qquad (\text{where, } r = PQ)$$

$$x = \frac{r}{\sqrt{3}} + 2, y = \frac{r}{\sqrt{3}} - 1, z = \frac{r}{\sqrt{3}} + 2,$$

$$\frac{2r}{\sqrt{3}} + 4 + \frac{r}{\sqrt{3}} - 1 + \frac{r}{\sqrt{3}} + 2 = 9$$

$$\Rightarrow \qquad \frac{4r}{\sqrt{3}} = 4 \Rightarrow r = \sqrt{3}$$

80. The shortest distance between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is given by

$$\text{Shortest distance} = \frac{\left\| \begin{matrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{matrix} \right\|}{\sqrt{\Sigma(mn' - nm')^2}}$$

$$= \frac{\left\| \begin{matrix} 6 & 15 & -3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{matrix} \right\|}{\sqrt{(-4-2)^2 + (12+3)^2 + (6-3)^2}}$$

$$= \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}$$

81. Direction cosines of given line are $\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$

$$\therefore \qquad AM = \left| (6-1)\cdot\frac{3}{\sqrt{17}} + (7-2)\cdot\frac{2}{\sqrt{17}} + (7-3)\cdot\frac{-2}{\sqrt{17}} \right|$$

$$= \sqrt{17}$$

$$AP = \sqrt{(6-1)^2 + (7-2)^2 + (7-3)^2}$$

$$= \sqrt{25 + 25 + 16} = \sqrt{66}$$

$\therefore$ Length of perpendicular

$$PM = \sqrt{AP^2 - AM^2}$$

$$= \sqrt{66 - 17}$$

$$= \sqrt{49} = 7$$

82. Equation of plane through three points $(2, 2, 1), (3, 0, 1)$ and $(4, -1, 0)$ is

$$[(\mathbf{r} - \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})]\cdot[(\hat{\mathbf{i}} - 2\hat{\mathbf{j}})\times(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})] = 0$$

i.e., $\quad \mathbf{r}\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 7$ or $2x + y + z - 7 = 0 \qquad$ ...(i)

Equation of line through $(3, -4, -5)$ and $(2, -3, 1)$ is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \qquad \text{...(ii)}$$

Any point on line (ii) is $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$. This point lies on plane (i). Therefore, $2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$, *i.e.*, $\lambda = z$.

Hence, the required point is $(1, -2, 7)$.

83. Given lines are respectively parallel to unit vector

$$\mathbf{b}_1 = l_1\,\hat{\mathbf{i}} + m_1\,\hat{\mathbf{j}} + n_1\,\hat{\mathbf{k}} \qquad \text{...(i)}$$

and $\quad \mathbf{b}_1 = l_2\hat{\mathbf{i}} + m_2\,\hat{\mathbf{j}} + n_2\hat{\mathbf{k}} \qquad$ ...(ii)

Now, $\mathbf{b}_1 \times \mathbf{b}_2$ is a vector which is at right angles to both $\mathbf{b}_1$ and $\mathbf{b}_2$ and is of magnitude unity.

Hence, components of $\mathbf{b}_1 \times \mathbf{b}_2$ are direction cosines of a line which is at right angles to both $\mathbf{b}_1$ and $\mathbf{b}_2$. So, we compute $\mathbf{b}_1 \times \mathbf{b}_2$.

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1 n_2 - m_2 n_1)\hat{\mathbf{i}} - (n_2 l_1 - n_1 l_2)\hat{\mathbf{j}}$$

$$+ (l_1 m_2 - l_2 m_1)\hat{\mathbf{k}}$$

$$= (m_1 n_2 - m_2 n_1)\,\hat{\mathbf{i}} + (n_1 l_2 - n_2 l_1)\,\hat{\mathbf{j}} + (l_1 m_2 - l_2 m_1)\,\hat{\mathbf{k}}$$

Thus, the direction cosines of the required line are

$$m_1 n_2 - m_2 n_1, n_1 l_2 - n_2 l_1, l_1 m_2 - l_2 m_1.$$

84. Let a, b and c be the direction ratios of the required line. Then, its equation is

$$\frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c} \qquad \text{...(i)}$$

Since, Eq. (i) is parallel to the planes $x + 2y + 0z = 0$ and $0x + 3y - z = 0$. Therefore, normal to the plane is perpendicular to the line.

$\therefore \quad a(1) + b(2) + c(0) = 0$ and $a(0) + b(3) + c(-1) = 0$

On solving these two equations by cross-multiplication, we obtain

$$\frac{a}{(2)(-1) - (0)(3)} = \frac{b}{(0)(0) - (1)(-1)} = \frac{c}{(1)(3) - (0)(2)}$$

$$\Rightarrow \qquad \frac{a}{-2} = \frac{b}{1} = \frac{c}{3} = \lambda \text{ (say)}$$

$$\Rightarrow \qquad a = -2\lambda,\ b = \lambda,\ c = 3\lambda$$

On substituting the values of a, b and c in Eq. (i), we obtain the equation of the required line as

$$\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$$

Alternate Method

The required line passes through the point having its position vector $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$ and is parallel to the planes $x + 2y = 0$ and

$3y - z = 0$. So, it is perpendicular to their normals $\mathbf{n}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\mathbf{n}_2 = 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ respectively.

Consequently, the required line is parallel to the vector.

$$\mathbf{b} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

Hence, the equation of the required line is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \text{ or } \mathbf{r} = (3\hat{\mathbf{i}} + \hat{\mathbf{k}}) + \lambda(-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

85. Here, $2x + 3y + 5z = 1$ divides (1, 0, –3) and (1, –5, 7) in the ratio of $k:1$ at point P.

Then, $P\left(\frac{k+1}{k+1}, \frac{-5k}{k+1}, \frac{7k-3}{k+1}\right)$ must satisfy the equation

$$2x + 3y + 5z = 1.$$

$$\therefore \quad 2\left(\frac{k+1}{k+1}\right) + 3\left(\frac{-5k}{k+1}\right) + 5\left(\frac{7k-3}{k+1}\right) = 1$$

$$\Rightarrow \quad 2k + 2 - 15k + 35k - 15 = k + 1$$

$$\Rightarrow \quad 21k = 14 \Rightarrow k = \frac{2}{3}$$

Thus, $2x + 3y + 5z = 1$ divides $(1, 0, -3)$ and $(1, -5, 7)$ in the ratio $2:3$.

86. Required volume

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \frac{abc}{6} \text{ cu units}$$

87. Here, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{n} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$$\therefore \quad \sin\theta = \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$= \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

88. Let the point P divide QR in the ratio $\lambda:1$, then the coordinate of P are

$$\left(\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$$

But x-coordinate of P is 4. Therefore,

$$\frac{5\lambda+2}{\lambda+1} \Rightarrow 4 \Rightarrow \lambda = 2$$

Hence, the z-coordinate of P is $\frac{-2\lambda+1}{\lambda+1} = -1$.

89. Eliminating m, we get

$$2(l+n)^2 + nl = 0 \qquad (\because \text{put } m = 2l + 2n)$$

$$\Rightarrow \quad (2l+n)(l+2n) = 0$$

$$\Rightarrow \quad n = -2l \Rightarrow m = -2l$$

or $\quad l = -2n \Rightarrow m = -2n$

The DR's are $1, -2, -2$ and $-2, -2, 1$.

Now, $\quad 1(-2) - 2(-2) - 2(1) = 0$

Hence, lines are perpendicular.

90. $l^2 + m^2 - n^2 = 0 \Rightarrow 1 - 2n^2 = 0 \Rightarrow n = \pm\frac{1}{\sqrt{2}}$

Eliminating n from the given relations, we get $lm = 0$

The two lines having direction cosines are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

Then, angle between them is $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.

91. Let OA, OB and OC be the coterminous edges of a cube, taken along the axes in such a way that $OA = OB = OC = a$.

Then, the coordinates of the vertices of the cube are

$O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $P(a, a, a)$, $L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.

The direction ratios of the diagonals OP, AL, BM, and CN are (a, a, a), $(-a, a, a)$, $(a, -a, a)$ and $(a, a, -a)$.

Direction cosines of OP, AL, BM and CN are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, respectively.

If θ be the angle between OP and AL, then

$$\cos\theta = \left\{\frac{1}{\sqrt{3}} \cdot \left(\frac{-1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right\} = \frac{1}{3}$$

or $\quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$

92. D divides BC in the ratio $AB:AC$ *i.e.*, 3 : 13. Therefore, coordinates of D are

$$\left[\frac{3\times(-9) + 13\times 5}{3+13}, \frac{3\times 6 + 13\times 3}{3+13}, \frac{3\times(-3) + 13\times 2}{3+13}\right]$$

or $\quad \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

93. The coordinates of any point on the given line are $(2r + 1, -3r - 1, r)$.

The distance of this point from the point $(1, -1, 0)$ is given to be $4\sqrt{14}$.

$$\Rightarrow \quad (2r)^2 + (-3r)^2 + (r)^2 = (4\sqrt{14})^2$$

$$\Rightarrow \quad 14r^2 = 16 \times 14 = r = \pm 4$$

So, the coordinates of the required point are $(9, -13, 4)$ or $(-7, 11, -4)$. Out of which nearer the origin is $(-7, 11, -4)$.

94. Let the components of the line vector be a, b, c. Then,

$$a^2 + b^2 + c^2 = (63)^2 \quad \ldots(i)$$

Also, $\frac{a}{3} = \frac{b}{-2} = \frac{c}{6} = \lambda$ (say), then $a = 3\lambda$, $b = -2\lambda$ and $c = 6\lambda$

and from Eq.(i), we have $9\lambda^2 + 4\lambda^2 + 36\lambda^2 = (63)^2$

$$\Rightarrow \quad 49\lambda^2 = (63)^2 \Rightarrow \lambda = \pm\frac{63}{7} = \pm 9$$

Since, $a = 3\lambda < 0$ as the line makes an obtuse angle with x-axis, $\lambda = -9$ and the required components are -27, 18, -54.

95. Equation of the given plane can be written as

$$\frac{x}{20} + \frac{y}{15} + \frac{z}{-12} = 1$$

which meets the coordinates axes in points $A(20, 0, 0)$, $B(0, 15, 0)$ and $C(0, 0, -12)$ and coordinates of the origin are (0, 0, 0).

$\therefore$ The volume of the tetrahedron $OABC$ is

$$\left|\frac{1}{6}\begin{vmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & -12 \end{vmatrix}\right| = \left|\frac{1}{6} \times 20 \times 15 \times (-12)\right| = 600$$

$$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$$

and $$\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$$

where, $$\mathbf{a}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\mathbf{b}_1 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Round II

1. Area of triangle $= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$

$$= \frac{1}{2}\begin{Vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{Vmatrix} = \begin{Vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{Vmatrix}$$

$$= |-\hat{\mathbf{i}} - \hat{\mathbf{j}}| = \sqrt{2}$$

2. Given plane is $2x - y + z = k$

It passes through $(2, -1, 1)$.

$\therefore \quad k = 6$

The distance of $(1, 2, 3)$ from

$$2x - y + z = 6 \text{ is } \frac{|2 - 2 + 3 - 6|}{\sqrt{2^2 + 1 + 1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

3. Let C and D divide AB internally and externally in the ratio $\lambda : 1$.

Then, $$AC = \frac{\lambda}{\lambda + 1} \cdot AB$$

$$AD = \frac{\lambda}{\lambda - 1} AB$$

$$\therefore \quad \frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$$

4. If three edges of the cube are along x, y and z, then diagonal has DR's 1, 1, 1 and edge along x-axis has DR's 1, 0, 0. The angle between them is

$$\cos^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\sqrt{2}$$

5. The DR's of the line joining $(3, 1, -3)$ and $(1, -2, 2)$ are $-2, -3, 5$.

Equation of plane passing through $(3, 1, -3)$ is

$$a(x - 3) + b(y - 1) + c(z + 3) = 0$$

It passes through $(1, -2, 2)$, then

$$-2a - 3b + 5c = 0$$

Also, $$a + b - 2c = 0$$

$\therefore \quad a = 1, b = 1, c = 1$

$\therefore$ Equation of plane is

$$(x - 3) \cdot 1 + (y - 1) \cdot 1 + (z + 3) \cdot 1 = 0$$

$$\therefore \quad x + y + z - 1 = 0$$

6. Since, $\cos^2\theta + \cos^2\theta + \cos^2\gamma = 1$

$$\Rightarrow \quad \cos^2\gamma = -\cos 2\theta$$

$$\Rightarrow \quad \cos 2\theta \le 0$$

$$\therefore \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

7. Let parallel plane be $2x - 2y + z + \lambda = 0$. It passes through $(1, -2, 3)$.

$$\therefore \quad \lambda = -9$$

The distance of $(-1, 2, 0)$ from the plane

$$2x - 2y + z - 9 = 0 \text{ is } \left|\frac{-2 - 4 - 9}{3}\right| = 5$$

8. Any point on the line is

$$P(1 + 2r, -1 - 3r, -10 + 8r)$$

Since, AP is perpendicular to the given line.

$$\therefore \quad 2r \times 2 + (1 + 3r)\,3 + 8(-10 + 8r) = 0$$

$$\Rightarrow \quad r = 1$$

The foot of perpendicular from A is $P(3, -4, -2)$. But P is the mid-point of A and its image whose coordinate will be $(5, -8, -4)$.

9. The vector joining the points is $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Its projection along the vector $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$$= \frac{|(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{|2 - 3 - 2|}{3} = 1$$

10. Substituting the coordinates of the points in the equation $2x + 3y - 2z - k = 0$, we get

$$(2 + 6 - 6 - k)(4 - 3 - k) < 0$$

$$\Rightarrow \quad (k - 1)(k - 2) < 0$$

$$\therefore \quad 1 < k < 2$$

11. Required equation of plane is

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ -3 & -3 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x - 3y - 6z + 8 = 0$$

Intercepts are $8, \frac{8}{3}, \frac{8}{6}$.

Their sum is 12.

12. Let $\text{OA} = \text{a}, \text{OB} = \text{b}, \text{OC} = \text{c}$. Then, the centroid G of ΔABC is $\frac{\text{a} + \text{b} + \text{c}}{3}$

$$\text{OG}^2 = \frac{1}{9}(3k^2 + 6k^2 \cos\theta) = \frac{k^2}{3}(1 + 2\cos\theta) > 0$$

$$\theta \in \left[0, \frac{2\pi}{3}\right]$$

13. Let $\text{OA} = \text{a}, \text{OB} = \text{b}, \text{OC} = \text{c}$

Then, $\quad |\text{a}| = |\text{b}| = |\text{c}| = 1;\ \text{a}\cdot\text{b} = \text{b}\cdot\text{c} = \text{c}\cdot\text{a} = \frac{1}{2}$

$$\text{Now, } [\text{a b c}]^2 = \begin{vmatrix} \text{a}\cdot\text{a} & \text{a}\cdot\text{b} & \text{a}\cdot\text{c} \\ \text{b}\cdot\text{a} & \text{b}\cdot\text{b} & \text{b}\cdot\text{c} \\ \text{c}\cdot\text{a} & \text{c}\cdot\text{b} & \text{c}\cdot\text{c} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}$$

$$\because \quad \text{Volume of tetrahedron} = \frac{1}{6}[\text{a b c}] = \frac{1}{6\sqrt{2}}$$

14. Consider the line through $(2, -1, 3)$ with DC's $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ is

$$\frac{x-2}{2/3} = \frac{y+1}{2/3} = \frac{z-3}{1/3} = r \quad \text{(say)}$$

$$\therefore \quad x = 2 + \frac{2r}{3}, y = -1 + \frac{2r}{3}, z = 3 + \frac{r}{3}$$

It lies on the plane $x + 2y - z = 2$

$$\therefore \quad 2 + \frac{2r}{3} - 2 + \frac{4r}{3} - 3 - \frac{r}{3} = 2 \Rightarrow r = 3$$

15. Now, $a_1a_2 + b_1b_2 + c_1c_2 = 2 - 1 + 2 > 0$.

The acute angle bisecting plane is

$x - y + 2z + 1 = -(2x + y + z + 2)$ *i.e.*, $x + z + 1 = 0$

16. The two normal vectors are $\mathbf{m} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{n} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

The line L is along

$$\therefore \quad \mathbf{m} \times \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

And DC's of x-axis is (1, 0, 0).

$$\therefore \quad \cos\alpha = \frac{3(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (\text{i})}{\sqrt{3^2(1+1+1)}\sqrt{1}} = \frac{1}{\sqrt{3}}$$

17. S divides OG in the ratio $3 : -1$

$$S = \left(\frac{9+3}{2}, \frac{-5+9}{2}, \frac{-3-1}{2}\right)$$

$$= (6, 2, -2)$$

18. Let plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Centroid of plane ABC is $D\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

Distance of the plane from the origin,

$$d = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 1 \quad \text{(given)}$$

$$\Rightarrow \quad 1 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

$$\therefore D(x, y, z) \Rightarrow a = 3x, b = 3y, c = 3z \quad \Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9$$

Hence, value of k is 9.

19. $\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

and $\cos^2\alpha = 1 - \sin^2\alpha$, similarly for all other angles.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$
$$= 4 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta)$$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \left(4 - \frac{4}{3}\right) = \frac{8}{3}$$

20. Let the direction cosine of the line perpendicular to two given lines is (l, m, n), then

$$l + 2m + 3n = 0 \text{ and } -2l + \frac{m}{2} + \frac{n}{3} = 0$$

From the above equation

$$\frac{l}{2 \times \frac{1}{3} - \frac{1}{2} \times 3} = -\frac{m}{1 \times \frac{1}{3} - 3 \times (-2)} = \frac{n}{1 \times \frac{1}{2} - 2(-2)}$$

$$\Rightarrow \quad \frac{l^2}{\frac{25}{36}} = \frac{m^2}{\frac{361}{9}} = \frac{n^2}{\frac{81}{4}} = \frac{1}{\frac{25}{36} + \frac{361}{9} + \frac{81}{4}}$$

$$\therefore \quad l = \sqrt{\frac{25}{2198}}, m = \sqrt{\frac{38}{1099}}, n = \sqrt{\frac{729}{2198}}$$

21. Here, $\quad \alpha = \beta = \gamma$

$$\because \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\therefore \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

DC's of PQ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

PM = Projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}} + (3-5)\cdot\frac{1}{\sqrt{3}} + (1-2)\cdot\frac{1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

and $\quad AP = \sqrt{(-2+3)^2 + (3-5)^2 + (1-2)^2} = \sqrt{6}$

$$AM = \sqrt{(AP)^2 - (PM)^2} = \sqrt{6 - \frac{4}{3}} = \sqrt{\frac{14}{3}}$$

22. Equation of plane through (3, 2, –1) is

$$a(x-3) + b(y-2) + c(z+1) = 0 \quad \text{...(i)}$$

Also, (3, 4, 2) and (7, 0, 6) lie on Eq. (i), then

$$0 \cdot a + 2b + 3c = 0 \quad \text{...(ii)}$$

and $$4a - 2b + 7c = 0 \quad \text{...(iii)}$$

On eliminating a, b, c from Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} x-3 & y-2 & z+1 \\ 0 & 2 & 3 \\ 4 & -2 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad 5x + 3y - 2z = 23$

$\therefore \quad \lambda = 23$

23. Let equation of plane is

$$lx + my + nz = p$$

or $$\frac{x}{\left(\frac{p}{l}\right)} + \frac{y}{\left(\frac{p}{m}\right)} + \frac{z}{\left(\frac{p}{n}\right)} = 1$$

Coordinates of A, B, C are $\left(\frac{p}{l}, 0, 0\right), \left(0, \frac{p}{m}, 0\right)$ and $\left(0, 0, \frac{p}{n}\right)$ respectively.

$\therefore$ Centroid of $OABC$ is $\left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right)$.

$$x_1 = \frac{p}{4l}, \quad y_1 = \frac{p}{4m}, \quad z_1 = \frac{p}{4n}$$

$\because \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \frac{p^2}{16x_1^2} + \frac{p^2}{16y_1^2} + \frac{p^2}{16z_1^2} = 1$

or $\quad x_1^2y_1^2 + y_1^2z_1^2 + z_1^2x_1^2 = \frac{16}{p^2}x_1^2y_1^2z_1^2$

$\therefore$ Locus is $x^2y^2 + y^2z^2 + z^2x^2 = \frac{16}{p^2}x^2y^2z^2$

Hence, $\quad k = \frac{16}{p^2}$

24. The straight line joining the points (1, 1, 2) and (3, –2, 1) is

$$\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-2}{-1} = r \quad \text{(say)}$$

$\therefore$ Point is $(2r+1, 1-3r, 2-r)$ which lies on $3x + 2y + z = 6$

$\therefore \quad 3(2r+1) + 2(1-3r) + 2 - r = 6$

$\Rightarrow \quad r = 1$

So, required point is (3, –2, 1).

25. DC's of the given line are $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$.

Hence, the equation of line can be point in the form

$$\frac{x-2}{1/3} = \frac{y+3}{-2/3} = \frac{z+5}{-2/3} = r$$

$\therefore$ Point is $\left(2 + \frac{r}{3}, -3 - \frac{2r}{3}, -5 - \frac{2r}{3}\right)$.

$\therefore \quad r = \pm 6$

Points are (4, –7, –9) and (0, 1, –1).

26. Equation of the plane through (5, 1, 2) is

$$a(x-5) + b(y-1) + c(z-2) = 0 \quad \text{...(i)}$$

Given plane (i) is perpendicular to the line

$$\frac{x-2}{1/2} = \frac{y-4}{1} = \frac{z-5}{1} \quad \text{...(ii)}$$

$\therefore$ Equation of normal of Eq. (i) and straight line (ii) are parallel

i.e., $$\frac{a}{1/2} = \frac{b}{1} = \frac{c}{1} = k \quad \text{(say)}$$

$\therefore \quad a = \frac{k}{2}, \quad b = k, \quad c = k$

From Eq. (i),

$$\frac{k}{2}(x-5) + k(y-1) + k(z-2) = 0$$

$\Rightarrow \quad x + 2y + 2z = 11$

Any point on Eq. (ii) is $\left(2 + \frac{\lambda}{2}, 4 + \lambda, 5 + \lambda\right)$.

Which lies on Eq. (iii), then $\lambda = -2$.

$\therefore$ Required point is (1, 2, 3).

27. Let point is (α, β, γ).

$$\therefore (\alpha - a)^2 + \beta^2 + \gamma^2 = \alpha^2 + (\beta - b)^2 + \gamma^2$$
$$= \alpha^2 + \beta^2 + (\gamma - c)^2$$
$$= \alpha^2 + \beta^2 + \gamma^2$$

We get, $\quad \alpha = \frac{a}{2}, \quad \beta = \frac{b}{2}$

and $\quad \gamma = \frac{c}{2}$

$\therefore$ Required point is $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

28. Let the vertices A, B, C, D of quadrilateral be $(x_1, y_1, z_1), (x_2, y_2, z_2)$, (x_3, y_3, z_3) and (x_4, y_4, z_4) and the equation of plane $PQRS$ be

$$u \equiv ax + by + cz + d = 0$$

Let $\quad u_r = a_r x + b_r y + c_r z + d,$

where $r = 1, 2, 3, 4$

Then, $\quad \dfrac{AP}{PB} \cdot \dfrac{BQ}{QC} \cdot \dfrac{CR}{RD} \cdot \dfrac{DS}{SA}$

$$= \left(-\frac{u_1}{u_2}\right)\left(-\frac{u_2}{u_3}\right)\left(-\frac{u_3}{u_4}\right)\left(-\frac{u_4}{u_1}\right) = 1$$

29. Eliminating l between the given relations, we have

$$\frac{u(bm + cn)^2}{a^2} + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)m^2 + 2ubcmn + (c^2u + a^2w)n^2 = 0 \qquad \ldots(i)$$

If the lines be parallel, their direction cosines are equal so that the two values of m/n must be equal. The condition for this is

$$u^2b^2c^2 = (b^2u + a^2v)(c^2u + a^2w)$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

Again, if l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the two lines, then Eq. (i) gives

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1m_2}{n_1n_2} = \frac{c^2u + a^2w}{b^2u + a^2v}$$

$$\Rightarrow \quad \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v}$$

Similarly, we have

$$\frac{l_1l_2}{b^2w + c^2v} = \frac{m_1m_2}{a^2w + c^2u}$$

Thus, we have $\dfrac{l_1l_2}{b^2w + c^2v} = \dfrac{m_1m_2}{a^2w + c^2u} = \dfrac{n_1n_2}{b^2u + a^2v} = k$, say

$$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = k(b^2w + c^2v + a^2w + c^2u + b^2u + a^2v)$$

For perpendicular lines

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Thus, the condition for perpendicularity is

$$a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$$

30. Since, $[l, m, n]$ and $[l + \delta l, m + \delta m, n + \delta n]$ are DC's ,hence

$$l^2 + m^2 + n^2 = 1 \qquad \ldots(i)$$

and $\quad (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$$\Rightarrow \quad \delta l^2 + \delta m^2 + \delta n^2 = -2(l\delta l + m\delta m + n\delta n) \qquad \ldots(ii)$$

Now, $\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$$= l^2 + m^2 + n^2 + l\delta l + m\delta m + n\delta n$$

$$= 1 - \frac{1}{2}\{\delta l^2 + \delta m^2 + \delta n^2\},$$

[from Eqs. (i) and (ii)]

$$\Rightarrow \delta l^2 + \delta m^2 + \delta n^2 = 2(1 - \cos\delta\theta)$$

$$= 2 \cdot 2\sin^2\frac{\delta\theta}{2} = 4\left(\frac{1}{2}\delta\theta\right)^2$$

$$= 4\left(\frac{1}{2}\delta\theta\right)^2 = \delta\theta^2 \qquad (\text{as } \delta\theta \text{ is small, } \sin\delta\theta \approx \delta\theta)$$

31. Let α, β and γ be the intercepts of the required plane with the axes. E and F are the mid-points of AC and BC. Therefore, EF is parallel and equal to half of AB.

$\because \quad EF^2 = OE^2 + OF^2$

$$= \alpha^2 + \beta^2$$

But $\quad EF = \dfrac{AB}{2} = \dfrac{c}{2} \Rightarrow \alpha^2 + \beta^2 = \dfrac{c^2}{4}$

Similarly, $\quad \beta^2 + \gamma^2 = \dfrac{a^2}{4}$

and $\quad \gamma^2 + \alpha^2 = \dfrac{b^2}{4}$

On adding, $\quad \alpha^2 + \beta^2 + \gamma^2 = \dfrac{a^2 + b^2 + c^2}{8}$

$$\Rightarrow \quad \gamma^2 = \frac{a^2 + b^2 + c^2}{8} - \frac{c^2}{4} = \frac{a^2 + b^2 - c^2}{8}$$

Similarly, $\alpha^2 = \dfrac{b^2 + c^2 - a^2}{8}, \beta^2 = \dfrac{c^2 + a^2 - b^2}{8}$

Hence, the equation of plane is $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$, where $\alpha^2, \beta^2, \gamma^2$ as given above.

32. Solving the given equations taking three planes at a time, we get the vertices of the tetrahedron as $O(0, 0, 0)$, $A\left(-\dfrac{p}{l}, \dfrac{p}{m}, \dfrac{p}{n}\right)$, $B\left(\dfrac{p}{l}, \dfrac{p}{m}, \dfrac{p}{n}\right)$ and $C\left(\dfrac{p}{l}, \dfrac{p}{m}, -\dfrac{p}{n}\right)$ with these points as vertices. The volume V of the tetrahedron is given by

$$V = \frac{1}{6}\begin{vmatrix} -p/l & p/m & p/n \\ p/l & -p/m & p/n \\ p/l & p/m & -p/n \end{vmatrix} = \frac{p^3}{6lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{2p^3}{3lmn}$$

33. Let the variable plane $lx + my + nz = p$ cut the axes in A, B and C, then

$$A = \left(\frac{p}{l}, 0, 0\right), B = \left(0, \frac{p}{m}, 0\right), C = \left(0, 0, \frac{p}{n}\right)$$

Planes through A, B, C parallel to coordinate planes are $x = \dfrac{p}{l}, y = \dfrac{p}{m}, z = \dfrac{p}{n}$ and if they intersect in the point (α, β, γ), then $\left(\alpha = \dfrac{p}{l}, \beta = \dfrac{p}{m}, \gamma = \dfrac{p}{n}\right)$

$$\Rightarrow \quad \left(l = \frac{p}{\alpha}, m = \frac{p}{\beta}, n = \frac{p}{\gamma}\right)$$

Since, $l^2 + m^2 + n^2 = 1$, we have

$$\left(\frac{p}{\alpha}\right)^2 + \left(\frac{p}{\beta}\right)^2 + \left(\frac{p}{\gamma}\right)^2 = 1$$

Hence, the locus of (α, β, γ) is

$$p^2(x^{-2} + y^{-2} + z^{-2}) = 1$$

$$\Rightarrow \quad x^{-2} + y^{-2} + z^{-2} = p^{-2}$$

34. Take O as the origin.

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of the vertices A, B, C of the triangle.

Since, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar there must exist a relation of the form

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0},$$

$(x, y, z \in R$, not all zero) ...(i)

$$\therefore \quad y\mathbf{b} + z\mathbf{c} = -x\mathbf{a}$$

$$\Rightarrow \quad \frac{y\mathbf{b} + z\mathbf{c}}{y + z} = -\frac{x}{y + z}\mathbf{a}$$

Now, $\dfrac{y\mathbf{b} + z\mathbf{c}}{y + z}$ is a point on the line BC.

Equation of $\mathbf{OA}$ is $\mathbf{r} = -\mathbf{ta}$

Thus, $-\dfrac{x}{y + a}\mathbf{a}$ is a point of AD.

From Eq. (ii), $\quad \mathbf{OD} = \dfrac{-x}{y + z}\mathbf{a}$

$$\therefore \quad \frac{OD}{AD} = \frac{|\mathbf{OD}|}{|\mathbf{AD}|} = \left(-\frac{x}{y + z}\right) \times \left(-\frac{y + z}{x + y + z}\right) = \frac{x}{x + y + z}$$

Similarly, $\dfrac{OE}{BE} = \dfrac{y}{x + y + z}$ and $\dfrac{OF}{CF} = \dfrac{z}{x + y + z}$

On adding, $\dfrac{OD}{AD} + \dfrac{OE}{BE} + \dfrac{OE}{CF} = \dfrac{x + y + z}{x + y + z} = 1$

35. Let one vertex of the parallelopiped be taken as the origin O of coordinates and the three edges OA, OB, OC meeting at O be taken as the coordinates axes. Then, the points O, A, B, C are respectively $(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c)$.

Also, the opposite vertices O', A', B', C' are respectively the points
$(a, b, c), (0, b, c), (a, 0, c), (a, b, 0)$.

Therefore, the direction numbers of the diagonals OO' and AA' are a, b, c and $a - b, -c$ and the angle between them is

$$\cos^{-1} \frac{a \cdot a + b(-b) + c(-c)}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a^2 + b^2 + c^2)}}$$

i.e., $$\cos^{-1} \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$$

Similarly, the angle between the diagonals OO' and BB' (direction numbers $a, -b, c$) is $\cos^{-1} \dfrac{a^2 - b^2 + c^2}{a^2 + b^2 + c^2}$ and the angle between OO' and CC' (direction numbers $a, b, -c$) is $\cos^{-1} \dfrac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2}$.

The angles between the diagonals AA' and BB', AA' and CC', BB' and CC' are similarly found to be given by one of the above expressions. It follows that the angle between any two diagonals of the parallelopiped is given by one of the expressions $\cos^{-1} \dfrac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}$, where the ambiguous sings are not both positive.

36. To find the values of l, m, n, from the given relation, we have to solve these equations. From the first, we have $l = 5m - 3n$.

On substituting this value in the second equation, we get

$$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$$

i.e., $30(2m - n)(3m - 2n) = 0$ *i.e.,* $2m = n$ and $3m = 2n$

Therefore, $$\frac{m}{1} = \frac{n}{2} = \frac{5m - 3n}{5 - 2 \cdot 3} = \frac{l}{-1}$$

$$= \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(1 + 4 + 1)}} = \frac{1}{\sqrt{6}}$$

and $\dfrac{m}{2} = \dfrac{n}{3} = \dfrac{5m - 3n}{5 \cdot 2 - 3 \cdot 3} = \dfrac{l}{1} = \dfrac{1}{\sqrt{14}}$, similarly

Thus, the direction cosines of the lines are

$$\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \text{ and } \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$$

37. Let h be the height of origin parallelopiped S.

Then, $\quad V_s = (\text{ar } ABCD) \times h$ ...(i)

Let equation of plane $ABCD$ be

$$ax + by + cz + d = 0$$

and $\quad A'' \equiv (\alpha, \beta, \gamma)$

Then, height of new parallelopiped T is the length of perpendicular from A'' to $ABCD$, *i.e.,*

$$\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad V_T = (\text{ar } ABCD) \times \frac{(a\alpha + b\beta + c\gamma + d)}{\sqrt{a^2 + b^2 + c^2}} \quad \text{...(ii)}$$

Given, $\quad V_T = \dfrac{90}{100} V_s$ ...(iii)

From Eqs. (i), (ii) and (iii), $\dfrac{(a\alpha + b\beta + c\gamma + d)}{\sqrt{a^2 + b^2 + c^2}} = 0.9h$

$$ax + by + cz + (d - 0.9h\sqrt{a^2 + b^2 + c^2}) = 0$$

which is a plane parallel to $ABCD$.

38. Take C as the origin of reference. Let a, b be the position vectors of the points A and B respectively. Now, the points D is b and the point E is a. The equation of AD and BE are $\mathbf{r} = \mathbf{a} + k\left(\frac{2}{5}\mathbf{b} - \mathbf{a}\right)$ and $\mathbf{r} = \mathbf{b} + t\left(\frac{1}{4}\mathbf{a} - \mathbf{b}\right)$ respectively. For the point of intersection H, we have

$$\mathbf{a} + k\left(\frac{2}{5}\mathbf{b} - \mathbf{a}\right) = b + t\left(\frac{1}{4}\mathbf{a} - \mathbf{b}\right) \quad \ldots(\text{i})$$

$$\Rightarrow \quad 1 - k = \frac{t}{4}, \frac{2}{5}k = 1 - t \Rightarrow k = \frac{5}{6}; t = \frac{2}{3}$$

On putting these values of k and t in the equations of the lines AD and BE, we see that the point H is $\frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

The equations of CH and AB being $\mathbf{r} = t\left(\frac{\mathbf{a}}{6} + \frac{\mathbf{b}}{3}\right)$ and $\mathbf{r} = \mathbf{a} + k(\mathbf{b} - \mathbf{a})$ respectively.

For the point of intersection $F, t\left(\frac{\mathbf{a}}{b} + \frac{\mathbf{b}}{3}\right) = \mathbf{a} + k(\mathbf{b} - \mathbf{a})$

$$\Rightarrow \quad \left(\frac{t}{6} - 1 + k\right) = 0, \left(\frac{t}{3} - k\right) = 0$$

These give $t = 2$; $\quad k = \frac{2}{3}$

Thus, the point F is $\quad \frac{\mathbf{a} + 2\mathbf{b}}{3}$.

So, that F divides AB in the ratio 2 : 1

39. The equation of the plane through the line of intersection of the planes

$ax + by + cz + d = 0$ and $\alpha x + \beta y + \gamma z + \delta = 0$ is

$$(ax + by + cz + d) + \lambda(\alpha x + \beta y + \gamma z + \delta) = 0 \quad \ldots(\text{i})$$

$$\Rightarrow \quad a(a + \alpha\lambda) + y(b + \beta\lambda) + z(c + \gamma\lambda) + (d + \gamma\lambda) = 0 \quad \ldots(\text{ii})$$

Now, the equation of the xy-plane *i.e.*, $z = 0$ $\quad \ldots(\text{iii})$

If the planes (ii) and (iii) are perpendicular to each other, we have

On putting this value of λ in Eq. (i), the equation of the required plane is

$$(ax + by + cz + d) - \frac{c}{\gamma}(\alpha x + \beta y + \gamma z + \delta) = 0$$

$$\Rightarrow \quad (a\gamma - c\alpha)x + (b\gamma - c\beta)y + (d\gamma - c\delta) = 0$$

40. The given equations

$$\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} \quad \ldots(\text{i})$$

$$\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4} \quad \ldots(\text{ii})$$

Any point P on Eq.(i) is $(3r_1 + 5, -r_1 + 7, r_1 - 2)$ and any point Q on Eq. (ii) is $(-3r_2 - 3, 2r_2 + 3, 4r_2 + 6)$ the direction ratios of PQ are

$$(3r_1 + 3r_2 + 8, -r_1 - 2r_2 + 4, r_1 - 4r_2 - 8) \quad \ldots(\text{iii})$$

Suppose the line with DR's 2, 7, –5 will be proportional to the DR's given by Eq.(iii)

$$\therefore \quad \frac{3r_1 + 3r_2 + 8}{2} = \frac{-r_1 - 2r_2 + 4}{7} = \frac{r_1 - 4r_2 - 8}{-5} \quad \ldots(\text{iv})$$

On solving Eq.(iv), we get $r_1 = r_2 = -1$

So, point of intersection are $P(2, 8, -3)$ and $Q(0, 1, 2)$.

41. DR of given line are $< 1, -2, 3 >$ and the attitude numbers of the given plane are $< 1, 2, 1 >$. Since,

$1 \times 1 + (-2) \times 2 + 3 \times 1 = 0$, therefore the line is parallel to the plane.

Also, the base point of the line (1, 2, 1) lies in the given plane ($1 + 2 \times 2 + 1 = 6$ is true).

Hence, the given line lies in the given plane.

Alternatively, any point on the given line is $(t + 1, -2t + 2, 3t + 1)$.

It lies in the given plane $x + 2y + z = 6$, if $t + 1 + 2(-2t + 2) + 3t + 1 = 6$

i.e., If $0t = 0$, which is true for all real t.

Hence, every point on the given line lies in the given plane *i.e.*, the line lies in the plane.

42. $l + m + n = 0$ and $2mn + 2ml - nl = 0$

$$l = -(m + n) \Rightarrow 2mn - (2m - n)(m + n) = 0$$

$$\Rightarrow \quad 2m^2 - n^2 - mn = 0 \Rightarrow \frac{m}{n} = 1 \text{ or } \frac{-1}{2}$$

when $m = n$, $l = -2n$

when $\quad m = \frac{-n}{2}, l = \frac{-n}{2}$

Hence, $\quad \frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$ or $\frac{l}{-1} = \frac{m}{-1} = \frac{n}{2}$

Since, $\quad l^2 + m^2 + n^2 = 1$

$$(l, m, n) = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \text{ or } \left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

or $\quad \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$ or $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

43. The given point $P(1, 3, 4)$ does not lie in the given plane. Hence, the image of P in the given plane is the point Q on a line through P at right angles to the given plane such that the mid-point of $[PQ]$ lies in the given plane. The equation of PQ are

$$\frac{x-1}{2} + \frac{y-3}{-1} = \frac{z-4}{1}$$

Since, Q lies in this line, we can take it, to be $(2t + 1, -t + 3, t + 4)$. Now, the mid-point of $[PQ]$ *i.e.*,

$$\left(\frac{2t+1+1}{2}, \frac{-t+3+3}{2}, \frac{t+4+4}{2}\right) = \left(t + 1, 3 - \frac{t}{2}, 4 + \frac{t}{2}\right)$$

44. Let A, B, C be $(\alpha, 0, 0), (0, \beta, 0)$ and $(0, 0, \gamma)$, then the plane ABC is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$. Since, it always passes through a, b, c

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$$

If p is (u, v, w), then $OP^2 = AP^2 = BP^2 = CP^2$

$\Rightarrow \quad u^2+v^2+w^2=(u-\alpha)^2+v^2+w^2=\ldots$

$\Rightarrow \quad \alpha=\frac{u}{2}, \beta=\frac{v}{2}, \gamma=\frac{w}{2}$

On putting α, β, γ in Eq. (i), we get

$\frac{a}{u}+\frac{b}{v}+\frac{c}{w}=2 \Rightarrow$ Locus of (u, v, w) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

45. Let $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$, $\mathbf{OC}=\mathbf{c}$, then

$$\mathbf{a}\cdot\mathbf{a}+(\mathbf{b}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})$$
$$=\mathbf{b}\cdot\mathbf{b}+(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})$$

$\Rightarrow \quad -2\mathbf{b}\cdot\mathbf{c}=-2\mathbf{c}\cdot\mathbf{a}$

$\Rightarrow \quad (\mathbf{a}-\mathbf{b})\cdot\mathbf{c}=0$ or $\mathbf{BA}\cdot\mathbf{OC}=0$

Hence, $AB \perp OC$

Similarly, $BC \perp OA$ and $CA \perp OB$

46. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ ...(i)

$\Rightarrow \quad 1-\sin^2\alpha+1-\sin^2\beta+\cos^2\gamma=1$

or $\quad \sin^2\alpha+\sin^2\beta=1+\cos^2\gamma$

Also, from Eq. (i), we get

$$\cos^2\beta+\cos^2\gamma=1-\cos^2\alpha$$

$\Rightarrow \quad \cos^2\beta+\cos^2\gamma=\sin^2\alpha$

$\Rightarrow \quad \sin^2\alpha=\cos^2\beta+\cos^2\gamma$

47. Both lines are coplanar

$$\begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix}=0$$

$\Rightarrow \quad 2(-2+3)+3(3+3)+\lambda(-3-2)=0$

$\therefore \quad \lambda=4$

$$\sin^{-1}\sin 4=\sin^{-1}\sin(\pi-4)=\pi-4$$

48. Let $\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{4}=r_1$

$\Rightarrow \quad x=3+2r_1, y=2+3r_1, z=1+4r_1$

It will lie on, $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z-2}{3}$

$\Rightarrow \quad \frac{3+2r_1-2}{3}=\frac{2+3r_1-3}{2} \Rightarrow r_1=1$

So, point of intersection is (5, 5, 5).

49. Equation of the plane contains both lines

$$\begin{vmatrix} x-3 & y-2 & z-1 \\ 2 & 3 & 4 \\ 3 & 2 & 3 \end{vmatrix}=0$$

$(x-3)(1)+(y-2)(12-6)+(z-1)(4-9)=0$

$\Rightarrow \quad x+6y-5z=10$

$\because \quad (1)(4)+(6)(1)+(-5)(2)=4+6-10=0$

Thus, the angle is $\frac{\pi}{2}$.

50. **Statement I** $\mathbf{r}\times(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}})=\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ 2 & -1 & 3 \end{vmatrix}$

$$=\hat{\mathbf{i}}(3y+z)-\hat{\mathbf{j}}(3x-2z)+\hat{\mathbf{k}}(-x-2y)$$
$$=3\hat{\mathbf{i}}+\hat{\mathbf{k}}$$

$3y+z=3, 3x-2z=0, -x-2y=1$

$\therefore \quad 3x+6y=6 \Rightarrow x+2y=2$ ...(i)

$x+2y=-1$ ...(ii)

From Eqs. (i) and (ii) are parallel planes not straight line

Statement II

$$\mathbf{r}\times(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})=\hat{\mathbf{i}}(-3y-2z)-\hat{\mathbf{j}}(3x-z)+\hat{\mathbf{k}}(2x-y)$$
$$=2\hat{\mathbf{i}}-\hat{\mathbf{j}}$$

i.e., $\quad -3y-2z=2, 3x+z=-1, 2x-y=0$

Straight line $2x-y=0, 3x+z=-1$ is true.

51. Since, orthocentre, nine point centre, centroid and circumcentre are collinear and centroid divides orthocentre and circumcentre in the ratio 2 : 1 (internally).

$\therefore \quad \alpha=\frac{x+2\gamma}{2+1}$

$\Rightarrow \quad x=3\alpha-2\gamma$

and $\quad \beta=\frac{\gamma+2\delta}{2+1}$

$\Rightarrow \quad y=3\beta-2\delta$

$\therefore$ Orthocentre is $(3\alpha-2\gamma, 3\beta-2\delta)$.

52. Let l, m, n be the DC's of the line of the common perpendicular (or SD) to the two given lines. Then, we have

$-4l+3m+2n=0$

and $\quad -4l+1m+1n=0$

On solving these, we get

$$\frac{l}{3-2}=\frac{m}{-8+4}=\frac{N}{-4+12}$$

$\Rightarrow \quad \frac{l}{1}=\frac{m}{-4}=\frac{n}{8}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1)^2+(-4)^2+(8)^2}}=\frac{1}{9}$

53. Thus, statement II is false.

Now, $\quad AB=\sqrt{(2-1)^2+(9-8)^2+(12-8)^2}$

$=\sqrt{18}=3\sqrt{2}$

$BC=\sqrt{(1+2)^2+(8-11)^2+(8-8)^2}$

$=\sqrt{18}=3\sqrt{2}$

$CD=\sqrt{\{(-2+1)^2+(11-12)^2+(8-12)^2\}}$

$=\sqrt{18}=3\sqrt{2}$

$DA=\sqrt{(-1-2)^2+(12-9)^2+(12-12)^2}$

$$= \sqrt{18} = 3\sqrt{2}$$

$$AC = \sqrt{(2+2)^2 + (9-11)^2 + (12-8)^2}$$

$$= \sqrt{36} = 6$$

and $$BD = \sqrt{(1+1)^2 + (8-12)^2 + (8-12)^2}$$

$$= \sqrt{36} = 6$$

Hence, $AB = BC = CD = DA$ and $AC = BC$

$\therefore$ DC's of SD are $\frac{1}{9}, \frac{4}{-9}, \frac{8}{9}$.

Also, $A(-3, 6, 0)$ is a point on first line and $B(-2, 0, 7)$ is a point on second line, then

$$\text{SD} = \left|(-2+3)\frac{1}{9} + (0-6)\left(-\frac{4}{9}\right) + (7-0)\left(\frac{8}{9}\right)\right| = 9$$

And two lines are said to be skew lines or non-intersecting lines, if they do not lie in the same plane.

54. **Statement I** $3y - 4z = 5 - 2k$

and $-2y + 4z = 7 - 3k$

$\therefore x = k, y = 12 - 5k$ and $z = \frac{31 - 13k}{4}$ is a point on the line for all real values of k.

Statement I is true.

Statement II Direction ratios of the straight line are $< bc' - kbc, kac - ac', 0 >$ direction ratios of normal to be plane $< 0, 0, 1 >$.

Now, $0 \times (bc' - kbc) + 0 \times (kac - ac') + 1 \times 0 = 0$

$\therefore$ The straight line is parallel to the plane.

55. The equation of the plane containing them is

$$\begin{vmatrix} x-1 & y & z+1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -(5x + zy - 3z - 8) = 0$$

Statement II Here, $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$

$\Rightarrow$ $\frac{1}{3} = \frac{1}{3} = \frac{1}{3}$

And $1(1) + 2(1) + 3(-1) = 0$

56. **Statement II** Lines L_1 and L_2 are parallel to the vectors $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, respectively. The unit vector perpendicular to both L_1 and L_2 is $\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{-\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}}{\sqrt{1 + 49 + 25}} = \frac{-\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}}{5\sqrt{3}}$.

Statement I Plane is $-(x+1) - 7(y+2) + 5(z+1) = 0$ whose distance from $(1, 1, 1)$ is $\frac{13}{5\sqrt{3}}$.

We observe that the line segment joining the points $A(3, 1, 6)$ and $B(1, 3, 4)$ has direction ratios $2, -2, 2$ which one proportional to $1, -1, 1$ the direction ratios of the normal to the plane. Hence, statement II is true.

Thus, the statement I and II are true and statement II is correct explanation of statement I.

57. Given planes are

$$2x + y + 2z - 8 = 0$$

and $$2x + y + 2z + \frac{5}{2} = 0$$

Distance between two planes

$$= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2 + c^2}} = \left|\frac{-8 - \frac{5}{2}}{\sqrt{2^2 + 1^2 + 2^2}}\right|$$

$$= \frac{\frac{21}{2}}{3} = \frac{7}{2}$$

58. Condition for two lines are coplanar.

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

where, (x_1, y_1, z_1) and (x_2, y_2, z_2) are the points lie on a line (i) and (ii) respectively and $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ are the direction cosines of the line (i) and (ii) respectively.

$\therefore$ $$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1(1 + 2k) + (1 + k^2) - (2 - k) = 0$

$\Rightarrow$ $k^2 + 2k + k = 0$

$\Rightarrow$ $k^2 + 3k = 0$

$\Rightarrow$ $k = 0, -3$

59. Let $L_1 : \frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = p$

and $L_2 : \frac{x-3}{1} = \frac{y-k}{2} = \frac{z-0}{1} = q$

$\Rightarrow$ Any point P on line L_1 is of type

$P(2p + 1, 3p - 1, 4p + 1)$ and

any point Q on line L_2 is of type

$Q(q + 3, 2q + k, q)$

Since, L_1 and L_2 are intersecting each other, hence both point P and Q should coincide at the point of intersection, *i.e.*, corresponding coordinates of P and Q should be same.

$$2p + 1 = q + 3,\ 4p + 1 = q \text{ and } 3p - 1 = 2q + k$$

On solving $2p + 1 = q + 3$ and $4p + 1 = q$, we get the values of p and q as

$$p = \frac{-3}{2} \text{ and } q = -5$$

On substituting the values of p and q in the third equation $3p - 1 = 2q + k$, we get

$\Rightarrow$ $3\left(\frac{-3}{2}\right) - 1 = 2(-5) + k \Rightarrow k = \frac{9}{2}$

60. Given, a plane $P : x - 2y + 2z - 5 = 0$

The equation of a plane parallel to given plane P and at a distance of 1 unit from origin. Equation of family of planes parallel to the given plane P is

$$Q : x - 2y + 2z + d = 0$$

Also, perpendicular distance of Q from origin is 1 unit.

$$\Rightarrow \quad \left| \frac{0 - 2(0) + 2(0) + d}{\sqrt{1^2 + 2^2 + 2^2}} \right| = 1$$

$$\Rightarrow \quad \left| \frac{d}{3} \right| = 1 \Rightarrow d = \pm 3$$

Hence, the required equation of the plane parallel to P and at unit distance from origin is

$$x - 2y + 2z \pm 3 = 0$$

61. Angle between straight line

$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and plane $\mathbf{r} \cdot \mathbf{n} = d$

$$\Rightarrow \quad \sin\theta = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

$$\therefore \quad \sin\theta = \frac{(\mathbf{i} + 2\mathbf{j} + \lambda \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}{\sqrt{1 + 4 + \lambda^2}\sqrt{1 + 4 + 9}}$$

$$\Rightarrow \quad \sin\theta = \frac{5 + 3\lambda}{\sqrt{\lambda^2 + 5} \cdot \sqrt{14}}$$

Given, $\quad \cos\theta = \sqrt{\frac{5}{14}}$

$$\therefore \quad \sin\theta = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \quad \frac{3}{\sqrt{14}} = \frac{5 + 3\lambda}{\sqrt{\lambda^2 + 5} \cdot \sqrt{14}}$$

$$\Rightarrow \quad 9(\lambda^2 + 5) = 9\lambda^2 + 30\lambda + 25$$

$$\Rightarrow \quad 9\lambda^2 + 45 = 9\lambda^2 + 30\lambda + 25 \Rightarrow 30\lambda = 20$$

$$\therefore \quad \lambda = \frac{2}{3}$$

62. Let the coordinate of Q be $(2\lambda, 3\lambda + 2, 4\lambda + 3)$, which is any point on the straight line AB.

$\therefore$ DR's of PQ, is $(2\lambda - 3, 3\lambda + 3, 4\lambda - 8)$ and it is perpendicular to straight line AB.

i.e., $\quad \frac{x}{2} = \frac{y - 2}{3} = \frac{z - 3}{4} = \lambda$ having DR's (2, 3, 4).

$$\Rightarrow \quad 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$$

$$\Rightarrow \quad 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$$

$$\therefore \quad 29\lambda - 29\lambda = 0$$

$$\lambda = 1$$

Coordinate of Q are (2, 5, 7).

$$\therefore \quad |PQ| = \sqrt{(3 - 2)^2 + (-1 - 5)^2 + (7 - 11)^2}$$

$$= \sqrt{1 + 36 + 16} = \sqrt{53}$$

63. Equation of PQ, is

$$\frac{x - 1}{1} = \frac{y + 5}{1} = \frac{z - 9}{1} = \lambda$$

$\therefore$ $x = \lambda + 1, y = \lambda - 5$ and $z = \lambda + 9$ lies on the plane

$$x - y + z = 5$$

$$\Rightarrow \quad \lambda + 1 - \lambda + 5 + \lambda + 9 = 5$$

$$\therefore \quad \lambda = -10$$

Coordinate of Q, is

$Q(-9, -15, -1)$ and coordinate of $P(1, -5, 9)$.

$$\Rightarrow \quad |PQ| = \sqrt{(10)^2 + (10)^2 + (10)^2} = 10\sqrt{3}$$

64. We know that, $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$

$$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{2}$$

$$\therefore \quad \theta = 60° \text{ or } 120°$$

Which shows that statement I is true.

65. DR's of given line are $(3, -5, 2)$.

DR's of normal to the plane $= (1, 3, -\alpha)$

$\therefore$ Line is perpendicular to the normal

$$\Rightarrow \quad 3(1) - 5(3) + 2(-\alpha) = 0$$

$$\Rightarrow \quad 3 - 15 - 2\alpha = 0$$

$$\Rightarrow \quad 2\alpha = -12$$

$$\Rightarrow \quad \alpha = -6$$

Also, point $(2, 1, -2)$ lies on the plane

$$2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$$

$$\therefore \quad (\alpha, \beta) = (-6, 7)$$

36 Advance Probability

JEE Main MILESTONE

- Event
- Probability
- Important Results Related to Probability
- Conditional Probability
- Important Results Related to Conditional Probability
- Law of Total Probability
- Baye's Theorem
- Random Variable and its Distribution
- Mean and Variance of a Random Variable
- Binomial Distribution
- Poisson Distribution
- Geometrical Probability

36.1 Event

Any particular performance of a random experiment is called a trial and outcome or combination of outcomes are termed as events.

e.g., If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces head or tail.

Thus, tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

Event is called **simple** if it corresponds to a single possible outcome of the experiment otherwise it is known as a **compound** or **composite event.**

e.g., In tossing of a single die the event of getting '6' is a simple event but the event of getting an even number is a composite event.

The probability as a measure of uncertainity of events in a random experiment. We have also established equivalence between the axiomatic theory and the classical theory of probability in case of equaly likely outcomes. On the basis of this relationship, we obtained probabilities of events associated with discrete sample spaces. Here, we discuss the important concept of conditional probability of an event given that another event has occurred, which will be helpful in understanding the Bayes's theorem, multiplication rule of probability and independence of events.

Important Type of Events

(i) Equally Likely Events

The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

e.g., In throwing an unbiased die, all the six faces are equally likely to come.

(ii) Mutually Exclusive Events

Events are said to be mutually exclusive or incompatible, if the happening of any one of them precludes the happening of all the others *i.e.,* if no two or more of them can happen simultaneously in the same trial.

OR

Two or more events associated to a random experiment are said to be mutually exclusive events, if the occurrence of one of them prevents or denies the occurrence of all others.

If E_1 and E_2 are mutually exclusive, then $P(E_1 \cap E_2) = 0$

e.g., In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive, since if any one of these faces comes, the possibility of others, in the same trial, is ruled out.

Note If $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive events, then $E_1 \cap E_2 \cap \ldots \cap E_n = \phi$

(iii) Exhaustive Events

The total number of possible outcomes of a random experiment is known as the exhaustive events or cases.

OR

Two or more events associated to a random experiments are exhaustive, if their union is the sample space.

i.e., Events $A_1, A_2, A_3, \ldots, A_n$ associated to a random experiment with sample space are exhaustive, if

$$A_1 \cup A_2 \cup \ldots \cup A_n = S.$$

i.e., $$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1$$

e.g., In throwing of two dice, the exhaustive number of cases is $6^2 = 36$, since any of the numbers 1 to 6 on the first die can be associated with any of the 6 numbers on the other die.

Note In throwing of n dice, the exhaustive number of case is 6^n.

(iv) Independent Events or Mutually Independent Events

Two events A and B are said to be independent events, if the happening (or non-happening) of any random experiment does not affect the happening (or non-happening) of the other.

i.e., $$P(A \cap B) = P(A)\, P(B)$$

Otherwise events are known as **dependent events.**

e.g., If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

(v) Complement of an Event

A sample space associated with a random experiment be S and event is E, then the complement of an event E is denoted by E', E^c or $\overline{E}$, E' means event E does not occur.

$$E' = \{n : n \in S, n \notin E\}$$

(i) $P(E \cup E') = S$

(ii) $P(E \cap E') = \phi$

(iii) $P(E')' = E$

e.g., When an unbiased die is thrown, then the sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

and $E = \{1, 3, 5\}$,

then $E' = \{2, 4, 6\}$

Note

- If E_1 and E_2 are independent events, then
- E_1 and $\overline{E}_2$ are independent events.
- $\overline{E}_1$ and E_2 are independent events.
- $\overline{E}_1$ and $\overline{E}_2$ are independent events.
- Let the sample space associated to a random experiment is S, then ϕ and S are subset of S. The event ϕ is known as impossible event and event S is known as certain event.

Partition of a Sample Space

The events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S, if they are pairwise disjoint, exhaustive and have non-zero probabilities, *i.e.,*

(i) $A_i \cap A_j = \phi; i \neq j; i, j = 1, 2, \ldots, n$

(ii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(iii) $P(A_i) > 0, \forall i = 1, 2, \ldots, n$

36.2 Probability

The probability of an event E occur is the ratio of the number of cases in its favour to the total number of cases.

$$\therefore \quad P(E) = \frac{\text{Number of cases favourable to event } E}{\text{Total number of cases}} = \frac{n(E)}{n(S)}$$

Probability of non-occurrence of event E is

$$P(\overline{E}) = 1 - P(E)$$

and $0 \le P(E) \le 1$. If $P(E) = 1$, then event E is known as certain event and if $P(E) = 0$, then E is known as impossible event.

***Sample* Problem 1** *Two cards are drawn at random from a pack of 52 cards. The probability of getting atleast a spade and an ace, is*

(a) $\frac{1}{34}$ (b) $\frac{8}{221}$ (c) $\frac{1}{26}$ (d) $\frac{2}{51}$

Interpret (c) Total number of cases, $n(S) = {}^{52}C_2$

$n(E)$ = The number of selections of 1 spade, 1 ace from 3 aces or selections of the ace of spade and 1 other spade.

$$= {}^{13}C_1 \times {}^{3}C_1 + {}^{12}C_1 \times {}^{1}C_1 = 51$$

$$\therefore \quad P(E) = \frac{51}{{}^{52}C_2} = \frac{1}{26}$$

Odds in favour and odds against an event If a is the number of cases favourable to the event E, b is the number of cases favourable to the event $\overline{E}$. (*i.e.,* number of cases

against to E). Then, odds infavour of E are $a : b$ and odds against of E are $b : a$. Then, $P(E) = \frac{a}{a+b}$ and $P(\overline{E}) = \frac{b}{a+b}$.

Thus, odds in favour of an event $E = \frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(\overline{E})}$

and odds against an event $E = \frac{b}{a} = \frac{P(\overline{E})}{P(E)}$

Sample **Problem 2** *What are the odds infavour of throwing atleast 8 in a single throw with two dice?*

(a) 5 : 1 (b) 5 : 3
(c) 5 : 7 (d) 5 : 6

Interpret (c) Let S be the sample space and E be the event of occurrence of a total of atleast 8, then

$$n(S) = 6 \times 6 = 36 \quad \text{and} \quad n(E) = 15$$

$$\therefore \quad P(E) = \frac{15}{36}$$

$$P(\overline{E}) = 1 - P(E) = 1 - \frac{15}{36} = \frac{21}{36}$$

Odd in favour of $E = P(E) : P(\overline{E})$
$= 5 : 7$

36.3 Important Results Related to Probability

(i) If E_1 and E_2 are events associated with a random experiment, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad \text{(addition theorem)}$$

If E_1 and E_2 are mutually exclusive events, *i.e.,* $E_1 \cap E_2 = \phi$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \qquad [\because P(\phi) = 0]$$

(ii) If E_1, E_2 and E_3 are events associated with a random experiment, then

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

If E_1, E_2 and E_3 are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

(iii) In general $P(E_1 \cup E_2 \cup \ldots \cup E_n)$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \ldots + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots \cap E_n)$$

If $E_1, E_2, \ldots, E_n$ are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$$

(iv) (a) $P(\overline{E}_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2)$
(b) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_1 \cap E_2)$
(c) $P[(E_1 \cap \overline{E}_2) \cup (\overline{E}_1 \cap E_2)] = P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$
(d) $P(\overline{E}_1 \cap \overline{E}_2) = 1 - P(E_1 \cup E_2)$
(e) $P(\overline{E}_1 \cup \overline{E}_2) = 1 - P(E_1 \cap E_2)$

(v) If $E_2 \subset E_1$, then
(a) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_2)$
(b) $P(E_2) \le P(E_1)$

(vi) **Booley's inequality** If events $E_1, E_2, E_3, \ldots, E_n$ are associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^{n} E_i\right) \ge \sum_{i=1}^{n} P(E_i) - (n-1)$

(b) $P\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} P(E_i)$

(vii) P (atleast two events from E_1, E_2 and E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 2P(E_1 \cap E_2 \cap E_3)$$

(viii) P (two events from E_1, E_2 and E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 3P(E_1 \cap E_2 \cap E_3)$$

(ix) P (only one event occur from E_1, E_2 and E_3)

$$= P(E_1) + P(E_2) + P(E_3) - 2P(E_1 \cap E_2) - 2P(E_2 \cap E_3) - 2P(E_3 \cap E_1) + 3P(E_1 \cap E_2 \cap E_3)$$

(x) If $E_1, E_2, E_3, \ldots, E_n$ are independent events, then

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n) = P(E_1)\,P(E_2) \ldots P(E_n)$$

(xi) If $E_1, E_2, E_3, \ldots, E_n$ are exhaustive events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1$$

(xii) If $E_1, E_2, E_3, \ldots, E_n$ are mutually exclusive and exhaustive events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n) = 1$$

(xiii) If $E_1, E_2, E_3, \ldots, E_n$ are n events, then
(a) $P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) \le P(E_1) + P(E_2) + \ldots + P(E_n)$
(b) $P(E_1 \cup E_2 \cup \ldots \cup E_n) \ge 1 - P(\overline{E}_1) - P(\overline{E}_2) - \ldots - P(\overline{E}_n)$

(xiv) If E_1 and E_2 are two events, then

$$P(E_1 \cap E_2) \le P(E_1) \le P(E_1 \cup E_2) \le P(E_1) + P(E_2)$$

(xv) (a) P (exactly one of E_1 and E_2 occurs)

$$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$$
$$= P(E_1 \cup E_2) - P(E_1 \cap E_2)$$

(b) P (neither E_1 nor E_2) $= P(E_1' \cap E_2') = 1 - P(E_1 \cup E_2)$

(xvi) If $E_1, E_2, \ldots, E_n$ are independent events associated with a random experiment, then probability of occurrence of atleast one $= P(E_1 \cup E_2 \cup \ldots \cup E_n)$

$$= 1 - P(\overline{E_1 \cup E_2 \cup \ldots \cup E_n})$$
$$= 1 - P(\overline{E}_1) \cdot P(\overline{E}_2) \ldots P(\overline{E}_n)$$

***Sample* Problem 3** *The probability that atleast one of the events A and B occurs is 0.6. If A and B occurs simultaneously with probability 0.2, then the value of $P(\overline{A}) + P(\overline{B})$ is*

(a) 1.8 (b) 1.2
(c) 0.6 (d) *None of these*

Interpret (*b*) We have,

P (atleast one of the events A and B occurs) *i.e.*, $P(A \cup B) = 0.6$

and P (A and B occur simultaneously) *i.e.*, $P(A \cap B) = 0.2$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow$ $0.6 = P(A) + P(B) - 0.2$

$\Rightarrow$ $0.6 = 1 - P(\overline{A}) + 1 - P(\overline{B}) - 0.2$

$\Rightarrow$ $0.6 = 2 - 0.2 - [P(\overline{A}) + P(\overline{B})]$

$\Rightarrow$ $0.6 = 1.8 - [P(\overline{A}) + P(\overline{B})]$

$\Rightarrow$ $P(\overline{A}) + P(\overline{B}) = 1.8 - 0.6 = 1.2$

Check Point **1**

1. On hundred cards are numbered from 1 to 100. The probability that a randomly chosen card has a digit 5 is
(a) $\frac{1}{100}$ (b) $\frac{9}{100}$
(c) $\frac{19}{100}$ (d) None of these

2. Two dice are rolled one after the other. The probability that the number on the first is smaller than the number on the second is
(a) $\frac{1}{2}$ (b) $\frac{7}{18}$
(c) $\frac{3}{4}$ (d) $\frac{5}{12}$

3. A car is parked by a driver amongst 25 cars in a row, not at either end. When he returns he finds that 10 places are empty. The probability that both the neighbouring places of driver's car are vacant, is
(a) $\frac{9}{92}$ (b) $\frac{15}{92}$
(c) $\frac{21}{92}$ (d) $\frac{27}{92}$

4. The probability of getting 10 in a single throw of three fair dice is
(a) $\frac{1}{6}$ (b) $\frac{1}{8}$
(c) $\frac{1}{9}$ (d) None of these

5. Odds in favour of an event A are 2 to 1 and odds in favour of $A \cup B$ are 3 to 1. Consistent with this information the smallest and largest values for the probability of event B are given by
(a) $\frac{1}{6} \le P(B) \le \frac{1}{3}$ (b) $\frac{1}{3} \le P(B) \le \frac{1}{2}$
(c) $\frac{1}{12} \le P(B) \le \frac{3}{4}$ (d) None of these

6. The probability that a teacher will give an unannounced test during any class meeting is $\frac{1}{5}$. If a student is absent twice, the probability that he will miss atleast one test, is
(a) 7/25 (b) 9/25
(c) 16/25 (d) 24/25

7. If two events A and B are such that $P(A^c) = 0.3$, $P(B) = 0.4$, $P(A \cap B^c) = 0.5$, then $P\left(\frac{B}{A \cup B^c}\right)$ is equal to
(a) 0.20 (b) 0.25 (c) 0.30 (d) 0.35

8. Probabilities of teams A, B and C winning are $\frac{1}{4}, \frac{1}{6}$ and $\frac{1}{8}$ respectively. Probability that one of these teams will win, is
(a) $\frac{13}{24}$ (b) $\frac{11}{24}$
(c) $\frac{23}{24}$ (d) None of these

36.4 Conditional Probability

The probability of occurrence of an event E_2, when it is known that some event E_1 has already occurred is called the conditional probability and is denoted by $P\left(\frac{E_2}{E_1}\right)$. The symbol $P\left(\frac{E_2}{E_1}\right)$ is usually read as probability of E_2, given E_1.

Consider two events E_1 and E_2. When it is known that event E_1 has occurred, it means that sample space would reduce to that sample points representing event E_1. Now, for $P\left(\frac{E_2}{E_1}\right)$ we must look for the sample points representing the simultaneous occurrence of E_1 and E_2, *i.e.*, sample points in $E_1 \cap E_2$.

$$\Rightarrow \quad P\left(\frac{E_2}{E_1}\right) = \frac{n(E_1 \cap E_2)}{n(E_1)} = \frac{\frac{n(E_1 \cap E_2)}{n(S)}}{\frac{n(E_1)}{n(S)}} = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

Thus, $P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$, where $0 < P(E_1) \le 1$.

Similarly, $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$, $0 < P(E_2) \le 1$.

Hence, $P(E_1 \cap E_2) = \begin{cases} P(E_1) \cdot P(E_2/E_1), & P(E_1) > 0 \\ P(E_2) \cdot P(E_1/E_2), & P(E_2) > 0 \end{cases}$

36.5 Important Results Related to Conditional Probability

(i) If E_1 and E_2 are independent events, then

$$P\left(\frac{E_2}{E_1}\right) = P(E_2)$$

(ii) If $E_1, E_2, E_3, \ldots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1 - P(E_1')\,P(E_2')\ldots P(E_n')$$

(iii) If E_1 and E_2 are events such that $E_2 \neq 0$, then

$$P\left(\frac{E_1}{E_2}\right) + P\left(\frac{E_1'}{E_2}\right) = 1$$

(iv) If E_1 and E_2 are events such that $E_1 \neq \phi$, then

$$P(E_2) = P(E_1)\cdot P\left(\frac{E_2}{E_1}\right) + P(E_1')\cdot P\left(\frac{E_2}{E_1'}\right)$$

(v) Multiplication theorem on probability if E_1, E_2 and E_3 are three events such that $E_1 \neq \phi$, $E_1E_2 \neq \phi$, then

$$P\,(E_1 \cap E_2 \cap E_3) = P(E_1)\cdot P\left(\frac{E_2}{E_1}\right)\cdot P\left(\frac{E_3}{E_1E_2}\right)$$

In general, if $E_1, E_2, \ldots, E_n$ are n events such that $E_1 \neq \phi$, $E_1E_2 \neq \phi$, $E_1E_2E_3 \neq \phi, \ldots, E_1E_2\ldots E_{n-1} \neq \phi$, then

$$P\,(E_1 \cap E_2 \cap \ldots \cap E_n)$$
$$= P(E_1)\,P\left(\frac{E_2}{E_1}\right)P\left(\frac{E_3}{E_1E_2}\right)\ldots P\left(\frac{E_n}{E_1E_2\ldots E_{n-1}}\right)$$

(vi) If E_1 and E_2 are independent events, then

(a) $P(E_1 \cap E_2) = P(E_1)\cdot P\left(\frac{E_2}{E_1}\right)$, If $P(E_1) \neq 0$

(b) $P(E_1 \cap E_2) = P(E_2)\cdot P\left(\frac{E_1}{E_2}\right)$, If $P(E_2) \neq 0$

(vii) If E_1 and E_2 are independent events, then

$$P(E_1 \cap E_2) = P(E_1)\cdot P(E_2)$$

(viii) If $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \ldots \cap E_n) = P(E_1)\,P(E_2)\ldots P(E_n)$$

(ix) **The events A and ϕ are always independent**

$\therefore \quad P(A \cap \phi) = P(A)\cdot P(\phi) = 0$

i.e., an event A and an impossible event (*i.e.,* null event) can never occur together.

(x) **The events A and S are always independent**

$\therefore \quad P(A \cap S) = P(A)\cdot P(S) = P(A)$

i.e., occurrence of an event A along with sample space event is the event A itself.

(xi) **Mutually exclusive events** (none of which is an impossible event) **are not independent.**

Let A and B be two non-impossible mutually exclusive events

$\therefore \quad P(A \cup B) = P(A) + P(B)$ ...(i)

Now, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ (additional theorem)

$\Rightarrow \quad P(A \cap B) = 0$ [using Eq. (i)]

But since A and B are non-impossible events

$\therefore \quad P(A)$ and $P(B) \neq 0$

$\Rightarrow \quad P(A)\cdot P(B) > 0$

{$\because$ probability of an event is never negative}

Hence, $P(A \cap B) \neq P(A)\cdot P(B)$

$\therefore$ Events A and B are not independent.

Hence, mutually exclusive events (none of which is impossible event) are not independent.

(xii) **Non-impossible independent events are not mutually exclusive**

Let A and B be non-impossible independent events, then

$P(A \cap B) = P(A)\cdot P(B)$ ...(i)

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (addition theorem)

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A)\cdot P(B)$ ...(ii)

Since, A and B are non-impossible events

$\therefore \quad P(A)$ and $P(B) \neq 0$

$\Rightarrow \quad P(A)\cdot P(B)$ exists finitely

Hence, $P(A \cup B) \neq P(A) + P(B)$ [from Eq. (ii)]

$\Rightarrow$ The events are not mutually exclusive.

Hot Spot 1

Word Type

Conditional Probability

Some word type problems are given and find the conditional probability of a given event. The nature of this problem is in average. These type of problem are directly asked in the various JEE Mains 2012, 2009, 2008.

To solve such type of problem following steps are given below.

Step I Firstly write the given condition in numerical problem.

Step II Using the important reslt and formula $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

Step III Simplify it and get the result.

***Sample* Problem 4** *If a pair of dice is thrown and it is known that sum of the number is even, then the probability that the sum is less than 6, is*

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{1}{3}$ (d) $\frac{4}{9}$

Interpret (b)

Step I Write the given condition in numerical form.

Let A be the given event and B be the event, whose probability is to be found. Then,

$$P(B \cap A) = \frac{4}{36},\ P(A) = \frac{18}{36}$$

Step II Using the formula $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

Step III Simplify it

Required probability, $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{4/36}{18/36} = \frac{2}{9}$

***Sample* Problem 5** *If A and B are independent events, then the following pairs of events are*

(i) A' *and* B (ii) A' *and* B'

(a) *dependent*

(b) *independent*

(c) *disjoint*

(d) *None of the above*

Interpret (c) (i) $P(A' \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$

($\because$ A and B are independent) $= P(B)\,(1 - P(A)) = P(B)\,P(A')$

(ii) $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= \{1 - P(A)\}\{1 - P(B)\} = P(A') \cdot P(B').$$

(xiii) **Mutually independent events**

Three events A, B and C are said to be mutually independent if,

$$P(A \cap B) = P(A) \cdot P(B),\ P(A \cap C) = P(A) \cdot P(C),$$

$$P(B \cap C) = P(B) \cdot P(C)$$

and $$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

These events would be said to be pairwise independent, if

$$P(A \cap B) = P(A) \cdot P(B),\ P(B \cap C) = P(B) \cdot P(C)$$

and $$P(A \cap C) = P(A) \cdot P(C).$$

Thus, mutually independent events are pairwise independent but the converse may not be true.

***Sample* Problem 6** *A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time with replacement. The events A, B and C are defined as under*

A = {The first bulb is defective},

B = {The second bulb is non-defective}

C = {The two bulbs are either both defective or both non-defective}

The events A, B and C to be pairwise,

(a) *dependent* (b) *independent*

(c) *mutually independent* (d) *None of these*

Interpret (b) $P(A) = \frac{1}{2} \times 1 = \frac{1}{2}, P(B) = 1 \times \frac{1}{2} = \frac{1}{2}$

and $$P(C) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$P(A \cap B) = P$ (the first bulb in defective and the second bulb is non-defective) $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

$P(B \cap C) = P$ (both the bulbs are non-defective) $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(C \cap A) = P$ (both the bulbs are defective)

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$\Rightarrow$ $$P(A \cap B \cap C) = P(\phi) = 0$$

As $$P(A \cap B) = P(A) \cdot P(B), P(B \cap C) = P(B) \cdot P(C)$$

and $P(C \cap A) = P(C) \cdot P(A)$, the events A, B and C are pairwise independent.

Since, $$P(A \cap B \cap C) = 0 \neq P(A) \cdot P(B) \cdot P(C),$$

A, B and C are not mutually independent.

***Sample* Problem 7** *$P_1, P_2, \ldots, P_8$ an eight players participating in a tournament. If $i < j$, then P_i will win the match against P_1. Players are paired up randomly for first round and winners of this round again paired up for the second round and so on. The probability that P_4 reaches in the final, is*

(a) $\frac{2}{7}$ (b) $\frac{4}{9}$

(c) $\frac{4}{35}$ (d) *None of these*

Interpret (c) Let A_1 be the event that in the first round the four winners are P_1, P_4, P_i, P_j where $i \in \{2, 3\}, j \in \{5, 6, 7\}$ and let A_2 be the event that out of the four winners in the first round, P_1 and P_4 reaches in the final.

The event A_1 will occur, if P_4 plays with any of P_5, P_6, P_7 or P_8 (say with P_6) and P_1, P_2 and P_3 are not paired with P_5, P_7 and P_8. Further A_2 will occur if P_1 plays with P_i.

The required probability $= P(A_1 \cap A_2) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right)$

$$= \frac{4 \times \left(\frac{6!}{(2!)^3\, 3!} - 3!\right)}{\frac{8!}{(2!)^4\, 4!}} \times \frac{1}{\frac{4!}{(2!)^2\, 2!}} = \frac{4}{35}.$$

(here, we have used the concept of division into groups).

36.6 Law of Total Probability

Let in a random experiment S is a sample space and $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events. If A is any event which occur with E_1 or E_2 or E_3 or ... or E_n, then

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + \ldots + P(E_n)\, P\left(\frac{A}{E_n}\right).$$

$$= \sum_{r=1}^{n} P(E_r)\, P\left(\frac{A}{E_r}\right)$$

***Sample* Problem 8** *Two sets of candidates are competing for the positions on the board of directors of a company. The probabilities that the first and second sets will win are 0.6 and 0.4 respectively. If the first set wins, the probability of introducing a new product is 0.8 and the corresponding probability, if the second set wins is 0.3. What is the probability that the new product will be introduced?*

(a) 0.1 (b) 0.3

(c) 0.5 (d) 0.6

Interpret (d) Let A_1 (A_2) denotes the event that first (second) set wins and let B be the event that a new product is introduced.

$\therefore$ $P(A_1) = 0.6, \quad P(A_2) = 0.4$

$$P\left(\frac{B}{A_1}\right) = 0.8, \quad P\left(\frac{B}{A_2}\right) = 0.3$$

$$P(B) = P(B \cap A_1) + P(B \cap A_2)$$

$$= P(A_1) \cdot P\left(\frac{B}{A_1}\right) + P(A_2)\, P\left(\frac{B}{A_2}\right)$$

$$= 0.6 \times 0.8 + 0.4 \times 0.3 = 0.6$$

36.7 Baye's Theorem

If $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$, then for any event E which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(E) > 0$, then

$$P\left(\frac{E_i}{E}\right) = \frac{P(E_i) \cdot P\left(\frac{E}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)}$$

Note The probabilities $P(E_1), P(E_2), \ldots, P(E_i)$ are called 'prior probabilities' because they exist before, we gain any information from the experiment itself.

$P\left(\frac{E}{E_1}\right)$, $i = 1, 2, \ldots, n$ are called the likelyhood probabilities.

The probability $P\left(\frac{E_i}{E}\right)$, $i = 1, 2, \ldots, n$ are called 'posterior probabilities' because they are determined after the results of the experiment are known.

***Sample* Problem 9** *Box I contains 2 white and 3 red balls and box II contains 4 white and 5 red balls. One ball is drawn at random from one of the boxes and is found to be red. Then, the probability that it was from box II, is*

(a) $\frac{25}{32}$ (b) $\frac{21}{32}$ (c) $\frac{7}{32}$ (d) $\frac{15}{32}$

Interpret (a) Let A denote the event that the drawn ball is red.

Let $A_1 \equiv$ The event that box I is selected and

let $A_2 \equiv$ The event that box II is selected

$$\therefore \quad P\left(\frac{A_2}{A}\right) = \frac{P(A_2) \cdot P\left(\frac{A}{A_2}\right)}{P(A_1)\, P\left(\frac{A}{A_1}\right) + P(A_2) \cdot P\left(\frac{A}{A_2}\right)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{3}{5}} = \frac{25}{32}$$

***Sample* Problem 10** *The chances of defective screws in three boxes A, B and C are $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$ respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, the probability that it came from box A is*

(a) $\frac{42}{107}$ (b) $\frac{4}{107}$ (c) $\frac{2}{107}$ (d) $\frac{1}{7}$

Interpret (a) Let E_1, E_2 and E_3 denote the events of selecting box A, B and C respectively and A be the event that a screw selected at random is defective. Then,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P\left(\frac{A}{E_1}\right) = \frac{1}{5}, \quad P\left(\frac{A}{E_2}\right) = \frac{1}{6}, \quad P\left(\frac{A}{E_3}\right) = \frac{1}{7}$$

By Baye's theorem

Required probability,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\, P\left(\frac{A}{E_1}\right)}{P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + P(E_3)\, P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{7}} = \frac{42}{107}$$

36.8 Random Variable and its Distribution

Random Variable

Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event $w \in S$ to a unique real number $X(w)$ is called a random variable.

A random variable is a function that associate a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated.

A random variable is usually denoted by the capital letters $X, Y, Z, \ldots$ etc.

e.g., A coin is tossed ten times. The random variable X is the number of tails that are noted. X can only take the values $0, 1, 2, \ldots, 10$. So, X is a discrete random variable.

There are two types of random variable

(i) **Discrete Random Variable** If the range of the real function $X : U \to R$ is a finite set or an infinite set of real numbers, it is called a discrete random variable.

(ii) **Continuous Random Variable** If the range of X is an interval (a, b) of R, then X is called a continuous random variable.

e.g., In tossing of two coins $S = \{HH, HT, TH, TT\}$, let X denotes number of heads in tossing of two coins, then

$$X(HH) = 2,\ X(TH) = 1,\ X(TT) = 0$$

Probability Distribution

If a random variable X takes values $X_1, X_2, \ldots, X_n$ with respective probabilities $P_1, P_2, \ldots, P_n$, then

X	X_1	X_2	X_3 …	X_n
$P(X)$	P_1	P_2	P_3 …	P_n

is known as the probability distribution of X.

OR

Probability distribution gives the values of the random variable along with the corresponding probabilities.

It satisfies the following conditions.

(i) $0 \le P(x_i) \le 1$

(ii) $\Sigma P(x_i) = 1$

e.g., Probability distribution when two coins are tossed.

Let X denote the number of heads occurred, then

$P(X = 0)$ = Probability of occurrence of zero head

$$= P(TT) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

$P(X - 1)$ = Probability of occurrence of one head

$$= HP(HT) + P(TH) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$P(X = 2)$ = Probability of occurrence of two heads

$$= P(HH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Thus, the probability distribution when two coins are tossed is as given below

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

36.9 Mean and Variance of a Random Variable

If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $P_1, P_2, P_3, \ldots, P_n$, then the mean $\overline{X}$ of X is defined as

$$\overline{X} = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

$$\Rightarrow \quad \overline{X} = \sum_{i=1}^{n} p_ix_i$$

and variance of X is defined as

$$\text{var}(X) = p_1(x_1 - \overline{X})^2 + p_2(x_2 - \overline{X})^2 + \ldots + p_n(x_n - \overline{X})^2$$

$$= \sum_{i=1}^{n} p_i\,(x_i - \overline{X})^2$$

where, $\overline{X} = \sum_{i=1}^{n} p_ix_i$ is the mean of X.

$$\Rightarrow \quad \text{var}(X) = \sum_{i=1}^{n} p_ix_i^2 - \left(\sum_{i=1}^{n} p_ix_i\right)^2$$

The square root of the variance gives the standard deviation *i.e.*,

$$\sqrt{\text{var}(X)} = \sqrt{\sigma^2} = \sigma$$

Note

- The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by $E(X)$.
- The variance and standard deviation of a random variable are always non-negative.

Important Results

(i) **Variance** $V(X) = \sigma_x^2 = E(X^2) - \{E(X)\}^2$

where, $E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$

(ii) **Standard Deviation**

$$\sqrt{V(X)} = \sigma_x = \sqrt{E(X^2) - \{E(X)\}^2}$$

(iii) If $Y = aX + b$, then

(a) $E(Y) = E(aX + b) = aE(X) + b$

(b) $\sigma_Y^2 = V(Y) = a^2 V(X) = a^2 \sigma_x^2$

(c) $\sigma_Y = \sqrt{V(Y)} = |a| \sigma_x$

(iv) If $Z = aX^2 + bX + c$, then

$E(Z) = E(aX^2 + bX + c)$

$= aE(X^2) + bE(X) + c$

36.10 Binomial Distribution

Bernoulli Trial

In a random experiment, if there are any two events, "Success and Failure" and the sum of the probabilities of these two events is one, then any outcome of such experiment is known as a Bernaulli Trial.

Let a binomial experiment has probability of success p and that of failure q (*i.e.*, $p + q = 1$). If E be an event and let X = number of success *i.e.*, number of times event E occurs in n trials.

Then, probability distribution of binomial distribution with parameters n and p is given by

$P(X = r) =$ Probability of r success in n trials

$= {}^nC_r p^r q^{n-r}$ $\quad (p + q = 1)$

$= (r + 1)$ th term in the expansion of $(q + p)^n$.

It is written as

$$X \sim B(n, p)$$

or

$$X \sim Bi(n, p).$$

Note The trials must meet the following requirements

- the total number of trials is fixed in advance.
- there are just two outcomes of each trials, success and failure.
- all the trials have the same probability of success.
- the outcomes of all the trials are statistically independent.

Mean and Variance of Binomial Distribution

Let $X \sim B(n, p)$, then

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where, $r = 0, 1, 2, \ldots, n$

and $p + q = 1$

$\therefore$ Mean, $\overline{X} = E(X) = np$

and variance, var $(X) = npq$

Standard deviation $= \sqrt{npq}$

Relation between Mean and Variance

Mean – Variance $= np - npq = np(1 - q) = np^2 > 0$

$\Rightarrow$ Mean > Variance

i.e., For binomial variable X, value of mean is always greater than its variance.

Mode of Binomial Distribution

In binomial distribution, the value of r for which $P(X = r)$ is maximum, is known as mode of binomial distribution.

$\therefore$ $(n + 1)\, p - 1 \le r \le (n + 1)\, P$

Hot Spot 2

Probability by using Binomial Distribution

Bernoulli's trials are given and find the probability by using the binomial distribution. These types of problem are directly asked in the Various JEE Main 2011, 2009, 2004.

To solve such type of problem following steps are given below.

Step I Firstly write the probability of success and failure. Also, write the number of trials in a event.

Step II Using the Binomial distribution formula, *i.e.*, $P(x = r) = {}^nC_r\, p^r q^{n-r}$

Step III Simplify it and get the result.

***Sample* Problem 11** *An ordinary die is rolled a certain number of times. The probability of getting an odd number 2 times is equal to the probability of getting an even number 3 times. Then, the probability of getting an odd number of times is*

(a) $\frac{1}{32}$ (b) $\frac{1}{2}$

(c) $\frac{1}{25}$ (d) 1

Interpret (b)

Step I Write the probability of sucess and failure.

The probability of getting an odd number in a throw, p

$$= \frac{3}{6} = \frac{1}{2}$$

and $$q = 1 - \frac{1}{2} = \frac{1}{2}$$

Step II Using the formula $P(X = r) = {}^nC_r\, p^r q^{n-r}$

The probability of getting odd number 2 times in n trials

$$= {}^nC_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{n-2}$$

Similarly, the probability of getting an even number 3 times in n trials

$$= {}^nC_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{n-3}$$

From the question, ${}^nC_2 \left(\frac{1}{2}\right)^n = {}^nC_3 \left(\frac{1}{2}\right)^n \Rightarrow n = 5$

Step III Simplify it and get the result.

$\therefore$ Required probability

$$= {}^5C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + {}^5C_5 \left(\frac{1}{2}\right)^5$$

$$= (5 + 10 + 1)\frac{1}{2^5} = \frac{16}{32} = \frac{1}{2}$$

36.11 Poisson Distribution

It is limiting case of binomial distribution. Let the number of events n is large ($n \to \infty$) and probability of success in each experiment is 0 and $np = \lambda$ (say) is finite, then

$$P(X = r) \text{ or } P(r) = \frac{e^{-\lambda}\lambda^r}{r!}, \text{ where } r = 0, 1, 2, \ldots$$

and $\lambda = np$. Here, λ is known as parameter of poisson distribution.

$$P(r + 1) = \frac{\lambda}{r + 1} P(r) \text{ is known as recurrence formula.}$$

Note

- $\sum_{r=0}^{\infty} P(r) = 1$
- If λ_1 and λ_2 are parameter of variables X and Y, then parameter of $(x + y)$ will be $(\lambda_1 + \lambda_2)$.
- In poisson distribution, mean = variance = $\lambda = np$

***Sample* Problem 12** *If mean of a poisson distribution of a random variable X is 2, then the value of $P(X > 1.5)$ is*

(a) $\frac{3}{e^2}$ (b) $\frac{3}{e}$

(c) $1 - \frac{3}{e}$ (d) $1 - \frac{3}{e^2}$

Interpret (d) Since, $P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$ (where, λ = mean)

$$\therefore \quad P(X = r > 1.5) = P(2) + P(3) + \ldots \infty$$

$$= 1 - P(X = r \le 1)$$

$$= 1 - P(0) - P(1)$$

$$= 1 - \left(e^{-2} + \frac{e^{-2} \times 2}{1!}\right)$$

$$= 1 - \frac{3}{e^2}$$

36.12 Geometrical Probability

If the total number of outcomes of a trial in a random experiment is infinite, in such cases, the definition of probability is modified and the general expression for the probability P of occurrence of an event is given by

$$P = \frac{\text{Measure of the specified part of the region}}{\text{Measure of the whole region}}$$

where, measure means length or area or volume of the region, if we are dealing with one, two or three dimensional space respectively.

Application Based Results

(i) When two dice are thrown, the number of ways of getting a total r is

(a) $(r-1)$, if $2 \le r \le 7$ and (b) $(13-r)$, if $8 \le r \le 12$

(ii) Experiment with insertion of n letters in n addressed envelopes.

(a) Probability of inserting all the n letters in right envelopes $= \frac{1}{n!}$

(b) Probability that all letters does not in right envelopes $= 1 - \frac{1}{n!}$

(c) Probability of keeping all the letters in wrong envelope $= \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$

(d) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

(iii) (a) **Selection of shoes from a cupboard** Out of n pair of shoes, if k shoes are selected at random, the probability that there is no pair is

$$p = \frac{{}^nC_k 2^k}{{}^{2n}C_k}$$

(b) The probability that there is atleast one pair is $(1-p)$.

(iv) **Selection of squares from the chessboard** If r squares are selected from a chessboard, then probability that they lie on a diagonal is

$$\frac{4[{}^7C_r + {}^6C_r + \ldots + {}^1C_r] + 2({}^8C_r)}{{}^{64}C_r}$$

(v) If A and B are two finite sets and if a mapping is selected at random from the set of all mapping from A into B, then the probability that the mapping is

(a) a one one function $= \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

(b) a many one function $= 1 - \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

(c) a constant function $= \frac{n(B)}{n(B)^{n(A)}}$

(d) a one one onto function $= \frac{n(A)!}{n(B)^{n(A)}}$

***Sample* Problem 13** *A point is selected at random from the interior of a circle. The probability that the point in closer to the centre than boundary of the circle is*

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) *None of these*

Interpret (c) In a circle, all the points which lie in the area of half the radius will be nearer to origin that to the boundary.

$n(S)$ = The area of circle of radius r.

$n(E)$ = The area of the circle of radius $\frac{r}{2}$

$\therefore$ Required probability $= \frac{n(E)}{n(S)}$

$\therefore$ Required probability $= \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$

WORKED OUT

Examples

Example 1 *From a city population, the probability of selecting* (i) *a male or a smoker is* $\frac{7}{10}$ (ii) *a male smoker is* $\frac{2}{5}$ *and* (iii) *a male, if a smoker is already selected is* $\frac{2}{3}$. *Then, probability of selecting a smoker, if a male is first selected, is*

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

Solution (b) Let A = Event that a male is selected

and B = a smoker is selected

Given, $P(A \cup B) = \frac{7}{10}$, $P(A \cap B) = \frac{2}{5}$, $P\left(\frac{A}{B}\right) = \frac{2}{3}$

Now, $P(A) = P(A \cup B) + P(A \cap B) - P(B)$

$$= \frac{7}{10} + \frac{2}{5} - \frac{3}{5} = \frac{1}{2}$$

Now, probability of selecting a smoker if a male is first selected

$$= P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{215}{112} = \frac{4}{5}$$

Example 2 *Ten coins are thrown simultaneously. Then, the probability of getting atleast seven heads, is*

(a) $\frac{150}{523}$ (b) $\frac{120}{504}$ (c) $\frac{176}{1024}$ (d) $\frac{167}{1024}$

Solution (c) p = Probability of getting a head $= \frac{1}{2}$

q = Probability of getting a tail $= \frac{1}{2}$

$\therefore$ Probability of getting atleast seven heads is $P(X \geq 7)$

$$= P(7) + P(8) + P(9) + P(10)$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= \left(\frac{1}{2}\right)^{10} \{{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\}$$

$$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024}$$

Example 3 *If two events A and B. If odds against A are as* 2 : 1 *and those in favour of* $A \cup B$ *are as* 3 : 1, *then*

(a) $\frac{1}{2} \leq P(B) \leq \frac{3}{4}$ (b) $\frac{5}{12} \leq P(B) \leq \frac{3}{4}$

(c) $\frac{1}{4} \leq P(B) \leq \frac{3}{5}$ (d) *None of these*

Solution (b) $\because P(A) = \frac{1}{3}, P(A \cup B) = \frac{3}{4}$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

$\Rightarrow$ $\frac{3}{4} \leq \frac{1}{3} + P(B)$

$\Rightarrow$ $P(B) \geq \frac{5}{12}$

Also, $B \subseteq A \cup B \Rightarrow P(B) \leq P(A \cup B) = \frac{3}{4}$

$\therefore$ $\frac{5}{12} \leq P(B) \leq \frac{3}{4}$

Example 4 *A lot contains* 50 *defective and* 50 *non-defective bulbs. Two bulbs are drawn at random one at a time with replacement. The events A, B and C are defined as the first bulb is defective, the second bulb is non-defective, the two bulbs are both defective or non-defective, respectively. Then,*

(a) *A, B and C are pairwise independent*

(b) *A, B and C are pairwise not independent*

(c) *A, B and C are independent*

(d) *None of the above*

Solution (a) $P(A) = \frac{50}{100} = \frac{1}{2}, P(B) = \frac{50}{100} = \frac{1}{2}$

$$P(C) = P(DD \text{ or } NN) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(A \cap B) = P(DN) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A) \cdot P(B)$$

$$P(A \cap C) = P(DD) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A) \cdot P(C)$$

$$P(B \cap C) = P(NN) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(B) \cdot P(C)$$

Since, A, B and C are pairwise independent.

$A \cap B \cap C$ is a null set since, if the first one defective, then both cannot be defective or both non-defective.

$$P(A \cap B \cap C) = P(\phi) = 0 \neq P(A) \cdot P(B) \cdot P(C)$$

Hence, A, B and C are not independent.

Example 5 *An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the sum of the numbers obtained is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4, …,12 is picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is*

(a) $\frac{192}{401}$ (b) $\frac{193}{401}$ (c) $\frac{193}{792}$ (d) $\frac{17}{75}$

Solution (c) The probability of getting the sum 7 or 8 from two dice is $\frac{6}{36}+\frac{5}{36}=\frac{11}{36}$.

The probability of getting the card with number 7 or 8 is $\frac{2}{11}$.

$\therefore$ Required probability $=\frac{1}{2}\cdot\frac{11}{36}+\frac{1}{2}\cdot\frac{2}{11}=\frac{11}{72}+\frac{2}{22}=\frac{193}{792}$

Example 6 *A person goes to office either by car, scooter, bus or train the probabilities of which being* $\frac{1}{7},\frac{3}{7},\frac{2}{7}$ and $\frac{1}{7}$, *respectively. The probability that he reaches office later if he takes car, scooter, bus or train is* $\frac{2}{9},\frac{1}{9},\frac{4}{9}$ and $\frac{1}{9}$, *respectively. If he reaches office in time, the probability that he travelled by car is*

(a) $\frac{1}{5}$ (b) $\frac{1}{9}$ (c) $\frac{2}{11}$ (d) $\frac{1}{7}$

Solution (d) Let C, S, B and T be the events of the person going by car, scooter, bus and train, respectively.

$$P(C)=\frac{1}{7},P(S)=\frac{3}{7},P(B)=\frac{2}{7},P(T)=\frac{1}{7}$$

Let L be the event that the person reaching the office in time.
Then, $\bar{L}$ be the event that the person reaching the office in late.

$$P(L/C)=\frac{7}{9},P(L/S)=\frac{8}{9},P(L/B)=\frac{5}{9},P(L/T)=\frac{8}{9}$$

$$\therefore \quad P(C/L)=\frac{P(C)\cdot P(L/C)}{\left[\begin{array}{l}P(C)\cdot P(L/C)+P(S)\cdot P(L/S)\\ \qquad +P(B)\cdot P(L/B)+P(T)\cdot P(\bar{L}/T)\end{array}\right]}$$

$$=\frac{\frac{1}{7}\cdot\frac{7}{9}}{\frac{1}{7}\cdot\frac{7}{9}+\frac{3}{7}\cdot\frac{8}{9}+\frac{2}{7}\cdot\frac{5}{9}+\frac{1}{7}\cdot\frac{8}{9}}$$

$$=\frac{7}{49}=\frac{1}{7}$$

Example 7 *If the letters of the word 'MATHEMATICS' are arranged arbitrarily, the probability that C comes before E, E before H, H before I and I before S, is*

(a) $\frac{1}{75}$ (b) $\frac{1}{24}$

(c) $\frac{1}{120}$ (d) $\frac{1}{720}$

Solution (c) Total number of arrangements is $\frac{11!}{2!2!2!}=\frac{11!}{8}$

Number of arrangement in which C, E, H, I and S appear in that order $=({}^{11}C_5)\frac{6!}{2!2!2!}=\frac{11!}{8\cdot 5!}$

$\therefore$ Required probability $=\frac{11!}{8\cdot 5!}\div\frac{11!}{8}=\frac{1}{5!}=\frac{1}{120}$

Example 8 *In a hurdle race, a runner has probability p of jumping over a specific hurdle. Given that in 5 trials, the runner succeeded 3 times, the conditionals probability that the runner had succeeded in the first trial, is*

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$

(c) $\frac{1}{5}$ (d) *None of these*

Solution (a) Let A and B denote the event that the runner succeeds exactly 3 times out of 5 times and event that the runner succeeds on the first trial.

$$\therefore \quad P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}$$

$$\therefore \quad P(B\cap A)={}^4C_2p^2(1-p)^2=6p^3(1-p)^2$$

$$P(A)={}^5C_3\,p^3(1-p)^2=10p^3\,(1-p)^2$$

$$\therefore \quad P\left(\frac{B}{A}\right)=\frac{6p^3(1-p)^2}{10p^3(1-p)^2}=\frac{3}{5}$$

Example 9 *A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing card is black, is*

(a) $\frac{2}{3}$ (b) $\frac{15}{26}$

(c) $\frac{16}{39}$ (d) $\frac{37}{52}$

Solution (a) Let B stand for the event that black card is missing, then $P(B)=P(\bar{B})=\frac{1}{2}$.

Let E be the event that all the first 13 cards are red.

$$\therefore \quad P(E/B)=\frac{26}{51}\cdot\frac{25}{50}\cdots\frac{14}{39},$$

$$P(E/\bar{B})=\frac{25}{51}\cdot\frac{24}{50}\cdots\frac{13}{39}$$

$$P(B/E)=\frac{P(B)\cdot P(E/B)}{P(B)\cdot P(E/B)+P(\bar{B})\cdot P(E/\bar{B})}$$

$$=\frac{26\cdot 25\ldots 14}{26\cdot 25\ldots 14+25\cdot 24\ldots 13}$$

$$=\frac{26}{26+13}=\frac{26}{39}=\frac{2}{3}$$

Start Practice for JEE Main

Round I (Topically Divided Problems)

Advance Probability

1. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale otherwise it is rejected. The probability that a box containing 15 oranges out of which 12 are good and 3 are bad one will be approved for sale, is [NCERT]

(a) $\frac{12}{15}$ (b) $\frac{11}{14}$

(c) $\frac{10}{13}$ (d) $\frac{44}{91}$

2. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'number is even' and B be the event, 'number is red', then A and B are [NCERT]

(a) mutually exclusive (b) dependent

(c) independent (d) None of these

3. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$, then A and B are [NCERT]

(a) independent (b) not independent

(c) mutually exclusive (d) None of these

4. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. The probability that [NCERT]

(i) both balls are red

(ii) first ball is black and second is red

(iii) one of them is black and other is red

are respectively

(a) $\frac{16}{81}, \frac{20}{81}$ and $\frac{40}{81}$ (b) $\frac{40}{81}, \frac{20}{81}$ and $\frac{16}{81}$

(c) $\frac{20}{81}, \frac{16}{81}$ and $\frac{40}{81}$ (d) None of these

5. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$, respectively. If both try to solve the problem independently, the probability that [NCERT]

(i) the problem is solved

(ii) exactly one of them solves the problem

are respectively

(a) $\frac{1}{2}$ and $\frac{2}{3}$ (b) $\frac{2}{3}$ and $\frac{1}{2}$

(b) $\frac{1}{4}$ and $\frac{3}{4}$ (d) None of these

6. Two events A and B are said to be independent, if [NCERT]

(a) A and B are mutually exclusive

(b) $P(A' \cap B') = [1 - P(A)][1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

7. A and B are two candidates seeking admission in a college. The probability that A is seleted is 0.7 and the probability that exactly one of them is selected is 0.6. The probability that B is selected, is [NCERT Exemplar]

(a) 0.1 (b) 0.3

(c) 0.5 (d) 0.25

8. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. The probability that the ball is drawn from the first bag. [NCERT]

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{3}{4}$ (d) $\frac{2}{3}$

9. If each element of a second order determinant is zero or one, what is the probability that the values of the determinant is positive? [NCERT]

(a) $\frac{3}{16}$ (b) $\frac{5}{16}$

(c) $\frac{15}{16}$ (d) None of these

10. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, then the following probabilities are assumed to be known

P (A fails) = 0.2, P (B fails alone) = 0.15, P (A and B fail) = 0.15

The probabilities of

(i) P (A fails/B has failed)

(ii) P (A fails alone)

are respectively [NCERT]

(a) 0.2 and 0.02 (b) 0.2 and 0.03

(c) 0.5 and 0.03 (d) 0.5 and 0.05

11. If $P\left(\frac{A}{B}\right) > P(A)$, then which of the following is correct? [NCERT]

(a) $P\left(\frac{B}{A}\right) < P(B)$ (b) $P(A \cap B) < P(A)\ P(B)$

(c) $P\left(\frac{B}{A}\right) > P(B)$ (d) $P\left(\frac{B}{A}\right) = P(B)$

12. Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are kings? [NCERT Exemplar]

(a) $\frac{1}{725}$ (b) $\frac{1}{125}$

(c) $\frac{1}{2025}$ (d) None of these

13. Ten coins are tossed. Then, the probability of getting atleast 8 heads is [NCERT Exemplar]

(a) $\frac{1}{128}$ (b) $\frac{1}{256}$

(c) $\frac{7}{128}$ (d) $\frac{3}{256}$

14. Two dice are tossed. The following two events A and B are

$$A = \{(x, y) : x + y = 11\}, B = \{(x, y) : x \neq 5\}$$

where (x, y) denotes a typical sample point. [NCERT Exemplar]

(a) Not independent

(b) independent

(c) Mutually exclusive

(d) None of the above

15. In a series of three trials the probability of exactly two successes in nine times as large as the probability of three successes. Then, the probability of success in each trial is

(a) 1/2 (b) 1/3

(c) 1/4 (d) 3/4

16. A natural number is chosen at random from the first one hundred natural numbers. The probability that $\frac{(x-20)(x-40)}{x-30} < 0$ is

(a) $\frac{1}{50}$ (b) $\frac{3}{50}$ (c) $\frac{3}{25}$ (d) $\frac{7}{25}$

17. For independent events $A_1, \ldots, A_n$. $P(A_i) = \frac{1}{i+1}$, $i = 1, 2, \ldots, n$. Then, the probability that none of the events will occur is

(a) $\frac{n}{(n+1)}$ (b) $\frac{(n-1)}{(n+1)}$

(c) $\frac{1}{(n+1)}$ (d) $n + \left(\frac{1}{(n+1)}\right)$

18. If two events A and B are such that $P(A') = 0.3$, $P(B) = 0.4$ and $(A \cap B') = 0.5$, then $P\left(\frac{B}{A \cup B'}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$

(c) $\frac{3}{5}$ (d) $\frac{2}{5}$

19. The probability that in a group of N (< 365) people, atleast two will have the same birthday is

(a) $1 - \frac{(365)!}{(365-N)!\,(365)!}$ (b) $\frac{(365)^N\,(365)!}{(365-N)!} - 1$

(c) $1 - \frac{(365)^N\,(365)!}{(365!\,N)!}$ (d) $1 - \frac{(365)!}{(365-N)!\,(365)^N}$

20. Let E and F be two independent events such that $P(E) > P(F)$. The probability that both E and F happen is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$, then

(a) $P(E) = \frac{1}{3}$, $P(F) = \frac{1}{4}$ (b) $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{6}$

(c) $P(E) = 1$, $P(F) = \frac{1}{12}$ (d) $P(E) = \frac{1}{3}$, $P(F) = \frac{1}{2}$

21. If the integers m and n are chosen at random from 1 to 100, then the probability that a number of the form $7^n + 7^m$ is divisible by 5 equals

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) $\frac{1}{3}$

22. Three numbers are chosen at random without replacement from the set $A = \{x \mid 1 \le x \le 10, x \in N\}$. The probability that the minimum of the chosen numbers is 3 and maximum is 7, is

(a) $\frac{1}{12}$ (b) $\frac{1}{15}$ (c) $\frac{1}{40}$ (d) $\frac{39}{40}$

Conditional Probability

23. If $2P(A) = P(B) = \frac{5}{13}$ and $P\left(\frac{A}{B}\right) = \frac{2}{5}$, then $P(A \cup B)$ is [NCERT]

(a) $\frac{11}{26}$ (b) $\frac{3}{26}$

(c) $\frac{5}{26}$ (d) None of these

24. A coin is tossed three times

E : atmost two tails

F : atleast one tail,

then $P\left(\frac{E}{F}\right)$ is [NCERT]

(a) $\frac{2}{7}$ (b) $\frac{6}{7}$ (c) $\frac{1}{7}$ (d) $\frac{3}{7}$

25. A die is thrown three times,

E : 4 appears on the third toss

F : 6 and 5 appears, respectively on first two tosses

then $P\left(\frac{E}{F}\right)$ is [NCERT]

(a) $\frac{1}{36}$ (b) $\frac{5}{6}$ (c) $\frac{5}{36}$ (d) $\frac{1}{6}$

26. A black and a red die are rolled, then the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4, is [NCERT]

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{2}{3}$ (d) 0

27. A fair die is rolled. Consider events $E = \{1, 3, 5\}$, $F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$, then $P\left(\frac{E \cup F}{G}\right)$ and $P\left(\frac{E \cap F}{G}\right)$ are [NCERT]

(a) $\frac{3}{4}$ and $\frac{1}{4}$ (b) $\frac{1}{2}$ and $\frac{1}{3}$

(c) 0 and 1 (d) None of these

28. Assume that each child born is equally likely to be boy or a girl. If a family has two children, what is the conditional probability that both are girls given that

(i) the youngest is a girl?

(ii) atleast one is a girl? [NCERT]

(a) $\frac{2}{3}$ and $\frac{1}{4}$ (b) $\frac{1}{2}$ and $\frac{1}{3}$

(c) $\frac{1}{3}$ and $\frac{1}{4}$ (d) $\frac{1}{2}$ and $\frac{1}{4}$

29. An instructor has a question bank consisting of 300 easy true/false questions, 200 difficult true/false question, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the test bank, what is the probability that it will be an easy question given that it is a multiple choice question? [NCERT]

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

30. Consider the experiment of throwing a die, if a multiple of 3 comes up throw the die again and if any other number comes, toss a coin. The conditional probability of the event the coin shows a tail, given that atleast one die shows a 3 is [NCERT]

(a) 0 (b) $\frac{7}{20}$ (c) $\frac{3}{20}$ (d) $\frac{4}{5}$

31. 10% of the bulbs produced in a factory are of red colour and 2% are red and defective. If one bulb is picked up at random, then the probability of its being defective if it is red, is [NCERT Exemplar]

(a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

32. A committee of 4 students is selected at random from a group consisting 8 boys and 4 girls. Given that there is atleast one girl on the committee, the probability that there are exactly 2 girls on the committee, is [NCERT Exemplar]

(a) $\frac{7}{99}$ (b) $\frac{13}{99}$

(c) $\frac{14}{99}$ (d) None of these

33. If A and B are two events such that $A \subset B$ and $P(B) \ne 0$, then which of the following is correct? [NCERT]

(a) $P\left(\frac{A}{B}\right) = \frac{P(A)}{P(B)}$ (b) $P\left(\frac{A}{B}\right) < P(A)$

(c) $P\left(\frac{A}{B}\right) \ge P(A)$ (d) None of these

34. A and B are two events such that $P(A) \ne 0$. $P\left(\frac{B}{A}\right)$ if

(i) A is a subset of B (ii) $A \cap B = \phi$

are respectively [NCERT]

(a) 0 and 1 (b) 1 and 0

(c) 1 and 1 (d) 0 and 0

35. A and B are two events such that $P(A) > 0$, $P(B) \neq 1$, then $P\left(\frac{\overline{A}}{\overline{B}}\right)$ is equal to [NCERT Exemplar]

(a) $1 - P\left(\frac{A}{B}\right)$ (b) $1 - P\left(\frac{\overline{A}}{B}\right)$

(c) $\frac{1 - P(A \cup B)}{P(\overline{B})}$ (d) $\frac{P(\overline{A})}{P(\overline{B})}$

36. Two cards are drawn one by one from a pack of cards. The probability of getting first card an ace and second a coloured one is (before drawing second card first card is not placed again in the pack)

(a) 1/26 (b) 5/52

(c) 5/221 (d) 4/13

37. For two events A and B, if $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$, then

(a) A and B are independent (b) $P\left(\frac{A'}{B}\right) = \frac{3}{4}$

(c) $P\left(\frac{B'}{A'}\right) = \frac{1}{2}$ (d) All of the above

38. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact present. However, the test also yields a false positive result for 0.5% of the healthy person tested (*i.e.,* if a healthy person is tested, then with probability 0.005, the test will imply he has the disease). If 0.1% of the population actually has the disease, what is the probability that a person has disease given that his test result is positive? [NCERT]

(a) $\frac{11}{133}$ (b) $\frac{22}{133}$

(c) $\frac{33}{133}$ (d) None of these

Law of Total Probability

39. Three machines E_1, E_2 and E_3 in a certain factory produce 50%, 25% and 25%, respectively, of the total daily output of electric tubes. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective and that 5% of those produced on E_3 are defective. If one tube is picked up at random from a day's production, the probability that it is defective, is [NCERT Exemplar]

(a) 0.025

(b) 0.125

(c) 0.325

(d) 0.0425

Baye's Theorem

40. In answering a question on a multiple choice test a student either knows the answer or guesses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability 1/4. What is the probability that a student knows the answer given that he answered it correctly? [NCERT]

(a) $\frac{4}{13}$ (b) $\frac{5}{13}$ (c) $\frac{9}{13}$ (d) $\frac{12}{13}$

41. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows head, what is the probability that it is was the two headed coin? [NCERT]

(a) $\frac{2}{9}$ (b) $\frac{4}{9}$ (c) $\frac{2}{3}$ (d) $\frac{2}{9}$

42. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accidents are 0.01, 0.03 and 0.15, respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver? [NCERT]

(a) $\frac{1}{52}$ (b) $\frac{1}{26}$ (c) $\frac{7}{52}$ (d) $\frac{3}{26}$

43. A factory has two machines A and B. Past record shows that machine A produced 60% of the item of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B? [NCERT]

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

44. Two groups are competing for the position on the board of directors of a corporation. The probability that the first and the second groups will win are 0.6 and 0.4, respectively. Further, if the first group wins the probability of introducing a new product is 0.7 and the corresponding probability is 0.3, if the second group wins. The probability that the new product introduce was by the second group, is [NCERT]

(a) $\frac{2}{9}$ (b) $\frac{1}{9}$

(c) $\frac{4}{9}$ (d) $\frac{1}{3}$

45. Suppose, a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die? [NCERT]

(a) $\frac{4}{11}$ (b) $\frac{2}{11}$ (c) $\frac{6}{11}$ (d) $\frac{8}{11}$

46. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be diamonds. The probability of the lost card being a diamond. [NCERT]

(a) $\frac{11}{50}$ (b) $\frac{2}{25}$ (c) $\frac{9}{50}$ (d) $\frac{43}{50}$

47. Probability that A speaks truth is 4/5. A coin is tossed, A reports that a head appears. The probability that actually there was head is [NCERT]

(a) $\frac{4}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$

48. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. The probability that the patient followed a course of meditation and yoga, is [NCERT]

(a) $\frac{3}{29}$ (b) $\frac{7}{29}$

(c) $\frac{14}{29}$ (d) None of these

49. A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATANAGAR? [NCERT Exemplar]

(a) $\frac{2}{11}$ (b) $\frac{5}{11}$

(c) $\frac{7}{11}$ (d) $\frac{10}{11}$

50. Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2 headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin? [NCERT Exemplar]

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) None of these

51. A company has two plants to manufacture televisions. Plant I manufacture 70% of televisions and plant II manufacture 30%. At plant I, 80% of the televisions are rated as of standard quality and at plant II, 90% of the televisions are rated as of standard quality. A television is chosen at random and is found to be of standard quality. The probability that it has come from plant II is

(a) $\frac{17}{50}$ (b) $\frac{27}{83}$ (c) $\frac{3}{5}$ (d) $\frac{9}{83}$

Random Variable and its Distribution

52. A random variable X has the following probability distribution. Then, the value of

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) k (ii) $P(X<3)$

(iii) $P(X>6)$ (iv) $P(0<X<3)$

are respectively [NCERT]

(a) $\frac{1}{10}, \frac{3}{10}, \frac{17}{100}$ and $\frac{3}{10}$ (b) $\frac{1}{10}, \frac{3}{10}, \frac{3}{10}$ and $\frac{17}{100}$

(c) $\frac{17}{100}, \frac{1}{10}, \frac{3}{10}$ and $\frac{3}{10}$ (d) None of these

53. Let X denote the sum of the numbers obtained when two fair dice are rolled. The variance and standard deviation of X are [NCERT]

(a) $\frac{31}{6}$ and $\sqrt{\frac{31}{6}}$ (b) $\frac{35}{6}$ and $\sqrt{\frac{35}{6}}$

(c) $\frac{17}{6}$ and $\sqrt{\frac{17}{6}}$ (d) None of these

54. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 yr. One student is selected in such a manner that each has the same chance of being of chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Mean, variance and Standard Deviation (SD) of X, are respectively [NCERT]

(a) 17.53, 4.8 and 2.19 (b) 2.19, 4.8 and 17.53

(c) 17.53, 2.19 and 4.8 (d) None of these

55. A discrete random variable X has the following probability distribution

X	1	2	3	4	5	6	7
$P(X)$	C	$2C$	$2C$	$3C$	C^2	$2C^2$	$7C^2+C$

The value of C the mean of the distribution are [NCERT Exemplar]

(a) $\frac{1}{10}$ and 3.66 (b) $\frac{1}{20}$ and 2.66

(c) $\frac{1}{15}$ and 1.33 (d) None of these

56. The variance and standard deviation of the number of heads in three tosses of a coin, are [NCERT Exemplar]

(a) $\frac{\sqrt{3}}{2}$ and $\frac{3}{4}$ (b) $\frac{3}{4}$ and $\frac{\sqrt{3}}{2}$

(c) $\frac{4}{9}$ and $\frac{3}{2}$ (d) $\frac{1}{2}$ and $\frac{1}{\sqrt{2}}$

57. In a dice game, a player pays a stake of ₹ 1 for each throw of a die. She receives ₹ 5, if the die shows a 3, ₹ 2, if the die shows a 1 or 6 and nothing otherwise. What is the player's expected profit per throw over a long series of throws? [NCERT Exemplar]

(a) 0.50 (b) 0.20 (c) 0.70 (d) 0.90

58. For a random variable X, $E(X) = 3$ and $E(X^2) = 11$. Then, variable of X is

(a) 8 (b) 5 (c) 2 (d) 1

Binomial Distribution

59. A pair of dice is thrown 4 times. If getting a doublet is considered a success, the probability of two successes is [NCERT]

(a) $\frac{25}{128}$ (b) $\frac{13}{216}$ (c) $\frac{25}{216}$ (d) $\frac{11}{128}$

60. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag. What is the probability that none is marked with the digit 0? [NCERT]

(a) $\left(\frac{9}{10}\right)^2$ (b) $\left(\frac{9}{10}\right)^3$ (c) $\left(\frac{9}{10}\right)^4$ (d) $\left(\frac{9}{10}\right)^5$

61. In an examination, 20 questions of true-false type are asked. Suppose a students tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers true, if it falls tails, he answers false. The probability that he answers atleast 12 questions correctly is [NCERT]

(a) $\left(\frac{1}{2}\right)^{20} ({}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20})$

(b) $\left(\frac{1}{2}\right)^{10} ({}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20})$

(c) $\left(\frac{1}{2}\right)^{20} ({}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20})$

(d) None of the above

62. Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$, then which of the outcome is most likely? [NCERT]

(a) $X = 0$ and $X = 6$ (b) $X = 3$

(c) $X = 0$ (d) $X = 6$

63. On a multiple choice examination with three possible answer for each of the five questions, what is the probability that a candidate would get four or more correct answer just by guessing? [NCERT]

(a) $\frac{3}{243}$ (b) $\frac{1}{243}$ (c) $\frac{25}{243}$ (d) $\frac{11}{243}$

64. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is [NCERT]

(a) 10^{-1} (b) $\left(\frac{1}{2}\right)^5$ (c) $\left(\frac{9}{10}\right)^5$ (d) $\frac{9}{10}$

Round II (Mixed Bag)

Only One Correct Option

1. A man throws a fair coin a number of times and gets 2 points for each head he throws and 1 point for each tail he throws. The probability that he gets exactly 6 points is

(a) $\frac{21}{32}$ (b) $\frac{23}{32}$

(c) $\frac{41}{64}$ (d) $\frac{43}{64}$

2. One function is selected from all the functions $F : S \to S$, where $S = \{1, 2, 3, 4, 5, 6\}$. The probability that it is onto function, is

(a) $\frac{5}{324}$ (b) $\frac{7}{324}$

(c) $\frac{5}{162}$ (d) $\frac{5}{81}$

3. A and B stand in ring along with 10 other persons. If the arrangement is at random, the probability that there are exactly 3 person between A and B, is

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$

(c) $\frac{3}{11}$ (d) $\frac{1}{12}$

4. If $x_1, x_2, \ldots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \ldots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle numbers, is

(a) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (b) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

(c) $\frac{{}^{19}C_2 \times {}^{31}C_2}{{}^{50}C_5}$ (d) None of these

5. A draws two cards at random from a pack of 52 cards. After returning them to the pack and shuffling it, B draws two cards at random. The probability that thier draws contain exactly one comman card is

(a) $\frac{25}{546}$ (b) $\frac{50}{663}$

(c) $\frac{25}{663}$ (d) None of these

6. In a random experiment, if the success is thrice that of failure. If the experiment is repeated 5 times, the probability that atleast 4 times favourable is

(a) $\frac{1053}{2048}$ (b) $\frac{1003}{2048}$

(c) $\frac{1203}{2048}$ (d) None of these

7. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

8. Three natural numbers are taken at random from the set $A = \{x \mid 1 \le x \le 100, x \in N\}$. The probability that the AM of the numbers taken is 75, is

(a) $\frac{{}^{77}C_2}{{}^{100}C_3}$ (b) $\frac{{}^{25}C_2}{{}^{100}C_3}$

(c) $\frac{{}^{74}C_{72}}{{}^{100}C_{97}}$ (d) $\frac{{}^{75}C_2}{{}^{100}C_3}$

9. In a multiple choice question there are four alternative answers of which one or more than one is correct. A candidate will get marks on the question only if he ticks the correct answer. The candidate decides to tick answers at random. If he is allowed up to three chances of answer the answer the question, then the probability that he will get marks on it is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{1}{5}$ (d) $\frac{2}{15}$

10. Three letters are written to there different persons and addresses on the three envelopes are also written. Without looking at the addresses, the letters are kept in these envelopes. The probability that all the letters are not placed into their right envelopes is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{6}$ (d) $\frac{5}{6}$

11. 7 white balls and 3 black balls are placed in a row at random. The probability that no two black balls are adjacent is

(a) $\frac{1}{2}$ (b) $\frac{7}{15}$ (c) $\frac{2}{15}$ (d) $\frac{1}{3}$

12. Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let A be the event of (x, y) satisfying $y^2 \le x$ and B be the event of (x, y) satisfying $x^2 \le y$. Then,

(a) $P(A \cap B) = \frac{1}{3}$

(b) A, B are exhaustive

(c) A, B are mutually exclusive

(d) A, B are independent

13. 10 apples are distributed at random among 6 persons. The probability that atleast one of them will receive none is

(a) $\frac{6}{143}$ (b) $\frac{{}^{14}C_4}{{}^{15}C_5}$

(c) $\frac{137}{143}$ (d) $\frac{135}{143}$

14. A second-order determinant is written down at random using the numbers 1, –1 as elements. The probability that the value of the determinant is non-zero is

(a) $\frac{1}{2}$ (b) $\frac{3}{8}$

(c) $\frac{5}{8}$ (d) $\frac{1}{3}$

15. If E and F are two events with $P(E) \le P(F) > 0$, then

(a) occurrence of $E \Rightarrow$ occurrence of F

(b) occurrence of $F \Rightarrow$ occurrence of E

(c) non-occurrence of $E \Rightarrow$ non-occurrence of F

(d) None of the above implications hold

More Than One Correct Option

16. A card is selected at random from cards numbered as 00, 01, 02,...,99. An event is said to have occured. If product of digits of the card number is 16. If card is selected 5 times with replacement each time, then the probability that the event occurs exactly three times is

(a) ${}^5C_3\left(\frac{3}{100}\right)^2\left(\frac{97}{100}\right)^3$ (b) ${}^5C_3\left(\frac{3}{100}\right)^3\left(\frac{97}{100}\right)^2$

(c) ${}^5C_3\left(\frac{0.3}{100}\right)^3\left(\frac{97}{100}\right)^3$ (d) $10(0.03)^3(0.97)^2$

17. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, a student has a 75% chance of passing in atleast one, a 50% chance of

passing in atleast two and a 40% chance of passing in exactly two subjects. Which of the following relations are true?

(a) $p + m + c = \frac{19}{20}$

(b) $p + m + c = \frac{27}{20}$

(c) $pmc = \frac{1}{10}$

(d) $pmc = \frac{1}{4}$

18. A coin is tossed repeatedly. A and B call alternately for winning a prize of ₹ 30. One who calls correctly first wins the prize. A starts the cell. Then, the expectation of

(a) A is ₹ 10 (b) B is ₹ 10

(c) A is ₹ 20 (d) B is ₹ 20

Comprehension Based Questions

Passage

Directions (Q. Nos. 19-21) *The probability of happening of an event in one trial being known, then the probability of its happening exactly x times in n trials is given by* ${}^nC_x\, q^{n-x} \cdot p^x$ *where*

p = probability of happening the event

q = probability of not happening the event $= 1 - p$

Now, ${}^nC_x\, q^{n-x} p^x$ *is* $(x + 1)$*th term in the expansion of* $(q + p)^n$ *whose expansion gives the happening of the event* 0, 1, 2, …, *n times, respectively.*

19. In four throws with a pair of dice, the chance of throwing doublets atleast twice is

(a) $\frac{19}{144}$ (b) $\frac{125}{144}$

(c) $\frac{17}{144}$ (d) $\frac{18}{144}$

20. A man takes a forward step with probability (.8) and backward step with probability (.2). What is the probability that at the end of 9 steps he is exactly three steps away from starting point.

(a) $\frac{69888}{5^8}$ (b) $\frac{5377}{5^8}$

(c) $\frac{5378}{5^8}$ (d) $\frac{5376}{5^8}$

21. Unbiased coin is tossed 6 times. The probability of getting atmost 4 heads is

(a) $\frac{7}{64}$ (b) $\frac{57}{64}$

(c) $\frac{21}{32}$ (d) $\frac{11}{32}$

Assertion and Reason

Directions (Q. Nos. 22-26) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

22. Let A and B are two events such that $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$, then

Statement I $\frac{4}{15} \le P(A \cap B) \le \frac{3}{5}$

Statement II $\frac{2}{5} \le P\left(\frac{A}{B}\right) \le \frac{9}{10}$

23. Consider the system of equations $ax + by = 0$, $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$

Statement I The probability that the system of equations has a unique solution 3/8 and

Statement II The probability that the system of equations has a solution is 1.

24. Consider a natural number x is chosen at random from the first 100 natural numbers.

Statement I The probability that $\frac{(x-10)(x-50)}{(x-30)} > 0$ is 0.69.

Statement II If A is an event, then $0 < P(A) < 1$.

25. Consider the relation $P(A) + P(\overline{A}) = 1$.

Statement I 20 persons are sitting in a row. These persons are selected at random. The probability that two selected persons are not together is 0.7.

Statement II If A is an event, then $P(\text{not } A) = 1 - P(A)$.

26. **Statement I** If 12 coins are thrown simultaneously, then probability of appearing exactly five head is equal to probability of appearing exactly 7 heads.

Statement II ${}^nC_r = {}^nC_s \Rightarrow$ either $r = s$ or $r + s = n$ and $P(H) = P(T)$ in a single trial.

Previous Years' Questions

27. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is **[JEE Main 2013]**

(a) $\frac{17}{3^5}$ (b) $\frac{13}{3^5}$

(c) $\frac{11}{3^5}$ (d) $\frac{10}{3^5}$

28. Three numbers are chosen at random without replacement from {1, 2, 3,…,8}. The probability that their minimum is 3, given that their maximum is 6, is **[AIEEE 2012]**

(a) $\frac{3}{8}$ (b) $\frac{1}{5}$ (c) $\frac{1}{4}$ (d) $\frac{2}{5}$

29. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of atleast one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval **[AIEEE 2011]**

(a) $\left(\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$

(c) $\left(\frac{11}{12}, 1\right]$ (d) $\left(\frac{1}{2}, \frac{3}{4}\right]$

30. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than **[AIEEE 2009]**

(a) $\frac{1}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1}{\log_{10} 4 + \log_{10} 3}$

(c) $\frac{9}{\log_{10} 4 - \log_{10} 3}$ (d) $\frac{4}{\log_{10} 4 - \log_{10} 3}$

31. One ticket is selected at random from 50 tickets numbered 00, 01, 02, …, 49. Then, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals to **[AIEEE 2009]**

(a) $\frac{1}{14}$ (b) $\frac{1}{7}$ (c) $\frac{5}{14}$ (d) $\frac{1}{50}$

Answers

Check Point 1

1. (c) **2.** (d) **3.** (b) **4.** (b) **5.** (c) **6.** (b) **7.** (b) **8.** (a)

Round I

1. (d)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (d)
11. (c)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (d)	**17.** (c)	**18.** (a)	**19.** (d)	**20.** (b)
21. (a)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (d)	**26.** (b)	**27.** (a)	**28.** (b)	**29.** (d)	**30.** (a)
31. (a)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (c)	**38.** (b)	**39.** (d)	**40.** (d)
41. (b)	**42.** (a)	**43.** (b)	**44.** (a)	**45.** (d)	**46.** (a)	**47.** (a)	**48.** (c)	**49.** (c)	**50.** (b)
51. (b)	**52.** (a)	**53.** (b)	**54.** (a)	**55.** (a)	**56.** (c)	**57.** (a)	**58.** (c)	**59.** (c)	**60.** (d)
61. (a)	**62.** (b)	**63.** (d)	**64.** (c)						

Round II

1. (d)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (c)	**9.** (c)	**10.** (b)
11. (b)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (d)	**16.** (b,d)	**17.** (b,c)	**18.** (b,c)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (b)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (b)	**29.** (b)	**30.** (a)
31. (a)									

the Guidance

Round I

1. Let A, B and C be the respective events that the first, second and third drawn orange is good.

Therefore, probability that first drawn orange is good, $P(A) = \frac{12}{15}$

Therefore, probability of getting second orange is good, $P(B) = \frac{11}{14}$

($\because$ The oranges are not replaced so number of good oranges left is 11)

Similarly, probability of getting third orange is good, $P(C) = \frac{10}{13}$

($\because$ the oranges are not replaced so number of good oranges left is 10)

The box is approved for sale, if all the three oranges are good.

Thus, probability of getting all the three oranges good

$$= \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13}$$

Therefore, the probability that the box is approved for sale $= \frac{44}{91}$

2. When a die is thrown, the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Also, A : number is even and B : number is red

$\therefore \quad A = \{2, 4, 6\}, B = \{1, 2, 3\}$ and $A \cap B = \{2\}$.

$\Rightarrow \quad n(A) = 3, n(B) = 3, n(A \cap B) = 1$

Now, $P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}$

and $P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \frac{1}{6} = P(A \cap B)$

$\Rightarrow \quad P(A \cap B) \neq P(A)\, P(B)$

Thus, A and B are not independent events.

3. Given, P (not A or not B) $= \frac{1}{4} \quad \Rightarrow \quad P(A' \text{ or } B') = \frac{1}{4}$

$$\Rightarrow \quad P(A' \cup B') = \frac{1}{4} \Rightarrow P[(A \cap B)'] = \frac{1}{4}$$

$$[\because P(A' \cup B') = P(A \cap B)']$$

$$\Rightarrow \quad 1 - P(A \cap B) = \frac{1}{4} \Rightarrow P(A \cap B) = 1 - \frac{1}{4} = \frac{3}{4} \qquad \ldots(i)$$

Also, $P(A) \times P(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \neq \frac{3}{4}$

$$\left[\text{Given}, P(A) = \frac{1}{2}, P(B) = \frac{7}{12}\right]$$

$\Rightarrow \quad P(A) \times P(B) \neq P(A \cap B)$

$\therefore$ A and B are not independent events.

4. Total number of balls $= 18$, number of red balls $= 8$ and number of black balls $= 10$

$\therefore$ Probability of drawing a red ball $= \frac{\text{Number of red balls}}{\text{Total number of balls}}$

$$= \frac{8}{18}$$

Similarly, probability of drawing a black ball

$$= \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{10}{18}$$

(i) Probability of getting both red balls $= P$ (both balls are red)

$= P$ (a red ball is drawn at first draw and again a red ball at second draw)

$$= \frac{8}{18} \times \frac{8}{18} = \frac{16}{81}$$

(ii) P (probability of getting first ball is black and second is red)

$$= \frac{10}{18} \times \frac{8}{18} = \frac{20}{81}$$

(iii) Probability of getting one black and other red ball

$= P$(first ball is black and second is red)

$+ P$ (first ball is red and second is black)

$$= \frac{10}{18} \times \frac{8}{18} + \frac{8}{18} \times \frac{10}{18} = \frac{40}{81}$$

5. Probability of solving the problem by A, $P(A) = \frac{1}{2}$

Probability of solving the problem by B, $P(B) = \frac{1}{3}$

Probability of not solving the problem by

$$A = P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

and probability of not solving the problem by

$$B = P(B') = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

(i) P (the problem is solved) $= 1 - P$ (none of them solve the problem)

$$= 1 - P(A' \cap B') = 1 - P(A')\, P(B')$$

($\because A$ and B are independent $\Rightarrow A'$ and B' are independent)

$$= 1 - \left(\frac{1}{2} \times \frac{2}{3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

(ii) P (exactly one of them solve the problem) $= P(A)\, P(B') + P(A')\, P(B)$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

6. A and B are independent, if $P(A \cap B) = P(A)\,P(B)$

$$\therefore \quad P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$[\because P(A \cup B)' = 1 - P(A \cup B)]$$
$$= 1 - P(A) - P(B) + P(A)P(B)$$
$$= [1 - P(A)][1 - P(B)]$$

Therefore, the correct answer is (b).

7. Let p be the probability that B gets selected.

P (exactly one of A, B is selected) $= 0.6$ (given)

P (A is selected, B is not selected; B is selected, A is not selected) $= 0.6$

$$P(A \cap B') + P(A' \cap B) = 0.6$$
$$P(A)\,P(B') + P(A')\,P(B) = 0.6$$
$$(0.7)(1 - p) + (0.3)p = 0.6$$
$$p = 0.25$$

Thus, the probability that B gets selected is 0.25.

8. Event that the ball drawn is red can be under two cases

(i) ball was drawn from first bag.

(ii) ball was drawn from second bag. Then, use Baye's theorem

Let E_1 : first bag is selected, E_2 : second bag is selected

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Let E : ball drawn is red.

$$\therefore \quad P\left(\frac{E}{E_1}\right) = P \text{ (drawing a red ball from first bag)} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow \quad P\left(\frac{E}{E_2}\right) = P \text{ (drawing a red ball from second bag)}$$
$$= \frac{2}{8} = \frac{1}{4}$$

$\therefore$ Required probability

$$= P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}} = \frac{\frac{1}{4}}{\frac{2+1}{8}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$$

9. A second order determinant has four entries which may be 0 or 1.

Total number of determinants $= 2^4 = 16$

The only positive determinants are $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$.

Since, each entry of the above determinant can be selected with probability $\frac{1}{2}$, therefore required probability.

$$= 3\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{16}$$

Alternatively Here, total number of cases $= 16$

Favourable number of cases $= 3$

$\therefore$ Required probability $= \frac{3}{16}$

10. Let the event in which A fails and B fails be denoted by A' and B'

$$P(A') = 0.2, P(A' \cap B') = 0.15$$

$$P\,(B \text{ fails alone}) = P(B') - P(A' \cap B')$$
$$\therefore \quad 0.15 = P(B') - 0.15$$
$$\Rightarrow \quad P(B') = 0.30$$

(i) $P(A$ fails/B has failed)

$$= P(A'/B') = \frac{P\,(A' \cap B')}{P(B')} = \frac{0.15}{0.30} = 0.5$$

(ii) $P(A$ fails alone$) = P(A$ fails$) - P$ (Both A and B *fails*)

$$= P(A') - P(A' \cap B') = 0.2 - 0.15 = 0.05$$

11. Given, $\quad P\left(\frac{A}{B}\right) > P(A)$

$$\Rightarrow \quad \frac{P\,(A \cap B)}{P(B)} > P(A)$$
$$\Rightarrow \quad P\,(A \cap B) > P(A) \cdot P(B)$$
$$\Rightarrow \quad \frac{P\,(A \cap B)}{P(A)} > P(B)$$
$$\Rightarrow \quad P\left(\frac{B}{A}\right) > P(B) \qquad \left[\because P\left(\frac{B}{A}\right) = \frac{P\,(A \cap B)}{P(A)}\right]$$

12. Total number of kings in a deck $= 4$

Total number of cards in a deck $= 52$

Let E be the event that all four cards drawn are kings.

Then, $\quad P(E) = \frac{{}^4C_4}{{}^{52}C_4}$

$$= \frac{4! \times 48!}{52 \times 51 \times 50 \times 49 \times 48!}$$
$$= \frac{4 \times 3 \times 2}{52 \times 51 \times 50 \times 49}$$
$$= \frac{24}{52 \times 51 \times 50 \times 49} = \frac{1}{270725}$$

13. Number of cases when 10 heads occur $= 1$

Number of cases when 9 heads occur $= 10$

Number of cases when 8 heads occur $= 45$

Total number of possible outcomes $= 2^{10}$

Let E be the event that atleast 8 heads occur.

Then, $P(E) = \dfrac{1+10+45}{2^{10}}$

$= \dfrac{56}{32 \times 32} = \dfrac{7}{128}$

14. Let S be the sample space having $6 \times 6 = 36$ elements.

Given, $A = \{x, y : x + y = 11\}$

$\Rightarrow$ $A = \{(5, 6), (6, 5)\}$

So, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{2}{36} = \dfrac{1}{18}$

and $B = \{(x, y) : x \neq 5\}$

$\Rightarrow$ $B = S - \{(1,5), (2,5), (3,5), (4,5), (6,5)\}$

$n(B) = 36 - 5 = 31$

$\therefore$ $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{31}{36}$

and $A \cap B = \{(x, y) : x + y = 11, x \neq 5\} = \phi$

$\Rightarrow$ $n(A \cap B) = 0$

$\Rightarrow$ $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{0}{36} = 0$

Clearly, $P(A) \cdot P(B) = \dfrac{1}{18} \times \dfrac{31}{36}$

$\neq P(A \cap B)$

Hence, A and B are not independent events.

15. Let x be the probability of success in each trial, then $(1 - x)$ will be the probability of failure in each trial.

Thus, probability of exactly successes in a series of three trials

$= P(\bar{E}_1 E_2 E_3 + E_1 \bar{E}_2 E_3 + E_1 E_2 \bar{E}_3)$

$= (1 - x)\, x \cdot x + x(1 - x)\, x + x \cdot x(1 - x) = 3x^2 (1 - x)$

and the probability of three success

$P(E_1 E_2 E_3) = x \cdot x \cdot x = x^3$

According to the question,

$9x^3 = 3x^2 (1 - x)$

$\Rightarrow$ $3x = 1 - x$

$\Rightarrow$ $4x = 1 \Rightarrow x = \dfrac{1}{4}$

Hence, the probability of success in each trial is $\dfrac{1}{4}$.

16. From the wavy curve method, given inequality is satisfied for $x < 20$ or $30 < x < 40$.

$\therefore$ Number of favourable outcomes $= 28$

$\therefore$ Required probability $= \dfrac{28}{100} = \dfrac{7}{25}$

17. P (non-occurrence of (A_1)) $= 1 - \dfrac{1}{(i+1)} = \dfrac{i}{(i+1)}$

$\therefore$ P (non-occurrence of any of events)

$= \left(\dfrac{1}{2}\right) \cdot \left(\dfrac{2}{3}\right) \cdots \left\{\dfrac{n}{(n+1)}\right\} = \dfrac{1}{(n+1)}$

18. $P(B/A \cup B') = \dfrac{P\{B \cap (A \cup B')\}}{P(A \cup B')}$

$= -\dfrac{P(A \cap B)}{P(A) + P(B') - P(A \cap B')}$

$= -\dfrac{P(A) - P(A \cap B')}{0.7 + 0.6 - 0.5} = \dfrac{0.7 - 0.5}{0.8} = \dfrac{1}{4}$

19. Let $\bar{A}$ be the event of different birthdays. Each can have birthday in 365 ways, so N persons can have their birthdays in 365^N ways. Number of ways in which all have different birthdays $= {}^{365}P_N$

$\therefore$ $P(A) = 1 - P(\bar{A}) = 1 - \dfrac{{}^{365}P_N}{(365)^N}$

$= 1 - \dfrac{(365)!}{(365)^N (365 - N)!}$

20. $P(E \cap F) = P(E)\, P(F) = \dfrac{1}{12}$ ...(i)

$P(E^c \cap F^c) = P(E^c) \cdot P(F^c) = \dfrac{1}{2}$

$\Rightarrow$ $(1 - P(E))(1 - P(F)) = \dfrac{1}{2}$ ...(ii)

On solving Eqs. (i) and (ii), we get

$P(E) = \dfrac{1}{2}$ and $P(F) = \dfrac{1}{6}$, as $P(E) > P(F)$

21. We observe that $7^1, 7^2, 7^3$ and 7^4 ends in 7, 9, 3 and 1 respectively. Thus, 7^l ends in 7, 9, 3 or 1 according as l is of the form $4k + 1$, $4k + 2$, $4k - 1$ or $4k$, respectively. If S is the sample space, then $n(S) = (100)^2 . 7^m + 7^n$ is divisible by 5 if (i) m is of the form $4k + 1$ and n is of the form $4k - 1$ or (ii) m is of the form $4k + 2$ and n is of the form $4k$ or (iii) m is of the form $4k - 1$ and n is of the form $4k + 1$ or (iv) m is of the form $4k$ and n is of the form $4k + 1$.

Thus, number of favourable ordered pairs $(m, n) = 4 \times 25 \times 25$. Hence, requried probability is $\dfrac{1}{4}$.

22. $n(S) = {}^{10}C_3$ and $n(E) = {}^3C_1$, because on selecting 3, 7 and we have to select one from 4, 5 and 6.

$\therefore$ $P(E) = \dfrac{{}^3C_1}{{}^{10}C_3} = \dfrac{1}{40}$

23. Given, $2P(A) = P(B) = \dfrac{5}{13}$

$\Rightarrow$ $P(A) = \dfrac{5}{26}$ and $P(B) = \dfrac{5}{13}$

Now, $P\left(\dfrac{A}{B}\right) = \dfrac{P(A \cap B)}{P(B)} \Rightarrow \dfrac{2}{5} = \dfrac{P(A \cap B)}{\frac{5}{13}}$

$\Rightarrow$ $P(A \cap B) = \dfrac{2}{5} \times \dfrac{5}{13} = \dfrac{2}{13}$

Again, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{5}{26} + \dfrac{5}{13} - \dfrac{2}{13} = \dfrac{5 + 10 - 4}{26} = \dfrac{11}{26}$

24. When a coin is tossed three times, the sample space S contains $2^3 = 8$ equally likely sample points.

$\therefore \quad S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Let E : set of events in which atmost two tails occur and

F : set of events in which atleast one tail occurs.

$\therefore \quad E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

(Here, we can consider the cases of the two tails, one tail and no tail because the condition is atmost not atleast)

and $F = \{TTT, THT, TTH, HTT, HHT, HTH, THH\}$

(Here, we can consider the cases of one or more tails because the condition given is atleast not atmost.)

$\Rightarrow \quad (E \cap F) = \{HHT, HTH, THH, HTT, THT, TTH\}$

Now, $\quad P(E) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{7}{8}$

Similarly, $P(F) = \dfrac{7}{8}$ and $P(E \cap F) = \dfrac{6}{8}$

$\therefore \quad P\left(\dfrac{E}{F}\right) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{6/8}{7/8} = \dfrac{6}{7}$

25. If a die is thrown three times, then the number of elements in the sample space will be $6 \times 6 \times 6 = 216$.

The sample space is

$$S = \{(x, y, z) : x, y, z \in \{1, 2, 3, 4, 5, 6\}\}.$$

Here, E : set of events in which 4 appears on the third toss and F : Set of events in which 6 and 5 appears respectively on first two tosses

$$E = \begin{Bmatrix} (1,1,4), (1,2,4), (1,3,4), (1,4,4), (1,5,4), (1,6,4) \\ (2,1,4), (2,2,4), (2,3,4), (2,4,4), (2,5,4), (2,6,4) \\ (3,1,4), (3,2,4), (3,3,4), (3,4,4), (3,5,4), (3,6,4) \\ (4,1,4), (4,2,4), (4,3,4), (4,4,4), (4,5,4), (4,6,4) \\ (5,1,4), (5,2,4), (5,3,4), (5,4,4), (5,5,4), (5,6,4) \\ (6,1,4), (6,2,4), (6,3,4), (6,4,4), (6,5,4), (6,6,4) \end{Bmatrix}$$

$\Rightarrow n(E) = 36$

and $F = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$

$\Rightarrow n(F) = 6 \Rightarrow E \cap F = \{(6,5,4)\} \Rightarrow n(E \cap F) = 1$

$\therefore \quad P(E) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{36}{216} = \dfrac{1}{6}$

Similarly, $\quad P(F) = \dfrac{6}{216} = \dfrac{1}{36}$ and $P(E \cap F) = \dfrac{1}{216}$

$\therefore$ Required probability

$$P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{216}}{\frac{1}{36}} = \frac{1}{216} \times \frac{36}{1} = \frac{1}{6}$$

26. Let the first observation be from the black die and second from the red die.

When two dice (one black and another red) are rolled, the sample space $S = 6 \times 6 = 36$ (equally likely sample events)

Let E : set of events having 8 as the sum of the observations,

F : set of events in which red die resulted in a (in any one die) number less than 4

$\therefore \quad E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\} \Rightarrow n(E) = 5$

and $\quad F = \begin{Bmatrix} (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) \\ (3,1), (3,2), (3,3), (4,1), (4,2), (4,3) \\ (5,1), (5,2), (5,3), (6,1), (6,2), (6,3) \end{Bmatrix}$

$\Rightarrow \quad n(F) = 18$

$\Rightarrow \quad E \cap F = \{(5,3), (6,2)\}\ n(E \cap F) = 2$

Now, $\quad P(E) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \dfrac{5}{36}$

Similarly, $\quad P(F) = \dfrac{18}{36} = \dfrac{1}{2}$ and $P(E \cap F) = \dfrac{2}{36} = \dfrac{1}{18}$

$\therefore$ Required probability

$$P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{18}}{\frac{1}{2}} = \frac{1}{18} \times \frac{2}{1} = \frac{1}{9}$$

27. Here, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$,

Given, $\quad E = \{1, 3, 5\}, F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$

$\Rightarrow \quad E \cap F = \{3\}, E \cap G = \{3, 5\},$

$E \cup F = \{1, 2, 3, 5\}, (E \cup F) \cap G = \{2, 3, 5\}$

and $(E \cap F) \cap G = \{3\}$.

$\Rightarrow \quad n(S) = 6, n(E) = 3, n(F) = 2, n(G) = 4$

$n(E \cap F) = 1, n(E \cap G) = 2, n(E \cap F) = 4$

$n[(E \cup F) \cap G] = 3$ and $n[(E \cap F) \cap G] = 1$

$\therefore \quad \text{Probability} = \dfrac{\text{Number of favourable events}}{\text{Total number of events}}$

We have, $P(E) = \dfrac{3}{6} = \dfrac{1}{2}, P(F) = \dfrac{2}{6} = \dfrac{1}{3}, P(G) = \dfrac{4}{6} = \dfrac{2}{3}$

$P(E \cap F) = \dfrac{1}{6}, P(E \cap G) = \dfrac{2}{6} = \dfrac{1}{3}$

$P[(E \cup F) \cap G] = \dfrac{3}{6} = \dfrac{1}{2}$ and $[P(E \cap F) \cap G] = \dfrac{1}{6}$

$$P\left(\frac{E \cup F}{G}\right) = \frac{P[(E \cup F) \cap G]}{P(G)} = \frac{1/2}{2/3} = \frac{3}{4}$$

and $\quad P\left(\dfrac{E \cap F}{G}\right) = \dfrac{P[(E \cap F) \cap G]}{P(G)} = \dfrac{1/6}{2/3} = \dfrac{1}{4}$

28. Let b and g represent the boy and the girl child, respectively. If a family has two children, the sample space will be

$$S = \{bb, bg, gb, gg\}$$

which contains four equally likely sample points *i.e.*, $n(S) = 4$

Let E : both children are girls, then $E = \{gg\} \Rightarrow n(E) = 1$

(i) Let F : the youngest is a girl, then $F = \{bg, gg\}$

$\Rightarrow n(F) = 2$

$\Rightarrow \quad E \cap F = \{gg\} \Rightarrow n(E \cap F) = 1$

$P(E) = \dfrac{1}{4}, P(F) = \dfrac{2}{4} = \dfrac{1}{2}$ and $P(E \cap F) = \dfrac{1}{4}$

$\therefore$ Required probability $= P\left(\dfrac{E}{F}\right) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/4}{2/4} = \dfrac{1}{2}$

(ii) Let F : atleast one is a girl, then $F = \{bg, gb, gg\}$

$\Rightarrow \quad E \cap F = \{gg\} = E$

$\Rightarrow \quad n(F) = 3, n(E \cap F) = 1$

$\therefore$ Required probability

$$= P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

29. Total number of questions

$$= 300 + 200 + 500 + 400 = 1400$$

Let E be the event that selected question is an easy question

Then, $\quad n(E) = 500 + 300 = 800$

$$\therefore \quad P(E) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{800}{1400} = \frac{4}{7}$$

Let F be the event that selected question is a multiple choice question.

Then, $\quad n(F) = 500 + 400 = 900$

$$\therefore \quad P(F) = \frac{\text{Total number of easy questions}}{\text{Total number of questions}} = \frac{900}{1400} = \frac{9}{14}$$

$$\therefore \quad P(E \cap F) = \frac{500}{1400} = \frac{5}{14}$$

$$\therefore \quad P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{5/14}{9/14} = \frac{5}{9}$$

30. The outcomes of the given experiment can be represented by the following set.

The sample space of the experiment is

$$S = \begin{Bmatrix} (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \\ (1,H), (1,T), (2,H), (2,T), (4,H) \\ (4,T), (5,H), (5,T) \end{Bmatrix}$$

$n(S) = 20$

Let E : the coin shows a tail,

F : atleast one die shows up a 3,

$E = \{(1,T), (2,T), (4,T), (5,T)\}, \Rightarrow n(E) = 4$

$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (6,3)\}$

$\Rightarrow n(F) = 7$

$\Rightarrow E \cap F = \phi$ because here is no common elements.

$$\therefore \quad P(E) = \frac{\text{Favourable outcomes}}{\text{Total number of outcomes}}$$

$$= \frac{n(E)}{n(S)} = \frac{4}{20} = \frac{1}{5}$$

Similarly, $\quad P(F) = \dfrac{n(F)}{n(S)} = \dfrac{7}{20}$

and $\quad P(E \cap F) = \dfrac{n(E \cap F)}{n(S)} = \dfrac{0}{20} = 0$

Hence, the required probability

$$P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{0}{7/20} = 0$$

31. Let A and B be the events that the bulb is red and defective, respectively.

$$P(A) = \frac{10}{100} = \frac{1}{10},$$

$$P(A \mid B) = \frac{2}{100} = \frac{1}{50}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{50} \times \frac{10}{1} = \frac{1}{5}$$

Thus, the probability of the picked up bulbs of its being defective, if it is red, is $\frac{1}{5}$.

32. Let A denote the event that atleast one girl will be chosen and B the event that exactly 2 girls will be chosen. We require $P(B \mid A)$.

Since, A denotes the event that atleast one girl will be chosen. A denotes that no girl is chosen *i.e.*, 4 boys are chosen. Then,

$$P(A') = \frac{{}^8C_4}{{}^{12}C_4} = \frac{70}{495} = \frac{14}{99}$$

$$P(A) = 1 - \frac{14}{99} = \frac{85}{99}$$

Now, $P(A \cap B) = P$ (2 boys and 2 girls) $= \dfrac{{}^8C_2 \cdot {}^4C_2}{{}^{12}C_4}$

$$= \frac{6 \times 28}{495} = \frac{56}{165}$$

Thus, $\quad P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{56}{165} \times \dfrac{99}{85} = \dfrac{168}{425}$

33. If $A \subset B$, then $A \cap B = A$

$\therefore \quad P(A \cap B) = P(A)$

We know that, $P\left(\dfrac{A}{B}\right) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)}{P(B)}$

But $\quad P(B) \le 1 \therefore P\left(\dfrac{A}{B}\right) \ge P(A)$.

34. It is given that, $P(A) \ne 0$

(i) Given, A is a subset of B *i.e.*, $(A \subset B)$

$\Rightarrow \quad A \cap B = A$

$\Rightarrow \quad P(A \cap B) = P(A)$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

(ii) Here, $\quad A \cap B = \phi \Rightarrow P(A \cap B) = 0$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{0}{P(A)} = 0$$

35. By conditional probability,

$$P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{A \cup B})}{P(\overline{B})}$$

($\because$ De-Morgan's law $\overline{A} \cap \overline{B} = \overline{A \cup B}$)

$$= \frac{1 - P(A \cup B)}{P(\overline{B})} \qquad (\because P(\overline{A}) + P(A) = 1)$$

36. Probability of getting an ace

$$P(E_1) = \frac{4}{52} = \frac{1}{13}, \quad P\left(\frac{E_2}{E_1}\right) = \frac{15}{51} = \frac{5}{17}$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$$

$$= \frac{1}{13} \cdot \frac{5}{17} = \frac{5}{221}$$

37. Given that, $P\left(\frac{B}{A}\right) = \frac{1}{2} \Rightarrow \frac{P(B \cap A)}{P(A)} = \frac{1}{2}$

$$\Rightarrow \quad P(B \cap A) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$P\left(\frac{A}{B}\right) = \frac{1}{4} \Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{1}{4},$$

$$\Rightarrow \quad P(B) = 4P(A \cap B) \Rightarrow P(B) = \frac{1}{2}$$

$$\therefore \quad P(A \cap B) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = P(A) \cdot P(B)$$

$\because$ Events A and B are independent.

Now, $$P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B')}{P(B)}$$

$$= \frac{P(A')\,P(B')}{P(B)} = \frac{3}{4}$$

and $$P\left(\frac{B'}{A'}\right) = \frac{P(B' \cap A')}{P(A')}$$

$$= \frac{P(B')\,P(A')}{P(A')} = \frac{1}{2}$$

38. Let E_1 : the event that the person has disease and E_2 : the events that the person is healthy.

Then, E_1 and E_2 are mutually exclusive and exhaustive.

Moreover, $$P(E_1) = 0.1\% = \frac{0.1}{100} = 0.001$$

and $$P(E_2) = 1 - 0.001 = 0.999$$

Let E : the event that test is positive,

$P\left(\frac{E}{E_1}\right) = P$ (result is positive given the person has disease)

$$= 99\% = \frac{99}{100} = 0.99$$

Probability that a person does not have disease and test result is positive.

$$\therefore \quad P\left(\frac{E}{E_2}\right) = 0.5\% = \frac{0.5}{100} = 0.005$$

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005}$$

$$= \frac{0.00099}{0.00099 + 0.004995}$$

$$= \frac{0.00099}{0.005985} = \frac{990}{5985} = \frac{110}{665} = \frac{22}{133}$$

39. Let D be the event that the picked up tube is defective

Let A_1, A_2 and A_3 be the events that the tube is produced on machines E_1, E_2 and E_3, respectively.

$$P(D) = P(A_1)P(D \mid A_1) + P(A_2)\,P(D \mid A_2) + P(A_3)\,P(D \mid A_3) \quad \ldots(i)$$

$$P(A_1) = \frac{50}{100} = \frac{1}{2}, \quad P(A_2) = \frac{1}{4}, \quad P(A_3) = \frac{1}{4}$$

Also, $$P(D \mid A_1) = P(D \mid A_2) = \frac{4}{100} = \frac{1}{25}$$

$$P(D \mid A_3) = \frac{5}{100} = \frac{1}{20}.$$

On putting these values in Eq. (i), we get

$$P(D) = \frac{1}{2} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{20}$$

$$= \frac{1}{50} + \frac{1}{100} + \frac{1}{80} = \frac{17}{400} = 0.425$$

40. Let E_1 : the event that the student knows the answer and

E_2 : the event that the student guesses the answer.

Therefore, E_1 and E_2 are mutually exclusive and exhaustive.

$$\therefore \quad P(E_1) = \frac{3}{4} \text{ and } P(E_2) = \frac{1}{4}$$

Let E : the answer is correct.

The probability that the student answered correctly, given that he knows the answer, is 1

i.e., $$P\left(\frac{E}{E_1}\right) = 1$$

Probability that the students answered correctly, given that the he guessed, is $\frac{1}{4}$

i.e., $$P\left(\frac{E}{E_2}\right) = \frac{1}{4}$$

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{1 \times \frac{3}{4}}{1 \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4}} = \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{16}} = \frac{\frac{3}{4}}{\frac{12+1}{16}} = \frac{3}{4} \times \frac{16}{13} = \frac{12}{13}$$

Note If two events are not mutually exclusive/exhaustive, then you do not use Baye's theorem.

41. Let E_1 : the event that the coin chosen is two headed,

E_2 : the event that the coin chosen is biased

and E_3 : the event that the coin chosen is unbiased

$\Rightarrow E_1, E_2$ and E_3 are mutually exclusive and exhaustive events. Moreover,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let E : tosses coin shows up a head,

$\therefore\ P\left(\frac{E}{E_1}\right) = P$ (coin showing heads, given that it is a two headed coin) $= 1$

$P\left(\frac{E}{E_2}\right) = P$ (coin showing heads, given that it is a biased coin)

$$75\% = \frac{75}{100} = \frac{3}{4}$$

$P\left(\frac{E}{E_3}\right) = P$ (coin showing heads, given that it is an unbiased coin) $= \frac{1}{2}$

The probability that the coin is two headed, given that it shows head, is given by $P\left(\frac{E_1}{E}\right)$

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{3}{4} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}}$$

$$= \frac{1}{1 + \frac{3}{4} + \frac{1}{2}} = \frac{1}{\frac{4+3+2}{4}} = \frac{4}{9}$$

42. There are 2000 scooter drivers, 4000 car drivers and 6000 truck drivers.

Total number of drivers $= 2000 + 4000 + 6000 = 12000$

Let E_1 : the event that insured person is a scooter driver,

E_2 : the event that insured person is a car driver

and E_3 : the event that insured person is a truck driver.

Then, E_1, E_2, E_3 are mutually exclusive and exhaustive events. Moreover,

$$P(E_1) = \frac{\text{Number of scooter drivers}}{\text{Total number of drivers}} = \frac{2000}{12000} = \frac{1}{6},$$

$$P(E_2) = \frac{\text{Number of car drivers}}{\text{Total number of drivers}} = \frac{4000}{12000} = \frac{1}{3}$$

and $P(E_3) = \frac{\text{Number of truck drivers}}{\text{Total number of drivers}} = \frac{6000}{12000} = \frac{1}{2}$

Let E : the events that insured person meets with an accident,

$P\left(\frac{E}{E_1}\right) = P$ (scooter driver met with an accident)

$$= 0.01 = \frac{1}{100},$$

$P\left(\frac{E}{E_2}\right) = P$ (car driver met with an accident) $= 0.03 = \frac{3}{100}$

$P\left(\frac{E}{E_3}\right) = P$ (truck driver met with an accident) $= 0.15 = \frac{15}{100}$

The probability that the driver is a scooter driver, given he met with an accident, is given by $P\left(\frac{E_1}{E}\right)$.

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2) + P\left(\frac{E}{E_3}\right)P(E_3)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{100} + \frac{1}{2} \times \frac{15}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + 1 + \frac{15}{2}} = \frac{1}{1 + 6 + 45} = \frac{1}{52}$$

43. Let E_1 : the event that the item is produced by machine A and

E_2 : the event that the item is produced by machine B.

Then, E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$$P(E_1) = 60\% = \frac{60}{100} = \frac{3}{5} \text{ and } P(E_2) = 40\% = \frac{40}{100} = \frac{2}{5}$$

Let E : the event that the item chosen is defective,

$\therefore\ P\left(\frac{E}{E_1}\right) = P$ (machine A produced defective items)

$$= 2\% = \frac{2}{100}$$

$P\left(\frac{E}{E_2}\right) = P$ (machine B produced defective items)

$$= 1\% = \frac{1}{100}$$

The probability that the randomly selected item was from machine B, given that it is defective, is given by $P\left(\frac{E_2}{E}\right)$.

By using Baye's theorem, we obtain

$$P\left(\frac{E_2}{E}\right) = \frac{P\left(\frac{E}{E_2}\right)P(E_2)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{1}{100} \times \frac{2}{5}}{\frac{2}{100} \times \frac{3}{5} + \frac{1}{100} \times \frac{2}{5}} = \frac{\frac{2}{500}}{\frac{6}{500} + \frac{2}{500}}$$

$$= \frac{2}{6+2} = \frac{1}{4}$$

44. Assume E_1 : first group coins,

E_2 : second group coins, E_3 : new product was introduced.

Then, the probability that new product was introduced by second group is $P\left(\frac{E_2}{E}\right)$ which can be found out by using Baye's theorem.

Let E_1 : the event that the first group wins and E_2 : the event that the second group wins.

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

$\therefore \quad P(E_1) = 0.6 = \frac{6}{10}$ and $P(E_2) = 0.4 = \frac{4}{10}$

Let E : the event that the new product is introduced.

$P\left(\frac{E}{E_1}\right) = P$ (introducing a new product, if the first group wins

$= 0.7 = \frac{7}{10}$

$\therefore \; P\left(\frac{E}{E_2}\right) = P$ (introducing a new product, if the second group wins) $= 0.3 = \frac{3}{10}$

The probability that the new product is introduced by the second group is given by $P\left(\frac{E_2}{E}\right)$.

By using Baye's theorem, we obtain

$$P\left(\frac{E_2}{E}\right) = \frac{P\left(\frac{E}{E_2}\right)P(E_2)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{3}{10}\times\frac{4}{10}}{\frac{7}{10}\times\frac{6}{10}+\frac{3}{10}\times\frac{4}{10}} = \frac{12}{42+12}$$

$$= \frac{12}{54} = \frac{2}{9}$$

45. Let E_1 : the event that 5 or 6 is shown on die

and E_2 : the event that 1, 2, 3, or 4 is shown on die.

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

and $\quad n(E_1) = 2, n(E_2) = 4$

Also, $\quad n(S) = 6$

$\therefore \quad P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = \frac{4}{6} = \frac{2}{3}$

Let E : The event that exactly one head show up,

$\therefore \; P\left(\frac{E}{E_1}\right) = P$ (exactly one head show up when coin is tossed thrice)

$$= P\{HTT, THT, TTH\} = \frac{3}{8}$$

($\because$ total number of events $= 2^3 = 8$)

$P\left(\frac{E}{E_2}\right) = P$ (head shows up when coin is tossed once)

$= \frac{1}{2}$

The probability that the girl threw, 1, 2, 3 or 4 with the die, if she obtained exactly one head, is given by $P\left(\frac{E_2}{E}\right)$.

By using Baye's theorem, we obtain

$$P\left(\frac{E_2}{E}\right) = \frac{P\left(\frac{E}{E_2}\right)P(E_2)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3}+\frac{3}{8}\times\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{8}}$$

$$= \frac{8}{8+3} = \frac{8}{11}$$

46. Let E_1 : the event that lost cards is a diamond $\Rightarrow n(E_1) = 13$

E_2 : lost cards is not a diamond $\Rightarrow n(E_2) = 52 - 13 = 39$

and $\quad n(S) = 52$

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

$\therefore \quad P(E_1) = \frac{13}{52} = \frac{1}{4}$

and $\quad P(E_2) = \frac{39}{52} = \frac{3}{4}$

Let E : the events that two cards drawn from the remaining pack are diamonds,

When one diamond card is lost, there are 12 diamond cards out of 51 cards.

The cards can be drawn out of 12 diamond cards in ${}^{12}C_2$ ways.

Similarly, 2 diamond cards can be drawn out of 51 cards in ${}^{51}C_2$ ways. The probability of getting two cards, when one diamond card is lost, is given by $P\left(\frac{E}{E_1}\right)$.

$$\therefore \quad P\left(\frac{E}{E_1}\right) = \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{\frac{12\times 11}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{12\times 11}{51\times 50}$$

$$\text{and} \quad P\left(\frac{E}{E_2}\right) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{\frac{13\times 12}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{13\times 12}{51\times 50}$$

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{12\times 11}{51\times 50}\times\frac{1}{4}}{\frac{12\times 11}{51\times 50}\times\frac{1}{4}+\frac{13\times 12}{51\times 50}\times\frac{3}{4}}$$

$$= \frac{12\times 11}{12\times 11+13\times 12\times 3} = \frac{132}{132+468} = \frac{132}{600} = \frac{11}{50}$$

47. Let E_1 : the event that coin comes up with a head,

E_2 : the event that coin comes up with a tail.

Then, E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$$P(E_1) = P(E_2) = \frac{1}{2}$$

E : The event that A reports that a head appears

$$\therefore \quad P\left(\frac{E}{E_1}\right) = P \text{ (head comes up and } A \text{ speaks truth)} = \frac{4}{5}$$

$$P\left(\frac{E}{E_2}\right) = P \text{ (tail comes up and } A \text{ speaks lie)} = \frac{1}{5}$$

The probability that there is actually a head is given by $P\left(\frac{E_1}{E}\right)$.

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1)+P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{4}{5}\times\frac{1}{2}}{\frac{4}{5}\times\frac{1}{2}+\frac{1}{5}\times\frac{1}{2}} = \frac{4}{4+1} = \frac{4}{5}$$

Hence, (a) is the correct option.

48. **Hint** Let E_1 : the event that the patient follows meditation and yoga and

E_2 : the event that the patient uses drug.

$\therefore$ E_1 and E_2 are mutually exclusive events and

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Let E : the event that the selected patient suffers a heart attack

$$\therefore \quad P\left(\frac{E}{E_1}\right) = \frac{40}{100}\left(1-\frac{30}{100}\right) = \frac{28}{100}$$

$$P\left(\frac{E}{E_2}\right) = \frac{40}{100}\left(1-\frac{25}{100}\right) = \frac{30}{100}$$

Using Baye's theorem and solve it

49. **Hint** E_1 = letter (*i.e.*, envelope) has come from CALCUTTA

E_2 = letter (*i.e.*, envelope) has come from TATANAGAR

E = two consecutive letters (*i.e.*, alphabets) TA are visible on envelope

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$$

$$P\left(\frac{E}{E_1}\right) = \frac{n(E\cap E_1)}{n(E_1)} = \frac{1}{7}$$

($\because$ 7 pairs of consecutive letters are CA, AL, LC, CU, UT, TT, TA)

$$P\left(\frac{E}{E_2}\right) = \frac{n(E\cap E_2)}{n(E_2)} = \frac{2}{8}$$

($\because$ 8 pairs of consecutive letters are TA, AT, TA, AN, NA, AG, GA, AR)

Using Baye's theorem and solve it.

50. **Hint** Let E be the event that we get a head,

E_1 be the event that fair coin was tossed and

E_2 be the event that 2 headed win was tossed.

Then, $\quad P(E_1) = P(E_2) = \frac{1}{2}, P\left(\frac{E}{E_1}\right) = \frac{1}{2}, P\left(\frac{E}{E_2}\right) = 1$

Using Baye's theorem and solve it

51. Let E be the event that a television chosen randomly is of standard quality. We have, to find

$$P(II/E) = \frac{P(E/II)\cdot P(II)}{P(E/I)\cdot P(I) + P(E/II)\cdot P(II)}$$

$$= \frac{(9/10)(3/10)}{(4/5)(7/10)+(9/10)(3/10)} = \frac{27}{83}$$

52. (i) It is known that the sum of a probability distribution of random variable is one *i.e.*, $\Sigma P(X) = 1$, therefore

$$P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$$

$$\Rightarrow \quad 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$\Rightarrow \quad 10k^2+9k-1 = 0$$

$$\Rightarrow \quad 10k^2+10k-k-1 = 0$$

$$\Rightarrow \quad 10k(k+1)-1(k+1) = 0$$

$$\Rightarrow \quad (k+1)(10k-1) = 0$$

$$\Rightarrow \quad k+1 = 0 \text{ or } 10k-1 = 0$$

$$\Rightarrow \quad k = -1 \text{ or } k = \frac{1}{10}$$

$k = -1$ is not possible as the probability of an event is never negative.

$$\therefore \quad k = \frac{1}{10}$$

(ii) $P(X<3) = P(0)+P(1)+P(2) = 0+k+2k$

$$= 0+\frac{1}{10}+\frac{2}{10} = \frac{3}{10} \qquad \left(\text{put } k = \frac{1}{10}\right)$$

(iii) $P(X>6) = P(7) = 7k^2+k = \frac{7}{100}+\frac{1}{10} = \frac{17}{100}$

$$\left(\text{put } k = \frac{1}{10}\right)$$

(iv) $P(0<X<3) = P(1)+P(2) = k+2k = \frac{1}{10}+\frac{2}{10} = \frac{3}{10}$

$$\left(\text{put } k = \frac{1}{10}\right)$$

53. Let X denote the sum of the numbers obtained when two fair dice are rolled. So, X may have values, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

(as 1 can't be the sum of two numbers on fair dice)

$P(X=2)=P[\{1,1\}]=\frac{1}{36}$, $P(X=3)=P[\{(1,2),(2,1)\}]=\frac{2}{36}$

$P(X=4)=P[\{(1,3),(2,2),(3,1)\}]=\frac{3}{36}$

$P(X=5)=P[\{(1,4),(2,3),(3,2),(4,1)\}]=\frac{4}{36}$

$P(X=6)=P[\{(1,5),(2,4),(3,3),(4,2),(5,1)\}]=\frac{5}{36}$

$P(X=7)=P[\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}]=\frac{6}{36}$

$P(X=8)=P[\{(2,6),(3,5),(4,4),(5,3),(6,2)\}]=\frac{5}{36}$

$P(X=9)=P[\{(3,6),(4,5),(5,4),(6,3)\}]=\frac{4}{36}$

$P(X=10)=P[\{(4,6),(5,5),(6,4)\}]=\frac{3}{36}$

$P(X=11)=P[\{(5,6),(6,5)\}]=\frac{2}{36}$

$P(X=12)=P[\{(6,6)\}]=\frac{1}{36}$

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean

$$X=\Sigma X P(X)=\frac{\begin{bmatrix}2\times1+3\times2+4\times3+5\times4+6\times5+7\times6\\+8\times5+9\times4+10\times3+11\times2+12\times1\end{bmatrix}}{36}$$

$$=\frac{252}{36}=7$$

Variance $X=\Sigma X^2P(X)-(\text{Mean})^2$

$$=\frac{\begin{bmatrix}2^2\times1+3^2\times2+4^2\times3+5^2\times4+6^2\times5+7^2\times6\\+8^2\times5+9^2\times4+10^2\times3+11^2\times2+12^2\times1\end{bmatrix}}{36}-7^2$$

$$=\frac{1974}{36}-49=\frac{1974-1764}{36}=\frac{210}{36}=\frac{35}{6}$$

Hence, $\text{SD}=\sqrt{\text{Variance}}=\sqrt{\frac{35}{6}}$

54. Here, total number of students $=15$

The ages of students in ascending order are 14, 14, 15, 16, 16, 17, 17, 17, 18, 19, 19, 20, 20, 20, 21

Now, $P(X=14)=\frac{2}{15}$, $P(X=15)=\frac{1}{15}$,

$P(X=16)=\frac{2}{15}$, $P(X=17)=\frac{3}{15}$

$P(X=18)=\frac{1}{15}$, $P(X=19)=\frac{2}{15}$,

$P(X=20)=\frac{3}{15}$, $P(X=21)=\frac{1}{15}$

Therefore, the probability distribution of random variable X is as follows

X	14	15	16	17	18	19	20	21
Number of students	2	1	2	3	1	2	3	1
P(X)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

The third row gives the probability distribution of X.

Mean $X=\Sigma X P(X)$

$$=\frac{14\times2+15\times1+16\times2+17\times3+18\times1+19\times2+20\times3+21\times1}{15}$$

$$=\frac{28+15+32+51+18+38+60+21}{15}=\frac{263}{15}=17.53$$

Variance $X=\Sigma X^2P(X)-(\text{Mean})^2$

$$=\frac{\begin{bmatrix}(14)^2\times2+(15)^2\times1+(16)^2\times2+(17)^2\times3\\+(18)^2\times1+(19)^2\times2+(20)^2\times3+(21)^2\times1\end{bmatrix}}{15}-\left(\frac{263}{15}\right)^2$$

$$=\frac{392+225+512+867+324+722+1200+441}{15}-\left(\frac{263}{15}\right)^2$$

$$=\frac{4683}{15}-\left(\frac{263}{15}\right)^2=312.2-307.4=4.8$$

SD of $X=\sqrt{\text{Variance}}=\sqrt{4.8}=2.19$

55. Since, $\Sigma p_i=1$, we have

$$C+2C+2C+3C+C^2+2C^2+7C^2+C=1$$

i.e., $10C^2+9C-1=0$

i.e., $(10C-1)(C+1)=0$

$\Rightarrow$ $C=\frac{1}{10}$, $C=-1$

Therefore, the permissible value of $C=\frac{1}{10}$

$$\text{Mean}=\sum_{i=1}^{n}x_ip_i=\sum_{i=1}^{7}x_ip_i$$

$$=1\times\frac{1}{10}+2\times\frac{2}{10}+3\times\frac{2}{10}+4\times\frac{3}{10}+5\left(\frac{1}{10}\right)^2+6\times2\left(\frac{1}{10}\right)^2+7\left(7\left(\frac{1}{10}\right)^2+\frac{1}{10}\right)$$

$$=\frac{1}{10}+\frac{4}{10}+\frac{6}{10}+\frac{12}{10}+\frac{5}{100}+\frac{12}{100}+\frac{49}{100}+\frac{7}{10}$$

$=3.66.$

56. Let X denote the number of heads tossed. So, X can take the values 0, 1, 2, 3. When a coin is tossed three times, we get Sample space,

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$P(X=0) = P \text{ (no head)} = P(TTT) = \frac{1}{8}$$

$$P(X=1) = P \text{ (one head)} = P(HTT, THT, TTH) = \frac{3}{8}$$

$$P(X=2) = P \text{ (two heads)} = P(HHT, HTH, THH) = \frac{3}{8}$$

$$P(X=3) = P \text{ (three heads)} = P(HHH) = \frac{1}{8}$$

Thus, the probability distribution of X is

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Variance of $X = \sigma^2 = \Sigma x_i^2 p_i - \mu^2$ ...(i)

where, μ is the mean of X given by

$$\mu = \Sigma x_i p_i = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2} \quad \text{...(ii)}$$

Now, $\Sigma x_i^2 p_i = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\sigma^2 = \left(3 - \frac{3}{2}\right)^2 = \frac{9}{4}$$

$\therefore$ Standard deviation $= \sqrt{\sigma^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

57. Let X be the money won in one throw.

Money lost in 1 throw = ₹ 1

Also, probability of getting 3 $= \frac{1}{6}$

Probability of getting 1 or 6

$$\Rightarrow \quad \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

Probability of getting any other number *i.e.*, 2 or 4 or 5

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

Then, probability distribution is

X	5	2	0
P(X)	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

Then, expectation of money that player can be won

$$E(X) = \frac{5}{6} + \frac{4}{6} + 0$$

$$= \frac{9}{6} = ₹\ 1.5$$

Then, player's expected profit = ₹ 1.5 − ₹ 1 = 0.50

58. Given that, $E(X) = 3$ and $(E(X^2) = 11$

Variance of $X = E(X^2) - [E(X)]^2$

$$= 11 - (3)^2 = 11 - 9 = 2$$

59. It is case of Bernoulli trials with $n = 4$. Here, success is getting a doublet.

When a pair of dice is thrown once there are

$6 \times 6 = 36$ equally likely outcomes.

$\Rightarrow$ Total number of cases = 36

And possible doublets are

$\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\Rightarrow$ Favourable number of cases = 6

$\therefore$ $p = P$ (success) = P (getting a doublet in a single throw of a pair of dice)

$$\frac{\text{Favourable number of cases}}{\text{Total number of cases}} = \frac{6}{36} = \frac{1}{6}$$

$$q = P \text{ (failure)} = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Clearly, X has the binomial distribution with $n = 4$, $p = \frac{1}{6}$ and $q = \frac{5}{6}$

Therefore, $P(X = r) = {}^nC_r\, p^r q^{n-r}$, where $r = 0, 1, 2, \ldots, n$.

$$P(X = r) = {}^4C_r \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{4-r} = {}^4C_r \frac{5^{4-2}}{6^4}$$

Now, $P(2 \text{ successes}) = {}^4C_2\, p^2 q^2 = \frac{4 \times 3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$

$$= 6 \times \frac{1}{36} \times \frac{25}{36} = \frac{25}{216}$$

60. Let X denote the number of balls marked with the digit 0 among the 4 balls drawn. Since, the balls are drawn with replacement, the trials are Bernoulli trials,

$$p = P \text{ (balls drawn is marked with 0)} = \frac{1}{10}$$

($\because$ only one ball out of 10 is marked with 0)

$$\Rightarrow \quad q = 1 - p = \frac{9}{10}$$

$\therefore$ X has a binomial distribution with $n = 4$, $p = \frac{1}{10}$ and $q = \frac{9}{10}$

$$\because \quad P(X = r) = {}^4C_r \cdot \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{4-r}$$

Required probability = P (none marked with 0)

$$= P(X = 0) = {}^4C_0 p^0 q^4 = q^4 = \left(\frac{9}{10}\right)^4$$

61. Let X denote the number of correct answers given by the student. The repeated tosses of a coin are Bernoulli trials. Since, head on a coin represents the true answer and tail represents the false answer, the correctly answered of the question are Bernoulli trials.

$\therefore \quad p = P$ (a success) $= P$ (coin show up a head) $= \frac{1}{2}$

$\therefore \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ X has a binomial distribution with

$$n = 20, \; p = \frac{1}{2}$$

and $$q = \frac{1}{2}$$

$$\therefore \quad P(X = r) = {}^{20}C_r . \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{20-r}$$

Hence, P (atleast 12 questions are answered as true)

$$= P(X \geq 12) = P(12) + P(13) + P(14) + P(15) + P(16) + P(17) + P(18) + P(19) + P(20)$$

$$= {}^{20}C_{12}\, p^{12}q^8 + {}^{20}C_{13}\, p^{13}q^7 + {}^{20}C_{14}\, p^{14}q^6 + {}^{20}C_{15}\, p^{15}q^5 + {}^{20}C_{16}\, p^{16}q^4 + {}^{20}C_{17}\, p^{17}q^3 + {}^{20}C_{18}\, p^{18}q^2 + {}^{20}C_{19}\, p^{19}q^1 + {}^{20}C_{20}\, p^{20}$$

$$= ({}^{20}C_{12} + {}^{20}C_{13} + {}^{20}C_{14} + {}^{20}C_{15} + {}^{20}C_{16} + {}^{20}C_{17} + {}^{20}C_{18} + {}^{20}C_{19} + {}^{20}C_{20}) . \frac{1}{2^{20}}$$

$$= \left(\frac{1}{2}\right)^{20} [{}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20}]$$

62. Therefore, $n = 6$ and $p = \frac{1}{2}$ and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Then, $$P(X = r) = {}^6C_r . \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{6-r}$$

Here, $$P(X = 0) = {}^6C_0\, p^0 q^6 = {}^6C_0 \left(\frac{1}{2}\right)^6 = \frac{1}{2^6} = \frac{1}{64},$$

$$P(X = 1) = {}^6C_1\, p^1 q^5 = 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^5 = 6\left(\frac{1}{2^6}\right) = \frac{6}{64}$$

$$P(X = 2) = {}^6C_2\, p^2 q^4 = \frac{6 \times 5}{1 \times 2} . \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = \frac{15}{64},$$

$$P(X = 3) = {}^6C_3\, p^3 q^3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 = \frac{20}{64},$$

$$P(X = 4) = {}^6C_4\, p^4 q^2 = {}^6C_2 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = \frac{15}{64}$$

$$P(X = 5) = {}^6C_5\, p^5 q^1 = {}^6C_1 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1 = \frac{6}{64}$$

and $$P(X = 6) = {}^6C_6\, p^6 q^0 = 1 \times \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

It is clear that $\frac{20}{64}$ is maximum of all the above values. This means that $X = 3$ is the most likely outcome.

63. Let X denotes the number of correct answers given by the candidate. It is a case of Bernoulli trials with $n = 5$, where success is guessing a correct answer to a multiple choice question with 3 options.

$$\therefore \quad p = \frac{1}{3} \text{ and } q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Clearly, X has a binomial distribution with $n = 5$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$

$$P(X = r) = {}^5C_r \left(\frac{1}{3}\right)^r . \left(\frac{2}{3}\right)^{5-r}$$

Required probability $= P$ (four or more correct answers)

$$= P(X \geq 4) = P(4) + P(5)$$

$$= {}^5C_4\, p^4 q + {}^5C_5\, p^5 q^0$$

$$= {}^5C_4 \left(\frac{1}{3}\right)^4 \frac{2}{3} + {}^5C_5 \left(\frac{1}{3}\right)^5 . \left(\frac{2}{3}\right)^0$$

$$= 5 \times \frac{2}{3} \times \frac{1}{3^4} + \frac{1}{3^5} = \frac{11}{3^5} = \frac{11}{243}$$

64. The repeated selection of defective bulbs from a box are Bernoulli trials. Let X denote the number of defective bulbs out of a sample of 5 bulbs.

Now, probability of getting a defective bulb, $p = \frac{10}{100} = \frac{1}{10}$

and $$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Clearly, X has binomial distribution with $n = 5$,

$$p = \frac{1}{10}$$

and $$q = \frac{9}{10}$$

$$\therefore \quad P(X = r) = {}^nC_r\, p^r q^{n-r}$$

$$= {}^5C_r . \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{5-r}$$

P (none of the bulbs is defective)

$$= P(X = 0) = {}^5C_0 . \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5$$

$$= 1 \times 1 \times \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

Round II

1. $P(HHH) + P(HHTT) + P(HTTTT) + P(TTTTTT)$

$$= \frac{1}{2^3} + ({}^4C_2)\frac{1}{2^4} + ({}^5C_1)\frac{1}{2^5} + \frac{1}{2^6} = \frac{43}{64}$$

2. Total number of function is 6^6

and number of onto function $= 6!$

$\therefore$ Required probability $= \dfrac{6!}{6^6} = \dfrac{5}{324}$

3. Number of ways of arranging 3 persons between A and B

$= 2 \times ({}^{10}P_3) \times 7! = 2 \times 10!$

Total number of arrangement of 12 person is $11!$.

$\therefore$ Required probability $= \dfrac{2 \times 10!}{11!} = \dfrac{2}{11}$

4. Here, $n(S) = {}^{50}C_5$, $n(E) = {}^{30}C_2 \times {}^{19}C_2$

$$\therefore \quad P(E) = \frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$$

5. The probability of both drawing the common cards x, $P(x)$ = (probability of A drawing the card x and any other card y) $\times$ (probability of B drawing the card x and a card other than y)

$$\therefore \quad P(x) = \frac{{}^{51}C_1}{{}^{52}C_2} \times \frac{{}^{50}C_1}{{}^{52}C_2}, \forall x, \text{ where } x \text{ has 52 values.}$$

$\therefore$ Required probability $= \Sigma P(x)$

$$= 52 \times \frac{51 \times 50 \times 4}{52 \times 51 \times 52 \times 51} = \frac{50}{663}$$

6. Let p be the probability of success and q be the probability of failure, then $p + q = 1$ and $p = 3q$

$$\therefore \quad 4q = 1 \Rightarrow q = \frac{1}{4} \text{ and } p = \frac{3}{4}$$

$\therefore$ The probability that atleast 4 times favourable

$$= P(4) + P(5) + P(6) + P(7)$$

$$= {}^5C_4(3/4)^4(1/4) + {}^5C_5(3/4)^5(1/4) = \frac{1053}{2048}$$

7. Let E = event when each American man is seated adjacent to his wife and A = event when Indian man is seated adjacent of his wife.

Now, $\quad n(A \cap E) = (4!) \times (2!)^5$

Event when each American man is seated adjacent to his wife.

Again, $\quad n(E) = (5!) \times (2!)^4$

$$\therefore \quad P\left(\frac{A}{E}\right) = \frac{n(A \cap E)}{n(E)}$$

$$= \frac{(4!) \times (2!)^5}{(5!) \times (2!)^4} = \frac{2}{5}$$

8. Here, $n(S) = {}^{100}C_3$

As the AM of three numbers is 25, their sum $= 75$

$\therefore$ $n(E)$ = the number of integral solutions of

$$x_1 + x_2 + x_3 = 75,$$

where, $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

$= $ coefficient of x^{75} in $(x + x^2 + x^3 + \ldots)^3$

$= $ coefficient of x^{72} in $\left(\dfrac{1}{1-x}\right)^2$

$= $ coefficient of x^{72} in $(1-x)^{-3}$

$= {}^{74}C_{72}$

$$\therefore \quad P(E) = \frac{{}^{74}C_{72}}{{}^{100}C_3} = \frac{{}^{74}C_{72}}{{}^{100}C_{97}}$$

9. The total number of ways of ticking one or more alternatives out of 4 is ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$. Out of these 15 combinations only one combination is correct. The probability of ticking the alternative correctly at the first trial is $\dfrac{1}{15}$ that at the second trial is $\left(\dfrac{14}{15}\right)\left(\dfrac{1}{14}\right) = \dfrac{1}{15}$ and that at the third trial is $\left(\dfrac{14}{15}\right)\left(\dfrac{13}{14}\right)\left(\dfrac{1}{13}\right) = \dfrac{1}{15}$

Thus, the probability that the candidate will get marks on the question, if he is allowed up to three trials is

$$\frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{1}{5}$$

10. Favourable ways $= 3!\left(1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!}\right) = 2$

Total ways $= 3!$

$\therefore$ Probability $= \dfrac{2}{3!} = \dfrac{1}{3}$

11. $\overset{B}{I}W\overset{B}{I}W\overset{B}{I}W\overset{B}{I}---\overset{B}{I}W\overset{B}{I}W\overset{B}{I}$

$$n(S) = \frac{10!}{(7!)(3!)} \quad n(E) = {}^8C_3 = \frac{8!}{(3!)(5!)},$$

(because there are 8 places for 3 black balls.)

$$\therefore \quad P(E) = \frac{\dfrac{18}{(3!)(5!)}}{\dfrac{10!}{(7!)(3!)}} = \frac{(8!)(7!)}{(10!)(5!)} = \frac{7 \cdot 6}{10 \cdot 9} = \frac{7}{15}$$

12. A = the event of (x, y) belonging to the area $OTQPO$

B = the event of (x, y) belonging to the area $OSQRO$

$$P(A) = \frac{\text{ar}(OTQPO)}{\text{ar}(OPQRO)}$$

$$= \frac{\int_0^1 \sqrt{x}\, dx}{1 \times 1} = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$$

$$P(B) = \frac{\text{ar}(OSQRO)}{\text{ar}(OPQRO)} = \frac{\int_0^1 \sqrt{y}\, dy}{1 \times 1} = \frac{2}{3}$$

$$P(A \cap B) = \frac{\text{ar}(OTQS)}{\text{ar}(OPQRO)} = \frac{\int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx}{1 \times 1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$P(A) + P(B) = \frac{2}{3} + \frac{2}{3} \neq 1.$

So, A and B are not exhaustive.

$$P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \neq P(A \cap B)$$

So, A and B are not independent.

$$P(A \cup B) = 1, P(A) + P(B) = \frac{2}{3} + \frac{2}{3} \neq P(A \cup B)$$

So, A and B are not mutually exclusive.

13. The required probability $= 1 -$ probability of each receiving atleast one $= 1 - \frac{n(E)}{n(S)}$

Now, the number of integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

such that $x_1 \geq 1, x_2 \geq 1, \ldots, x_6 \geq 1$ gives $n(E)$ and the number of integral solutions of $x_1 + x_2 + \ldots x_5 + x_6 = 10$ such that $x_1 \geq 0$, $x_2 \geq 0, \ldots, x_6 \geq 0$ gives $n(S)$.

$\therefore$ The required probability

$$= 1 - \frac{{}^{10-1}C_{6-1}}{{}^{10+6-1}C_{6-1}} = 1 - \frac{{}^9C_5}{{}^{15}C_5} = \frac{137}{143}$$

14. $n(S) = 2 \times 2 \times 2 \times 2 = 16$, because each of the four places can be filled in 2 ways,

The zero determinants are

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix},$$

$$\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$\therefore \quad P(E) = \frac{8}{16} = \frac{1}{2}$

15. $P(E) \leq P(F) \Rightarrow \frac{n(E)}{n(S)} \leq \frac{n(F)}{n(S)} \Rightarrow n(E) \leq n(F)$

$P(E \cap F) > 0$

$\Rightarrow E \cap F \neq \phi$

Clearly, $x \in E \not\Rightarrow x \in F$, $x \in F$

$\not\Rightarrow x \in E, x \in E \not\Rightarrow x \notin F.$

16. Out of the numbers 00, 01, 02, 03, ..., 99 those numbers the product of whose digits is 16 are 28, 44, 82 *i.e.*, only 3.

$\therefore \quad P = P(E) = \frac{3}{100}$

and $q = P(\bar{E}) = 1 - P(E) = 1 - p = 1 - \frac{3}{100} = \frac{97}{100}$

$\therefore$ The binomial distribution is $\left(\frac{97}{100} + \frac{3}{100}\right)^5$

$\therefore$ The probability that the event occurs exactly three times

$$= {}^5C_3 \left(\frac{3}{100}\right)^3 \left(\frac{97}{100}\right)^2 = 10\,(0.03)^3\,(0.97)^2$$

17. Here, $P(M) = m, P(P) = p, P(C) = c$

The probability of passing in atleast one subject

$$= 1 - P(\bar{M}\bar{P}\bar{C})$$

$\therefore \quad \frac{75}{100} = 1 - (1-m)(1-p)(1-c)$

$\Rightarrow \quad \frac{3}{4} = m + p + c - mp - pc - mpc \quad \ldots(i)$

The probability of passing in atleast two subjects

$$= P(MPC) + P(MP\bar{C}) + P(M\bar{P}C) + P(\bar{M}PC)$$

$\therefore \quad \frac{1}{2} = mpc + mp(1-c) + m(1-p)\,c + (1-m)pc$

$\therefore \quad 2mpc = pm + mc + pc - \frac{1}{2} \quad \ldots(ii)$

The probability of passing in exactly two subjects

$$= P(MP\bar{C}) + P(M\bar{P}C) + P(\bar{M}PC)$$

$\therefore \quad \frac{2}{5} = mp(1-c) + m(1-p)\,c + (1-m)pc$

$= mp + mc + pc - 3mpc \quad \ldots(iii)$

From Eqs. (ii) and (iii),

$$2mpc + \frac{1}{2} = \frac{2}{5} + 3mpc \Rightarrow mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$\therefore$ From Eqs. (ii),

$$\frac{1}{5} = mp + mc + pc - \frac{1}{2}$$

$\Rightarrow \quad mp + mc + pc = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$

$\therefore$ From Eq. (i),

$$\frac{3}{4} = m + p + c - \frac{7}{10} + \frac{1}{10}$$

$\therefore \quad m + p + c = \frac{27}{20}$

18. The probability of A winning $= P(A) + P(\bar{A}\bar{B}A)$

$$+ P(\bar{A}\bar{B}\bar{A}\bar{B}A) + \ldots$$

$$= \frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\frac{1}{2} \ldots = \frac{1}{2} + \left(\frac{1}{3}\right)^3 + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$$

$\therefore$ The probability of B winning $= 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ The expectation of $A = ₹\,30 \times \frac{2}{3} = ₹\,20$

and the expectation of $B = ₹\,30 \times \frac{1}{3} = ₹\,10$

19. $P(2) + P(3) + P(4) = {}^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$

$$+ {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right) + {}^4C_4 \left(\frac{1}{6}\right)^4$$

$$= \frac{6 \times 25 + 20 + 1}{1296} = \frac{171}{1296} = \frac{19}{144}$$

20. $P(x = x) = {}^9C_x(.2)^{9-x}(.8)^x = {}^9C_6(.2)^3(.8)^6 + {}^9C_3(.2)^6(.8)^3$

$= {}^9C_6(.2)^3(.8)^6 + {}^9C_3(.2)^6(.8)^3$

$= {}^9C_6(.2)^3(.8)^3[(.8)^3 + (.2)^3] = {}^9C_6(.2)^3(.8)^3\left[\frac{4^3}{5^3} + \frac{1}{5^3}\right]$

$= {}^9C_6\left(\frac{1}{5}\right)^3\left(\frac{4^3}{5^3}\right)\cdot\frac{65}{5^3} = \frac{69888}{5^8}$

21. $P(x = x) = {}^6C_x\left(\frac{1}{2}\right)^6$

$P(x \le 4) = 1 - P(x > 4) = 1 - \frac{{}^6C_5}{26} - \frac{1}{26} = \frac{57}{64}$

22. $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$

$\Rightarrow \quad P(A \cap B) \ge P(A) + P(B) - 1 = \frac{3}{5} + \frac{2}{3} - 1 = \frac{4}{15}$

$P(A \cap B) \le \text{minima } \{P(A), P(B)\}$

$P(A \cap B) \le \frac{3}{5}$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \quad \frac{4}{15} \le P(A \cap B) \le \frac{3}{5}$ and $\frac{2}{5} \le P(A/B) \le \frac{9}{10}$

23. If $\left.\begin{matrix} ax + by = 0 \\ cx + dy = 0 \end{matrix}\right\}$ has a unique solution,

then $ad \ne bc$ which can occur in 6 ways.

Statement I is true

Statement II is always true

because the given equation has $x = 0$, $y = 0$ as trivial solution.

24. Since, $\frac{(x-10)(x-50)}{(x-30)} > 0$

$\Rightarrow \quad 10 < x < 30$ or $x > 50$

$\therefore \quad x = 11, \ldots, 29$ or $x = 51, 52, \ldots, 100$

$n(x) = 69$

$\therefore$ Required probability $= \frac{69}{100} = 0.69$

25. The number of ways of selecting two persons out of 20 is ${}^{20}C_2 = 190$.

The number of ways in which two selected persons together is 19.

$\therefore \quad$ Required probability $= 1 - \frac{19}{190} = 0.9$

26. Here, $p = q = \frac{1}{2}$

$\therefore$ Probability of appearing exactly five heads

$= {}^{12}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7 = {}^{12}C_{12-5}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5 = {}^{12}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5$

Probability of appearing exactly seven heads.

27. Probability of guessing a correct answer, $p = \frac{1}{3}$ and probability of guessing a wrong answer, $q = \frac{2}{3}$

$\therefore$ The probability of guessing a 4 or more correct answer

$= {}^5C_4\left(\frac{1}{3}\right)^4\cdot\frac{2}{3} + {}^5C_5\left(\frac{1}{3}\right)^5 = 5\cdot\frac{2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}$

28. Let A denotes the event that the minimum of the three selected numbers is 3 and B denotes the event that the maximum of the three selected numbers is 6.

If the maximum number is 6, then

(i) 6 should be one of the three selected numbers (1 way), and

(ii) The remaining two numbers should be less than 6 (*i.e.*, any 2 from 1 to 5) (${}^5C_2$ ways).

$\therefore \quad P(B) = \frac{1 \times {}^5C_2}{{}^8C_3} = \frac{10}{{}^8C_3}$

Similarly, $P(A \cap B) =$ The probability that the minimum number is 3 and the maximum number is 6. If the minimum number is 3 and the maximum number 6 is, then

(i) 6 should be one of the 3 selected numbers (1 way).

(ii) 3 should be one of the three selected numbers (1 ways) and

(iii) The remaining 1 number should lie between 3 and 6 (*i.e.*, any one of 4 and 5) (2 ways).

$\therefore \quad P(A \cap B) = \frac{1 \times 1 \times {}^2C_1}{{}^8C_3}$

$\therefore \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2}{10} = \frac{1}{5}$

29. $n = 5$ and $r \ge 1$

$\therefore \quad P(X = r) = {}^nC_r p^{n-r} q^r$,

$\therefore \quad P(X \ge 1) = 1 - P(X = 0)$

$= 1 - {}^5C_0 . p^5 . q^0 \ge \frac{31}{32} \Rightarrow p^5 \le 1 - \frac{31}{32} = \frac{1}{32}$

$\therefore \quad p \le \frac{1}{2}$ and $p \ge 0 \Rightarrow p \in \left[0, \frac{1}{2}\right]$

30. According to the given condition,

$$1 - \left(\frac{3}{4}\right)^n \ge \frac{9}{10}$$

$\Rightarrow \quad \left(\frac{3}{4}\right)^n \le 1 - \frac{9}{10} = \frac{1}{10} \Rightarrow \left(\frac{4}{3}\right)^n \ge 10$

$\Rightarrow \quad n(\log_{10} 4 - \log_{10} 3) \ge \log_{10} 10 = 1$

$\therefore \quad n \ge \frac{1}{\log_{10} 4 - \log_{10} 3}$

31. $S = \{00, 01, 02, \ldots, 49\}$

Let A be the event that sum of the digits on the selected ticket is 8, then

$$A = \{08, 17, 26, 35, 44\}$$

Let B be the event that the product of the digits is zero.

$$B = \{00, 01, 02, 03, \ldots, 09, 10, 20, 30, 40\}$$

$\therefore \quad A \cap B = \{08\}$

$\therefore$ Required probability $= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{50}}{\frac{14}{50}} = \frac{1}{14}$

Online Exam Held on 9-4-2013

JEE Main

Solved Paper 2013

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. If the lines $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z+1}{3}$ and $\frac{x+2}{2} = \frac{y-k}{3} = \frac{z}{4}$ are coplanar, then the value of k is

(a) $\frac{11}{2}$ (b) $-\frac{11}{2}$

(c) $\frac{9}{2}$ (d) $-\frac{9}{2}$

2. **Statement I** The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement II The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

(a) Statement I is true, Statement II is false

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(d) Statement I is false, Statement II is true

3. If $Z_1 \neq 0$ and Z_2 be two complex numbers such that $\frac{Z_2}{Z_1}$ is a purely imaginary number, then $\left|\frac{2Z_1 + 3Z_2}{2Z_1 - 3Z_2}\right|$ is equal to

(a) 2 (b) 5

(c) 3 (d) 1

4. If $\int \frac{dx}{x + x^7} = p(x)$ then, $\int \frac{x^6}{x + x^7}\, dx$ is equal to

(a) $\ln|x| - p(x) + c$ (b) $\ln|x| + p(x) + c$

(c) $x - p(x) + c$ (d) $x + p(x) + c$

5. If each of the lines $5x = 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then

(a) $a = 5$ and $b \notin (-1, 1)$

(b) $a = 1$ and $b \notin (-1, 1)$

(c) $a = 2$ and $b \in (-\infty, 1)$

(d) $a = 5$ and $b \in (-\infty, 1)$

6. If a, b, c are sides of a scalene triangle, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) non-negative (b) negative
(c) positive (d) non-positive

7. If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to

(a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2

8. Let $a_1, a_2, a_3, \ldots$ be an AP such that $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + a_3 + \ldots + a_q} = \frac{p^3}{q^3}$; $p \neq q$. then $\frac{a_6}{a_{21}}$ is equal to

(a) $\frac{41}{11}$ (b) $\frac{121}{1681}$
(c) $\frac{11}{41}$ (d) $\frac{121}{1861}$

9. **Statement I** The equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between I and II.

Statement II The function $f(x) = x \log x$ is an increasing function in $[1, 2]$ and $g(x) = 2 - x$ is a decreasing function in $[1, 2]$ and the graphs represented by these functions intersect at a point in $[1, 2]$.

(a) Statement I is true, Statement II is false
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is false, Statement II is true

10. Let $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ be three vectors. A vector of the type $\mathbf{b} + \lambda\,\mathbf{c}$ for some scalar λ, whose projection on $\mathbf{a}$ is of magnitude $\sqrt{\frac{2}{3}}$, is

(a) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (d) $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

11. The area bounded by the curve $y = \ln(x)$ and the lines $y = 0$, $y = \ln(3)$ and $x = 0$ is equal to

(a) 3 (b) $3 \ln(3) - 2$
(c) $3 \ln(3) + 2$ (d) 2

12. The values of a for which one root of the equation $x^2 - (a+1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is lesser than 2, are given by

(a) $3 < a < 10$ (b) $a \geq 10$
(c) $-2 < a < 3$ (d) $a \leq -2$

13. If the surface area of a sphere of radius r is increasing uniformly at the rate 8 cm^2/s, then the rate of change of its volume is

(a) constant (b) proportional to $\sqrt{r}$
(c) proportional to r^2 (d) proportional to r

14. The probability of a man hitting a target is $\frac{2}{5}$. He fires at the target k times (k, a given number). Then the minimum k, so that the probability of hitting the target at least once is more than $\frac{7}{10}$, is

(a) 3 (b) 5
(c) 2 (d) 4

15. Equation of the line passing through the points of intersection of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is

(a) $y - 3 = 0$ (b) $y + 3 = 0$
(c) $3y + 1 = 0$ (d) $3y - 1 = 0$

16. The sum of the series

$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ upto 10 terms is

(a) $\frac{18}{11}$ (b) $\frac{22}{13}$
(c) $\frac{20}{11}$ (d) $\frac{16}{9}$

17. A value of x for which $\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1} x)$, is

(a) $-\frac{1}{2}$ (b) 1
(c) 0 (d) $\frac{1}{2}$

18. If a and c are positive real numbers and the ellipse $\frac{x^2}{4c^2} + \frac{y^2}{c^2} = 1$ has four distinct points in common with the circle $x^2 + y^2 = 9a^2$, then

(a) $9ac - 9a^2 - 2c^2 < 0$ (b) $6ac - 9a^2 - 2c^2 < 0$
(c) $9ac - 9a^2 - 2c^2 > 0$ (d) $6ac + 9a^2 - 2c^2 > 0$

19. The vector $(\hat{\mathbf{i}} \times \mathbf{a} \cdot \mathbf{b})\,\hat{\mathbf{i}} + (\hat{\mathbf{j}} \times \mathbf{a} \cdot \mathbf{b})\,\hat{\mathbf{j}} + (\hat{\mathbf{k}} \times \mathbf{a} \cdot \mathbf{b})\,\hat{\mathbf{k}}$ is equal to

(a) $\mathbf{b} \times \mathbf{a}$ (b) $\mathbf{a}$
(c) $\mathbf{a} \times \mathbf{b}$ (d) $\mathbf{b}$

20. A light ray emerging from the point source placed at $P(1, 3)$ is reflected at a point Q in the axis of x. If the reflected ray passes through the point $R(6, 7)$, then the abscissa of Q is

(a) 1 (b) 3
(c) $\frac{7}{2}$ (d) $\frac{5}{2}$

21. The mean of a data set consisting of 20 observations is 40. If one observation 53 was wrongly recorded as 33, then the correct mean will be
(a) 41 (b) 49
(c) 40.5 (d) 42.5

22. Let $A = \{1, 2, 3, 4\}$ and $R : A \to A$ be the relation defined by $R = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$. The correct statement is
(a) R does not have an inverse.
(b) R is not a one to one function.
(c) R is an onto function.
(d) R is not a function.

23. If the three lines $x - 3y = p$, $ax + 2y = q$ and $ax = y = r$ from a right-angled triangle then
(a) $a^2 - 9a + 18 = 0$ (b) $a^2 - 6a - 12 = 0$
(c) $a^2 - 6a - 18 = 0$ (d) $a^2 - 9a + 12 = 0$

24. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, equals
(a) $4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $4\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$
(c) $32\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (d) $32\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

25. The ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^{15}$ is
(a) 7 : 16 (b) 7 : 64
(c) 1 : 4 (d) 1 : 32

26. The value of $\lim_{x \to 0} \frac{1}{x}\left[\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \frac{\pi}{4}\right]$ is
(a) 1 (b) $-\frac{1}{2}$
(c) 2 (d) 0

27. A vector **n** is inclined to x-axis at 45°, to y-axis at 60° and at an acute angle to z-axis. If **n** is a normal to a plane passing through the point $(\sqrt{2}, -1, 1)$, then the equation of the plane is
(a) $4\sqrt{2}x + 7y + z = 2$
(b) $\sqrt{2}\,x + y + z = 2$
(c) $3\sqrt{2}\,x - 4y - 3z = 7$
(d) $\sqrt{2}x - y - z = 2$

28. **Statement I** The statement $A \to (B \to A)$ is equivalent to $A \to (A \vee B)$.

Statement II The statement $\sim [(A \wedge B) \to (\sim A \vee B)]$ is a Tautology.
(a) Statement-I is true, Statement II is false.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(d) Statement I is false, Statement II is true.

29. A committee of 4 persons is to be formed from 2 ladies, 2 old men and 4 young men such that in includes at least 1 lady, at least 1 old man and at most 2 young men. Then the total number of ways in which this committee can be formed is
(a) 40 (b) 41
(c) 16 (d) 32

30. Let $f(x) = \frac{x^2 - x}{x^2 + 2x}$, $x \neq 0, -2$. then $\frac{d}{dx}[f^{-1}(x)]$ (wherever it is defined) is equal to
(a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$
(c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$

Answers

1. (a)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (b)
11. (d)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (d)
21. (a)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (b)

Hints & Solutions

1. Since, the lines $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+1}{3}$

and $\frac{x+2}{2}=\frac{y-k}{3}=\frac{z}{4}$ are coplanar.

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -2+1 & k-1 & 0+1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & k-1 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = -(4-0)-(k-1)(8-6)+1(6-2)=0$$

$$\Rightarrow \quad 5-2k+2+4=0$$

$$\Rightarrow \quad 2k=11 \quad \Rightarrow \quad k=\frac{11}{2}$$

2. **Statement I** let the equation of parabola whose axis is the axis of x and vertex at the origin is

$y^2=4ax$

$$2y\frac{dy}{dx}=4a \quad \Rightarrow \quad \frac{dy}{dx}=\frac{2a}{y} \quad \Rightarrow \quad \frac{dy}{dx}\propto\frac{1}{y}$$

(where $a \to$ parameter)

Statement II $y^2=4ax$ ...(i)

$$\Rightarrow \quad 2y\frac{dy}{dx}=4a \quad \Rightarrow \quad a=\frac{y}{2}\frac{dy}{dx} \text{ from (i)}$$

$$y^2=4x\cdot\frac{y}{2}\frac{dy}{dx}$$

$$\Rightarrow \quad y^2=2xy\frac{dy}{dx} \quad \Rightarrow \quad y=2x\frac{dy}{dx}$$

$\therefore$ Order $=1$ and degree $=1$

3. Given, $\frac{z_2}{z_1}$ is a purely imaginary *ie*, ni

Now, $$\left|\frac{2+3\frac{z_2}{z_1}}{2-3\frac{z_2}{z_1}}\right| = \frac{|2+3ni|}{|2-3ni|} = \frac{\sqrt{4+9n^2}}{\sqrt{4+9n^2}} = 1$$

4. Let $I=\int\frac{x^6}{x+x^7}dx=\int\frac{x^6}{x(1+x^6)}dx$

$$=\int\frac{(1+x^6)-1}{x(1+x^6)}dx$$

$$\Rightarrow \quad I=\int\frac{dx}{x}-\int\frac{dx}{x+x^7}=\log|x|-P(x)+c$$

5. On solving the equation of diameters, we get

$x=1$ and $y=1$

ie, centre of circle is $(1,1)$

Given also, $x^2+y^2-2(a^2-7a+11)x$

$$-2(a^2-6a+6)y+b^3+1=0 \quad \text{...(i)}$$

Centre of the circle is $\{a^2-7a+11, a^2-6a+6\}$

$$\therefore \quad a^2-7a+11=1 \text{ and } a^2-6a+6=1$$

$$\Rightarrow \quad a^2-7a+10=0 \text{ and } a^2-6a+5=0$$

$$\Rightarrow \quad a=2,5 \quad \text{and} \quad a=1,5$$

put $a=5$ in Eq. (i) , we get

$$x^2+y^2-2x-2y+b^3+1=0$$

$$\Rightarrow \quad (x-1)^2+(y-1)^3=(\sqrt{1-b^3})^2$$

Here, Radius $=\sqrt{1-b^3}>0$

$$\Rightarrow \quad b^3-1<0$$

$$\Rightarrow \quad (b-1)(b^2+1+b)<0$$

$$\Rightarrow \quad b<1 \text{ ie}, b\in(-\infty,1)$$

6. Let $\Delta=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$=a(bc-a^2)-b(b^2-ac)+c(ab-c^2)$$

$$=abc-a^3-b^3+abc+abc-c^3$$

$$=-(a^3+b^3+c^3-3abc)$$

$$=-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$=-\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

<0; where $a\neq b\neq c$

7. $x=\int_0^y\frac{dt}{\sqrt{1+t^2}}$, by Leibnitz Rule, we get

$$1=\frac{1}{\sqrt{1+y^2}}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\sqrt{1+y^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{y}{\sqrt{1+y^2}}\cdot\frac{dy}{dx}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{y}{\sqrt{1+y^2}}\cdot\sqrt{1+y^2}=y$$

8. Given that; $\frac{a_1+a_2+\ldots+a_p}{a_1+a_2+\ldots+a_r}=\frac{p^3}{q^3}$

$$\Rightarrow \quad \frac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_2+(q-1)d]}=\frac{p^3}{q^3}$$

where d be a common difference of an AP

$$\Rightarrow \quad \frac{(2a_1-d)+pd}{(2a_1-d)+qd}=\frac{p^2}{q^2}$$

9. Curve of $g(x) = 2 - x$

and $f(x) = x\log x$

from graph we observe that $f(x)$ and $g(x)$ are increasing and decreasing in the interval [1, 2] respectively atleast one point in [1, 2] will be exist where the both curve intersect each other.

10. Given, $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Now, we have; $\mathbf{b} + \lambda\mathbf{c} = (1+\lambda)\hat{\mathbf{i}} + (2+\lambda)\hat{\mathbf{j}} + (-1-2\lambda)\hat{\mathbf{k}}$

$\therefore$ Projection of $(\mathbf{b} + \lambda\mathbf{c})$ on $\mathbf{a} = \left|\dfrac{(\mathbf{b}+\lambda\mathbf{c})\cdot\mathbf{a}}{|\mathbf{a}|}\right| = \sqrt{\dfrac{2}{3}}$ (given)

$$\Rightarrow \quad \left|\frac{2(1-\lambda) - (2+\lambda) + (-1-2\lambda)}{\sqrt{4+1+1}}\right| = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \left|\frac{-\lambda - 1}{\sqrt{6}}\right| = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \lambda + 1 = 2 \Rightarrow \lambda = 1$$

$$\therefore \quad \mathbf{b} + \lambda\mathbf{c} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

11. $\therefore$ Required Area $= \int_0^{\log 3} x\,dy = \int_0^{\log 3} e^y dy$

$$= [e^y]_0^{\log 3} = [e^{\log 3} - e^0]$$

$$= 3 - 1 = 2$$

12. If one root is less than α other root is greater than β.

then; $D \geq 0$ and $f(\alpha) < 0$, $f(\beta) < 0$

Here, equation is $x^2 - (a+1)x + a^2 + a - 8 = 0$

Now, $D \geq 0 \Rightarrow (a+1)^2 - 4(a^2 + a - 8) \geq 0$

$$\Rightarrow \quad a^2 + 1 + 2a - 4a^2 - 4a + 32 \geq 0$$

$$\Rightarrow \quad -3a^2 - 2a + 33 \geq 0$$

$$\Rightarrow \quad 3a^2 + 2a - 33 \leq 0$$

$$\Rightarrow \quad 3a^2 + 11a - 9a - 33 \leq 0$$

$$\Rightarrow \quad a(3a + 11) - 3(3a + 11) \leq 0$$

$$\Rightarrow \quad (3a + 11)(a - 3) \leq 0$$

$$\begin{array}{ccccc} & + & & - & & + \\ \hline -\infty & & -11/3 & & 3 & & +\infty \end{array}$$

$$\Rightarrow \quad a \in \left[\frac{-11}{3}, 3\right] \quad \ldots(i)$$

and $\quad f(2) < 0$

$$\Rightarrow \quad 4 - (a+1)\cdot 2 + a^2 + a - 8 < 0$$

$$\Rightarrow \quad 4 - 2a - 2 + a^2 + a - 8 < 0$$

$$\Rightarrow \quad a^2 - a - 6 < 0$$

$$\begin{array}{ccccc} & + & & - & & + \\ \hline -\infty & & -2 & & 3 & & +\infty \end{array}$$

$$\Rightarrow \quad a^2 - 3a + 2a - 6 < 0$$

$$\Rightarrow \quad a(a-3) + 2(a-3) < 0$$

$$\Rightarrow \quad (a-3)(a+2) < 0$$

$$\Rightarrow \quad a \in (-2, 3) \quad \ldots(ii)$$

$\therefore$ From Eq. (i) and (ii), we get

$$a \in (-2, 3)$$

13. Surface area, $s = 4\pi r^2$

$$\frac{ds}{dt} = 8\pi r\frac{dr}{dt} \Rightarrow 8\pi r\frac{dr}{dt} = 8 \qquad \left[\because \frac{ds}{dt} = 8 \text{ cm}^2/\text{s}\right]$$

$$\frac{1}{dt} = \frac{1}{\pi r}$$

Now, $\quad v = \dfrac{4\pi r^3}{3} \Rightarrow \dfrac{dv}{dt} = 4\pi r^2\dfrac{dr}{dt} = 4\pi r^2 \cdot \dfrac{1}{\pi r} = 4r$

$$\therefore \quad \frac{dv}{dt} \propto r$$

15. On solving both the equations, we get

$$\frac{8y}{3} + y^2 = 1$$

$$\Rightarrow \quad 3y^2 + 8y - 3 = 0$$

$$\Rightarrow \quad (3y - 1)(y + 3) = 0$$

$$\Rightarrow \quad y = -3, \frac{1}{3} \text{ here } y \neq -3$$

$$\therefore \quad \text{At } y = \frac{1}{3}$$

$$\Rightarrow \quad x = \pm 2\sqrt{\frac{2}{3}}$$

$\therefore$ Point of intersection is $\left(2\sqrt{\dfrac{2}{3}}, \dfrac{1}{3}\right)$ and $\left(-2\sqrt{\dfrac{2}{3}}, \dfrac{1}{3}\right)$

From option (d); $3y - 1 = 0$ is the required equation which satisfied the intersection points.

16. nth term of the series is, $T_n = \dfrac{1}{\dfrac{n(n+1)}{2}} = \dfrac{2}{n(n+1)}$

$$\Rightarrow \quad T_n = 2\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$$\Rightarrow \quad T_1 = 2\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right), T_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\ldots T_{10} = 2\left(\frac{1}{10} - \frac{1}{11}\right)$$

$$\therefore \quad S_{10} = T_1 + T_2 + \ldots + T_{20}$$

$$S_{10} = 2\left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{10} - \frac{1}{11}\right]$$

$$= 2\left(1 - \frac{1}{11}\right) = 2 \cdot \frac{10}{11} = \frac{20}{11}.$$

19. $(\hat{\mathbf{i}} \times \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{i}} + (\hat{\mathbf{j}} \times \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{j}} + (\hat{\mathbf{k}} \times \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{k}}$

$$= [\hat{\mathbf{i}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{i}} + [\hat{\mathbf{j}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{j}} + [\hat{\mathbf{k}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{k}}$$

Let $\quad \mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

$$\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$$

$$\therefore \quad [\hat{\mathbf{i}}\ \mathbf{a}\ \mathbf{b}] = \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)$$

$$[\hat{\mathbf{j}}\ \mathbf{a}\ \mathbf{b}] = \begin{vmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_1a_3 - a_1b_3)$$

and $$[\hat{\mathbf{k}}\ \mathbf{a}\ \mathbf{b}] = \begin{vmatrix} 0 & 0 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_1b_2 - a_2b_1)$$

$$\therefore \quad [\hat{\mathbf{i}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{i}} + [\hat{\mathbf{j}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{j}}\ [\hat{\mathbf{k}}\ \mathbf{a}\ \mathbf{b}]\hat{\mathbf{k}}$$

$$= (a_2b_3 - b_2a_3)\hat{\mathbf{i}} + (b_1a_3 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}$$

$$= \mathbf{a} \times \mathbf{b}$$

20. Here, $AS \perp OX$

It means AS bisect the angle PAR.

then $\quad PAS = RAS$

$\Rightarrow \quad RAX = PAO = \theta$ (let)

$\Rightarrow \quad XAP = 180° - \theta$

Slope of $AR = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-0}{6-k}$ ...(i)

Slope of $AP = \tan(180° - \theta) = -\tan\theta$

$= \dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{3-0}{1-k}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii),

$$\frac{7}{6-k} = -\frac{3}{1-k} \Rightarrow 7 - 7k = -18 + 3k$$

$$10k = 25 \quad \Rightarrow \quad k = \frac{5}{2}$$

Hence, the coordinate of A is $\left(\dfrac{5}{2}, 0\right)$

21. Given, $\dfrac{\Sigma x_{20}}{20} = 40 \Rightarrow \Sigma x_{20} = 20 \times 40 = 800$

$\therefore \quad \Sigma x_{20} = 800 - 33 + 53 = 820$

$\Rightarrow \quad \dfrac{\Sigma x_{20}}{20} = \dfrac{820}{20}$

$\therefore$ New mean = 41

22. Here, we see that every element of codomain there exist a preimage, hence it is onto.

23. ***Case* I** Let line $l_1 \equiv x - 3y = p$ and $l_2 \equiv ax + 2y = p$ are perpendicular, then

$$\frac{1}{3} \times -\frac{a}{2} = -1 \quad \Rightarrow \quad a = 6.$$

***Case* II** Let line $l_2 \equiv ax + 2y = p$ and $l_3 \equiv ax + y = r$ are perpendicular, then

$$\frac{-a}{2} \times -a = -1 \Rightarrow a^2 = -2 \text{ (not possible)}$$

***Case* III** let line $l_3 \equiv ax + y = r$ and $l_1 \equiv x - 3y = p$ are perpendicular, then

$$-a \times \frac{1}{3} = -1 \Rightarrow a = 3$$

$\therefore$ Formation of quadratic equation in a, whose roots are 3 and 6.

$a^2 - (6+3)a + (6 \cdot 3) = 0 \Rightarrow a^2 - 9a + 18 = 0$

24. $\because A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}, A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & +7 \end{bmatrix}$

$\therefore$

$$A^2 + 4A - 5I = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix} = 4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

25. $T_{r+1} = {}^{15}C_r (x^2)^{15-r} \cdot \left(\dfrac{2}{x}\right)^r = {}^{15}C_r\, x^{30-2r} \cdot 2^r \cdot x^{-r}$

$= {}^{15}C_r \cdot x^{30-3r} \cdot 2^r$ ...(i)

For coefficient of x^{15}, put $30 - 3r = 15 \Rightarrow 3r = 15 \Rightarrow r = 5$

$\therefore$ Coefficient of $x^{15} = {}^{15}C_5 \cdot 2^5$

For coefficient of independent of x *ie*, x^0, put $30 - 3r = 0 \Rightarrow r = 10$

$\therefore$ Coefficient of $x^0 = {}^{15}C_{10} \cdot 2^{10}$

By condition $\Rightarrow \dfrac{\text{Coefficient of } x^{15}}{\text{Coefficient of } x^0} = \dfrac{{}^{15}C_6 \cdot 2^5}{{}^{15}C_{10} \cdot 2^{10}}$

$= \dfrac{{}^{15}C_{10} \cdot 2^5}{{}^{15}C_{10} \cdot 2^{10}} = 1:32$

26. $\lim\limits_{x \to 0} \dfrac{1}{x}\left\{\tan^{-1}\left(\dfrac{x+1}{2x+1}\right) - \dfrac{\pi}{4}\right\} = \lim\limits_{x \to 0} \dfrac{1}{x}\left\{\tan^{-1}\left(\dfrac{x+1}{2x+1}\right) - \tan^{-1}(1)\right\}$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \tan^{-1}\left\{\frac{\dfrac{x+1}{2x+1} - 1}{1 + \dfrac{x+1}{2x+1}}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \tan^{-1}\left(\frac{x+1-2x-1}{2x+1+x+1}\right)$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \tan^{-1}\left(\frac{-x}{4x+2}\right) \to \text{form}\left(\frac{0}{0}\right)$$

Use L-hospital rule,

$$= \lim_{x \to 0} \frac{1}{1 + \dfrac{x^2}{(4x+2)^2}} \times \left[-\left(\frac{4x+2-4x}{(4x+2)^2}\right)\right]$$

$$= \lim_{x \to 0} -\frac{(+2)}{x^2 + (4x+2)^2} = -\frac{(+2)}{0 + (0+2)^2} = \frac{-2}{4} = \frac{-1}{2}$$

27. $\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2 45° + \cos^2 60° + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow \cos\gamma = \frac{1}{2}$

$\therefore$ Direction Ratio's of normal to the plane is $< \cos 45°, \cos 60°, \frac{1}{2} > = < \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} >$

$\therefore$ Equation of plane passing through $(\sqrt{2}, -1, 1)$

$$\frac{1}{\sqrt{2}}(x - \sqrt{2}) + \frac{1}{2}(y + 1) + \frac{1}{2}(z - 1) = 0$$

$\Rightarrow \quad 2(x - \sqrt{2}) + \sqrt{2}(y + 1) + \sqrt{2}(z - 1) = 0$

$\Rightarrow \quad \sqrt{2}(x - \sqrt{2}) + (y + 1) + d(z - 1) = 0$

$\Rightarrow \quad \sqrt{2}x - 2 + y + 1 + z - 1 = 0$

$\Rightarrow \quad \sqrt{2}x + y + z = 2$

28.

A	B	$A \vee B$	$B \to A$	$A \wedge B$	$\sim A$	$\sim A \vee B$	$A \to (A \vee B)$
T	T	T	T	T	F	T	T
T	F	T	T	F	F	F	T
F	T	T	F	F	T	T	T
F	F	F	T	F	T	T	T

$A \to (B \to A)$	$(A \wedge B) \to (\sim A \vee B)$	$\sim [(A \wedge B) \to (\sim A \vee B)]$
T	T	F
T	T	F
T	T	F
T	T	F

$\therefore A \to (B \to A)$ is equivalent to $A \to (A \vee B)$

But $\sim[(A \wedge B) \to (\sim A \vee B)]$ is not a tautology *ie*, it is contradiction.

29. Ladies = 2, old men = 2, young men = 4

***Case* I** 1 lady, 1 old man, 2 young man

$$= {}^2C_1 \cdot {}^2C_1 \cdot {}^4C_2 = 2 \cdot 2 \cdot 6 = 24$$

***Case* II** 2 ladies, 1 old man, 1 young man

$$= {}^2C_2 \cdot {}^2C_1 \cdot {}^4C_1 = 1 \cdot 2 \cdot 4 = 8$$

***Case* III** 1 lady, 2 old man, 1 young man

$$= {}^2C_1 \cdot {}^2C_2 \cdot {}^4C_1 = 2 \cdot 1 \cdot 4 = 8$$

***Case* IV** 2 lady, 2 old man, 0 young man

$$= {}^2C_2 \cdot {}^2C_2 \cdot {}^4C_0 = 1 \cdot 1 \cdot 1 = 1$$

$\therefore$ Required noise of ways = 24 + 8 + 8 + 1 = 41

30. Let $y = \frac{x^2 - x}{x^2 + 2x}$

$\Rightarrow \quad x = \frac{2y + 1}{-y + 1}; x \neq 0$

$\Rightarrow \quad f^{-1}(x) = \frac{2x + 1}{-x + 1}$

$\therefore \quad \frac{d}{dx}\{f^{-1}(x)\} = \frac{(-x + 1) \cdot 2 - (2x + 1)(-1)}{(-x + 1)^2}$

$$= \frac{3}{(-x + 1)^2}$$

Online Exam Held on 22-4-2013

JEE Main

Solved Paper 2013

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. The number of ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question, is

(a) ${}^{30}C_7$ (b) ${}^{21}C_8$

(c) ${}^{21}C_7$ (d) ${}^{30}C_8$

2. If the system of linear equations

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then

(a) $a = 8$, b can be any real number

(b) $b = 15$, a can be any real number

(c) $a \in R - \{8\}$ and $b \in R - \{15\}$

(d) $a = 8$, $b = 15$

3. Given sum of the first n terms of an AP is $2n + 3n^2$. Another AP is formed with the same first term and double of the common difference, the sum of n terms of the new AP is

(a) $n + 4n^2$ (b) $6n^2 - n$

(c) $n^2 + 4n$ (d) $3n + 2n^2$

4. **Statement I** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$

Statement II The function x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b)

(a) Statement I is true, Statement II is false

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(d) Statement I is false, Statement II is true

5. Mean of 5 observations is 7. If four of these observations are 6, 7, 8, 10 and one is missing then the variance of all the five observations is

(a) 4 (b) 6

(c) 8 (d) 2

6. The area of the region (in sq. units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$, is

(a) $\frac{7}{9}$ (b) $\frac{14}{3}$ (c) $\frac{7}{3}$ (d) $\frac{14}{9}$

7. If the x-intercept of some line L is double as that of the line, $3x + 4y = 12$ and the y-intercept of L is half as that of the same line, then the slope of L is

(a) -3 (b) $-\frac{3}{8}$

(c) $-\frac{3}{2}$ (d) $-\frac{3}{16}$

8. The sum $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \ldots$ upto 11-terms is

(a) $\frac{7}{2}$ (b) $\frac{11}{4}$

(c) $\frac{11}{2}$ (d) $\frac{60}{11}$

9. The integral $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\, dx$ is equal to

(a) $\log 2\sqrt{2}$ (b) $\log 2$

(c) $2 \log 2$ (d) $\log \sqrt{2}$

10. Let $R = \{(3, 3), (5, 5), (9, 9), (12, 12), (5, 12), (3, 9), (3, 12)$ $(3, 5)\}$ be a relation on the set $A = \{3, 5, 9, 12\}$. Then, R is

(a) reflexive, symmetric but not transitive

(b) Symmetric, transitive but not reflexive

(c) an equivalence relation

(d) reflexive, transitive but not symmetric.

11. If a complex number z satisfies the equation $z + \sqrt{2}\,|z+1| + i = 0$, then $|z|$ is equal to

(a) 2 (b) $\sqrt{3}$

(c) $\sqrt{5}$ (d) 1

12. If the 7th term in the binomial expansion of $\left(\frac{3}{\sqrt[3]{84}} + \sqrt{3}\ln x\right)^9$, $x > 0$, is equal to 729, then x can be

(a) e^2 (b) e

(c) $\frac{e}{2}$ (d) $2e$

13. **Statement I** The line $x - 2y = 2$ meets the parabola, $y^2 + 2x = 0$ only at the point $(-2, -2)$

Statement II The line $y = mx - \frac{1}{2m}$ $(m \neq 0)$ is tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$

(a) Statement I is true, Statement II is false

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(d) Statement I is false, Statement II is true

14. If a circle C passing through $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$, externally at a point $(1, -1)$, then the radius of the circle C is

(a) 5 (b) $2\sqrt{5}$

(c) 4 (d) $\sqrt{57}$

15. Let Q be the foot of perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and R be a point $(-1, 1, -6)$ on the plane. Then, length QR is

(a) $\sqrt{14}$ (b) $\sqrt{\frac{19}{2}}$

(c) $3\sqrt{\frac{7}{2}}$ (d) $\frac{3}{\sqrt{2}}$

16. Given two independent events, if the probability that exactly one of them occurs is $\frac{26}{49}$ and the probability that none of them occurs is $\frac{15}{49}$, then the probability of more probable of the two events is

(a) $\frac{4}{7}$ (b) $\frac{6}{7}$

(c) $\frac{3}{7}$ (d) $\frac{5}{7}$

17. The statement $p \to (q \to p)$ is equivalent to

(a) $p \to q$ (b) $p \to (p \vee q)$

(c) $p \to (p \to q)$ (d) $p \to (p \wedge q)$

18. The maximum area of a right angled triangle with hypotenuse h is

(a) $\frac{h^3}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$

(c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$

19. If $\int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1} x}\, dx = A(x)\, e^{\cot^{-1} x} + C$, then $A(x)$ is equal to

(a) $-x$ (b) x

(c) $\sqrt{1-x}$ (d) $\sqrt{1+x}$

20. If two vertices of an equilateral triangle are $A(-a, 0)$ and $B(a, 0)$, $a > 0$, and the third vertex C lies above x-axis, then the equation of the circumcircle of ΔABC is

(a) $3x^2 + 3y^2 - 2\sqrt{3}ay = 3a^2$

(b) $3x^2 + 3y^2 - 2ay = 3a^2$

(c) $x^2 + y^2 - 2ay = a^2$

(d) $x^2 + y^2 - \sqrt{3}ay = a^2$

21. The acute angle between two lines such that the direction cosines 1, m, n of each of them satisfy the equations $1 + m + n = 0$ and $1^1 + m^2 - n^2 = 0$ is

(a) 15°
(b) 30°
(c) 60°
(d) 45°

22. Consider the differential equation $\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)}$

Statement I The substitution $z = y^2$ transforms the above equation into a first order homogenous differential equation.

Statement II The solution of this differential equation is $y^2 e^{-y^2/x} = C$.

(a) Both Statement are false
(b) Statement I is true and Statement II is false
(c) Statement I is false and Statement II is true
(d) Both Statement are true

23. The number of solutions of the equation, $\sin^{-1} x = 2 \tan^{-1} x$ (in principal values) is

(a) 1
(b) 4
(c) 2
(d) 3

24. For $a > 0, t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$.

Then, $1 + \left(\frac{dy}{dx}\right)^2$ equals

(a) $\frac{x^2}{y^2}$
(b) $\frac{y^2}{x^2}$
(c) $\frac{x^2 + y^2}{y^2}$
(d) $\frac{x^2 + y^2}{x^2}$

25. If p, q, r are 3 real numbers satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ \ 0\ \ 1]$, then $2p + q - r$ equals

(a) -3
(b) -1
(c) 4
(d) 2

26. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors satisfying $\hat{\mathbf{a}} - \sqrt{3}\,\hat{\mathbf{b}} + \hat{\mathbf{c}} = \mathbf{0}$, then the angle between the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ is

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$
(d) $\frac{\pi}{2}$

27. Let the equations of two ellipses be

$E_1 : \frac{x^2}{3} + \frac{y^2}{2} = 1$ and

$E_2 : \frac{x^2}{16} + \frac{y^2}{b^2} = 1$. If the product of their eccentricities is $\frac{1}{2}$, then the length of the minor axis of ellipse E_2 is

(a) 8
(b) 9
(c) 4
(d) 2

28. If α and β are roots of the equation $x^2 + px + 3\frac{p}{4} = 0$, such that $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set

(a) $\{2, -5\}$
(b) $\{-3, 2\}$
(c) $\{-2, 5\}$
(d) $\{3, -5\}$

29. **Statement I** The number of common solution of the trigonometric equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is two.

Statement II The number of solutions of the equation, $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, \pi]$ is two.

(a) Statement I is true, Statement II is false
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(d) Statement I is false, Statement II is true

30. Let $f(x) = -1 + |x - 2|$ and $g(x) = 1 - |x|$, then the set of all points where fog is discontinuous is

(a) $\{0, 2\}$
(b) $\{0, 1, 2\}$
(c) $\{0\}$
(d) an empty set

Answers

1. (c)	**2.** (d)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (a)	**10.** (d)
11. (c)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (a)
21. (c)	**22.** (d)	**23.** (d)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (d)

Hints & Solutions

1. Let $x_1, x_2, \ldots, x_8$ denote the question.

$\therefore \quad x_1 + x_2 + \ldots + x_8 = 30$

Also, $\quad x_1, x_2, \ldots, x_8 \geq 2$

Let $\quad u_1 = x_1 - 2, u_2 = x_2 - 2 \ldots u_8 = x_8 - 2$

$\therefore \quad (u_1 + 2 + u_2 + 2 + \ldots + u_8 + 2) = 30$

$\Rightarrow \quad u_1 + u_2 + \ldots + u_8 = 14$

$\therefore$ Total number of solutions $= {}^{14+8-1}C_{8-1}$

$= {}^{21}C_7$

2. For consistent in $|A| = 0$ and (adj A) $B = 0$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{vmatrix} = 0$$

$\Rightarrow \quad 1(3a - 25) - 2(a - 10) + 3(5 - 6) = 0$

$\Rightarrow \quad a = 8$

On solving (adj A)$B = 0$, we get, $b = 15$

3. Here, $T_1 = S_1 = 2(1) + 3(1)^2 = 5$

$T_2 = S_2 - S_1$

$= 16 - 5 = 11$

$T_3 = S_3 - S_2$

$= 33 - 16 = 17$

$\therefore$ Sequence 5, 11, 17

$a = 5, d = 6$

For new AP,

$A = 5, D = 2 \times 6 = 12$

$S'_n = \frac{n}{2}[2 \times 5 + (n-1)12] = 6n^2 - n$

4. **Statement I** x^2 is an increasing function and $e^x + e^{-x}$ is also an increasing function. Therefore, $x^2(e^x + e^{-x})$ is an increasing function.

Statement II Here x^2e^x is an increasing function, but x^2e^{-x} is not always increasing.

5. Mean, $7 = \frac{6 + 7 + 8 + 10 + x}{5}$

$\Rightarrow \quad x = 4$

Variance, $= \frac{(6-7)^2 + (7-7)^2 + (8-7)^2 + (10-7)^2 + (4-7)^2}{5}$

$= \frac{1^2 + 0 + 1^2 + 3^2 + 3^2}{5}$

$= \frac{20}{5} = 4$

6. Area $= \int_1^4 \frac{\sqrt{y}}{3}\, dy = \frac{1}{3}\left[\frac{y^{3/2}}{3/2}\right]_1^4$

$= \frac{2}{9}(4^{3/2} - 1^{3/2}) = \frac{2}{9} \times 7 = \frac{14}{9}$

7. $\frac{x}{4} + \frac{y}{3} = 1$

For line L, x intercept $= 2 \times 4 = 8$

y intercept $= \frac{1}{2} \times 3 = \frac{3}{2}$

$\therefore$ Line L is $\frac{x}{8} + \frac{y}{3/2} = 1$

Slope, $m = -\frac{3}{16}$

8. $T_n = \frac{2n+1}{(1^2 + 2^2 + \ldots n^2)} = \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$

$= 6\left(\frac{1}{n} - \frac{1}{(n+1)}\right)$

$T_1 = 6\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 6\left[\frac{1}{2} - \frac{1}{3}\right] \ldots T_{11} = 6\left[\frac{1}{11} - \frac{1}{12}\right]$

$\therefore \quad S = 6\left[\frac{1}{1} - \frac{1}{12}\right] = \frac{6 \times 11}{12} = \frac{11}{2}$

9. $I = \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\, dx = \int_{7\pi/4}^{2\pi} |\tan x|\, dx + \int_{2\pi}^{7\pi/3} |\tan x|\, dx$

$= -\int_{7\pi/4}^{2\pi} \tan x\, dx + \int_{2\pi}^{7\pi/3} \tan x\, dx$

10. $\forall a \in A, (a, a) \in R$, so it is reflexive $(5, 12) \in R$ but $(12, 5) \notin R$, so it is not symmetric.

$(3, 5), (5, 12) \in R \Rightarrow (3, 12) \in R$, so it is transitive.

11. $(x + iy) + \sqrt{2}\,|x + iy + 1| + i = 0$ (put $z = x + iy$)

$\Rightarrow \quad (x + iy) + \sqrt{2}\sqrt{(x+1)^2 + y^2} + i = 0$

$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2 + y^2} = 0$ and $y + 1 = 0$

$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2 + (-1)^2} = 0$ and $y = -1$

$\Rightarrow \quad x^2 = 2[(x+1)^2+1] \quad \Rightarrow \quad x^2$

$\Rightarrow \quad x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x+2)^2 = 0 \Rightarrow x = -2$

$\therefore \quad z = -2 - i$

$\therefore \quad |z| = \sqrt{4+1} = \sqrt{5}$

12. $T_7 = {}^9C_6 \left(\frac{3}{3\sqrt{84}}\right)^3 (\sqrt{3}\ln x)^6 = 729$

$\Rightarrow \quad \frac{84 \times 3^3}{84} \times 3^3 \times (\ln x)^6 = 729$

$= (\ln x)^6 = 1 \Rightarrow x = e$

13. I. Intersection point of $x - 2y = 2$ and $y^2 + 2x = 0$ is $(-2, -2)$

II. Here, $a = -\frac{1}{2}$

Equation of tangent $y = mx + \frac{a}{m}$

$y = mx - \frac{1}{2m}$

And point of contact is $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$.

14. Let centre and radius of circle C is (h, k), and r_1 respectively.

$\therefore$ centre and radius of circle

$(-2, 3)$

and $\quad r = \sqrt{4+9+12} = 5$

15. Let foot of perpendicular $Q(x, y, z)$ from $O(0, 0, 0)$

$$\frac{x-0}{4} = \frac{y-0}{-3} = \frac{z-0}{1} = -\frac{\{4(0) - 3(0) + 1(0) + 13)\}}{4^2 + 3^2 + 1^2}$$

$\Rightarrow \quad \frac{x}{4} = \frac{y}{-3} = \frac{z}{1} = \frac{-13}{26} = -\frac{1}{2}$

$x = -2, y = \frac{3}{2}, z = -\frac{1}{2}$

$\therefore \quad Q\left(-2, \frac{3}{2}, -\frac{1}{2}\right)$

$\therefore \quad PQ = \sqrt{(-1+2)^2 + \left(1 - \frac{3}{2}\right)^2 + \left(-6 + \frac{1}{2}\right)^2}$

$= \sqrt{1 + \frac{1}{4} + \frac{121}{4}} = \frac{\sqrt{126}}{2} = 3\sqrt{\frac{7}{2}}$

16. $P(A \cap \overline{B}) + P(\overline{A} \cap B) = \frac{26}{49}$

$\Rightarrow P(A)\,P(\overline{B}) + P(\overline{A})\,P(B) = \frac{26}{49}$ ...(i)

Also, $\quad P(\overline{A} \cap \overline{B}) = \frac{15}{49}$

$\Rightarrow \quad P(\overline{A})\,P(\overline{B}) = \frac{15}{49}$

17.

p	q	$q \to p$	$p \to (q \to p)$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T

18. Area of triangle, $\Delta = \frac{1}{2} x\sqrt{h^2 - x^2}$

$$\frac{d\Delta}{dx} = \frac{1}{2}\left[\sqrt{h^2 - x^2} + \frac{x(-2x)}{2\sqrt{h^2 - x^2}}\right] = 0$$

$\Rightarrow \quad x = \frac{h}{\sqrt{2}}$

$\frac{d^2\Delta}{dx^2} < 0$ at $x = \frac{h}{\sqrt{2}}$

$\therefore \quad \Delta = \frac{1}{2} + \frac{h}{\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}} = \frac{h^2}{4}$

19. LHS $= \int \left[\frac{x^2+1}{x^2+1} - \frac{x}{x^2+1}\right] e^{\cot^{-1} x} dx$

$= \int 1 \cdot e^{\cot^{-1} x} dx - \int \frac{x}{x^2+1} e^{\cot^{-1} x}\, dx$

$= xe^{\cot^{-1} x} - \int x \cdot e^{\cot^{-1} x}\left(-\frac{1}{1+x^2}\right) dx$

$- \int \frac{x}{1+x^2} e^{\cot^{-1} x} dx + c = xe^{\cot^{-1} x} + C$

20. In an equilateral triangle circumcentre lies on the centroid of a triangle.

$\therefore$ Centre of circle = Centroid of triangle $= \left(0, \frac{a}{\sqrt{2}}\right)$

$r = \frac{2a}{\sqrt{3}}$

$\therefore$

$(x-0)^2 + \left(y - \frac{a}{\sqrt{3}}\right)^2 = \left(\frac{2a}{\sqrt{3}}\right)^2$

21. $\because l + m + n = 0, l^2 + m^2 - n^2 = 0$ and $l^2 + m^2 + n^2 = 1$

On solving, we get, $m = \pm\frac{1}{\sqrt{2}}, n = \pm\frac{1}{\sqrt{2}}$ and $l = 0$

$\therefore \quad \theta = \frac{\pi}{3}$ or $\frac{\pi}{2}$

22. **Statement I** Put $z = y^2 \Rightarrow \frac{dz}{dx} = 2y\frac{dy}{dx}$

$\therefore \quad \frac{dz}{dx} = \frac{2y^4}{2(xy^2 - x^2)} = \frac{z^2}{(xz - x^2)}$

Put $\quad z = vx \Rightarrow \frac{dz}{dx} = v + \frac{xdv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{v^2x^2}{xvx - x^2} = \frac{v^2}{v-1}$

$\Rightarrow \quad \frac{x\,dv}{dx} = \frac{v^2 - v^2 + v}{v-1} \Rightarrow \frac{x\,dv}{dx} = \frac{v}{(v-1)}$

$\Rightarrow \quad \left(\frac{v-1}{v}\right) dv = \frac{dx}{x}$

$\Rightarrow \quad \int \left(1 - \frac{1}{v}\right) dv = \int \frac{dx}{x} - \log c$

$\Rightarrow \quad v - \log v = \log x - \log c$

$\Rightarrow \quad = \frac{z}{x} - \log \frac{z}{x} = \log x - \log c$

$\Rightarrow \quad = \frac{y^2}{x} - \log \frac{y^2}{x} = \log x - \log c$

$\Rightarrow \quad = \frac{y^2}{x} = \log \frac{y^2}{x} \cdot x - \log c \Rightarrow e^{y^2/x} = y^2/c$

$\Rightarrow \quad y^2 = c \cdot e^{y^2/x} \Rightarrow y^2 \cdot e^{-y^2/x} = c$

23. $\sin^{-1} x = 2 \tan^{-1} x$

$\Rightarrow \quad \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \frac{\tan^{-1} 2x}{1-x^2}$

$\Rightarrow \quad x(1-x^2) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad x\sqrt{1-x^2}\,[\sqrt{1-x^2} - 2] = 0$

$\Rightarrow \quad x\sqrt{1-x^2} = 0$ and $\sqrt{1-x^2} = 2$

$\Rightarrow \quad x = 0, \pm 1$ and $1 - x^2 = 4$

$\Rightarrow \quad x^2 = -3$ which in not possible

Hence, three solutions exist.

24. $\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1} t}}} \left(a^{\sin^{-1} t} \times \frac{1}{\sqrt{1-t^2}} \right)$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \left(a^{\cos^{-1} t} \times \frac{-1}{\sqrt{1-t^2}} \right)$$

$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{a^{\sin^{-1} t}}}{\sqrt{a^{\cos^{-1} t}}} \left(\frac{a^{\cos^{-1} t}}{a^{\sin^{-1} t}} \times 1 \right) = -\frac{a^{\sqrt{\cos^{-1} t}}}{a^{\sqrt{\sin^{-1} t}}}$

$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^{\cos^{-1} t}}{a^{\sin^{-1} t}} = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2}$

25. $[3p + 3q + 2r \quad 4p + 2q + 0 \quad p + 3q + 2r] = [3 \quad 0 \quad 1]$

$\Rightarrow 3p + 3q + 2r = 3, 4p + 2q = 0, p + 3q + 2r = 1$

$\Rightarrow \quad p = 1, q = -2, r = 3$

$\therefore \; 2p + q - r = 2 - z - 3 = -3$

26. $\sqrt{3}\,\hat{\mathbf{b}} = (\hat{\mathbf{a}} + \hat{\mathbf{c}})$

$\Rightarrow \quad 3|\hat{\mathbf{b}}|^2 = |\hat{\mathbf{a}}|^2 + |\hat{\mathbf{c}}|^2 + 2\,\hat{\mathbf{a}} \cdot \hat{\mathbf{c}}$

$\Rightarrow \quad 3(1) = 1 + 1 + 2\,\hat{\mathbf{a}} \cdot \hat{\mathbf{c}} \Rightarrow 2\,\hat{\mathbf{a}} \cdot \hat{\mathbf{c}} = 1$

$\Rightarrow \quad |\hat{\mathbf{a}}||\hat{\mathbf{c}}| \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

27. E_1 : Eccentricity $e_1 = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$

E_2 : Eccentricity $e_2 = \sqrt{1 - \frac{b^2}{16}} = \frac{\sqrt{16 - b^2}}{4}$

Also, $\quad \frac{1}{\sqrt{3}} \times \frac{\sqrt{16 - b^2}}{4} = \frac{1}{2}$

$\Rightarrow \quad b^2 = 4$

Minor axis of ellipse $E_2 = 2b = 2 \times 2 = 4$

28. $\alpha + \beta = -p, \alpha\beta = \frac{3p}{4}$

Also, $\quad (\alpha - \beta)^2 = 10$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 10$

$\Rightarrow \quad p^2 - 3p = 10$

$\Rightarrow \quad (p + 2)(p - 5) = 0$

29. **Statement I** $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow \quad 2\sin^2\theta - (1 - 2\sin^2\theta) = 0$

$\Rightarrow \quad 4\sin^2\theta - 1 = 0 \Rightarrow \sin^2\theta = \frac{1}{4}$

$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{6}$

$\therefore \quad \theta = n\pi + \frac{\pi}{6} \quad \ldots(i)$

and $\quad 2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2(1 - \sin^2\theta) - 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow \quad (\sin\theta + 2)(2\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta + 2 \neq 0, \sin\theta = \frac{1}{2}$

$\theta = n\pi + (-1)^n \frac{\pi}{3} \quad \ldots(ii)$

$\therefore$ Common solutions of Eqs. (i) and (ii) is 2.

Statement II From Eq. (ii), $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \in [0, \pi]$

30. $g(x) = \begin{cases} 1 + x, & x < 0 \\ 1 - x, & x \geq 0 \end{cases}$

$\therefore \quad f\{g(x)\} = \begin{cases} 1 + |x - 1|, & x < 0 \\ 1 + |-x - 1|, & x \geq 0 \end{cases}$

$= \begin{cases} 1 + 1 - x, & x < 0 \\ 1 + x + 1, & x \geq 0 \end{cases}$

$= \begin{cases} 2 - x, & x < 0 \\ 2 + x, & x \geq 0 \end{cases}$

It is a polynomial function, so it is continuous in everywhere except at $x = 0$.

Now, LHL $= \lim_{x \to 0} (2 - x) = 2$

RHL $= \lim_{x \to 0} (2 + x) = 2$

Also, $\quad f(0) = 2 + 0 = 2$

Hence, it is continuous everywhere.

Online Exam Held on 23-4-2013

JEE Main

Solved Paper 2013

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. The integral $\int \frac{x\,dx}{2 - x^2 + \sqrt{2 - x^2}}$ is equals to

(a) $\log|1 + \sqrt{2 + x^2}| + C$

(b) $-\log|1 + \sqrt{2 - x^2}| + C$

(c) $-x\log|1 - \sqrt{2 - x^2}| + C$

(d) $x\log|1 - \sqrt{2 + x^2}| + C$

2. If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then a value of α is

(a) 2 (b) $\frac{4}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

3. For integers m and n, both greater than 1, consider the following three statements

P : m divides n

Q : m divides n^2

R : m is prime, then

(a) $Q \wedge R \rightarrow P$ (b) $P \wedge Q \rightarrow R$

(c) $Q \rightarrow R$ (d) $Q \rightarrow P$

4. The sum of the rational terms in the binomial expansion of $(2^{1/2} + 3^{1/5})^{10}$ is

(a) 25

(b) 32

(c) 9

(d) 41

5. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle, in square units, is

(a) $\frac{5}{4}a^2$

(b) $\frac{5}{2}a^2$

(c) $\frac{25a^2}{4}$

(d) $5a^2$

6. The sum of the series $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$ upto 10 terms is

(a) 11300 (b) 11200

(c) 12100 (d) 12300

Directions *(Q. Nos. 7 to 9) Each of these questions contains two statements. Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c), (d) given below*

(a) *Statement* I *is false, Statement* II *is true.*

(b) *Statement* I *is true, Statement* II *is true; Statement* II *is a correct explanation for Statement* I.

(c) *Statement* I *is true, Statement* II *is true; Statement* II *is not a correct explanation for Statement* I.

(d) *Statement* I *is true, Statement* II *is false.*

7. **Statement I** The system of linear equations

$$x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos a)y + (\sin a)z = 0$$
$$x - (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution for only one value of α lying in the interval $\left(0, \frac{\pi}{2}\right)$.

Statement II The equation in α

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

has only one solution lying in the interval $\left(0, \frac{\pi}{2}\right)$.

8. Let θ_1 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_2 = 0$, and θ_2 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$, where c_1, c_2, c_3 are any real numbers

Statement I If c_2 and c_3 are proportional, then $\theta_1 = \theta_2$.

Statement II $\theta_1 = \theta_2$ for all c_2 and c_3.

9. Let A, other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let $Tr(A)$ be the sum of diagonal elements of A

Statement I $\text{Tr}(A) = 0$

Statement II $\det(A) = -1$

10. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then 'a' equals.

(a) 0 (b) ± 4

(c) ± 2 (d) ± 3

11. If $S = \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right) + \tan^{-1}\left(\frac{1}{n^2 + 3n + 3}\right) + \ldots$

$+ \tan^{-1}\left\{\frac{1}{1 + (n+19)(n+20)}\right\}$, then $\tan S$ is equal to

(a) $\frac{20}{401 + 20n}$ (b) $\frac{n}{n^2 + 20n + 1}$

(c) $\frac{20}{n^2 + 20n + 1}$ (d) $\frac{n}{401 + 20n}$

12. If **a** and **b** are non-collinear vectors, then the value of α for which the vectors $\mathbf{u} = (\alpha - 2)\mathbf{a} + \mathbf{b}$ and $\mathbf{v} = (2 + 3\alpha)\mathbf{a} - 3\mathbf{b}$ are collinear is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) $\frac{-3}{2}$ (d) $\frac{-2}{3}$

13. The least integral value α of x such that $\frac{x-5}{x^2 + 5x - 14} > 0$, satisfies

(a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$

(c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$

14. A, B, C try to hit a target simultaneously but independently. Their respective probabilities of hitting the targets are $\frac{3}{4}, \frac{1}{2}, \frac{5}{8}$. The probability that the target is hit by A or B but not by C is

(a) $\frac{21}{64}$ (b) $\frac{7}{8}$

(c) $\frac{7}{32}$ (d) $\frac{9}{64}$

15. If two lines L_1 and L_2 in space, are defined by

$L_1 = \{x = \sqrt{\lambda}\, y + (\sqrt{\lambda} - 1), z = (\sqrt{\lambda} - 1)\, y + \sqrt{\lambda}\}$ and

$L_2 = \{x = \sqrt{\mu}\, y + (1 - \sqrt{\mu}),\ z = (1 - \sqrt{\mu})\, y + \sqrt{\mu}\}$, then L_1 is perpendicular to L_2, for all non-negative reals λ and μ, such that :

(a) $\sqrt{\lambda} + \sqrt{\mu} = 1$ (b) $\lambda \neq \mu$

(c) $\lambda + \mu = 0$ (d) $\lambda = \mu$

16. The number of solutions of the equation $\sin 2x - 2\cos x + 4\sin x = 4$ in the interval $[0, 5\pi]$ is

(a) 3 (b) 5

(c) 4 (d) 6

17. If the projections of a line segment on the x, y and z-axes in 3-dimensional space are 2, 3 are 6 respectively, then the length of the line segment is

(a) 12 (b) 7 (c) 9 (d) 6

18. Let $a = \text{Im}\left(\frac{1 + z^2}{2iz}\right)$, where z is any non-zero complex number.

The set $A = \{a : |z| = 1 \text{ and } z \neq -1\}$ is equal to

(a) $(-1, 1)$ (b) $[-1, 1]$

(c) $[0, 1)$ (d) $(-1, 0]$

19. If $f(x) = \sin(\sin x)$ and $f''(x) + \tan x\, f'(x) + g(x) = 0$, then $g(x)$ is

(a) $\cos^2 x \cos(\sin x)$ (b) $\sin^2 x \cos(\cos x)$

(c) $\sin^2 x \sin(\cos x)$ (d) $\cos^2 x \sin(\sin x)$

20. If $a_1, a_2, a_3, \ldots, a_n$ are in AP such that $a_4 - a_7 + a_{10} = m$, then the sum of first 13 terms of this AP is
(a) 10 m (b) 12 m
(c) 13 m (d) 15 m

21. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin). Then, R lies on
(a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$
(b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$
(c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$
(d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$

22. Let f be a composite function of x defined by $f(u) = \frac{1}{u^2 + u - 2}$, $u(x) = \frac{1}{x-1}$. Then, the number of points x where f is discontinuous is
(a) 4 (b) 3
(c) 2 (d) 1

23. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$ is
(a) π (b) $\frac{\pi}{2}$
(c) 4π (d) $\frac{\pi}{4}$

24. The cost of running a bus from A to B, is Rs. $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be ₹ 75 while at 40 km/h, it is ₹ 65. Then, the most economical speed (in km/h) of the bus is
(a) 45 (b) 50
(c) 60 (d) 40

25. The area under the curve $y = |\cos x - \sin x|, 0 \le x \le \frac{\pi}{2}$, and above x-axis is
(a) $2\sqrt{2}$ (b) $2\sqrt{2} - 2$
(c) $2\sqrt{2} + 2$ (d) 0

26. On the sides AB, BC, BC, CA of a ΔABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are
(a) 210 (b) 205
(c) 215 (d) 220

27. If the median and the range of four numbers $\{x, y, 2x + y, x - y\}$, where $0 < y < x < 2y$, are 10 and 28 respectively, then the mean of the numbers is
(a) 18 (b) 10
(c) 5 (d) 14

28. If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 - \frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve whose abscissa is -2 is
(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$
(c) $\frac{5}{2}$ (d) $-\frac{5}{2}$

29. The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is
(a) (0, 2) (b) (3, 0)
(c) (0, 3) (d) (2, 0)

30. Let $R = \{(x, y) : x, y \in N$ and $x^2 - 4xy + 3y^2 = 0\}$, where N is the set of all natural numbers. Then, the relation R is
(a) reflexive but neither symmetric nor transitive
(b) symmetric and transitive
(c) reflexive and symmetric
(d) reflexive and transitive

Answers

1. (b)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (d)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (d)	**20.** (c)
21. (d)	**22.** (b)	**23.** (b)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (a)

Hints & Solutions

1. Put $t^2 = 2 - x^2$

$$t\,dt = -x\,dx$$

$$\int \frac{dt}{t+1} = -\log|1+t| + C = -\log|1 + t\sqrt{2-x^2}| + C$$

2. Slope of the curve $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ is

$$m_1 = \frac{-4x}{\alpha y}$$

Now, slope of the curve $y^3 = 16x$ is

$$m_2 = \frac{16}{3y^2}$$

Now, apply the condition of perpendicularity of two curves,

ie, $m_1 \cdot m_2 = -1$

and get $\alpha = \frac{4}{3}\cdot$ with the help of equation of curves.

3. $P : \frac{n}{m}; Q : \frac{n^2}{m}; R : m$ is prime.

let $m = 5, n = 10, n^2 = 100$

$m = 3, n = 12, n^2 = 144$

$m = 7, n = 14, n^2 = 196$

$\therefore P, Q$ and R are true statements

So, $Q \wedge R = T \wedge T \rightarrow T = P$

4. General term, $T_{r+1} = {}^{10}C_r (2^{1/2})^{10-r} (3^{1/5})^r = {}^{10}C_r\, 2^{\left(5 - \frac{r}{2}\right)} \cdot 3^{r/5}$

clearly, T_{r+1} will be independent of radical sign if $\pi/2$ and $r/5$ are integers, where $0 \le r \le 10$

$\therefore$ $r = 0, 10.$

$$T_{0+1} = T_1 = {}^{10}C_0\, 2^5 \cdot 3^0 = 32, T_{10+1} = T_u = {}^{10}C_{10} \cdot 2^0 \cdot 3^2 = 9$$

$\therefore$ Required Sum $= 32 + 9 = 41$

6. Series : $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$

$$= 4\{1 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + \ldots\}$$

$$\text{Tn} = 4n \cdot n^2$$

$$S_n = \Sigma T_n = 4\Sigma n^2 = 4\left[\frac{n(n+1)}{2}\right]^2$$

$\Rightarrow$ $S_{10} = [10 \cdot (10+1)]^2 = (110)^2 = 12100$

7. **Statement I** The coefficient matrix $A = \begin{bmatrix} 1 & \sin x & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{bmatrix}$

for non-trivial solution, Put $|A| = 0$

$\Rightarrow$ $\cos 2\alpha = 0 = \cos\frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}\cdot$

Statement II $\begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$

$\Rightarrow$ $2\cos\alpha(2\cos^2\alpha - 1) = 0$

$\Rightarrow$ $\cos\alpha = 0, \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\alpha = \frac{\pi}{2}, \pm\frac{\pi}{4}$

8. Here, angle between the lines $2x + 3y + c_1 = 0$ and $-x + 5 + c_2 = 0$ is θ_1

$\therefore$ $\tan\theta_1 = \left|\frac{1/5 + 2/3}{1 - 2/15}\right| = \left|\frac{13/15}{13/15}\right| = 1 = \tan 45°$

$\Rightarrow$ $\theta_1 = 45°$

Also, the angle between the lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$ is θ_2

$\therefore$ $\tan\theta_2 = \left|\frac{1/5 + 2/3}{1 - 2/15}\right| = \left|\frac{13/15}{13/15}\right| = 1 = \tan 45°$

$\Rightarrow$ $\theta_2 = 45°$

Here, we observe that the value of c_1, c_2 and c_3 is not depend to measuring the angle between the lines.

So, c_2 and c_3 are proportional or for all c_2 and c_3

$$\theta_1 = \theta_2.$$

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^2 = I$ (given)

$\therefore$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $b(a + d) = 0, c(a + d) = 0$

and $a^2 + bc = 1, bc + d^2 = 0$

$\Rightarrow$ $a = 1, d = -1, b = c = 0$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A \ne I, A \ne -I$

$\det(A) = -1$ and $Tr(A) = 1 - 1 = 0$

10. When the circle touches abscissa (x-axis) the equation of circle is

$(x - h)^2 + (y - k)^2 = k^2$ where $(h, k) \rightarrow$ centre

$\Rightarrow$ $x^2 + y^2 - 2hx - 2ky + h^2 = 0$ ...(i)

Compair with, $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$

$\Rightarrow$ $h^2 = 25 - a^2$ $[\because h = -3]$

$\Rightarrow$ $a^2 = 25 - 9 = 16 \Rightarrow a = +4$

11. $S = \tan^{-1}\left\{\dfrac{(n+1)-(n+0)}{1+(n+0)(n+1)}\right\} + \tan^{-1}\left\{\dfrac{(n+2)-(n+1)}{1+(n+1)(n+2)}\right\}$

$$+ \ldots + \tan^{-1}\left\{\frac{(n+20)-(n+19)}{1+(n+19)(n+20)}\right\}$$

$$= \tan^{-1}(n+1) - \tan^{-1} n + \tan^{-1}(n+2) - \tan^{-1}(n+1)$$

$$+ \ldots + \tan^{-1}(n+20) - \tan^{-1}(n+19)$$

$$= \tan^{-1}(n+20) - \tan^{-1} n$$

$$= \tan^{-1}\left\{\frac{n+20-n}{1+n(n+20)}\right\} = \tan^{-1}\left(\frac{20}{n^2+20n+1}\right)$$

$$\Rightarrow \quad \tan S = \tan\left\{\tan^{-1}\left(\frac{20}{n^2+20n+1}\right)\right\}$$

$$\Rightarrow \quad \tan S = \frac{20}{n^2+20n+1}$$

12. Since vectors **u** and **v** are collinear

$\therefore \quad \mathbf{u} = \lambda \mathbf{v}$

$\Rightarrow \quad \{(\alpha-2)\mathbf{a} + \mathbf{b}\} = \lambda(2+3\alpha)\mathbf{a} - 3\mathbf{b}\}$

On comparing, we get

$$(2+3\alpha)\lambda = \alpha - 2 \quad \ldots(\text{i})$$

and $\quad 1 = -3\lambda$

$\Rightarrow \quad \lambda = \dfrac{-1}{3}$

$\therefore \quad (2+3\alpha)\left(\dfrac{-1}{3}\right) = \alpha - 2 \Rightarrow \alpha = \dfrac{2}{3}$

13. $\dfrac{x-5}{x^2+5x-14} > 0 \Rightarrow \dfrac{(x+7)(x-2)(x-5)}{(x-2)^2(x+7)^2}$

$\Rightarrow \quad x \in (-7, 2) \cup [5, \infty)$

So, the least integral value α of x is -6, which satisfy the equation $\alpha^2 + 5\alpha - 6 = 0$.

14. $P(A \cup B \cap \overline{C}) = P(A \cup B) \times P(\overline{C})$

$$= [P(A) + P(B) - P(A \cap B)] \times P(\overline{C})$$

$$= \left\{\frac{3}{4} + \frac{1}{2} - \frac{3}{8}\right\} \times \frac{3}{8}$$

$$\left[\because P(A \cap B) = P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}\right]$$

$$= \frac{7}{8} \cdot \frac{3}{8} = \frac{21}{64}$$

15. We can written the given lines in symmetric form

$$L_1 \equiv \frac{x-(\sqrt{\lambda}-1)}{\sqrt{\lambda}} = \frac{y-0}{1} = \frac{z-\sqrt{\lambda}}{\sqrt{\lambda}-1},$$

$$L_2 \equiv \frac{x-(1-\sqrt{\mu})}{\sqrt{\mu}} = \frac{y-0}{1} = \frac{z-\sqrt{\mu}}{1-\sqrt{\mu}}$$

Since, L and L_2 are perpendicular

$\therefore \quad \sqrt{\lambda} \cdot \sqrt{\mu} + 1 \cdot 1 + (\sqrt{\lambda} - 1)(1 - \sqrt{\mu}) = 0$

$\Rightarrow \quad \sqrt{\lambda} + \sqrt{\mu} = 0 \Rightarrow \lambda = \mu$

16. $\sin 2x - 2\cos x + 4\sin x = 4$

$\Rightarrow \quad (\sin x - 1)(2\cos x + 4) = 0$

$\Rightarrow \quad \sin x = 1, \cos x \neq -2$

$\Rightarrow \quad x = n\pi + (-1)^n \dfrac{\pi}{2}$, where $n = 0, 1, 2, \ldots$

Here, required value of $x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$ in interval $[0, 5\pi]$.

17. Given that, the projections of a line segment on the x, y and z-axis in $3D$-space are, $lr = 2, mr = 3$ and $nr = 6$

$\because \quad (lr)^2 + (mr)^2 + (nr)^2 = (2)^2 + (3)^2 + (6)^2$

$\Rightarrow \quad (l^2 + m^2 + n^2)r^2 = 4 + 9 + 36$

$\Rightarrow \quad r^2 = 49 \Rightarrow r = 7$

18. Let $z = x + iy \quad \Rightarrow \quad |z| = x^2 + y^2$

$\Rightarrow \quad 1 = x^2 + y^2 \quad \ldots(\text{i})$

Now,

$$a = I_m\left(\frac{1+(x+iy)^2}{2iz}\right)$$

$$= I_m\left(\frac{1+x^2-y^2+2ixy}{2i(x+iy)}\right)$$

$$= I_m\left[\frac{(1+x^2-y^2+2ixy)}{2(ix-y)}\right]$$

$$= I_m\left[\frac{1+x^2-y^2+2ixy}{2(ix-y)} \times \frac{ix+y}{(ix+y)}\right]$$

$$= I_m\left[\frac{(1+x^2-y^2+2ixy)(ix+y)}{2(-x^2-y^2)}\right]$$

$$= \frac{x(1+x^2-y^2)+2xy^2}{-2(x^2+y^2)}$$

$$= \frac{x(1+x^2-(1-x^2))+2x(1-x^2)}{-2(1)}$$

$$= \frac{x(2x^2)+2x-2x^3}{-2}$$

$$a = -x$$

where, $x = -1$

$$y = \sqrt{1-(-1)^2} = 0$$

$\therefore \quad z = -1$, but it is given

$$z \neq -1$$

$\therefore \quad a = 1$, is not possible

where, $x = 1$,

$$y = \sqrt{1-1^2} = 0$$

$\therefore \quad z = 1$, which is exact

$\therefore \quad a = -1$, is possible

Hence, option (a) is correct.

19. $f(x) = \sin(\sin x)$

$\Rightarrow \quad f'(x) = \cos x \cdot \cos(\sin x)$

$\Rightarrow \quad f''(x) = -\sin x \cdot \cos(\sin x) - \cos^2 x \cdot \sin(\sin x)$

Now, $\quad g(x) = -[f''(x) + f'(x) \cdot \tan x]$

$= +\sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x)$

$- \tan x \cdot \cos x \cdot \cos(\sin x)$

$= \sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x) - \sin x \cdot \cos(\sin x)$

$= \cos^2 x \cdot \sin(\sin x)$

20. Let 'a' and 'd' be the first term and common difference of an AP, respectively.

$\because \quad a_4 - a_7 + a_{10} = m$

$\Rightarrow \quad (a + 3d) - (a + 6d) + (a + 9d) = m$

$\Rightarrow \quad a + 6d = m \quad \ldots(i)$

$\because \quad S_{13} = \frac{13}{2}[2a + 12d]$

$= 13(a + 6d) = 13m$

22. The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point $x = 1$.

The function $y = g(u) = \frac{1}{u^2 + u - 2} = \frac{1}{(u+2)(v-1)}$

is discontinuous at $u = -2$ and $u = 1$

when, $\quad u = -2 \Rightarrow \frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$,

when, $\quad u = 1 \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2.$

Hence, the composite function $y = gf(x)$ is discontinuous at three points, $x = \frac{1}{2}$, $x = 1$ and $x = 2$.

23. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$

$$\left[\because \int_a^b f(x)\,dx - \int_a^b f(a + b - x)\,dx\right]$$

$= I = \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1 + 2^x} dx$

$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} 2\sin^2 x\,dx$

$= \int_0^{\pi/2} (1 - \cos 2x)\,dx$

$\Rightarrow \quad 2I = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$

$\Rightarrow \quad I = \frac{\pi}{4}$

24. Let $c = av + \frac{b}{v} \quad \ldots(i)$

When $v = 30$ Km/h, then $c = ₹75$

$\therefore \quad 75 = 30a + \frac{b}{30} \quad \ldots(ii)$

When $v = 40$ Km/h, then $c = ₹65$

$\therefore \quad 65 = 40a + \frac{b}{40} \quad \ldots(iii)$

On solving Eqs. (ii) and (iii), we get :

$a = \frac{1}{2}$

and $\quad b = 1800$

On differentiating Eq. (i) w.r.t v we get,

$\frac{dc}{dv} = a - \frac{b}{v^2}$

For Max of minimum c

$\frac{dc}{dv} = 0$

$\Rightarrow \quad a - \frac{b}{v^2} = 0$

$\Rightarrow \quad v = \pm\sqrt{\frac{b}{a}}$

$\Rightarrow \quad \frac{d^2c}{dv^2} = \frac{2b}{v^3}$

at $\quad v = \sqrt{\frac{b}{a}}, \frac{d^2c}{dv^2} > 0$

So, at $v = \sqrt{\frac{b}{a}}$ the speed is most economical.

$\therefore$ Most economical speed is

$c = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} = 2\sqrt{ab}$

$c = 2\sqrt{\frac{1}{2} \times 1800} = 2 \times 30$

$c = ₹60$

25. $\therefore$ Required area $= \int_0^{\pi/2} |\cos x - \sin x|\,dx$

$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$

$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$

$= (\sqrt{2} - 1) - (1 - \sqrt{2})$

$= 2\sqrt{2} - 2$

26. Required number of triangles that can be constructed using these chosen points as vertices

$= {}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3$

(here, we subtract those triangle in which points are collinear)

$= 220 - 1 - 4 - 10$

$= 220 - 15 = 205$

27. First we arrange four numbers according to the condition $0 < y < x < 2y$.

ie, $x - y, y, x, 2x + y$

$$\text{Median} = \frac{\text{2nd term} + \text{3rd term}}{2} = 10$$

$\Rightarrow$ $y + x = 20$ ...(i)

Range $= (2x + y) - (x - y) = 28$

$\Rightarrow$ $x + 2y = 28$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = 12, y = 8$$

$\therefore$ For numbers are, $4, 8, 12, 32$.

$$\therefore \quad \text{Mean} = \frac{4 + 8 + 12 + 32}{4}$$

$$= \frac{56}{4} = 14$$

28. Given, $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

$\Rightarrow$ $dy = \left(1 - \frac{1}{x^2}\right)dx$

$\Rightarrow$ $y = x + \frac{1}{x} + C$ (on integration)

since curve passing through the point $\left(2, \frac{7}{2}\right)$

ie, $\frac{7}{2} = 2 + \frac{1}{2} + C$

$\Rightarrow$ $C = 1$

$\therefore$ $y = x + \frac{1}{x} + 1$ ...(i)

Given, also $x = -2$

$\therefore$ (ordinate) $y = -2 - \frac{1}{2} + 1 = -3/2$

29. Given, $y^2 = 4x$

$$\frac{dy}{dx} = \frac{2}{y} \quad \text{...(i)}$$

$\because$ S is the mid-point of PP'.

$\therefore$ $t^2 = 1 \Rightarrow t = \pm 1$

Equation of normal PQ is $(y - 2) = \left(\frac{-2}{2}\right)(x - 1)$

$\Rightarrow$ $x + y = 3$ ...(ii)

Equation of normal $P'Q$

$$(y + 2) = 1(x - 1)$$

$\Rightarrow$ $x - y = 3$ ...(iii)

On solving Eqs. (ii) and (iii), we get

$$2x = 6 \Rightarrow x = 3$$

and $y = 0$

$\therefore$ Required intersection point is $(3, 0)$

30. $\because a^2 - 4a \cdot a + 3a^2 = 4a^2 - 4a^2 = 0$

$\therefore$ $(a, a) \in N$ is reflexive.

Also $(a, b) \in N$

$\Rightarrow$ $a^2 - 4ab + 3b^2 = 0$

$\Rightarrow$ $b^2 - 4ba + 3a^2 \neq 0$

$\therefore R$ is not symmetric.

Now, $(a, b) \in N$ and $(b, c) \in N \not\Rightarrow (a, c) \in N$

$\therefore R$ is not transitive

Online Exam Held on 25-4-2013

JEE Main

Solved Paper 2013

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. Given a sequence of 4 numbers, first three of which are in GP and the last three are in AP with common difference six. If first and last terms of this sequence are equal, then the last term is

(a) 16
(b) 8
(c) 4
(d) 2

2. **Statement I** The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$

Statement II $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

3. If a circle of unit radius is divided into two parts by an arc of another circle subtending an angle 60° on the circumference of the first circle, then the radius of the arc is

(a) $\sqrt{3}$
(b) $\frac{1}{2}$
(c) 1
(d) $\sqrt{2}$

4. If the image of point $P(2, 3)$ in a line L is $Q(4, 5)$, then the image of point $R(0, 0)$ in the same line is

(a) (2, 2)
(b) (4, 5)
(c) (3, 4)
(d) (7, 7)

5. Consider the system of equations

$x + ay = 0$, $y + az = 0$ and $z + ax = 0$. Then, the set of all real values of 'a' for which the system has a unique solution is

(a) $R - \{1\}$
(b) $R - \{-1\}$
(c) $\{1, -1\}$
(d) $\{1, 0, -1\}$

6. A common tangent to the conics $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is

(a) $x - y = \frac{3}{2}$
(b) $x + y = 1$
(c) $x + y = \frac{9}{2}$
(d) $x - y = 1$

7. Let $S = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22}\right\}$

Then, the number of non-singular matrices in the set S is

(a) 27 (b) 24
(c) 10 (d) 20

8. Let $x \in (0, 1)$. The set of all x such that $\sin^{-1} x > \cos^{-1} x$, is the interval

(a) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, 1\right)$
(c) $(0, 1)$ (d) $\left(0, \frac{\sqrt{3}}{2}\right)$

9. Let $A(-3, 2)$ and $B(-2, 1)$ be the vertices of a ΔABC. If the centroid of this Δ lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line

(a) $4x + 3y + 5 = 0$ (b) $3x + 4y + 2 = 0$
(c) $4x + 3y + 3 = 0$ (d) $3x + 4y + 5 = 0$

10. Let ABC be a triangle with vertices at points $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then (λ, μ) is equal to

(a) (10, 7) (b) (7, 5)
(c) (7, 10) (d) (5, 7)

11. In the integral

$\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A \cos 8x + k$, where k is an arbitrary constant, then A is equal to

(a) $-\frac{1}{16}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

12. The equation of the curve passing through the origin and satisfying the differential equation

$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ is

(a) $(1 + x^2)y = x^3$
(b) $3(1 + x^2)y = 2x^3$
(c) $(1 + x^2)y = 3x^3$
(d) $3(1 + x^2)y = 4x^3$

13. For $0 \le x \le \frac{\pi}{2}$, the value of

$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$ is equal to

(a) $\frac{\pi}{4}$ (b) 0
(c) 1 (d) $-\frac{\pi}{4}$

14. Let $f : [-2, 3] \to [0, \infty)$ be a continuous function such that $f(1 - x) = f(x)$ for all $x \in [-2, 3]$.

If R_1 is the numerical value of the area of the region bounded by $y = f(x)$, $x = -2$, $x = 3$ and the axis of x and $R_2 = \int_{-2}^{3} xf(x)\, dx$, then

(a) $3R_1 = 2R_2$ (b) $2R_1 = 3R_2$
(c) $R_1 = R_2$ (d) $R_1 = 2R_2$

15. Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$. If $\mathbf{c}$ is a vector such that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$, $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ is $30°$, then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{3\sqrt{3}}{2}$
(c) 3 (d) $\frac{3}{2}$

16. Let p and q be any two logical statements and $r : p \to (\sim p \vee q)$. If r has a truth value F, then the truth values of p and q are respectively

(a) F, F (b) T, T
(c) T, F (d) F, T

17. If the events A and B are mutually exclusive events such that $P(A) = \frac{3x + 1}{3}$ and $P(B) = \frac{1 - x}{4}$, then the set of possible values of x lies in the interval

(a) $[0, 1]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$
(c) $\left[-\frac{1}{3}, \frac{5}{9}\right]$ (d) $\left[-\frac{7}{9}, \frac{4}{9}\right]$

18. The equation of a plane through the line of intersection of the planes $x + 2y = 3$, $y - 2z + 1 = 0$, and perpendicular to the first plane is

(a) $2x - y - 10z = 9$ (b) $2x - y + 7z = 11$
(c) $2x - y + 10z = 11$ (d) $2x - y - 9z = 10$

19. If for positive intergers $r > 1$, $n > 2$, the coefficients of the $(3r)$th and $(r + 2)$th powers of x in the expansion of $(1 + x)^{2n}$ are equal, then n is equal to

(a) $2r + 1$ (b) $2r - 1$
(c) $3r$ (d) $r + 1$

20. If p and q are non-zero real numbers and $\alpha^3 + \beta^3 = -p$, $\alpha\beta = q$, then a quadratic equation whose roots are $\frac{\alpha^2}{\beta}, \frac{\beta^2}{\alpha}$ is

(a) $px^2 - qx + p^2 = 0$
(b) $qx^2 + px + q^2 = 0$
(c) $px^2 + qx + p^2 = 0$
(d) $qx^2 - px + q^2 = 0$

21. A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in the surface area (in cm^2/min) of the balloon when its diameter is 14 cm, is

(a) 10 (b) $\sqrt{10}$

(c) 100 (d) $10\sqrt{10}$

22. Let $A = \{\theta : \sin(\theta) = \tan(\theta)\}$ and $B = \{\theta : \cos(\theta) = 1\}$ be two sets. Then,

(a) $A = B$ (b) $A \not\subset B$

(c) $B \not\subset A$ (d) $A \subset B$ and $B - A \neq \phi$

23. Let z satisfy $|z| = 1$ and $z = 1 - \bar{z}$

Statement I z is a real number.

Statement II Principal argument of z is $\frac{\pi}{3}$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; statement II is false.

(d) Statement I is false; Statement II is true.

24. Consider the function

$f(x) = [x] + |1 - x|, -1 \le x \le 3$ where, $[x]$ is the greatest integer function.

Statement I f is not continuous at $x = 0, 1, 2$ and 3.

Statement II $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ 1 + x, & 1 \le x < 2 \\ 2 + x, & 2 \le x \le 3 \end{cases}$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

25. Let $f(1) = -2$ and $f'(x) \ge 4.2$ for $1 \le x \le 6$. The possible value of $f(6)$ lies in the interval

(a) $[15, 19)$ (b) $(-\infty, 12)$

(c) $[12, 15)$ (d) $[19, \infty)$

26. In a set of $2n$ observations, half of them are equal to 'a' and the remaining half are equal to '$-a$'. If the standard deviation of all the observations is 2; then the value of $|a|$ is

(a) 2 (b) $\sqrt{2}$

(c) 4 (d) $2\sqrt{2}$

27. The value of $1^2 + 3^2 + 5^2 + \ldots + 25^2$ is

(a) 2925 (b) 1469

(c) 1728 (d) 1456

28. If an equation of a tangent to the curve, $y = \cos(x + y), -1 \le x \le 1 + \pi$, is $x + 2y = k$ then, k is equal to

(a) 1 (b) 2

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

29. 5-digit numbers are to be formed using 2, 3, 5, 7, 9 without repeating the digits. If p be the number of such numbers that exceed 20000 and q be the number of those that lie between 30000 and 90000, then $p : q$ is

(a) 6 : 5 (b) 3 : 2

(c) 4 : 3 (d) 5 : 3

30. A point on the ellipse, $4x^2 + 9y^2 = 36$, where the normal is parallel to the line, $4x - 2y - 5 = 0$, is

(a) $\left(\frac{9}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, \frac{-9}{5}\right)$

(c) $\left(-\frac{9}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{5}, \frac{9}{5}\right)$

Answers

1. (b)	**2.** (d)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (c)
11. (a)	**12.** (d)	**13.** (a)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (b)
21. (a)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (a)	**28.** (d)	**29.** (d)	**30.** (a)

Hints & Solutions

1. Let the numbers $\frac{a}{r}, a, ar, 2ar - a$ ...(i)

Given, common difference of an AP is 6.

$\therefore \quad ar - a = 6$ ...(ii)

Also, $\frac{a}{r} = 2ar - a \Rightarrow \frac{a}{r} = 2(ar - a) + a$

$\Rightarrow \quad \frac{a}{r} - a = 12$

$\Rightarrow \quad a(1 - r) = 12r$

$\Rightarrow \quad r = -\frac{1}{2}$

From Eq. (ii), we get

$$a\left[\left(-\frac{1}{2}\right) - 1\right] = 6 \Rightarrow a = -4$$

$\therefore$ Required series are $8, -4, 2, 8$

2. (a) **Statement I** Centre of circle $(3, -1)$

Now, $2(3) + (-1) = 5 = 5$, true

Statement II Centre $(3, -1)$, which lies on given line.

Simplify it and get the result.

3. (a) In ΔC_1MC_2,

$$\tan 30° = \frac{C_1M}{C_2M}$$

$$\frac{1}{\sqrt{3}} = \frac{1}{r}$$

$\Rightarrow \quad r = \sqrt{3}$

4. Mid-point of P and Q is $(3, 4)$. Slope of P and Q is $\frac{5-3}{4-2} = 1$

$\therefore$ Equation of line l is $y - 4 = -1(x - 3)$

$\Rightarrow \quad x + y = 7$

5. $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} \neq 0$

Simplify it and get the result.

6. $\frac{x^2}{9/2} - \frac{y^2}{9/4} = 1$

Equation of a tangent in $y = mx \pm \sqrt{\frac{9}{2}m^2 - \frac{9}{4}}$

This equation also, intersect the conic $x^2 = 6y$. This will give us a quadratic equation. whose discriminant is zero.

Simplify it and get the result.

7. A matrix whose determinant is non-zero is called a non-singular matrix. There are total 27 combinations, in those of them 20 such combination, in which determinant is non-zero.

8. By properties of inverse function

$$\sin^{-1} x > \cos^{-1} x, \forall x \in \left(\frac{1}{\sqrt{2}}, 1\right)$$

9. Let third vertex be $C(x_1, y_1)$

$\therefore$ Centroid $\left(\frac{-3-2+x_1}{3}, \frac{2+1+y_1}{3}\right)$ lies on line

$$3x + 4y + 2 = 0$$

10. Centroid of ΔABC, $G = \left(\frac{2-1+\lambda}{3}, \frac{3+3+5}{3}, \frac{5+2+\mu}{3}\right)$

$$= \left(\frac{1+\lambda}{3}, \frac{11}{3}, \frac{7+\mu}{3}\right)$$

Since Median is always passes through centroid.

Since, they are equally inclined

$\therefore \quad \frac{1+\lambda}{3} - 2 = \frac{11}{3} - 3 = \frac{7+\mu}{3} - 5$

$\Rightarrow \quad \frac{\lambda - 5}{3} = \frac{2}{3} = \frac{\mu - 8}{3}$

$\Rightarrow \quad \lambda = 7, \mu = 10$

11. LHS $= \int \frac{2\cos^2 4x}{\frac{\cos^2 2x - \sin^2 2x}{\cos 2x \sin 2x}} dx = \int \frac{2\cos^2 4x \times \cos 2x \sin 2x}{\cos 4x} dx$

$$= \int \cos 4x \times \sin 4x dx = \frac{1}{2}\int \sin 8x dx = \frac{-1}{2}\frac{\cos 8x}{8} + k$$

12. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$

$$\text{If} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log x} = x$$

$\therefore \quad y \times x = \int \frac{4x^3}{1+x^2}$

Simplify it and get the result.

13. Put $t = \sin^2 z$ in Ist integral and $t = \cos^2 u$ in IInd integral, we get

$dt = 2\sin z \cos z\, dz$ and $dt = -2\cos u \sin u\, du$

$\therefore \quad I = \int_0^x z_2 \sin z \cos z\, dz + \int_{\pi/2}^x -u2\cos u \sin u\, du$

$$= \int_0^x \underset{\text{I}}{z} \underset{\text{II}}{\sin 2z}\, dz - \int_{\pi/2}^x u \sin 2u\, du$$

$$= \left[-z\frac{\cos 2z}{2} + \frac{\sin 2z}{4}\right]_0^x - \left[\frac{-u\cos 2u}{2} + \frac{\sin 2u}{4}\right]_{\pi/2}^x$$

$$= \left[-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \{0 + 0\}\right]$$

$$-\left[-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \left(\frac{\pi}{4} + 0)\right)\right] = \frac{\pi}{4}$$

14. Given, $R_1 = \int_{-2}^{3} f(x)\,dx$

$\therefore \quad R_2 = \int_{-2}^{3} xf(x)\,dx = \int_{-2}^{3} (1-x)\,f(1-x)\,dx$

$= \int_{-2}^{3} (1-x)\,f(x)\,dx = \int_{-2}^{3} f(x)\,dx - \int_{-2}^{3} x\,f(x)\,dx$

$\Rightarrow \quad R_2 = R_1 - R_2 \Rightarrow 2R_2 = R_1$

15. $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2} \Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{c} = 8$

$\Rightarrow \quad |\mathbf{c}|^2 + (\sqrt{9})^2 - 2|\mathbf{c}| = 8$

$\Rightarrow \quad |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0 \Rightarrow (|\mathbf{c}| - 1)^2 = 0$

$\Rightarrow \quad |\mathbf{c}| = 1$

Now, $\quad \mathbf{a}\times\mathbf{b} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad |\mathbf{a}\times\mathbf{b}| = \sqrt{4+4+1} = 3$

$\therefore \quad |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = |\mathbf{a}\times\mathbf{b}|\,|\mathbf{c}|\sin 30°$

$= 3\times 1\times\frac{1}{2} = \frac{3}{2}$

16.

p	q	$\sim p$	$\sim p \vee q$	$p \to (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

17. $\because 0 \le \frac{3x+1}{3} \le 1, 0 \le \frac{1-x}{4} \le 1$

and $\quad 0 \le \frac{3x+1}{3} + \frac{1-x}{4} \le 1$

$\Rightarrow \quad 0 \le 3x+1 \le 3, 0 \le 1-x \le 4,\ 0 \le \frac{12x+4+3-3x}{12} \le 1$

$\Rightarrow \quad -1 \le 3x \le 2, 0 \le 1-x \le 4, 0 \le 9x+7 \le 12$

$\Rightarrow \quad -\frac{1}{3} \le x \le \frac{2}{3}, -1 \le -x \le 3, -7 \le 9x \le 5$

$\Rightarrow \quad -\frac{1}{3} \le x \le \frac{2}{3}, -3 \le x \le 1,\ -\frac{7}{9} \le x \le \frac{5}{9}$

$\therefore \quad -\frac{1}{3} \le x \le \frac{5}{9}$

18. Intersection of two planes is

$(x+2y-3) + \lambda(y-2z+1)$

$\Rightarrow \quad x + (2+\lambda)y - 2\lambda z + \lambda - 3 = 0$

$\therefore \quad 1(1) + 2(2+\lambda) + 0(-2\lambda) = 0$

$\Rightarrow \quad \lambda = -\frac{5}{2}$

Simplify it and get the result.

20. Sum $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{-p}{q}$

$\therefore$ Product $= \frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha} = \alpha\beta = q$

$\therefore \quad x^2 + \frac{p}{q}x + q = 0$

$\Rightarrow \quad qx^2 + px + q^2 = 0$

21. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi^3 3r^2\frac{dr}{dt}$

$\Rightarrow \quad 35 = 4\pi(7)^2\frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{5}{28\pi}$

Surface area of balloon, $S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$= 8\pi\times 7\times\frac{5}{28\pi} = 10\ cm^2/\text{min}$

22. $A\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}\ B = \{\ldots, -2\pi, 0, 2\pi, \ldots\}$

$\therefore \quad A \not\subset B$

23. Let $z = x + iy$

$\therefore \quad x^2 + y^2 = 1$ and $x + iy = 1 - (x - iy)$

$\Rightarrow \quad x^2 + y^2 = 1$

and $\quad 2x = 1$

$\Rightarrow \quad x = \frac{1}{2}$

$\Rightarrow \quad y = \pm\frac{\sqrt{3}}{2}$

$\therefore \quad z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Now, $\quad z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$

24. I. We know $[x]$ is discontinuous for every integer values of x.

$\therefore f(x)$ is discontinuous for $x = 0, 1, 2,$ and

II. $$f(x) = \begin{cases} -1+1-x & -\infty \le x < 0 \\ 0+1-x & 0 \le x < 1 \\ 1+x-1 & 1 \le x < 2 \\ 2+x-1 & 2 \le x < 3 \\ 3+2 & x = 3 \end{cases}$$

$$= \begin{cases} -x & -\infty \le x < 0 \\ 1-x & 0 \le x < 1 \\ x & 1 \le x < 2 \\ x+1 & 2 \le x < 3 \\ 5 & x = 3 \end{cases}$$

25. $\because f'(x) \ge 4.2$

$\Rightarrow \quad f(x) \ge 4.2\,x$

$\Rightarrow \quad f(6) \ge 4.2\times 6$

$\Rightarrow \quad f(6) \ge 25.2 \in [19, \infty)$

26. Mean $= \underbrace{(a + a + \ldots + n \text{ times}) + (-a - a - \ldots - n \text{ times})}_{2n}$

$$= 0$$

$$\therefore \quad SD = \sqrt{\frac{(x_i - \bar{x})^2}{2n}}$$

$$= \sqrt{\frac{a^2 + a^2 + \ldots 2n \text{ times}}{2n}}$$

$$2 = \sqrt{\frac{(a)^2\, 2n}{2n}} = |a|$$

27. $T_n = (2n-1)^2 = 4n^2 + 1 - 4n$

$$S_n = \sum_{n=1}^{13} (4n^2 + 1 - 4n)$$

$$= 4\left[\frac{13\,(13+1)\,(26+1)}{6} + 13 - \frac{4 \times 13 \times 14}{2}\right]$$

$$= 4\,(819) + 13 - 364 = 2925$$

28. $y = \cos(x + y)$

$$\frac{dy}{dx} = -\sin(x+y)\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}[1 + \sin(x+y)] = -\sin(x+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin(x+y)}{1 + \sin(x+y)} = -\frac{1}{2}$$

$$\left(\because \quad x + 2y = k \Rightarrow m = \frac{dy}{dx} = \frac{-1}{2}\right)$$

$$\Rightarrow \quad \sin(x+y) = 1$$

$$\Rightarrow \quad x + y = \frac{\pi}{2}$$

$$\therefore \quad y = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad x = \frac{\pi}{2}$$

$$\because \quad x + 2y = k$$

$$\therefore \quad \frac{\pi}{2} + 2(0) = k \Rightarrow k = \frac{\pi}{2}$$

29. p = The number of such numbers that exceeds 20000

$$= 5! = 120$$

q = The number of those that lie between 30000 and 90000, then,

$$= 5! - 4! - 4!$$

$$= 120 - 24 - 24$$

$$= 72$$

$$\therefore \quad \frac{p}{q} = \frac{120}{72} = \frac{5}{3}$$

30. Let (x_1, y_1) be a point on the ellipse

$$\therefore \quad 4x_1^2 + 9y_1^2 = 36 \qquad \ldots(i)$$

Now, $\quad 4(2x) + 18y\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4x}{9y}$$

$$= \frac{dy}{dx}_{(x_1, y_1)} = -\frac{4x_1}{9y_1}$$

$\therefore$ Slope of normal $= \dfrac{9y_1}{4x_1}$

which is parallel to $4x - 2y - 5 = 0$

$$\therefore \quad \frac{9y_1}{4x_1} = 2$$

$$\Rightarrow \quad 9y_1 = 8x_1 \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\left(\frac{9}{5}, \frac{8}{5}\right)$$

JEE Main

Solved Paper 2013

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$

(c) $\frac{7}{2}$ (d) $\frac{9}{2}$

2. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by

$$\frac{dP}{dx} = 100 - 12\sqrt{x}.$$

If the firm employees 25 more workers, then the new level of production of items is

(a) 2500 (b) 3000

(c) 3500 (d) 4500

3. Let A and B two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is

(a) 256

(b) 220

(c) 219

(d) 211

4. If the lines

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k} \text{ and } \frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$$

are coplanar, then k can have

(a) any value (b) exactly one value

(c) exactly two values (d) exactly three values

5. If the vectors $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC, then the length of the median through A is

(a) $\sqrt{18}$ (b) $\sqrt{72}$ c) $\sqrt{33}$ (d) $\sqrt{45}$

6. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

(a) lies between 1 and 2

(b) lies between 2 and 3

(c) lies between -1 and 0

(d) does not exist

7. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777,…, is

(a) $\frac{7}{81}(179 - 10^{-20})$ (b) $\frac{7}{9}(99 - 10^{-20})$

(c) $\frac{7}{81}(179 + 10^{-20})$ (d) $\frac{7}{9}(99 + 10^{-20})$

8. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is

(a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$

(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$

9. The number of values of k, for which the system of equations

$$(k+1)x + 8y = 4k$$
$$kx + (k+3)y = 3k - 1$$

has no solution, is

(a) infinite (b) 1 (c) 2 (d) 3

10. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is

(a) 1 : 2 : 3 (b) 3 : 2 : 1

(c) 1 : 3 : 2 (d) 3 : 1 : 2

11. The circle passing through (1, – 2) and touching the axis of x at (3, 0) also passes through the point

(a) (– 5, 2) (b) (2, – 5)

(c) (5, – 2) (d) (– 2, 5)

12. If x, y and z are in AP and $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are also in AP, then

(a) $x = y = z$

(b) $2x = 3y = 6z$

(c) $6x = 3y = 2z$

(d) $6x = 4y = 3z$

13. Consider

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.

Statement II $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tautology.

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

14. If $\int f(x)\,dx = \psi(x)$, then $\int x^5 f(x^3)dx$ is equal to

(a) $\frac{1}{3}[x^3\psi(x^3) - \int x^2\psi(x^3)dx] + C$

(b) $\frac{1}{3}x^3\psi(x^3) - 3\int x^3\psi(x^3)\,dx + C$

(c) $\frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\,dx + C$

(d) $\frac{1}{3}[x^3\psi(x^3) - \int x^3\psi(x^3)\,dx] + C$

15. $\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

16. **Statement I** The value of the integral

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \text{ is equal to } \pi/6 .$$

Statement II $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

17. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is

(a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 6y + 7 = 0$

(c) $x^2 + y^2 - 6y - 5 = 0$ (d) $x^2 + y^2 - 6y + 5 = 0$

18. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is

(a) $\frac{17}{3^5}$ (b) $\frac{13}{3^5}$

(c) $\frac{11}{3^5}$ (d) $\frac{10}{3^5}$

19. The x-coordinate of the incentre of the triangle that has the coordinates of mid-points of its sides as (0, 1), (1, 1) and (1, 0) is

(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$

(c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$

20. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is

(a) 4 (b) 120

(c) 210 (d) 310

21. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in the first quadrant is

(a) 9 (b) 6

(c) 18 (d) $\frac{27}{4}$

22. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is

(a) 7 (b) 5

(c) 10 (d) 8

23. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to

(a) $-\theta$ (b) $\frac{\pi}{2}-\theta$

(c) θ (d) $\pi-\theta$

24. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(c) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

25. If $P=\begin{bmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

(a) 4 (b) 11

(c) 5 (d) 0

26. The intercepts on x-axis made by tangents to the curve, $y=\int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to

(a) ± 1

(b) ± 2

(c) ± 3

(d) ± 4

27. **Given** A circle, $2x^2+2y^2=5$ and a parabola, $y^2=4\sqrt{5}x$.

Statement I An equation of a common tangent to these curves is $y = x+\sqrt{5}$.

Statement II If the line, $y = mx+\frac{\sqrt{5}}{m}(m\neq 0)$ is the common tangent, then m satisfies $m^4-3m^2+2=0$.

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

28. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) 1 (d) $\sqrt{2}$

29. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as

(a) $\sin A\cos A+1$ (b) $\sec A\,\text{cosec}\,A+1$

(c) $\tan A+\cot A$ (d) $\sec A+\text{cosec}\,A$

30. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given?

(a) Mean (b) Median

(c) Mode (d) Variance

Answers

1. (c)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (d)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (c)
21. (a)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (d)

Hints & Solution

1. Given planes are

$$2x + y + 2z - 8 = 0$$

and $$2x + y + 2z + \frac{5}{2} = 0$$

Distance between two planes

$$= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2 + c^2}} = \left| \frac{-8 - \frac{5}{2}}{\sqrt{2^2 + 1^2 + 2^2}} \right|$$

$$= \frac{\frac{21}{2}}{3} = \frac{7}{2}$$

2. Given, $\frac{dP}{dx} = (100 - 12\sqrt{x}) \Rightarrow dP = (100 - 12\sqrt{x})\, dx$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})\, dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000 \Rightarrow C = 2000$

Now, when $x = 25$, then is

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$

$$= 2500 - 8 \times 125 + 2000$$

$$= 4500 - 1000 = 3500$$

3. Given, $n(A) = 2, n(B) = 4$

$\therefore \quad n(A \times B) = 8$

The number of subsets of $A \times B$ having 3 or more elements

$= {}^8C_3 + {}^8C_4 + \ldots + {}^8C_8$

$= ({}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$

$\therefore \quad 2^n = {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n$

$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$

$= 256 - 1 - 8 - 28 = 219$

4. Condition for two lines are coplanar.

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

where, (x_1, y_1, z_1) and (x_2, y_2, z_2) are the points lie on a line (i) and (ii) respectively and $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ are the direction cosines of the line (i) and line (ii) respectively.

$$\therefore \quad \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1 + 2k) + (1 + k^2) - (2 - k) = 0$

$\Rightarrow \quad k^2 + 2k + k = 0$

$\Rightarrow \quad k^2 + 3k = 0$

$\Rightarrow \quad k = 0, -3$

If 0 appears in the denominator, then the correct way of representing the equation of straight line is

$$\frac{x-2}{1} = \frac{y-3}{1};\ z = 4$$

5. We know that, the sum of three vectors of a triangle is zero.

$\therefore \quad \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$

$\Rightarrow \quad \mathbf{BC} = \mathbf{AC} - \mathbf{AB}$

$\Rightarrow \quad \mathbf{BM} = \frac{\mathbf{AC} - \mathbf{AB}}{2}$ ($\because$ M is a mid-point of **BC**)

Also, $\mathbf{AB} + \mathbf{BM} + \mathbf{MA} = 0$

(by properties of a triangle)

$\Rightarrow \quad \mathbf{AB} + \frac{\mathbf{AC} - \mathbf{AB}}{2} = \mathbf{AM}$

$\Rightarrow \quad \mathbf{AM} = \frac{\mathbf{AB} + \mathbf{AC}}{2} = \frac{3\hat{\mathbf{i}} + 4\hat{\mathbf{k}} + 5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} + 4\hat{\mathbf{k}}}{2}$

$= 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{AM}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

6. Let $f(x) = 2x^3 + 3x + k$

On differentiating w.r.t x, we get $f'(x) = 6x^2 + 3 > 0, \forall\, x \in R$

$\Rightarrow f(x)$ is strictly increasing function.

$\Rightarrow f(x) = 0$ has only one real root, so two roots are not possible.

7. Let $S = 0.7 + 0.77 + 0.777 + \ldots$

$= \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \ldots +$ upto 20 terms

$= 7\left[\frac{1}{10} + \frac{11}{10^2} + \frac{111}{10^3} + \ldots + \text{upto 20 terms}\right]$

$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots + \text{upto 20 terms}\right]$

$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \text{upto 20 terms}\right]$

$= \frac{7}{9}\left[(1 + 1 + \ldots + \text{upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \text{upto 20 terms}\right)\right]$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^{20}\right\}}{1-\frac{1}{10}}\right]$$

$$\left[\because \sum_{i=1}^{20} = 20 \text{ and sum of } n \text{ terms of GP } S_n = \frac{a(1-r^n)}{1-r} \text{ when } (r<1)\right]$$

$$= \frac{7}{9}\left[20 - \frac{1}{9}\left\{1-\left(\frac{1}{10}\right)^{20}\right\}\right]$$

$$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + (10)^{-20}]$$

8. Take any point $B(0, 1)$ on given line.

Equation of AB'

$$y - 0 = \frac{-1-0}{0-\sqrt{3}}(x - \sqrt{3})$$

$\Rightarrow \quad -\sqrt{3}y = -x + \sqrt{3}$

$\Rightarrow \quad x - \sqrt{3}y = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}y = x - \sqrt{3}$

9. Given equations can be written in matrix form

$$AX = B$$

where, $A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

and $B = \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$

For no solution, $|A| = 0$ and (ady A) $B \neq 0$

Now, $|A| = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix} = 0$

$\Rightarrow \quad (k+1)(k+3) - 8k = 0$

$\Rightarrow \quad k^2 + 4k + 3 - 8k = 0$

$\Rightarrow \quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-1)(k-3) = 0$

$\Rightarrow \quad k = 1, k = 3$

Now, adj $A = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}$

Now, (adj A) $B = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}\begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$

$$= \begin{bmatrix} (k+3)(4k) - 8(3k-1) \\ -4k^2 + (k+1)(3k-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4k^2 - 12k + 8 \\ -k^2 + 2k - 1 \end{bmatrix}$$

Put $k = 1$

(adj A) $B = \begin{bmatrix} 4-12+8 \\ -1+2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ not true

Put $k = 3$

(adj A) $B = \begin{bmatrix} 36-36+8 \\ -9+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \neq 0$ true.

Hence, required value of k is 3.

Alternate Solution

Condition for the system of equations has no solution,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

$\therefore \quad \frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$

Take $\frac{k+1}{k} = \frac{8}{k+3}$

$\Rightarrow \quad k^2 + 4k + 3 = 8k$

$\Rightarrow \quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-1)(k-3) = 0$

$k = 1, 3$

If $k = 1$, then $\frac{8}{1+3} \neq \frac{4 \cdot 1}{2}$, false

And if $k = 3$, then $\frac{8}{6} \neq \frac{4 \cdot 3}{9-1}$, true

Therefore $k = 3$

Hence, only one value of k exist.

10. Given equations are

$$x^2 + 2x + 3 = 0 \quad \text{...(i)}$$

and $$ax^2 + bx + c = 0 \quad \text{...(ii)}$$

Since, Eq. (i) has imaginary roots so, Eq. (ii) will also have both roots same as Eq. (i).

Thus, $\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

Hence, $a : b : c$ is $1 : 2 : 3$

11. Let the equation of circle be

$$(x-3)^2 + (y-0)^2 + \lambda y = 0$$

As it passes through $(1, -2)$

$\therefore \quad (1-3)^2 + (-2)^2 + \lambda(-2) = 0$

$\Rightarrow \quad 4 + 4 - 2\lambda = 0 \quad \Rightarrow \lambda = 4$

$\therefore$ Equation of circle is

$$(x-3)^2 + y^2 + 4y = 0$$

By hit and trial method, we see that point $(5, -2)$ satisfies equation of circle.

12. Since, x, y and z are in AP.

$\therefore \quad 2y = x + z$

Also, $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are in AP.

$\therefore \quad 2\tan^{-1} y = \tan^{-1} x + \tan^{-1}(z)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \quad \frac{x+z}{1-y^2} = \frac{x+z}{1-xz}$$

$$\Rightarrow \quad y^2 = xz$$

Since x, y and z are in AP as well as in GP

$\therefore \quad x = y = z$

13. **Statement II** $(p \to q) \leftrightarrow (\sim q \to \sim p)$

$\equiv (p \to q) \leftrightarrow (p \to q)$

which is always true, so statement II is true.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$

$\equiv p \wedge \sim q \wedge \sim p \wedge q$

$\equiv p \wedge \sim p \wedge \sim q \wedge q$

$\equiv f \wedge f \equiv f$

Hence, it is a fallacy statement.

So, statement I is true.

Alternate Solution

Statement II $(p \to q) \leftrightarrow (\sim q \to \sim p)$

$\sim q \to \sim p$ is contrapositive of $p \to q$ hence $(p \to q) \leftrightarrow (p \to q)$ will be a tautology.

Statement II $(p \wedge \sim q) \wedge (\sim p \wedge q)$

p	q	~p	~q	p ∧ ~q	~p ∧ q	(p ∧ ~q) ∧ (~p ∧ q)
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, it is a fallacy.

14. Given, $\int f(x)\, dx = \psi(x)$

Let $\quad I = \int x^5 f(x^3)\, dx$

Put $\quad x^3 = t \quad \Rightarrow \quad x^2 dx = \dfrac{dt}{3} \qquad \ldots(i)$

$$\therefore \quad I = \frac{1}{3}\int t\, f(t)\, dt$$

$$= \frac{1}{3}\left[t \cdot \int f(t)\, dt - \int \left\{\frac{d}{dt}(t) \int f(t)\, dt\right\} dt\right] \qquad \text{(by parts)}$$

$$= \frac{1}{3}[t\, \psi(t) - \int \psi(t)\, dt]$$

$$= \frac{1}{3}[x^3\psi(x^3) - 3\int x^2\psi(x^3)\, dx] + C \qquad [\because \text{from Eq. (i)}]$$

$$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\, dx + C$$

15. Let $I = \lim\limits_{x \to 0} \dfrac{(1-\cos 2x)}{x^2} \dfrac{(3+\cos x)}{1} \cdot \dfrac{x}{\tan 4x}$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3+\cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x \to 0}(3+\cos x) \cdot \lim_{x \to 0}\frac{4x}{4\tan 4x}$$

$$= 2 \cdot (1)^2 \cdot (3 + \cos 0°) \cdot \frac{1}{4}(1)$$

$$= 2 \cdot 1 \cdot (3+1) \cdot \frac{1}{4} = 2 \cdot 4 \cdot \frac{1}{4} = 2$$

16. Let $\quad I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} \qquad \ldots(i)$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2} - x\right)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1+\sqrt{\tan x}} \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12}$$

Statement I is false.

But $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$ is a true statement by property of definite integrals.

17. Given equation of ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Here, $a = 4, b = 3, e = \sqrt{1 - \dfrac{9}{16}} \Rightarrow \dfrac{\sqrt{7}}{4}$

$\therefore$ Foci is $(\pm ae, 0) = \left(\pm 4 \times \dfrac{\sqrt{7}}{4}, 0\right) = (\pm\sqrt{7}, 0)$

$\therefore$ Radius of the circle, $r = \sqrt{(ae)^2 + b^2}$

$$= \sqrt{7+9} = \sqrt{16} = 4$$

Now, equation of circle is

$$(x-0)^2 + (y-3)^2 = 16$$

$$\therefore \quad x^2 + y^2 - 6y - 7 = 0$$

18. Probability of guessing a correct answer, $p = \frac{1}{3}$ and probability of guessing a wrong answer, $q = \frac{2}{3}$

$\therefore$ The probability of guessing a 4 or more correct answer

$$= {}^5C_4\left(\frac{1}{3}\right)^4 \cdot \frac{2}{3} + {}^5C_5\left(\frac{1}{3}\right)^5$$

$$= 5 \cdot \frac{2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}$$

19. Given mid-points of a triangle are (0, 1), (1, 1) and (1, 0). Plotting these points on a graph paper and make a triangle. So, the sides of a triangle will be 2, 2 and $\sqrt{2^2 + 2^2}$

ie, $2\sqrt{2}$

$$x\text{-coordinate of incentre} = \frac{2 \times 0 + 2\sqrt{2} \cdot 0 + 2 \cdot 2}{2 + 2 + 2\sqrt{2}}$$

$$= \frac{2}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}} = 2 - \sqrt{2}$$

20. $\therefore \left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$

$$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}\,(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[\frac{(x^{1/3} + 1)(x^{2/3} + 1 - x^{1/3})}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}\,(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}}\right]^{10}$$

$$= (x^{1/3} - x^{-1/2})^{10}$$

$\therefore$ The general term is

$$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$$

$$= {}^{10}C_r (-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$$

For independent of x, put

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$$\Rightarrow \quad 20 - 2r - 3r = 0$$

$$\Rightarrow \quad 20 = 5r$$

$$\Rightarrow \quad r = 4$$

$$\therefore \quad T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

21. Given curves are $\quad y = \sqrt{x} \quad$ ...(i)

and $\quad 2y - x + 3 = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$$\Rightarrow (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$$

$$\Rightarrow \sqrt{x} = 3, \qquad (\because \sqrt{x} = -1 \text{ is not possible})$$

$$\therefore \quad y = 3$$

$$\therefore \quad \text{Required area} = \int_0^3 (x_2 - x_1)\, dy$$

$$= \int_0^3 \{(2y+3) - y^2\}\, dy$$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9$$

22. Given, $T_n = {}^nC_3$

$$T_{n+1} = {}^{n+1}C_3$$

$$\therefore \quad T_{n+1} - T_n = {}^{n+1}C_3 - {}^nC_3 = 10 \qquad \text{(given)}$$

$$\Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 10$$

$$(\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$$

$$\Rightarrow \quad {}^nC_2 = 10$$

$$\Rightarrow \quad n = 5$$

23. Given, $|z| = 1$, $\arg z = \theta \therefore z = e^{i\theta}$

But $\quad \bar{z} = \frac{1}{z}$

$$\therefore \quad \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta$$

24. Let $AB = x$

In ΔDAM, $\tan(\pi - \theta - \alpha) = \dfrac{p}{x-q}$

$\Rightarrow \quad \tan(\theta + \alpha) = \dfrac{p}{q-x}$

$\Rightarrow \quad q - x = p \cot(\theta + \alpha)$

$\Rightarrow \quad x = q - p \cot(\theta + \alpha)$

$$= q - p\left(\frac{\cot\theta \cot\alpha - 1}{\cot\alpha + \cot\theta}\right) \qquad \left(\because \cot\alpha = \frac{q}{p}\right)$$

$$= q - p\left(\frac{\frac{q}{p}\cot\theta - 1}{\frac{q}{p} + \cot\theta}\right) = q - p\left(\frac{q\cot\theta - p}{q + p\cot\theta}\right)$$

$$= q - p\left(\frac{q\cos\theta - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$$

$$\Rightarrow \quad x = \frac{q^2\sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta + q\sin\theta}$$

$$\Rightarrow \quad AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Alternate Solution

Applying Sine rule in ΔABD,

$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin\{\pi - (\theta+\alpha)\}}$$

$$\Rightarrow \quad \frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{\sqrt{p^2+q^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$= \frac{(p^2+q^2)\sin\theta}{q\sin\theta + p\cos\theta} \quad \left(\because \cos\alpha = \frac{q}{\sqrt{p^2+q^2}}\right)$$

$$= \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta} \text{ and } \sin\alpha = \frac{p}{\sqrt{p^2+q^2}}$$

25. Given, $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

$\therefore \quad |P| = 1(12-12) - \alpha(4-6) + 3(4-6)$

$= 2\alpha - 6$

$\because \quad P = \text{adj}(A)$ (given)

$\therefore \quad |P| = |\text{adj } A|$

$= |A|^2 = 16 \qquad (\because |\text{ady } A| = |A|^{n-1})$

$\therefore \quad 2\alpha - 6 = 16$

$\Rightarrow \quad 2\alpha = 22$

$\Rightarrow \quad \alpha = 11$

26. Given, $y = \int_0^x |t|\, dt$

$\therefore \quad \dfrac{dy}{dx} = |x| \cdot 1 - 0 = |x|$ (by Leibnitz rule)

$\because$ Tangent to the curve $y = \int_0^x (t)\, dt$, $x \in R$ are parallel to the line $y = 2x$

$\therefore$ Slope of both are equal

$\Rightarrow \quad x = \pm 2$

$\therefore$ Points, $y = \int_0^{\pm 2} |t|\, dt = \pm 2$

$\therefore$ Equation of tangent is

$y - 2 = 2(x-2)$

and $\quad y + 2 = 2(x+2)$

For x-intercept put $y = 0$, we get

$0 - 2 = 2(x-2)$

and $\quad 0 + 2 = 2(x+2)$

$\Rightarrow \quad x = \pm 1$

27. Equation of circle can be rewritten as

$$x^2 + y^2 = \frac{5}{2},$$

centre $\rightarrow (0,)$ and radius $\rightarrow \sqrt{\dfrac{5}{2}}$

Let common tangent be

$$y = mx + \frac{\sqrt{5}}{m}$$

$\Rightarrow \quad m^2x - my + \sqrt{5} = 0$

The perpendicular from centre to the tangent is equal to radius of the circle.

$$\therefore \quad \frac{\frac{\sqrt{5}}{m}}{\sqrt{1+m^2}} = \sqrt{\frac{5}{2}}$$

$\Rightarrow \quad m\sqrt{1+m^2} = \sqrt{2}$

$\Rightarrow \quad m^2(1+m^2) = 2$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = \pm 1 \quad (\because m^2 + 2 \neq 0, \text{ as } m \in R)$

$\therefore\ y = \pm (x + \sqrt{5})$, both statements are correct as $m = \pm 1$ satisfies the given equation of statement II.

28. Given, $y = \sec(\tan^{-1} x)$

Let $\tan^{-1} x = \theta \Rightarrow x = \tan\theta$

$\therefore \quad y = \sec\theta = \sqrt{1 + x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 + x^2}} \cdot 2x$$

At $x = 1$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

29. Given expression is

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A} = 1 + \sec A \operatorname{cosec} A$$

30. If initially all marks were x_i, then

$$\sigma_1^2 = \frac{\sum (x_i - \bar{x})^2}{N}$$

Now, each is increased by 10

$$\therefore \quad \sigma_2^2 = \frac{\sum (x_i + 10) - (\bar{x} + 10)]^2}{N} = \sigma_1^2$$

So, variance will not change whereas mean, median and mode will increase by 10.

Solved Paper 2014

JEE Main

Joint Entrance Examination

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

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1. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$; where N is the set of natural numbers, then $X \cup Y$ is equal to

(a) N (b) $Y - X$
(c) X (d) Y

2. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(a) is equal to $\frac{5}{2}$
(b) lies in the interval (1, 2)
(c) is strictly greater than $\frac{5}{2}$
(d) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

3. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where, [x] denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval

(a) $(-1, 0) \cup (0, 1)$ (b) $(1, 2)$
(c) $(-2, -1)$ (d) $(-\infty, -2) \cup (2, \infty)$

4. Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q and r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is

(a) $\frac{\sqrt{61}}{9}$ (b) $\frac{2\sqrt{17}}{9}$ (c) $\frac{\sqrt{34}}{9}$ (d) $\frac{2\sqrt{13}}{9}$

5. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$= K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to

(a) $\alpha\beta$ (b) $\frac{1}{\alpha\beta}$
(c) 1 (d) -1

6. If A is a 3×3 non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then BB^T is equal to

(a) $I + B$ (b) I
(c) B^{-1} (d) $(B^{-1})^T$

7. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to

(a) $\left(16, \frac{251}{3}\right)$ (b) $\left(14, \frac{251}{3}\right)$
(c) $\left(14, \frac{272}{3}\right)$ (d) $\left(16, \frac{272}{3}\right)$

8. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

9. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + ... + 10(11)^9 = k(10)^9$,

then k is equal to

(a) $\frac{121}{10}$ (b) $\frac{441}{100}$
(c) 100 (d) 110

10. Three positive numbers form an increasing GP. If the middle term in this GP is doubled, then new numbers are in AP. Then, the common ratio of the GP is

(a) $\sqrt{2} + \sqrt{3}$ (b) $3 + \sqrt{2}$
(c) $2 - \sqrt{3}$ (d) $2 + \sqrt{3}$

11. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

(a) $\frac{\pi}{2}$ (b) 1
(c) $-\pi$ (d) π

12. If g is the inverse of a function f and $f'(x) = \frac{1}{1 + x^5}$,

then $g'(x)$ is equal to

(a) $1 + x^5$ (b) $5x^4$
(c) $\frac{1}{1 + \{g(x)\}^5}$ (d) $1 + \{g(x)\}^5$

13. If f and g are differentiable functions in (0, 1) satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$
(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

14. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

(a) $\alpha = -6, \beta = \frac{1}{2}$ (b) $\alpha = -6, \beta = -\frac{1}{2}$
(c) $\alpha = 2, \beta = -\frac{1}{2}$ (d) $\alpha = 2, \beta = \frac{1}{2}$

15. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$ is equal to

(a) $(x - 1)e^{x + \frac{1}{x}} + C$ (b) $xe^{x + \frac{1}{x}} + C$
(c) $(x + 1)e^{x + \frac{1}{x}} + C$ (d) $-xe^{x + \frac{1}{x}} + C$

16. The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin \frac{x}{2}}\, dx$ is equal to

(a) $\pi - 4$ (b) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$
(c) $4\sqrt{3} - 4$ (d) $4\sqrt{3} - 4 - \frac{\pi}{3}$

17. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is

(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$
(c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$

18. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ is equal to

(a) $400 - 300e^{\frac{t}{2}}$ (b) $300 - 200e^{-\frac{t}{2}}$
(c) $600 - 500e^{\frac{t}{2}}$ (d) $400 - 300e^{-\frac{t}{2}}$

19. If PS is the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$, then equation of the line passing through $(1, -1)$ and parallel to PS is

(a) $4x - 7y - 11 = 0$
(b) $2x + 9y + 7 = 0$
(c) $4x + 7y + 3 = 0$
(d) $2x - 9y - 11 = 0$

20. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then

(a) $2bc - 3ad = 0$
(b) $2bc + 3ad = 0$
(c) $2ad - 3bc = 0$
(d) $3bc + 2ad = 0$

21. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
(b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
(d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

22. Let C be the circle with centre at (1, 1) and radius 1. If T is the circle centred at (0, y) passing through origin and touching the circle C externally, then the radius of T is equal to

(a) $\frac{\sqrt{3}}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

23. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) $\frac{1}{8}$ (d) $\frac{2}{3}$

24. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line

(a) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$

(b) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

(c) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$

(d) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

25. If $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \lambda [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$, then λ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

26. Let A and B be two events such that $P(\overline{A \cup B}) = 1/6$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then, the events A and B are

(a) independent but not equally likely

(b) independent and equally likely

(c) mutually exclusive and independent

(d) equally likely but not independent

27. The variance of first 50 even natural numbers is

(a) $\frac{833}{4}$ (b) 833 (c) 437 (d) $\frac{437}{4}$

28. If $f_k(x) = 1/k\,(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$, then $f_4(x) - f_6(x)$ is equal to

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{12}$

29. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After 1s, the elevation of the bird from O is reduced to 30°. Then, the speed (in m/s) of the bird is

(a) $40(\sqrt{2} - 1)$ (b) $40(\sqrt{3} - \sqrt{2})$

(c) $20\sqrt{2}$ (d) $20(\sqrt{3} - 1)$

30. The statement $\sim(p \leftrightarrow \sim q)$ is

(a) equivalent to $p \leftrightarrow q$ (b) equivalent to $\sim p \leftrightarrow q$

(c) a tautology (d) a fallacy

Answers

1. (d)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (c)	**10.** (d)
11. (d)	**12.** (d)	**13.** (d)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (c)
21. (c)	**22.** (d)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (d)	**30.** (a)

Solutions

1. We have, $X = \{4^n - 3n - 1 : n \in N\}$

$\Rightarrow$ $X = \{0, 9, 54, 243, \ldots\}$ (Put $n = 1, 2, 3, \ldots$)

and $Y = \{9(n-1) : n \in N\}$

$\Rightarrow$ $Y = \{0, 9, 18, 27, \ldots\}$ (Put $n = 1, 2, 3, \ldots$)

It is clear that $X \subset Y$

$\therefore$ $X \cup Y = Y$

2. $|z| \geq 2$ is the region on or outside circle whose centre is (0, 0) and radius is 2.

Minimum $\left|z + \frac{1}{2}\right|$ is distance of z, which lie on circle $|z| = 2$ from $\left(-\frac{1}{2}, 0\right)$.

$\therefore$ Minimum $\left|z + \frac{1}{2}\right|$ = Distance of $\left(-\frac{1}{2}, 0\right)$ from (–2, 0)

$$= \sqrt{\left(-2 + \frac{1}{2}\right)^2 + 0} = \frac{3}{2}$$

$$= \sqrt{\left(\frac{-1}{2} + 2\right)^2 + 0} = \frac{3}{2}$$

Geometrically Min $\left|z + \frac{1}{2}\right| = AD$

Hence, minimum value of $\left(z + \frac{1}{2}\right)$ lies in the interval (1, 2).

3. *Put $t = x - [x] = \{X\}$, which is a fractional part function and lie between $0 \leq \{X\} < 1$ and then solve it.*

Given, $a \in R$ and equation is

$$-3\{x - [x]\}^2 + 2\{x - [x]\} + a^2 = 0$$

Let $t = x - [x]$, then equation is

$\therefore$ $-3t^2 + 2t + a^2 = 0$

$\Rightarrow$ $t = \dfrac{1 \pm \sqrt{1 + 3a^2}}{3}$

$\because$ $t = x - [x] = \{X\}$ (Fractional part)

$\therefore$ $0 \leq t \leq 1$

$\Rightarrow$ $0 \leq \dfrac{1 \pm \sqrt{1 + 3a^2}}{3} \leq 1$

Taking positive sign

$$0 \leq \frac{1 + \sqrt{1 + 3a^2}}{3} < 1 \qquad (\because \{x\} > 0)$$

$\Rightarrow$ $\sqrt{1 + 3a^2} < 2$

$\Rightarrow$ $1 + 3a^2 < 4 \Rightarrow a^2 - 1 < 0$

$\Rightarrow$ $(a + 1)(a - 1) < 0$

$a \in (-1, 1)$

For no integral solution of a, we consider the interval $(-1, 0) \cup (0, 1)$.

Note *Here, we figure out the integral solution, we get $a = 0$. This implies any interval excluding zero should be correct answer as it gives either no solution or no integral solution.*

4. *If $ax^2 + bx + c = 0$ has roots α and β, then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$. Find the values of $\alpha + \beta$ and $\alpha\beta$ and then put in $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to get required value.*

Given, α and β are roots of $px^2 + qx + r = 0$, $p \neq 0$

$\therefore$ $\alpha + \beta = \dfrac{-q}{p}, \quad \alpha\beta = \dfrac{r}{p}$ ...(i)

Since, p, q and r are in AP.

$\therefore$ $2q = p + r$ ...(ii)

Also, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 4$

$\Rightarrow$ $\dfrac{\alpha + \beta}{\alpha\beta} = 4 \Rightarrow \alpha + \beta = 4\alpha\beta$

$\Rightarrow$ $\dfrac{-q}{p} = \dfrac{4r}{p}$ [From Eq. (i)]

$\Rightarrow$ $q = -4r$

On putting the value of q in Eq. (ii), we get

$2(-4r) = p + r$

$\Rightarrow$ $p = -9r$

Now, $\alpha + \beta = \dfrac{-q}{p} = \dfrac{4r}{p} = \dfrac{4r}{-9r} = -\dfrac{4}{9}$

and $\alpha\beta = \dfrac{r}{p} = \dfrac{r}{-9r} = \dfrac{1}{-9}$

$\therefore$ $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= \dfrac{16}{81} + \dfrac{4}{9} = \dfrac{16 + 36}{81}$

$\Rightarrow$ $(\alpha - \beta)^2 = \dfrac{52}{81}$

$\Rightarrow$ $|\alpha - \beta| = \dfrac{2}{9}\sqrt{13}$

5. *Use the property that two determinants it can be multiplied row-to-row or row-to-column, to write the given determinant as the product of two determinants and then expand.*

Given, $f(n) = \alpha^n + \beta^n$

$\therefore$ $f(1) = \alpha + \beta$, $f(2) = \alpha^2 + \beta^2$,

$f(3) = \alpha^3 + \beta^3$, $f(4) = \alpha^4 + \beta^4$

Let $\Delta = \begin{vmatrix} 3 & 1 + f(1) & 1 + f(2) \\ 1 + f(1) & 1 + f(2) & 1 + f(3) \\ 1 + f(2) & 1 + f(3) & 1 + f(4) \end{vmatrix}$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\cdot1+1\cdot1+1\cdot1 & 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+\alpha\cdot\alpha+\beta\cdot\beta & 1\cdot1+\alpha\cdot\alpha^2+\beta\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 & 1\cdot1+\alpha^2\cdot\alpha+\beta^2\cdot\beta & 1\cdot1+\alpha^2\cdot\alpha^2+\beta^2\cdot\beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

On expanding, we get

$$\Delta = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

But given, $\Delta = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

Hence, $K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

$\therefore \quad K = 1$

6. *Use the following properties of transpose $(AB)^T = B^TA^T, (A^T)^T = A$, and $A^{-1}A = I$ and simplify.*

If A is non-singular matrix, then $|A| \neq 0$

Given, $AA^T = A^TA$ and $B = A^{-1}A^T$

Now, $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$

$= A^{-1}A^TA(A^{-1})^T \quad (\because (AB)^T = B^TA^T)$

$= A^{-1}AA^T(A^{-1})^T \quad (\because AA^T = A^TA)$

$= IA^T(A^{-1})^T \quad (\because A^{-1}A = I)$

$= A^T(A^{-1})^T$

$= (A^{-1}A)^T \quad (\because (AB)^T = B^TA^T)$

$= I^T = I$

7. *To find the coefficient of x^3 and x^4, use the formula of coefficient of x^r in $(1-x)^n$ is $(-1)^r\,{}^nC_r$ and then simplify.*

In expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$

Coefficient of x^3 = Coefficient of x^3 in $(1-2x)^{18}$

+ Coefficient of x^2 in $a(1-2x)^{18}$

+ Coefficient of x in $b(1-2x)^{18}$

$$= -{}^{18}C_3\cdot2^3 + a\,{}^{18}C_2\cdot2^2 - b\,{}^{18}C_1\cdot2$$

Given, coefficient of $x^3 = 0$

$$\Rightarrow \quad -{}^{18}C_3\cdot2^3 + a\,{}^{18}C_2\cdot2^2 - b\,{}^{18}C_1\cdot2 = 0$$

$$\Rightarrow \quad -\frac{18\times17\times16}{3\times2}\cdot8 + a\cdot\frac{18\times17}{2}\cdot2^2 - b\cdot18\cdot2 = 0$$

$$\Rightarrow \quad 17a - b = \frac{34\times16}{3} \quad \text{...(i)}$$

Similarly, coefficient of x^4

$${}^{18}C_4\cdot2^4 - a\cdot{}^{18}C_3 2^3 + b\cdot{}^{18}C_2\cdot2^2 = 0$$

$$\Rightarrow \quad 32a - 3b = 240 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$a = 16, \quad b = \frac{272}{3}$$

8. We know that angle between two lines is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Given, $l + m + n = 0 \Rightarrow l = -(m+n)$

$\Rightarrow \quad (m+n)^2 = l^2$

$\Rightarrow \quad m^2 + n^2 + 2mn = m^2 + n^2 \quad (\because l^2 = m^2 + n^2$, given)

$\Rightarrow \quad 2mn = 0$

When, $m = 0 \Rightarrow l = -n$

Hence, (l_1, m_1, n_1) is $(1, 0, -1)$.

When $n = 0$, then $l = -m$.

Hence, (l_2, m_2, n_2) is $(1, -1, 0)$.

$$\therefore \quad \cos\theta = \frac{1+0+0}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

9. Given, $k\cdot10^9 = 10^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9$

$$\Rightarrow \quad k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \ldots + 10\left(\frac{11}{10}\right)^9 \quad \text{...(i)}$$

$$\Rightarrow \quad \left(\frac{11}{10}\right)k = 1\left(\frac{11}{10}\right) + 2\left(\frac{11}{10}\right)^2 + \ldots + 9\left(\frac{11}{10}\right)^9 + 10\left(\frac{11}{10}\right)^{10} \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$k\left(1 - \frac{11}{10}\right) = 1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 + \ldots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}$$

$$\Rightarrow \quad k\left(\frac{10-11}{10}\right) = \frac{1\left[\left(\frac{11}{10}\right)^{10} - 1\right]}{\left(\frac{11}{10} - 1\right)} - 10\left(\frac{11}{10}\right)^{10}$$

$\left[\because \text{In GP, sum of } n \text{ terms} = \frac{a(r^n - 1)}{r-1}, \text{ when } r > 1\right]$

$$\Rightarrow \quad -k = 10\left[10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}\right]$$

$$\Rightarrow \quad k = 100$$

10. Let a, ar, ar^2 are in GP $(r > 1)$.

On multiplying middle term by 2, $a, 2ar, ar^2$ are in AP.

$\therefore \quad 4ar = a + ar^2$

$\Rightarrow \quad r^2 - 4r + 1 = 0$

$$\Rightarrow \quad r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$\Rightarrow \quad r = 2 + \sqrt{3} \quad (\because$ AP is increasing)

11. $$\lim_{x\to0}\frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to0}\frac{\sin\pi(1-\sin^2 x)}{x^2} = \lim_{x\to0}\frac{\sin(\pi - \pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to0}\frac{\sin(\pi\sin^2 x)}{x^2} \quad (\because \sin(\pi - \theta) = \sin\theta)$$

$$= \lim_{x\to0}\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times(\pi)\left(\frac{\sin^2 x}{x^2}\right) = \pi$$

$$\left(\because \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1\right)$$

12. Here, 'g' is the inverse of $f(x)$.

$\Rightarrow \quad fog(x) = x$

On differentiating w.r.t. x, we get

$$f'\{g(x)\} \times g'(x) = 1$$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

$$= \frac{1}{\frac{1}{1+\{g(x)\}^5}} \qquad \left[\because f'(x) = \frac{1}{1+x^5}\right]$$

$$\Rightarrow \quad g'(x) = 1 + \{g(x)\}^5$$

13. Given, $f(0) = 2 = g(1), g(0) = 0$ and $f(1) = 6$

f and g are differentiable in (0, 1).

Let $\quad h(x) = f(x) - 2g(x) \quad$ ...(i)

Then, $\quad h(0) = f(0) - 2g(0)$

$\Rightarrow \quad h(0) = 2 - 0$

$\Rightarrow \quad h(0) = 2$

and $\quad h(1) = f(1) - 2g(1) = 6 - 2(2)$

$\Rightarrow \quad h(1) = 2$

Thus, $h(0) = h(1) = 2$

Hence, using Rolle's theorem

$$h'(c) = 0 \text{ such that } c \in (0, 1)$$

Differentiating Eq. (i) at c, we get

$\Rightarrow \quad f'(c) - 2g'(c) = 0 \quad \Rightarrow \quad f'(c) = 2g'(c)$

14. Here, $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha\log|x| + \beta x^2 + x$, then

$$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$$

$$f'(-1) = -\alpha - 2\beta + 1 = 0 \quad \text{...(i)}$$

[At extreme point, $f'(x) = 0$]

$$f'(2) = \frac{\alpha}{2} + 4\beta + 1 = 0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\alpha = 2, \ \beta = -\frac{1}{2}$$

15. $\int\left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$$= \int e^{x+\frac{1}{x}}dx + \int x\left(1 - \frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$$

$$= \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int \frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$$

$$\left[\because \int\left(1 - \frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx = e^{x+\frac{1}{x}}\right]$$

$$= \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}dx$$

$$= xe^{x+\frac{1}{x}} + C$$

16. Use the formula

$|x - a| = \begin{cases} x - a, & x \geq a \\ -(x - a), & x < a \end{cases}$ to break given integral in two parts and then integrate separately.

$$\int_0^\pi \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}dx$$

$$= \int_0^\pi \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2}dx = \int_0^\pi \left|1 - 2\sin\frac{x}{2}\right|dx$$

$$= \int_0^{\frac{\pi}{3}}\left(1 - 2\sin\frac{x}{2}\right)dx - \int_{\frac{\pi}{3}}^{\pi}\left(1 - 2\sin\frac{x}{2}\right)dx$$

$$= \left(x + 4\cos\frac{x}{2}\right)_0^{\frac{\pi}{3}} - \left(x + 4\cos\frac{x}{2}\right)_{\frac{\pi}{3}}^{\pi}$$

$$= 4\sqrt{3} - 4 - \frac{\pi}{3}$$

17. Given, $A = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$

Required area $= \frac{1}{2}\pi r^2 + 2\int_0^1 (1 - y^2)dy$

$$= \frac{1}{2}\pi(1)^2 + 2\left(y - \frac{y^3}{3}\right)_0^1 = \frac{\pi}{2} + \frac{4}{3}$$

18. Given differential equation $\frac{dp}{dt} - \frac{1}{2}p(t) = -200$ is a linear differential equation.

Here, $P(t) = -\frac{1}{2}, Q(t) = -200$

$\therefore \quad \text{IF} = e^{\int -\left(\frac{1}{2}\right)dt} = e^{-\frac{t}{2}}$

Hence, solution is

$$P(t) \cdot \text{IF} = \int Q(t) \cdot \text{IF}dt$$

$$\Rightarrow \quad p(t) \cdot e^{-\frac{t}{2}} = \int -200 \cdot e^{-\frac{t}{2}}dt$$

$$\Rightarrow \quad p(t) \cdot e^{-\frac{t}{2}} = 400e^{-\frac{t}{2}} + K$$

$\Rightarrow \quad p(t) = 400 + ke^{-1/2}$

If $p(0) = 100$, then $\quad K = -300$

$\therefore \quad p(t) = 400 - 300e^{\frac{t}{2}}$

19. Coordinate of $S = \left(\frac{7+6}{2}, \frac{3-1}{2}\right) = \left(\frac{13}{2}, 1\right)$

($\because S$ is mid-point of line QR)

Slope of the line PS is $\frac{2-1}{2-\frac{13}{2}} = \frac{-2}{9}$.

Required equation passes through (1, – 1) and parallel to PS is

$$y + 1 = \frac{-2}{9}(x - 1) \quad \Rightarrow \quad 2x + 9y + 7 = 0$$

20. Let coordinate of the intersection point in fourth quadrant be $(\alpha, -\alpha)$.

Since, $(\alpha, -\alpha)$ lies on both lines

$$4ax + 2ay + c = 0$$

and $$5bx + 2by + d = 0$$

$\therefore$ $$4a\alpha - 2a\alpha + c = 0$$

$\Rightarrow$ $$\alpha = \frac{-c}{2a} \quad \text{...(i)}$$

and $$5b\alpha - 2b\alpha + d = 0$$

$\Rightarrow$ $$\alpha = \frac{-d}{3b} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{-c}{2a} = \frac{-d}{3b}$$

$\Rightarrow$ $$3bc = 2ad$$

$\Rightarrow$ $$2ad - 3bc = 0$$

21. Equation of ellipse is $x^2 + 3y^2 = 6$.

or $$\frac{x^2}{6} + \frac{y^2}{2} = 1$$

Equation of the tangent is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

Let (h, k) be any point on the locus.

$\therefore$ $$\frac{h}{a}\cos\theta + \frac{k}{b}\sin\theta = 1 \quad \text{...(i)}$$

Slope of the tangent line is $\frac{-b}{a}\cot\theta$.

Slope of perpendicular drawn from centre (0,0) to (h, k) is $\frac{k}{h}$.

Since, both the lines are perpendicular.

$\therefore$ $$\left(\frac{k}{h}\right) \times \left(-\frac{b}{a}\cot\theta\right) = -1$$

$\Rightarrow$ $$\frac{\cos\theta}{ha} = \frac{\sin\theta}{kb} = \alpha \quad \text{(Say)}$$

$\Rightarrow$ $$\cos\theta = \alpha ha$$
$$\sin\theta = \alpha kb$$

From Eq. (i), we get

$$\frac{h}{a}(\alpha ha) + \frac{k}{b}(\alpha kb) = 1$$

$\Rightarrow$ $$h^2\alpha + k^2\alpha = 1$$

$\Rightarrow$ $$\alpha = \frac{1}{h^2 + k^2}$$

Also, $$\sin^2\theta + \cos^2\theta = 1$$

$\Rightarrow$ $$(\alpha kb)^2 + (\alpha ha)^2 = 1$$

$\Rightarrow$ $$\alpha^2k^2b^2 + \alpha^2h^2a^2 = 1$$

$\Rightarrow$ $$\frac{k^2b^2}{(h^2 + k^2)^2} + \frac{h^2a^2}{(h^2 + k^2)^2} = 1$$

$\Rightarrow$ $$\frac{2k^2}{(h^2 + k^2)^2} + \frac{6h^2}{(h^2 + k^2)^2} = 1 \quad (\because a^2 = 6, b^2 = 2)$$

$\Rightarrow$ $$6x^2 + 2y^2 = (x^2 + y^2)^2$$

(Replacing k by y and h by x)

22. *Use the property, when two circles touch each other externally, then distance between their centre is equal to sum of their radii, to get required radius.*

Let the coordinate of the centre of T be $(0, k)$.

Distance between their centre, $k + 1 = \sqrt{1 + (k-1)^2}$

$(\because C_1C_2 = k + 1)$

$\Rightarrow$ $$k + 1 = \sqrt{1 + k^2 + 1 - 2k}$$

$\Rightarrow$ $$k + 1 = \sqrt{k^2 + 2 - 2k}$$

$\Rightarrow$ $$k^2 + 1 + 2k = k^2 + 2 - 2k$$

$\Rightarrow$ $$k = \frac{1}{4}$$

So, the radius of circle T is k *i.e.*, $\frac{1}{4}$.

23. *Let the tangent to parabola is $y = mx + a/m$, since it touches the other curve, then $D = 0$ to get the value of m.*

For parabola, $y^2 = 4x$

Let $y = mx + \frac{1}{m}$ be tangent line and it touches the parabola $x^2 = -32y$.

$\therefore$ $$x^2 = -32\left(mx + \frac{1}{m}\right)$$

$\Rightarrow$ $$x^2 + 32mx + \frac{32}{m} = 0$$

Now, $$D = 0$$

$\Rightarrow$ $$(32m)^2 - 4 \cdot \left(\frac{32}{m}\right) = 0$$

$\Rightarrow$ $$m^3 = \frac{1}{8}$$

$\therefore$ $$m = \frac{1}{2}$$

24. *Here, plane, line and its image are parallel to each other. So, find any point on the normal to the plane from which the image line will be passed and then find equation of image line.*

Here, plane and line are parallel to each other. Equation of normal to the plane through the point (1, 3, 4) is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = k \quad \text{(Say)}$$

Any point on this normal is $(2k + 1, -k + 3, 4 + k)$.

Then, $\left(\frac{2k+1+1}{2}, \frac{3-k+3}{2}, \frac{4+k+4}{2}\right)$ lies on plane.

$\Rightarrow \quad 2(k+1) - \left(\frac{6-k}{2}\right) + \left(\frac{8+k}{2}\right) + 3 = 0$

$\Rightarrow \quad k = -2$

Hence, point through which this image pass is

$(2k+1, 3-k, 4+k)$

i.e., $(2(-2)+1, 3+2, 4-2) = (-3, 5, 2)$.

Hence, equation of image line is

$$\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$$

25. Use the following formula to simplify

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = [\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$$

and $[\mathbf{a}\,\mathbf{a}\,\mathbf{b}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{b}] = [\mathbf{a}\,\mathbf{c}\,\mathbf{c}] = 0$

We have, $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \mathbf{a} \times \mathbf{b} \cdot ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}))$

$= \mathbf{a} \times \mathbf{b} \cdot (\mathbf{k} \times (\mathbf{c} \times \mathbf{a}))$ (Let, $\mathbf{k} = \mathbf{b} \times \mathbf{c}$)

$= \mathbf{a} \times \mathbf{b} \cdot ((\mathbf{k} \cdot \mathbf{a})\mathbf{c} - (\mathbf{k} \cdot \mathbf{c})\mathbf{a})$

$= (\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{b} \times \mathbf{c} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \times \mathbf{c} \cdot \mathbf{c})\mathbf{a})$

$= (\mathbf{a} \times \mathbf{b}) \cdot ([\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\,\mathbf{c}) - 0$

$(\because [\mathbf{b} \times \mathbf{c} \cdot \mathbf{c}] = 0)$

$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\,[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]$

$= [\mathbf{a}\,\mathbf{b}\,\mathbf{c}][\mathbf{b}\,\mathbf{c}\,\mathbf{a}]$

$= [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$ $\{\because [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\}$

But given, $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \lambda[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]^2$

So, $[\mathbf{a}\,\mathbf{b}\mathbf{c}]^2 = \lambda[\mathbf{a}\,\mathbf{b}\mathbf{c}]^2$

$\Rightarrow \quad \lambda = 1$

26. Given, $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$, $P(\overline{A}) = \frac{1}{4}$

$\therefore \quad P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$

and $P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad \frac{5}{6} = \frac{3}{4} + P(B) - \frac{1}{4}$

$\Rightarrow P(B) = \frac{1}{3} \Rightarrow A$ and B are not equally likely.

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4}$$

So, events are independent.

27. Here, $\overline{X} = \frac{\Sigma X_i}{n} = \frac{2+4+6+8+\ldots+100}{50}$

$= \frac{50 \times 51}{50}$ $[\because \sum 2n = n(n+1)$, here $n = 50]$

$= 51$

Variance, $\sigma^2 = \frac{1}{n}\Sigma X_i^2 - (\overline{X})^2$

$\Rightarrow \quad \sigma^2 = \frac{1}{50}(2^2 + 4^2 + \ldots + 100^2) - (51)^2 = 833$

28. Given, $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k > 1$

$f_4(x) - f_6(x) = \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$

$= \frac{1}{4}(1 - 2\sin^2 x \cdot \cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x \cdot \cos^2 x)$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

29. In ΔOA_1B_1,

$$\tan 45° = \frac{A_1B_1}{OB_1} \Rightarrow \frac{20}{OB_1} = 1 \Rightarrow OB_1 = 20$$

In ΔOA_2B_2, $\tan 30° = \frac{20}{OB_2} \Rightarrow OB_2 = 20\sqrt{3}$

$\Rightarrow \quad B_1B_2 + OB_1 = 20\sqrt{3} \Rightarrow B_1B_2 = 20\sqrt{3} - 20$

$\Rightarrow \quad B_1B_2 = 20(\sqrt{3} - 1)$ m

Now, Speed $= \frac{\text{Distance}}{\text{Time}} = \frac{20(\sqrt{3}-1)}{1} = 20(\sqrt{3} - 1)$ m/s

30.

p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$p \leftrightarrow \sim q$	$\sim p \leftrightarrow q$	$\sim (p \leftrightarrow \sim q)$
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	F
T	T	F	F	T	F	F	T
F	F	T	T	T	F	F	T

$\sim (p \leftrightarrow \sim q)$ is equivalent to $(p \leftrightarrow q)$.

Solved Paper 2015

JEE Main

Joint Entrance Examination

Instructions

- This test consists of 30 questions.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. Let A and B be two sets containing four and two elements respectively. Then, the number of subsets of the set $A \times B$, each having atleast three elements are
   (a) 219 (b) 256 (c) 275 (d) 510

2. A complex number z is said to be unimodular, if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then, the point z_1 lies on a
   (a) straight line parallel to X-axis
   (b) straight line parallel to Y-axis
   (c) circle of radius 2
   (d) circle of radius $\sqrt{2}$

3. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to
   (a) 6 (b) –6
   (c) 3 (d) –3

4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to
   (a) (2, – 1) (b) (–2, 1)
   (c) (2, 1) (d) (–2, – 1)

5. The set of all values of λ for which the system of linear equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$ and $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution,
   (a) is an empty set
   (b) is a singleton set
   (c) contains two elements
   (d) contains more than two elements

6. The number of integers greater than 6000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition, is
   (a) 216
   (b) 192
   (c) 120
   (d) 72

7. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is
   (a) $\frac{1}{2}(3^{50} + 1)$ (b) $\frac{1}{2}(3^{50})$
   (c) $\frac{1}{2}(3^{50} - 1)$ (d) $\frac{1}{2}(2^{50} + 1)$

8. If m is the AM of two distinct real numbers l and $n (l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals
   (a) $4\, l^2mn$
   (b) $4\, lm^2n$
   (c) $4\, lmn^2$
   (d) $4l^2m^2n^2$

9. The sum of first 9 terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ is
(a) 71 (b) 96
(c) 142 (d) 192

10. $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to
(a) 4 (b) 3
(c) 2 (d) $\frac{1}{2}$

11. If the function $g(x)=\begin{cases} k\sqrt{x+1} & , 0\le x\le 3 \\ mx+2 & , 3<x\le 5\end{cases}$ is differentiable, then the value of $k+m$ is
(a) 2 (b) $\frac{16}{5}$
(c) $\frac{10}{3}$ (d) 4

12. The normal to the curve $x^2+2xy-3y^2=0$ at $(1,1)$
(a) does not meet the curve again
(b) meets the curve again in the second quadrant
(c) meets the curve again in the third quadrant
(d) meets the curve again in the fourth quadrant

13. Let $f(x)$ be a polynomial of degree four having extreme values at $x=1$ and $x=2$. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then $f(2)$ is equal to
(a) –8 (b) –4 (c) 0 (d) 4

14. The integral $\int\frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals
(a) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$ (b) $(x^4+1)^{\frac{1}{4}}+c$
(c) $-(x^4+1)^{\frac{1}{4}}+c$ (d) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$

15. The integral $\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to
(a) 2 (b) 4
(c) 1 (d) 6

16. The area (in sq units) of the region described by $\{(x,y): y^2\le 2x \text{ and } y\ge 4x-1\}$ is
(a) $\frac{7}{32}$ (b) $\frac{5}{64}$ (c) $\frac{15}{64}$ (d) $\frac{9}{32}$

17. Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx}+y=2x\log x, (x\ge 1)$. Then, $y(e)$ is equal to
(a) e (b) 0 (c) 2 (d) $2e$

18. The number of points having both coordinates as integers that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is
(a) 901 (b) 861
(c) 820 (d) 780

19. Locus of the image of the point (2, 3) in the line $(2x-3y+4)+k(x-2y+3)=0, k\in R$, is a
(a) straight line parallel to X-axis.
(b) straight line parallel to Y-axis.
(c) circle of radius $\sqrt{2}$.
(d) circle of radius $\sqrt{3}$.

20. The number of common tangents to the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ is
(a) 1 (b) 2
(c) 3 (d) 4

21. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is
(a) $\frac{27}{4}$ (b) 18
(c) $\frac{27}{2}$ (d) 27

22. Let O be the vertex and Q be any point on the parabola $x^2=8y$. If the point P divides the line segment OQ internally in the ratio $1:3$, then the locus of P is
(a) $x^2=y$ (b) $y^2=x$
(c) $y^2=2x$ (d) $x^2=2y$

23. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=16$ is
(a) $2\sqrt{14}$ (b) 8
(c) $3\sqrt{21}$ (d) 13

24. The equation of the plane containing the line $2x-5y+z=3$, $x+y+4z=5$ and parallel to the plane $x+3y+6z=1$ is
(a) $2x+6y+12z=13$
(b) $x+3y+6z=-7$
(c) $x+3y+6z=7$
(d) $2x+6y+12z=-13$

25. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of them are collinear and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is
(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$
(c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$

26. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls, is

(a) $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$ (b) $55\left(\frac{2}{3}\right)^{10}$ (c) $220\left(\frac{1}{3}\right)^{12}$ (d) $22\left(\frac{1}{3}\right)^{11}$

27. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data is

(a) 16.8 (b) 16.0
(c) 15.8 (d) 14.0

28. If the angles of elevation of the top of a tower from three collinear points A, B and C on a line leading to the foot of the tower are 30°, 45° and 60° respectively, then the ratio $AB : BC$ is

(a) $\sqrt{3} : 1$ (b) $\sqrt{3} : \sqrt{2}$
(c) $1 : \sqrt{3}$ (d) $2 : 3$

29. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$.

Then, a value of y is

(a) $\frac{3x - x^3}{1 - 3x^2}$ (b) $\frac{3x + x^3}{1 - 3x^2}$
(c) $\frac{3x - x^3}{1 + 3x^2}$ (d) $\frac{3x + x^3}{1 + 3x^2}$

30. The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to

(a) $s \wedge \sim r$
(b) $s \wedge (r \wedge \sim s)$
(c) $s \vee (r \vee \sim s)$
(d) $s \wedge r$

Answer *with* Explanation

1. *(a)* Given,

$$n(A) = 4,\ n(B) = 2$$

$$\Rightarrow \quad n(A \times B) = 8$$

Total number of subsets of set $(A \times B) = 2^8$

Number of subsets of set $A \times B$ having no element (i.e. ϕ) = 1

Number of subsets of set $A \times B$ having one element = ${}^8C_1$

Number of subsets of set $A \times B$ having two elements = ${}^8C_2$

$\therefore$ Number of subsets having atleast three elements

$$= 2^8 - (1 + {}^8C_1 + {}^8C_2)$$
$$= 2^8 - 1 - 8 - 28$$
$$= 2^8 - 37$$
$$= 256 - 37 = 219$$

2. *(c)* ***Central Idea*** *If z is unimodular, then $|z| = 1$. Also, use property of modulus i.e. $z\bar{z} = |z|^2$.*

Given, z_2 is not unimodular i.e. $|z_2| \neq 1$

and $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular

$$\Rightarrow \quad \left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$$

$$\Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$$

$$\Rightarrow \quad (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2) \qquad (\because z\bar{z} = |z|^2)$$

$$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2$$
$$= 4 + |z_1|^2|z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2$$

$$\Rightarrow \quad (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$$

$$\because \quad |z_2| \neq 1$$

$$\therefore \quad |z_1| = 2$$

Let $\quad z_1 = x + iy$

$$\Rightarrow \quad x^2 + y^2 = (2)^2$$

$\therefore$ Point z_1 lies on a circle of radius 2.

3. *(c)* Given, α and β are the roots of the equation $x^2 - 6x - 2 = 0$.

$$\because \quad a_n = \alpha^n - \beta^n \text{ for } n \geq 1$$

$$\therefore \quad a_{10} = \alpha^{10} - \beta^{10}$$
$$a_8 = \alpha^8 - \beta^8$$
$$a_9 = \alpha^9 - \beta^9$$

Now, consider

$$\frac{a_{10} - 2a_8}{2a_9} = \frac{\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$$
$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)}$$
$$= \frac{\alpha^8 \cdot 6\alpha - \beta^8 \cdot 6\beta}{2(\alpha^9 - \beta^9)}$$
$$= \frac{6\alpha^9 - 6\beta^9}{2(\alpha^9 - \beta^9)} = \frac{6}{2} = 3$$

[$\because \alpha$ and β are the roots of $x^2 - 6x - 2 = 0$
or $\quad x^2 = 6x + 2$
$\Rightarrow \quad \alpha^2 = 6\alpha + 2$
$\Rightarrow \quad \alpha^2 - 2 = 6\alpha$
and $\quad \beta^2 = 6\beta + 2$
$\Rightarrow \quad \beta^2 - 2 = 6\beta$]

Aliter

Since, α and β are the roots of the equation $x^2 - 6x - 2 = 0$.

or $\quad x^2 = 6x + 2$

$$\therefore \quad \alpha^2 = 6\alpha + 2$$

$$\Rightarrow \quad \alpha^{10} = 6\alpha^9 + 2\alpha^8 \qquad \ldots(i)$$

Similarly, $\quad \beta^{10} = 6\beta^9 + 2\beta^8 \qquad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$\alpha^{10} - \beta^{10} = 6(\alpha^9 - \beta^9) + 2(\alpha^8 - \beta^8)$$

$$\Rightarrow \quad a_{10} = 6a_9 + 2a_8 \qquad (\because a_n = \alpha^n - \beta^n)$$

$$\Rightarrow \quad a_{10} - 2a_8 = 6a_9$$

$$\Rightarrow \quad \frac{a_{10} - 2a_8}{2a_9} = 3$$

4. *(d)* Given,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$$

It is given that

$$AA^T = 9I$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

On comparing, we get

$$a + 4 + 2b = 0$$

$$\Rightarrow \quad a + 2b = -4 \quad \ldots(i)$$

$$2a + 2 - 2b = 0$$

$$\Rightarrow \quad a - b = -1 \quad \ldots(ii)$$

and $\quad a^2 + 4 + b^2 = 9 \quad \ldots(iii)$

On solving Eqs. (i) and (ii), we get

$$a = -2, b = -1$$

This satisfies Eq. (iii)

Hence, $(a, b) \equiv (-2, -1)$

5. *(c)* Given system of linear equations

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$

$$\Rightarrow \quad (2-\lambda)x_1 - 2x_2 + x_3 = 0 \quad \ldots(i)$$

$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$

$$\Rightarrow \quad 2x_1 - (3+\lambda)x_2 + 2x_3 = 0 \quad \ldots(ii)$$

$$-x_1 + 2x_2 = \lambda x_3$$

$$\Rightarrow \quad -x_1 + 2x_2 - \lambda x_3 = 0 \quad \ldots(iii)$$

Since, the system has non-trivial solution.

$$\therefore \quad \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3\lambda + \lambda^2 - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$$

$$\Rightarrow \quad (2-\lambda)(\lambda^2 + 3\lambda - 4) + 4(1-\lambda) + (1-\lambda) = 0$$

$$\Rightarrow \quad (2-\lambda)(\lambda+4)(\lambda-1) + 5(1-\lambda) = 0$$

$$\Rightarrow \quad (\lambda-1)[(2-\lambda)(\lambda+4) - 5] = 0$$

$$\Rightarrow \quad (\lambda-1)(\lambda^2 + 2\lambda - 3) = 0$$

$$\Rightarrow \quad (\lambda-1)[(\lambda-1)(\lambda+3)] = 0$$

$$\Rightarrow \quad (\lambda-1)^2(\lambda+3) = 0$$

$$\Rightarrow \quad \lambda = 1, 1, -3$$

6. *(b)* The integer greater than 6000 may be of 4 digit or 5 digit. So, here two cases arise.

***Case* I** When number is of 4 digit.

Four digit number can starts from 6, 7 or 8

Thus, total number of 4 digit number, which are greater than $6000 = 3 \times 4 \times 3 \times 2 = 72$

***Case* II** When number is of 5 digit.

Total number of five digit number which are greater than $6000 = 5! = 120$

$\therefore$ Total number of integers $= 72 + 120 = 192$

7. *(a)* Let T_{r+1} be the general term in the expansion of $(1 - 2\sqrt{x})^{50}$

$$\therefore \quad T_{r+1} = {}^{50}C_r (1)^{50-r}(-2x^{1/2})^r$$

$$= {}^{50}C_r 2^r x^{r/2}(-1)^r$$

For the integral power of x, r should be even integer.

$$\therefore \text{ Sum of coefficients} = \sum_{r=0}^{25} {}^{50}C_{2r}(2)^{2r}$$

$$= \frac{1}{2}[(1+2)^{50} + (1-2)^{50}]$$

$$= \frac{1}{2}[3^{50} + 1]$$

Aliter

We have,

$$(1 - 2\sqrt{x})^{50} = C_0 - C_1 2\sqrt{x} + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50} \quad \ldots(i)$$

$$(1 + 2\sqrt{x})^{50} = C_0 + C_1 2\sqrt{x} + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50} \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$(1 - 2\sqrt{x})^{50} + (1 + 2\sqrt{x})^{50} = 2[C_0 + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}]$$

$$\Rightarrow \frac{(1 - 2\sqrt{x})^{50} + (1 + 2\sqrt{x})^{50}}{2} = C_0 + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}$$

On putting $x = 1$, we get

$$\frac{(1 - 2\sqrt{1})^{50} + (1 + 2\sqrt{1})^{50}}{2} = C_0 + C_2(2)^2 + \ldots + C_{50}(2)^{50}$$

$$\Rightarrow \quad \frac{(-1)^{50} + (3)^{50}}{2} = C_0 + C_2(2)^2 + \ldots + C_{50}(2)^{50}$$

$$\Rightarrow \quad \frac{1 + 3^{50}}{2} = C_0 + C_2(2)^2 + \ldots + C_{50}(2)^{50}$$

8. *(b)* Given, m is the AM of l and n

$$\therefore \quad l + n = 2m \quad \ldots(i)$$

and G_1, G_2, G_3 are geometric means between l and n

$\therefore \quad l, G_1, G_2, G_3, n$ are in GP.

Let r be the common ratio of this GP.

$$\therefore \quad G_1 = lr$$

$$G_2 = lr^2$$

$$G_3 = lr^3$$

$$n = lr^4$$

$$\Rightarrow \quad r = \left(\frac{n}{l}\right)^{\frac{1}{4}}$$

Now, $G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$

$$= l^4 \times r^4(1 + 2r^4 + r^8) = l^4 \times r^4(r^4 + 1)^2$$

$$= l^4 \times \frac{n}{l}\left(\frac{n+l}{l}\right)^2$$

$$= ln \times 4m^2 = 4\,lm^2 n$$

9. *(b)* ***Central Idea*** *Write the nth term of the given series and simplify it to get its lowest form. Then, apply,* $S_n = \Sigma T_n$.

Given series is

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots \infty$$

Let T_n be the nth term of the given series.

$$\therefore \quad T_n = \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + \text{to } n \text{ terms}}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{n^2}$$

$$= \frac{(n+1)^2}{4}$$

$$S_9 = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}[(2^2 + 3^2 + \ldots + 10^2) + 1^2 - 1^2]$$

$$= \frac{1}{4}\left[\frac{10\,(10+1)(20+1)}{6} - 1\right] = \frac{384}{4} = 96$$

10. *(c)* We have,

$$\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x} = \lim_{x \to 0} \frac{2\sin^2 x(3 + \cos x)}{x \times \frac{\tan 4x}{4x} \times 4x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{(3 + \cos x)}{4} \times \frac{1}{\lim_{x \to 0} \frac{\tan 4x}{4x}}$$

$$= 2 \times \frac{4}{4} \times 1 \qquad \left(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right)$$

$$= 2$$

11. *(a)* Since, $g(x)$ is differentiable $\Rightarrow g(x)$ must be continuous.

$$\therefore \quad g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$$

At $x = 3$, RHL $= 3m + 2$

and at $x = 3$, LHL $= 2k$

$$\therefore \quad 2k = 3m + 2 \qquad \ldots(\text{i})$$

$$\text{Also,} \quad g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}, & 0 \le x < 3 \\ m, & 3 < x \le 5 \end{cases}$$

$$\therefore \quad L\{g'(3)\} = \frac{k}{4} \text{ and } R\{g'(3)\} = m$$

$$\Rightarrow \quad \frac{k}{4} = m \text{ i.e. } k = 4m \qquad \ldots(\text{ii})$$

On solving Eqs. (i) and (ii), we get

$$k = \frac{8}{5}, m = \frac{2}{5}$$

$$\Rightarrow \quad k + m = 2$$

12. *(d)* Given equation of curve is

$$x^2 + 2xy - 3y^2 = 0 \qquad \ldots(\text{i})$$

On differentiating w.r.t. x, we get

$$2x + 2xy' + 2y - 6yy' = 0$$

$$\Rightarrow \quad y' = \frac{x + y}{3y - x}$$

At $\quad x = 1, y = 1, \; y' = 1$

i.e. $\quad \left(\frac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at (1, 1) is

$$y - 1 = -\frac{1}{1}(x - 1)$$

$$\Rightarrow \quad y - 1 = -(x - 1)$$

$$\Rightarrow \quad x + y = 2 \qquad \ldots(\text{ii})$$

On solving Eqs. (i) and (ii) simultaneously, we get

$$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$$

$$\Rightarrow \quad x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$$

$$\Rightarrow \quad -x^2 + 4x - 12 - 3x^2 + 12x = 0$$

$$\Rightarrow \quad -4x^2 + 16x - 12 = 0$$

$$\Rightarrow \quad 4x^2 - 16x + 12 = 0$$

$$\Rightarrow \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 1)(x - 3) = 0$$

$$\Rightarrow \quad x = 1, 3$$

Now, when $\quad x = 1$, then $y = 1$

and when $\quad x = 3$, then $y = -1$

$\therefore \quad P = (1, 1)$ and $Q = (3, -1)$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

Aliter

Given,

$$x^2 + 2xy - 3y^2 = 0$$

$$\Rightarrow \quad (x - y)(x + 3y) = 0$$

$$\Rightarrow \quad x - y = 0 \text{ or } x + 3y = 0$$

Equation of normal at (1, 1) is

$$y - 1 = -1(x - 1)$$

$$\Rightarrow \quad x + y - 2 = 0$$

It intersects $x + 3y = 0$ at $(3, -1)$

and hence normal meet the curve in fourth quadrant.

13. *(c)* ***Central Idea*** *Any function have extreme values (maximum or minimum) at its critical points, where* $f'(x) = 0$.

Since, the function have extreme values at $x = 1$ and $x = 2$.

$$\therefore \quad f'(x) = 0 \text{ at } x = 1 \text{ and } x = 2$$

$$\Rightarrow \quad f'(1) = 0 \text{ and } f'(2) = 0$$

Also it is given that

$$\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$$

$\Rightarrow \quad 1 + \lim_{x \to 0} \frac{f(x)}{x^2} = 3$

$\Rightarrow \quad \lim_{x \to 0} \frac{f(x)}{x^2} = 2$

$\Rightarrow$ $f(x)$ will be of the form

$ax^4 + bx^3 + 2x^2$ [$\because f(x)$ is of four degree polynomial]

Let $\quad f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow \quad f'(x) = 4ax^3 + 3bx^2 + 4x$

$\Rightarrow \quad f'(1) = 4a + 3b + 4 = 0 \quad \ldots$(i)

and $f'(2) = 32a + 12b + 8 = 0$

$\Rightarrow \quad 8a + 3b + 2 = 0 \quad \ldots$(ii)

On solving Eqs. (i) and (ii), we get

$a = \frac{1}{2}, b = -2$

$\therefore \quad f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$

$\Rightarrow \quad f(2) = 8 - 16 + 8 = 0$

14. *(d)* $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

Put $\quad 1 + \frac{1}{x^4} = t^4$

$\Rightarrow \quad -\frac{4}{x^5}dx = 4t^3 dt$

$\Rightarrow \quad \frac{dx}{x^5} = -t^3 dt = \int \frac{-t^3 dt}{t^3}$

$= -\int dt = -t + c = -\left(1 + \frac{1}{x^4}\right)^{\frac{1}{4}} + c$

15. *(c)* ***Central Idea*** *Apply the property* $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ *and then add.*

Let $\quad I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}dx$

$= \int_2^4 \frac{2\log x}{2\log x + \log(6-x)^2}dx$

$= \int_2^4 \frac{2\log x\, dx}{2[\log x + \log(6-x)]}$

$\Rightarrow \quad I = \int_2^4 \frac{\log x\, dx}{[\log x + \log(6-x)]} \quad \ldots$(i)

$\Rightarrow \quad I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x}dx \quad \ldots$(ii)

$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

On adding Eqs. (i) and (ii), we get

$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)}dx$

$\Rightarrow \quad 2I = \int_2^4 dx = [x]_2^4$

$\Rightarrow \quad 2I = 2$

$\Rightarrow \quad I = 1$

16. *(d)* Given region is $\{(x, y): y^2 \le 2x \text{ and } y \ge 4x - 1\}$

$y^2 \le 2x$ represents a region inside the parabola

$y^2 = 2x \quad \ldots$(i)

and $y \ge 4x - 1$ represents a region to the left of the line

$y = 4x - 1 \quad \ldots$(ii)

The point of intersection of the curve (i) and (ii) is

$(4x-1)^2 = 2x$

$\Rightarrow \quad 16x^2 + 1 - 8x = 2x$

$\Rightarrow \quad 16x^2 - 10x + 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}, \frac{1}{8}$

$\therefore$ The points where these curves intersect, are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, -\frac{1}{2}\right)$.

Hence, required area $= \int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy$

$= \frac{1}{4}\left(\frac{y^2}{2} + y\right)_{-1/2}^{1} - \frac{1}{6}(y^3)_{-1/2}^{1}$

$= \frac{1}{4}\left\{\left(\frac{1}{2}+1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$

$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\}$

$= \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32}$

17. *(c)* Given differential equation is

$(x\log x)\frac{dy}{dx} + y = 2x\log x, \quad (x \ge 1)$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x\log x} = 2$

This is a linear differential equation.

$\therefore \quad \text{IF} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

Now, the solution of given differential equation is given by

$y \cdot \log x = \int \log x \cdot 2dx$

$\Rightarrow \quad y \cdot \log x = 2\int \log x\, dx$

$\Rightarrow \quad y \cdot \log x = 2[x\log x - x] + c$

At $x = 1$,

$c = 2$

$\Rightarrow$ $y \cdot \log x = 2[x \log x - x] + 2$

At $x = e$,

$y = 2(e - e) + 2 \Rightarrow y = 2$

18. *(d)* Required points (x, y) are such that it satisfy $x + y < 41$

and $x > 0, y > 0$

Number of positive integral solution of the equation $x + y + k = 41$ will be number of integral coordinates in the bounded region.

$\therefore$ Total number of integral coordinates

$$= {}^{41-1}C_{3-1} = {}^{40}C_2 = \frac{40!}{2!\,38!} = 780$$

Aliter

Consider the following figure :

Clearly, the number of required points

$$= 1 + 2 + 3 + \ldots + 39$$

$$= \frac{39}{2}(39 + 1) = 780$$

19. *(c)* ***Central Idea*** *First of all find the point of intersection of the lines $2x - 3y + 4 = 0$ and $x - 2y + 3 = 0$ (say A). Now, the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$ is the perpendicular bisector of the line joining points $P(2, 3)$ and image $P'(h, k)$. Now, $AP = AP'$ and simplify.*

Given line is

$$(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R \quad \ldots(\text{i})$$

This line will pass through the point of intersection of the lines

$$2x - 3y + 4 = 0 \quad \ldots(\text{ii})$$

and $$x - 2y + 3 = 0 \quad \ldots(\text{iii})$$

On solving Eqs. (ii) and (iii), we get

$$x = 1, y = 2$$

$\therefore$ Point of intersection of lines (ii) and (iii) is $(1, 2)$.

Let M be the mid-point of PP', then AM is perpendicular bisector of PP' (where, A is the point of intersection of given lines).

$\therefore$ $AP = AP'$

$\Rightarrow$ $\sqrt{(2-1)^2 + (3-2)^2} = \sqrt{(h-1)^2 + (k-2)^2}$

$\Rightarrow$ $\sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 1 + 4}$

$\Rightarrow$ $\sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 5}$

$\Rightarrow$ $h^2 + k^2 - 2h - 4k + 5 = 2$

$\Rightarrow$ $h^2 + k^2 - 2h - 4k + 3 = 0$

Thus, the required locus is

$$x^2 + y^2 - 2x - 4y + 3 = 0$$

which is a equation of circle with

$$\text{radius} = \sqrt{1 + 4 - 3} = \sqrt{2}$$

Aliter

$(2x - 3y + 4) + k(x - 2y + 3) = 0$ is family of lines passing through (1, 2). By congruency of triangles, we can prove that mirror image (h, k) and the point (2, 3) will be equidistant from (1, 2).

$\therefore$ Locus of (h, k) is $PR = PQ$

$\Rightarrow$ $(h-1)^2 + (k-2)^2 = (2-1)^2 + (3-2)^2$

or $(x-1)^2 + (y-2)^2 = 2$

$\therefore$ Locus is a circle of radius $= \sqrt{2}$

20. *(c)* ***Central Idea*** *Number of common tangents depend on the position of the circle with respect to each other.*

(i) If circles touch externally $\Rightarrow C_1C_2 = r_1 + r_2$, 3 common tangents

(ii) If circles touch internally $\Rightarrow C_1C_2 = r_2 - r_1$, 1 common tangent

(iii) If circles do not touch each other, 4 common tangents

Given equations of circles are

$$x^2 + y^2 - 4x - 6y - 12 = 0 \quad \ldots(i)$$

$$x^2 + y^2 + 6x + 18y + 26 = 0 \quad \ldots(ii)$$

Centre of circle (i) is $C_1(2, 3)$ and radius $= \sqrt{4 + 9 + 12}$

$= 5(r_1)$ (say)

Centre of circle (ii) is $C_2(-3, -9)$ and radius $= \sqrt{9 + 81 - 26}$

$= 8(r_2)$ (say)

Now, $C_1C_2 = \sqrt{(2 + 3)^2 + (3 + 9)^2}$

$\Rightarrow$ $C_1C_2 = \sqrt{5^2 + 12^2}$

$\Rightarrow$ $C_1C_2 = \sqrt{25 + 144} = 13$

$\therefore$ $r_1 + r_2 = 5 + 8 = 13$

Also, $C_1C_2 = r_1 + r_2$

Thus, both circles touch each other externally. Hence, there are three common tangents.

21. *(d)* Given equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1 \quad \ldots(i)$$

$\therefore$ $a^2 = 9, b^2 = 5$

$\Rightarrow$ $a = 3, b = \sqrt{5}$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

foci $= (\pm ae, 0) = (\pm 2, 0)$

and $\frac{b^2}{a} = \frac{5}{3}$

$\therefore$ Extremities of one of latusrectum are

$$\left(2, \frac{5}{3}\right) \text{ and } \left(2, \frac{-5}{3}\right).$$

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is,

$$\frac{x(2)}{9} + \frac{y(5/3)}{5} = 1$$

or $2x + 3y = 9 \quad \ldots(ii)$

Eq.(ii) intersects X and Y-axes at $\left(\frac{9}{2}, 0\right)$ and $(0, 3)$, respectively.

$\therefore$ Area of quadrilateral

$= 4 \times$ Area of ΔPOQ

$= 4 \times \left(\frac{1}{2} \times \frac{9}{2} \times 3\right) = 27$ sq units

22. *(d)* ***Central Idea*** *Any point on the parabola $x^2 = 8y$ is $(4t, 2t^2)$. Point P divides the line segment joining of $O(0, 0)$ and $Q(4t, 2t^2)$ in the ratio 1 : 3. Apply the section formula for internal division.*

Equation of parabola is

$$x^2 = 8y \quad \ldots(i)$$

Let any point Q on the parabola (i) is $(4t, 2t^2)$.

Let $P(h, k)$ be the point which divides the line segment joining (0,0) and $(4t, 2t^2)$ in the ratio 1 : 3.

$\therefore$ $h = \frac{1 \times 4t + 3 \times 0}{4} \Rightarrow h = t$

and $k = \frac{1 \times 2t^2 + 3 \times 0}{4} \Rightarrow k = \frac{t^2}{2}$

$\Rightarrow$ $k = \frac{1}{2}h^2 \quad (\because t = h)$

$\Rightarrow$ $2k = h^2 \Rightarrow 2y = x^2$, which is required locus.

23. *(d)* Given equation of line is

$$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{12} = \lambda \quad \text{(say)} \quad \ldots(i)$$

and equation of plane is

$$x - y + z = 16 \quad \ldots(ii)$$

Any point on the line (i) is

$$(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$$

Let this point be point of intersection of the line and plane.

$\therefore$ $(3\lambda + 2) - (4\lambda - 1) + (12\lambda + 2) = 16$

$\Rightarrow$ $11\lambda + 5 = 16$

$\Rightarrow$ $11\lambda = 11$

$\Rightarrow$ $\lambda = 1$

$\therefore$ Point of intersection is (5, 3, 14).

Now, distance between the points (1, 0, 2) and (5, 3, 14)

$= \sqrt{(5 - 1)^2 + (3 - 0)^2 + (14 - 2)^2}$

$= \sqrt{16 + 9 + 144}$

$= \sqrt{169} = 13$

24. *(c)* Let equation of plane containing the lines $2x - 5y + z = 3$ and $x + y + 4z = 5$ be

$$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$$

$$\Rightarrow \quad (2 + \lambda)x + (\lambda - 5)y + (4\lambda + 1)z - 3 - 5\lambda = 0 \quad \ldots(i)$$

This plane is parallel to the plane $x + 3y + 6z = 1$.

$$\therefore \quad \frac{2 + \lambda}{1} = \frac{\lambda - 5}{3} = \frac{4\lambda + 1}{6}$$

On taking first two equalities, we get

$$6 + 3\lambda = \lambda - 5$$

$$\Rightarrow \quad 2\lambda = -11$$

$$\Rightarrow \quad \lambda = -\frac{11}{2}$$

On taking last two equalities, we get

$$6\lambda - 30 = 3 + 12\lambda$$

$$\Rightarrow \quad -6\lambda = 33$$

$$\Rightarrow \quad \lambda = -\frac{11}{2}$$

So, the equation of required plane is

$$\left(2 - \frac{11}{2}\right)x + \left(\frac{-11}{2} - 5\right)y + \left(-\frac{44}{2} + 1\right)z - 3 + 5 \times \frac{11}{2} = 0$$

$$\Rightarrow \quad -\frac{7}{2}x - \frac{21}{2}y - \frac{42}{2}z + \frac{49}{2} = 0$$

$$\Rightarrow \quad x + 3y + 6z - 7 = 0$$

25. *(a)* Given,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad -(\mathbf{c} \cdot \mathbf{b}) \cdot \mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\left[\frac{1}{3}|\mathbf{b}||\mathbf{c}| + (\mathbf{c} \cdot \mathbf{b})\right]\mathbf{a} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

Since, $\mathbf{a}$ and $\mathbf{b}$ are not collinear.

$$\mathbf{c} \cdot \mathbf{b} + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0 \text{ and } \mathbf{c} \cdot \mathbf{a} = 0$$

$$\Rightarrow \quad |\mathbf{c}||\mathbf{b}|\cos\theta + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$$

$$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\left(\cos\theta + \frac{1}{3}\right) = 0$$

$$\Rightarrow \quad \cos\theta + \frac{1}{3} = 0 \quad (\because |\mathbf{b}| \neq 0, |\mathbf{c}| \neq 0)$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{3}$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

26. *(a)* There seems to be ambiguity in this question. It should be mentioned that boxes are different and one particular box has 3 balls.

Then, number of ways $= \dfrac{{}^{12}C_3 \times 2^9}{3^{12}}$

$$= \frac{55}{3}\left(\frac{2}{3}\right)^{11}$$

According to the question,

$$\frac{{}^3C_1 \times {}^{12}C_3 2^9 - {}^3C_2\,{}^{12}C_3\,{}^9C_3 + \dfrac{12! \times 3!}{3!3!6!3!}}{3^{12}}$$

27. *(d)* Given, $\dfrac{x_1 + x_2 + x_3 + \ldots + x_{16}}{16} = 16$

$$\Rightarrow \quad \sum_{i=1}^{16} x_i = 16 \times 16$$

Sum of new observations $= \displaystyle\sum_{i=1}^{18} y_i = (16 \times 16 - 16) + (3 + 4 + 5)$

$$= 252$$

Number of observations = 18

$$\therefore \quad \text{New mean} = \frac{\sum_{i=1}^{18} y_i}{18}$$

$$= \frac{252}{18} = 14$$

28. *(a)* According to the given information, the figure should be as follows.

Let the height of tower = h

In Right ΔEDA,

$$\tan 30° = \frac{ED}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{ED}{AD} = \frac{h}{AD}$$

$$\Rightarrow \quad AD = h\sqrt{3}$$

In Right ΔEDB,

$$\tan 45° = \frac{h}{BD}$$

$$\Rightarrow \quad BD = h$$

In Right ΔEDC,

$$\tan 60° = \frac{h}{CD} \Rightarrow CD = \frac{h}{\sqrt{3}}$$

Now, $$\frac{AB}{BC} = \frac{AD - BD}{BD - CD} \Rightarrow \frac{AB}{BC} = \frac{h\sqrt{3} - h}{h - \dfrac{h}{\sqrt{3}}}$$

$$\Rightarrow \quad \frac{AB}{BC} = \frac{h(\sqrt{3} - 1)}{\dfrac{h(\sqrt{3} - 1)}{\sqrt{3}}} \Rightarrow \frac{AB}{BC} = \frac{\sqrt{3} - 1}{(\sqrt{3} - 1)} \times \sqrt{3}$$

$$\Rightarrow \quad \frac{AB}{BC} = \frac{\sqrt{3}}{1}$$

$$\therefore \quad AB : BC = \sqrt{3} : 1$$

29. *(a)* Given,

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right),$$

where $|x| < \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \tan^{-1} y = \tan^{-1}\left\{\frac{x + \frac{2x}{1-x^2}}{1 - x\left(\frac{2x}{1-x^2}\right)}\right\}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),\right.$$

$$\left. x > 0,\ y > 0,\ xy < 1\right]$$

$$= \tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right)$$

$$\tan^{-1} y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad y = \frac{3x - x^3}{1 - 3x^2}$$

Aliter

$$|x| < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Let $\quad x = \tan\theta$

$$\Rightarrow \quad -\frac{\pi}{6} < \theta < \frac{\pi}{6}$$

$$\therefore \quad \tan^{-1} y = \theta + \tan^{-1}(\tan 2\theta) = \theta + 2\theta = 3\theta$$

$$\Rightarrow \quad y = \tan 3\theta$$

$$\Rightarrow \quad y = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$$

30. *(d)* $\sim(\sim s \vee (\sim r \wedge s))$

$$\equiv s \wedge (\sim(\sim r \wedge s)) \equiv s \wedge (r \vee \sim s)$$

$$\equiv (s \wedge r) \vee (s \wedge \sim s)$$

$$\equiv (s \wedge r) \vee \mathrm{F} \qquad (\because s \wedge \sim s \text{ is false})$$

$$\equiv s \wedge r$$

SOLVED PAPER 2016 **{OFFLINE}**

JEE MAIN

Joint Entrance Examination

INSTRUCTIONS

1. This test consists of 30 questions.
2. Each question is allotted 4 mark for correct response.
3. Candidates will be awarded mark as stated above in instruction no. 2 for correct response of each question. 1 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
4. There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

Mathematics

1. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x$, $x \neq 0$ and $S = \{x \in R : f(x) = f(-x)\}$; then S
   (a) is an empty set
   (b) contains exactly one element
   (c) contains exactly two elements
   (d) contains more than two elements

2. A value of θ for which $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, is
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
   (c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

3. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$ is
   (a) 3 (b) – 4
   (c) 6 (d) 5

4. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \operatorname{adj} A = AA^T$, then $5a + b$ is equal to
   (a) – 1 (b) 5
   (c) 4 (d) 13

5. The system of linear equations
   $x + \lambda y - z = 0;\ \lambda x - y - z = 0;\ x + y - \lambda z = 0$
   has a non-trivial solution for
   (a) infinitely many values of λ
   (b) exactly one value of λ
   (c) exactly two values of λ
   (d) exactly three values of λ

6. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is
   (a) 46th (b) 59th
   (c) 52nd (d) 58th

7. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is
   (a) 64 (b) 2187 (c) 243 (d) 729

8. If the 2nd, 5th and 9th terms of a non-constant AP are in GP, then the common ratio of this GP is
   (a) $\frac{8}{5}$ (b) $\frac{4}{3}$
   (c) 1 (d) $\frac{7}{4}$

9. If the sum of the first ten terms of the series

$$\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots, \quad \text{is} \quad \frac{16}{5}m,$$

then m is equal to

(a) 102 (b) 101
(c) 100 (d) 99

10. Let $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$, then $\log p$ is equal to

(a) 2 (b) 1
(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

11. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(a) g is not differentiable at $x = 0$
(b) $g'(0) = \cos(\log 2)$
(c) $g'(0) = -\cos(\log 2)$
(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

12. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point

(a) (0, 0) (b) $\left(0, \frac{2\pi}{3}\right)$
(c) $\left(\frac{\pi}{6}, 0\right)$ (d) $\left(\frac{\pi}{4}, 0\right)$

13. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then

(a) $2x = (\pi + 4)r$ (b) $(4 - \pi)x = \pi r$
(c) $x = 2r$ (d) $2x = r$

14. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to

(a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$ (b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$
(c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$ (d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

where, C is an arbitrary constant.

15. $\lim_{n \to \infty} \left(\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$
(c) $\frac{9}{e^2}$ (d) $3\log 3 - 2$

16. The area (in sq units) of the region $\{(x, y) : y^2 \geq 2x$ and $x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is

(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$
(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

17. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$
(c) $\frac{2}{5}$ (d) $\frac{4}{5}$

18. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?

(a) $(-3, -9)$ (b) $(-3, -8)$
(c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$

19. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the X-axis, lie on

(a) a circle
(b) an ellipse which is not a circle
(c) a hyperbola
(d) a parabola

20. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$, then the radius of S is

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$
(c) 5 (d) 10

21. Let P be the point on the parabola, $y^2 = 8x$, which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then, the equation of the circle, passing through C and having its centre at P is

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$

22. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$

23. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is

(a) $3\sqrt{10}$ (b) $10\sqrt{3}$
(c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$

24. If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to

(a) 26 (b) 18
(c) 5 (d) 2

25. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three unit vectors such that $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$. If $\hat{\mathbf{b}}$ is not parallel to $\hat{\mathbf{c}}$, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

26. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?

(a) $3a^2 - 26a + 55 = 0$ (b) $3a^2 - 32a + 84 = 0$
(c) $3a^2 - 34a + 91 = 0$ (d) $3a^2 - 23a + 44 = 0$

27. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is not true?

(a) E_1 and E_2 are independent
(b) E_2 and E_3 are independent
(c) E_1 and E_3 are independent
(d) E_1, E_2 and E_3 are independent

28. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0, \text{ is}$$

(a) 3 (b) 5
(c) 7 (d) 9

29. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then, the time taken (in minutes) by him, from B to reach the pillar, is

(a) 6 (b) 10
(c) 20 (d) 5

30. The Boolean expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to

(a) $\sim p \wedge q$ (b) $p \wedge q$ (c) $p \vee q$ (d) $p \vee \sim q$

ANSWER WITH EXPLANATION

1. *(c)* We have, $f(x) + 2f\left(\frac{1}{x}\right) = 3x,\ x \neq 0$ ...(i)

On replacing x by $\frac{1}{x}$ in the above equation, we get

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$$

$$\Rightarrow \quad 2f(x) + f\left(\frac{1}{x}\right) = \frac{3}{x} \quad \text{...(ii)}$$

On multiplying Eq. (ii) by 2 and subtracting Eq. (i) from Eq. (ii), we get

$$4f(x) + 2f\left(\frac{1}{x}\right) = \frac{6}{x}$$

$$\underline{-\quad f(x) + 2f\left(\frac{1}{x}\right) = 3x \quad -}$$

$$3f(x) = \frac{6}{x} - 3x$$

$$\Rightarrow \quad f(x) = \frac{2}{x} - x$$

Now, consider $f(x) = f(-x)$

$$\Rightarrow \quad \frac{2}{x} - x = -\frac{2}{x} + x$$

$$\Rightarrow \quad \frac{4}{x} = 2x$$

$$\Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow \quad x^2 = 2$$

$$\Rightarrow \quad x = \pm\sqrt{2}$$

Hence, S contains exactly two elements.

2. *(d)* Let $z = \frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary. Then, we have

$\text{Re}(z) = 0$

Now, consider
$$z = \frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$$
$$= \frac{(2 + 3i\sin\theta)(1 + 2i\sin\theta)}{(1 - 2i\sin\theta)(1 + 2i\sin\theta)}$$
$$= \frac{2 + 4i\sin\theta + 3i\sin\theta + 6i^2\sin^2\theta}{1^2 - (2i\sin\theta)^2}$$
$$= \frac{2 + 7i\sin\theta - 6\sin^2\theta}{1 + 4\sin^2\theta}$$
$$= \frac{2 - 6\sin^2\theta}{1 + 4\sin^2\theta} + i\,\frac{7\sin\theta}{1 + 4\sin^2\theta}$$

$\because \quad \text{Re}(z) = 0$

$$\therefore \quad \frac{2 - 6\sin^2\theta}{1 + 4\sin^2\theta} = 0 \Rightarrow 2 = 6\sin^2\theta$$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{3} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\pm\frac{1}{\sqrt{3}}\right) = \pm\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

3. *(a)* Given, $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$

Clearly, this is possible when

I. $x^2 + 4x - 60 = 0$ and $x^2 - 5x + 5 \neq 0$

or

II. $x^2 - 5x + 5 = 1$

or

III. $x^2 - 5x + 5 = -1$ and $x^2 + 4x - 60 =$ Even integer.

***Case* I** When $x^2 + 4x - 60 = 0$

$\Rightarrow \quad x^2 + 10x - 6x - 60 = 0$

$\Rightarrow \quad x(x + 10) - 6(x + 10) = 0$

$\Rightarrow \quad (x + 10)(x - 6) = 0$

$\Rightarrow \quad x = -10$ or $x = 6$

Note that, for these two values of x, $x^2 - 5x + 5 \neq 0$

***Case* II** When $\quad x^2 - 5x + 5 = 1$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$\Rightarrow \quad x^2 - 4x - x + 4 = 0$

$\Rightarrow \quad x(x - 4) - 1(x - 4) = 0$

$\Rightarrow \quad (x - 4)(x - 1) = 0$

$\Rightarrow \quad x = 4$ or $x = 1$

***Case* III** When $\quad x^2 - 5x + 5 = -1$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$\Rightarrow \quad x^2 - 2x - 3x + 6 = 0$

$\Rightarrow \quad x(x - 2) - 3(x - 2) = 0$

$\Rightarrow \quad (x - 2)(x - 3) = 0$

$\Rightarrow \quad x = 2$ or $x = 3$

Now, when $x = 2$, $x^2 + 4x - 60 = 4 + 8 - 60 = -48$, which is an even integer.

When $x = 3$, $x^2 + 4x - 60 = 9 + 12 - 60 = -39$, which is not an even integer.

Thus, in this case, we get $x = 2$.

Hence, the sum of all real values of $x = -10 + 6 + 4 + 1 + 2 = 3$

4. ***(b)*** Given, $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = AA^T$

Clearly, $A(\text{adj } A) = |A| I_2$

[$\because$ if A is square matrix of order n, then $A(\text{adj } A) = (\text{adj } A) \cdot A = |A| I_n$]

$$= \begin{vmatrix} 5a & -b \\ 3 & 2 \end{vmatrix} I_2 = (10a + 3b) I_2 = (10a + 3b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} \quad \text{...(i)}$$

and $$AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} \quad \text{...(ii)}$$

$\because \quad A(\text{adj } A) = AA^T$

$$\therefore \quad \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$$

[using Eqs. (i) and (ii)]

$\Rightarrow \quad 15a - 2b = 0 \Rightarrow a = \dfrac{2b}{15}$ ...(iii)

and $\quad 10a + 3b = 13$ ...(iv)

On substituting the value of 'a' from Eq. (iii) in Eq. (iv), we get

$$10 \cdot \left(\frac{2b}{15}\right) + 3b = 13 \Rightarrow \frac{20b + 45b}{15} = 13 \Rightarrow \frac{65b}{15} = 13$$

$\Rightarrow \quad b = 3$

Now, substituting the value of b in Eq. (iii), we get $5a = 2$

Hence, $5a + b = 2 + 3 = 5$

5. ***(d)*** Given, system of linear equation is

$x + \lambda y - z = 0$; $\lambda x - y - z = 0$; $x + y - \lambda z = 0$

Note that, given system will have a non-trivial solution only if determinant of coefficient matrix is zero, i.e. $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad 1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$

$\Rightarrow \quad \lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0$

$\Rightarrow \quad \lambda^3 - \lambda = 0$

$\Rightarrow \quad \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \quad \lambda = 0$ or $\lambda = \pm 1$

Hence, given system of linear equation has a non-trivial solution for exactly three values of λ.

6. ***(d)*** Clearly, number of words start with A $= \dfrac{4!}{2!} = 12$

Number of words start with L $= 4! = 24$

Number of words start with M $= \dfrac{4!}{2!} = 12$

Number of words start with SA $= \dfrac{3!}{2!} = 3$

Number of words start with SL $= 3! = 6$

Note that, next word will be "SMALL".

Hence, the position of word "SMALL" is 58th.

7. ***(d)*** Clearly, number of terms in the expansion of

$\left(1 - \dfrac{2}{x} + \dfrac{4}{x^2}\right)^n$ is $\dfrac{(n + 2)(n + 1)}{2}$ or ${}^{n+2}C_2$.

[assuming $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ distinct]

$$\therefore \quad \frac{(n + 2)(n + 1)}{2} = 28$$

$\Rightarrow (n + 2)(n + 1) = 56 = (6 + 1)(6 + 2) \Rightarrow n = 6$

Hence, sum of coefficients $= (1 - 2 + 4)^6 = 3^6 = 729$

Note As $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ are functions of same variables, therefore number of dissimilar terms will be $2n + 1$, i.e. odd, which is not possible. Hence, it contains error.

8. ***(b)*** Let a be the first term and d be the common difference. Then, we have $a + d$, $a + 4d$, $a + 8d$ in GP,

i.e. $\quad (a + 4d)^2 = (a + d)(a + 8d)$

$\Rightarrow \quad a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$

$\Rightarrow \quad 8d^2 = ad$

$\Rightarrow \quad 8d = a \quad [\because d \neq 0]$

Now, common ratio, $\quad r = \dfrac{a + 4d}{a + d} = \dfrac{8d + 4d}{8d + d} = \dfrac{12d}{9d} = \dfrac{4}{3}$

9. ***(b)*** Let S_{10} be the sum of first ten terms of the series. Then, we have

$$S_{10} = \left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \text{... to 10 terms}$$

$$= \left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + 4^2 + \left(\frac{24}{5}\right)^2 + \text{... to 10 terms}$$

$$= \frac{1}{5^2}(8^2 + 12^2 + 16^2 + 20^2 + 24^2 + \text{... to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \text{... to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2)$$
$$= \frac{16}{25}((1^2 + 2^2 + \ldots + 11^2) - 1^2)$$
$$= \frac{16}{25}\left(\frac{11 \cdot (11+1)(2 \cdot 11 + 1)}{6} - 1\right)$$
$$= \frac{16}{25}(506 - 1) = \frac{16}{25} \times 505$$
$$\Rightarrow \quad \frac{16}{5} m = \frac{16}{25} \times 505$$
$$\Rightarrow \quad m = 101$$

10. ***(c)*** Given, $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$ (1^∞ form)

$$= e^{\lim_{x \to 0^+} \frac{\tan^2 \sqrt{x}}{2x}} = e^{\frac{1}{2} \lim_{x \to 0^+} \left(\frac{\tan \sqrt{x}}{\sqrt{x}}\right)^2} = e^{\frac{1}{2}}$$
$$\therefore \quad \log p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

11. ***(b)*** We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, $x \in R$

Note that, for $x \to 0$, $\log 2 > \sin x$

$$\therefore \quad f(x) = \log 2 - \sin x$$
$$\Rightarrow \quad g(x) = \log 2 - \sin(f(x))$$
$$= \log 2 - \sin(\log 2 - \sin x)$$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is differentiable.

Now, $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$
$$= \cos x \cdot \cos(\log 2 - \sin x)$$
$$\Rightarrow \quad g'(0) = 1 \cdot \cos(\log 2)$$

12. ***(b)*** We have, $f(x) = \tan^{-1}\sqrt{\frac{1 + \sin x}{1 - \sin x}}$, $x \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \quad f(x) = \tan^{-1}\sqrt{\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}$$
$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$$
$$\left(\because \cos\frac{x}{2} > \sin\frac{x}{2} \text{ for } 0 < \frac{x}{2} < \frac{\pi}{4}\right)$$
$$= \tan^{-1}\left(\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right)$$
$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$
$$= \frac{\pi}{4} + \frac{x}{2}$$
$$\Rightarrow \quad f'(x) = \frac{1}{2}$$
$$\Rightarrow \quad f'\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Now, equation of normal at $x = \frac{\pi}{6}$ is given by

$$\left(y - f\left(\frac{\pi}{6}\right)\right) = -2\left(x - \frac{\pi}{6}\right)$$
$$\Rightarrow \quad \left(y - \frac{\pi}{3}\right) = -2\left(x - \frac{\pi}{6}\right)$$
$$\left[\because f\left(\frac{\pi}{6}\right) = \frac{\pi}{4} + \frac{\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}\right]$$

which passes through $\left(0, \frac{2\pi}{3}\right)$.

13. ***(c)*** According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$$\Rightarrow \quad 4x + 2\pi r = 2$$
$$\Rightarrow \quad r = \frac{1 - 2x}{\pi} \quad \ldots(i)$$

Now, let A be the sum of the areas of the square and the circle. Then,

$$A = x^2 + \pi r^2$$
$$= x^2 + \pi \frac{(1 - 2x)^2}{\pi^2}$$
$$\Rightarrow \quad A(x) = x^2 + \frac{(1 - 2x)^2}{\pi}$$

Now, for minimum value of $A(x)$, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 2x + \frac{2(1 - 2x)}{\pi} \cdot (-2) = 0$$
$$\Rightarrow \quad x = \frac{2 - 4x}{\pi}$$
$$\Rightarrow \quad \pi x + 4x = 2$$
$$\Rightarrow \quad x = \frac{2}{\pi + 4} \quad \ldots(ii)$$

Now, from Eq. (i), we get

$$r = \frac{1 - 2 \cdot \frac{2}{\pi + 4}}{\pi}$$
$$= \frac{\pi + 4 - 4}{\pi(\pi + 4)} = \frac{1}{\pi + 4} \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$x = 2r$$

14. ***(b)*** Let $I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$

$$= \int \frac{2x^{12} + 5x^9}{x^{15}(1 + x^{-2} + x^{-5})^3} dx$$
$$= \int \frac{2x^{-3} + 5x^{-6}}{(1 + x^{-2} + x^{-5})^3} dx$$

Now, put $1 + x^{-2} + x^{-5} = t$

$$\Rightarrow \quad (-2x^{-3} - 5x^{-6})\, dx = dt$$
$$\Rightarrow \quad (2x^{-3} + 5x^{-6})\, dx = -dt$$
$$\therefore \quad I = -\int \frac{dt}{t^3} = -\int t^{-3}\, dt$$
$$= -\frac{t^{-3+1}}{-3+1} + C = \frac{1}{2t^2} + C$$
$$= \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$$

15. *(b)* Let $l = \lim_{n \to \infty} \left(\frac{(n+1)\cdot(n+2)\ldots(3n)}{n^{2n}} \right)^{\frac{1}{n}}$

$$= \lim_{n \to \infty} \left(\frac{(n+1)\cdot(n+2)\ldots(n+2n)}{n^{2n}} \right)^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \left(\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\left(\frac{n+2n}{n}\right) \right)^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log l = \lim_{n \to \infty} \frac{1}{n}\left[\log \left\{ \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{2n}{n}\right) \right\} \right]$$

$$\Rightarrow \log l = \lim_{n \to \infty} \frac{1}{n} \left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \ldots + \log\left(1+\frac{2n}{n}\right) \right]$$

$$\Rightarrow \log l = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log l = \int_0^2 \log(1+x)\, dx$$

$$\Rightarrow \log l = \left[\log(1+x)\cdot x - \int \frac{1}{1+x}\cdot x\, dx \right]_0^2$$

$$\Rightarrow \log l = [\log(1+x)\cdot x]_0^2 - \int_0^2 \frac{x+1-1}{1+x}\, dx$$

$$\Rightarrow \log l = 2\cdot\log 3 - \int_0^2 \left(1 - \frac{1}{1+x}\right) dx$$

$$\Rightarrow \log l = 2\cdot\log 3 - \left[x - \log|1+x| \right]_0^2$$

$$\Rightarrow \log l = 2\cdot\log 3 - [2 - \log 3]$$

$$\Rightarrow \log l = 3\cdot\log 3 - 2$$

$$\Rightarrow \log l = \log 27 - 2$$

$$\therefore \quad l = e^{\log 27 - 2} = 27\cdot e^{-2} = \frac{27}{e^2}$$

16. *(b)* Given equations of curves are

$$y^2 = 2x$$

which is a parabola with vertex (0, 0) and axis parallel to X-axis. ...(i)

And $\quad x^2 + y^2 = 4x$

which is a circle with centre (2, 0) and radius = 2 ...(ii)

On substituting $y^2 = 2x$ in Eq. (ii), we get

$$x^2 + 2x = 4x$$

$$\Rightarrow \quad x^2 = 2x$$

$$\Rightarrow \quad x = 0 \text{ or } x = 2$$

$$\Rightarrow \quad y = 0 \text{ or } y = \pm 2 \qquad \text{[using Eq. (i)]}$$

Now, the required area is the area of shaded region, i.e.

$$\text{Required area} = \frac{\text{Area of a circle}}{4} - \int_0^2 \sqrt{2x}\, dx$$

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 x^{1/2} dx = \pi - \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= \pi - \frac{2\sqrt{2}}{3}[2\sqrt{2} - 0] = \left(\pi - \frac{8}{3}\right) \text{sq unit}$$

17. *(d)* Given differential equation is

$$y(1+xy)\, dx = x\, dy$$

$$\Rightarrow \quad y\, dx + xy^2\, dx = x\, dy$$

$$\Rightarrow \quad \frac{x\, dy - y\, dx}{y^2} = x\, dx$$

$$\Rightarrow \quad -\frac{(y\, dx - x\, dy)}{y^2} = x\, dx$$

$$\Rightarrow \quad -d\left(\frac{x}{y}\right) = x\, dx$$

On integrating both sides, we get

$$-\frac{x}{y} = \frac{x^2}{2} + C \qquad \ldots\text{(i)}$$

$\because$ It passes through (1, −1).

$$\therefore \quad 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Now, from Eq. (i) $\quad -\frac{x}{y} = \frac{x^2}{2} + \frac{1}{2}$

$$\Rightarrow \quad x^2 + 1 = -\frac{2x}{y} \Rightarrow y = -\frac{2x}{x^2+1}$$

$$\therefore \quad f\left(-\frac{1}{2}\right) = \frac{4}{5}$$

18. *(c)* As the given lines $x - y + 1 = 0$ and $7x - y - 5 = 0$ are not parallel, therefore they represent the adjacent sides of the rhombus.

On solving $x - y + 1 = 0$ and $7x - y - 5 = 0$, we get $x = 1$ and $y = 2$. Thus, one of the vertex is $A(1, 2)$.

Let the coordinate of point C be (x, y).

Then, $\quad -1 = \frac{x+1}{2}$ and $-2 = \frac{y+2}{2}$

$$\Rightarrow \quad x + 1 = -2 \text{ and } y = -4 - 2$$

$$\Rightarrow \quad x = -3 \text{ and } y = -6$$

Hence, coordinates of $C = (-3, -6)$

Note that, vertices B and D will satisfy $x - y + 1 = 0$ and $7x - y - 5 = 0$, respectively.

Since, option (c) satisfies $7x - y - 5 = 0$, therefore coordinate of vertex D is $\left(\frac{1}{3}, \frac{-8}{3}\right)$.

19. *(d)* Given equation of circle is
$x^2 + y^2 - 8x - 8y - 4 = 0$, whose centre is $C(4, 4)$ and radius

$$= \sqrt{4^2 + 4^2 + 4} = \sqrt{36} = 6$$

Let the centre of required circle be $C_1(x, y)$. Now, as it touch the X-axis, therefore its radius $= |y|$. Also, it touch the circle $x^2 + y^2 - 8x - 8y - 4 = 0$, therefore $CC_1 = 6 + |y|$

$\Rightarrow \quad \sqrt{(x-4)^2 + (y-4)^2} = 6 + |y|$

$\Rightarrow \quad x^2 + 16 - 8x + y^2 + 16 - 8y = 36 + y^2 + 12|y|$

$\Rightarrow \quad x^2 - 8x - 8y + 32 = 36 + 12|y|$

$\Rightarrow \quad x^2 - 8x - 8y - 4 = 12|y|$

***Case* I** If $y > 0$, then we have

$x^2 - 8x - 8y - 4 = 12y$

$\Rightarrow \quad x^2 - 8x - 20y - 4 = 0$

$\Rightarrow \quad x^2 - 8x - 4 = 20y$

$\Rightarrow \quad (x-4)^2 - 20 = 20y$

$\Rightarrow \quad (x-4)^2 = 20(y+1)$, which is a parabola.

***Case* II** If $y < 0$, then we have

$x^2 - 8x - 8y - 4 = -12y$

$\Rightarrow x^2 - 8x - 8y - 4 + 12y = 0$

$\Rightarrow \quad x^2 - 8x + 4y - 4 = 0$

$\Rightarrow \quad x^2 - 8x - 4 = -4y$

$\Rightarrow \quad (x-4)^2 = 20 - 4y$

$\Rightarrow (x-4)^2 = -4(y-5)$, which is again a parabola.

20. *(b)* Given equation of a circle is $x^2 + y^2 - 4x + 6y - 12 = 0$, whose centre is $(2, -3)$ and radius

$$= \sqrt{2^2 + (-3)^2 + 12} = \sqrt{4 + 9 + 12} = 5$$

Now, according to given information, we have the following figure.

$x^2 + y^2 - 4x + 6y - 12 = 0$

Clearly, $AO \perp BC$, as O is mid-point of the chord.
Now, in ΔAOB, we have

$OA = \sqrt{(-3-2)^2 + (2+3)^2}$

$= \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$

and $\quad OB = 5$

$\therefore \quad AB = \sqrt{OA^2 + OB^2} = \sqrt{50 + 25} = \sqrt{75} = 5\sqrt{3}$

21. *(a)* Centre of circle $x^2 + (y+6)^2 = 1$ is $C(0, -6)$.

Let the coordinates of point P be $(2t^2, 4t)$.

Now, let $\quad D = CP = \sqrt{(2t^2)^2 + (4t+6)^2}$

$\Rightarrow \quad D = \sqrt{4t^4 + 16t^2 + 36 + 48t}$

Squaring on both sides

$\Rightarrow \quad D^2(t) = 4t^4 + 16t^2 + 48t + 36$

Let $\quad F(t) = 4t^4 + 16t^2 + 48t + 36$

For minimum, $\quad F'(t) = 0$

$\Rightarrow \quad 16t^3 + 32t + 48 = 0$

$\Rightarrow \quad t^3 + 2t + 3 = 0$

$\Rightarrow \quad (t+1)(t^2 - t + 3) = 0$

$\Rightarrow \quad t = -1$

Thus, coordinate of point P are $(2, -4)$.

Now, $\quad CP = \sqrt{2^2 + (-4+6)^2} = \sqrt{4+4} = 2\sqrt{2}$

Hence, the required equation of circle is

$(x-2)^2 + (y+4)^2 = (2\sqrt{2})^2$

$\Rightarrow x^2 + 4 - 4x + y^2 + 16 + 8y = 8$

$\Rightarrow \quad x^2 + y^2 - 4x + 8y + 12 = 0$

22. *(c)* We have,

$\frac{2b^2}{a} = 8$ and $2b = ae \Rightarrow b^2 = 4a$ and $2b = ae$

Consider, $2b = ae$

$\Rightarrow \quad 4b^2 = a^2e^2$

$\Rightarrow \quad 4a^2(e^2 - 1) = a^2e^2$

$\Rightarrow \quad 4e^2 - 4 = e^2 \qquad [\because a \neq 0]$

$\Rightarrow \quad 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}} \qquad [\because e > 0]$

23. *(b)* Equation of line passing through the point $(1, -5, 9)$ and parallel to $x = y = z$ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda \qquad \text{(say)}$$

Thus, any point on this line is of the form $(\lambda + 1, \lambda - 5, \lambda + 9)$.

Now, if $P(\lambda + 1, \lambda - 5, \lambda + 9)$ is the point of intersection of line and plane, then

$(\lambda + 1) - (\lambda - 5) + \lambda + 9 = 5$

$\Rightarrow \quad \lambda + 15 = 5$

$\Rightarrow \quad \lambda = -10$

$\therefore$ Coordinates of point P are $(-9, -15, -1)$.

Hence, required distance

$= \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2} = \sqrt{10^2 + 10^2 + 10^2} = 10\sqrt{3}$

24. *(d)* Since, the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane $lx + my - z = 9$, therefore we have

$2l - m - 3 = 0$

[$\because$ normal will be perpendicular to the line]

$\Rightarrow \quad 2l - m = 3 \qquad \ldots(i)$

and $3l - 2m + 4 = 9$

[$\because$ point $(3, -2, -4)$ lies on the plane]

$\Rightarrow \quad 3l - 2m = 5 \qquad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$l = 1$ and $m = -1$

$\therefore \quad l^2 + m^2 = 2$

25. *(d)* Given, $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = 1$

and $\quad \hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

Now, consider $\quad \hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

$\Rightarrow \quad (\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})\hat{\mathbf{b}} - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{c}} = \frac{\sqrt{3}}{2}\hat{\mathbf{b}} + \frac{\sqrt{3}}{2}\hat{\mathbf{c}}$

On comparing, we get

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = -\frac{\sqrt{3}}{2} \Rightarrow |\hat{\mathbf{a}}|\,|\hat{\mathbf{b}}| \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\theta = -\frac{\sqrt{3}}{2} \qquad [\because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1]$$

$$\Rightarrow \quad \cos\theta = \cos\left(\pi - \frac{\pi}{6}\right) \Rightarrow \theta = \frac{5\pi}{6}$$

26. *(b)* We know that, if $x_1, x_2, \ldots, x_n$ are n observations, then their standard deviation is given by

$$\sqrt{\frac{1}{n}\Sigma x_i^2 - \left(\frac{\Sigma x_i}{n}\right)^2}$$

We have, $(3.5)^2 = \dfrac{(2^2 + 3^2 + a^2 + 11^2)}{4} - \left(\dfrac{2 + 3 + a + 11}{4}\right)^2$

$$\Rightarrow \quad \frac{49}{4} = \frac{4 + 9 + a^2 + 121}{4} - \left(\frac{16 + a}{4}\right)^2$$

$$\Rightarrow \quad \frac{49}{4} = \frac{134 + a^2}{4} - \frac{256 + a^2 + 32a}{16}$$

$$\Rightarrow \quad \frac{49}{4} = \frac{4a^2 + 536 - 256 - a^2 - 32a}{16}$$

$$\Rightarrow \quad 49 \times 4 = 3a^2 - 32a + 280$$

$$\Rightarrow \quad 3a^2 - 32a + 84 = 0$$

27. *(d)* Clearly, $E_1 = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$E_2 = \{(1, 2),(2, 2),(3, 2),(4, 2),(5, 2),(6, 2)\}$

and $E_3 = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5)\}$

$$\Rightarrow \quad P(E_1) = \frac{6}{36} = \frac{1}{6}, \; P(E_2) = \frac{6}{36} = \frac{1}{6}$$

and $P(E_3) = \dfrac{18}{36} = \dfrac{1}{2}$

Now, $P(E_1 \cap E_2) = P$ (getting 4 on die A and 2 on die B)

$$= \frac{1}{36} = P(E_1) \cdot P(E_2)$$

$P(E_2 \cap E_3) = P$ (getting 2 on die B and sum of numbers on both dice is odd)

$$= \frac{3}{36} = P(E_2) \cdot P(E_3)$$

$P(E_1 \cap E_3) = P$ (getting 4 on die A and sum of numbers on both dice is odd)

$$= \frac{3}{36} = P(E_1) \cdot P(E_3)$$

and $P(E_1 \cap E_2 \cap E_3) = P$ [getting 4 on die A, 2 on die B and sum of numbers is odd]

$= P$ (impossible event) $= 0$

Hence, E_1, E_2 and E_3 are not independent.

28. *(c)* Given equation is

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$$

$$\Rightarrow \quad (\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0$$

$$\Rightarrow \quad 2\cos 2x \cos x + 2\cos 3x \cos x = 0$$

$$\Rightarrow \quad 2\cos x (\cos 2x + \cos 3x) = 0$$

$$\Rightarrow \quad 2\cos x \left(2\cos\frac{5x}{2}\cos\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \cos x \cdot \cos\frac{5x}{2} \cdot \cos\frac{x}{2} = 0$$

$$\Rightarrow \quad \cos x = 0 \text{ or } \cos\frac{5x}{2} = 0 \text{ or } \cos\frac{x}{2} = 0$$

Now, $\cos x = 0$

$$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad [\because 0 \le x < 2\pi]$$

$$\cos\frac{5x}{2} = 0$$

$$\Rightarrow \quad \frac{5x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}\ldots,$$

$$\Rightarrow \quad x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5} \qquad [\because 0 \le x < 2\pi]$$

and $\cos\dfrac{x}{2} = 0$

$$\Rightarrow \quad \frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\Rightarrow \quad x = \pi \qquad [\because 0 \le x < 2\pi]$$

Hence, $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \pi, \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}$

29. *(d)* According to given information, we have the following figure

Now, from ΔACD and ΔBCD, we have

$$\tan 30° = \frac{h}{x + y}$$

and $\tan 60° = \dfrac{h}{y}$

$$\Rightarrow \quad h = \frac{x + y}{\sqrt{3}} \qquad \ldots(i)$$

and $h = \sqrt{3}\, y$ ...(ii)

From Eqs. (i) and (ii),

$$\frac{x + y}{\sqrt{3}} = \sqrt{3}\, y$$

$$\Rightarrow \quad x + y = 3y$$

$$\Rightarrow \quad x - 2y = 0$$

$$\Rightarrow \quad y = \frac{x}{2}$$

$\because$ Speed is uniform.

$\therefore$ Distance y will be cover in 5 min.

$\because$ Distance x covered in 10 min.

$\therefore$ Distance $\dfrac{x}{2}$ will be cover in 5 min.

30. *(c)* Consider, $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$

$$\equiv [(p \wedge \sim q) \vee q] \vee (\sim p \wedge q)$$

$$\equiv [(p \vee q) \wedge (\sim q \vee q)] \vee (\sim p \wedge q)$$

$$\equiv [(p \vee q) \wedge t] \vee (\sim p \wedge q)$$

$$\equiv (p \vee q) \vee (\sim p \wedge q)$$

$$\equiv (p \vee q \vee \sim p) \wedge (p \vee q \vee q)$$

$$\equiv (q \vee t) \wedge (p \vee q)$$

$$\equiv t \wedge (p \vee q)$$

$$\equiv p \vee q$$

SOLVED PAPER 2017

JEE MAIN

Joint Entrance Examination

OFFLINE

INSTRUCTIONS

1. This test consists of 30 questions.
2. Each question is allotted 4 mark for correct response.
3. Candidates will be awarded mark as stated above in instruction no. 2 for correct response of each question. 1 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
4. There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

Mathematics

1. If S is the set of distinct values of b for which the following system of linear equations

$$x + y + z = 1,$$
$$x + ay + z = 1$$
and $$ax + by + z = 0$$

has no solution, then S is
(a) an infinite set
(b) a finite set containing two or more elements
(c) singleton set
(d) a empty set

2. The statement $(p \to q) \to [(\sim p \to q) \to q]$ is
(a) a tautology
(b) equivalent to $\sim p \to q$
(c) equivalent to $p \to \sim q$
(d) a fallacy

3. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is
(a) $-\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $-\frac{7}{9}$

4. For three events A, B and C, if P(exactly one of A or B occurs) = P(exactly one of B or C occurs) = P (exactly one of C or A occurs) = $\frac{1}{4}$ and P (all the three events occur simultaneously) = $\frac{1}{16}$, then the probability that atleast one of the events occurs, is
(a) $\frac{7}{32}$ (b) $\frac{7}{16}$ (c) $\frac{7}{64}$ (d) $\frac{3}{16}$

5. Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to
(a) $-z$ (b) z (c) -1 (d) 1

6. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq units. Then, the orthocentre of this triangle is at the point
(a) $\left(2, -\frac{1}{2}\right)$ (b) $\left(1, \frac{3}{4}\right)$ (c) $\left(1, -\frac{3}{4}\right)$ (d) $\left(2, \frac{1}{2}\right)$

7. If 20 m of wire is available for fencing off a flower-bed in the form of a circular sector, then the maximum area (in sq m) of the flower-bed is
(a) 12.5 (b) 10
(c) 25 (d) 30

8. The area (in sq units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is

(a) $\frac{59}{12}$ (b) $\frac{3}{2}$
(c) $\frac{7}{3}$ (d) $\frac{5}{2}$

9. If the image of the point $P(1, -2, 3)$ in the plane $2x + 3y - 4z + 22 = 0$ measured parallel to the line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to

(a) $3\sqrt{5}$ (b) $2\sqrt{42}$
(c) $\sqrt{42}$ (d) $6\sqrt{5}$

10. For $x \in \left(0, \frac{1}{4}\right)$, if the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1 - 9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals

(a) $\frac{9}{1 + 9x^3}$ (b) $\frac{3x\sqrt{x}}{1 - 9x^3}$
(c) $\frac{3x}{1 - 9x^3}$ (d) $\frac{3}{1 + 9x^3}$

11. If $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$
(c) $-\frac{1}{3}$ (d) $\frac{4}{3}$

12. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to

(a) $\frac{6}{7}$ (b) $\frac{1}{4}$
(c) $\frac{2}{9}$ (d) $\frac{4}{9}$

13. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to

(a) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$
(c) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

14. For any three positive real numbers a, b and c, if $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, then

(a) b, c and a are in GP
(b) b, c and a are in AP
(c) a, b and c are in AP
(d) a, b and c are in GP

15. The distance of the point $(1, 3, -7)$ from the plane passing through the point $(1, -1, -1)$ having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is

(a) $\frac{20}{\sqrt{74}}$ units (b) $\frac{10}{\sqrt{83}}$ units
(c) $\frac{5}{\sqrt{83}}$ units (d) $\frac{10}{\sqrt{74}}$ units

16. Let $I_n = \int \tan^n x\, dx\ (n > 1)$. If $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to

(a) $\left(-\frac{1}{5}, 1\right)$ (b) $\left(\frac{1}{5}, 0\right)$
(c) $\left(\frac{1}{5}, -1\right)$ (d) $\left(-\frac{1}{5}, 0\right)$

17. The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is

(a) $2y - x = 2$ (b) $4x - 2y = 1$
(c) $4x + 2y = 7$ (d) $x + 2y = 4$

18. If a hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$, then the tangent to this hyperbola at P also passes through the point

(a) $(3\sqrt{2}, 2\sqrt{3})$
(b) $(2\sqrt{2}, 3\sqrt{3})$
(c) $(\sqrt{3}, \sqrt{2})$
(d) $(-\sqrt{2}, -\sqrt{3})$

19. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1 + x^2}$ is

(a) invertible
(b) injective but not surjective
(c) surjective but not injective
(d) neither injective nor surjective

20. $\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals

(a) $\frac{1}{24}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

21. Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to

(a) $\frac{25}{8}$ (b) 2
(c) 5 (d) $\frac{1}{8}$

22. The normal to the curve $y(x-2)(x-3) = x+6$ at the point, where the curve intersects the Y-axis passes through the point

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

23. If two different numbers are taken from the set $\{0, 1, 2, 3, \ldots, 10\}$, then the probability that their sum as well as absolute difference are both multiple of 4, is

(a) $\frac{6}{55}$ (b) $\frac{12}{55}$

(c) $\frac{14}{45}$ (d) $\frac{7}{55}$

24. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

(a) 485 (b) 468

(c) 469 (d) 484

25. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

(a) $2^{21} - 2^{11}$ (b) $2^{21} - 2^{10}$

(c) $2^{20} - 2^{9}$ (d) $2^{20} - 2^{10}$

26. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn one-by-one with replacement, then the variance of the number of green balls drawn is

(a) $\frac{12}{5}$ (b) 6

(c) 4 (d) $\frac{6}{25}$

27. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ be such that $a + b + c = 3$ and $f(x+y) = f(x) + f(y) + xy$, $\forall\ x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to

(a) 330 (b) 165

(c) 190 (d) 255

28. The radius of a circle having minimum area, which touches the curve $y = 4 - x^2$ and the lines $y = |x|$, is

(a) $2(\sqrt{2}+1)$ (b) $2(\sqrt{2}-1)$

(c) $4(\sqrt{2}-1)$ (d) $4(\sqrt{2}+1)$

29. For a positive integer n, if the quadratic equation, $x(x+1) + (x+1)(x+2) + \ldots + (x+\overline{n-1})(x+n) = 10n$ has two consecutive integral solutions, then n is equal to

(a) 12 (b) 9

(c) 10 (d) 11

30. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to

(a) -2 (b) 2

(c) 4 (d) -1

Answer with Explanation

1. *(d)* $\because\ \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = 1(a-b) - 1(1-a) + 1(b-a^2)$

$= -(a-1)^2$

$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{vmatrix} = 1(a-b) - 1(1) + 1(b) = -(a-1)$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{vmatrix} = 1(1) - 1(1-a) + 1(0-a) = 0$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{vmatrix} = 1(-b) - 1(-a) + 1(b-a^2) = -a(a-1)$

For $a = 1$

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

Δ for $b = 1$ only

$x + y + z = 1,$

$x + y + z = 1$

and $x + y + z = 0$

i.e. no solution ($\because$ RHS is not equal)

Hence, for no solution $b = 1$ only

2. *(a)* The truth table of the given expression is given below :

p	q	$x \equiv p \to q$	$\sim p$	$\sim p \to q$	$y \equiv (\sim p \to q) \to q$	$x \to y$
T	T	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

Hence, it is a tautology.

3. *(d)* Given, $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

$\Rightarrow\ 5\left(\frac{1-\cos 2x}{1+\cos 2x} - \frac{1+\cos 2x}{2}\right) = 2\cos 2x + 9$

Put $\cos 2x = y$, we have

$5\left(\frac{1-y}{1+y} - \frac{1+y}{2}\right) = 2y + 9$

$\Rightarrow \quad 5(2-2y-1-y^2-2y)=2(1+y)(2y+9)$

$\Rightarrow \quad 5(1-4y-y^2)=2(2y+9+2y^2+9y)$

$\Rightarrow \quad 5-20y-5y^2=22y+18+4y^2$

$\Rightarrow \quad 9y^2+42y+13=0$

$\Rightarrow \quad 9y^2+3y+39y+13=0$

$\Rightarrow \quad 3y(3y+1)+13(3y+1)=0$

$\Rightarrow \quad (3y+1)(3y+13)=0$

$\Rightarrow \quad y=-\frac{1}{3},-\frac{13}{3}$

$\therefore \quad \cos 2x=-\frac{1}{3},-\frac{13}{3}$

$\Rightarrow \quad \cos 2x=-\frac{1}{3} \qquad \left[\because \cos 2x \neq -\frac{13}{3}\right]$

Now, $\quad \cos 4x=2\cos^2 2x-1=2\left(-\frac{1}{3}\right)^2-1$

$=\frac{2}{9}-1=-\frac{7}{9}$

4. ***(b)*** We have, P (exactly one of A or B occurs)

$=P(A\cup B)-P(A\cap B)$

$=P(A)+P(B)-2P(A\cap B)$

According to the question,

$P(A)+P(B)-2P(A\cap B)=\frac{1}{4} \quad \ldots$(i)

$P(B)+P(C)-2P(B\cap C)=\frac{1}{4} \quad \ldots$(ii)

and $\quad P(C)+P(A)-2P(C\cap A)=\frac{1}{4} \quad \ldots$(iii)

On adding Eqs. (i), (ii) and (iii), we get

$2[P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)]=\frac{3}{4}$

$\Rightarrow P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)=\frac{3}{8}$

$\therefore P$ (atleast one event occurs)

$=P(A\cup B\cup C)$

$=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

$=\frac{3}{8}+\frac{1}{16}=\frac{7}{16} \qquad \left[\because P(A\cap B\cap C)=\frac{1}{16}\right]$

5. ***(a)*** Given, $\quad 2\omega+1=z$

$\Rightarrow \quad 2\omega+1=\sqrt{-3} \qquad [\because z=\sqrt{-3}]$

$\Rightarrow \quad \omega=\frac{-1+\sqrt{3}i}{2}$

Since, ω is cube root of unity.

$\therefore \quad \omega^2=\frac{-1-\sqrt{3}i}{2}$ and $\omega^{3n}=1$

Now, $\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=3k$

$\Rightarrow \quad \begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}=3k$

$[\because 1+\omega+\omega^2=0$ and $\omega^7=(\omega^3)^2\cdot\omega=\omega]$

On applying $R_1\to R_1+R_2+R_3$, we get

$\begin{vmatrix}3 & 1+\omega+\omega^2 & 1+\omega+\omega^2\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}=3k \Rightarrow \begin{vmatrix}3 & 0 & 0\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}=3k$

$\Rightarrow \quad 3(\omega^2-\omega^4)=3k$

$\Rightarrow \quad (\omega^2-\omega)=k$

$\therefore \quad k=\left(\frac{-1-\sqrt{3}i}{2}\right)-\left(\frac{-1+\sqrt{3}i}{2}\right)$

$=-\sqrt{3}i=-z$

6. ***(d)*** Given, vertices of triangle are $(k,-3k)$, $(5,k)$ and $(-k,2)$.

$\therefore \quad \frac{1}{2}\begin{vmatrix}k & -3k & 1\\ 5 & k & 1\\ -k & 2 & 1\end{vmatrix}=\pm 28$

$\Rightarrow \quad \begin{vmatrix}k & -3k & 1\\ 5 & k & 1\\ -k & 2 & 1\end{vmatrix}=\pm 56$

$\Rightarrow \quad k(k-2)+3k(5+k)+1(10+k^2)=\pm 56$

$\Rightarrow \quad k^2-2k+15k+3k^2+10+k^2=\pm 56$

$\Rightarrow \quad 5k^2+13k+10=\pm 56$

$\Rightarrow \quad 5k^2+13k-66=0$ or $5k^2+13k-46=0$

$\Rightarrow \quad k=2 \qquad [\because k\in I]$

Thus, the coordinates of vertices of triangle are $A(2,-6)$, $B(5,2)$ and $C(-2,2)$.

Now, equation of altitude from vertex A is

$y-(-6)=\frac{-1}{\left(\frac{2-2}{-2-5}\right)}(x-2)\Rightarrow x=2 \quad \ldots$(i)

Equation of altitude from vertex C is

$y-2=\frac{-1}{\left[\frac{2-(-6)}{5-2}\right]}[x-(-2)]$

$\Rightarrow \quad 3x+8y-10=0 \quad \ldots$(ii)

On solving Eqs. (i) and (ii), we get $x=2$ and $y=\frac{1}{2}$.

$\therefore$ Orthocentre $=\left(2,\frac{1}{2}\right)$

7. ***(c)*** Total length $=2r+r\theta=20$

$\Rightarrow \quad \theta=\frac{20-2r}{r}$

Now, area of flower-bed,

$A=\frac{1}{2}r^2\theta$

$\Rightarrow \quad A=\frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$

$\Rightarrow \quad A=10r-r^2$

$\therefore \qquad \dfrac{dA}{dr} = 10 - 2r$

For maxima or minima, put $\dfrac{dA}{dr} = 0$.

$\Rightarrow \qquad 10 - 2r = 0$

$\Rightarrow \qquad r = 5$

$\therefore \qquad A_{\max} = \dfrac{1}{2}(5)^2\left[\dfrac{20-2(5)}{5}\right]$

$= \dfrac{1}{2} \times 25 \times 2 = 25$ sq m

8. *(d)* Required area

$= \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \dfrac{x^2}{4}dx$

$= \left[x + \dfrac{x^{3/2}}{3/2}\right]_0^1 + \left[3x - \dfrac{x^2}{2}\right]_1^2 - \left[\dfrac{x^3}{12}\right]_0^2$

$= \left(1 + \dfrac{2}{3}\right) + \left(6 - 2 - 3 + \dfrac{1}{2}\right) - \left(\dfrac{8}{12}\right)$

$= \dfrac{5}{3} + \dfrac{3}{2} - \dfrac{2}{3} = 1 + \dfrac{3}{2} = \dfrac{5}{2}$ sq units

9. *(b)* Any line parallel to $\dfrac{x}{1} = \dfrac{y}{4} = \dfrac{z}{5}$ and passing through $P(1, -2, 3)$ is

$$\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5} = \lambda \qquad \text{(say)}$$

Any point on above line can be written as

$(\lambda + 1, 4\lambda - 2, 5\lambda + 3)$.

$\therefore$ Coordinates of R are $(\lambda + 1, 4\lambda - 2, 5\lambda + 3)$.

Since, point R lies on the above plane.

$\therefore \quad 2(\lambda + 1) + 3(4\lambda - 2) - 4(5\lambda + 3) + 22 = 0$

$\Rightarrow \qquad \lambda = 1$

So, point R is (2, 2, 8).

Now, $PR = \sqrt{(2-1)^2 + (2+2)^2 + (8-3)^2} = \sqrt{42}$

$\therefore \quad PQ = 2PR = 2\sqrt{42}$

10. *(a)* Let $y = \tan^{-1}\left(\dfrac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left[\dfrac{2\cdot(3x^{3/2})}{1-(3x^{3/2})^2}\right]$

$= 2\tan^{-1}(3x^{3/2})\left[\because 2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}\right]$

$\therefore \quad \dfrac{dy}{dx} = 2\cdot\dfrac{1}{1+(3x^{3/2})^2}\cdot 3 \times \dfrac{3}{2}(x)^{1/2}$

$= \dfrac{9}{1+9x^3}\cdot\sqrt{x}$

$\therefore \quad g(x) = \dfrac{9}{1+9x^3}$

11. *(a)* We have, $(2+\sin x)\dfrac{dy}{dx} + (y+1)\cos x = 0$

$\Rightarrow \qquad \dfrac{dy}{dx} + \dfrac{\cos x}{2+\sin x}y = \dfrac{-\cos x}{2+\sin x}$

which is a linear differential equation.

$\therefore \quad \text{IF} = e^{\int \frac{\cos x}{2+\sin x}dx} = e^{\log(2+\sin x)} = 2 + \sin x$

$\therefore$ Required solution is given by

$y\cdot(2+\sin x) = \int \dfrac{-\cos x}{2+\sin x}\cdot(2+\sin x)dx + C$

$\Rightarrow \quad y(2+\sin x) = -\sin x + C$

Also, $\qquad y(0) = 1$

$\therefore \quad 1(2+\sin 0) = -\sin 0 + C$

$\Rightarrow \qquad C = 2$

$\therefore \qquad y = \dfrac{2-\sin x}{2+\sin x}$

$\Rightarrow \qquad y\left(\dfrac{\pi}{2}\right) = \dfrac{2-\sin\frac{\pi}{2}}{2+\sin\frac{\pi}{2}} = \dfrac{1}{3}$

12. *(c)* Let $AB = h$, then $AD = 2h$ and $AC = BC = \dfrac{h}{2}$

Again, let $\angle CPA = \alpha$

Now, in ΔABP, $\tan(\alpha+\beta) = \dfrac{AB}{AP} = \dfrac{h}{2h} = \dfrac{1}{2}$

Also, in ΔACP, $\tan\alpha = \dfrac{AC}{AP} = \dfrac{\frac{h}{2}}{2h} = \dfrac{1}{4}$

Now, $\tan\beta = \tan[(\alpha+\beta) - \alpha]$

$= \dfrac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \dfrac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2}\times\frac{1}{4}} = \dfrac{\frac{1}{4}}{\frac{9}{8}}$

$= \dfrac{2}{9}$

13. *(b)* We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

Now, $3A^2 + 12A = 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} + 12\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$

$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

$\therefore \text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

14. *(c)* We have,

$$225a^2 + 9b^2 + 25c^2 - 75ac - 45ab - 15bc = 0$$

$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$\Rightarrow \frac{1}{2}[(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2] = 0$

$\Rightarrow \quad 15a = 3b, 3b = 5c \text{ and } 5c = 15a$

$\therefore \quad 15a = 3b = 5c$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{5} = \frac{c}{3} = \lambda \text{ (say)}$

$\Rightarrow \quad a = \lambda, b = 5\lambda, c = 3\lambda$

Hence, a, b and c are in AP.

15. *(b)* Given, equations of lines are

$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and

$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$

Let $\mathbf{n}_1 = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{n}_2 = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$

$\therefore$ Any vector $\mathbf{n}$ perpendicular to both $\mathbf{n}_1, \mathbf{n}_2$ is given by

$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2$

$\Rightarrow \quad \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$\therefore$ Equation of a plane passing through $(1, -1, -1)$ and perpendicular to $\mathbf{n}$ is given by

$5(x-1) + 7(y+1) + 3(z+1) = 0$

$\Rightarrow \quad 5x + 7y + 3z + 5 = 0$

$\therefore$ Required distance $= \left|\frac{5 + 21 - 21 + 5}{\sqrt{5^2 + 7^2 + 3^2}}\right| = \frac{10}{\sqrt{83}}$ units

16. *(b)* We have, $I_n = \int \tan^n x \, dx$

$\therefore \quad I_n + I_{n+2} = \int \tan^n x \, dx + \int \tan^{n+2} x \, dx$

$= \int \tan^n x(1 + \tan^2 x) \, dx$

$= \int \tan^n x \sec^2 x \, dx = \frac{\tan^{n+1} x}{n+1} + C$

Put $n = 4$, we get $I_4 + I_6 = \frac{\tan^5 x}{5} + C$

$\therefore \quad a = \frac{1}{5}$ and $b = 0$

17. *(b)* We have, $e = \frac{1}{2}$ and $\frac{a}{e} = 4$

$\therefore \quad a = 2$

Now, $b^2 = a^2(1 - e^2) = (2)^2\left[1 - \left(\frac{1}{2}\right)^2\right] = 4\left(1 - \frac{1}{4}\right) = 3$

$\Rightarrow \quad b = \sqrt{3}$

$\therefore$ Equation of the ellipse is $\frac{x^2}{(2)^2} + \frac{y^2}{(\sqrt{3})^2} = 1$

$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1$

Now, the equation of normal at $\left(1, \frac{3}{2}\right)$ is

$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

$\Rightarrow \quad \frac{4x}{1} - \frac{3y}{(3/2)} = 4 - 3$

$\Rightarrow \quad 4x - 2y = 1$

18. *(b)* Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore \quad ae = 2 \Rightarrow a^2e^2 = 4$

$\Rightarrow \quad a^2 + b^2 = 4 \Rightarrow b^2 = 4 - a^2$

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{4 - a^2} = 1$

Since, $(\sqrt{2}, \sqrt{3})$ lie on hyperbola.

$\therefore \quad \frac{2}{a^2} - \frac{3}{4 - a^2} = 1$

$\Rightarrow \quad 8 - 2a^2 - 3a^2 = a^2(4 - a^2)$

$\Rightarrow \quad 8 - 5a^2 = 4a^2 - a^4$

$\Rightarrow \quad a^4 - 9a^2 + 8 = 0$

$\Rightarrow \quad (a^4 - 8)(a^4 - 1) = 0$

$\Rightarrow \quad a^4 = 8, a^4 = 1$

$\therefore \quad a = 1$

Now, equation of hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$.

$\therefore$ Equation of tangent at $(\sqrt{2}, \sqrt{3})$ is given by

$\sqrt{2}x - \frac{\sqrt{3}y}{3} = 1 \Rightarrow \sqrt{2}x - \frac{y}{\sqrt{3}} = 1$

which passes through the point $(2\sqrt{2}, 3\sqrt{3})$.

19. *(c)* We have, $f(x) = \frac{x}{1 + x^2}$

$\therefore \quad f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} = \frac{x}{1 + x^2} = f(x)$

$\therefore \quad f\left(\frac{1}{2}\right) = f(2) \text{ or } f\left(\frac{1}{3}\right) = f(3)$ and so on.

So, $f(x)$ is many-one function.

Again, let $y = f(x) \Rightarrow y = \dfrac{x}{1+x^2}$

$\Rightarrow \quad y + x^2 y = x \Rightarrow yx^2 - x + y = 0$

As, $\quad x \in R$

$\therefore \quad (-1)^2 - 4(y)(y) \geq 0 \Rightarrow 1 - 4y^2 \geq 0$

$\Rightarrow \quad y \in \left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$

$\therefore$ Range = Codomain = $\left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$

So, $f(x)$ is surjective.

Hence, $f(x)$ is surjective but not injective.

20. *(b)* $\lim\limits_{x \to \pi/2} \dfrac{\cot x - \cos x}{(\pi - 2x)^3} = \lim\limits_{x \to \pi/2} \dfrac{1}{8} \cdot \dfrac{\cos x(1 - \sin x)}{\sin x\left(\dfrac{\pi}{2} - x\right)^3}$

$$= \lim_{h \to 0} \frac{1}{8} \cdot \frac{\cos\left(\frac{\pi}{2} - h\right)\left[1 - \sin\left(\frac{\pi}{2} - h\right)\right]}{\sin\left(\frac{\pi}{2} - h\right)\left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)^3}$$

$$= \frac{1}{8} \lim_{h \to 0} \frac{\sin h\,(1 - \cos h)}{\cos h \cdot h^3}$$

$$= \frac{1}{8} \lim_{h \to 0} \frac{\sin h\left(2\sin^2 \frac{h}{2}\right)}{\cos h \cdot h^3}$$

$$= \frac{1}{4} \lim_{h \to 0} \frac{\sin h \cdot \sin^2\left(\frac{h}{2}\right)}{h^3 \cos h}$$

$$= \frac{1}{4} \lim_{h \to 0} \left(\frac{\sin h}{h}\right)\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \cdot \frac{1}{\cos h} \cdot \frac{1}{4} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

21. *(b)* We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{a}| = \sqrt{4 + 1 + 4} = 3$

and $\quad \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \Rightarrow |\mathbf{b}| = \sqrt{1 + 1} = \sqrt{2}$

Now, $\quad |\mathbf{c} - \mathbf{a}| = 3 \Rightarrow |\mathbf{c} - \mathbf{a}|^2 = 9$

$\Rightarrow \quad (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 9$

$\Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{c} \cdot \mathbf{a} = 9 \quad \ldots$(i)

Again, $\quad |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}||\mathbf{c}| \sin 30° = 3 \Rightarrow |\mathbf{c}| = \dfrac{6}{|\mathbf{a} \times \mathbf{b}|}$

But $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad |\mathbf{c}| = \dfrac{6}{\sqrt{4 + 4 + 1}} = 2 \quad \ldots$(ii)

From Eqs. (i) and (ii), we get

$(2)^2 + (3)^2 - 2\mathbf{c} \cdot \mathbf{a} = 9 \Rightarrow 4 + 9 - 2\mathbf{c} \cdot \mathbf{a} = 9$

$\Rightarrow \quad \mathbf{c} \cdot \mathbf{a} = 2$

22. *(b)* Given curve is

$$y(x - 2)(x - 3) = x + 6 \quad \ldots(i)$$

Put $x = 0$ in Eq. (i), we get

$$y(-2)(-3) = 6 \Rightarrow y = 1$$

So, point of intersection is (0, 1).

Now, $\quad y = \dfrac{x + 6}{(x - 2)(x - 3)}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1(x - 2)(x - 3) - (x + 6)(x - 3 + x - 2)}{(x - 2)^2(x - 3)^2}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(0, 1)} = \dfrac{6 + 30}{4 \times 9} = \dfrac{36}{36} = 1$

$\therefore$ Equation of normal at (0, 1) is given by

$$y - 1 = \frac{-1}{1}(x - 0) \Rightarrow x + y - 1 = 0$$

which passes through the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

23. *(a)* Total number of ways of selecting 2 different numbers from $\{0, 1, 2, \ldots, 10\} = {}^{11}C_2 = 55$

Let two numbers selected be x and y.

Then, $\quad x + y = 4m \quad \ldots$(i)

and $\quad x - y = 4n \quad \ldots$(ii)

$\Rightarrow \quad 2x = 4(m + n)$ and $2y = 4(m - n)$

$\Rightarrow \quad x = 2(m + n)$ and $y = 2(m - n)$

$\therefore$ x and y both are even numbers.

x	y
0	4, 8
2	6, 10
4	0, 8
6	2, 10
8	0, 4
10	2, 6

$\therefore$ Required probability = $\dfrac{6}{55}$

24. *(a)* Given, X has 7 friends, 4 of them are ladies and 3 are men while Y has 7 friends, 3 of them are ladies and 4 are men.

$\therefore$ Total number of required ways

$= {}^3C_3 \times {}^4C_0 \times {}^4C_0 \times {}^3C_3 + {}^3C_2 \times {}^4C_1 \times {}^4C_1 \times {}^3C_2$

$\quad + {}^3C_1 \times {}^4C_2 \times {}^4C_2 \times {}^3C_1 + {}^3C_0 \times {}^4C_3 \times {}^4C_3 \times {}^3C_0$

$= 1 + 144 + 324 + 16 = 485$

25. *(d)* $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$

$= ({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10})$

$= \dfrac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{20}) - (2^{10} - 1)$

$= \dfrac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{21} - 1) - (2^{10} - 1)$

$= \dfrac{1}{2}(2^{21} - 2) - (2^{10} - 1) = 2^{20} - 1 - 2^{10} + 1$

$= 2^{20} - 2^{10}$

26. *(a)* Given box contains 15 green and 10 yellow balls.

$\therefore$ Total number of balls = 15 + 10 = 25

P(green balls) = $\dfrac{15}{25} = \dfrac{3}{5} = p$ = Probability of success

P(yellow balls) = $\dfrac{10}{25} = \dfrac{2}{5} = q$ = Probability of unsuccess and

$n = 10$ = Number of trials.

$\therefore$ Variance = $npq = 10 \times \dfrac{3}{5} \times \dfrac{2}{5} = \dfrac{12}{5}$

27. *(a)* We have, $f(x) = ax^2 + bx + c$

Now, $f(x + y) = f(x) + f(y) + xy$

Put $y = 0 \Rightarrow f(x) = f(x) + f(0) + 0$

$\Rightarrow \quad f(0) = 0$

$\Rightarrow \quad c = 0$

Again, put $y = -x$

$\therefore \quad f(0) = f(x) + f(-x) - x^2$

$\Rightarrow \quad 0 = ax^2 + bx + ax^2 - bx - x^2$

$\Rightarrow \quad 2ax^2 - x^2 = 0$

$\Rightarrow \quad a = \frac{1}{2}$

Also, $a + b + c = 3$

$\Rightarrow \quad \frac{1}{2} + b + 0 = 3 \Rightarrow b = \frac{5}{2}$

$\therefore \quad f(x) = \frac{x^2 + 5x}{2}$

Now, $f(n) = \frac{n^2 + 5n}{2} = \frac{1}{2}n^2 + \frac{5}{2}n$

$$\therefore \quad \sum_{n=1}^{10} f(n) = \frac{1}{2}\sum_{n=1}^{10} n^2 + \frac{5}{2}\sum_{n=1}^{10} n$$

$$= \frac{1}{2} \cdot \frac{10 \times 11 \times 21}{6} + \frac{5}{2} \times \frac{10 \times 11}{2}$$

$$= \frac{385}{2} + \frac{275}{2} = \frac{660}{2} = 330$$

28. *(c)* Let the radius of circle with least area be r. Then, then coordinates of centre $= (0, 4 - r)$.

Since, circle touches the line $y = x$ in first quadrant.

$$\therefore \quad \left|\frac{0 - (4 - r)}{\sqrt{2}}\right| = r \Rightarrow r - 4 = \pm r\sqrt{2}$$

$$\Rightarrow \quad r = \frac{4}{\sqrt{2} + 1} \text{ or } \frac{4}{1 - \sqrt{2}}$$

But $\quad r \neq \frac{4}{1 - \sqrt{2}} \quad \left[\because \frac{4}{1 - \sqrt{2}} < 0\right]$

$$\therefore \quad r = \frac{4}{\sqrt{2} + 1} = 4(\sqrt{2} - 1)$$

29. *(d)* Given quadratic equation is

$x(x + 1) + (x + 1)(x + 2) + \ldots + (x + \overline{n - 1}) \quad (x + n) = 10n$

$\Rightarrow (x^2 + x^2 + \ldots + x^2) + [(1 + 3 + 5 + \ldots + (2n - 1)]x$

$+ [(1 \cdot 2 + 2 \cdot 3 + \ldots + (n - 1)n] = 10n$

$$\Rightarrow nx^2 + n^2x + \frac{n(n^2 - 1)}{3} - 10n = 0$$

$$\Rightarrow x^2 + nx + \frac{n^2 - 1}{3} - 10 = 0$$

$$\Rightarrow 3x^2 + 3nx + n^2 - 31 = 0$$

Let α and β be the roots.

Since, α and β are consecutive.

$\therefore \quad |\alpha - \beta| = 1 \quad \Rightarrow \quad (\alpha - \beta)^2 = 1$

Again, $\quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$\Rightarrow \quad 1 = \left(\frac{-3n}{3}\right)^2 - 4\left(\frac{n^2 - 31}{3}\right)$$

$$\Rightarrow \quad 1 = n^2 - \frac{4}{3}(n^2 - 31) \Rightarrow 3 = 3n^2 - 4n^2 + 124$$

$\Rightarrow \quad n^2 = 121 \quad \Rightarrow \quad n = \pm 11$

$\therefore \quad n = 11 \quad [\because n > 0]$

30. *(b)* Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x} = \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{1 - \cos^2 x} dx$

$$= \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{\sin^2 x} dx$$

$$= \int_{\pi/4}^{3\pi/4} (\text{cosec}^2 x - \text{cosec } x \cot x) dx$$

$$= [-\cot x + \text{cosec } x]_{\pi/4}^{3\pi/4}$$

$$= [(1 + \sqrt{2}) - (-1 + \sqrt{2})] = 2$$

JEE MAIN

SOLVED PAPER 2018

1. Two sets A and B are as under

$A = \{(a,b) \in \mathbf{R} \times \mathbf{R} : |a-5| < 1 \text{ and } |b-5| < 1\}$;

$B = \{(a,b) \in \mathbf{R} \times \mathbf{R} : 4(a-6)^2 + 9(b-5)^2 \le 36\}$. Then,

(a) $B \subset A$
(b) $A \subset B$
(c) $A \cap B = \phi$ (an empty set)
(d) neither $A \subset B$ nor $B \subset A$

2. Let $S = \{x \in \mathbf{R} : x \ge 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$. Then, S

(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains exactly four elements

3. If $\alpha, \beta \in \mathbf{C}$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

(a) –1 (b) 0 (c) 1 (d) 2

4. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to

(a) (–4, – 5) (b) (–4, 3)
(c) (–4, 5) (d) (4, 5)

5. If the system of linear equations

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

(a) –10 (b) 10 (c) –30 (d) 30

6. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf, so that the dictionary is always in the middle. The number of such arrangements is

(a) atleast 1000 (b) less than 500
(c) atleast 500 but less than 750
(d) atleast 750 but less than 1000

7. The sum of the coefficients of all odd degree terms in the expansion of $\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$, $(x > 1)$ is

(a) –1 (b) 0 (c) 1 (d) 2

8. Let $a_1, a_2, a_3, \ldots, a_{49}$ be in AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots + a_{17}^2 = 140$ m, then m is equal to

(a) 66 (b) 68
(c) 34 (d) 33

9. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$

If $B - 2A = 100\lambda$, then λ is equal to

(a) 232 (b) 248 (c) 464 (d) 496

10. For each $t \in \mathbf{R}$, let $[t]$ be the greatest integer less than or equal to t. Then,

$$\lim_{x \to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots + \left[\frac{15}{x}\right]\right)$$

(a) is equal to 0 (b) is equal to 15
(c) is equal to 120 (d) does not exist (in $\mathbf{R}$)

11. Let $S = (t \in \mathbf{R} : f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$ is not differentiable at t}. Then, the set S is equal to

(a) ϕ (an empty set) (b) $\{0\}$
(c) $\{\pi\}$ (d) $\{0, \pi\}$

12. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(a) 6 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

13. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbf{R} - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is

(a) 3 (b) –3 (c) $-2\sqrt{2}$ (d) $2\sqrt{2}$

14. The integral

$$\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$$

is equal to

(a) $\frac{1}{3(1 + \tan^3 x)} + C$ (b) $\frac{-1}{3(1 + \tan^3 x)} + C$
(c) $\frac{1}{1 + \cot^3 x} + C$ (d) $\frac{-1}{1 + \cot^3 x} + C$

(where C is a constant of integration)

15. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

16. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area (in sq units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is

(a) $\frac{1}{2}(\sqrt{3} - 1)$ (b) $\frac{1}{2}(\sqrt{3} + 1)$ (c) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$ (d) $\frac{1}{2}(\sqrt{2} - 1)$

17. Let $y = y(x)$ be the solution of the differential equation $\sin x \frac{dy}{dx} + y\cos x = 4x$, $x \in (0, \pi)$.

If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(a) $\frac{4}{9\sqrt{3}}\pi^2$ (b) $\frac{-8}{9\sqrt{3}}\pi^2$ (c) $-\frac{8}{9}\pi^2$ (d) $-\frac{4}{9}\pi^2$

18. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle OPRQ is completed, then the locus of R is

(a) $3x + 2y = 6$ (b) $2x + 3y = xy$
(c) $3x + 2y = xy$ (d) $3x + 2y = 6xy$

19. Let the orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3), respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(a) $\sqrt{10}$ (b) $2\sqrt{10}$ (c) $3\sqrt{\frac{5}{2}}$ (d) $\frac{3\sqrt{5}}{2}$

20. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is

(a) 195 (b) 185 (c) 85 (d) 95

21. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is

(a) $\frac{1}{2}$ (b) 2 (c) 3 (d) $\frac{4}{3}$

22. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point T(0, 3), then the area (in sq units) of ΔPTQ is

(a) $45\sqrt{5}$ (b) $54\sqrt{3}$ (c) $60\sqrt{3}$ (d) $36\sqrt{5}$

23. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is

(a) $\frac{1}{4\sqrt{2}}$ (b) $\frac{1}{3\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

24. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x + y + z = 7$ is

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\sqrt{\frac{2}{3}}$

25. Let $\mathbf{u}$ be a vector coplanar with the vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$. If $\mathbf{u}$ is perpendicular to $\mathbf{a}$ and $\mathbf{u} \cdot \mathbf{b} = 24$, then $|\mathbf{u}|^2$ is equal to

(a) 336 (b) 315 (c) 256 (d) 84

26. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

(a) $\frac{3}{10}$ (b $\frac{2}{5}$ (c) $\frac{1}{5}$ (d) $\frac{3}{4}$

27. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, \ldots, x_9$ is

(a) 9 (b) 4 (c) 2 (d) 3

28. If sum of all the solutions of the equation

$$8\cos x \cdot \left(\cos\left(\frac{\pi}{6} + x\right) \cdot \cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2}\right)$$

$= 1$ in $[0, \pi]$ is $k\pi$, then k is equal to

(a) $\frac{2}{3}$ (b) $\frac{13}{9}$ (c) $\frac{8}{9}$ (d) $\frac{20}{9}$

29. PQR is a triangular park with PQ = PR = 200 m. A TV tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is

(a) 100 (b) 50 (c) $100\sqrt{3}$ (d) $50\sqrt{2}$

30. The boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to

(a) $\sim p$ (b) p (c) q (d) $\sim q$

Answers

1.	(b)	2.	(c)	3.	(c)	4.	(c)	5.	(b)	6.	(a)	7.	(d)	8.	(c)	9.	(b)	10.	(c)
11.	(a)	12.	(d)	13.	(d)	14.	(b)	15.	(d)	16.	(a)	17.	(c)	18.	(c)	19.	(c)	20.	(d)
21.	(b)	22.	(a)	23.	(b)	24.	(d)	25.	(a)	26.	(b)	27.	(c)	28.	(b)	29.	(a)	30.	(a)

Answer *with* Explanations

1. *(b)* We have,

$$|a-5|<1 \text{ and } |b-5|<1$$

$\therefore \quad -1<a-5<1$ and $-1<b-5<1$

$\Rightarrow \quad 4<a<6$ and $4<b<6$

Now, $4(a-6)^2+9(b-5)^2 \le 36$

$$\Rightarrow \quad \frac{(a-6)^2}{9}+\frac{(b-5)^2}{4} \le 1$$

Taking axes as a-axis and b-axis

The set A represents square PQRS inside set B representing ellipse and hence $A \subset B$.

2. *(c)* We have,

$$2|\sqrt{x}-3|+\sqrt{x}(\sqrt{x}-6)+6=0$$

Let $\sqrt{x}-3=y$

$\Rightarrow \quad \sqrt{x}=y+3$

$\therefore \quad 2|y|+(y+3)(y-3)+6=0$

$\Rightarrow \quad 2|y|+y^2-3=0$

$\Rightarrow \quad |y|^2+2|y|-3=0$

$\Rightarrow \quad (|y|+3)(|y|-1)=0$

$\Rightarrow \quad |y| \ne -3 \Rightarrow |y|=1$

$\Rightarrow \quad y=\pm 1$

$\Rightarrow \quad \sqrt{x}-3=\pm 1$

$\Rightarrow \quad \sqrt{x}=4, 2$

$\Rightarrow \quad x=16, 4$

3. *(c)* We have,

α, β are the roots of $x^2-x+1=0$

$\because$ Roots of $x^2-x+1=0$ are $-\omega, -\omega^2$

$\therefore$ Let $\alpha=-\omega$ and $\beta=-\omega^2$

$\Rightarrow \alpha^{101}+\beta^{107}=(-\omega)^{101}+(-\omega^2)^{107}=-(\omega^{101}+\omega^{214})$

$=-(\omega^2+\omega) \qquad (\because \omega^3=1)$

$=-(-1) \qquad [\because 1+\omega+\omega^2=0]$

$=1$

4. *(c)* Given,

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix}=(A+Bx)(x-A)^2$$

$\Rightarrow$ Apply $C_1 \to C_1+C_2+C_3$

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix}=(A+Bx)(x-A)^2$$

Taking common $(5x-4)$ from C_1, we get

$$(5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix}=(A+Bx)(x-A)^2$$

Apply $R_2 \to R_2-R_1$ and $R_3 \to R_3-R_1$

$$\therefore (5x-4)\begin{vmatrix} 1 & 2x & 0 \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix}=(A+Bx)(x-A)^2$$

Expanding along C_1, we get

$$(5x-4)(x+4)^2=(A+Bx)(x-A)^2$$

Equating, we get

$$A=-4 \text{ and } B=5$$

5. *(b)* We have,

$x+ky+3z=0;\ 3x+ky-2z=0;\ 2x+4y-3z=0$

System of equation has non-zero solution, if

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix}=0$$

$\Rightarrow (-3k+8)-k(-9+4)+3(12-2k)=0$

$\Rightarrow \quad -3k+8+9k-4k+36-6k=0$

$\Rightarrow \quad -4k+44=0 \Rightarrow k=11$

Let $z=\lambda$, then we get

$$x+11y+3\lambda=0 \qquad \ldots(i)$$

$$3x+11y-2\lambda=0 \qquad \ldots(ii)$$

and $$2x+4y-3\lambda=0 \qquad \ldots(iii)$$

Solving Eqs. (i) and (ii), we get

$$x=\frac{5\lambda}{2},\ y=\frac{-\lambda}{2},\ z=\lambda \Rightarrow \frac{xz}{y^2}=\frac{5\lambda^2}{2\times\left(-\frac{\lambda}{2}\right)^2}=10$$

6. *(a)* Given 6 different novels and 3 different dictionaries.

Number of ways of selecting 4 novels from 6 novels is

$${}^6C_4=\frac{6!}{2!4!}=15$$

Number of ways of selecting 1 dictionary is from 3 dictionaries is ${}^3C_1=\frac{3!}{1!2!}=3$

$\therefore$ Total number of arrangement of 4 novels and 1 dictionary where dictionary is always in the middle, is

$$15\times 3\times 4!=45\times 24=1080$$

7. *(d)* ***Key idea*** $=(a+b)^n+(a-b)^n$

$$=2({}^nC_0a^n+{}^nC_2a^{n-2}b^2+{}^nC_4a^{n-4}b^4\ldots)$$

We have,

$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5,\ x>1$

$=2({}^5C_0x^5+{}^5C_2x^3(\sqrt{x^3-1})^2+{}^5C_4x(\sqrt{x^3-1})^4)$

$=2(x^5+10x^3(x^3-1)+5x(x^3-1)^2)$

$=2(x^5+10x^6-10x^3+5x^7-10x^4+5x)$

Sum of coefficients of all odd degree terms is

$$2(1-10+5+5)=2$$

8. *(c)* We have, $a_1, a_2, a_3, \ldots a_{49}$ are in AP.

$$\sum_{k=0}^{12} a_{4k+1} = 416 \text{ and } a_9 + a_{43} = 66$$

Let $a_1 = a$ and d = common difference

$\because \quad a_1 + a_5 + a_9 + \cdots + a_{49} = 416$

$\therefore \quad a + (a + 4d) + (a + 8d) + \ldots (a + 48d) = 416$

$\Rightarrow \quad \frac{13}{2}(2a + 48d) = 416$

$\Rightarrow \quad a + 24d = 32 \quad \ldots(i)$

Also $\quad , a_9 + a_{43} = 66$

$\therefore \quad a + 8d + a + 42d = 66$

$\Rightarrow \quad 2a + 50d = 66$

$\Rightarrow \quad a + 25d = 33 \quad \ldots(ii)$

Solving Eqs. (i) and (ii), we get

$$a = 8 \text{ and } d = 1$$

Now, $a_1^2 + a_2^2 + a_3^2 + \cdots + a_{17}^2 = 140m$

$$8^2 + 9^2 + 10^2 + \ldots + 24^2 = 140m$$

$\Rightarrow \quad (1^2 + 2^2 + 3^2 + \ldots + 24^2) - (1^2 + 2^2 + 3^2 + \ldots + 7^2) = 140m$

$\Rightarrow \quad \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$

$\Rightarrow \quad \frac{3 \times 7 \times 8 \times 5}{6}(7 \times 5 - 1) = 140m$

$\Rightarrow \quad 7 \times 4 \times 5 \times 34 = 140m$

$\Rightarrow \quad 140 \times 34 = 140m$

$\Rightarrow \quad m = 34$

9. *(b)* We have,

$$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$$

A = sum of first 20 terms

B = sum of first 40 terms

$\therefore A = 1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots + 2 \cdot 20^2$

$A = (1^2 + 2^2 + 3^2 + \ldots + 20^2) + (2^2 + 4^2 + 6^2 + \ldots + 20^2)$

$A = (1^2 + 2^2 + 3^2 + \ldots + 20^2) + 4(1^2 + 2^2 + 3^2 + \ldots + 10^2)$

$A = \frac{20 \times 21 \times 41}{6} + \frac{4 \times 10 \times 11 \times 21}{6}$

$A = \frac{20 \times 21}{6}(41 + 22) = \frac{20 \times 41 \times 63}{6}$

Similarly

$B = (1^2 + 2^2 + 3^2 + \ldots + 40^2) + 4(1^2 + 2^2 + \ldots + 20^2)$

$B = \frac{40 \times 41 \times 81}{6} + \frac{4 \times 20 \times 21 \times 41}{6}$

$B = \frac{40 \times 41}{6}(81 + 42) = \frac{40 \times 41 \times 123}{6}$

Now, $B - 2A = 100\lambda$

$\therefore \quad \frac{40 \times 41 \times 123}{6} - \frac{2 \times 20 \times 21 \times 63}{6} = 100\lambda$

$\Rightarrow \quad \frac{40}{6}(5043 - 1323) = 100\lambda$

$\Rightarrow \quad \frac{40}{6} \times 3720 = 100\lambda$

$\Rightarrow \quad 40 \times 620 = 100\lambda$

$\Rightarrow \quad \lambda = \frac{40 \times 620}{100} = 248$

10. *(c)* Key idea Use property of greatest integer function $[x] = x - \{x\}$.

We have,

$$\lim_{x \to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots + \left[\frac{15}{x}\right]\right)$$

We know, $[x] = x - \{x\}$

$\therefore \quad \left[\frac{1}{x}\right] = \frac{1}{x} - \left\{\frac{1}{x}\right\}$

Similarly, $\quad \left[\frac{n}{x}\right] = \frac{n}{x} - \left\{\frac{n}{x}\right\}$

$\therefore$ Given limit $= \lim_{x \to 0^+} x\left(\frac{1}{x} - \left\{\frac{1}{x}\right\} + \frac{2}{x} - \left\{\frac{2}{x}\right\} + \ldots \frac{15}{x} - \left\{\frac{15}{x}\right\}\right)$

$= \lim_{x \to 0^+} (1 + 2 + 3 + \ldots + 15) - x\left(\left\{\frac{1}{x}\right\} + \left\{\frac{2}{x}\right\} + \ldots + \left\{\frac{15}{x}\right\}\right)$

$= 120 - 0 = 120$

$$\left[\begin{array}{l} \because 0 \le \left\{\frac{n}{x}\right\} < 1, \text{therefore} \\ 0 \le x\left\{\frac{n}{x}\right\} < x \Rightarrow \lim_{x \to 0^+} x\left\{\frac{n}{x}\right\} = 0 \end{array}\right]$$

11. *(a)* We have,

$$f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$$

$$f(x) = \begin{cases} (x - \pi)(e^{-x} - 1)\sin x, & x < 0 \\ -(x - \pi)(e^{x} - 1)\sin x, & 0 \le x < \pi \\ (x - \pi)(e^{x} - 1)\sin x, & x \ge \pi \end{cases}$$

We check the differentiability at $x = 0$ and π.

We have,

$$f'(x) = \begin{cases} (x - \pi)(e^{-x} - 1)\cos x + (e^{-x} - 1)\sin x \\ \qquad + (x - \pi)\sin x e^{-x}(-1), x < 0 \\ -[(x - \pi)(e^{x} - 1)\cos x + (e^{x} - 1)\sin x \\ \qquad + (x - \pi)\sin x e^{x}], 0 < x < \pi \\ (x - \pi)(e^{x} - 1)\cos x + (e^{x} - 1)\sin x \\ \qquad + (x - \pi)\sin x e^{x}, x > \pi \end{cases}$$

Clearly,

$$\lim_{x \to 0^-} f'(x) = 0 = \lim_{x \to 0^+} f'(x)$$

and $\quad \lim_{x \to \pi^-} f'(x) = 0 = \lim_{x \to \pi^+} f'(x)$

$\therefore$ f is differentiable at $x = 0$ and $x = \pi$

Hence, f is differentiable for all x.

12. *(d)* We have, $y^2 = 6x$

$\Rightarrow \quad 2y\frac{dy}{dx} = 6 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3}{y}$

Slope of tangent at (x_1, y_1) is $m_1 = \frac{3}{y_1}$

Also, $\quad 9x^2 + by^2 = 16$

$\Rightarrow \quad 18x + 2by\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-9x}{by}$

Slope of tangent at (x_1, y_1) is $m_2 = \frac{-9x_1}{by_1}$

Since, these are intersection at right angle.

$\therefore \quad m_1 m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1$

$\Rightarrow \quad \frac{27x_1}{6bx_1} = 1 \qquad [\because y_1^2 = 6x_1]$

$\Rightarrow \quad b = \frac{9}{2}$

13. *(d)* We have,

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x} \Rightarrow h(x) = \frac{f(x)}{g(x)}$$

$$\therefore \quad h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$$

$$\Rightarrow \quad h(x) = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

$$x - \frac{1}{x} > 0, \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in [2\sqrt{2}, \infty)$$

$$x - \frac{1}{x} < 0, \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in (-\infty, 2\sqrt{2}]$$

$\therefore$ Local minimum value is $2\sqrt{2}$.

14. *(b)* We have,

$$I = \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^5 x + \cos^3 x \cdot \sin^2 x + \sin^3 x \cdot \cos^2 x + \cos^5 x)^2} dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{\{\sin^3 x(\sin^2 x + \cos^2 x) + \cos^3 x(\sin^2 x + \cos^2 x)\}^2} dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \int \frac{\sin^2 x \cos^2 x}{\cos^6 x(1 + \tan^3 x)^2} dx$$

$$= \int \frac{\tan^2 x \sec^2 x}{(1 + \tan^3 x)^2} dx$$

Put $\quad \tan^3 x = t$

$\Rightarrow \quad 3\tan^2 x \sec^2 x dx = dt$

$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(1 + t)^2}$

$\Rightarrow \quad I = \frac{-1}{3(1 + t)} + C \quad \Rightarrow I = \frac{-1}{3(1 + \tan^3 x)} + C$

15. *(d)* ***Key idea***

Use property $= \int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2\left(-\frac{\pi}{2} + \frac{\pi}{2} - x\right)}{1 + 2^{-\frac{\pi}{2} + \frac{\pi}{2} - x}} dx \left[\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^{-x}} dx$$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{2^x \sin^2 x}{2^x + 1} dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\left(\frac{2^x + 1}{2^x + 1}\right)dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$$

$$\Rightarrow \quad 2I = 2\int_0^{\pi/2} \sin^2 x \, dx \qquad [\because \sin^2 x \text{ is an even function}]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \sin^2 x dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \cos^2 x dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow \quad 2I = [x]_0^{\pi/2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

16. *(a)* We have,

$\Rightarrow \quad 18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow \quad 18x^2 - 6\pi x - 3\pi x + \pi^2 = 0$

$(6x - \pi)(3x - \pi) = 0$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{3}$

Now, $\alpha < \beta \qquad \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

Given, $g(x) = \cos x^2$ and $f(x) = \sqrt{x}$

$y = gOf(x)$

$\therefore \quad y = g(f(x)) = \cos x$

Area of region bounded by $x = \alpha, x = \beta$, $y = 0$ and curve $y = g(f(x))$ is

$$A = \int_{\pi/6}^{\pi/3} \cos x \, dx$$

$$A = [\sin x]_{\pi/6}^{\pi/3}$$

$$A = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$A = \left(\frac{\sqrt{3} - 1}{2}\right)$$

17. *(c)* We have,

$$\sin x \frac{dy}{dx} + y\cos x = 4x \Rightarrow \frac{dy}{dx} + y\cot x = 4x\text{cosec } x$$

This is a linear differential equation of form

$$\frac{dy}{dx} + Py = Q$$

where $P = \cot x$, $Q = 4x \text{ cosec } x$

Now, $IF = e^{\int Pdx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Solution of the differential equation is

$$y \cdot \sin x = \int 4x \text{ cosec } x \sin x dx + C$$

$\Rightarrow \quad y\sin x = \int 4x dx + C = 2x^2 + C$

Put $x = \frac{\pi}{2}, y = 0$, we get

$$C = -\frac{\pi^2}{2}$$

$\Rightarrow \quad y\sin x = 2x^2 - \frac{\pi^2}{2}$

Put $\quad x = \frac{\pi}{6}$

$\therefore \quad y\left(\frac{1}{2}\right) = 2\left(\frac{\pi^2}{36}\right) - \frac{\pi^2}{2}$

$\Rightarrow \quad y = \frac{\pi^2}{9} - \pi^2$

$\Rightarrow \quad y = -\frac{8\pi^2}{9}$

Alternate Method

We have, $\sin x \frac{dy}{dx} + y\cos x = 4x$, which can be written as

$\frac{d}{dx}(\sin x \cdot y) = 4x$

On integrating both sides, we get

$$\int \frac{d}{dx}(\sin x \cdot y) \cdot dx = \int 4x \cdot dx$$

$\Rightarrow \quad y \cdot \sin x = \frac{4x^2}{2} + C \Rightarrow y \cdot \sin x = 2x^2 + C$

Now, as $y = 0$ when $x = \frac{\pi}{2}$

$\therefore \quad C = -\frac{\pi^2}{2}$

$\Rightarrow \quad y \cdot \sin x = 2x^2 - \frac{\pi^2}{2}$

Now, putting $x = \frac{\pi}{6}$, we get

$$y\left(\frac{1}{2}\right) = 2\left(\frac{\pi^2}{36}\right) - \frac{\pi^2}{2} \Rightarrow y = \frac{\pi^2}{9} - \pi^2 = -\frac{8\pi^2}{9}$$

18. *(c)*

Equation of line PQ is $\frac{x}{\alpha} + \frac{y}{\beta} = 1$

Since this line is passes through fixed point (2, 3).

$\therefore \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1$

$\therefore$ Locus of R is $\quad 2\beta + 3\alpha = \alpha\beta$

i.e. $\quad 2y + 3x = xy \Rightarrow 3x + 2y = xy$

19. *(c)* ***Key idea*** Otrhocentre, centroid and circumcentre are collinear and centroid divide orthocentre and circumcentre in 2 : 1 ratio.

We have orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3) respectively and C circumcentre.

Clearly, $AB = \sqrt{(3+3)^2 + (3-5)^2} = \sqrt{36+4} = 2\sqrt{10}$

We know that, AB : BC = 2 : 1

$\Rightarrow \quad BC = \sqrt{10}$

Now, $\quad AC = AB + BC = 2\sqrt{10} + \sqrt{10} = 3\sqrt{10}$

Since, AC is a diameter of circle.

$\therefore \quad r = \frac{AC}{2}$

$\Rightarrow \quad r = \frac{3\sqrt{10}}{2} = 3\sqrt{\frac{5}{2}}$

20. *(d)* ***Key idea*** Equation of tangent to the curve $x^2 = 4ay$ at (x_1, y_1) is $xx_1 = 4a\left(\frac{y + y_1}{2}\right)$

Tangent to the curve $x^2 = y - 6$ at (1, 7) is

$$x = \frac{y+7}{2} - 6$$

$\Rightarrow \quad 2x - y + 5 = 0$

…(i)

Equation of circle is $x^2 + y^2 + 16x + 12y + c = 0$

Centre (–8, – 6)

$$r = \sqrt{8^2 + 6^2 - c} = \sqrt{100 - c}$$

Since, line $2x - y + 5 = 0$ also touches the circle.

$\therefore \quad \sqrt{100 - c} = \left|\frac{2(-8) - (-6) + 5}{\sqrt{2^2 + 1^2}}\right|$

$\Rightarrow \quad \sqrt{100 - c} = \left|\frac{-16 + 6 + 5}{\sqrt{5}}\right|$

$\Rightarrow \quad \sqrt{100 - c} = |-\sqrt{5}|$

$\Rightarrow \quad 100 - c = 5 \Rightarrow c = 95$

21. *(b)* Equation of tangent and normal to the curve $y^2 = 16x$ at (16, 16) is $x - 2y + 16 = 0$ and $2x + y - 48 = 0$, respectively.

A = (–16, 0) ; B = (24, 0)

C is the centre of circle passing through PAB

i.e. $\quad C = (4, 0)$

Slope of $PC = \frac{16 - 0}{16 - 4} = \frac{16}{12} = \frac{4}{3} = m_1$

Slope of $PB = \frac{16 - 0}{16 - 24} = \frac{16}{-8} = -2 = m_2$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{4}{3}+2}{1-\left(\frac{4}{3}\right)(2)}\right|$$

$$\Rightarrow \qquad \tan\theta = 2$$

22. *(a)* Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q.

Tangent intersects at point T(0, 3)

Clearly, PQ is chord of contact.

$\therefore$ Equation of PQ is $-3y = 36$

$\Rightarrow \qquad y = -12$

Solving the curve $4x^2 - y^2 = 36$ and $y = -12$,

we get $\qquad x = \pm 3\sqrt{5}$

Area of $\qquad \Delta PQT = \frac{1}{2} \times PQ \times ST$

$$= \frac{1}{2}(6\sqrt{5} \times 15) = 45\sqrt{5}$$

23. *(b)* L_1 is the line of intersection of the plane $2x - 2y + 3z - 2 = 0$ and $x - y + z + 1 = 0$ and L_2 is the line of intersection of the plane $x + 2y - z - 3 = 0$ and $3x - y + 2z - 1 = 0$

Since L_i is parallel to $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} + \hat{j}$

L_2 is parallel to $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 7\hat{k}$

Also, L_2 passes through $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$.

[put z = 0 in last two planes]

So, equation of plane is

$$\begin{vmatrix} x - \frac{5}{7} & y - \frac{8}{7} & z \\ 1 & 1 & 0 \\ 3 & -5 & -7 \end{vmatrix} = 0$$

$$\Rightarrow \qquad 7x - 7y + 8z + 3 = 0$$

Now, perpendicular distance from origin is

$$\left|\frac{3}{\sqrt{7^2 + 7^2 + 8^2}}\right| = \frac{3}{\sqrt{162}} = \frac{1}{3\sqrt{2}}$$

24. *(d)* ***Key idea*** length of projection of the line segment joining a_1 and a_2 on the plane $r \cdot n = d$ is $\left|\frac{(a_2 - a_1) \times n}{|n|}\right|$

Length of projection the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane $x + y + z = 7$ is

$$AC = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{n}}{|\mathbf{n}|}\right| = \frac{|(-\hat{i} - \hat{k}) \times (\hat{i} + \hat{j} + \hat{k})|}{|i + j + k|}$$

$$AC = \frac{|\hat{i} - \hat{k}|}{\sqrt{3}}$$

$$\Rightarrow \qquad AC = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

Alternative Method

Clearly, DR's of AB are $4 - 5, -1 + 1, 3 - 4$,

i.e. $-1, 0, -1$

and DR's of normal to plane are 1, 1, 1.

Now, let θ be the angle between the line and plane, then θ is given by

$$\sin\theta = \frac{|-1 + 0 - 1|}{\sqrt{(-1)^2 + (-1)^2}\sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{2}{\sqrt{2}\sqrt{3}} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \sin\theta = \sqrt{\frac{2}{3}} \Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

Clearly, length of projection $= AB\cos\theta = \sqrt{2}\frac{1}{\sqrt{3}}$ $\qquad [\because AB = \sqrt{2}]$

$$= \sqrt{\frac{2}{3}}$$

25. *(a)* ***Key idea*** If any vector $\boldsymbol{x}$ is coplanar with the vector $\mathbf{y}$ and $\mathbf{z}$, then $\mathbf{x} = \lambda\mathbf{y} + \mu\mathbf{z}$

Here, $\mathbf{u}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.

$\therefore \qquad \mathbf{u} = \lambda\mathbf{a} + \mu\mathbf{b}$

Dot product with $\mathbf{a}$, we get

$$\mathbf{u} \cdot \mathbf{a} = \lambda(\mathbf{a} \cdot \mathbf{a}) + \mu(\mathbf{b} \cdot \mathbf{a}) \Rightarrow 0 = 14\lambda + 2\mu \qquad \ldots(i)$$

$$[\because \mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{u} \cdot \mathbf{a} = 0]$$

Dot product with **b**, we get

$$\mathbf{u}\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) + \mu(\mathbf{b}\cdot\mathbf{b})$$

$$24 = 2\lambda + 2\mu \qquad \ldots(\text{ii}) \; [\because \mathbf{u}\cdot\mathbf{b} = 24]$$

Solving Eqs. (i) and (ii), we get

$$\lambda = -2, \mu = 14$$

Dot product with **u**, we get

$$|\mathbf{u}|^2 = \lambda(\mathbf{u}\cdot\mathbf{a}) + \mu(\mathbf{u}\cdot\mathbf{b})$$

$$|\mathbf{u}|^2 = -2(0) + 14(24) \Rightarrow |\mathbf{u}|^2 = 336$$

26. *(b)* ***Key idea*** Use the theorem of total probability

Let E_1 = Event that first ball drawn is red

E_2 = Event that first ball drawn is black

A = Event that second ball drawn is red

$$P(E_1) = \frac{4}{10}, P\left(\frac{A}{E_1}\right) = \frac{6}{12}$$

$$\Rightarrow \quad P(E_2) = \frac{6}{10}, P\left(\frac{A}{E_2}\right) = \frac{4}{12}$$

By law of total probability

$$P(A) = P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right)$$

$$= \frac{4}{10}\times\frac{6}{12} + \frac{6}{10}\times\frac{4}{12} = \frac{24+24}{120} = \frac{48}{120} = \frac{2}{5}$$

27. *(c)* ***Key idea*** Standard deviation is remain unchanged, if observations are added or subtracted by a fixed number

We have,

$$\sum_{i=1}^{9}(x_1 - 5) = 9 \text{ and } \sum_{i=1}^{9}(x_1 - 5)^2 = 45$$

$$SD = \sqrt{\frac{\sum_{i=1}^{9}(x_1-5)^2}{9} - \left(\frac{\sum_{i=1}^{9}(x_1-5)}{9}\right)^2}$$

$$\Rightarrow \quad SD = \sqrt{\frac{45}{9} - \left(\frac{9}{9}\right)^2} \Rightarrow SD = \sqrt{5-1} = \sqrt{4} = 2$$

28. *(b)* ***Key idea*** Apply the identity

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

and $\cos 3x = 4\cos^3 x - 3\cos x$

We have, $8\cos x\left(\cos\left(\frac{\pi}{6}+x\right)\cos\left(\frac{\pi}{6}-x\right) - \frac{1}{2}\right) = 1$

$$\Rightarrow \quad 8\cos x\left(\cos^2\frac{\pi}{6} - \sin^2 x - \frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{3}{4} - \sin^2 x - \frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{3}{4} - \frac{1}{2} - 1 + \cos^2 x\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{-3+4\cos^2 x}{4}\right) = 1$$

$$\Rightarrow \quad 2(4\cos^3 x - 3\cos x) = 1$$

$$\Rightarrow \quad 2\cos 3x = 1 \Rightarrow \cos 3x = \frac{1}{2}$$

$$\Rightarrow \quad 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \qquad [0 \le 3x \le 3\pi]$$

$$\Rightarrow \quad x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

Sum $= \frac{\pi}{9} + \frac{5\pi}{9} + \frac{7\pi}{9} = \frac{13\pi}{9} \Rightarrow k\pi = \frac{13\pi}{9}$

Hence, $k = \frac{13}{9}$

29. *(a)*

Let height of tower TM be h.

In ΔPMT, $\tan 45° = \frac{TM}{PM}$

$$\Rightarrow \quad 1 = \frac{h}{PM}$$

$$\Rightarrow \quad PM = h$$

In ΔTQM, $\tan 30° = \frac{h}{QM}$; $QM = \sqrt{3}h$

In ΔPMQ, $PM^2 + QM^2 = PQ^2$

$$h^2 + (\sqrt{3}h)^2 = (200)^2$$

$$\Rightarrow \quad 4h^2 = (200)^2$$

$$\Rightarrow \quad h = 100\text{ m}$$

30. *(a)* ***Key idea*** Use De-Morgan's and distributive law.

We have, $\sim(p\vee q)\vee(\sim p\wedge q)$

$\equiv (\sim p\wedge\sim q)\vee(\sim p\wedge q)$

$[\because$ By De-Morgan's law $\sim(p\vee q) = (\sim p\wedge\sim q)]$

$\equiv \quad \sim p\wedge(\sim q\vee q)$ [By distributive law]

$\equiv \quad \sim p\wedge t$ $[\sim q\vee q = t]$

$\equiv \quad \sim p$

ONLINE QUESTION PAPERS

JEE Main 2019 (April & January Attempt)

APRIL ATTEMPT

8 April, Shift-I

1 The shortest distance between the line $y = x$ and the curve $y^2 = x - 2$ is

(a) 2 (b) $\frac{7}{8}$

(c) $\frac{7}{4\sqrt{2}}$ (d) $\frac{11}{4\sqrt{2}}$

2 $\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$ equals

(a) $4\sqrt{2}$ (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) 4

3 The greatest value of $c \in R$ for which the system of linear equations

$$x - cy - cz = 0,$$
$$cx - y + cz = 0,$$
$$cx + cy - z = 0$$

has a non-trivial solution, is

(a) -1 (b) $\frac{1}{2}$ (c) 2 (d) 0

4 The contrapositive of the statement "If you are born in India, then you are a citizen of India", is

(a) If you are not a citizen of India, then you are not born in India.

(b) If you are a citizen of India, then you are born in India.

(c) If you are born in India, then you are not a citizen of India.

(d) If you are not born in India, then you are not a citizen of India.

5 All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is

(a) 180 (b) 175

(c) 160 (d) 162

6 Let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, (\alpha \in R)$ such that $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, a value of α is

(a) $\frac{\pi}{32}$ (b) 0

(c) $\frac{\pi}{64}$ (d) $\frac{\pi}{16}$

7 If $\cos(\alpha + \beta) = \frac{3}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to

(a) $\frac{63}{52}$ (b) $\frac{63}{16}$

(c) $\frac{21}{16}$ (d) $\frac{33}{52}$

8 The sum of the coefficients of all even degree terms is x in the expansion of $(x + \sqrt{x^3 - 1})^6 + (x - \sqrt{x^3 - 1})^6$, $(x > 1)$ is equal to

(a) 29 (b) 32

(c) 26 (d) 24

9 $\int \frac{\sin \frac{5x}{2}}{\sin \frac{x}{2}} dx$ is equal to

(where, C is a constant of integration)

(a) $2x + \sin x + 2 \sin 2x + C$

(b) $x + 2\sin x + 2 \sin 2x + C$

(c) $x + 2\sin x + \sin 2x + C$

(d) $2x + \sin x + \sin 2x + C$

10 The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is

(a) 45 (b) 49

(c) 48 (d) 40

11 The equation of a plane containing the line of intersection of the planes $2x - y - 4 = 0$ and $y + 2z - 4 = 0$ and passing through the point (1, 1, 0) is

(a) $x - 3y - 2z = -2$

(b) $2x - z = 2$

(c) $x - y - z = 0$

(d) $x + 3y + z = 4$

12 The magnitude of the projection of the vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ on the vector perpendicular to the plane containing the vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, is

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{6}$

(c) $3\sqrt{6}$ (d) $\sqrt{\frac{3}{2}}$

13 The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is

(a) 320 (b) 105

(c) 160 (d) 210

14 Let A and B be two non-null events such that $A \subset B$. Then, which of the following statements is always correct.

(a) $P(A/B) = P(B) - P(A)$

(b) $P(A/B) \geq P(A)$

(c) $P(A/B) \leq P(A)$

(d) $P(A/B) = 1$

15 If α and β are the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is

(a) 2 (b) 5

(c) 4 (d) 3

16 The area (in sq units) of the region $A = \{(x, y) \in R \times R \mid 0 \leq x \leq 3, 0 \leq y \leq 4, y \leq x^2 + 3x\}$ is

(a) $\frac{53}{6}$ (b) 8

(c) $\frac{59}{6}$ (d) $\frac{26}{3}$

17 If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in R$, then

(a) $S_1 = \{-2\}$; $S_2 = \{0, 1\}$

(b) $S_1 = \{-2, 0\}$; $S_2 = \{1\}$

(c) $S_1 = \{-2, 1\}$; $S_2 = \{0\}$

(d) $S_1 = \{-1\}$; $S_2 = \{0, 2\}$

18 If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to

(a) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

(c) $\tan^{-1}\left(\frac{9}{14}\right)$ (d) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

19 The sum of the series $2 \cdot {}^{20}C_0 + 5 \cdot {}^{20}C_1 + 8 \cdot {}^{20}C_2 + 11 \cdot {}^{20}C_3 + ... + 62 \cdot {}^{20}C_{20}$ is equal to

(a) 2^{26} (b) 2^{25}

(c) 2^{23} (d) 2^{24}

20 The sum of the solutions of the equation

$|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0 (x > 0)$ is equal to

(a) 9 (b) 12

(c) 4 (d) 10

21 If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points (1, 2) and (a, b) are perpendicular to each other, then a^2 is equal to

(a) $\frac{128}{17}$ (b) $\frac{64}{17}$

(c) $\frac{4}{17}$ (d) $\frac{2}{17}$

22 Let $y = y(x)$ be the solution of the differential equation,

$(x^2 + 1)^2 \frac{dy}{dx} + 2x(x^2 + 1)y = 1$ such that $y(0) = 0$. If $\sqrt{a}\, y(1) = \frac{\pi}{32}$, then the value of '$a$' is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{16}$

23 The sum of all natural numbers 'n' such that $100 < n < 200$ and HCF $(91, n) > 1$ is

(a) 3203 (b) 3303

(c) 3221 (d) 3121

24 The length of the perpendicular from the point (2, – 1, 4) on the straight line, $\frac{x+3}{10} = \frac{y-2}{-7} = \frac{z}{1}$ is

(a) greater than 3 but less than 4

(b) less than 2

(c) greater than 2 but less than 3

(d) greater than 4

25 A point on the straight line, $3x + 5y = 15$ which is equidistant from the coordinate axes will lie only in

(a) IV quadrant

(b) I quadrant

(c) I and II quadrants

(d) I, II and IV quadrants

26 Let $O(0, 0)$ and $A(0, 1)$ be two fixed points, then the locus of a point P such that the perimeter of ΔAOP is 4, is

(a) $8x^2 - 9y^2 + 9y = 18$

(b) $9x^2 - 8y^2 + 8y = 16$

(c) $9x^2 + 8y^2 - 8y = 16$

(d) $8x^2 + 9y^2 - 9y = 18$

27 If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0, \frac{\pi}{2}\right)$ then $\frac{dy}{dx}$ is equal to

(a) $\frac{\pi}{6} - x$ (b) $x - \frac{\pi}{6}$

(c) $\frac{\pi}{3} - x$ (d) $2x - \frac{\pi}{3}$

28 If $f(x) = \log_e\left(\frac{1-x}{1+x}\right)$, $|x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $2f(x)$ (b) $2f(x^2)$

(c) $(f(x))^2$ (d) $-2f(x)$

29 Let $f : [0, 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then ϕ is

(a) increasing on (0, 1) and decreasing on (1, 2)

(b) decreasing on (0, 2)

(c) decreasing on (0, 1) and increasing on (1, 2)

(d) increasing on (0, 2)

30 If $f(x) = \frac{2 - x\cos x}{2 + x\cos x}$ and $g(x) = \log_e x$, $(x > 0)$ then the value of the integral $\int_{-\pi/4}^{\pi/4} g(f(x))dx$ is

(a) $\log_e 3$ (b) $\log_e e$ (c) $\log_e 2$ (d) $\log_e 1$

ANSWERS

1. (c)	2. (a)	3. (b)	4. (a)	5. (a)	6. (c)	7. (b)	8. (d)	9. (c)	10. (c)
11. (c)	12. (d)	13. (d)	14. (b)	15. (c)	16. (c)	17. (c)	18. (d)	19. (b)	20. (d)
21. (d)	22. (d)	23. (d)	24. (a)	25. (c)	26. (c)	27. (b)	28. (a)	29. (c)	30. (d)

For Detailed Solutions
Visit : http://tinyurl.com/y4t86zrd
Or Scan :

8 April, Shift-II

1 If the system of linear equations

$$x - 2y + kz = 1,$$
$$2x + y + z = 2,$$
$$3x - y - kz = 3$$

has a solution (x, y, z), $z \neq 0$, then (x, y) lies on the straight line whose equation is

(a) $3x - 4y - 4 = 0$

(b) $3x - 4y - 1 = 0$

(c) $4x - 3y - 4 = 0$

(d) $4x - 3y - 1 = 0$

2 The sum $\sum_{k=1}^{20} k\frac{1}{2^k}$ is equal to

(a) $2 - \frac{11}{2^{19}}$ (b) $1 - \frac{11}{2^{20}}$

(c) $2 - \frac{3}{2^{17}}$ (d) $2 - \frac{21}{2^{20}}$

3 Let the numbers 2, b, c be in an AP and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$. If $\det(A) \in [2, 16]$, then c lies in the interval

(a) $[3, 2 + 2^{3/4}]$ (b) $(2 + 2^{3/4}, 4)$

(c) $[4, 6]$ (d) $[2, 3)$

4 Let $f(x) = \int_0^x g(t)dt$, where g is a non-zero even function. If $f(x + 5) = g(x)$, then $\int_0^x f(t)dt$ equals

(a) $5\int_{x+5}^{5} g(t)dt$ (b) $\int_{5}^{x+5} g(t)dt$

(c) $2\int_{5}^{x+5} g(t)dt$ (d) $\int_{x+5}^{5} g(t)dt$

5 If the lengths of the sides of a triangle are in AP and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is

(a) 3 : 4 : 5 (b) 4 : 5 : 6
(c) 5 : 9 : 13 (d) 5 : 6 : 7

6 If $\int \frac{dx}{x^3(1+x^6)^{2/3}} = xf(x)(1+x^6)^{\frac{1}{3}} + C$

where, C is a constant of integration, then the function $f(x)$ is equal to

(a) $-\frac{1}{6x^3}$ (b) $-\frac{1}{2x^3}$
(c) $-\frac{1}{2x^2}$ (d) $\frac{3}{x^2}$

7 Which one of the following statements is not a tautology?

(a) $(p \wedge q) \to (\sim p) \vee q$
(b) $(p \wedge q) \to p$
(c) $p \to (p \vee q)$
(d) $(p \vee q) \to (p \vee (\sim q))$

8 If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is

(a) 12 (b) 9 (c) 15 (d) 33

9 If three distinct numbers a, b and c are in GP and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct?

(a) d, e and f are in GP
(b) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in AP
(c) d, e and f are in AP
(d) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in GP

10 The minimum number of times one has to toss a fair coin so that the probability of observing atleast one head is atleast 90% is

(a) 2 (b) 3 (c) 5 (d) 4

11 Suppose that the points (h, k), (1, 2) and (–3, 4) lie on the line L_1. If a line L_2 passing through the points (h, k) and (4, 3) is perpendicular to L_1, then k/h equals

(a) $-\frac{1}{7}$ (b) $\frac{1}{3}$
(c) 3 (d) 0

12 Given that the slope of the tangent to a curve $y = y(x)$ at any point (x, y) is $\frac{2y}{x^2}$. If the curve passes through the centre of the circle $x^2 + y^2 - 2x - 2y = 0$, then its equation is

(a) $x^2 \log_e|y| = -2(x-1)$
(b) $x \log_e|y| = x - 1$
(c) $x \log_e|y| = 2(x-1)$
(d) $x \log_e|y| = -2(x-1)$

13 Let $f : [-1, 3] \to R$ be defined as

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3, \end{cases}$$

where, $[t]$ denotes the greatest integer less than or equal to t. Then, f is discontinuous at

(a) four or more points
(b) only two points
(c) only three points
(d) only one point

14 The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is

(a) $\sqrt{6}$ (b) $2\sqrt{3}$ (c) $\sqrt{3}$ (d) $\frac{2}{3}\sqrt{3}$

15 If $z = \frac{\sqrt{3}}{2} + \frac{i}{2} (i = \sqrt{-1})$, then $(1 + iz + z^5 + iz^8)^9$ is equal to

(a) 1 (b) $(-1 + 2i)^9$
(c) –1 (d) 0

16 A student scores the following marks in five tests 45, 54, 41, 57, 43. His score is not known for the sixth test. If the mean score is 48 in the six tests, then the standard deviation of the marks in six tests is

(a) $\frac{10}{3}$ (b) $\frac{10}{\sqrt{3}}$ (c) $\frac{100}{\sqrt{3}}$ (d) $\frac{100}{3}$

17 The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, has no real root is

(a) 3 (b) infinitely many
(c) 1 (d) 2

18 In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latus rectum is

(a) 5 (b) 10 (c) 8 (d) 6

19 Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, for some real x. Then $|\mathbf{a} \times \mathbf{b}| = r$ is possible if

(a) $0 < r \le \sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{3}{2}} < r \le 3\sqrt{\frac{3}{2}}$
(c) $3\sqrt{\frac{3}{2}} < r < 5\sqrt{\frac{3}{2}}$ (d) $r \ge 5\sqrt{\frac{3}{2}}$

20 Let $f : R \to R$ be a differentiable function satisfying $f'(3) + f'(2) = 0$. Then $\lim_{x \to 0} \left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)} \right)^{\frac{1}{x}}$ is equal to

(a) e (b) e^{-1}
(c) e^2 (d) 1

21 The vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$, which is perpendicular to the plane $x - y + z = 0$ is

(a) $\mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{k}}) - 2 = 0$
(b) $\mathbf{r} \times (\hat{\mathbf{i}} + \hat{\mathbf{k}}) + 2 = 0$
(c) $\mathbf{r} \times (\hat{\mathbf{i}} - \hat{\mathbf{k}}) + 2 = 0$
(d) $\mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{k}}) + 2 = 0$

22 If the fourth term in the binomial expansion of $\left(\sqrt{x^{\left(\frac{1}{1+\log_{10} x}\right)}} + x^{\frac{1}{12}} \right)^6$ is equal to 200, and $x > 1$, then the value of x is

(a) 100 (b) 10^4
(c) 10 (d) 10^3

23 The tangent to the parabola $y^2 = 4x$ at the point where it intersects the circle $x^2 + y^2 = 5$ in the first quadrant, passes through the point

(a) $\left(\frac{1}{4}, \frac{3}{4}\right)$ (b) $\left(\frac{3}{4}, \frac{7}{4}\right)$
(c) $\left(-\frac{1}{3}, \frac{4}{3}\right)$ (d) $\left(-\frac{1}{4}, \frac{1}{2}\right)$

24 If the eccentricity of the standard hyperbola passing through the point (4, 6) is 2, then the equation of the tangent to the hyperbola at (4, 6) is

(a) $3x - 2y = 0$
(b) $x - 2y + 8 = 0$
(c) $2x - y - 2 = 0$
(d) $2x - 3y + 10 = 0$

25 Let $S(\alpha) = \{(x, y) : y^2 \le x, 0 \le x \le \alpha\}$ and $A(\alpha)$ is area of the region $S(\alpha)$. If for λ, $0 < \lambda < 4$, $A(\lambda) : A(4) = 2 : 5$, then λ equals

(a) $2\left(\frac{4}{25}\right)^{\frac{1}{3}}$ (b) $4\left(\frac{2}{5}\right)^{\frac{1}{3}}$
(c) $4\left(\frac{4}{25}\right)^{\frac{1}{3}}$ (d) $2\left(\frac{2}{5}\right)^{\frac{1}{3}}$

26 Let $f(x) = a^x\ (a > 0)$ be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. Then $f_1(x + y) + f_1(x - y)$ equals

(a) $2f_1(x + y) \cdot f_2(x - y)$
(b) $2f_1(x + y) \cdot f_1(x - y)$
(c) $2f_1(x) \cdot f_2(y)$
(d) $2f_1(x) \cdot f_1(y)$

27 The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed) is

(a) 306 (b) 310
(c) 360 (d) 288

28 Two vertical poles of heights, 20 m and 80 m stand apart on a horizontal plane. The height (in m) of the point of intersection of the lines joining the top of each pole to the foot of the other, from this horizontal plane is

(a) 15 (b) 16
(c) 12 (d) 18

29 If a point $R(4, y, z)$ lies on the line segment joining the points $P(2, -3, 4)$ and $Q(8, 0, 10)$, then the distance of R from the origin is

(a) $2\sqrt{21}$ (b) $\sqrt{53}$
(c) $2\sqrt{14}$ (d) 6

30 The tangent and the normal lines at the point $(\sqrt{3}, 1)$ to the circle $x^2 + y^2 = 4$ and the X-axis form a triangle. The area of this triangle (in square units) is

(a) $\frac{1}{3}$
(b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$
(d) $\frac{1}{\sqrt{3}}$

ANSWERS

1. (c)	2. (a)	3. (c)	4. (d)	5. (b)	6. (b)	7. (d)	8. (d)	9. (b)	10. (d)
11. (c)	12. (c)	13. (c)	14. (b)	15. (c)	16. (b)	17. (b)	18. (a)	19. (d)	20. (d)
21. (d)	22. (c)	23. (b)	24. (c)	25. (c)	26. (d)	27. (b)	28. (b)	29. (c)	30. (c)

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9 April, Shift-I

1 Slope of a line passing through $P(2, 3)$ and intersecting the line, $x + y = 7$ at a distance of 4 units from P, is

(a) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$ (b) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$
(c) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (d) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$

2 If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x = -1, 0, 1$, then the set $S = \{x \in R : f(x) = f(0)\}$ contains exactly

(a) four rational numbers
(b) two irrational and two rational numbers
(c) four irrational numbers
(d) two irrational and one rational number

3 Four persons can hit a target correctly with probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{8}$ respectively. If all hit at the target independently, then the probability that the target would be hit, is

(a) $\frac{1}{192}$ (b) $\frac{25}{32}$
(c) $\frac{7}{32}$ (d) $\frac{25}{192}$

4 Let $\sum_{k=1}^{10} f(a + k) = 16(2^{10} - 1)$, where the function f satisfies $f(x + y) = f(x)\, f(y)$ for all natural numbers x, y and $f(1) = 2$. Then, the natural number 'a' is

(a) 2 (b) 4 (c) 3 (d) 16

5 If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ meets the plane, $x + 2y + 3z = 15$ at a point P, then the distance of P from the origin is

(a) $7/2$ (b) $9/2$
(c) $\sqrt{5}/2$ (d) $2\sqrt{5}$

6 If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{\sqrt{15}}{2}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{2}$

7 If a tangent to the circle $x^2 + y^2 = 1$ intersects the coordinate axes at distinct points P and Q, then the locus of the mid-point of PQ is

(a) $x^2 + y^2 - 2x^2y^2 = 0$
(b) $x^2 + y^2 - 2xy = 0$
(c) $x^2 + y^2 - 4x^2y^2 = 0$
(d) $x^2 + y^2 - 16x^2y^2 = 0$

8 If the function $f : \mathbf{R} - \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1-x^2}$, is surjective, then A is equal to

(a) $\mathbf{R} - \{-1\}$ (b) $[0, \infty)$
(c) $\mathbf{R} - [-1, 0)$ (d) $\mathbf{R} - (-1, 0)$

9 The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}\, dx$ is

(a) $\frac{\pi - 1}{2}$ (b) $\frac{\pi - 2}{8}$
(c) $\frac{\pi - 1}{4}$ (d) $\frac{\pi - 2}{4}$

10 If one end of a focal chord of the parabola, $y^2 = 16x$ is at $(1, 4)$, then the length of this focal chord is

(a) 22 (b) 25 (c) 24 (d) 20

11 The solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 (x \neq 0)$ with $y(1) = 1$, is

(a) $y = \frac{x^2}{4} + \frac{3}{4x^2}$ (b) $y = \frac{x^3}{5} + \frac{1}{5x^2}$
(c) $y = \frac{3}{4}x^2 + \frac{1}{4x^2}$ (d) $y = \frac{4}{5}x^3 + \frac{1}{5x^2}$

12 All the points in the set $S = \left\{\frac{\alpha + i}{\alpha - i} : \alpha \in \mathbf{R}\right\}$ $(i = \sqrt{-1})$ lie on a

(a) circle whose radius is $\sqrt{2}$.
(b) straight line whose slope is -1.
(c) circle whose radius is 1.
(d) straight line whose slope is 1.

13 If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \dfrac{\pi}{4} \\ k, & x = \dfrac{\pi}{4} \end{cases}$$ is

continuous,
then k is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) $\frac{1}{\sqrt{2}}$

14 Let
$S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$,
then the sum of the elements of S is

(a) 2π (b) π
(c) $\frac{5\pi}{3}$ (d) $\frac{13\pi}{6}$

15 If the tangent to the curve,
$y = x^3 + ax - b$ at the point $(1, -5)$ is perpendicular to the line,
$-x + y + 4 = 0$, then which one of the following points lies on the curve ?

(a) $(-2, 2)$ (b) $(2, -2)$
(c) $(-2, 1)$ (d) $(2, -1)$

16 If the standard deviation of the numbers $-1, 0, 1, k$ is $\sqrt{5}$ where $k > 0$, then k is equal to

(a) $2\sqrt{\frac{10}{3}}$ (b) $2\sqrt{6}$
(c) $4\sqrt{\frac{5}{3}}$ (d) $\sqrt{6}$

17 Let $p, q \in \mathbf{R}$. If $2 - \sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then

(a) $q^2 - 4p - 16 = 0$
(b) $p^2 - 4q - 12 = 0$
(c) $p^2 - 4q + 12 = 0$
(d) $q^2 + 4p + 14 = 0$

18 A plane passing through the points $(0, -1, 0)$ and $(0, 0, 1)$ and making an angle $\frac{\pi}{4}$ with the plane $y - z + 5 = 0$, also passes through the point

(a) $(\sqrt{2}, 1, 4)$ (b) $(-\sqrt{2}, 1, -4)$
(c) $(-\sqrt{2}, -1, -4)$ (d) $(\sqrt{2}, -1, 4)$

19 If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$,

then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 13 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$

20 Let $f(x) = 15 - |x - 10|$; $x \in \mathbf{R}$. Then, the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is

(a) $\{5, 10, 15, 20\}$ (b) $\{5, 10, 15\}$
(c) $\{10\}$ (d) $\{10, 15\}$

21 Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$, then S is equal to

(a) $\left\{\frac{1}{3}, -1\right\}$ (b) $\left\{\frac{1}{3}, 1\right\}$
(c) $\left\{-\frac{1}{3}, 1\right\}$ (d) $\left\{-\frac{1}{3}, -1\right\}$

22 For any two statements p and q, the negation of the expression
$p \vee (\sim p \wedge q)$ is

(a) $\sim p \wedge \sim q$ (b) $\sim p \vee \sim q$
(c) $p \wedge q$ (d) $p \leftrightarrow q$

23 If the fourth term in the binomial expansion of $\left(\frac{2}{x} + x^{\log_8 x}\right)^6$ $(x > 0)$ is 20×8^7, then the value of x is

(a) 8^{-2} (b) 8^3
(c) 8 (d) 8^2

24 The value of
$\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is

(a) $\frac{3}{2}(1 + \cos 20°)$ (b) $\frac{3}{4} + \cos 20°$
(c) $3/2$ (d) $3/4$

25 Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then, for $y \neq 0$ in $\mathbf{R}$,

$$\begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$ is equal to

(a) $y(y^2 - 1)$ (b) $y(y^2 - 3)$
(c) $y^3 - 1$ (d) y^3

26 The area (in sq units) of the region $A = \{(x, y) : x^2 \leq y \leq x + 2\}$ is

(a) $\frac{13}{6}$ (b) $\frac{9}{2}$
(c) $\frac{31}{6}$ (d) $\frac{10}{3}$

27 The integral $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ is equal to (here C is a constant of integration)

(a) $3\tan^{-1/3} x + C$
(b) $-3\tan^{-1/3} x + C$
(c) $-3\cot^{-1/3} x + C$
(d) $-\frac{3}{4}\tan^{-4/3} x + C$

28 A committee of 11 members is to be formed from 8 males and 5 females. If m is the number of ways the committee is formed with at least 6 males and n is the number of ways the committee is formed with atleast 3 females, then

(a) $m = n = 68$ (b) $m + n = 68$
(c) $m = n = 78$ (d) $n = m - 8$

29 Let the sum of the first n terms of a non-constant AP a_1, a_2, a_3.....be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this AP, then the ordered pair (d, a_{50}) is equal to

(a) $(A, 50 + 46A)$ (b) $(50, 50 + 45A)$
(c) $(50, 50 + 46A)$ (d) $(A, 50 + 45A)$

30 Let $\vec{\alpha} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\vec{\beta} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. If $\vec{\beta} = \vec{\beta}_1 - \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1 \times \vec{\beta}_2$ is equal to

(a) $\frac{1}{2}(3\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$
(b) $\frac{1}{2}(-3\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$
(c) $-3\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$
(d) $3\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

ANSWERS

1. (c)	2. (d)	3. (b)	4. (c)	5. (b)	6. (c)	7. (c)	8. (c)	9. (c)	10. (b)
11. (a)	12. (c)	13. (a)	14. (a)	15. (b)	16. (b)	17. (b)	18. (a)	19. (b)	20. (b)
21. (c)	22. (a)	23. (d)	24. (d)	25. (d)	26. (b)	27. (b)	28. (c)	29. (a)	30. (b)

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9 April, Shift-II

1 Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then

(a) $4\,\text{Im}(\omega) > 5$ (b) $5\,\text{Re}(\omega) > 1$
(c) $5\,\text{Im}(\omega) < 1$ (d) $5\,\text{Re}(\omega) > 4$

2 If a unit vector **a** makes angles $\frac{\pi}{3}$ with $\hat{\mathbf{i}}$, $\frac{\pi}{4}$ with $\hat{\mathbf{j}}$ and $\theta \in (0, \pi)$ with $\hat{\mathbf{k}}$, then a value of θ is

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{5\pi}{12}$ (d) $\frac{2\pi}{3}$

3 The value of $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$ is

(a) $\frac{1}{36}$ (b) $\frac{1}{32}$
(c) $\frac{1}{16}$ (d) $\frac{1}{18}$

4 Let P be the plane, which contains the line of intersection of the planes, $x + y + z - 6 = 0$ and $2x + 3y + z + 5 = 0$ and it is perpendicular to the XY-plane. Then, the distance of the point (0, 0, 256) from P is equal to

(a) $63\sqrt{5}$ (b) $205\sqrt{5}$
(c) $\frac{11}{\sqrt{5}}$ (d) $\frac{17}{\sqrt{5}}$

5 A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 cu m/min. Then, the rate (in m/min) at which the level of water is rising at the instant when the depth of water in the tank is 10 m is

(a) $\frac{2}{\pi}$ (b) $\frac{1}{5\pi}$ (c) $\frac{1}{15\pi}$ (d) $\frac{1}{10\pi}$

6 If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to

(a) $\sqrt{2} + 1$ (b) $\sqrt{2} - 1$
(c) $2\sqrt{2} + 1$ (d) $2\sqrt{2} - 1$

7 A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are (–8, 5) and (6, 5), then the area of the rectangle (in sq units) is

(a) 72 (b) 84
(c) 98 (d) 56

8 The area (in sq units) of the region $A = \left\{(x, y) : \frac{y^2}{2} \le x \le y + 4\right\}$ is

(a) 30 (b) $\frac{53}{3}$
(c) 16 (d) 18

9 If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively

(a) T, T, F (b) T, F, F
(c) F, F, F (d) F, T, T

10 If the two lines $x + (a-1)y = 1$ and $2x + a^2y = 1$, $(a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is

(a) $\frac{2}{5}$ (b) $\frac{\sqrt{2}}{5}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\sqrt{\frac{2}{5}}$

11 If some three consecutive coefficients in the binomial expansion of $(x+1)^n$ in powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficients is

(a) 964 (b) 227 (c) 232 (d) 625

12 If $f: R \to R$ is a differentiable function and $f(2) = 6$, then $\lim_{x \to 2} \int_6^{f(x)} \frac{2t\,dt}{(x-2)}$ is

(a) $12f'(2)$ (b) 0
(c) $24f'(2)$ (d) $2f'(2)$

13 The mean and the median of the following ten numbers in increasing order 10, 22, 26, 29, 34, x, 42, 67, 70, y are 42 and 35 respectively, then $\frac{y}{x}$ is equal to

(a) $\frac{7}{3}$ (b) $\frac{7}{2}$
(c) $\frac{8}{3}$ (d) $\frac{9}{4}$

14 The sum of the series $1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + \ldots$ upto 11th term is

(a) 915 (b) 946
(c) 916 (d) 945

15 If m is chosen in the quadratic equation $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is

(a) $10\sqrt{5}$ (b) $8\sqrt{5}$
(c) $8\sqrt{3}$ (d) $4\sqrt{3}$

16 The vertices B and C of a ΔABC lie on the line, $\frac{x+2}{3} = \frac{y-1}{0} = \frac{z}{4}$ such that $BC = 5$ units. Then, the area (in sq units) of this triangle, given that the point $A(1, -1, 2)$ is

(a) $\sqrt{34}$ (b) $2\sqrt{34}$
(c) $5\sqrt{17}$ (d) 6

17 If $\cos x \frac{dy}{dx} - y \sin x = 6x$, $\left(0 < x < \frac{x}{2}\right)$ and $y\left(\frac{\pi}{3}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(a) $\frac{\pi^2}{2\sqrt{3}}$ (b) $-\frac{\pi^2}{2\sqrt{3}}$
(c) $-\frac{\pi^2}{4\sqrt{3}}$ (d) $-\frac{\pi^2}{2}$

18 Two poles standing on a horizontal ground are of heights 5 m and 10 m, respectively. The line joining their tops makes an angle of 15° with the ground. Then, the distance (in m) between the poles, is

(a) $5(\sqrt{3}+1)$
(b) $\frac{5}{2}(2+\sqrt{3})$
(c) $10(\sqrt{3}-1)$
(d) $5(2+\sqrt{3})$

19 Some identical balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row consists of two balls and so on. If 99 more identical balls are added to the total number of balls used in forming the equilateral triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than the number of balls each side of the triangle contains. Then, the number of balls used to form the equilateral triangle is

(a) 262 (b) 190
(c) 225 (d) 157

20 If $\int e^{\sec x} (\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x))\,dx = e^{\sec x} f(x) + C$, then a possible choice of $f(x)$ is

(a) $x \sec x + \tan x + \frac{1}{2}$
(b) $\sec x + \tan x + \frac{1}{2}$
(c) $\sec x + x \tan x - \frac{1}{2}$
(d) $\sec x - \tan x - \frac{1}{2}$

21 Two newspapers A and B are published in a city. It is known that 25% of the city population reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then, the percentage of the population who look into advertisements is

(a) 13.5 (b) 13 (c) 12.8 (d) 13.9

22 The area (in sq units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the X-axis is

(a) $8\pi(3-2\sqrt{2})$ (b) $4\pi(3+\sqrt{2})$
(c) $8\pi(2-\sqrt{2})$ (d) $4\pi(2-\sqrt{2})$

23 If the sum and product of the first three terms in an AP are 33 and 1155, respectively, then a value of its 11th term is

(a) 25 (b) –36 (c) –25 (d) –35

24 If $f(x) = [x] - \left[\frac{x}{4}\right]$, $x \in R$ where $[x]$ denotes the greatest integer function, then

(a) $\lim_{x\to 4+} f(x)$ exists but $\lim_{x\to 4-} f(x)$ does not exist
(b) f is continuous at $x = 4$
(c) Both $\lim_{x\to 4-} f(x)$ and $\lim_{x\to 4+} f(x)$ exist but are not equal
(d) $\lim_{x\to 4-} f(x)$ exists but $\lim_{x\to 4+} f(x)$ does not exist

25 The domain of the definition of the function $f(x) = \frac{1}{4-x^2} + \log_{10}(x^3 - x)$ is

(a) $(-1, 0) \cup (1, 2) \cup (3, \infty)$
(b) $(-2, -1) \cup (-1, 0) \cup (2, \infty)$
(c) $(-1, 0) \cup (1, 2) \cup (2, \infty)$
(d) $(1, 2) \cup (2, \infty)$

26 If the system of equations $2x + 3y - z = 0$, $x + ky - 2z = 0$ and $2x - y + z = 0$ has a non-trivial solution (x, y, z), then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to

(a) –4 (b) $\frac{1}{2}$
(c) $-\frac{1}{4}$ (d) $\frac{3}{4}$

27 The value of the integral $\int_0^1 x\cot^{-1}(1 - x^2 + x^4)dx$ is

(a) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$ (b) $\frac{\pi}{2} - \frac{1}{2}\log_e 2$
(c) $\frac{\pi}{4} - \log_e 2$ (d) $\frac{\pi}{2} - \log_e 2$

28 The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point

(a) (6, – 2)
(b) (4, – 2)
(c) (–6, 4)
(d) (–4, 6)

29 If the function $f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x - \pi| + 3, & x > 5 \end{cases}$ is continuous at $x = 5$, then the value of $a - b$ is

(a) $\frac{-2}{\pi + 5}$ (b) $\frac{2}{\pi + 5}$
(c) $\frac{2}{\pi - 5}$ (d) $\frac{2}{5 - \pi}$

30 The total number of matrices $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$, $(x, y \in R, x \ne y)$ for which $A^T A = 3I_3$ is

(a) 2 (b) 4
(c) 3 (d) 6

ANSWERS

1. (b)	2. (d)	3. (c)	4. (c)	5. (b)	6. (a)	7. (b)	8. (d)	9. (b)	10. (d)
11. (c)	12. (a)	13. (a)	14. (b)	15. (b)	16. (a)	17. (b)	18. (d)	19. (b)	20. (b)
21. (d)	22. (a)	23. (c)	24. (b)	25. (c)	26. (b)	27. (a)	28. (a)	29. (d)	30. (b)

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10 April, Shift-I

1 If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x - 5$	$x^2 - 3x$	x

Then, the mean of the marks is

(a) 3.0 (b) 2.8
(c) 2.5 (d) 3.2

2 If $Q(0, -1, -3)$ is the image of the point P in the plane $3x - y + 4z = 2$ and R is the point $(3, -1, -2)$, then the area (in sq units) of ΔPQR is

(a) $\frac{\sqrt{91}}{2}$ (b) $2\sqrt{13}$
(c) $\frac{\sqrt{91}}{4}$ (d) $\frac{\sqrt{65}}{2}$

3 If the circles $x^2 + y^2 + 5Kx + 2y + K = 0$ and $2(x^2 + y^2) + 2Kx + 3y - 1 = 0$, $(K \in R)$, intersect at the points P and Q, then the line $4x + 5y - K = 0$ passes through P and Q, for

(a) no values of K
(b) exactly one value of K
(c) exactly two values of K
(d) infinitely many values of K

4 Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$. Then, the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is

(a) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$
(b) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$
(c) $[0, \infty)$
(d) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$

5 If $\int \frac{dx}{(x^2 - 2x + 10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2 - 2x + 10}\right) + C$, where, C is a constant of integration, then

(a) $A = \frac{1}{27}$ and $f(x) = 9(x - 1)$
(b) $A = \frac{1}{81}$ and $f(x) = 3(x - 1)$
(c) $A = \frac{1}{54}$ and $f(x) = 3(x - 1)$
(d) $A = \frac{1}{54}$ and $f(x) = 9(x - 1)^2$

6 If a directrix of a hyperbola centred at the origin and passing through the point $(4, -2\sqrt{3})$ is $5x = 4\sqrt{5}$ and its eccentricity is e, then
(a) $4e^4 - 12e^2 - 27 = 0$
(b) $4e^4 - 24e^2 + 27 = 0$
(c) $4e^4 + 8e^2 - 35 = 0$
(d) $4e^4 - 24e^2 + 35 = 0$

7 If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left(3, \frac{-9}{2}\right)$, then the length of the latusrectum of the ellipse is
(a) $8\sqrt{3}$ (b) 9
(c) 5 (d) $12\sqrt{2}$

8 If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0, \theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12} + \beta^{12}}{(\alpha^{-12} + \beta^{-12})(\alpha - \beta)^{24}}$ is equal to
(a) $\frac{2^{12}}{(\sin\theta + 8)^{12}}$ (b) $\frac{2^6}{(\sin\theta + 8)^{12}}$
(c) $\frac{2^{12}}{(\sin\theta - 4)^{12}}$ (d) $\frac{2^{12}}{(\sin\theta - 8)^6}$

9 The number of 6 digits numbers that can be formed using the digits 0, 1, 2,5, 7 and 9 which are divisible by 11 and no digit is repeated, is
(a) 60 (b) 72
(c) 48 (d) 36

10 If $a_1, a_2, a_3, \ldots, a_n$ are in AP and $a_1 + a_4 + a_7 + \ldots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to
(a) 64 (b) 76
(c) 98 (d) 38

11 ABC is a triangular park with $AB = AC = 100$ m. A vertical tower is situated at the mid-point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}(3\sqrt{2})$ and $\text{cosec}^{-1}(2\sqrt{2})$ respectively, then the height of the tower (in m) is
(a) 25 (b) 20
(c) $10\sqrt{5}$ (d) $\frac{100}{3\sqrt{3}}$

12 $\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \ldots + \frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to
(a) $\frac{4}{3}(2)^{4/3}$ (b) $\frac{3}{4}(2)^{4/3} - \frac{4}{3}$
(c) $\frac{3}{4}(2)^{4/3} - \frac{3}{4}$ (d) $\frac{4}{3}(2)^{3/4}$

13 The region represented by $|x - y| \le 2$ and $|x + y| \le 2$ is bounded by a
(a) rhombus of side length 2 units
(b) rhombus of area $8\sqrt{2}$ sq units
(c) square of side length $2\sqrt{2}$ units
(d) square of area 16 sq units

14 If $a > 0$ and $z = \frac{(1+i)^2}{a - i}$, has magnitude $\sqrt{\frac{2}{5}}$, then $\bar{z}$ is equal to
(a) $\frac{1}{5} - \frac{3}{5}i$ (b) $-\frac{1}{5} - \frac{3}{5}i$
(c) $-\frac{1}{5} + \frac{3}{5}i$ (d) $-\frac{3}{5} - \frac{1}{5}i$

15 If the length of the perpendicular from the point $(\beta, 0, \beta)$ $(\beta \ne 0)$ to the line, $\frac{x}{1} = \frac{y-1}{0} = \frac{z+1}{-1}$ is $\sqrt{\frac{3}{2}}$, then β is equal to
(a) 2 (b) -2
(c) -1 (d) 1

16 If $\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}$, $x \ne 0$,

then for all $\theta \in \left(0, \frac{\pi}{2}\right)$
(a) $\Delta_1 + \Delta_2 = -2(x^3 + x - 1)$
(b) $\Delta_1 - \Delta_2 = -2x^3$
(c) $\Delta_1 + \Delta_2 = -2x^3$
(d) $\Delta_1 - \Delta_2 = x(\cos 2\theta - \cos 4\theta)$

17 Let $f(x) = x^2, x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R : f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true?
(a) $f(g(S)) = S$ (b) $g(f(S)) \ne S$
(c) $g(f(S)) = g(S)$ (d) $f(g(S)) \ne f(S)$

18 If $\lim_{x\to 1}\frac{x^4 - 1}{x - 1} = \lim_{x\to k}\frac{x^3 - k^3}{x^2 - k^2}$, then k is
(a) $\frac{4}{3}$ (b) $\frac{3}{8}$
(c) $\frac{3}{2}$ (d) $\frac{8}{3}$

19 All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the equation
(a) $2|\sin x| = 3\sin y$
(b) $\sin x = |\sin y|$
(c) $\sin x = 2\sin y$
(d) $2\sin x = \sin y$

20 The line $x = y$ touches a circle at the point (1, 1). If the circle also passes through the point (1, – 3), then its radius is
(a) $3\sqrt{2}$ (b) $2\sqrt{2}$
(c) 2 (d) 3

21 Let $A(3, 0, -1)$, $B(2, 10, 6)$ and $C(1, 2, 1)$ be the vertices of a triangle and M be the mid-point of AC. If divides BM in the ratio 2 : 1, then cos ($\angle$ OA) (O being the origin) is equal to
(a) $\frac{1}{\sqrt{15}}$ (b) $\frac{1}{2\sqrt{15}}$
(c) $\frac{1}{\sqrt{30}}$ (d) $\frac{1}{6\sqrt{10}}$

22 Which one of the following Boolean expressions is a tautology ?
(a) $(p \vee q) \vee (p \vee \sim q)$
(b) $(p \wedge q) \vee (p \wedge \sim q)$
(c) $(p \vee q) \wedge (p \vee \sim q)$
(d) $(p \vee q) \wedge (\sim p \vee \sim q)$

23 Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at $x = c$, g is
(a) not differentiable
(b) differentiable if $f'(c) \ne 0$
(c) not differentiable if $f'(c) = 0$
(d) differentiable if $f'(c) = 0$

24 If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1 + ax + bx^2)(1 - 3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to
(a) (28, 315) (b) (– 21, 714)
(c) (28, 861) (d) (– 54, 315)

25 The sum of series $\frac{3 \times 1^3}{1^2} + \frac{5 \times (1^3 + 2^3)}{1^2 + 2^2} + \frac{7 \times (1^3 + 2^3 + 3^3)}{1^2 + 2^2 + 3^2} + \ldots\ldots\ldots +$ upto 10th term, is
(a) 680 (b) 600
(c) 660 (d) 620

26 Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls; is
(a) $\frac{1}{17}$ (b) $\frac{1}{12}$ (c) $\frac{1}{10}$ (d) $\frac{1}{11}$

27 If the system of linear equations

$x + y + z = 5$

$x + 2y + 2z = 6$

$x + 3y + \lambda z = \mu, (\lambda, \mu \in R)$, has infinitely many solutions, then the value of $\lambda + \mu$ is

(a) 7 (b) 12 (c) 10 (d) 9

28 If $f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$

is continuous at $x = 0$, then the ordered pair (p, q) is equal to

(a) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, \frac{3}{2}\right)$

(c) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

29 If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} = (\tan x - y)\sec^2 x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $y(0) = 0$, then $y\left(-\frac{\pi}{4}\right)$ is equal to

(a) $\frac{1}{e} - 2$ (b) $\frac{1}{2} - e$

(c) $2 + \frac{1}{e}$ (d) $e - 2$

30 The value of $\int_0^{2\pi} [\sin 2x (1 + \cos 3x)]\, dx$, where $[t]$ denotes the greatest integer function, is

(a) $-\pi$ (b) 2π

(c) -2π (d) π

ANSWERS

1. (b)	2. (a)	3. (a)	4. (a)	5. (c)	6. (d)	7. (b)	8. (a)	9. (a)	10. (b)
11. (b)	12. (c)	13. (c)	14. (b)	15. (c)	16. (c)	17. (c)	18. (d)	19. (b)	20. (b)
21. (a)	22. (a)	23. (d)	24. (a)	25. (c)	26. (d)	27. (c)	28. (d)	29. (d)	30. (a)

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10 April, Shift-II

1 The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x\, dx$ is equal to

(a) $3^{5/6} - 3^{2/3}$

(b) $3^{7/6} - 3^{5/6}$

(c) $3^{5/3} - 3^{1/3}$

(d) $3^{4/3} - 3^{1/3}$

2 The sum of series

$$1 + \frac{1^3 + 2^3}{1 + 2} + \frac{1^3 + 2^3 + 3^3}{1 + 2 + 3} + \ldots + \frac{1^3 + 2^3 + 3^3 + \ldots + 15^3}{1 + 2 + 3 + \ldots + 15} - \frac{1}{2}(1 + 2 + 3 + \ldots + 15)$$ is equal to

(a) 620 (b) 660

(c) 1240 (d) 1860

3 The area (in sq units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is

(a) $\frac{3}{2}$ (b) $\log_e 2 + \frac{3}{2}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2} - \frac{1}{\log_e 2}$

4 If $5x + 9 = 0$ is the directrix of the hyperbola $16x^2 - 9y^2 = 144$, then its corresponding focus is

(a) $\left(-\frac{5}{3}, 0\right)$ (b) $(-5, 0)$

(c) $\left(\frac{5}{3}, 0\right)$ (d) $(5, 0)$

5 The locus of the centres of the circles, which touch the circle, $x^2 + y^2 = 1$ externally, also touch the Y-axis and lie in the first quadrant, is

(a) $y = \sqrt{1 + 2x}, x \geq 0$

(b) $y = \sqrt{1 + 4x}, x \geq 0$

(c) $x = \sqrt{1 + 2y}, y \geq 0$

(d) $x = \sqrt{1 + 4y}, y \geq 0$

6 The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meets the X-axis at Q and R, respectively. Then, the area (in sq units) of the ΔPQR is

(a) $\frac{16}{3}$ (b) $\frac{14}{3}$

(c) $\frac{34}{15}$ (d) $\frac{68}{15}$

7 If the tangent to the curve $y = \frac{x}{x^2 - 3}$, $x \in R$, $(x \neq \pm\sqrt{3})$, at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then

(a) $|6\alpha + 2\beta| = 19$

(b) $|6\alpha + 2\beta| = 9$

(c) $|2\alpha + 6\beta| = 19$

(d) $|2\alpha + 6\beta| = 11$

8 If $\lim_{x \to 1} \frac{x^2 - ax + b}{x - 1} = 5$, then $a + b$ is equal to

(a) -4 (b) 1

(c) -7 (d) 5

9 Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number of beams is

(a) 180 (b) 210

(c) 170 (d) 190

10 Let λ be a real number for which the system of linear equations

$x + y + z = 6$, $4x + \lambda y - \lambda z = \lambda - 2$ and

$3x + 2y - 4z = -5$

has infinitely many solutions. Then λ is a root of the quadratic equation

(a) $\lambda^2 - 3\lambda - 4 = 0$ (b) $\lambda^2 + 3\lambda - 4 = 0$

(c) $\lambda^2 - \lambda - 6 = 0$ (d) $\lambda^2 + \lambda - 6 = 0$

11 The angles A, B and C of a ΔABC are in AP and $a : b = 1 : \sqrt{3}$. If $c = 4$ cm, then the area (in sq cm) of this triangle is

(a) $\frac{2}{\sqrt{3}}$ (b) $4\sqrt{3}$

(c) $2\sqrt{3}$ (d) $\frac{4}{\sqrt{3}}$

12 The distance of the point having position vector $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ from the straight line passing through the point $(2, 3, -4)$ and parallel to the vector, $6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ is

(a) $2\sqrt{13}$ (b) $4\sqrt{3}$

(c) 6 (d) 7

13 If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \le x \le 1, -2 \le y \le 2, x \le \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to
(a) $2\sin^2\alpha$ (b) $4\cos^2\alpha + 2x^2y^2$
(c) $4\sin^2\alpha$ (d) $4\sin^2\alpha - 2x^2y^2$

14 If both the mean and the standard deviation of 50 observations $x_1, x_2, \ldots, x_{50}$ are equal to 16, then the mean of $(x_1 - 4)^2$, $(x_2 - 4)^2, \ldots, (x_{50} - 4)^2$ is
(a) 480 (b) 400
(c) 380 (d) 525

15 If the plane $2x - y + 2z + 3 = 0$ has the distances $\frac{1}{3}$ and $\frac{2}{3}$ units from the planes $4x - 2y + 4z + \lambda = 0$ and $2x - y + 2z + \mu = 0$, respectively, then the maximum value of $\lambda + \mu$ is equal to
(a) 13 (b) 15 (c) 5 (d) 9

16 The sum of the real roots of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, is equal to
(a) 0 (b) -4 (c) 6 (d) 1

17 Lines are drawn parallel to the line $4x - 3y + 2 = 0$, at a distance $\frac{3}{5}$ from the origin. Then which one of the following points lies on any of these lines?
(a) $\left(-\frac{1}{4}, -\frac{2}{3}\right)$ (b) $\left(-\frac{1}{4}, \frac{2}{3}\right)$
(c) $\left(\frac{1}{4}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{4}, \frac{1}{3}\right)$

18 A perpendicular is drawn from a point on the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{1}$ to the plane $x + y + z = 3$ such that the foot of the perpendicular Q also lies on the plane $x - y + z = 3$. Then, the coordinates of Q are
(a) $(-1, 0, 4)$ (b) $(4, 0, -1)$
(c) $(2, 0, 1)$ (d) $(1, 0, 2)$

19 Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and $g(x) = \sin^{-1}(e^{-x})$, $(x \ge 0)$. If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then
(a) $a\alpha^2 - b\alpha - a = 0$
(b) $a\alpha^2 - b\alpha - a = 1$
(c) $a\alpha^2 + b\alpha - a = -2\alpha^2$
(d) $a\alpha^2 + b\alpha + a = 0$

20 If the line $ax + y = c$, touches both the curves $x^2 + y^2 = 1$ and $y^2 = 4\sqrt{2}x$, then $|c|$ is equal to
(a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) $\sqrt{2}$ (d) $\frac{1}{2}$

21 If $\int x^5 e^{-x^2} dx = g(x)e^{-x^2} + C$, where C is a constant of integration, then $g(-1)$ is equal to
(a) -1 (b) 1
(c) $-\frac{1}{2}$ (d) $-\frac{5}{2}$

22 Let $a_1, a_2, a_3, \ldots$ be an AP with $a_6 = 2$. Then, the common difference of this AP, which maximises the product a_1, a_4, a_5, is
(a) $\frac{8}{5}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{6}{5}$

23 If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then
(a) $\bar{z}w = -i$ (b) $z\bar{w} = \frac{1-i}{\sqrt{2}}$
(c) $\bar{z}w = i$ (d) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

24 A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is
(a) $\frac{1}{9\pi}$ (b) $\frac{1}{18\pi}$
(c) $\frac{1}{36\pi}$ (d) $\frac{5}{6\pi}$

25 The negation of the boolean expression $\sim s \vee (\sim r \wedge s)$ is equivalent to
(a) $s \wedge r$ (b) $\sim s \wedge \sim r$
(c) $s \vee r$ (d) r

26 Minimum number of times a fair coin must be tossed so that the probability of getting atleast one head is more than 99% is
(a) 8 (b) 6
(c) 7 (d) 5

27 The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is
(a) 35 (b) 23
(c) 58 (d) 38

28 The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is
(a) 1 (b) 3
(c) 4 (d) 2

29 Let $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $y(0) = 1$. Then
(a) $y'\left(\frac{\pi}{4}\right) - y'\left(-\frac{\pi}{4}\right) = \pi - \sqrt{2}$
(b) $y'\left(\frac{\pi}{4}\right) + y'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$
(c) $y\left(\frac{\pi}{4}\right) + y\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{2} + 2$
(d) $y\left(\frac{\pi}{4}\right) - y\left(-\frac{\pi}{4}\right) = \sqrt{2}$

30 Let a, b and c be in GP with common ratio r, where $a \ne 0$ and $0 < r \le \frac{1}{2}$. If $3a$, $7b$ and $15c$ are the first three terms of an AP, then the 4th term of this AP is
(a) $5a$ (b) $\frac{2}{3}a$
(c) a (d) $\frac{7}{3}a$

ANSWERS

1. (b)	2. (a)	3. (d)	4. (b)	5. (a)	6. (d)	7. (a)	8. (c)	9. (c)	10. (c)
11. (c)	12. (d)	13. (c)	14. (b)	15. (a)	16. (a)	17. (b)	18. (c)	19. (b)	20. (c)
21. (d)	22. (a)	23. (a)	24. (b)	25. (a)	26. (c)	27. (d)	28. (a)	29. (a)	30. (c)

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12 April, Shift-I

1 If A is a symmetric matrix and B is a skew-symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to

(a) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$

2 If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at $x = 0$ is equal to

(a) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (b) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$

(c) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ (d) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

3 If the angle of intersection at a point where the two circles with radii 5 cm and 12 cm intersect is 90°, then the length (in cm) of their common chord is

(a) $\frac{13}{5}$ (b) $\frac{120}{13}$

(c) $\frac{60}{13}$ (d) $\frac{13}{2}$

4 If the area (in sq units) of the region $\{(x, y): y^2 \le 4x, x + y \le 1, x \ge 0, y \ge 0\}$ is $a\sqrt{2} + b$, then $a - b$ is equal to

(a) $\frac{10}{3}$ (b) 6

(c) $\frac{8}{3}$ (d) $-\frac{2}{3}$

5 For $x \in R$, let $[x]$ denote the greatest integer $\le x$, then the sum of the series

$$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{99}{100}\right] \text{ is}$$

(a) -153 (b) -133 (c) -131 (d) -135

6 The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is

(a) $2^{20} - 1$ (b) 2^{21}

(c) 2^{20} (d) $2^{20} + 1$

7 The integral $\int \frac{2x^3 - 1}{x^4 + x}\, dx$ is equal to

(here C is a constant of integration)

(a) $\frac{1}{2}\log_e \frac{|x^3 + 1|}{x^2} + C$

(b) $\frac{1}{2}\log_e \frac{(x^3 + 1)^2}{|x^3|} + C$

(c) $\log_e \left|\frac{x^3 + 1}{x}\right| + C$

(d) $\log_e \frac{|x^3 + 1|}{x^2} + C$

8 The equation $y = \sin x \sin(x + 2) - \sin^2(x + 1)$ represents a straight line lying in

(a) second and third quadrants only
(b) first, second and fourth quadrants
(c) first, third and fourth quadrants
(d) third and fourth quadrants only

9 Let $f : R \to R$ be a continuously differentiable function such that $f(2) = 6$ and $f'(2) = \frac{1}{48}$. If $\int_6^{f(x)} 4t^3\, dt = (x - 2)g(x)$, then $\lim_{x \to 2} g(x)$ is equal to

(a) 18 (b) 24 (c) 12 (d) 36

10 The coefficient of x^{18} in the product $(1 + x)(1 - x)^{10}(1 + x + x^2)^9$ is

(a) 84 (b) -126

(c) -84 (d) 126

11 If there of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is

(a) $\frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$ (d) $\frac{3}{20}$

12 Consider the differential equation, $y^2 dx + \left(x - \frac{1}{y}\right)dy = 0$. If value of y is 1 when $x = 1$, then the value of x for which $y = 2$, is

(a) $\frac{5}{2} + \frac{1}{\sqrt{e}}$ (b) $\frac{3}{2} - \frac{1}{\sqrt{e}}$

(c) $\frac{1}{2} + \frac{1}{\sqrt{e}}$ (d) $\frac{3}{2} - \sqrt{e}$

13 Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ be two vectors. If a vector perpendicular to both the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ has the magnitude 12, then one such vector is

(a) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $4(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(c) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $4(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

14 Let a random variable X have a binomial distribution with mean 8 and variance 4. If $P(X \le 2) = \frac{k}{2^{16}}$, then k is equal to

(a) 17 (b) 121

(c) 1 (d) 137

15 The number of solutions of the equation

$1 + \sin^4 x = \cos^2 3x, x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is

(a) 3 (b) 5

(c) 7 (d) 4

16 The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents

(a) a circle of radius $\frac{1}{2}$
(b) the line passing through the origin with slope 1
(c) a circle of radius 1
(d) the line passing through the origin with slope -1

17 If the truth value of the statement $p \to (\sim q \vee r)$ is false (F), then the truth values of the statements p, q and r are respectively

(a) T, T and F (b) T, F and F

(c) T, F and T (d) F, T and T

18 If $\int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx = m(\pi + n)$, then $m \cdot n$ is equal to

(a) $-\frac{1}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) -1

19 For $x \in \left(0, \frac{3}{2}\right)$, let

$f(x) = \sqrt{x}$, $g(x) = \tan x$ and

$h(x) = \frac{1 - x^2}{1 + x^2}$. If $\phi(x) = ((hof)og)(x)$,

then $\phi\left(\frac{\pi}{3}\right)$ is equal to

(a) $\tan\frac{\pi}{12}$ (b) $\tan\frac{11\pi}{12}$

(c) $\tan\frac{7\pi}{12}$ (d) $\tan\frac{5\pi}{12}$

20 The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to

(a) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$ (b) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

(c) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$ (d) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

21 A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/s, then the rate (in cm/s) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is

(a) $25\sqrt{3}$ (b) $\frac{25}{\sqrt{3}}$

(c) $\frac{25}{3}$ (d) 25

22 If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim_{n\to\infty}\sum_{r=1}^{n}\alpha^r + \lim_{n\to\infty}\sum_{r=1}^{n}\beta^r$ is equal to

(a) $\frac{21}{346}$ (b) $\frac{29}{358}$ (c) $\frac{1}{12}$ (d) $\frac{7}{116}$

23 If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through $Q(4, 4)$ then PQ is equal to

(a) $\frac{5\sqrt{5}}{2}$ (b) $\frac{\sqrt{61}}{2}$
(c) $\frac{\sqrt{221}}{2}$ (d) $\frac{\sqrt{157}}{2}$

24 If m is the minimum value of k for which the function $f(x) = x\sqrt{kx - x^2}$ is increasing in the interval [0, 3] and M is the maximum value of f in the interval [0, 3] when $k = m$, then the ordered pair (m, M) is equal to

(a) $(4, 3\sqrt{2})$ (b) $(4, 3\sqrt{3})$
(c) $(3, 3\sqrt{3})$ (d) $(5, 3\sqrt{6})$

25 Let S_n denote the sum of the first n terms of an AP. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to

(a) -260 (b) -410
(c) -320 (d) -380

26 If the data $x_1, x_2, \ldots, x_{10}$ is such that the mean of first four of these is 11, the mean of the remaining six is 16 and the sum of squares of all of these is 2000, then the standard deviation of this data is

(a) $2\sqrt{2}$ (b) 2
(c) 4 (d) $\sqrt{2}$

27 Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and S' denotes the foci of the hyperbola where S lies on the positive X-axis then P divides SS' in a ratio

(a) 13 : 11 (b) 14 : 13
(c) 5 : 4 (d) 2 : 1

28 If $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which det $(A) + 1 = 0$, is

(a) 0 (b) -1
(c) 1 (d) 2

29 If the volume of parallelopiped formed by the vectors $\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}} + \hat{\mathbf{k}}$ is minimum, then λ is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) $-\sqrt{3}$

30 If the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the plane $2x + 3y - z + 13 = 0$ at a point P and the plane $3x + y + 4z = 16$ at a point Q, then PQ is equal to

(a) 14 (b) $\sqrt{14}$
(c) $2\sqrt{7}$ (d) $2\sqrt{14}$

ANSWERS

1. (b)	2. (b)	3. (b)	4. (b)	5. (b)	6. (c)	7. (c)	8. (d)	9. (a)	10. (a)
11. (a)	12. (b)	13. (b)	14. (d)	15. (b)	16. (b)	17. (a)	18. (d)	19. (b)	20. (b)
21. (b)	22. (c)	23. (a)	24. (b)	25. (c)	26. (b)	27. (c)	28. (c)	29. (b)	30. (d)

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12 April, Shift-II

1 The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x \in \left(0, \frac{\pi}{2}\right)\right)$ is

(a) 1 (b) $\frac{2}{3}$
(c) $\frac{1}{2}$ (d) 2

2 For an initial screening of an admission test, a candidate is given fifty problems to solve. If the probability that the candidate can solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problem is

(a) $\frac{201}{5}\left(\frac{1}{5}\right)^{49}$ (b) $\frac{316}{25}\left(\frac{4}{5}\right)^{48}$
(c) $\frac{54}{5}\left(\frac{4}{5}\right)^{49}$ (d) $\frac{164}{25}\left(\frac{1}{5}\right)^{48}$

3 A value of α such that

$$\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right) \text{ is}$$

(a) -2 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 2

4 Let $\alpha \in (0, \pi/2)$ be fixed. If the integral

$$\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha}\, dx = A(x)\cos 2\alpha + B(x)\sin 2\alpha + C,$$

where C is a constant of integration, then the functions $A(x)$ and $B(x)$ are respectively

(a) $x + \alpha$ and $\log_e|\sin(x+\alpha)|$
(b) $x - \alpha$ and $\log_e|\sin(x-\alpha)|$
(c) $x - \alpha$ and $\log_e|\cos(x-\alpha)|$
(d) $x + \alpha$ and $\log_e|\sin(x-\alpha)|$

5 The angle of elevation of the top of a vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point A. If the angle of elevation of the top of the tower from B be 30°, then the distance (in m) of the foot of the tower from the point A is

(a) $15(3 + \sqrt{3})$ (b) $15(5 - \sqrt{3})$
(c) $15(3 - \sqrt{3})$ (d) $15(1 + \sqrt{3})$

6 Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then, S is equal to

(a) R (b) [1, 4]
(c) [3, 7] (d) [2, 6]

7 A plane which bisects the angle between the two given planes $2x - y + 2z - 4 = 0$ and $x + 2y + 2z - 2 = 0$, passes through the point

(a) (1, – 4, 1) (b) (1, 4, – 1)
(c) (2, 4, 1) (d) (2, – 4, 1)

8 $\lim_{x\to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ is

(a) 6 (b) 2
(c) 3 (d) 1

9 A group of students comprises of 5 boys and n girls. If the number of ways, in which a team of 3 students can randomly be selected from this group such that there is at least one boy and at least one girl in each team, is 1750, then n is equal to
(a) 28 (b) 27
(c) 25 (d) 24

10 An ellipse, with foci at (0, 2) and (0, – 2) and minor axis of length 4, passes through which of the following points?
(a) $(\sqrt{2}, 2)$ (b) $(2, \sqrt{2})$
(c) $(2, 2\sqrt{2})$ (d) $(1, 2\sqrt{2})$

11 The boolean expression $\sim (p \Rightarrow (\sim q))$ is equivalent to
(a) $p \wedge q$ (b) $q \Rightarrow \sim p$
(c) $p \vee q$ (d) $(\sim p) \Rightarrow q$

12 A circle touching the X-axis at (3, 0) and making a intercept of length 8 on the Y-axis passes through the point
(a) (3, 10) (b) (3, 5)
(c) (2, 3) (d) (1, 5)

13 If ${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \ldots\ldots + (20^2)\,{}^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to
(a) (420, 19) (b) (420, 18)
(c) (380, 18) (d) (380, 19)

14 A value of $\theta \in (0, \pi/3)$, for which

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0, \text{ is}$$

(a) $\frac{\pi}{9}$ (b) $\frac{\pi}{18}$
(c) $\frac{7\pi}{24}$ (d) $\frac{7\pi}{36}$

15 The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$, is
(a) $x - y + 4 = 0$
(b) $x + y + 4 = 0$
(c) $x - 2y + 16 = 0$
(d) $2x - y + 2 = 0$

16 Let $z \in C$ with $\text{Im}(z) = 10$ and it satisfies $\frac{2z-n}{2z+n} = 2i - 1$ for some natural number n, then
(a) $n = 20$ and $\text{Re}(z) = -10$
(b) $n = 40$ and $\text{Re}(z) = 10$
(c) $n = 40$ and $\text{Re}(z) = -10$
(d) $n = 20$ and $\text{Re}(z) = 10$

17 A triangle has a vertex at (1, 2) and the mid-points of the two sides through it are (–1, 1) and (2, 3). Then, the centroid of this triangle is
(a) $\left(1, \frac{7}{3}\right)$ (b) $\left(\frac{1}{3}, 2\right)$
(c) $\left(\frac{1}{3}, 1\right)$ (d) $\left(\frac{1}{3}, \frac{5}{3}\right)$

18 If $a_1, a_2, a_3, \ldots$ are in AP such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this AP is
(a) 200
(b) 280
(c) 120
(d) 150

19 If $[x]$ denotes the greatest integer $\le x$, then the system of liner equations $[\sin\theta]x + [-\cos\theta]y = 0$, $[\cot\theta]x + y = 0$
(a) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$.
(b) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$
(c) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$
(d) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

20 A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line $x + y = 0$. Then, an equation of the line L is
(a) $x + \sqrt{3}y = 8$
(b) $(\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 8\sqrt{2}$
(c) $\sqrt{3}x + y = 8$
(d) $(\sqrt{3} - 1)x + (\sqrt{3} + 1)y = 8\sqrt{2}$

21 Let $f(x) = 5 - |x - 2|$ and $g(x) = |x + 1|$, $x \in R$. If $f(x)$ attains maximum value at α and $g(x)$ attains minimum value of β, then $\lim_{x \to -\alpha\beta} \frac{(x-1)(x^2 - 5x + 6)}{x^2 - 6x + 8}$ is equal to
(a) 1/2 (b) – 3/2
(c) – 1/2 (d) 3/2

22 Let $\alpha \in R$ and the three vectors
$\mathbf{a} = \alpha\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}$ and $\mathbf{c} = \alpha\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Then, the set $S = \{\alpha : \mathbf{a}, \mathbf{b} \text{ and } \mathbf{c} \text{ are coplanar}\}$
(a) is singleton
(b) is empty
(c) contains exactly two positive numbers
(d) contains exactly two numbers only one of which is positive

23 A person throws two fair dice. He wins ₹ 15 for throwing a doublet (same numbers on the two dice), wins ₹ 12 when the throw results in the sum of 9, and loses ₹ 6 for any other outcome on the throw. Then, the expected gain/loss (in ₹) of the person is
(a) $\frac{1}{2}$ gain (b) $\frac{1}{4}$ loss
(c) $\frac{1}{2}$ loss (d) 2 gain

24 The tangents to the curve $y = (x - 2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point
(a) $\left(\frac{5}{2}, 1\right)$ (b) $\left(-\frac{5}{2}, -1\right)$
(c) $\left(\frac{5}{2}, -1\right)$ (d) $\left(-\frac{5}{2}, 1\right)$

25 If α, β and γ are three consecutive terms of a non-constant GP such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then, $\alpha(\beta + \gamma)$ is equal to
(a) 0 (b) $\alpha\beta$
(c) $\alpha\gamma$ (d) $\beta\gamma$

26 Let A, B and C be sets such that $\phi \ne A \cap B \subseteq C$. Then, which of the following statements is not true?
(a) $B \cap C \ne \phi$
(b) If $(A - B) \subseteq C$, then $A \subseteq C$
(c) $(C \cup A) \cap (C \cup B) = C$
(d) If $(A - C) \subseteq B$, then $A \subseteq B$

27 The general solution of the differential equation $(y^2 - x^3)dx - xydy = 0$ $(x \ne 0)$ is (where, C is a constant of integration)
(a) $y^2 - 2x^2 + Cx^3 = 0$
(b) $y^2 + 2x^3 + Cx^2 = 0$
(c) $y^2 + 2x^2 + Cx^3 = 0$
(d) $y^2 - 2x^3 + Cx^2 = 0$

28 If the area (in sq units) bounded by the parabola $y^2 = 4\lambda x$ and the line $y = \lambda x$, $\lambda > 0$, is $\frac{1}{9}$, then λ is equal to

(a) $2\sqrt{6}$ (b) 48
(c) 24 (d) $4\sqrt{3}$

29 The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines
$\mathbf{r} = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$
and $\mathbf{r} = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \mu\,(-\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ is

(a) 3 (b) $\frac{1}{3}$ (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

30 The term independent of x in the expansion of $\left(\frac{1}{60} - \frac{x^8}{81}\right).\left(2x^2 - \frac{3}{x^2}\right)^6$ is equal to

(a) – 72 (b) 36
(c) – 36 (d) – 108

ANSWERS

1. *(d)*	2. *(c)*	3. *(a)*	4. *(b)*	5. *(a)*	6. *(d)*	7. *(d)*	8. *(b)*	9. *(c)*	10. *(a)*
11. *(a)*	12. *(a)*	13. *(b)*	14. *(a)*	15. *(a)*	16. *(c)*	17. *(b)*	18. *(a)*	19. *(a)*	20. *(d)*
21. *(a)*	22. *(b)*	23. *(c)*	24. *(c)*	25. *(d)*	26. *(d)*	27. *(b)*	28. *(c)*	29. *(c)*	30. *(c)*

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JANUARY ATTEMPT

9 January, Shift-I

1 If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to

(a) $\frac{\sqrt{145}}{10}$ (b) $\frac{\sqrt{146}}{12}$
(c) $\frac{\sqrt{145}}{12}$ (d) $\frac{\sqrt{145}}{11}$

2 The value of $\int_0^{\pi} |\cos x|^3\, dx$ is

(a) $\frac{2}{3}$ (b) $-\frac{4}{3}$
(c) 0 (d) $\frac{4}{3}$

3 For $x^2 \neq n\pi + 1$, $n \in N$ (the set of natural numbers), the integral

$\int x\sqrt{\frac{2\sin(x^2-1) - \sin 2(x^2-1)}{2\sin(x^2-1) + \sin 2(x^2-1)}}\, dx$ is

equal to (where C is a constant of integration)

(a) $\frac{1}{2}\log_e|\sec(x^2-1)| + C$

(b) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right| + C$

(c) $\log_e\left|\frac{1}{2}\sec^2(x^2-1)\right| + C$

(d) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right| + C$

4 If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{13}{16}$ (b) $\frac{1}{4}$
(c) $\frac{49}{16}$ (d) $\frac{7}{64}$

5 The equation of the line passing through (–4, 3, 1), parallel to the plane $x + 2y - z - 5 = 0$ and intersecting the line

$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z-2}{-1}$ is

(a) $\frac{x+4}{3} = \frac{y-3}{-1} = \frac{z-1}{1}$

(b) $\frac{x+4}{-1} = \frac{y-3}{1} = \frac{z-1}{1}$

(c) $\frac{x+4}{1} = \frac{y-3}{1} = \frac{z-1}{3}$

(d) $\frac{x-4}{2} = \frac{y+3}{1} = \frac{z+1}{4}$

6 Let $f: R \to R$ be a function defined as

$$f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$$

Then, f is

(a) continuous if $a = -5$ and $b = 10$
(b) continuous if $a = 5$ and $b = 5$
(c) continuous if $a = 0$ and $b = 5$
(d) not continuous for any values of a and b

7 Axis of a parabola lies along X-axis. If its vertex and focus are at distances 2 and 4 respectively from the origin, on the positive X-axis, then which of the following points does not lie on it?

(a) (4, –4) (b) $(6, 4\sqrt{2})$
(c) (8, 6) (d) $(5, 2\sqrt{6})$

8 Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true?

(a) Each line passes through the origin.
(b) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$
(c) The lines are all parallel
(d) The lines are not concurrent

9 Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to

(a) 256 (b) 512 (c) –256 (d) –512

10 If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to

(a) 14 (b) 6 (c) 4 (d) 8

11 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is

(a) 16 (b) 22
(c) 20 (d) 18

12 If a, b and c be three distinct real numbers in GP and $a + b + c = xb$, then x cannot be

(a) 4 (b) 2 (c) –2 (d) –3

13 The plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to Y-axis also passes through the point

(a) (3, 3, –1) (b) (–3, 1, 1)
(c) (3, 2, 1) (d) (–3, 0, –1)

14 The maximum volume (in cu.m) of the right circular cone having slant height 3m is

(a) $\frac{4}{3}\pi$ (b) $2\sqrt{3}\pi$

(c) $3\sqrt{3}\pi$ (d) 6π

15 If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to

(a) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

16 For $x \in R - \{0, 1\}$, let

$f_1(x) = \frac{1}{x}, f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$

be three given functions. If a function, $J(x)$ satisfies $(f_2 \circ J \circ f_1)(x) = f_3(x)$, then $J(x)$ is equal to

(a) $f_2(x)$ (b) $f_3(x)$

(c) $f_1(x)$ (d) $\frac{1}{x} f_3(x)$

17 Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team, is

(a) 350 (b) 500

(c) 200 (d) 300

18 If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to

(a) $\frac{7}{17}$ (b) $\frac{8}{15}$ (c) $\frac{4}{9}$ (d) $\frac{8}{17}$

19 Three circles of radii $a, b, c (a < b < c)$ touch each other externally. If they have X-axis as a common tangent, then

(a) a, b, c are in AP

(b) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$

(c) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in AP

(d) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

20 For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the expression

$3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals

(a) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

(b) $13 - 4\cos^2\theta + 6\cos^4\theta$

(c) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

(d) $13 - 4\cos^6\theta$

21 If the Boolean expression

$(p \oplus q) \wedge (\sim p \cdot q)$ is equivalent to $p \wedge q$, where $\oplus, \cdot \in \{\wedge, \vee\}$, then the ordered pair$(\oplus, \cdot)$ is

(a) $(\wedge, \vee)$ (b) $(\wedge, \wedge)$

(c) $(\vee, \wedge)$ (d) $(\vee, \vee)$

22 Equation of a common tangent to the circle, $x^2 + y^2 - 6x = 0$ and the parabola, $y^2 = 4x$, is

(a) $\sqrt{3}y = 3x + 1$ (b) $2\sqrt{3}y = 12x + 1$

(c) $\sqrt{3}y = x + 3$ (d) $2\sqrt{3}y = -x - 12$

23 Let $A = \left\{\theta \in \left(-\frac{\pi}{2}, \pi\right) : \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ is purely imaginary}\right\}$

Then, the sum of the elements in A is

(a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{6}$

(c) π (d) $\frac{2\pi}{3}$

24 Let $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c}$ be a vector such that $\mathbf{a} \times \mathbf{c} + \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{c} = 4$, then $|\mathbf{c}|^2$ is equal to

(a) 8 (b) $\frac{19}{2}$

(c) 9 (d) $\frac{17}{2}$

25 The system of linear equations

$$x + y + z = 2$$
$$2x + 3y + 2z = 5$$
$$2x + 3y + (a^2 - 1)z = a + 1$$

(a) has infinitely many solutions for $a = 4$

(b) is inconsistent when $a = 4$

(c) has a unique solution for $|a| = \sqrt{3}$

(d) is inconsistent when $|a| = \sqrt{3}$

26 The area (in sq units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the Y-axis is

(a) $\frac{8}{3}$ (b) $\frac{56}{3}$ (c) $\frac{32}{3}$ (d) $\frac{14}{3}$

27 $\lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$

(a) exists and equals $\frac{1}{4\sqrt{2}}$

(b) does not exist

(c) exists and equals $\frac{1}{2\sqrt{2}}$

(d) exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2} + 1)}$

28 Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval

(a) $(1, \frac{3}{2}]$ (b) $(3, \infty)$ (c) $(\frac{3}{2}, 2]$ (d) $(2, 3]$

29 Let $a_1, a_2, \ldots a_{30}$ be an AP, $S = \sum_{i=1}^{30} a_i$ and

$T = \sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = 27$ and

$S - 2T = 75$,

then a_{10} is equal to

(a) 42 (b) 57

(c) 52 (d) 47

30 Two cards are drawn successively with replacement from a well shuffled deck of 52 cards. Let X denote the random variable of number of aces obtained in the two drawn cards. Then, $P(X = 1) + P(X = 2)$ equals

(a) $\frac{25}{169}$ (b) $\frac{52}{169}$

(c) $\frac{49}{169}$ (d) $\frac{24}{169}$

ANSWERS

1. (c)	2. (d)	3. (b)	4. (c)	5. (a)	6. (d)	7. (c)	8. (b)	9. (c)	10. (d)
11. (c)	12. (b)	13. (c)	14. (b)	15. (c)	16. (b)	17. (d)	18. (b)	19. (b)	20. (d)
21. (a)	22. (c)	23. (d)	24. (b)	25. (d)	26. (a)	27. (a)	28. (b)	29. (c)	30. (a)

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9 January, Shift-II

1 Let the equations of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at (1, 1) then the equation of its third side is

(a) $122y - 26x - 1675 = 0$
(b) $26x - 122y - 1675 = 0$
(c) $122y + 26x + 1675 = 0$
(d) $26x + 61y + 1675 = 0$

2 Let z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$, If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then arg z is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) 0 (d) $\frac{\pi}{3}$

3 If $A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$ then A is

(a) invertible only when $t = \pi$
(b) invertible for every $t \in R$
(c) not invertible for any $t \in R$
(d) invertible only when $t = \frac{\pi}{2}$

4 A pot contain 5 red and 2 green balls. At random a ball is drawn from this pot. If a drawn ball is green then put a red ball in the pot and if a drawn ball is red, then put a green ball in the pot, while drawn ball is not replace in the pot. Now we draw another ball randomnly, the probability of second ball to be red is

(a) $\frac{27}{49}$ (b) $\frac{26}{49}$
(c) $\frac{21}{49}$ (d) $\frac{32}{49}$

5 Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2|x - y|^{\frac{3}{2}}$, for all $x, y \in R$. If $f(0) = 1$, then $\int_0^1 f^2(x)\,dx$ is equal to

(a) 2 (b) $\frac{1}{2}$
(c) 1 (d) 0

6 If lines $x = ay + b$, $z = cy + d$ and $x = a'z + b'$, $y = c'z + d'$ are perpendicular, then

(a) $ab' + bc' + 1 = 0$
(b) $bb' + cc' + 1 = 0$
(c) $aa' + c + c' = 0$
(d) $cc' + a + a' = 0$

7 The area of the region $A = \{(x, y); 0 \le y \le x|x| + 1$ and $-1 \le x \le 1\}$ in sq. units, is

(a) 2 (b) $\frac{4}{3}$
(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

8 The equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(a) $5x + 2y - 4z = 0$ (b) $x + 2y - 2z = 0$
(c) $3x + 2y - 3z = 0$ (d) $x - 2y + z = 0$

9 Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ be three vectors such that the projection vector of $\mathbf{b}$ on $\mathbf{a}$ is $\mathbf{a}$. If $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{c}$, then $|\mathbf{b}|$ is equal to

(a) 6 (b) 4
(c) $\sqrt{22}$ (d) $\sqrt{32}$

10 If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to

(a) 0 (b) 10
(c) 7π (d) π

11 Let a, b and c be the 7th, 11th and 13th terms respectively of a non-constant AP. If these are also the three consecutive terms of a GP, then $\frac{a}{c}$ is equal to

(a) 2 (b) $\frac{7}{13}$
(c) 4 (d) $\frac{1}{2}$

12 Let $A = \{x \in R : x$ is not a positive integer$\}$. Define a function $f : A \to R$ as $f(x) = \frac{2x}{x-1}$, then f is

(a) injective but not surjective
(b) not injective
(c) surjective but not injective
(d) neither injective nor surjective

13 In a group of data, there are n observations, $x, x_2, \ldots, x_n$. If $\sum_{i=1}^{n}(x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i - 1)^2 = 5n$, the standard deviation of the data is

(a) 2 (b) $\sqrt{7}$
(c) 5 (d) $\sqrt{5}$

14 If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is

(a) 2 (b) 3
(c) 1 (d) 4

15 If the system of linear equations

$$x - 4y + 7z = g$$
$$3y - 5z = h$$
$$-2x + 5y - 9z = k$$

is consistent, then

(a) $2g + h + k = 0$
(b) $g + 2h + k = 0$
(c) $g + h + k = 0$
(d) $g + h + 2k = 0$

16 Let A (4, – 4) and B(9, 6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is

(a) $31\frac{1}{4}$ (b) 32
(c) $31\frac{3}{4}$ (d) $30\frac{1}{2}$

17 The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is

(a) 5 (b) 2
(c) 4 (d) 3

18 For each $x \in R$, let $[x]$ be the greatest integer less than or equal to x. Then, $\lim_{x \to 0^-} \frac{x([x] + |x|)\sin[x]}{|x|}$ is equal to

(a) 0 (b) sin 1
(c) – sin 1 (d) 1

19 The coefficient of t^4 in the expansion of $\left(\frac{1 - t^6}{1 - t}\right)^3$ is

(a) 12 (b) 10
(c) 15 (d) 14

20 Let S be the set of all triangles in the xy-plane, each having one vertex at the origin and the other two vertices

lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is

(a) 36 (b) 32
(c) 18 (d) 9

21 The logical statement
$[\sim(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$
is equivalent to

(a) $\sim p \vee r$ (b) $(p \wedge \sim q) \vee r$
(c) $(p \wedge r) \wedge \sim q$ (d) $(\sim p \wedge \sim q) \wedge r$

22 If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x-4)^2 + (y-7)^2 = 36$ intersect at two distinct points, then

(a) $0 < r < 1$ (b) $r > 11$
(c) $1 < r < 11$ (d) $r = 11$

23 If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval [1, 5] then m lies in the interval

(a) (4, 5) (b) (–5, – 4)
(c) (5, 6) (d) (3, 4)

24 The number of natural numbers less than 7,000 which can be formed by using the digits 0, 1, 3, 7, 9 (repitition of digits allowed) is equal to

(a) 374 (b) 375
(c) 372 (d) 250

25 If $\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}}\, d\theta = 1 - \frac{1}{\sqrt{2}}$, $(k > 0)$, then the value of k is

(a) 1 (b) $\frac{1}{2}$
(c) 2 (d) 4

26 If $f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2}\, dx$, $(x \geq 0)$, and $f(0) = 0$, then the value of $f(1)$ is

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$
(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

27 Let $f : [0, 1] \to R$ be such that $f(xy) = f(x).\, f(y)$, for all $x, y \in [0, 1]$ and $f(0) \neq 0$. If $y = y(x)$ satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with $y(0) = 1$, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to

(a) 5 (b) 3
(c) 2 (d) 4

28 The sum of the following series
$1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + \ldots + 5^2)}{11} + \ldots$ up to 15 terms is

(a) 7510 (b) 7820
(c) 7830 (d) 7520

29 If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is

(a) $\frac{1}{6}$ (b) $\frac{1}{6\sqrt{2}}$
(c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{3}{2\sqrt{2}}$

30 A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the X-axis. Then the eccentricity of the hyperbola is

(a) 2 (b) $\frac{2}{\sqrt{3}}$
(c) $\frac{3}{2}$ (d) $\sqrt{3}$

ANSWERS

1 (b)	2 (a)	3 (b)	4 (d)	5 (c)	6 (c)	7 (a)	8 (d)	9 (a)	10 (d)
11 (c)	12 (a)	13 (d)	14 (a)	15 (a)	16 (a)	17 (d)	18 (c)	19 (c)	20 (a)
21 (c)	22 (c)	23 (a)	24 (a)	25 (c)	26 (c)	27 (b)	28 (b)	29 (b)	30 (b)

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10 January, Shift-I

1 The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is

(a) $x - y - 3 = 0$ (b) $x - y + 9 = 0$
(c) $x - y + 1 = 0$ (d) $x - y + 7 = 0$

2 Consider a triangular plot ABC with sides $AB = 7$ m, $BC = 5$ m and $CA = 6$ m. A vertical lamp-post at the mid-point D of AC subtends an angle 30° at B. The height (in m) of the lamp-post is

(a) $\frac{2}{3}\sqrt{21}$ (b) $2\sqrt{21}$ (c) $7\sqrt{3}$ (d) $\frac{3}{2}\sqrt{21}$

3 If $\frac{dy}{dx} + \frac{3}{\cos^2 x} y = \frac{1}{\cos^2 x}$, $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals

(a) $\frac{1}{3} + e^6$ (b) $-\frac{4}{3}$
(c) $\frac{1}{3} + e^3$ (d) $\frac{1}{3}$

4 Let $\mathbf{a} = 2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + (3 - \lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}$ be three vectors such that $\mathbf{b} = 2\mathbf{a}$ and $\mathbf{a}$ is perpendicular to $\mathbf{c}$. Then a possible value of $(\lambda_1, \lambda_2, \lambda_3)$ is

(a) (1, 3, 1) (b) (1, 5, 1)
(c) $\left(-\frac{1}{2}, 4, 0\right)$ (d) $\left(\frac{1}{2}, 4, -2\right)$

5 Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, $x \in R$. Then, $f(2)$ equals

(a) 30 (b) – 4
(c) – 2 (d) 8

6 If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c)?

(a) $\left(\frac{1}{2}, 2, 0\right)$ (b) (1, 1, 0)
(c) (1, 1, 3) (d) $\left(\frac{1}{2}, 2, 3\right)$

7 Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$, then

(a) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (b) $\text{Im}(z) = 0$
(c) $\text{Re}(z) = 0$ (d) $|z| = \sqrt{\frac{5}{2}}$

8 A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line

(a) with slope $\frac{2}{3}$

(b) with slope $\frac{3}{2}$

(c) parallel to Y-axis

(d) parallel to X-axis

9 Consider the statement : "$P(n) : n^2 - n + 41$ is prime." Then, which one of the following is true?

(a) Both $P(3)$ and $P(5)$ are true.

(b) $P(3)$ is false but $P(5)$ is true.

(c) Both $P(3)$ and $P(5)$ are false.

(d) $P(5)$ is false but $P(3)$ is true.

10 Let $n \geq 2$ be a natural number and $0 < \theta < \frac{\pi}{2}$. Then,

$\int \frac{(\sin^n\theta - \sin\theta)^{\frac{1}{n}} \cos\theta}{\sin^{n+1}\theta} d\theta$ is equal to (where C is a constant of integration)

(a) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}} + C$

(b) $\frac{n}{n^2-1}\left(1 + \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(c) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(d) $\frac{n}{n^2+1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

11 Let $I = \int_a^b (x^4 - 2x^2)\, dx$. If I is minimum, then the ordered pair (a, b) is

(a) $(-\sqrt{2}, 0)$ (b) $(0, \sqrt{2})$

(c) $(\sqrt{2}, -\sqrt{2})$ (d) $(-\sqrt{2}, \sqrt{2})$

12 The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is

(a) 1256 (b) 1465

(c) 1356 (d) 1365

13 An unbiased coin is tossed. If the outcome is a head, then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail, then a card from a well-shuffled pack of nine cards numbered 1, 2, 3, …, 9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is

(a) $\frac{15}{72}$ (b) $\frac{13}{36}$

(c) $\frac{19}{72}$ (d) $\frac{19}{36}$

14 If a circle C passing through the point $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point $(1, -1)$, then the radius of C is

(a) 5 (b) $2\sqrt{5}$

(c) $\sqrt{57}$ (d) 4

15 In a class of 140 students numbered 1 to 140, all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then, the number of students who did not opt for any of the three courses is

(a) 42 (b) 102

(c) 38 (d) 1

16 If the third term in the binomial expansion of $(1 + x^{\log_2 x})^5$ equals 2560, then a possible value of x is

(a) $4\sqrt{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{8}$ (d) $2\sqrt{2}$

17 If the system of equations

$$x + y + z = 5$$
$$x + 2y + 3z = 9$$
$$x + 3y + \alpha z = \beta$$

has infinitely many solutions, then $\beta - \alpha$ equals

(a) 8 (b) 18 (c) 21 (d) 5

18 If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i + {}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals

(a) 100 (b) 400 (c) 200 (d) 50

19 If the area enclosed between the curves $y = kx^2$ and $x = ky^2$, $(k > 0)$, is 1 square unit. Then, k is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

20 If the line $3x + 4y - 24 = 0$ intersects the X-axis at the point A and the Y-axis at the point B, then the incentre of the triangle OAB, where O is the origin, is

(a) (4, 3) (b) (3, 4)

(c) (4, 4) (d) (2, 2)

21 The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is

(a) 4 : 9 (b) 6 : 7

(c) 10 : 3 (d) 5 : 8

22 Consider the quadratic equation $(c - 5)x^2 - 2cx + (c - 4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0, 2) and its other root lies in the interval (2, 3). Then, the number of elements in S is

(a) 11 (b) 10 (c) 12 (d) 18

23 Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \leq 2 \\ 8 - 2|x|, & 2 < |x| \leq 4 \end{cases}$

Let S be the set of points in the interval $(-4, 4)$ at which f is not differentiable. Then, S

(a) equals $\{-2, -1, 0, 1, 2\}$ (b) equals $\{-2, 2\}$

(c) is an empty set (d) equals $\{-2, -1, 1, 2\}$

24 Let A be a point on the line $\mathbf{r} = (1 - 3\mu)\hat{\mathbf{i}} + (\mu - 1)\hat{\mathbf{j}} + (2 + 5\mu)\hat{\mathbf{k}}$ and $B(3, 2, 6)$ be a point in the space. Then, the value of μ for which the vector $\mathbf{AB}$ is parallel to the plane $x - 4y + 3z = 1$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

25 Let $d \in R$, and

$$A = \begin{bmatrix} -2 & 4 + d & (\sin\theta) - 2 \\ 1 & (\sin\theta) + 2 & d \\ 5 & (2\sin\theta) - d & (-\sin\theta) + 2 + 2d \end{bmatrix},$$

$\theta \in [\theta, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is

(a) -5 (b) -7

(c) $2(\sqrt{2} + 1)$ (d) $2(\sqrt{2} + 2)$

26 If $5, 5r, 5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to

(a) $\frac{5}{4}$ (b) $\frac{7}{4}$

(c) $\frac{3}{2}$ (d) $\frac{3}{4}$

27 The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is

(a) $\frac{3\pi}{8}$ (b) $\frac{5\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

28 The plane passing through the point $(4, -1, 2)$ and parallel to the lines $\frac{x+2}{3} = \frac{y-2}{-1} = \frac{z+1}{2}$ and $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-4}{3}$ also passes through the point

(a) $(-1, -1, -1)$ (b) $(1, 1, -1)$
(c) $(1, 1, 1)$ (d) $(-1, -1, 1)$

29 The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}$, $(x > 0)$, is

(a) $\frac{3}{2}$ (b) $\frac{5}{4}$
(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{5}}{2}$

30 For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then,

$$\lim_{x \to 1+} \frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|\,[1-x]}$$

(a) equals 0 (b) does not exist
(c) equals -1 (d) equals 1

ANSWERS

1 (c)	2 (a)	3 (a)	4 (c)	5 (c)	6 (c)	7 (*)	8 (a)	9 (a)	10 (c)
11 (d)	12 (c)	13 (c)	14 (a)	15 (c)	16 (b)	17 (a)	18 (a)	19 (b)	20 (d)
21 (a)	22 (a)	23 (a)	24 (a)	25 (a)	26 (b)	27 (c)	28 (c)	29 (d)	30 (a)

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10 January, Shift-II

1 The positive value of λ for which the coefficient of x^2 in the expression $x^2\left(\sqrt{x} + \frac{\lambda}{x^2}\right)^{10}$ is 720, is

(a) 3 (b) $\sqrt{5}$
(c) $2\sqrt{2}$ (d) 4

2 Let $S = \left\{(x, y) \in R^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1\right\}$, where $r \neq \pm 1$. Then, S represents

(a) a hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$.
(b) a hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0 < r < 1$.
(c) an ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$.
(d) an ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, when $r > 1$.

3 With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is

(a) 7 : 1 (b) 3 : 1
(c) 9 : 7 (d) 5 : 3

4 Let $\alpha = (\lambda - 2)\,\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\,\mathbf{a} + 3\mathbf{b}$ be two given vectors where vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. The value of λ for which vectors α and β are collinear, is

(a) 4 (b) -3
(c) 3 (d) -4

5 A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then, this nearest distance is

(a) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (b) $\frac{\sqrt{5}}{6}$
(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

6 The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$, where $[t]$ denotes the greatest integer less than or equal to t, is

(a) $\frac{1}{12}(7\pi - 5)$ (b) $\frac{1}{12}(7\pi + 5)$
(c) $\frac{3}{10}(4\pi - 3)$ (d) $\frac{3}{20}(4\pi - 3)$

7 If the probability of hitting a target by a shooter in any shot, is $\frac{1}{3}$, then the minimum number of independent shots at the target required by him so that the probability of hitting the target at least once is greater than $\frac{5}{6}$, is

(a) 6 (b) 3
(c) 5 (d) 4

8 Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at $(2, 4)$, then one of its vertex is

(a) $(3, 6)$ (b) $(2, 6)$
(c) $(2, 1)$ (d) $(3, 5)$

9 The value of $\cos\frac{\pi}{2^2} \cdot \cos\frac{\pi}{2^3} \ldots\ldots \cos\frac{\pi}{2^{10}} \cdot \sin\frac{\pi}{2^{10}}$ is

(a) $\frac{1}{1024}$ (b) $\frac{1}{2}$
(c) $\frac{1}{512}$ (d) $\frac{1}{256}$

10 Let N be the set of natural numbers and two functions f and g be defined as $f, g : N \longrightarrow N$ such that

$$f(n) = \begin{cases} \frac{n+1}{2}; & \text{if } n \text{ is odd} \\ \frac{n}{2}; & \text{if } n \text{ is even} \end{cases}$$

and $g(n) = n - (-1)^n$. Then, fog is

(a) one-one but not onto
(b) onto but not one-one
(c) both one-one and onto
(d) neither one-one nor onto

11 If $\sum_{r=0}^{25} \{{}^{50}C_r \cdot {}^{50-r}C_{25-r}\} = K({}^{50}C_{25})$, then, K is equal to

(a) 2^{24}
(b) $2^{25} - 1$
(c) 2^{25}
(d) $(25)^2$

12 Let f be a differentiable function such that $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}$, $(x > 0)$ and $f(1) \neq 4$. Then, $\lim_{x \to 0^+} x f\left(\frac{1}{x}\right)$

(a) does not exist
(b) exists and equals $\frac{4}{7}$
(c) exists and equals 0
(d) exists and equals 4

13 Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$, where $b > 0$.

Then, the minimum value of $\frac{\det(A)}{b}$ is

(a) $-\sqrt{3}$ (b) $-2\sqrt{3}$
(c) $2\sqrt{3}$ (d) $\sqrt{3}$

14 Let $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$. If $R(z)$ and $I(z)$ respectively denote the real and imaginary parts of z, then

(a) $R(z) > 0$ and $I(z) > 0$
(b) $I(z) = 0$
(c) $R(z) < 0$ and $I(z) > 0$
(d) $R(z) = -3$

15 Two vertices of a triangle are (0, 2) and (4, 3). If its orthocentre is at the origin, then its third vertex lies in which quadrant?

(a) Fourth (b) Third
(c) Second (d) First

16 If $\int x^5 e^{-4x^3}\, dx = \frac{1}{48} e^{-4x^3} f(x) + C$, where C is a constant of integration, then $f(x)$ is equal to

(a) $-4x^3 - 1$ (b) $4x^3 + 1$
(c) $-2x^3 - 1$ (d) $-2x^3 + 1$

17 The value of $\cot\left(\sum_{n=1}^{19} \cot^{-1}\left(1 + \sum_{p=1}^{n} 2p\right)\right)$ is

(a) $\frac{23}{22}$ (b) $\frac{21}{19}$ (c) $\frac{19}{21}$ (d) $\frac{22}{23}$

18 The tangent to the curve, $y = xe^{x^2}$ passing through the point $(1, e)$ also passes through the point

(a) $\left(\frac{4}{3}, 2e\right)$ (b) $(3, 6e)$
(c) $(2, 3e)$ (d) $\left(\frac{5}{3}, 2e\right)$

19 On which of the following lines lies the point of intersection of the line, $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane, $x + y + z = 2$?

(a) $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-5}{-1}$
(b) $\frac{x+3}{3} = \frac{4-y}{3} = \frac{z+1}{-2}$
(c) $\frac{x-2}{2} = \frac{y-3}{2} = \frac{z+3}{3}$
(d) $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+4}{-5}$

20 Let $f : (-1, 1) \longrightarrow R$ be a function defined by $f(x) = \max\{-|x|, -\sqrt{1-x^2}\}$. If K be the set of all points at which f is not differentiable, then K has exactly

(a) three elements (b) five elements
(c) two elements (d) one element

21 The curve amongst the family of curves represented by the differential equation, $(x^2 - y^2)dx + 2xydy = 0$, which passes through (1, 1), is

(a) a circle with centre on the Y-axis
(b) a circle with centre on the X-axis
(c) an ellipse with major axis along the Y-axis
(d) a hyperbola with transverse axis along the X-axis.

22 If $\int_0^x f(t)\, dt = x^2 + \int_x^1 t^2 f(t)dt$, then $f'\left(\frac{1}{2}\right)$ is

(a) $\frac{24}{25}$ (b) $\frac{18}{25}$
(c) $\frac{6}{25}$ (d) $\frac{4}{5}$

23 The number of values of $\theta \in (0, \pi)$ for which the system of linear equations

$$x + 3y + 7z = 0,$$
$$-x + 4y + 7z = 0,$$
$$(\sin 3\theta)x + (\cos 2\theta)y + 2z = 0$$

has a non-trivial solution, is

(a) two (b) three
(c) four (d) one

24 The plane which bisects the line segment joining the points (–3, – 3, 4) and (3, 7, 6) at right angles, passes through which one of the following points ?

(a) (4, – 1, 7) (b) (2, 1, 3)
(c) (–2, 3, 5) (d) (4, 1, – 2)

25 If mean and standard deviation of 5 observations x_1, x_2, x_3, x_4, x_5 are 10 and 3, respectively, then the variance of 6 observations $x_1, x_2, \ldots, x_5$ and – 50 is equal to

(a) 507.5 (b) 586.5
(c) 582.5 (d) 509.5

26 Let $a_1, a_2, a_3 \ldots, a_{10}$ be in GP with $a_i > 0$ for $i = 1, 2, \ldots, 10$ and S be the set of pairs (r, k), $r, k \in N$ (the set of natural numbers) for which

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$$

Then, the number of elements in S, is

(a) 4 (b) 2
(c) 10 (d) infinitely many

27 Consider the following three statements:

P : 5 is a prime number.
Q : 7 is a factor of 192.
R : LCM of 5 and 7 is 35.

Then, the truth value of which one of the following statements is true ?

(a) $(P \wedge Q) \vee (\sim R)$ (b) $P \vee (\sim Q \wedge R)$
(c) $(\sim P) \vee (Q \wedge R)$ (d) $(\sim P) \wedge (\sim Q \wedge R)$

28 The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is

(a) $8\sqrt{2}$ (b) $2\sqrt{11}$
(c) $3\sqrt{2}$ (d) $6\sqrt{3}$

29 If the area of an equilateral triangle inscribed in the circle,

$$x^2 + y^2 + 10x + 12y + c = 0$$

is $27\sqrt{3}$ sq units, then c is equal to

(a) 20 (b) –25
(c) 13 (d) 25

30 The value of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is

(a) $\frac{4}{9}$ (b) 1
(c) $\frac{15}{8}$ (d) 2

ANSWERS

1. (d)	2. (c)	3. (a)	4. (d)	5. (c)	6. (d)	7. (c)	8. (a)	9. (c)	10. (b)
11. (c)	12. (d)	13. (c)	14. (b)	15. (c)	16. (a)	17. (b)	18. (a)	19. (d)	20. (a)
21. (b)	22. (a)	23. (a)	24. (d)	25. (a)	26. (d)	27. (b)	28. (d)	29. (d)	30. (d)

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11 January, Shift-I

1 Let $[x]$ denote the greatest integer less than or equal to x. Then,

$$\lim_{x\to 0}\frac{\tan(\pi\sin^2 x)+(|x|-\sin(x[x]))^2}{x^2}$$

(a) equals π (b) equals $\pi+1$
(c) equals 0 (d) does not exist

2 Let $f(x)=\begin{cases}-1, & -2\le x<0\\ x^2-1, & 0\le x\le 2\end{cases}$ and

$g(x)=|f(x)|+f(|x|)$. Then, in the interval $(-2, 2)$, g is
(a) not differentiable at one point
(b) not differentiable at two points
(c) differentiable at all points
(d) not continuous

3 If the system of linear equations

$$2x+2y+3z=a$$
$$3x-y+5z=b$$
$$x-3y+2z=c$$

where a, b, c are non-zero real numbers, has more than one solution, then
(a) $b-c-a=0$ (b) $a+b+c=0$
(c) $b-c+a=0$ (d) $b+c-a=0$

4 The area (in sq units) of the region bounded by the curve $x^2=4y$ and the straight line $x=4y-2$ is
(a) $\frac{7}{8}$ (b) $\frac{9}{8}$
(c) $\frac{5}{4}$ (d) $\frac{3}{4}$

5 If $\int\frac{\sqrt{1-x^2}}{x^4}dx=A(x)(\sqrt{1-x^2})^m+C$,

for a suitable chosen integer m and a function $A(x)$, where C is a constant of integration, then $(A(x))^m$ equals
(a) $\frac{1}{9x^4}$ (b) $\frac{-1}{3x^3}$ (c) $\frac{-1}{27x^9}$ (d) $\frac{1}{27x^6}$

6 The maximum value of the function $f(x)=3x^3-18x^2+27x-40$ on the set $S=\{x\in R: x^2+30\le 11x\}$ is
(a) 122 (b) -122
(c) -222 (d) 222

7 Let $f: R\to R$ be defined by $f(x)=\frac{x}{1+x^2}$, $x\in R$. Then, the range of f is
(a) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (b) $(-1, 1)-\{0\}$
(c) $R-\left[-\frac{1}{2},\frac{1}{2}\right]$ (d) $R-[-1, 1]$

8 The outcome of each of 30 items was observed ; 10 items gave an outcome $\frac{1}{2}-d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2}+d$ each. If the variance of this outcome data is $\frac{4}{3}$, then $|d|$ equals
(a) $\frac{2}{3}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\sqrt{2}$ (d) 2

9 Two integers are selected at random from the set $\{1, 2, \ldots\ldots, 11\}$. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is
(a) $\frac{2}{5}$ (b) $\frac{1}{2}$ (c) $\frac{7}{10}$ (d) $\frac{3}{5}$

10 The straight line $x+2y=1$ meets the coordinate axes at A and B. A circle is drawn through A, B and the origin. Then, the sum of perpendicular distances from A and B on the tangent to the circle at the origin is
(a) $2\sqrt{5}$ (b) $\frac{\sqrt{5}}{4}$ (c) $4\sqrt{5}$ (d) $\frac{\sqrt{5}}{2}$

11 The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is
(a) 4 (b) 0 (c) 6 (d) 8

12 Let $a_1, a_2, \ldots, a_{10}$ be a GP. If $\frac{a_3}{a_1}=25$, then $\frac{a_9}{a_5}$ equals
(a) 5^3 (b) $2(5^2)$ (c) $4(5^2)$ (d) 5^4

13 Let $\left(-2-\frac{1}{3}i\right)^3=\frac{x+iy}{27}$ $(i=\sqrt{-1})$,

where x and y are real numbers, then $y-x$ equals
(a) 91 (b) 85
(c) -85 (d) -91

14 If $x\log_e(\log_e x)-x^2+y^2=4(y>0)$, then $\frac{dy}{dx}$ at $x=e$ is equal to
(a) $\frac{e}{\sqrt{4+e^2}}$ (b) $\frac{(2e-1)}{2\sqrt{4+e^2}}$
(c) $\frac{(1+2e)}{\sqrt{4+e^2}}$ (d) $\frac{(1+2e)}{2\sqrt{4+e^2}}$

15 A square is inscribed in the circle $x^2+y^2-6x+8y-103=0$ with its sides parallel to the coordinate axes. Then, the distance of the vertex of this square which is nearest to the origin is
(a) 6 (b) 13 (c) $\sqrt{41}$ (d) $\sqrt{137}$

16 If q is false and $p\wedge q\longleftrightarrow r$ is true, then which one of the following statements is a tautology?
(a) $p\vee r$ (b) $(p\wedge r)\to(p\vee r)$
(c) $(p\vee r)\to(p\wedge r)$ (d) $p\wedge r$

17 If $y(x)$ is the solution of the differential equation

$\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}$, $x>0$,

where $y(1)=\frac{1}{2}e^{-2}$, then
(a) $y(x)$ is decreasing in $\left(\frac{1}{2}, 1\right)$
(b) $y(x)$ is decreasing in $(0, 1)$
(c) $y(\log_e 2)=\log_e 4$
(d) $y(\log_e 2)=\frac{\log_e 2}{4}$

18 The value of r for which

$${}^{20}C_r\,{}^{20}C_0+{}^{20}C_{r-1}\,{}^{20}C_1+{}^{20}C_{r-2}\,{}^{20}C_2+\ldots+{}^{20}C_0\,{}^{20}C_r$$

is maximum, is
(a) 15 (b) 10
(c) 11 (d) 20

19 Two circles with equal radii are intersecting at the points (0, 1) and (0, –1). The tangent at the point (0,1) to one of the circles passes through the centre of the other circle. Then, the distance between the centres of these circles is
(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) 1 (d) 2

20 Equation of a common tangent to the parabola $y^2=4x$ and the hyperbola $xy=2$ is
(a) $x+2y+4=0$ (b) $x-2y+4=0$
(c) $4x+2y+1=0$ (d) $x+y+1=0$

21 The value of the integral $\int_{-2}^{2}\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$

(where, $[x]$ denotes the greatest integer less than or equal to x) is
(a) $4-\sin 4$ (b) 4
(c) $\sin 4$ (d) 0

22 If one real root of the quadratic equation $81x^2+kx+256=0$ is cube of the other root, then a value of k is
(a) 100 (b) 144
(c) -81 (d) -300

23 The direction ratios of normal to the plane through the points (0, – 1, 0) and (0, 0, 1) and making an angle $\frac{\pi}{4}$ with the plane $y - z + 5 = 0$ are

(a) 2, – 1, 1 (b) $\sqrt{2}$, 1, – 1
(c) 2, $\sqrt{2}$, – $\sqrt{2}$ (d) $2\sqrt{3}$, 1, – 1

24 In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is

(a) $\frac{c}{3}$ (b) $\frac{c}{\sqrt{3}}$ (c) $\frac{3}{2}y$ (d) $\frac{y}{\sqrt{3}}$

25 The plane containing the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-1}{3}$ and also containing its projection on the plane $2x + 3y - z = 5$, contains which one of the following points?

(a) (–2, 2, 2) (b) (2, 2, 0)
(c) (2, 0, – 2) (d) (0,– 2,2)

26 Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I_3$, then $|p|$ is

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

27 If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices, then the mid-points of the tangents intercepted between the coordinate axes lie on the curve

(a) $\frac{x^2}{4} + \frac{y^2}{2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
(c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

28 Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for $k = 1, 2, 3 \ldots$. Then, for all $x \in R$, the value of $f_4(x) - f_6(x)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) $\frac{-1}{12}$ (d) $\frac{1}{4}$

29 The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{27}{19}$. Then, the common ratio of this series is

(a) $\frac{4}{9}$ (b) $\frac{2}{3}$ (c) $\frac{2}{9}$ (d) $\frac{1}{3}$

30 Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + (\lambda^2 - 1)\hat{\mathbf{k}}$ be coplanar vectors. Then, the non-zero vector $\mathbf{a} \times \mathbf{c}$ is

(a) $-10\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (b) $-10\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$
(c) $-14\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ (d) $-14\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

ANSWERS

1 (d)	2 (a)	3 (a)	4 (b)	5 (c)	6 (a)	7 (a)	8 (c)	9 (a)	10 (d)
11 (b)	12 (d)	13 (a)	14 (b)	15 (c)	16 (b)	17 (a)	18 (d)	19 (d)	20 (a)
21 (d)	22 (d)	23 (b,c)	24 (b)	25 (c)	26 (b)	27 (d)	28 (a)	29 (b)	30 (a)

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11 January, Shift-II

1 Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$, $x \in R$, where a, b and d are non-zero real constants. Then,

(a) f is an increasing function of x
(b) f' is not a continuous function of x
(c) f is a decreasing function of x
(d) f is neither increasing nor decreasing function of x

2 Let K be the set of all real values of x, where the function $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$ is not differentiable. Then, the set K is equal to

(a) {0} (b) ϕ (an empty set)
(c) {π} (d) {0, π}

3 Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$).

Then, $|z|$ is equal to

(a) $\frac{\sqrt{34}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{\sqrt{41}}{4}$ (d) $\frac{5}{4}$

4 Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1 + x^{2m})(1 + y^{2n})}$ is

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{1}{4}$ (d) $\frac{m + n}{6mn}$

5 Let a function $f : (0, \infty) \longrightarrow (0, \infty)$ be defined by $f(x) = \left|1 - \frac{1}{x}\right|$. Then, f is

(a) injective only
(b) both injective as well as surjective
(c) not injective but it is surjective
(d) neither injective nor surjective

6 Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, for all $x \in R$; then $\frac{a_2}{a_0}$ is equal to

(a) 12.25 (b) 12.50
(c) 12.00 (d) 12.75

7 Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{16}$ (d) 16

8 The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals

(a) $\frac{1}{5}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$
(b) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$
(c) $\frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$
(d) $\frac{\pi}{40}$

9 The area (in sq units) in the first quadrant bounded by the parabola, $y = x^2 + 1$, the tangent to it at the point (2, 5) and the coordinate axes is

(a) $\frac{14}{3}$ (b) $\frac{187}{24}$
(c) $\frac{8}{3}$ (d) $\frac{37}{24}$

10 If 19th term of a non-zero AP is zero, then its (49th term) : (29th term) is

(a) 1 : 3 (b) 4 : 1
(c) 2 : 1 (d) 3 : 1

11 If the point $(2, \alpha, \beta)$ lies on the plane which passes through the points (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x - 5y = 15$, then $2\alpha - 3\beta$ is equal to

(a) 17 (b) 7
(c) 5 (d) 12

12 A circle cuts a chord of length $4a$ on the X-axis and passes through a point on the Y-axis, distant $2b$ from the origin. Then, the locus of the centre of this circle, is
(a) a parabola (b) an ellipse
(c) a straight line (d) a hyperbola

13 Let $S_n = 1 + q + q^2 + \ldots + q^n$ and
$T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$,
where q is a real number and $q \neq 1$. If
${}^{101}C_1 + {}^{101}C_2 \cdot S_1 + \ldots + {}^{101}C_{101} \cdot S_{100} = \alpha T_{100}$, then α is equal to
(a) 2^{100} (b) 202 (c) 200 (d) 2^{99}

14 All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval
(a) $(-\infty, \cot 5) \cup (\cot 2, \infty)$
(b) $(\cot 5, \cot 4)$
(c) $(\cot 2, \infty)$
(d) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

15 The number of functions f from $\{1, 2, 3, \ldots, 20\}$ onto $\{1, 2, 3, \ldots, 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is
(a) $(15)! \times 6!$ (b) $5^6 \times 15$
(c) $5! \times 6!$ (d) $6^5 \times (15)!$

16 Let the length of the latus rectum of an ellipse with its major axis along X-axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it?
(a) $(4\sqrt{2}, 2\sqrt{3})$ (b) $(4\sqrt{3}, 2\sqrt{2})$
(c) $(4\sqrt{2}, 2\sqrt{2})$ (d) $(4\sqrt{3}, 2\sqrt{3})$

17 Let α and β be the roots of the quadratic equation
$x^2 \sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$
$(0 < \theta < 45°)$ and $\alpha < \beta$. Then,
$\sum_{n=0}^{\infty}\left(\alpha^n + \frac{(-1)^n}{\beta^n}\right)$ is equal to
(a) $\frac{1}{1-\cos\theta} - \frac{1}{1+\sin\theta}$
(b) $\frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$
(c) $\frac{1}{1+\cos\theta} - \frac{1}{1-\sin\theta}$
(d) $\frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta}$

18 If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$
$= (a+b+c)(x+a+b+c)^2$, $x \neq 0$ and $a+b+c \neq 0$, then x is equal to
(a) $-(a+b+c)$ (b) $-2(a+b+c)$
(c) $2(a+b+c)$ (d) abc

19 The solution of the differential equation, $\frac{dy}{dx} = (x-y)^2$, when $y(1) = 1$, is
(a) $\log_e\left|\frac{2-y}{2-x}\right| = 2(y-1)$
(b) $-\log_e\left|\frac{1+x-y}{1-x+y}\right| = x + y - 2$
(c) $\log_e\left|\frac{2-x}{2-y}\right| = x - y$
(d) $-\log_e\left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

20 Contrapositive of the statement "If two numbers are not equal, then their squares are not equal" is
(a) If the squares of two numbers are not equal, then the numbers are not equal.
(b) If the squares of two numbers are equal, then the numbers are equal.
(c) If the squares of two numbers are not equal, then the numbers are equal.
(d) If the squares of two numbers are equal, then the numbers are not equal.

21 $\lim_{x \to 0} \frac{x\cot(4x)}{\sin^2 x \cot^2(2x)}$ is equal to
(a) 0 (b) 1 (c) 4 (d) 2

22 A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X be the number of white balls drawn, then
$\left(\frac{\text{mean of } X}{\text{standard deviation of } X}\right)$ is equal to
(a) $\frac{4\sqrt{3}}{3}$ (b) 4 (c) $3\sqrt{2}$ (d) $4\sqrt{3}$

23 If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and Y-axis, is 250 sq units, then a value of 'a' is
(a) $5\sqrt{5}$ (b) 5
(c) $5(2^{1/3})$ (d) $(10)^{2/3}$

24 Two lines $\frac{x-3}{1} = \frac{y+1}{3} = \frac{z-6}{-1}$ and $\frac{x+5}{7} = \frac{y-2}{-6} = \frac{z-3}{4}$ intersect at the point R. The reflection of R in the xy-plane has coordinates
(a) $(2, -4, -7)$ (b) $(2, -4, 7)$
(c) $(-2, 4, 7)$ (d) $(2, 4, 7)$

25 If $\int \frac{x+1}{\sqrt{2x-1}}\,dx = f(x)\sqrt{2x-1} + C$,
where C is a constant of integration, then $f(x)$ is equal to
(a) $\frac{2}{3}(x+2)$ (b) $\frac{1}{3}(x+4)$
(c) $\frac{2}{3}(x-4)$ (d) $\frac{1}{3}(x+1)$

26 If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is
(a) $\frac{13}{12}$ (b) 2 (c) $\frac{13}{8}$ (d) $\frac{13}{6}$

27 Let $S = \{1, 2, \ldots, 20\}$. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then, the probability that a randomly chosen subset of S is "nice", is
(a) $\frac{6}{2^{20}}$ (b) $\frac{4}{2^{20}}$ (c) $\frac{7}{2^{20}}$ (d) $\frac{5}{2^{20}}$

28 If in a parallelogram $ABDC$, the coordinates of A, B and C are respectively (1, 2), (3, 4) and (2, 5), then the equation of the diagonal AD is
(a) $3x + 5y - 13 = 0$ (b) $3x - 5y + 7 = 0$
(c) $5x - 3y + 1 = 0$ (d) $5x + 3y - 11 = 0$

29 Let $\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}$ and $\beta\hat{\mathbf{i}} + (1-\beta)\hat{\mathbf{j}}$ respectively be the position vectors of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is
(a) 1 (b) 3 (c) 4 (d) 2

30 Given, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value
(a) (19, 7, 25) (b) (3, 4, 5)
(c) (5, 12, 13) (d) (7, 19, 25)

ANSWERS

1. (a)	2. (b)	3. (b)	4. (c)	5. (d)	6. (a)	7. (c)	8. (c)	9. (d)	10. (d)
11. (b)	12. (a)	13. (a)	14. (c)	15. (a)	16. (b)	17. (b)	18. (b)	19. (d)	20. (b)
21. (b)	22. (d)	23. (b)	24. (a)	25. (b)	26. (a)	27. (d)	28. (c)	29. (a)	30. (d)

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12 January, Shift-I

1 Let $P(4, -4)$ and $Q(9, 6)$ be two points on the parabola, $y^2 = 4x$ and let X be any point on the arc POQ of this parabola, where O is the vertex of this parabola, such that the area of ΔPXQ is maximum. Then, this maximum area (in sq units) is

(a) $\frac{125}{2}$ (b) $\frac{75}{2}$ (c) $\frac{625}{4}$ (d) $\frac{125}{4}$

2 Consider three boxes, each containing 10 balls labelled 1, 2, …, 10. Suppose one ball is randomly drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the ith box, $(i = 1, 2, 3)$. Then, the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is

(a) 82 (b) 120 (c) 240 (d) 164

3 Let $y = y(x)$ be the solution of the differential equation, $x\frac{dy}{dx} + y = x\log_e x$, $(x > 1)$. If $2y(2) = \log_e 4 - 1$, then $y(e)$ is equal to

(a) $-\frac{e}{2}$ (b) $-\frac{e^2}{2}$ (c) $\frac{e}{4}$ (d) $\frac{e^2}{4}$

4 The sum of the distinct real values of μ, for which the vectors, $\mu\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \mu\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ are coplanar, is

(a) 2 (b) 0 (c) 1 (d) – 1

5 Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$ respectively. If P and Q are the points of intersection of these circles, then the area (in sq units) of the quadrilateral PC_1QC_2 is

(a) 8 (b) 4 (c) 6 (d) 9

6 If the straight line, $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points (7, 17) and $(15, \beta)$, then β equals

(a) $\frac{35}{3}$ (b) – 5 (c) $-\frac{35}{3}$ (d) 5

7 A ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is

(a) $1 : 2(6)^{1/3}$ (b) $1 : 4(16)^{1/3}$

(c) $4(36)^{1/3} : 1$ (d) $2(36)^{\frac{1}{3}} : 1$

8 Let $S = \{1, 2, 3, \ldots, 100\}$. The number of non-empty subsets A of S such that the product of elements in A is even, is

(a) $2^{50}(2^{50} - 1)$ (b) $2^{50} - 1$

(c) $2^{50} + 1$ (d) $2^{100} - 1$

9 The maximum area (in sq. units) of a rectangle having its base on the X-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is

(a) 36 (b) $20\sqrt{2}$ (c) 32 (d) $18\sqrt{3}$

10 The integral $\int \cos(\log_e x)\, dx$ is equal to (where C is a constant of integration)

(a) $\frac{x}{2}\left[\cos(\log_e x) + \sin(\log_e x)\right] + C$

(b) $x\left[\cos(\log_e x) + \sin(\log_e x)\right] + C$

(c) $x\left[\cos(\log_e x) - \sin(\log_e x)\right] + C$

(d) $\frac{x}{2}\left[\sin(\log_e x) - \cos(\log_e x)\right] + C$

11 $\lim_{x \to \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ is

(a) $4\sqrt{2}$ (b) 4 (c) 8 (d) $8\sqrt{2}$

12 An ordered pair (α, β) for which the system of linear equations

$$(1 + \alpha)x + \beta y + z = 2$$
$$\alpha x + (1 + \beta)y + z = 3$$
$$\alpha x + \beta y + 2z = 2$$

has a unique solution, is

(a) (2, 4) (b) (– 4, 2)

(c) (1, – 3) (d) (–3, 1)

13 If the sum of the deviations of 50 observations from 30 is 50, then the mean of these observations is

(a) 50 (b) 30

(c) 51 (d) 31

14 For $x > 1$, if $(2x)^{2y} = 4e^{2x - 2y}$, then $(1 + \log_e 2x)^2 \frac{dy}{dx}$ is equal to

(a) $\frac{x\log_e 2x + \log_e 2}{x}$

(b) $\frac{x\log_e 2x - \log_e 2}{x}$

(c) $x\log_e 2x$

(d) $\log_e 2x$

15 The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is

(a) $\frac{\sqrt{79}}{2}$ (b) $\sqrt{34}$ (c) $\sqrt{31}$ (d) $\sqrt{19}$

16 The area (in sq units) of the region bounded by the parabola, $y = x^2 + 2$ and lines, $y = x + 1$, $x = 0$ and $x = 3$, is

(a) $\frac{15}{2}$ (b) $\frac{17}{4}$ (c) $\frac{21}{2}$ (d) $\frac{15}{4}$

17 The Boolean expression $((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to

(a) $p \wedge q$ (b) $p \vee (\sim q)$

(c) $p \wedge (\sim q)$ (d) $(\sim p) \wedge (\sim q)$

18 If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

and $x^2 + y^2 - 18x - 2y + 78 = 0$

are on its opposite sides, then the set of all values of λ is the interval

(a) [13, 23] (b) (2, 17)

(c) [12, 21] (d) (23, 31)

19 The perpendicular distance from the origin to the plane containing the two lines,

$\frac{x+2}{3} = \frac{y-2}{5} = \frac{z+5}{7}$

and $\frac{x-1}{1} = \frac{y-4}{4} = \frac{z+4}{7}$, is

(a) $11\sqrt{6}$ (b) $\frac{11}{\sqrt{6}}$ (c) 11 (d) $6\sqrt{11}$

20 If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m-4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is

(a) $-2 + \sqrt{2}$ (b) $4 - 2\sqrt{3}$

(c) $4 - 3\sqrt{2}$ (d) $2 - \sqrt{3}$

21 Considering only the principal values of inverse functions, the set

$A = \left\{x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}\right\}$

(a) is an empty set

(b) is a singleton

(c) contains more than two elements

(d) contains two elements

22 If the vertices of a hyperbola be at (–2, 0) and (2, 0) and one of its foci be at (–3, 0), then which one of the following points does not lie on this hyperbola?

(a) $(2\sqrt{6}, 5)$ (b) $(6, 5\sqrt{2})$

(c) $(4, \sqrt{15})$ (d) $(-6, 2\sqrt{10})$

23 If $\frac{z - \alpha}{z + \alpha}$ $(\alpha \in R)$ is a purely imaginary number and $|z| = 2$, then a value of α is

(a) $\sqrt{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

24 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then, $\frac{q_{21} + q_{31}}{q_{32}}$ is equal to

(a) 10 (b) 135 (c) 9 (d) 15

25 In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to

(a) $\frac{175}{6^5}$ (b) $\frac{225}{6^5}$ (c) $\frac{200}{6^5}$ (d) $\frac{150}{6^5}$

26 Let S be the set of all points in $(-\pi, \pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then, S is a subset of which of the following?

(a) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

(b) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$

(c) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

(d) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

27 A tetrahedron has vertices $P(1, 2, 1)$, $Q(2, 1, 3)$, $R(-1, 1, 2)$ and $O(0, 0, 0)$. The angle between the faces OPQ and PQR is

(a) $\cos^{-1}\left(\frac{7}{31}\right)$ (b) $\cos^{-1}\left(\frac{9}{35}\right)$

(c) $\cos^{-1}\left(\frac{19}{35}\right)$ (d) $\cos^{-1}\left(\frac{17}{31}\right)$

28 The product of three consecutive terms of a GP is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an AP. Then, the sum of the original three terms of the given GP is

(a) 36 (b) 28

(c) 32 (d) 24

29 Let $S_k = \frac{1 + 2 + 3 + \ldots + k}{k}$. If $S_1^2 + S_2^2 + \ldots + S_{10}^2 = \frac{5}{12}A$, then A is equal to

(a) 156 (b) 301

(c) 283 (d) 303

30 Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$, then $\int_0^a f(x)\, g(x)\, dx$ is equal to

(a) $4\int_0^a f(x)\, dx$ (b) $\int_0^a f(x)\, dx$

(c) $2\int_0^a f(x)\, dx$ (d) $-3\int_0^a f(x)\, dx$

ANSWERS

1. (d)	2. (b)	3. (c)	4. (d)	5. (b)	6. (d)	7. (c)	8. (a)	9. (c)	10. (a)
11. (c)	12. (a)	13. (d)	14. (b)	15. (d)	16. (a)	17. (d)	18. (c)	19. (b)	20. (c)
21. (b)	22. (b)	23. (d)	24. (a)	25. (a)	26. (c)	27. (c)	28. (b)	29. (d)	30. (c)

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12 January, Shift-II

1 $\lim_{x \to 1-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1 - x}}$ is equal to

(a) $\sqrt{\frac{\pi}{2}}$ (b) $\sqrt{\frac{2}{\pi}}$

(c) $\sqrt{\pi}$ (d) $\frac{1}{\sqrt{2\pi}}$

2 Let S be the set of all real values of λ such that a plane passing through the points $(-\lambda^2, 1, 1)$, $(1, -\lambda^2, 1)$ and $(1, 1, -\lambda^2)$ also passes through the point $(-1, -1, 1)$. Then, S is equal to

(a) $\{\sqrt{3}, -\sqrt{3}\}$ (b) $\{3, -3\}$

(c) $\{1, -1\}$ (d) $\{\sqrt{3}\}$

3 If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is

(a) $(x^2 + y^2)^2 = 4R^2x^2y^2$

(b) $(x^2 + y^2)^3 = 4R^2x^2y^2$

(c) $(x^2 + y^2)(x + y) = R^2xy$

(d) $(x^2 + y^2)^2 = 4Rx^2y^2$

4 If an angle between the line, $\frac{x + 1}{2} = \frac{y - 2}{1} = \frac{z - 3}{-2}$ and the plane, $x - 2y - kz = 3$ is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$, then a value of k is

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{3}{5}}$

(c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

5 The expression $\sim(\sim p \to q)$ is logically equivalent to

(a) $p \wedge \sim q$ (b) $p \wedge q$

(c) $\sim p \wedge q$ (d) $\sim p \wedge \sim q$

6 The set of all values of λ for which the system of linear equations
$x - 2y - 2z = \lambda x$, $x + 2y + z = \lambda y$ and $-x - y = \lambda z$
has a non-trivial solution

(a) contains exactly two elements.

(b) contains more than two elements.

(c) is a singleton.

(d) is an empty set.

7 Let Z be the set of integers. If $A = \{x \in Z : 2^{(x+2)(x^2 - 5x + 6)} = 1\}$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is

(a) 2^{12} (b) 2^{18}

(c) 2^{15} (d) 2^{10}

8 In a class of 60 students, 40 opted for NCC, 30 opted for NSS and 20 opted for both NCC and NSS. If one of these students is selected at random, then the probability that the student selected has opted neither for NCC nor for NSS is

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{5}{6}$

9 If a curve passes through the point $(1, -2)$ and has slope of the tangent at any point (x, y) on it as $\frac{x^2 - 2y}{x}$, then the curve also passes through the point

(a) $(\sqrt{3}, 0)$ (b) $(-1, 2)$

(c) $(-\sqrt{2}, 1)$ (d) $(3, 0)$

10 $\lim_{n \to \infty}\left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \frac{n}{n^2 + 3^2} + \ldots + \frac{1}{5n}\right)$ is equal to

(a) $\tan^{-1}(3)$ (b) $\tan^{-1}(2)$

(c) $\pi/4$ (d) $\pi/2$

11 In a game, a man wins ₹100 if he gets 5 or 6 on a throw of a fair die and loses ₹50 for getting any other number on the die. If he decides to throw the die either till he gets a five or a six or to a maximum of three

throws, then his expected gain/loss (in rupees) is

(a) $\frac{400}{3}$ loss (b) $\frac{400}{9}$ loss

(c) 0 (d) $\frac{400}{3}$ gain

12 If $\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cos\beta$; $\alpha, \beta \in [0, \pi]$, then $\cos(\alpha + \beta) - \cos(\alpha - \beta)$ is equal to

(a) -1 (b) $\sqrt{2}$ (c) $-\sqrt{2}$ (d) 0

13 If the function f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x) - 14}{(x-1)^2} = 0\ (x \neq 1)$ is

(a) -7 (b) 6 (c) 7 (d) 5

14 If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in AP, then n can be

(a) 9 (b) 11 (c) 14 (d) 12

15 Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = f(x)$ for all $x \in R$. If $h(x) = f(f(x))$, then $h'(1)$ is equal to

(a) $4e^2$ (b) $4e$ (c) $2e$ (d) $2e^2$

16 The mean and the variance of five observations are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4, then the absolute value of the difference of the other two observations, is

(a) 1 (b) 7 (c) 5 (d) 3

17 The number of integral values of m for which the quadratic expression, $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$, is always positive, is

(a) 6 (b) 8 (c) 7 (d) 3

18 The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is

(a) 49 (b) 48 (c) 54 (d) 55

19 If $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$; then for all $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, det($A$) lies in the interval

(a) $\left(\frac{3}{2}, 3\right]$ (b) $\left[\frac{5}{2}, 4\right)$

(c) $\left(0, \frac{3}{2}\right]$ (d) $\left(1, \frac{5}{2}\right]$

20 Let **a**, **b** and **c** be three unit vectors, out of which vectors **b** and **c** are non-parallel. If α and β are the angles which vector **a** makes with vectors **b** and **c** respectively and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}\mathbf{b}$, then $|\alpha - \beta|$ is equal to

(a) 30° (b) 45°

(c) 90° (d) 60°

21 If the sum of the first 15 terms of the series

$$\left(\frac{3}{4}\right)^3 + \left(1\frac{1}{2}\right)^3 + \left(2\frac{1}{4}\right)^3 + 3^3 + \left(3\frac{3}{4}\right)^3 + \ldots$$

is equal to 225 k, then k is equal to

(a) 108 (b) 27

(c) 54 (d) 9

22 The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right\} \log_e x\, dx$ is equal to

(a) $\frac{3}{2} - e - \frac{1}{2e^2}$ (b) $-\frac{1}{2} + \frac{1}{e} - \frac{1}{2e^2}$

(c) $\frac{1}{2} - e - \frac{1}{e^2}$ (d) $\frac{3}{2} - \frac{1}{e} - \frac{1}{2e^2}$

23 The integral $\int \frac{3x^{13} + 2x^{11}}{(2x^4 + 3x^2 + 1)^4}\, dx$ is equal to (where C is a constant of integration)

(a) $\frac{x^4}{6(2x^4 + 3x^2 + 1)^3} + C$

(b) $\frac{x^{12}}{6(2x^4 + 3x^2 + 1)^3} + C$

(c) $\frac{x^4}{(2x^4 + 3x^2 + 1)^3} + C$

(d) $\frac{x^{12}}{(2x^4 + 3x^2 + 1)^3} + C$

24 The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point

(a) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{1}{4}\right)$

(c) $\left(-\frac{1}{8}, 7\right)$ (d) $\left(\frac{1}{8}, -7\right)$

25 Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then, the minimum value of $|z_1 - z_2|$ is

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 0

26 The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of X-axis, is

(a) $y = x\tan\theta - 2\cot\theta$

(b) $x = y\cot\theta + 2\tan\theta$

(c) $y = x\tan\theta + 2\cot\theta$

(d) $x = y\cot\theta - 2\tan\theta$

27 If the angle of elevation of a cloud from a point P which is 25 m above a lake be 30° and the angle of depression of reflection of the cloud in the **lake** from P be 60°, then the height of the cloud (in meters) from the surface of the lake is

(a) 50 (b) 60

(c) 45 (d) 42

28 Let S and S' be the foci of an ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and area ($\Delta S'BS$) = 8 sq units, then the length of a latus rectum of the ellipse is

(a) $2\sqrt{2}$ (b) $4\sqrt{2}$

(c) 2 (d) 4

29 There are m men and two women participating in a chess tournament. Each participant plays two games with every other participant. If the number of games played by the men between themselves exceeds the number of games played between the men and the women by 84, then the value of m is

(a) 12 (b) 11

(c) 9 (d) 7

30 If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is

(a) $x - y + 7 = 0$ (b) $4x - 3y + 24 = 0$

(c) $3x - 4y + 25 = 0$ (d) $4x + 3y = 0$

ANSWERS

1. (b)	2. (a)	3. (b)	4. (a)	5. (d)	6. (c)	7. (c)	8. (a)	9. (a)	10. (b)
11. (c)	12. (c)	13. (c)	14. (c)	15. (b)	16. (b)	17. (c)	18. (c)	19. (a)	20. (a)
21. (b)	22. (a)	23. (b)	24. (d)	25. (d)	26. (b)	27. (a)	28. (d)	29. (a)	30. (b)

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