KATHMANDU UNIVERSITY

DHULIKHEL, KAVRE

Assignment-4

Subject- MATH 104

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ASSIGNMENT IV - MATH 104

1. Define limit, continuity and derivative of the vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. State the component test for continuity of the vector function. $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. Let $\vec{r}(t) = \sqrt{1-t^2}\,\vec{i} + 8t\vec{j} - 7\vec{k}$. At what value of t is the vector function $\vec{r}$ continuous? Explain reason.

Solution:-

Limit of a vector function:-

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and $\vec{L}$ a vector. We say that $\vec{r}$ has a limit $\vec{L}$ as t approaches to t_0 and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - L| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

Continuity of a vector function :-

A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

The function is continuous if it is continuous at every point in its domain.

Derivative of a vector function:-

The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has a derivative (is differentiable) at t if f, g and h have derivatives at t.

The derivative is the vector function

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}$$

~~Continuity :-~~

Component test for continuity of vector Function :-

The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is continuous at $t = t_0$ if and only if f, g and h are continuous at t_0.

Here,

We have,

$$\vec{r}(t) = \sqrt{1-t^2}\,\vec{i} + 3t\vec{j} - 7\vec{k}$$

Here, $g(t) = 3t$ and $h(t) = -7$ are continuous over every values of k.

But,

$f(t) = \sqrt{1-t^2}$ is only defined over the interval $[-1,1]$ and beyond this interval, there exist no functional value of given component in vector function.

$\therefore$ Vector function $\vec{r}$ is continuous in $[-1,1]$.

2. Find $\frac{d\vec{r}}{dt}$ if

(a) $\vec{r}(t) = \ln\sqrt{(1-t)}\,\vec{i} + \sqrt{1-t^2}\,\vec{j}$

Solution :-

$$\frac{d\vec{r}(t)}{dt} = \frac{d \ln\sqrt{1-t}}{dt}\vec{i} + \frac{d \sqrt{1-t^2}}{dt}\vec{j}$$

$$= \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{1-t}}(-1)\vec{i} + \frac{1}{2\sqrt{1-t^2}}\cdot(-2t)\vec{j}$$

$$= \frac{-1}{2(1-t)}\vec{i} - \frac{t}{\sqrt{1-t^2}}\vec{j}$$

(b) $\vec{r}(t) = (\sin^{-1} 2t)\vec{i} + (\tan^{-1} 3t)\vec{j} + \frac{1}{t}\vec{k}$

Solution:

$$\frac{d\vec{r}(t)}{dt} = \frac{d\sin^{-1}2t}{d2t} \times \frac{d2t}{dt}\vec{i} + \frac{d\tan^{-1}3t}{d3t} \times \frac{d3t}{dt}\vec{j} + \frac{d}{dt}\frac{1}{t}\vec{k}$$

$$= \frac{1\times 2}{\sqrt{1-4t^2}}\vec{i} + \frac{1}{1+9t^2} \times 3\,\vec{j} + \left(-\frac{1}{t^2}\right)\vec{k}$$

$$= \frac{2}{\sqrt{1-4t^2}}\vec{i} + \frac{3}{1+9t^2}\vec{j} - \frac{1}{t^2}\vec{k}$$

(c) $\vec{r}(t) = \frac{2t-1}{2t+1}\vec{i} + \ln(1-4t^2)\vec{j} + (\sec t)\vec{k}$

Solution :-

$$\frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left(\frac{2t-1}{2t+1}\right)\vec{i} + \frac{d}{dt}\ln(1-4t^2)\vec{j} + \frac{d\sec t}{dt}\vec{k}$$

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$$= \left[\frac{(2t+1)\frac{d}{dt}(2t-1) - (2t-1)\frac{d}{dt}(2t+1)}{(2t+1)^2}\right]\vec{i} + \frac{d \ln 1-4t^2}{d(1-4t^2)} \times (-8t)\vec{j}$$

$$+ \sec t \tan t \,\vec{k}$$

$$= \left(\frac{(2t-1)\times 2 - 2(2t-1)}{(2t+1)^2}\right)\vec{i} - \frac{8t}{1-4t^2}\vec{j} + \sec t \cdot \tan t \,\vec{k}$$

$$= \frac{4}{(2t+1)^2}\vec{i} - \frac{8t}{1-4t^2}\vec{j} + \sec t \cdot \tan t \,\vec{k}$$

(3). The vector $\vec{r}(t)$ defines the position of a particle moving in the plane / space at time t. Find the particles velocity, acceleration, speed and direction of motion of particle at time specified.

(a) $\vec{r}(t) = (t^2+1)\vec{i} + (2t-1)\vec{j}$, $t = \frac{1}{2}$

Solution:-

$$\vec{r}(t) = (t^2+1)\vec{i} + (2t-1)\vec{j}$$

Velocity of particle $(\vec{v}(t)) = \frac{d\vec{r}(t)}{dt}$

$$= 2t\,\vec{i} + 2\vec{j}$$

when $t = 1/2$

$$\vec{v} = 2 \times \frac{1}{2}\vec{i} + 2\vec{j}$$

$$= \vec{i} + 2\vec{j}$$

$\therefore$ Particle's velocity $= \vec{i} + 2\vec{j}$

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Speed $= |\vec{v}|$

$= \sqrt{1^2+2^2}$

$= \sqrt{5}$

Direction of motion $(\hat{v}) = \frac{\vec{v}}{|\vec{v}|}$

$= \frac{\vec{i}+2\vec{j}}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$

Acceleration $(\vec{a}) = \frac{d\vec{v}}{dt}$

$= 2\vec{i}$

(b) $\vec{r}(t) = (\cos 2t)\vec{i} + (3\sin 2t)\vec{j}$, $t=0$

Solution :-

Velocity of particle $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$

$= (-2\sin 2t)\vec{i} + (6\cos 2t)\vec{j}$

At $t=0$,

$\vec{v} = 6\vec{j}$

Speed of particle $= |\vec{v}|$

$= 6$

Direction of motion $= \frac{\vec{v}}{|\vec{v}|}$

$= \frac{6\vec{j}}{6}$

$= \vec{j}$

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Acceleration of particle $\vec{a} = \frac{d\vec{v}}{dt}$

$$= (-4\sin 2t)\vec{i} + (-12\cos 2t)\vec{j}$$

At $t=0$

$$\vec{a} = -12\vec{j}$$

4. Solve the initial value problem:

$$\frac{d^2\vec{r}}{dt^2} = -32\vec{k}$$

with the initial conditions:

$$\vec{r}(0) = 100\vec{k} \quad \text{and} \quad \left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

Solution:

Given,

$$\frac{d^2\vec{r}}{dt^2} = -32\vec{k} \quad \text{—(1)}$$

Integrating both sides w.r.t. 't', we get,

$$\frac{d\vec{r}(t)}{dt} = -32t\vec{k} + \vec{c}_1 \quad \text{—(2)}$$

We have,

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

or, $-32 \times 0 + c_1 = 8\vec{i} + 8\vec{j}$

or, $c_1 = 8\vec{i} + 8\vec{j}$

Putting value of c_1 in eq (2)

$$\frac{d\vec{r}}{dt} = -32t\vec{k} + 8\vec{i} + 8\vec{j}$$

$$= 8\vec{i} + 8\vec{j} - 32t\vec{k} \quad \text{—(3)}$$

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Integrating both sides w.r.t. we get,

$$\vec{r} = 8t\vec{i} + 8t\vec{j} - \frac{32t^2}{2}\vec{k} + c_2$$

$$= 8t\vec{i} + 8t\vec{j} - 16t^2\vec{k} + c_2 \quad -(4)$$

Again,

We have,

$$\vec{r}(0) = 100\vec{k}$$

or, $8\times 0\vec{i} + 8\times 0\vec{j} - 16\times 0^2\vec{k} + c_2 = 100\vec{k}$

or, $c_2 = 100\vec{k}$

Putting value of c_2 in eq (4)

$$\vec{r} = 8t\vec{i} + 8t\vec{j} - 16t^2\vec{k} + 100\vec{k}$$

$$= 8t\vec{i} + 8t\vec{j} + (100-16t^2)\vec{k}$$

5. Find the arc length parameter along the curve $\vec{r}(t) = (e^t\cos t)\vec{i} + (e^t\sin t)\vec{j} + (e^t)\vec{k}$, from the point where $t=0$ by evaluating the integrals $s = \int_0^t |v(\tau)|d\tau$ and then find the length of the curve for $-\ln 4 \le t \le 0$.

Solution:-

$$\vec{r}(t) = (e^t\cos t)\vec{i} + (e^t\sin t)\vec{j} + (e^t)\vec{k}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$= (e^t\cos t - e^t\sin t)\vec{i} + (e^t\sin t + e^t\cos t)\vec{j} + e^t\vec{k}$$

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$$|\vec{v}(t)| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2}$$
$$= \sqrt{3e^{2t}}$$
$$= \sqrt{3}\, e^t$$

So,

$$s(t) = \int_0^t \sqrt{3}\, e^t\, dt$$

$$= \sqrt{3}\, [e^t]_0^t$$
$$= \sqrt{3}\, [e^t - e^0]$$
$$= \sqrt{3}\, (e^t - 1)$$

Now,

$$\text{Length} = \cancel{s(t)} - \int_0^{-\ln 4} \sqrt{3}\, e^t$$

$$= \sqrt{3}\, [e^{-\ln 4} - e^0]$$
$$= \sqrt{3}\, (e^{-\ln 4} - 1)$$
$$= \sqrt{3}\, \left(\frac{1}{4} - 1\right)$$

$$= -\frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} \text{ units}$$

6. Find $\vec{T}$, $\vec{N}$, K, τ for the following space curves:

(a) $\vec{r}(t) = (e^t\cos t)\vec{i} + (e^t\sin t)\vec{j} + 2\vec{k}$

Here,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$= (e^t(-\sin t) + e^t\cos t)\vec{i} + (e^t\cos t + e^t\sin t)\vec{j} + 0$$
$$= (e^t(\cos t - \sin t))\vec{i} + (e^t(\cos t + \sin t))\vec{j}$$

$$|\vec{v}(t)| = \sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\cos t + \sin t))^2}$$
$$= \sqrt{(e^t)^2(1 - \sin 2t) + (e^t)^2(1 + \sin 2t)}$$

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$$= e^t \sqrt{1 - \cancel{\sin 2t} + 1 + \cancel{\sin 2t}}$$
$$= e^t \sqrt{2}$$

Unit Tangential vector $\vec{T} = \dfrac{\vec{v}'(t)}{|\vec{v}'(t)|}$

$$= \frac{e^t(\cos t - \sin t)\vec{i}}{e^t\sqrt{2}} + \frac{e^t(\cos t + \sin t)\vec{j}}{e^t\sqrt{2}}$$

$$= \frac{(\cos t - \sin t)\vec{i}}{\sqrt{2}} + \frac{(\cos t + \sin t)\vec{j}}{\sqrt{2}}$$

Principal unit normal vector $\vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

Now,

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\frac{(\cos t - \sin t)\vec{i}}{\sqrt{2}} + \frac{d}{dt}\frac{(\cos t + \sin t)\vec{j}}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(-\sin t - \cos t)\vec{i} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\vec{j}$$

$$\left|\frac{d\vec{T}}{dT}\right| = \sqrt{\left[\frac{1}{\sqrt{2}}(-\sin t - \cos t)\right]^2 + \left[\frac{1}{\sqrt{2}}(\cos t - \sin t)\right]^2}$$

$$= \sqrt{\frac{1}{2}(1 + \sin 2t) + \frac{1}{2}(1 - \sin 2t)}$$

$$= \sqrt{\frac{1}{2} \times 2}$$

$$= 1$$

$$\vec{N} = \frac{1}{\sqrt{2}}(-\sin t - \cos t)\vec{i} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\vec{j}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}}(1) = \frac{1}{e^t\sqrt{2}}$$

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(b) $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + \vec{3k}$

Solution,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$= (-\sin t + t(\cos t) + \sin t)\vec{i} + (\cos t + t\sin t - \cos t)\vec{j}$$

$$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$$

$$|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2}$$

$$= t$$

Unit Tangential vector $(\vec{T}) = \frac{\vec{v}}{|\vec{v}|}$

$$= \cos t\,\vec{i} + \sin t\,\vec{j}$$

Now,

$$\frac{d\vec{T}}{dt} = -\sin t\vec{i} + \cos t\,\vec{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

Normal vector $(\vec{N}) = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$$= -\sin t\,\vec{i} + \cos t\,\vec{j}$$

Curvature $K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$

$$= \frac{1}{t}(1)$$

$$= \frac{1}{t}$$

(c) $\vec{r}(t) = (\cosh t)\vec{i} + (\sinh t)\vec{j} + t\vec{k}$

Solution

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$= \sinh t\,\vec{i} + \cosh t\,\vec{j} + \vec{k}$$

$$|\vec{v}(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$= \sqrt{\sinh^2 t + \cosh^2 t + \cosh^2 t - \sinh^2 t}$$

$$= \cosh t\sqrt{2}$$

Now,

Unit Tangent vector $(\vec{T}) = \frac{\vec{v}}{|\vec{v}|}$

$$= \frac{\sinh t\,\vec{i}}{\cosh t\,\sqrt{2}} + \frac{\cosh t\,\vec{j}}{\cosh t\sqrt{2}} + \frac{\vec{k}}{\cosh t\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\tanh t\,\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\operatorname{sech} t\,\vec{k}$$

Then,

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{2}}\frac{d(\tanh t)}{dt}\vec{i} + \frac{1}{\sqrt{2}}\frac{d}{dt}(1)\vec{j} + \frac{1}{\sqrt{2}}\frac{d\operatorname{sech} t}{dt}\vec{k}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech}^2 t\,\vec{i} + 0 + \frac{1}{\sqrt{2}}(-\operatorname{sech} t\cdot\tanh t)\vec{k}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech}^2 t\,\vec{i} + \frac{1}{\sqrt{2}}(-\operatorname{sech} t\cdot\tanh t)\vec{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t\cdot\tanh^2 t}$$

$$= \sqrt{\frac{1}{2}\operatorname{sech}^2 t\,(\operatorname{sech}^2 t + \tanh^2 t)}$$

$$= \sqrt{\frac{1}{2}\operatorname{sech}^2 t}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech} t$$

Now,

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

$$= \frac{\frac{1}{\sqrt{2}} \operatorname{sech}^2 t \, \vec{i} + \frac{1}{\sqrt{2}} (-\operatorname{sech} t \cdot \tanh t) \vec{K}}{\frac{1}{\sqrt{2}} \operatorname{sech} t}$$

$$= \operatorname{sech} t \, \vec{i} - \tanh t \, \vec{K}$$

$$K = \frac{1}{|\vec{V}|} \left(\frac{d\vec{T}}{dt}\right)$$

$$= \frac{1}{\cosh t \sqrt{2}} \cdot \frac{1}{\sqrt{2}} \operatorname{sech} t$$

$$= \frac{1}{2} \operatorname{sech}^2 t$$