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AN INTRINSIC PROOF OF AN EXTENSION OF ITÔ'S ISOMETRY FOR ANTICIPATING STOCHASTIC INTEGRALS

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Abstract. Itô's isometry forms the cornerstone of the definition of Itô's integral and consequently the theory of stochastic calculus. Therefore, for any theory which extends Itô's theory, it is important to know if the isometry holds. In this paper, we use probabilistic arguments to demonstrate that the extension of the isometry formula contains an extra term for the anticipating stochastic integral defined by Ayed and Kuo. We give examples to illustrate the usage of this formula and to show that the extra term can be positive or negative.

1. Introduction

Let $B_t, t \geq 0$, be a Brownian motion and $[a, b]$ a fixed interval with $a \geq 0$. Suppose f and ϕ are continuous functions on $\mathbb{R}$. In [1] the following anticipating stochastic integral is defined as

$$\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(B_{t_{i-1}})\phi(B_b - B_{t_i})\Delta B_i \tag{1.1}$$

provided that the limit exists in probability. Here $\Delta_n = \{a = t_0, t_1, t_2, \ldots, t_n = b\}$ is a partition of $[a, b]$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$. Note that when $\phi \equiv 1$ this stochastic integral is an Itô integral (see Theorem 5.3.3 in [6].) It is proved in Theorem 3.1 [8] that when f and ϕ are C^1-functions we have the equality:

$$\begin{aligned}\mathbb{E}\left[\left(\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t\right)^2\right] &= \int_a^b \mathbb{E}\Big[f(B_t)^2\phi(B_b - B_t)^2\Big]\, dt \\ &\quad + 2\int_a^b\int_a^t \mathbb{E}\Big[f(B_s)\phi'(B_b - B_s)f'(B_t)\phi(B_b - B_t)\Big]\, ds\, dt,\end{aligned} \tag{1.2}$$

provided that the integrals in the right-hand side exist. In particular, when $\phi \equiv 1$, the equality in equation (1.2) is the well-known Itô isometry.

We need to point out that the proof of equation (1.2) in [8] is too lengthy and involves rather tedious computations by using the binomial expansion. Moreover,

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