Communications on Stochastic Analysis
Vol. 12, No. 4 (2018) 473-485

# STOCHASTIC DIFFERENTIAL EQUATIONS WITH ANTICIPATING INITIAL CONDITIONS

HUI-HSIUNG KUO, SUDIP SINHA, AND JIAYU ZHAI*

Abstract. In this paper, we study the solutions of a stochastic differential equation with various anticipating initial conditions. We show that the conditional expectation of the solution of such a stochastic differential equation is not simply the solution of the corresponding stochastic differential equation with initial condition taken as the conditional expectation of the anticipating initial condition. We derive the conditional expectation of the solution in general, and apply it to the special case of anticipating initial condition given by Hermite polynomials. We also extend the class of initial conditions to functions of Wiener integrals.

## 1. Introduction

In 1942 [7], Kiyosi Itô published his pioneering paper on stochastic integration, which enabled integration of stochastic processes with respect to a Brownian motion. In 1944 [8], his efforts to model Markov processes led him to construct stochastic differential equations of the form $dX_t = \alpha(X_t)\,dB(t)+\beta(X_t)\,dt, X_0 = x$, which subsequently led him to publish what is now known as the Itô formula. In 1973, Black and Scholes [4], and Merton [12] used Itô's calculus to give a framework for option pricing, which rapidly expanded the interest of stochastic calculus to practitioners in other fields.

Even though it is extremely useful, the Itô calculus cannot handle *anticipating* conditions. For example, consider the following simple stochastic differential equation with anticipating initial condition

$$\begin{cases} dX_t = X_t\,dB(t), \quad t \in [0,1], \\ X_0 = B(1). \end{cases} \tag{1.1}$$

To solve the equation analytically, we have to assign a meaning to the integral

$$\int_0^t B(1)\,dB(s), \quad t \in [0,1], \tag{1.2}$$

which is outside the theory of Itô calculus since the integral is not adapted with respect to the filtration generated by the Brownian motion $B(t)$. This is the primary motivation for extending the Itô integral to anticipating integrands.

Received 2018-11-11; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60H05; Secondary 60H10, 60H20.
*Key words and phrases.* Stochastic integral, near-martingale, Itô formula, stochastic differential equations, conditional expectation.
* Corresponding author.