Anticipating Stochastic Integrals and Related Linear Stochastic Differential Equations

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Main contributions

1. Extension of Itô's isometry
2. Near-martingale optional stopping theorem
3. LSDEs with anticipating initial conditions
   3.1 Solutions
   3.2 Conditionals
4. LSDEs with anticipating coefficients
   4.1 Solutions in Ayed–Kuo theory
   4.2 Solutions via a novel braiding technique
   4.3 Large deviation principles

Outline

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The Ayed–Kuo integral
- Essential ideas
- Extension of Itô's isometry
- Near-martingales

LSDEs with anticipating initial conditions
- Solutions
- Conditionals

LSDEs with anticipating coefficients
- Solutions via ansatz
- Solutions using a novel braiding technique
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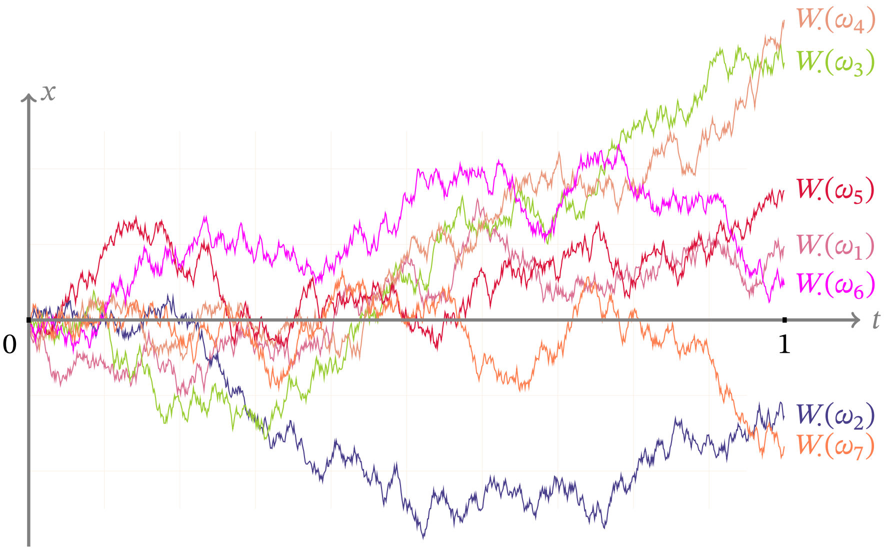
Wiener process / Brownian motion



Stochastic integration

Setup

- $t \in [0,1]$ and $(\Omega, \Sigma, \mathcal{F}_\cdot, \mathbb{P})$ is a filtered space
- $W_\cdot$ is a Wiener process on $(\Omega, \Sigma, \mathcal{F}_\cdot, \mathbb{P})$
- A stochastic process X is called *adapted* if X_t is $\mathcal{F}_t$-measurable $\forall t$

Integration with respect to W

- Naive integration: not possible
- Wiener's integral: deterministic integrands
- Itô's integral: adapted integrands

Anticipating integrands

- Itô's idea of enlargement of filtration
- Skorokhod integral and Malliavin calculus
- White-noise distribution theory
- Ayed–Kuo integral

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Itô's integral (Itô 1944)

Basics

» Definition. Let $\Delta W_i = W_{t_i} - W_{t_{i-1}}$. For $X \in L^2_{\mathrm{ad}}([0,1] \times \Omega)$ as integrand: take $L^2(\Omega)$ limits of the left endpoint evaluation of Riemann sums

$$M_t \triangleq \int_0^t X_s \, \mathrm{d}W_s \triangleq \lim_{n\to\infty} \sum_{i=1}^{n} X_{t_{i-1}} \, \Delta W_i \quad \text{in } L^2(\Omega).$$

» Example. $\int_0^t W_s \, \mathrm{d}W_s = \frac{1}{2}\left(W_t^2 - t\right)$.

Properties

» Linearity

» Mean: 0

» Variance: $\|M\|_{L^2(\Omega)} = \|X\|_{L^2_{\mathrm{ad}}([0,t]\times\Omega)}$ (Itô's isometry)

» Martingale: $\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s$ for any $s \leq t$

Linear stochastic differential equations (LSDEs)

» Linear differential equations incorporating "noise", for example

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \beta_t X_t + \alpha_t X_t \dot{W}_t.$$

» But $\dot{W}_t$ is meaningless. Heuristically, we multiply by $\mathrm{d}t$, write $\dot{W}_t \,\mathrm{d}t = \mathrm{d}W_t$, and interpret the second expression as an Itô integral.

» Example. For adapted α and β, the following is an LSDE

$$\begin{cases} \mathrm{d}X_t = \alpha_t X_t \,\mathrm{d}W_t + \beta_t X_t \,\mathrm{d}t, \\ X_0 = 1. \end{cases}$$

» The solution is given by the *exponential process*

$$\mathcal{E}_t = \exp\left(\int_0^t \alpha_s \,\mathrm{d}W_s + \int_0^t \left(\beta_s - \frac{1}{2}\alpha_s^2\right) \mathrm{d}s\right).$$

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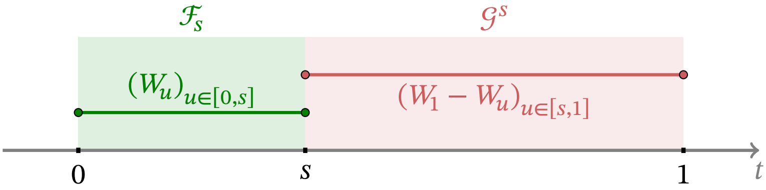
Idea

- A stochastic process Y is called *instantly-independent* (i.i.) if Y^t and $\mathcal{F}_t$ are independent $\forall t$.
- Decompose the integrand into adapted and i.i. components.
- Left endpoint evaluation for adapted processes.
- Right endpoint evaluation for i.i. processes.



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Example
Ayed and Kuo 2008, equation 1.6

$$
\begin{aligned}
N(t) = \int_0^t W_1 \, \mathrm{d}W_s &= \int_0^t [W_s + (W_1 - W_s)] \, \mathrm{d}W_s \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[W_{t_{i-1}} + \left(W_1 - W_{t_i}\right)\right] \Delta W_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (W_1 - \Delta W_i) \, \Delta W_i \\
&= W_1 \cdot \lim_{n\to\infty} \sum_{i=1}^{n} \Delta W_i - \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta W_i)^2 \\
&= W_1 W_t - t.
\end{aligned}
$$

Definition (Ayed and Kuo 2008)

» For X adapted and Y instantly-independent, define

$$\int_0^1 X_t \, Y^t \, \mathrm{d}W_t = \lim_{m\to\infty} \sum_{j=1}^{m} X_{t_{j-1}} Y^{t_j} \, \Delta W_i \quad \text{in } L^2(\Omega).$$

Extend to linear combinations.

» Let Z be a stochastic process such that a sequence $(Z_n)_{n=1}^{\infty}$ of stochastic processes each of the form above (or linear combinations thereof) satisfies

1. $\int_0^1 |Z_n(t) - Z(t)|^2 \, \mathrm{d}t \to 0$ as $n \to \infty$ almost surely, and
2. $\int_0^1 Z_n(t) \, \mathrm{d}W_t$ converges in $L^2(\Omega)$ as $n \to \infty$.

Then the stochastic integral of Z is defined by the following (if it exists):

$$\int_0^1 Z(t) \, \mathrm{d}W_t = \lim_{n\to\infty} \int_0^1 Z_n(t) \, \mathrm{d}W_t \quad \text{in } L^2(\Omega).$$

Differential formula

C.-R. Hwang, Kuo, Saitô, and Zhai 2016, theorem 3.2

type	definition	representation
Itô	$X_\cdot = X_0 + \int_0^\cdot m_t \,\mathrm{d}t + \int_0^\cdot \sigma_t \,\mathrm{d}W_t$	$\mathrm{d}X_t = m_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t$
i.i.	$Y^\cdot = Y^1 + \int_\cdot^1 \eta_t \,\mathrm{d}t + \int_\cdot^1 \zeta_t \,\mathrm{d}W_t$	$\mathrm{d}Y^t = -\eta_t \,\mathrm{d}t - \zeta_t \,\mathrm{d}W_t$

Here η_t and ζ_t are i.i. such that Y is also i.i.
Assume $\theta(t, x, y) \in C^{1,2,2}([0,1] \times \mathbb{R} \times \mathbb{R})$. Then

$$\mathrm{d}\theta\left(t, X_t, Y^t\right) = \theta_t \,\mathrm{d}t + \theta_x \,\mathrm{d}X_t + \frac{1}{2}\theta_{xx} \left(\mathrm{d}X_t\right)^2 + \theta_y \,\mathrm{d}Y^t - \frac{1}{2}\theta_{yy} \left(\mathrm{d}Y^t\right)^2,$$

where $(\mathrm{d}W_t)^2 = \mathrm{d}t$, all other products being zero.

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Identity for a simple case

Theorem (Kuo, Shrestha, and Sinha 2021a, theorem 3.1)

Suppose $f, \phi \in C^1(\mathbb{R})$ such that

$$f(W_t)\,\phi(W_1 - W_t),\; f(W_t)\,\phi'(W_1 - W_t),\; f'(W_t)\,\phi(W_1 - W_t) \in L^2([0,1] \times \Omega).$$

Then $\mathbb{E}\left[\int_0^1 f(W_t)\,\phi(W_1 - W_t)\,\mathrm{d}W_t\right] = 0$, and

$$\mathbb{E}\left[\left(\int_0^1 f(W_t)\,\phi(W_1 - W_t)\,\mathrm{d}W_t\right)^2\right] = \int_0^1 \mathbb{E}\left[f(W_t)^2\,\phi(W_1 - W_t)^2\right]\mathrm{d}t$$
$$+ 2\int_0^1 \int_0^t \mathbb{E}[f(W_s)\,\phi'(W_1 - W_s)\,f'(W_t)\,\phi(W_1 - W_t)]\,\mathrm{d}s\,\mathrm{d}t.$$

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$$+ 2\int_0^1 \int_0^t \mathbb{E}[f(W_s)\,\phi'(W_1 - W_s)\,f'(W_t)\,\phi(W_1 - W_t)]\,\mathrm{d}s\,\mathrm{d}t.$$

Remark. The double integral term can take any real value (ibid., example 3.9).

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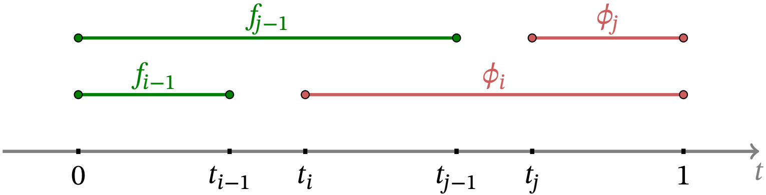
Proof idea

- Write integral as $L^2(\Omega)$-limit of sums over partitions of $[0, 1]$.
- Diagonal: Use quadratic variation of W.



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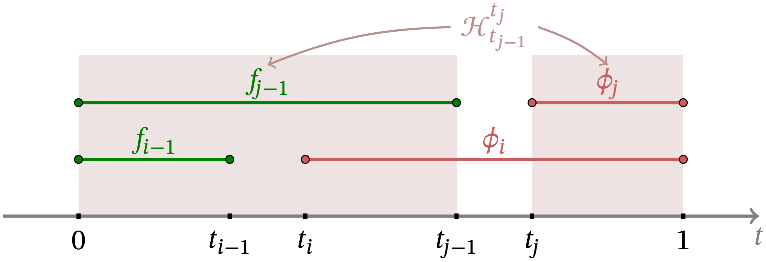
Proof idea

- Write integral as $L^2(\Omega)$-limit of sums over partitions of $[0,1]$.
- Diagonal: Use quadratic variation of W.
- Off-diagonal: Use the expectation and approximation identities.
  - $\mathbb{E}\left[f\left(W_{t_{i-1}}\right)\phi\left(W_b - W_{t_i} - \Delta W_j\right) f\left(W_{t_{j-1}}\right)\phi\left(W_1 - W_{t_j}\right)\Delta W_i \Delta W_j\right] = 0.$
  - $\phi\left(W_b - W_{t_i}\right) - \phi\left(W_b - W_{t_i} - \Delta W_j\right) \simeq \phi'\left(W_b - W_{t_i} - \Delta W_j\right)\Delta W_j.$
  - Conditioning w.r.t. $\mathcal{H}_{t_{j-1}}^{t_j}$, noting ΔW_j is independent of $\mathcal{H}_{t_{j-1}}^{t_j}$.



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» Write integral as $L^2(\Omega)$-limit of sums over partitions of $[0, 1]$.

» Diagonal: Use quadratic variation of W.

» Off-diagonal: Use the expectation and approximation identities.

- $\mathbb{E}\left[f(W_{t_{i-1}})\,\phi(W_b - W_{t_i} - \Delta W_j)\, f(W_{t_{j-1}})\,\phi(W_1 - W_{t_j})\,\Delta W_i\,\Delta W_j\right] = 0.$
- $\phi(W_b - W_{t_i}) - \phi(W_b - W_{t_i} - \Delta W_j) \simeq \phi'(W_b - W_{t_i} - \Delta W_j)\,\Delta W_j.$
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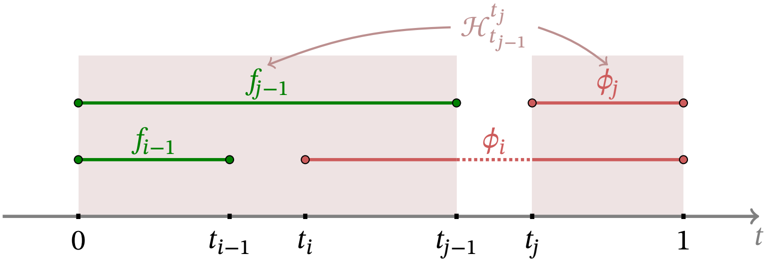

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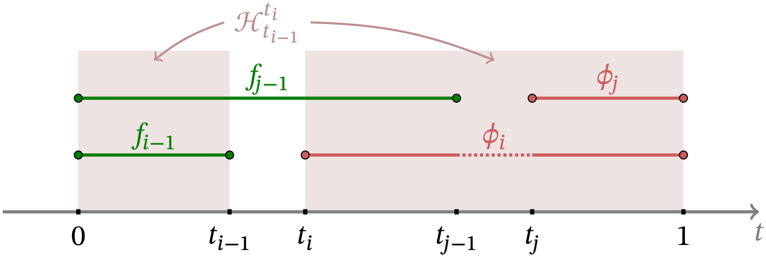

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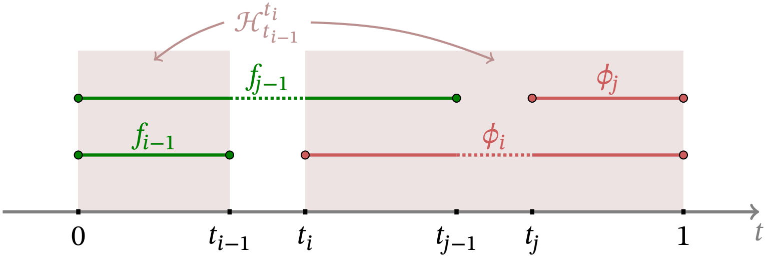

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Discussion

- Shown before under restrictive conditions (Kuo, Sae-Tang, and Szozda 2013, theorem 3.1).
- Vast improvement over the previous result.
- Minimal restrictions on f and ϕ.
- Short, direct, probabilistic proof.
- Utilize the left and right evaluation point definition of the integral.
- Introduce the separation σ-algebra as the canonical σ-algebra to condition on for the Ayed–Kuo integral.

General form (Kuo, Shrestha, and Sinha 2021a, theorem 3.6)

Let $\Theta(x,y),\ \Lambda(x,y) \in C^1(\mathbb{R}^2)$ and assume that

1. $\Theta(W_t, W_1 - W_t),\ \Theta_x(W_t, W_1 - W_t),\ \Theta_y(W_t, W_1 - W_t) \in L^2([0,1] \times \Omega)$, and
2. $\Lambda(W_t, W_1 - W_t),\ \Lambda_x(W_t, W_1 - W_t),\ \Lambda_y(W_t, W_1 - W_t) \in L^2([0,1] \times \Omega)$.

Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^1 \Theta(W_t, W_1 - W_t)\,\mathrm{d}W_t\right)\left(\int_0^1 \Lambda(W_t, W_1 - W_t)\,\mathrm{d}W_t\right)\right] \\
&= \int_0^1 \mathbb{E}[\Theta(W_t, W_1 - W_t)\,\Lambda(W_t, W_1 - W_t)]\,\mathrm{d}t \\
&\quad + \int_0^1 \int_0^t \mathbb{E}\Big[\Theta_y(W_s, W_1 - W_s)\,\Lambda_x(W_t, W_1 - W_t) \\
&\qquad\qquad + \Theta_x(W_t, W_1 - W_t)\,\Lambda_y(W_t, W_1 - W_t)\Big]\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

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Setup

Motivation

- Process defined by Itô integrals $M_t = \int_0^t X_s \, \mathrm{d}W_s$ are martingales.
- Are Ayed–Kuo integrals martingales?
- Example. $N(t) = \int_0^t W_1 \, \mathrm{d}W_s = W_1 W_t - t$.
  Now, $\mathbb{E}(N(t) \mid \mathcal{F}_s) = W_s^2 - s \neq W_1 W_s - s = N(s)$, so not a martingale.
  However, $\mathbb{E}(N(s) \mid \mathcal{F}_s) = W_s^2 - s = \mathbb{E}(N(t) \mid \mathcal{F}_s)$.

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Definition (C.-R. Hwang, Kuo, Saitô, and Zhai 2017, definition 2.1)

An integrable stochastic process N is called a *near-martingale* if $\mathbb{E}(N(t) - N(s) \mid \mathcal{F}_s) = 0$ almost surely for every $s \leq t$.

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Definition (C.-R. Hwang, Kuo, Saitô, and Zhai 2017, definition 2.1)

An integrable stochastic process N is called a *near-martingale* if $\mathbb{E}(N(t) - N(s) \mid \mathcal{F}_s) = 0$ almost surely for every $s \leq t$.

Theorem (ibid., theorem 2.5)

A process N is a near-martingale if and only if the conditioned process M given by $M_t = \mathbb{E}(N(t) \mid \mathcal{F}_t)$ is a martingale.

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Optional stopping theorem

Theorem (Kuo, Shrestha, Sinha, and Sundar 2022, theorem 3.3)

Suppose $\Theta : \mathbb{R}^2 \to \mathbb{R}$ is measurable. Then the processes

$$N(t) = \int_0^t \Theta(W_u, W_1 - W_u)\,\mathrm{d}W_u \quad \text{and} \quad \widetilde{N}(t) = \int_t^1 \Theta(W_u, W_1 - W_u)\,\mathrm{d}W_u$$

are near-martingales.

Theorem (ibid., theorem 3.10)

Let N be a near-submartingale with right-continuous sample paths. Suppose $\sigma \leq \tau$ are two bounded stopping. If N is either non-negative or uniformly integrable, then $N(\sigma)$ and $N(\tau)$ are integrable, and

$$\mathbb{E}(N(\tau) - N(\sigma) \mid \mathcal{F}_\sigma) \geq 0 \text{ almost surely.}$$

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Motivation

For $x \in \mathbb{R}$, the solution of

$$\begin{cases} \mathrm{d}X_t = X_t \,\mathrm{d}W_t \\ X_0 = x \end{cases}$$

is $X_t = x \exp\left(W_t - \frac{1}{2}t\right)$.

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$$\begin{cases} \mathrm{d}X_t = X_t \,\mathrm{d}W_t \\ X_0 = x \end{cases}$$

is $X_t = x \exp\left(W_t - \frac{1}{2}t\right)$.

However, the solution of

$$\begin{cases} \mathrm{d}Z(t) = Z(t) \,\mathrm{d}W_t \\ Z(0) = W_1 \end{cases}$$

is not $Z(t) = W_1 \exp\left(W_t - \frac{1}{2}t\right)$.

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Non-intuitive nature

Example (Khalifa, Kuo, Ouerdiane, and Szozda 2013, section 3)

The solution of

$$\begin{cases} \mathrm{d}Z(t) = Z(t)\,\mathrm{d}W_t \\ Z(0) = W_1 \end{cases}$$

is given by $Z(t) = (W_1 - t)\exp\left(W_t - \frac{1}{2}t\right)$.

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Example (Ayed and Kuo 2008, example 4.1)

The solution of

$$\begin{cases} \mathrm{d}Z(t) = Z(t)\,\mathrm{d}W_t + \dfrac{1}{W_1}Z(t)\,\mathrm{d}t \\ Z(0) = 1 \end{cases}$$

is given by $Z(t) = W_1 \exp\left(W_t - \frac{1}{2}t\right)$.

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Generalization

Theorem (Kuo, Sinha, and Zhai 2018, theorem 5.1)

Let $\alpha, h \in L^2[0,1]$, $\beta \in L^1[0,1]$ and $\phi \in C^2(\mathbb{R})$. Then the solution of

$$\begin{cases} \mathrm{d}Z(t) = \alpha(t)\, Z(t)\, \mathrm{d}W_t + \beta(t)\, Z(t)\, \mathrm{d}t, \quad t \in [0,1] \\ Z(0) = \phi\Big(\int_0^1 h(s)\, \mathrm{d}W_s\Big), \end{cases}$$

is given by

$$Z(t) = \phi\Big(\int_0^1 h(s)\, \mathrm{d}W_s - \int_0^t \alpha(s)\, h(s)\, \mathrm{d}s\Big)\, \mathcal{E}_t.$$

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Proof idea. Use an ansatz and then apply the differential formula.

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Further generalization

Theorem (Kuo, Shrestha, and Sinha 2021b, theorem 4.2)

Let $\alpha \in L^2_{\mathrm{ad}}([0,1] \times \Omega)$, $\beta \in L^1_{\mathrm{ad}}([0,1] \times \Omega)$ be stochastic processes. Suppose $h \in L^2[0,1]$ and $\phi \in C^2(\mathbb{R})$ are deterministic functions. Then the solution of

$$\begin{cases} \mathrm{d}Z(t) = \alpha_t\, Z(t)\,\mathrm{d}W_t + \beta_t\, Z(t)\,\mathrm{d}t, \\ Z(0) = \phi\Big(\int_0^1 h(s)\,\mathrm{d}W_s\Big), \end{cases}$$

is given by

$$Z(t) = \phi\Big(\int_0^1 h(s)\,\mathrm{d}W_s - \int_0^t h(s)\,\alpha_s\,\mathrm{d}s\Big)\,\mathcal{E}_t.$$

Proof idea. Use an ansatz and then apply the differential formula.

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Motivation and setup

For $\alpha \in L^2_{\text{ad}}([0,1] \times \Omega)$, $\beta \in L^1_{\text{ad}}([0,1] \times \Omega)$, and $h \in L^2[0,1]$, consider

$$\begin{cases} \mathrm{d}Z(t) = \alpha_t\, Z(t)\, \mathrm{d}W_t + \beta_t\, Z(t)\, \mathrm{d}t \\ Z(0) = \phi\Big(\int_0^1 h(s)\, \mathrm{d}W_s\Big), \end{cases}$$

and the conditioned process

$$X_t = \mathbb{E}(Z(t) \mid \mathcal{F}_t)\,.$$

Questions

- What SDE does X satisfy?
- What is the relationship between the SDEs of X and Z?
- Is there a formula to obtain X from Z directly?

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Analytic functions (Kuo, Shrestha, and Sinha 2021b, theorem 5.1)

Let ϕ is an analytic function on $\mathbb{R}$ with derivative ϕ'. Furthermore, let $Z(t)$ be the solution of

$$\mathrm{d}Z(t) = \alpha_t\, Z(t)\,\mathrm{d}W_t + \beta_t\, Z(t)\,\mathrm{d}t, \quad Z(0) = \phi\Big(\int_0^1 h(s)\,\mathrm{d}W_s\Big),$$

and $X_t = \mathbb{E}(Z(t) \mid \mathcal{F}_t)$ is the conditioned process. Then X satisfies

$$\mathrm{d}X_t = \alpha_t\, X_t\,\mathrm{d}W_t + \beta_t\, X_t\,\mathrm{d}t + h(t)\,\widetilde{X}_t\,\mathrm{d}W_t, \quad X_0 = \mathbb{E}\Big[\phi\Big(\int_0^1 h(s)\,\mathrm{d}W_s\Big)\Big],$$

where $\widetilde{X}_t = \mathbb{E}\big(\widetilde{Z}(t) \mid \mathcal{F}_t\big)$, and $\widetilde{Z}$ is the solution of

$$\mathrm{d}\widetilde{Z}(t) = \alpha_t\, \widetilde{Z}(t)\,\mathrm{d}W_t + \beta_t\, \widetilde{Z}(t)\,\mathrm{d}t, \quad \widetilde{Z}(0) = \phi'\Big(\int_0^1 h(s)\,\mathrm{d}W_s\Big).$$

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Special case: the exponential function

Example (Kuo, Shrestha, and Sinha 2021b, example 5.3)

The solution of

$$\begin{cases} \mathrm{d}X_t = (\alpha_t + h(t))\, X_t \,\mathrm{d}W_t + \beta_t \, X_t \,\mathrm{d}t, \\ X_0 = 1. \end{cases}$$

is given by

$$X_t = \mathcal{E}_t \exp\left(\int_0^t h(s) \,\mathrm{d}W_s - \int_0^t h(s)\, \alpha_s \,\mathrm{d}s \right).$$

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Hermite polynomials

Kuo, Shrestha, and Sinha 2021b, theorem 5.5

For a fixed $n \in \mathbb{N}$, suppose Z is the solution of

$$\mathrm{d}Z(t) = \alpha_t Z(t)\,\mathrm{d}W_t + \beta_t Z(t)\,\mathrm{d}t, \quad Z(0) = H_n\Big(\int_0^1 h(s)\,\mathrm{d}W_s\,;\int_0^1 h(s)^2\,\mathrm{d}s\Big).$$

Then $X_t = \mathbb{E}(Z(t) \mid \mathcal{F}_t)$ is given by

$$X_t = H_n\Big(\int_0^t h(s)\,\mathrm{d}W_s - \int_0^t h(s)\,\alpha_s\,\mathrm{d}s\,;\int_0^t h(s)^2\,\mathrm{d}s\Big)\,\mathcal{E}_t.$$

Moreover, X_t satisfies

$$\begin{cases} \mathrm{d}X_t = \alpha_t X_t\,\mathrm{d}W_t + \beta_t X_t\,\mathrm{d}t \\ \qquad + nH_{n-1}\Big(\int_0^t h(s)\,\mathrm{d}W_s - \int_0^t h(s)\,\alpha_s\,\mathrm{d}s\,;\int_0^t h(s)^2\,\mathrm{d}s\Big)\,\mathcal{E}_t\,h(t)\,\mathrm{d}W_t \\ X_0 = 0. \end{cases}$$

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Motivation

» In Itô's theory, for an adapted process H_t, the solution of

$$\begin{cases} \mathrm{d}X_t = H_t\, X_t\, \mathrm{d}W_t \\ X_0 = 1 \end{cases}$$

is given by

$$X_t = \exp\left(\int_0^t H_s\, \mathrm{d}W_s - \frac{1}{2}\int_0^t H_s^2\, \mathrm{d}s\right).$$

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» In Itô's theory, for an adapted process H_t, the solution of

$$\begin{cases} \mathrm{d}X_t = H_t\, X_t\, \mathrm{d}W_t \\ X_0 = 1 \end{cases}$$

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$$X_t = \exp\left(\int_0^t H_s\, \mathrm{d}W_s - \frac{1}{2}\int_0^t H_s^2\, \mathrm{d}s\right).$$

» On the other hand, the solution of

$$\begin{cases} \mathrm{d}Z(t) = W_1\, Z(t)\, \mathrm{d}W_t \\ Z(0) = 1 \end{cases}$$

for the anticipating coefficient W_1 is not given by

$$Z(t) = \exp\left(\int_0^t W_1\, \mathrm{d}W_s - \frac{1}{2}\int_0^t W_1^{\,2}\, \mathrm{d}s\right) = \exp\left(W_1\, W_t - t - \frac{1}{2} t W_1^{\,2}\right).$$

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Non-trivial nature

Example (C. R. Hwang, Kuo, and Saitô 2019, theorem 3.1)

The solution of

$$\begin{cases} \mathrm{d}Z(t) = W_1\, Z(t)\, \mathrm{d}W_t \\ Z(0) = 1. \end{cases}$$

is given by

$$Z(t) = \exp\left[W_1 \int_0^t e^{-(t-s)}\, \mathrm{d}W_s - t - \frac{1}{4}\left(1 - e^{-2t}\right)\, W_1^2 \right].$$

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Example (ibid., theorem 3.3)

The process $Z(t) = \exp\left(W_1\, W_t - t - \frac{1}{2} W_1^2 t \right)$ is a solution of

$$\mathrm{d}Z(t) = W_1\, Z(t)\, \mathrm{d}W_t + W_1\, (W_t - tW_1)\, Z(t)\, \mathrm{d}t, \quad Z(0) = 1.$$

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Generalization for anticipating drift

Kuo, Shrestha, Sinha, and Sundar 2022, theorem 4.2

Suppose $\sigma \in L^2_{\mathrm{ad}}([0,1]\times\Omega)$, $\gamma \in L^2[0,1]$, and ξ is independent of W. Moreover, assume $f \in C^2(\mathbb{R})$ along with $f, f', f'' \in L^1(\mathbb{R})$. Then the solution of

$$\begin{cases} \mathrm{d}Z(t) = f\left(\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z(t)\,\mathrm{d}t + \sigma_t\, Z(t)\,\mathrm{d}W_t \\ Z(0) = \xi \end{cases}$$

in the Ayed–Kuo theory is given by

$$Z(t) = \xi \exp\left[\int_0^t \sigma_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2\,\mathrm{d}s + \int_0^t f\left(\int_0^1 \gamma(u)\,\mathrm{d}W_u - \int_s^t \gamma(u)\,\sigma_u\,\mathrm{d}u\right)\mathrm{d}s\right].$$

Proof idea. Use an ansatz and then apply the differential formula.

The squared process

Theorem (Kuo, Shrestha, Sinha, and Sundar 2022, theorem 4.3) Under identical conditions as before,

$$\begin{cases} \mathrm{d}V(t) = \Big[\sigma_t^2 + f\Big(\int_0^1 \gamma(s)\,\mathrm{d}W_s\Big) \\ \qquad\qquad +2\,\gamma(t)\,\sigma_t \int_0^t f'\Big(\int_0^1 \gamma(u)\,\mathrm{d}W_u - \int_s^t \gamma(u)\,\sigma_u\,\mathrm{d}u\Big)\,\mathrm{d}s\Big]\,V(t)\,\mathrm{d}t \\ \qquad + 2\,\sigma_t\,V(t)\,\mathrm{d}W_t, \\ V(0) = \xi^2 \end{cases}$$

is solved by Z^2, where Z is given as before.

Proof idea. Use an ansatz and then apply the differential formula.

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Setup

» **Goal.** Under reasonable conditions on γ, σ, f, and ξ, find the solution of

$$\begin{cases} \mathrm{d}Z(t) = f\left(\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z(t)\,\mathrm{d}t + \sigma(t)\, Z(t)\,\mathrm{d}W_t \\ Z(0) = \xi. \end{cases}$$

» In Ayed–Kuo theory, the solution is

$$Z(t) = \xi \exp\Bigg[\int_0^t \sigma(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s + \int_0^t f\left(\int_0^1 \gamma(u)\,\mathrm{d}W_u - \int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\Bigg].$$

» **Question.** Can we do this without an ansatz?

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Setup

- **Goal.** Under reasonable conditions on γ, σ, f, and ξ, find the solution of

$$\begin{cases} \mathrm{d}Z(t) = f\left(\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z(t)\,\mathrm{d}t + \sigma(t)\, Z(t)\,\mathrm{d}W_t \\ Z(0) = \xi. \end{cases}$$

- In Ayed–Kuo theory, the solution is

$$\begin{aligned} Z(t) = \xi \exp\Bigg[&\int_0^t \sigma(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s \\ &+ \int_0^t f\left(\int_0^1 \gamma(u)\,\mathrm{d}W_u - \int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\Bigg]. \end{aligned}$$

- **Question.** Can we do this without an ansatz?
- **Inspiration.** Trotter's product formula (Trotter 1959).

Skorokhod integral

» The *stochastic derivative* D allows us to differentiate certain random variables w.r.t. ω.

» The *Skorokhod integral* δ is defined as the adjoint of D.

» $L^2_{\mathrm{ad}}([0,1] \times \Omega) \subset \mathrm{dom}(\delta)$, and for any $u \in L^2_{\mathrm{ad}}([0,1] \times \Omega)$, we have

$$\delta(u) = \int_0^1 u_t \, \mathrm{d}W_t,$$

where the right side is in the sense of Itô (Nualart 2006, proposition 1.3.4).

» Ayed–Kuo integral $\equiv$ Skorokhod integral (Parczewski 2017, theorem 2.3).

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Simple anticipating SDE

For $\sigma \in L^2[0,1]$, fix the family of translation $A_t : \mathcal{C}_0 \to \mathcal{C}_0$ in the Cameron–Martin direction

$$(A_t(\omega))_s = \omega_s - \int_0^{t \wedge s} \sigma(u)\,\mathrm{d}u.$$

Theorem (Kuo, Shrestha, Sinha, and Sundar 2022, lemma 4.8)

Suppose $\sigma \in L^2[0,1]$ and $\xi \in L^p(\Omega)$ for some $p > 2$. Then

$$\begin{cases} \mathrm{d}Z(t) = \sigma(t)\, Z(t)\,\mathrm{d}W_t \\ Z(0) = \xi, \end{cases}$$

in the Skorokhod sense has the unique solution

$$Z(t) = (\xi \circ A_t)\, \mathcal{E}_t.$$

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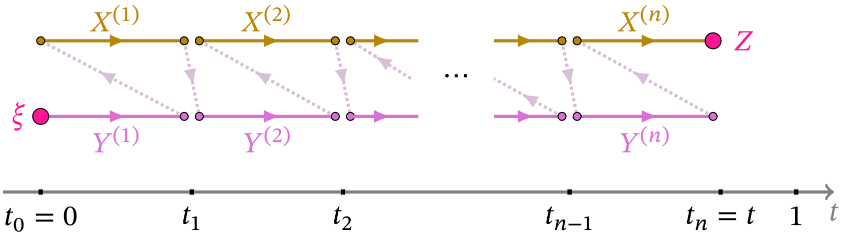
The braiding technique: idea

Kuo, Shrestha, Sinha, and Sundar 2022, section 4.2



For $u \in [t_{k-1}, t_k]$,

$$\mathrm{d}Y_u^{(k)} = \sigma(u)\, Y_u^{(k)}\, \mathrm{d}W_u \qquad \mathrm{d}X_u^{(k)} = f\left(\int_0^1 \gamma(u)\, \mathrm{d}W_u\right) X_u^{(k)}\, \mathrm{d}s$$

$$Y_{t_{k-1}}^{(k)} = X_{t_{k-1}}^{(k-1)} \qquad X_{t_{k-1}}^{(k)} = Y_{t_k}^{(k)}$$

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The braiding technique: algorithm

Kuo, Shrestha, Sinha, and Sundar 2022, section 4.2

1. Consider a partition $\Delta_n = \{t_0 = 0, t_1, \dots, t_n = t\}$ of $[0, t]$.
2. On each subinterval, iteratively solve the following:
   2.1 the SDE with only diffusion

   $$\begin{cases} \mathrm{d}\, Y_u^{(k)} = \sigma(u)\, Y_u^{(k)}\, \mathrm{d}W_u, & u \in [t_{k-1}, t_k], \\ Y_{t_{k-1}}^{(k)} = X_{t_{k-1}}^{(k-1)}, \text{ and} \end{cases}$$

   2.2 the ODE with only the drift

   $$\begin{cases} \mathrm{d}X_u^{(k)} = f\left(\int_0^1 \gamma(u)\, \mathrm{d}W_u\right) X_u^{(k)}\, \mathrm{d}s, & u \in [t_{k-1}, t_k], \\ X_{t_{k-1}}^{(k)} = Y_{t_k}^{(k)}. \end{cases}$$

   For the first step, use $Y_0^{(1)} = \xi$.
3. The limit of $X^{(n)}$ as $n \to \infty$ gives us the required solution Z.

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Existence lemma

Kuo, Shrestha, Sinha, and Sundar 2022, lemma 4.9

Let $\xi \in L^p(\Omega)$ for some $p > 2$. Consider the kth subinterval $u \in [t_{k-1}, t_k]$ for any $k \in [n]$, and define $Y^{(k)}$ and $X^{(k)}$ as above. Then there exists a set $\Omega_k \subseteq \Omega$ with $\mathbb{P}(\Omega_k) = 1$ such that on Ω_k, we have

$$X^{(k)}_{t_k} = (\xi \circ A^{t_k}_0)\, E^{t_k}_0 \prod_{i=1}^{k} (g^{t_i}_{t_{i-1}} \circ A^{t_k}_{t_i}),$$

where

$$g^v_u = \exp\left[(v-u)\, f\left(\int_0^1 \gamma(u)\,\mathrm{d}W_u\right)\right], \quad \text{and}$$

$$E^v_u = \exp\left[\int_u^v \sigma(s)\,\mathrm{d}W_s - \frac{1}{2}\int_u^v \sigma(s)^2\,\mathrm{d}s\right].$$

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General result

Kuo, Shrestha, Sinha, and Sundar 2022, theorem 4.10

Suppose $\sigma, \gamma \in L^2[0,1]$, $f : \mathbb{R} \to \mathbb{R}$, and $\xi \in L^p(\Omega)$ for some $p > 2$. Then the unique solution of

$$\begin{cases} \mathrm{d}Z(t) = f\left(\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z(t)\,\mathrm{d}t + \sigma(t)\,Z(t)\,\mathrm{d}W_t \\ Z(0) = \xi. \end{cases}$$

in the Skorokhod sense is

$$Z(t) = (\xi \circ A_0^t) \exp\left[\int_0^t \sigma(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s + \int_0^t f\left(\int_0^1 \gamma(u)\,\mathrm{d}W_u - \int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\right].$$

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Basics

- Allows us to calculate probabilities of rare events that decay exponentially.
- Heuristically, $\mathbb{P}\{X^\epsilon \in \mathrm{d}x\} \asymp \exp\left(-\frac{I(x)}{\epsilon}\right)\mathrm{d}x$.
- More rigorously, $\epsilon \log \mathbb{P}\{X^\epsilon \in E\} \to -\inf_E I(x)$ as $\epsilon \to 0$.

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Define

- $\mathcal{C}_\kappa = \{f : [0,1] \to \mathbb{R} \mid f \text{ continuous}, f(0) = \kappa\}$
- $\mathcal{H}^1 = \left\{f \in \mathcal{C}_0 \mid f' \in L^2[0,1]\right\}$

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Theorem (Schilder 1966)

The family $\left(\sqrt{\epsilon}W\right)_{\epsilon>0}$ on $\left(\mathcal{C}_0, \|\cdot\|_\infty\right)$ satisfies LDP with rate function

$$I(\omega) = \begin{cases} \frac{1}{2}\int_0^1 |\omega'(t)|^2 \,\mathrm{d}t & \text{if } \omega \in \mathcal{H}^1, \\ \infty & \text{otherwise.} \end{cases}$$

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Constant initial conditions: setup

Suppose σ and γ are deterministic functions of bounded variation on $[0,1]$. Moreover, suppose $f \in C^2(\mathbb{R})$ is Lipschitz continuous along with $f, f', f'' \in L^1(\mathbb{R})$. For a fixed $\kappa \in \mathbb{R}$, consider the family of linear stochastic differential equations with parameter $\epsilon > 0$ given by

$$\begin{cases} \mathrm{d}Z_\kappa^\epsilon(t) = f\left(\sqrt{\epsilon}\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z_\kappa^\epsilon(t)\,\mathrm{d}t + \sqrt{\epsilon}\sigma(t) Z_\kappa^\epsilon(t)\,\mathrm{d}W_t \\ Z_\kappa^\epsilon(0) = \kappa, \end{cases}$$

Then the unique solutions are given by

$$\begin{aligned} Z_\kappa^\epsilon(t) = \kappa \exp\Bigg[&\sqrt{\epsilon}\int_0^t \sigma(s)\,\mathrm{d}W_s - \frac{\epsilon}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s \\ &+ \int_0^t f\left(\sqrt{\epsilon}\int_0^1 \gamma(u)\,\mathrm{d}W_u - \epsilon\int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\Bigg]. \end{aligned}$$

LSDEs with anticipating coefficients

Constant initial conditions: large deviation principle

Theorem (Kuo, Shrestha, Sinha, and Sundar 2022, theorem 5.7)

The family $(Z_\kappa^\epsilon)_{\epsilon>0}$ follows LDP on $\left(\mathcal{C}_\kappa, \|\cdot\|_\infty\right)$ with the rate function

$$J(y) = \inf\left\{I \circ \theta^{-1}(y)\right\},$$

where I is the Schilder's rate function and the continuous function $\theta : \mathcal{C}_0 \to \mathcal{C}_\kappa$ is defined by

$$\begin{aligned}\theta(x) = \kappa \exp\Bigg[&\int_0^t \sigma(s)\,\mathrm{d}x(s) - \frac{\epsilon}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s \\ &+ \int_0^t f\left(\int_0^1 \gamma(u)\,\mathrm{d}x(u) - \epsilon\int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\Bigg],\end{aligned}$$

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Random initial conditions: setup

Suppose σ, γ, f are as before. Consider the family of linear stochastic differential equations with parameter $\epsilon > 0$ given by

$$\begin{cases} \mathrm{d}Z_\xi^\epsilon(t) = f\left(\sqrt{\epsilon}\int_0^1 \gamma(s)\,\mathrm{d}W_s\right) Z_\xi^\epsilon(t)\,\mathrm{d}t + \sqrt{\epsilon}\sigma(t)\, Z_\xi^\epsilon(t)\,\mathrm{d}W_t \\ Z_\xi^\epsilon(0) = \xi^\epsilon, \end{cases}$$

where each ξ^ϵ is a random variable independent of the Wiener process W. Then the unique solutions are given by

$$\begin{aligned} Z_\xi^\epsilon(t) = \xi^\epsilon \exp\Bigg[\sqrt{\epsilon}\int_0^t \sigma(s)\,\mathrm{d}W_s - \frac{\epsilon}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s \\ + \int_0^t f\left(\sqrt{\epsilon}\int_0^1 \gamma(u)\,\mathrm{d}W_u - \epsilon\int_s^t \gamma(u)\,\sigma(u)\,\mathrm{d}u\right)\mathrm{d}s\Bigg]. \end{aligned}$$

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Theorem (Kuo, Shrestha, Sinha, and Sundar 2022, theorem 5.8)

Let $\kappa \in \mathbb{R}$ and

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}(\xi^\epsilon - \kappa)^2 = -\infty.$$

Moreover, assume that the functions f, f', σ, γ are all bounded. Then the family $\left(Z_\xi^\epsilon\right)_{\epsilon>0}$ follows LDP on $\left(\mathcal{C}_\kappa, \|\cdot\|_\infty\right)$ with the rate function

$$J(y) = \inf\left\{I \circ \theta^{-1}(y)\right\},$$

where I is the Schilder's rate function and $\theta : \mathcal{C}_0 \to \mathcal{C}_\kappa$ is the continuous function shown before.

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Summary

property	classical theory	Ayed–Kuo theory
definition	Itô's integral	Ayed–Kuo integral
well-defined	✓	✓
linearity	✓	✓
mean 0	✓	✓
isometry	Itô's isometry	extension
martingale	martingales	near-martingales
stopped processes	Doob's OST	near-martingale OST
differential equations	SDEs	anticipating SDEs
LDP	Freidlin–Wentzell theory	specific results
inequalities	Doob's martingale inequality	open problem
memory	Markov processes	open problem
measure equivalence	Girsanov's theorem	open problem

Main contributions

1. Extension of Itô's isometry
2. Near-martingale optional stopping theorem
3. LSDEs with anticipating initial conditions
   3.1 Solutions
   3.2 Conditionals
4. LSDEs with anticipating coefficients
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Thank you!

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Appendix A: Malliavin calculus

- Goal. Differentiate a stochastic process w.r.t. ω.
- Formalized by Malliavin calculus.
- Let $W(h) = \int_0^1 h(t)\,\mathrm{d}W_t$. Let $\mathcal{S}$ denote the class of *smooth random variables* such that a random variable $F \in \mathcal{S}$ has the form

$$F = f\left(W(h_1), \dots, W(h_n)\right),$$

  where $f \in C_p^\infty(\mathbb{R}^n)$, and $h_1, \dots, h_n \in L^2[0,1]$ for any natural number n.
- Then the *stochastic derivative* of $F \in \mathcal{S}$ is given by

$$\mathrm{D}F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left(W(h_1), \dots, W(h_n)\right) h_i.$$

- *Example 1.* $\mathrm{D}W_{\frac{1}{2}} = \mathbb{1}_{[0,\frac{1}{2}]}$, since $W_{\frac{1}{2}} = \int_0^1 \mathbb{1}_{[0,\frac{1}{2}]}\,\mathrm{d}W_t$.
- Example. $\mathrm{D}W(h) = h$ and $\mathrm{D}W(h)^2 = 2W(h)\,h$.
- (Integration-by-parts) For $F \in \mathcal{S}$ and $h \in L^2[0,1]$, we have

$$\mathbb{E}(\langle \mathrm{D}F, h\rangle) = \mathbb{E}(FW(h)).$$

Appendix B: Skorokhod integral

» The *Skorokhod stochastic integral* or *divergence operator* δ is defined as the adjoint of the stochastic derivative operator D.

» Let $F_i \in \mathcal{S}$ and $h_i \in L^2[0,1]$ for all $i \in [n]$. For $u(t) = \sum_{i=1}^n F_i\, h_i(t)$, we have

$$\delta(u) = \sum_{i=1}^n F_i\, W(h_i(t)) - \sum_{i=1}^n (\mathrm{D}F_i)(t)\, h_i(t)\, \mathrm{d}t.$$

» $L^2_{\mathrm{ad}}([0,1] \times \Omega) \subset \operatorname{dom}(\delta)$, and for any $u \in L^2_{\mathrm{ad}}([0,1] \times \Omega)$, we have

$$\delta(u) = \int_0^1 u_t\, \mathrm{d}W_t,$$

where the right side is in the sense of Itô (Nualart 2006, proposition 1.3.4).

» Ayed–Kuo integral ≡ Skorokhod integral (Parczewski 2017, theorem 2.3).