Pricing exotic path-dependent options

The Singular Points method

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Introduction

- Assets
  - basic assets
    - riskless: deterministic: $S^0_t = e^{rt}$
    - risky: stochastic process: $(S_t)_t$
  - derivatives – contracts on other assets
    - futures & forwards: symmetric risk
    - options: asymmetric risk
- Problem: pricing derivatives – finding a fair price
- Assumptions
  - viable / no arbitrage / no free lunch
  - frictionless
  - infinitely divisible assets

Option types

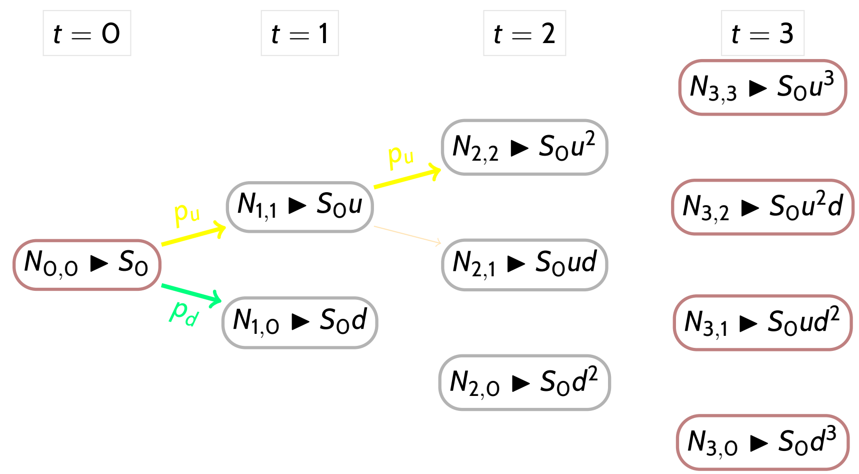
- Simple options – classification of the basis of:
  - exercise time: European or American
  - right of owner: call or put

Example (European call)

payoff function: $(S_T - K)_+ = \max\{S_T - K, 0\}$, exercise only at maturity

- Exotic options
  - Asian: payoff: function of average of the underlying.
  - lookback: payoff: function of extrema of the underlying.
  - cliquet: A series of globally or locally, capped or floored, at-the-money options.
  - digital: existence depends on pre-set barriers.

Visual representation of risky asset



Market models – discrete vs continuous

Parameter	Discrete	Continuous
Example	[CRR79]	[BS73]
Theoretical complexity	Easy	Hard
Ease of implementation	Hard	Easy
Closed-form formula	No[1]	Yes
Computational complexity	Hard: $O(2^n)$[1]	Easy: $O(1)$
Universality	Yes	No

Theorem (Convergence in distribution of CRR to BS)

$CRR \xrightarrow{d} BS$

Quest: Find algorithms with reduced computational complexity under discrete models converging to continuous ones.

[1] CRR: backward recursive

Asian options
Introduction

Definition (Asian options)

Payoff is a function of some form of averaging on the underlying's price.

Example (fixed-strike Asian call of European type)

A fixed-strike Asian call of European type, with a strike-price K would imply that the payoff at maturity is given by $(A_T \quad K)_+$, and the owner of the option may only exercise it at maturity.

Average	Discrete	Continuous
AM	$A_n = \frac{1}{n+1} \sum_{i=0}^{n} S_n$	$A_T = \frac{1}{T} \int_0^T S_t \, dt$
GM	$G_n = \left(\prod_{i=0}^{n} S_n\right)^{\frac{1}{n+1}}$	$G_T = \exp\left(\frac{1}{T} \int_0^T \log(S_t) \, dt\right)$

Asian option
Pre-existing methods

Arithmetic mean

Method	Type	Complex	Remarks
[CRR79]	Tree	$O(2^n)$	simple, accurate, convergence
[HW93]	Tree	$O(n^3)$	accuracy & convergence problems
[BP96]	Tree	$O(n^3)$	accuracy & convergence problems
[Cha+99]	Tree	$O(n^4)$	thin bounds, very large memory
[Vec01]	PDE	$O(n^2)$	not universally applicable
[dFL05]	PDE		more general than [Vec01]

Geometric mean

Closed-form formula exist under BS.

Singular points method for Asian options
Introduction

Idea: At any node $N_{i,j}$:

- payoff P: continuous, convex function of the underlying's average A
- number of possible averages = number of paths to $N_{i,j}$ = $\binom{i}{j}$
- these averages completely characterise the payoff; no other payoff being possible under the given tree
- the above points $((A^l_{i,j}, P^l_{i,j}))_l$ are called singular points
- payoff is represented as continuous, convex, and piecewise-linear function of the underlying's possible averages, found by joining the singular points
- minimum possible average $A^{\min}_{i,j} = \frac{S_0}{i+1} \left(\frac{1 \quad d^{i \quad j+1}}{1 \quad d} + d^{i \quad j} u \frac{1 \quad u^j}{1 \quad u} \right)$
- maximum possible average $A^{\max}_{i,j} = \frac{S_0}{i+1} \left(\frac{1 \quad u^{j+1}}{1 \quad u} + u^j d \frac{1 \quad d^{i \quad j \quad 1}}{1 \quad d} \right)$

Singular points method for Asian options
Start: at maturity ($i = n$)

At any node $N_{n,j}$, $\forall$ three possibilities:

1. $j \in \{0, n\}$: end points, only one path possible, only one singular point $(A, (A - K)_+)$
2. $j \notin \{0, n\}$ and $K \in \left(A_{n,j}^{\min}, A_{n,j}^{\max}\right)$: three singular points:
   1. $(A_{n,j}^{\min}, 0)$
   2. $(K, 0)$
   3. $(A_{n,j}^{\max}, A_{n,j}^{\max} - K)$
3. $j \notin \{0, n\}$ and $K \notin \left(A_{n,j}^{\min}, A_{n,j}^{\max}\right)$: two singular points:
   1. $\left(A_{n,j}^{\min}, (A_{n,j}^{\min} - K)_+\right)$
   2. $\left(A_{n,j}^{\max}, (A_{n,j}^{\max} - K)_+\right)$

- We discuss only the European case.
- The American case needs minor modifications.

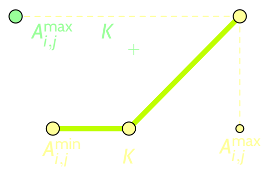
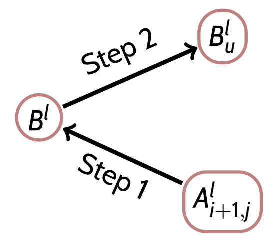


Figure: Case 2

Singular points method for Asian options
Up movement at node $N_{i,j}$, $\forall i < n$

1. Take a singular point $A_{i+1,j}^{l}$.
2. $B^l = \frac{(i+2)A_{i+1,j}^{l} - S_{i+1,j}}{i+1}$.
3. Each $B^l \in \left[A_{i,j}^{\min}, A_{i,j}^{\max}\right]$ is a singular average of $N_{i,j}$.
4. For each such B^l, $B_u^l = \frac{(i+1)B^l + S_{i+1,j+1}}{i+2}$
5. $v_{i,j}(B^l) =$ $\frac{1}{R}\left[p_u v_{i+1,j+1}\left(B_u^l\right) + p_d v_{i+1,j}\left(A_{i+1,j}^{l}\right)\right]$.
6. $\left(B^l, v_{i,j}(B^l)\right)$ is singular point of $N_{i,j}$.

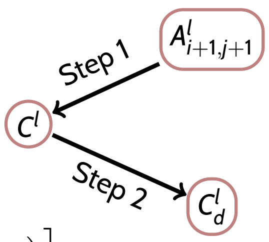
$t = i$ $t = i + 1$



Singular points method for Asian options
Down movement at node $N_{i,j}$, $\forall i < n$

1. Take a singular point $A_{i+1,j+1}^{l}$.
2. $C^l = \frac{(i+2)A_{i+1,j+1}^{l} - S_{i+1,j+1}}{i+1}$.
3. Each $C^l \in \left[A_{i,j}^{\min}, A_{i,j}^{\max}\right]$ is a singular average of $N_{i,j}$.
4. For each such C^l, $C_d^l = \frac{(i+1)C^l + S_{i+1,j}}{i+2}$
5. $v_{i,j}(C^l) =$ $\frac{1}{R}\left[p v_{i+1,j+1}\left(A_{i+1,j+1}^{l}\right) + (1 - p) v_{i+1,j}\left(C_d^l\right)\right]$.
6. $\left(C^l, v_{i,j}(C^l)\right)$ is singular point of $N_{i,j}$.

$t = i$ $t = i + 1$



Singular points method for Asian options

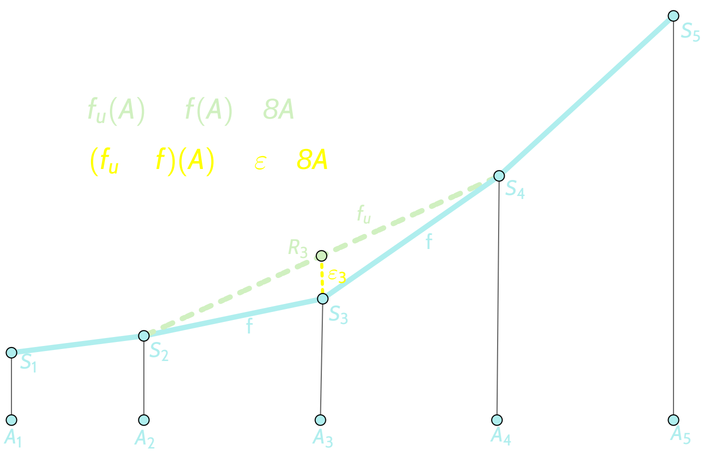
Aggregation and final price

- Sort the singular points obtained according to the abscissa.
- Repeat the procedure for each node in a backward fashion.
- $P_{0,0}^1$ is the exact binomial price.

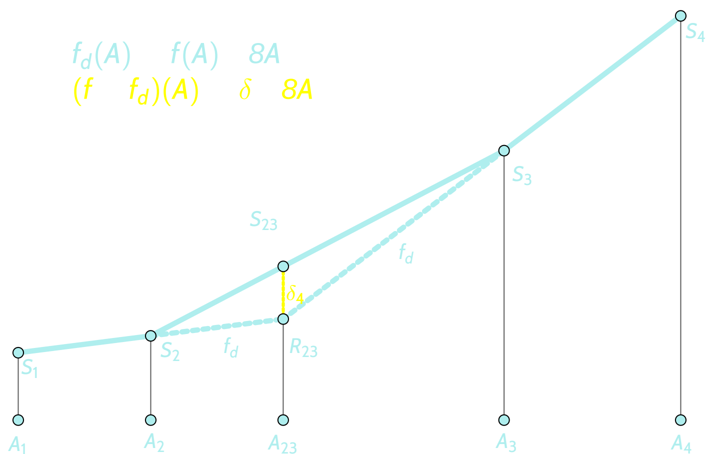
Introduction to approximation

- Ability to approximate id a key strength of the method.
- Not all points necessary $\implies$ selectively remove points.
- Net result: reduction of complexity from exponential time to polynomial time (experimental).
- Experimental order of complexity: $O(n^3)$.

Singular points method for Asian options
Approximations: upper estimates



Singular points method for Asian options
Approximations: lower estimates



Singular points method for Asian options
Numerical results

Data: $s_0 = 100, T = 1, r = 0.1, q = 0.03$.

		$K = 90$		$K = 110$	
	n	$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.4$
Bin	10	14.5912	17.8033	2.5100	6.6523
	25	15.1535	18.6786	2.6270	7.3451
SP	10	14.5925	17.8068	2.5090	6.6511
	25	15.1535	18.6785	2.6270	7.3449
	50	15.3524	19.0420	2.6673	7.4563
	100	15.4732	19.2696	2.6886	7.5174
	200	15.5453	19.4065	2.6996	7.5502
	400	15.5861	19.4845	2.7053	7.5674

Singular points method for Asian options
Summary

- ☐ Introduced by Gaudenzi *et al* [GZA10] in 2010.
- ☑ Convergent to exact CRR and thus BS.
- ☑ Approximation – *A priori* error bounds.
- ☑ Fast – experimental order of complexity $O(n^3)$.
- ☒ Difficult to compute theoretical complexity.
- ☒ Depends on the recombinant nature of the underlying's tree.
- ☒ Not extensible to GM, since the price function is non-linear.

  $$G_u = \left(G^{i+1} S_{i+1,j}\right)^{\frac{1}{i+2}} \;\; / \;\; G^{\frac{i+1}{i+2}}$$

- ☒ Assumes constant volatility, hence not applicable to local volatility models.

Cliquet options: introduction I

forward start option An option that starts at a future date u, with an expiration date T set further in the future. The payoff is given by $(S_T \quad S_u)_+$.

cliquet option An exotic option consisting of a series of consecutive at-the-money forward start options, where the return may be locally or globally capped and floored.

Pros and cons

- ☑ Safety against downside risks.
- ☑ Significant upside potential.
- ☒ Unbounded gains not possible.

Cliquet options: introduction II

Some terminology

- Observation times: time points at which one forward start option expires, and another starts. Assumption: observation times are equidistant.
- Return: $R_i = \frac{S_i \quad S_{i \quad 1}}{S_{i \quad 1}} = \frac{S_i}{S_{i \quad 1}} \quad 1$.
- Running sum: $Z_i = \sum_{k=1}^{i} \max f F_{loc}, \min f C_{loc}, R_k gg$.
- $\text{Payoff} = \text{notional} \quad \max f F_{glob}, \min f C_{glob}, Z_N gg$.

Cliquet options: introduction III

Pre-existing methods for pricing cliquet options

- No prominent tree-based method.
- [Wil02]: PDE based, FD approach.
- [WFV06]: PDE based, FD approach; many more generalisations.

Singular points method for cliquet options

Introduction

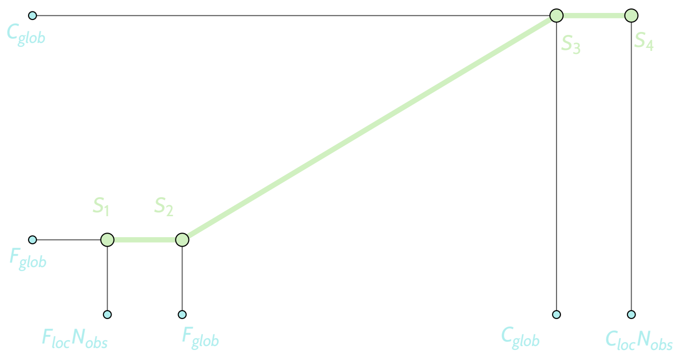
Idea: At every observational time i:

- The possible paths are 2^m, where m is the number of steps in each duration.
- The probability of choosing a path is binomially distributed.
- The running sums Z depend on the possible paths and respective probabilities.

Start with maturity, and proceed backwards.

Singular points method for cliquet options
At maturity ($i = N$)

The price function is not convex, but is piecewise-linear and continuous. It is characterised by four singular points, as depicted in the figure below.

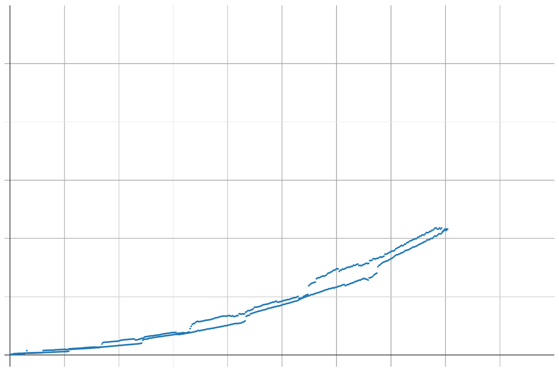


Singular points method for cliquet options
At all other times ($i < N$)

Not all paths are possible due to local floors and caps. The realizable paths and associated quantities are denoted by primed variables.

1. From all the singular points Z^l_{i+1}, subtract all the possible returns R'_j to get $B_{l,j}$.
2. Each $B_{l,j}$ 2 $[iF_{loc}, iC_{loc}]$ becomes a singular point at time i.
3. The corresponding price function is: $V_i(B_{l,j}) = e^{\ \frac{rT}{N}} \sum_{j=0}^{j_0} \left[p'_j V_{i+1}(Z + R'_j)\right]$.
4.

Singular points method for cliquet options
Experimental computational complexity – $O(m^2)$



Singular points method for cliquet options
Summary

- ☐ Introduced by Gaudenzi *et al* [GZ11] in 2011.
- ☑ Convergent to exact CRR and thus BS.
- ☑ Approximation – *A priori* error bounds.
- ☑ Significant speed improvement in low volatility cases against binomial model.
- ☑ Can be used for local volatility models and varying interest rates in each period.
- ☑ Fast – experimental order of complexity $O(m^2)$.
- ☒ Difficult to compute theoretical complexity.

Recapitulation

- The singular points method is a new efficient technique to evaluate path-dependent exotic options.
- The theory varies with option type.
- In the Asian case, the method is quite complicated and is not very flexible, although it is fast and efficient. It is easily generalised to the American case and to lookback options. We have shown that it fails to be generalised for geometric mean and local volatility models.
- In the cliquet case, the method is quite flexible, and takes care of local volatility models and varying interest rates. We have found out that the experimental order of computational complexity is approximately $O(m^2)$ for low m, which is a marked improvement over pre-existing methods.

Further research

- Theoretical complexity: dependence of singular point redundancy on initial data.
- Customising the method for other path-dependent options (like?).

Questions?

Thank you!

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