The Singular Points method for Asian American options for local volatility models

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1 Introduction

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2 Notations

$[n] = \{0, 1, 2, \ldots, n\}$

3 Basic formulae

Arithmetic average

$$A_n = \frac{\sum_{i=0}^{n} S_i}{n+1} \tag{3.1a}$$

$$\implies (n+1)A_n = \sum_{i=0}^{n} S_i \tag{3.1b}$$

4 Results

Definition 1 (Path). *A path is a sequence $(j_i)_{i\in[n]}$ such that $j_{i+1} \in \{j_i, j_i + 1\}$.*

Lemma 1. *Let there be two paths α and β, such that $S_{i,j_i^\alpha} >= S_{i,j_i^\beta}\ \forall i$. Then $A^\alpha >= A^\beta$.*

Proof. Clearly if $S_{i,j_i^\alpha} = S_{i,j_i^\beta}\ \forall i$, then $A^\alpha = A^\beta$.
We only need to show the case of inequality. Let $S_{i,j_i^\alpha} = S_{i,j_i^\beta}\ \forall i \in [n] \setminus \{l\}$. That is, $S_{l,j_l^\alpha} > S_{l,j_l^\beta}$.
Now, from equation 3.1, we have:

$$\begin{aligned}
(n+1)A^\alpha_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\alpha} + \sum_{i=l+1}^{n} S_{i,j_i} \\
(n+1)A^\beta_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\beta} + \sum_{i=l+1}^{n} S_{i,j_i} \\
\implies (n+1)\left(A^\alpha_{n,j} - A^\beta_{n,j}\right) &= S_{l,j_l^\alpha} - S_{l,j_l^\beta} \\
&= S_{l-1,j_{l-1}} u_l - S_{l-1,j_{l-1}} d_l \\
&= S_{l-1,j_{l-1}}(u_l - d_l) > 0 \\
\implies A^\alpha_{n,j} &> A^\beta_{n,j}
\end{aligned}$$

□

Remark. *The path α signifies the path above and β signifies the path below. Thus, the path above always has a higher arithmetic mean.*

Corollary 1. *At each node $N(n, j)$, the average values vary between a minimum average $A_{n,j}^{\min}$ (corresponding to the path with $(n-j)$ down movements followed by j up movements) and a maximum average $A_{n,j}^{\max}$ (corresponding to the path with j up movements followed by $(n-j)$ down movements).*

Proof. The path 'min' is the bottom-most one and 'max' is the topmost one. □

Lemma 2 (Lemma 3). *The price function at maturity $v_{n,j}$ is convex and piecewise linear.*

Proof. By construction. See the paper. □

Lemma 3 (Lemma 4). *The price function $v_{i,j}$ is concave and non-linear.*

Proof. 3.1

□

5 Conclusion

The singular points method may not be used to price Geometric Asian options.