Pricing exotic path-dependent options

The Singular Points method [GZA10; GZ11]

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(Financial) Market models: introduction

- Financial assets
  - basic assets

    riskless (deterministic) treasury bonds: $S_t^0 = e^{rt}$

    risky (stochastic) stocks: $(S_t)_t$, parameter σ characterises risk
  - derivatives – contracts on other assets (*underlying*), till *maturity* T

    futures symmetric risk

    options asymmetric risk – historically used for insurance
- Problem: pricing derivatives
- Assumptions
  - viable / no arbitrage / no free lunch
  - frictionless
  - infinitely divisible assets

Option types

- Simple options – classification of the basis of:

  exercise time European or American

  right of owner call or put

Example (European call)

payoff: $h(S_T) = (S_T - K)_+ = \max\{S_T - K, 0\}$, exercise at maturity

- Exotic options – usually path-dependent

  Asian payoff: function of average of the underlying.

  lookback payoff: function of extrema of the underlying.

  cliquet a series of globally or locally bounded at-the-money options.

  digital existence depends on pre-set barriers.

Evolution of risky asset: [CRR79] model (discrete)

$t = 0$	$t = 1$	$t = 2$	$t = 3$
			$N_{3,3} \blacktriangleright S_0 u^3$
		$N_{2,2} \blacktriangleright S_0 u^2$	

[BS73] model (continuous)

riskless $S_t^0 = e^{rt}$

risky $S_t = s_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}$

Theorem (Convergence of prices from CRR to BS)

Prices of basic assets under CRR $\xrightarrow{d}$ prices of basic assets under BS.

Corollary (Convergence of evaluation formulae)

The previous theorem implies that evaluation formulae under CRR converge in distribution to evaluation formulae for BS.

Approximate BS price by using CRR model.

Quest: Find algorithms with reduced computational complexity.

Market models: discrete vs. continuous

Parameter	Discrete	Continuous
Example	[CRR79]	[BS73]
Theoretical complexity	Easy	Hard
Ease of implementation	Hard	Easy
Closed-form formula	No[1]	Yes
Computational complexity	Hard: $O(2^n)$[1]	Easy: $O(1)$
Versatile	Yes	No

[1] CRR: backward recursive

Asian options: introduction

Payoff: function of some form of average price.

Average	Discrete	Continuous
AM	$A_n = \frac{1}{n+1} \sum_{i=0}^{n} S_n$	$A_T = \frac{1}{T} \int_0^T S_t \, \mathrm{d}t$
GM	$G_n = \left(\prod_{i=0}^{n} S_n\right)^{\frac{1}{n+1}}$	$G_T = \exp\left(\frac{1}{T} \int_0^T \log(S_t) \, \mathrm{d}t\right)$

Example (fixed-strike Asian call of European type)

Given strike-price K, payoff $= (A_T - K)_+$, exercised only at maturity.

Asian option: pre-existing methods

Arithmetic mean

Method	Type	Complex	Remarks
[CRR79]	Tree	$O(2^n)$	simple, accurate, convergence
[HW93]	Tree	$O(n^3)$	accuracy & convergence problems
[BP96]	Tree	$O(n^3)$	accuracy & convergence problems
[Cha+99]	Tree	$O(n^4)$	thin bounds, very large memory
[Vec01]	PDE	$O(n^2)$	not universally applicable
[dFL05]	PDE		more general than [Vec01]

Geometric mean

Closed-form formula exist under BS.

SPM for Asian options: idea

$N_{i,j}$ Node of the binomial tree

$A^l_{i,j}$ Average upto $N_{i,j}$, $l \in \binom{i}{j}$

$P^l_{i,j}$ Corresponding option price

$\left\{ \left(A^l_{i,j}, P^l_{i,j} \right) \right\}_l$ singular points (SPs) – completely characterise price

price function continuous, convex, piecewise-linear; found by joining SPs.

SPM for Asian options: start at maturity ($i = n$)

For $N_{n,j}$: calculate $A_{i,j}^{\min}$ and $A_{i,j}^{\max}$.

Singular points:

1. $j \in \{0, n\}$: $(A, (A-K)_+)$
2. $j \notin \{0, n\}$ and $K \in \left(A_{n,j}^{\min}, A_{n,j}^{\max}\right)$:
    1. $(A_{n,j}^{\min}, 0)$
    2. $(K, 0)$
    3. $(A_{n,j}^{\max}, A_{n,j}^{\max} - K)$
3. $j \notin \{0, n\}$ and $K \notin \left(A_{n,j}^{\min}, A_{n,j}^{\max}\right)$:
    1. $\left(A_{n,j}^{\min}, (A_{n,j}^{\min} - K)_+\right)$
    2. $\left(A_{n,j}^{\max}, (A_{n,j}^{\max} - K)_+\right)$

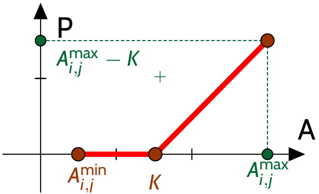
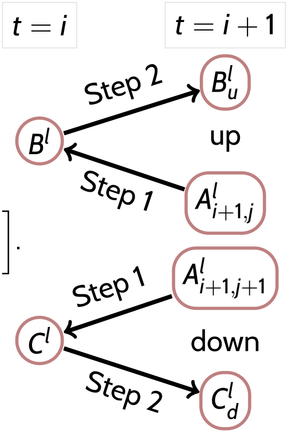


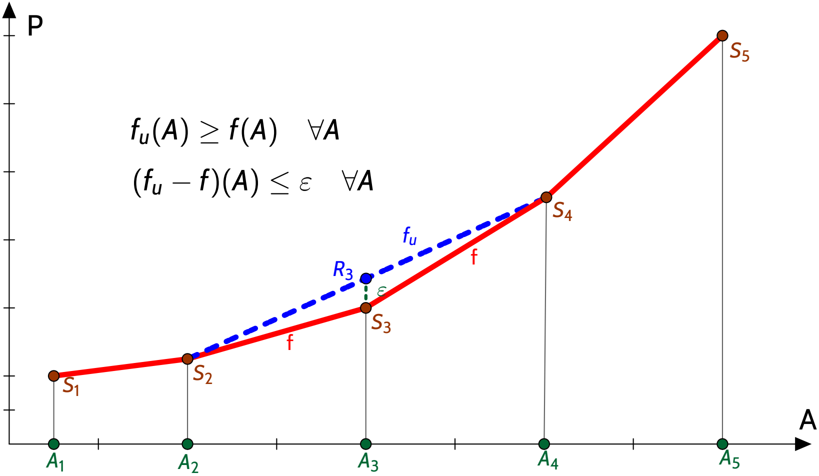
Figure: Case 2

SPM for Asian options: exact price

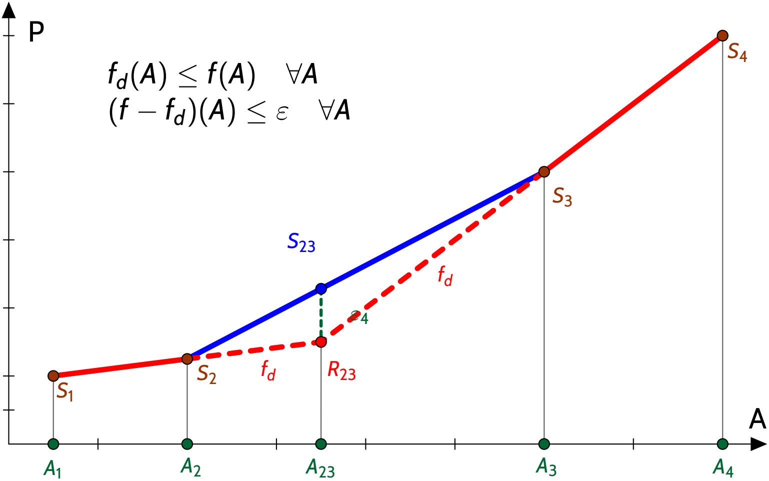
- Up movement (for $N_{i,j}$):
  1. $\forall A^l_{i+1,j} : B^l = \frac{(i+2)A^l_{i+1;j} - S_{i+1;j}}{i+1}$.
  2. If $B^l \in \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right] \implies B^l$ is SA.
  3. For each SA, $B^l_u = \frac{(i+1)B^l + S_{i+1;j+1}}{i+2}$.
  4. $v_{i,j}(B^l) =$ $\frac{1}{R}\left[p_u v_{i+1,j+1}\left(B^l_u\right) + p_d v_{i+1,j}\left(A^l_{i+1,j}\right)\right]$.
  5. $\left(B^l, v_{i,j}(B^l)\right)$ is a SP.
- Down movement (for $N_{i,j}$).
- Aggregate and sort by SAs.
- Repeat for all j.
- Iterate backward till $i = 0$. $P^1_{0,0}$ is the exact binomial price.



SPM for Asian options: approximation
Upper estimates



SPM for Asian options: approximation
Lower estimates



SPM for Asian options: numerical results

Data: $s_0 = 100, T = 1, r = 0.1, q = 0.03$.

		$K = 90$		$K = 110$	
	n	$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.4$
Bin	10	14.5912	17.8033	2.5100	6.6523
	25	15.1535	18.6786	2.6270	7.3451
SP	10	14.5925	17.8068	2.5090	6.6511
	25	15.1535	18.6785	2.6270	7.3449
	50	15.3524	19.0420	2.6673	7.4563
	100	15.4732	19.2696	2.6886	7.5174
	200	15.5453	19.4065	2.6996	7.5502
	400	15.5861	19.4845	2.7053	7.5674

SPM for Asian options: summary

- ☐ Introduced by Gaudenzi *et al* [GZA10] in 2010.
- ☑ Convergent to exact CRR and thus BS.
- ☑ Easily generalised to American case and lookback options.
- ☑ Approx: unnecessary points $\implies$ selective removal.
- ☑ Approx: *A priori* error bounds.
- ☑ Approx $\implies$ fast: reduction of complexity from exponential time to polynomial time (experimental) of $O(n^3)$.
- ☒ Difficult to compute theoretical complexity.
- ☒ Depends on the recombinant nature of the underlying's tree.
- ☒ Not extensible to GM, since the price function is non-linear.

  $$G_u = \left(G^{i+1} S_{i+1,j}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}}$$
- ☒ Constant volatility assumption $\implies$ local volatility models fail.

Cliquet options: introduction

Definitions

forward start option price option today with payoff $= (S_T - S_u)_+$, $0 \leq u < T$.

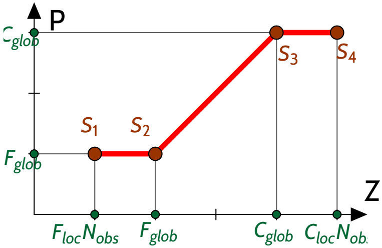
cliquet option a series of consecutive at-the-money forward start options, with bounded returns.

Pre-existing methods for pricing

- No prominent tree-based method.
- [Wil02]: PDE based, FD approach.
- [WFV06]: PDE based, FD approach; generalisations.

Cliquet options: terminology

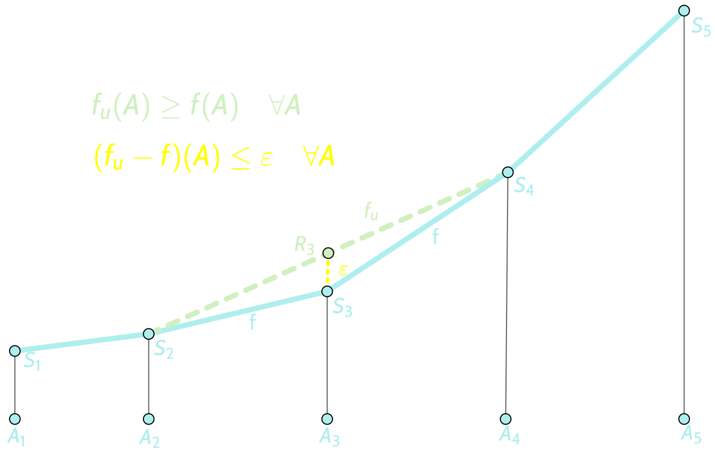
- N: observation times (equidistant).
- Return: $R_i = \frac{S_i - S_{i-1}}{S_{i-1}} = \frac{S_i}{S_{i-1}} - 1$.
- Running sum: $Z_i = \sum_{k=1}^{i} \max\{F_{loc}, \min\{C_{loc}, R_k\}\}$.
- Payoff = $\max\{F_{glob}, \min\{C_{glob}, Z_N\}\}$.
- m: computational time steps.
- 2^m possible paths; $\sim \mathrm{Bin}(p)$.
- Z depends on paths, probs.
- $(Z, P) \implies$ SP (P is price).
- Price function at maturity.
- Proceed as in the Asian case.
- Iterate backwards. P_0^1 is the exact binomial price.



SPM for cliquet options: approximation

$f_u(A) \geq f(A) \quad \forall A$

$(f_u - f)(A) \leq \varepsilon \quad \forall A$



SPM for cliquet options: numerical results

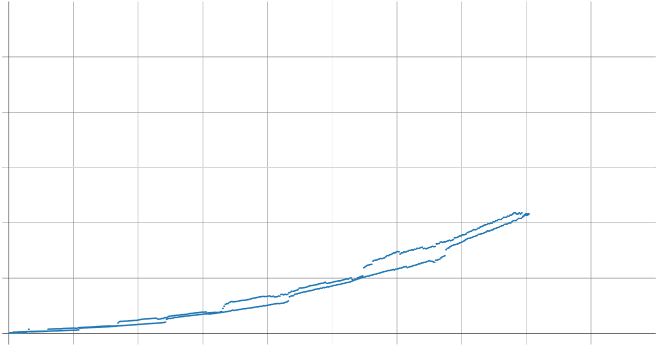
Data

- $F_{loc} = 0, C_{loc} = 0.08, F_{glob} = 0.16, C_{glob} = \infty$
- $T = 5, N = 5, r = 0.03$

σ	m	Price		Time (s) [2]	
		Bin	SP	Bin	SP
	200	0.173716366	0.173716366	0.0165	0.00828
0.2	500	0.173922597	0.173922671	0.0875	0.0437
	1000	0.174051949	0.174051983	2.38	0.183
	200	0.150465004	0.150466828	600	6.09
0.02	500	0.150508871	0.150510526	∞	24.2
	1000	0.150522368	0.150524027	∞	55

[2] ∞ means time taken is more than an hour.

SPM for cliquet options: complexity $O(m^2)$



SPM for cliquet options: summary

- ☑ Approximation: *A priori* error bounds.
- ☒ Approximation: converges to binomial price, no bounds.
- ☑ Significant speed improvement in low volatility cases against binomial model.
- ☑ Can be used for local volatility models and varying interest rates in each period.
- ☑ Fast – experimental order of complexity $O(m^2)$.
- ☒ Difficult to compute theoretical complexity.

Recapitulation

- Efficient polynomial-time technique.
- Theory and flexibility varies with option type.

Further research

- Theoretical complexity: dependence of singular point redundancy on initial data.
- Verify cliquet complexity for large m.
- Customising the method for other path-dependent options.

Questions?

Thank you!

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