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The Singular Points method for pricing exotic path-dependent options

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1 Financial instruments

A *financial instrument* or a *financial asset* is an intangible asset whose value is derived from a contractual claim, such as bank deposits, bonds, stocks and derivatives. Financial assets are usually more liquid than other tangible assets, such as commodities or real estate, and may be traded on financial markets. Every financial asset is characterised by its return. When the return is deterministic, we call it a *risk-free* or *riskless* asset. When the return is contingent on the market and external conditions, it is called *risky*. It must be kept in mind that no instrument is fundamentally risk-free, it has only negligible risk compared to its risky counterparts.

1.1 Riskless instruments

Bonds A *bond* is an instrument of indebtedness of the bond issuer to the holders. It is a *debt security*, under which the issuer owes the holders a debt and, depending on the terms of the bond, is obliged to pay them interest (the coupon) and/or to repay the principal at a later date, termed the maturity date. Bonds can also be thought of as a *loan* given to the issuer of the bond by the holder. A bond issued by a reliable institution like the United States government is a good illustration of a risk-free asset. This is because the probability of such an organisation defaulting is close to zero, or in other words, the bond has negligible *credit default risk*. Such bonds are only subject to fluctuations of the current interest rate, called *interest rate risk*. If we assume that the interest rate is deterministic (the fluctuations are not random), the value of the bond is consequently computable at any given future date, making them riskless. Such an assumption is quite reasonable in short periods of time and for low credit risk institutions.

Compounding The idea of compounding is the first idea that we must be familiar with. Essentially, a riskless asset will increase in monetary value in a deterministic manner if we keep it in the market. The increase depends on the compounding frequency and the duration of investment. The term compounding is used because the interest earned in each period also contributes to the principal in the successive periods.

Let the compounding frequency is n times per year, the total time is t, and the annual rate of interest is r. Then

$$B_t = B_0 \left(1 + \frac{r}{n}\right)^{\lfloor nt \rfloor}, \tag{1.1}$$

where B_0 is the starting value of the asset, and B_t is the value of the asset at time t.

If the compounding is continuous, we let $n \to \infty$ to obtain

$$B_t = B_0 e^{rt}. \tag{1.2}$$

1.2 Risky instruments

Stocks A *stock* of a corporation is an ownership certificate, and constitutes the equity stake of its owners. It represents the residual assets of the company that would be due to stockholders after discharge of all senior claims such as secured and unsecured debt. A *share* of a stock is a unit of ownership of the organisation. Stocks are inherently risky, since the value of the organisation may change from time to time due to various internal and external factors.

Derivatives A *derivative* is a contract between two parties that specify conditions (starting and termination dates, resulting values and definitions of the underlying variables, the parties' contractual obligations, and the notional amount) under which transactions are to be made between the parties. The most common underlying assets include commodities, stocks, bonds, interest rates and currencies, but they can also be other derivatives, which adds another layer of complexity to proper valuation. Essentially the value of a derivative is a function of the value of the *underlying* asset. Derivatives are traded in their own right, and a fair price must be found for a derivative at each time of its existence. This problem is known as the *pricing problem*. One of the primary motivations for creation of derivatives was to hedge one's position from fluctuations in the market. A *hedge* is an investment strategy intended to offset potential losses or gains that may be incurred by a companion investment. Finding a hedging strategy is called the *hedging problem*. These are the two problems that must be looked at when defining a market model. In this thesis, we shall mainly concern ourselves with the pricing problem of a particular class of derivatives, called *exotic options*.

1.2 Classification of derivatives

Derivatives may be classified on the basis of various factors. One important factor is whether the risk is shared, or taken up by only one party. Another factor is the nature of the function (of the underlying) that the derivative depends on. This function may either be dependent only on the final value of the underlying (*path-independent*), or on the path that it took to reach this final value (*path-dependent*). The function may be discrete (*digital* or *binary*), or continous. In this section, we briefly look at some of the more important types of derivatives. [1]

Futures and forwards

Definition 1.1 (Futures and forwards). Futures and forwards are contracts between two parties, the seller and the buyer, to exchange a certain asset at a predetermined future time at a agreed upon price. Futures are *exchange-traded derivatives* (ETDs), whereas forwards are traded *over-the-counter* (OTC).

Such derivatives obligate the contractual parties to the terms over the life of the contract. Futures are in some sense ‘safer’ compared to forwards, since the involved parties must go through standard protocols of the exchange. The contract contains the following details.

T: The maturity, or the duration of the contract

F_0: The delivery price, or the price prefixed (at the initial time) at which trades must take place at maturity

r: The rate of interest

underlying: The asset(s) of trade at maturity

S_0: The initial value of the underlying asset(s)

There are, of course, other possibilities, for instance a variable interest rate, dividends yielded by the underlying, but these may be seen as generalised cases of the stated simple case.

Let us assume that the compounding is continuous. We may show that under the condition of a *viable market*[2], the fair delivery price of a future with underlying prices $(S_t)_{t\in[0,T]}$ at any time $t \in [0, T]$ is given by the following equation.

$$F_t = S_t e^{r(T-t)} \tag{1.3}$$

[1] A more interested reader should consult the following extensive Wikipedia articles.

- https://en.wikipedia.org/wiki/Option_(finance)#Types
- https://en.wikipedia.org/wiki/Option_style

[2] see Section 1.3 for definitions of the term

Options

Definition 1.2 (option). An *option* is a derivative which provides the buyer *the right, but not the obligation* to enter the contract under the specified terms.

Thus, the owner of the option may choose whether to exercise his right or not. Thus, on the one hand, the owner of the option bears no risk, since all the choice is his. On the other hand, the seller of the option is *obligated* to honour the terms of the contract, whether it benefits him or not – essentially making him bear all the risks. This asymmetry is primarily what sets options apart from the locks discussed earlier. The contract contains the following details

- T: The maturity, or the duration of the contract
- K: The strike price, or the prefixed price at which trades may take place at maturity
- r: The rate of interest
- **underlying:** The asset(s) which may be traded at maturity
- S_0: The initial value of the underlying asset at the initial time
- **right:** The exact right that the owner of the options has (see below)
- **exercise time:** European or American

According to the right of the owner, a simple option may be of two types.

- **call:** The owner has the right to buy. In this case, the price of the option at maturity is given by $c_T = (S_T - K)_+$, where $(x)_+ := \max\{0, x\}$.
- **put:** The owner has the right to sell. In this case, the price of the option at maturity is given by $p_T = (K - S_T)_+$.

Of course, other complicated ownership rights may be constructed, but we shall concern ourselves with these basic ones for time being.

Proposition 1.1 (Equality of portfolios). *In a* viable *and* frictionless market[3]*, if the values of two portfolios coincide at a time* T*, they have to coincide at* 0 *(or any other intermediate time* t*).*

Proof. Let us denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the two portfolios and by $v(\mathcal{P})$ the value of a portfolio $\mathcal{P}$ at time t. By assumption $v_T(\mathcal{P}_1) = v_T(\mathcal{P}_2)$, so we assume by contradiction that $v_T(\mathcal{P}_1) > v_T(\mathcal{P}_2)$.

Under this hypothesis it is possible to construct the following arbitrage strategy. At time 0, one can borrow the portfolio $\mathcal{P}_1$ and sell it right away to buy portfolio $\mathcal{P}_2$. One can pocket the difference $v_T(\mathcal{P}_1) - v_T(\mathcal{P}_2) > 0$. At $t = T$ the values of the two portfolios coincide, so selling $\mathcal{P}_2$ one gets the exact money to buy $\mathcal{P}_1$ to be returned to the original lender. An profit is achieved, without investing any money, implying an arbitrage and violating the viable market hypothesis. Similarly, we can show that $v_0(\mathcal{P}_1) < v_0(\mathcal{P}_2)$ would also enable an arbitrage opportunity. $\square$

[3]see Section 1.3 for definitions of the terms

Looking at the call and put prices at maturity, we note that they are related. In fact, $S_T - K = (S_T - K)_+ + (S_T - K)_- = (S_T - K)_+ - (K - S_T)_+ = c_T - p_T$. For any general time t, using Proposition 1.1, it holds that $c_t - p_t = S_t - Ke^{-r(T-t)}$. This is known as the *call-put parity*.

According to the time at which the option may be exercised, an option may be of two types

European: The owner may exercise the option only at maturity

American: The owner may exercise the option at any time up to the maturity

Since American options allow for more flexibility for the owner, and thus more risk for the seller, they are more expensive as compared to their European counterparts. Let c_t, p_t denote the prices of an European call and put, and C_t, P_t denote the prices of an American call and put, respectively. Then, we must have $C_t \geqslant c_t$ and $P_t \geqslant p_t$.

European options are path-independent and the simplest type of options available. Hence, they are popularly known as *vanilla options*. The American options are path-dependent. Typically, other options which are more complex in nature are collectively called *exotic options*. These are usually path-dependent, and may be either European, American or have more complex exercise times. A few such options are described in brief.

Asian: The payoff depends on the average of the underlying's prices

lookback: The payoff depends on one of the extrema of the underlying's prices

cliquet or ratchet: A series of globally or locally, capped or floored, at-the-money options, but where the total premium is determined in advance.

barrier: The price of the underlying reaching the pre-set barrier level either springs the option into existence (*knock-in*) or extinguishes an already existing option (*knock-out*).

Return We denote the *spot price* of a risky asset $\forall t \in [0, T]$ by the stochastic process $(S_t)_t$. Since the future value of the asset is adventitious, we use the following quantities to measure the return of the risky asset in a time interval.

Definition 1.3 (absolute and relative returns). The absolute return on an asset for the time interval $[0, t]$, $t \in [0, T]$ is given by

$$\tilde{R}_t = S_t - S_0$$

The relative return on the asset is given by

$$R_t = \frac{S_t - S_0}{S_0}$$

1.3 Financial Markets – Characteristics

The idea of financial markets is intricately linked to that of financial transactions. Analogous to the ordinary markets, a financial market is a human construct to allow transaction between investors. The assets in the financial market are typically financial instruments such as bonds, stocks and derivatives discussed in the previous section. In this section we will primarily concern ourselves with the nature of financial markets and the assumptions we make while modelling them. Some of the jargon used in the previous section will become clear after this section.

Viable market absence of arbitrage opportunities / viable / No Free Lunch

Frictionless market

Infinitely divisible assets

Small investor hypothesis

Chapter 2

Cox Ross Rubinstein model

1 Introduction

.... Since continuous models are mathematically more complex than their discrete time counterparts (also known as 'lattice models'), we shall discuss the latter first. We shall then show that under certain convergence conditions, the discrete models discusses converges to the continuous models.

2 The binomial model

We start our discussion with one of the simplest model used for pricing of assets, the binomial model. This model was first introduced by Cox, Ross and Rubinstein [CRR79] in the paper titled "Option pricing: A simplified approach" in 1979. Even though it is very simple, it does contain all the necessary ingredients to construct a viable market model, and to solve the problems of pricing and hedging of derivatives.

In this model, we essentially have

- two time points, $t = 0$ (present) and $t = 1$ (future)
- two traded assets
  - the riskless asset, usually a bond, which is compounded at a constant rate of interest $r > 0$
  - the risky asset, usually a stock, which may either go up with a factor u, or down with a factor d.

The binomial model is so called because there are two times, two assets and two possible movements of the risky asset.

We denote as S_0 the value of the risky asset at time 0, and by S_1 its value at time 1. Firstly, in order to have no arbitrage opportunities, we must have $d < R < u$, where $R := 1 + r$. Secondly, for fairness, there must exist a probability distribution $p, (1-p)$ – signifying the

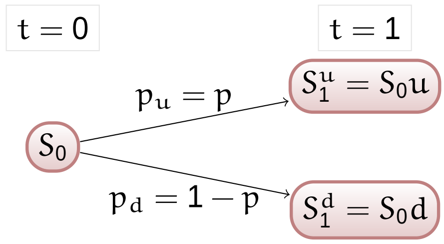

Figure 1. Binomial tree for the underlying

probabilities of the up and down movements – such that the expected value of the asset remains the same as that of the riskless asset given the same time. (See Figure ??.)

Thus, we have

$$S_1 = \begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } (1-p) \end{cases}$$

We may also write $S_1 = TS_0$, where T is a random variable taking values in $\{u, d\}$ with associated probability distribution $(p, 1-p)$.

$$T = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } (1-p) \end{cases}$$

We want $p \in [0, 1]$ such that $E(S_1) = S_0 R$. Thus

$$E\left(\frac{S_1}{R}\right) = S_0 \tag{2.1}$$

$$\implies \frac{1}{R}(puS_0 + (1-p)dS_0) = S_0$$

$$\implies \qquad p = \frac{R-d}{u-d} \tag{2.2}$$

The probability thus obtained is called the *risk neutral probability*, because under this probability, it is equivalent for the investor whether he invests in a risky or a riskless asset. Note that this probability is completely objective as it is determined completely by the parameters u, d and r.

Now we impose the condition $p \in [0, 1]$ to obtain

$$d < R < u \tag{2.3}$$

Pricing a call Let us use the above model to price a call. Recall that the pay-off of a call is given by $h(x) = (x - K)_+$, where K is the strike price, a fixed value specified in the contract. Thus, we know the values of the call at maturity. To ensure fairness, we may again write the following Note that for financial viability, we must have $K \in (S_1^d, S_1^u)$, implying $c_1^u = (S_1^u - K)_+ = (S_1^u - K)$ and $c_1^d = (S_1^d - K)_+ = 0$ (See Figure 2).

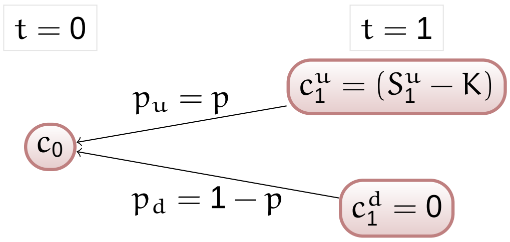

Figure 2. Binomial tree for the underlying

$$\begin{aligned} c_0 &= E(\frac{c_1}{R}) \\ &= pc_1^u + (1-p)c_1^d \\ &= pc_1^u \\ &= \frac{R-d}{u-d}(uS_0 - K) \end{aligned}$$

Thus we have been able to price the call uniquely at all times. This is an implication of the completeness of the market, whose randomness is totally characterized by the unique probability measure p.

For the sake of completeness, we comment here that the call is also completely hedgable in the market model.

3 The Cox-Ross-Rubinstein model

In this section we extend the binomial model introduced in Section 2 to a sequence of integer times $[N] := \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_n, P)$ be a finite probability space ($|\Omega| < \infty$), such that $P(\omega) > 0$ $\forall \omega \in \Omega$, endowed with with a filtration $(\mathcal{F}_n)_n$, such that $\mathcal{F}_0$ is trivial ($\mathcal{F}_0 = \{\emptyset, \Omega\}$).

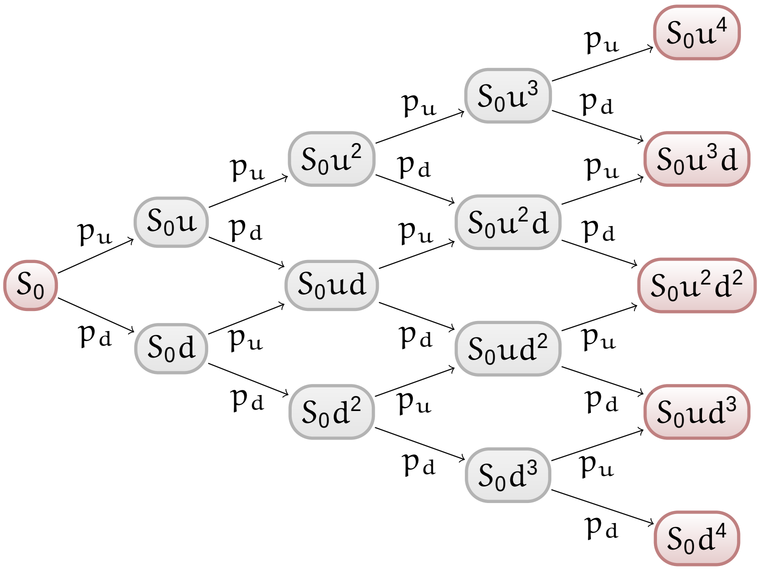

Figure 3. A 4 step lattice ($N = 4$)

Proof. See [LL96, page 9, Theorem 1.3.4] □

Augmented by these theorems, we may now seek to price options in this model. If the pay-off of an option at maturity is given by the function $h(x)$ (for example, in the case of a call, $h(x) = (x - K)_+$), then the price of the option at any time step n is given by $E^*(h(S_N)|\mathcal{F}_n)$

4 European options

5 American options

Chapter 3

Black Scholes model

Chapter 4

The Singular Points method applied to Asian options

1 Introduction

As we have seen in the earlier chapters, European and American options may be priced using the CRR and BS models. Even though the BS model has a very high degree of computability, it does not allow us to find closed-form pricing formulae for many path-dependent options. The way out is by using numerical methods. A class of numerical methods use tree or lattice methods in the CRR model. One simple idea is to apply an explicit pricing scheme using CRR, which tends to BS as the number of time steps increases to infinity. But the exponential number of paths (2^n to be exact, where n is the number of time steps) make the method very slow and memory intensive, making it impractical in terms of computability. A logical step would be to modify the basic CRR model to allow for approximations. In this direction, Gaudenzi et al[GZA10] introduced a new method called the 'singular points method' for pricing certain path-dependent options in an e□cient manner. The chapter is a study on this method.

We will mainly focus on Asian options, in which the price is expressed as a function of some form of averaging on the underlying's price. Popular Asian options use the arithmetic or geometric means as the average. Again, Asian options may be exercised only at maturity (European) or at any time till the maturity (American). They may give the owner of the option the right to either sell (put) or buy (call). Theoretically, we may study either a call or a put, because the framework for the other one may be derived in the exact same way.

TODO: Push this defn to an earlier chapter.

Definition 4.1 (Path-dependent option). A path-dependent option is an option for which the value of the option is dependent not only on the final value of the underlying, but also on the path taken to reach that value.

2 Existing methods

Before we go into the details of the singular points method, we shall look into the pre-existing methods, and discuss their advantages and disadvantages briefly.

Ref: Forsyth et al (2002)

- American Asian options with arithmetic mean
  - Tree based
    * CRR binomial method
    * Hull and White (1993)
    * Barraquand and Pudet (1996)
    * Chalasani et al (1999a, b)
  - PDE based
    * Vecer (2001)
    * D'Halluin et al (2005)
- American lookback options
  - Hull and White (1993)
  - Barraquand and Pudet (1996)
  - Babbs (2000), using a 'change of numeraire' approach, which cannot be applied to the fixed-strike case

TODO: Discuss the advantages and disadvantages of each method.

A number of these algorithms has been implemented in Premia 13. Premia is a software designed for option pricing, hedging and financial model calibration. It has been developed by the 'MathFi' team in INRIA. It is provided with it's C/C++ source code and an extensive scientific documentation. More information about Premia can be found at https://www.rocq.inria.fr/mathfi/Premia/.

3 The Singular Points method

The price of an Asian option at each instance is a continuous function of the underlying's average. Since the number of paths to a node in a binomial tree is finite, we have that at each node of the underlying's binomial tree, the option price may be represented as a piecewise-linear, continuous, convex function of the average. We shall develop the theoretical idea in this section. In the subsequent section, we shall see that the nature of the function allows us to make approximations with *a priori* error bounds.

Definition 4.2 (Singular points and singular values). Let $P = (P_i)_{i\in[n]} = ((x_i, y_i))_{i\in[n]}$, $n \in \mathbb{N}$ be a sequence of points such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \tag{3.1a}$$

$$m_{i+1} := \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \leqslant \frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}} = m_{i+2} \qquad \forall i \in \{1, \dots, n-1\} \tag{3.1b}$$

Let $f : [a, b] \to [0, \infty)$ be the function obtained by linear interpolation of the points in P. From the definition of f and 3.1b, the function is continuous, piecewise-linear and convex.

Then, the elements of P are called *singular points of* f and the abscissae $\{x_i\}_{i\in[n]}$ are called *singular values of* f.

Remark 4.1. We note that the singular points characterise such a function completely. This can be seen from the following representation of the function.

$$f(x) = y_0 + \sum_{i=1}^{n} [m_i(\min\{x_i, x\} - \min\{x_{i-1}, x\})] \tag{3.2}$$

Where $m_{i+1} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ represents the slope of the function between (x_i, y_i) and (x_{i+1}, y_i).

Remark 4.2. From the conditions 3.1, we get

$$y_0 < y_1 < \cdots < y_{n-1} < y_n$$

So it is equivalent to sort points using either abscissae or ordinates.

3.1 Upper estimates

The following lemmas shall provide us with the necessary framework for upper and lower estimates for approximations on the functions generated by singular points.

Lemma 4.1 (Upper estimate). *Let $f : [a, b] \to [0, \infty)$ be a continuous, piecewise-linear, convex function characterised by the singular points $P = ((x_i, y_i))_{i\in[n]}$. Then, if a point $(x_j, y_j), j \in \{1, \dots, n-1\}$ is removed from the sequence, the function $f_u : [a, b] \to [0, \infty)$ obtained by the new sequence $(P_i)_{i\in[n]\setminus\{j\}}$ is also continuous, piecewise-linear and convex, and*

$$f_u(x) \geqslant f(x) \qquad \forall x \in [a, b] \tag{3.3}$$

Proof. By construction, $\forall x \notin (x_{j-1}, x_{j+1})$, $f_u(x) = f(x)$.

Again, by construction, $\forall x \in (x_{j-1}, x_{j+1}), f_u(x) = (1 - t)f(x_{j-1}) + tf(x_{j+1})$, where $t = \frac{x-x_{j-1}}{x_{j+1}-x_{j-1}}$.

Now, we have:

$$\begin{aligned} x_{j-1} < \quad & x \quad < x_{j+1} \\ \implies \quad 0 < \quad & x - x_{j-1} \quad < x_{j+1} - x_{j-1} \\ \implies \quad 0 < \quad & \frac{x - x_{j-1}}{x_{j+1} - x_{j-1}} \quad < 1 \\ \implies \quad 0 < \quad & t \quad < 1 \end{aligned}$$

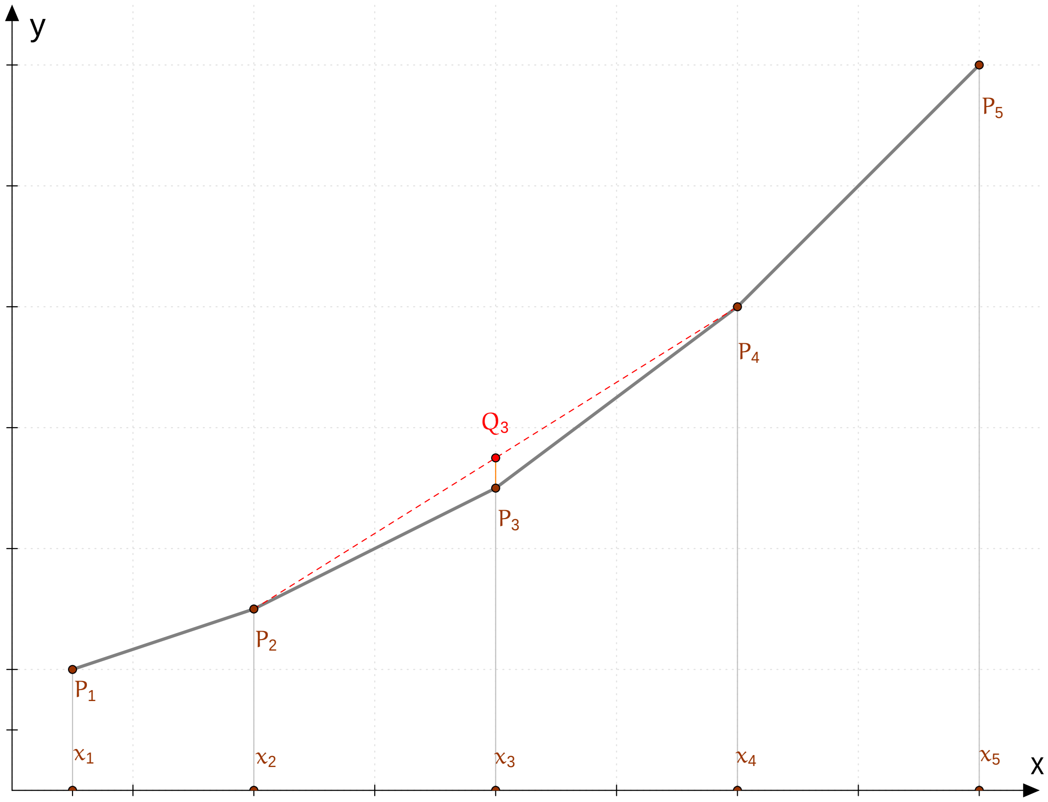

Figure 1. Upper estimate: Illustration of Lemma 4.1 with $j = 3$

f is convex $\implies \forall t \in (0,1),\ f((1-t)x_{j-1} + tx_{j+1}) < (1-t)f(x_{j-1}) + tf(x_{j+1})$.

Thus, $f_u(x) \geqslant f(x)\ \forall x \in [a,b]$. □

3.2 Lower estimates

Lemma 4.2 (Lower estimate). *Let $f : [a,b] \to [0,\infty)$ be a continuous, piecewise-linear, convex function characterised by the singular points $P = ((x_i, y_i))_{i\in[n]}$. Let l_j be the line joining points P_{j-1} and P_j. Similarly, let l_{j+2} be the line joining points P_{j+1} and P_{j+2}. Denote the intersection of the lines l_j and l_{j+2} by $\overline{P} = (\overline{x}, \overline{y})$.*

Then the function $f_d : [a,b] \to [0,\infty)$ characterised by $(P_0, \ldots, P_{j-1}, \overline{P}, P_{j+2}, \ldots, P_n)$ is also continuous, piecewise-linear and convex, and

(3.4) $$f_d(x) \leqslant f(x) \qquad \forall x \in [a,b]$$

Proof. First we show the convexity of f_d. We know that f satisfies the property of increasing slopes, that is $m_i \leqslant m_{i+1} \leqslant m_{i+2}$. Since f_d is obtained from f by removing the line l_{j+1}, for f_d we have that $m_i \leqslant m_{i+2}$, which implies that the function f_d is still convex.

Secondly, to prove the inequality, we may look at the convex function f as if it has been obtained by removing point $\overline{P}$ from the convex function f_d. Then, if $\overline{x} \in (x_j, x_{j+1})$, we have, using Lemma 4.1, that $f_d(x) \leqslant f(x) \qquad \forall x \in [a,b]$. □

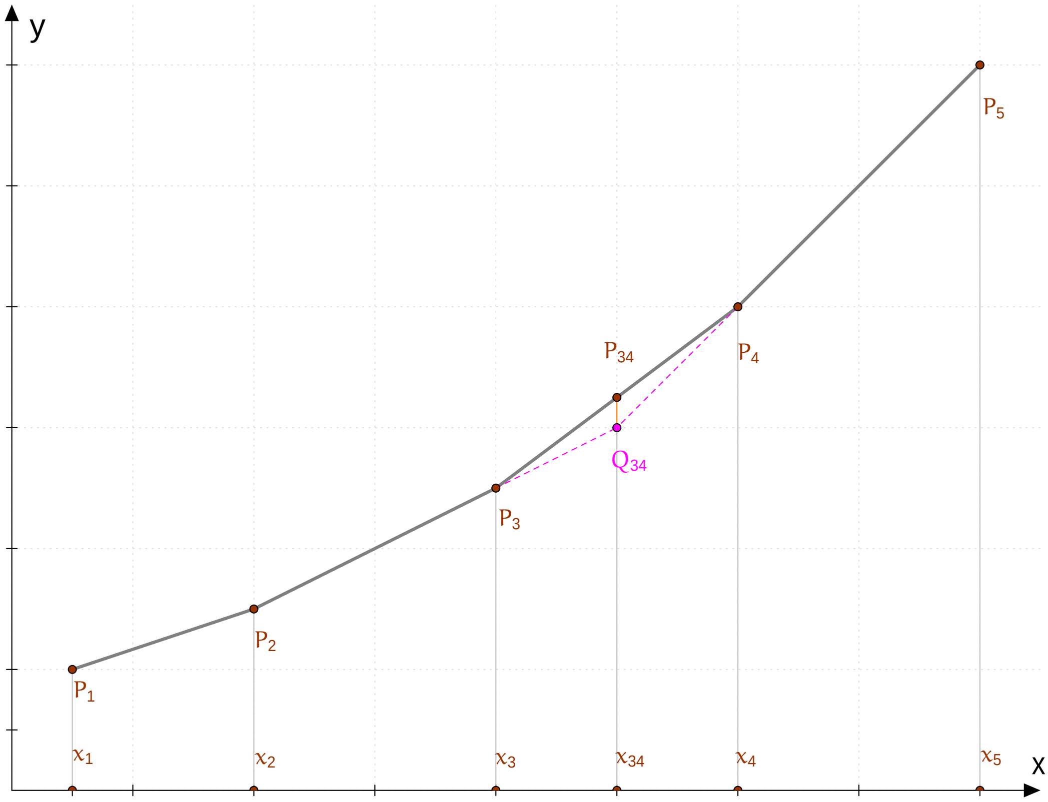

Figure 2. Lower estimate: Illustration of Lemma 4.2 with $j = 3, x_{34} = \overline{x}, P_{34} = \overline{P}$

The lemmas 4.1 and 4.2, will be used later to reduce the memory requirement of the algorithm by removing points or edges to simplify the function.

4 Notations and conventions

In this and subsequent sections, we shall use the convention that $[n] = \{0, 1, 2, \ldots, n\}$.

Let the number of time steps be n. Let i denote the highlighted time step, and j represent the number of up movements. In this way, we may represent any node by $N_{i,j}$.

The price of the underlying at each node $N_{i,j}$ is denoted by $S_{i,j}$. Since there are j up movements, there must be $i - j$ down movements, and thus

$$S_{i,j} = S_0 u^j d^{i-j} = S_0 u^j u^{-(i-j)} = S_0 u^{-i+2j} \qquad \forall i \in [n],\ \forall j \in [i] \tag{4.1}$$

Proposition 4.1. *The number of paths to a node* $N_{i,j}$ *is* $\binom{i}{j}$.

Proof. At each point in a path, we may choose either an up movement or a down movement. To reach node $N_{i,j}$, we much choose j up movements among i possibilities. The result follows immediately. □

Any number of paths among the possible paths may give zero as the price for the option. We denote the number of singular points in a node $N_{i,j}$ by $L_{i,j}$, where $L_{i,j} \in \left[\binom{i}{j}\right]$.

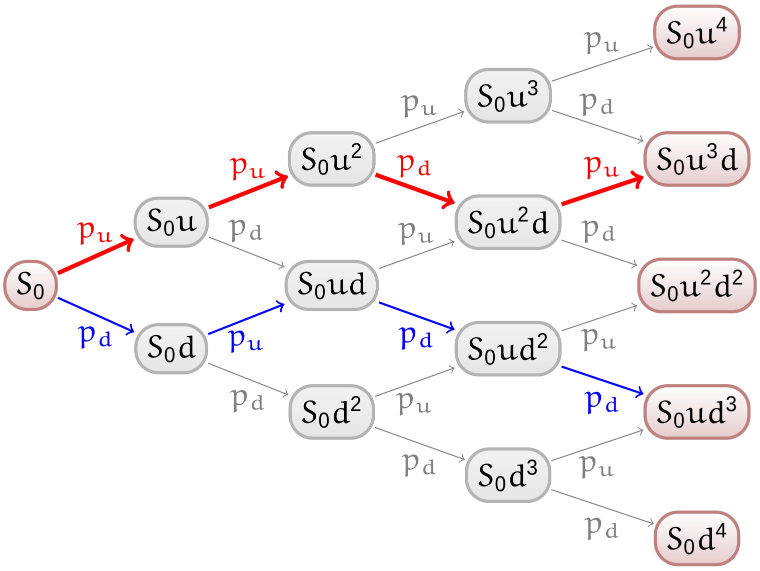

Figure 3. Two paths shown using red/thicker and blue/thick arrows. The other arrows are in grey/thin.

The l^{th} average (in ascending order) ($l \in \{1, \ldots, L_{i,j}\}$) is denoted by $A^l_{i,j}$, and the corresponding price by $P^l_{i,j}$. Thus the singular points characterising the price function are $((A^l_{i,j}, P^l_{i,j}))_{l \in \{1,\ldots,L_{i,j}\}}$.

Definition 4.3 (singular average and singular price). In the particular case of Asian options with arithmetic mean, the $A^l_{i,j}$s are called 'singular averages' and the $P^l_{i,j}$s are called 'singular prices'.

We recall some basic definitions and derive simple results for the maximum and minimum attainable value of the averages on each node.

Let the spot rate of interest be r (constant) and the compounding be continous. Then, the eective compounding rate in each time period Δt is given by R as

$$R = e^{r\Delta t} \tag{4.2}$$

We note that the R is not an instantaneous quantity, but one which is constant on an interval of time.

Definition 4.4 (Arithmetic mean). The arithmetic mean of a set of numbers $\{S_i\}_{i \in [n]}$ is given by:

$$A_n = \frac{\sum_{i=0}^{n} S_i}{n+1} \tag{4.3}$$

Definition 4.5 (Path). A path is a sequence $(j_i)_{i \in [n]}$ such that $j_{i+1} \in \{j_i, j_i + 1\}$.

Example 4.1. See Figure 3.

Theorem 4.1 (Path inequality). *Let there be two paths α and β, such that $S_{i,j_i^\alpha} \geqslant S_{i,j_i^\beta}$ $\forall i$, where $(j_i^\alpha)_{i \in [n]}$ and $(j_i^\beta)_{i \in [n]}$ denote the paths as defined above. Denote the corresponding averages by A^α and A^β, respectively. Then $A^\alpha \geqslant A^\beta$.*

Proof. Clearly if $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i$, then $A^\alpha = A^\beta$.

We only need to show the result in the case of inequality. Let $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i \in [n] \setminus \{l\}$, and $S_{l,j_l^\alpha} > S_{l,j_l^\beta}$.

Now, from equation 4.3, we have:

$$\begin{aligned}
(n+1)A^\alpha_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\alpha} + \sum_{i=l+1}^{n} S_{i,j_i} \\
(n+1)A^\beta_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\beta} + \sum_{i=l+1}^{n} S_{i,j_i} \\
\implies (n+1) \quad A^\alpha_{n,j} - A^\beta_{n,j} &= S_{l,j_l^\alpha} - S_{l,j_l^\beta} \\
&= S_{l-1,j_{l-1}} u_l - S_{l-1,j_{l-1}} d_l \\
&= S_{l-1,j_{l-1}}(u_l - d_l) > 0 \qquad (u_l > d_l \text{ by definition}) \\
\implies A^\alpha_{n,j} > A^\beta_{n,j}
\end{aligned}$$

□

Remark 4.3. The path α signifies a path 'above' and β a path 'below' in the usual depiction of the binomial tree (the up movement shown above the down movement). Thus, any path above has a higher arithmetic mean than the one below.

Corollary 4.1. *At each node $N(i, j)$, the following hold:*

(1) *The minimum average possible $A^{\text{min}}_{i,j}$ is attained by the path corresponding to the path corresponding to the path with $(i - j)$ down movements followed by j up movements, and*

$$A^{\text{min}}_{i,j} = \frac{S_0}{i+1} \left(\frac{1 - d^{i-j+1}}{1-d} + d^{i-j} u \frac{1-u^j}{1-u} \right) \tag{4.4}$$

(2) *The maximum average possible $A^{\text{max}}_{i,j}$ is attained by the path corresponding to the path with j up movements followed by $(i - j)$ down movements, and*

$$A^{\text{max}}_{i,j} = \frac{S_0}{i+1} \left(\frac{1 - u^{j+1}}{1-u} + u^j d \frac{1-d^{i-j-1}}{1-d} \right) \tag{4.5}$$

Proof. We show the proof only for the case of the maximum, since the case of the minimum can be shown using the exact same argument.

From Theorem 4.1, the result about path with the maximum average holds directly, since there cannot be a path above the one given by j up movements followed by $(i - j)$ down movements.

Table 1. Summary of notations

Symbol	Range	Formula	Description
i	$[n]$		highlighted time step
j	$[i]$		number of up movements
$N_{i,j}$			node fixed by (i,j)
$S_{i,j}$	$[0,\infty)$	Eq 4.1	value of the underlying at node $N_{i,j}$
$L_{i,j}$	$\binom{i}{j}$		number of singular points in node $N_{i,j}$
l	$\{1,\dots,L\}$		index for points in ascending order of averages
$A^{\text{min}}_{i,j}$	$[0,\infty)$	Eq 4.4	minimum average attainable for node $N_{i,j}$
$A^{\text{max}}_{i,j}$	$[0,\infty)$	Eq 4.5	maximum average attainable for node $N_{i,j}$
$A^{l}_{i,j}$	$\left[A^{\text{min}}_{i,j}, A^{\text{max}}_{i,j}\right]$	Eq 4.3	l^{th} singular average of node $N_{i,j}$
$P^{l}_{i,j}$			price corresponding to the average $A^{l}_{i,j}$
$(A^{l}_{i,j}, P^{l}_{i,j})$			l^{th} singular point of node $N_{i,j}$

The subsequent formula may be derived as follows.

$$
\begin{aligned}
(i+1)A^{\text{max}}_{i,j} &= \underbrace{(S_0 + S_0u + S_0u^2 + \cdots + S_0u^j)}_{\text{up movement}} + \underbrace{(S_0u^jd + S_0u^jd^2 + \cdots + S_0u^jd^{i-j})}_{\text{down movement}} \\
&= S_0\left((1+u+u^2+\cdots+u^j) + u^jd(1+d+\cdots+d^{i-j-1})\right) \\
&= S_0\left(\sum_{k=0}^{j} u^k + u^jd\sum_{k=0}^{i-j-1} d^k\right) \\
&= S_0\left(\frac{1-u^{j+1}}{1-u} + u^jd\frac{1-d^{i-j-1}}{1-d}\right) \qquad \text{(Geometric series)} \\
\implies A^{\text{max}}_{i,j} &= \frac{S_0}{i+1}\left(\frac{1-u^{j+1}}{1-u} + u^jd\frac{1-d^{i-j-1}}{1-d}\right)
\end{aligned}
$$

□

Table ?? summarises the discussion above.

5 Fixed-strike European Asian options

For this type of options, the pay-off at maturity is dependent only on (some type of) average A_T at maturity T and a fixed constant K, and is given by the function

$$P_T = (A_T - K)_+ \tag{5.1}$$

We shall focus on this case in this section because it is the easiest to handle.

In each node of the binomial tree, we have a set of possible averages depending on the paths which may be taken to arrive at the node, and prices corresponding to each such average. We shall show these points to satisfy condition 3.1, so they completely characterise a price function. Thus we focus not on the averages and corresponding prices possible under a particular binomial tree, but on the continuous representation of prices. The intuitive idea is that as the time step is reduced to zero, this function converges to the price function of the continuous time model.

5.1 The price function at maturity

From equations 4.4 and 4.5, putting $i = n$, we get

$$A^{\text{min}}_{n,j} = \frac{S_0}{n+1}\left(\frac{1-d^{n-j+1}}{1-d} + d^{n-j}u\frac{1-u^j}{1-u}\right)$$
$$A^{\text{max}}_{n,j} = \frac{S_0}{n+1}\left(\frac{1-u^{j+1}}{1-u} + u^j d\frac{1-d^{n-j-1}}{1-d}\right)$$

In defining the price function, we note that three cases may arise.

- $j \in \{0, n\}$
  There exists only one path to these nodes, so there is only one average, implying one price and one singular point.
- $j \notin \{0, n\}$ and $K \in (A^{\text{min}}_{n,j}, A^{\text{max}}_{n,j})$
  The price function is characterised by three singular points ($L_{i,j} = 3$), $(A^l_{n,j}, P^l_{n,j})_{l\in\{1,2,3\}}$, which are

$$(5.2)\qquad \begin{aligned} (A^1_{n,j}, P^1_{n,j}) &= (A^{\text{min}}_{n,j}, 0) \\ (A^2_{n,j}, P^2_{n,j}) &= (K, 0) \\ (A^3_{n,j}, P^3_{n,j}) &= (A^{\text{max}}_{n,j}, A^{\text{max}}_{n,j} - K) \end{aligned}$$

- $j \notin \{0, n\}$ and $K \notin (A^{\text{min}}_{n,j}, A^{\text{max}}_{n,j})$
  The price function is characterised by only two singular points ($L_{i,j} = 2$), $(A^l_{n,j}, P^l_{n,j})_{l\in\{1,2\}}$, which are

$$(5.3)\qquad \begin{aligned} (A^1_{n,j}, P^1_{n,j}) &= (A^{\text{min}}_{n,j}, (A^{\text{min}}_{n,j} - K)_+) \\ (A^2_{n,j}, P^2_{n,j}) &= (A^{\text{max}}_{n,j}, (A^{\text{max}}_{n,j} - K)_+) \end{aligned}$$

Lemma 4.3 (Price function at maturity). *At each node at maturity, the price function* $v_{n,j} : \left[A^{\text{min}}_{n,j}, A^{\text{max}}_{n,j}\right] \to \left[(A^{\text{min}}_{n,j} - K)_+, (A^{\text{max}}_{n,j} - K)_+\right]$ *defined as* $v_{n,j}(A) = (A - K)_+$ *is continuous, piecewise-linear and convex.*

Proof. The singular points satisfy the conditions 3.1. So for each $A \in \left[A^{\text{min}}_{n,j}, A^{\text{max}}_{n,j}\right]$, the price function $v_{n,j}(A)$ characterised by the singular points is continuous, piecewise-linear and convex by remark 5.1. □

5.2 The price function before maturity

Lemma 4.4 (Price function at any node). *At any node $N_{i,j}$, the price function $\nu_{i,j} : \left[A^{min}_{i,j}, A^{max}_{i,j}\right] \rightarrow [0,\infty)$ is continuous, piecewise-linear and convex.*

Proof. We shall prove this using backward induction, the base case at maturity being true by virtue of Lemma 4.3. We now consider step $i = n-1$. Let A_u and A_d respectively represent the averages after an up and down movement corresponding to the average A. From equation 4.3, we get

$$A_u = \frac{(i+1)A + S_0 u^{-i+2j+1}}{i+1} A_d \qquad = \frac{(i+1)A + S_0 u^{-i+2j-1}}{i+1} \tag{5.4a}$$

The price function $\nu_{i,j} : \left[A^{min}_{i,j}, A^{max}_{i,j}\right] \rightarrow [0,\infty)$ is obtained by considering the discounted expectation value.

$$\nu_{i,j}(A) = \frac{1}{R} \quad \pi\nu_{i+1,j+1}(A_u) + (1-\pi)\nu_{i+1,j}(A_d) \tag{5.5}$$

From equation 5.4, we get that A_u and A_d are linear functions of A. Thus, $\nu_{i+1,j+1}(A_u) = \nu_{n,j+1}(A_u)$ and $\nu_{i+1,j}(A_d) = \nu_{n,j}(A_d)$ are piecewise-linear convex continuous functions of A_u and A_d respectively. Thus, $\nu_{i+1,j+1}$ and $\nu_{i+1,j}$ may be seen as composition of the above functions, and is thus piecewise-linear, convex and continuous itself. Again, from equation 5.5, we get that $\nu_{i,j}$ is a convex combination of such functions, and the proof is complete. □

TODO: Insert picture for this.

Remark 4.4. From Lemma 4.4, we see that the price function may be characterised by singular points.

5.3 Evaluation of singular points

The evaluation of singular points for any node $N_{i,j}$ is done by the following algorithm, which works in a backward fashion in time, starting from the maturity.

We note that for the only influencing nodes for the node $N_{i,j}$ are $N_{i+1,j+1}$ and $N_{i+1,j}$. Thus we need to calculate the price of the option for each singular average belonging to either of these nodes.

Up movement First we take each singular average $A^l_{i+1,j}$ belonging to $N_{i+1,j}$ and project it to $N_{i,j}$ via the relation

$$B^l = \frac{(i+2)A^l_{i+1,j} - S_0 u^{-i+2j-1}}{i+1} \tag{5.6}$$

Thus, B^l is that average which after a down movement of the asset gives us the average $A^l_{i+1,j}$.

Next, we note that B^l is an increasing function of l, since a higher average at time step i would yield a higher average at time $i+1$. This in turn implies the following:

- $B^{L_{i+1,j}} = A^{\max}_{i+i,j}\ \forall j$
- $B^1 \notin \left[A^{\min}_{i+i,j}, A^{\max}_{i+i,j}\right]\ \forall j \in \{1, \dots, i-1\}$

Each $B^l \in \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right]$ becomes the singular average of $N_{i,j}$.

In this way, we have determined the first coordinate of the singular points. We need to determine the second coordinate, or the prices $v_{i,j}(B^l)$, $\forall \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right]$, in order to determine the singular points completely. The idea is to calculate the discounted expected value of the price corresponding to each average B^l at $N_{i,j}$. In order to be able to do this, we need the prices corresponding to the average projected to the node $N_{i+1,j+1}$.

We consider an up movement of the underlying asset from node $N_{i,j}$. In this case, B^l transforms into the average $B^l_u = \left((i+1)B^l + S_0 u^{-i+2j+1}\right)/(i+2)$. Clearly, this average cannot belong to the set of averages associated with the node $N_{i+1,j+1}$. Thus, we need to find the index s such that $B^l_u \in \left[A^s_{i+1,j+1}, A^{s+1}_{i+1,j+1}\right]$. In the intervals the price function is linear, and thus we have

$$v_{i+1,j+1}\left(B^l_u\right) = \frac{P^{s+1}_{i+1,j+1} - P^s_{i+1,j+1}}{A^{s+1}_{i+1,j+1} - A^s_{i+1,j+1}}\left(B^l_u - A^s_{i+1,j+1}\right) + P^s_{i+1,j+1} \tag{5.7}$$

We follow this up by calculating the price associated with the singular value B^l by evaluation the discounted expectation value.

$$v_{i,j}(B^l) = \frac{1}{R}\left(\pi v_{i+1,j+1}\left(B^l_u\right) + (1-\pi) v_{i+1,j}\left(A^l_{i+1,j}\right)\right) \tag{5.8}$$

Down movement We now proceed to formulate the theory for the downward movement in the exact same fashion. Define the new average C^l at the node $N_{i,j}$ via the relation

$$C^l = \frac{(i+2)A^l_{i+1,j+1} - S_0 u^{-i+2j+1}}{i+1} \tag{5.9}$$

Again, we note that

- $C^1 = A^{\min}_{i,j}\ \forall j$
- $C^{L_{i+1,j+1}} \notin \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right]\ \forall j \in \{1, \dots, i-1\}$
- $C^l_d = \left((i+1)C^l + S_0 u^{-i+2j-1}\right)/(i+2)$

Each $C^l \in \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right]$ becomes the singular average of $N_{i,j}$.

For $v_{i,j}(C^l)$, $\forall \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right]$, we now have the following.

$$v_{i+1,j+1}\left(C^l_d\right) = \frac{P^{s+1}_{i+1,j} - P^s_{i+1,j}}{A^{s+1}_{i+1,j} - A^s_{i+1,j}}\left(C^L_d - A^s_{i+1,j}\right) + P^s_{i+1,j} \tag{5.10}$$

$$v_{i,j}(C^l) = \frac{1}{R}\left(\pi v_{i+1,j+1}\left(A^l_{i+1,j+1}\right) + (1-\pi)v_{i+1,j}\left(C^l_d\right)\right) \tag{5.11}$$

Aggregation Now we have the singular points for both up and down movements. We sort these points in ascending order of the first coordinate, i.e. the averages B^l and C^l that belong to $\left[A^{\text{min}}_{i,j}, A^{\text{max}}_{i,j}\right]$. These is an exhaustive list of all the singular points in the node (by construction). We note that $L_{i,j} \leqslant L_{i+1,j} + L_{i+1,j+1} - 2$.

This procedure is applied to all nodes, starting from maturity and proceeding backwards. At the 'edge' nodes $N_{i,0}$ and $N_{i,i}$, there is only one singular point whose price is given as follows

$$P^1_{i,0} = \frac{1}{R}\left[\pi P^1_{i+1,0} + (1-\pi)P^1_{i+1,1}\right] \tag{5.12a}$$

$$P^1_{i,i} = \frac{1}{R}\left[\pi P^1_{i+1,i+1} + (1-\pi)P^{L_{i+1,i}}_{i+1,i}\right] \tag{5.12b}$$

Thus we have a complete description of the price function at each node of the binomial tree. The price $P^1_{0,0}$ is exactly the binomial price relative to the tree with n steps of a fixed-strike European call option.

6 Fixed-strike American Asian options

TODO

7 Extensions

Let us recapitulate the conditions required for the singular points method to work in the case of Asian options with arithmetic mean.

- The ability to calculate the upper and lower bounds of the mean for all nodes of the tree.
- The recombinant nature of the tree for the underlying. Note that the tree for the option prices are *not* recombinant.
- Convexity and piecewise-linearity of the price function on the mean of the underlying.
- Fixed volatility

Keeping these in mind, let us look at the possibility of extending the singular points method to the following cases:

(1) Asian options with geometric mean and fixed volatility.

(2) Asian options with arithmetic mean and local volatility.

7.1 Geometric mean and fixed volatility

In the case of geometric options, we have a closed form formula under the Black-Scholes market model. We try to extend the singular points method.

Firstly, we show that the result about the maximum and minimum paths still hold in the geometric case.

Definition 4.6 (Geometric mean). The geometric mean of the risky asset's prices $(S_i)_{i\in[n]}$ is given by:

$$G_n = \left(\prod_{i=0}^{n} S_i\right)^{\frac{1}{n+1}} \tag{7.1}$$

Lemma 4.5. *At each node* $N(i,j)$, *the following hold:*

(1) *The maximum average possible* $G^{\max}_{i,j}$ *is attained by the path corresponding to the path with* j *up movements followed by* $(i-j)$ *down movements.*

(2) *The minimum average possible* $G^{\min}_{i,j}$ *is attained by the path corresponding to the path corresponding to the path with* $(i-j)$ *down movements followed by* j *up movements.*

Proof. The proof is the same as 4.1, with A replaced by G and relevant modifications. □

One of the central ideas behind the singular points method is that the price of the option is a convex, piecewise-linear function of the average A. But in the geometric case, this no longer holds true. For example, take a node $N_{i,j}$ with $i = n - 1$. The price function given by $v_{i,j}(G)$, with $G \in [G^{min}, G^{max}]$, can be calculated by the discounted expectation value.

$$v_{i,j}(G) = \frac{1}{R} \; p v_{i+1,j+1}(G_u) + (1-p) v_{i+1,j}(G_d) \tag{7.2}$$

$$G_u = \; G^{i+1} S_0 u^{-i+2j+1} \; {}^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{7.3}$$

$$G_d = \; G^{i+1} S_0 u^{-i+2j-1} \; {}^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{7.4}$$

Clearly, the final function $v_{i,j}$ is not linear in G. Rather it is piecewise-concave. Thus we cannot use the singular points method in this case. TODO: Insert a graph of the function here.

7.2 Arithmetic mean with local volatility

In this case, the tree for the underlying is not recombinant, so we do not have more than one singular point in one (non-recombining) node. Clearly, we cannot use the singular points method.

8 Conclusion

We conclude the chapter by noting the pros and cons of the singular points method.

Advantages

- Fast – Experimental order of complexity = $O(n^3)$
- It allows us to specify an *a priori* error bound.

Disadvantages

- Very specific method – only applicable to a few specific cases.

Chapter 5

The Singular Points method applied to Cliquet options

1 Introduction

A ‘cliquet option’ or a ‘ratchet option’ is an exotic option consisting of a series of consecutive forward start options. The first such option is active immediately, and once it expires the second comes into existence, and so on. Each option is struck at-the-money when it becomes active. Therefore, such an option periodically settles and resets its strike price at the level of the underlying during the time of settlement. Investors can opt to receive their payout either when each option expires or wait until the entire time has elapsed.

Usually, the return on a cliquet option is capped and floored. The capping and flooring may be local or global (or both). The motivation behind bounding the return is to provide the investor safety against downside risks, yet allowing significant upside potential. Consequently, the investor is also constrained from having unbounded gains. Capping the maximum ensures that the payoff is never too extreme and therefore that the value of the contract is not too outrageous. Some variants of cliquet options are as follows:

Reverse cliquet: Amounts to a cash flow minus a capped cliquet of puts.

Digital cliquet: The forward-starting options are digital options.

Example 5.1. A three-year cliquet option with a strike of 1,000 would expire worthless on the first year if the underlying was to be 900. This value (900) would then be the new strike price for the following year and should the underlying on the settlement be 1,200, the contract holder would receive a payout and the strike would reset to this new level. Higher volatility provides better conditions for investors to earn profits.

Literature review [ToDo] The literature for pricing of cliquet options is primarily based on partial differential equations (PDE) techniques. Notable mentions include Wilmott’s[TODO:reference] finite difference (FD) approach in a non-linear uncertain volatility model (UVM) in 2002.

Later, Windcliff et al.[TODO:reference] explored a variety of modelling alternatives including jump diffusion models, local volatility and UVM models, again using finite differences methods in 2006.

Lattice method Gaudenzi *et al*[GZ11] introduced a discrete method for pricing cliquet options in a binomial Cox Ross Rubinstein model [TODO:reference]. This Singular Points method is faster than the alternative lattice based approaches available, while retaining a fair bit of flexibility to handle varying volatilities and rates of interest. Most of the chapter is inspired from the former paper.

2 Cliquet contracts and models

Notation review: In this and the following sections, we will use the notation $[n] := \{0, 1, \dots, n\}$.

We consider a market model, in which the price of the underlying risky asset is governed by the Black-Scholes stochastic differential equation.

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma\,dB_t \tag{2.1a}$$

$$S_0 = s_0 \tag{2.1b}$$

where the quantities are as follows

S_t: Price of the underlying risky asset at time t.

B_t: A standard Brownian motion under the risk-neutral probability measure Q.

$r > 0$: Rate of interest

$q \geqslant 0$: Continuous divident yield

$\sigma > 0$: Volatility of the risky asset

Solving the equation, at any time t, the price of the underlying risky asset is

$$S_t = s_0 e^{((r-q)-\frac{\sigma^2}{2})t+\sigma B_t}$$

Let T be the maturity of the cliquet contract. Let the payoffs depend on the N preordained observation times $t_1, t_2, \dots, t_N$ ($t_0 = 0$). At these observation times, the value of the underlying are $(S_i)_i, S_i = S_{t_i}, i \in [N]$. The returns for the time interval $(t_{i-1}, t_i]$ are given by

$$R_i = \frac{S_i - S_{i-1}}{S_{i-1}} = \frac{S_i}{S_{i-1}} - 1 \tag{2.2}$$

During each time interval, the return is capped and floored locally by the quantities C_{loc} and F_{loc}. In other words, we consider the quantity $\max\{F_{loc}, \min\{C_{loc}, R_i\}\}$ rather than the return itself. The sum of these quantities till time t_i is called the 'running sum' and is given by

$$Z_i = \sum_{k=1}^{i} \max\{F_{loc}, \min\{C_{loc}, R_k\}\} \tag{2.3}$$

We also consider a global cap C_{glob} and floor F_{glob}. Thus, the expression for the payoff finally becomes

$$\text{payoff} = \text{notional} \cdot \max\{F_{glob}, \min\{C_{glob}, Z_N\}\} \tag{2.4}$$

For ease of notation, we take notional $= 1$.

We note that the case $C_{glob} > NC_{loc}$ is equivalent to $C_{glob} = NC_{loc}$. Similarly, the case $F_{glob} < NF_{loc}$ is equivalent to $F_{glob} = NF_{loc}$. We simply say

$$F_{glob} = \max\{NF_{loc}, F_{glob}\}$$
$$C_{glob} = \min\{NC_{loc}, C_{glob}\}$$

Now, we consider the binomial approach in order to price the cliquet option in the Black-Scholes framework. Let the number of intervals be n and the corresponding time step be $\Delta T = \frac{T}{n}$. Then the lognormal diffusion process is approximated by the Cox-Ross-Rubinstein binomial process.

$$\tilde{S}_j = s_0 \prod_{k=1}^{j} Y_k \qquad \forall j \in [n] \tag{2.6}$$

where the random variables Y_k are independent and identically distributed with values in $\{d, u\}$, representing a down and up movement, respectively. Let p be the probability of an up movement of the asset, that is, $p = \Pr(Y_n = u)$. The Cox-Ross-Rubinstein corresponds to the choice $u = 1/d = e^{\sigma\sqrt{\Delta T}}$. Thus,

$$p = \frac{e^{r\Delta T} - d}{u - d} = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}} \tag{2.7}$$

We assume that the difference between two observation times is constant and we denote by m the number of steps of the binomial tree in every period (so that the total number of steps of the binomial tree is $n = mN$).

To recapitulate, the following table highlights the notations in a concise manner. [TODO]

3 The Singular Points method for cliquet options

The binomial method may always be used to price any option, including path-dependent ones. The binomial method looks through all possible paths of the underlying in order to price the option. The number of possible paths are 2^{mN}. Thus, the method is inherently extremely computationally expensive due to the exponential dependence of the number of paths on m and N. The theoretical computational complexity is $O(m^N)$, as in Page 128 of [GZ11].

A modification of the Singular Points method described in the previous chapter solves this problem for cliquet options by the process of approximation. The method of approximation selectively removes certain paths that would be normally considered, but which do not aect the result in a significant manner. This may be done by putting an *a priori* error bound while removing points. The method turns out to be significantly faster and memory ecient compared to known binomial techniques. Moreover, its flexible is evinced by the fact that it is adaptable for varying interest rate and volatility in each observational period.

In this section, we will note some elementary definitions and results, and give a detailed exposition of the method. The reader is advised to keep in mind the notation introduced in section 4 of chapter 4.

Definition 5.1 (Singular points and singular values). Let $P = (P_i)_{i\in[n]} = ((x_i, y_i))_{i\in[n]}$, $n \in \mathbb{N}$ be a sequence of points such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \forall i \in [n]$

Let $f : [a, b] \to [0, \infty)$ be the function obtained by linear interpolation of the points in P. The definition of f ensures that the function is continuous and piecewise-linear.

Then, the elements of P are called *singular points of* f and the abscissae $\{x_i\}_{i\in[n]}$ are called *singular values of* f.

Remark 5.1. We note that the singular points characterise such a function completely. This can be seen from the following representation of the function.

$$f(x) = y_0 + \sum_{i=1}^{n} [m_i(\min\{x_i, x\} - \min\{x_{i-1}, x\})] \tag{3.1}$$

Where $m_{i+1} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ represents the slope of the function between (x_i, y_i) and (x_{i+1}, y_i).

Remark 5.2. Note that there is no mention of convexity in the foregoing discussion, since the price function in the cliquet case is not necessarily convex, as opposed to the Asian case. The problem comes from the fact that due to the presence of a global cap, the price function at maturity is no longer convex (see Figure 1). Since the algorithm starts from maturity, it follows that we cannot assume convexity at any point of time. It was primarily the convexity of the price functions in the Asian case that allowed us to obtain simple upper and lower bounds of the exact binomial price. Nevertheless, in the case of cliquet options, the singular points approach still provides an ecient binomial framework, even in the absence of convexity of the price functions, as we shall see.

3.1 The method

Our aim is to look at every possible value taken by the running sum Z within the bounds $[F_{loc}, C_{loc}]$ for each time interval $t_1, \ldots, t_N$. If we know the price function at maturity, we may use a backward procedure (in time) in order to obtain a continuous representation of the cliquet price as a piecewise-linear function of the running sum Z. Since it is piecewise-linear and continuous, the function may be represented using its singular points. Thus, we see an evolution of singular points as we go back in time. Since the number of singular points may be significantly high for any computer, we shall introduce an error controlled approximation procedure to reduce the number of singular points.

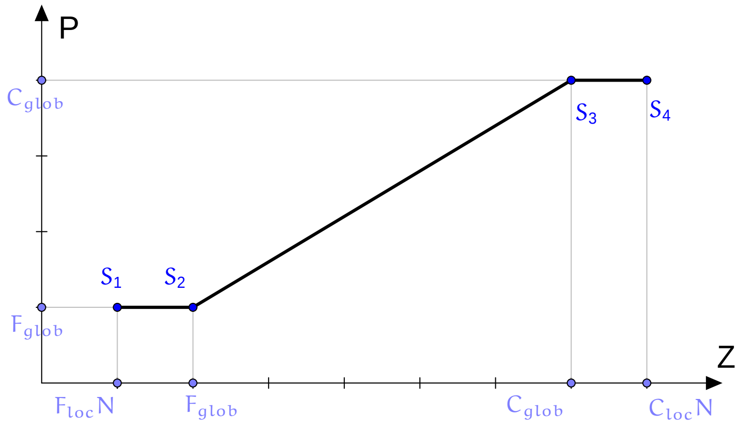

Figure 1. The price function at maturity

Let the number of singular points at each observational time t_i be L_i, where $i \in \{1, \dots, N\}$. For each singular point $l \in \{1, \dots, L_i\}$, the abscissa is called the *singular running sum* Z_i^l and the ordinate is called the *singular price* P_i^l. Thus, the singular points are denoted by

$$(Z_i^l, P_i^l) \qquad \forall l \in \{1, \dots, L_i\}$$

At maturity At maturity ($t_{Nobs} = T$), for every running sum Z, the price of the cliquet option $V_N(Z)$ as function of Z, is given by

$$V_N(Z) = \max\{Fglob, \min\{Cglob, Z\}\} \tag{3.2}$$

We note the following.

(1) $V_N(Z)$ is a continuous and piecewise-linear function

(2) $V_N(Z)$ is defined in the interval $[NF_{loc}, NC_{loc}]$

(3) There are only four points where the function changes slope, namely NF_{loc}, F_{glob}, C_{glob} and NC_{loc}. This is because if $Z < F_{glob}$, the price is constant. Same can be said about $Z > C_{glob}$. Thus the four numbers enumerated above form the complete set of singular running sums at maturity ($L_N = 4$).

Focusing further on point 3 above, we note down the singular running sums and corresponding prices.

Table 1. Singular points at maturity

l	Z_N^l	P_N^l
1	NF_{loc}	F_{glob}
2	F_{glob}	F_{glob}
3	C_{glob}	C_{glob}
4	NC_{loc}	C_{glob}

A more visual representation of the price function at maturity is given in Figure 1.

The penultimate time step We consider the time step $N-1$. If the running sum at time t_{N-1} is denoted by Z, then the corresponding price depends on the possible returns of the underlying asset during the time interval $[t_{N-1}, T]$.

We revisit equation 2.2 once more. Now we note that since the number of time steps in each interval is m, there will be m up movements or down movements of the asset. Thus, there are $m+1$ possible outcomes, given by $S_i = u^{-m+2j} S_{i-1}, j \in [m]$. Corresponding to these cases, there are also $m+1$ returns, given by

$$R_j = u^{-m+2j} - 1 \qquad j \in [m] \tag{3.3}$$

Since the probability of an up movement in time ΔT is p (refer equation 2.7), we have that the probability of each return is distributed binomially, and is given by

$$p_j = \binom{m}{j} p^j (1-p)^{m-j} \tag{3.4}$$

The above derivations assume that the there has been no local flooring or capping of the returns. In the case of such bounds, the actual possibilities are fewer in number, and we need to put bounds on j. We can do this in the following fashion.

In the case of a local floor, we must have

$$\begin{aligned}
& & F_{loc} &\geqslant u^{-m+2j_{\text{min}}} - 1 & & \\
&\implies & \log(F_{loc}+1) &\geqslant (-m+2j_{\text{min}})\log(u) & & \ldots(\log \text{ is monotonic}) \\
&\implies & \frac{\log(F_{loc}+1)}{\log(u)} &\geqslant (-m+2j_{\text{min}}) & & \ldots(u > 1 \implies \log(u) > 0) \\
&\implies & j_{\text{min}} &\leqslant \frac{\log(F_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} & & \ldots(u = e^{\sigma\Delta T}) \\
&\implies & j_{\text{min}} &= \frac{\log(F_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} & & \ldots(j \in [m])
\end{aligned}$$

Where $\lfloor\cdot\rfloor$ denotes the floor function.

Similarly, in the case of local cap, we have

$$\begin{aligned}
& & C_{loc} &\leqslant u^{-m+2j_{\text{max}}} - 1 & & \\
&\implies & \log(C_{loc}+1) &\leqslant (-m+2j_{\text{max}})\log(u) & & \ldots(\log \text{ is monotonic}) \\
&\implies & \frac{\log(C_{loc}+1)}{\log(u)} &\leqslant (-m+2j_{\text{max}}) & & \ldots(u > 1 \implies \log(u) > 0) \\
&\implies & j_{\text{max}} &\geqslant \frac{\log(C_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} & & \ldots(u = e^{\sigma\Delta T}) \\
&\implies & j_{\text{max}} &= \frac{\log(C_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} & & \ldots(j \in [m])
\end{aligned}$$

Where $\lceil\cdot\rceil$ denotes the ceiling function.

We represent by j_0 the number of possibilities of the return after enforcing the local floor and cap. Summarising

$$j_{\text{min}} = \frac{\log(F_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} \tag{3.5a}$$

$$j_{\text{max}} = \frac{\log(C_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} \tag{3.5b}$$

$$j_0 = j_{\text{max}} - j_{\text{min}} \tag{3.5c}$$

$\forall j \leqslant j_{\text{min}}$, the return is F_{loc}, and $\forall j \geqslant j_{\text{max}}$, the return is C_{loc}. For the other indices, the return remains unchanged.

For ease of implementation, we shift the indices from $\{j_{\text{min}}, \dots, j_{\text{max}}\}$ to $\{0, \dots, j_0\}$ by putting $j' = j - j_{\text{min}}$. Table 2 highlights the shifted indices, and the corresponding returns and probabilities.

Table 2. Shifted returns and probabilities

Range(j)	j'	R'_j	p'_j
$j \leqslant j_{\text{min}}$	0	F_{loc}	$\sum_{k=0}^{j_{\text{min}}} p_k$
$\{j_{\text{min}}+1, \dots, j_{\text{max}}-1\}$	$j - j_{\text{min}}$	$R_{j+j_{\text{min}}}$	$p_{j+j_{\text{min}}}$
$j \geqslant j_{\text{max}}$	j_0	C_{loc}	$\sum_{k=j_{\text{max}}}^{m} p_k$

Now we focus on how to determine the price function at time t_{N-1}. Recall that at maturity, the function $V_N(Z)$ giving the price of the cliquet option as a function of the running sum Z at maturity, is the piecewise linear function whose singular points are presented in Table 1. At the penultimate time step t_{N-1}, we must have $Z \in [(N-1)F_{loc}, (N-1)C_{loc}]$. Note that Z is the running sum till the penultimate time step. The price function at this time is given as a discounted conditional expectation of the price function at maturity given the information at penultimate time. Thus

$$V_{N-1}(Z) = e^{-m\Delta T} \sum_{j=0}^{j_0} \Big[p'_j V_N(Z + R'_j) \Big] \tag{3.6}$$

Since V_N is piecewise-linear and continuous, and the above is just a linear combination of such functions, the resulting function is also piecewise-linear and continuous, and thus may be completely represented by singular points. The next logical step is of course to figure out a method to compute these singular points.

From equation 3.6, we note that each singular point l at maturity may have j_0+1 possible returns, given by

$$B_{l,j} = Z_N^l - R'_j \qquad j \in [j_0] \tag{3.7}$$

Then the maximum number of singular points at time t_{N-1} is $(j_0+1)L_N$. But not all the running sums at time t_{N-1} would belong to the interval $[(N-1)F_{loc}, (N-1)C_{loc}$. All the $B_{l,j}$ which belong to the interval become the abscissa of a singular point at time t_{N-1}. The corresponding singular price is determined by formula 3.6. The term $V_N(B_{l,j} + R'_k)$, required by the formula, is computed using linearity of the price function at maturity. We just need to figure out the interval l_0 such that $(B_{l,j} + R'_k) \in [Z_N^{l_0}, Z_N^{l_0+1}]$, and evaluate $V_N(B_{l,j} + R'_k)$ by linear interpolation of the extrema. Such an interpolation gives not an approximation by the exact value due to the piecewise-linear nature of the original function.

Finally, the singular points thus obtained are sorted in ascending order on the basis of the running sums. This ordered sequence of singular points $((Z_{N-1}^l, P_{N-1}^l))_l, l \in \{1, \ldots, L_{N-1}\}$ completely characterises the price function at time $V_{N-1}(Z)$.

At all times The previous argument may be applied iteratively in a backward fashion at each step $N-2, N-3, \ldots, 1, 0$ to obtain the singular points at each time step. At time 0, there is only one singular point $(0, P_0^1)$, and P_0^1 provides the exact binomial price of the cliquet option. The equivalence of this price and the exact binomial price is proved in Proposition 5.1.

Proposition 5.1 (Equality of price obtained by the singular points method and the binomial price). *TODO*

[TODO] Insert remark 2

3.2 Approximation

The method demonstrated in Section 3.1 gives us the actual binomial price of the cliquet option. But as was mentioned earlier, the computational complexity of the method is theoretically the same as that of the binomial method, which is m^N. But contrary to the binomial method, the singular point method enables us to use precise, e□cient and controlled approximation to accelerate the procedure significantly. Thus its e□cacy become apparent when we have time and memory constraints, or we wish to substantially increase the number of intermediate time steps.

The prime idea of the approximation procedure is to remove singular points in a controlled manner in order to simplify the computations. To this end, we fix a given maximal level $h > 0$ of the error in each period. Now, we eliminate singular points in a fashion so that the function after deletion of the points ($\tilde{V}_i$) is di□erent from the original function by less than (h) at each point. This may be achieved as follows.

Start with the point (Z_i^1, P_i^1). Find the largest index $l > 1$ such that the distances between the straight line joining (Z_i^1, P_i^1) and (Z_i^l, P_i^l) and the points $(Z_i^2, P_i^2), (Z_i^3, P_i^3), \ldots, (Z_i^{l-1}, P_i^{l-1})$ are always less than h. Note that this, coupled with the fact that the original function is piecewise-linear, ensures that the di□erences in the values of the functions for any value of the running sum would be bounded over by h. Now delete the points $(Z_i^2, P_i^2), (Z_i^3, P_i^3), \ldots, (Z_i^{l-1}, P_i^{l-1})$. The point (Z_i^l, P_i^l) now becomes the second singular point, and we continue the procedure iteratively starting with this point till we cover all the points. Figure 2 elaborates on this graphically.

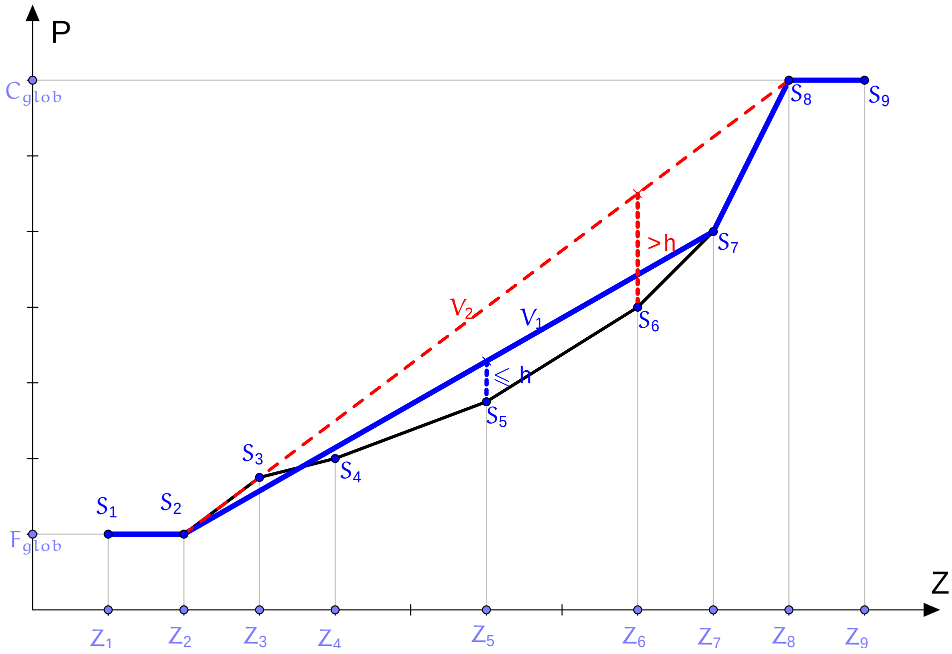

Figure 2. Approximation procedure: Elimination of points S_3 to S_6 gives function V_1 (bold, blue), which has maximum error $\leqslant h$. Elimination of points S_3 to S_7 gives function V_2 (dashed, red), which has maximum error $> h$. Thus we choose V_1 as the approximation function.

Since at each N the upper bound for the error is h, if we repeat the approximation procedure at every observational time, the total error infused in the price of the option is bounded by Nh.

Remark 5.3 (Comparison with Asian options). While using approximations for Asian options, the error bound was nh, where n was the number of time steps considered for computation (refer to Section [TODO: reference] of Chapter 4). But in the case of the cliquet options, the error depends only on the h and the number of observations - precluding any dependence on the computational parameters. This technique gives a precise approximation of the price without considering upper and lower estimates.

The following proposition tells us that the approximation of the price obtained by this method does converge to the price of the cliquet option in the continuous model.

[TODO: Proposition 2]

4 The program

4.1 Algorithm

[TODO: Include approximation procedure]

Algorithm 1: The Singular Points method for cliquet options

Input:

Contract details
- time to maturity: T
- number of observations: N
- local floor and cap: F_{loc}, C_{loc}
- global floor and cap: F_{glob}, C_{glob}

Details of the underlying asset
- initial price: s_0
- volatility: σ
- continuous dividend rate: q

Market parameters – spot interest rate: r

Computational parameters – time steps within each observed period: m

Output: The price of the option at the initial time

begin
- Update F_{glob} and C_{glob} using Equations 2.5.
- Set $\Delta T, u, p$ from the formulae in Section 2.
- Compute the returns and probabilities using Equation 3.3 and Equation 3.4.
- Compute j_{min}, j_{max}, j_0 (Equations 3.5) and shifted returns (Table 2).
- `//` S_i `and` S_+ `denotes the current (`i^{th}`) and next (`$(i+1)^{th}$`) list of singular points.`
- $S_N \leftarrow \{(NF_{loc}, F_{glob}), (F_{glob}, F_{glob}), (C_{glob}, C_{glob}), (NC_{loc}, C_{glob})\}$ `// Table 1.`
- $S_+ \leftarrow S_N$
- **for** $i \in \{N-1, \dots, 0\}$ **do**
  - $S_i \leftarrow \emptyset$, $\quad L_+ \leftarrow \text{length}(S_+)$
  - **forall the** $(l, j) \in [L_+] \times [j_0 + 1]$ **do**
    - Compute $B_{l,j}$ using Equation 3.7, replacing N by i.
    - **if** $(B_{l,j} + R'_k) \notin [Z_N^{l_0}, Z_N^{l_0+1}]$ **then**
      - Continue `// to the next item in the loop`
    - Find l_0 such that $(B_{l,j} + R'_k) \in [Z_{i+1}^{l_0}, Z_{i+1}^{l_0+1}]$.
    - Evaluate $V_{i+1}(B_{l,j} + R'_k)$ by linear interpolation of the extrema.
    - Evaluate $V_i(B_{l,j})$ by using Equation 3.6, replacing N by i.
    - $S_i \leftarrow S_i \cup (B_{l,j}, V_{N-1}(B_{l,j}))$
  - $S_+ \leftarrow S_i$
- **return** $(S_i)_{1,2}$ `// Singular price at time 0`

Analysis of algorithm [TODO if time permits] Consider computational complexity, space complexity.

4.2 Implementation

The algorithm was implemented in Python 3.5.0 (2015-09-13). The system specifications are as follows [TODO: Shift this to some common region for both Asian and cliquet]

<TODO: The actual implementation goes here>

5 Numerical results

6 Conclusions

Chapter 6

Epilogue

Bibliography

[CRR79] John C Cox, Stephen A Ross, and Mark Rubinstein. “Option pricing: A simplified approach”. In: *Journal of financial Economics* 7.3 (1979), pp. 229–263.

[GZ11] Marcellino Gaudenzi and Antonino Zanette. “Pricing cliquet options by tree methods”. In: *Computational Management Science* 8.1-2 (2011), pp. 125–135.

[GZA10] Marcellino Gaudenzi, Antonino Zanette, and Maria Antonietta Lepellere. “The singular points binomial method for pricing American path-dependent options”. In: *Journal of Computational Finance* 14.1 (2010), p. 29.

[LL96] D. Lamberton and B. Lapeyre. *Introduction to Stochastic Calculus Applied to Finance*. London: Chapman and Hall, 1996.