



MathMods



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Notations

Mathematical

- $[n] = \{0, 1, 2, \dots, n\}$
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$
- $\mathbb{R}_{0+} = [0, \infty)$, $\mathbb{R}_+ = (0, \infty)$
- $a \wedge b = \inf(\{a, b\})$
- $a \vee b = \sup(\{a, b\})$

General

CRR: The Cox-Ross-Rubinstein model

BS: The Black-Scholes model

Chapter 1

The Singular Points method

1 Introduction

As we have seen in the earlier chapters, European and American options may be priced using the CRR and BS models. Even though the BS model has a very high degree of computability, it does not allow us to find closed-form pricing formulae for many path-dependent options. The way out is by using numerical methods. A class of numerical methods use tree or lattice methods in the CRR model. One simple idea is to apply an explicit pricing scheme using CRR, which tends to BS as the number of time steps increases to infinity. But the exponential number of paths (2^n to be exact, where n is the number of time steps) make the method very slow and memory intensive, making it impractical in terms of computability. A logical step would be to modify the basic CRR model to allow for approximations. In this direction, Gaudenzi et al[GZAL10] introduced a new method called the ‘singular points method’ for pricing certain path-dependent options in an efficient manner. The chapter is a study on this method.

We will mainly focus on Asian options, in which the price is expressed as a function of some form of averaging on the underlying’s price. Popular Asian options use the arithmetic or geometric means as the average. Again, Asian options may be exercised only at maturity (European) or at any time till the maturity (American). They may give the owner of the option the right to either sell (put) or buy (call). Theoretically, we may study either a call or a put, because the framework for the other one may be derived in the exact same way.

TODO: Push this defn to an earlier chapter.

DEFINITION 1.1 (Path-dependent option). A path-dependent option is an option for which the value of the option is dependent not only on the final value of the underlying, but also on the path taken to reach that value.

2 Existing methods

Before we go into the details of the singular points method, we shall look into the pre-existing methods, and discuss their advantages and disadvantages briefly.

Ref: Forsyth et al (2002)

- American Asian options with arithmetic mean
 - Tree based
 - * CRR binomial method
 - * Hull and White (1993)
 - * Barraquand and Pudet (1996)
 - * Chalasani et al (1999a, b)
 - PDE based
 - * Vecer (2001)
 - * D'Halluin et al (2005)
- American lookback options
 - Hull and White (1993)
 - Barraquand and Pudet (1996)
 - Babbs (2000), using a 'change of numeraire' approach, which cannot be applied to the fixed-strike case

TODO: Discuss the advantages and disadvantages of each method.

3 The Singular Points method

The singular points method looks at the price of the option as a continuous function of the underlying's average at each node of the binomial tree.

DEFINITION 1.2 (Singular points and singular values). Let $P = (P_i)_{i \in [n]} = ((x_i, y_i))_{i \in [n]}$, $n \in \mathbb{N}$ be a sequence of points such that

$$(3.1a) \quad a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

$$(3.1b) \quad m_i := \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \leq \frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}} = m_{i+1} \quad \forall i \in \{1, \dots, n-1\}$$

Let $f : [a, b] \rightarrow \mathbb{R}_{0+}$ be the function obtained by linear interpolation of the points in P . From the definition of f and 3.1b, the function is continuous, piecewise-linear and convex.

Then, the elements of P are called *singular points of f* and the abscissae $\{x_i\}_{i \in [n]}$ are called *singular values of f* .

REMARK 1.1. We note that the singular points characterise such a function completely. This can be seen from the following representation of the function.

$$(3.2) \quad f(x) = y_0 + \sum_{i=0}^{n-1} [m_i \{(x_{i+1} \wedge x) - (x_i \wedge x)\}]$$

Where $m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ represents the slope of the function between (x_i, y_i) and (x_{i+1}, y_{i+1}) .

REMARK 1.2. From the conditions 3.1, we get

$$y_0 < y_1 < \cdots < y_{n-1} < y_n$$

So it is equivalent to sort points using either abscissae or ordinates.

3.1 Upper and lower bounds

The following lemmas shall provide us with the necessary framework for upper and lower bounds for approximations on the functions generated by singular points.

LEMMA 1.1 (Upper bound). *Let $f : [a, b] \rightarrow \mathbb{R}_{0+}$ be a continuous, piecewise-linear, convex function characterised by the singular points $P = ((x_i, y_i))_{i \in [n]}$. Then, if a point $(x_j, y_j), j \in \{1, \dots, n-1\}$ is removed from the sequence, the function $f_u : [a, b] \rightarrow \mathbb{R}_{0+}$ obtained by the new sequence $(P_i)_{i \in [n] \setminus \{j\}}$ is also continuous, piecewise-linear and convex, and*

$$(3.3) \quad f_u(x) \geq f(x) \quad \forall x \in [a, b]$$

PROOF. Graphical or analytical? □

LEMMA 1.2 (Lower bound). *Let $f : [a, b] \rightarrow \mathbb{R}_{0+}$ be a continuous, piecewise-linear, convex function characterised by the singular points $P = ((x_i, y_i))_{i \in [n]}$. Let l_j be the line joining points P_{i-1} and P_i . Similarly, let l_{j+1} be the line joining points P_{i+1} and P_{i+2} . Denote the intersection of the lines l_j and l_{j+1} by $\bar{P} = (\bar{x}, \bar{y})$.*

Then the function $f_d : [a, b] \rightarrow \mathbb{R}_{0+}$ characterised by $(P_0, \dots, P_{j-1}, \bar{P}, P_{j+2}, \dots, P_n)$ is also continuous, piecewise-linear and convex, and

$$(3.4) \quad f_d(x) \leq f(x) \quad \forall x \in [a, b]$$

PROOF. Graphical or analytical? □

The lemmas 1.1 and 1.2, will be used later to reduce the memory requirement of the algorithm by removing points or edges to simplify the function and reduce memory requirements.

3.2 Fixed-strike European Asian options

For this type of options, the pay-off at maturity is dependent only on the arithmetic mean A_T at maturity T , and is given by the function

$$(3.5) \quad P_T = (A_T - K)_+$$

We shall focus on this case in this section because it is the easiest to handle.

In each node of the binomial tree, we have a set of possible averages depending on the paths which may be taken to arrive at the node, and prices corresponding to each such average. We shall show these points to satisfy condition 3.1, so they completely characterise a price function. Thus we focus not on the averages and corresponding prices possible under a particular binomial tree, but on the continuous representation of prices. The intuitive idea is that as the time step is reduced to zero, this function converges to the price function of the continuous time model.

3.3 Some notations and conventions

Let the number of time steps be n . Let i denote the highlighted time step, and j represent the number of up movements. In this way, we may represent any node by $N_{i,j}$.

The price of the underlying at each node $N_{i,j}$ is denoted by $S_{i,j}$. Since there are j up movements, there must be $i - j$ down movements, and thus

$$(3.6) \quad S_{i,j} = S_0 u^j d^{i-j} = S_0 u^j u^{-(i-j)} = S_0 u^{-i+2j} \quad \forall i \in [n], \forall j \in [i]$$

PROPOSITION 1.1. *The number of paths to a node $N_{i,j}$ is $\binom{i}{j}$.*

PROOF. At each point in a path, we may choose either an up movement or a down movement. To reach node $N_{i,j}$, we must choose j up movements among i possibilities. The result follows immediately. \square

Any number of paths among the possible paths may give zero as the price for the option. We denote the effective number of singular points in a node by $N_{i,j}$ by $L_{i,j}$, where $L_{i,j} \in \left[\binom{i}{j}\right]$. The l^{th} average (in ascending order) ($l \in \{1, \dots, L_{i,j}\}$) is denoted by $A_{i,j}^l$, and the corresponding price by $P_{i,j}^l$. Thus the singular points characterising the price function are $((A_{i,j}^l, P_{i,j}^l))_{l \in \{1, \dots, L_{i,j}\}}$.

DEFINITION 1.3 (singular average and singular price). In the particular case of Asian options with arithmetic mean, the $A_{i,j}^l$ s are called ‘singular averages’ and the $P_{i,j}^l$ s are called ‘singular prices’.

We recall some basic definitions and derive simple results for the maximum and minimum attainable value of the averages on each node.

DEFINITION 1.4 (Arithmetic mean). The arithmetic mean of a set of numbers $\{S_i\}_{i \in [n]}$ is given by:

$$(3.7) \quad A_n = \frac{\sum_{i=0}^n S_i}{n+1}$$

DEFINITION 1.5 (Path). A path is a sequence $(j_i)_{i \in [n]}$ such that $j_{i+1} \in \{j_i, j_i + 1\}$.

THEOREM 1.1 (Path inequality). *Let there be two paths α and β , such that $S_{i,j_i^\alpha} \geq S_{i,j_i^\beta} \forall i$. Denote the corresponding averages by A^α and A^β , respectively. Then $A^\alpha \geq A^\beta$.*

PROOF. Clearly if $S_{i,j_i^\alpha} = S_{i,j_i^\beta} \forall i$, then $A^\alpha = A^\beta$.

We only need to show the result in the case of inequality. Let $S_{i,j_i^\alpha} = S_{i,j_i^\beta} \forall i \in [n] \setminus \{l\}$, and $S_{l,j_l^\alpha} > S_{l,j_l^\beta}$.

Now, from equation 3.7, we have:

$$\begin{aligned}
(n+1)A_{n,j}^{\alpha} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^{\alpha}} + \sum_{i=l+1}^n S_{i,j_i} \\
(n+1)A_{n,j}^{\beta} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^{\beta}} + \sum_{i=l+1}^n S_{i,j_i} \\
\Rightarrow (n+1)(A_{n,j}^{\alpha} - A_{n,j}^{\beta}) &= S_{l,j_l^{\alpha}} - S_{l,j_l^{\beta}} \\
&= S_{l-1,j_{l-1}}u_l - S_{l-1,j_{l-1}}d_l \\
&= S_{l-1,j_{l-1}}(u_l - d_l) > 0 \quad (u_l > d_l \text{ by definition}) \\
\Rightarrow A_{n,j}^{\alpha} &> A_{n,j}^{\beta}
\end{aligned}$$

□

REMARK 1.3. The path α signifies a path ‘above’ and β a path ‘below’ in the usual depiction of the binomial tree (the up movement shown above the down movement). Thus, any path above has a higher arithmetic mean than the one below.

COROLLARY 1.1. At each node $N(i, j)$, the following hold:

- (1) The minimum average possible $A_{i,j}^{\min}$ is attained by the path corresponding to the path corresponding to the path with $(i - j)$ down movements followed by j up movements, and

$$(3.8) \quad A_{i,j}^{\min} = \frac{S_0}{i+1} \left(\frac{1 - d^{i-j+1}}{1 - d} + d^{i-j}u \frac{1 - u^j}{1 - u} \right)$$

- (2) The maximum average possible $A_{i,j}^{\max}$ is attained by the path corresponding to the path with j up movements followed by $(i - j)$ down movements, and

$$(3.9) \quad A_{i,j}^{\max} = \frac{S_0}{i+1} \left(\frac{1 - u^{j+1}}{1 - u} + u^j d \frac{1 - d^{i-j-1}}{1 - d} \right)$$

PROOF. We show the proof only for the case of the maximum, since the case of the minimum can be shown using the exact same argument.

From Theorem 1.1, the result about path with the maximum average holds directly, since there cannot be a path above the one given by j up movements followed by $(i - j)$ down movements.

The subsequent formula may be derived as follows.

$$\begin{aligned}
(i+1)A_{i,j}^{\max} &= \underbrace{(S_0 + S_0u + S_0u^2 + \dots + S_0u^j)}_{\text{up movement}} + \underbrace{(S_0u^j d + S_0u^j d^2 + \dots + S_0u^j d^{i-j})}_{\text{down movement}} \\
&= S_0((1 + u + u^2 + \dots + u^j) + u^j d(1 + d + \dots + d^{i-j-1})) \\
&= S_0 \left(\sum_{k=0}^j u^k + u^j d \sum_{k=0}^{i-j-1} d^k \right) \\
&= S_0 \left(\frac{1 - u^{j+1}}{1 - u} + u^j d \frac{1 - d^{i-j-1}}{1 - d} \right) \quad (\text{Geometric series}) \\
\Rightarrow A_{i,j}^{\max} &= \frac{S_0}{i+1} \left(\frac{1 - u^{j+1}}{1 - u} + u^j d \frac{1 - d^{i-j-1}}{1 - d} \right)
\end{aligned}$$

□

The following table highlights the discussion above.

Symbol	Range	Formula	Description
i	$[n]$		highlighted time step
j	$[i]$		number of up movements
$N_{i,j}$			node fixed by (i, j)
$S_{i,j}$	\mathbb{R}_{0+}	Eq 3.6	value of the underlying at node $N_{i,j}$
$L_{i,j}$	$\left[\binom{i}{j} \right]$		number of singular points in node $N_{i,j}$
l	$\{1, \dots, L\}$		index for points in ascending order of averages
$A_{i,j}^{\max}$	\mathbb{R}_{0+}	Eq 3.9	maximum average attainable for node $N_{i,j}$
$A_{i,j}^{\min}$	\mathbb{R}_{0+}	Eq 3.8	minimum average attainable for node $N_{i,j}$
$A_{i,j}^l$	$[A_{i,j}^{\min}, A_{i,j}^{\max}]$	Eq 3.7	l^{th} singular average of node $N_{i,j}$
$P_{i,j}^l$			price corresponding to the average $A_{i,j}^l$
$(A_{i,j}^l, P_{i,j}^l)$			l^{th} singular point of node $N_{i,j}$

3.3 At maturity

From equations 3.9 and 3.8, putting $i = n$, we get

$$\begin{aligned}
A_{n,j}^{\min} &= \frac{S_0}{n+1} \left(\frac{1 - d^{n-j+1}}{1 - d} + d^{n-j} u \frac{1 - u^j}{1 - u} \right) \\
A_{n,j}^{\max} &= \frac{S_0}{n+1} \left(\frac{1 - u^{j+1}}{1 - u} + u^j d \frac{1 - d^{n-j-1}}{1 - d} \right)
\end{aligned}$$

The price function

In defining the price function, we note that three cases may arise.

- $j \in \{0, n\}$

There exists only one path to these nodes, so there is only one average, implying one price and one singular point.

- $j \notin \{0, n\} \wedge K \in (A_{n,j}^{\min}, A_{n,j}^{\max})$

The price function is characterised by three singular points $(L_{i,j} = 3)$, $(A_{n,j}^l, P_{n,j}^l)_{l \in \{1,2,3\}}$, which are

$$(3.10) \quad \begin{aligned} (A_{n,j}^1, P_{n,j}^1) &= (A_{n,j}^{\min}, 0) \\ (A_{n,j}^2, P_{n,j}^2) &= (K, 0) \\ (A_{n,j}^3, P_{n,j}^3) &= (A_{n,j}^{\max}, A_{n,j}^{\max} - K) \end{aligned}$$

- $j \notin \{0, n\} \wedge K \notin (A_{n,j}^{\min}, A_{n,j}^{\max})$

The price function is characterised by only two singular points $(L_{i,j} = 2)$, $(A_{n,j}^l, P_{n,j}^l)_{l \in \{1,2\}}$, which are

$$(3.11) \quad \begin{aligned} (A_{n,j}^1, P_{n,j}^1) &= (A_{n,j}^{\min}, (A_{n,j}^{\min} - K)_+) \\ (A_{n,j}^2, P_{n,j}^2) &= (A_{n,j}^{\max}, (A_{n,j}^{\max} - K)_+) \end{aligned}$$

LEMMA 1.3 (Price function at maturity). *At each node at maturity, the price function $v_{n,j} : [A_{n,j}^{\min}, A_{n,j}^{\max}] \rightarrow [(A_{n,j}^{\min} - K)_+, (A_{n,j}^{\max} - K)_+]$ defined as $v_{n,j}(A) = (A - K)_+$ is continuous, piecewise-linear and convex.*

PROOF. The singular points satisfy the conditions 3.1. So for each $A \in [A_{n,j}^{\min}, A_{n,j}^{\max}]$, the price function $v_{n,j}(A)$ characterised by the singular points is continuous, piecewise-linear and convex by remark 1.1. \square

3.3 Before maturity

TODO

LEMMA 1.4 (Price function at any node $N_{i,j}$). *The price function $v_{i,j} : [A_{i,j}^{\min}, A_{i,j}^{\max}] \rightarrow \mathbb{R}_{0+}$ is continuous, piecewise-linear and convex.*

PROOF. TODO

\square

TODO

4 Extensions

Let us recapitulate the conditions required for the singular points method to work in the case of Asian options with arithmetic mean.

- The ability to calculate the upper and lower bounds of the mean for all nodes of the tree.
- The recombinant nature of the tree for the underlying. Note that the tree for the option prices are *not* recombinant.
- Convexity and piecewise-linearity of the price function on the mean of the underlying.
- Fixed volatility

Keeping these in mind, let us look at the possibility of extending the singular points method to the following cases:

- (1) Asian options with geometric mean and fixed volatility.
- (2) Asian options with arithmetic mean and local volatility.

4.1 Geometric mean and fixed volatility

In the case of geometric options, we have a closed form formula under the Black-Scholes market model. We try to extend the singular points method.

Firstly, we show that the result about the maximum and minimum paths still hold in the geometric case.

DEFINITION 1.6 (Geometric mean). The geometric mean of the risky asset's prices $(S_i)_{i \in [n]}$ is given by:

$$(4.1) \quad G_n = \left(\prod_{i=0}^n S_i \right)^{\frac{1}{n+1}}$$

LEMMA 1.5. *At each node $N(i, j)$, the following hold:*

- (1) *The maximum average possible $G_{i,j}^{\max}$ is attained by the path corresponding to the path with j up movements followed by $(i - j)$ down movements.*
- (2) *The minimum average possible $G_{i,j}^{\min}$ is attained by the path corresponding to the path with $(i - j)$ down movements followed by j up movements.*

PROOF. The proof is the same as 1.1, with A replaced by G and relevant modifications. \square

One of the central ideas behind the singular points method is that the price of the option is a convex, piecewise-linear function of the average A . But in the geometric case, this no

longer holds true. For example, take a node $N_{i,j}$ with $i = n - 1$. The price function given by $v_{i,j}(G)$, with $G \in [G^{\min}, G^{\max}]$, can be calculated by the discounted expectation value.

$$(4.2) \quad v_{i,j}(G) = \frac{1}{R} [pv_{i+1,j+1}(G_u) + (1-p)v_{i+1,j}(G_d)]$$

$$(4.3) \quad G_u = (G^{i+1}S_0u^{-i+2j+1})^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}}$$

$$(4.4) \quad G_d = (G^{i+1}S_0u^{-i+2j-1})^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}}$$

Clearly, the final function $v_{i,j}$ is not linear in G . Rather it is piecewise-concave. Thus we cannot use the singular points method in this case. TODO: Insert a graph of the function here.

4.2 Arithmetic mean with local volatility

In this case, the tree for the underlying is not recombining, so we do not have more than one singular point in one (non-recombining) node. Clearly, we cannot use the singular points method.

5 Conclusion

We conclude the chapter by noting the pros and cons of the singular points method.

Advantages

- Fast – Experimental order of complexity = $O(n^3)$
- It allows us to specify an *a priori* error bound.

Disadvantages

- Very specific method – only applicable to a few specific cases.

Bibliography

- [GZAL10] Marcellino Gaudenzi, Antonino Zanette, and Maria Antonietta Lepellere. “The singular points binomial method for pricing American path-dependent options”. In: *Journal of Computational Finance* 14.1 (2010), p. 29.