MathMods

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Notations

- $[n] = \{0, 1, 2, \ldots, n\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $\mathbb{R}_{0+} = [0, \infty)$

Chapter 1
Cox Ross Rubenstein model

1 Introduction

2 The binomial model

3 The first fundamental theorem of asset pricing

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6 European options

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Chapter 2

Black Scholes model

Chapter 3
The Singular Points method

1 Introduction

As we have seen in the earlier chapters, European and American options may be priced using the CRR and BS models. But for many path-dependent options, we cannot find a closed-form pricing formula in the BS model. One way to overcome this difficulty is the use of tree or lattice methods in the CRR model. But such methods are slow and very memory intensive owing to the exponential nature of the number of possible paths. Gaudenzi et al[**Gaudenzi2010**] introduced a new method for pricing path-path dependent options in an efficient manner.

2 Existing methods

Ref: Forsyth et al (2002)

- American Asian options with arithmetic mean
  - Tree based
    - CRR binomial method
    - Hull and White (1993)
    - Barraquand and Pudet (1996)
    - Chalasani et al (1999a, b)
  - PDE based
    - Vecer (2001)
    - D'Halluin et al (2005)
- American lookback options
  - Hull and White (1993)
  - Barraquand and Pudet (1996)
  - Babbs (2000), using a 'change of numeraire' approach, which cannot be applied to the fixed-strike case

3 The Singular Points method

First we recollect some basic definitions, and derive simple results for upper and lower bound for the averages.

Definition 3.1 (Arithmetic mean). The arithmetic mean of the risky asset's prices $(S_i)_{i\in[n]}$ is given by:

$$A_n = \frac{\sum_{i=0}^{n} S_i}{n+1} \tag{1}$$

Definition 3.2 (Path). A path is a sequence $(j_i)_{i\in[n]}$ such that $j_{i+1} \in \{j_i, j_i+1\}$.

Theorem 3.1. *Let there be two paths α and β, such that $S_{i,j_i^\alpha} >= S_{i,j_i^\beta}$ $\forall i$. Denote the corresponging averages by A^α and A^β, respectively. Then $A^\alpha >= A^\beta$.*

Proof. Clearly if $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i$, then $A^\alpha = A^\beta$.

We only need to show the result in the case of inequality. Let $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i \in [n] \setminus \{l\}$, and $S_{l,j_l^\alpha} > S_{l,j_l^\beta}$.

Now, from equation 1, we have:

$$\begin{aligned}
(n+1)A^\alpha_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\alpha} + \sum_{i=l+1}^{n} S_{i,j_i} \\
(n+1)A^\beta_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\beta} + \sum_{i=l+1}^{n} S_{i,j_i} \\
\implies (n+1)\left(A^\alpha_{n,j} - A^\beta_{n,j}\right) &= S_{l,j_l^\alpha} - S_{l,j_l^\beta} \\
&= S_{l-1,j_{l-1}} u_l - S_{l-1,j_{l-1}} d_l \\
&= S_{l-1,j_{l-1}} (u_l - d_l) > 0 \\
\implies A^\alpha_{n,j} &> A^\beta_{n,j}
\end{aligned}$$

□

Remark. The path α signifies a path above and β signifies a path below in the usual depiction of the binomial tree. Thus, any path above has a higher arithmetic mean than the one below.

Corollary 3.1.1. *At each node $N(i,j)$, the following hold:*

(1) *The maximum average possible $A^{\max}_{i,j}$ is attained by the path corresponding to the path with j up movements followed by $(i-j)$ down movements.*
(2) *The minimum average possible $A^{\min}_{i,j}$ is attained by the path corresponding to the path corresponding to the path with $(i-j)$ down movements followed by j up movements.*

Definition 3.3 (Singular points, singular values). Let $P = \{(x_i, y_i)\}_{i\in[n]}$, $n \in \mathbb{N}$ be a set of points such that $a = x_1 < \cdots < x_n < b$ and

$$m_{i-1} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \leqslant \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = m_i \qquad \forall i \in \{2, \ldots, n-1\}$$

Let $f : [a,b] \to \mathbb{R}_{0+}$, $f(x) = y_1 + \sum_{i=1}^{n-1} (x_{i+1} \wedge x - x_i \wedge x)$ be the function obtained by linear interpolation of the points in P. Then, the elmments of P are called *singular points of f* and the abscissae $\{x_i\}_{i\in[n]}$ are called *singular values of f.*

Lemma 3.2 (Lemma 1). *TODO*

Lemma 3.3 (Lemma 2). *TODO*

Lemma 3.4 (Lemma 3). *TODO*

Lemma 3.5 (Lemma 4). *TODO*

4 Extensions

Let us recapitulate the conditions required for the singular points method to work in the case of Asian options with arithmetic mean.

- The ability to calculate the upper and lower bounds of the mean for all nodes of the tree.
- The recombinant nature of the tree for the underlying. Note that the tree for the option prices are *not* recombinant.
- Convexity and piecewise-linearity of the price function on the mean of the underlying.
- Fixed volatility

Keeping these in mind, let us look at the possibility of extending the singular points method to the following cases:

(1) Asian options with geometric mean and fixed volatility.
(2) Asian options with arithmetic mean and local volatility.

4.1 Pros and cons

Advantages

- Fast – Experimental order of complexity $= O(n^3)$
- It allows us to specify an *a priori* error bound.

Disadvantages

- Very specific method – only applicable to a few specific cases.

4.2 Geometric mean and fixed volatility

In the case of geometric options, we have a closed form formula under the Black-Scholes market model. We try to extend the singular points method in this case.

Firstly, we show that the result about the maximum and minimum paths still hold in the geometric case.

Definition 3.4 (Geometric mean). The geometric mean of the risky asset's prices $(S_i)_{i\in[n]}$ is given by:

$$G_n = \left(\prod_{i=0}^{n} S_i\right)^{\frac{1}{n+1}} \tag{2}$$

Lemma 3.6. *At each node $N(i,j)$, the following hold:*

(1) *The maximum average possible $G_{i,j}^{max}$ is attained by the path corresponding to the path with j up movements followed by $(i-j)$ down movements.*
(2) *The minimum average possible $G_{i,j}^{min}$ is attained by the path corresponding to the path corresponding to the path with $(i-j)$ down movements followed by j up movements.*

Proof. The proof is the same as 3.1.1, with A replaced by G and relevant modifications. □

One of the central ideas behind the singular points method is that the price of the option is a convex, piecewise-linear function of the average A. But in the geometric case, this no longer holds true. For example, take a node $N(i,j)$ with $i = n-1$. The price function given by $v_{i,j}(G)$, with $G \in [G^{min}, G^{max}]$, can be calculated by the discounted expectation value.

$$v_{i,j}(G) = \frac{1}{R}\left[p v_{i+1,j+1}(G_u) + (1-p) v_{i+1,j}(G_d)\right] \tag{3}$$

$$G_u = \left(G^{i+1} S_0 u^{-i+2j+1}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{4}$$

$$G_d = \left(G^{i+1} S_0 u^{-i+2j-1}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{5}$$

Clearly, the final function $v_{i,j}$ is not linear in G. Rather it is piecewise-concave. Thus we cannot use the singular points method in this case.

4.3 Arithmetic mean with local volatility

In this case, the tree for the underlying is not recombinant, so we do not have more than one singular point in one (non-recombining) node. Clearly, we cannot use the singular points method in this case.