Universität Hamburg

Università degli Studi dell'Aquila

Universitat Autònoma de Barcelona

Erasmus Mundus Consortium

MathMods

Joint Degree of Master of Science in
Mathematical Modelling in Engineering: Theory, Numerics, Applications

In the framework of the
Consortium Agreement and Award of a Joint/Multiple Degree 2013-2019

Master's Thesis

The Singular Points method for pricing exotic path-dependent options

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September 30, 2015

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Preface

A path-dependent option is a type of exotic option in which the price of the option depends not only on the price of the underlying asset at maturity, but also on the history of the asset's price till that point. Typical examples of popular exotic options are Asian options, lookback options, barrier options, and digital options.

In the Black-Scholes market model, it is not possible to find closed-form analytical formulae for the prices of most exotic options. This inspires the requirement of fast numerical algorithms to determine the fair price of such options. Such algorithms may be clustered into categories. Some algorithms utilise the convergence of the prices found by discrete models like the Cox-Ross-Rubinstein to the Black-Scholes prices as a limit when the time step is reduced. Other approaches include use of numerical methods to solve partial differential equation, or simulation using Monte Carlo methods.

Under the Cox-Ross-Rubinstein model, Gaudenzi, Zanette and Lepellere introduced the Singular Points method. It is a numerical method to price Asian and lookback options. A modification enables pricing of cliquet options. In the method, in each node of the binomial tree of the underlying risky asset, the price is represented as a continuous function of the path-dependent parameter. An advantage of this method over pre-existing methods is its low order of computational and space complexity. It is convergent, and allows us to set *a priori* upper and lower bounds on the error.

In the master's thesis, we present an exposition on the Singular Points method and how it may be used to price exotic options. We also explore the extensibility of the method to similar types of options, like Asian options with geometric mean as opposed to arithmetic mean, and whether it may be used for variable local volatilities and interest rates.

Prerequisites The reader is expected to be familiar about basic Probability Theory and Stochastic Processes. In particular, (s)he should be comfortable with probability spaces and measures, filtrations, random variables, stochastic processes, martingales, Brownian motion, and elementary stochastic calculus. There is no strict requirement of knowledge of financial concepts, as we introduce the required terminologies in the introductory chapters. [TODO: Appendix]

Note to the reader The theory part of the thesis borrows heavily from the book titled Introduction to Stochastic Calculus Applied to Finance by Damien Lamberton and Bernard Lapeyre [LL96], as well as from the lecture notes of Mathematical Finance authored by Prof. Fabio Antonelli. This course was offered in the Fall semester of 2014-15 in Università degli Studi dell'Aquila, Italy.

The chapters on the Singular Point method has been motivated by the series of papers published by Gaudenzi, Zanette and Lepellere on the same topic.

Notations

$[n]$: $\{0, 1, 2, \dots, n\}$

$\mathbb{N}$: $\{1, 2, 3, \dots\}$

$\mathbb{N}_0$: $\mathbb{N} \cup \{0\}$

$\mu\mathcal{F}$: The class of all $\mathcal{F}$-measurable functions.

Chapter 1

Prologue

[TODO] One page history

- The thesis of Louis Bachelier (1900) on the "Theory of Speculation"
- Introduction of Brownian motion to model fluctuating prices in the Paris stock exchange
- Black and Scholes
- Cox Ross Rubinstein

1.1 Financial instruments

A *financial instrument* or a *financial asset* is an intangible asset whose value is derived from a contractual claim, such as bank deposits, bonds, stocks and derivatives. Financial assets are usually more liquid than other tangible assets, such as commodities or real estate, and may be traded on financial markets. Every financial asset is characterised by its return. When the return is deterministic, we call it a *risk-free* or *riskless* asset. When the return is contingent on the market and external conditions, it is called *risky*. It must be kept in mind that no instrument is fundamentally risk-free, it has only negligible risk compared to its risky counterparts.

1.1.1 Return on an asset

Return on a riskless asset – Compounding Compounding is the first idea that we must be familiar with. Essentially, a riskless asset will increase in monetary value in a deterministic manner if we keep it in the market. The increase depends on the compounding frequency and the duration of investment. The term compounding is used because the interest earned in each period also contributes to the principal in the successive periods.

Let the compounding frequency is n times per year, the total time is t, and the annual rate of interest is r. Then

$$B_t = B_0 \left(1 + \frac{r}{n}\right)^{\lfloor nt \rfloor}, \tag{1.1}$$

where B_0 is the starting value of the asset, and B_t is the value of the asset at time t.

If the compounding is continuous, we let $n \to \infty$ to obtain

$$B_t = B_0 e^{rt}. \tag{1.2}$$

Return on a risky asset At any point of time in the future, the value of a risky asset is not known with certainty, so it is a random variable taking values in $[0, \infty)$. Since the values change with time, we denote the *spot price* of a risky asset by the stochastic process $(S_t)_t$, where $t \in [0, T]$ denotes the time and T is the maturity. Since the future value of the asset is adventitious, we use the following quantities to measure the return of the risky asset in the given time interval.

Definition 1.1 (absolute and relative returns). The absolute return on an asset for the time interval $[0, t]$, $t \in [0, T]$ is given by

$$\tilde{R}_t = S_t - S_0$$

The relative return on the asset is given by

$$R_t = \frac{S_t - S_0}{S_0}$$

1.1.2 Common asset types

Bonds A *bond* is an instrument of indebtedness of the bond issuer to the holders. It is a *debt security*, under which the issuer owes the holders a debt and, depending on the terms of the bond, is obliged to pay them interest (the coupon) and/or to repay the principal at a later date, termed the maturity date. Bonds can also be thought of as a *loan* given to the issuer of the bond by the holder. A bond primarily has two kinds of risks, *credit default risk* and *interest rate risk*. A bond issued by a reliable institution like the United States government is a good illustration of a risk-free asset. This is because the probability of such an organisation defaulting is close to zero, or in other words, the bond has *negligible credit default risk*. Such bonds are only subject to fluctuations of the current interest rate, called *interest rate risk*. The interest rate risk may also be nullified if the bond is held till maturity. If we assume that the interest rate is deterministic (the fluctuations are not random), the value of the bond is computable at any given future date, making it riskless. Such an assumption is quite reasonable in short periods of time and for institutions with a low default risk rating.

Stocks A *stock* of a corporation is an ownership certificate, and constitutes the equity stake of its owners. It represents the residual assets of the company that would be due to stockholders after discharge of all senior claims such as secured and unsecured debt. A *share* of a stock is a unit of ownership of the organisation. Stocks are inherently risky, since the value of the organisation may change from time to time due to various internal and external factors. The value of a stock in time is usually represented by a stochastic process $(S_t)_t$.

Derivatives A *derivative* is a contract between two parties that specify conditions (starting and termination dates, resulting values and definitions of the underlying variables, the parties' contractual obligations, and the notional amount) under which transactions are to be made between the parties. The most common underlying assets include commodities, stocks, bonds, interest rates and currencies, but they can also be other derivatives, which adds another layer of complexity to proper valuation. Essentially, the value of a derivative is a function of the value of the *underlying* asset(s). Derivatives are traded in their own right, and a fair price must be found for a derivative at each time of its existence. This problem is known as the *pricing problem*. One of the primary motivations for creation of derivatives was to hedge one's position from fluctuations in the market. A *hedge* is an investment strategy intended to offset potential losses or gains that may be incurred by a companion investment. Finding a hedging strategy is called the *hedging problem*. These are the two problems that must be looked at when defining a market model. In this thesis, our main focus shall be the pricing problem of a particular class of derivatives, called *exotic options*.

1.1.3 Classification of derivatives

Derivatives may be classified on the basis of various factors. One important factor is whether the risk is shared, or taken up by only one party. Another factor is the nature of the function (of the underlying) that the derivative depends on. This function may either be dependent only on the final value of the underlying (*path-independent*), or on the path that it took to reach this final value (*path-dependent*). The function may be discrete (*digital* or *binary*), or continous. In this section, we briefly look at some of the more important types of derivatives. [1]

Futures and forwards

Definition 1.2 (Futures and forwards). Futures and forwards are contracts between two parties, the seller and the buyer, to exchange a certain asset at a predetermined future time at a agreed upon price. Futures are *exchange-traded derivatives* (ETDs), whereas forwards are traded *over-the-counter* (OTC).

[1] A more interested reader should consult the following extensive Wikipedia articles.

- https://en.wikipedia.org/wiki/Option_(finance)#Types
- https://en.wikipedia.org/wiki/Option_style

Such derivatives obligate the contractual parties to the terms over the life of the contract. Futures are in some sense 'safer' compared to forwards, since the involved parties must go through standard protocols of the exchange. The contract contains the following details.

T: The maturity, or the duration of the contract

F_0: The delivery price, or the price prefixed (at the initial time) at which trades must take place at maturity

r: The rate of interest

underlying: The asset(s) of trade at maturity

S_0: The initial value of the underlying asset(s)

There are, of course, other possibilities, for instance a variable interest rate, dividends yielded by the underlying, but these may be viewed as generalisations of this simple case.

Let us assume that the compounding is continuous. We may show that under the condition of a *viable market*[2], the fair delivery price of a future with underlying prices $(S_t)_{t\in[0,T]}$ at any time $t \in [0, T]$ is given by the following equation.

$$F_t = S_t e^{r(T-t)} \tag{1.3}$$

Options

DEFINITION 1.3 (option). An *option* is a derivative which provides the buyer *the right, but not the obligation* to enter the contract under the specified terms.

Thus, the owner of the option may choose whether to exercise his right or not. Thus, on the one hand, the owner of the option bears no risk, since all the choice is his. On the other hand, the seller of the option is *obligated* to honour the terms of the contract – whether it benefits him or not – essentially making him bear all the risks. This asymmetry is primarily what sets options apart from the futures and forwards discussed earlier. The contract contains the following details.

T: The maturity, or the duration of the contract

K: The strike price, or the prefixed price at which trades may take place at maturity

r: The rate of interest

underlying: The asset(s) which may be traded at maturity

S_0: The initial value of the underlying asset at the initial time

right: The exact right that the owner of the options has (see below)

exercise time: European or American (see below)

Options may be categorised by the right of the owner and the exercise time.

[2] see Section 1.2 for definitions of the term

According to the right of the owner Options may be of two main types.

call: The owner has the right to buy. In this case, the price of the option at maturity is given by $c_T = (S_T - K)_+$, where $(x)_+ := \max\{0, x\}$.

put: The owner has the right to sell. In this case, the price of the option at maturity is given by $p_T = (K - S_T)_+$.

Of course, other complicated ownership rights may be constructed, but we shall not go into the details of those.

According to the time at which the option may be exercised Again, option may be of two main types.

European: The owner may exercise the option only at maturity

American: The owner may exercise the option at any time up to the maturity

Again, more complicated options exist, which allow exercising rights only at certain time points, but we exclude them from out discussion.

Since American options allow for more flexibility for the owner, and thus more risk for the seller, they are more expensive as compared to their European counterparts. Let c_t, p_t denote the prices of an European call and put, and C_t, P_t denote the prices of an American call and put, respectively. Then, we must have $C_t \geqslant c_t$ and $P_t \geqslant p_t$.

Call-put parity Call and put prices are connected to each other. We need the following proposition to explore the relationship.

Proposition 1.1 (Equality of portfolios). *In a* viable *and* frictionless market[3], *if the values of two portfolios coincide at a time* T, *they have to coincide at* 0 *(or any other intermediate time* t*).*

Proof. Let us denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the two portfolios and by $v(\mathcal{P})$ the value of a portfolio $\mathcal{P}$ at time t. By assumption $v_T(\mathcal{P}_1) = v_T(\mathcal{P}_2)$, so we assume by contradiction that $v_T(\mathcal{P}_1) > v_T(\mathcal{P}_2)$.

Under this hypothesis it is possible to construct the following arbitrage strategy. At time 0, one can borrow the portfolio $\mathcal{P}_1$ and sell it right away to buy portfolio $\mathcal{P}_2$. One can pocket the difference $v_T(\mathcal{P}_1) - v_T(\mathcal{P}_2) > 0$. At $t = T$ the values of the two portfolios coincide, so selling $\mathcal{P}_2$ one gets the exact money to buy $\mathcal{P}_1$ to be returned to the original lender. An profit is achieved, without investing any money, implying an arbitrage and violating the viable market hypothesis. Similarly, we can show that $v_0(\mathcal{P}_1) < v_0(\mathcal{P}_2)$ would also enable an arbitrage opportunity. □

We now look at the relationship. $S_T - K = (S_T - K)_+ + (S_T - K)_- = (S_T - K)_+ - (K - S_T)_+ = c_T - p_T$. For any general time t, using Proposition 1.1, it holds that $c_t - p_t = S_t - Ke^{-r(T-t)}$. This is known as the *call-put parity*.

[3] see Section 1.2 for definitions of the terms

Exotic options European options are path-independent and the simplest type of options available. Hence, they are popularly known as *vanilla options*. The American options are path-dependent. Typically, other options which are more complex in nature are collectively called *exotic options*. These are usually path-dependent, and may be either European, American or have more complex exercise times. A few such options are described in brief.

Asian: The payoff depends on the average of the underlying's prices

lookback: The payoff depends on one of the extrema of the underlying's prices

cliquet or ratchet: A series of globally or locally, capped or floored, at-the-money options, but where the total premium is determined in advance.

barrier: The price of the underlying reaching the pre-set barrier level either springs the option into existence (*knock-in*) or extinguishes an already existing option (*knock-out*).

1.2 Financial Markets

The idea of financial markets is intricately linked to that of financial transactions. Analogous to the ordinary markets, a financial market is a human construct to allow transaction between investors. The assets in the financial market are typically financial instruments such as bonds, stocks and derivatives discussed in the previous section. In this section we will primarily concern ourselves with the nature of financial markets and the assumptions we make while modelling them. Some of the jargon used in the previous section will become clear after this section.

Pricing of financial assets is one of the most pressing aims of the subject of Financial Mathematics. In order to do so, we need to understand and characterise the fundamental mechanisms of the market that shape the pricing of assets. In doing so, we must note which dynamics of the market are most fundamental and must be incorporated in every model, and which are more debatable may be excluded from simpler models.

Viable market The term viability here refers to the fairness of a market. To interpret viability, we need to familiarise ourselves with the following definitions.

In what follows, we assume the following.

- There is a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, endowed with the filtration $(\mathcal{F}_t)_t$.
- $\forall t \in [0,T], T \in [0,\infty)$, there is one riskless asset worth $S_t^0 = e^{rt}$ (Take $S_0^0 = 1$).
- $\forall t \in [0,T], T \in [0,\infty)$, there are d risky assets each worth S_t^i, where $i \in \{1,2,\dots,d\}$ is the index of the risky asset. These may be represented as a d-dimensional (vector) stochastic process $(S_t^1, S_t^2, \dots, S_t^d)$.

DEFINITION 1.4 (investment strategy). A $d+1$ dimensional (vector) stochastic process $\Phi = (\boldsymbol{\phi}_t)_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^d)_t$ is called an *investment strategy* or *trading strategy* if ϕ_t^i is $\mathcal{F}_t$-measurable $\forall i \in [d]$.

This means that there is a procedure to allocate resources within the portfolio at all times. We shall write strategy to mean investment strategy in the rest of the document.

The following definition gives the value of a strategy at a point in time.

Definition 1.5 (value of a strategy). The value of a strategy Φ at a time t is given by $V_t(\Phi) = \boldsymbol{\phi_t} \cdot S_t$.

Definition 1.6 (self-financing strategy). A portfolio is self-financing if its changes in value are only due to changes in prices of the assets. This can be represented as follows.

$$dV_t = \boldsymbol{\phi_t} \cdot dS_t \qquad \forall t \in [0, T] \tag{1.4a}$$

$$\implies \qquad d\tilde{V}_t = \boldsymbol{\phi_t} \cdot d\tilde{S}_t \qquad \forall t \in [0, T] \tag{1.4b}$$

This implies that we do not put in any fresh money in the strategy at any point of time, apart from what is generated due to the change is values of the underlying assets.

Definition 1.7 (admissible strategy). A self-financing strategy Φ is said to be admissible if $V_t(\Phi) \geqslant 0 \ \forall t \in [0, T]$.

This implies that we do not run out of money at any point of time.

Definition 1.8 (arbitrage strategy). An admissible strategy Φ is said to be an arbitrage strategy if $V_0(\Phi) = 0$ and $P(V_T(\Phi) > 0) > 0$.

A arbitrage strategy basically means that we generate value at time T without any initial investment. For the sake of fairness, we do not want a market in which there exist arbitrage opportunities. The next definition addresses this issue.

Definition 1.9 (viable market, no free lunch). A market is called *viable*, or there is *no free lunch* derivative is replicable, so its price at all times may be written as the value of a portfolio employing basic assets only, that is to say it is a linear combination of the prices of the basic assets. On the other hand, in a viable market the discounted value of a portfolio is a martingale under the risk neutral probability, so if a contingent claim is attainable, then its price, by the martingale property, belongs to the linear span of the basic assets' prices under this probability prices. Vice versa, if the market is incomplete, then there must be sources of randomness that cannot be totally represented as linear combinations of the prices of the basic tradeable assets, which means that the basic titles are not sufficient to construct all the necessary hedging strategies.

We will see in Theorem 2.2 of Chapter 2 how the financial concept of market completeness translates mathematically to the existence of a unique equivalent martingale measure.

Frictionless market For any transaction (sale or purchase) in the market, one usually pays some *commission*. The commission is a very small fraction of the current value of the traded assets, and it seems reasonable to assume that it is not a factor that affects the dynamics of the prices in a direct fashion. Furthermore, the computational difficulty of including such transactional costs in the market is quite high. Thus, we choose to ignore such costs in the simple market models that we shall deal with, and call the market as *frictionless*.

Definition 1.11 (frictionless market). A market is called frictionless if there are no transaction costs.

Infinitely divisible assets In a market, usually only discrete units of assets may be bought or sold. This would pose an additional constraint in the modelling of the market. But it is quite evident that this constraint does in no way affect prices of individual assets. Furthermore, markets are usually so varied that one might think to combine stocks and bonds to a value that is roughly equivalent to a fraction of a different asset. Thus, we ignore this constraint in our market models, and say that we may have *infinitely divisible assets* in our market.

Small investor hypothesis An investor who has virtually unlimited funds might decide to buy massively quantities of an asset to make its price rise in order to sell it later at the new higher price. We shall ignore such cases, and assume that all agents are trifling with respect to the market dimension, meaning that they cannot influence prices uniquely by means of their investing strategies, hence prices are determined only by the combined actions of all agents. This assumption is called the *small investor hypothesis* and it is totally realistic in bigger stock markets like for those in the United States, even though it is less so for much smaller markets.

Borrowing Lastly, we assume that an investor may borrow assets, whether they are financial instruments such as money, bonds and stocks, or otherwise.

Chapter 2

Market models

Now that we are familiar the basics of financial assets and markets, we may delve into market models under which we may price options. Since continuous models are mathematically more complex than their discrete time counterparts (also known as 'lattice models'), we shall discuss the latter first. We shall then show that under certain convergence conditions, the discrete model converges to the continuous one.

2.1 Discrete models – The binomial model

We start our discussion with one of the simplest model used for pricing of assets, the binomial model. This model was first introduced by Cox, Ross and Rubinstein [CRR79] in the paper titled "Option pricing: A simplified approach" in 1979. Even though it is quite simplistic, it does contain all the necessary ingredients to construct a viable market model, and to solve the problems of pricing and hedging of derivatives.

We assume that the following are true.

- The market is frictionless
- All assets are infinitely divisible
- The small investor hypothesis holds
- The annual interest rate is constant and it is applied both for borrowing or lending, hence there are no spreads

In this model, we essentially have

- two time points, $t = 0$ (present) and $t = 1$ (future)
- two traded assets
  - the riskless asset, usually a bond, which is compounded at a constant rate of interest $r > 0$
  - the risky asset, usually a stock, which may either go up with a factor u, or down with a factor d

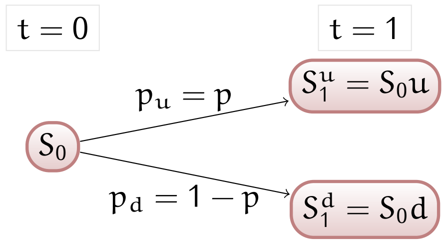

Figure 1. Binomial tree for the underlying

The binomial model is so called because there are two times, two assets and two possible movements of the risky asset.

We denote as S_0 the value of the risky asset at time 0, and by S_1 its value at time 1. Firstly, in order to have no arbitrage opportunities, we must have $d < R < u$, where $R := 1 + r$. Secondly, for fairness, there must exist a probability distribution $p, (1-p)$ – signifying the probabilities of the up and down movements – such that the expected value of the asset remains the same as that of the riskless asset given the same time. (See Figure 1.)

Thus, we have

$$S_1 = \begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } (1-p) \end{cases}$$

We may also write $S_1 = T_1 S_0$, where T_1 is a random variable taking values in $\{u, d\}$ with associated probability distribution $(p, 1-p)$.

$$T_1 = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } (1-p) \end{cases}$$

We want $p \in [0, 1]$ such that $\mathbb{E}(S_1) = S_0 R$. This would make the system fair because the expected gain from either asset should be the same.

$$\begin{aligned} & & \mathbb{E}(S_1) &= S_0 R && (2.1) \\ & \implies & puS_0 + (1-p)dS_0 &= S_0 R \\ & \implies & p &= \frac{R-d}{u-d} && (2.2) \end{aligned}$$

The probability thus obtained is called the ***risk neutral probability***, because under this probability, it is equivalent for the investor whether he invests in a risky or a riskless asset. Note that this probability is completely objective as it is determined completely by the parameters u, d and r. Since the probability distribution is uniquely determinable from the market parameters, the market is viable and complete (See Theorems 2.1 and 2.2 in the next section).

Now we impose the condition $p \in [0, 1]$ to obtain

$$d < R < u \qquad (2.3)$$

Volatility A natural way to express the risk associated with an asset is by its variance. In the case of binomial model, at each time, the variance is associated with the gap between u and d, the up and down factors. It is reasonable to assume that when comparing to assets, an asset with the propensity of a higher up movement will also have a similar disposition towards a large down movement. If it were not so, the investors would invest in the asset with higher up movement potential, driving up the prices, and naturally recalibrating the current value of the asset, so that the assumption is true. Thus we may reduce the requirement of two variables u and d by expressing them as a function of only the variance of the return T_i. This may be achieved by making the model symmetrical by setting $u = d^{-1}$. In this case, the log return is symmetrical w.r.t. the origin since $\log(u) = -\log(d)$.

Moreover it is reasonable to think that this variance stays constant at each time unit, but directly proportional to time.

$$\mathbb{V}\mathrm{ar}(\log(T_i)) = \sigma^2 \Delta T$$

Here $\sigma > 0$ is called the *volatility* of the asset, and is used as a constant of proportionality.

With this choice it is natural to take

$$u = e^{\sigma\sqrt{\Delta T}} \tag{2.4a}$$

$$d = e^{-\sigma\sqrt{\Delta T}} \tag{2.4b}$$

Now we require only one parameter, σ, to get u and d, and subsequently to generate the whole tree. The value of this parameter must be estimated from the market.

Pricing a call Let us use the above model to price a call. Recall that the pay-off of a call is given by $h(x) = (x - K)_+$, where K is the strike price, a fixed value specified in the contract. Thus, we know the values of the call at maturity. Note that for financial viability, we must have $K \in (S_1^d, S_1^u)$, implying $c_1^u = (S_1^u - K)_+ = (S_1^u - K)$ and $c_1^d = (S_1^d - K)_+ = 0$ (See Figure 2). To ensure fairness, we may again write the following

$$\begin{aligned}
c_0 &= \mathbb{E}(\frac{c_1}{R}) \\
&= \frac{1}{R}(pc_1^u + (1-p)c_1^d) \\
&= p\frac{c_1^u}{R} \\
&= \frac{R-d}{u-d}\frac{uS_0 - K}{R}
\end{aligned}$$

Thus we have been able to price the call uniquely at all times. This is an implication of the completeness of the market, whose randomness is totally characterized by the unique probability measure p.

For the sake of completeness, we comment here that a call is also completely hedgeable in the binomial model.

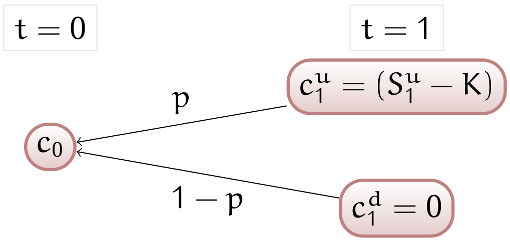

FIGURE 2. Binomial tree for the underlying

2.2 Discrete models – The Cox Ross Rubinstein model

In this section we extend the binomial model introduced in Section 2.1 to a sequence of integer times $[n] \coloneqq \{0, 1, \dots, n\}$, $n \in \mathbb{N}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_i)_i, P)$ be a finite probability space ($|\Omega| < \infty$), such that $P(\omega) > 0 \; \forall \omega \in \Omega$, endowed with a filtration $(\mathcal{F}_i)_{i \in [n]}$, such that $\mathcal{F}_0$ is trivial ($\mathcal{F}_0 = \{\emptyset, \Omega\}$).

We assume that the following are true.

- The market is frictionless
- All assets are infinitely divisible
- The small investor hypothesis holds
- The annual interest rate is constant and it is applied both for borrowing or lending, hence there are no spreads
- The market is viable
- The market is complete

From the binomial model, we have at any time $i \in [n-1]$

$$S_{i+1} = \begin{cases} uS_i & \text{with probability } p \\ dS_i & \text{with probability } (1-p) \end{cases}.$$

And, $S_{i+1} = T_{i+1} S_n$, where $(T_{i+1})_{i \in [n-1]}$ is a sequence of independent and identically distributed random variables taking values in $\{u, d\}$ with associated probability distribution $(p, 1-p)$.

$$T_{i+1} = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } (1-p) \end{cases}$$

Using the above, we may write

$$S_n = s_0 \prod_{j=1}^{i} T_j \qquad \forall i \in [n]. \tag{2.5}$$

Using exactly the same computations as in the binomial case, we again find the risk-neutral probability as $p = \frac{R-d}{u-d}$, such that $d < R < u$. Again, since the probability distribution is uniquely determinable from the market parameters, the market is viable and complete (See Theorems 2.1 and 2.2 in the next section).

2.2.1 Fundamental Theorems of Asset Pricing

We now need to invoke two cornerstone theorems, which will allow us to price the option at any time step i.

THEOREM 2.1 (First Fundamental Theorem of Asset Pricing). *The market model is viable if and only if there exists a probability measure P^* equivalent to the historic probability measure P under which the discounted prices of the basic risky assets are martingales.*

Mathematically, Viable market $\iff \exists\ P^* \sim P$ such that $\mathbb{E}^*(\tilde{S}_{i+1}|\mathcal{F}_n) = \tilde{S}_i$, where $\mathbb{E}^*$ is the expectation computed under P^*..

PROOF. See [LL96, page 6, Theorem 1.2.7] □

Theorem 2.1 guarantees the existence of an equivalent martingale measure for viable markets. This implies that under this probability measure, we can calculate the fair price of an option, although they may not be unique. To ensure uniqueness, we need the subsequent theorem.

THEOREM 2.2 (Second Fundamental Theorem of Asset Pricing). *The market model is complete if and only if there exists a **unique** probability measure P^* equivalent to the historic probability measure P under which the discounted prices of the basic risky assets are martingales.*

Mathematically, Complete market $\iff \exists!\ P^* \sim P$ such that $\mathbb{E}^*(\tilde{S}_{i+1}|\mathcal{F}_n) = \tilde{S}_i$, where $\mathbb{E}^*$ is the expectation computed under P^*.

PROOF. See [LL96, page 9, Theorem 1.3.4] □

Theorem 2.2 guarantees the uniqueness of an equivalent martingale measure for complete markets. This implies that we can uniquely and objectively compute the fair price of an option.

2.2.2 European options

Augmented by these theorems, we may now seek to price path-independent options in this model. Let E^* be the expectation computed under the risk-neutral martingale measure P^*. If the pay-off of an option at maturity is given by the function $h(x)$, then the price of the option v_n at any time step i is given by the following formula.

$$v_i = \mathbb{E}^*(R^{-(n-i)}h(S_n)|\mathcal{F}_i) \tag{2.6}$$

In particular, the price a European call at any time i is as follows.

$$c_i = \mathbb{E}^*(R^{-(n-i)}(S_n - K)_+|\mathcal{F}_i) \tag{2.7}$$

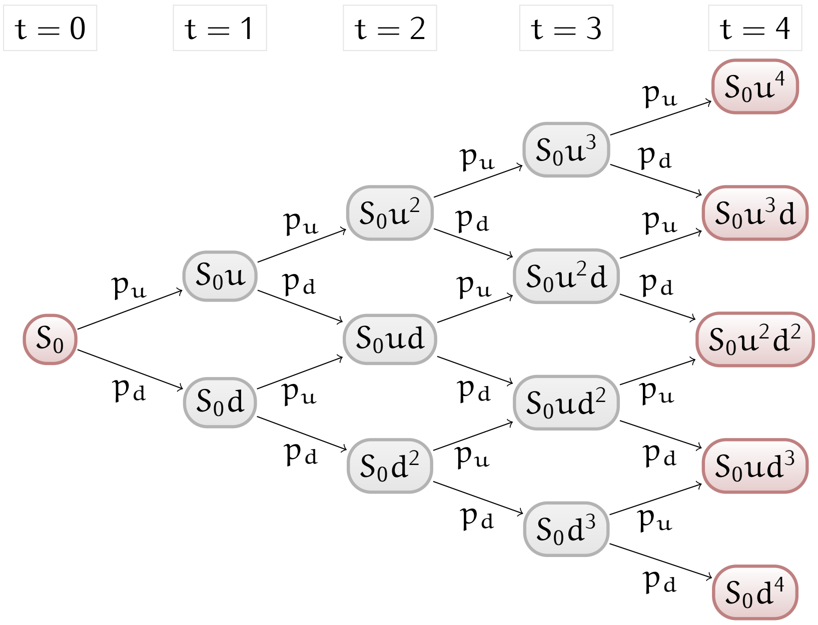

FIGURE 3. A 4 step lattice ($n = 4$)

Computational details If we are given the contingency claim, and the option is of the European type, it becomes very simple to evaluate it value in this model. While going through the algorithm, it is suggested to keep in mind Figure 3.

Algorithm 1: Evaluation of European options using the Cox-Ross-Rubinstein model

---

Input: Claim $h(x)$, Number of time steps n, Volatility σ

Output: The price of the option at the initial time

begin
- Calculate p using Equation 2.2.
- Calculate u and d using Equations 2.4.
- Construct the binomial tree.
- Use the claim to calculate the values that the option might take at maturity.
- **for** $i \in \{n-1, \ldots, 0\}$ **do**
  - **foreach** *node* **do**
    - Calculate the expected value of the option using the weighted mean of the prices at the $(i+1)^{\text{th}}$ time which are connected to this node. The weighting is done by the probabilities p and $1-p$.
- **return** *Value at time 0*

2.2.3 American options

Since an American option can be exercised at any time $i \in [n]$, at any time, the bearer of the option has two options – to exercise or to hold. The exercise value at any time is given by $Z_i = h(S_i)$, which in the case of a call is $Z_i = (S_i - K)_+$. Let the value of the options at a time i be denoted by U_i. We start by backward induction. At maturity, we must have $U_n = Z_n$. The value of holding the option at the time just before the maturity ($i = n-1$)

is given by the discounted risk-neutral price $\mathbb{E}^*(R^{-1}h(S_n)|\mathcal{F}_{n-1}) = \mathbb{E}^*(R^{-1}U_n|\mathcal{F}_{n-1})$. The value of the option is thus given by the maximum of these two values, or

$$U_{n-1} = \max\left\{Z_{n-1}, \mathbb{E}^*(R^{-1}U_n|\mathcal{F}_{n-1})\right\}. \tag{2.8}$$

By induction, we write the price of an American option at any time $i \in [n]$.

$$\begin{cases} U_i = \max\left\{Z_i, \mathbb{E}^*(R^{-1}U_{i+1}|\mathcal{F}_i)\right\} & \forall i \in [n-1] \\ U_n = Z_n := h(S_n). \end{cases} \tag{2.9}$$

Note that Equation 2.9 may be equivalently represented using discounted values.

$$\begin{cases} \tilde{U}_i = \max\left\{\tilde{Z}_i, \mathbb{E}^*(\tilde{U}_{i+1}|\mathcal{F}_i)\right\} & \forall i \in [n-1] \\ \tilde{U}_n = \tilde{Z}_n. \end{cases} \tag{2.10}$$

This is sufficient to understand Algorithm 2, which tells the mechanism of pricing American options under the Cox-Ross-Rubinstein model. But we will go ahead with some theoretical results regarding the American options, because we will use the results in Chapter ??.

Super-martingales and discounted prices The discounted prices of American options are super-martingales under P^*, in contrast to European options. This is evinced from Equation 2.10, since $\tilde{U}_i \geqslant \tilde{Z}_i$ $\forall i$. In fact, this is the smallest super-martingale that dominates $(Z_n)_n$ (see [LL96, Proposition 1.3.6]). Historically, $\tilde{U}_i$ is called the *Snell envelope* of the process $(\tilde{Z}_i)_i$.

Stopping time The owner of an American options may exercise his/her right at any point of time. For the decision to be fair, the owner should be able to make the choice with the information available only at till that time i. To model this, we need the definition of a stopping time.

DEFINITION 2.1 (stopping time). A random variable τ, taking values in $[n]$ is called a stopping time, if for any $i \in [n]$,

$$\{\tau = i\} \in \mathcal{F}_i$$

An equivalent characterisation of a stopping time is $\{\tau \leqslant i\} \in \mathcal{F}_i$.

Process stopped at a stopping time A process is $(X_i)_i$ is said to be adapted to a filtration $(\mathcal{F}_i)_i$ if X_i is $\mathcal{F}_i$-measurable $\forall i$.

Let $(X_i)_i$ be a process adapted to the filtration $(\mathcal{F}_i)_i$ and let τ be a stopping time. Then, the process stopped at τ is defined as

$$X_i^\tau(\omega) := X_{\tau(\omega)\wedge i}(\omega).$$

On the set $\{\tau = m\}$, this is equivalent to

$$X_i^\tau(\omega) = \begin{cases} X_i & \forall i < m \\ X_m & \forall i \geqslant m \end{cases}.$$

[TODO: Introduce completely stopping times and sup over stopping times to motivate the discussion on the theory of Asian options.]

Computational details If we are given the contingency claim, and the option is of the American type, it becomes very simple to evaluate it value in this model. The only difference with the European options is that at each node, we select the higher value among the the price for immediate exercise and holding the option till the next time. While going through the algorithm, it is suggested to keep in mind Figure 3.

Algorithm 2: Evaluation of American options using the Cox-Ross-Rubinstein model

Input: Claim $h(x)$, Number of time steps n, Volatility σ
Output: The price of the option at the initial time

```
begin
    Calculate p using Equation 2.2.
    Calculate u and d using Equations 2.4.
    Construct the binomial tree.
    Use the claim to calculate the values that the option might take at maturity.
    for i ∈ {n − 1, ..., 0} do
        foreach node do
            Calculate the expected value of the option using the weighted mean of the
            prices at the (i + 1)th time which are connected to this node. The weighting is
            done by the probabilities p and 1 − p.
            Calculate the value of exercising the option.
            Select the higher value among the two numbers obtained above.
    return Value at time 0
```

2.3 Continuous models – the Black-Scholes model

As in the previous chapter, we consider two assets – one riskless and one risky.

The value of the riskless asset is proportional to the instantaneous rate of interest, the current value of the asset, and the time. Therefore, the dynamics of the riskless asset is given as follows.

$$dS_t^0 = rS_t^0\,dt \tag{2.11}$$

$$S_0^0 = 1 \tag{2.12}$$

Solving this initial value problem, we get

$$S_t^0 = e^{rt}. \tag{2.13}$$

The risky asset is dependent on both deterministic and stochastic factors. The deterministic part is proportional to the current value of the asset and the time. The non-deterministic part is dependent on its current value and a stochastic process, whose mean should be zero, and variance should be proportional to the time. We choose these criteria because we do not expect the process to have a bias on going up or down (mean zero) and we expect that the process has a higher probability of going away from the current value with more time. It so happens that the best candidate for such a process is a standard Brownian motion. [TODO: Brownian motion in Appendix]

Let B_t be a Brownian process. Let μ (drift) and σ (volatility) be proportionality constants.

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \tag{2.14a}$$

$$S_0 = s_0 \tag{2.14b}$$

This is a stochastic differential equation with initial value. The solution to this problem is the geometric Brownian motion

$$S_t = s_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}. \tag{2.15}$$

Before we proceed further in this section, we need to recall the Girsanov theorem.

THEOREM 2.3 (Girsanov). *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ *be a filtered probability space,* $(B_t)_t$ *be a* $(\mathcal{F}_t)_t$*-Brownian motion on the bounded interval* $[0, T]$*, and* $\theta \in \mathbb{R}$*. Let the probability* P^* *be defined by the Radon-Nikodym derivative*

$$\frac{dP^*}{dP} = e^{\theta B_T - \frac{\theta^2}{2} T}. \tag{2.16}$$

Then, the following holds.

- $P^* \sim P$
- *The process* $Z_t := \mathbb{E}\left(\frac{dP^*}{dP} \mid \mathcal{F}_t\right)$ *is a* P*-martingale*
- *The process* $W_t := B_t + \theta t$ *is a* P^**-Brownian motion*

PROOF. Refer to [LL96, Theorem 4.2.2 and Chapter 4: Exercise 19] □

First, we need to show that there exists a probability measure P^* equivalent to P under which the discounted share price $\tilde{S}_t = e^{-rt} S_t$ is a martingale [TODO]. From Equation 2.14 and Ito's Lemma [TODO], we get

$$\begin{aligned}
d\tilde{S}_t &= -re^{-rt} S_t \, dt + e^{-rt} \, dS_t \\
&= -r\tilde{S}_t \, dt + e^{-rt}(\mu S_t \, dt + \sigma S_t \, dB_t) \\
&= (-r\tilde{S}_t + \mu \tilde{S}_t) \, dt + \sigma \tilde{S}_t \, dB_t \\
&= \tilde{S}_t((\mu - r) \, dt + \sigma \, dB_t)
\end{aligned}$$

Let $W_t = B_t + \frac{\mu - r}{\sigma} t$. The we derive the following.

$$
\begin{aligned}
&\text{(2.17a)} & & W_t = B_t + \frac{\mu - r}{\sigma} t \\
& & \implies & dW_t = dB_t + \frac{\mu - r}{\sigma} dt \\
& & \implies & \sigma \tilde{S}_t \, dW_t = \tilde{S}_t((\mu - r)\, dt + \sigma \, dB_t) \\
&\text{(2.17b)} & \implies & \sigma \tilde{S}_t \, dW_t = d\tilde{S}_t
\end{aligned}
$$

From Theorem 2.3, with $\theta = \frac{\mu - r}{\sigma}$, we know that there exists probability $P^* \sim P$ given by $\frac{dP^*}{dP} = e^{\theta B_t - \frac{\theta^2}{2} t}$, under which $(W_t)_t$ is a standard Brownian motion. The definition of stochastic integral is invariant by change of equivalent probability. Therefore, from Equation 2.17b, under the probability P^*, the discounted risky asset prices become

$$
\begin{aligned}
&\text{(2.18a)} & & \tilde{S}_t = s_0 e^{\sigma W_t - \frac{\sigma^2}{2} t} \\
&\text{(2.18b)} & \implies & S_t = s_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}.
\end{aligned}
$$

To derive a SDE under the risk-neutral probability, we use Ito's lemma.

$$
\text{(2.19)} \qquad \frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t
$$

We verify that $\tilde{S}_t$ is a P^*-martingale.

$$
\begin{aligned}
\mathbb{E}^* \left(\tilde{S}_{t+s} \mid \mathcal{F}_t \right) &= \mathbb{E}^* \left(s_0 e^{\sigma W_{t+s} - \frac{\sigma^2}{2}(t+s)} \mid \mathcal{F}_t \right) \\
&= \mathbb{E}^* \left(e^{\sigma \overbrace{(W_{t+s} - W_t)}^{\perp \mathcal{F}_t}} \cdot \overbrace{e^{\sigma W_t}}^{\in \mu \mathcal{F}_t} \cdot \overbrace{s_0 e^{-\frac{\sigma^2}{2}(t+s)}}^{\in \mu \mathcal{F}_0} \,\middle|\, \mathcal{F}_t \right) \\
&= s_0 e^{\sigma W_t - \frac{\sigma^2}{2} t} e^{-\frac{\sigma^2}{2} s} \, \mathbb{E}^* \left(e^{\sigma (W_{t+s} - W_t)} \right) \\
&= \tilde{S}_t e^{-\frac{\sigma^2}{2} s} e^{\frac{\sigma^2}{2} s} \\
&= \tilde{S}_t
\end{aligned}
$$

The last expectation was calculated using the fact that $(W_{t+s} - W_t) \sim \mathcal{N}(0, s)$, and that the characteristic function of $\mathcal{N}(\mu, \sigma)$ is given by $\varphi(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$.

Remark 2.1 (Continuous dividend yield). If the risky asset pays continuous dividends, the only modification that needs to be done is to replace r in all the equations by the effective interest rate $r - q$.

Pricing the European call From Equation 2.18a, we have:

$$\begin{aligned}
\frac{S_T}{S_t} &= \frac{s_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}}{s_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}} \\
&= e^{(r-\frac{\sigma^2}{2})\overbrace{(T-t)}^{:=\Delta t}+\sigma\overbrace{(W_T-W_t)}^{:=\Delta W_t}} \\
\implies \qquad S_T &= S_t e^{(r-\frac{\sigma^2}{2})\Delta t+\sigma \Delta W_t}
\end{aligned}$$

The value of a contingency claim then becomes

$$\begin{aligned}
V_t &= \mathbb{E}^*\left(e^{-r\Delta t}\ h(S_T)\middle|\mathcal{F}_t\right) \\
&= \mathbb{E}^*\left(e^{-r\Delta t}\ h(S_t e^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t})\middle|\mathcal{F}_t\right).
\end{aligned}$$

Since S_t is $\mathcal{F}_t$-measurable, we can treat it as a constant x. The value of the call can then be represented by $V_t = F(t, S_t)$, where $F(t,x) = \mathbb{E}^*\left(e^{-r\Delta t}\ h\left(xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}\right)\middle|\mathcal{F}_t\right)$.

For the call, we put $h(x) = (x-K)_+$, and denote $F(t,x)$ by $c(t,x)$.

$$\begin{aligned}
c(t,x) &= \mathbb{E}^*\left(e^{-r\Delta t}\ h\left(xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}\right)\middle|\mathcal{F}_t\right) \\
&= \mathbb{E}^*\left(e^{-r\Delta t}\ \left(xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}-K\right)_+\middle|\mathcal{F}_t\right) \\
&= \underbrace{x\,\mathbb{E}^*\left(e^{\sigma W_t-\frac{\sigma^2}{2}t}\mathbb{1}_{xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}>K}\middle|\mathcal{F}_t\right)}_{A} + Ke^{-r\Delta t}\underbrace{\mathbb{E}^*\left(\mathbb{1}_{xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}>K}\middle|\mathcal{F}_t\right)}_{B}
\end{aligned}$$

We deal with B first as it is more manageable.

$$\begin{aligned}
B &= \mathbb{E}^*\left(\mathbb{1}_{xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}>K}\middle|\mathcal{F}_t\right) \\
&= P^*\left(xe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma\Delta W_t}>K\middle|\mathcal{F}_t\right) \\
&= P^*\left(\frac{\Delta W_t}{\sqrt{\Delta t}} > \frac{\log\frac{K}{x}-(r-\frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\middle|\mathcal{F}_t\right) \\
&= P^*\left(\frac{\Delta W_t}{\sqrt{\Delta t}} < -\frac{\log\frac{K}{x}-(r-\frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\middle|\mathcal{F}_t\right) \\
&= P^*\left(\frac{\Delta W_t}{\sqrt{\Delta t}} < \frac{\log\frac{x}{K}+(r-\frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\middle|\mathcal{F}_t\right)
\end{aligned}$$

Now, $\Delta W_t \sim N(0, \Delta t)$, implying $\frac{\Delta W_t}{\sqrt{\Delta t}} \sim N(0,1)$. Also note that $\Delta W_t \perp \mathcal{F}_t$. Writing

$$d_-(\Delta t, x; K, \sigma, r) = \frac{\log \frac{x}{K} + (r - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}},$$

and using independence, we get:

$$\begin{aligned} & B = P^* \left(N(0,1) < d_-\right) \\ \implies \quad & B = \mathcal{N}(d_-(\Delta t, x; K, \sigma, r)), \end{aligned}$$

where $\mathcal{N}$ is the (cumulative) distribution function of the standard normal distribution, given by

$$\mathcal{N}(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz. \tag{2.20}$$

Similarly, we now evaluate A.

[TODO: for the final draft.]

Writing

$$d_+(\Delta t, x; K, \sigma, r) = \frac{\log \frac{x}{K} + (r + \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}},$$

and using independence, we get:

$$\begin{aligned} & A = P^* \left(N(0,1) < d_+\right) \\ \implies \quad & A = \mathcal{N}(d_+(\Delta t, x; K, \sigma, r)). \end{aligned}$$

Henceforth, we shall write $d_\pm$ to denote $d_\pm(\Delta t, x; K, \sigma, r)$.

Assembling all the pieces together we obtain the famous Black-Scholes formula for a European call.

$$c_t = c(t, S_t) = S_t\mathcal{N}(d_+) - Ke^{-r\Delta t}\mathcal{N}(d_-) \tag{2.21}$$

Pricing the put Employing the call-put parity to get

$$p_t = p(t, S_t) = Ke^{-r\Delta t}\mathcal{N}(-d_-) - S_t\mathcal{N}(-d_+). \tag{2.22}$$

Remark 2.2 (Hedging). Even though we shall not derive the hedging strategy, we comment that in the Black-Scholes model, derivatives are hedgeable, and is determined by the so called *delta*, where $\Delta := \frac{\partial c}{\partial x}(t, S_t) = \mathcal{N}(d_+)$. The hedging strategy which uses the delta is called *delta hedging*. In the real-world, there are other sources of risk which needs different hedging strategies, as highlighted in the limitations of the model.

Limitations

- Finding closed-form formulae for prices of American options and most exotic options is not possible.
- Security prices in reality do not follow a strict stationary log-normal process.
- The risk-free interest rate is not known, and is not constant over time.

- The volatility is not constant and depends on current value of the underlying (the curve is called the *volatility smile*), and on the time to maturity (the curve is called the *volatility skew*).
- The model underestimates extreme moves, yielding *tail risk*, which can be hedged with out-of-the-money options.
- Its assumption of instant, cost-less trading yields *liquidity risk*, which is difficult to hedge.
- Its assumption of a stationary process yields volatility risk, which can be hedged with *volatility hedging*.
- Its assumption of continuous time and continuous trading yields *gap risk*, which can be hedged with *Gamma hedging*.

Benefits In spite of its limitations, the Black-Scholes model remain one of the most useful models in Mathematical Finance. Below are some advantages of the model.

computability: The ease of computation is one of the biggest advantages of the model.

useful approximation: It serves as decent approximation of the real world, particularly when analysing the direction in which prices move when crossing critical points.

basis: It serves as a bedrock for more refined models, which try to address its drawbacks.

reversible: the model's original output, price, can be used as an input and one of the other variables solved for; the implied volatility calculated in this way is often used to quote option prices.

2.3.1 Discrete and continuous – different worlds?

[TODO: Proof of convergence in distribution of CRR to BS]

2.4 Other options

We have seen that it is quite easy to compute the price of European options in the Black-Scholes model. On the other hand, pricing most other options is not so straight forward. For most options, we do not have closed-form solutions in the Black-Scholes model. Examples of such options include American options and most exotic options.

On the other hand, the Cox-Ross-Rubinstein, even though computationally much more demanding, does allow pricing of all options in a straightforward manner. In fact, even though the theoretical foundations are quite different between European and American options, the similarity among the Algorithms 1 and 2 testifies to this fact.

A good balance has been struck by algorithms which develop on the Cox-Ross-Rubinstein model in order to decrease the computational complexity by allowing for approximations. The sacrifice in accuracy is justified by the significant improvement in computability. One such algorithm is the *Singular Points method* introduced by Gaudenzi, Zanette and Lepellere.

In the next two chapters, we shall deal with Asian options of the American type, and cliquet options, respectively. These options do not lend themselves to be priced in closed-form formulae in the Black-Scholes model. The Singular Points method is a viable alternative method for these options, and we focus our study on its theory and extensibility, stating also the numerical results.

Chapter 3

The Singular Points method applied to Asian options

3.1 Introduction

As we have seen in the earlier chapters, European options may be priced using the Cox-Ross-Rubinstein and the Black-Scholes models. Even though the Black-Scholes model has a very high degree of computability, it does not allow us to find closed-form pricing formulae for many path-dependent options, including American options. The way out is by using numerical methods. Numerical methods using discrete time structures were introduced in Chapter 2. One simple idea is to apply an explicit pricing scheme using a lattice method, which converges to the Black-Scholes model as the number of time steps increase to infinity. But the exponential number of paths (2^n to be exact, where n is the number of time steps) make the method very slow and memory intensive, making it computably impractical. A logical step would be to modify the basic Cox-Ross-Rubinstein model to allow for approximations. In this direction, Gaudenzi et al[GZA10] introduced a new method called the 'singular points method' for pricing certain path-dependent options in an efficient manner. The chapter is a study on how this method and its applications.

We will mainly focus on Asian options, in which the price is expressed as a function of some form of averaging on the underlying's price. Popular Asian options use the arithmetic or geometric means as the average. Again, Asian options may be exercised only at maturity (European) or at any time till the maturity (American). They may give the owner of the option the right to either sell (put) or buy (call). Theoretically, we may study either a call or a put, because the framework for the other one may be derived in the exact same way.

3.2 Existing methods

Before we go into the details of the singular points method, we shall look into the pre-existing methods, and discuss their advantages and disadvantages briefly.

Ref: Forsyth et al (2002)

- American Asian options with arithmetic mean
  - Tree based
    - CRR binomial method
    - Hull and White (1993)
    - Barraquand and Pudet (1996)
    - Chalasani et al (1999a, b)
  - PDE based
    - Vecer (2001)
    - D'Halluin et al (2005)
- American lookback options
  - Hull and White (1993)
  - Barraquand and Pudet (1996)
  - Babbs (2000), using a 'change of numeraire' approach, which cannot be applied to the fixed-strike case

TODO: Discuss the advantages and disadvantages of each method.

A number of these algorithms has been implemented in Premia 13. Premia is a software designed for option pricing, hedging and financial model calibration. It has been developed by the 'MathFi' team in INRIA. It is provided with its C/C++ source code and an extensive scientific documentation. More information about Premia can be found at the website[1].

3.3 The exact binomial algorithm

The evolution of the risky asset price $(S_t)_t$ is governed by the Black-Scholes stochastic differential equation

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma\,dW_t$$
$$S_0 = s_0$$

Here W_t is a standard Brownian motion under the risk-neutral probability P^*, r is the instantaneous interest rate, q is the continuous divident yield, σ is the volatility of the underlying risky asset. (Refer also to Equation 2.19 and Remark 2.1 in Section 2.3 of Chapter 2.)

[1] https://www.rocq.inria.fr/mathfi/Premia/

Asian options Asian options are dependent on the averaging of the underlying The price of an American Asian option of initial time 0 and maturity T is

$$P(0, S_0, A_0) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}^* \left(e^{-r\tau} \, h(S_\tau, A_\tau) \mid S_0 = s_0, A_0 = s_0 \right) \tag{3.2}$$

The quantities used in the formula are explained below.

$\mathcal{T}_{[0,T]}$: the set of all stopping times with values in $[0, T]$

h: the payoff function

A_τ: the average of the price of the underlying asset over the period $[0, \tau]$

Let K denote the strike price. The price function may be one of the following

fixed Asian call: $h = (A_T - K)_+$

fixed Asian put: $h = (K - A_T)_+$

floating Asian call: $h = (S_T - A_T)_+$

floating Asian put: $h = (A_T - S_T)_+$

Now we consider the discrete model. If the number of time steps in the binomial tree is n, then the corresponding time step is $\Delta T = \frac{T}{n}$. The lognormal diffusion process $(S_{i\Delta T})_{i \in [n]}$ is approximated by the Cox–Ross–Rubinstein binomial process (refer Equation 2.5).

$$S_i = s_0 \prod_{j=1}^{i} T_j \qquad \forall i \in [n].$$

As usual, we represent the risk-neutral probability by $p = \frac{R-d}{u-d}$, where $u = d^{-1} = e^{\sigma \Delta T}$. We denote the effective rate of interest in each period as $R := e^{r\Delta T}$.

In the Cox–Ross–Rubinstein model, the price at time 0 of the Asian option of the American and European types with payoff function h is given by $v(0, s_0, s_0)$, where the functions $v(i, x, y)$ can be computed by the following backward dynamic programming equations.

$$v(n, x, y) = h(x, y) \tag{3.3a}$$

$$v(i, x, y) = \frac{1}{R} \left(pv \left(i+1, xu, \frac{(i+1)y + xu}{i+2} \right) + (1-p)v \left(i+1, xd, \frac{(i+1)y + xd}{i+2} \right) \right) \qquad \forall i \in [n-1] \tag{3.3b}$$

In case of Asian options of the American type, we modify the equations accordingly.

$$v(n, x, y) = h(x, y) \tag{3.4a}$$

$$v(i, x, y) = \max \left\{ h(x, y), \frac{1}{R} \left(pv \left(i+1, xu, \frac{(i+1)y + xu}{i+2} \right) + (1-p)v \left(i+1, xd, \frac{(i+1)y + xd}{i+2} \right) \right) \right\} \qquad \forall i \in [n-1] \tag{3.4b}$$

The payoff is a function of the average, which is clearly path-dependent. Thus, the option is path-dependent, and the corresponding price tree is non-recombinant. This makes the classical binomial method infeasible after a small number of steps. Note that the binomial tree for the underlying is always recombinant for constant volatility.

3.4 The Singular Points method

The price of an Asian option at each instance is a continuous function of the underlying's average. Since the number of paths to a node in a binomial tree is finite, we have that at each node of the underlying's binomial tree, the option price may be represented as a piecewise-linear, continuous, convex function of the average. We shall develop the theoretical idea in this section. In the subsequent section, we shall see that the nature of the function allows us to make approximations with ***a priori*** error bounds.

Definition 3.1 (Singular points and singular values). Let $P = (P_i)_{i\in[n]} = ((x_i, y_i))_{i\in[n]}$, $n \in \mathbb{N}$ be a sequence of points such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \tag{3.5a}$$

$$m_{i+1} := \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \leqslant \frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}} = m_{i+2} \qquad \forall i \in \{1, \ldots, n-1\} \tag{3.5b}$$

Let $f : [a, b] \to [0, \infty)$ be the function obtained by linear interpolation of the points in P. From the definition of f and 3.5b, the function is continuous, piecewise-linear and convex.

Then, the elements of P are called *singular points of* f and the abscissae $\{x_i\}_{i\in[n]}$ are called *singular values of* f.

Remark 3.1. We note that the singular points characterise such a function completely. This can be seen from the following representation of the function.

$$f(x) = y_0 + \sum_{i=1}^{n} [m_i(\min\{x_i, x\} - \min\{x_{i-1}, x\})] \tag{3.6}$$

Where $m_{i+1} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ represents the slope of the function between x_i and x_{i+1}.

Remark 3.2. From the conditions 3.5, we get

$$y_0 < y_1 < \cdots < y_{n-1} < y_n$$

So it is equivalent to sort points using either abscissae or ordinates.

3.4.1 Upper estimates

The following lemmas shall provide us with the necessary framework for upper and lower estimates for approximations on the functions generated by singular points.

Lemma 3.1 (Upper estimate). *Let* $f : [a, b] \to [0, \infty)$ *be a continuous, piecewise-linear, convex function characterised by the singular points* $P = ((x_i, y_i))_{i\in[n]}$. *Then, if a point* $(x_j, y_j), j \in \{1, \ldots, n-1\}$ *is removed from the sequence, the function* $f_u : [a, b] \to [0, \infty)$

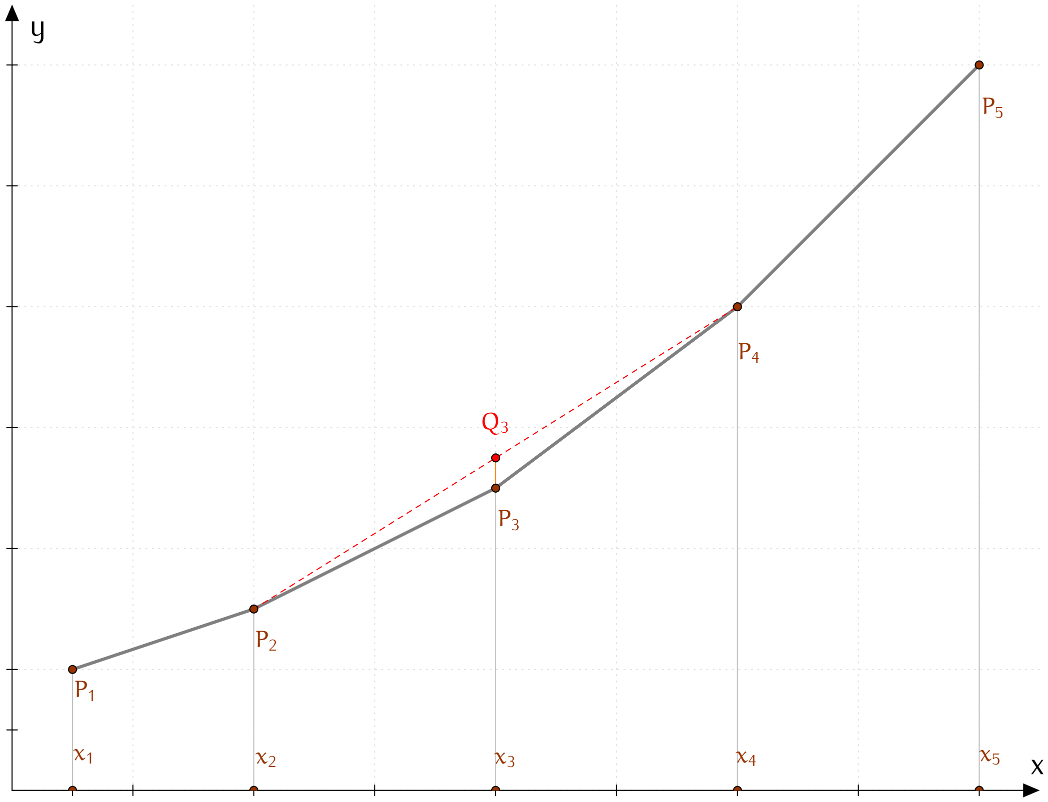

FIGURE 1. Upper estimate: Illustration of Lemma 3.1 with $j = 3$

obtained by the new sequence $(P_i)_{i\in[n]\setminus\{j\}}$ *is also continuous, piecewise-linear and convex, and*

$$f_u(x) \geqslant f(x) \qquad \forall x \in [a,b] \tag{3.7}$$

PROOF. By construction, $\forall x \notin (x_{j-1}, x_{j+1})$, $f_u(x) = f(x)$.

Again, by construction, $\forall x \in (x_{j-1}, x_{j+1}), f_u(x) = (1-t)f(x_{j-1}) + tf(x_{j+1})$, where $t = \frac{x-x_{j-1}}{x_{j+1}-x_{j-1}}$.

Now, we have:

$$\begin{aligned} x_{j-1} < {} & \quad x \quad < x_{j+1} \\ \implies \quad 0 < {} & \; x - x_{j-1} < x_{j+1} - x_{j-1} \\ \implies \quad 0 < {} & \frac{x - x_{j-1}}{x_{j+1} - x_{j-1}} < 1 \\ \implies \quad 0 < {} & \quad t \quad < 1 \end{aligned}$$

f is convex $\implies \forall t \in (0,1),\; f((1-t)x_{j-1} + tx_{j+1}) < (1-t)f(x_{j-1}) + tf(x_{j+1})$.

Thus, $f_u(x) \geqslant f(x)\ \forall x \in [a,b]$. □

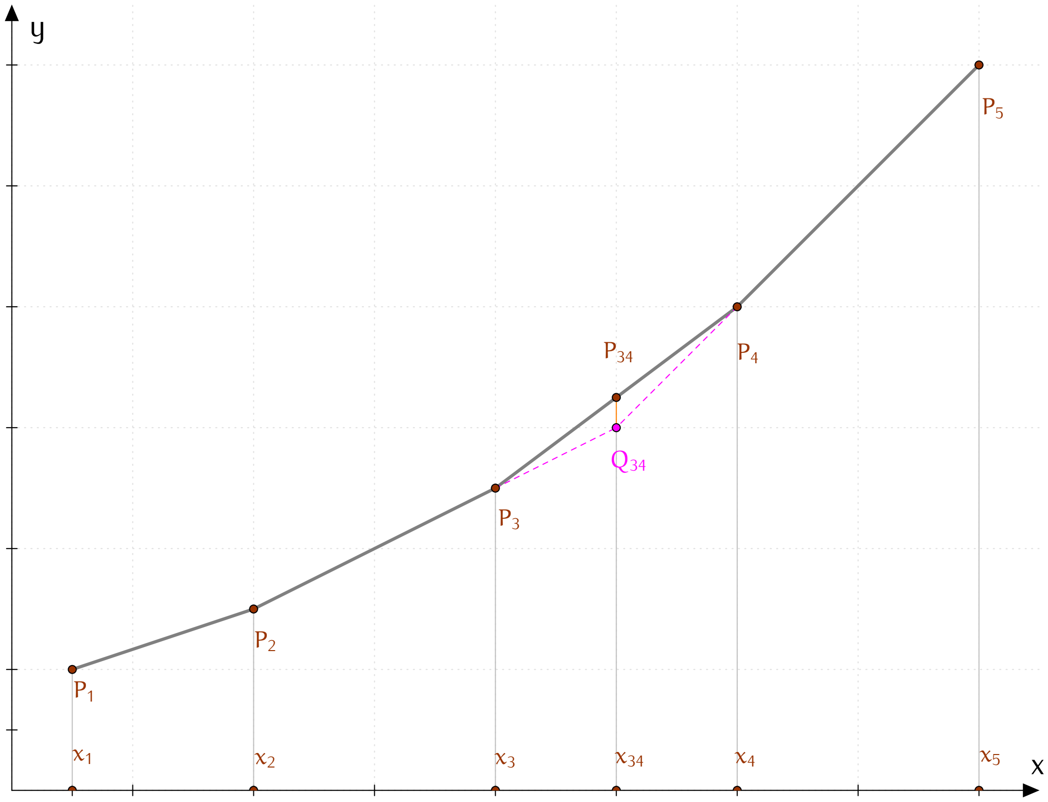

Figure 2. Lower estimate: Illustration of Lemma 3.2 with $j = 3, x_{34} = \bar{x}, P_{34} = \bar{P}$

3.4.2 Lower estimates

Lemma 3.2 (Lower estimate). *Let* $f : [a, b] \to [0, \infty)$ *be a continuous, piecewise-linear, convex function characterised by the singular points* $P = ((x_i, y_i))_{i \in [n]}$. *Let* l_j *be the line segment joining points* P_{j-1} *and* P_j. *Similarly, let* l_{j+2} *be the line segment joining points* P_{j+1} *and* P_{j+2}. *Denote the intersection of the line segments* l_j *and* l_{j+2} *by* $\bar{P} = (\bar{x}, \bar{y})$.

Then the function $f_d : [a, b] \to [0, \infty)$ *characterised by* $(P_0, \dots, P_{j-1}, \bar{P}, P_{j+2}, \dots, P_n)$ *is also continuous, piecewise-linear and convex, and*

$$f_d(x) \leqslant f(x) \qquad \forall x \in [a, b] \tag{3.8}$$

Proof. First we show the convexity of f_d. We know that f satisfies the property of increasing slopes, that is $m_i \leqslant m_{i+1} \leqslant m_{i+2}$. Since f_d is obtained from f by removing the line segment l_{j+1}, for f_d we have that $m_i \leqslant m_{i+2}$, which implies that the function f_d is still convex.

Secondly, to prove the inequality, we may look at the convex function f as if it has been obtained by removing point $\bar{P}$ from the convex function f_d. Then, if $\bar{x} \in (x_j, x_{j+1})$, we have, using Lemma 3.1, that $f_d(x) \leqslant f(x) \qquad \forall x \in [a, b]$. □

The lemmas 3.1 and 3.2, will be used later to reduce the memory requirement of the algorithm by removing points or edges to simplify the function.

3.5 Notations and conventions

In this and subsequent sections, we shall use the convention that $[n] = \{0, 1, 2, \ldots, n\}$.

Let the number of time steps be n. Let i denote the highlighted time step, and j represent the number of up movements. In this way, we may represent any node by $N_{i,j}$.

The price of the underlying at each node $N_{i,j}$ is denoted by $S_{i,j}$. Since there are j up movements, there must be $i - j$ down movements, and thus

$$S_{i,j} = S_0 u^j d^{i-j} = S_0 u^j u^{-(i-j)} = S_0 u^{-i+2j} \qquad \forall i \in [n],\ \forall j \in [i] \tag{3.9}$$

Proposition 3.1. *The number of paths to a node* $N_{i,j}$ *is* $\binom{i}{j}$.

Proof. At each point in a path, we may choose either an up movement or a down movement. To reach node $N_{i,j}$, we much choose j up movements among i possibilities. The result follows immediately. □

Any number of paths among the possible paths may give zero as the price for the option. We denote the number of singular points in a node $N_{i,j}$ by $L_{i,j}$, where $L_{i,j} \in \left[\binom{i}{j}\right]$. The l^{th} average (in ascending order) ($l \in \{1, \ldots, L_{i,j}\}$) is denoted by $A^l_{i,j}$, and the corresponding price by $P^l_{i,j}$. Thus the singular points characterising the price function are $((A^l_{i,j}, P^l_{i,j}))_{l \in \{1,\ldots,L_{i,j}\}}$.

Definition 3.2 (singular average and singular price). In the particular case of Asian options with arithmetic mean, the $A^l_{i,j}$s are called 'singular averages' and the $P^l_{i,j}$s are called 'singular prices'.

We recall some basic definitions and derive simple results for the maximum and minimum attainable value of the averages on each node.

Let the spot rate of interest be r (constant) and the compounding be continous. Then, the effective compounding rate in each time period Δt is given by R as

$$R = e^{r\Delta t} \tag{3.10}$$

We note that the R is not an instantaneous quantity, but one which is constant on an interval of time.

Definition 3.3 (Arithmetic mean). The arithmetic mean of a set of numbers $\{S_i\}_{i \in [n]}$ is given by:

$$A_n = \frac{\sum_{i=0}^{n} S_i}{n+1} \tag{3.11}$$

Definition 3.4 (Path). A path is a sequence $(j_i)_{i \in [n]}$ such that $j_{i+1} \in \{j_i, j_i + 1\}$.

Example 3.1. See Figure 3.

Theorem 3.1 (Path inequality). *Let there be two paths* α *and* β, *such that* $S_{i,j^\alpha_i} \geqslant S_{i,j^\beta_i}\ \forall i$, *where* $(j^\alpha_i)_{i \in [n]}$ *and* $(j^\beta_i)_{i \in [n]}$ *denote the paths as defined above. Denote the corresponding averages by* A^α *and* A^β, *respectively. Then* $A^\alpha \geqslant A^\beta$.

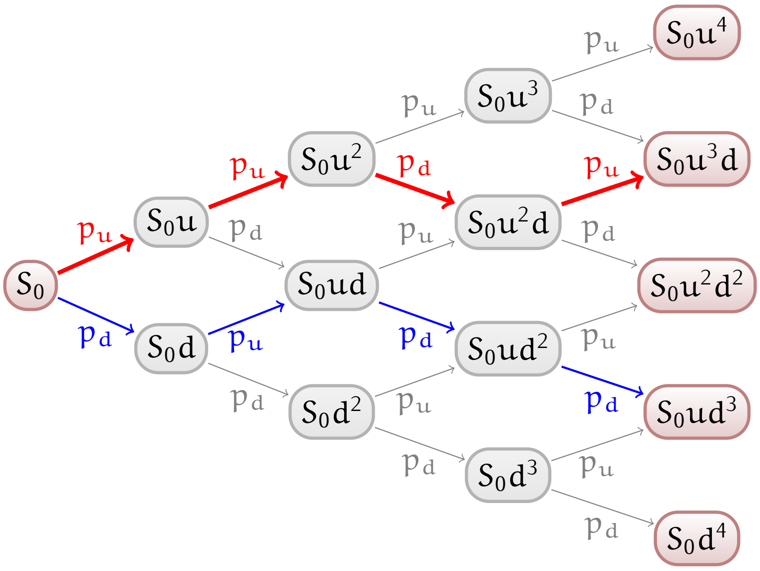

FIGURE 3. Two paths shown using red/thicker and blue/thick arrows. The other arrows are in grey/thin.

PROOF. Clearly if $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i$, then $A^\alpha = A^\beta$.

We only need to show the result in the case of strict inequality. Let $S_{i,j_i^\alpha} = S_{i,j_i^\beta}$ $\forall i \in [n]\backslash\{l\}$, and $S_{l,j_l^\alpha} > S_{l,j_l^\beta}$.

Now, from equation 3.11, we have:

$$\begin{aligned}
(n+1)A^\alpha_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\alpha} + \sum_{i=l+1}^{n} S_{i,j_i} \\
(n+1)A^\beta_{n,j} &= \sum_{i=0}^{l-1} S_{i,j_i} + S_{l,j_l^\beta} + \sum_{i=l+1}^{n} S_{i,j_i} \\
\implies (n+1)\left(A^\alpha_{n,j} - A^\beta_{n,j}\right) &= S_{l,j_l^\alpha} - S_{l,j_l^\beta} \\
&= S_{l-1,j_{l-1}} u_l - S_{l-1,j_{l-1}} d_l \\
&= S_{l-1,j_{l-1}}(u_l - d_l) > 0 \qquad (u_l > d_l \text{ by definition}) \\
\implies A^\alpha_{n,j} > A^\beta_{n,j}
\end{aligned}$$

Iterating this procedure, we obtain the general case. □

REMARK 3.3. The path α signifies a path 'above' and β a path 'below' in 3(case-30ru(al)-304depicition)]TJ/

movements, and

$$A_{i,j}^{\min} = \frac{S_0}{i+1}\left(\frac{1-d^{i-j+1}}{1-d} + d^{i-j}u\frac{1-u^j}{1-u}\right) \tag{3.12}$$

(2) *The maximum average possible $A_{i,j}^{\max}$ is attained by the path corresponding to the path with j up movements followed by $(i-j)$ down movements, and*

$$A_{i,j}^{\max} = \frac{S_0}{i+1}\left(\frac{1-u^{j+1}}{1-u} + u^j d\frac{1-d^{i-j-1}}{1-d}\right) \tag{3.13}$$

Proof. We show the proof only for the case of the maximum, since the case of the minimum can be shown by using the exact same argument.

From Theorem 3.1, the result about path with the maximum average holds directly, since there cannot be a path above the one given by j up movements followed by $(i-j)$ down movements.

The subsequent formula may be derived as follows.

$$\begin{aligned}
(i+1)A_{i,j}^{\max} &= \underbrace{(S_0 + S_0u + S_0u^2 + \cdots + S_0u^j)}_{\text{up movement}} + \underbrace{(S_0u^jd + S_0u^jd^2 + \cdots + S_0u^jd^{i-j})}_{\text{down movement}} \\
&= S_0((1+u+u^2+\cdots+u^j) + u^jd(1+d+\cdots+d^{i-j-1})) \\
&= S_0\left(\sum_{k=0}^{j} u^k + u^jd\sum_{k=0}^{i-j-1} d^k\right) \\
&= S_0\left(\frac{1-u^{j+1}}{1-u} + u^jd\frac{1-d^{i-j-1}}{1-d}\right) \qquad \text{(Geometric series)} \\
\implies A_{i,j}^{\max} &= \frac{S_0}{i+1}\left(\frac{1-u^{j+1}}{1-u} + u^jd\frac{1-d^{i-j-1}}{1-d}\right)
\end{aligned}$$

□

Table 1 summarises the discussion above.

3.6 Fixed-strike Asian options of the European type

For this type of options, the pay-off at maturity is dependent only on (some type of) average A_T at maturity T and a fixed constant K, and is given by the function

$$P_T = (A_T - K)_+ \tag{3.14}$$

We shall focus on this case in this section because it is the easiest to handle.

In each node of the binomial tree, we have a set of possible averages depending on the paths which may be taken to arrive at the node, and prices corresponding to each of those averages. We shall show these points to satisfy condition 3.5, so they completely characterise a price function. Thus we focus not on the averages and corresponding prices

Table 1. Summary of notations

Symbol	Range	Formula	Description
i	$[n]$		highlighted time step
j	$[i]$		number of up movements
$N_{i,j}$			node fixed by (i,j)
$S_{i,j}$	$[0,\infty)$	Eq 3.9	value of the underlying at node $N_{i,j}$
$L_{i,j}$	$\left[\binom{i}{j}\right]$		number of singular points in node $N_{i,j}$
l	$\{1,\dots,L\}$		index for points in ascending order of averages
$A^{min}_{i,j}$	$[0,\infty)$	Eq 3.12	minimum average attainable for node $N_{i,j}$
$A^{max}_{i,j}$	$[0,\infty)$	Eq 3.13	maximum average attainable for node $N_{i,j}$
$A^{l}_{i,j}$	$\left[A^{min}_{i,j},A^{max}_{i,j}\right]$	Eq 3.11	l^{th} singular average of node $N_{i,j}$
$P^{l}_{i,j}$			price corresponding to the average $A^{l}_{i,j}$
$(A^{l}_{i,j},P^{l}_{i,j})$			l^{th} singular point of node $N_{i,j}$

possible under a particular binomial tree, but on the continuous representation of prices. The intuitive idea is that as the time step is reduced to zero, this function converges to the price function of the continuous time model.

3.6.1 The price function at maturity

From equations 3.12 and 3.13, putting $i=n$, we get

$$A^{min}_{n,j}=\frac{S_0}{n+1}\left(\frac{1-d^{n-j+1}}{1-d}+d^{n-j}u\frac{1-u^j}{1-u}\right)$$

$$A^{max}_{n,j}=\frac{S_0}{n+1}\left(\frac{1-u^{j+1}}{1-u}+u^jd\frac{1-d^{n-j-1}}{1-d}\right)$$

In defining the price function, we note that three cases may arise.

- $j\in\{0,n\}$
  There exists only one path to these nodes, so there is only one average, implying one price and one singular point.
- $j\notin\{0,n\}$ and $K\in(A^{min}_{n,j},A^{max}_{n,j})$
  The price function is characterised by three singular points ($L_{i,j}=3$), $(A^{l}_{n,j},P^{l}_{n,j})_{l\in\{1,2,3\}}$, which are

$$\begin{aligned}(A^1_{n,j},P^1_{n,j})&=(A^{min}_{n,j},0)\\ (A^2_{n,j},P^2_{n,j})&=(K,0)\\ (A^3_{n,j},P^3_{n,j})&=(A^{max}_{n,j},A^{max}_{n,j}-K)\end{aligned}\tag{3.15}$$

- $j\notin\{0,n\}$ and $K\notin(A^{min}_{n,j},A^{max}_{n,j})$
  The price function is characterised by only two singular points ($L_{i,j}=2$), $(A^{l}_{n,j},P^{l}_{n,j})_{l\in\{1,2\}}$,

which are

$$(3.16)\qquad \begin{aligned}(A^1_{n,j}, P^1_{n,j}) &= (A^{\min}_{n,j}, (A^{\min}_{n,j} - K)_+)\\ (A^2_{n,j}, P^2_{n,j}) &= (A^{\max}_{n,j}, (A^{\max}_{n,j} - K)_+)\end{aligned}$$

Lemma 3.3 (Price function at maturity). *At each node at maturity, the price function* $v_{n,j} : \left[A^{\min}_{n,j}, A^{\max}_{n,j}\right] \to \left[(A^{\min}_{n,j} - K)_+, (A^{\max}_{n,j} - K)_+\right]$ *defined as* $v_{n,j}(A) = (A - K)_+$ *is continuous, piecewise-linear and convex.*

Proof. The singular points satisfy the conditions 3.5. So for each $A \in \left[A^{\min}_{n,j}, A^{\max}_{n,j}\right]$, the price function $v_{n,j}(A)$ characterised by the singular points is continuous, piecewise-linear and convex by remark 3.1. □

3.6.2 The price function before maturity

Lemma 3.4 (Price function at any node). *At any node* $N_{i,j}$*, the price function* $v_{i,j} : \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right] \to [0, \infty)$ *is continuous, piecewise-linear and convex.*

Proof. We shall prove this using backward induction, the base case at maturity being true by virtue of Lemma 3.3. We now consider step $i = n - 1$. Let A_u and A_d respectively represent the averages after an up and down movement corresponding to an average A. From equation 3.11, we get

$$(3.17a)\qquad A_u = \frac{(i+1)A + S_0 u^{-i+2j+1}}{i+1}$$

$$(3.17b)\qquad A_d = \frac{(i+1)A + S_0 u^{-i+2j-1}}{i+1}$$

The price function $v_{i,j} : \left[A^{\min}_{i,j}, A^{\max}_{i,j}\right] \to [0, \infty)$ is obtained by considering the discounted expectation value.

$$(3.18)\qquad v_{i,j}(A) = \frac{1}{R}\left[\pi v_{i+1,j+1}(A_u) + (1-\pi) v_{i+1,j}(A_d)\right]$$

From equation 3.17, we get that A_u and A_d are linear functions of A. Thus, $v_{i+1,j+1}(A_u) = v_{n,j+1}(A_u)$ and $v_{i+1,j}(A_d) = v_{n,j}(A_d)$ are piecewise-linear convex continuous functions of A_u and A_d respectively. Thus, $v_{i+1,j+1}$ and $v_{i+1,j}$ may be seen as a linear combination of the above functions, and is thus piecewise-linear, convex and continuous itself. Again, from equation 3.18, we get that $v_{i,j}$ is a convex combination of such functions, and the proof is complete. □

TODO: Insert picture for this.

Remark 3.4. From Lemma 3.4, we see that the price function may be characterised by singular points.

3.6.3 Evaluation of singular points

The evaluation of singular points for any node $N_{i,j}$ is done by the following algorithm, which works in a backward fashion in time, starting from the maturity.

We note that for the only influencing nodes for the node $N_{i,j}$ are $N_{i+1,j+1}$ and $N_{i+1,j}$. Thus we need to calculate the price of the option for each singular average belonging to either of these nodes.

Up movement First we take each singular average $A^l_{i+1,j}$ belonging to $N_{i+1,j}$ and project it to $N_{i,j}$ via the relation

$$B^l = \frac{(i+2)A^l_{i+1,j} - S_0 u^{-i+2j-1}}{i+1} \tag{3.19}$$

Thus, B^l is that average which after a down movement of the asset gives us the average $A^l_{i+1,j}$.

Next, we note that B^l is an increasing function of l, since a higher average at time step i would yield a higher average at time $i+1$. This in turn implies the following:

- $B^{L_{i+1,j}} = A^{max}_{i+i,j}\ \forall j$
- $B^1 \notin \left[A^{min}_{i+i,j}, A^{max}_{i+i,j}\right]\ \forall j \in \{1,\ldots,i-1\}$

Each $B^l \in \left[A^{min}_{i,j}, A^{max}_{i,j}\right]$ becomes the singular average of $N_{i,j}$.

In this way, we have determined the first coordinate of the singular points. We need to determine the second coordinate, or the prices $\nu_{i,j}(B^l)$, $\forall \left[A^{min}_{i,j}, A^{max}_{i,j}\right]$, in order to determine the singular points completely. The idea is to calculate the discounted expected value of the price corresponding to each average B^l at $N_{i,j}$. In order to be able to do this, we need the prices corresponding to the average projected to the node $N_{i+1,j+1}$.

We consider an up movement of the underlying asset from node $N_{i,j}$. In this case, B^l transforms into the average $B^l_u = \left((i+1)B^l + S_0 u^{-i+2j+1}\right)/(i+2)$. Clearly, this average cannot belong to the set of averages associated with the node $N_{i+1,j+1}$. Thus, we need to find the index s such that $B^l_u \in \left[A^s_{i+1,j+1}, A^{s+1}_{i+1,j+1}\right]$. In the intervals the price function is linear, and thus we have

$$\nu_{i+1,j+1}\left(B^l_u\right) = \frac{P^{s+1}_{i+1,j+1} - P^s_{i+1,j+1}}{A^{s+1}_{i+1,j+1} - A^s_{i+1,j+1}}\left(B^l_u - A^s_{i+1,j+1}\right) + P^s_{i+1,j+1} \tag{3.20}$$

We follow this up by calculating the price associated with the singular value B^l by evaluating the discounted expectation value.

$$\nu_{i,j}(B^l) = \frac{1}{R}\left[\pi\nu_{i+1,j+1}\left(B^l_u\right) + (1-\pi)\nu_{i+1,j}\left(A^l_{i+1,j}\right)\right] \tag{3.21}$$

Down movement We now proceed to formulate the theory for the downward movement in the exact same fashion. Define the new average C^l at the node $N_{i,j}$ via the relation

$$C^l = \frac{(i+2)A^l_{i+1,j+1} - S_0 u^{-i+2j+1}}{i+1} \tag{3.22}$$

Again, we note that

- $C^1 = A^{min}_{i,j} \; \forall j$
- $C^{L_{i+1,j+1}} \notin \left[A^{min}_{i,j}, A^{max}_{i,j}\right] \;\; \forall j \in \{1, \ldots, i-1\}$
- $C^l_d = \left((i+1)C^l + S_0 u^{-i+2j-1}\right)/(i+2)$

Each $C^l \in \left[A^{min}_{i,j}, A^{max}_{i,j}\right]$ becomes the singular average of $N_{i,j}$.

For $\nu_{i,j}(C^l)$, $\forall \left[A^{min}_{i,j}, A^{max}_{i,j}\right]$, we now have the following.

$$\nu_{i+1,j+1}\left(C^l_d\right) = \frac{P^{s+1}_{i+1,j} - P^s_{i+1,j}}{A^{s+1}_{i+1,j} - A^s_{i+1,j}}\left(C^l_d - A^s_{i+1,j}\right) + P^s_{i+1,j} \tag{3.23}$$

$$\nu_{i,j}(C^l) = \frac{1}{R}\left[\pi \nu_{i+1,j+1}\left(A^l_{i+1,j+1}\right) + (1-\pi)\nu_{i+1,j}\left(C^l_d\right)\right] \tag{3.24}$$

Aggregation Now we have the singular points for both up and down movements. We sort these points in ascending order of the first coordinate, i.e. the averages B^l and C^l that belong to $\left[A^{min}_{i,j}, A^{max}_{i,j}\right]$. These is an exhaustive list of all the singular points in the node (by construction). We note that $L_{i,j} \leqslant L_{i+1,j} + L_{i+1,j+1} - 2$.

This procedure is applied to all nodes, starting from maturity and proceeding backwards. At the 'edge' nodes $N_{i,0}$ and $N_{i,i}$, there is only one singular point whose price is given as follows

$$P^1_{i,0} = \frac{1}{R}\left[\pi P^1_{i+1,0} + (1-\pi)P^1_{i+1,1}\right] \tag{3.25a}$$

$$P^1_{i,i} = \frac{1}{R}\left[\pi P^1_{i+1,i+1} + (1-\pi)P^{L_{i+1,i}}_{i+1,i}\right] \tag{3.25b}$$

Thus we have a complete description of the price function at each node of the binomial tree. The price $P^1_{0,0}$ is exactly the binomial price relative to the tree with n steps of a fixed-strike European call option.

3.7 Fixed-strike American Asian options

TODO

3.8 Extensions

Let us recapitulate the conditions required for the singular points method to work in the case of Asian options with arithmetic mean.

- The ability to calculate the upper and lower bounds of the mean for all nodes of the tree.
- The recombinant nature of the tree for the underlying. Note that the tree for the option prices are *not* recombinant.
- Convexity and piecewise-linearity of the price function on the mean of the underlying.
- Fixed volatility

Keeping these in mind, let us look at the possibility of extending the singular points method to the following cases:

(1) Asian options with geometric mean and fixed volatility.

(2) Asian options with arithmetic mean and local volatility.

3.8.1 Geometric mean and fixed volatility

In the case of geometric options, we have a closed form formula under the Black-Scholes market model. We try to extend the singular points method.

Firstly, we show that the result about the maximum and minimum paths still hold in the geometric case.

Definition 3.5 (Geometric mean). The geometric mean of the risky asset's prices $(S_i)_{i\in[n]}$ is given by:

$$G_n = \left(\prod_{i=0}^{n} S_i\right)^{\frac{1}{n+1}} \tag{3.26}$$

Lemma 3.5. *At each node* $N(i,j)$, *the following hold:*

(1) *The maximum average possible* $G^{max}_{i,j}$ *is attained by the path corresponding to the path with* j *up movements followed by* $(i-j)$ *down movements.*

(2) *The minimum average possible* $G^{min}_{i,j}$ *is attained by the path corresponding to the path corresponding to the path with* $(i-j)$ *down movements followed by* j *up movements.*

Proof. The proof is the same as 3.1, with A replaced by G and relevant modifications. □

One of the central ideas behind the singular points method is that the price of the option is a convex, piecewise-linear function of the average A. But in the geometric case, this no longer holds true. For example, take a node $N_{i,j}$ with $i = n - 1$. The price function given by $v_{i,j}(G)$, with $G \in [G^{min}, G^{max}]$, can be calculated by the discounted expectation value.

$$v_{i,j}(G) = \frac{1}{R}\left[p v_{i+1,j+1}(G_u) + (1-p) v_{i+1,j}(G_d)\right] \tag{3.27}$$

$$G_u = \left(G^{i+1} S_0 u^{-i+2j+1}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{3.28}$$

$$G_d = \left(G^{i+1} S_0 u^{-i+2j-1}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}} \tag{3.29}$$

Clearly, the final function $v_{i,j}$ is not linear in G. Rather it is piecewise-concave. Thus we cannot use the singular points method in this case. TODO: Insert a graph of the function here.

3.8.2 Arithmetic mean with local volatility

In this case, the tree for the underlying is not recombinant, so we do not have more than one singular point in one (non-recombining) node. Clearly, we cannot use the singular points method.

3.9 Conclusion

We conclude the chapter by noting the pros and cons of the singular points method.

Advantages

- Fast – Experimental order of complexity $= O(n^3)$
- It allows us to specify an *a priori* error bound.

Disadvantages

- Very specific method – only applicable to a few specific cases.

Chapter 4

The Singular Points method applied to Cliquet options

4.1 Introduction

A 'cliquet option' or a 'ratchet option' is an exotic option consisting of a series of consecutive forward start options. The first such option is active immediately, and once it expires the second comes into existence, and so on. Each option is struck at-the-money when it becomes active. Therefore, such an option periodically settles and resets its strike price at the level of the underlying during the time of settlement. Investors can opt to receive their payout either when each option expires or wait until the entire time has elapsed.

Usually, the return on a cliquet option is capped and floored. The capping and flooring may be local or global (or both). The motivation behind bounding the return is to provide the investor safety against downside risks, yet allowing significant upside potential. Consequently, the investor is also constrained from having unbounded gains. Capping the maximum ensures that the payoff is never too extreme and therefore that the value of the contract is not too outrageous. Some variants of cliquet options are as follows:

> **Reverse cliquet:** Amounts to a cash flow minus a capped cliquet of puts.
>
> **Digital cliquet:** The forward-starting options are digital options.

The following example, copied verbatim from Investopedia[1], provides a good illustration.

Example 4.1. A three-year cliquet option with a strike of 1000 would expire worthless on the first year if the underlying was to be 900. This value (900) would then be the new strike price for the following year and should the underlying on the settlement be 1200, the contract holder would receive a payout and the strike would reset to this new level. Higher volatility provides better conditions for investors to earn profits.

Literature review [ToDo] The literature for pricing of cliquet options is primarily based on partial differential equations (PDE) techniques. Notable mentions include Wilmott's[TODO:reference]

[1] http://www.investopedia.com/terms/c/cliquet.asp

finite difference (FD) approach in a non-linear uncertain volatility model (UVM) in 2002. Later, Windcliff et al.[TODO:reference] explored a variety of modelling alternatives including jump diffusion models, local volatility and UVM models, again using finite differences methods in 2006.

Lattice method Gaudenzi *et al*[GZ11] introduced a discrete method for pricing cliquet options in a binomial Cox Ross Rubinstein model [TODO:reference]. This Singular Points method is faster than the alternative lattice based approaches available, while retaining a fair bit of flexibility to handle varying volatilities and rates of interest. Most of the chapter is inspired from the former paper.

4.2 Cliquet contracts and models

Notation review: In this and the following sections, we will use the notation $[n] := \{0, 1, \dots, n\}$.

We consider a market model, in which the price of the underlying risky asset is governed by the Black-Scholes stochastic differential equation.

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma\,dB_t \tag{4.1a}$$

$$S_0 = s_0 \tag{4.1b}$$

The quantities used above are defined as follows.

S_t: Price of the underlying risky asset at time t

B_t: A standard Brownian motion under the risk-neutral probability measure Q

$r > 0$: Rate of interest

$q \geqslant 0$: Continuous divident yield

$\sigma > 0$: Volatility of the risky asset

Solving the equation, at any time t, the price of the underlying risky asset is given by

$$S_t = s_0 e^{((r-q)-\frac{\sigma^2}{2})t+\sigma B_t}.$$

Let T be the maturity of the cliquet contract. Let the payoffs depend on the N preordained observation times $t_1, t_2, \dots, t_N$ ($t_0 = 0$). At these observation times, the value of the underlying are $(S_i)_i, S_i = S_{t_i}, i \in [N]$. The returns for the time interval $(t_{i-1}, t_i]$ are given by

$$R_i = \frac{S_i - S_{i-1}}{S_{i-1}} = \frac{S_i}{S_{i-1}} - 1\ . \tag{4.2}$$

During each time interval, the return is capped and floored locally by the quantities C_{loc} and F_{loc}. In other words, we consider the quantity $\max\{F_{loc}, \min\{C_{loc}, R_i\}\}$ rather than the return itself. The sum of these quantities till time t_i is called the 'running sum' and is given by

$$Z_i = \sum_{k=1}^{i} \max\{F_{loc}, \min\{C_{loc}, R_k\}\}. \tag{4.3}$$

We also consider a global cap C_{glob} and floor F_{glob}. Thus, the expression for the payoff finally becomes

$$\text{payoff} = \text{notional} \cdot \max\{F_{glob}, \min\{C_{glob}, Z_N\}\}. \tag{4.4}$$

For ease of notation, we take notional $= 1$.

We note that the case $C_{glob} > NC_{loc}$ is equivalent to $C_{glob} = NC_{loc}$. Similarly, the case $F_{glob} < NF_{loc}$ is equivalent to $F_{glob} = NF_{loc}$. In general, we may write the following.

$$F_{glob} = \max\{NF_{loc}, F_{glob}\}$$
$$C_{glob} = \min\{NC_{loc}, C_{glob}\}$$

Now, we consider the binomial approach in order to price the cliquet option in the Black-Scholes framework. Let the number of intervals be n and the corresponding time step be $\Delta T = \frac{T}{n}$. Then the lognormal diffusion process is approximated by the Cox-Ross-Rubinstein binomial process:

$$\tilde{S}_j = s_0 \prod_{k=1}^{j} Y_k \qquad \forall j \in [n], \tag{4.6}$$

where the random variables Y_k are independent and identically distributed with values in $\{d, u\}$, representing a down and up movement, respectively. Let p be the probability of an up movement of the asset, that is, $p = \Pr(Y_n = u)$. The Cox-Ross-Rubinstein corresponds to the choice $u = 1/d = e^{\sigma\sqrt{\Delta T}}$. Thus,

$$p = \frac{e^{r\Delta T} - d}{u - d} = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}. \tag{4.7}$$

We assume that the difference between two observation times is constant and we denote by m the number of steps of the binomial tree in every period (so that the total number of steps of the binomial tree is $n = mN$).

To recapitulate, the following table highlights the notations in a concise manner. [TODO]

4.3 The Singular Points method for cliquet options

The binomial method may always be used to price any option, including path-dependent ones. The binomial method looks through all possible paths of the underlying in order to price the option. The number of possible paths are 2^{mN}. Thus, the method is inherently extremely computationally expensive due to the exponential dependence of the number of paths on m and N. The theoretical computational complexity is $O(m^N)$, as in [GZ11, Page 128].

A modification of the Singular Points method described in the previous chapter solves this problem for cliquet options by the process of approximation. The method of approximation selectively removes certain paths that would be normally considered, but which do not affect the result in a significant manner. This may be done by putting an *a priori* error bound while removing points. The method turns out to be significantly faster and memory efficient compared to known binomial techniques. Moreover, its flexible is evinced by the fact that it is adaptable for varying interest rate and volatility in each observational period.

In this section, we will note some elementary definitions and results, and give a detailed exposition of the method. The reader is advised to keep in mind the notation introduced in section 3.5 of chapter 3.

DEFINITION 4.1 (Singular points and singular values). Let $P = (P_i)_{i\in[n]} = ((x_i, y_i))_{i\in[n]}$, $n \in \mathbb{N}$ be a sequence of points such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \forall i \in [n]$

Let $f : [a, b] \to [0, \infty)$ be the function obtained by linear interpolation of the points in P. The definition of f ensures that the function is continuous and piecewise-linear.

Then, the elements of P are called *singular points of* f and the abscissae $\{x_i\}_{i\in[n]}$ are called *singular values of* f.

REMARK 4.1. We note that the singular points characterise such a function completely. This can be seen from the following representation of the function.

$$f(x) = y_0 + \sum_{i=1}^{n} [m_i(\min\{x_i, x\} - \min\{x_{i-1}, x\})] \tag{4.8}$$

Where $m_{i+1} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ represents the slope of the function between (x_i, y_i) and (x_{i+1}, y_i).

REMARK 4.2. Note that there is no mention of convexity in the foregoing discussion, since the price function in the cliquet case is not necessarily convex, as opposed to the Asian case. The problem comes from the fact that due to the presence of a global cap, the price function at maturity is no longer convex (see Figure 1). Since the algorithm starts from maturity, it follows that we cannot assume convexity at any point of time. It was primarily the convexity of the price functions in the Asian case that allowed us to obtain simple upper and lower bounds of the exact binomial price. Nevertheless, in the case of cliquet options, the singular points approach still provides an efficient binomial framework, even in the absence of convexity of the price functions, as we shall see.

4.3.1 The method

Our aim is to look at every possible value taken by the running sum Z within the bounds $[F_{loc}, C_{loc}]$ for each time interval $t_1, \ldots, t_N$. If we know the price function at maturity, we may use a backward procedure (in time) in order to obtain a continuous representation of the cliquet price as a piecewise-linear function of the running sum Z. Since it is piecewise-linear and continuous, the function may be represented using its singular points. Thus, we see an evolution of singular points as we go back in time. Since the number of singular points may be significantly high for any computer, we shall introduce an error controlled approximation procedure to reduce the number of singular points.

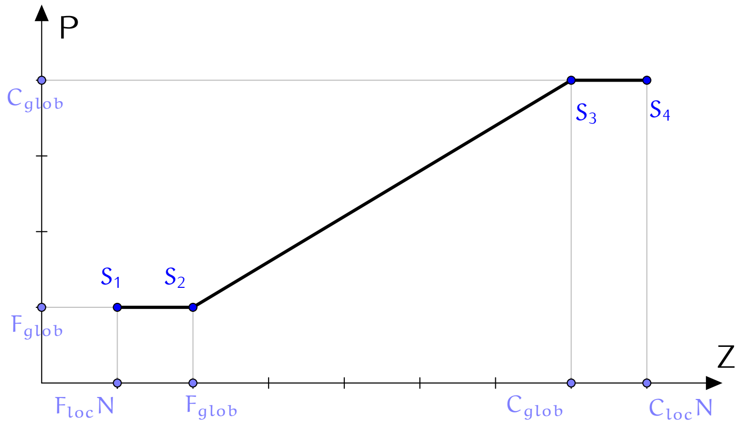

FIGURE 1. The price function at maturity

Let the number of singular points at each observational time t_i be L_i, where $i \in \{1, \dots, N\}$. For each singular point $l \in \{1, \dots, L_i\}$, the abscissa is called the *singular running sum* Z_i^l and the ordinate is called the *singular price* P_i^l. Thus, the singular points are denoted by

$$(Z_i^l, P_i^l) \qquad \forall l \in \{1, \dots, L_i\}$$

At maturity At maturity ($t_{Nobs} = T$), for every running sum Z, the price of the cliquet option $V_N(Z)$ as function of Z, is given by

$$V_N(Z) = \max\{Fglob, \min\{Cglob, Z\}\} \tag{4.9}$$

We note the following.

(1) $V_N(Z)$ is a continuous and piecewise-linear function

(2) $V_N(Z)$ is defined in the interval $[NF_{loc}, NC_{loc}]$

(3) There are only four points where the function changes slope, namely NF_{loc}, F_{glob}, C_{glob} and NC_{loc}. This is because if $Z < F_{glob}$, the price is constant. Same can be said about $Z > C_{glob}$. Thus the four numbers enumerated above form the complete set of singular running sums at maturity ($L_N = 4$).

Focusing further on point 3 above, we note down the singular running sums and corresponding prices.

TABLE 1. Singular points at maturity

l	Z_N^l	P_N^l
1	NF_{loc}	F_{glob}
2	F_{glob}	F_{glob}
3	C_{glob}	C_{glob}
4	NC_{loc}	C_{glob}

A more visual representation of the price function at maturity is given in Figure 1.

The penultimate time step We consider the time step $N-1$. If the running sum at time t_{N-1} is denoted by Z, then the corresponding price depends on the possible returns of the underlying asset during the time interval $[t_{N-1}, T]$.

We revisit equation 4.2 once more. Now we note that since the number of time steps in each interval is m, there will be m up movements or down movements of the asset. Thus, there are $m+1$ possible outcomes, given by $S_i = u^{-m+2j}S_{i-1}, j \in [m]$. Corresponding to these cases, there are also $m+1$ returns, given by

$$R_j = u^{-m+2j} - 1 \qquad j \in [m] \tag{4.10}$$

Since the probability of an up movement in time ΔT is p (refer equation 4.7), we have that the probability of each return is distributed binomially, and is given by

$$p_j = \binom{m}{j} p^j (1-p)^{m-j} \tag{4.11}$$

The above derivations assume that the there has been no local flooring or capping of the returns. In the case of such bounds, the actual possibilities are fewer in number, and we need to put bounds on j. We can do this in the following fashion.

In the case of a local floor, we must have

$$\begin{array}{rrcll}
 & F_{loc} & \geqslant & u^{-m+2j_{min}} - 1 & \\
\implies & \log(F_{loc}+1) & \geqslant & (-m+2j_{min})\log(u) & \ldots(\log \text{is monotonic}) \\
\implies & \dfrac{\log(F_{loc}+1)}{\log(u)} & \geqslant & (-m+2j_{min}) & \ldots(u > 1 \implies \log(u) > 0) \\
\implies & j_{min} & \leqslant & \dfrac{\log(F_{loc}+1)}{2\sigma\Delta T} + \dfrac{m}{2} & \ldots(u = e^{\sigma\Delta T}) \\
\implies & j_{min} & = & \left\lfloor \dfrac{\log(F_{loc}+1)}{2\sigma\Delta T} + \dfrac{m}{2} \right\rfloor & \ldots(j \in [m])
\end{array}$$

Where $\lfloor\cdot\rfloor$ denotes the floor function.

Similarly, in the case of local cap, we have

$$\begin{array}{rrcll}
 & C_{loc} & \leqslant & u^{-m+2j_{max}} - 1 & \\
\implies & \log(C_{loc}+1) & \leqslant & (-m+2j_{max})\log(u) & \ldots(\log \text{is monotonic}) \\
\implies & \dfrac{\log(C_{loc}+1)}{\log(u)} & \leqslant & (-m+2j_{max}) & \ldots(u > 1 \implies \log(u) > 0) \\
\implies & j_{max} & \geqslant & \dfrac{\log(C_{loc}+1)}{2\sigma\Delta T} + \dfrac{m}{2} & \ldots(u = e^{\sigma\Delta T}) \\
\implies & j_{max} & = & \left\lceil \dfrac{\log(C_{loc}+1)}{2\sigma\Delta T} + \dfrac{m}{2} \right\rceil & \ldots(j \in [m])
\end{array}$$

Where $\lceil\cdot\rceil$ denotes the ceiling function.

We represent by j_0 the number of possibilities of the return after enforcing the local floor and cap. Summarising

$$j_{min} = \left\lfloor \frac{\log(F_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} \right\rfloor \tag{4.12a}$$

$$j_{max} = \left\lceil \frac{\log(C_{loc}+1)}{2\sigma\Delta T} + \frac{m}{2} \right\rceil \tag{4.12b}$$

$$j_0 = j_{max} - j_{min} \tag{4.12c}$$

$\forall j \leqslant j_{min}$, the return is F_{loc}, and $\forall j \geqslant j_{max}$, the return is C_{loc}. For the other indices, the return remains unchanged.

For ease of implementation, we shift the indices from $\{j_{min}, \dots, j_{max}\}$ to $\{0, \dots, j_0\}$ by putting $j' = j - j_{min}$. Table 2 highlights the shifted indices, and the corresponding returns and probabilities.

TABLE 2. Shifted returns and probabilities

Range(j)	j'	R'_j	p'_j
$j \leqslant j_{min}$	0	F_{loc}	$\sum_{k=0}^{j_{min}} p_k$
$\{j_{min}+1, \dots, j_{max}-1\}$	$j - j_{min}$	$R_{j+j_{min}}$	$p_{j+j_{min}}$
$j \geqslant j_{max}$	j_0	C_{loc}	$\sum_{k=j_{max}}^{m} p_k$

Now we focus on how to determine the price function at time t_{N-1}. Recall that at maturity, the function $V_N(Z)$ giving the price of the cliquet option as a function of the running sum Z at maturity, is the piecewise linear function whose singular points are presented in Table 1. At the penultimate time step t_{N-1}, we must have $Z \in [(N-1)F_{loc}, (N-1)C_{loc}]$. Note that Z is the running sum till the penultimate time step. The price function at this time is given as a discounted conditional expectation of the price function at maturity given the information at penultimate time. Thus

$$V_{N-1}(Z) = e^{-m\Delta T} \sum_{j=0}^{j_0} \left[p'_j V_N(Z + R'_j) \right] \tag{4.13}$$

Since V_N is piecewise-linear and continuous, and the above is just a linear combination of such functions, the resulting function is also piecewise-linear and continuous, and thus may be completely represented by singular points. The next logical step is of course to figure out a method to compute these singular points.

From equation 4.13, we note that each singular point l at maturity may have $j_0 + 1$ possible returns, given by

$$B_{l,j} = Z_N^l - R'_j \qquad j \in [j_0] \tag{4.14}$$

Then the maximum number of singular points at time t_{N-1} is $(j_0+1)L_N$. But not all the running sums at time t_{N-1} would belong to the interval $[(N-1)F_{loc}, (N-1)C_{loc}$. All the $B_{l,j}$ which belong to the interval become the abscissa of a singular point at time t_{N-1}. The corresponding singular price is determined by formula 4.13. The term $V_N(B_{l,j} + R'_k)$, required by the formula, is computed using linearity of the price function at maturity. We just need to figure out the interval l_0 such that $(B_{l,j} + R'_k) \in [Z_N^{l_0}, Z_N^{l_0+1}]$, and evaluate $V_N(B_{l,j} + R'_k)$ by linear interpolation of the extrema. Such an interpolation gives not an approximation by the exact value due to the piecewise-linear nature of the original function.

Finally, the singular points thus obtained are sorted in ascending order on the basis of the running sums. This ordered sequence of singular points $((Z_{N-1}^l, P_{N-1}^l))_l, l \in \{1, \dots, L_{N-1}\}$ completely characterises the price function at time $V_{N-1}(Z)$.

At all times The previous argument may be applied iteratively in a backward fashion at each step $N-2, N-3, \dots, 1, 0$ to obtain the singular points at each time step. At time 0, there is only one singular point $(0, P_0^1)$, and P_0^1 provides the exact binomial price of the cliquet option. The equality of this price and the exact binomial price is proved in Proposition 4.1.

Proposition 4.1 (Equality of price obtained by the singular points method and the binomial price). P_0^l *coincides with the exact binomial price with* n *time steps of the cliquet option.*

Proof. Let the running sum $Z_{i_1,\dots,i_k} = \sum_{j=1}^k R'_{i_j}$, $(i_1, \dots, i_k) \in [j_0]^k, k \in \{1, \dots, N\}$. Then the exact binomial price Q_n of the cliquet option is given by

$$Q_n = e^{-rT} \sum_{i_1,\dots,i_N=0}^{j_0} \left(p'_{i_1} \cdot \dots \cdot p'_{i_N} \max \left\{ F_{glob}, \min \left\{ C_{glob}, Z_{i_1,\dots,i_N} \right\} \right\} \right).$$

□

[TODO] Insert remark 2

4.3.2 Approximation

The method demonstrated in Section 4.3.1 gives us the actual binomial price of the cliquet option. But as was mentioned earlier, the computational complexity of the method is theoretically the same as that of the binomial method, which is m^N. But contrary to the binomial method, the singular point method enables us to use precise, efficient and controlled approximation to accelerate the procedure significantly. Thus its efficacy become apparent when we have time and memory constraints, or we wish to substantially increase the number of intermediate time steps.

The prime idea of the approximation procedure is to remove singular points in a controlled manner in order to simplify the computations. To this end, we fix a given maximal level $h > 0$ of the error in each period. Now, we eliminate singular points in a fashion so that the function after deletion of the points $(\tilde{V}_i)$ is different from the original function by less than (h) at each point. This may be achieved as follows.

Start with the point (Z_i^1, P_i^1). Find the largest index $l > 1$ such that the distances between the straight line joining (Z_i^1, P_i^1) and (Z_i^l, P_i^l) and the points $(Z_i^2, P_i^2), (Z_i^3, P_i^3), \dots, (Z_i^{l-1}, P_i^{l-1})$ are always less than h. Note that this, coupled with the fact that the original function is piecewise-linear, ensures that the differences in the values of the functions for

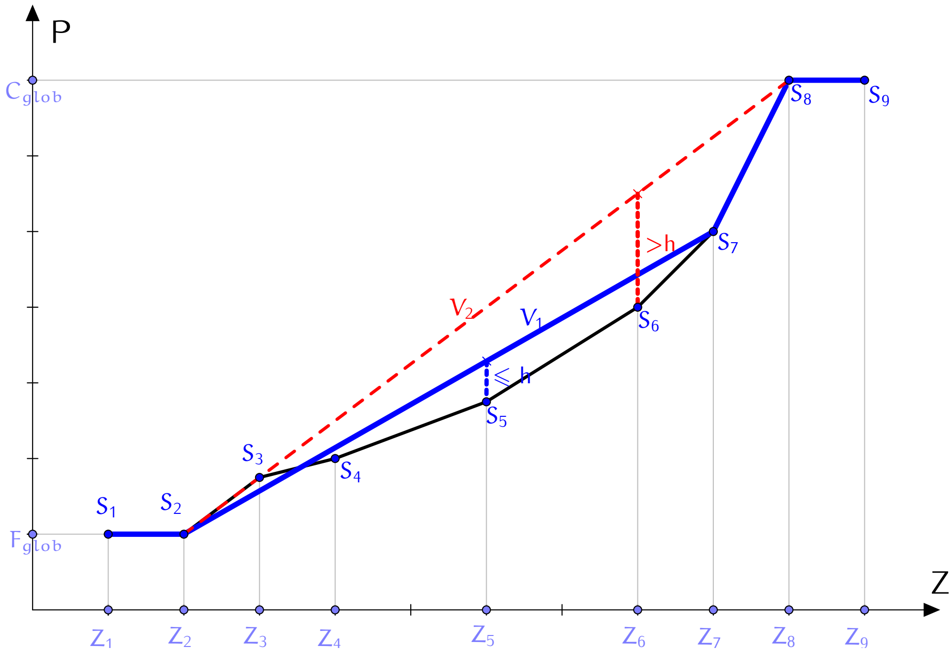

FIGURE 2. Approximation procedure: Elimination of points S_3 to S_6 gives function V_1 (bold, blue), which has maximum error $\leqslant h$. Elimination of points S_3 to S_7 gives function V_2 (dashed, red), which has maximum error $> h$. Thus we choose V_1 as the approximation function.

any value of the running sum would be bounded over by h. Now delete the points $(Z_i^2, P_i^2), (Z_i^3, P_i^3), \ldots, (Z_i^{l-1}, P_i^{l-1})$. The point (Z_i^l, P_i^l) now becomes the second singular point, and we continue the procedure iteratively starting with this point till we cover all the points. Figure 2 elaborates on this graphically.

Since at each N the upper bound for the error is h, if we repeat the approximation procedure at every observational time, the total error infused in the price of the option is bounded by Nh.

REMARK 4.3 (Comparison with Asian options). While using approximations for Asian options, the error bound was nh, where n was the number of time steps considered for computation (refer to Section [TODO: reference] of Chapter 3). But in the case of the cliquet options, the error depends only on the h and the number of observations - precluding any dependence

4.4.1 Algorithm

[TODO: Include approximation procedure]

Algorithm 3: The Singular Points method for cliquet options

Input:

Contract details

time to maturity: T
number of observations: N
local floor and cap: F_{loc}, C_{loc}
global floor and cap: F_{glob}, C_{glob}

Details of the underlying asset

initial price: s_0
volatility: σ
continuous dividend rate: q

Market parameters – spot interest rate: r

Computational parameters – time steps within each observed period: m

Output: The price of the option at the initial time

begin
  Update F_{glob} and C_{glob} using Equations 4.5.
  Set $\Delta T, u, p$ from the formulae in Section 4.2.
  Compute the returns and probabilities using Equation 4.10 and Equation 4.11.
  Compute j_{min}, j_{max}, j_0 (Equations 4.12) and shifted returns (Table 2).
  `//` S_i and S_+ denotes the current (i^{th}) and next ($(i+1)^{th}$) list of singular points.
  $S_N \leftarrow \{(NF_{loc}, F_{glob}), (F_{glob}, F_{glob}), (C_{glob}, C_{glob}), (NC_{loc}, C_{glob})\}$ `//` Table 1.
  $S_+ \leftarrow S_N$
  for $i \in \{N-1, \ldots, 0\}$ **do**
    $S_i \leftarrow \emptyset, \qquad L_+ \leftarrow \text{length}(S_+)$
    forall the $(l, j) \in [L_+] \times [j_0 + 1]$ **do**
      Compute $B_{l,j}$ using Equation 4.14, replacing N by i.
      if $(B_{l,j} + R'_k) \notin [Z_N^{l_0}, Z_N^{l_0+1}]$ **then**
        Continue `// to the next item in the loop`
      Find l_0 such that $(B_{l,j} + R'_k) \in [Z_{i+1}^{l_0}, Z_{i+1}^{l_0+1}]$.
      Evaluate $V_{i+1}(B_{l,j} + R'_k)$ by linear interpolation of the extrema.
      Evaluate $V_i(B_{l,j})$ by using Equation 4.13, replacing N by i.
      $S_i \leftarrow S_i \cup (B_{l,j}, V_{N-1}(B_{l,j}))$
    $S_+ \leftarrow S_i$
  return $(S_i)_{1,2}$ `// Singular price at time 0`

Analysis of algorithm [TODO if time permits] Consider computational complexity, space complexity.

4.4.2 Implementation

The algorithm was implemented in Python 3.5.0 (2015-09-13). The system specifications are as follows [TODO: Shift this to some common region for both Asian and cliquet]

<TODO: The actual implementation goes here>

4.5 Numerical results

In this section, we provide some numerical results that we have obtained from our program, along with the reported values of the same from [GZ11, Section 4]. They compare the results with other techniques, namely Monte Carlo and finite differences. The cases considered are as follows:

(1) Constant volatility of $\sigma = 0.2$

(2) Constant volatility of $\sigma = 0.02$, in order to test the efficiency of the methods for low volatility cases

(3) Varying volatility for each observational period, as $\sigma(i) = 0.05 + 0.04i, i = 1, \ldots, 8$ (denoted by *Var* in the σ column of Table 3).

In order to obtain corroboration with the continuous value, we also report the price obtained by using Monte Carlo method with 1,000,000 of trials, along with the values in the 95% confidence interval.

The specifications of the cliquet contract are as follows.

- $F_{loc} = 0, C_{loc} = 0.08, F_{glob} = 0.16, C_{glob} = \infty$
- $T = 5$ years
- $N = 5$
- $r = 0.03$

[TODO: Include computer specifications and working environment.] [TODO: Insert remark 3]

4.6 Conclusions

Table 3. Results

σ	m	Price: Bin	Price: SP	Price: MC	Time (s): Bin	Time (s): SP
	10		0.165661911			
	20		0.172300056			
	50		0.165661911	0.174106		
0.2	100		0.173927464	(0.17398 –		
	200	0.173716366	0.173716366	0.17423)		
	500	0.173922597	0.173922597			
	1000	0.174051949	0.174051949			
	10		0.151234115			
	20		0.149734212			
	50		0.150228984	0.150525		
0.02	100		0.150386306	(0.150472 –		
	200	0.150465004	0.150465004	0.150578)		
	500	0.150508871				
	1000	0.150522368				
	10					
	20					
	50			0.226169		
Var	100			(0.225978 –		
	200	0.225819395		0.226360)		
	500	0.226028534				
	1000	0.226039040				

Chapter 5

Epilogue

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