Pricing exotic path-dependent options

The Singular Points method

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Market models: introduction

- Assets
  - basic assets

    riskless deterministic: $S_t^0 = e^{rt}$

    risky stochastic process: $(S_t)_t$
  - derivatives – contracts on other assets

    futures & forwards symmetric risk

    options asymmetric risk
- Problem: pricing derivatives – finding a fair price
- Assumptions
  - viable / no arbitrage / no free lunch
  - frictionless
  - infinitely divisible assets

Option types

- Simple options – classification of the basis of:

  exercise time European or American
  right of owner call or put

Example (European call)

payoff: $h(S_T) = (S_T - K)_+ = \max\{S_T - K, 0\}$, exercise at maturity

- Exotic options – usually path-dependent

Asian	payoff: function of average of the underlying.
lookback	payoff: function of extrema of the underlying.
cliquet	A series of globally or locally, capped or floored, at-the-money options.
digital	existence depends on pre-set barriers.

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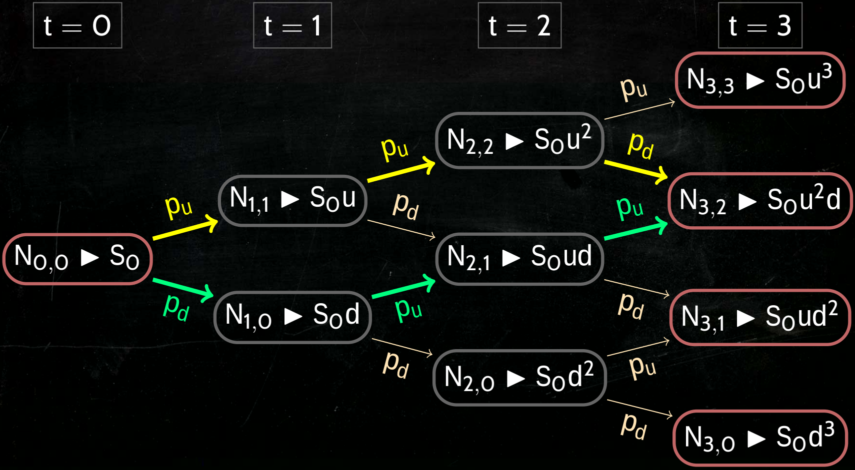
Evolution of risky asset: [CRR79] model (discrete)



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[BS73] model (continuous)

$$\text{riskless} \quad S_t^0 = e^{rt}$$

$$\text{risky} \quad S_t = s_0 e^{(r \;\; \frac{\sigma^2}{2})t+\sigma W_t}$$

Theorem (Convergence of prices from CRR to BS)

Prices of basic assets under CRR $\xrightarrow{d}$ prices of basic assets under BS.

Corollary (Convergence of evaluation formulae)

The previous theorem implies that evaluation formulae under CRR converge in distribution to evaluation formulae for BS.

Approximate BS price by using CRR model.

Quest: Find algorithms with reduced computational complexity.

Market models: discrete vs. continuous

Parameter	Discrete	Continuous
Example	[CRR79]	[BS73]
Theoretical complexity	Easy	Hard
Ease of implementation	Hard	Easy
Closed-form formula	No[1]	Yes
Computational complexity	Hard: $O(2^n)$[1]	Easy: $O(1)$
Universality	Yes	No

[1] CRR: backward recursive

Asian options

Payoff: function of some form of average price.

Average	Discrete	Continuous
AM	$A_n = \frac{1}{n+1} \sum_{i=0}^{n} S_n$	$A_T = \frac{1}{T} \int_0^T S_t \, dt$
GM	$G_n = \left(\prod_{i=0}^{n} S_n\right)^{\frac{1}{n+1}}$	$G_T = \exp\left(\frac{1}{T} \int_0^T \log(S_t) \, dt\right)$

Example (fixed-strike Asian call of European type)

Given strike-price K, payoff $= (A_T - K)_+$, exercised only at maturity.

Asian option

Pre-existing methods

Arithmetic mean

Method	Type	Complex	Remarks
[CRR79]	Tree	$O(2^n)$	simple, accurate, convergence
[HW93]	Tree	$O(n^3)$	accuracy & convergence problems
[BP96]	Tree	$O(n^3)$	accuracy & convergence problems
[Cha+99]	Tree	$O(n^4)$	thin bounds, very large memory
[Vec01]	PDE	$O(n^2)$	not universally applicable
[dFL05]	PDE		more general than [Vec01]

Geometric mean

Closed-form formula exist under BS.

SPs method for Asian options: Introduction

Idea: At any node $N_{i,j}$:

- payoff P: continuous, convex function of the underlying's average A
- number of possible averages = number of paths to $N_{i,j}$ = $\binom{i}{j}$
- these averages completely characterise the payoff; no other payoff being possible under the given tree
- the above points $((A^{l}_{i,j}, P^{l}_{i,j}))_{l}$ are called singular points
- payoff is represented as continuous, convex, and piecewise-linear function of the underlying's possible averages, found by joining the singular points
- minimum possible average $A^{min}_{i,j} = \frac{S_0}{i+1}\left(\frac{1 \quad d^{i-j+1}}{1 \quad d} + d^{i \quad j} u \frac{1 \quad u^{j}}{1 \quad u}\right)$
- maximum possible average $A^{max}_{i,j} = \frac{S_0}{i+1}\left(\frac{1 \quad u^{j+1}}{1 \quad u} + u^{j} d \frac{1 \quad d^{i-j-1}}{1 \quad d}\right)$

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Singular Points method (SPM) for Asian option: Idea

$N_{i,j}$ Node of the binomial tree

$A_{i,j}$ Average upto $N_{i,j}$, $\binom{i}{j}$ choices

$P_{i,j}$ Option price at $N_{i,j}$

$\left\{ \left(A^l_{i,j}, P^l_{i,j} \right) \right\}_l$ singular points (SPs) – completely characterise price

price continuous, convex, piecewise-linear function found by joining SPs

SPM for Asian

Start: at maturity ($i = n$)

At any node $N_{n,j}$, $\exists$ three possibilities:

1. $j \in \{0, n\}$: $(A, (A - K)_+)$
2. $j \notin \{0, n\}$ and $K \notin \left(A^{min}_{n,j}, A^{max}_{n,j}\right)$:

   2.1 $\left(A^{min}_{n,j}, (A^{min}_{n,j} - K)_+\right)$

   2.2 $\left(A^{max}_{n,j}, (A^{max}_{n,j} - K)_+\right)$
3. $j \notin \{0, n\}$ and $K \in \left(A^{min}_{n,j}, A^{max}_{n,j}\right)$:

   3.1 $(A^{min}_{n,j}, 0)$

   3.2 $(K, 0)$

   3.3 $(A^{max}_{n,j}, A^{max}_{n,j} - K)$

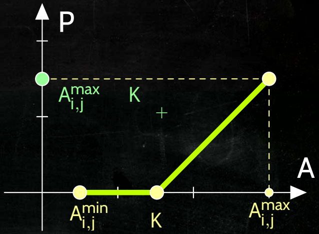

Figure: Case 3

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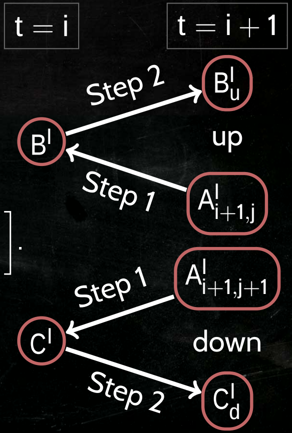
SPM for Asian options: exact price

- Up movement (for $N_{i,j}$):
  1. $\forall A^l_{i+1,j} : B^l = \frac{(i+2)A^l_{i+1;j} - S_{i+1;j}}{i+1}$.
  2. If $B^l \in \left[A^{min}_{i,j}, A^{max}_{i,j}\right] \implies B^l$ is SA.
  3. For each SA, $B^l_u = \frac{(i+1)B^l + S_{i+1;j+1}}{i+2}$.
  4. $v_{i,j}(B^l) =$
     $\frac{1}{R}\left[p_u v_{i+1,j+1}\left(B^l_u\right) + p_d v_{i+1,j}\left(A^l_{i+1,j}\right)\right]$.
  5. $\left(B^l, v_{i,j}(B^l)\right)$ is a SP.
- Down movement (for $N_{i,j}$).
- Aggregate and sort by SAs.
- Repeat for all j.
- Iterate backward till $i = 0$. $P^1_{0,0}$ is the exact binomial price.



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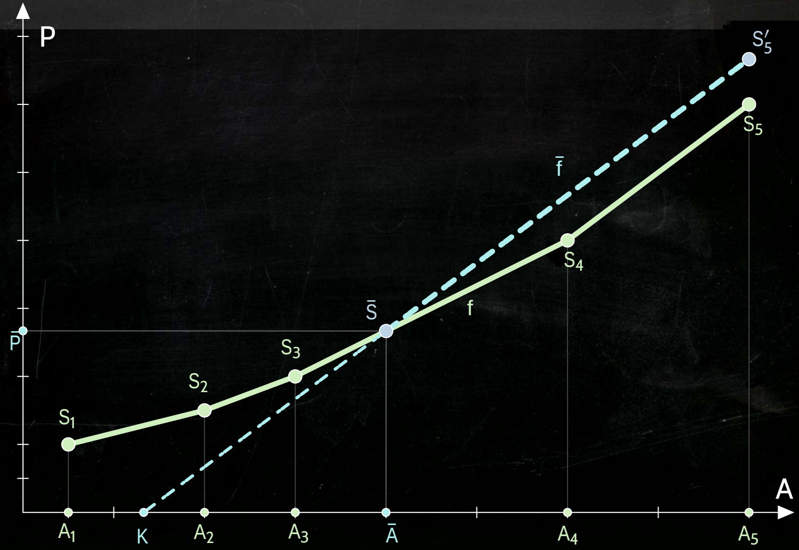
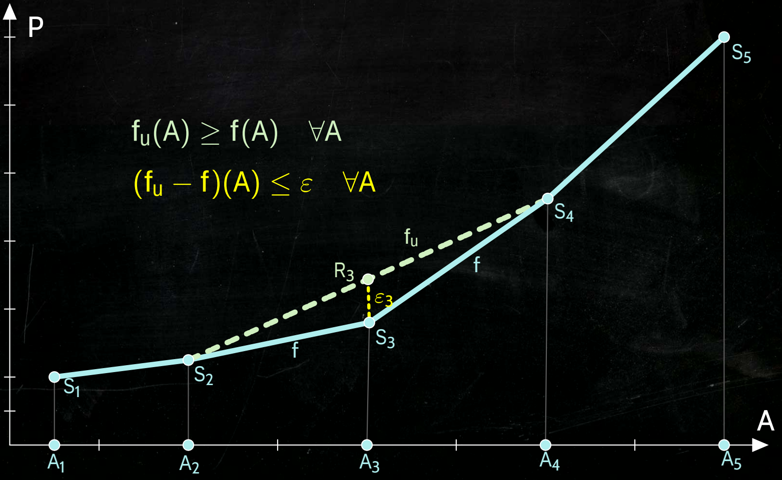
SPM for Asian options: American case



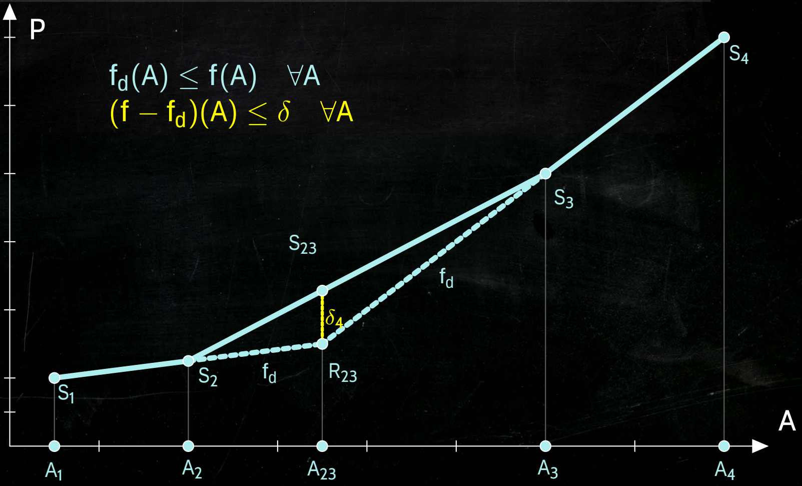
SPM for Asian options: Approximation

Upper estimates



SPM for Asian options: Approximation

Lower estimates



Singular points method for Asian options
Numerical results

Data: $s_0 = 100, T = 1, r = 0.1, q = 0.03.$

		$K = 90$		$K = 110$	
	n	$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.4$
Bin	10	14.5912	17.8033	2.5100	6.6523
	25	15.1535	18.6786	2.6270	7.3451
SP	10	14.5925	17.8068	2.5090	6.6511
	25	15.1535	18.6785	2.6270	7.3449
	50	15.3524	19.0420	2.6673	7.4563
	100	15.4732	19.2696	2.6886	7.5174
	200	15.5453	19.4065	2.6996	7.5502
	400	15.5861	19.4845	2.7053	7.5674

SPM for Asian options: summary

- □ Introduced by Gaudenzi et al [GZA10] in 2010.
- ☑ Convergent to exact CRR and thus BS.
- ☑ Easily generalised to American case and lookback options.
- ☑ Approx: unnecessary points $\implies$ selective removal.
- ☑ Approx: A priori error bounds.
- ☑ Approx $\implies$ fast: reduction of complexity from exponential time to polynomial time (experimental) of $O(n^3)$.
- ☒ Difficult to compute theoretical complexity.
- ☒ Depends on the recombinant nature of the underlying's tree.
- ☒ Not extensible to GM, since the price function is non-linear.

  $$G_u = \left(G^{i+1}S_{i+1,j}\right)^{\frac{1}{i+2}} \propto G^{\frac{i+1}{i+2}}$$
- ☒ Constant volatility assumption $\implies$ local volatility models fail.

Cliquet options: introduction

Definitions

forward start option price option today with payoff $= (S_T - S_u)_+$, $0 \leq u < T$.

cliquet option a series of consecutive at-the-money forward start options, with bounded returns.

Pros and cons

- ☑ Safety against downside risks.
- ☑ Significant upside potential.
- ☒ Unbounded gains not possible.

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Cliquet options: terminology & literature review

Terminology

- Observation times: time points at which the forward start options expire. Assumption: equidistant.
- Return: $R_i = \frac{S_i \quad S_{i-1}}{S_{i-1}} = \frac{S_i}{S_{i-1}} - 1$.
- Running sum: $Z_i = \sum_{k=1}^{i} \max\{F_{loc}, \min\{C_{loc}, R_k\}\}$.
- $\mathrm{Payoff} = \mathrm{notional} \cdot \max\{F_{glob}, \min\{C_{glob}, Z_N\}\}$.

Pre-existing methods for pricing

- No prominent tree-based method.
- [Wil02]: PDE based, FD approach.
- [WFV06]: PDE based, FD approach; generalisations.

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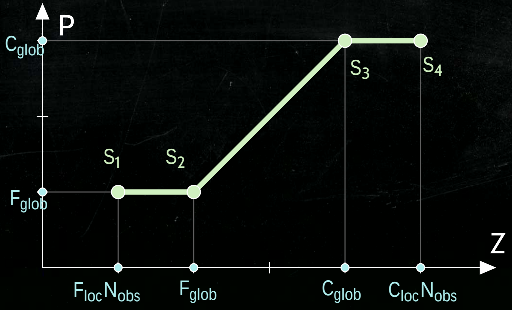
SPM for cliquet options: details

Idea: At the i^{th} interval:

- m: number of computational steps
- 2^m possible paths; probability – binomially distributed.
- Z (running sum) depends on paths and their probabilities.
- $(Z, P) \implies$ SP, where $P \implies$ price function.

Price function at maturity: piecewise-linear, continuous (not convex).



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SPM for cliquet options: other times ($i < N$)

The realizable paths and associated quantities are denoted by primed variables.

1. From running sums Z^{l}_{i+1}, subtract possible returns R^{ℓ}_{j} to get $B_{l,j}$.
2. If $B_{l,j} \in [iF_{loc}, iC_{loc}] \implies$ singular point at time i.
3. Price function: $v_i(B_{l,j}) = e^{\;\frac{rT}{N}} \sum_{j=0}^{j_0} \left[p^{\ell}_{j} v_{i+1}(Z + R^{\ell}_{j})\right]$.
4. Linear combination of piecewise-linear continuous functions is piecewise-linear and continuous. Iterate backward.
5. $v_0(B_{0,0})$ is the exact binomial price.

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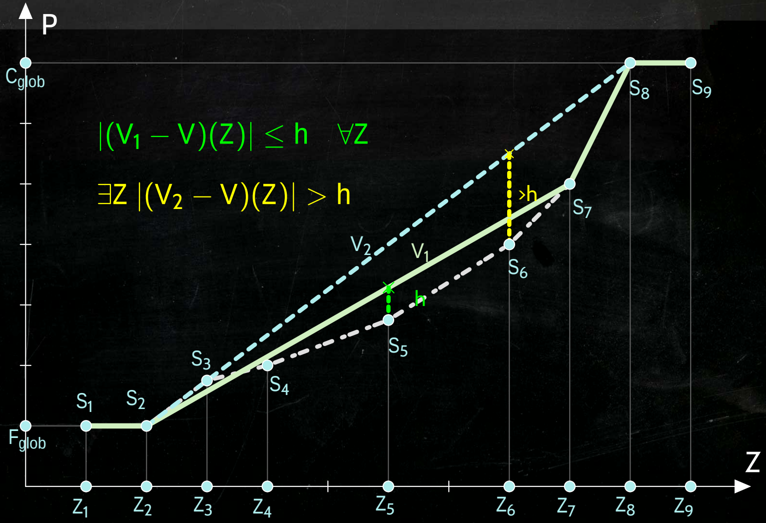
SPM for cliquet options: Approximation



SPM for cliquet options: Numerical results

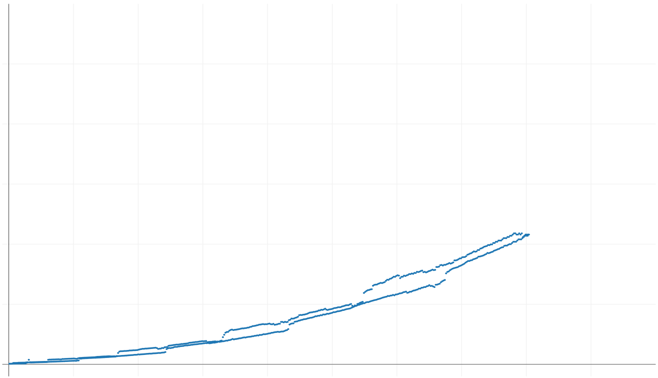
Data

- $F_{loc} = 0, C_{loc} = 0.08, F_{glob} = 0.16, C_{glob} = \infty$
- $T = 5, N = 5, r = 0.03$

σ	m	Price		Time (s) [2]	
		Bin	SP	Bin	SP
0.2	200	0.173716366	0.173716366	0.0165	0.00828
	500	0.173922597	0.173922671	0.0875	0.0437
	1000	0.174051949	0.174051983	2.38	0.183
0.02	200	0.150465004	0.150466828	600	6.09
	500	0.150508871	0.150510526	∞	24.2
	1000	0.150522368	0.150524027	∞	55

[2] ∞ means time taken is more than an hour.

SPM for cliquet options: complexity – $O(m^2)$



SPM for cliquet options: Summary

- □ Introduced by Gaudenzi et al [GZ11] in 2011.
- ☑ Convergent to exact CRR and thus BS.
- ☑ Approximation – A priori error bounds.
- ☑ Significant speed improvement in low volatility cases against binomial model.
- ☑ Can be used for local volatility models and varying interest rates in each period.
- ☑ Fast – experimental order of complexity $O(m^2)$.
- ☒ Difficult to compute theoretical complexity.

Recapitulation

- Efficient technique to evaluate path-dependent exotic options.
- Theory varies with option type.
- Asian: the method is complicated and is not flexible, although being fast and efficient. Easily generalised to the American case and to lookback options. Fails to be generalised for geometric mean and local volatility models.
- Cliquet: flexible method; takes care of local volatility and interest rates. We found out that computational complexity (experimental) is approximately $O(m^2)$ for low m.

Further research

- Theoretical complexity: dependence of singular point redundancy on initial data.
- Customising the method for other path-dependent options.

Questions?

Thank you!

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