Stochastic Differential Equations with Anticipating Initial Conditions

Sudip Sinha

Joint work with
Hui-Hsiung Kuo
Jiayu Zhai

Joint Mathematics Meetings 2020
Denver, CO, USA
2020-01-17

Outline

Section 1
Introduction and motivation

Review and notations

- Let $T \in (0, \infty)$, $t \in [0, T]$, and B_t is a Brownian motion.
- The Itô integral for an adapted process X_t w.r.t. B_t is defined as

$$\int_0^t X_s \,\mathrm{d}B_s = \mathbb{P}\lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} \Delta B_j, \quad \text{where } \Delta_n \text{ is a partition of } [0, t] \text{ and } \Delta B_j = B_{t_{j+1}} - B_{t_j}.$$

- The process $Y_t = \int_0^t X_s \,\mathrm{d}B_s$ is a martingale.
- An Itô process is an adapted process of the form $X_t = X_0 + \int_0^t m_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s$. Equivalently expressed as $\mathrm{d}X_t = m_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t$.
- Itô formula: If X_t is an Itô process and $f(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R})$, then $Y_t = f(t, X_t)$ is also an Itô process given by

$$\mathrm{d}Y_t = \mathrm{d}f(t, X_t) = \partial_t f(t, X_t) \,\mathrm{d}t + \partial_x f(t, X_t) \,\mathrm{d}X_t + \frac{1}{2} \partial_x^2 f(t, X_t) \,(\mathrm{d}X_t)^2.$$

Remarks on the Itô integral

- The integrand must an adapted stochastic process.

- Iterated integrals: $\int_0^T \int_0^T \mathrm{d}B_u \, \mathrm{d}B_v = \int_0^T B_T \, \mathrm{d}B_v = ?$

- What about anticipating stochastic differential equations like

$$\begin{cases} \mathrm{d}X_t = & X_t \, \mathrm{d}B_t \\ X_0 = & B_T \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{d}Y_t = & B_T \, \mathrm{d}B_t \\ Y_0 = & 1 \end{cases} \quad ?$$

- Problem: We want to define $\int_0^T Z_t \, \mathrm{d}B_t$, where Z_t is not (necessarily) adapted.

- Some approaches
  - Enlargement of filtration [Itô78]
  - White noise theory
  - Malliavin calculus
  - Numerous others

Section 2
The generalized integral

Definition of the integral [AK08; AK10]

- A process Y^t and filtration $\mathcal{F}_t$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$. Example: The process $B_T - B_t$ is instantly independent of the filtration generated by B_t.

- Idea

  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.

- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_t$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is defined as

$$\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \mathbb{P} \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j.$$

- Now, for any stochastic process $Z(t) = \sum_{k=1}^{n} X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity. This is well-defined [HKS+16].

A simple example

$$\int_0^t B_T \, \mathrm{d}B_s = \int_0^t (B_s + (B_T - B_s)) \, \mathrm{d}B_s = \int_0^t B_s \, \mathrm{d}B_s + \int_0^t (B_T - B_s) \, \mathrm{d}B_s$$

$$= L^2 \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + L^2 \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j$$

$$= L^2 \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} \left(B_T - \Delta B_j\right) \Delta B_j$$

$$= B_T \cdot L^2 \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} \Delta B_j - L^2 \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} (\Delta B_j)^2$$

$$= B_T B_t - t$$

The near-martingale property

- Question: What is the analogues of the martingale property?
- Example: $\mathbb{E}(B_T B_t - t \mid \mathcal{F}_s) = B_s^2 - s \neq B_T B_s - s$. ☹
  But $\mathbb{E}(B_T B_s - s \mid \mathcal{F}_s) = B_s^2 - s$. ☺
- Let Z_t be a process such that $\mathbb{E}\,|Z_t| < \infty$ $\forall t$, and $0 \le s \le t \le T$. Then Z_t is called a near-martingale if $\mathbb{E}(Z_t \mid \mathcal{F}_s) = \mathbb{E}(Z_s \mid \mathcal{F}_s)$.

Theorem 1 ([KSS12b]) *Let f and ϕ be continuous functions on $\mathbb{R}$. Under integrability conditions, the processes $X_t = \int_0^t f(B_t)\phi(B_T - B_t)\, dB_t$ and $Y^t = \int_t^T f(B_t)\phi(B_T - B_t)\, dB_t$ are near-martingales.*

Theorem 2 ([HKS+17]) *Let Z_t be a stochastic process bounded in L^1, and $X_t = \mathbb{E}(Z_t \mid \mathcal{F}_t)$. Then X_t is a martingale if and only if Z_t is a near-martingale.*

The general Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_t = X_0 + \int_0^t m_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s$	$\mathrm{d}X_t = \quad m_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t$
instantly independent	$Y^t = Y^T + \int_t^T \eta^s \, \mathrm{d}s + \int_t^T \varsigma^s \, \mathrm{d}B_s$	$\mathrm{d}Y^t = -\eta^t \, \mathrm{d}t - \varsigma^t \, \mathrm{d}B_t$

Here η^t and ς^t are instantly independent such that Y^t is also instantly independent.

Theorem 3 ([HKS+16]) *Let $dX_t = m_t \, dt + \sigma_t \, dB_t$ be an Itô process, and $dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$ be a instantly independent process. If $f(t,x,y) \in C^{1,2,2}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, then*

$$\begin{aligned} df(t, X_t, Y^t) = \partial_t f(t, X_t, Y^t) \, dt &+ \partial_x f(t, X_t, Y^t) \, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t, Y^t)(dX_t)^2 \\ &+ \partial_y f(t, X_t, Y^t) \, dY^t - \frac{1}{2}\partial_y^2 f(t, X_t, Y^t)(dY^t)^2. \end{aligned}$$

Section 3
Conditional expectation

Linear stochastic differential equations

Definition 4 *Define the exponential process with parameters α and β by*

$$\mathcal{E}_t^{(\alpha,\beta)} = \exp\left(\int_0^t \alpha_s \, dB_s + \int_0^t \left(\beta_s - \frac{1}{2}\alpha_s^2 \right) ds \right).$$

Theorem 5 ([HKS+16]) *The solution of the stochastic differential equation*

$$\begin{cases} dX_t = & \alpha_t X_t \, dB_t + \beta_t X_t \, dt \\ X_0 = & \psi(B_T) \end{cases}$$

is given by $X_t = \psi\left(B_T - \int_0^t \alpha_s \, ds \right) \mathcal{E}_t^{(\alpha,\beta)}$.

Motivating question

What can we say about the conditional expectation of the solution of the stochastic differential equation

$$\begin{cases} dX_t = & \alpha_t X_t \, dB_t + \beta_t X_t \, dt \\ X_0 = & \psi(B_T) \end{cases} \qquad ?$$

In particular, if $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$, can we expect Y_t to be the solution of the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t \, dB_t + \beta_t Y_t \, dt \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases} \qquad ?$$

Unexpected behaviour

Theorem 6 ([KSZ18]) *Suppose $\alpha_t \in L^2[0,T]$, β_t is adapted with $\mathbb{E}\int_0^T |\beta_t|^2\, dt < \infty$, and $\psi : \mathbb{R} \to \mathbb{R}$ has power series expansion at 0 with infinite radius of convergence, and ψ' denotes the derivative of ψ. Consider the two stochastic differential equations*

$$\begin{cases} dX_t = & \alpha_t X_t\, dB_t + \beta_t X_t\, dt \\ X_0 = & \psi(B_T) \end{cases} \quad \text{and} \quad \begin{cases} d\overline{X}_t = & \alpha_t \overline{X}_t\, dB_t + \beta_t \overline{X}_t\, dt \\ \overline{X}_0 = & \psi'(B_T) \end{cases}.$$

Denote $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$ and $\overline{Y}_t = \mathbb{E}(\overline{X}_t \mid \mathcal{F}_t)$.
Then Y_t satisfies the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t\, dB_t + \beta_t Y_t\, dt + \overline{Y}_t\, dB_t \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases}.$$

A brief detour: Hermite polynomials

- An Hermite polynomial of degree n with parameter ρ is given by

$$H_n(x;\rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \partial_x^n e^{-\frac{x^2}{2\rho}}.$$

- The first few Hermite polynomials are: $1, x, x^2 - \rho, x^3 - 3\rho x, x^4 - 6\rho x^2 + 3\rho^2, \ldots$.

- Hermite polynomials form an orthonormal basis of $L^2(\mathbb{R}, \gamma)$, where γ is the Gaussian measure with mean 0 and variance ρ.

- For fixed $n \in \mathbb{N}$, the stochastic process $X_t = H_n(B_t; t)$ is a martingale, and

$$\mathrm{d}X_t = nH_{n-1}(B_t; t)\, \mathrm{d}B_t.$$

Initial condition: Hermite polynomials

Theorem 7 ([KSZ18]) *Suppose $\alpha_t \in L^2[0,T]$, β_t is adapted with $\mathbb{E}\int_0^T |\beta_t|^2\,dt < \infty$, and let n be a fixed natural number. Let X_t be the solution of*

$$\begin{cases} dX_t = & \alpha_t X_t\,dB_t + \beta_t X_t\,dt \\ X_0 = & H_n(B_T;\,T), \end{cases}$$

and $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$.
Then Y_t satisfies the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t\,dB_t + \beta_t Y_t\,dt + nH_{n-1}\Big(B_t - \int_0^t \alpha_s\,ds;\ t\Big)\mathcal{E}_t^{(\alpha,\beta)}\,dB_t \\ Y_0 = & 0 \end{cases},$$

and is explicitly given by

$$Y_t = H_n\left(B_t - \int_0^t \alpha_s\,ds;\ t\right)\mathcal{E}_t^{(\alpha,\beta)}.$$

Initial condition: differentiable function in $L^2(\mathbb{R}, \gamma)$

Theorem 8 ([KSZ18]) *Suppose $\alpha_t \in L^2[0,T]$, β_t is adapted with $\mathbb{E}\int_0^T |\beta_t|^2\, dt < \infty$.*
Let $\psi(x) = \sum_{n=0}^{\infty} c_n H_n(x;T)$ be a differentiable function in $L^2(\mathbb{R},\gamma)$.
Consider the two stochastic differential equations

$$\begin{cases} dX_t = & \alpha_t X_t\, dB_t + \beta_t X_t\, dt \\ X_0 = & \psi(B_T) \end{cases} \quad \text{and} \quad \begin{cases} d\overline{X}_t = & \alpha_t \overline{X}_t\, dB_t + \beta_t \overline{X}_t\, dt \\ \overline{X}_0 = & \psi'(B_T) \end{cases}.$$

Denote $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$ and $\overline{Y}_t = \mathbb{E}(\overline{X}_t \mid \mathcal{F}_t)$.
Then Y_t satisfies the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t\, dB_t + \beta_t Y_t\, dt + \overline{Y}_t\, dB_t \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases},$$

and is explicitly given by

$$Y_t = \sum_{n=0}^{\infty} c_n H_n\left(B_t - \int_0^t \alpha_s\, ds;\ t \right) \mathcal{E}_t^{(\alpha,\beta)}.$$

Section 4
A larger class of initial conditions

Initial condition: Wiener integral

Question: Can we extend the class of initial conditions?

Theorem 9 ([KSZ18]) *Let* $\alpha_t \in L^2[0,T], \beta_t \in L^1[0,T], h_t \in L^2[0,T], \psi(t) \in C^2(\mathbb{R})$. *Then the (unique) solution of the stochastic differential equation*

$$\begin{cases} dX_t = & \alpha_t X_t \, dB_t + \beta_t X_t \, dt \\ X_0 = & \psi\left(\int_0^T h_s \, dB_s\right) \end{cases}$$

is given by

$$X_t = \psi\left(\int_0^T h_s \, dB_s - \int_0^t \alpha_s h_s \, ds\right) \mathcal{E}_t^{(\alpha,\beta)}.$$

An example

Consider the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = & X_t \,\mathrm{d}B_t \\ X_0 = & \psi\left(\int_0^T B_s \,\mathrm{d}s\right) \end{cases} .$$

Using Itô lemma, we rewrite $\int_0^T B_s \,\mathrm{d}s = \int_0^T (T-s)\,\mathrm{d}B_s$.
Now using the previous theorem, we get

$$X_t = \psi\left(\int_0^T B_s \,\mathrm{d}s - \left(Tt - \frac{1}{2}t^2\right)\right) e^{B_t - \frac{1}{2}t}.$$

Thank you!

Appendix

Iterated integrals

Theorem 10 ([Itô51]) *Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then*

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1,\dots,t_n)\, dB_{t_n} \right) dB_{t_{n-1}} \dots dB_{t_1}.$$

Theorem 11 ([AK10]) *Let $f \in L^2([0,T]^n)$. Then*

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n} = \int_0^T \cdots \int_0^T f(t_1,\dots,t_n)\, dB_{t_n} \dots dB_{t_1}.$$

Example[HKS+16]: For the new integral, $\int_0^T \left(\int_0^T B_u \, \mathrm{d}u \right) \mathrm{d}B_v = \int_0^T \left(\int_0^T B_u \, \mathrm{d}B_v \right) \mathrm{d}u$.

A generalization of Itô isometry

Theorem 12 ([KSS12b]) *Let ϕ be an analytic function on $\mathbb{R}$. Then under integrability conditions and for each t,*

$$\mathbb{E}\left[\left(\int_0^t \phi(B_T - B_s)\, dB_s\right)^2\right] = \int_0^t \mathbb{E}\left[(\phi(B_T - B_s))^2\right] ds$$

Theorem 13 ([KSS13]) *Let f and ϕ be C^1 functions on $\mathbb{R}$. Then*

$$\mathbb{E}\left[\left(\int_0^T f(B_t)\phi(B_T - B_t)\, dB_t\right)^2\right] = \int_0^T \mathbb{E}\left[(f(B_t)\phi(B_T - B_t))^2\right] dt$$
$$+ 2\int_0^T \int_0^t \mathbb{E}\left[f(B_s)\phi'(B_T - B_s)f'(B_s)\phi(B_T - B_s)\right] ds\, dt.$$

A generalization of Girsanov theorem

Theorem 14 ([KPS13]) *Let X_t and Y^t be continuous square-integrable stochastic processes such that X_t is adapted and Y^t is instantly independent.*
Then the translated stochastic process $W_t = B_t - \int_0^t (X_t + Y^t)\, dt$ is a near-martingale under the probability measure $\widetilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(X+Y,0)}$.

Thank you!

Bibliography

Ayed, W. & Kuo, H. H. (2008). An extension of the Itô integral. *Communications on Stochastic Analysis*, *2*(3). doi:10.31390/cosa.2.3.05

——— (2010). An extension of the Itô integral: Toward a general theory of stochastic integration. *Theory of Stochastic Processes*, *16*(32), 17–28. Retrieved from http://mi.mathnet.ru/thsp56

Hwang, C. R., Kuo, H. H., Saitô, K., & Zhai, J. (2016). A general Itô formula for adapted and instantly independent stochastic processes. *Communications on Stochastic Analysis*, *10*(3). doi:10.31390/cosa.10.3.05

——— (2017). Near-martingale Property of Anticipating Stochastic Integration. *Communications on Stochastic Analysis*, *11*(4). doi:10.31390/cosa.11.4.06

Itô, K. (1951). Multiple Wiener Integral. *J. Math. Soc. Japan*, *3*(1), 157–169. doi:10.2969/jmsj/00310157

——— (1978). Extension of stochastic integrals. In Extension of stochastic integrals., *Proceedings of the International Symposium on Stochastic Differential Equations*. Kinokuniya.

Kuo, H. H., Peng, Y., & Szozda, B. (2013). Generalization of the anticipative Girsanov theorem. *Communications on Stochastic Analysis*, *7*(4). doi:10.31390/cosa.7.4.06

Kuo, H. H., Sae-Tang, A., & Szozda, B. (2012b). A stochastic integral for adapted and instantly independent stochastic processes. In A stochastic integral for adapted and instantly independent stochastic processes., *Stochastic Processes, Finance and Control*. (Vol. 29, pp. 53–71). Author. doi:10.1142/9789814383318_0003

——— (2013). And isometry formula for a new stochastic integral. In And isometry formula for a new stochastic integral., *Quantum Probability and Related Topics*. (Vol. 29, pp. 222–232). Author. doi:10.1142/9789814447546_0014

Kuo, H. H., Sinha, S., & Zhai, J. (2018). Stochastic Differential Equations with Anticipating Initial Conditions. *Communications on Stochastic Analysis*, *12*(4). doi:10.31390/cosa.12.4.06