Stochastic Differential Equations with Anticipating Initial Conditions

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Joint Mathematics Meetings 2020
Denver, CO, USA
2020-01-17

Outline

Section 1
Introduction and motivation

Quick revision and notations

- Let $T \in (0, \infty)$, $t \in [0, T]$, and $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ be a filtered probability space with a Brownian motion $B_\bullet$.

- The Itô integral

  1. For adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$, define $\int_0^T X_t \, \mathrm{d}B_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
  2. For adapted process $X_\bullet$ such that $\int_0^T |X_t|^2 \, \mathrm{d}t < \infty$ a.s., use adapted step processes approximating X to extend the integral using limit in probability.

- An Itô process is a process of the form $X_\bullet = X_0 + \int_0^\bullet \alpha_t \, \mathrm{d}t + \int_0^\bullet \beta_t \, \mathrm{d}B_t$, equivalently expressed as $\mathrm{d}X_t = \alpha_t \, \mathrm{d}t + \beta_t \, \mathrm{d}B_t$.

- Itô formula: If $X_\bullet$ is an Itô process and $f(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R})$, then $f(t, X_t)$ is also an Itô process given by $\mathrm{d}f(t, X_t) = \mathrm{D}_t f(t, X_t) \, \mathrm{d}t + \mathrm{D}_x f(t, X_t) \, \mathrm{d}X_t + \frac{1}{2} \mathrm{D}_x^2 f(t, X_t) \, (\mathrm{d}X_t)^2$, where $(\mathrm{d}B_t)^2 = \mathrm{d}t$ with all other products being zero.

Limitations of the Itô integral

- Adaptedness of the integrand is a primary requirement in Itô theory.
- Iterated integrals: Consider the iterated integral $\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = \int_0^t B_t \, \mathrm{d}B_v \stackrel{?}{=} B_t^2$.
- Note that $\mathbb{E}(B_t^2) = t \neq 0$, so no martingale property ☹.
- Stochastic differential equations with anticipation

$$\begin{cases} \mathrm{d}X_t = & X_t \, \mathrm{d}B_t \\ X_0 = & B_T \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{d}Y_t = & B_T \, \mathrm{d}B_t \\ Y_0 = & 1 \end{cases}.$$

- Problem: We want to define $\int_0^T Z(t) \, \mathrm{d}B_t$, where $Z(\bullet)$ is not (necessarily) adapted.
- Some approaches
  - Enlargement of filtration [Itô78]
  - White noise theory
  - Malliavin calculus
  - Numerous others

Section 2
The generalized integral

Definition of the integral [AK08; AK10]

- A process $Y^{\bullet}$ and filtration $\mathcal{F}_{\bullet}$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$. Example: The process $(B_T - B_{\bullet})$ is instantly independent of the filtration generated by $B_{\bullet}$.
- Idea
  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.
- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\bullet}$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is defined as

$$\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \mathbb{P}\lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j.$$

- Now, for any stochastic process $Z(t) = \sum_{k=1}^{n} X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity. This is well-defined [HKS+16].

A simple example

$$\int_0^t B_T \,\mathrm{d}B_t = \int_0^t (B_t + (B_T - B_t)) \,\mathrm{d}B_t = \int_0^t B_t \,\mathrm{d}B_t + \int_0^t (B_T - B_t) \,\mathrm{d}B_t$$

$$= L^2 \lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + L^2 \lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j$$

$$= L^2 \lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} \left(B_T - \Delta B_j\right) \Delta B_j$$

$$= B_T \cdot L^2 \lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} \Delta B_j - L^2 \lim_{\|\Delta_n\|\to 0} \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t.$$

- ▷ Note that $\mathbb{E}(B_T B_t - t) = 0$.
- ▷ In general, $\mathbb{E} \int_0^t Z(s) \,\mathrm{d}B_s = 0$. ☺

The general Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_{\bullet} = X_0 + \int_0^{\bullet} \alpha_t \,\mathrm{d}t + \int_0^{\bullet} \beta_t \,\mathrm{d}B_t$	$\mathrm{d}X_t = \alpha_t \,\mathrm{d}t + \beta_t \,\mathrm{d}B_t$
instantly independent	$Y^{\bullet} = Y^0 + \int_{\bullet}^{T} \eta^t \,\mathrm{d}t + \int_{\bullet}^{T} \varsigma^t \,\mathrm{d}B_t$	$\mathrm{d}Y^t = -\eta^t \,\mathrm{d}t - \varsigma^t \,\mathrm{d}B_t$

Here η^t and ς^t are instantly independent such that Y^t is also instantly independent.

Theorem 1 ([HKS+16]) *Let $dX_t = \alpha_t \, dt + \beta_t \, dB_t$ be an Itô process, and $dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$ be a instantly independent process. If $f(t,x,y) \in C^{1,2,2}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, then*

$$
\begin{aligned}
df(t, X_t, Y^t) = {} & D_t f(t, X_t, Y^t)\, dt + D_x f(t, X_t, Y^t)\, dX_t + \frac{1}{2} D_x^2 f(t, X_t, Y^t) (dX_t)^2 \\
& + D_y f(t, X_t, Y^t)\, dY^t - \frac{1}{2} D_y^2 f(t, X_t, Y^t)(dY^t)^2,
\end{aligned}
$$

where $(dB_t)^2 = dt$ with all other products being zero.

Section 3
Conditional expectation

Motivating question

What can we say about the conditional expectation of the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = & \alpha_t X_t \,\mathrm{d}B_t + \beta_t X_t \,\mathrm{d}t \\ X_0 = & \psi(B_T) \end{cases} \qquad ?$$

In particular, if $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$, can we expect $Y_\bullet$ to be the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}Y_t = & \alpha_t Y_t \,\mathrm{d}B_t + \beta_t Y_t \,\mathrm{d}t \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases} \qquad ?$$

Linear stochastic differential equations

Definition 2 *Define the exponential process with parameters α and β by*

$$\mathcal{E}_t^{(\alpha,\beta)} = \exp\left(\int_0^t \alpha_s \, dB_s + \int_0^t \left(\beta_s - \frac{1}{2}\alpha_s^2 \right) ds \right).$$

Theorem 3 ([HKS+16]) *The solution of the stochastic differential equation*

$$\begin{cases} dX_t = & \alpha_t X_t \, dB_t + \beta_t X_t \, dt \\ X_0 = & \psi(B_T) \end{cases}$$

is given by $X_t = \psi\left(B_T - \int_0^t \alpha_s \, ds\right) \mathcal{E}_t^{(\alpha,\beta)}$.

Unexpected behaviour

Theorem 4 ([KSZ18]) *Suppose $\alpha_{\bullet} \in L^2[0,T]$, $\beta_{\bullet}$ is adapted with $\mathbb{E}\int_0^T |\beta_t|^2\, dt < \infty$, and $\psi : \mathbb{R} \to \mathbb{R}$ has power series expansion at 0 with infinite radius of convergence, and ψ' denotes the derivative of ψ. Consider the two stochastic differential equations*

$$\begin{cases} dX_t = & \alpha_t X_t\, dB_t + \beta_t X_t\, dt \\ X_0 = & \psi(B_T) \end{cases} \quad \text{and} \quad \begin{cases} d\overline{X}_t = & \alpha_t \overline{X}_t\, dB_t + \beta_t \overline{X}_t\, dt \\ \overline{X}_0 = & \psi'(B_T) \end{cases}.$$

Denote $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$ and $\overline{Y}_t = \mathbb{E}(\overline{X}_t \mid \mathcal{F}_t)$.
Then $Y_{\bullet}$ satisfies the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t\, dB_t + \beta_t Y_t\, dt + \overline{Y}_t\, dB_t \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases}.$$

A brief detour: Hermite polynomials

- An Hermite polynomial of degree n with parameter ρ is given by

$$H_n(x;\rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \mathrm{D}_x^n e^{-\frac{x^2}{2\rho}}.$$

- Some useful equalities for Hermite polynomials:

  1. $\mathrm{D}_x H_n(x;\rho) = nH_{n-1}(x;\rho)$
  2. $\mathrm{D}_x^2 H_n(x;\rho) = -2\,\mathrm{D}_\rho H_n(x;\rho)$
  3. $H_n(x+y;\rho) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x;\rho) y^k$

- For fixed $n \in \mathbb{N}$, the stochastic process $X_t = H_n(B_t;t)$ is a martingale, and

$$\mathrm{d}X_t = nH_{n-1}(B_t;t)\,\mathrm{d}B_t.$$

- Hermite polynomials form an orthonormal basis of $L^2(\mathbb{R},\gamma)$, where γ is the Gaussian measure with mean 0 and variance ρ.

Initial condition: Hermite polynomials

Theorem 5 ([**KSZ18**]) *Suppose* $\alpha_\bullet \in L^2[0,T]$, $\beta_\bullet$ *is adapted with* $\mathbb{E}\int_0^T |\beta_t|^2\,dt < \infty$, *and let* n *be a fixed natural number. Let* $X_\bullet$ *be the solution of*

$$\begin{cases} dX_t = & \alpha_t X_t\,dB_t + \beta_t X_t\,dt \\ X_0 = & H_n(B_T;\,T), \end{cases}$$

and $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$.
Then $Y_\bullet$ *satisfies the stochastic differential equation*

$$\begin{cases} dY_t = & \left[\alpha_t Y_t + nH_{n-1}\left(B_t - \int_0^t \alpha_s\,ds;\ t\right)\mathcal{E}_t^{(\alpha,\beta)}\right] dB_t + \beta_t Y_t\,dt \\ Y_0 = & 0 \end{cases},$$

and is explicitly given by

$$Y_t = H_n\left(B_t - \int_0^t \alpha_s\,ds;\ t\right)\mathcal{E}_t^{(\alpha,\beta)}.$$

Initial condition: $\psi \in L^2(\mathbb{R}, \gamma)$ with ψ differentiable

Theorem 6 (**[KSZ18]**) *Suppose $\alpha_\cdot \in L^2[0,T]$, $\beta_\cdot$ is adapted with $\mathbb{E}\int_0^T |\beta_t|^2\, dt < \infty$. Let $\psi(x) = \sum_{n=0}^\infty c_n H_n(x;T)$ be a differentiable function in $L^2(\mathbb{R},\gamma)$. Consider the two stochastic differential equations*

$$\begin{cases} dX_t = & \alpha_t X_t\, dB_t + \beta_t X_t\, dt \\ X_0 = & \psi(B_T) \end{cases} \quad \text{and} \quad \begin{cases} d\overline{X}_t = & \alpha_t \overline{X}_t\, dB_t + \beta_t \overline{X}_t\, dt \\ \overline{X}_0 = & \psi'(B_T) \end{cases}.$$

Denote $Y_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$ and $\overline{Y}_t = \mathbb{E}(\overline{X}_t \mid \mathcal{F}_t)$. Then $Y_\cdot$ satisfies the stochastic differential equation

$$\begin{cases} dY_t = & \alpha_t Y_t\, dB_t + \beta_t Y_t\, dt + \overline{Y}_t\, dB_t \\ Y_0 = & \mathbb{E}\psi(B_T) \end{cases},$$

and is explicitly given by

$$Y_t = \sum_{n=0}^{\infty} c_n H_n\left(B_t - \int_0^t \alpha_s\, ds;\ t\right) \mathcal{E}_t^{(\alpha,\beta)}.$$

Section 4

A larger class of initial conditions

Wiener integrals as initial conditions

Question: Can we extend the class of initial conditions?

Theorem 7 ([KSZ18]) *Let $\alpha_\bullet \in L^2[0,T], \beta_\bullet \in L^1[0,T], h_\bullet \in L^2[0,T], \psi(\cdot) \in C^2(\mathbb{R})$. Then the (unique) solution of the stochastic differential equation*

$$\begin{cases} dX_t = & \alpha_t X_t \, dB_t + \beta_t X_t \, dt \\ X_0 = & \psi\left(\int_0^T h_s \, dB_s\right) \end{cases}$$

is given by

$$X_t = \psi\left(\int_0^T h_s \, dB_s - \int_0^t \alpha_s h_s \, ds\right) \mathcal{E}_t^{(\alpha,\beta)}.$$

A simple example

Consider the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = & X_t\,\mathrm{d}B_t \\ X_0 = & \psi\left(\int_0^T B_s\,\mathrm{d}s\right) \end{cases}.$$

Using Itô lemma, we rewrite $\int_0^T B_s\,\mathrm{d}s = \int_0^T (T-s)\,\mathrm{d}B_s$, we get

$$X_t = \psi\left(\int_0^T B_s\,\mathrm{d}s - \left(Tt - \frac{1}{2}t^2\right)\right) e^{B_t - \frac{1}{2}t}.$$

Thank you!

Appendix

Misc results on nearmartingales, Girsanov theorem, exponential processes

1.

2.

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Anticipated questions

1. Look at Theorem 4.7 of [HKS+17]!
2. How to prove the uniqueness of solution stochastic differential equations? Standard method?
3. In Section 4, how much can we extend? In particular, can we have an Itô or anticipating integral? Is that even meaningful?
4. In Section 3, what is the difference between ψ analytic and $\psi : \mathbb{R} \to \mathbb{R}$ has power series expansion at 0 with infinite radius of convergence?
5. In Section 3, why do we need α to be deterministic?
6. In Section 3, why does the extra term appear intuitively?
7. What results go in the Appendix?
8. Applications apart from modeling insider trading in finance
9. Better motivation for the conditional expectation part