Mathematical Logic

Notes and Exercises

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Contents

1 Sudip Sinha PHIL 4010: HW1 2019-09-10

Exercise 1.1 **(Notes, 1.8)** *For any sets A and B, we have $A \cap B \subseteq A$.*

Solution. Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A, we have $A \cap B \subseteq A$. □

Exercise 1.2 **(Notes, 1.10)** *For any set A, we have $A \cap \emptyset = \emptyset$.*

Solution. ($\subseteq$) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.
($\supseteq$) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. □

Exercise 1.3 **(Notes, 1.13)** *For any sets A and B, if $A \subseteq B$, then $A \cup B = B$.*

Solution. ($\subseteq$) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.
($\supseteq$) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

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Note: We shall say that a truth assigment v satisfies Σ iff it satisfies every member of Σ.

Exercise 2.1 (Enderton, 1.2.1) *Show that neither of the following two formulas tautologically implies the other:*

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Solution. We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$.

Consider v such that $v(A) = v(B) = F$ and $v(C) = T$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below. (Here the truth assignments by $\bar{v}$ is denoted under each symbol.)

$$\begin{aligned} &\alpha = (A \leftrightarrow (B \leftrightarrow C)) \\ &T \quad\ \ F\ \ T\quad F\ \ F\ \ T \end{aligned}$$

$$\begin{aligned} &\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C)))) \\ &F \qquad F\ F \qquad\qquad F \qquad\qquad F \qquad\qquad F \quad FT \end{aligned}$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = T$ and $\bar{v}(\alpha) = F$.

Consider v such that $v(A) = v(B) = v(C) = F$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below.

$$\begin{aligned} &\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C)))) \\ &T = \qquad\qquad\qquad\qquad T \quad TF \quad T \quad TF \quad T \quad TF \end{aligned}$$

$$\begin{aligned} &\alpha = (A \leftrightarrow (B \leftrightarrow C)) \\ &F = \ \ F\ \ F\quad F\ \ T\ \ F \end{aligned}$$

$\square$

Exercise 2.2 (Enderton, 1.2.4(a)) *Show that $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \to \beta)$.*

Solution. We show each direction separately.

($\Longrightarrow$) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ. We have to show that v satisfies $(\alpha \to \beta)$. We have two cases.

i. $\bar{v}(\alpha) = T$: In this case, from the supposition, we get $\bar{v}(\beta) = T$. So $\bar{v}(\alpha \to \beta) = T$.

ii. $\bar{v}(\alpha) = F$: In this case, $\bar{v}(\alpha \to \beta) = T$ since the antecedent is F.

Since v was arbitrary, we have $\Sigma \models (\alpha \to \beta)$.

($\Longleftarrow$) We suppose $\Sigma \models (\alpha \to \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β. Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ. Therefore, by our supposition, v satisfies $(\alpha \to \beta)$. Now, since v satisfies α, it can only be that v satisfies β, since the only other way the material implication can be satisfied is when v does not satisfies α. This proves our claim. □

Exercise 2.3 (Enderton, 1.2.5) *Prove or refute each of the following assertions:*

a. If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.

Solution. (**T**) There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α. This implies $\bar{v}(\alpha \vee \beta) = T$ by the definition of extension of $\bar{v}$ for $\vee$. □

b. If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.

Solution. (**F**) We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \vee (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$.

For an explicit example, consider two truth assignments v_1 and v_2, such that $v_1(\alpha) = T$ and $v_2(\alpha) = F$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α, and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$. □

Exercise 2.4 (Enderton, 1.2.6)

a. *Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.*

Solution. Let G be the set of sentence symbols used in α, and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$.

Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α.

Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ. Let $\boxdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for $\bar{v}_1$ and $\bar{v}_2$, we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \boxdot \psi) = \bar{v}_2(\phi \boxdot \psi)$. Hence $(\neg\phi), (\phi \boxdot \psi) \in B$, that is, B is closed with respect to the formula building operations.

Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done. □

b. *Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ.*

Solution. In this part, we use v to denote truth assignments and "v on a set" means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ. Clearly, $G \subseteq S$.

We show each direction separately.

($\Longrightarrow$) From the definition of tautological implication,

$$\begin{aligned} & \Sigma \models \tau \\ \iff & (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \Longrightarrow & (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \; [\text{Part (a)}] \end{aligned}$$

($\Longleftarrow$) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ. Therefore,

$$\begin{aligned} & (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \Longrightarrow & (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \iff & \Sigma \models \tau \end{aligned}$$

□

3 Sudip Sinha PHIL 4010: Prelim 2019-10-08

Exercise 3.1 (Set Theory) *Prove the following. 10 points each.*

Note: *Let A and B are sets. In order to prove $A = B$, it is enough to show $A \subseteq B$ and $A \supseteq B$. In each of the following problems, we show each inclusion separately.*
Moreover, to show $A \subseteq B$, it suffices to show that for x arbitrary, $x \in A \implies x \in B$.

i. *If $A \subseteq B$, then $A \cap B = A$.*

Solution.

($\subseteq$) Let x be arbitrary. Then
$$x \in A \cap B \iff x \in A \text{ and } x \in B \implies x \in A.$$

($\supseteq$) Let $x \in A$ be arbitrary. Then by the hypothesis $x \in B$ since $A \subseteq B$. Therefore, $x \in A$ and $x \in B$, and thus $x \in A \cap B$.

□

ii. *If $A \cap B = \emptyset$, then $A \setminus B = A$.*

Solution.

($\subseteq$) Let $x \in A \setminus B$ be arbitrary. Then $x \in A$ and $x \notin B$. It is enough to show that $x \in A$ implies $x \notin B$. But must be true since if $x \in A$ and $x \in B$, then $x \in A \cap B = \emptyset$, which is absurd.

($\supseteq$) Let $x \in A$ be arbitrary. Now, either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A \cap B$ since $x \in A$ by hypothesis. But this is an impossibility since $A \cap B = \emptyset$. Therefore, it must be that $x \notin B$. So $x \in A \setminus B$.

□

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

($\subseteq$) Let $x \in A \cap (B \cup C)$ be arbitrary. Then $x \in A$ and $x \in B \cup C$. Note that $x \in B \cup C$ means $x \in B$ or $x \in C$. Now, either $x \in B$ or $x \notin B$, so have two cases.

- ($x \in B$) In this case, $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.

- ($x \notin B$) Since $x \in B$ or $x \in C$, and $x \notin B$, it is necessary that $x \in C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

(⊇) Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. This means $x \in A \cap B$ or $x \in A \cap C$. As above, we have two cases, either $x \in A \cap B$ or $x \notin A \cap B$.

- $(x \in A \cap B)$ In this case, $x \in A$ and $x \in B$. Now, so $x \in B$ implies $x \in B$ or $x \in C$, that is, $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.
- $(x \notin A \cap B)$ Again, since $x \in A \cap B$ or $x \notin A \cap B$, and $x \notin A \cap B$, it is necessary that $x \in A \cap C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

□

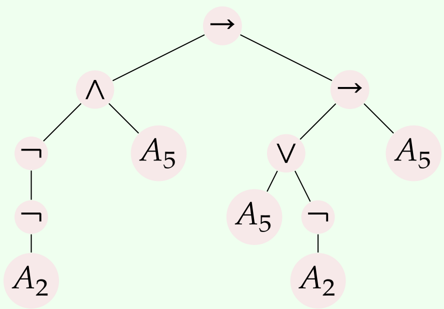
Exercise 3.2 (Construction) *10 points each.*

i. Write down a construction sequence for $((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))$.

Solution. $\langle A_1, A_3, A_4, A_7, (\neg A_1), ((\neg A_1) \vee A_4), (\neg((\neg A_1) \vee A_4)), (A_1 \to A_3), ((A_1 \to A_3) \leftrightarrow A_7), ((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))\rangle$. □

ii. Write down a construction tree for $(((\neg(\neg A_2)) \wedge A_5) \to ((A_5 \vee (\neg A_2)) \to A_5))$.

Solution. □



Exercise 3.3 (Truth Assignments)

i. Let S be the set of all sentence symbols, and assume that $v : S \to \{F, T\}$ is a truth assignment. Show there is at most one extension v meeting conditions 0–5 on pp. 20–21. (Hint: Show that if v_1 and v_2 are such extensions, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α. Use the induction principle.) 20 points.

Solution. We show this via induction on the complexity of any arbitrary wff α.

- **(Base case)** Assume α be a sentence symbol. Then $\bar{v}_1(\alpha) = v(\alpha) = \bar{v}_2(\alpha)$ since $\bar{v}_1$ and $\bar{v}_2$ are both extensions of v.
- **(Induction step)** We assume that the result holds for all wffs less complex than α (induction hypothesis). Now, we show that the result holds under all the formula building operations.

($\neg$) Assume $\alpha = (\neg\beta)$ for some wff β. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T && \\ \iff & \bar{v}_1(\neg\beta) = T && [\text{Def of } \alpha] \\ \iff & \bar{v}_1(\beta) = F && [\text{Def of } \bar{v} \text{ under } \neg] \\ \iff & \bar{v}_2(\beta) = F && [\text{Induction hypothesis}] \\ \iff & \bar{v}_2(\neg\beta) = T && [\text{Def of } \bar{v} \text{ under } \neg] \\ \iff & \bar{v}_2(\alpha) = T && [\text{Def of } \alpha] \end{aligned}$$

($\wedge$) Assume $\alpha = (\beta \wedge \gamma)$ for some wffs β, γ. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T && \\ \iff & \bar{v}_1(\beta \wedge \gamma) = T && [\text{Def of } \alpha] \\ \iff & \bar{v}_1(\beta) = T \text{ and } \bar{v}_1(\gamma) = T && [\text{Def of } \bar{v} \text{ under } \wedge] \\ \iff & \bar{v}_2(\beta) = T \text{ and } \bar{v}_2(\gamma) = T && [\text{Induction hypothesis}] \\ \iff & \bar{v}_2(\beta \wedge \gamma) = T && [\text{Def of } \bar{v} \text{ under } \wedge] \\ \iff & \bar{v}_2(\alpha) = T && [\text{Def of } \alpha] \end{aligned}$$

($\vee$) Assume $\alpha = (\beta \vee \gamma)$ for some wffs β, γ. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T && \\ \iff & \bar{v}_1(\beta \vee \gamma) = T && [\text{Def of } \alpha] \\ \iff & \bar{v}_1(\beta) = T \text{ or } \bar{v}_1(\gamma) = T && [\text{Def of } \bar{v} \text{ under } \vee] \\ \iff & \bar{v}_2(\beta) = T \text{ or } \bar{v}_2(\gamma) = T && [\text{Induction hypothesis}] \\ \iff & \bar{v}_2(\beta \vee \gamma) = T && [\text{Def of } \bar{v} \text{ under } \vee] \\ \iff & \bar{v}_2(\alpha) = T && [\text{Def of } \alpha] \end{aligned}$$

$(\to)$ Assume $\alpha = (\beta \to \gamma)$ for some wffs β, γ. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = T \\
\iff & \bar{v}_1(\beta \to \gamma) = T && \text{[Def of } \alpha\text{]} \\
\iff & \bar{v}_1(\beta) = F \text{ or } \bar{v}_1(\gamma) = T && \text{[Def of } \bar{v} \text{ under } \to\text{]} \\
\iff & \bar{v}_2(\beta) = F \text{ or } \bar{v}_2(\gamma) = T && \text{[Induction hypothesis]} \\
\iff & \bar{v}_2(\beta \to \gamma) = T && \text{[Def of } \bar{v} \text{ under } \to\text{]} \\
\iff & \bar{v}_2(\alpha) = T && \text{[Def of } \alpha\text{]}
\end{aligned}$$

$(\leftrightarrow)$ Assume $\alpha = (\beta \leftrightarrow \gamma)$ for some wffs β, γ. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = T \\
\iff & \bar{v}_1(\beta \leftrightarrow \gamma) = T && \text{[Def of } \alpha\text{]} \\
\iff & \bar{v}_1(\beta) = \bar{v}_1(\gamma) && \text{[Def of } \bar{v} \text{ under } \leftrightarrow\text{]} \\
\iff & \bar{v}_2(\beta) = \bar{v}_2(\gamma) && \text{[Induction hypothesis]} \\
\iff & \bar{v}_2(\beta \leftrightarrow \gamma) = T && \text{[Def of } \bar{v} \text{ under } \leftrightarrow\text{]} \\
\iff & \bar{v}_2(\alpha) = T && \text{[Def of } \alpha\text{]}
\end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α, which proves the uniqueness of the extension. □

ii. Show that for a set of wffs Σ and a wff α: $\Sigma \cup \{\neg\neg\alpha\}$ is satisfiable $\iff$ $\Sigma \cup \{\alpha\}$ is satisfiable. 10 points.

Solution. First, note that for any wff α and truth assignment v,

$$\bar{v}(\alpha) = T \iff \bar{v}(\neg\alpha) = F \iff \bar{v}(\neg\neg\alpha) = T.$$

Therefore, we have the following (v always represents a truth assignment):

$$\begin{aligned}
& \Sigma \cup \{\alpha\} \text{ is satisfiable.} \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\alpha) = T. \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\alpha) = F. \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\neg\alpha) = T. \\
\iff & \Sigma \cup \{\neg\neg\alpha\} \text{ is satisfiable.}
\end{aligned}$$

□

Exercise 3.4 (Compactness) *Recall the Compactness Theorem: A set of wffs is satisfiable iff it is finitely satisfiable.*
Recall Corollary 17A: If $\Sigma \models \tau$, then $\Sigma_0 \models \tau$ for some finite $\Sigma_0 \subseteq \Sigma$.
Prove that they are equivalent, i.e., prove that the Compactness Theorem holds iff Corollary 17A holds.
(Hint: Use the fact that $\Gamma \models \sigma$ iff $\Gamma \cup \{\neg\sigma\}$ is unsatisfiable and 3.3.ii above.) 20 points.

Solution. The proof of Corollary 17A in the book shows that the Compactness Theorem implies Corollary 17A. Therefore, we are left to show that Corollary 17A implies the Compactness Theorem.

For this, we assume Corollary 17A and prove Compactness Theorem. Note that if a set of wffs is satisfiable with a truth assignment, then it is finitely satified with the same truth assignment. Therefore, we only have to show that finite satisfiability implies satisfiability.

Suppose not. That is, assume that Σ is a set of wffs such that Σ is finitely satisfiable but Σ is unsatisfiable. Fix a wff τ. Since Σ is unsatisfiable, both $\Sigma \models \tau$ and $\Sigma \models \neg\tau$. Since $\Sigma \models \tau$, using Corollary 17A, there is a finite subset $\Sigma_1 \subseteq \Sigma$ such that $\Sigma_1 \models \tau$. Similarly, there exists $\Sigma_2 \subseteq \Sigma$ finite such that $\Sigma_2 \models \neg\tau$. Now, since $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$ is finite, it is finitely satisfiable by a truth assignment, say v. Clearly, since $\Sigma_1, \Sigma_2 \subseteq \Sigma$, v satisfies Σ_1 and Σ_2. Then v satisfies both τ and $\neg\tau$, which is an impossibility. This contradiction shows Σ is finitely satisfiable implies Σ is satisfiable. This concludes the proof. $\square$

Exercise 3.5 (Substitution) *Let $\alpha_1, \alpha_2, \ldots$ be a sequence of wffs. For each wff ϕ and $n \in \mathbb{N}$, let ϕ^* be the result of replacing the sentence symbol A_n in ϕ by the wff α_n. Suppose that v is a truth assignment for the set of all sentence symbols and that u is a truth assignment defined by $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\phi) = \bar{v}(\phi^*)$.*
(Hint: Use the induction principle.) 20 points

Solution. We show this via induction on the complexity of any arbitrary wff ϕ.

- **(Base case)** Assume $\phi = A_n$ for some $n \in \mathbb{N}$, so $\phi^* = \alpha_n$. Now $\bar{u}(\phi) = \bar{u}(A_n) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\phi^*)$, so the result holds when ϕ is a sentence symbol.
- **(Induction step)** We assume that the result holds for all wffs less complex than ϕ (induction hypothesis). Now, we show that the result holds under all the formula building operations.

  ($\neg$) Assume $\phi = (\neg\psi)$ for some wff ψ, so $\phi^* = (\neg\psi^*)$. Then

$$\begin{aligned}
&\bar{u}(\phi) = T && \\
\iff\ & \bar{u}(\neg\psi) = T && [\text{Def of } \phi] \\
\iff\ & \bar{u}(\psi) = F && [\text{Def of } \bar{u} \text{ under } \neg] \\
\iff\ & \bar{v}(\psi^*) = F && [\text{Induction hypothesis}] \\
\iff\ & \bar{v}(\neg\psi^*) = T && [\text{Def of } \bar{v} \text{ under } \neg] \\
\iff\ & \bar{v}(\phi^*) = T && [\text{Def of } \phi^*]
\end{aligned}$$

  ($\wedge$) Assume $\phi = (\psi \wedge \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \wedge \theta^*)$. Then

$$\begin{aligned}
& \bar{u}(\phi) = T && \\
\iff\ & \bar{u}(\psi \wedge \theta) = T && [\text{Def of } \phi] \\
\iff\ & \bar{u}(\psi) = T \text{ and } \bar{u}(\theta) = T && [\text{Def of } \bar{u} \text{ under } \wedge] \\
\iff\ & \bar{v}(\psi^*) = T \text{ and } \bar{v}(\theta^*) = T && [\text{Induction hypothesis}] \\
\iff\ & \bar{v}(\psi^* \wedge \theta^*) = T && [\text{Def of } \bar{v} \text{ under } \wedge] \\
\iff\ & \bar{v}(\phi^*) = T && [\text{Def of } \phi^*]
\end{aligned}$$

($\vee$) Assume $\phi = (\psi \vee \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \vee \theta^*)$. Then

$$\begin{aligned}
& \bar{u}(\phi) = T && \\
\iff\ & \bar{u}(\psi \vee \theta) = T && [\text{Def of } \phi] \\
\iff\ & \bar{u}(\psi) = T \text{ or } \bar{u}(\theta) = T && [\text{Def of } \bar{u} \text{ under } \vee] \\
\iff\ & \bar{v}(\psi^*) = T \text{ or } \bar{v}(\theta^*) = T && [\text{Induction hypothesis}] \\
\iff\ & \bar{v}(\psi^* \vee \theta^*) = T && [\text{Def of } \bar{v} \text{ under } \vee] \\
\iff\ & \bar{v}(\phi^*) = T && [\text{Def of } \phi^*]
\end{aligned}$$

($\rightarrow$) Assume $\phi = (\psi \rightarrow \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \rightarrow \theta^*)$. Then

$$\begin{aligned}
& \bar{u}(\phi) = T && \\
\iff\ & \bar{u}(\psi \rightarrow \theta) = T && [\text{Def of } \phi] \\
\iff\ & \bar{u}(\psi) = F \text{ or } \bar{u}(\theta) = T && [\text{Def of } \bar{u} \text{ under } \rightarrow] \\
\iff\ & \bar{v}(\psi^*) = F \text{ or } \bar{v}(\theta^*) = T && [\text{Induction hypothesis}] \\
\iff\ & \bar{v}(\psi^* \rightarrow \theta^*) = T && [\text{Def of } \bar{v} \text{ under } \rightarrow] \\
\iff\ & \bar{v}(\phi^*) = T && [\text{Def of } \phi^*]
\end{aligned}$$

($\leftrightarrow$) Assume $\phi = (\psi \leftrightarrow \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \leftrightarrow \theta^*)$. Then

$$\begin{aligned}
& \bar{u}(\phi) = T && \\
\iff\ & \bar{u}(\psi \leftrightarrow \theta) = T && [\text{Def of } \phi] \\
\iff\ & \bar{u}(\psi) = \bar{u}(\theta) && [\text{Def of } \bar{u} \text{ under } \leftrightarrow] \\
\iff\ & \bar{v}(\psi^*) = \bar{v}(\theta^*) && [\text{Induction hypothesis}] \\
\iff\ & \bar{v}(\psi^* \leftrightarrow \theta^*) = T && [\text{Def of } \bar{v} \text{ under } \leftrightarrow] \\
\iff\ & \bar{v}(\phi^*) = T && [\text{Def of } \phi^*]
\end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{u}(\phi) = \bar{v}(\phi)$ for every wff ϕ. □

Bibliography