

# A generalization of Itô calculus and large deviations theory

Sudip Sinha

2019-04-05

Advisors

Prof. Hui-Hsiung Kuo

Prof. Padmanabhan Sundar

# Outline

1	Introduction and motivation	3
2	A Generalization of Itô calculus	14
3	Large deviations theory	23
4	The way forward	31

# SECTION 1

## INTRODUCTION AND MOTIVATION

# Quick revision and notations

- ▷ Let  $T \in (0, \infty)$ , and denote  $t \in [0, T]$ .
- ▷ Let  $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$  be a filtered probability space.
- ▷  $B_\bullet$  is a Brownian motion on  $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ .
- ▷ Properties of  $B_\bullet$ 
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t - s)$
  - continuous paths (a.s.)
  - is a.s. nowhere differentiable
  - has **unbounded linear variation** 😞
  - has **bounded quadratic variation** 😊
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- ▷ **Naive integration w.r.t.  $B_t$ : not possible.**
- ▷ A stochastic process  $X_\bullet$  is called **adapted** to  $\mathcal{F}_\bullet$  if  $X_t$  is measurable w.r.t.  $\mathcal{F}_t \forall t$ .

# Martingales and Markov processes

**Definition** Let  $X_\bullet$  is a  $L^1$ -bounded  $\mathcal{F}_\bullet$ -adapted stochastic process, and let  $0 \leq s \leq t \leq T$ . Then  $X_\bullet$  is called a

- martingale if  $\mathbb{E}(X_t | \mathcal{F}_s) = X_s$ ,
- submartingale if  $\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s$ , and
- supermartingale if  $\mathbb{E}(X_t | \mathcal{F}_s) \leq X_s$ .

**Remark** If  $(X_n)$  is a discrete-time martingale and  $(H_n)$  is an adapted process, then the process  $Y_n = \sum_{j=0}^{n-1} H_j(X_{j+1} - X_j) =: (H \bullet X)_n$  is itself a martingale and called a **martingale transform** of  $(X_n)$ .

**Definition** A stochastic process  $X_\bullet$  is called **Markov** if for any  $0 \leq s \leq t \leq T$ , we have

$$\mathbb{P}(X_t \in E | \mathcal{F}_s) = \mathbb{P}(X_t \in E | X_s).$$

# Wiener integral for $f \in L^2[0, T]$

## ▷ Definition of the integral:

1. Step functions  $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$ : Define  $\int_0^T f(t) \, dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$ , where  $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ .
2.  $f \in L^2[0, T]$ : Use step functions approximating  $f$  to extend the integral a.s.

## ▷ Properties of the integral:

- Linear.
- **Gaussian distribution** with mean 0 and variance  $\|f\|_{L^2[0, T]}^2$  (Itô isometry).
- Agrees with the Riemann–Stieltjes integral for continuous functions of bounded variation.

## ▷ Properties of the associated process $I_\bullet = \int_0^\bullet f(t) \, dB_t$ :

- continuity.
- martingale.

## ▷ Problem: Cannot integrate stochastic processes.

# Trying to integrate stochastic processes

▷  $\int_0^T B_t \, dB_t \stackrel{?}{=}$

Since  $B_t$  is continuous, let us try Riemann–Stieltjes integral. Consider a sequence of partitions  $\Delta_n$  such that  $\|\Delta_n\| \rightarrow 0$ . Then

$$\int_0^T B_t \, dB_t = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

▷ Choosing different endpoints for  $t_j^*$  gives us different results.

$t_j^*$	$\int_0^t B_s \, dB_s$	Intuitive?	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2} (B_t^2 - t)$	☹	0	☺	Itô
mid	$\frac{1}{2} (B_t^2)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2} (B_t^2 + t)$	☹	$t$	☹	

▷ Which one do we choose?

# Itô integral [Itô44] for $X_{\bullet} \in L^2_{\text{ad}}([0, T] \times \Omega)$

## ▷ Definition of the integral:

1. Adapted step processes  $X_t(\omega) = \sum_{j=0}^{n-1} \tilde{\zeta}_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$ : define  $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \tilde{\zeta}_j \Delta B_j$ .
2.  $X \in L^2_{\text{ad}}([0, T] \times \Omega)$ : use step processes approximating  $X$  to extend the integral in  $L^2(\Omega)$ .

## ▷ Properties of the integral:

- Linear.
- Mean 0 and variance  $\|f\|_{L^2[0, T]}^2$  (Itô isometry).
- For  $X_{\bullet}$  continuous,  $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$ .

## ▷ Properties of the associated process $I_{\bullet} = \int_0^{\bullet} X_t \, dB_t$ :

- continuity.
- martingale.

## ▷ Example: $\int_0^t B_u \, dB_u = \frac{1}{2}(B_t^2 - t) \quad \forall t$ .



Itô integral for  $X_\cdot$  such that  $\int_0^T X_t^2 dt < \infty$  a.s.

- ▷ Definition: Use sequences of processes in  $L_{\text{ad}}^2([0, T] \times \Omega)$  approximating  $X$  in probability to extend the integral in probability.
- ▷ Properties of the integral:
  - Linear.
  - Mean and variance? ☹
- ▷ Properties of the associated process  $I_\cdot = \int_0^\cdot X_t dB_t$ :
  - continuity.
  - local martingale.

# The Itô formula

- ▷ An Itô process is a process of the form  $X_\bullet = X_0 + \int_0^\bullet m_t \, dt + \int_0^\bullet \sigma_t \, dB_t$ .  
Equivalently expressed as  $dX_t = m_t \, dt + \sigma_t \, dB_t$ .

**Theorem ([Itô44])** Let  $X_t$  be a  $d$ -dimensional Itô process, and let  $Y_t = f(X_t)$ , where  $f \in C^2(\mathbb{R})$ . Then  $f(X_t)$  is also a  $d$ -dimensional Itô process, and

$$df(X_t) = \langle (Df)(X_t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D^2 f)(X_t) dX_t \rangle,$$

where we use the rule  $dB_t \otimes dB_t = I_d \, dt$ .

- ▷ Example: For  $\sigma$  constant,  $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$ ,  $d\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, dt + \sigma \mathcal{E}_t \, dB_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (dB_t)^2$ .

# Exponential processes and the Girsanov theorem

- ▷ Let  $h_\bullet$  be a stochastic process. The associated exponential process is defined as

$$\mathcal{E}_\bullet^{(h)} = \exp \left( \int_0^\bullet h_t \, dB_t - \frac{1}{2} \int_0^\bullet h_t^2 \, dt \right).$$

- ▷ The exponential process is a martingale if and only if  $\mathbb{E} \mathcal{E}_t^{(h)} = 1 \, \forall t$ .
- ▷ The Novikov condition: The exponential process is a martingale if  $\mathbb{E} \exp \left( \frac{1}{2} \int_0^T h_t^2 \, dt \right) < \infty$ .
- ▷ The Girsanov theorem [Gir60]: The translated stochastic process  $W_\bullet = B_\bullet + \int_0^\bullet h_t \, dt$  is a Brownian motion under the probability measure  $\widetilde{\mathbb{P}}$  defined by the Radon-Nikodym derivative  $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(h)}$ .

# Stochastic differential equations

- ▷ Let  $\zeta \in L^2(\Omega)$  be independent of  $B_\cdot$ , and  $m, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  have ‘nice’ measurability. Then a  $\mathcal{F}_t$ -adapted stochastic process  $X_t$  is called a solution of the **stochastic *integral* equation**  $X_t = \zeta + \int_0^t m(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$  if for each  $t$ , the  $X_t$  satisfies the integral equation a.s.
- ▷ Stochastic *differential* equation  $dX_t = m(t, X_t) dt + \sigma(t, X_t) dB_t$ ,  $X_0 = \zeta$  is a *formal representation*.

**Theorem (Existence and uniqueness, Markov property)** The SDE above has a unique solution if there exists an  $M > 0$  such that the following two conditions are satisfied:

- (Lipschitz condition)  $|m(t, x) - m(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq M|x - y|^2$  a.s.
- (growth condition)  $|m(t, x)|^2 + |\sigma(t, y)|^2 \leq M(1 + |x|^2)$  a.s.

The solution is a Markov process.

Moreover if  $\zeta \in \mathbb{R}$  and  $m, \sigma$  are function of only  $x$ , then the solution is also stationary.

- Example: For  $\sigma$  constant,  $d\mathcal{E}_t = \sigma \mathcal{E}_t dB_t$ ,  $\mathcal{E}_0 = 1$  is solved by  $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$ .

# Multiple Wiener–Itô integrals

- ▷ How do we define the double integral?
- ▷ Naive idea:  $\int_0^t \int_0^t dB_u dB_v = \int_0^t dB_u \int_0^t dB_v = B_t^2$ .  
But  $\mathbb{E}B_t^2 = t \neq 0$ , so **no martingale property**. ☹
- ▷ Itô's idea: remove the diagonal to get

$$\int_0^t \int_0^t dB_u dB_v = 2 \int_0^t \int_0^v dB_u dB_v = 2 \int_0^t B_v dB_v = B_t^2 - t.$$

**Theorem** ([Itô51]) Let  $f \in L^2([0, T]^n)$  and  $\hat{f}$  be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) dB_{t_1} \cdots dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left( \int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) dB_{t_n} \right) dB_{t_{n-1}} \cdots dB_{t_1}.$$

## SECTION 2

# A GENERALIZATION OF ITÔ CALCULUS

# Motivation

▷ Iterated integrals: Consider the iterated integral  $\int_0^T \int_0^T dB_s dB_t = \int_0^T B_T dB_t \stackrel{?}{=} B_T^2$ .

▷ Note that  $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$ , so **no martingale property** ☹.

▷ Stochastic differential equations with anticipation:

$$dX_t = X_t dB_t$$

$$X_0 = B_T$$

$$dY_t = B_T dB_t$$

$$Y_0 = 1$$

▷ Problem: We want to define  $\int_0^T Z(\cdot) dB_t$ , where  $Z(\cdot)$  is not (necessarily) adapted.

▷ Some approaches:

- Enlargement of filtration  $\mathcal{G}_\bullet = \mathcal{F}_\bullet \vee B_T$ , with Itô's decomposition of integrand [[Itô78](#)]

$$B_t = \left( B_t - \int_0^t \frac{B_T - B_s}{T-s} ds \right) + \int_0^t \frac{B_T - B_s}{T-s} ds.$$

- White noise theory
- Malliavin calculus



# The new integral [AK08; AK10]: Idea

- A process  $Y^\bullet$  and filtration  $\mathcal{F}_\bullet$  are called **instantly independent** if  $Y^t$  and  $\mathcal{F}_t$  are independent  $\forall t$ .  
Example: The process  $(B_T - B_\bullet)$  is instantly independent of the filtration generated by  $B_\bullet$ .
- Ideas
  1. Decompose the integrand into **adapted** and **instantly independent** parts.
  2. Evaluate the **adapted** and the **instantly independent** parts at the **left** and **right** endpoints.
- Consider two continuous stochastic processes,  $X_t$  **adapted** and  $Y^t$  **instantly independent** w.r.t.  $\mathcal{F}_\bullet$ . Then the integral  $\int_0^T X_t Y^t dB_t$  is **defined** as

$$\int_0^T X_t Y^t dB_t \triangleq \lim_{\|\Delta_n\| \rightarrow 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

provided that the limit exists in probability.

- Now, for any stochastic process  $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$  we extend the definition by linearity.
- This is well-defined [HKS+16].



# A simple example

▷ In the following,  $\lim$  is the limit in  $L^2$ .

$$\begin{aligned}\int_0^t B_T \, dB_t &= \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t \\&= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\&= \lim \sum_{j=0}^{n-1} (B_T - \Delta B_j) \Delta B_j \\&= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t\end{aligned}$$

▷ Note that  $\mathbb{E}(B_T B_t - t) = 0$ .

▷ In general,  $\mathbb{E} \int_0^t Z(s) \, dB_s = 0$ . 😊

# The near-martingale property

- ▷ Question: What are the analogues of the martingale property and the Markov property?
- ▷ Example:  $\mathbb{E}(B_T B_t - t \mid \mathcal{F}_s) = B_s^2 - s \neq B_T B_s - s \text{ ☹}$   
But  $\mathbb{E}(B_T B_s - s \mid \mathcal{F}_s) = B_s^2 - s. \text{ ☺}$
- ▷ Let  $Z(t)$  be a process such that  $\mathbb{E} |Z(t)| < \infty \forall t$ , and  $0 \leq s \leq t \leq T$ . Then  $Z(t)$  is called a
  - near-martingale if  $\mathbb{E}(Z(t) \mid \mathcal{F}_s) = \mathbb{E}(Z(s) \mid \mathcal{F}_s)$ ,
  - near-submartingale if  $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \geq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$ , and
  - near-supermartingale if  $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \leq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$ .

**Theorem ([KSS12b])** Let  $f, \phi$  be continuous functions on  $\mathbb{R}$ . Under integrability conditions, the processes  $X_\bullet = \int_0^\bullet f(B_t) \phi(B_T - B_t) dB_t$  and  $Y^\bullet = \int_\bullet^T f(B_t) \phi(B_T - B_t) dB_t$  are near-martingales.

**Theorem ([HKS+17])** Let  $Z(\cdot)$  be a stochastic process bounded in  $L^1$ , and  $X_\bullet = \mathbb{E}(Z(\cdot) \mid \mathcal{F}_\bullet)$ . Then  $X_\bullet$  is a (sub/super)martingale if and only if  $Z(\cdot)$  is a near-(sub/super)martingale.

# A generalized Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_t = X_0 + \int_0^t m_s \, ds + \int_0^t \sigma_s \, dB_s$	$dX_t = m_t \, dt + \sigma_t \, dB_t$
instantly independent	$Y^t = Y^0 + \int_t^T \eta^s \, ds + \int_t^T \zeta^s \, dB_s$	$dY^t = -\eta^t \, dt - \zeta^t \, dB_t$

Here  $\eta^t$  and  $\zeta^t$  are instantly independent such that  $Y^t$  is also instantly independent.

**Theorem ([HKS+16])** Let  $dX_t = m_t \, dt + \sigma_t \, dB_t$  be an  $d$ -dimensional Itô process, and  $dY^t = -\eta^t \, dt - \zeta^t \, dB_t$  be a  $\tilde{d}$ -dimensional instantly independent process. If  $f(x, y) \in C^2(\mathbb{R}^2)$ , then

$$\begin{aligned}
 df(X_t, Y^t) = & \langle (D_x f)(X_t, Y^t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D_x^2 f)(X_t, Y^t) dX_t \rangle \\
 & + \langle (D_y f)(X_t, Y^t), dY^t \rangle - \frac{1}{2} \langle dY^t, (D_y^2 f)(X_t, Y^t) dY^t \rangle,
 \end{aligned}$$

where we use the rules  $dB_t \otimes dB_t = I_d \, dt$ .

# Iterated integrals

**Theorem ([Itô51])** Let  $f \in L^2([0, T]^n)$  and  $\hat{f}$  be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = n! \int_0^T \dots \int_0^{t_{n-2}} \left( \int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, dB_{t_n} \right) \, dB_{t_{n-1}} \dots dB_{t_1}.$$

**Theorem ([AK10])** Let  $f \in L^2([0, T]^n)$ . Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = \int_0^T \dots \int_0^T f(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1}.$$

Example[[HKS+16](#)]: For the new integral,  $\int_0^T \left( \int_0^T B_u \, du \right) \, dB_v = \int_0^T \left( \int_0^T B_u \, dB_v \right) \, du$ .

# Generalization of Itô isometry

**Theorem ([KSS12b])** Let  $\phi$  be an analytic function on  $\mathbb{R}$ . Then under integrability conditions and for each  $t$ ,

$$\mathbb{E} \left[ \left( \int_0^t \phi(B_T - B_s) \, dB_s \right)^2 \right] = \int_0^t \mathbb{E} \left[ (\phi(B_T - B_s))^2 \right] \, ds$$

**Theorem ([KSS13])** Let  $f, \phi$  be  $C^1$  function on  $\mathbb{R}$ . Then

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^T f(B_t) \phi(B_T - B_t) \, dB_t \right)^2 \right] &= \int_0^T \mathbb{E} \left[ (f(B_t) \phi(B_T - B_t))^2 \right] \, dt \\ &\quad + 2 \int_0^T \int_0^t \mathbb{E} \left[ f(B_s) \phi'(B_T - B_s) f'(B_s) \phi(B_T - B_s) \right] \, ds \, dt. \end{aligned}$$

# A generalization of Girsanov theorem

**Theorem** ([KPS13]) Let  $X_\bullet$  and  $Y^\bullet$  be continuous square-integrable stochastic processes such that  $X_\bullet$  is adapted and  $Y^\bullet$  is instantly independent.

Then translated stochastic process  $W_\bullet = B_\bullet + \int_0^\bullet (X_t + Y^t) dt$  is a near-martingale under the probability measure  $\widetilde{\mathbb{P}}$  defined by the Radon-Nikodym derivative  $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(X+Y)}$ .

# SECTION 3

## LARGE DEVIATIONS THEORY



# What is it about?

- ▷ A theory to find probabilities of rare events that decay exponentially.
- ▷ Started by Swedish actuarials Fredrik Esscher, Harald Cramér, Filip Lundberg.
- ▷ Unified by Varadhan in his 1966 paper [[Var66](#)].
- ▷ Example: A problem faced by the insurance industry.
  - Value of claims received on the  $n$ th day:  $X_n$  \$.
  - Steady income from premium:  $x$  \$/day.
  - Planning period:  $n$  days.
  - Average expenditure:  $\bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j$  \$/day.
  - *Question*: How should the company decide on the premium?
  - *Idea*: Determine  $x$  such that  $\mathbb{P} \{ \bar{X}_n > x \} < \varepsilon$  (specified).



# Insurance problem: setup

1. Let the following hold:

- $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space.
- $(X_n)$  is a sequence of i.i.d. random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with finite moment generating function  $M$ .
- $\mathbb{E}X_1 = m$ ,  $\mathbb{V}X_1 = \sigma^2$ , and  $X_1 \sim \mu$ .
- $\bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j$ .

2. Asymptotic behavior of  $\bar{X}_n$ :

- Weak law of large numbers:  $\bar{X}_n \xrightarrow{\mathbb{P}} m$ .
- Central limit theorem:  $\sqrt{n}(\bar{X}_n - m) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ .

3. What is the rate for LLN?

# Insurance problem: large deviation bounds

1. For  $x > m$  and an arbitrary  $\theta > 0$ , we get

$$\mathbb{P} \left\{ \bar{X}_n \geq x \right\} = \mathbb{P} \left\{ e^{\theta n \bar{X}_n} \geq e^{\theta n x} \right\} \leq e^{-\theta n x} \mathbb{E} \left( e^{\theta n \bar{X}_n} \right) = e^{-\theta n x} M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}.$$

2. Since  $\theta$  was arbitrary, we have

$$\mathbb{P} \left\{ \bar{X}_n \geq x \right\} \leq \inf_{\theta} e^{-n(\theta x - \log M_X(\theta))} = e^{-n \sup_{\theta} (\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim} \frac{1}{n} \log \mathbb{P} \left\{ \bar{X}_n \in F \right\} \leq - \inf_F I \quad \forall F \text{ closed.}$$

4. We can also obtain a large deviation lower bound using an exponential change of measure

$$\underline{\lim} \frac{1}{n} \log \mathbb{P} \left\{ \bar{X}_n \in G \right\} \geq - \inf_G I \quad \forall G \text{ open.}$$

5. We informally write  $\mathbb{P} \left\{ \bar{X}_n \in dx \right\} \asymp e^{-nI(x)} dx$  for  $x \in \mathbb{R}$ .

# Definition of large deviation principle

- ▷ The setup:  $(X_n)$  is a stochastic process on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in a Polish space  $(\mathcal{X}, d)$ .
- ▷ A function  $I : \mathcal{X} \rightarrow [0, \infty]$  is called a **rate function** if it has compact level sets.

**Definition**  $(X_n)$  is said to satisfy the **large deviation principle** on  $\mathcal{X}$  with rate function  $I$  if the large deviation **upper** and **lower** bounds hold.

## Example

**Theorem** ([Cra38]) Let  $(X_n)$  be a sequence of i.i.d. real random variables with finite moment generating function  $M$ . Then  $(X_n)$  follows large deviation principle with rate function  $I(x) = \sup_{\theta} (\theta x - \log M(\theta))$ .

# Applications of the Cramér theorem

Rate functions for some common distributions

Distribution	$M(\theta)$	$I(x)$
Bernoulli( $p$ )	$1 - p + pe^\theta$	$\left( x \log x + (1 - x) \log(1 - x) - \left( x \log \frac{1 - p}{p} + \log p \right) \right) \mathbb{1}_{[0,1]}(x) + \infty \mathbb{1}_{[0,1]^c}(x)$
Poisson( $\lambda$ )	$e^{\lambda(e^\theta - 1)}$	$\left( \lambda - x + x \log \frac{x}{\lambda} \right) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
Exp( $\lambda$ )	$\left( 1 - \frac{\theta}{\lambda} \right)^{-1}$	$(\lambda x - 1 + x \log(\lambda x)) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
$\mathcal{N}(m, \sigma^2)$	$e^{m\theta + \frac{1}{2}\sigma^2\theta^2}$	$\frac{(x - m)^2}{2\sigma^2}$
$\chi^2(k)$	$(1 - 2\theta)^{-\frac{k}{2}}$	$\frac{1}{2} \left( x - k + k \log \frac{k}{x} \right)$

# The Schilder theorem

- ▷ Aim: Estimate the probability that a scaled-down sample path of a **Brownian motion** will stray far from the mean path.
- ▷ Let  $C_x$  denote the set of continuous functions from  $[0, T]$  to  $\mathbb{R}^d$  starting at  $x$ , and let  $\text{CM}_x = \{\omega \in C_x : \omega \text{ is absolutely continuous and } \omega'_t \in L^2[0, T]\}$ .
- ▷ Let  $B_\cdot$  be a  $d$ -dimensional Brownian motion, so  $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- ▷ Let  $\frac{1}{\sqrt{n}}B_t \sim W^{(n)}$ . Then  $W^{(n)} = \mathcal{N}\left(0, \frac{t}{n}\right) \xrightarrow{\mathcal{D}} \delta_0$  as  $n \rightarrow \infty$ .

**Theorem ([Sch66])** On the Banach space  $(C_0, \|\cdot\|_\infty)$ , the sequence of probability measures  $(W^{(n)})$  satisfies LDP with the rate function  $I : C_0 \rightarrow \overline{\mathbb{R}}$  given by

$$I(\omega) = \left( \frac{1}{2} \int_0^T |\omega'_t|^2 dt \right) \mathbb{1}_{\text{CM}_0}(\omega) + \infty \mathbb{1}_{\text{CM}_0^c}(\omega)$$

# The Freidlin–Wentzell theorem

- ▷ Aim: Estimate the probability that a scaled-down sample path of an **Itô diffusion** will stray far from the mean path.
- ▷ Let  $X_{\bullet}^{(n)}$  be the solution of the  $d$ -dimensional stochastic differential equation  $dX_t^{(n)} = m(X_t^{(n)}) dt + \frac{1}{\sqrt{n}}\sigma(X_t^{(n)}) dB_t$ ,  $X_0^{(n)} = x$ , where  $m, \sigma$  are sufficiently nice.
- ▷ Let  $W_x^{(n)}$  denote the law of  $X_{\bullet}^{(n)}$  starting at  $x$ .
- ▷ As  $n \rightarrow \infty$ ,  $W_x^{(n)} \xrightarrow{\mathcal{D}} \delta_{\tilde{\zeta}}$ , where  $\tilde{\zeta}$  solves the ODE  $\dot{\tilde{\zeta}}(t) = m(\tilde{\zeta}(t))$ ,  $\tilde{\zeta}(0) = x$ .

**Theorem ([FW12])** For any fixed  $x$ , the sequence of probability measures  $(W_x^{(n)})$  satisfies LDP with the rate function  $I_x : C_0 \rightarrow \overline{\mathbb{R}}$  given by

$$I_x(\omega) = \left( \frac{1}{2} \int_0^T \langle \omega'_t - b(\omega_t), A^{-1}(\omega_t)(\omega'_t - b(\omega_t)) \rangle dt \right) \mathbb{1}_{\text{CM}_x}(\omega) + \infty \mathbb{1}_{\text{CM}_x^c}(\omega),$$

where  $A = \sigma\sigma^*$ .

# SECTION 4

## THE WAY FORWARD



# Possible research directions

- ▷ Identify the class of integrable processes under the new integral.
- ▷ Give a broader generalization of the Itô isometry for the new integral.
- ▷ Provide a broader generalization of the Girsanov theorem.
- ▷ Formulate the extension to stochastic differential equations with anticipating coefficients.
- ▷ Develop the near-Markov property for the new integral.
- ▷ Prove Freidlin–Wentzell type results for stochastic differential equations with anticipating initial conditions.
- ▷ Study LDP results for SDEs with anticipating coefficients.
- ▷ Analyze LDP for linear SPDEs with anticipating initial conditions.



Thank you!

# APPENDIX

# Laplace principle and equivalence to LDP

**Definition (Laplace principle)**  $(X_n)$  is said to satisfy the Laplace principle on  $\mathcal{X}$  with rate function  $I$  if for all bounded continuous functions  $h$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp(-nh(X_n)) = \inf_{\mathcal{X}} (h + I)$$

**Theorem**  $(X_n)$  satisfies LP on  $\mathcal{X}$  with rate function  $I$  if and only if  $(X_n)$  satisfies LDP on  $\mathcal{X}$  with the same rate function  $I$ .

## Some important results

- Uniqueness of the rate function.
- Continuity principle.
- Superexponential approximation preserves Laplace principle.

Thank you!

# Bibliography

- Ayed, W. & Kuo, H. H. (2008). An extension of the Itô integral. *Communications on Stochastic Analysis*, 2(3). doi:10.31390/cosa.2.3.05
- (2010). An extension of the Itô integral: Toward a general theory of stochastic integration. *Theory of Stochastic Processes*, 16(32), 17–28. Retrieved from <http://mi.mathnet.ru/thsp56>
- Cramér, H. (1938). Sur un nouveau théorème-limite de la théorie des probabilités. *Actualités Scientifiques et Industrielles*, 736, 5–23.
- Freidlin, M. I. & Wentzell, A. D. (2012). *Random Perturbations of Dynamical Systems*. (3 ed.). Springer-Verlag. doi:10.1007/978-3-642-25847-3
- Girsanov, I. V. (1960). On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures. *Theory of Probability & Its Applications*, 5, 285–301. doi:10.1137/1105027
- Hwang, C. R., Kuo, H. H., Saitô, K., & Zhai, J. (2016). A general Itô formula for adapted and instantly independent stochastic processes. *Communications on Stochastic Analysis*, 10(3). doi:10.31390/cosa.10.3.05

- \_\_\_\_\_ (2017). Near-martingale Property of Anticipating Stochastic Integration. *Communications on Stochastic Analysis*, 11(4). doi:10.31390/cosa.11.4.06
- Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad.*, 20(8), 519–524. doi:10.3792/pia/1195572786
- \_\_\_\_\_ (1951). Multiple Wiener Integral. *J. Math. Soc. Japan*, 3(1), 157–169. doi:10.2969/jmsj/00310157
- \_\_\_\_\_ (1978). Extension of stochastic integrals. In Extension of stochastic integrals., *Proceedings of the International Symposium on Stochastic Differential Equations*. Kinokuniya.
- Kuo, H. H., Peng, Y., & Szozda, B. (2013). Generalization of the anticipative Girsanov theorem. *Communications on Stochastic Analysis*, 7(4). doi:10.31390/cosa.7.4.06
- Kuo, H. H., Sae-Tang, A., & Szozda, B. (2012b). A stochastic integral for adapted and instantly independent stochastic processes. In A stochastic integral for adapted and instantly independent stochastic processes., *Stochastic Processes, Finance and Control*. (Vol. 29, pp. 53–71). Author. doi:10.1142/9789814383318\_0003
- \_\_\_\_\_ (2013). And isometry formula for a new stochastic integral. In And isometry formula for a new stochastic integral., *Quantum Probability and Related Topics*. (Vol. 29, pp. 222–232). Author. doi:10.1142/9789814447546\_0014

Schilder, M. (1966). Some Asymptotic Formulas for Wiener Integrals. *Transactions of the American Mathematical Society*, 125(1), 63–85. doi:10.2307/1994588

Varadhan, S. R. S. (1966). Asymptotic probabilities and differential equations. *Communications on Pure and Applied Mathematics*, 19, 261–286. doi:https://doi.org/10.1002/cpa.3160190303