An introduction to Itô calculus and anticipating integrals

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Outline

Section 1
History of Probability Theory

History: Probability theory

1. 1564: Gerolamo Cardano published *Liber de ludo aleae* (Book on Games of Chance).
2. 1654: Pascal and Fermat corresponded about the *problem of points* floated by the gambler Chevalier de Méré. This is understood to be the origin of systematic study of probability.
3. 1657: Christiaan Huygens published a book.
4. 1800s: Pierre Laplace completed what is today considered the classic interpretation.
5. Applications in annuities, statistics of mortality, life insurance, assessing evidence, etc.
6. 1904: Henri Lebesgue published what is now knows as the Lebesgue integral.
   The idea was generalized into abstract integrals (over arbitrary spaces).
7. 1933: Andrey Kolmogorov published *Foundations of the Theory of Probability*.
   This axiomatic approach unified the theories of discrete and continuous probability.

History: Brownian motion

1. 1827: Discovered by the biologist Robert Brown while studying pollen particles floating in water in the microscope.

2. 1900: Louis Bachelier used Brownian motion to model financial markets in his PhD thesis *The theory of speculation*.

3. 1905: Albert Einstein tried to explain Brownian motion using a probabilistic model for diffusion transport.

4. 1923: Norbert Wiener rigorously constructed the Brownian motion, proving its existence.

5. 1944: Kiyosi Itô published his integral w.r.t. a Brownian motion.

6. 1973: Black and Scholes used Brownian motion and the Itô integral to model the stock market.

Brownian motions in one dimension

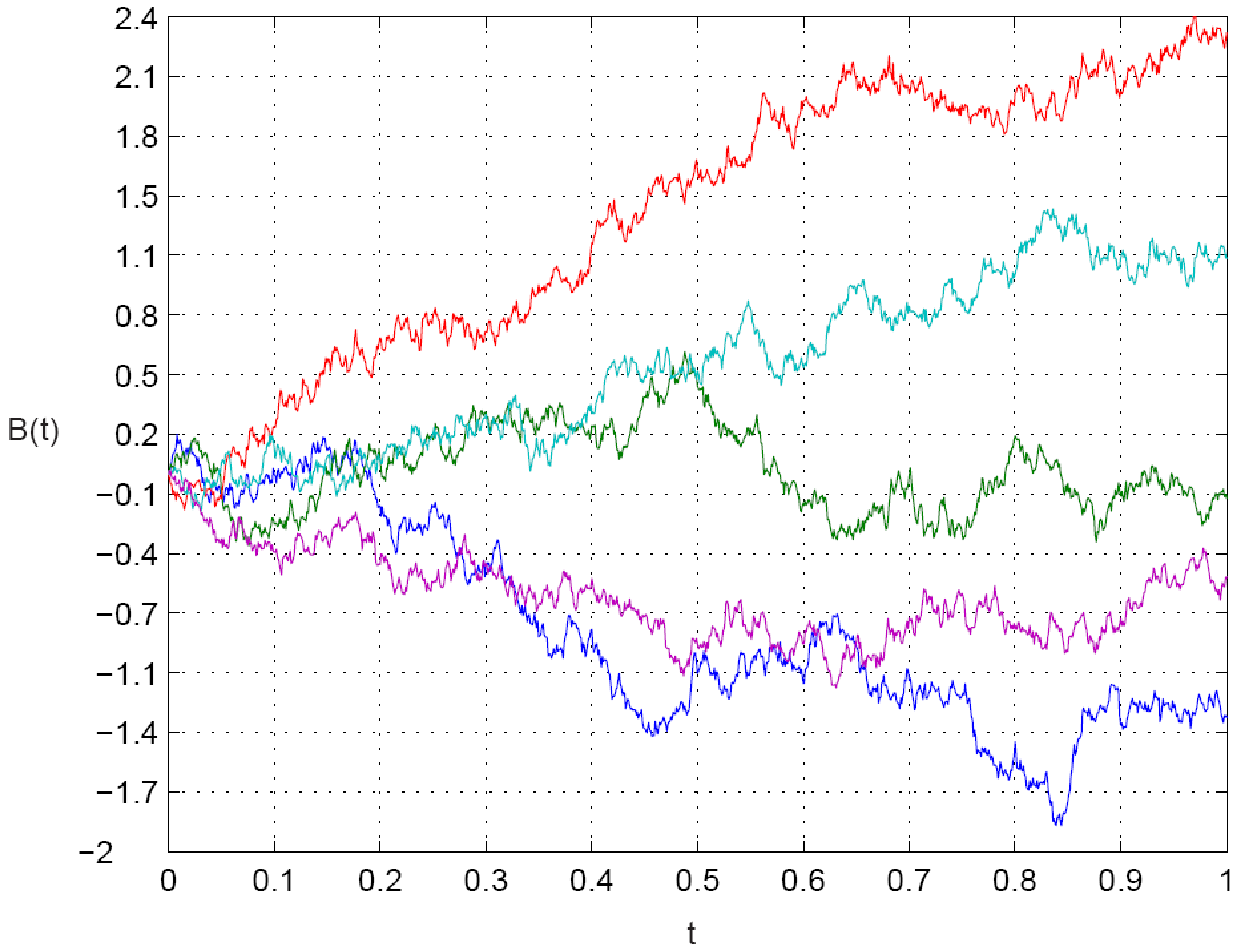

Figure 1

Section 2
Introduction to the Theory

Axiomatic probability theory

Definition *A probability space is a triple* $(\Omega, \mathcal{F}, \mathbb{P})$, *where*

- Ω *is a set containing the elementary outcomes.*
- $\mathcal{F} \subseteq 2^{\Omega}$ *is a* σ*-algebra on* Ω, *i.e.*
  - $\emptyset \in \mathcal{F}$,
  - $E \in \mathcal{F} \implies E^{\complement} \in \mathcal{F}$, *and*
  - $(E_n)_{n\in\mathbb{N}} \subset \mathcal{F} \implies \bigcup E_n \in \mathcal{F}$.
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ *is the probability measure on the measurable space* $(\Omega, \mathcal{F})$, *i.e.*
  - $\mathbb{P}(\emptyset) = 0$,
  - (σ*-additivity*) *If* $(E_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ *are a disjoint sequence of sets in* $\mathcal{F}$, *then* $\mathbb{P}(\bigsqcup E_n) = \sum P(E_n)$, *and*
  - (probability *measure*) $\mathbb{P}(\Omega) = 1$.

Remarks

Elements of $\mathcal{F}$ (sets) are the *events* to which we can assign a *probability* in a meaningful way.
Thus, the σ-algebra represents "information" in the system.
The finer the σ-algebra, the more information we have.

Martingales

- A **random variable** is a $\mathcal{F}$-*measurable* function $X : \Omega \to \mathbb{R}$.
- A **stochastic process** is a *parameterized family* of random variables $(X_t)_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}$.
  We usually think of t as time and (X_t) as the process evolving in time.
- A **filtration** is an increasing *parameterized family* $(\mathcal{F}_t)_{t\in[0,T]}$ of σ-algebras.
  We think of the system evolving in time, so it has more information as time passes.
- Let $0 \leq s \leq t \leq T$. Then a stochastic process $(X_t)_t$ is called a **martingale** if $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$.
  Martingales represent *fair games*.
  Example: A fair coin is tossed at each unit of time. I win 1$ if heads turn up and lose 1$ when tails turn up. Then my wealth is a martingale, because at any point in time my conditional expected fortune after the next trial, given the history, is equal to their present fortune.
- A stochastic process $(X_t)_t$ is called **adapted** to the filtration $(\mathcal{F}_t)_t$ if X_t is $\mathcal{F}_t$-measurable $\forall t$.

Brownian motion

1. A Brownian motion $(B_t)_{t\in[0,T]}$ is a stochastic process which has the following properties:
   1. Starts at 0 (a.s.)
   2. Has independent increments
   3. $B_t - B_s \sim \mathcal{N}(0, t-s)$
   4. Has continuous sample paths (a.s.)
2. Other properties of Brownian motion (B_t)
   1. It is a.s. nowhere differentiable
   2. It has unbounded linear variation ☹, so naive integration w.r.t. B_t is not possible
   3. It has bounded quadratic variation ☺
   4. (B_t) a martingale
   5. $(B_t^2 - t)$ is a martingale

Section 3
Itô calculus

Trying to integrate stochastic processes

- Question: $\int_0^T B_t \, \mathrm{d}B_t \stackrel{?}{=}$
  Since B_t is continuous, let us try the Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \to 0$. Then

$$\int_0^T B_t \, \mathrm{d}B_t = \lim \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

- Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, \mathrm{d}B_s$	Intuitive?	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2}\left(B_t^2 - t\right)$	☹	0	☺	Itô
mid	$\frac{1}{2}\left(B_t^2\right)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2}\left(B_t^2 + t\right)$	☹	t	☹	

- Which one do we choose?

Itô integral [Itô44] for (X_t) with continuous paths

- Definition of the integral: $\int_0^T X_t \, \mathrm{d}B_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
- Properties of the integral:
  - Linear.
  - Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).
- Properties of the associated process $I_\bullet = \int_0^\bullet X_t \, \mathrm{d}B_t$:
  - continuity
  - martingale
- Example: $\int_0^t B_u \, \mathrm{d}B_u = \frac{1}{2}(B_t^2 - t) \quad \forall t$.
- Remark: We can only integrate over processes which are adapted.

Multiple integrals

- Question: How do we define the double integral?
- Naive idea: $\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = \int_0^t \mathrm{d}B_u \int_0^t \mathrm{d}B_v = B_t^2$.
  But $\mathbb{E}B_t^2 = t \neq 0$, so no martingale property. ☹
- Itô's idea: remove the diagonal to get

$$\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = 2 \int_0^t \int_0^v \mathrm{d}B_u \, \mathrm{d}B_v = 2 \int_0^t B_v \, \mathrm{d}B_v = B_t^2 - t.$$

Theorem ([Itô51]) *Let $f \in L^2\left([0,T]^n\right)$ and $\hat{f}$ be its symmetrization. Then*

$$\int_{[0,T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \cdots dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, dB_{t_n} \right) dB_{t_{n-1}} \cdots dB_{t_1}.$$

- Feels non-intuitive ☹.

Section 4
A Generalization of Itô calculus

Motivation

- Iterated integrals: Consider the iterated integral $\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = \int_0^t B_t \, \mathrm{d}B_v \overset{?}{=} B_t^2$.
- Note that $\mathbb{E}(B_t^2) = t \neq 0$, so no martingale property ☹.
- Problem: We want to define $\int_0^T Z(\bullet) \, \mathrm{d}B_t$, where $Z(\bullet)$ is not (necessarily) adapted.
- Some approaches:
  - Enlargement of filtration $\mathcal{G}_{\bullet} = \mathcal{F}_{\bullet} \vee \sigma(B_T)$, with Itô's decomposition of integrand [Itô78] $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} \mathrm{d}s\right) + \int_0^t \frac{B_T - B_s}{T-s} \mathrm{d}s$.
  - White noise theory
  - Malliavin calculus

The new integral [AK08; AK10]: Idea

- A process $Y^{\bullet}$ and filtration $\mathcal{F}_{\bullet}$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$.
  Example: The process $(B_T - B_{\bullet})$ is instantly independent of the filtration generated by $B_{\bullet}$.

- Idea

  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.

- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\bullet}$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is defined as

$$\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

provided that the limit exists in probability.

A simple example

- In the following, lim is the limit in L^2.

$$\begin{aligned}
\int_0^t B_T \, \mathrm{d}B_t &= \int_0^t (B_t + (B_T - B_t)) \, \mathrm{d}B_t = \int_0^t B_t \, \mathrm{d}B_t + \int_0^t (B_T - B_t) \, \mathrm{d}B_t \\
&= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\
&= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j \right) \Delta B_j \\
&= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t
\end{aligned}$$

- In general, $\mathbb{E} \int_0^t Z(s) \, \mathrm{d}B_s = 0$. ☺
- This motivates the definition of the *near-martingale* property, which we shall not cover.

Appendix

Thank you!

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