Functional analysis

Mostly operator theory for now

Sudip Sinha

February 25, 2020

Contents

Part 1

Convergence

1.1 Strong, weak and weak* convergence

Disclaimer: This section is shamelessly copied from Christopher Heil's notes.

Definition 1.1 *Let X be a normed vector space, and $x_n, x \in X$. We define the following convergences as $n \to \infty$.*

$$\text{(strong)} \qquad x_n \to x \qquad \Longleftrightarrow \qquad \|x_n - x\| \to 0$$

$$\text{(weak)} \qquad x_n \xrightarrow{w} x \qquad \Longleftrightarrow \qquad \forall \phi \in X^*, \quad (x_n - x, \phi) \to 0$$

Definition 1.2 *Let X be a normed vector space, and $\phi_n, \phi \in X^*$. We define the following convergences as $n \to \infty$.*

$$\text{(strong)} \qquad \phi_n \to \phi \qquad \Longleftrightarrow \qquad \|\phi_n - \phi\| \to 0$$

$$\text{(weak)} \qquad \phi_n \xrightarrow{w} \phi \qquad \Longleftrightarrow \qquad \forall \xi \in X^{**}, \quad (\phi_n - \phi, \xi) \to 0$$

$$\text{(weak*)} \qquad \phi_n \xrightarrow{w^*} \phi \qquad \Longleftrightarrow \qquad \forall x \in X, \quad (x, \phi_n - \phi) \to 0$$

Remark 1.3 *Weak* convergence is simply* pointwise convergence *for the functionals ϕ_n.*

Proposition 1.4 (strong $\Longrightarrow$ weak $\Longrightarrow$ weak* for convergence) *Suppose $\phi_n, \phi \in X^*$.*

Then $\phi_n \to \phi \Longrightarrow \phi_n \xrightarrow{w} \phi \Longrightarrow \phi_n \xrightarrow{w^*} \phi$.

The second implication reverses if X is reflexive.

Proof. *strong $\Longrightarrow$ weak*: $\qquad (x_n - x, \phi) \le \|x_n - x\| \, \|\phi\| \to 0$.

*weak $\Longrightarrow$ weak**: $\qquad (x, \phi_n - \phi) = (\phi_n - \phi, x^{**}) \to 0$.

The claim about the reverse implication is now obvious.

Counterexample for converse of the first implication: Suppose $X = \ell^2(\mathbb{N})$. Then $e_n \xrightarrow{w} 0$, but $\|e_n - 0\| = 1 \nrightarrow 0$. □

Proposition 1.5 *In Hilbert spaces, weak convergence plus convergence of norms ($\|x_n\| \to \|x\|$) is equivalent to strong convergence.*

Proof. $\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle = \langle x_n - x, x_n \rangle - \langle x_n - x, x \rangle \to 0.$ □

Proposition 1.6 *Let H and K be Hilbert spaces, and let $T \in B(H, K)$ be a compact operator.*

Show that $x_n \xrightarrow{w} x \Longrightarrow Tx_n \to Tx$.

Thus, a compact operator maps weakly convergent sequences to strongly convergent sequences.

Proof. *Disclaimer*: Stolen from MSx1142451.

$Tx_n \xrightarrow{w} Tx$ by continuity. Thus if any subsequence has a strong limit, it certainly is Tx. But compactness guarantees every subsequence has a subsequence that converges to something: that something is Tx by uniqueness, and so by our above equivalence with convergence, we have $Tx_n \to Tx$. □

Part 2

Operator theory

2.1 Elementary ideas

A great source is Trace class operators and Hilbert-Schmidt operators by Jordan Bell.

2.1.1 Intuition

On a separable Hilbert space, we have

- ▷ $T \in \mathcal{B}^\infty \iff \lambda \in \ell^\infty$ (bounded)
  Example $I : \ell^2 \to \ell^2 : e_n \mapsto e_n$.
- ▷ $T \in \mathcal{B}_0 \iff \lambda \in c_0$ (compact)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{\sqrt{n}} e_n$.
- ▷ $T \in \mathcal{B}^2 \iff \lambda \in \ell^2$ (Hilbert-Schmidt)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n} e_n$.
- ▷ $T \in \mathcal{B}^1 \iff \lambda \in \ell^1$ (trace-class)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n^2} e_n$.
- ▷ $T \in \mathcal{B}_{00} \iff \lambda \in c_{00}$ (degenerate or finite rank)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \alpha_n e_n \mathbb{1}_{[N]}(n)$ for $\alpha_n \in \mathbb{C}$ and $N \in \mathbb{N}$.

Remark 1.1 *Since the dual of c_0 is ℓ^1 and the dual of ℓ^1 is ℓ^∞, we have $\mathcal{B}_0^* = \mathcal{B}^1$ and $(\mathcal{B}^1)^* = \mathcal{B}^\infty$. Similarly, $(\mathcal{B}^2)^* = \mathcal{B}^2$.*

Theorem 1.2 (Operator inclusions) $\mathcal{B}_{00} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{B}_0 \subset \mathcal{B}^\infty$

Proof.

- ▷ $\mathcal{B}_{00} \subset \mathcal{B}^1$ Trivial
- ▷ $\mathcal{B}^1 \subset \mathcal{B}^2$
- ▷ $\mathcal{B}^2 \subset \mathcal{B}_0$
- ▷ $\mathcal{B}_0 \subset \mathcal{B}^\infty$ ((<BMC2009>), Proposition 4.6) If T is unbounded, we can find a sequence of unit vectors (e_n) such that $\|Te_n\| \nearrow \infty$. So Te_n cannot have a convergent subsequence, for if $Te_n \to x$, then $\|Te_n\| \to \|x\|$.

□

Proposition 1.3 *For $T \in \mathcal{B}^\infty$, $\|T\|_\infty = \sup\{|\langle Tx, y\rangle|\} : \|x\| = 1, \|y\| = 1$.*

Proof.

($\leq$) Since $\|Tx\| = \frac{\|Tx\|^2}{\|Tx\|} = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \left\langle Tx, \frac{Tx}{\|Tx\|}\right\rangle$, we have

$$\|T\|_\infty = \sup\{\|Tx\| : \|x\| = 1\} \leq \sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\}.$$

($\geq$) Since $\langle Tx, y\rangle \leq \|Tx\| \|y\| \leq \|T\|_\infty \|x\| \|y\|$, we have

$$\sup \{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\} \leq \|T\|_\infty .$$

□

2.1.2 Projection operators

Proposition 1.4 $\|P\|_\infty \leq 1$.

Proof. Since $\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PPx, x\rangle = \langle Px, x\rangle \leq \|Px\| \|x\|$, we have $\|P\|_\infty \leq 1$. □

Proposition 1.5 *A projection operator is compact iff its image is finite dimensional.*

Proof.

($\Longrightarrow$) Let $P : H \to H$ be a projection operator, so that $P^2 = P$, or $P(P - I) = 0$.

($\Longleftarrow$) Since the image is finite dimensional, fix an orthonormal basis $e_1, \ldots, e_n$ of $\operatorname{im} T$.

□

2.2 Optimization

2.2.1 Duality in optimization is the same as duality in functional analysis

For an various intuitions of duality in optimization, see MSx223235.

Let X and Y be Banach spaces, and X^* and Y^* be their (algebraic?) duals. Consider the two problems, with ϕ_0, y_0 fixed. Here $(\cdot, \cdot)$ denotes the canonical duality pairing.

$$
\begin{array}{lll} & \max & (\phi_0, x) \\ \text{(Primal)} & \text{s.t.} & Tx \leq y_0 \\ & & x \geq 0 \end{array}
\qquad\qquad
\begin{array}{lll} & \min & (\psi, y_0) \\ \text{(Dual)} & \text{s.t.} & T^*\psi \geq \phi_0 \\ & & \psi \geq 0 \end{array}
$$

See the following diagram for more details.

$$
\begin{tikzcd}
 & x \arrow[r, mapsto, "T"] & Tx & \\
x \in & X \arrow[r, "T"] \arrow[d, dash] & y_0 \arrow[d, dash] & \ni Tx, y_0 \\
\phi_0, T^*\psi \in & X^* & y_0^* \arrow[l, "T^*"'] & \ni \psi \\
 & T^*\psi & \psi \arrow[l, mapsto, "T^*"'] &
\end{tikzcd}
$$

Bibliography