Generalization of stochastic calculus and its applications in large deviations theory

Sudip Sinha

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Advisors
Prof Hui-Hsiung Kuo
Prof Padmanabhan Sundar

Outline

§ 1
Introduction and motivation

Quick revision and notations

- Let $T \in (0, \infty)$, and denote $\mathbb{T} = [0, T]$ as the index set for t.
- Let $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$ be a filtered probability space.
- $B_\cdot$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$.
- Properties of $B_\cdot$
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t-s)$
  - continuous paths
  - has unbounded linear variation ☹
  - has bounded quadratic variation ☺
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- Naive stochastic integration w.r.t. B_t: not possible.
- A stochastic process X_t is called $(\mathcal{F}_t)$-adapted if $\forall t$, X_t is measurable w.r.t. $\mathcal{F}_t$.

Wiener integral ($f \in L^2[0,T]$)

- Definition

  1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j,t_{j+1})}(t)$: Define $\int_0^T f(t)\,\mathrm{d}B_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
  2. $f \in L^2[0,T]$: Use step functions approximating f to extend the integral a.s.

- Properties

  ⋆ Linear.

  ⋆ Gaussian distribution with mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).

  ⋆ Corresponds to the Riemann–Stieltjes integral for $f \in C[0,T]$.

- The associated process $I_t = \int_0^t X_s\,\mathrm{d}B_s$ has the following properties.

  ⋆ continuity

  ⋆ martingale

- Problem: Cannot integrate stochastic processes.

Trying to integrate stochastic processes naively

- $\int_0^T B_t \, \mathrm{d}B_t \overset{?}{=}$.
  Since B_t is continuous, let us try Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \to 0$. Then
  $$\int\limits_0^T B_t \, \mathrm{d}B_t = \lim \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$
- Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, \mathrm{d}B_s$	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2}\left(B_t^2 - t\right)$	0	☺	Itô
mid	$\frac{1}{2}\left(B_t^2\right)$	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2}\left(B_t^2 + t\right)$	t	☹	[AK08]

- Which one do we choose?

Itô integral ($X \in L^2_{\text{ad}}([0,T]\times\Omega)$)

- Definition

  1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega)\mathbb{1}_{[t_j,t_{j+1})}(t)$: define $\int_0^T X_t \, \mathrm{d}B_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
  2. $X \in L^2_{\text{ad}}([0,T]\times\Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

- Properties

  ⋆ Linear.

  ⋆ Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).

  ⋆ For $X_\cdot$ continuous, $\int_0^T X_t \, \mathrm{d}B_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, \mathrm{d}B_t = \lim \sum_{j=0}^{n-1} X_{\lfloor \frac{tn}{n} \rfloor} \Delta B_j$.

- The associated process $I_t = \int_0^t X_s \, \mathrm{d}B_s$ has the following properties.

  ⋆ continuity

  ⋆ martingale

- Example: $\int_0^T B_t \, \mathrm{d}B_t = \frac{1}{2}(B_T^2 - T)$.

Itô integral ($\int_0^T X_t^2 \, \mathrm{d}t < \infty$ a.s.)

- Definition: Use sequences of processes in $L^2_{\mathrm{ad}}([0,T] \times \Omega)$ approximating X in probability to extend the integral in probability.

- Properties

  ⋆ Linear.
  ⋆ Mean 0, but variance? 😐.

- The associated process $I_t = \int_0^t X_s \, \mathrm{d}B_s$ has the following properties.

  ⋆ continuity
  ⋆ local martingale

- Example: $\int_0^T e^{B_t^2} \, \mathrm{d}B_t = \int_0^{B_1} e^{t^2} \, \mathrm{d}t - \int_0^T B_t e^{B_t^2} \, \mathrm{d}t.$

Itô formula

- An Itô process is a process of the form $X_t = X_0 + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$, equivalently expressed as $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t$.
  [Only makes sense when $\int_0^T \left(|m(s, X_s)| + |\sigma(s, X_s)|^2\right)\mathrm{d}s < \infty$ a.s.]

Theorem ([Itô44]) Let X_t be a d-dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d-dimensional Itô process, and

$$\mathrm{d}f(X_t) = \langle (\mathrm{D}f)(X_t), \mathrm{d}X_t \rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D^2 f)(X_t)\,\mathrm{d}X_t \right\rangle,$$

where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2\mathcal{E}_t\,\mathrm{d}t + \sigma\mathcal{E}_t\,\mathrm{d}B_t + \frac{1}{2}\sigma^2\mathcal{E}_t(\mathrm{d}B_t)^2$.

Exponential processes and Girsanov theorem

- Let $h_\cdot$ be a stochastic process. The associated exponential process is defined as

$$\mathcal{E}_t^{(h)} = \exp\left(\int_0^t h_s \,\mathrm{d}B_s - \frac{1}{2}\int_0^t h_s^2 \,\mathrm{d}s \right).$$

- The exponential process is a martingale if and only if $\mathbb{E}\mathcal{E}_t = 1 \;\forall t$.

- (Novikov condition) The exponential process is a martingale if $\mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_s^2 \,\mathrm{d}s\right) < \infty \;\forall t$.

- (Girsanov theorem) The translated stochastic process $W_t = B_t - \int_0^t h(s)\,\mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^h$. Moreover the process $Z_t := \mathbb{E}\left(\mathcal{E}_T^h \mid \mathcal{F}_t\right)$ is a martingale.

Stochastic differential equations

- Let $\xi \in L^2(\Omega)$ be independent of $B_\cdot$, and $m, \sigma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}[0,T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable such that $m(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable $\forall t$.
  Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \xi + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$ if for each t, the X_t satisfies the integral equation a.s.

- The stochastic *differential* equation $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t,\ X_0 = \xi$ is a *symbolic representation* of the stochastic integral equation.

Theorem (Existence and uniqueness, Markov property) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

⋆ (Lipschitz condition) $|m(t,x) - m(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \leq M|x-y|^2$ a.s.

⋆ (growth condition) $|m(t,x)|^2 + |\sigma(t,y)|^2 \leq M\left(1 + |x|^2\right)$ a.s.

The solution is a Markov process. Moreover if $\xi \in \mathbb{R}$, then the solution is also stationary.

- Example: For σ constant, $\mathrm{d}\mathcal{E}_t = \sigma \mathcal{E}_t\,\mathrm{d}B_t, \mathcal{E}_0 = 1$ is solved by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

- How to define the double integral?
- $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T \mathrm{d}B_s \int_0^T \mathrm{d}B_t = B_T^2$.
  But $\mathbb{E} \int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = t \neq 0$, so no martingale property. ☹
- Itô's idea: remove the diagonal to get $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = 2 \int_0^T \int_0^t \mathrm{d}B_s \, \mathrm{d}B_t = B_T^2 - T$. ☺

Theorem ([Itô51]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$I_n(f) = \int_{[0,T]^n} f(t_1, \dots, t_n) \, \mathrm{d}B_{t_1} \dots \mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, \mathrm{d}B_{t_n} \dots \mathrm{d}B_{t_1},$$

- TODO

§ 2

Generalization of Itô calculus

Motivation

- Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T B_T \, \mathrm{d}B_t \stackrel{?}{=} B_T B_t$.
- Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so no martingale property ☹.
- Stochastic differential equations with anticipation

$$\begin{aligned} \mathrm{d}X_t &= X_t \, \mathrm{d}B_t \\ X_0 &= B_1 \end{aligned} \qquad\qquad \begin{aligned} \mathrm{d}Y_t &= B_T \, \mathrm{d}B_t \\ Y_0 &= 1 \end{aligned}$$

- Problem: We want to define $\int_0^T X_t \, \mathrm{d}B_t$, where $X_\cdot$ is not adapted (anticipating).
- Some approaches
  - ⋆ Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s \right) + \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s$
  - ⋆ Enlargement of filtration
  - ⋆ White noise theory
  - ⋆ ...

The new integral [AK08; AK10]: Idea

- A process $Y^{\cdot}$ and filtration $\mathcal{F}_{\cdot}$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$.
- Ideas

  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.

- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\cdot}$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is defined as

$$\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

  provided that the limit exists in probability.

- Now, for any stochastic process $Z(t) = \sum_{k=1}^{n} X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.
- This is well-defined [HKS+16].

A simple example

- In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and lim is the limit in L^2.

$$\int_0^t B_T \, \mathrm{d}B_t = \int_0^t (B_t + (B_T - B_t)) \, \mathrm{d}B_t = \int_0^t B_t \, \mathrm{d}B_t + \int_0^t (B_T - B_t) \, \mathrm{d}B_t$$

$$= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j$$

$$= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j \right) \Delta B_j$$

$$= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t$$

- Note that $\mathbb{E}(B_T B_t - t) = 0$.
- In general, $\mathbb{E} \int_0^t Z(t) \, \mathrm{d}B_t = 0$. ☺

Generalized Itô formula [HKS+16]

- Let $\mathrm{d}X_t = m(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$ be an d-dimensional Itô process, $Y^t = \tilde{m}(t)\,\mathrm{d}t + \tilde{\sigma}(t)\,\mathrm{d}B_t$ be a $\tilde{d}$-dimensional instantly independent process, $f(x,y) \in C^2(\mathbb{R}^2)$. Then

$$\mathrm{d}f(X_t, Y^t) = \langle (\mathrm{D}_x f)(X_t, Y^t),\, \mathrm{d}X_t \rangle + \frac{1}{2}\Big\langle \mathrm{d}X_t, (D_x^2 f)(X_t, Y^t)\ \mathrm{d}X_t \Big\rangle$$
$$+ \Big\langle (\mathrm{D}_y f)(X_t, Y^t),\, \mathrm{d}Y^t \Big\rangle - \frac{1}{2}\Big\langle \mathrm{d}Y^t, (D_y^2 f)(X_t, Y^t)\ \mathrm{d}Y^t \Big\rangle,$$

  where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: TODO

Exponential processes and generalized Girsanov theorem

- TODO

Iterated integrals

Theorem ([Itô51]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\,\mathrm{d}B_{t_1}\dots\mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-1}} \hat{f}(t_1,\dots,t_n)\,\mathrm{d}B_{t_n}\dots\mathrm{d}B_{t_1},$$

Theorem ([AK10]) Let $f \in L^2([0,T]^n)$. Then

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\,\mathrm{d}B_{t_1}\dots\mathrm{d}B_{t_n} = \int_0^T \cdots \int_0^T f(t_1,\dots,t_n)\,\mathrm{d}B_{t_n}\dots\mathrm{d}B_{t_1}.$$

Near-martingale property [HKS+17]

- Question: What are the analogues of the martingale property and the Markov property?
- Answer for martingales: "near-martingales".
- Let $Z(t)$ be a stochastic process such that $\mathbb{E}\,|Z(t)| < \infty\ \forall t$, and $0 \le s \le t \le T$. Then, with respect to $\mathcal{F}_\cdot$, the process $Z(t)$ is called a
  - ⋆ near-martingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) = 0$,
  - ⋆ near-submartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \ge 0$, and
  - ⋆ near-supermartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \le 0$.
- TODO

§ 3
Conclusion

Open areas for research

- Extension to SDEs with anticipating coefficients
- Near-Markov property
- Girsanov theorem for anticipating integrals
- Freidlin-Wintzell type result for SDEs with anticipation

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