Mathematical Logic

Notes and Exercises

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1 Sudip Sinha PHIL 4010: HW1 2019-09-10

Exercise 1.1 **(Notes, 1.8)** *For any sets A and B, we have $A \cap B \subseteq A$.*

Solution. Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A, we have $A \cap B \subseteq A$. $\square$

Exercise 1.2 **(Notes, 1.10)** *For any set A, we have $A \cap \emptyset = \emptyset$.*

Solution. ($\subseteq$) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.
($\supseteq$) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. $\square$

Exercise 1.3 **(Notes, 1.13)** *For any sets A and B, if $A \subseteq B$, then $A \cup B = B$.*

Solution. ($\subseteq$) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.
($\supseteq$) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. $\square$

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Note: We shall say that a truth assigment v satisfies Σ iff it satisfies every member of Σ.

Exercise 2.1 (Enderton, 1.2.1) *Show that neither of the following two formulas tautologically implies the other:*

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$

Solution. We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = \top$ and $\bar{v}(\beta) = \bot$.

Consider v such that $v(A) = v(B) = \bot$ and $v(C) = \top$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below. (Here the truth assignments by $\bar{v}$ is denoted under each symbol.)

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$\top \quad \bot \; \top \; \bot \; \bot \; \top$$

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$
$$\bot \quad \bot \; \bot \quad \bot \quad \bot \quad \bot \;\; \bot \top$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = \top$ and $\bar{v}(\alpha) = \bot$.

Consider v such that $v(A) = v(B) = v(C) = \bot$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below.

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$
$$\top = \quad \top \quad \top \bot \; \top \quad \top \bot \; \top \; \top \bot$$

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$\bot = \; \bot \; \bot \; \bot \; \top \; \bot$$

$\square$

Exercise 2.2 (Enderton, 1.2.4(a)) *Show that* $\Sigma \cup \{\alpha\} \models \beta$ *iff* $\Sigma \models (\alpha \to \beta)$.

Solution. We show each direction separately.

($\Longrightarrow$) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ. We have to show that v satisfies $(\alpha \to \beta)$. We have two cases.

i. $\bar{v}(\alpha) = \top$: In this case, from the supposition, we get $\bar{v}(\beta) = \top$. So $\bar{v}(\alpha \to \beta) = \top$.

ii. $\bar{v}(\alpha) = \bot$: In this case, $\bar{v}(\alpha \to \beta) = \top$ since the antecedent is $\bot$.

Since v was arbitrary, we have $\Sigma \models (\alpha \to \beta)$.

($\Longleftarrow$) We suppose $\Sigma \models (\alpha \to \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β. Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ. Therefore, by our supposition, v satisfies $(\alpha \to \beta)$. Now, since v satisfies α, it can only be that v satisfies β, since the only other way the material implication can be satisfied is when v does not satisfies α. This proves our claim. □

Exercise 2.3 (Enderton, 1.2.5) *Prove or refute each of the following assertions:*

a. If either $\Sigma \models \alpha$ *or* $\Sigma \models \beta$, *then* $\Sigma \models (\alpha \vee \beta)$.

Solution. ($\top$) There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α. This implies $\bar{v}(\alpha \vee \beta) = \top$ by the definition of extension of $\bar{v}$ for $\vee$. □

b. If $\Sigma \models (\alpha \vee \beta)$, *then either* $\Sigma \models \alpha$ *or* $\Sigma \models \beta$.

Solution. ($\bot$) We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \vee (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$.

For an explicit example, consider two truth assignments v_1 and v_2, such that $v_1(\alpha) = \top$ and $v_2(\alpha) = \bot$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α, and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$. □

Exercise 2.4 **(Enderton, 1.2.6)**

a. *Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.*

Solution. Let G be the set of sentence symbols used in α, and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$.

Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α.

Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ. Let $\boxdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for $\bar{v}_1$ and $\bar{v}_2$, we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \boxdot \psi) = \bar{v}_2(\phi \boxdot \psi)$. Hence $(\neg\phi), (\phi \boxdot \psi) \in B$, that is, B is closed with respect to the formula building operations.

Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done. □

b. *Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ.*

Solution. In this part, we use v to denote truth assignments and "v on a set" means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ. Clearly, $G \subseteq S$.

We show each direction separately.

($\Longrightarrow$) From the definition of tautological implication,

$$\begin{aligned} & \Sigma \models \tau \\ \iff & (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \implies & (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \text{ [Part (a)]} \end{aligned}$$

($\Longleftarrow$) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ. Therefore,

$$\begin{aligned} & (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \implies & (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \iff & \Sigma \models \tau \end{aligned}$$

□

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Exercise 3.1 (Set Theory (10 × 3)) *Prove the following.*

Note: *Let A and B are sets. In order to prove $A = B$, it is enough to show $A \subseteq B$ and $A \supseteq B$. Moreover, to show $A \subseteq B$, it suffices to show that for an arbitrary x, we have $x \in A \implies x \in B$.*

i. If $A \subseteq B$, then $A \cap B = A$.

Solution. We show $A \cap B \subseteq A$ and $A \cap B \supseteq A$ separately.

($\subseteq$) Let x be arbitrary. Then

$$x \in A \cap B \iff x \in A \text{ and } x \in B \implies x \in A$$

($\supseteq$)

□

ii. If $A \cap B = \emptyset$, then $A \setminus B = A$.

Solution. ABCD □

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution. We show each inclusion separately.

($\subseteq$) Let x be arbitrary. Then

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in B \cup C \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \end{aligned}$$

If $x \in B$ or $x \in C$, we have two cases:

a. ($x \in B$) In this case, $x \in A$ and $x \in B$. Therefore

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in B \\ &\iff x \in A \cap B \\ &\implies x \in A \cap B \text{ or } x \in A \cap C \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

b. ($x \in C$) Interchanging the roles of B and C in Case 1, we get the exact same result.

Hence, from the above $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

($\supseteq$) Let x be arbitrary. Then $x \in (A \cap B) \cup (A \cap C) \iff x \in (A \cap B)$ or $x \in (A \cap C)$.

$$
\begin{aligned}
x \in (A \cap B) \cup (A \cap C) &\iff x \in (A \cap B) \text{ or } x \in (A \cap C) \\
&\iff x \in A \cap B \\
&\implies x \in A \cap B \text{ or } x \in A \cap C \\
&\iff
\end{aligned}
$$

□

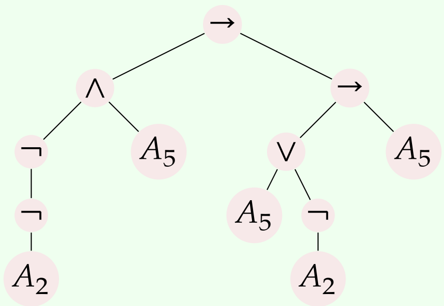
Exercise 3.2 (Construction (10 × 2))

i. Write down a construction sequence for $((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))$.

Solution. $\langle A_1, A_3, A_4, A_7, (\neg A_1), ((\neg A_1) \vee A_4), (\neg((\neg A_1) \vee A_4)), (A_1 \to A_3), ((A_1 \to A_3) \leftrightarrow A_7), ((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))\rangle$. □

ii. Write down a construction tree for $(((\neg(\neg A_2)) \wedge A_5) \to ((A_5 \vee (\neg A_2)) \to A_5))$.

Solution. □



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