Generalization of stochastic calculus and its applications in large deviations theory

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2019-04-05

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Outline

Section 1
Introduction and motivation

Quick revision and notations

- Let $T \in (0, \infty)$, and denote $[0, T]$ as the index set for t.
- Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ be a filtered probability space.
- $B_\bullet$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$.
- Properties of $B_\bullet$
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t-s)$
  - continuous paths
  - has unbounded linear variation ☹
  - has bounded quadratic variation ☺
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- Naive integration w.r.t. B_t: not possible.
- A stochastic process $X_\bullet$ is called adapted to $\mathcal{F}_\bullet$ if X_t is measurable w.r.t. $\mathcal{F}_t$ $\forall t$.

Wiener integral for $f \in L^2[0,T]$

- Definition

  1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j,t_{j+1})}(t)$: Define $\int_0^T f(t)\,\mathrm{d}B_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
  2. $f \in L^2[0,T]$: Use step functions approximating f to extend the integral a.s.

- Properties

  - Linear.
  - Gaussian distribution with mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).
  - Corresponds to the Riemann–Stieltjes integral for continuous functions of bounded variation.

- The associated process $I_\bullet = \int_0^\bullet f(t)\,\mathrm{d}B_t$ has the following properties.

  - continuity
  - martingale

- Problem: Cannot integrate stochastic processes.

Trying to integrate stochastic processes

- ▷ $\int_0^T B_t \, \mathrm{d}B_t \overset{?}{=}$
  Since B_t is continuous, let us try Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \to 0$. Then

$$\int_0^T B_t \, \mathrm{d}B_t = \lim \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

- ▷ Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, \mathrm{d}B_s$	Intuitive?	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2}\left(B_t^2 - t\right)$	☹	0	☺	Itô
mid	$\frac{1}{2}\left(B_t^2\right)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2}\left(B_t^2 + t\right)$	☹	t	☹	

- ▷ Which one do we choose?

Itô integral for $X_\cdot \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$

- Definition

  1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega)\mathbb{1}_{[t_j,t_{j+1})}(t)$: define $\int_0^T X_t \,\mathrm{d}B_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
  2. $X \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

- Properties

  - Linear.
  - Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).
  - For $X_\bullet$ continuous, $\int_0^T X_t \,\mathrm{d}B_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \,\mathrm{d}B_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$.

- The associated process $I_\bullet = \int_0^\bullet X_t \,\mathrm{d}B_t$ has the following properties.

  - continuity
  - martingale

- Example: $\int_0^t B_s \,\mathrm{d}B_s = \frac{1}{2}(B_t^2 - t) \quad \forall t.$

Itô integral for $X_{\cdot}$ such that $\int_0^T X_t^2 \, \mathrm{d}t < \infty$ a.s.

- Definition: Use sequences of processes in $L^2_{\mathrm{ad}}([0,T] \times \Omega)$ approximating X in probability to extend the integral in probability.
- Properties
  - Linear.
  - Mean and variance? ☹
- The associated process $I_{\cdot} = \int_0^{\bullet} X_t \, \mathrm{d}B_t$ has the following properties.
  - continuity
  - local martingale
- Example: $\int_0^T e^{B_t^2} \, \mathrm{d}B_t = \int_0^{B_1} e^{t^2} \, \mathrm{d}t - \int_0^T B_t e^{B_t^2} \, \mathrm{d}t$.

Itô formula

- An Itô process is a process of the form $X_t = X_0 + \int_0^t m_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s$. Equivalently expressed as $\mathrm{d}X_t = m_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t$.

Theorem ([Itô44]) Let X_t be a d-dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d-dimensional Itô process, and

$$\mathrm{d}f(X_t) = \langle (\mathrm{D}f)(X_t), \mathrm{d}X_t \rangle + \frac{1}{2} \left\langle \mathrm{d}X_t, (D^2 f)(X_t)\, \mathrm{d}X_t \right\rangle,$$

where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d \,\mathrm{d}t$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \,\mathrm{d}t + \sigma \mathcal{E}_t \,\mathrm{d}B_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (\mathrm{d}B_t)^2$.

Exponential processes and Girsanov theorem

- Let $h_\bullet$ be a stochastic process. The associated exponential process is defined as

$$\mathcal{E}_t^{(h)} = \exp\left(\int_0^t h_s \, \mathrm{d}B_s - \frac{1}{2} \int_0^t h_s^2 \, \mathrm{d}s \right).$$

- The exponential process is a martingale if and only if $\mathbb{E}\mathcal{E}_t^{(h)} = 1 \ \forall t$.

- (Novikov condition) The exponential process is a martingale if $\mathbb{E} \exp\left(\frac{1}{2}\int_0^T h_t^2 \, \mathrm{d}t\right) < \infty$.

- (Girsanov theorem) The translated stochastic process $W_t = B_t - \int_0^t h(s) \, \mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^{(h)}$.

Stochastic differential equations

- Let $\xi \in L^2(\Omega)$ be independent of $B_\bullet$, and $m, \sigma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}[0,T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable such that $m(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable $\forall t$.
  Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \xi + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$ if for each t, the X_t satisfies the integral equation a.s.

- The stochastic *differential* equation $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t,\ X_0 = \xi$ is a *symbolic representation* of the stochastic integral equation.

Theorem (Existence and uniqueness, Markov property) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

- (Lipschitz condition) $|m(t,x) - m(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \leq M|x-y|^2$ a.s.
- (growth condition) $|m(t,x)|^2 + |\sigma(t,y)|^2 \leq M\left(1 + |x|^2\right)$ a.s.

The solution is a Markov process.
Moreover if $\xi \in \mathbb{R}$ and m, σ are function of only x, then the solution is also stationary.

- Example: For σ constant, $\mathrm{d}\mathcal{E}_t = \sigma \mathcal{E}_t\,\mathrm{d}B_t, \mathcal{E}_0 = 1$ is solved by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

- How do we define the double integral?

- Naive idea: $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T \mathrm{d}B_s \int_0^T \mathrm{d}B_t = B_T^2$.
  But $\mathbb{E}B_T^2 = t \neq 0$, so no martingale property. ☹

- Itô's idea: remove the diagonal to get $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = 2\int_0^T \int_0^t \mathrm{d}B_s \, \mathrm{d}B_t = B_T^2 - T$. ☺

Theorem ([Itô51]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1, \dots, t_n) \, \mathrm{d}B_{t_1} \dots \mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, \mathrm{d}B_{t_n} \right) \mathrm{d}B_{t_{n-1}} \dots \mathrm{d}B_{t_1}.$$

- TODO

Section 2
Generalization of Itô calculus

Motivation

- Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T B_T \, \mathrm{d}B_t \stackrel{?}{=} B_T B_t$.
- Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so no martingale property ☹.
- Stochastic differential equations with anticipation

$$\begin{aligned} \mathrm{d}X_t &= X_t \, \mathrm{d}B_t & \qquad \mathrm{d}Y_t &= B_T \, \mathrm{d}B_t \\ X_0 &= B_1 & \qquad Y_0 &= 1 \end{aligned}$$

- Problem: We want to define $\int_0^T X_t \, \mathrm{d}B_t$, where $X_\bullet$ is not adapted (anticipating).
- Some approaches
  - Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s\right) + \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s$
  - Enlargement of filtration
  - White noise theory
  - ...

The new integral [AK08; AK10]: Idea

- A process $Y^{\bullet}$ and filtration $\mathcal{F}_{\bullet}$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$.
  Example: The process $(B_T - B_{\bullet})$ is instantly independent of the filtration generated by $B_{\bullet}$.
- Ideas
  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.
- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\bullet}$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is defined as

$$\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

  provided that the limit exists in probability.
- Now, for any stochastic process $Z(t) = \sum_{k=1}^{n} X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.
- This is well-defined [HKS+16].

A simple example

- In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and lim is the limit in L^2.

$$\int_0^t B_T \,\mathrm{d}B_t = \int_0^t (B_t + (B_T - B_t))\,\mathrm{d}B_t = \int_0^t B_t \,\mathrm{d}B_t + \int_0^t (B_T - B_t)\,\mathrm{d}B_t$$
$$= \lim \sum_{j=0}^{n-1} B_{t_j}\Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}})\Delta B_j$$
$$= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j\right) \Delta B_j$$
$$= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t$$

- Note that $\mathbb{E}(B_T B_t - t) = 0$.
- In general, $\mathbb{E} \int_0^t Z(s)\,\mathrm{d}B_s = 0$. ☺

Generalized Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_t = X_0 + \int_0^t m_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s$	$\mathrm{d}X_t = m_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t$
instantly independent	$Y^t = Y^0 + \int_t^T \eta^s \,\mathrm{d}s + \int_t^T \varsigma^s \,\mathrm{d}B_s$	$\mathrm{d}Y^t = -\eta^t \,\mathrm{d}t - \varsigma^t \,\mathrm{d}B_t$

Theorem ([HKS+16]) Let $\mathrm{d}X_t = m_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t$ be an d-dimensional Itô process, $\mathrm{d}Y^t = -\eta^t \,\mathrm{d}t - \varsigma^t \,\mathrm{d}B_t$ be a $\tilde{d}$-dimensional instantly independent process. If $f(x,y) \in C^2(\mathbb{R}^2)$, then

$$\mathrm{d}f(X_t, Y^t) = \langle (\mathrm{D}_x f)(X_t, Y^t), \mathrm{d}X_t \rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D_x^2 f)(X_t, Y^t)\ \mathrm{d}X_t \right\rangle$$
$$+ \left\langle (\mathrm{D}_y f)(X_t, Y^t), \mathrm{d}Y^t \right\rangle - \frac{1}{2}\left\langle \mathrm{d}Y^t, (D_y^2 f)(X_t, Y^t)\ \mathrm{d}Y^t \right\rangle,$$

where we use the rules $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d \,\mathrm{d}t$.

Exponential processes and generalized Girsanov theorem

- TODO

Iterated integrals

Theorem ([Itô51]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1, \dots, t_n)\, \mathrm{d}B_{t_1} \dots \mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n)\, \mathrm{d}B_{t_n} \right) \mathrm{d}B_{t_{n-1}} \dots \mathrm{d}B_{t_1}.$$

Theorem ([AK10]) Let $f \in L^2([0,T]^n)$. Then

$$\int_{[0,T]^n} f(t_1, \dots, t_n)\, \mathrm{d}B_{t_1} \dots \mathrm{d}B_{t_n} = \int_0^T \cdots \int_0^T f(t_1, \dots, t_n)\, \mathrm{d}B_{t_n} \dots \mathrm{d}B_{t_1}.$$

Near-martingale property [HKS+17]

- Question: What are the analogues of the martingale property and the Markov property?
- Answer for martingales: "near-martingales".
- Let $Z(t)$ be a stochastic process such that $\mathbb{E}\,|Z(t)| < \infty\ \forall t$, and $0 \leq s \leq t \leq T$. Then, with respect to $\mathcal{F}_\bullet$, the process $Z(t)$ is called a
  - near-martingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) = \mathbb{E}(Z(s) \mid \mathcal{F}_s)$,
  - near-submartingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \geq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$, and
  - near-supermartingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \leq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$.

Theorem ([HKS+17]) Let $Z(\cdot)$ be a stochastic process bounded in L^1, and $X_\bullet = \mathbb{E}(Z(\cdot) \mid \mathcal{F}_\bullet)$. Then $X_\bullet$ is a martingale if and only if $Z(\cdot)$ is a near-martingale.

Section 3
Large deviations theory

What is it about?

- A theory of characterization of rare events by providing exponential bounds on their probability.
- Started by Swedish actuarials Fredrik Esscher, Harald Cramér, Filip Lundberg.
- Unified by Varadhan in his 1966 paper [Var66].
- Example: A problem faced by the insurance industry.
  - Value of claims received on the nth day: X_n \$.
  - Steady income from premium: x \$/day.
  - Planning period: n days.
  - Average expenditure: $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$ \$/day.
  - *Question*: How should the company decide on the premium?
  - *Idea*: Choose x such that $\mathbb{P}\left\{\bar{X}_n > x\right\}$ is exponentially low.

Insurance problem: setup

1. Let the following hold:

    - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
    - (X_n) is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moment generating function M.
    - $\mathbb{E}X_1 = m$, $\mathbb{V}X_1 = \sigma^2$, and $X_1 \sim \mu$.
    - $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$.

2. Asymptotic behavior of $\bar{X}_n$:

    - Weak law of large numbers: $\bar{X}_n \xrightarrow{\mathbb{P}} m$.
    - Central limit theorem: $\sqrt{n}(\bar{X}_n - m) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$.

3. But at what rate?

4. We want to “control large deviations from the mean”.

Insurance problem: large deviation bounds

1. For $x > m$ and an arbitrary $\theta > 0$, we get

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} = \mathbb{P}\left\{e^{\theta n \bar{X}_n} \geq e^{\theta n x}\right\} \leq e^{-\theta n x}\mathbb{E}\left(e^{\theta n \bar{X}_n}\right) = e^{-\theta n x} M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}.$$

2. Since θ was arbitrary, we have

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} \leq \inf_{\theta} e^{-n(\theta x - \log M_X(\theta))} = e^{-n \sup_{\theta}(\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \leq -\inf_{\bar{E}} I \qquad \forall E \in \mathscr{B}.$$

4. We can also obtain a lower bound using an exponential change of measure

$$\underline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -\inf_{E^\circ} I \qquad \forall E \in \mathscr{B}.$$

5. We formally write $\mathbb{P}\left\{\bar{X}_n \in \mathrm{d}x\right\} \asymp e^{-nI(x)}\,\mathrm{d}x$ for $x \in \mathbb{R}$.

Definition of large deviation principle

- The setup: (X_n) is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space $(\mathcal{X}, d)$.
- A function $I : \mathcal{X} \to [0, \infty]$ is called a rate function if it has compact level sets.

Definition (X_n) is said to satisfy the large deviation principle on $\mathcal{X}$ with rate function I if the following two conditions hold.

$$\text{(upper bound)} \qquad \overline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in F\right\} \leq -\inf_F I \qquad \forall F \text{ closed}$$

$$\text{(lower bound)} \qquad \underline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -\inf_G I \qquad \forall G \text{ open}$$

Laplace principle and equivalence to LDP

Definition (Laplace principle) (X_n) is said to satisfy the Laplace principle on $\mathcal{X}$ with rate function I if for all bounded continuous functions h, we have

$$\lim \frac{1}{n} \log \mathbb{E} \exp(-nh(X_n)) = \inf_{\mathcal{X}} (h + I)$$

Theorem (X_n) satisfies LP on $\mathcal{X}$ with rate function I if and only if (X_n) satisfies LDP on $\mathcal{X}$ with the same rate function I.

Some important results

- Uniqueness of the rate function.
- Continuity principle.
- Superexponential approximation preserves Laplace principle.

Cramér theorem

Theorem ([Cra38]) Let (X_n) be a sequence of i.i.d. real random variables with finite moment generating function M. Then (X_n) follows LDP with rate function $I(x) = \sup_\theta (\theta x - \log M(\theta))$.

Rate functions for some common distributions

Distribution	$M(\theta)$	$I(x)$
$Bern(p)$	$1-p+pe^\theta$	$\left(x\log\frac{x}{1-p} + (x-1)\log\frac{p}{x-1}\right)\mathbb{1}_{[0,1]}(x) + \infty\mathbb{1}_{[0,1]^\complement}(x)$
$Pois(\lambda)$	$e^{\lambda(e^\theta-1)}$	$\left(\lambda - x + x\log\frac{x}{\lambda}\right)\mathbb{1}_{[0,\infty)}(x) + \infty\mathbb{1}_{(-\infty,0)}(x)$
$Exp(\lambda)$	$\left(1-\frac{\theta}{\lambda}\right)^{-1}$	$(\lambda x - 1 + x\log(\lambda x))\,\mathbb{1}_{[0,\infty)}(x) + \infty\mathbb{1}_{(-\infty,0)}(x)$
$\mathcal{N}(m,\sigma^2)$	$e^{m\theta+\frac{1}{2}\sigma^2\theta^2}$	$\frac{(x-m)^2}{2\sigma^2}$
$\chi^2(k)$	$(1-2\theta)^{-\frac{k}{2}}$	$\frac{1}{2}\left(x-k+k\log\frac{k}{x}\right)$

Schilder theorem

- Aim: Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path.

- Setup
  - Let $B_\bullet$ be a d-dimensional Brownian motion, so $B_\bullet \in C_0 = C_0([0,T];\mathbb{R}^d)$
  - $\forall \varepsilon > 0$, let $\sqrt{\varepsilon} B_t \sim W^{(\varepsilon)}$. Then $W^{(\varepsilon)} = \mathcal{N}(0, \varepsilon t) \xrightarrow{\mathcal{D}} \delta_0$ as $\varepsilon \to 0$.
  - Let $\mathrm{CM} = \{\omega \in C_0 : \omega$ is absolutely continuous and $\dot{\omega}_t \in L^2[0,T]\}$.

Theorem On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W^{(\varepsilon)} : \varepsilon > 0\}$ satisfies LDP with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 \, \mathrm{d}t \right) \mathbb{1}_{\mathrm{CM}}(\omega) + \infty \mathbb{1}_{\mathrm{CM}^{\mathrm{c}}}(\omega)$$

Freidlin–Wentzell theorem

- Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

- Setup

  - $\forall \varepsilon > 0$, let $X^{(\varepsilon)}_\bullet$ be the solution of the SDE $\mathrm{d}X^{(\varepsilon)}_t = m(X^{(\varepsilon)}_t)\,\mathrm{d}t + \sigma(X^{(\varepsilon)}_t)\sqrt{\varepsilon}\,\mathrm{d}B_t,\ X^{(\varepsilon)}_0 = x$, where m, σ are sufficiently nice.
  - Let $W^{(\varepsilon)}_x$ denote the law of $X^{(\varepsilon)}_\bullet$ starting at x.
  - As $\varepsilon \to 0$, $W^{(\varepsilon)}_x \xrightarrow{\mathscr{D}} \delta_\xi$, where ξ solves the ODE $\dot{\xi}(t) = m(\xi(t)),\ \xi(0) = x$.

Theorem For x fixed, the family of probability measures $\left\{W^{(\varepsilon)}_x : \varepsilon > 0\right\}$ satisfies LDP with the rate function $I_x : C_0 \to \overline{\mathbb{R}}$ given by

$$I_x(\omega) = \left(\frac{1}{2}\int_0^T \left\langle \dot{\omega}_t - b(\omega_t), A^{-1}(\omega_t)(\dot{\omega}_t - b(\omega_t))\right\rangle \mathrm{d}t\right) \mathbb{1}_{\mathrm{CM}\cap\{\omega_0 = x\}}(\omega) + \infty \mathbb{1}_{CM^{\mathrm{C}}\cup\{\omega_0 \neq x\}}(\omega),$$

where $A = \sigma\sigma^*$.

Section 4
Possible directions

Open problems

- ▷ Near-Markov property.
- ▷ Extension to stochastic differential equations with anticipating coefficients.
- ▷ Freidlin–Wentzell type results for stochastic differential equations with anticipation.
- ▷ Identification of the class of integrable processes under the new integral.
- ▷ Girsanov theorem for anticipating integrals.

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