Functional analysis

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Contents

Part 1

Preliminaries

1.1 Relationships between structures

Let X be a set.

Definition 1.1

1. *A* basis *of a topology is is a collection* $\mathcal{B}$ *of subsets of* X *satisfying the following properties:*

   i. (*cover*) *The base elements cover* X.
   ii. (*intersection*) *For every* $B_1, B_2 \in \mathcal{B}$, *if* $x \in B_1 \cap B_2$, *then there is a* $B_x \in \mathcal{B}$ *such that* $x \in B_x \subseteq B_1 \cap B_2$.

2. *A* metric *is a function* $d(\cdot,\cdot) : X \times X \to [0, \infty)$ *such that for all vectors* $x, y, z \in X$, *we have*

   i. (*identity of indiscernibles*) $d(x, y) = 0$ *iff* $x = y$.
   ii. (*symmetry*) $d(x, y) = d(y, x)$.
   iii. (*triangle inequality*) $d(x, z) \leq d(x, y) + d(y, z)$.

3. *A* norm *is a function* $\|\cdot\| : X \to [0, \infty)$ *such that for all vectors* $x, y \in X$ *and scalar* $\alpha \in \mathbb{C}$, *we have*

   i. (*identity of indiscernibles*) $\|x\| = 0$ *iff* $x = 0$.
   ii. (*scaling*) $\|\alpha x\| = |\alpha| \, \|x\|$.
   iii. (*triangle inequality*) $\|x + y\| \leq \|x\| + \|y\|$.

4. *An* inner product *is a function* $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ *such that for all vectors* $x, y, z \in X$ *and scalar* $\alpha \in \mathbb{C}$, *and we have*

   i. (*positive-definiteness*) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *iff* $x = 0$.
   ii. (*conjugate symmetry a.k.a. Hermitian*) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
   iii. (*sesquilinearity*) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.

Proposition 1.2 *Inner product* $\Longrightarrow$ *norm* $\Longrightarrow$ *metric* $\Longrightarrow$ *topology.*

Proof.

A. *inner product* $\Longrightarrow$ *norm*. Define the norm as $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

   i. $\|x\| = 0 \iff \|x\|^2 = 0 \iff \langle x, x \rangle = 0 \iff x = 0$ using 4.i.
   ii. $\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle = \langle \overline{\alpha} \alpha x, x \rangle = \overline{\alpha} \alpha \langle x, x \rangle = |\alpha|^2 \, \|x\|^2$ using 4.ii and 4.iii.
   iii.

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle && [4.iii] \\
&= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2 && [4.ii] \\
&= \|x\|^2 + 2\Re\langle x,y\rangle + \|y\|^2 \\
&\le \|x\|^2 + 2\,|\langle x,y\rangle| + \|y\|^2 \\
&\le \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 && [1.2] \\
&= (\|x\| + \|y\|)^2 .
\end{aligned}$$

B. *norm* $\Longrightarrow$ *metric*. Define the metric as $d(x,y) = \|x-y\|$.

i. $d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y=0 \iff x=y$ using 3.i.
ii. $d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\,\|y-x\| = d(y,x)$ using 3.ii.
iii. $d(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \le \|x-y\| + \|y-z\| = d(x,y)+d(y,z)$ using 3.iii.

C. *metric* $\Longrightarrow$ *topology*. A good description is in this MSx1409687 answer.

Define the basis of the topology as open balls of the form

$$D_r(x_0) = \{x \in X \mid d(x,x_0) < r\}, \qquad x_0 \in X, r > 0.$$

That is, $\mathcal{T} = \tau(\mathcal{B})$, where $\mathcal{B} = \{D_r(x_0) \mid x_0 \in X, r > 0\}$.

All we have to do is show that $\mathcal{B}$ is a basis. The cover is obvious. Note that for any $B_1, B_2 \in \mathcal{B}$, we can write $B_1 = D_{r_1}(x_1), B_2 = D_{r_2}(x_2)$. Suppose $x \in B_1 \cap B_2$. Then $x \in D_r(x) \subseteq D_{r_1}(x_1) \cap D_{r_2}(x_2)$ if $r \le \min\{r_1 - d(x,x_1), r_2 - d(x,x_2)\}$, and we are done.

The topology induced by the metric is called the *metric topology*.

$\square$

1.2 Strong, weak and weak* convergence

Disclaimer: This section is shamelessly copied from Christopher Heil's notes.

Definition 2.1 *Let X be a normed vector space, and $x_n, x \in X$. We define the following convergences as $n \to \infty$.*

$$\begin{array}{rccl} (strong) & x_n \to x & \iff & \|x_n - x\| \to 0 \\ (weak) & x_n \xrightarrow{w} x & \iff & \forall \phi \in X^*, \quad (x_n - x, \phi) \to 0 \end{array}$$

Definition 2.2 *Let X be a normed vector space, and $\phi_n, \phi \in X^*$. We define the following convergences as $n \to \infty$.*

$$\begin{array}{rccl} (strong) & \phi_n \to \phi & \iff & \|\phi_n - \phi\| \to 0 \\ (weak) & \phi_n \xrightarrow{w} \phi & \iff & \forall \xi \in X^{**}, \quad (\phi_n - \phi, \xi) \to 0 \\ (weak^*) & \phi_n \xrightarrow{w^*} \phi & \iff & \forall x \in X, \quad (x, \phi_n - \phi) \to 0 \end{array}$$

Remark 2.3 *Weak* convergence is simply* pointwise convergence *for the functionals ϕ_n.*

Proposition 2.4 (strong $\implies$ weak $\implies$ weak* for convergence) *Suppose $\phi_n, \phi \in X^*$. Then $\phi_n \to \phi \implies \phi_n \xrightarrow{w} \phi \implies \phi_n \xrightarrow{w^*} \phi$.*
The second implication reverses if X is reflexive.

Proof. *strong $\implies$ weak*: $\quad (x_n - x, \phi) \le \|x_n - x\| \, \|\phi\| \to 0.$
*weak $\implies$ weak**: $\quad (x, \phi_n - \phi) = (\phi_n - \phi, x^{**}) \to 0.$
The claim about the reverse implication is now obvious.

Counterexample for converse of the first implication: Suppose $X = \ell^2(\mathbb{N})$. Then $e_n \xrightarrow{w} 0$, but $\|e_n - 0\| = 1 \nrightarrow 0$. □

Proposition 2.5 *In Hilbert spaces, weak convergence plus convergence of norms ($\|x_n\| \to \|x\|$) is equivalent to strong convergence.*

Proof. $\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \langle x_n - x, x_n\rangle - \langle x_n - x, x\rangle \to 0.$ □

Proposition 2.6 *Let H and K be Hilbert spaces, and let $T \in B(H, K)$ be a compact operator.*
Show that $x_n \xrightarrow{w} x \implies Tx_n \to Tx$.
Thus, a compact operator maps weakly convergent sequences to strongly convergent sequences.

Proof. *Disclaimer*: Stolen from MSx1142451.

$Tx_n \xrightarrow{w} Tx$ by continuity. Thus if any subsequence has a strong limit, it certainly is Tx. But compactness guarantees every subsequence has a subsequence that converges to something: that something is Tx by uniqueness, and so by our above equivalence with convergence, we have $Tx_n \to Tx$. □

Part 2

Hilbert spaces

2.1 Basics

In what follows, $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space, and we write $x \perp y$ iff $\langle x, y\rangle = 0$.

Theorem 1.1 (Pythagorean) *If $x, y \in H$ and $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Proof.

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 . \end{aligned}$$

□

Theorem 1.2 (Cauchy–Schwarz inequality) *If $x, y \in H$, then $|\langle x,y\rangle| \leq \|x\|\,\|y\|$.*

Proof. (*norm expansion*) Note that $0 \leq \|x-\lambda y\|^2 = \|x\|^2 - 2\Re\left(\overline{\lambda}\langle x,y\rangle\right) + |\lambda|^2\|y\|^2$, so if we take $\lambda = \frac{\langle x,y\rangle}{\|y\|^2}$, we get $0 \leq \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$, which gives us the required result. □

Proof. (*projection*) Note that we can write $x = x_\parallel + x_\perp$, where $x_\parallel$ is the component of x in the direction of y and $x_\perp$ is the component of x in the direction perpendicular to y. Explicitly, $x_\parallel = \langle x, \hat{y}\rangle \hat{y} = \langle x,y\rangle \frac{y}{\|y\|^2}$. Using the Pythagorean theorem (1.1), we get

$$\|x\|^2 = \|x_\parallel\|^2 + \|x_\perp\|^2 \geq \|x_\parallel\|^2 = \frac{|\langle x,y\rangle|^2}{\|y\|^2}.$$

□

Theorem 1.3 (Riesz–Fischer) *$L^p(X,\mu)$ is complete for $p \in [0,\infty]$.*

Proof. Let (f_n) be a Cauchy sequence in L^p. We have to show that there exists $f \in L^p$ such that $f_n \to f$ in L^p.

Since (f_n) is Cauchy, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n, m > N$, we have $\|f_n - f_m\|_p < \varepsilon$. Therefore, there exists a subsequence $\left(f_{n_k}\right)$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p < 2^{-(k+1)}$ for every $k \in \mathbb{N}_0$, where we adopt the convention that $f_{n_0} \equiv 0$. Note that $f_{n_k} = \sum_{j=0}^{k-1}\left(f_{n_{j+1}} - f_{n_j}\right)$ for each $k \in \mathbb{N}$.

Define $f = \sum_{j=0}^{\infty}\left(f_{n_{j+1}} - f_{n_j}\right)$. Clearly, $f_{n_k} \to f$ pointwise. Moreover, if $g = \sum_{j=0}^{\infty}\left|f_{n_{j+1}} - f_{n_j}\right|$, then $\left|f_{n_k}\right| \leq g$ and $\|g\|_p \leq \sum_{j=0}^{\infty}\left\|f_{n_{j+1}} - f_{n_j}\right\|_p \leq 1$ using the triangle inequality. Therefore, by Lebesgue's dominated convergence theorem, $f_{n_k} \to f$ in L^p. Similar to g, we get $\|f\|_p \leq 1$, showing $f \in L^p$. All that is left to show is that $f_n \to f$ in L^p. Using the fact that the sequence is Cauchy, we get

$$\|f_n - f\|_p \leq \left\|f_n - f_{n_k}\right\|_p + \left\|f_{n_k} - f\right\|_p \to 0 \text{ as } n \to \infty.$$

□

Part 3

Operator theory

3.1 Elementary ideas

A great source is Trace class operators and Hilbert-Schmidt operators by Jordan Bell.

3.1.1 Intuition

On a separable Hilbert space, we have

- ▷ $T \in \mathcal{B}^\infty \iff \lambda \in \ell^\infty$ (bounded)
  Example $I : \ell^2 \to \ell^2 : e_n \mapsto e_n$.
- ▷ $T \in \mathcal{B}_0 \iff \lambda \in c_0$ (compact)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{\sqrt{n}} e_n$.
- ▷ $T \in \mathcal{B}^2 \iff \lambda \in \ell^2$ (Hilbert-Schmidt)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n} e_n$.
- ▷ $T \in \mathcal{B}^1 \iff \lambda \in \ell^1$ (trace-class)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n^2} e_n$.
- ▷ $T \in \mathcal{B}_{00} \iff \lambda \in c_{00}$ (degenerate or finite rank)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \alpha_n e_n \mathbb{1}_{[N]}(n)$ for $\alpha_n \in \mathbb{C}$ and $N \in \mathbb{N}$.

Remark 1.1 *Since the dual of c_0 is ℓ^1 and the dual of ℓ^1 is ℓ^∞, we have $\mathcal{B}_0^* = \mathcal{B}^1$ and $(\mathcal{B}^1)^* = \mathcal{B}^\infty$. Similarly, $(\mathcal{B}^2)^* = \mathcal{B}^2$.*

Theorem 1.2 (Operator inclusions) $\mathcal{B}_{00} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{B}_0 \subset \mathcal{B}^\infty$

Proof.

- ▷ $\mathcal{B}_{00} \subset \mathcal{B}^1$ Trivial
- ▷ $\mathcal{B}^1 \subset \mathcal{B}^2$
- ▷ $\mathcal{B}^2 \subset \mathcal{B}_0$
- ▷ $\mathcal{B}_0 \subset \mathcal{B}^\infty$ ((<BMC2009>), Proposition 4.6) If T is unbounded, we can find a sequence of unit vectors (e_n) such that $\|Te_n\| \nearrow \infty$. So Te_n cannot have a convergent subsequence, for if $Te_n \to x$, then $\|Te_n\| \to \|x\|$.

□

Proposition 1.3 *For $T \in \mathcal{B}^\infty$, $\|T\|_\infty = \sup\{|\langle Tx, y\rangle|\} : \|x\| = 1, \|y\| = 1$.*

Proof.

$(\leq)$ Since $\|Tx\| = \frac{\|Tx\|^2}{\|Tx\|} = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \left\langle Tx, \frac{Tx}{\|Tx\|}\right\rangle$, we have

$$\|T\|_\infty = \sup\{\|Tx\| : \|x\| = 1\} \leq \sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\}.$$

$(\geq)$ Since $\langle Tx, y\rangle \leq \|Tx\|\,\|y\| \leq \|T\|_\infty \|x\|\,\|y\|$, we have

$$\sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\} \leq \|T\|_\infty .$$

□

3.1.2 Projection operators

Proposition 1.4 $\|P\|_\infty \leq 1$.

Proof. Since $\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PPx, x\rangle = \langle Px, x\rangle \leq \|Px\|\,\|x\|$, we have $\|P\|_\infty \leq 1$. □

Proposition 1.5 *A projection operator is compact iff its image is finite dimensional.*

Proof.

$(\Longrightarrow)$ Let $P : H \to H$ be a projection operator, so that $P^2 = P$, or $P(P - I) = 0$.

$(\Longleftarrow)$ Since the image is finite dimensional, fix an orthonormal basis $e_1, \ldots, e_n$ of $\operatorname{im} T$.

□

3.2 Optimization

3.2.1 Duality in optimization is the same as duality in functional analysis

For an various intuitions of duality in optimization, see MSx223235.

Let X and Y be Banach spaces, and X^* and Y^* be their (algebraic?) duals. Consider the two problems, with ϕ_0, y_0 fixed. Here $(\cdot,\cdot)$ denotes the canonical duality pairing.

$$
\text{(Primal)} \quad \begin{aligned} \max \quad & (\phi_0, x) \\ \text{s.t.} \quad & Tx \leq y_0 \\ & x \geq 0 \end{aligned}
\qquad\qquad
\text{(Dual)} \quad \begin{aligned} \min \quad & (\psi, y_0) \\ \text{s.t.} \quad & T^*\psi \geq \phi_0 \\ & \psi \geq 0 \end{aligned}
$$

See the following diagram for more details.

$$
\begin{tikzcd}
 & x \arrow[r, mapsto, "T"] & Tx & \\
x \in & X \arrow[r, "T"] \arrow[d, dash] & y_0 \arrow[d, dash] & \ni Tx, y_0 \\
\phi_0, T^*\psi \in & X^* & y_0^* \arrow[l, "T^*"'] & \ni \psi \\
 & T^*\psi & \psi \arrow[l, mapsto, "T^*"'] &
\end{tikzcd}
$$

Bibliography