Anticipating stochastic integrals and its large deviations

Sudip Sinha

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Our aim is to formulate a large deviations principle for an SDE with anticipating initial conditions. We start of with a very simple case

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\,\mathrm{d}W_t,$$

where $t \in [0,T]$ for some $T < \infty$, and conditions on σ shall be imposed as necessary.
We shall look at the method of enlargement of filtration by [<Itô1978>]. We denote the enlarged filtration by $\tilde{\mathscr{F}}_t = \mathscr{F}_t \vee \sigma(W_T)$. For $t \in [0,T]$, define the process $A_t = \int_0^t \frac{W_T - W_u}{T-u}\,\mathrm{d}u$. Then $W_t = \tilde{W}_t + A_t$, where $\tilde{W}_\cdot$ is a Wiener process w.r.t. $\tilde{\mathscr{F}}_\cdot$.
Using this, we write our original SDE as

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\,\mathrm{d}\tilde{W}_t + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\frac{W_T - W_s}{T-s}\,\mathrm{d}s.$$

Now, let $Y_t^\varepsilon = \sqrt{\varepsilon}(W_T - W_t)$. Then X_t^ε is given by

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\,\mathrm{d}\tilde{W}_t + \int_0^t \sigma(X_t^\varepsilon)\frac{Y_s^\varepsilon}{T-s}\,\mathrm{d}s.$$

Moreover,

$$\begin{aligned} Y_t^\varepsilon &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}W_t \\ &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}\left(\tilde{W}_t + \int_0^t \frac{W_T - W_s}{T-s}\,\mathrm{d}s\right) \\ &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}\tilde{W}_t - \int_0^t \frac{Y_s^\varepsilon}{T-s}\,\mathrm{d}s \end{aligned}$$

Therefore, we have the joint process

$$\begin{matrix} A & B & C \\ a & b & c \end{matrix}$$

2.2.1 $\tilde{W}_\cdot$ is a Wiener process

We show that $(\tilde{W}_t)$ is a $(\tilde{\mathscr{F}}_t)$-martingale with quadratic variation t. Then by Lévy's Characterization of Wiener process, we obtain that $\tilde{W}_\cdot$ is a Wiener process.
First we prove two lemmas.

Lemma 2.1 *The σ-algebras $\mathscr{F}_s \vee \sigma(W_T)$ and $\mathscr{F}_s \vee \sigma(W_T - W_s)$ are the same.*

Proof. For any Borel set B, the set $\{W_T \in B\} = \{(W_T - W_t) + W_t \in B\}$.
TODO

Lemma 2.2 *For $0 \le s \le t \le T$, we have*

$$\mathbb{E}(W_t - W_s \mid W_T - W_s) = \frac{t-s}{T-s}(W_T - W_s).$$

Proof. We partition the interval $[0,T]$ into $n = kn_0$ equal parts, where $n_0 = (\min\{s, t-s, T-t\})^{-1}$ and $k \in \mathbb{N}$. Let $n_s = s\frac{n}{T}$ and $n_t = t\frac{n}{T}$. That is, the partition is

$$P = \left\{0, \frac{T}{n}, \ldots, \frac{n_s T}{n} = s, \ldots, \frac{n_t T}{n} = t, \ldots, \frac{(n-1)T}{n}, T\right\}.$$

Let $\Delta_i W = W_{\frac{(i+1)T}{n}} - W_{\frac{iT}{n}}$.

Firstly, note that the $\Delta_i W$s are independent and identically distributed from the definition of Wiener process. Now, using the linearity of conditional expectation, we have

$$\begin{aligned}
\mathbb{E}(W_t - W_s \mid W_T - W_s) &= \mathbb{E}\left(\sum_{i=n_s}^{n_t-1} \Delta_i W \mid \sum_{i=n_s}^{n_t-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \mathbb{E}\left(\sum_{i=n_s}^{n-1} \Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \Delta_i W \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} (W_T - W_s) \\
&= \frac{n_t - n_s}{n - n_s}(W_T - W_s) \quad = \quad \frac{t-s}{T-s}(W_T - W_s).
\end{aligned}$$

Proposition 2.3 *$\tilde{W}_\cdot$ is a $\tilde{\mathscr{F}}_\cdot$-martingale.*

Proof. Let $0 \le s \le t \le T$. Then

$$\tilde{W}_t - \tilde{W}_s = (W_t - W_s) - \int_s^t \frac{W_T - W_u}{T-u} du = (W_t - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{W_u - W_s}{T-u}\right) du.$$

Moreover, since $W_t - W_s$ is independent of $\mathscr{F}_s$ for every $t \ge s$, using lemmas 2.1 and 2.2, we get

$$\mathbb{E}(W_t - W_s \mid \tilde{\mathscr{F}}_s) = \mathbb{E}\left(W_t - W_s \mid \mathscr{F}_s \vee \sigma(W_T - W_s)\right) = \mathbb{E}\left(W_t - W_s \mid W_T - W_s\right) = \frac{t-s}{T-s}(W_T - W_s).$$

Therefore, using the fact that W_T and W_s are $\tilde{\mathscr{F}}_s$-measurable with conditional Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) &= \mathbb{E}\left(W_t - W_s \mid \tilde{\mathscr{F}}_s\right) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{\mathbb{E}\left(W_u - W_s \mid \tilde{\mathscr{F}}_s\right)}{T-u}\right) du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{u-s}{T-s}\frac{W_T - W_s}{T-u}\right) du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \frac{W_T - W_s}{T-s} du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \frac{t-s}{T-s}(W_T - W_s) \quad = \quad 0.
\end{aligned}$$

Now, since $\tilde{W}_s$ is $\tilde{\mathscr{F}}_s$-measurable, $\mathbb{E}(\tilde{W}_t \mid \tilde{\mathscr{F}}_s) = \mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) + \tilde{W}_s = \tilde{W}_s$.

Proposition 2.4 *The quadratic variation of $\tilde{W}_t$ is t.*

Bibliography