

# Generalization of stochastic calculus and its applications in large deviations theory

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# Outline

1	Introduction and motivation	3
2	Generalization of Itô calculus	12
3	Large deviations theory	19
4	Conclusion	33
5	Sample slides	35

# § 1

## INTRODUCTION AND MOTIVATION

# Quick revision and notations

- Let  $T \in (0, \infty)$ , and denote  $\mathbb{T} = [0, T]$  as the index set for  $t$ .
- Let  $(\Omega, \mathcal{F}, \mathcal{F}_., \mathbb{P})$  be a filtered probability space.
- $B_.$  is a Brownian motion on  $(\Omega, \mathcal{F}, \mathcal{F}_., \mathbb{P})$ .
- Properties of  $B$ .
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t - s)$
  - continuous paths
  - has **unbounded linear variation** ☹️
  - has **bounded quadratic variation** 😊
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- **Naive stochastic integration w.r.t.  $B_t$ : not possible.**
- A stochastic process  $X_t$  is called  $(\mathcal{F}_t)$ -adapted if  $\forall t$ ,  $X_t$  is measurable w.r.t.  $\mathcal{F}_t$ .

# Wiener integral ( $f \in L^2[0, T]$ )

- Definition

1. Step functions  $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$ : Define  $\int_0^T f(t) dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$ , where  $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ .
2.  $f \in L^2[0, T]$ : Use step functions approximating  $f$  to extend the integral **a.s.**

- Properties

- ★ Linear
- ★ **Gaussian distribution** with mean 0 and variance  $\|f\|_{L^2[0, T]}^2$  (Itô isometry)
- ★ Corresponds to the Riemann–Stieltjes integral for  $f \in C[0, T]$

- The associated process:  $I_t = \int_0^t X_t dB_t$

- ★ continuous
- ★ martingale

- Problem: Cannot integrate stochastic processes.

# Itô integral ( $X \in L^2_{\text{ad}}([0, T] \times \Omega)$ )

- Definition

1. Adapted step processes  $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$ : define  $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$ .
2.  $X \in L^2_{\text{ad}}([0, T] \times \Omega)$ : use step processes approximating  $X$  to extend the integral in  $L^2(\Omega)$ .

- Properties

- ★ Linear
- ★ Mean 0 and variance  $\|f\|_{L^2[0, T]}^2$  (Itô isometry)
- ★ For  $X$  continuous,  $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t$ , for example  $\int_0^t B_s \, dB_s = \frac{1}{2} (B_t^2 - t)$

- The associated process:  $I_t = \int_0^t X_s \, dB_s$

- ★ continuous
- ★ martingale

- Example:  $\int_0^T B_t \, dB_t = \frac{1}{2} (B_T^2 - T)$ .

## Itô integral ( $\int_0^T X_t^2 dt < \infty$ a.s.)

- Definition: Use sequences of processes in  $L^2_{\text{ad}}([0, T] \times \Omega)$  approximating  $X$  in probability to extend the integral in probability.
- Properties
  - ★ Linear
  - ★ Mean 0, but variance? ☹
- The associated process:  $I_t = \int_0^t X_t dB_t$ 
  - ★ continuous
  - ★ local martingale
- Example:  $\int_0^T e^{B_t^2} dB_t = \int_0^{B_1} e^{t^2} dt - \int_0^T B_t e^{B_t^2} dt.$

## Itô formula

- An **Itô process** is a process of the form  $X_t = X_0 + \int_0^t m(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s$ , equivalently expressed as  $dX_t = m(t, X_t) \, dt + \sigma(t, X_t) \, dB_t$ .  
[Only makes sense when  $\int_0^T (|m(s, X_s)| + |\sigma(s, X_s)|^2) \, ds < \infty$  a.s.]

**Theorem ([Itô44])** Let  $X_t$  be a  $d$ -dimensional Itô process, and let  $Y_t = f(X_t)$ , where  $f \in C^2(\mathbb{R})$ . Then  $f(X_t)$  is also a  $d$ -dimensional Itô process, and

$$df(X_t) = \langle (Df)(X_t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D^2 f)(X_t) dX_t \rangle,$$

where we use the rule  $dB_t \otimes dB_t = I_d \, dt$ .

- Example: For  $\sigma$  constant,  $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$ , we have  $d\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, dt + \sigma \mathcal{E}_t \, dB_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (dB_t)^2$ .



# Exponential processes and Girsanov theorem

TODO

Let  $h \in L^2[0, T]$ . Then the translated stochastic process  $W_t = B_t - \int_0^t h(s) \, ds$  is a Brownian motion under the probability measure  $\tilde{\mathbb{P}}$  defined by the Radon-Nikodym derivative  $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\sigma B_T - \frac{1}{2}\sigma^2 T\right) =: \mathcal{E}_T^h$ .

Then  $\tilde{\mathbb{P}} \sim \mathbb{P}$  and the process  $Z_t := \mathbb{E}\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \mid \mathcal{F}_t\right)$  is a martingale.

# Stochastic differential equations

- Let  $\zeta \in L^2(\Omega)$  be independent of  $B$ , and  $m, \sigma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  be  $\mathcal{B}[0, T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$  measurable such that  $m(t, \cdot, \cdot)$  and  $\sigma(t, \cdot, \cdot)$  are  $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$  measurable  $\forall t$ .  
Then a  $\mathcal{F}_t$ -adapted stochastic process  $X_t$  is called a solution of the stochastic *integral* equation  $X_t = \zeta + \int_0^t m(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$  if for each  $t$ , the  $X_t$  satisfies the integral equation a.s.
- The stochastic differential equation  $dX_t = m(t, X_t) dt + \sigma(t, X_t) dB_t$ ,  $X_0 = \zeta$  is a symbolic representation of the stochastic integral equation.

**Theorem (Existence and uniqueness, Markov property)** The stochastic differential equation above has a unique solution if there exists an  $M > 0$  such that the following two conditions are satisfied:

- ★ (Lipschitz condition)  $|m(t, x) - m(t, y)| + |\sigma(t, x) - \sigma(t, y)|^2 \leq M(1 + |x|^2)$  a.s.
- ★ (growth condition)  $|m(t, x)| + |\sigma(t, y)|^2 \leq M(1 + |x|^2)$  a.s.

The solution is a Markov process. Moreover if  $\zeta \in \mathbb{R}$ , then the solution is also stationary.

- Example: For  $\sigma$  constant, the solution of  $d\mathcal{E}_t = \sigma \mathcal{E}_t dB_t$ ,  $\mathcal{E}_0 = 1$  is given by  $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$ .

# Multiple Wiener–Itô integrals

## § 2

# GENERALIZATION OF ITÔ CALCULUS

# Motivation

- Iterated integrals: Consider the iterated integral  $\int_0^T \int_0^T dB_s dB_t = \int_0^T B_T dB_t \stackrel{?}{=} B_T B_t$ .
- Note that  $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$ , so **no martingale property** ☹.
- Stochastic differential equations with anticipation

$$dX_t = X_t dB_t$$

$$X_0 = B_1$$

$$dY_t = B_T dB_t$$

$$Y_0 = 1$$

- Problem: We want to define  $\int_0^T X_t dB_t$ , where  $X_\cdot$  is not adapted (anticipating).
- Some approaches
  - ★ Itô's decomposition of integrand  $B_t = \left( B_t - \int_0^t \frac{B_T - B_s}{T-s} ds \right) + \int_0^t \frac{B_T - B_s}{T-s} ds$
  - ★ Enlargement of filtration
  - ★ White noise theory
  - ★ ...

## The new integral [AK08; AK10]: Idea

- A process  $Y^\cdot$  and filtration  $\mathcal{F}_\cdot$  are called **instantly independent** if  $Y^t$  and  $\mathcal{F}_t$  are independent  $\forall t$ .
- Ideas
  1. Decompose the integrand into **adapted** and **instantly independent** parts.
  2. Evaluate the **adapted** and the **instantly independent** parts at the **left** and **right** endpoints.
- Consider two continuous stochastic processes,  $X_t$  **adapted** and  $Y^t$  **instantly independent** w.r.t.  $\mathcal{F}_\cdot$ . Then the integral  $\int_0^T X_t Y^t dB_t$  is **defined** as

$$\int_0^T X_t Y^t dB_t \triangleq \lim_{\|\Delta_n\| \rightarrow 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

provided that the limit exists in probability.

- Now, for any stochastic process  $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$  we extend the definition by linearity.
- This is well-defined [HKS+16].

## A simple example

- In the following, denote  $\Delta B_j = B_{t_{j+1}} - B_{t_j}$  and  $\lim$  is the limit in  $L^2$ .

$$\begin{aligned}\int_0^t B_T \, dB_t &= \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t \\&= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\&= \lim \sum_{j=0}^{n-1} (B_T - \Delta B_j) \Delta B_j \\&= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t\end{aligned}$$

- Note that  $\mathbb{E}(B_T B_t - t) = 0$ .
- In general,  $\mathbb{E} \int_0^t Z(t) \, dB_t = 0$ . ☺

## Generalized Itô formula [HKS+16]

- Let  $dX_t = m(t) dt + \sigma(t) dB_t$  be an  $d$ -dimensional **Itô** process,  $Y^t = \tilde{m}(t) dt + \tilde{\sigma}(t) dB_t$  be a  $\tilde{d}$ -dimensional instantly independent process,  $f(x, y) \in C^2(\mathbb{R}^2)$ . Then

$$\begin{aligned} df(X_t, Y^t) = & \langle (D_x f)(X_t, Y^t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D_x^2 f)(X_t, Y^t) dX_t \rangle \\ & + \langle (D_y f)(X_t, Y^t), dY^t \rangle - \frac{1}{2} \langle dY^t, (D_y^2 f)(X_t, Y^t) dY^t \rangle, \end{aligned}$$

where we use the rule  $dB_t \otimes dB_t = I_d dt$ .

- Example: TODO



# Iterated integrals

**Theorem ([Itô51])** Let  $f \in L^2([0, T]^n)$  and  $\hat{f}$  be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = n! \int_0^T \dots \int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1},$$

**Theorem ([AK10])** Let  $f \in L^2([0, T]^n)$ . Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = \int_0^T \dots \int_0^T f(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1}.$$

# Near-martingale property [HKS+17]

- Question: What are the analogues of the martingale property and the Markov property?
- Partial answer: near-martingales
- Let  $Z(t)$  be a stochastic process such that  $\mathbb{E} |Z(t)| < \infty \forall t$ , and  $0 \leq s \leq t \leq T$ . Then, with respect to  $\mathcal{F}_\cdot$ , the process  $Z(t)$  is called a
  - ★ near-martingale if  $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) = 0$ ,
  - ★ near-submartingale if  $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \geq 0$ , and
  - ★ near-supermartingale if  $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \leq 0$ .

§ 3

# LARGE DEVIATIONS THEORY

# Introduction

# Weak convergence of measures

# Laplace principle

# Motivation: towards Cramér theorem

## Setup

- $(X_n)$ : i.i.d. random variables
- $X_n \sim \mu$  and  $\mathbb{E}X_n = m$
- $\overline{S}_n = \frac{1}{n} \sum_{j=1}^n X_j$
- $B \in \mathcal{B}$  such that  $m \notin \overline{B}$
- By LLN, we have  $\overline{S}_n \rightarrow m$  as  $n \rightarrow \infty$  (a.s.)
- So  $\mathbb{P}(S_n \in B) \rightarrow 0$  as  $n \rightarrow \infty$
- But at what speed?
- $\asymp$

# Motivation: towards Cramér theorem



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# Cramér theorem

**Theorem (Cramér, 1938)** Let  $X_1, X_2, \dots$  be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example  $\Lambda(t) < \infty \forall t \in \mathbb{R}$ . Then the Legendre transform of  $\Lambda$ ,  $\Lambda^* = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$  satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{i=1}^n X_i \geq nx \right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

# Cramér theorem applied to common distributions

# Motivation: towards Cramér theorem

# Sanov theorem

# LD in $\infty$ -dimensions — Schilder theorem

**Aim:** Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path (the 0 function).

## Setup

- Let  $B_\cdot$  be a  $d$ -dimensional Brownian motion, so  $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- $\forall \varepsilon > 0$ , let  $W_\varepsilon$  denote the law of  $\sqrt{\varepsilon}B_\cdot$ .
- Let  $\text{CM} = \{\omega \in C_0 : \omega \in \text{AC}, \text{ and } \dot{\omega}_t \in L^2[0, T]\}$

**Theorem** On the Banach space  $(C_0, \|\cdot\|_\infty)$ , the family of probability measures  $\{W_\varepsilon : \varepsilon > 0\}$  satisfy the large deviations principle with the rate function  $I : C_0 \rightarrow \overline{\mathbb{R}}$  given by

$$I(\omega) = \left( \frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 dt \right) \mathbb{1}_{\text{AC}}(\omega) + \infty \mathbb{1}_{\text{AC}^c}(\omega)$$

# Freidlin–Wentzell theorem

**Aim:** Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

## Setup

- Let  $B_\cdot$  be a  $d$ -dimensional Brownian motion, so  $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- $\forall \varepsilon > 0$ , let  $X^{(\varepsilon)}$  be a  $\mathbb{R}^d$ -valued Itô diffusion solving an Itô SDE of the form

$$dX_t^{(\varepsilon)} = b(X_t^{(\varepsilon)}) dt + \sigma(X_t^{(\varepsilon)}) \sqrt{\varepsilon} dB_t, \quad X_0^{(\varepsilon)} = 0.$$

- $\forall \varepsilon > 0$ , let  $W_\varepsilon$  denote the law of  $X_\cdot^{(\varepsilon)}$ .

**Theorem (Freidlin, Wentzell (year?))** On the Banach space  $(C_0, \|\cdot\|_\infty)$ , the family of probability measures  $\{W_\varepsilon : \varepsilon > 0\}$  satisfy the large deviations principle with the rate function  $I : C_0 \rightarrow \overline{\mathbb{R}}$  given by

$$I(\omega) = \left( \frac{1}{2} \int_0^T |\dot{\omega}_t - b(\omega_t)|^2 dt \right) \mathbb{1}_{H^1([0, T]; \mathbb{R}^d)}(\omega) + \infty \mathbb{1}_{H^1([0, T]; \mathbb{R}^d)^c}(\omega)$$



§ 4

## CONCLUSION

# Open areas for research

- ★ Extension to SDEs with anticipating coefficients
- ★ Near-Markov property
- ★ Girsanov theorem for anticipating integrals
- ★ Freidlin-Wintzell type result for SDEs with anticipation

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

§ 5

**SAMPLE SLIDES**

## Possible areas of interest

- ★ Extension to SDEs with anticipating coefficients
- ★ Near-Markov property
- ★ Girsanov theorem for generalized integration
- ★ Freidlin-Wintzell type result for SDEs with anticipating initial conditions

**Theorem (Cramér, 1938)** Let  $X_1, X_2, \dots$  be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example  $\Lambda(t) < \infty \forall t \in \mathbb{R}$ . Then the Legendre transform of  $\Lambda$ ,  $\Lambda^* = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$  satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{i=1}^n X_i \geq nx \right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

# Freidlin–Wentzell theorem

## Column 1

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

## Column 2

$\times$	$dt$	$dB_t$
$dt$	0	0
$dB_t$	0	$dt$

# Something

☐ One

☐ Two

☐ Three

☐ Four

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