Generalization of stochastic calculus and its applications in large deviations theory

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Outline

§ 1
Introduction and motivation

Quick revision and notations

- Let $T \in (0, \infty)$, and denote $\mathbb{T} = [0, T]$ as the index set for t.
- Let $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$ be a filtered probability space.
- $B_\cdot$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$.
- Properties of $B_\cdot$
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t-s)$
  - continuous paths
  - has unbounded linear variation ☹
  - has bounded quadratic variation ☺
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- Naive stochastic integration w.r.t. B_t: not possible.
- A stochastic process X_t is called $(\mathcal{F}_t)$-adapted if $\forall t$, X_t is measurable w.r.t. $\mathcal{F}_t$.

Wiener integral ($f \in L^2[0,T]$)

- Definition

  1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t)\,\mathrm{d}B_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
  2. $f \in L^2[0,T]$: Use step functions approximating f to extend the integral a.s.

- Properties

  ⋆ Linear
  ⋆ Gaussian distribution with mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry)
  ⋆ Corresponds to the Riemann–Stieltjes integral for $f \in C[0,T]$

- The associated process: $I_t = \int_0^t X_t\,\mathrm{d}B_t$

  ⋆ continuous
  ⋆ martingale

- Problem: Cannot integrate stochastic processes.

Itô integral ($X \in L^2_{\text{ad}}([0,T] \times \Omega)$)

- Definition

  1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, \mathrm{d}B_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
  2. $X \in L^2_{\text{ad}}([0,T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

- Properties

  * Linear
  * Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry)
  * For $X_\cdot$ continuous, $\int_0^T X_t \, \mathrm{d}B_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, \mathrm{d}B_t$, for example $\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}\left(B_t^2 - t\right)$

- The associated process: $I_t = \int_0^t X_t \, \mathrm{d}B_t$

  * continuous
  * martingale

- Example: $\int_0^T B_t \, \mathrm{d}B_t = \frac{1}{2}(B_T - T)$.

Itô integral ($\int_0^T X_t^2 \, \mathrm{d}t < \infty$ a.s.)

- Definition: Use sequences of processes in $L^2_{\mathrm{ad}}([0,T] \times \Omega)$ approximating X in probability to extend the integral in probability.

- Properties
  - ⋆ Linear
  - ⋆ Mean 0, but variance? 😐

- The associated process: $I_t = \int_0^t X_t \, \mathrm{d}B_t$
  - ⋆ continuous
  - ⋆ local martingale

- Example: $\int_0^T e^{B_t^2} \, \mathrm{d}B_t = \int_0^{B_1} e^{t^2} \, \mathrm{d}t - \int_0^T B_t e^{B_t^2} \, \mathrm{d}t.$

Itô formula

- An Itô process is a process of the form $X_t = X_0 + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$, equivalently expressed as $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t$.
  Only makes sense when $\int_0^T \left(|m(s, X_s)| + |\sigma(s, X_s)|^2\right)\,\mathrm{d}s < \infty$ a.s.

- Let X_t be a d-dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d-dimensional Itô process, and

$$\mathrm{d}f(X_t) = \langle (\mathrm{D}f)(X_t), \mathrm{d}X_t \rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D^2 f)(X_t)\,\mathrm{d}X_t \right\rangle,$$

  where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, we have $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t\,\mathrm{d}t + \sigma \mathcal{E}_t\,\mathrm{d}B_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (\mathrm{d}B_t)^2$.

Exponential processes and Girsanov theorem

TODO

Let $h \in L^2[0,T]$. Then the translated stochastic process $W_t = B_t - \int_0^t h(s)\,\mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(\sigma B_T - \frac{1}{2}\sigma^2 T\right) =: \mathcal{E}_T^h$. Then $\tilde{\mathbb{P}} \sim \mathbb{P}$ and the process $Z_t := \mathbb{E}\left(\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \mid \mathcal{F}_t\right)$ is a martingale.

Stochastic differential equations

- Let ξ be independent of $B_\cdot$, and $m, \sigma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}[0,T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable with
  - ⋆ $\forall t,\, m(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable, and
  - ⋆ $\int_0^T \left(|m(s,X_s)| + |\sigma(s,X_s)|^2\right) \mathrm{d}s < \infty$ a.s.

  Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \xi + \int_0^t m(s,X_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s)\,\mathrm{d}B_s$ if for each t, the X_t satisfies the integral equation a.s.
- The stochastic differential equation $\mathrm{d}X_t = m(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}B_t,\ X_0 = \xi$ is a symbolic representation of the stochastic integral equation.

Theorem (Existence and uniqueness) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:
- ⋆ (Lipschitz condition) $|m(t,x) - m(t,y)| + |\sigma(t,x) - \sigma(t,y)|^2 \leq M(1 + |x|^2)$ a.s.
- ⋆ (growth condition) $|m(t,x)| + |\sigma(t,y)|^2 \leq M(1 + |x|^2)$ a.s.

- Example: For σ constant, the solution of $\mathrm{d}\mathcal{E}_t = \sigma \mathcal{E}_t\,\mathrm{d}B_t, \mathcal{E}_0 = 1$ is given by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

§ 2
Generalization of Itô calculus

Why?

- Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T B_T \, \mathrm{d}B_t \stackrel{?}{=} B_T B_t$.
- Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so no martingale property ☹.
- Stochastic differential equations with anticipation

$$\begin{aligned} \mathrm{d}X_t &= X_t \, \mathrm{d}B_t & \qquad \mathrm{d}Y_t &= B_T \, \mathrm{d}B_t \\ X_0 &= B_1 & \qquad Y_0 &= 1 \end{aligned}$$

- Problem: We want to define $\int_0^T X_t \, \mathrm{d}B_t$, where $X_.$ is not adapted (anticipating).
- Some approaches
  - ⋆ Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s \right) + \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s$
  - ⋆ Enlargement of filtration
  - ⋆ White noise theory
  - ⋆ ...

The new integral (Ayed & Kuo, 2008): Idea

- A process $Y^{\cdot}$ and filtration $\mathcal{F}_{\cdot}$ are called instantly independent if Y^t and filtration $\mathcal{F}_t$ are independent for every t.
- Decompose the integrand into adapted and instantly independent parts.
- Evaluate the adapted part at the left endpoint, and the instantly independent part at the right endpoint.

A simple example

In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and lim is the limit in L^2.

$$\begin{aligned}
\int_0^t B_T \,\mathrm{d}B_t &= \int_0^t (B_t + (B_T - B_t))\,\mathrm{d}B_t = \int_0^t B_t\,\mathrm{d}B_t + \int_0^t (B_T - B_t)\,\mathrm{d}B_t \\
&= \lim \sum_{j=0}^{n-1} B_{t_j}\Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}})\Delta B_j \\
&= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j\right)\Delta B_j \\
&= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t
\end{aligned}$$

$\mathbb{E}(B_T B_t - t) = 0$ ☺.

Generalized Itô formula

- Let $\mathrm{d}X_t = m(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$ be an d-dimensional Itô process, $Y^t = \tilde{m}(t)\,\mathrm{d}t + \tilde{\sigma}(t)\,\mathrm{d}B_t$ be a $\tilde{d}$-dimensional instantly independent process, $f(x,y) \in C^2(\mathbb{R}^2)$. Then

$$\mathrm{d}f(X_t, Y^t) = \langle (\mathrm{D}_x f)(X_t, Y^t), \mathrm{d}X_t \rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D_x^2 f)(X_t, Y^t)\,\mathrm{d}X_t \right\rangle$$
$$+ \left\langle (\mathrm{D}_y f)(X_t, Y^t), \mathrm{d}Y^t \right\rangle - \frac{1}{2}\left\langle \mathrm{d}Y^t, (D_y^2 f)(X_t, Y^t)\,\mathrm{d}Y^t \right\rangle,$$

  where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, we have $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t\,\mathrm{d}t + \sigma \mathcal{E}_t\,\mathrm{d}B_t + \frac{1}{2}\sigma^2 \mathcal{E}_t(\mathrm{d}B_t)^2$.

Differential formula (Itô, 1944 TODO:ref)

bla bla bla

§ 3
Large deviations theory

Introduction

Weak convergence of measures

Laplace principle

Motivation: towards Cramér theorem

Setup

- (X_n): i.i.d. random variables
- $X_n \sim \mu$ and $\mathbb{E}X_n = m$
- $\overline{S_n} = \frac{1}{n}\sum_{j=1}^{n} X_j$
- $B \in \mathcal{B}$ such that $m \notin \overline{B}$
- By LLN, we have $\overline{S_n} \to m$ as $n \to \infty$ (a.s.)
- So $\mathbb{P}(S_n \in B) \to 0$ as $n \to \infty$
- But at what speed?
- $\asymp$

Motivation: towards Cramér theorem

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Cramér theorem

Theorem (Cramér, 1938) Let $X_1, X_2, \ldots$ be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example $\Lambda(t) < \infty \, \forall t \in \mathbb{R}$.
Then the Legendre transform of Λ, $\Lambda^* = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq nx \right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Cramér theorem applied to common distributions

Motivation: towards Cramér theorem

Sanov theorem

LD in ∞-dimensions — Schilder theorem

Aim: Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path (the 0 function).

Setup

- Let $B_\cdot$ be a d-dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0,T];\mathbb{R}^d)$
- $\forall \varepsilon > 0$, let W_ε denote the law of $\sqrt{\varepsilon} B_\cdot$
- Let $\mathrm{CM} = \{\omega \in C_0 : \omega \in \mathrm{AC}, \text{ and } \dot{\omega}_t \in L^2[0,T]\}]$

Theorem (Schilder, 1966 (<Schilder1966>)) On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 \,\mathrm{d}t\right) \mathbb{1}_{\mathrm{AC}}(\omega) + \infty \mathbb{1}_{\mathrm{AC}^\complement}(\omega)$$

Freidlin–Wentzell theorem

Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

Setup

- Let $B_\cdot$ be a d-dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0,T];\mathbb{R}^d)$
- $\forall \varepsilon > 0$, let $X^{(\varepsilon)}$ be a $\mathbb{R}^d$-valued Itô diffusion solving an Itô SDE of the form

$$\mathrm{d}X_t^{(\varepsilon)} = b(X_t^{(\varepsilon)})\,\mathrm{d}t + \sigma(X_t^{(\varepsilon)})\sqrt{\varepsilon}\,\mathrm{d}B_t, \quad X_0^{(\varepsilon)} = 0.$$

- $\forall \varepsilon > 0$, let W_ε denote the law of $X_\cdot^{(\varepsilon)}$.

Theorem (Freidlin, Wentzell (year?)) On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2}\int_0^T |\dot{\omega}_t - b(\omega_t)|^2\,\mathrm{d}t\right)\mathbb{1}_{H^1([0,T];\mathbb{R}^d)}(\omega) + \infty\mathbb{1}_{H^1([0,T];\mathbb{R}^d)^\complement}(\omega)$$

§ 4
Conclusion

Open areas for research

- ⋆ Extension to SDEs with anticipating coefficients
- ⋆ Near-Markov property
- ⋆ Girsanov theorem for anticipating integrals
- ⋆ Freidlin-Wintzell type result for SDEs with anticipation

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

§ 5
Sample slides

Possible areas of interest

- ⋆ Extension to SDEs with anticipating coefficients
- ⋆ Near-Markov property
- ⋆ Girsanov theorem for generalized integration
- ⋆ Freidlin-Wintzell type result for SDEs with anticipating initial conditions

Theorem (Cramér, 1938) Let $X_1, X_2, \ldots$ be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example $\Lambda(t) < \infty \; \forall t \in \mathbb{R}$.
Then the Legendre transform of Λ, $\Lambda^* = \sup_{t\in\mathbb{R}}(tx - \Lambda(t))$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq nx\right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Freidlin–Wentzell theorem

Column 1

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

Column 2

$\times$	$\mathrm{d}t$	$\mathrm{d}B_t$
$\mathrm{d}t$	0	0
$\mathrm{d}B_t$	0	$\mathrm{d}t$

Something

This is a citation (<HKSZ2016>).

- ☐ One
- ☐ Two
- ☐ Three
- ☐ Four

Bibliography

Ayed, W. & Kuo, H. H. (2008). An extension of the Itô integral. *Communications on Stochastic Analysis*, *2*(3). doi:10.31390/cosa.2.3.05