Contents

1.1 Counting

Proposition (**Basic principle of counting**) *Suppose two independent experiments are performed, and there are m possible outcomes of the first experiment and n possible outcomes of the second experiment. Then the total possible outcomes of of the two experiments combined is mn.*

Proof Let (i, j) denote the case when the first experiment gives the ith outcome and the second experiment gives the jth outcome. Enumerating, we get

$$\begin{matrix} (1,1) & (1,2) & \dots & (1,n) \\ (2,1) & (2,2) & \dots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (m,1) & (m,2) & \dots & (m,n) \end{matrix}$$

Since there are m rows and n columns, we have total mn entries. □

Remark *This can be generalized to a finite number of experiments.*

Theorem (**Binomial theorem**)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof (*Inductive*) Homework. □

Proof (*Combinatorial*) Consider the product $(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$.
First, note that the expansion consists of 2^n terms, each being a product of n factors.
Secondly, each product contains either x_j xor y_j for each $j \in [n]$.
For example, $(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$.
Now, we can we choose k of the x_js and $n-k$ of the y_js in $\binom{n}{k}$ ways, so there are precisely those many terms with mk x_js and $n-k$ y_js in the expansion.
Finally, letting $x_j = x$ and $y_j = y$ for each $j \in [n]$, we get the result. □

Remark *This can be generalized to a finite number of experiments.*

2.1 Discrete Probability Spaces

Notations

Term	Description	Symbol/Idea	Coin toss Example
sample space	set of outcomes	Ω	$\{H, T\}$
outcome	arbitrary outcome	$\omega \in \Omega$	H
event	subset of sample space	E	$\emptyset, \{H\}, \{T\}, \{H, T\}$
mutually exclusive events	events with empty intersection	$E_1 \cap E_2 = \emptyset$	$\{H\}$ *and* $\{T\}$
probability mass function	weightage of each outcome	$p : \Omega \to [0,1]$, with $\sum_{\omega} p(\omega) = 1$	$p(H) = \frac{1}{3}, p(T) = \frac{2}{3}$
probability	(of an event)	$\mathbb{P} : 2^{\Omega} \to [0,1]$, $\mathbb{P}(E) = \sum_{\omega \in \Omega} p(\omega)$	$\mathbb{P}(\emptyset) = 0, \mathbb{P}(\{H, T\}) = 1$
random variable	a function	$X : \Omega \to \mathbb{R}$	$X(H) = 1, X(T) = 0$

Definition (Probability axioms) *A* non-negative valued *function* $\mathbb{P}$ *defined on the set of events is called a* probability measure *if the following hold.*

1. *(null empty set)* $\mathbb{P}(\emptyset) = 0$.
2. *(countable additivity) For any sequence of mutually exclusive events* $E_1, E_2, \cdots$, *we have* $\mathbb{P}\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$.
3. *(probability)* $\mathbb{P}(\Omega) = 1$.

Draw Venn diagrams for all of the following.

Proposition $\mathbb{P}(E^{\mathsf{C}}) = 1 - \mathbb{P}(E)$.

Proof Since $E \cap E^{\mathsf{C}} = \emptyset$, by Axiom 2 we have $1 = \mathbb{P}(\Omega) = \mathbb{P}(E \sqcup E^{\mathsf{C}}) = \mathbb{P}(E) + \mathbb{P}(E^{\mathsf{C}})$. $\square$

Proposition *If* $E \subset F$, *then* $\mathbb{P}(E) \leq \mathbb{P}(F)$.

Proof Note that $F = E \sqcup (F \setminus E)$. So by Axiom 2 we have $\mathbb{P}(F) = \mathbb{P}(E \sqcup (F \setminus E)) = \mathbb{P}(E) + \mathbb{P}(F \setminus E)$. Therefore, $\mathbb{P}(F) - \mathbb{P}(E) = \mathbb{P}(F \setminus E)$, which is non-negative since probability is a non-negative set function. $\square$

Proposition (Inclusion-Exclusion) $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$..

Proof

1. $E \cup F = (E \setminus F) \sqcup (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(E \cup F) = \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F)$.
2. $E = (E \setminus F) \sqcup (E \cap F)$, so $\mathbb{P}(E) = \mathbb{P}(E \setminus F) + \mathbb{P}(E \cap F)$, and similarly
3. $F = (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(F) = \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F)$.

Combining the above,

$$\begin{aligned}\mathbb{P}(E \cup F) &= \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F) \\ &= (\mathbb{P}(E) - \mathbb{P}(E \cap F)) + (\mathbb{P}(F) - \mathbb{P}(E \cap F)) + \mathbb{P}(E \cap F) \\ &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).\end{aligned}$$

□

Bibliography