

Mathematical Logic

Notes and Exercises

Sudip Sinha

October 15, 2019

CONTENTS

1	Sudip Sinha	PHIL 4010: PS1	2019-09-10	1
2	Sudip Sinha	PHIL 4010: PS2	2019-09-24	2
3	Sudip Sinha	PHIL 4010: Prelim	2019-10-08	5
4	Sudip Sinha	PHIL 4010: PS3	2019-10-15	11
	Bibliography			14

Exercise 1.1 (Notes, 1.8) For any sets A and B , we have $A \cap B \subseteq A$.

Solution. Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A , we have $A \cap B \subseteq A$. \square

Exercise 1.2 (Notes, 1.10) For any set A , we have $A \cap \emptyset = \emptyset$.

Solution. (\subseteq) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.

(\supseteq) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. \square

Exercise 1.3 (Notes, 1.13) For any sets A and B , if $A \subseteq B$, then $A \cup B = B$.

Solution. (\subseteq) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.

(\supseteq) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. \square

Note: We shall say that a truth assignment v satisfies Σ iff it satisfies every member of Σ .

Exercise 2.1 (Enderton, 1.2.1) Show that neither of the following two formulas tautologically implies the other:

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Solution. We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$.

Consider v such that $vitem = vitem = F$ and $vitem = T$. Under \bar{v} , we get exactly what is required as is shown in the computations below. (Here the truth assignments by \bar{v} is denoted under each symbol.)

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$T \quad F \quad T \quad F \quad F \quad T$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

$$F \quad F \quad F \quad F \quad F \quad FT$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = T$ and $\bar{v}(\alpha) = F$.

Consider v such that $vitem = vitem = vitem = F$. Under \bar{v} , we get exactly what is required as is shown in the computations below.

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

$$T = \quad T \quad TF \quad T \quad TF \quad T \quad TF$$

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$F = \quad F \quad F \quad F \quad T \quad F$$

□

Exercise 2.2 (Enderton, 1.2.4item) Show that $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \rightarrow \beta)$.

Solution. We show each direction separately.

(\Rightarrow) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ . We have to show that v satisfies $(\alpha \rightarrow \beta)$. We have two cases.

- i. $\bar{v}(\alpha) = T$: In this case, from the supposition, we get $\bar{v}(\beta) = T$. So $\bar{v}(\alpha \rightarrow \beta) = T$.
- ii. $\bar{v}(\alpha) = F$: In this case, $\bar{v}(\alpha \rightarrow \beta) = T$ since the antecedent is F .

Since v was arbitrary, we have $\Sigma \models (\alpha \rightarrow \beta)$.

(\Leftarrow) We suppose $\Sigma \models (\alpha \rightarrow \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β . Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ . Therefore, by our supposition, v satisfies $(\alpha \rightarrow \beta)$. Now, since v satisfies α , it can only be that v satisfies β , since the only other way the material implication can be satisfied is when v does not satisfy α . This proves our claim. \square

Exercise 2.3 (Enderton, 1.2.5) Prove or refute each of the following assertions:

- a. If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.

Solution. (T) There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α . This implies $\bar{v}(\alpha \vee \beta) = T$ by the definition of extension of \bar{v} for \vee . \square

- b. If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.

Solution. (F) We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \vee (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$.

For an explicit example, consider two truth assignments v_1 and v_2 , such that $v_1(\alpha) = T$ and $v_2(\alpha) = F$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α , and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$. \square

Exercise 2.4 (Enderton, 1.2.6)

- a. Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α , then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.

Solution. Let G be the set of sentence symbols used in α , and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$.
 Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α .
 Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ . Let $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for \bar{v}_1 and \bar{v}_2 , we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \Box \psi) = \bar{v}_2(\phi \Box \psi)$. Hence $(\neg\phi), (\phi \Box \psi) \in B$, that is, B is closed with respect to the formula building operations.
 Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done. \square

- b. Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ .

Solution. In this part, we use v to denote truth assignments and “ v on a set” means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ . Clearly, $G \subseteq S$.

We show each direction separately.

(\Rightarrow) From the definition of tautological implication,

$$\begin{aligned} \Sigma \models \tau & \\ \Leftrightarrow (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) & \\ \Rightarrow (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \text{ [Part (a)]} & \end{aligned}$$

(\Leftarrow) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ . Therefore,

$$\begin{aligned} & (\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \Rightarrow & (\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\ \Leftrightarrow & \Sigma \models \tau \end{aligned}$$

\square

Exercise 3.1 (Set Theory) Prove the following. 10 points each.

Note: Let A and B are sets. In order to prove $A = B$, it is enough to show $A \subseteq B$ and $A \supseteq B$.

In each of the following problems, we show each inclusion separately.

Moreover, to show $A \subseteq B$, it suffices to show that for x arbitrary, $x \in A \implies x \in B$.

i. If $A \subseteq B$, then $A \cap B = A$.

Solution.

(\subseteq) Let $x \in A \cap B$ be arbitrary. This mean $x \in A$ and $x \in B$. So $x \in A$.

(\supseteq) Let $x \in A$ be arbitrary. Then by the hypothesis $x \in B$ since $A \subseteq B$. Therefore, $x \in A$ and $x \in B$, and thus $x \in A \cap B$. □

ii. If $A \cap B = \emptyset$, then $A \setminus B = A$.

Solution.

(\subseteq) Let $x \in A \setminus B$ be arbitrary. Then $x \in A$ and $x \notin B$. It is enough to show that $x \in A$ implies $x \notin B$. But must be true since if $x \in A$ and $x \in B$, then $x \in A \cap B = \emptyset$, which is absurd.

(\supseteq) Let $x \in A$ be arbitrary. Now, either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A \cap B$ since $x \in A$ by hypothesis. But this is an impossibility since $A \cap B = \emptyset$. Therefore, it must be that $x \notin B$. So $x \in A \setminus B$. □

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

(\subseteq) Let $x \in A \cap (B \cup C)$ be arbitrary. Then $x \in A$ and $x \in B \cup C$. Note that $x \in B \cup C$ means $x \in B$ or $x \in C$. Now, either $x \in B$ or $x \notin B$, so have two cases.

- ($x \in B$) In this case, $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.
- ($x \notin B$) Since $x \in B$ or $x \in C$, and $x \notin B$, it is necessary that $x \in C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

(\supset) Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. This means $x \in A \cap B$ or $x \in A \cap C$. As above, we have two cases, either $x \in A \cap B$ or $x \notin A \cap B$.

- $(x \in A \cap B)$ In this case, $x \in A$ and $x \in B$. Now, so $x \in B$ implies $x \in B$ or $x \in C$, that is, $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.
- $(x \notin A \cap B)$ Again, since $x \in A \cap B$ or $x \in A \cap C$, and $x \notin A \cap B$, it is necessary that $x \in A \cap C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

□

Exercise 3.2 (Construction) 10 points each.

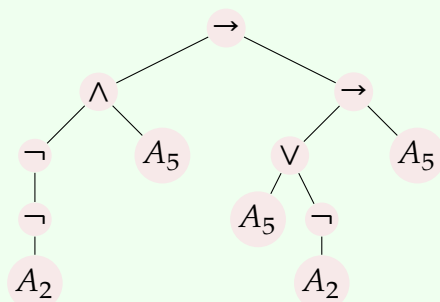
- i. Write down a construction sequence for $((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \rightarrow A_3) \leftrightarrow A_7))$.

Solution. $\langle A_1, A_3, A_4, A_7, (\neg A_1), ((\neg A_1) \vee A_4), (\neg((\neg A_1) \vee A_4)), (A_1 \rightarrow A_3), ((A_1 \rightarrow A_3) \leftrightarrow A_7), ((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \rightarrow A_3) \leftrightarrow A_7)) \rangle$. □

- ii. Write down a construction tree for $((\neg(\neg A_2)) \wedge A_5) \rightarrow ((A_5 \vee (\neg A_2)) \rightarrow A_5)$.

Solution.

□



Exercise 3.3 (Truth Assignments)

- i. Let S be the set of all sentence symbols, and assume that $v : S \rightarrow \{F, T\}$ is a truth assignment. Show there is at most one extension \bar{v} meeting conditions 0–5 on pp. 20–21. (Hint: Show that if v_1 and v_2 are such extensions, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α . Use the induction principle.) 20 points.

Solution. We show this via induction on the complexity of any arbitrary wff α .

- **(Base case)** Assume α be a sentence symbol. Then $\bar{v}_1(\alpha) = v(\alpha) = \bar{v}_2(\alpha)$ since \bar{v}_1 and \bar{v}_2 are both extensions of v .
- **(Induction step)** We assume that the result holds for all wffs less complex than α (induction hypothesis). Now, we show that the result holds under all the formula building operations.

(\neg) Assume $\alpha = (\neg\beta)$ for some wff β . Then

$$\begin{aligned}
 \bar{v}_1(\alpha) &= T \\
 \iff \bar{v}_1(\neg\beta) &= T && [\text{Def of } \alpha] \\
 \iff \bar{v}_1(\beta) &= F && [\text{Def of } \bar{v} \text{ under } \neg] \\
 \iff \bar{v}_2(\beta) &= F && [\text{Induction hypothesis}] \\
 \iff \bar{v}_2(\neg\beta) &= T && [\text{Def of } \bar{v} \text{ under } \neg] \\
 \iff \bar{v}_2(\alpha) &= T && [\text{Def of } \alpha]
 \end{aligned}$$

(\wedge) Assume $\alpha = (\beta \wedge \gamma)$ for some wffs β, γ . Then

$$\begin{aligned}
 \bar{v}_1(\alpha) &= T \\
 \iff \bar{v}_1(\beta \wedge \gamma) &= T && [\text{Def of } \alpha] \\
 \iff \bar{v}_1(\beta) = T \text{ and } \bar{v}_1(\gamma) &= T && [\text{Def of } \bar{v} \text{ under } \wedge] \\
 \iff \bar{v}_2(\beta) = T \text{ and } \bar{v}_2(\gamma) &= T && [\text{Induction hypothesis}] \\
 \iff \bar{v}_2(\beta \wedge \gamma) &= T && [\text{Def of } \bar{v} \text{ under } \wedge] \\
 \iff \bar{v}_2(\alpha) &= T && [\text{Def of } \alpha]
 \end{aligned}$$

(\vee) Assume $\alpha = (\beta \vee \gamma)$ for some wffs β, γ . Then

$$\begin{aligned}
 \bar{v}_1(\alpha) &= T \\
 \iff \bar{v}_1(\beta \vee \gamma) &= T && [\text{Def of } \alpha] \\
 \iff \bar{v}_1(\beta) = T \text{ or } \bar{v}_1(\gamma) &= T && [\text{Def of } \bar{v} \text{ under } \vee] \\
 \iff \bar{v}_2(\beta) = T \text{ or } \bar{v}_2(\gamma) &= T && [\text{Induction hypothesis}] \\
 \iff \bar{v}_2(\beta \vee \gamma) &= T && [\text{Def of } \bar{v} \text{ under } \vee] \\
 \iff \bar{v}_2(\alpha) &= T && [\text{Def of } \alpha]
 \end{aligned}$$

(\rightarrow) Assume $\alpha = (\beta \rightarrow \gamma)$ for some wffs β, γ . Then

$$\begin{aligned}
 & \bar{v}_1(\alpha) = T \\
 \Leftrightarrow & \bar{v}_1(\beta \rightarrow \gamma) = T && [\text{Def of } \alpha] \\
 \Leftrightarrow & \bar{v}_1(\beta) = F \text{ or } \bar{v}_1(\gamma) = T && [\text{Def of } \bar{v} \text{ under } \rightarrow] \\
 \Leftrightarrow & \bar{v}_2(\beta) = F \text{ or } \bar{v}_2(\gamma) = T && [\text{Induction hypothesis}] \\
 \Leftrightarrow & \bar{v}_2(\beta \rightarrow \gamma) = T && [\text{Def of } \bar{v} \text{ under } \rightarrow] \\
 \Leftrightarrow & \bar{v}_2(\alpha) = T && [\text{Def of } \alpha]
 \end{aligned}$$

(\leftrightarrow) Assume $\alpha = (\beta \leftrightarrow \gamma)$ for some wffs β, γ . Then

$$\begin{aligned}
 & \bar{v}_1(\alpha) = T \\
 \Leftrightarrow & \bar{v}_1(\beta \leftrightarrow \gamma) = T && [\text{Def of } \alpha] \\
 \Leftrightarrow & \bar{v}_1(\beta) = \bar{v}_1(\gamma) && [\text{Def of } \bar{v} \text{ under } \leftrightarrow] \\
 \Leftrightarrow & \bar{v}_2(\beta) = \bar{v}_2(\gamma) && [\text{Induction hypothesis}] \\
 \Leftrightarrow & \bar{v}_2(\beta \leftrightarrow \gamma) = T && [\text{Def of } \bar{v} \text{ under } \leftrightarrow] \\
 \Leftrightarrow & \bar{v}_2(\alpha) = T && [\text{Def of } \alpha]
 \end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α , which proves the uniqueness of the extension. \square

- ii. Show that for a set of wffs Σ and a wff α : $\Sigma \cup \{\neg\neg\alpha\}$ is satisfiable $\Leftrightarrow \Sigma \cup \{\alpha\}$ is satisfiable. 10 points.

Solution. First, note that for any wff α and truth assignment v ,

$$\bar{v}(\alpha) = T \quad \Leftrightarrow \quad \bar{v}(\neg\alpha) = F \quad \Leftrightarrow \quad \bar{v}(\neg\neg\alpha) = T.$$

Therefore, we have the following (v always represents a truth assignment):

$$\begin{aligned}
 & \Sigma \cup \{\alpha\} \text{ is satisfiable.} \\
 \Leftrightarrow & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\alpha) = T. \\
 \Leftrightarrow & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\alpha) = F. \\
 \Leftrightarrow & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\neg\alpha) = T. \\
 \Leftrightarrow & \Sigma \cup \{\neg\neg\alpha\} \text{ is satisfiable.}
 \end{aligned}$$

\square

Exercise 3.4 (Compactness) Recall the Compactness Theorem: A set of wffs is satisfiable iff it is finitely satisfiable.

Recall Corollary 17A: If $\Sigma \models \tau$, then $\Sigma_0 \models \tau$ for some finite $\Sigma_0 \subseteq \Sigma$.

Prove that they are equivalent, i.e., prove that the Compactness Theorem holds iff Corollary 17A holds.

(Hint: Use the fact that $\Gamma \models \sigma$ iff $\Gamma \cup \{\neg\sigma\}$ is unsatisfiable and 3.3.ii above.) 20 points.

Solution. The proof of Corollary 17A in the book shows that the Compactness Theorem implies Corollary 17A. Therefore, we are left to show that Corollary 17A implies the Compactness Theorem.

For this, we assume Corollary 17A and prove Compactness Theorem. Note that if a set of wffs is satisfiable with a truth assignment, then it is finitely satisfied with the same truth assignment. Therefore, we only have to show that finite satisfiability implies satisfiability.

Suppose not. That is, assume that Σ is a set of wffs such that Σ is finitely satisfiable but Σ is unsatisfiable. Fix a wff τ . Since Σ is unsatisfiable, it is vacuously true that $\Sigma \models \tau$ and $\Sigma \models \neg\tau$ (as in page 23 of Enderton). Since $\Sigma \models \tau$, using Corollary 17A, there is a finite subset $\Sigma_1 \subseteq \Sigma$ such that $\Sigma_1 \models \tau$. Similarly, there exists $\Sigma_2 \subseteq \Sigma$ finite such that $\Sigma_2 \models \neg\tau$. Now, since $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$ is finite, it is satisfiable by a truth assignment, say v . Clearly, since $\Sigma_1, \Sigma_2 \subseteq \Sigma$, the assignment v satisfies both Σ_1 and Σ_2 . Then v satisfies both τ and $\neg\tau$, which is an impossibility. This contradiction shows that if Σ is finitely satisfiable then Σ is satisfiable. This concludes the proof. \square

Exercise 3.5 (Substitution) Let $\alpha_1, \alpha_2, \dots$ be a sequence of wffs. For each wff ϕ and $n \in \mathbb{N}$, let ϕ^* be the result of replacing the sentence symbol A_n in ϕ by the wff α_n . Suppose that v is a truth assignment for the set of all sentence symbols and that u is a truth assignment defined by $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\phi) = \bar{v}(\phi^*)$.
(Hint: Use the induction principle.) 20 points

Solution. We show this via induction on the complexity of any arbitrary wff ϕ .

- **(Base case)** Assume $\phi = A_n$ for some $n \in \mathbb{N}$, so $\phi^* = \alpha_n$. Now $\bar{u}(\phi) = \bar{u}(A_n) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\phi^*)$, so the result holds when ϕ is a sentence symbol.
- **(Induction step)** We assume that the result holds for all wffs less complex than ϕ (induction hypothesis). Now, we show that the result holds under all the formula building operations.

(\neg) Assume $\phi = (\neg\psi)$ for some wff ψ , so $\phi^* = (\neg\psi^*)$. Then

$$\begin{aligned}
 \bar{u}(\phi) &= T \\
 \iff \bar{u}(\neg\psi) &= T && [\text{Def of } \phi] \\
 \iff \bar{u}(\psi) &= F && [\text{Def of } \bar{u} \text{ under } \neg] \\
 \iff \bar{v}(\psi^*) &= F && [\text{Induction hypothesis}] \\
 \iff \bar{v}(\neg\psi^*) &= T && [\text{Def of } \bar{v} \text{ under } \neg] \\
 \iff \bar{v}(\phi^*) &= T && [\text{Def of } \phi^*]
 \end{aligned}$$

(\wedge) Assume $\phi = (\psi \wedge \theta)$ for some wffs ψ, θ , so $\phi^* = (\psi^* \wedge \theta^*)$. Then

$$\bar{u}(\phi) = T$$

$$\Leftrightarrow \bar{u}(\psi \wedge \theta) = T \quad [\text{Def of } \phi]$$

$$\Leftrightarrow \bar{u}(\psi) = T \text{ and } \bar{u}(\theta) = T \quad [\text{Def of } \bar{u} \text{ under } \wedge]$$

$$\Leftrightarrow \bar{v}(\psi^*) = T \text{ and } \bar{v}(\theta^*) = T \quad [\text{Induction hypothesis}]$$

$$\Leftrightarrow \bar{v}(\psi^* \wedge \theta^*) = T \quad [\text{Def of } \bar{v} \text{ under } \wedge]$$

$$\Leftrightarrow \bar{v}(\phi^*) = T \quad [\text{Def of } \phi^*]$$

(\vee) Assume $\phi = (\psi \vee \theta)$ for some wffs ψ, θ , so $\phi^* = (\psi^* \vee \theta^*)$. Then

$$\bar{u}(\phi) = T$$

$$\Leftrightarrow \bar{u}(\psi \vee \theta) = T \quad [\text{Def of } \phi]$$

$$\Leftrightarrow \bar{u}(\psi) = T \text{ or } \bar{u}(\theta) = T \quad [\text{Def of } \bar{u} \text{ under } \vee]$$

$$\Leftrightarrow \bar{v}(\psi^*) = T \text{ or } \bar{v}(\theta^*) = T \quad [\text{Induction hypothesis}]$$

$$\Leftrightarrow \bar{v}(\psi^* \vee \theta^*) = T \quad [\text{Def of } \bar{v} \text{ under } \vee]$$

$$\Leftrightarrow \bar{v}(\phi^*) = T \quad [\text{Def of } \phi^*]$$

(\rightarrow) Assume $\phi = (\psi \rightarrow \theta)$ for some wffs ψ, θ , so $\phi^* = (\psi^* \rightarrow \theta^*)$. Then

$$\bar{u}(\phi) = T$$

$$\Leftrightarrow \bar{u}(\psi \rightarrow \theta) = T \quad [\text{Def of } \phi]$$

$$\Leftrightarrow \bar{u}(\psi) = F \text{ or } \bar{u}(\theta) = T \quad [\text{Def of } \bar{u} \text{ under } \rightarrow]$$

$$\Leftrightarrow \bar{v}(\psi^*) = F \text{ or } \bar{v}(\theta^*) = T \quad [\text{Induction hypothesis}]$$

$$\Leftrightarrow \bar{v}(\psi^* \rightarrow \theta^*) = T \quad [\text{Def of } \bar{v} \text{ under } \rightarrow]$$

$$\Leftrightarrow \bar{v}(\phi^*) = T \quad [\text{Def of } \phi^*]$$

(\leftrightarrow) Assume $\phi = (\psi \leftrightarrow \theta)$ for some wffs ψ, θ , so $\phi^* = (\psi^* \leftrightarrow \theta^*)$. Then

$$\bar{u}(\phi) = T$$

$$\Leftrightarrow \bar{u}(\psi \leftrightarrow \theta) = T \quad [\text{Def of } \phi]$$

$$\Leftrightarrow \bar{u}(\psi) = \bar{u}(\theta) \quad [\text{Def of } \bar{u} \text{ under } \leftrightarrow]$$

$$\Leftrightarrow \bar{v}(\psi^*) = \bar{v}(\theta^*) \quad [\text{Induction hypothesis}]$$

$$\Leftrightarrow \bar{v}(\psi^* \leftrightarrow \theta^*) = T \quad [\text{Def of } \bar{v} \text{ under } \leftrightarrow]$$

$$\Leftrightarrow \bar{v}(\phi^*) = T \quad [\text{Def of } \phi^*]$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{u}(\phi) = \bar{v}(\phi)$ for every wff ϕ . \square

Note: \nexists abbreviates $\neg\exists$, and \nsubseteq abbreviates $\neg\subseteq$. We shall also use the convention that grouping for conditionals is from the right. That is, $(p \rightarrow q \rightarrow r) = ((p \rightarrow (q \rightarrow r)))$.

Exercise 4.1 (Enderton, 2.1.1) Assume that we have a language with the following parameters: \forall , intended to mean “for all things”; N , intended to mean “is a number”; I , intended to mean “is interesting”; $<$, intended to mean “is less than”; and 0 , a constant symbol intended to denote zero. Translate into this language the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.

- a. Zero is less than any number.

Solution. $\forall x(Nx \rightarrow < 0x)$ □

- b. If any number is interesting, then zero is interesting.

Solution. The word *any number* can be interpreted as *every number* or *some number*. Using the exportation tautology, the corresponding translations are

(every) $\forall x(Nx \rightarrow Ix \rightarrow I0)$

(some) $\exists x(Nx \rightarrow Ix) \rightarrow I0$ □

- c. No number is less than zero.

Solution.

$$\begin{aligned} & \nexists x(Nx \wedge < x0) \\ \Leftrightarrow & \neg\exists x(Nx \wedge < x0) \\ \Leftrightarrow & \neg\neg\forall x(\neg(Nx \wedge < x0)) \\ \Leftrightarrow & \neg\neg\forall x(\neg\neg(Nx \rightarrow \neg < x0)) \\ \Leftrightarrow & \forall x(Nx \rightarrow \neg < x0) \end{aligned}$$

□

- d. Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.

Solution. $\forall x(Nx \rightarrow \neg Ix \rightarrow \forall y(Ny \rightarrow < yx \rightarrow Iy) \rightarrow Ix)$ □

- e. There is no number such that all numbers are less than it.

Solution.

$$\begin{aligned}
 & \neg \exists x (Nx \rightarrow \forall y (Ny \rightarrow (x < y))) \\
 \Leftrightarrow & \neg \exists x (Nx \rightarrow \forall y (Ny \rightarrow \neg (x < y))) \\
 \Leftrightarrow & \neg \neg \forall x \neg (Nx \rightarrow \forall y (Ny \rightarrow \neg (x < y))) \\
 \Leftrightarrow & \forall x \neg (Nx \rightarrow \forall y (Ny \rightarrow \neg (x < y)))
 \end{aligned}$$

□

f. *There is no number such that no number is less than it.*

Solution.

$$\begin{aligned}
 & \neg \exists x (Nx \rightarrow \neg \exists y (Ny \rightarrow (y < x))) \\
 \Leftrightarrow & \neg \exists x (Nx \rightarrow \neg \exists y (Ny \rightarrow \neg (y < x))) \\
 \Leftrightarrow & \neg \neg \forall x \neg (Nx \rightarrow \neg \neg \forall y \neg (Ny \rightarrow \neg (y < x))) \\
 \Leftrightarrow & \forall x \neg (Nx \rightarrow \forall y \neg (Ny \rightarrow \neg (y < x)))
 \end{aligned}$$

□

Exercise 4.2 (Enderton, 2.1.3) *Translate the English sentence into the first-order language specified by \forall , for all sets; \in , is a member of; a , a ; b , b .*

“Neither a nor b is a member of every set.”

Solution.

Neither a nor b is a member of every set.

\Leftrightarrow a is not a member of every set and b is not a member of every set.

\Leftrightarrow There is a set that a is not a member of and there is a set that b is not a member of.

$\Leftrightarrow (\exists x (a \notin x)) \wedge (\exists y (b \notin y))$

$\Leftrightarrow (\exists x (\neg \in ax)) \wedge (\exists y (\neg \in by))$

$\Leftrightarrow (\neg \forall x (\neg \neg \in ax)) \wedge (\neg \forall y (\neg \neg \in by))$

$\Leftrightarrow (\neg \forall x \in ax) \wedge (\neg \forall y \in by)$

□

Exercise 4.3 (Enderton, page 87) *Prove that $\models_{\mathcal{A}} \alpha \vee \beta [s]$ iff $\models_{\mathcal{A}} \alpha [s]$ or $\models_{\mathcal{A}} \beta [s]$.*

Solution.

$$\begin{aligned}
 & \models_{\mathcal{A}} (\alpha \vee \beta) [s] \\
 \Leftrightarrow & \models_{\mathcal{A}} (\neg \alpha \rightarrow \beta) [s] && [\text{Expansion of } \vee] \\
 \Leftrightarrow & \text{either } \not\models_{\mathcal{A}} \neg \alpha [s] \text{ or } \models_{\mathcal{A}} \beta [s] && [\text{Definition of } s \text{ for } \rightarrow] \\
 \Leftrightarrow & \text{either } \models_{\mathcal{A}} \alpha [s] \text{ or } \models_{\mathcal{A}} \beta [s] && [\text{Definition of } s \text{ for } \neg]
 \end{aligned}$$



BIBLIOGRAPHY