Mathematical Logic

Notes and Exercises

Sudip Sinha

September 23, 2019

Contents

1 Sudip Sinha PHIL 4010 2019-09-10

Exercise (Notes, 1.8) *For any sets A and B, we have $A \cap B \subseteq A$.*

Solution Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A, we have $A \cap B \subseteq A$. □

Exercise (Notes, 1.10) *For any set A, we have $A \cap \emptyset = \emptyset$.*

Solution ($\subseteq$) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.

($\supseteq$) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. □

Exercise (Notes, 1.13) *For any sets A and B, if $A \subseteq B$, then $A \cup B = B$.*

Solution ($\subseteq$) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.

($\supseteq$) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

2 Sudip Sinha PHIL 4010 2019-09-24

Note: We shall say that a truth assigment v satisfies Σ iff it satisfies every member of Σ.

Exercise (Enderton, 1.2.1) *Show that neither of the following two formulas tautologically implies the other:*

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$

Solution We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$.
Consider v such that $v(A) = v(B) = F$ and $v(C) = T$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below. (Here the truth assignments by $\bar{v}$ is denoted under each symbol.)

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$T \qquad F \;\; T \quad F \;\; F \;\; T$$

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$
$$F \qquad F \;\; F \qquad\qquad F \qquad\qquad F \qquad\qquad F \quad FT$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = T$ and $\bar{v}(\alpha) = F$.
Consider v such that $v(A) = v(B) = v(C) = F$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below.

$$\beta = ((A \land (B \land C)) \lor ((\neg A) \land ((\neg B) \land (\neg C))))$$
$$T = \qquad\qquad\qquad\qquad T \quad TF \quad T \quad TF \quad T \quad TF$$

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$F = \quad F \;\; F \quad F \;\; T \;\; F$$

$\square$

Exercise (Enderton, 1.2.4(a)) *Show that $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \to \beta)$.*

Solution We show each direction separately.

($\Longrightarrow$) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ. We have to show that v satisfies $(\alpha \to \beta)$. We have two cases.

i. $\bar{v}(\alpha) = T$: In this case, from the supposition, we get $\bar{v}(\beta) = T$. So $\bar{v}(\alpha \to \beta) = T$.

ii. $\bar{v}(\alpha) = F$: In this case, $\bar{v}(\alpha \to \beta) = T$ since the antecedent is F.

Since v was arbitrary, we have $\Sigma \models (\alpha \to \beta)$.

($\Longleftarrow$) We suppose $\Sigma \models (\alpha \to \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β. Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ. Therefore, by our supposition, v satisfies $(\alpha \to \beta)$. Now, since v satisfies α, it can only be that v satisfies β, since the only other way the material implication can be satisfied is when v does not satisfies α. This proves our claim. $\square$

Exercise (Enderton, 1.2.5) *Prove or refute each of the following assertions:*

a. If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \lor \beta)$.

b. If $\Sigma \models (\alpha \lor \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.

Solution

a. (**T**) There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α. This implies $\bar{v}(\alpha \lor \beta) = T$ by the definition of extension of $\bar{v}$ for $\lor$.

b. (**F**) We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \lor (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$. For an explicit example, consider two truth assignments v_1 and v_2, such that $v_1(\alpha) = T$ and $v_2(\alpha) = F$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α, and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$.

$\square$

Exercise (Enderton, 1.2.6)

a. *Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.*

b. *Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ.*

Solution

a. Let G be the set of sentence symbols used in α, and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$. Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α.
Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ. Let $\boxdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for $\bar{v}_1$ and $\bar{v}_2$, we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \boxdot \psi) = \bar{v}_2(\phi \boxdot \psi)$. Hence $(\neg\phi), (\phi \boxdot \psi) \in B$, that is, B is closed with respect to the formula building operations. Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done.

b. In this part, we use v to denote truth assignments and "v on a set" means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ. Clearly, $G \subseteq S$.
We show each direction separately.
($\Longrightarrow$) From the definition of tautological implication,

$$\begin{aligned}
&\Sigma \models \tau \\
\iff &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\Longrightarrow &(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \text{ [Part (a)]}
\end{aligned}$$

($\Longleftarrow$) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ. Therefore,

$$\begin{aligned}
&(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\Longrightarrow &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\iff &\Sigma \models \tau
\end{aligned}$$

$\square$

Bibliography