Mathematical Logic

Notes and Exercises

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Contents

1 Sudip Sinha PHIL 4010: HW1 2019-09-10

Exercise 1.1 **(Notes, 1.8)** *For any sets A and B, we have $A \cap B \subseteq A$.*

Solution. Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A, we have $A \cap B \subseteq A$. □

Exercise 1.2 **(Notes, 1.10)** *For any set A, we have $A \cap \emptyset = \emptyset$.*

Solution. ($\subseteq$) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.
($\supseteq$) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. □

Exercise 1.3 **(Notes, 1.13)** *For any sets A and B, if $A \subseteq B$, then $A \cup B = B$.*

Solution. ($\subseteq$) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.
($\supseteq$) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

2 Sudip Sinha PHIL 4010: HW2 2019-09-24

Note: We shall say that a truth assigment v satisfies Σ iff it satisfies every member of Σ.

Exercise 2.1 (Enderton, 1.2.1) *Show that neither of the following two formulas tautologically implies the other:*

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Solution. We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = \top$ and $\bar{v}(\beta) = \bot$.

Consider v such that $v(A) = v(B) = \bot$ and $v(C) = \top$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below. (Here the truth assignments by $\bar{v}$ is denoted under each symbol.)

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$\top \qquad \bot \;\; \top \;\; \bot \;\; \bot \;\; \top$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$
$$\bot \qquad \bot \; \bot \qquad\qquad \bot \qquad\qquad \bot \qquad\qquad \bot \quad \bot \top$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = \top$ and $\bar{v}(\alpha) = \bot$.

Consider v such that $v(A) = v(B) = v(C) = \bot$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below.

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$
$$\top = \qquad\qquad\qquad\qquad \top \quad \top\bot \;\; \top \quad \top\bot \;\; \top \;\; \top\bot$$

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$\bot = \;\; \bot \;\; \bot \;\; \bot \;\; \top \;\; \bot$$

∎

Exercise 2.2 **(Enderton, 1.2.4(a))** *Show that* $\Sigma \cup \{\alpha\} \models \beta$ *iff* $\Sigma \models (\alpha \to \beta)$.

Solution. We show each direction separately.

($\Longrightarrow$) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ. We have to show that v satisfies $(\alpha \to \beta)$. We have two cases.

i. $\bar{v}(\alpha) = \top$: In this case, from the supposition, we get $\bar{v}(\beta) = \top$. So $\bar{v}(\alpha \to \beta) = \top$.

ii. $\bar{v}(\alpha) = \bot$: In this case, $\bar{v}(\alpha \to \beta) = \top$ since the antecedent is $\bot$.

Since v was arbitrary, we have $\Sigma \models (\alpha \to \beta)$.

($\Longleftarrow$) We suppose $\Sigma \models (\alpha \to \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β. Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ. Therefore, by our supposition, v satisfies $(\alpha \to \beta)$. Now, since v satisfies α, it can only be that v satisfies β, since the only other way the material implication can be satisfied is when v does not satisfies α. This proves our claim. $\square$

Exercise 2.3 **(Enderton, 1.2.5)** *Prove or refute each of the following assertions:*

a. *If either* $\Sigma \models \alpha$ *or* $\Sigma \models \beta$, *then* $\Sigma \models (\alpha \vee \beta)$.

Solution. ($\top$) There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α. This implies $\bar{v}(\alpha \vee \beta) = \top$ by the definition of extension of $\bar{v}$ for $\vee$. $\square$

b. *If* $\Sigma \models (\alpha \vee \beta)$, *then either* $\Sigma \models \alpha$ *or* $\Sigma \models \beta$.

Solution. ($\bot$) We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \vee (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$.

For an explicit example, consider two truth assignments v_1 and v_2, such that $v_1(\alpha) = \top$ and $v_2(\alpha) = \bot$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α, and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$. $\square$

Exercise 2.4 **(Enderton, 1.2.6)**

a. *Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.*

Solution. Let G be the set of sentence symbols used in α, and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$.

Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α.

Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ. Let $\boxdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for $\bar{v}_1$ and $\bar{v}_2$, we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \boxdot \psi) = \bar{v}_2(\phi \boxdot \psi)$. Hence $(\neg\phi), (\phi \boxdot \psi) \in B$, that is, B is closed with respect to the formula building operations.

Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done. □

b. *Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ.*

Solution. In this part, we use v to denote truth assignments and "v on a set" means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ. Clearly, $G \subseteq S$.

We show each direction separately.

($\Longrightarrow$) From the definition of tautological implication,

$$
\begin{aligned}
&\Sigma \models \tau \\
\iff &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\Longrightarrow &(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \text{ [Part (a)]}
\end{aligned}
$$

($\Longleftarrow$) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ. Therefore,

$$
\begin{aligned}
&(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\Longrightarrow &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \rightarrow (v \text{ satisfies } \tau)) \\
\iff &\Sigma \models \tau
\end{aligned}
$$

□

3 Sudip Sinha PHIL 4010: Prelim 2019-10-08

Exercise 3.1 (Set Theory) *Prove the following. 10 points each.*

Note: *Let A and B are sets. In order to prove $A = B$, it is enough to show $A \subseteq B$ and $A \supseteq B$. In each of the following problems, we show each inclusion separately.*
Moreover, to show $A \subseteq B$, it suffices to show that for x arbitrary, $x \in A \implies x \in B$.

i. *If $A \subseteq B$, then $A \cap B = A$.*

Solution.

($\subseteq$) Let x be arbitrary. Then
$x \in A \cap B \iff x \in A$ and $x \in B \implies x \in A$.

($\supseteq$) Let $x \in A$ be arbitrary. Then by the hypothesis $x \in B$ since $A \subseteq B$. Therefore, $x \in A$ and $x \in B$, and thus $x \in A \cap B$.

□

ii. *If $A \cap B = \emptyset$, then $A \setminus B = A$.*

Solution.

($\subseteq$) Let $x \in A \setminus B$ be arbitrary. Then $x \in A$ and $x \notin B$. It is enough to show that $x \in A$ implies $x \notin B$. But must be true since if $x \in A$ and $x \in B$, then $x \in A \cap B = \emptyset$, which is absurd.

($\supseteq$) Let $x \in A$ be arbitrary. Now, either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A \cap B$ since $x \in A$ by hypothesis. But this is an impossibility since $A \cap B = \emptyset$. Therefore, it must be that $x \notin B$. So $x \in A \setminus B$.

□

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

($\subseteq$) Let $x \in A \cap (B \cup C)$ be arbitrary. Then $x \in A$ and $x \in B \cup C$. Note that $x \in B \cup C$ means $x \in B$ or $x \in C$. Now, either $x \in B$ or $x \notin B$, so have two cases.

- ($x \in B$) In this case, $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.

- ($x \notin B$) Since $x \in B$ or $x \in C$, and $x \notin B$, it is necessary that $x \in C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

(⊇) Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. This means $x \in A \cap B$ or $x \in A \cap C$. As above, we have two cases, either $x \in A \cap B$ or $x \notin A \cap B$.

- $(x \in A \cap B)$ In this case, $x \in A$ and $x \in B$. Now, so $x \in B$ implies $x \in B$ or $x \in C$, that is, $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.
- $(x \notin A \cap B)$ Again, since $x \in A \cap B$ or $x \notin A \cap B$, and $x \notin A \cap B$, it is necessary that $x \in A \cap C$. Therefore we get the exact same result by interchanging the roles of B and C in the previous case.

$\square$

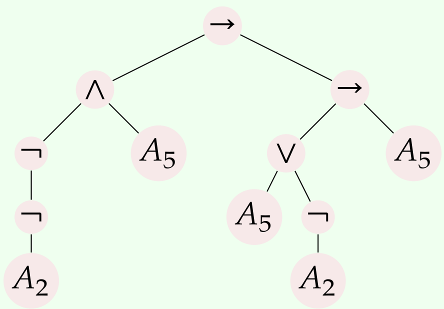
Exercise 3.2 (Construction) *10 points each.*

i. Write down a construction sequence for $((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))$.

Solution. $\langle A_1, A_3, A_4, A_7, (\neg A_1), ((\neg A_1) \vee A_4), (\neg((\neg A_1) \vee A_4)), (A_1 \to A_3), ((A_1 \to A_3) \leftrightarrow A_7), ((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))\rangle$. $\square$

ii. Write down a construction tree for $(((\neg(\neg A_2)) \wedge A_5) \to ((A_5 \vee (\neg A_2)) \to A_5))$.

Solution. $\square$



Exercise 3.3 (Truth Assignments)

i. *Let S be the set of all sentence symbols, and assume that $v : S \to \{F, T\}$ is a truth assignment. Show there is at most one extension v meeting conditions 0–5 on pp. 20–21. (Hint: Show that if v_1 and v_2 are such extensions, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α. Use the induction principle.) 20 points.*

Solution. We show this via induction on the complexity of any arbitrary wff α.

- **(Base case)** Assume α be a sentence symbol. Then $\bar{v}_1(\alpha) = v(\alpha) = \bar{v}_2(\alpha)$ since $\bar{v}_1$ and $\bar{v}_2$ are both extensions of v.

- **(Induction step)** We assume that the result holds for all wffs less complex than α (induction hypothesis). Now, we show that the result holds under all the formula building operations.

  ($\neg$) Assume $\alpha = (\neg\beta)$ for some wff β. Then

$$
\begin{aligned}
& \bar{v}_1(\alpha) = \top & & \\
\iff & \bar{v}_1(\neg\beta) = \top & & [\text{Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = \bot & & [\text{Def of } \bar{v} \text{ under } \neg] \\
\iff & \bar{v}_2(\beta) = \bot & & [\text{Induction hypothesis}] \\
\iff & \bar{v}_2(\neg\beta) = \top & & [\text{Def of } \bar{v} \text{ under } \neg] \\
\iff & \bar{v}_2(\alpha) = \top & & [\text{Def of } \alpha]
\end{aligned}
$$

  ($\wedge$) Assume $\alpha = (\beta \wedge \gamma)$ for some wffs β, γ. Then

$$
\begin{aligned}
& \bar{v}_1(\alpha) = \top & & \\
\iff & \bar{v}_1(\beta \wedge \gamma) = \top & & [\text{Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = \top \text{ and } \bar{v}_1(\gamma) = \top & & [\text{Def of } \bar{v} \text{ under } \wedge] \\
\iff & \bar{v}_2(\beta) = \top \text{ and } \bar{v}_2(\gamma) = \top & & [\text{Induction hypothesis}] \\
\iff & \bar{v}_2(\beta \wedge \gamma) = \top & & [\text{Def of } \bar{v} \text{ under } \wedge] \\
\iff & \bar{v}_2(\alpha) = \top & & [\text{Def of } \alpha]
\end{aligned}
$$

  ($\vee$) Assume $\alpha = (\beta \vee \gamma)$ for some wffs β, γ. Then

$$
\begin{aligned}
& \bar{v}_1(\alpha) = \top & & \\
\iff & \bar{v}_1(\beta \vee \gamma) = \top & & [\text{Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = \top \text{ or } \bar{v}_1(\gamma) = \top & & [\text{Def of } \bar{v} \text{ under } \vee] \\
\iff & \bar{v}_2(\beta) = \top \text{ or } \bar{v}_2(\gamma) = \top & & [\text{Induction hypothesis}] \\
\iff & \bar{v}_2(\beta \vee \gamma) = \top & & [\text{Def of } \bar{v} \text{ under } \vee] \\
\iff & \bar{v}_2(\alpha) = \top & & [\text{Def of } \alpha]
\end{aligned}
$$

$(\to)$ Assume $\alpha = (\beta \to \gamma)$ for some wffs β, γ. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = \top & \\
\iff & \bar{v}_1(\beta \to \gamma) = \top & [\text{Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = \bot \text{ or } \bar{v}_1(\gamma) = \top & [\text{Def of } \bar{v} \text{ under } \to] \\
\iff & \bar{v}_2(\beta) = \bot \text{ or } \bar{v}_2(\gamma) = \top & [\text{Induction hypothesis}] \\
\iff & \bar{v}_2(\beta \to \gamma) = \top & [\text{Def of } \bar{v} \text{ under } \to] \\
\iff & \bar{v}_2(\alpha) = \top & [\text{Def of } \alpha]
\end{aligned}$$

$(\leftrightarrow)$ Assume $\alpha = (\beta \leftrightarrow \gamma)$ for some wffs β, γ. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = \top & \\
\iff & \bar{v}_1(\beta \leftrightarrow \gamma) = \top & [\text{Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = \bar{v}_1(\gamma) & [\text{Def of } \bar{v} \text{ under } \leftrightarrow] \\
\iff & \bar{v}_2(\beta) = \bar{v}_2(\gamma) & [\text{Induction hypothesis}] \\
\iff & \bar{v}_2(\beta \leftrightarrow \gamma) = \top & [\text{Def of } \bar{v} \text{ under } \leftrightarrow] \\
\iff & \bar{v}_2(\alpha) = \top & [\text{Def of } \alpha]
\end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α, which proves the uniqueness of the extension. $\square$

ii. *Show that for a set of wffs Σ and a wff α: $\Sigma \cup \{\neg\neg\alpha\}$ is satisfiable $\iff$ $\Sigma \cup \{\alpha\}$ is satisfiable. 10 points.*

Solution. First, note that for any wff α and truth assignment v,

$$\bar{v}(\alpha) = \top \quad \iff \quad \bar{v}(\neg\alpha) = \bot \quad \iff \quad \bar{v}(\neg\neg\alpha) = \top.$$

Therefore, we have the following (v always represents a truth assignment):

$$\begin{aligned}
& \Sigma \cup \{\alpha\} \text{ is satisfiable.} \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\alpha) = \top. \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\alpha) = \bot. \\
\iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\neg\alpha) = \top. \\
\iff & \Sigma \cup \{\neg\neg\alpha\} \text{ is satisfiable.}
\end{aligned}$$

$\square$

Exercise 3.4 (Compactness) *Recall the Compactness Theorem: A set of wffs is satisfiable iff it is finitely satisfiable.*
Recall Corollary 17A: If $\Sigma \models \tau$, then $\Sigma_0 \models \tau$ for some finite $\Sigma_0 \subseteq \Sigma$.
Prove that they are equivalent, i.e., prove that the Compactness Theorem holds iff Corollary 17A holds.
(Hint: Use the fact that $\Gamma \models \sigma$ iff $\Gamma \cup \{\neg\sigma\}$ is unsatisfiable and 3.3.ii above.) 20 points.

Exercise 3.5 (Substitution) *Let $\alpha_1, \alpha_2, \ldots$ be a sequence of wffs. For each wff ϕ and $n \in \mathbb{N}$, let ϕ^* be the result of replacing the sentence symbol A_n in ϕ by the wff α_n. Suppose that v is a truth assignment for the set of all sentence symbols and that u is a truth assignment defined by $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\phi) = \bar{v}(\phi^*)$.*
(Hint: Use the induction principle.) 20 points

Solution. We show this via induction on the complexity of any arbitrary wff ϕ.

- **(Base case)** Assume $\phi = A_n$ for some $n \in \mathbb{N}$, so $\phi^* = \alpha_n$. Now $\bar{u}(\phi) = \bar{u}(A_n) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\phi^*)$, so the result holds when ϕ is a sentence symbol.

- **(Induction step)** We assume that the result holds for all wffs less complex than ϕ (induction hypothesis). Now, we show that the result holds under all the formula building operations.

  $(\neg)$ Assume $\phi = (\neg\psi)$ for some wff ψ, so $\phi^* = (\neg\psi^*)$. Then

$$
\begin{aligned}
& \bar{u}(\phi) = \top & \\
\iff & \bar{u}(\neg\psi) = \top & \text{[Def of } \phi\text{]} \\
\iff & \bar{u}(\psi) = \bot & \text{[Def of } \bar{u} \text{ under } \neg\text{]} \\
\iff & \bar{v}(\psi^*) = \bot & \text{[Induction hypothesis]} \\
\iff & \bar{v}(\neg\psi^*) = \top & \text{[Def of } \bar{v} \text{ under } \neg\text{]} \\
\iff & \bar{v}(\phi^*) = \top & \text{[Def of } \phi^*\text{]}
\end{aligned}
$$

  $(\wedge)$ Assume $\phi = (\psi \wedge \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \wedge \theta^*)$. Then

$$
\begin{aligned}
& \bar{u}(\phi) = \top & \\
\iff & \bar{u}(\psi \wedge \theta) = \top & \text{[Def of } \phi\text{]} \\
\iff & \bar{u}(\psi) = \top \text{ and } \bar{u}(\theta) = \top & \text{[Def of } \bar{u} \text{ under } \wedge\text{]} \\
\iff & \bar{v}(\psi^*) = \top \text{ and } \bar{v}(\theta^*) = \top & \text{[Induction hypothesis]} \\
\iff & \bar{v}(\psi^* \wedge \theta^*) = \top & \text{[Def of } \bar{v} \text{ under } \wedge\text{]} \\
\iff & \bar{v}(\phi^*) = \top & \text{[Def of } \phi^*\text{]}
\end{aligned}
$$

  $(\vee)$ Assume $\phi = (\psi \vee \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \vee \theta^*)$. Then

$$
\begin{aligned}
& \bar{u}(\phi) = \top \\
\iff\ & \bar{u}(\psi \vee \theta) = \top && \text{[Def of } \phi] \\
\iff\ & \bar{u}(\psi) = \top \text{ or } \bar{u}(\theta) = \top && \text{[Def of } \bar{u} \text{ under } \vee] \\
\iff\ & \bar{v}(\psi^*) = \top \text{ or } \bar{v}(\theta^*) = \top && \text{[Induction hypothesis]} \\
\iff\ & \bar{v}(\psi^* \vee \theta^*) = \top && \text{[Def of } \bar{v} \text{ under } \vee] \\
\iff\ & \bar{v}(\phi^*) = \top && \text{[Def of } \phi^*]
\end{aligned}
$$

($\to$) Assume $\phi = (\psi \to \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \to \theta^*)$. Then

$$
\begin{aligned}
& \bar{u}(\phi) = \top \\
\iff\ & \bar{u}(\psi \to \theta) = \top && \text{[Def of } \phi] \\
\iff\ & \bar{u}(\psi) = \bot \text{ or } \bar{u}(\theta) = \top && \text{[Def of } \bar{u} \text{ under } \to] \\
\iff\ & \bar{v}(\psi^*) = \bot \text{ or } \bar{v}(\theta^*) = \top && \text{[Induction hypothesis]} \\
\iff\ & \bar{v}(\psi^* \to \theta^*) = \top && \text{[Def of } \bar{v} \text{ under } \to] \\
\iff\ & \bar{v}(\phi^*) = \top && \text{[Def of } \phi^*]
\end{aligned}
$$

($\leftrightarrow$) Assume $\phi = (\psi \leftrightarrow \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \leftrightarrow \theta^*)$. Then

$$
\begin{aligned}
& \bar{u}(\phi) = \top \\
\iff\ & \bar{u}(\psi \leftrightarrow \theta) = \top && \text{[Def of } \phi] \\
\iff\ & \bar{u}(\psi) = \bar{u}(\theta) && \text{[Def of } \bar{u} \text{ under } \leftrightarrow] \\
\iff\ & \bar{v}(\psi^*) = \bar{v}(\theta^*) && \text{[Induction hypothesis]} \\
\iff\ & \bar{v}(\psi^* \leftrightarrow \theta^*) = \top && \text{[Def of } \bar{v} \text{ under } \leftrightarrow] \\
\iff\ & \bar{v}(\phi^*) = \top && \text{[Def of } \phi^*]
\end{aligned}
$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{u}(\phi) = \bar{v}(\phi)$ for every wff ϕ. □

Bibliography