Generalization of stochastic calculus and its applications in large deviations theory

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Outline

§ 1
Stochastic calculus

Quick revision

1. Properties of Brownian motion $B(t)$
   - starts at 0
   - continuous paths
   - Markov (independence of increments)
   - $B(t) - B(s) \sim N(0, t-s)$
   - infinite linear variation
   - finite quadratic variation
   - martingale
2. Naive stochastic integration w.r.t. $B(t)$: not possible

Itô integral: $f \in L^2$

Definition (Itô integral) $f \in C[a,b]$ is called a continuous function.

Properties of the associated process $X_t = \int_0^t f(t)\,\mathrm{d}B(t)$

1. continuity

2. martingale

Itô integral: $f \in \mathcal{L}^2$

Definition
Properties of the associated process $X_t = \int_0^t f(t)\,\mathrm{d}B(t)$

1. continuity

2. local martingale

Itô formula

Stochastic differential equations

Girsanov theorem

Itô integral: $f \in \mathcal{L}^2$

Itô isometry: $f \in L^2$

bla bla bla

Differential formula (Itô, 1944 TODO:ref)

bla bla bla

§ 2
Large deviations theory

Introduction

Weak convergence of measures

Laplace principle

Motivation: An example

Setup

- (X_n): i.i.d. random variables
- $X_n \sim \mu$ and $\mathbb{E}X_n = m$
- $\overline{S_n} = \frac{1}{n}\sum_{j=1}^{n} X_j$
- $B \in \mathcal{B}$ such that $m \notin \overline{B}$
- By LLN, we have $\overline{S_n} \to m$ as $n \to \infty$ (a.s.)
- So $\mathbb{P}(S_n \in B) \to 0$ as $n \to \infty$
- But at what speed?
- $\asymp$

Cramér theorem

Theorem (Cramér, 1938) Let $X_1, X_2, \ldots$be a series of i.i.d. real random variables with finite logarithmic moment generating function, e.g. $\Lambda(t) < \infty \; \forall t \in \mathbb{R}$.
Then the Legendre transform of Λ, $\Lambda^* = \sup_{t\in\mathbb{R}}(tx - \Lambda(t))$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq nx\right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Sanov theorem

Schilder theorem

Freidlin–Wentzell theorem

§ 3
Conclusion

Possible areas of interest

- ⋆ Extension to SDEs with anticipating coefficients
- ⋆ Near-Markov property
- ⋆ Girsanov theorem for generalized integration
- ⋆ Freidlin-Wintzell type result for SDEs with anticipating initial conditions

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

§ 4
Sample slides

Possible areas of interest

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- ⋆ Girsanov theorem for generalized integration
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Freidlin–Wentzell theorem

Column 1

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Column 2

Since the mid-1990s, humans have taken an unprecedented step in Earthly annals by introducing not just exotic flora or fauna from one ecosystem into another, but actually inserting exotic genes into the operating systems of individual plants and animals, where they're intended to do exactly the same thing: copy themselves, over and over.

Something

This is a citation [1].

- ☐ One
- ☐ Two
- ☐ Three
- ☐ Four

Bibliography

1 C.R. Hwang, H.H. Kuo, K. Saitô et al., "A general Itô formula for adapted and instantly independent stochastic processes", *Communications on Stochastic Analysis* 10(3), 2016.