Anticipating stochastic integrals and its large deviations

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Part 1

Anticipating integrals

1.1 Interpretation of exponential processes

In classical Itô theory, there are three equivalent interpretations of exponential processes.

i. renormalization
ii. martingale
iii. SDEs

This is not so in the two-sided stochastic integral theory.

1.2 Miscellaneous

Example 2.1 *The process X given by $X_t = B_{\frac{1}{2}(t+T)} - B_t)$ cannot be expressed as a Borel function of $B_T - B_\cdot$.*

Part 2

Large deviations theory

2.1 Friedlin-Wentzell theorem

2.2 Friedlin-Wentzell theorem for anticipating initial condition with extension of filtration

Our aim is to formulate a large deviations principle for an SDE with anticipating initial conditions. We start of with a very simple case

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\mathrm{d}W_t,$$

where $t \in [0,T]$ for some $T < \infty$, and conditions on σ shall be imposed as necessary.
We shall look at the method of enlargement of filtration by [<Itô1978>]. We denote the enlarged filtration by $\tilde{\mathscr{F}}_t = \mathscr{F}_t \vee \sigma(W_T)$. For $t \in [0,T]$, define the process $A_t = \int_0^t \frac{W_T - W_u}{T-u}\mathrm{d}u$. Then $W_t = \tilde{W}_t + A_t$, where $\tilde{W}_\cdot$ is a Wiener process w.r.t. $\tilde{\mathscr{F}}_\cdot$.
For now, we shall bound $t \in [0,T_b]$, where $T_b \in [0,T)$. What happens when $t \to T$?
Using this, we write our original SDE as

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\mathrm{d}\tilde{W}_t + \sqrt{\varepsilon}\int_0^t \sigma(X_t^\varepsilon)\frac{W_T - W_s}{T-s}\mathrm{d}s.$$

Let $\tilde{X}_t^\varepsilon = X_t^\varepsilon - W_T$ and $Y_t^\varepsilon = \sqrt{\varepsilon}(W_T - W_t)$. Then we have

$$\tilde{X}_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right)\mathrm{d}\tilde{W}_t + \int_0^t \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right)\frac{Y_s^\varepsilon}{T-s}\mathrm{d}s, \text{ and}$$

$$Y_t^\varepsilon = \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}\int_0^t \mathrm{d}\tilde{W}_t - \int_0^t \frac{Y_s^\varepsilon}{T-s}\mathrm{d}s.$$

Therefore, we have the joint process

$$Z_t^\varepsilon := \begin{pmatrix}\tilde{X}_t^\varepsilon \\ Y_t^\varepsilon\end{pmatrix} = \begin{pmatrix}0 \\ \sqrt{\varepsilon}W_T\end{pmatrix} + \sqrt{\varepsilon}\int_0^t \begin{pmatrix}\sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \\ -1\end{pmatrix}\mathrm{d}\tilde{W}_s + \int_0^t \begin{pmatrix}\sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \\ -1\end{pmatrix}\frac{Y_s^\varepsilon}{T-s}\mathrm{d}s.$$

Note that we expect $Z_t^\varepsilon \to 0$ as $\varepsilon \to 0$. In order to obtain a large deviation principle for X_t^ε, we first obtain the LDP for Z_t^ε.
Firstly, we show that there is a unique solution of Z_t^ε. For convinience, let $\tilde{\sigma}_t(x,y) = (\sigma(x), -1)$ and $\tilde{b}(x,y) = \frac{y}{T-s}(\sigma(x), -1)$.

Proposition 2.1 *$\tilde{\sigma}$ and $\tilde{b}$ satisfy the linear growth and Lipshitz conditions.*

Proof.

We want a large deviation principle of the joint process $Z_t^\varepsilon = (X_t^\varepsilon - W_T, Y_t^\varepsilon)$.
We take a sequence of stopping times (τ_n) such that $\tau_n \nearrow T_b$. Then

2.2.1 $\tilde{W}_\cdot$ is a Wiener process

We show that $(\tilde{W}_t)$ is a $(\tilde{\mathscr{F}}_t)$-martingale with quadratic variation t. Then by Lévy's Characterization of Wiener process, we obtain that $\tilde{W}_\cdot$ is a Wiener process.
First we prove two lemmas.

Lemma 2.2 *The σ-algebras $\mathscr{F}_s \vee \sigma(W_T)$ and $\mathscr{F}_s \vee \sigma(W_T - W_s)$ are the same.*

Proof. For any Borel set B, the set $\{W_T \in B\} = \{(W_T - W_t) + W_t \in B\}$.
TODO

Lemma 2.3 *For $0 \le s \le t \le T$, we have*

$$\mathbb{E}(W_t - W_s \mid W_T - W_s) = \frac{t-s}{T-s}(W_T - W_s).$$

Proof. We partition the interval $[0, T]$ into $n = kn_0$ equal parts, where $n_0 = (\min\{s, t-s, T-t\})^{-1}$ and $k \in \mathbb{N}$. Let $n_s = s\frac{n}{T}$ and $n_t = t\frac{n}{T}$. That is, the partition is

$$P = \left\{0, \frac{T}{n}, \ldots, \frac{n_s T}{n} = s, \ldots, \frac{n_t T}{n} = t, \ldots, \frac{(n-1)T}{n}, T\right\}.$$

Let $\Delta_i W = W_{\frac{(i+1)T}{n}} - W_{\frac{iT}{n}}$.

Firstly, note that the $\Delta_i W$s are independent and identically distributed from the definition of Wiener process. Now, using the linearity of conditional expectation, we have

$$\begin{aligned}
\mathbb{E}(W_t - W_s \mid W_T - W_s) &= \mathbb{E}\left(\sum_{i=n_s}^{n_t-1} \Delta_i W \mid \sum_{i=n_s}^{n_t-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \mathbb{E}\left(\sum_{i=n_s}^{n-1} \Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \Delta_i W \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} (W_T - W_s) \\
&= \frac{n_t - n_s}{n - n_s}(W_T - W_s) \quad = \quad \frac{t-s}{T-s}(W_T - W_s).
\end{aligned}$$

Proposition 2.4 *$\tilde{W}_\cdot$ is a $\tilde{\mathscr{F}}_\cdot$-martingale.*

Proof. Let $0 \le s \le t \le T$. Then

$$\tilde{W}_t - \tilde{W}_s = (W_t - W_s) - \int_s^t \frac{W_T - W_u}{T-u} du = (W_t - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{W_u - W_s}{T-u}\right) du.$$

Moreover, since $W_t - W_s$ is independent of $\mathscr{F}_s$ for every $t \ge s$, using lemmas 2.2 and 2.3, we get

$$\mathbb{E}(W_t - W_s \mid \tilde{\mathscr{F}}_s) = \mathbb{E}\left(W_t - W_s \mid \mathscr{F}_s \vee \sigma(W_T - W_s)\right) = \mathbb{E}\left(W_t - W_s \mid W_T - W_s\right) = \frac{t-s}{T-s}(W_T - W_s).$$

Therefore, using the fact that W_T and W_s are $\tilde{\mathscr{F}}_s$-measurable with conditional Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) &= \mathbb{E}\left(W_t - W_s \mid \tilde{\mathscr{F}}_s\right) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{\mathbb{E}\left(W_u - W_s \mid \tilde{\mathscr{F}}_s\right)}{T-u}\right) du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{u-s}{T-s}\frac{W_T - W_s}{T-u}\right) du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \frac{W_T - W_s}{T-s} du \\
&= \frac{t-s}{T-s}(W_T - W_s) - \frac{t-s}{T-s}(W_T - W_s) \quad = \quad 0.
\end{aligned}$$

Now, since $\tilde{W}_s$ is $\tilde{\mathscr{F}}_s$-measurable, $\mathbb{E}(\tilde{W}_t \mid \tilde{\mathscr{F}}_s) = \mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) + \tilde{W}_s = \tilde{W}_s$.

Proposition 2.5 *The quadratic variation of $\tilde{W}_t$ is t.*

Bibliography