A generalization of Itô calculus and large deviations theory

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Outline

Section 1
Introduction and motivation

Quick revision and notations

▷ Let $T \in (0, \infty)$, and denote $t \in [0, T]$.

▷ Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ be a filtered probability space.

▷ $B_\bullet$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$.

▷ Properties of $B_\bullet$
  ◦ starts at 0
  ◦ has independent increments
  ◦ $B_t - B_s \sim \mathcal{N}(0, t - s)$
  ◦ continuous paths (a.s.)
  ◦ is a.s. nowhere differentiable

  ◦ has unbounded linear variation ☹
  ◦ has bounded quadratic variation ☺
  ◦ $\mathbb{E}(B_t B_s) = s \wedge t$
  ◦ martingale

▷ Naive integration w.r.t. B_t: not possible.

▷ A stochastic process $X_\bullet$ is called adapted to $\mathcal{F}_\bullet$ if X_t is measurable w.r.t. $\mathcal{F}_t$ $\forall t$.

Martingales and Markov processes

Definition   Let $X_\bullet$ be an integrable $\mathcal{F}_\bullet$-adapted stochastic process and let $0 \le s \le t \le T$. Then $X_\bullet$ is called a martingale if $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$.

Remark   If (X_n) is a discrete-time martingale and (H_n) is an adapted process, then the process $Y_n = \sum_{j=0}^{n-1} H_j(X_{j+1} - X_j) =: (H \bullet X)_n$ is itself a martingale and called a martingale transform of (X_n).

Definition   A stochastic process $X_\bullet$ is called Markov if for any $0 \le s \le t \le T$, we have

$$\mathbb{P}(X_t \in E \mid \mathcal{F}_s) = \mathbb{P}(X_t \in E \mid X_s).$$

Wiener integral for $f \in L^2[0, T]$

▷ Definition of the integral:

1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t)\, dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.

2. $f \in L^2[0, T]$: Use step functions approximating f to extend the integral a.s.

▷ Properties of the integral:

○ Linear.

○ Gaussian distribution with mean 0 and variance $\|f\|_{L^2[0,T]}^2$ (Itô isometry).

○ Agrees with the Riemann–Stieltjes integral for continuous functions of bounded variation.

▷ Properties of the associated process $I_{\bullet} = \int_0^{\bullet} f(t)\, dB_t$:

○ continuity.

○ martingale.

▷ Problem: Cannot integrate stochastic processes.

Trying to integrate stochastic processes

▷ $\int_0^T B_t \, dB_t \overset{?}{=}$

Since B_t is continuous, let us try the Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \to 0$. Then

$$\int_0^T B_t \, dB_t = \lim \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

▷ Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, dB_s$	Intuitive?	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2}\left(B_t^2 - t\right)$	☹	0	☺	Itô
mid	$\frac{1}{2}\left(B_t^2\right)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2}\left(B_t^2 + t\right)$	☹	t	☹	

▷ Which one do we choose?

Itô integral [Itô44] for $X_\bullet \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$

▷ Definition of the integral:

1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
2. $X \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

▷ Properties of the integral:

- Linear.
- Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).
- For $X_\bullet$ continuous, $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$.

▷ Properties of the associated process $I_\bullet = \int_0^\bullet X_t \, dB_t$:

- continuity.
- martingale.

▷ Example: $\int_0^t B_u \, dB_u = \frac{1}{2}(B_t^2 - t) \quad \forall t$.

Itô integral for $X_\bullet$ such that $\int_0^T X_t^2\, dt < \infty$ a.s.

▷ Definition: Use sequences of processes in $L_{ad}^2([0, T] \times \Omega)$ approximating X in probability to extend the integral in probability.

▷ Properties of the integral:

   ○ Linear.
   ○ Mean and variance? ☹

▷ Properties of the associated process $I_\bullet = \int_0^\bullet X_t\, dB_t$:

   ○ continuity.
   ○ local martingale.

▷ Example: $\int_0^T e^{B_t^2}\, dB_t = \int_0^{B_T} e^{t^2}\, dt - \int_0^T B_t e^{B_t^2}\, dt$.

The Itô formula

▷ An Itô process is a process of the form $X_\bullet = X_0 + \int_0^\bullet m_t \, \mathrm{d}t + \int_0^\bullet \sigma_t \, \mathrm{d}B_t$. Equivalently expressed as $\mathrm{d}X_t = m_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t$.

Theorem ([**Itô44**]) Let X_t be a d-dimensional Itô process, and assume $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$. Then $f(t, X_t)$ is also a d-dimensional Itô process given by

$$\mathrm{d}f(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) \, \mathrm{d}t + \langle (\mathrm{D}f)(t, X_t), \mathrm{d}X_t \rangle + \frac{1}{2} \langle \mathrm{d}X_t, (\mathrm{D}^2 f)(t, X_t) \, \mathrm{d}X_t \rangle,$$

where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d \, \mathrm{d}t$.

▷ Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, \mathrm{d}t + \sigma \mathcal{E}_t \, \mathrm{d}B_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (\mathrm{d}B_t)^2$.

Exponential processes and the Girsanov theorem

▷ Let $h_\bullet$ be an adapted stochastic process. The associated **exponential process** is defined as

$$\mathcal{E}_\bullet^{(h)} = \exp\left(\int\limits_0^\bullet h_t \, \mathrm{d}B_t - \frac{1}{2} \int\limits_0^\bullet h_t^2 \, \mathrm{d}t \right).$$

▷ The exponential process is a martingale if and only if $\mathbb{E}\mathcal{E}_t^{(h)} = 1 \; \forall t$.

▷ The **Novikov condition**: The exponential process is a martingale if $\mathbb{E}\exp\left(\frac{1}{2} \int_0^T h_t^2 \, \mathrm{d}t \right) < \infty$.

▷ The **Girsanov theorem** [Gir60]: The process $W_\bullet = B_\bullet - \int_0^\bullet h_t \, \mathrm{d}t$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^{(h)}$.

Stochastic differential equations

▷ Let $\tilde{\xi} \in L^2(\Omega)$ be independent of $B_\bullet$, and $m, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ have 'nice' measurability. Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \tilde{\xi} + \int_0^t m(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$ if for each t, the X_t satisfies the integral equation a.s.

▷ Stochastic *differential* equation $dX_t = m(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$, $X_0 = \tilde{\xi}$ is a *formal representation*.

Theorem (**Existence and uniqueness, Markov property**)   The SDE above has a **unique** solution if m and σ are Lipschitz and satisfy the linear growth condition. The solution is a Markov process. Moreover if $\tilde{\xi} \in \mathbb{R}$ and m, σ are functions of only x, then the solution is also stationary.

● Example: For σ constant, $d\mathcal{E}_t = \sigma \mathcal{E}_t\, dB_t$, $\mathcal{E}_0 = 1$ is solved by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

▷ How do we define the double integral?

▷ Naive idea: $\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = \int_0^t \mathrm{d}B_u \int_0^t \mathrm{d}B_v = B_t^2$.
But $\mathbb{E}B_t^2 = t \neq 0$, so no martingale property. ☹

▷ Itô's idea: remove the diagonal to get

$$\int_0^t \int_0^t \mathrm{d}B_u \, \mathrm{d}B_v = 2 \int_0^t \int_0^v \mathrm{d}B_u \, \mathrm{d}B_v = 2 \int_0^t B_v \, \mathrm{d}B_v = B_t^2 - t.$$

Theorem (**[Itô51]**)   Let $f \in L^2\left([0,T]^n\right)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1, \ldots, t_n) \, \mathrm{d}B_{t_1} \cdots \mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \ldots, t_n) \, \mathrm{d}B_{t_n} \right) \mathrm{d}B_{t_{n-1}} \cdots \mathrm{d}B_{t_1}.$$

Section 2
A Generalization of Itô calculus

Motivation

▷ Iterated integrals: Consider the iterated integral $\int_0^t \int_0^t dB_u \, dB_v = \int_0^t B_t \, dB_v \overset{?}{=} B_t^2$.

▷ Note that $\mathbb{E}(B_t^2) = t \neq 0$, so no martingale property ☹.

▷ Stochastic differential equations with anticipation:

$$dX_t = X_t \, dB_t \qquad\qquad\qquad dY_t = B_T \, dB_t$$

$$X_0 = B_T \qquad\qquad\qquad\qquad Y_0 = 1$$

▷ Problem: We want to define $\int_0^T Z(\cdot) \, dB_t$, where $Z(\cdot)$ is not (necessarily) adapted.

▷ Some approaches:

  ○ Enlargement of filtration $\mathcal{G}_\cdot = \mathcal{F}_\cdot \vee \sigma(B_T)$, with Itô's decomposition of integrand [Itô78]
  $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T - s} \, ds \right) + \int_0^t \frac{B_T - B_s}{T - s} \, ds$.
  ○ White noise theory
  ○ Malliavin calculus

The new integral [AK08; AK10]: Idea

- A process $Y^{\bullet}$ and filtration $\mathcal{F}_{\bullet}$ are called **instantly independent** if Y^t and $\mathcal{F}_t$ are independent $\forall t$. Example: The process $(B_T - B_{\bullet})$ is instantly independent of the filtration generated by $B_{\bullet}$.

- Ideas

  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.

- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\bullet}$. Then the integral $\int_0^T X_t Y^t \, \mathrm{d}B_t$ is **defined** as

$$
\int_0^T X_t Y^t \, \mathrm{d}B_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,
$$

  provided that the limit exists in probability.

- Now, for any stochastic process $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.

- This is well-defined [HKS+16].

A simple example

▷ In the following, lim is the limit in L^2.

$$\int_0^t B_T \, dB_t = \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t$$

$$= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j$$

$$= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j \right) \Delta B_j$$

$$= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t$$

▷ Note that $\mathbb{E}(B_T B_t - t) = 0$.

▷ In general, $\mathbb{E} \int_0^t Z(s) \, dB_s = 0$. ☺

The near-martingale property

▷ Question: What are the analogues of the martingale property and the Markov property?

▷ Example: $\mathbb{E}(B_T B_t - t \mid \mathcal{F}_s) = B_s^2 - s \neq B_T B_s - s.$ ☹
But $\mathbb{E}(B_T B_s - s \mid \mathcal{F}_s) = B_s^2 - s.$ ☺

▷ Let $Z(t)$ be a process such that $\mathbb{E}|Z(t)| < \infty \ \forall t$, and $0 \leq s \leq t \leq T$. Then $Z(t)$ is called a near-martingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) = \mathbb{E}(Z(s) \mid \mathcal{F}_s)$.

Theorem ([**KSS12b**]) Let f and ϕ be continuous functions on $\mathbb{R}$. Under integrability conditions, the processes $X_{\bullet} = \int_0^{\bullet} f(B_t)\phi(B_T - B_t)\,\mathrm{d}B_t$ and $Y^{\bullet} = \int_{\bullet}^T f(B_t)\phi(B_T - B_t)\,\mathrm{d}B_t$ are near-martingales.

Theorem ([**HKS+17**]) Let $Z(\cdot)$ be a stochastic process bounded in L^1, and $X_{\bullet} = \mathbb{E}(Z(\cdot) \mid \mathcal{F}_{\bullet})$. Then $X_{\bullet}$ is a (sub/super)martingale if and only if $Z(\cdot)$ is a near-(sub/super)martingale.

A generalized Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_\bullet = X_0 + \int_0^\bullet m_t \, dt + \int_0^\bullet \sigma_t \, dB_t$	$dX_t = m_t \, dt + \sigma_t \, dB_t$
instantly independent	$Y^\bullet = Y^0 + \int_\bullet^T \eta^t \, dt + \int_\bullet^T \varsigma^t \, dB_t$	$dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$

Here η^t and ς^t are instantly independent such that Y^t is also instantly independent.

Theorem ([**HKS+16**]) Let $dX_t = m_t \, dt + \sigma_t \, dB_t$ be an d-dimensional Itô process, and $dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$ be a k-dimensional instantly independent process. If $f(t, x, y) \in C^{1,2,2}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, then

$$df(t, X_t, Y^t) = \frac{\partial f}{\partial t}(t, X_t, Y^t) \, dt + \left\langle (D_x f)(t, X_t, Y^t), dX_t \right\rangle + \frac{1}{2}\left\langle dX_t, (D_x^2 f)(t, X_t, Y^t) \, dX_t \right\rangle$$

$$+ \left\langle (D_y f)(t, X_t, Y^t), dY^t \right\rangle - \frac{1}{2}\left\langle dY^t, (D_y^2 f)(t, X_t, Y^t) \, dY^t \right\rangle,$$

where we use the rules $dB_t \otimes dB_t = I_d \, dt$.

Iterated integrals

Theorem ([**Itô51**]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1,...,t_n)\, dB_{t_1}... dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1,...,t_n)\, dB_{t_n} \right) dB_{t_{n-1}}... dB_{t_1}.$$

Theorem ([**AK10**]) Let $f \in L^2([0,T]^n)$. Then

$$\int_{[0,T]^n} f(t_1,...,t_n)\, dB_{t_1}... dB_{t_n} = \int_0^T \cdots \int_0^T f(t_1,...,t_n)\, dB_{t_n}... dB_{t_1}.$$

Example[HKS+16]: For the new integral, $\int_0^T \left(\int_0^T B_u\, du \right) dB_v = \int_0^T \left(\int_0^T B_u\, dB_v \right) du$.

A generalization of Itô isometry

Theorem ([**KSS12b**]) Let ϕ be an analytic function on $\mathbb{R}$. Then under integrability conditions and for each t,

$$\mathbb{E}\left[\left(\int_0^t \phi(B_T - B_s)\, \mathrm{d}B_s\right)^2\right] = \int_0^t \mathbb{E}\left[(\phi(B_T - B_s))^2\right]\mathrm{d}s$$

Theorem ([**KSS13**]) Let f and ϕ be C^1 functions on $\mathbb{R}$. Then

$$\mathbb{E}\left[\left(\int_0^T f(B_t)\phi(B_T - B_t)\, \mathrm{d}B_t\right)^2\right] = \int_0^T \mathbb{E}\left[(f(B_t)\phi(B_T - B_t))^2\right]\mathrm{d}t$$

$$+ 2\int_0^T \int_0^t \mathbb{E}\left[f(B_s)\phi'(B_T - B_s)f'(B_s)\phi(B_T - B_s)\right]\mathrm{d}s\,\mathrm{d}t.$$

A generalization of Girsanov theorem

Theorem ([**KPS13**])   Let $X_\bullet$ and $Y^\bullet$ be continuous square-integrable stochastic processes such that $X_\bullet$ is adapted and $Y^\bullet$ is instantly independent.

Then the translated stochastic process $W_\bullet = B_\bullet - \int_0^\bullet (X_t + Y^t)\, \mathrm{d}t$ is a near-martingale under the probability measure $\widetilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^{(X+Y)}$.

Section 3
Large deviations theory

What is it about?

▷ A theory to find probabilities of rare events that decay exponentially.

▷ Started by Swedish actuarials Fredrik Esscher, Harald Cramér, Filip Lundberg.

▷ Unified by Varadhan in his 1966 paper [Var66].

▷ Example: A problem faced by the insurance industry.

- Value of claims received on the nth day: X_n \$.
- Steady income from premium: x \$/day.
- Planning period: n days.
- Average expenditure: $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$ \$/day.
- *Question*: How should the company decide on the premium?
- *Idea*: Determine x such that $\mathbb{P}\left\{\overline{X}_n > x\right\} < \varepsilon$ (specified).

Insurance problem: setup

1. Let the following hold:

   - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
   - (X_n) is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moment generating function M.
   - $\mathbb{E}X_1 = m$, $\mathbb{V}X_1 = \sigma^2$, and $X_1 \sim \mu$.
   - $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$.

2. Asymptotic behavior of $\overline{X}_n$:

   - Weak law of large numbers: $\overline{X}_n \xrightarrow{\mathbb{P}} m$.

   - Central limit theorem: $\sqrt{n}(\overline{X}_n - m) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$.

3. What is the rate for LLN?

Insurance problem: large deviation bounds

1. For $x > m$ and an arbitrary $\theta > 0$, we get

$$\mathbb{P}\left\{\overline{X}_n \geq x\right\} = \mathbb{P}\left\{e^{\theta n \overline{X}_n} \geq e^{\theta n x}\right\} \leq e^{-\theta n x}\mathbb{E}\left(e^{\theta n \overline{X}_n}\right) = e^{-\theta n x}M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}.$$

2. Since θ was arbitrary, we have

$$\mathbb{P}\left\{\overline{X}_n \geq x\right\} \leq \inf_\theta e^{-n(\theta x - \log M_X(\theta))} = e^{-n\sup_\theta(\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim}\frac{1}{n}\log \mathbb{P}\left\{\overline{X}_n \in F\right\} \leq -\inf_F I \qquad \forall F \text{ closed.}$$

4. We can also obtain a large deviation lower bound using an exponential change of measure

$$\underline{\lim}\frac{1}{n}\log \mathbb{P}\left\{\overline{X}_n \in G\right\} \geq -\inf_G I \qquad \forall G \text{ open.}$$

5. We informally write $\mathbb{P}\left\{\overline{X}_n \in \mathrm{d}x\right\} \asymp e^{-nI(x)}\,\mathrm{d}x$ for $x \in \mathbb{R}$.

Definition of large deviation principle

▷ The setup: (X_n) is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space $(\mathcal{X}, d)$.

▷ A function $I : \mathcal{X} \to [0, \infty]$ is called a rate function if it has compact lower level sets.

Definition (X_n) is said to satisfy the large deviation principle on $\mathcal{X}$ with rate function I if the large deviation upper and lower bounds hold.

Example

Theorem ([**Cra38**]) Let (X_n) be a sequence of i.i.d. real random variables with finite moment generating function M. Then $(\overline{X}_n)$ follows the large deviation principle with rate function $I(x) = \sup_\theta (\theta x - \log M(\theta))$.

Applications of the Cramér theorem

Rate functions for some common distributions

Distribution	$M(\theta)$	$I(x)$
Bernoulli(p)	$1 - p + pe^\theta$	$\left(x \log x + (1-x) \log(1-x) - \left(x \log \dfrac{1-p}{p} + \log p \right) \right) \mathbb{1}_{[0,1]}(x) + \infty \mathbb{1}_{[0,1]^\complement}(x)$
Poisson(λ)	$e^{\lambda(e^\theta - 1)}$	$\left(\lambda - x + x \log \dfrac{x}{\lambda} \right) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
Exp(λ)	$\left(1 - \dfrac{\theta}{\lambda} \right)^{-1}$	$(\lambda x - 1 + x \log(\lambda x)) \, \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
$\mathcal{N}(m, \sigma^2)$	$e^{m\theta + \frac{1}{2}\sigma^2\theta^2}$	$\dfrac{(x-m)^2}{2\sigma^2}$

The Schilder theorem

▷ Aim: Estimate the probability that a scaled-down sample path of a **Brownian motion** will stray far from the mean path.

▷ Let C_x denote the set of continuous functions from $[0, T]$ to $\mathbb{R}^d$ starting at x, and let $\mathrm{CM}_x = \{\omega \in C_x : \omega \text{ is absolutely continuous and } \omega'_t \in L^2[0, T]\}$.

▷ Let $B_\bullet$ be a d-dimensional Brownian motion, so $B_\bullet \in C_0 = C_0([0, T]; \mathbb{R}^d)$

▷ Let $\frac{1}{\sqrt{n}} B_t \sim W^{(n)}$. Then $W^{(n)} = \mathcal{N}\left(0, \frac{t}{n}\right) \Longrightarrow \delta_0$ as $n \to \infty$.

Theorem ([**Sch66**]) On the Banach space $(C_0, \|\cdot\|_\infty)$, the sequence of probability measures $(W^{(n)})$ satisfies LDP with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\omega'_t|^2 \, dt\right) \mathbb{1}_{\mathrm{CM}_0}(\omega) + \infty \mathbb{1}_{\mathrm{CM}_0^\complement}(\omega)$$

The Freidlin–Wentzell theorem

▷ Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

▷ Let $X_\bullet^{(n)}$ be the solution of the d-dimensional stochastic differential equation
$$dX_t^{(n)} = m(X_t^{(n)})\, dt + \frac{1}{\sqrt{n}}\sigma(X_t^{(n)})\, dB_t,\ X_0^{(n)} = x,$$ where m and σ are sufficiently nice.

▷ Let $W_x^{(n)}$ denote the law of $X_\bullet^{(n)}$ starting at x.

▷ As $n \to \infty$, $W_x^{(n)} \Longrightarrow \delta_\xi$, where ξ solves the ODE $\dot{\xi}(t) = m(\xi(t))$, $\xi(0) = x$.

Theorem ([**FW12**]) For any fixed x, the sequence of probability measures $\left(W_x^{(n)}\right)$ satisfies LDP with the rate function $I_x : C_0 \to \overline{\mathbb{R}}$ given by

$$I_x(\omega) = \left(\frac{1}{2}\int_0^T \left\langle \omega_t' - m(\omega_t), A^{-1}(\omega_t)\left(\omega_t' - m(\omega_t)\right)\right\rangle dt \right) \mathbb{1}_{\mathrm{CM}_x}(\omega) + \infty \mathbb{1}_{\mathrm{CM}_x^\complement}(\omega),$$

where $A = \sigma\sigma^*$.

Section 4
The way forward

Possible research directions

▷ Identify the class of integrable processes under the new integral.

▷ Give a broader generalization of the Itô isometry for the new integral.

▷ Provide a broader generalization of the Girsanov theorem.

▷ Formulate an extension of the new integral to stochastic differential equations with anticipating coefficients.

▷ Develop the near-Markov property for the new integral.

▷ Prove Freidlin–Wentzell type results for SDEs with anticipating initial conditions.

▷ Study LDP results for SDEs with anticipating coefficients involving the new integral.

▷ Analyze LDP for linear SPDEs with anticipating initial conditions.

Thank you!

Appendix

Laplace principle and equivalence to LDP

Definition (**Laplace principle**) (X_n) is said to satisfy the Laplace principle on $\mathcal{X}$ with rate function I if for all bounded continuous functions h, we have

$$\lim \frac{1}{n} \log \mathbb{E} \exp(-nh(X_n)) = \inf_{\mathcal{X}}(h + I)$$

Theorem (X_n) satisfies LP on $\mathcal{X}$ with rate function I if and only if (X_n) satisfies LDP on $\mathcal{X}$ with the same rate function I.

Some important results
- Uniqueness of the rate function.
- Continuity principle.
- Superexponential approximation preserves Laplace principle.

Thank you!

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