Generalization of stochastic calculus and its applications in large deviations theory

Sudip Sinha

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Advisors

Prof Hui-Hsiung Kuo

Prof Padmanabhan Sundar

Outline

Section 1
Introduction and motivation

Quick revision and notations

▷ Let $T \in (0, \infty)$, and denote $[0, T]$ as the index set for t.

▷ Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ be a filtered probability space.

▷ $B_\bullet$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$.

▷ Properties of $B_\bullet$
  ○ starts at 0
  ○ has independent increments
  ○ $B_t - B_s \sim \mathcal{N}(0, t - s)$
  ○ continuous paths

  ○ has unbounded linear variation ☹
  ○ has bounded quadratic variation ☺
  ○ $\mathbb{E}(B_t B_s) = s \wedge t$
  ○ martingale

▷ Naive integration w.r.t. B_t: not possible.

▷ A stochastic process $X_\bullet$ is called adapted to $\mathcal{F}_\bullet$ if X_t is measurable w.r.t. $\mathcal{F}_t$ $\forall t$.

Wiener integral for $f \in L^2[0, T]$

▷ Definition

1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t)\, dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.

2. $f \in L^2[0, T]$: Use step functions approximating f to extend the integral a.s.

▷ Properties

- Linear.
- Gaussian distribution with mean 0 and variance $\|f\|_{L^2[0,T]}^2$ (Itô isometry).
- Corresponds to the Riemann–Stieltjes integral for continuous functions of bounded variation.

▷ The associated process $I_\bullet = \int_0^\bullet f(t)\, dB_t$ has the following properties.

- continuity
- martingale

▷ Problem: Cannot integrate stochastic processes.

Trying to integrate stochastic processes

▷ $\int_0^T B_t \, dB_t \overset{?}{=}$

  Since B_t is continuous, let us try Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \to 0$. Then

$$\int_0^T B_t \, dB_t = \lim \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

▷ Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, dB_s$	Intuitive?	$\mathbb{E}$	Martingale?	Theory
left	$\frac{1}{2}\left(B_t^2 - t\right)$	☹	0	☺	Itô
mid	$\frac{1}{2}\left(B_t^2\right)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2}\left(B_t^2 + t\right)$	☹	t	☹	

▷ Which one do we choose?

Itô integral for $X_\bullet \in L^2_{\text{ad}}([0, T] \times \Omega)$

▷ Definition

1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
2. $X \in L^2_{\text{ad}}([0, T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

▷ Properties

○ Linear.

○ Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry).

○ For $X_\bullet$ continuous, $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$.

▷ The associated process $I_\bullet = \int_0^\bullet X_t \, dB_t$ has the following properties.

○ continuity
○ martingale

▷ Example: $\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t) \quad \forall t.$

Itô integral for $X_\bullet$ such that $\int_0^T X_t^2 \, dt < \infty$ a.s.

▷ Definition: Use sequences of processes in $L^2_{ad}([0, T] \times \Omega)$ approximating X in probability to extend the integral in probability.

▷ Properties

  ○ Linear.
  ○ Mean and variance? ☹

▷ The associated process $I_\bullet = \int_0^\bullet X_t \, dB_t$ has the following properties.

  ○ continuity
  ○ local martingale

▷ Example: $\int_0^T e^{B_t^2} \, dB_t = \int_0^{B_1} e^{t^2} \, dt - \int_0^T B_t e^{B_t^2} \, dt.$

Itô formula

▷ An Itô process is a process of the form $X_t = X_0 + \int_0^t m_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s$. Equivalently expressed as $\mathrm{d}X_t = m_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t$.

Theorem ([**Itô44**]) Let X_t be a d-dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d-dimensional Itô process, and

$$\mathrm{d}f(X_t) = \langle (\mathrm{D}f)(X_t), \mathrm{d}X_t \rangle + \frac{1}{2} \langle \mathrm{d}X_t, (D^2 f)(X_t) \, \mathrm{d}X_t \rangle,$$

where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d \, \mathrm{d}t$.

▷ Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t \right)$, $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, \mathrm{d}t + \sigma \mathcal{E}_t \, \mathrm{d}B_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (\mathrm{d}B_t)^2$.

Exponential processes and Girsanov theorem

▷ Let $h_\bullet$ be a stochastic process. The associated exponential process is defined as

$$\mathcal{E}_t^{(h)} = \exp\left(\int_0^t h_s \, \mathrm{d}B_s - \frac{1}{2}\int_0^t h_s^2 \, \mathrm{d}s\right).$$

▷ The exponential process is a martingale if and only if $\mathbb{E}\mathcal{E}_t^{(h)} = 1 \; \forall t$.

▷ (Novikov condition) The exponential process is a martingale if $\mathbb{E} \exp\left(\frac{1}{2}\int_0^T h_t^2 \, \mathrm{d}t\right) < \infty$.

▷ (Girsanov theorem) The translated stochastic process $W_t = B_t - \int_0^t h(s) \, \mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^{(h)}$.

Stochastic differential equations

▷ Let $\xi \in L^2(\Omega)$ be independent of $B_\bullet$, and $m, \sigma : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}[0, T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable such that $m(t, \cdot, \cdot)$ and $\sigma(t, \cdot, \cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable $\forall t$. Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \xi + \int_0^t m(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}B_s$ if for each t, the X_t satisfies the integral equation a.s.

▷ The stochastic *differential* equation $\mathrm{d}X_t = m(t, X_t) \, \mathrm{d}t + \sigma(t, X_t) \, \mathrm{d}B_t$, $X_0 = \xi$ is a *symbolic representation* of the stochastic integral equation.

Theorem (**Existence and uniqueness, Markov property**) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

- ○ (Lipschitz condition) $|m(t, x) - m(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \le M |x - y|^2$ a.s.
- ○ (growth condition) $|m(t, x)|^2 + |\sigma(t, y)|^2 \le M \left(1 + |x|^2\right)$ a.s.

The solution is a Markov process.

Moreover if $\xi \in \mathbb{R}$ and m, σ are function of only x, then the solution is also stationary.

● Example: For σ constant, $\mathrm{d}\mathcal{E}_t = \sigma \mathcal{E}_t \, \mathrm{d}B_t$, $\mathcal{E}_0 = 1$ is solved by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

▷ How do we define the double integral?

▷ Naive idea: $\int_0^T \int_0^T dB_s\, dB_t = \int_0^T dB_s \int_0^T dB_t = B_T^2$.
But $\mathbb{E} B_T^2 = t \neq 0$, so no martingale property. ☹

▷ Itô's idea: remove the diagonal to get $\int_0^T \int_0^T dB_s\, dB_t = 2 \int_0^T \int_0^t dB_s\, dB_t = B_T^2 - T$. ☺

> **Theorem** (**[Itô51]**) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then
>
> $$\int_{[0,T]^n} f(t_1, ..., t_n)\, dB_{t_1} ... dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, ..., t_n)\, dB_{t_n} \right) dB_{t_{n-1}} ... dB_{t_1}.$$

▷ TODO

Section 2
Generalization of Itô calculus

Motivation

▷ Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T \mathrm{d}B_s\, \mathrm{d}B_t = \int_0^T B_T\, \mathrm{d}B_t \overset{?}{=} B_T B_t$.

▷ Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so no martingale property ☹.

▷ Stochastic differential equations with anticipation

$$\mathrm{d}X_t = X_t\, \mathrm{d}B_t \qquad\qquad\qquad \mathrm{d}Y_t = B_T\, \mathrm{d}B_t$$

$$X_0 = B_1 \qquad\qquad\qquad\qquad Y_0 = 1$$

▷ Problem: We want to define $\int_0^T X_t\, \mathrm{d}B_t$, where $X_{\bullet}$ is not adapted (anticipating).

▷ Some approaches

   ○ Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s}\, \mathrm{d}s \right) + \int_0^t \frac{B_T - B_s}{T-s}\, \mathrm{d}s$

   ○ Enlargement of filtration

   ○ White noise theory

   ○ …

The new integral [AK08; AK10]: Idea

- A process $Y^\bullet$ and filtration $\mathcal{F}_\bullet$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$. Example: The process $(B_T - B_\bullet)$ is instantly independent of the filtration generated by $B_\bullet$.

- Ideas

  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.

- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_\bullet$. Then the integral $\int_0^T X_t Y^t \, dB_t$ is defined as

$$\int_0^T X_t Y^t \, dB_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

  provided that the limit exists in probability.

- Now, for any stochastic process $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.

- This is well-defined [HKS+16].

A simple example

▷ In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and lim is the limit in L^2.

$$\int_0^t B_T \, dB_t = \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t$$

$$= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j$$

$$= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j \right) \Delta B_j$$

$$= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t$$

▷ Note that $\mathbb{E}(B_T B_t - t) = 0$.

▷ In general, $\mathbb{E} \int_0^t Z(s) \, dB_s = 0$. ☺

Generalized Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_t = X_0 + \int_0^t m_s \, ds + \int_0^t \sigma_s \, dB_s$	$dX_t = m_t \, dt + \sigma_t \, dB_t$
instantly independent	$Y^t = Y^0 + \int_t^T \eta^s \, ds + \int_t^T \varsigma^s \, dB_s$	$dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$

Theorem ([**HKS+16**]) Let $dX_t = m_t \, dt + \sigma_t \, dB_t$ be an d-dimensional Itô process, $dY^t = -\eta^t \, dt - \varsigma^t \, dB_t$ be a $\tilde{d}$-dimensional instantly independent process. If $f(x,y) \in C^2(\mathbb{R}^2)$, then

$$df(X_t, Y^t) = \langle (D_x f)(X_t, Y^t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D_x^2 f)(X_t, Y^t) \, dX_t \rangle$$

$$+ \langle (D_y f)(X_t, Y^t), dY^t \rangle - \frac{1}{2} \langle dY^t, (D_y^2 f)(X_t, Y^t) \, dY^t \rangle,$$

where we use the rules $dB_t \otimes dB_t = I_d \, dt$.

Exponential processes and generalized Girsanov theorem

▷ TODO

Iterated integrals

Theorem ([**Itô51**])   Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1,...,t_n)\, dB_{t_1}...dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1,...,t_n)\, dB_{t_n} \right) dB_{t_{n-1}}...dB_{t_1}.$$

Theorem ([**AK10**])   Let $f \in L^2([0,T]^n)$. Then

$$\int_{[0,T]^n} f(t_1,...,t_n)\, dB_{t_1}...dB_{t_n} = \int_0^T \cdots \int_0^T f(t_1,...,t_n)\, dB_{t_n}...dB_{t_1}.$$

Near-martingale property [HKS+17]

▷ Question: What are the analogues of the martingale property and the Markov property?

▷ Answer for martingales: "near-martingales".

▷ Let $Z(t)$ be a stochastic process such that $\mathbb{E}\,|Z(t)| < \infty\ \forall t$, and $0 \leq s \leq t \leq T$. Then, with respect to $\mathcal{F}_\bullet$, the process $Z(t)$ is called a

  ○ near-martingale if $\mathbb{E}(Z(t)\mid\mathcal{F}_s) = \mathbb{E}(Z(s)\mid\mathcal{F}_s)$,
  ○ near-submartingale if $\mathbb{E}(Z(t)\mid\mathcal{F}_s) \geq \mathbb{E}(Z(s)\mid\mathcal{F}_s)$, and
  ○ near-supermartingale if $\mathbb{E}(Z(t)\mid\mathcal{F}_s) \leq \mathbb{E}(Z(s)\mid\mathcal{F}_s)$.

Theorem (**[HKS+17]**) Let $Z(\cdot)$ be a stochastic process bounded in L^1, and $X_\bullet = \mathbb{E}(Z(\cdot)\mid\mathcal{F}_\bullet)$. Then $X_\bullet$ is a martingale if and only if $Z(\cdot)$ is a near-martingale.

Section 3
Large deviations theory

What is it about?

▷ Informally a theory of characterization of rare events by providing exponential bounds on their probability.

▷ Started by Swedish actuarials Fredrik Esscher, Harald Cramér, Filip Lundberg

▷ Unified by Varadhan in his 1966 paper [Var66].

▷ Example: A problem faced by the insurance industry.

  ○ Value of claims received on day n: X_n \$.
  ○ Steady income from premium: p \$/day.
  ○ Planning period: n days.
  ○ Average expenditure: $\bar{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j$ \$/day.
  ○ *Question*: How do we decide on the premium?
  ○ *Idea*: Choose p such that the probability of the average expenditure to the company going above p is exponentially low.

Insurance problem: setup

1. Let the following hold:

   - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
   - (X_n) is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moment generating function M.
   - $\mathbb{E}X_1 = m$, $\mathbb{V}X_1 = \sigma^2$, and $X_1 \sim \mu$.
   - $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$.

2. Asymptotic behavior of $\bar{X}_n$ given by

   - Weak law of large numbers: $\bar{X}_n \xrightarrow{\mathbb{P}} m$.

   - Central limit theorem: $\sqrt{n}(\bar{X}_n - m) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$.

3. But at what rate?

4. We want to "control large deviations from the mean".

Insurance problem: large deviation bounds

1. Fixing $x > m$ and forcing the exponential with a free parameter $\theta > 0$, we get

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} = \mathbb{P}\left\{e^{\theta n \bar{X}_n} \geq e^{\theta n x}\right\} \leq e^{-\theta n x}\mathbb{E}\left(e^{\theta n \bar{X}_n}\right) = e^{-\theta n x}M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}$$

2. Since θ was arbitrary, we have

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} \leq \inf_\theta e^{-n(\theta x - \log M_X(\theta))} = e^{-n\sup_\theta(\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim}\frac{1}{n}\log\mathbb{P}\left\{\bar{X}_n \in E\right\} \leq -\inf_{\overline{E}} I \qquad \forall E \in \mathcal{B}.$$

4. We can also obtain a lower bound using an exponential change of measure

$$\underline{\lim}\frac{1}{n}\log\mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -\inf_{E^\circ} I \qquad \forall E \in \mathcal{B}.$$

5. So we "formally" write $\mathbb{P}\left\{\bar{X}_n \in \mathrm{d}x\right\} \asymp e^{-nI(x)}\,\mathrm{d}x$ for $x \in \mathbb{R}$.

Definition of large deviation principle

▷ The setup: (X_n) is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space $(\mathcal{X}, d)$.

▷ A function $I : \mathcal{X} \to [0, \infty]$ is called a rate function if it has compact level sets.

▷ I is lower semicontinuous and attains its infimum on every nonempty closed set.

Definition (X_n) is said to satisfy the large deviation principle on $\mathcal{X}$ with rate function I if the following two conditions hold.

$$\text{(upper bound)} \qquad \overline{\lim}\frac{1}{n}\log \mathbb{P}\left\{\bar{X}_n \in F\right\} \leq -\inf_F I \qquad \forall F \text{ closed}$$

$$\text{(lower bound)} \qquad \underline{\lim}\frac{1}{n}\log \mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -\inf_G I \qquad \forall G \text{ open}$$

Laplace principle and equivalence to LDP

Definition (**Laplace principle**) (X_n) is said to satisfy the Laplace principle on $\mathcal{X}$ with rate function I if for all bounded continuous functions h, we have

$$\lim \frac{1}{n} \log \mathbb{E} \left(\exp(-nh(X_n))\right) = \inf_{\mathcal{X}}(h + I)$$

Theorem (X_n) satisfies the Laplace principle on $\mathcal{X}$ with rate function I if and only if (X_n) satisfies the large deviation principle on $\mathcal{X}$ with rate function I.

Some other results
- Uniqueness of the rate function.
- Continuity principle.
- Superexponential approximation preserves Laplace principle.

Cramér theorem

Theorem ([**Cra38**]) Let (X_n) be a sequence of i.i.d. real random variables with finite moment generating function M. Then (X_n) follows large deviation principle with rate function $I(x) = \sup_\theta (\theta x - \log M(\theta))$.

Rate function for some common distributions for (X_n)

Distribution	$M(\theta)$	$I(x)$
$Bern(p)$	$1 - p + pe^\theta$	$\left(x \log \dfrac{x}{1-p} + (x-1) \log \dfrac{p}{x-1} \right) \mathbb{1}_{[0,1]}(x) + \infty \mathbb{1}_{[0,1]^{\mathsf{C}}}(x)$
$Pois(\lambda)$	$e^{\lambda(e^\theta - 1)}$	$\left(\lambda - x + x \log \dfrac{x}{\lambda} \right) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
$Exp(\lambda)$	$\left(1 - \dfrac{\theta}{\lambda} \right)^{-1}$	$(\lambda x - 1 + x \log(\lambda x)) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
$\mathcal{N}(m, \sigma^2)$	$e^{m\theta + \frac{1}{2}\sigma^2 \theta^2}$	$\dfrac{(x-m)^2}{2\sigma^2}$
$\chi^2(k)$	$(1 - 2\theta)^{-\frac{k}{2}}$	$\dfrac{1}{2} \left(x - k + k \log \dfrac{k}{x} \right)$

Schilder theorem

▷ Aim: Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path.

▷ Setup

- Let $B_{\bullet}$ be a d-dimensional Brownian motion, so $B_{\bullet} \in C_0 = C_0([0, T]; \mathbb{R}^d)$

- $\forall \varepsilon > 0$, let $\sqrt{\varepsilon} B_t \sim W_\varepsilon$. Then $W_\varepsilon = \mathcal{N}(0, \varepsilon t) \overset{\mathcal{D}}{\to} \delta_0$ as $\varepsilon \to 0$.
- Let $\mathrm{CM} = \{\omega \in C_0 : \omega \text{ is absolutely continuous and } \dot{\omega}_t \in L^2[0, T]\}$.

Theorem On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 \, \mathrm{d}t \right) \mathbb{1}_{\mathrm{CM}}(\omega) + \infty \mathbb{1}_{\mathrm{CM}^{\complement}}(\omega)$$

Freidlin–Wentzell theorem

▷ Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

▷ Setup

  ○ Let $B_\bullet$ be a d-dimensional Brownian motion.
  ○ $\forall \varepsilon > 0$, let $X_\bullet^{(\varepsilon)}$ be the solution of the stochastic differential equation
    $$dX_t^{(\varepsilon)} = m(X_t^{(\varepsilon)})\,dt + \sigma(X_t^{(\varepsilon)})\sqrt{\varepsilon}\,dB_t,\ X_0^{(\varepsilon)} = x, \text{ where } m, \sigma \text{ are sufficiently nice. Let } A = \sigma\sigma^*.$$
  ○ Let $W_x^{(\varepsilon)}$ denote the law of $X_\bullet^{(\varepsilon)}$ starting at x.
  ○ As $\varepsilon \to 0$, $W_x^{(\varepsilon)} \xrightarrow{\mathcal{D}} \delta_{\tilde{\zeta}}$, where $\tilde{\zeta}$ solves the ordinary differential equation
    $$\dot{\tilde{\zeta}}(t) = m(\tilde{\zeta}(t)),\ \tilde{\zeta}(0) = x.$$

Theorem   For x fixed, the family of probability measures $\{W_x^{(\varepsilon)} : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I_x : C_0 \to \overline{\mathbb{R}}$ given by

$$I_x(\omega) = \left(\frac{1}{2}\int_0^T \left\langle \dot{\omega}_t - b(\omega_t), A^{-1}(\omega_t)(\dot{\omega}_t - b(\omega_t))\right\rangle dt\right) \mathbb{1}_{CM \cap \{\omega_0 = x\}}(\omega) + \infty \mathbb{1}_{CM^\complement \cup \{\omega_0 \neq x\}}(\omega).$$

Section 4
Summary

Open problems

▷ Near-Markov property

▷ Extension to stochastic differential equations with anticipating coefficients

▷ Freidlin-Wintzell type result for stochastic differential equations with anticipation

▷ Identification of class of integrable processes under the generalized integral

▷ Girsanov theorem for anticipating integrals

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