Algebra

Groups, rings, linear algebra

Sudip Sinha

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Part 1

Multilinear algebra

1.1 Tensor products

1.1.1 Examples

1. $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \simeq \mathbb{Z}_2$, because firstly $0 \otimes b = 0(1 \otimes b) = 0 = 0(a \otimes 1) = a \otimes 0$, and
   - $1 \otimes 1 = 1 \otimes 1$,
   - $1 \otimes 2 = 2 \otimes 1 = 0 \otimes 1 = 0$,
   - $1 \otimes 3 = 3 \otimes 1 = 1 \otimes 1 + 2 \otimes 1 = 1 \otimes 1$,

   so $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 = \{0 \otimes 0, 1 \otimes 1\} \simeq \mathbb{Z}_2$.

2. $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \simeq \mathbb{Z}_2$, because
   - $a \otimes 2 = a \otimes (4+2) = a \otimes 6 = 6 \otimes a = 0$,
   - $2 \otimes b = (2+6) \otimes b = 8 \otimes b = b \otimes 8 = 0$,
   - $4 \otimes b = b \otimes 4 = 0$,
   - $(2n+1) \otimes 1 = 2n \otimes 1 + 1 \otimes 1 = 0 + 1 \otimes 1 = 1 \otimes 1$,
   - $(2n+1) \otimes 3 = 2n \otimes 3 + 1 \otimes (4+3) = 0 + 1 \otimes 7 = 7 \otimes 1 = 1 \otimes 1$.

3. In general, $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q \simeq \mathbb{Z}_{\gcd(p,q)}$.

4. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$, because $\forall \left(\frac{p}{q}, b\right) \in \mathbb{Q} \times \mathbb{Z}_n, \frac{p}{q} \otimes b = \frac{np}{nq} \otimes b = \frac{p}{nq} \otimes (nb) = 0$.
   Remember that we can only move integers around, not fractions, because the base field is $\mathbb{Z}$.

5. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$, because $\forall \left(\frac{p}{q}, \frac{a}{b}\right) \in \mathbb{Q} \times (\mathbb{Q}/\mathbb{Z}), \frac{p}{q} \otimes \frac{a}{b} = \frac{bp}{bq} \otimes \frac{a}{b} = \frac{p}{bq} \otimes \frac{ab}{b} = \frac{p}{bq} \otimes a = \frac{p}{bq} \otimes 0 = 0$.

6. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \simeq \mathbb{C}$, because $(a + b\imath) \otimes r = (ar)(1 \otimes 1) + (br)(\imath \otimes 1)$, so $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} = \operatorname{span}\{1 \otimes 1, \imath \otimes 1\} \simeq \mathbb{C}$. Moreover $2 = \dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}) = \dim_{\mathbb{R}} \mathbb{C} \cdot \dim_{\mathbb{R}} \mathbb{R} = 2 \cdot 1$.

7. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{span}\{1 \otimes 1, \imath \otimes 1, 1 \otimes \imath, \imath \otimes \imath\} \simeq \mathbb{C} \oplus \mathbb{C}$, because $(a + b\imath) \otimes (c + d\imath) = (ac)(1 \otimes 1) + (ad)(1 \otimes \imath) + (bc)(\imath \otimes 1) + (bd)(\imath \otimes \imath)$. Moreover $4 = \dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{R}} \mathbb{C} \cdot \dim_{\mathbb{R}} \mathbb{C} = 2 \cdot 2$.

8. $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \operatorname{span}\{1 \otimes 1\} \simeq \mathbb{C}$, because $(a + b\imath) \otimes (c + d\imath) = (a + b\imath)(c + d\imath)(1 \otimes 1)$. Moreover $1 = \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{C} \cdot \dim_{\mathbb{C}} \mathbb{C} = 1 \cdot 1$.

9. $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R}) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \oplus \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \simeq \mathbb{C} \oplus \mathbb{C}$.

10. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \simeq \mathbb{C}^n$.

11. $\mathbb{C} \otimes_{\mathbb{R}} M(n, \mathbb{R}) \simeq M(n, \mathbb{C})$. This is how $GL(n, \mathbb{R}) \hookrightarrow GL(n, \mathbb{C})$.

12. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^n \simeq \bigoplus(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \simeq \bigoplus(\mathbb{C} \oplus \mathbb{C}) \simeq \mathbb{C}^{2n}$.
    This is the beginning of Clifford algebras.

Bibliography