

# Functional analysis

Mostly operator theory for now

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# PART 1

## OPERATOR THEORY

## 1.1 ELEMENTARY IDEAS

### 1.1.1 Definitions

**Compact**  $T \in \mathcal{K} \iff \lambda \rightarrow 0$

**Hilbert-Schmidt**  $T \in \mathcal{B}^2 \iff \lambda \in \ell^2$

**Trace-class**  $T \in \mathcal{B}^1 \iff \lambda \in \ell^1$

### 1.1.2 Inclusions: $\mathcal{D} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{K} \subset \mathcal{B}^\infty$

$\mathcal{K} \subseteq \mathcal{B}^\infty$  ((<BMC2009>), Proposition 4.6) If  $T$  is unbounded, we can find a sequence of unit vectors  $(e_n)$  such that  $\|Te_n\| \nearrow \infty$ . So  $Te_n$  cannot have a convergent subsequence, for if  $Te_n \rightarrow x$ , then  $\|Te_n\| \rightarrow \|x\|$ .

$\mathcal{K} \neq \mathcal{B}^\infty$  The identity operator  $I \in \mathcal{B}^\infty$  is not compact because for the bounded sequence of unit vectors  $(e_n)$ ,  $Ie_n = e_n$  does not converge as  $\|e_n - e_m\| = \sqrt{2} \forall n \neq m$ .

$\mathcal{B}^2 \subseteq \mathcal{K}$  TODO

$\mathcal{B}^2 \neq \mathcal{K}$   $T : \ell^2 \rightarrow \ell^2, Te_n = \frac{1}{\sqrt{n}}e_n; T \in \mathcal{K} \setminus \mathcal{B}^2$ .

$\mathcal{B}^1 \subseteq \mathcal{B}^2$  TODO

$\mathcal{B}^1 \neq \mathcal{B}^2$   $T : \ell^2 \rightarrow \ell^2, Te_n = \frac{1}{n}e_n; T \in \mathcal{B}^2 \setminus \mathcal{B}^1$ .

### 1.1.3 For $T \in \mathcal{B}^\infty$ , $\|T\|_\infty = \sup \{|\langle Tx, y \rangle| : \|x\| = 1, \|y\| = 1\}$

( $\leq$ ) Since  $\|Tx\| = \frac{\|Tx\|^2}{\|Tx\|} = \frac{\langle Tx, Tx \rangle}{\|Tx\|} = \left\langle Tx, \frac{Tx}{\|Tx\|} \right\rangle$ , we have  $\|T\|_\infty = \sup \{\|Tx\| : \|x\| = 1\} \leq \sup \{|\langle Tx, y \rangle| : \|x\| = 1, \|y\| = 1\}$ .

( $\geq$ ) On the other hand,  $\langle Tx, y \rangle \leq \|Tx\| \|y\| \leq \|T\|_\infty \|x\| \|y\|$ , so  $\sup \{|\langle Tx, y \rangle| : \|x\| = 1, \|y\| = 1\} \leq \|T\|_\infty$ .

### 1.1.4 $\|P\|_\infty \leq 1$

Since  $\|Px\|^2 = \langle Px, Px \rangle = \langle P^*Px, x \rangle = \langle PPx, x \rangle = \langle Px, x \rangle \leq \|Px\| \|x\|$ , we have  $\|P\|_\infty \leq 1$ .

### 1.1.5 Projection operator is compact iff its image is finite dimensional

( $\implies$ ) Let  $P : H \rightarrow H$  be a projection operator, so that  $P^2 = P$ , or  $P(P - I) = 0$ .

( $\Leftarrow$ ) Since the image is finite dimensional, fix an orthonormal basis  $e_1, \dots, e_n$  of  $\text{im } T$ .

## 1.2 OPTIMIZATION

### 1.2.1 Duality in optimization is the same as duality in functional analysis

For an various intuitions of duality in optimization, see [MSx223235](#).

Let  $X$  and  $Y$  be Banach spaces, and  $X^*$  and  $Y^*$  be their (algebraic?) duals. Consider the two problems, with  $\phi_0, y_0$  fixed. Here  $(\cdot, \cdot)$  denotes the canonical duality pairing.

$$\begin{array}{ll}
 \text{(Primal)} & \begin{array}{ll} \max & (\phi_0, x) \\ \text{s.t.} & Tx \leq y_0 \\ & x \geq 0 \end{array}
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{(Dual)} & \begin{array}{ll} \min & (\psi, y_0) \\ \text{s.t.} & T^* \psi \geq \phi_0 \\ & \psi \geq 0 \end{array}
 \end{array}$$

See the following diagram for more details.

$$\begin{array}{ccccc}
 & x & \xrightarrow{T} & Tx & \\
 & \downarrow & & \downarrow & \\
 x \in & X & \xrightarrow{T} & y_0 & \ni Tx, y_0 \\
 & \downarrow & & \downarrow & \\
 \phi_0, T^* \psi \in & X^* & \xleftarrow{T^*} & y_0^* & \ni \psi \\
 & \downarrow & & \downarrow & \\
 & T^* \psi & \xleftarrow{T^*} & \psi &
 \end{array}$$

## BIBLIOGRAPHY