Functional analysis

Mostly operator theory for now

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Part 1

Operator theory

1.1 Elementary ideas

1.1.1 Definitions

Compact $T \in \mathcal{K} \iff \lambda \to 0$

Hilbert-Schmidt $T \in \mathcal{B}^2 \iff \lambda \in \ell^2$

Trace-class $T \in \mathcal{B}^1 \iff \lambda \in \ell^1$

1.1.2 Inclusions: $\mathcal{D} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{K} \subset \mathcal{B}^\infty$

$\boldsymbol{\mathcal{K} \subseteq \mathcal{B}^\infty}$ ((<BMC2009>), Proposition 4.6) If T is unbounded, we can find a sequence of unit vectors (e_n) such that $\|Te_n\| \nearrow \infty$. So Te_n cannot have a convergent subsequence, for if $Te_n \to x$, then $\|Te_n\| \to \|x\|$.

$\boldsymbol{\mathcal{K} \neq \mathcal{B}^\infty}$ The identity operator $I \in \mathcal{B}^\infty$ is not compact because for the bounded sequence of unit vectors (e_n), $Ie_n = e_n$ does not converge as $\|e_n - e_m\| = \sqrt{2}\ \forall n \neq m$.

$\boldsymbol{\mathcal{B}^2 \subseteq \mathcal{K}}$ TODO

$\boldsymbol{\mathcal{B}^2 \neq \mathcal{K}}$ $T : \ell^2 \to \ell^2, Te_n = \frac{1}{\sqrt{n}}e_n; T \in \mathcal{K} \setminus \mathcal{B}^2$.

$\boldsymbol{\mathcal{B}^1 \subseteq \mathcal{B}^2}$ TODO

$\boldsymbol{\mathcal{B}^1 \neq \mathcal{B}^2}$ $T : \ell^2 \to \ell^2, Te_n = \frac{1}{n}e_n; T \in \mathcal{B}^2 \setminus \mathcal{B}^1$.

1.1.3 For $T \in \mathcal{B}^\infty$, $\|T\|_\infty = \sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\}$

(≤) Since $\|Tx\| = \frac{\|Tx\|^2}{\|Tx\|} = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \left\langle Tx, \frac{Tx}{\|Tx\|}\right\rangle$, we have $\|T\|_\infty = \sup\{\|Tx\| : \|x\| = 1\} \leq \sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\}$.

(≥) On the other hand, $\langle Tx, y\rangle \leq \|Tx\|\,\|y\| \leq \|T\|_\infty \|x\|\,\|y\|$, so $\sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\} \leq \|T\|_\infty$.

1.1.4 $\|P\|_\infty \leq 1$

Since $\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PPx, x\rangle = \langle Px, x\rangle \leq \|Px\|\,\|x\|$, we have $\|P\|_\infty \leq 1$.

1.1.5 Projection operator is compact iff its image is finite dimensional

(⟹) Let $P : H \to H$ be a projection operator, so that $P^2 = P$, or $P(P - I) = 0$.

($\Longleftarrow$) Since the image is finite dimensional, fix an orthonormal basis $e_1, \ldots, e_n$ of $\operatorname{im} T$.

Bibliography