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# 1.1 TENSOR PRODUCTS

## 1.1.1 Examples

1.  $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \simeq \mathbb{Z}_2$ , because firstly  $0 \otimes b = 0(1 \otimes b) = 0 = 0(a \otimes 1) = a \otimes 0$ , and
  - $1 \otimes 1 = 1 \otimes 1$ ,
  - $1 \otimes 2 = 2 \otimes 1 = 0 \otimes 1 = 0$ ,
  - $1 \otimes 3 = 3 \otimes 1 = 1 \otimes 1 + 2 \otimes 1 = 1 \otimes 1$ ,
 so  $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 = \{0 \otimes 0, 1 \otimes 1\} \simeq \mathbb{Z}_2$ .
2.  $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \simeq \mathbb{Z}_2$ , because
  - $a \otimes 2 = a \otimes (4 + 2) = a \otimes 6 = 6 \otimes a = 0$ ,
  - $2 \otimes b = (2 + 6) \otimes b = 8 \otimes b = b \otimes 8 = 0$ ,
  - $4 \otimes b = b \otimes 4 = 0$ ,
  - $(2n + 1) \otimes 1 = 2n \otimes 1 + 1 \otimes 1 = 0 + 1 \otimes 1 = 1 \otimes 1$ ,
  - $(2n + 1) \otimes 3 = 2n \otimes 3 + 1 \otimes (4 + 3) = 0 + 1 \otimes 7 = 7 \otimes 1 = 1 \otimes 1$ .
3. In general,  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q \simeq \mathbb{Z}_{\gcd(p,q)}$ .
4.  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$ , because  $\forall \left(\frac{p}{q}, b\right) \in \mathbb{Q} \times \mathbb{Z}_n, \frac{p}{q} \otimes b = \frac{np}{nq} \otimes b = \frac{p}{nq} \otimes (nb) = 0$ .  
Remember that we can only move integers around, not fractions, because the base field is  $\mathbb{Z}$ .
5.  $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$ , because  $\forall \left(\frac{p}{q}, \frac{a}{b}\right) \in \mathbb{Q} \times (\mathbb{Q}/\mathbb{Z}), \frac{p}{q} \otimes \frac{a}{b} = \frac{bp}{bq} \otimes \frac{a}{b} = \frac{p}{bq} \otimes \frac{ab}{b} = \frac{p}{bq} \otimes a = \frac{p}{bq} \otimes 0 = 0$ .
6.  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \simeq \mathbb{C}$ , because  $(a + bi) \otimes r = (ar)(1 \otimes 1) + (br)(i \otimes 1)$ , so  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} = \text{span}\{1 \otimes 1, i \otimes 1\} \simeq \mathbb{C}$ . Moreover  $2 = \dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}) = \dim_{\mathbb{R}} \mathbb{C} \cdot \dim_{\mathbb{R}} \mathbb{R} = 2 \cdot 1$ .
7.  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \text{span}\{1 \otimes 1, i \otimes 1, 1 \otimes i, i \otimes i\} \simeq \mathbb{C} \oplus \mathbb{C}$ , because  $(a + bi) \otimes (c + di) = (ac)(1 \otimes 1) + (ad)(1 \otimes i) + (bc)(i \otimes 1) + (bd)(i \otimes i)$ . Moreover  $4 = \dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{R}} \mathbb{C} \cdot \dim_{\mathbb{R}} \mathbb{C} = 2 \cdot 2$ .
8.  $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \text{span}\{1 \otimes 1\} \simeq \mathbb{C}$ , because  $(a + bi) \otimes (c + di) = (a + bi)(c + di)(1 \otimes 1)$ . Moreover  $1 = \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{C} \cdot \dim_{\mathbb{C}} \mathbb{C} = 1 \cdot 1$ .
9.  $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R}) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \oplus \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \simeq \mathbb{C} \oplus \mathbb{C}$ .
10.  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \simeq \mathbb{C}^n$ .
11.  $\mathbb{C} \otimes_{\mathbb{R}} M(n, \mathbb{R}) \simeq M(n, \mathbb{C})$ . This is how  $GL(n, \mathbb{R}) \hookrightarrow GL(n, \mathbb{C})$ .
12.  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^n \simeq \bigoplus (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \simeq \bigoplus (\mathbb{C} \oplus \mathbb{C}) \simeq \mathbb{C}^{2n}$ .  
This is the beginning of Clifford algebras.

## BIBLIOGRAPHY