Logic

Notes and Exercises

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Contents

1 Set theory

For any sets A, B and C, prove the following.
Note: Let A and B are sets. In order to prove $A = B$, it is enough to show $A \subseteq B$ and $A \supseteq B$. In each of the following problems, we show each inclusion separately. Moreover, to show $A \subseteq B$, it suffices to show that for x arbitrary, $x \in A \implies x \in B$.

Exercise 1.1 (Notes, 1.8) $A \cap B \subseteq A$.

Solution. Let $x \in A \cap B$ be arbitrary. This means $x \in A$ and $x \in B$. Therefore $x \in A$. Since every element in $A \cap B$ is also an element of A, we have $A \cap B \subseteq A$. □

Exercise 1.2 (Notes, 1.10) *we have* $A \cap \emptyset = \emptyset$.

Solution. ($\subseteq$) Let $x \in A \cap \emptyset$ be arbitrary. This means $x \in A$ and $x \in \emptyset$. But there does not exist $x \in \emptyset$. Therefore, the statement is vacuously true.
($\supseteq$) Now, let $x \in \emptyset$ be arbitrary. Again, since there does not exist $x \in \emptyset$, the statement vacuously true. □

Exercise 1.3 (Notes, 1.13) *If* $A \subseteq B$, *then* $A \cup B = B$.

Solution. ($\subseteq$) Let $x \in A \cup B$ be arbitrary. This means $x \in A$ or $x \in B$. If $x \in A$, then by the condition $A \subseteq B$, we obtain $x \in B$. Therefore, in either case, $x \in B$.
($\supseteq$) Let $x \in B$ be arbitrary. Therefore, $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

Exercise 1.4 *If* $A \subseteq B$, *then* $A \cap B = A$.

Solution.

($\subseteq$) Let $x \in A \cap B$ be arbitrary. This mean $x \in A$ and $x \in B$. So $x \in A$.

($\supseteq$) Let $x \in A$ be arbitrary. Then by the hypothesis $x \in B$ since $A \subseteq B$. Therefore, $x \in A$ and $x \in B$, and thus $x \in A \cap B$.

□

Exercise 1.5 *If* $A \cap B = \emptyset$, *then* $A \setminus B = A$.

Solution.

($\subseteq$) Let $x \in A \setminus B$ be arbitrary. Then $x \in A$ and $x \notin B$, so $x \in A$.

($\supseteq$) Let $x \in A$ be arbitrary. Now, either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A \cap B$ since $x \in A$ by hypothesis. But this is an impossibility since $A \cap B = \emptyset$. Therefore, it must be that $x \notin B$. So $x \in A \setminus B$.

□

Exercise 1.6 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

($\subseteq$) Let $x \in A \cap (B \cup C)$ be arbitrary. Then $x \in A$ and $x \in B \cup C$. Note that $x \in B \cup C$ means $x \in B$ or $x \in C$. Now, either $x \in B$ or $x \notin B$, so have two cases.

- $(x \in B)$ In this case, $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.

- $(x \in C)$ We get the exact same result by interchanging the roles of B and C in the previous case.

($\supseteq$) Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. This means $x \in A \cap B$ or $x \in A \cap C$. So we have two cases:

- $(x \in A \cap B)$ In this case, $x \in A$ and $x \in B$. Now, so $x \in B$ implies $x \in B$ or $x \in C$, that is, $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.

- $(x \in A \cap C)$ Again, we get the exact same result by interchanging the roles of B and C in the previous case.

$\square$

2 Zeroth-order logic

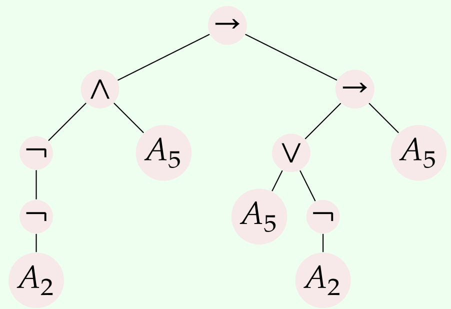
Exercise 2.1 (Construction)

i. Write down a construction sequence for $((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))$.

Solution. $\langle A_1, A_3, A_4, A_7, (\neg A_1), ((\neg A_1) \vee A_4), (\neg((\neg A_1) \vee A_4)), (A_1 \to A_3), ((A_1 \to A_3) \leftrightarrow A_7), ((\neg((\neg A_1) \vee A_4)) \wedge ((A_1 \to A_3) \leftrightarrow A_7))\rangle$. □

ii. Write down a construction tree for $(((\neg(\neg A_2)) \wedge A_5) \to ((A_5 \vee (\neg A_2)) \to A_5))$.

Solution. □



Exercise 2.2 (Enderton, 1.2.1) *Show that neither of the following two formulas tautologically implies the other:*

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Solution. We have to show that $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

($\alpha \not\models \beta$) For this, it suffices to produce a truth assignment v such that $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$.

Consider v such that $v(A) = v(B) = F$ and $v(C) = T$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below. (Here the truth assignments by $\bar{v}$ is denoted under each symbol.)

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$T \quad F \; T \quad F \; F \; T$$

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$
$$F \quad F \; F \quad F \quad F \quad F \; FT$$

($\beta \not\models \alpha$) Again, it suffices to produce v such that $\bar{v}(\beta) = T$ and $\bar{v}(\alpha) = F$.

Consider v such that $v(A) = v(B) = v(C) = F$. Under $\bar{v}$, we get exactly what is required as is shown in the computations below.

$$\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$
$$T = \quad T \quad TF \quad T \quad TF \quad T \quad TF$$

$$\alpha = (A \leftrightarrow (B \leftrightarrow C))$$
$$F = \quad F \; F \quad F \; T \; F$$

□

Exercise 2.3 (Enderton, 1.2.4) *Show that $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \to \beta)$.*

Solution. We show each direction separately.
($\Longrightarrow$) We suppose $\Sigma \cup \{\alpha\} \models \beta$. Let v be an arbitrary truth assignment that satisfies Σ. We have to show that v satisfies $(\alpha \to \beta)$. We have two cases.
i. $\bar{v}(\alpha) = T$: In this case, from the supposition, we get $\bar{v}(\beta) = T$. So $\bar{v}(\alpha \to \beta) = T$.
ii. $\bar{v}(\alpha) = F$: In this case, $\bar{v}(\alpha \to \beta) = T$ since the antecedent is F.
Since v was arbitrary, we have $\Sigma \models (\alpha \to \beta)$.
($\Longleftarrow$) We suppose $\Sigma \models (\alpha \to \beta)$. Let v be an arbitrary truth assignment that satisfies $\Sigma \cup \{\alpha\}$. We have to show that v satisfies β. Since v satisfies $\Sigma \cup \{\alpha\}$, it satisfies Σ. Therefore, by our supposition, v satisfies $(\alpha \to \beta)$. Now, since v satisfies α, it can only be that v satisfies β, since the only other way the material implication can be satisfied is when v does not satisfies α. This proves our claim. □

Exercise 2.4 (Enderton, 1.2.5) *Prove or refute each of the following assertions:*

a. If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.

Solution. **(T)** There are two cases: $\Sigma \models \alpha$ and $\Sigma \models \beta$. Without loss of generality, we can assume that $\Sigma \models \alpha$, as the argument for other case is exactly the same. This means any arbitrary truth assignment v satisfying Σ also satisfies α. This implies $\bar{v}(\alpha \vee \beta) = T$ by the definition of extension of $\bar{v}$ for $\vee$. □

b. If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.

Solution. **(F)** We give a counterexample. Let α be a sentence symbol and $\Sigma = \emptyset$. Then it is always true that $\models (\alpha \vee (\neg\alpha))$. But it does not follow that $\models \alpha$ or $\models (\neg\alpha)$.
For an explicit example, consider two truth assignments v_1 and v_2, such that $v_1(\alpha) = T$ and $v_2(\alpha) = F$. In this case, $\models \alpha$ is not true since v_2 does not satisfy α, and $\models (\neg\alpha)$ is not true since v_1 does not satisfy $(\neg\alpha)$. □

Exercise 2.5 (Enderton, 1.2.6) *See Roland's solution for this problem.*

a. Show that if v_1 and v_2 are truth assignments which agree on all the sentence symbols in the wff α, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$.

Solution. Let G be the set of sentence symbols used in α, and let $B = \{\phi \text{ wff} : \bar{v}_1(\phi) = \bar{v}_2(\phi)\}$. All we need to show is that $\alpha \in B$.
Firstly, $G \subseteq B$ since v_1 and v_2 agree on the sentence symbols used in α.
Secondly, let $\phi, \psi \in B$ (arbitrary), so v_1 and v_2 agree on ϕ and ψ. Let $\boxdot \in \{\wedge, \vee, \to, \leftrightarrow\}$. Since conditions 1–5 on page 20–21 are the same for $\bar{v}_1$ and $\bar{v}_2$, we have $\bar{v}_1(\neg\phi) = \bar{v}_2(\neg\phi)$ and $\bar{v}_1(\phi \boxdot \psi) = \bar{v}_2(\phi \boxdot \psi)$. Hence

$(\neg\phi), (\phi \boxdot \psi) \in B$, that is, B is closed with respect to the formula building operations.
Therefore, by the induction principle, B is the set of *all* wffs generated by the formula building operations. So $\alpha \in B$, and we are done. □

b. *Let S be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for S which satisfies every member of Σ also satisfies τ.*

Solution. In this part, we use v to denote truth assignments and "v on a set" means v is defined on that set. Let G be the set of sentence symbols used in Σ and τ. Clearly, $G \subseteq S$.
We show each direction separately.
($\Longrightarrow$) From the definition of tautological implication,

$$\begin{aligned} &\Sigma \models \tau \\ \iff &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \to (v \text{ satisfies } \tau)) \\ \implies &(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \to (v \text{ satisfies } \tau)) \text{ [Part (a)]} \end{aligned}$$

($\Longleftarrow$) Since Σ and τ does not depend on any element of $S \setminus G$, restricting the definition of v from S to G will not change anything on Σ and τ. Therefore,

$$\begin{aligned} &(\forall v \text{ on } S)((v \text{ satisfies } \Sigma) \to (v \text{ satisfies } \tau)) \\ \implies &(\forall v \text{ on } G)((v \text{ satisfies } \Sigma) \to (v \text{ satisfies } \tau)) \\ \iff &\Sigma \models \tau \end{aligned}$$

□

Exercise 2.6 (Enderton, 1.2.8(a) (Substitution)) *Let $\alpha_1, \alpha_2, \ldots$ be a sequence of wffs. For each wff ϕ and $n \in \mathbb{N}$, let ϕ^* be the result of replacing the sentence symbol A_n in ϕ by the wff α_n. Suppose that v is a truth assignment for the set of all sentence symbols and that u is a truth assignment defined by $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\phi) = \bar{v}(\phi^*)$.*
(Hint: Use the induction principle.)

Solution. We show this via induction on the complexity of any arbitrary wff ϕ.

- **(Base case)** Assume $\phi = A_n$ for some $n \in \mathbb{N}$, so $\phi^* = \alpha_n$. Now $\bar{u}(\phi) = \bar{u}(A_n) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\phi^*)$, so the result holds when ϕ is a sentence symbol.

- **(Induction step)** We assume that the result holds for all wffs less complex than ϕ (induction hypothesis). Now, we show that the result holds under all the formula building operations.

  ($\neg$) Assume $\phi = (\neg\psi)$ for some wff ψ, so $\phi^* = (\neg\psi^*)$. Then

$$\begin{aligned}
&\bar{u}(\phi) = T && \\
\iff\ &\bar{u}(\neg\psi) = T && \text{[Def of } \phi] \\
\iff\ &\bar{u}(\psi) = F && \text{[Def of } \bar{u} \text{ under } \neg] \\
\iff\ &\bar{v}(\psi^*) = F && \text{[Induction hypothesis]} \\
\iff\ &\bar{v}(\neg\psi^*) = T && \text{[Def of } \bar{v} \text{ under } \neg] \\
\iff\ &\bar{v}(\phi^*) = T && \text{[Def of } \phi^*]
\end{aligned}$$

($\wedge$) Assume $\phi = (\psi \wedge \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \wedge \theta^*)$. Then

$$\begin{aligned}
&\bar{u}(\phi) = T && \\
\iff\ &\bar{u}(\psi \wedge \theta) = T && \text{[Def of } \phi] \\
\iff\ &\bar{u}(\psi) = T \text{ and } \bar{u}(\theta) = T && \text{[Def of } \bar{u} \text{ under } \wedge] \\
\iff\ &\bar{v}(\psi^*) = T \text{ and } \bar{v}(\theta^*) = T && \text{[Induction hypothesis]} \\
\iff\ &\bar{v}(\psi^* \wedge \theta^*) = T && \text{[Def of } \bar{v} \text{ under } \wedge] \\
\iff\ &\bar{v}(\phi^*) = T && \text{[Def of } \phi^*]
\end{aligned}$$

($\vee$) Assume $\phi = (\psi \vee \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \vee \theta^*)$. Then

$$\begin{aligned}
&\bar{u}(\phi) = T && \\
\iff\ &\bar{u}(\psi \vee \theta) = T && \text{[Def of } \phi] \\
\iff\ &\bar{u}(\psi) = T \text{ or } \bar{u}(\theta) = T && \text{[Def of } \bar{u} \text{ under } \vee] \\
\iff\ &\bar{v}(\psi^*) = T \text{ or } \bar{v}(\theta^*) = T && \text{[Induction hypothesis]} \\
\iff\ &\bar{v}(\psi^* \vee \theta^*) = T && \text{[Def of } \bar{v} \text{ under } \vee] \\
\iff\ &\bar{v}(\phi^*) = T && \text{[Def of } \phi^*]
\end{aligned}$$

($\rightarrow$) Assume $\phi = (\psi \rightarrow \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \rightarrow \theta^*)$. Then

$$\begin{aligned}
&\bar{u}(\phi) = T && \\
\iff\ &\bar{u}(\psi \rightarrow \theta) = T && \text{[Def of } \phi] \\
\iff\ &\bar{u}(\psi) = F \text{ or } \bar{u}(\theta) = T && \text{[Def of } \bar{u} \text{ under } \rightarrow] \\
\iff\ &\bar{v}(\psi^*) = F \text{ or } \bar{v}(\theta^*) = T && \text{[Induction hypothesis]} \\
\iff\ &\bar{v}(\psi^* \rightarrow \theta^*) = T && \text{[Def of } \bar{v} \text{ under } \rightarrow] \\
\iff\ &\bar{v}(\phi^*) = T && \text{[Def of } \phi^*]
\end{aligned}$$

($\leftrightarrow$) Assume $\phi = (\psi \leftrightarrow \theta)$ for some wffs ψ, θ, so $\phi^* = (\psi^* \leftrightarrow \theta^*)$. Then

$$\begin{aligned}
& \bar{u}(\phi) = T & \\
\iff & \bar{u}(\psi \leftrightarrow \theta) = T & \text{[Def of } \phi] \\
\iff & \bar{u}(\psi) = \bar{u}(\theta) & \text{[Def of } \bar{u} \text{ under } \leftrightarrow] \\
\iff & \bar{v}(\psi^*) = \bar{v}(\theta^*) & \text{[Induction hypothesis]} \\
\iff & \bar{v}(\psi^* \leftrightarrow \theta^*) = T & \text{[Def of } \bar{v} \text{ under } \leftrightarrow] \\
\iff & \bar{v}(\phi^*) = T & \text{[Def of } \phi^*]
\end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{u}(\phi) = \bar{v}(\phi)$ for every wff ϕ. □

Exercise 2.7 (Enderton, 1.2.14)

i. Let S be the set of all sentence symbols, and assume that $v : S \to \{F, T\}$ is a truth assignment. Show there is at most one extension v meeting conditions 0–5 on pp. 20–21. (Hint: Show that if v_1 and v_2 are such extensions, then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α. Use the induction principle.)

Solution. We show this via induction on the complexity of any arbitrary wff α.

- **(Base case)** Assume α is a sentence symbol. Then $\bar{v}_1(\alpha) = v(\alpha) = \bar{v}_2(\alpha)$ since $\bar{v}_1$ and $\bar{v}_2$ are both extensions of v.

- **(Induction step)** We assume that the result holds for all wffs less complex than α (induction hypothesis). Now, we show that the result holds under all the formula building operations.

  $(\neg)$ Assume $\alpha = (\neg\beta)$ for some wff β. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = T & \\
\iff & \bar{v}_1(\neg\beta) = T & \text{[Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = F & \text{[Def of } \bar{v} \text{ under } \neg] \\
\iff & \bar{v}_2(\beta) = F & \text{[Induction hypothesis]} \\
\iff & \bar{v}_2(\neg\beta) = T & \text{[Def of } \bar{v} \text{ under } \neg] \\
\iff & \bar{v}_2(\alpha) = T & \text{[Def of } \alpha]
\end{aligned}$$

  $(\wedge)$ Assume $\alpha = (\beta \wedge \gamma)$ for some wffs β, γ. Then

$$\begin{aligned}
& \bar{v}_1(\alpha) = T & \\
\iff & \bar{v}_1(\beta \wedge \gamma) = T & \text{[Def of } \alpha] \\
\iff & \bar{v}_1(\beta) = T \text{ and } \bar{v}_1(\gamma) = T & \text{[Def of } \bar{v} \text{ under } \wedge] \\
\iff & \bar{v}_2(\beta) = T \text{ and } \bar{v}_2(\gamma) = T & \text{[Induction hypothesis]} \\
\iff & \bar{v}_2(\beta \wedge \gamma) = T & \text{[Def of } \bar{v} \text{ under } \wedge] \\
\iff & \bar{v}_2(\alpha) = T & \text{[Def of } \alpha]
\end{aligned}$$

(∨) Assume $\alpha = (\beta \vee \gamma)$ for some wffs β, γ. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T \\ \iff & \bar{v}_1(\beta \vee \gamma) = T && \text{[Def of } \alpha\text{]} \\ \iff & \bar{v}_1(\beta) = T \text{ or } \bar{v}_1(\gamma) = T && \text{[Def of } \bar{v} \text{ under } \vee\text{]} \\ \iff & \bar{v}_2(\beta) = T \text{ or } \bar{v}_2(\gamma) = T && \text{[Induction hypothesis]} \\ \iff & \bar{v}_2(\beta \vee \gamma) = T && \text{[Def of } \bar{v} \text{ under } \vee\text{]} \\ \iff & \bar{v}_2(\alpha) = T && \text{[Def of } \alpha\text{]} \end{aligned}$$

(→) Assume $\alpha = (\beta \to \gamma)$ for some wffs β, γ. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T \\ \iff & \bar{v}_1(\beta \to \gamma) = T && \text{[Def of } \alpha\text{]} \\ \iff & \bar{v}_1(\beta) = F \text{ or } \bar{v}_1(\gamma) = T && \text{[Def of } \bar{v} \text{ under } \to\text{]} \\ \iff & \bar{v}_2(\beta) = F \text{ or } \bar{v}_2(\gamma) = T && \text{[Induction hypothesis]} \\ \iff & \bar{v}_2(\beta \to \gamma) = T && \text{[Def of } \bar{v} \text{ under } \to\text{]} \\ \iff & \bar{v}_2(\alpha) = T && \text{[Def of } \alpha\text{]} \end{aligned}$$

(↔) Assume $\alpha = (\beta \leftrightarrow \gamma)$ for some wffs β, γ. Then

$$\begin{aligned} & \bar{v}_1(\alpha) = T \\ \iff & \bar{v}_1(\beta \leftrightarrow \gamma) = T && \text{[Def of } \alpha\text{]} \\ \iff & \bar{v}_1(\beta) = \bar{v}_1(\gamma) && \text{[Def of } \bar{v} \text{ under } \leftrightarrow\text{]} \\ \iff & \bar{v}_2(\beta) = \bar{v}_2(\gamma) && \text{[Induction hypothesis]} \\ \iff & \bar{v}_2(\beta \leftrightarrow \gamma) = T && \text{[Def of } \bar{v} \text{ under } \leftrightarrow\text{]} \\ \iff & \bar{v}_2(\alpha) = T && \text{[Def of } \alpha\text{]} \end{aligned}$$

Therefore, the induction step holds under all the formula building operations. By the method of induction, $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ for every wff α, which proves the uniqueness of the extension. □

ii. *Show that for a set of wffs Σ and a wff α: $\Sigma \cup \{\neg\neg\alpha\}$ is satisfiable $\iff$ $\Sigma \cup \{\alpha\}$ is satisfiable.*

Solution. First, note that for any wff α and truth assignment v,

$$\bar{v}(\alpha) = T \iff \bar{v}(\neg\alpha) = F \iff \bar{v}(\neg\neg\alpha) = T.$$

Therefore, we have the following (v always represents a truth assignment):

$$\begin{aligned} & \Sigma \cup \{\alpha\} \text{ is satisfiable.} \\ \iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\alpha) = T. \\ \iff & \exists v \text{ such that } v \text{ satisfies } \Sigma \text{ and } \bar{v}(\neg\alpha) = F. \end{aligned}$$

$\iff$ $\exists v$ such that v satisfies Σ and $\bar{v}(\neg\neg\alpha) = T$.
$\iff$ $\Sigma \cup \{\neg\neg\alpha\}$ is satisfiable.

□

Exercise 2.8 (Enderton, 1.7.1) *Assume that every finite subset of Σ is satisfiable. Show that the same is true of at least one of the sets $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.*
(This is part of the proof of the compactness theorem.)
Suggestion*: If not, then $\Sigma_1 \cup \{\alpha\}$ and $\Sigma_2 \cup \{\neg\alpha\}$ are unsatisfiable for some finite $\Sigma_1 \subseteq \Sigma$ and $\Sigma_2 \subseteq \Sigma$. Look at $\Sigma_1 \cup \Sigma_2$.*

Solution. Since $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$ is finite, it must be satisfiable with some truth assignment, say v. But then v satisfies Σ_1 since $\Sigma_1 \subseteq \Sigma_1 \cup \Sigma_2$. Therefore the only way to make $\Sigma_1 \cup \{\alpha\}$ unsatisfiable would be to have $\bar{v}(\alpha) = F$. By arguing similarly about Σ_2, we get $\bar{v}(\neg\alpha) = F$, which gives a contradiction. □

Exercise 2.9 (Enderton, 1.7.2) *Let Δ be a set of wffs such that (i) every finite subset of Δ is satisfiable, and (ii) for every wff α, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$. Define the truth assignment v:*

$$v(A) = \begin{cases} T & \text{if } A \in \Delta \\ F & \text{if } A \notin \Delta \end{cases}$$

for each sentence symbol A. Show that for every wff ϕ, $v(\phi) = T$ iff $\phi \in \Delta$.
(This is part of the proof of the compactness theorem.)
Suggestion*: Use induction on ϕ.*

Solution. This has to be copied from the notes. TODO! □

Exercise 2.10 (Enderton, 1.7.3) *Recall the Compactness Theorem:* A set of wffs is satisfiable iff it is finitely satisfiable.
Recall Corollary 17A: If $\Sigma \models \tau$, then $\Sigma_0 \models \tau$ for some finite $\Sigma_0 \subseteq \Sigma$.
Prove that they are equivalent, i.e., prove that the Compactness Theorem holds iff Corollary 17A holds.
(Hint: Use the fact that $\Gamma \models \sigma$ iff $\Gamma \cup \{\neg\sigma\}$ is unsatisfiable and 3.3.ii above.)

Solution. The proof of Corollary 17A in the book shows that the Compactness Theorem implies Corollary 17A. Therefore, we are left to show that Corollary 17A implies the Compactness Theorem.
For this, we assume Corollary 17A and prove Compactness Theorem. Note that if a set of wffs is satisfiable with a truth assignment, then it is finitely satified with the same truth assignment. Therefore, we only have to show that finite satisfiability implies satisfiability.
Suppose not. That is, assume that Σ is a set of wffs such that Σ is finitely satisfiable but Σ is unsatisfiable. Fix a wff τ. Since Σ is unsatisfiable, it is vacously true that $\Sigma \models \tau$ and $\Sigma \models \neg\tau$ (as in page 23 of Enderton). Since $\Sigma \models \tau$, using Corollary 17A, there is a finite subset $\Sigma_1 \subseteq \Sigma$ such that $\Sigma_1 \models \tau$. Similarly, there exists $\Sigma_2 \subseteq \Sigma$ finite such that $\Sigma_2 \models \neg\tau$. Now, since $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$ is finite, it is satisfiable by a truth assignment,

say v. Clearly, since $\Sigma_1, \Sigma_2 \subseteq \Sigma$, the assignment v satisfies both Σ_1 and Σ_2. Then v satisfies both τ and $\neg\tau$, which is an impossibility. This contradiction shows that if Σ is finitely satisfiable then Σ is satisfiable. This concludes the proof. $\square$

3 First-order logic

Note: $\nexists$ abbreviates $\neg\exists$, and $\notin$ abbreviates $\neg\in$. We shall also use the convention that grouping for conditionals is from the right. That is, $(p \to q \to r) = ((p \to (q \to r)))$.

Exercise 3.1 (Enderton, 2.1.1) *Assume that we have a language with the following parameters: $\forall$, intended to mean "for all things"; N, intended to mean "is a number"; I, intended to mean "is interesting"; $<$, intended to mean "is less than"; and 0, a constant symbol intended to denote zero. Translate into this language the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.*

a. *Zero is less than any number.*

Solution. $\forall x\,(Nx \to < 0x)$ □

b. *If any number is interesting, then zero is interesting.*

Solution. The word *any number* can be interpreted as *every number* or *some number*. Using the exportation tautology, the corresponding translations are

(every)	$\forall x\,(Nx \to Ix \to I0)$
(some)	$\exists x(Nx \to Ix) \to I0$

□

c. *No number is less than zero.*

Solution.

$$\begin{aligned}
& \nexists x(Nx \wedge < x0) \\
\iff & \neg\exists x(Nx \wedge < x0) \\
\iff & \neg\neg\forall x\,(\neg(Nx \wedge < x0)) \\
\iff & \neg\neg\forall x\,(\neg\neg(Nx \to \neg < x0)) \\
\iff & \forall x\,(Nx \to \neg < x0))
\end{aligned}$$

□

d. *Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.*

Solution. $\forall x\,(Nx \to \neg Ix \to \forall y\,(Ny \to < yx \to Iy) \to Ix)$ □

e. *There is no number such that all numbers are less than it.*

Solution.

$$
\begin{aligned}
& \nexists x(Nx \wedge \forall y\,(Ny \rightarrow < xy)) \\
\iff & \neg\exists x(\neg(Nx \rightarrow \neg\forall y\,(Ny \rightarrow < xy))) \\
\iff & \neg\neg\forall x\,\neg(\neg(Nx \rightarrow \neg\forall y\,(Ny \rightarrow < xy))) \\
\iff & \forall x(Nx \rightarrow \neg\forall y\,(Ny \rightarrow < xy))
\end{aligned}
$$

□

f. *There is no number such that no number is less than it.*

Solution.

$$
\begin{aligned}
& \nexists x(Nx \wedge \nexists y(Ny \rightarrow (y < x))) \\
\iff & \neg\exists x(Nx \wedge \neg\exists y(Ny \rightarrow < xy)) \\
\iff & \neg\neg\forall x\,\neg(\neg(Nx \rightarrow \neg\neg\exists y(Ny \rightarrow < xy))) \\
\iff & \forall x(Nx \rightarrow \exists y(Ny \rightarrow < xy)) \\
\iff & \forall x(Nx \rightarrow \neg\forall y\,\neg(Ny \rightarrow < xy))
\end{aligned}
$$

□

Exercise 3.2 (Enderton, 2.1.3) *Translate the English sentence into the first-order language specified by $\forall$, for all sets; $\in$, is a member of; a, a; b, b.*
"Neither a nor b is a member of every set."

Solution.

Neither a nor b is a member of every set.

$\iff$ a is not a member of every set and b is not a member of every set.

$\iff$ There is a set that a is not a member of and there is a set that b is not a member of.

$\iff (\exists x(a \notin x)) \wedge (\exists y(b \notin y))$

$\iff (\exists x(\neg \in ax)) \wedge (\exists y(\neg \in bx))$

$\iff (\neg\forall x\,(\neg\neg \in ax)) \wedge (\neg\forall y\,(\neg\neg \in by))$

$\iff (\neg\forall x\, \in ax) \wedge (\neg\forall y\, \in by)$

□

Exercise 3.3 (Enderton, page 87) *Prove that $\models_{\mathfrak{A}} \alpha \vee \beta\ [s]$ iff $\models_{\mathfrak{A}} \alpha\ [s]$ or $\models_{\mathfrak{A}} \beta\ [s]$.*

Solution.

$$
\begin{aligned}
& \models_{\mathfrak{A}} (\alpha \vee \beta)\ [s] \\
\iff & \models_{\mathfrak{A}} (\neg\alpha \rightarrow \beta)\ [s] && [\text{Expansion of } \vee] \\
\iff & \not\models_{\mathfrak{A}} \neg\alpha\ [s] \text{ or } \models_{\mathfrak{A}} \beta\ [s] && [\text{Definition of } s \text{ for } \rightarrow] \\
\iff & \models_{\mathfrak{A}} \alpha\ [s] \text{ or } \models_{\mathfrak{A}} \beta\ [s] && [\text{Definition of } s \text{ for } \neg]
\end{aligned}
$$

$\square$

Exercise 3.4 (Enderton, 2.2.1) *Show that*

a. $\Gamma \cup \{\alpha\} \vDash \phi$ *iff* $\Gamma \vDash (\alpha \to \phi)$.

Solution.

($\Longrightarrow$) $\Gamma \cup \{\alpha\} \vDash \phi$ means that any structure $\mathfrak{A}$ and satisfaction function $s : V \to |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every member of Γ and α with s, the structure $\mathfrak{A}$ also satisfies ϕ with s. Now, let $\mathfrak{A}$ and s be such a combination. Now, either $\nvDash_{\mathfrak{A}} \alpha\ [s]$ or $\vDash_{\mathfrak{A}} \alpha\ [s]$. If $\nvDash_{\mathfrak{A}} \alpha\ [s]$, then $\vDash_{\mathfrak{A}} \alpha \to \phi\ [s]$ by definition. If $\vDash_{\mathfrak{A}} \alpha\ [s]$, then $\Gamma \cup \{\alpha\} \vDash \phi$ and $\vDash_{\mathfrak{A}} \alpha\ [s]$ implies $\vDash_{\mathfrak{A}} (\alpha \to \phi)\ [s]$. Therefore, $\Gamma \vDash (\alpha \to \phi)$.

($\Longleftarrow$) Since we have $\Gamma \vDash (\alpha \to \phi)$, any structure $\mathfrak{A}$ and satisfaction function $s : V \to |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every member of Γ and α with s, since we have $\vDash_{\mathfrak{A}} \alpha\ [s]$ and $\Gamma \vDash (\alpha \to \phi)$, it must be that $\vDash_{\mathfrak{A}} \phi\ [s]$.

$\square$

b. $\phi \mathrel{=\!\mid\!\vDash} \psi$ *iff* $\vDash (\phi \leftrightarrow \psi)$.

Solution.

$$
\begin{aligned}
& \phi \mathrel{=\!\mid\!\vDash} \psi \\
\Longleftrightarrow\ & \vDash (\phi \to \psi) \text{ and } \vDash (\psi \to \phi) && [\textit{Using}\,(a)] \\
\Longleftrightarrow\ & \text{for every } (\mathfrak{A}, s), \vDash_{\mathfrak{A}} (\phi \to \psi)\ [s] \text{ and } \vDash_{\mathfrak{A}} (\psi \to \phi)\ [s] \\
\Longleftrightarrow\ & \text{for every } (\mathfrak{A}, s), \vDash_{\mathfrak{A}} (\phi \to \psi) \wedge (\psi \to \phi)\ [s] && [\text{Definition of } \wedge] \\
\Longleftrightarrow\ & \text{for every } (\mathfrak{A}, s), \vDash_{\mathfrak{A}} (\phi \leftrightarrow \psi)\ [s] && [\text{Definition of } \leftrightarrow] \\
\Longleftrightarrow\ & \vDash (\phi \leftrightarrow \psi)
\end{aligned}
$$

$\square$

Exercise 3.5 (Enderton, 2.2.3) *Show that* $\{\forall x\, \alpha, \forall x\, (\alpha \to \beta)\} \vDash \forall x\, \beta$.

Solution. Let $\mathfrak{A}$ be an arbitrary structure and $s : V \to |\mathfrak{A}|$ be an arbitrary satisfaction function such that $\mathfrak{A}$ satisfies $\forall x\, \alpha$ and $\forall x\, (\alpha \to \beta)$ with s.
Now, by the definition of the satisfaction function,

$$
\begin{aligned}
\vDash_{\mathfrak{A}} \forall x\, \alpha\ [s] &\Longleftrightarrow \text{for every } a \in |\mathfrak{A}|, \vDash_{\mathfrak{A}} \alpha\ [s(x \mid a)], \qquad \text{and} && (3.1) \\
\vDash_{\mathfrak{A}} \forall x\, (\alpha \to \beta)\ [s] &\Longleftrightarrow \text{for every } a \in |\mathfrak{A}|, \vDash_{\mathfrak{A}} \alpha \to \beta\ [s(x \mid a)] \\
&\Longleftrightarrow \text{for every } a \in |\mathfrak{A}|, \nvDash_{\mathfrak{A}} \alpha\ [s(x \mid a)] \text{ or } \vDash_{\mathfrak{A}} \beta\ [s(x \mid a)] && (3.2)
\end{aligned}
$$

Fix an arbitrary $a \in |\mathfrak{A}|$. If $\nvDash_{\mathfrak{A}} \alpha\ [s(x \mid a)]$ in (3.2), then combining it with (3.1) gives a contradiction. Therefore, we must have $\vDash_{\mathfrak{A}} \beta\ [s(x \mid a)]$. This means for every $a \in |\mathfrak{A}|$, we have $\vDash_{\mathfrak{A}} \beta\ [s(x \mid a)]$, which says $\vDash_{\mathfrak{A}} \forall x\, \beta\ [s]$. $\square$

Exercise 3.6 (Enderton, 2.2.4) *Show that if x does not occur free in α, then $\alpha \models \forall x\, \alpha$.*

Solution. Let $\mathfrak{A}$ be an arbitrary structure and $s : V \to |\mathfrak{A}|$ be an arbitrary satisfaction function such that $\mathfrak{A}$ satisfies α with s. Now, fix an arbitrary $a \in |\mathfrak{A}|$. Then the satisfaction functions s and $s(x \mid a)$ agree on all variables except possibly at x. But x does not occur free in α, so s and $s(x \mid a)$ agree on all variables that occur free in α. Now by Theorem 22A, $\models_{\mathfrak{A}} \alpha\ [s(x \mid a)]$. Since a was arbitrary, this holds for every $a \in |\mathfrak{A}|$. Therefore by the definition of satisfaction function, $\models_{\mathfrak{A}} \forall x\, \alpha\ [s]$. Therefore $\alpha \models \forall x\, \alpha$. □

Exercise 3.7 (Enderton, 2.2.6) *Show that a formula θ is valid iff $\forall x\, \theta$ is valid.*

Solution.

$$\begin{aligned}
& \theta \text{ is valid} \\
\iff & \text{for every } \mathfrak{A} \text{ and } s : V \to |\mathfrak{A}|,\ \text{we have } \models_{\mathfrak{A}} \theta\ [s] \\
\iff & \text{for every } \mathfrak{A} \text{ and } s : V \to |\mathfrak{A}| \text{ and } a \in |\mathfrak{A}|,\ \text{we have } \models_{\mathfrak{A}} \theta\ [s(x \mid a)] \\
\iff & \text{for every } \mathfrak{A} \text{ and } s : V \to |\mathfrak{A}|,\ \text{we have } \models_{\mathfrak{A}} \forall x\, \theta\ [s] \\
\iff & \forall x\, \theta \text{ is valid}
\end{aligned}$$

The second $\iff$ above works because
($\Longrightarrow$) $s(x \mid a) : V \to |\mathfrak{A}|$ is also a satisfaction function, and
($\Longleftarrow$) $s = s(x \mid s(x))$.

□

Exercise 3.8 (Enderton, 2.2.8) *Assume that Σ is a set of sentences such that for any sentence τ, either $\Sigma \models \tau$ or $\Sigma \models \neg\tau$. Assume that $\mathfrak{A}$ is a model of Σ. Show that for any sentence τ, we have $\models_{\mathfrak{A}} \tau$ iff $\Sigma \models \tau$.*

Solution.

($\Longleftarrow$) Since $\Sigma \models \tau$, and $\mathfrak{A}$ is a model of Σ, it follows from definition of logical implication that $\models_{\mathfrak{A}} \tau$.

($\Longrightarrow$) We prove this side by proving the contrapositive. If $\Sigma \not\models \tau$, then $\Sigma \models \neg\tau$ by the given condition. Since $\mathfrak{A}$ is a model of Σ, $\models_{\mathfrak{A}} \neg\tau$, and therefore $\not\models_{\mathfrak{A}} \tau$.

□

Exercise 3.9 (Enderton, 2.2.13) *Prove part (a) of the homomorphism theorem.*

Solution. We prove this by induction of the complexity of a term t.

i. (base case: t is a variable) In this case,

$$
\begin{aligned}
(h \circ \bar{s})(t) &= h(\bar{s}(t)) \\
&= h(s(t)) && [\text{Definition of } \bar{s}] \\
&= (h \circ s)(t) \\
&= (\overline{h \circ s})(t). && [\text{Definition of } \overline{h \circ s}]
\end{aligned}
$$

ii. (base case: t is a constant) In this case,

$$
\begin{aligned}
(h \circ \bar{s})(t) &= h(\bar{s}(t)) \\
&= h(t^{\mathfrak{A}}) && [\text{Definition of } \bar{s}] \\
&= t^{\mathfrak{B}} && [\text{Definition of homomorphism}] \\
&= (\overline{h \circ s})(t). && [\text{Definition of } \overline{h \circ s}]
\end{aligned}
$$

iii. (induction step: $t = ft_1 \cdots t_n$) In this case, we use the induction hypothesis that the result holds for all terms less complex than t. Now,

$$
\begin{aligned}
(h \circ \bar{s})(t) &= h(\bar{s}(ft_1 \cdots t_n)) \\
&= h(f^{\mathfrak{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))) && [\text{Definition of } \bar{s}] \\
&= h(f^{\mathfrak{B}}(h(\bar{s}(t_1)), \ldots, h(\bar{s}(t_n)))) && [\text{Definition of homomorphism}] \\
&= h(f^{\mathfrak{B}}(\overline{h \circ s})(t_1), \ldots, (\overline{h \circ s})(t_n))) && [\text{Induction hypothesis}] \\
&= (\overline{h \circ s})(ft_1 \cdots t_n) && [\text{Definition of } \overline{h \circ s}] \\
&= (\overline{h \circ s})(t). && [\text{Definition of } \overline{h \circ s}]
\end{aligned}
$$

Therefore, $\overline{h \circ s} \equiv h \circ \bar{s}$ on T. □

Exercise 3.10 (Enderton, 2.2.18) *A universal* ($\forall_1$) *formula is one of the form* $\forall x_1 \cdots \forall x_n\, \theta$, *where* θ *is quantifier-free. An existential* ($\exists_1$) *formula is of the dual form* $\exists x_1 \cdots \exists x_n \theta$. *Let* $\mathfrak{A}$ *be a substructure of* $\mathfrak{B}$, *and let* $s : V \to |\mathfrak{A}|$.

a. Show that if $\models_{\mathfrak{A}} \psi\ [s]$ *and* ψ *is existential, then* $\models_{\mathfrak{B}} \psi\ [s]$. *And if* $\models_{\mathfrak{B}} \phi\ [s]$ *and* ϕ *is universal, then* $\models_{\mathfrak{A}} \phi\ [s]$.

Solution. Firstly, note that since $\mathfrak{A}$ be a substructure of $\mathfrak{B}$, the identity map $i : |\mathfrak{A}| \to |\mathfrak{B}|$ is a isomorphism. Therefore, parts (a), (b), and (c) of the Homomorphism Theorem are true.

i. **If $\models_{\mathfrak{B}} \phi\ [s]$ and ϕ is universal, then $\models_{\mathfrak{A}} \phi\ [s]$.**
Since ϕ is universal, $\phi = \forall x_1 \cdots \forall x_n\, \theta$ for some quantifier-free wff θ. Therefore, $\models_{\mathfrak{B}} \forall x_1 \cdots \forall x_n\, \theta\ [s]$ means that for every $b_1, \ldots, b_n \in |\mathfrak{B}|$, we have $\models_{\mathfrak{B}} \theta\ [s(\langle x_1, \ldots, x_n \rangle \mid \langle b_1, \ldots, b_n \rangle)]$. Therefore, in particular, for every $a_1, \ldots, a_n \in |\mathfrak{A}| \subseteq |\mathfrak{B}|$, we have $\models_{\mathfrak{B}} \theta\ [s(\langle x_1, \ldots, x_n \rangle \mid \langle a_1, \ldots, a_n \rangle)]$. By the Homomorphism Theorem, for every $a_1, \ldots, a_n \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} \theta\ [s(\langle x_1, \ldots, x_n \rangle \mid \langle a_1, \ldots, a_n \rangle)]$, that is, $\models_{\mathfrak{A}} \forall x_1 \cdots \forall x_n\, \theta\ [s]$.

ii. **If $\models_{\mathfrak{A}} \psi\ [s]$ and ψ is existential, then $\models_{\mathfrak{B}} \psi\ [s]$.**
Since ψ is existential, $\psi = \exists x_1 \cdots \exists x_n \theta = \neg\forall x_1 \neg\neg\forall x_2 \neg \cdots \neg\forall x_2 \neg\theta$, which is logically equivalent to saying $\psi = \neg\forall x_1 \forall x_2 \cdots \forall x_n \neg\theta$. Therefore, $\models_{\mathfrak{A}} \psi\ [s]$ is the same as $\models_{\mathfrak{A}} \neg\forall x_1 \forall x_2 \cdots \forall x_n \neg\theta\ [s]$, which is in turn the same as $\not\models_{\mathfrak{A}} \forall x_1 \forall x_2 \cdots \forall x_n \neg\theta\ [s]$. Now, using the above result, we get that this implies $\not\models_{\mathfrak{B}} \forall x_1 \forall x_2 \cdots \forall x_n \neg\theta\ [s]$. Reversing the above steps, we get $\models_{\mathfrak{B}} \psi\ [s]$.

□

b. Conclude that the sentence $\exists x\, Px$ is not logically equivalent to any universal sentence, nor $\forall x\, Px$ to any existential sentence.

Solution. □

Exercise 3.11 (Enderton, 2.2.20(a)) *Assume the language has equality and a two-place predicate symbol P. Consider the two structures $(\mathbb{N}; <)$ and $(\mathbb{R}; <)$ for the language. Find a sentence true in one structure and false in the other.*

Solution.

i. True in $(\mathbb{N}; <)$ but false in $(\mathbb{R}; <)$: There is an smallest object, i.e.,

$$\exists x\, \forall y\, ((y \neq x) \rightarrow (x < y)).$$

ii. True in $(\mathbb{R}; <)$ but false in $(\mathbb{N}; <)$: There is an object between any two objects, i.e.,

$$\forall x\, \forall y\, \exists z\, ((z < y \wedge z > x) \vee (z < x \wedge z > y)).$$

□

Exercise 3.12 (Enderton, 2.4.2) *To which axiom groups, if any, do each of the following formulas belong?*

a. $[(\forall x\, Px \rightarrow \forall y\, Py) \rightarrow Pz] \rightarrow [\forall x\, Px \rightarrow (\forall y\, Py \rightarrow Pz)]$

Solution. Group 1.
This is of the form $((p \rightarrow q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$, which is a tautology. If not, there would be a truth assignment v of p, q, r such that $\bar{v}(((p \rightarrow q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))) = F$. Now, this is only possible iff $\bar{v}((p \rightarrow q) \rightarrow r) = T$ and $\bar{v}(p \rightarrow (q \rightarrow r)) = F$. Looking at the consequent, $\bar{v}(p \rightarrow (q \rightarrow r)) = F$ iff $v(p) = T$ and $\bar{v}(q \rightarrow r) = F$. Furthermore, $\bar{v}(q \rightarrow r) = F$ iff $v(q) = T$ and $v(r) = F$. Now, using these values, $\bar{v}((p \rightarrow q) \rightarrow r) = F$, which gives a contradiction. □

b. $\forall y\, [\forall x\, (Px \rightarrow Px) \rightarrow (Pc \rightarrow Pc)]$

Solution. Generalization of group 2. □

c. $\forall x\, \exists y\, Pxy \rightarrow \exists y\, Pyy$

Solution. None (x is not substitutable by y in $\exists y\, Pxy$).
In fact, this is false. A counterexample is given by $\mathfrak{A} = \langle \mathbb{N}, < \rangle$ as every number is smaller than some number but no number is smaller than itself. □

Exercise 3.13 (Enderton, 2.4.6(a)) *Show that if* $\vdash \alpha \to \beta$, *then* $\vdash \forall x\, \alpha \to \forall x\, \beta$.

Solution. If $\vdash \alpha \to \beta$, using the Generalization Theorem, we have $\vdash \forall x\, (\alpha \to \beta)$. Now using axiom group 3, we have $\vdash \forall x\, (\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$. Using these and modus ponens, we get $\vdash \forall x\, \alpha \to \forall x\, \beta$. □

Exercise 3.14 (Enderton, 2.4.3)

a. *Let* $\mathfrak{A}$ *be a structure and let* $s : V \to |\mathfrak{A}|$. *Define a truth assignment* v *on the set of prime formulas by*

$$v(\alpha) = T \qquad \textit{iff} \qquad \models_{\mathfrak{A}} \alpha\ [s]\,.$$

Show that for any formula (prime or not),

$$\bar{v}(\alpha) = T \qquad \textit{iff} \qquad \models_{\mathfrak{A}} \alpha\ [s]\,.$$

Solution. We show this by induction on the complexity of α, with the induction hypothesis that the result holds for all wffs less complex than α.

i. **Base case: α is prime.** In this case, $\bar{v}(\alpha) = v(\alpha) = T$ iff $\models_{\mathfrak{A}} \alpha\ [s]$ by the given condition.

ii. **Induction step: α is $\neg\beta$ for some β.**

$$\begin{aligned}
\bar{v}(\alpha) = T &\iff \bar{v}(\neg\beta) = T \\
&\iff \bar{v}(\beta) = F && [\text{Definition of } \bar{v}] \\
&\iff \not\models_{\mathfrak{A}} \beta\ [s] && [\text{Induction hypothesis}] \\
&\iff \models_{\mathfrak{A}} \neg\beta\ [s] && [\text{Definition of satisfaction}] \\
&\iff \models_{\mathfrak{A}} \alpha\ [s]
\end{aligned}$$

iii. **Induction step: α is $\beta \to \gamma$ for some β and γ.**

$$\begin{aligned}
\bar{v}(\alpha) = T &\iff \bar{v}(\beta \to \gamma) = T \\
&\iff \bar{v}(\beta) = F \text{ or } \bar{v}(\gamma) = T && [\text{Definition of } \bar{v}] \\
&\iff \not\models_{\mathfrak{A}} \beta\ [s] \text{ or } \models_{\mathfrak{A}} \gamma\ [s] && [\text{Induction hypothesis}] \\
&\iff \models_{\mathfrak{A}} \beta \to \gamma\ [s] && [\text{Definition of satisfaction}] \\
&\iff \models_{\mathfrak{A}} \alpha\ [s]
\end{aligned}$$

□

b. Conclude that if Γ tautologically implies ϕ, then Γ logically implies ϕ.

Solution. *Note*: Satisfying a set means to satisfy each element of that set.

Suppose Γ tautologically implies ϕ.

Let $\mathfrak{A}$ by an arbitrary structure and $s : V \to |\mathfrak{A}|$ by an arbitrary satisfaction function. Corresponding to $\langle \mathfrak{A}, s \rangle$, we have a truth assignment, say v. Now,

$$\begin{aligned} \mathfrak{A} \text{ satisfies } \Gamma \text{ with } s &\iff v \text{ satisfies } \Gamma && [\text{Part (a)}] \\ &\implies v \text{ satisfies } \phi && [\Gamma \text{ tautologically implies } \phi] \\ &\iff \mathfrak{A} \text{ satisfies } \phi \text{ with } s && [\text{Part (a)}] \end{aligned}$$

Since this holds for any arbitrary structure and satisfaction function, Γ logically implies ϕ. ☐

Exercise 3.15 (Enderton, 2.4.9(b)) *Show that if y does not occur at all in ϕ, then x is substitutable for y in ϕ^x_y and $(\phi^x_y)^y_x = \phi$.*

Solution. We use induction on the complexity of ϕ, with the induction hypothesis that the result holds for all wffs less complex than ϕ.

i. **Atomics: ϕ is atomic.** Since ϕ is atomic, so is ϕ^x_y, and therefore x is substitutable for y in ϕ^x_y.
   To show that $(\phi^x_y)^y_x = \phi$, we consider each case separately.

   ▷ *Constants: ϕ is a constant symbol.* Since $\phi^x_y = \phi$, so $(\phi^x_y)^y_x = \phi$, as desired.

   ▷ *Variables: ϕ is a variable (that is not y).* If $\phi = x$, then $\phi^x_y = y$, so $(\phi^x_y)^y_x = x = \phi$, and if $\phi \neq x$, then $\phi^x_y = \phi$, so $(\phi^x_y)^y_x = \phi$ since y does not occur at all in ϕ.

   ▷ *Terms: ϕ is $ft_1 \cdots t_n$ for some function symbol f and terms $t_1, \ldots, t_n$.* Since $\phi^x_y = f(t_1)^x_y \cdots (t_n)^x_y$, so by the induction hypothesis, $(\phi^x_y)^y_x = f((t_1)^x_y)^y_x \cdots ((t_n)^x_y)^y_x = ft_1 \cdots t_n = \phi$.

   ▷ *Predicates: $Pt_1 \cdots t_n$ for some predicate symbol P and terms $t_1, \ldots, t_n$.* Since $\phi^x_y = P(t_1)^x_y \cdots (t_n)^x_y$, so by the induction hypothesis, $(\phi^x_y)^y_x = P((t_1)^x_y)^y_x \cdots ((t_n)^x_y)^y_x = Pt_1 \cdots t_n = \phi$. Note that this includes the case for equality.

ii. **Negation: ϕ is $(\neg\psi)$ for some ψ.** We know that x is substitutable for y in ϕ^x_y iff x is substitutable for y in ψ^x_y, which it is by the induction principle.
   Moreover, $\phi^x_y = (\neg\psi^x_y)$, so using the induction hypothesis, $(\phi^x_y)^y_x = (\neg\psi^x_y)^y_x = (\neg(\psi^x_y)^y_x) = (\neg\psi) = \phi$.

iii. **Implication: ϕ is $(\psi \to \theta)$ for some ψ and θ.** We know that x is substitutable for y in ϕ^x_y iff x is substitutable for y in both ψ^x_y and θ^x_y, which they are by the induction principle.

Moreover, $\phi^x_y = (\psi \to \theta)^x_y = (\psi^x_y \to \theta^x_y)$, so using the induction hypothesis, $(\phi^x_y)^y_x = (\psi^x_y \to \theta^x_y)^y_x = ((\psi^x_y)^y_x \to (\theta^x_y)^y_x) = (\psi \to \theta) = \phi$.

iv. **Quantifiers: ϕ is $\forall z\, \psi$ for some ψ.** We have two cases.

▷ $x = z$. In this case, $\phi^x_y = \forall z\, \psi = \phi$. Since y does not occur free in ϕ, x is substitutable for y in ϕ^x_y. Moreover, $(\phi^x_y)^y_x = (\forall z\, \psi)^y_x = \forall z\, \psi = \phi$ since y does not occur at all in ϕ.

▷ $x \neq z$. In this case, $\phi^x_y = \forall z\, \psi^x_y$. Using the induction hypothesis, x is substitutable for y in ψ^x_y. Moreover, since z does not occur in x, so x is substitutable for y in ϕ^x_y. Moreover, $\phi^x_y = \forall z\, \psi^x_y$, so by the induction hypothesis, $(\phi^x_y)^y_x = (\forall z\, \psi^x_y)^y_x = \forall z\, (\psi^x_y)^y_x = \forall z\, \psi = \phi$.

□

Exercise 3.16 **(Enderton, 2.4.12)** *Show that any consistent set Γ of formulas can be extended to a consistent set Δ having the property that for any formula α, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$. (Assume that the language is countable. Do not use the compactness theorem of sentential logic.)*

Solution. Since the language is countable, let $\{\alpha_1, \alpha_2, \ldots\}$ an enumeration of all wffs of the language. Define by recursion (on the natural numbers)

$$\Delta_0 = \Gamma,$$

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if this is consistent,} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise .} \end{cases}$$

Let $\Delta = \lim_{n\to\infty} \Delta_n = \bigcup_{n\in\mathbb{N}} \Delta_n$. Clearly, for any formula α, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$ since $\alpha = \alpha_n$ for some $n \in \mathbb{N}$. Notice that the procedure ensures that all finite subsets of Δ are consistent, since every subset of Δ_n is consistent for every $n \in \mathbb{N}$.
Therefore we just have to prove that Δ is consistent. Suppose not. Say $\Delta \vdash \perp$, where $\perp$ is some unsatisfiable, refutable formula like $\neg\forall x x = x$ or $\beta \wedge \neg\beta$. Since $\Delta \vdash \perp$, there exists a deduction for $\perp$. Let Σ the wffs in the deduction that are in Δ. Now, Σ is inconsistent since it produces $\perp$, and is a finite subset of Δ since deductions are necessarily finite. This gives us an inconsistent finite subset of Δ, which is a contradiction. Hence Δ must be consistent. □

Exercise 3.17 **(Enderton, 2.5.2)** *Prove the equivalence of parts (a) and (b) of the completeness theorem.*
Suggestion*: $\Gamma \vDash \phi$ iff $\Gamma \cup \neg\phi$ is unsatisfiable. And Δ is satisfiable iff $\Delta \vDash \perp$, where $\perp$ is some unsatisfiable, refutable formula like $\neg\forall x x = x$.*
Remark*: Similarly, the soundness theorem is equivalent to the statement that every satisfiable set of formulas is consistent.*

Solution. First we prove the suggestion. That is, we prove that $\Gamma \models \phi$ iff $\Gamma \cup \neg\phi$ is unsatisfiable. This is equivalent to proving $\Gamma \not\models \phi$ iff $\Gamma \cup \neg\phi$ is satisfiable.

Lemma 3.18 *$\Gamma \models \phi$ iff $\Gamma \cup \{\neg\phi\}$ is unsatisfiable.*

Proof. Notation: In what follows, we use "$\exists \langle \mathfrak{A}, s\rangle\ P$" to denote "there exists a structure $\mathfrak{A}$ and a satisfaction function $s : V \to |\mathfrak{A}|$ such that P". We shall also write $\models_{\mathfrak{A}} \Gamma\ [s]$ to mean that $\models_{\mathfrak{A}} \gamma\ [s]$ for every $\gamma \in \Gamma$.

$$\begin{aligned}
\Gamma \not\models \phi &\iff \exists \langle \mathfrak{A}, s\rangle \left(\models_{\mathfrak{A}} \Gamma\ [s] \text{ and } \not\models_{\mathfrak{A}} \phi\ [s]\right) && \text{[Definition of satisfiability]}\\
&\iff \exists \langle \mathfrak{A}, s\rangle \left(\models_{\mathfrak{A}} \Gamma\ [s] \text{ and } \models_{\mathfrak{A}} \neg\phi\ [s]\right) && \text{[Definition of satisfaction function]}\\
&\iff \exists \langle \mathfrak{A}, s\rangle \left(\models_{\mathfrak{A}} \Gamma \cup \{\neg\phi\}\ [s]\right) &&\\
&\iff \Gamma \cup \{\neg\phi\} \text{ is satisfiable.} && \text{[Definition of satisfiability]}
\end{aligned}$$

Moreover, Δ is satisfiable iff $\Delta \cup \{\neg\bot\}$ is satisfiable iff $\Delta \not\models \bot$, since $\neg\bot$ is always satisfiable. □

Now we prove the equivalence.

i. **((a) $\implies$ (b))** Let Γ be consistent. Then $\Gamma \nvdash \bot$, where $\bot$ is as defined in the problem. Using (a), we have $\Gamma \not\models \bot$. By the claim above, Γ is satisfiable.

ii. **((b) $\implies$ (a))** Let $\Gamma \models \phi$. So $\Gamma \cup \{\neg\phi\}$ is unsatisfiable. Using the contrapositive of (b), we have $\Gamma \cup \{\neg\phi\}$ is inconsistent. Using RAA and Rule T, we get $\Gamma \vdash \phi$.

□

Bibliography