Functional analysis

Mostly operator theory for now

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Contents

Part 1

Convergence

1.1 Strong, weak and weak* convergence

Disclaimer: This section is shamelessly copied from Christopher Heil's notes.

Definition 1.1 *Let X be a normed vector space, and $x_n, x \in X$. We define the following convergences as $n \to \infty$.*

$$\begin{aligned} &(strong) && x_n \to x && \iff && \|x_n - x\| \to 0 \\ &(weak) && x_n \xrightarrow{w} x && \iff && \forall \phi \in X^*, \quad (x_n - x, \phi) \to 0 \end{aligned}$$

Definition 1.2 *Let X be a normed vector space, and $\phi_n, \phi \in X^*$. We define the following convergences as $n \to \infty$.*

$$\begin{aligned} &(strong) && \phi_n \to \phi && \iff && \|\phi_n - \phi\| \to 0 \\ &(weak) && \phi_n \xrightarrow{w} \phi && \iff && \forall \xi \in X^{**}, \quad (\phi_n - \phi, \xi) \to 0 \\ &(weak^*) && \phi_n \xrightarrow{w^*} \phi && \iff && \forall x \in X, \quad (x, \phi_n - \phi) \to 0 \end{aligned}$$

Remark 1.3 *Weak* convergence is simply* pointwise convergence *for the functionals ϕ_n.*

Proposition 1.4 (strong $\implies$ weak $\implies$ weak* for convergence) *Suppose $\phi_n, \phi \in X^*$. Then $\phi_n \to \phi \implies \phi_n \xrightarrow{w} \phi \implies \phi_n \xrightarrow{w^*} \phi$.*
The second implication reverses if X is reflexive.

Proof. *strong $\implies$ weak*: $\quad (x_n - x, \phi) \le \|x_n - x\| \, \|\phi\| \to 0.$
*weak $\implies$ weak**: $\quad (x, \phi_n - \phi) = (\phi_n - \phi, x^{**}) \to 0.$
The claim about the reverse implication is now obvious.

Counterexample for converse of the first implication: Suppose $X = \ell^2(\mathbb{N})$. Then $e_n \xrightarrow{w} 0$, but $\|e_n - 0\| = 1 \nrightarrow 0$. □

Proposition 1.5 *In Hilbert spaces, weak convergence plus convergence of norms ($\|x_n\| \to \|x\|$) is equivalent to strong convergence.*

Proof. $\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle = \langle x_n - x, x_n \rangle - \langle x_n - x, x \rangle \to 0.$ □

Proposition 1.6 *Let H and K be Hilbert spaces, and let $T \in B(H, K)$ be a compact operator.*

Show that $x_n \xrightarrow{w} x \implies Tx_n \to Tx$.

Thus, a compact operator maps weakly convergent sequences to strongly convergent sequences.

Proof. *Disclaimer*: Stolen from MSx1142451.

$Tx_n \xrightarrow{w} Tx$ by continuity. Thus if any subsequence has a strong limit, it certainly is Tx. But compactness guarantees every subsequence has a subsequence that converges to something: that something is Tx by uniqueness, and so by our above equivalence with convergence, we have $Tx_n \to Tx$. □

Part 2

Operator theory

2.1 Elementary ideas

2.1.1 Intuition

- ▷ $T \in \mathcal{B}^\infty \iff \lambda \in \ell^\infty$ (bounded)
  Example $I : \ell^2 \to \ell^2 : e_n \mapsto e_n$.
- ▷ $T \in \mathcal{K} \iff \lambda \in c_0$ (compact)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{\sqrt{n}} e_n$.
- ▷ $T \in \mathcal{B}^2 \iff \lambda \in \ell^2$ (Hilbert-Schmidt)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n} e_n$.
- ▷ $T \in \mathcal{B}^1 \iff \lambda \in \ell^1$ (trace-class)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \frac{1}{n^2} e_n$.
- ▷ $T \in \mathcal{D} \iff \lambda \in c_{00}$ (degenerate)
  Example $T : \ell^2 \to \ell^2 : e_n \mapsto \alpha_n e_n \mathbb{1}_{[N]}(n)$ for $\alpha_n \in \mathbb{C}$ and $N \in \mathbb{N}$.

Theorem 1.1 (Operator inclusions) $\mathcal{D} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{K} \subset \mathcal{B}^\infty$

Proof.

- ▷ $\mathcal{D} \subset \mathcal{B}^1$ Trivial
- ▷ $\mathcal{B}^1 \subset \mathcal{B}^2$
- ▷ $\mathcal{B}^2 \subset \mathcal{K}$
- ▷ $\mathcal{K} \subset \mathcal{B}^\infty$ ((<BMC2009>), Proposition 4.6) If T is unbounded, we can find a sequence of unit vectors (e_n) such that $\|Te_n\| \nearrow \infty$. So Te_n cannot have a convergent subsequence, for if $Te_n \to x$, then $\|Te_n\| \to \|x\|$.

□

Proposition 1.2 *For* $T \in \mathcal{B}^\infty$, $\|T\|_\infty = \sup\{|\langle Tx, y\rangle|\} : \|x\| = 1, \|y\| = 1$.

Proof.

($\leq$) Since $\|Tx\| = \frac{\|Tx\|^2}{\|Tx\|} = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \left\langle Tx, \frac{Tx}{\|Tx\|}\right\rangle$, we have

$$\|T\|_\infty = \sup\{\|Tx\| : \|x\| = 1\} \leq \sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\}.$$

($\geq$) Since $\langle Tx, y\rangle \leq \|Tx\| \|y\| \leq \|T\|_\infty \|x\| \|y\|$, we have

$$\sup\{|\langle Tx, y\rangle| : \|x\| = 1, \|y\| = 1\} \leq \|T\|_\infty .$$

□

2.1.2 Projection operators

Proposition 1.3 $\|P\|_\infty \leq 1$.

Proof. Since $\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PPx, x\rangle = \langle Px, x\rangle \leq \|Px\| \|x\|$, we have $\|P\|_\infty \leq 1$. □

Proposition 1.4 *A projection operator is compact iff its image is finite dimensional.*

Proof.

($\Longrightarrow$) Let $P : H \to H$ be a projection operator, so that $P^2 = P$, or $P(P - I) = 0$.

($\Longleftarrow$) Since the image is finite dimensional, fix an orthonormal basis $e_1, \ldots, e_n$ of $\operatorname{im} T$.

□

2.2 Optimization

2.2.1 Duality in optimization is the same as duality in functional analysis

For an various intuitions of duality in optimization, see MSx223235.

Let X and Y be Banach spaces, and X^* and Y^* be their (algebraic?) duals. Consider the two problems, with ϕ_0, y_0 fixed. Here $(\cdot,\cdot)$ denotes the canonical duality pairing.

$$
\begin{aligned}
&& \max \quad & (\phi_0, x) \\
&(\text{Primal}) & \text{s.t.} \quad & Tx \le y_0 \\
&&& x \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
&& \min \quad & (\psi, y_0) \\
&(\text{Dual}) & \text{s.t.} \quad & T^*\psi \ge \phi_0 \\
&&& \psi \ge 0
\end{aligned}
$$

See the following diagram for more details.

$$
\begin{tikzcd}
 & x \arrow[r, mapsto, "T"] & Tx & \\
x \in & X \arrow[r, "T"] \arrow[d, dash] & y_0 \arrow[d, dash] & \ni Tx, y_0 \\
\phi_0, T^*\psi \in & X^* & y_0^* \arrow[l, "T^*"'] & \ni \psi \\
 & T^*\psi & \psi \arrow[l, mapsto, "T^*"'] &
\end{tikzcd}
$$

Bibliography