

Anticipating stochastic integrals and its large deviations

Sudip Sinha

November 23, 2019

CONTENTS

Part 1	Anticipating integrals	1
1.1	Elementary ideas	2
Part 2	Large deviations theory	3
2.1	Friedlin-Wentzell theorem	4
2.2	Friedlin-Wentzell theorem for anticipating initial condition with extension of filtration	5
2.2.1	\tilde{W}_\cdot is a Wiener process	5
	Bibliography	8

PART 1

ANTICIPATING INTEGRALS

1.1 ELEMENTARY IDEAS

ABCD

PART 2

LARGE DEVIATIONS THEORY

2.1 FRIEDLIN-WENTZELL THEOREM

2.2 FRIEDLIN-WENTZELL THEOREM FOR ANTICIPATING INITIAL CONDITION WITH EXTENSION OF FILTRATION

Our aim is to formulate a large deviations principle for an SDE with anticipating initial conditions. We start of with a very simple case

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) dW_t,$$

where $t \in [0, T]$ for some $T < \infty$, and conditions on σ shall be imposed as necessary.

We shall look at the method of enlargement of filtration by [Itô1978]. We denote the enlarged filtration by $\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(W_T)$. For $t \in [0, T]$, define the process $A_t = \int_0^t \frac{W_T - W_u}{T-u} du$. Then $W_t = \tilde{W}_t + A_t$, where \tilde{W} is a Wiener process w.r.t. $\tilde{\mathcal{F}}$.

For now, we shall bound $t \in [0, T_b]$, where $T_b \in [0, T)$. What happens when $t \rightarrow T$?

Using this, we write our original SDE as

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) d\tilde{W}_t + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) \frac{W_T - W_s}{T-s} ds.$$

Let $Y_t^\varepsilon = \sqrt{\varepsilon}(W_T - W_t)$. Then X_t^ε is given by

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) d\tilde{W}_t + \int_0^t \sigma(X_t^\varepsilon) \frac{Y_s^\varepsilon}{T-s} ds.$$

Moreover,

$$\begin{aligned} Y_t^\varepsilon &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}W_t \\ &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon} \left(\tilde{W}_t + \int_0^t \frac{W_T - W_s}{T-s} ds \right) \\ &= \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}\tilde{W}_t - \int_0^t \frac{Y_s^\varepsilon}{T-s} ds \end{aligned}$$

Therefore, we have the joint process

$$\begin{pmatrix} X_t^\varepsilon \\ Y_t^\varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{\varepsilon} \end{pmatrix} W_T + \sqrt{\varepsilon} \int_0^t \begin{pmatrix} \sigma(X_s^\varepsilon) \\ -1 \end{pmatrix} d\tilde{W}_s + \int_0^t \begin{pmatrix} \sigma(X_s^\varepsilon) \\ -1 \end{pmatrix} \frac{Y_s^\varepsilon}{T-s} ds.$$

Firstly, we show that there is a unique solution to the above system of SDEs. For convinience, let $\tilde{\sigma}(x, y) = (\sigma(x), -1)$ and $\tilde{b}(x, y) = \frac{y}{T-s}(\sigma(x), -1)$.

Proposition 2.1 *$\tilde{\sigma}$ and \tilde{b} satisfy the linear growth and Lipschitz conditions.*

Proof.

Note that $Y_t^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, we expect

$$\begin{pmatrix} X_t^\varepsilon - W_T \\ Y_t^\varepsilon \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } \varepsilon \rightarrow 0.$$

We want a large deviation principle of the joint process $Z_t^\varepsilon = (X_t^\varepsilon - W_T, Y_t^\varepsilon)$.

We take a sequence of stopping times (τ_n) such that $\tau_n \nearrow T_b$. Then

2.2.1 \tilde{W} is a Wiener process

We show that (\tilde{W}_t) is a $(\tilde{\mathcal{F}}_t)$ -martingale with quadratic variation t . Then by Lévy's Characterization of Wiener process, we obtain that \tilde{W} is a Wiener process.

First we prove two lemmas.

Lemma 2.2 *The σ -algebras $\mathcal{F}_s \vee \sigma(W_T)$ and $\mathcal{F}_s \vee \sigma(W_T - W_s)$ are the same.*

Proof. For any Borel set B , the set $\{W_T \in B\} = \{(W_T - W_t) + W_t \in B\}$.
TODO

Lemma 2.3 *For $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}(W_t - W_s | W_T - W_s) = \frac{t-s}{T-s} (W_T - W_s).$$

Proof. We partition the interval $[0, T]$ into $n = kn_0$ equal parts, where $n_0 = (\min\{s, t-s, T-t\})^{-1}$ and $k \in \mathbb{N}$. Let $n_s = s \frac{n}{T}$ and $n_t = t \frac{n}{T}$. That is, the partition is

$$P = \left\{0, \frac{T}{n}, \dots, \frac{n_s T}{n} = s, \dots, \frac{n_t T}{n} = t, \dots, \frac{(n-1)T}{n}, T\right\}.$$

Let $\Delta_i W = W_{\frac{(i+1)T}{n}} - W_{\frac{iT}{n}}$.

Firstly, note that the $\Delta_i W$ s are independent and identically distributed from the definition of Wiener process. Now, using the linearity of conditional expectation, we have

$$\begin{aligned} \mathbb{E}(W_t - W_s | W_T - W_s) &= \mathbb{E}\left(\sum_{i=n_s}^{n_t-1} \Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\ &= \sum_{i=n_s}^{n_t-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\ &= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\ &= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \mathbb{E}\left(\sum_{i=n_s}^{n-1} \Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\ &= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \Delta_i W \\ &= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} (W_T - W_s) \\ &= \frac{n_t - n_s}{n - n_s} (W_T - W_s) = \frac{t-s}{T-s} (W_T - W_s). \end{aligned}$$

Proposition 2.4 *\tilde{W} is a $\tilde{\mathcal{F}}$ -martingale.*

Proof. Let $0 \leq s \leq t \leq T$. Then

$$\tilde{W}_t - \tilde{W}_s = (W_t - W_s) - \int_s^t \frac{W_T - W_u}{T-u} du = (W_t - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{W_u - W_s}{T-u} \right) du.$$

Moreover, since $W_t - W_s$ is independent of \mathcal{F}_s for every $t \geq s$, using lemmas 2.2 and 2.3, we get

$$\mathbb{E}(W_t - W_s | \tilde{\mathcal{F}}_s) = \mathbb{E}(W_t - W_s | \mathcal{F}_s \vee \sigma(W_T - W_s)) = \mathbb{E}(W_t - W_s | W_T - W_s) = \frac{t-s}{T-s} (W_T - W_s).$$

Therefore, using the fact that W_T and W_s are $\tilde{\mathcal{F}}$ -measurable with conditional Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}(\tilde{W}_t - \tilde{W}_s | \tilde{\mathcal{F}}_s) &= \mathbb{E}(W_t - W_s | \tilde{\mathcal{F}}_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{\mathbb{E}(W_u - W_s | \tilde{\mathcal{F}}_s)}{T-u} \right) du \\
&= \frac{t-s}{T-s} (W_T - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{u-s}{T-s} \frac{W_T - W_s}{T-u} \right) du \\
&= \frac{t-s}{T-s} (W_T - W_s) - \int_s^t \frac{W_T - W_s}{T-s} du \\
&= \frac{t-s}{T-s} (W_T - W_s) - \frac{t-s}{T-s} (W_T - W_s) = 0.
\end{aligned}$$

Now, since \tilde{W}_s is $\tilde{\mathcal{F}}_s$ -measurable, $\mathbb{E}(\tilde{W}_t | \tilde{\mathcal{F}}_s) = \mathbb{E}(\tilde{W}_t - \tilde{W}_s | \tilde{\mathcal{F}}_s) + \tilde{W}_s = \tilde{W}_s$.

Proposition 2.5 *The quadratic variation of \tilde{W}_t is t .*

BIBLIOGRAPHY