Contents

Introduction and revision

In the first two days we will review what we learned in Combinatorics 1 and try to solve interesting problems related to them.
The structure of the course is going to be as follows (the ordered pair represents (week, day)):

1. (week 1, day 1) Revision of counting
    a. what is combinatorics?
    b. addition and multiplication rules (+ proofs)
    c. permutations and combinations
    d. binomial and multinomial coefficients and theorem
    e. proofs using combinatorial methods
    f. proofs of basic results
2. (week 1, day 2)
    a. Multinomial type problems
    b. Revision of graph theory
    c.
    d.
3. (week 1, day 3)
4. (week 1, day 4)
5. (week 2, day 1)
6. (week 2, day 2)
7. (week 2, day 3)
8. (week 2, day 4)
9. (week 3, day 1)
10. (week 3, day 2)
11. (week 3, day 3)
12. (week 3, day 4)

1.1 Combinatorics basics

1.1.1 What is combinatorics?

The study of
i. *discrete structures*: graphs, strings, distributions, partitions
ii. *enumerations*: permutations, combinations, inclusion and exclusion, generating functions, recurrence relations
iii. *algorithms and optimization*: sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, shortest path, bipartite matching

We will focus on enumerations and discrete structures in this course. The algorithms can be interesting for the projects.

Why is it interesting?

- Concerns with counting, which is very fundamental.
- We live in a finite world, and every problem is essentially combinatorial in a sense.
- Which means it is ubiquitious in mathematics
- Gives us fun visual proofs of results that can be proved algebraic.
- ...

1.1.2 Enumerations

- Multiplication rule: if E and F are finite sets, then $|E \times F| = |E||F|$.
  Number of ways of constructing a 100 character string out of the 26 letters of the English alphabet.
- Addition rule: if E and F are finite *disjoint* sets, then $|E \sqcup F| = |E| + |F|$.
  If there are two roads from Baton Rouge to New Orleans, and three roads from Baton Rouge to Lafayette, in how many ways can you go from Baton Rouge to either of the places?
- Factorials
  Numbers of ways to arrange 5 people in a row.
- Permutations
  Numbers of ways to arrange any 4 people in a row when there are 7 people.
- Combinations
  Numbers of ways to form a committee of 3 people from a group of 7 employees.
- Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Exercise (AC Ex 2.7) *How many strings of the form $l_1 l_2 d_1 d_2 d_3 l_3 l_4 d_4 l_5 l_6$ are there where*

1. *for $1 \le i \le 6$, l_i is an uppercase letter in the English alphabet;*
2. *for $1 \le i \le 4$, d_i is a decimal digit;*
3. *l_2 is not a vowel (i.e., $l_2 \notin \{A, E, I, O, U\}$); and*
4. *the digits d_1, d_2, and d_3 are distinct (i.e., $d_1 \ne d_2 \ne d_3 \ne d_1$).*

Solution $(26^5 \cdot (26 - 5)) \cdot (10 \cdot (10 \cdot 9 \cdot 8))$. □

Exercise (AC Ex 2.9) *A database uses 20-character strings as record identifiers. The valid characters in these strings are upper-case letters in the English alphabet and decimal digits. (Recall there are 26 letters in the English alphabet and 10 decimal digits.) How many valid record identifiers are possible if a valid record identifier must meet* all *of the following criteria:*

1. *Letter(s) from the set $\{A, E, I, O, U\}$ occur in* exactly *three positions of the string.*
2. *The last three characters in the string are* distinct *decimal digits that do not appear elsewhere in the string.*
3. *The remaining characters of the string may be filled with any of the remaining letters or decimal digits.*

Solution $(10 \cdot 9 \cdot 8) \cdot (5^3) \cdot ((26 - 5) + (10 - 3))^{20-3-3}$. □

1.1.3 Combinatorial proofs are fun!

References: (<KT2016>) and (<Nelsen1993>).

1. sum of first n natural numbers: x3
2. sum of first n odd numbers: x2
3. $1 + 2 + \cdots +$ (n-1) + n + (n-1) $+ \cdots + 2 + 1$
4. $1 + 3 + \cdots +$ (2n-3) + (2n-1) + (2n-3) $+ \cdots + 2 + 1$
5. sum of binomial coefficients

1.1.4 Background of basic results

Proposition (Basic principle of counting) *Suppose two independent experiments are performed, and there are m possible outcomes of the first experiment and n possible outcomes of the second experiment. Then the total possible outcomes of of the two experiments combined is mn.*

Proof Let (i, j) denote the case when the first experiment gives the ith outcome and the second experiment gives the jth outcome. Enumerating, we get

$$\begin{matrix} (1,1) & (1,2) & \dots & (1,n) \\ (2,1) & (2,2) & \dots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (m,1) & (m,2) & \dots & (m,n) \end{matrix}$$

Since there are m rows and n columns, we have total mn entries. □

Remark *This can be generalized to a finite number of experiments.*

Theorem (Binomial theorem) *Let x and y be real numbers with x, y and $x+y$ nonzero. Then for every non-negative integer n,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof (Inductive) Homework. □

Proof (Combinatorial) Consider the product $(x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)$.
First, note that the expansion consists of 2^n terms, each being a product of n factors.
Secondly, each product contains either x_j xor y_j for each $j \in [n]$.
For example, $(x_1+y_1)(x_2+y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$.
Now, we can we choose k of the x_js and $n-k$ of the y_js in $\binom{n}{k}$ ways, so there are precisely those many terms with mk x_js and $n-k$ y_js in the expansion.
Finally, letting $x_j = x$ and $y_j = y$ for each $j \in [n]$, we get the result. □

Remark *This can be generalized to the* multinomial theorem *and* multinomial coefficients, *which we will revise if the need arises.*

1.2 Exercises

1.2.1 2019-06-12

Exercise (recursion formula for binomial coefficients) *Prove that* $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$. *How does this relate to the Pascal's triangle?*

Exercise (sum of binomial coefficients) *Prove that* $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Exercise (variation of the handshake problem) *Each person on campus has 1 secret. Every time two people converse, they exchange secrets. If those two people move on to talk to other people, they will share all the secrets they have learnt in addition to their own. How many conversations must be had for everyone on campus to know everyone elses' secrets if there are n students?*
What we want is a formula *for the minimum number of interactions dependent on the number of students. That is, we want to find a function* $f : \mathbb{N} \to \mathbb{N}$ *in terms of* n.

Exercise (Venn diagrams) *Let X be the set of 63 students in an applied combinatorics course at a large technological university. Suppose there are 47 computer science majors and 51 male students. Also, we know there are 45 male students majoring in computer science. How many students in the class are female students not majoring in computer science?*

Exercise (counting integers) *How many integers in* $\{1, \cdots, 100\}$ *are divisible by 2, 3 or 5?*

Exercise (stars and bars problems) *Calculate the number of integer solutions of each the following problems. Note that the problems progress sequentially, and you should be able to use the ideas/results from the previous problem to the next.*

1. $x_1 + x_2 + x_3 + x_4 = 64; \qquad x_1, x_2, x_3, x_4 \geq 1.$
2. $x_1 + x_2 + x_3 + x_4 \leq 64; \qquad x_1, x_2, x_3, x_4 \geq 1.$
3. $x_1 + x_2 + x_3 + x_4 \leq 64; \qquad x_1, x_2 \geq 1;\ x_3 \geq 0;\ x_4 \geq 16.$
4. $x_1 + x_2 + x_3 + x_4 \leq 64; \qquad x_1 \geq 1;\ 1 \leq x_2 \leq 8;\ x_3 \geq 0;\ x_4 \geq 16.$
5. $x_1 + x_2 + x_3 + x_4 \leq 64; \qquad 1 \leq x_1 \leq 8;\ 1 \leq x_2 \leq 8;\ x_3 \geq 0;\ x_4 \geq 16.$

2.1 Discrete Probability Spaces

Notations

Term	Description	Symbol/Idea	Coin toss Example
sample space	set of outcomes	Ω	$\{H, T\}$
outcome	arbitrary outcome	$\omega \in \Omega$	H
event	subset of sample space	E	$\emptyset, \{H\}, \{T\}, \{H, T\}$
mutually exclusive events	events with empty intersection	$E_1 \cap E_2 = \emptyset$	$\{H\}$ *and* $\{T\}$
probability mass function	weightage of each outcome	$p : \Omega \to [0, 1]$, with $\sum_{\omega} p(\omega) = 1$	$p(H) = \frac{1}{3}, p(T) = \frac{2}{3}$
probability	(of an event)	$\mathbb{P} : 2^{\Omega} \to [0, 1]$, $\mathbb{P}(E) = \sum_{\omega \in \Omega} p(\omega)$	$\mathbb{P}(\emptyset) = 0, \mathbb{P}(\{H, T\}) = 1$
random variable	a function	$X : \Omega \to \mathbb{R}$	$X(H) = 1, X(T) = 0$

2.2 The algebra of sets

Suppose Ω be a set, and E, F, G are subsets of Ω. We have the following operators:

- complement(E^{C})
- union ($E \cup F$)
- intersection($E \cap F$)
- set difference ($E \setminus F = E \cap F^{\mathsf{C}}$)

Then the following *laws* hold.

- Commutativity of union ($E \cup F = F \cup E$) and intersection ($E \cap F = F \cap E$).
- Associativity of union ($(E \cup F) \cup G = E \cup (F \cup G)$) and intersection ($(E \cap F) \cap G = E \cap (F \cap G)$).
- Distributivity of union over intersection ($E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$).
- Distributivity of intersection over union ($E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$).
- Idempotence: $E \cup E = E$, $E \cap E = E$.
- Domination: $\Omega \cup E = \Omega$, $\Omega \cap E = E$.
- Absorption: $E \cup (E \cap F) = E$, $E \cap (E \cup F) = E$.
- De Morgan: $(E \cup F)^{\mathsf{C}} = E^{\mathsf{C}} \cap F^{\mathsf{C}}$, $(E \cap F)^{\mathsf{C}} = E^{\mathsf{C}} \cup F^{\mathsf{C}}$.
- Involution: $(E^{\mathsf{C}})^{\mathsf{C}} = E$.
- ...

2.3 Axiomatic probability theory

Definition (Probability axioms) *A non-negative valued function* $\mathbb{P}$ *defined on the set of events is called a* probability measure *if the following hold.*

1. *(null empty set)* $\mathbb{P}(\emptyset) = 0$.
2. *(countable additivity) For any sequence of mutually exclusive events* $E_1, E_2, \cdots$*, we have* $\mathbb{P}\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$.
3. *(probability)* $\mathbb{P}(\Omega) = 1$.

Draw Venn diagrams for all of the following.

Proposition $\mathbb{P}(E^{\mathsf{C}}) = 1 - \mathbb{P}(E)$.

Proof Since $E \cap E^{\mathsf{C}} = \emptyset$, by Axiom 2 we have $1 = \mathbb{P}(\Omega) = \mathbb{P}(E \sqcup E^{\mathsf{C}}) = \mathbb{P}(E) + \mathbb{P}(E^{\mathsf{C}})$. □

Proposition *If* $E \subset F$*, then* $\mathbb{P}(E) \leq \mathbb{P}(F)$.

Proof Note that $F = E \sqcup (F \setminus E)$. So by Axiom 2 we have $\mathbb{P}(F) = \mathbb{P}(E \sqcup (F \setminus E)) = \mathbb{P}(E) + \mathbb{P}(F \setminus E)$. Therefore, $\mathbb{P}(F) - \mathbb{P}(E) = \mathbb{P}(F \setminus E)$, which is non-negative since probability is a non-negative set function. □

Proposition (Inclusion-Exclusion) $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$..

Proof

1. $E \cup F = (E \setminus F) \sqcup (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(E \cup F) = \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F)$.
2. $E = (E \setminus F) \sqcup (E \cap F)$, so $\mathbb{P}(E) = \mathbb{P}(E \setminus F) + \mathbb{P}(E \cap F)$, and similarly
3. $F = (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(F) = \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F)$.

Combining the above,

$$
\begin{aligned}
\mathbb{P}(E \cup F) &= \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F) \\
&= (\mathbb{P}(E) - \mathbb{P}(E \cap F)) + (\mathbb{P}(F) - \mathbb{P}(E \cap F)) + \mathbb{P}(E \cap F) \\
&= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).
\end{aligned}
$$

□

2.4 Exercises

2.4.1 2019-06-19

Exercise (Monty Hall) *Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to switch to door number 2?" Is it to your advantage to switch your choice of the door?*

~Marilyn vos Savant, Parade (1990)

Choose one:

1. *no*
2. *yes*
3. *switching is irrelevant and does not change anything*

Exercise (strange dice) *There are three dice, A, B, and C. The dice are numbered* strangely, *as shown below:*

A. 2, 6, 7
B. 1, 5, 9
C. 3, 4, 8

with the numbers on opposite faces being the same.

The rules are simple. You pick one of the three dice, and then I pick one of the two remainders. We both roll and the player with the higher number wins. Which die do you choose?

Exercise (Bertrand's box) *There are three boxes. Each box contains two coins, which can be either gold(G) or silver(S). Their composition is as follows:*

A. G, G
B. G, S
C. S, S

After choosing a box at random, you pick a coin, and find that it is a G. What is the probability that the next coin is also a G?

Choose one:

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{2}{3}$
4. *none of the above*

2.4.2 2019-06-26

Exercise (Bulb factory) *Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?*

Exercise (Law of total probability) *Let $E_1, E_2, \cdots, E_n$ is a mutually exclusive and exhaustive set of events, that is,*

- (mutually exclusive) $E_i \cap E_j = \emptyset$ *for every combination of i and j, and*
- (exhaustive) $E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n = \Omega$.

If F is an event, then prove that

$$\mathbb{P}(F) = \sum_{i=1}^{n} \left[\mathbb{P}(F \mid E_i) \mathbb{P}(E_i) \right].$$

Exercise (Bayes formula) *Use the* law of total probability *to prove the Bayes formula*

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(F \mid E)\mathbb{P}(E)}{\mathbb{P}(F \mid E)\mathbb{P}(E) + \mathbb{P}(F \mid E^{\mathsf{C}})\mathbb{P}(E^{\mathsf{C}})}.$$

Exercise (Blood test) *A blood test for the HIV virus is developed. When the virus is present, it reports positive with a probability of 99%. When the virus is absent, it reports negative with a probability of 99%. By 2017 estimates, 0.5% of the world population is infected. What is the probability that a person whose test results come out positive is infected?*

Exercise (Mutual and pairwise independence) *Two events A and B are called* (pairwise) *indendent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Two events A, B, and C are called* mutually *indendent if $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$.*
If the events A, B, and C are pairwise independent, does it mean that they are mutually independent? If yes, argue why. If no, try to find a counterexample.

Bibliography