

Generalization of stochastic calculus and its applications in large deviations theory

Sudip Sinha

2019-04-05

Advisors

Prof Hui-Hsiung Kuo

Prof Padmanabhan Sundar

Outline

1	Introduction and motivation	3
2	Generalization of Itô calculus	12
3	Large deviations theory	19
4	Conclusion	33
5	Sample slides	35

§ 1

INTRODUCTION AND MOTIVATION

Quick revision and notations

- Let $T \in (0, \infty)$, and denote $\mathbb{T} = [0, T]$ as the index set for t .
- Let $(\Omega, \mathcal{F}, \mathcal{F}_., \mathbb{P})$ be a filtered probability space.
- $B_.$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_., \mathbb{P})$.
- Properties of B .
 - starts at 0
 - has independent increments
 - $B_t - B_s \sim \mathcal{N}(0, t - s)$
 - continuous paths
 - has **unbounded linear variation** 😞
 - has **bounded quadratic variation** 😊
 - $\mathbb{E}(B_t B_s) = s \wedge t$
 - martingale
- **Naive stochastic integration w.r.t. B_t : not possible.**
- A stochastic process X_t is called (\mathcal{F}_t) -adapted if $\forall t$, X_t is measurable w.r.t. \mathcal{F}_t .

Wiener integral ($f \in L^2[0, T]$)

- Definition

1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t) dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
2. $f \in L^2[0, T]$: Use step functions approximating f to extend the integral **a.s.**

- Properties

- ★ Linear
- ★ **Gaussian distribution** with mean 0 and variance $\|f\|_{L^2[0, T]}^2$ (Itô isometry)
- ★ Corresponds to the Riemann–Stieltjes integral for $f \in C[0, T]$

- The associated process: $I_t = \int_0^t X_t dB_t$

- ★ continuous
- ★ martingale

- Problem: Cannot integrate stochastic processes.

Itô integral ($X \in L^2_{\text{ad}}([0, T] \times \Omega)$)

- Definition

1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
2. $X \in L^2_{\text{ad}}([0, T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

- Properties

- ★ Linear
- ★ Mean 0 and variance $\|f\|_{L^2[0, T]}^2$ (Itô isometry)
- ★ For X continuous, $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t$, for example $\int_0^t B_s \, dB_s = \frac{1}{2} (B_t^2 - t)$

- The associated process: $I_t = \int_0^t X_s \, dB_s$

- ★ continuous
- ★ martingale

- Example: $\int_0^T B_t \, dB_t = \frac{1}{2} (B_T^2 - T)$.

Itô integral ($\int_0^T X_t^2 dt < \infty$ a.s.)

- Definition: Use sequences of processes in $L^2_{\text{ad}}([0, T] \times \Omega)$ approximating X in probability to extend the integral in probability.
- Properties
 - ★ Linear
 - ★ Mean 0, but variance? ☹
- The associated process: $I_t = \int_0^t X_t dB_t$
 - ★ continuous
 - ★ local martingale
- Example: $\int_0^T e^{B_t^2} dB_t = \int_0^{B_1} e^{t^2} dt - \int_0^T B_t e^{B_t^2} dt.$

Itô formula

- An **Itô process** is a process of the form $X_t = X_0 + \int_0^t m(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s$, equivalently expressed as $dX_t = m(t, X_t) \, dt + \sigma(t, X_t) \, dB_t$.
[Only makes sense when $\int_0^T (|m(s, X_s)| + |\sigma(s, X_s)|^2) \, ds < \infty$ a.s.]

Theorem ([Itô44]) Let X_t be a d -dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d -dimensional Itô process, and

$$df(X_t) = \langle (Df)(X_t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D^2 f)(X_t) dX_t \rangle,$$

where we use the rule $dB_t \otimes dB_t = I_d \, dt$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, we have $d\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, dt + \sigma \mathcal{E}_t \, dB_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (dB_t)^2$.

Exponential processes and Girsanov theorem

TODO

Let $h \in L^2[0, T]$. Then the translated stochastic process $W_t = B_t - \int_0^t h(s) \, ds$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\sigma B_T - \frac{1}{2}\sigma^2 T\right) =: \mathcal{E}_T^h$.

Then $\tilde{\mathbb{P}} \sim \mathbb{P}$ and the process $Z_t := \mathbb{E}\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \mid \mathcal{F}_t\right)$ is a martingale.

Stochastic differential equations

- Let $\zeta \in L^2(\Omega)$ be independent of B , and $m, \sigma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be $\mathcal{B}[0, T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable such that $m(t, \cdot, \cdot)$ and $\sigma(t, \cdot, \cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable $\forall t$.
Then a \mathcal{F}_t -adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \zeta + \int_0^t m(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$ if for each t , the X_t satisfies the integral equation a.s.
- The stochastic differential equation $dX_t = m(t, X_t) dt + \sigma(t, X_t) dB_t$, $X_0 = \zeta$ is a symbolic representation of the stochastic integral equation.

Theorem (Existence and uniqueness, Markov property) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

- ★ (Lipschitz condition) $|m(t, x) - m(t, y)| + |\sigma(t, x) - \sigma(t, y)|^2 \leq M(1 + |x|^2)$ a.s.
- ★ (growth condition) $|m(t, x)| + |\sigma(t, y)|^2 \leq M(1 + |x|^2)$ a.s.

The solution is a Markov process. Moreover if $\zeta \in \mathbb{R}$, then the solution is also stationary.

- Example: For σ constant, the solution of $d\mathcal{E}_t = \sigma \mathcal{E}_t dB_t$, $\mathcal{E}_0 = 1$ is given by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

§ 2

GENERALIZATION OF ITÔ CALCULUS

Motivation

- Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T dB_s dB_t = \int_0^T B_T dB_t \stackrel{?}{=} B_T B_t$.
- Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so **no martingale property** ☹.
- Stochastic differential equations with anticipation

$$dX_t = X_t dB_t$$

$$X_0 = B_1$$

$$dY_t = B_T dB_t$$

$$Y_0 = 1$$

- Problem: We want to define $\int_0^T X_t dB_t$, where X_\cdot is not adapted (anticipating).
- Some approaches
 - ★ Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} ds \right) + \int_0^t \frac{B_T - B_s}{T-s} ds$
 - ★ Enlargement of filtration
 - ★ White noise theory
 - ★ ...

The new integral [AK08; AK10]: Idea

- A process Y^\cdot and filtration \mathcal{F}_\cdot are called **instantly independent** if Y^t and \mathcal{F}_t are independent $\forall t$.
- Ideas
 1. Decompose the integrand into **adapted** and **instantly independent** parts.
 2. Evaluate the **adapted** and the **instantly independent** parts at the **left** and right endpoints.
- Consider two continuous stochastic processes, X_t **adapted** and Y^t **instantly independent** w.r.t. \mathcal{F}_\cdot . Then the integral $\int_0^T X_t Y^t dB_t$ is **defined** as

$$\int_0^T X_t Y^t dB_t \triangleq \lim_{\|\Delta_n\| \rightarrow 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

provided that the limit exists in probability.

- Now, for any stochastic process $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.
- This is well-defined [HKS+16].

A simple example

- In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and \lim is the limit in L^2 .

$$\begin{aligned}\int_0^t B_T \, dB_t &= \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t \\ &= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\ &= \lim \sum_{j=0}^{n-1} (B_T - \Delta B_j) \Delta B_j \\ &= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t\end{aligned}$$

- Note that $\mathbb{E}(B_T B_t - t) = 0$.
- In general, $\mathbb{E} \int_0^t Z(t) \, dB_t = 0$. 😊

Generalized Itô formula [HKS+16]

- Let $dX_t = m(t) dt + \sigma(t) dB_t$ be an d -dimensional **Itô** process, $Y^t = \tilde{m}(t) dt + \tilde{\sigma}(t) dB_t$ be a \tilde{d} -dimensional instantly independent process, $f(x, y) \in C^2(\mathbb{R}^2)$. Then

$$\begin{aligned} df(X_t, Y^t) = & \langle (D_x f)(X_t, Y^t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D_x^2 f)(X_t, Y^t) dX_t \rangle \\ & + \langle (D_y f)(X_t, Y^t), dY^t \rangle - \frac{1}{2} \langle dY^t, (D_y^2 f)(X_t, Y^t) dY^t \rangle, \end{aligned}$$

where we use the rule $dB_t \otimes dB_t = I_d dt$.

- Example: TODO

Iterated integrals

Theorem ([Itô51]) Let $f \in L^2([0, T]^n)$ and \hat{f} be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = n! \int_0^T \dots \int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1},$$

Theorem ([AK10]) Let $f \in L^2([0, T]^n)$. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = \int_0^T \dots \int_0^T f(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1}.$$

Near-martingale property [HKS+17]

- Question: What are the analogues of the martingale property and the Markov property?
- Partial answer: near-martingales
- Let $Z(t)$ be a stochastic process such that $\mathbb{E} |Z(t)| < \infty \forall t$, and $0 \leq s \leq t \leq T$. Then, with respect to \mathcal{F}_\cdot , the process $Z(t)$ is called a
 - ★ near-martingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) = 0$,
 - ★ near-submartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \geq 0$, and
 - ★ near-supermartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \leq 0$.

§ 3

LARGE DEVIATIONS THEORY

Introduction

Weak convergence of measures

Laplace principle

Motivation: towards Cramér theorem

Setup

- (X_n) : i.i.d. random variables
- $X_n \sim \mu$ and $\mathbb{E}X_n = m$
- $\overline{S}_n = \frac{1}{n} \sum_{j=1}^n X_j$
- $B \in \mathcal{B}$ such that $m \notin \overline{B}$
- By LLN, we have $\overline{S}_n \rightarrow m$ as $n \rightarrow \infty$ (a.s.)
- So $\mathbb{P}(S_n \in B) \rightarrow 0$ as $n \rightarrow \infty$
- But at what speed?
- \asymp

Motivation: towards Cramér theorem

Motivation: towards Cramér theorem

Motivation: towards Cramér theorem

Cramér theorem

Theorem (Cramér, 1938) Let X_1, X_2, \dots be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example $\Lambda(t) < \infty \forall t \in \mathbb{R}$. Then the Legendre transform of Λ , $\Lambda^* = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$ satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \left(\sum_{i=1}^n X_i \geq nx \right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Cramér theorem applied to common distributions

Motivation: towards Cramér theorem

Sanov theorem

LD in ∞ -dimensions — Schilder theorem

Aim: Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path (the 0 function).

Setup

- Let B_\cdot be a d -dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- $\forall \varepsilon > 0$, let W_ε denote the law of $\sqrt{\varepsilon}B_\cdot$.
- Let $\text{CM} = \{\omega \in C_0 : \omega \in \text{AC}, \text{ and } \dot{\omega}_t \in L^2[0, T]\}$

Theorem On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \rightarrow \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 dt \right) \mathbb{1}_{\text{AC}}(\omega) + \infty \mathbb{1}_{\text{AC}^c}(\omega)$$

Freidlin–Wentzell theorem

Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

Setup

- Let B_\cdot be a d -dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- $\forall \varepsilon > 0$, let $X^{(\varepsilon)}$ be a \mathbb{R}^d -valued Itô diffusion solving an Itô SDE of the form

$$dX_t^{(\varepsilon)} = b(X_t^{(\varepsilon)}) dt + \sigma(X_t^{(\varepsilon)}) \sqrt{\varepsilon} dB_t, \quad X_0^{(\varepsilon)} = 0.$$

- $\forall \varepsilon > 0$, let W_ε denote the law of $X_\cdot^{(\varepsilon)}$.

Theorem (Freidlin, Wentzell (year?)) On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \rightarrow \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\dot{\omega}_t - b(\omega_t)|^2 dt \right) \mathbb{1}_{H^1([0, T]; \mathbb{R}^d)}(\omega) + \infty \mathbb{1}_{H^1([0, T]; \mathbb{R}^d)^c}(\omega)$$

§ 4

CONCLUSION

Open areas for research

- ★ Extension to SDEs with anticipating coefficients
- ★ Near-Markov property
- ★ Girsanov theorem for anticipating integrals
- ★ Freidlin-Wintzell type result for SDEs with anticipation

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

§ 5

SAMPLE SLIDES

Possible areas of interest

- ★ Extension to SDEs with anticipating coefficients
- ★ Near-Markov property
- ★ Girsanov theorem for generalized integration
- ★ Freidlin-Wintzell type result for SDEs with anticipating initial conditions

Theorem (Cramér, 1938) Let X_1, X_2, \dots be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example $\Lambda(t) < \infty \forall t \in \mathbb{R}$. Then the Legendre transform of Λ , $\Lambda^* = \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$ satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \left(\sum_{i=1}^n X_i \geq nx \right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Freidlin–Wentzell theorem

Column 1

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

Column 2

\times	dt	dB_t
dt	0	0
dB_t	0	dt

Something

☐ One

☐ Two

☐ Three

☐ Four

Bibliography

Ayed, W. & Kuo, H. H. (2008). An extension of the Itô integral. *Communications on Stochastic Analysis*, 2(3). doi:10.31390/cosa.2.3.05

_____ (2010). An extension of the Itô integral: Toward a general theory of stochastic integration. *Theory of Stochastic Processes*, 16(32), 17–28. Retrieved from <http://mi.mathnet.ru/thsp56>

Hwang, C. R., Kuo, H. H., Saitô, K., & Zhai, J. (2016). A general Itô formula for adapted and instantly independent stochastic processes. *Communications on Stochastic Analysis*, 10(3). doi:10.31390/cosa.10.3.05

_____ (2017). Near-martingale Property of Anticipating Stochastic Integration. *Communications on Stochastic Analysis*, 11(4). doi:10.31390/cosa.11.4.06

Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad.*, 20(8), 519–524. doi:10.3792/pia/1195572786

_____ (1951). Multiple Wiener Integral. *J. Math. Soc. Japan*, 3(1), 157–169. doi:10.2969/jmsj/00310157