CONTENTS

Introduction and revision

In the first two days we will review what we learned in Combinatorics 1 and try to solve interesting problems related to them.

The structure of the course is going to be as follows (the ordered pair represents (week, day)):

1. (week 1, day 1) Revision of counting
   a. what is combinatorics?
   b. addition and multiplication rules (+ proofs)
   c. permutations and combinations
   d. binomial and multinomial coefficients and theorem
   e. proofs using combinatorial methods
   f. proofs of basic results
2. (week 1, day 2)
   a. Multinomial type problems
   b. Revision of graph theory
   c.
   d.
3. (week 1, day 3)
4. (week 1, day 4)
5. (week 2, day 1)
6. (week 2, day 2)
7. (week 2, day 3)
8. (week 2, day 4)
9. (week 3, day 1)
10. (week 3, day 2)
11. (week 3, day 3)
12. (week 3, day 4)

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1.1 Combinatorics basics

1.1.1 What is combinatorics?

The study of
i.   *discrete structures*: graphs, strings, distributions, partitions
ii. *enumerations*: permutations, combinations, inclusion and exclusion, generating functions, recurrence relations
iii. *algorithms and optimization*: sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, shortest path, bipartite matching

We will try to focus on enumerations and discrete structures in this course. The algorithms can be interesting for the projects.

Why is it interesting?

- We live in a finite world, and every problem is essentially combinatorial in a sense.
- Which means it is ubiquitious in mathematics
- Gives us fun visual proofs of results that can be proved algebraic.
- Concerns with counting, which is very fundamental.
- ...

1.1.2 Enumerations

- Multiplication rule: if E and F are finite sets, then $|E \times F| = |E\|F|$.
  Number of ways of constructing a 100 character string out of the 26 letters of the English alphabet.
- Addition rule: if E and F are finite *disjoint* sets, then $|E \sqcup F| = |E| + |F|$.
  If there are two roads from Baton Rouge to New Orleans, and three roads from Baton Rouge to Lafayette, in how many ways can you go from Baton Rouge to either of the places?
- Factorials
  Numbers of ways to arrange 5 people in a row.
- Permutations
  Numbers of ways to arrange any 4 people in a row when there are 7 people.
- Combinations
  Numbers of ways to form a committee of 3 people from a group of 7 employees.
- Exercise: (<KT2016>), Exercise 9
  A database uses 20-character strings as record identifiers. The valid characters in these strings are upper-case letters in the English alphabet and decimal digits. (Recall there are 26 letters in the English alphabet and 10 decimal digits.) How many valid record identifiers are possible if a valid record identifier must meet *all* of the following criteria:

1. Letter(s) from the set $\{A, E, I, O, U\}$ occur in *exactly* three positions of the string.
2. The last three characters in the string are *distinct* decimal digits that do not appear elsewhere in the string.
3. The remaining characters of the string may be filled with any of the remaining letters or decimal digits.

Solution: $(10 \cdot 9 \cdot 8) \cdot \left(\binom{17}{3} \cdot 5^3\right) \cdot ((26 - 5) + (10 - 3))^{20-3-3}$.

- Binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

1.1.3 Combinatorial proofs are fun!

References: (`<KT2016>`) and (`<Nelsen1993>`).

1. sum of first n natural numbers: x3
2. sum of first n odd numbers: x2
3. $1 + 2 + \cdots + (n\text{-}1) + n + (n\text{-}1) + \cdots + 2 + 1$
4. $1 + 3 + \cdots + (2n\text{-}3) + (2n\text{-}1) + (2n\text{-}3) + \cdots + 2 + 1$
5. sum of binomial coefficients

1.1.4 Background of basic results

Proposition (Basic principle of counting)   *Suppose two independent experiments are performed, and there are m possible outcomes of the first experiment and n possible outcomes of the second experiment. Then the total possible outcomes of of the two experiments combined is mn.*

Proof   Let (i, j) denote the case when the first experiment gives the ith outcome and the second experiment gives the jth outcome. Enumerating, we get

$$
\begin{array}{cccc}
(1,1) & (1,2) & \dots & (1,n) \\
(2,1) & (2,2) & \dots & (2,n) \\
\vdots & \vdots & \ddots & \vdots \\
(m,1) & (m,2) & \dots & (m,n)
\end{array}
$$

Since there are m rows and n columns, we have total mn entries.   □

Remark   *This can be generalized to a finite number of experiments.*

Theorem (Binomial theorem)   *Let x and y be real numbers with x, y and x + y nonzero. Then for every non-negative integer n,*

$$
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
$$

Proof (Inductive)   Homework.   □

Proof (Combinatorial)   Consider the product $(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$.

First, note that the expansion consists of 2^n terms, each being a product of n factors.

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Secondly, each product contains either x_j xor y_j for each $j \in [n]$.

For example, $(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2$.

Now, we can we choose k of the x_js and $n - k$ of the y_js in $\binom{n}{k}$ ways, so there are precisely those many terms with mk x_js and $n - k$ y_js in the expansion.

Finally, letting $x_j = x$ and $y_j = y$ for each $j \in [n]$, we get the result. $\qquad\square$

Remark *This can be generalized to the multinomial theorem and multinomial coefficients, which we will revise if the need arises.*

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2.1 DISCRETE PROBABILITY SPACES

Notations

Term	Description	Symbol/Idea	Coin toss Example
sample space	set of outcomes	Ω	$\{H, T\}$
outcome	arbitrary outcome	$\omega \in \Omega$	H
event	subset of sample space	E	$\emptyset, \{H\}, \{T\}, \{H, T\}$
mutually exclusive events	events with empty intersection	$E_1 \cap E_2 = \emptyset$	$\{H\}$ and $\{T\}$
probability mass function	weightage of each outcome	$p : \Omega \to [0, 1]$, with $\sum_{\omega} p(\omega) = 1$	$p(H) = \frac{1}{3}, p(T) = \frac{2}{3}$
probability	(of an event)	$\mathbb{P} : 2^{\Omega} \to [0, 1]$, $\mathbb{P}(E) = \sum_{\omega \in \Omega} p(\omega)$	$\mathbb{P}(\emptyset) = 0, \mathbb{P}(\{H, T\}) = 1$
random variable	a function	$X : \Omega \to \mathbb{R}$	$X(H) = 1, X(T) = 0$

2.2 AXIOMATIC PROBABILITY THEORY

Definition (Probability axioms) *A non-negative valued function* $\mathbb{P}$ *defined on the set of events is called a* probability measure *if the following hold.*
1. *(null empty set)* $\mathbb{P}(\emptyset) = 0.$
2. *(countable additivity) For any sequence of mutually exclusive events* $E_1, E_2, \cdots$, *we have* $\mathbb{P}\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n).$
3. *(probability)* $\mathbb{P}(\Omega) = 1.$

Draw Venn diagrams for all of the following.

Proposition $\mathbb{P}(E^{\mathsf{C}}) = 1 - \mathbb{P}(E).$

Proof Since $E \cap E^{\mathsf{C}} = \emptyset$, by Axiom 2 we have $1 = \mathbb{P}(\Omega) = \mathbb{P}(E \sqcup E^{\mathsf{C}}) = \mathbb{P}(E) + \mathbb{P}(E^{\mathsf{C}}).$ □

Proposition *If* $E \subset F$, *then* $\mathbb{P}(E) \leq \mathbb{P}(F).$

Proof Note that $F = E \sqcup (F \setminus E)$. So by Axiom 2 we have $\mathbb{P}(F) = \mathbb{P}(E \sqcup (F \setminus E)) = \mathbb{P}(E) + \mathbb{P}(F \setminus E)$. Therefore, $\mathbb{P}(F) - \mathbb{P}(E) = \mathbb{P}(F \setminus E)$, which is non-negative since probability is a non-negative set function. □

Proposition (Inclusion-Exclusion) $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)..$

Proof
1. $E \cup F = (E \setminus F) \sqcup (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(E \cup F) = \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F).$
2. $E = (E \setminus F) \sqcup (E \cap F)$, so $\mathbb{P}(E) = \mathbb{P}(E \setminus F) + \mathbb{P}(E \cap F)$, and similarly
3. $F = (F \setminus E) \sqcup (E \cap F)$, so $\mathbb{P}(F) = \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F).$

Combining the above,

$$\begin{aligned}
\mathbb{P}(E \cup F) &= \mathbb{P}(E \setminus F) + \mathbb{P}(F \setminus E) + \mathbb{P}(E \cap F) \\
&= (\mathbb{P}(E) - \mathbb{P}(E \cap F)) + (\mathbb{P}(F) - \mathbb{P}(E \cap F)) + \mathbb{P}(E \cap F) \\
&= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).
\end{aligned}$$

□

Bibliography

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