Anticipating stochastic integrals and its large deviations

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Contents

Part 1

Anticipating integrals

1.1 Interpretation of exponential processes

In classical Itô theory, there are three equivalent interpretations of exponential processes.

i. renormalization
ii. martingale
iii. SDEs

This is not so in the two-sided stochastic integral theory.

1.2 Some examples of integrals

Remark 2.1 *For $p \in \mathbb{N}$, we have $\int_0^t W_T^p dW_t = W_T^p W_t - rtW_T^{p-1}$.*

Proof.

i. **From the definition**. In the following, lim denotes the limit in $L^2(\Omega)$. We use the quadratic variation of Wiener process, that is, $(\Delta_i W)^2 \to \Delta_i t$ and $(\Delta_i W)^{>2} \to 0$ in L^2.

$$
\begin{aligned}
I_t = \int_0^t W_T^p \mathrm{d}W_u &= \int_0^t (W_u + (W_T - W_u))^p \, \mathrm{d}W_u \\
&= L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} \left(W_{t_i} + \left(W_T - W_{t_{i+1}}\right)\right)^p (\Delta_i W) \\
&= L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} (W_T - \Delta_i W)^p \, (\Delta_i W) \\
&= L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} \sum_{k=0}^{p} \binom{p}{k} (-1)^k W_T^{p-k} \, (\Delta_i W)^{k+1} \\
&= \sum_{k=0}^{p} \left[\binom{p}{k} (-1)^k \cdot L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} W_T^{p-k} \, (\Delta_i W)^{k+1} \right] \\
&= W_T^p \cdot L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} (\Delta_i W) - W_T^{p-1} \cdot L^2 \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n-1} (\Delta_i W)^2 \\
&= W_T^p W_t - ptW_T^{p-1}.
\end{aligned}
$$

ii. **From the two-sided Itô formula**. Since we have to find $\int_0^t W_T^p \mathrm{d}W_t$, we guess the solution to be $W_t W_T^p$. Decomposing this into adapted and instantly independent parts, we get $W_t W_T^p = W_t \left(W_t + (W_T - W_t)\right)^p$. Let $X_t = W_t, Y^t = W_T - W_t$. Define $\phi(x,y) = x(x+y)^p$. We have the following partial derivatives.

$$
\begin{aligned}
\phi_x &= (x+y)^p + px(x+y)^{p-1} & \phi_y &= px(x+y)^{p-1} \\
\phi_{xx} &= p(x+y)^{p-1} + p(x+y)^{p-1} + p(p-1)x(x+y)^{p-2} & \phi_{yy} &= p(p-1)x(x+y)^{p-2}
\end{aligned}
$$

Using the two-sided Itô formula on $\phi(X_t, Y^t)$ and noting that $\mathrm{d}Y^t = -\mathrm{d}X_t$, we get

$$
\mathrm{d}(W_t W_T^p) = \mathrm{d}\phi(X_t, Y^t) = (X_t + Y^t)^p \mathrm{d}W_t + \frac{1}{2} \cdot 2p(X_t + Y^t)^{p-1} \mathrm{d}t = W_T^p \mathrm{d}W_t + pW_T^{p-1} \mathrm{d}t,
$$

which implies

$$
\int_0^t W_T^p \mathrm{d}W_s = W_t W_T^p - ptW_T^{p-1}.
$$

Remark 2.2 *For $p \in \mathbb{N}$, we have $\int_0^t W_T^p dW_t$ is a near-martingale.*

Proof. All we need to do is to show that $Z_t = \mathbb{E}(I_t \mid \mathscr{F}_t)$ is a martingale, where $I_t = W_t W_T^p - ptW_T^{p-1}$. For convenience, we write $Y_t = W_T - W_t$ (note the ad hoc change in notation). Now,

$$
I_t = W_t (W_t + Y_t)^p - pt(W_t + Y_t)^{p-1} = W_t \sum_{k=0}^{p} \binom{p}{k} W_t^{p-k} Y_t^k - pt \sum_{k=0}^{p-1} \binom{p-1}{k} W_t^{p-1-k} Y_t^k.
$$

Now, note that W_t is $\mathscr{F}_t$-measurable and Y_t is independent of $\mathscr{F}_t$, so

$$\mathbb{E}(Y_t^k \mid \mathscr{F}_t) = \mathbb{E}(Y_t^k) = (k-1)!!(T-t)^k \mathbb{1}_{k \text{ even}}$$

Now,

$$\begin{aligned}
Z_t &= \mathbb{E}(I_t \mid \mathscr{F}_t) \\
&= W_t \mathbb{E}(Y_t^p) + (W_t^2 - pt) \sum_{k=0}^{p-1} \left(\binom{p}{k} + \binom{p-1}{k} \right) W_t^{p-k-1} \mathbb{E}(Y_t^k) \\
&= W_t^2 \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2k} (2k-1)!!(T-t)^{2k} W_t^{p-2k-1} - pt \sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \binom{p-1}{2k} (2k-1)!!(T-t)^{2k} W_t^{p-2k-1}
\end{aligned}$$

ToDo

1.3 Miscellaneous

Example 3.1 *The process X given by $X_t = W_{\frac{1}{2}(t+T)} - W_t$ cannot be expressed as a Borel function of $W_T - W_t$.*

Part 2

Large deviations theory

2.1 Friedlin-Wentzell theorem

2.2 Friedlin-Wentzell theorem for anticipating initial condition with extension of filtration

Notation: In what follows, $T < \infty$ and $t \in [0, T]$.
Our aim is to formulate a large deviations principle for a stochastic differential equation with anticipating initial conditions. Consider a very simple case

$$X_t^\varepsilon = W_T + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) \mathrm{d}W_t, \quad t \in [0, T],$$

and σ satisfies the following:

- *bounded*: $|\sigma(x)| \le M_\sigma$.
- *Lipschitz*: $|\sigma(x) - \sigma(y)| \le L_\sigma |x - y|$.
- *linear growth*: $|\sigma(x)|^2 \le G_\sigma(1 + |x|^2)$.

We shall look at the method of enlargement of filtration by [<Itô1978>]. We denote the enlarged filtration by $\tilde{\mathscr{F}}_t = \mathscr{F}_t \vee \sigma(W_T)$. For $t \in [0, T]$, define the process $A_t = \int_0^t \frac{W_T - W_u}{T - u} \mathrm{d}u$. Then $W_t = \tilde{W}_t + A_t$, where $\tilde{W}_\cdot$ is a Wiener process w.r.t. $\tilde{\mathscr{F}}_\cdot$.

2.2.1 $\tilde{W}_\cdot$ is a Wiener process

We show that $(\tilde{W}_t)$ is a $(\tilde{\mathscr{F}}_t)$-martingale with quadratic variation t. Then by Lévy's Characterization of Wiener process, we obtain that $\tilde{W}_\cdot$ is a Wiener process.
First we prove two lemmas.

Lemma 2.1 *The σ-algebras $\mathscr{F}_s \vee \sigma(W_T)$ and $\mathscr{F}_s \vee \sigma(W_T - W_s)$ are the same.*

Proof. For any Borel set B, the set $\{W_T \in B\} = \{(W_T - W_t) + W_t \in B\}$.
TODO

Lemma 2.2 *For $0 \le s \le t \le T$, we have*

$$\mathbb{E}(W_t - W_s \mid W_T - W_s) = \frac{t - s}{T - s}(W_T - W_s).$$

Proof. We partition the interval $[0, T]$ into $n = kn_0$ equal parts, where $n_0 = (\min\{s, t - s, T - t\})^{-1}$ and $k \in \mathbb{N}$. Let $n_s = s\frac{n}{T}$ and $n_t = t\frac{n}{T}$. That is, the partition is

$$P = \left\{0, \frac{T}{n}, \dots, \frac{n_s T}{n} = s, \dots, \frac{n_t T}{n} = t, \dots, \frac{(n-1)T}{n}, T\right\}.$$

Let $\Delta_i W = W_{\frac{(i+1)T}{n}} - W_{\frac{iT}{n}}$.
Firstly, note that the $\Delta_i W$s are independent and identically distributed from the definition of Wiener process. Now, using the linearity of conditional expectation, we have

$$
\begin{aligned}
\mathbb{E}(W_t - W_s \mid W_T - W_s) &= \mathbb{E}\left(\sum_{i=n_s}^{n_t-1} \Delta_i W \mid \sum_{i=n_s}^{n_t-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \mathbb{E}\left(\Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \mathbb{E}\left(\sum_{i=n_s}^{n-1} \Delta_i W \mid \sum_{i=n_s}^{n-1} \Delta_i W\right) \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} \sum_{i=n_s}^{n-1} \Delta_i W \\
&= \sum_{i=n_s}^{n_t-1} \frac{1}{n-n_s} (W_T - W_s) \\
&= \frac{n_t - n_s}{n - n_s}(W_T - W_s) \quad = \quad \frac{t-s}{T-s}(W_T - W_s).
\end{aligned}
$$

Proposition 2.3 *$\tilde{W}_\cdot$ is a $\tilde{\mathscr{F}}_\cdot$-martingale.*

Proof. Let $0 \le s \le t \le T$. Then

$$
\tilde{W}_t - \tilde{W}_s = (W_t - W_s) - \int_s^t \frac{W_T - W_u}{T-u} \mathrm{d}u = (W_t - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{W_u - W_s}{T-u}\right) \mathrm{d}u.
$$

Moreover, since $W_t - W_s$ is independent of $\mathscr{F}_s$ for every $t \ge s$, using lemmas 2.1 and 2.2, we get

$$
\mathbb{E}(W_t - W_s \mid \tilde{\mathscr{F}}_s) = \mathbb{E}\left(W_t - W_s \mid \mathscr{F}_s \vee \sigma(W_T - W_s)\right) = \mathbb{E}\left(W_t - W_s \mid W_T - W_s\right) = \frac{t-s}{T-s}(W_T - W_s).
$$

Therefore, using the fact that W_T and W_s are $\tilde{\mathscr{F}}_s$-measurable with conditional Fubini's theorem, we get

$$
\begin{aligned}
\mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) &= \mathbb{E}\left(W_t - W_s \mid \tilde{\mathscr{F}}_s\right) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{\mathbb{E}\left(W_u - W_s \mid \tilde{\mathscr{F}}_s\right)}{T-u}\right) \mathrm{d}u \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \left(\frac{W_T - W_s}{T-u} - \frac{u-s}{T-s}\frac{W_T - W_s}{T-u}\right) \mathrm{d}u \\
&= \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \frac{W_T - W_s}{T-s} \mathrm{d}u \\
&= \frac{t-s}{T-s}(W_T - W_s) - \frac{t-s}{T-s}(W_T - W_s) \quad = \quad 0.
\end{aligned}
$$

Now, since $\tilde{W}_s$ is $\tilde{\mathscr{F}}_s$-measurable, $\mathbb{E}(\tilde{W}_t \mid \tilde{\mathscr{F}}_s) = \mathbb{E}(\tilde{W}_t - \tilde{W}_s \mid \tilde{\mathscr{F}}_s) + \tilde{W}_s = \tilde{W}_s$.

Proposition 2.4 *The quadratic variation of $\tilde{W}_t$ is t.*

Proof. TODO

2.2.2 Reformulating the problem

For now, we shall bound $t \in [0, T_b]$, where $T_b \in [0, T)$. What happens when $t \to T$?
Using this, we write our original stochastic differential equation as

$$
X_t^\varepsilon = W_T + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) \mathrm{d}\tilde{W}_t + \sqrt{\varepsilon} \int_0^t \sigma(X_t^\varepsilon) \frac{W_T - W_s}{T-s} \mathrm{d}s.
$$

Let $\tilde{X}_t^\varepsilon = X_t^\varepsilon - W_T$ and $Y_t^\varepsilon = \sqrt{\varepsilon}(W_T - W_t)$. Then we have

$$\tilde{X}_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \mathrm{d}\tilde{W}_t + \int_0^t \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \frac{Y_s^\varepsilon}{T-s}\mathrm{d}s, \text{ and}$$
$$Y_t^\varepsilon = \sqrt{\varepsilon}W_T - \sqrt{\varepsilon}\int_0^t \mathrm{d}\tilde{W}_t - \int_0^t \frac{Y_s^\varepsilon}{T-s}\mathrm{d}s.$$

So together we have the joint process

$$Z_t^\varepsilon := \begin{pmatrix} \tilde{X}_t^\varepsilon \\ Y_t^\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{\varepsilon}W_T \end{pmatrix} + \sqrt{\varepsilon}\int_0^t \begin{pmatrix} \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \\ -1 \end{pmatrix} \mathrm{d}\tilde{W}_s + \int_0^t \begin{pmatrix} \sigma\left(\tilde{X}_t^\varepsilon + \frac{Y_0^\varepsilon}{\sqrt{\varepsilon}}\right) \\ -1 \end{pmatrix} \frac{Y_s^\varepsilon}{T-s}\mathrm{d}s. \tag{2.1}$$

Note that we expect $Z_t^\varepsilon \to 0$ as $\varepsilon \searrow 0$. We first obtain a large deviation principle for Z_t^ε.

2.2.3 Existence and uniqueness of Z^ε in equation (2.1)

First we aim for local existence. Fix $R > 0$, and define the exit time from the R-ball centered at the origin as

$$\tau_R = \inf\{t : Y_0^\varepsilon > R\} \wedge T_b.$$

Clearly $\tau_R \nearrow T_b$ as $R \nearrow \infty$.
We now show that that there is a unique solution of Z_t^ε on $t \in [0, \tau_R]$. For convenience, let

$$\tilde{\sigma}_{t,v}^\varepsilon(x,y) = \left(\sigma\left(x + \frac{v}{\sqrt{\varepsilon}}\right), -1\right) \qquad \text{and} \qquad \tilde{b}_{t,v}^\varepsilon(x,y) = \frac{y}{T-s}\left(\sigma\left(x + \frac{v}{\sqrt{\varepsilon}}\right), -1\right).$$

Lemma 2.5 *The process Z_t^ε exists and is unique for $t \in [0, \tau_R]$.*

Proof. Let $v = Y_0^\varepsilon$. We first show that $\tilde{\sigma}$ and $\tilde{b}$ satisfy the linear growth and Lipshitz conditions locally.

- *Lipschitz condition for $\tilde{\sigma}$:* Since σ is Lipschitz, we have

$$\begin{aligned}\left\|\tilde{\sigma}_{t,v_2}^\varepsilon(x_2,y_2) - \tilde{\sigma}_{t,v_1}^\varepsilon(x_1,y_1)\right\| &= \left|\sigma\left(x_2 + \frac{v_2}{\sqrt{\varepsilon}}\right) - \sigma\left(x_1 + \frac{v_1}{\sqrt{\varepsilon}}\right)\right| \\ &\le L_\sigma\left(|x_2 - x_1| + \frac{1}{\sqrt{\varepsilon}}|v_2 - v_1|\right) \\ &\le L_\sigma\left(1 \vee \frac{1}{\sqrt{\varepsilon}}\right)(|x_2 - x_1| + |v_2 - v_1|).\end{aligned}$$

- *Lipschitz condition for $\tilde{b}$:* Using the boundedness of σ, we get

$$\begin{aligned}&\left\|\tilde{b}_{t,v_2}^\varepsilon(x_2,y_2) - \tilde{b}_{t,v_1}^\varepsilon(x_1,y_1)\right\| \\ &\le \frac{1}{T-t}\left(\left|\sigma\left(x_2 + \frac{v_2}{\sqrt{\varepsilon}}\right)y_2 - \sigma\left(x_1 + \frac{v_1}{\sqrt{\varepsilon}}\right)y_1\right| + |y_2 - y_1|\right) \\ &\le \frac{1}{T-t}\left(\left|\sigma\left(x_2 + \frac{v_2}{\sqrt{\varepsilon}}\right)\right||y_2 - y_1| + \left|\sigma\left(x_2 + \frac{v_2}{\sqrt{\varepsilon}}\right) - \sigma\left(x_1 + \frac{v_1}{\sqrt{\varepsilon}}\right)\right||y_1| + |y_2 - y_1|\right) \\ &\le \frac{1}{T-t}\left(M_\sigma|y_2 - y_1| + L_\sigma\left(1 \vee \frac{1}{\sqrt{\varepsilon}}\right)|y_1|(|x_2 - x_1| + |v_2 - v_1|) + |y_2 - y_1|\right) \\ &\le \frac{1}{T-t}\left((M_\sigma + 1) \vee \left(L_\sigma\left(1 \vee \frac{1}{\sqrt{\varepsilon}}\right)R\right)\right)(|x_2 - x_1| + |y_2 - y_1| + |v_2 - v_1|),\end{aligned}$$

  where $|y_1| \le R$ since $t \in [0, \tau_R]$.

- *Linear growth condition for $\tilde{\sigma}$:*

$$
\begin{aligned}
\left\|\tilde{\sigma}^{\varepsilon}_{t,\upsilon}(x,y)\right\|^2 &= 1 + \left|\sigma\left(x + \frac{\upsilon}{\sqrt{\varepsilon}}\right)\right|^2 \\
&\le 1 + G_\sigma\left(1 + \left|x + \frac{\upsilon}{\sqrt{\varepsilon}}\right|^2\right) \\
&\le 1 + G_\sigma\left(1 + 2|x|^2 + 2\frac{|\upsilon|^2}{\varepsilon}\right) \\
&\le 2G_\sigma\left(1 \vee \varepsilon^{-1}\right)\left(1 + |x|^2 + |\upsilon|^2\right).
\end{aligned}
$$

- *Linear growth condition for* $\tilde{b}$:

$$
\begin{aligned}
\left\|\tilde{b}^{\varepsilon}_{t,\upsilon}(x,y)\right\|^2 &= 1 + \left|\sigma\left(x + \frac{\upsilon}{\sqrt{\varepsilon}}\right)\frac{y}{T-t}\right|^2 \\
&\le \frac{1}{T-t} 2G_\sigma\left(1 \vee \varepsilon^{-1}\right) R^2 \left(1 + |x|^2 + |\upsilon|^2\right).
\end{aligned}
$$

The above implies that Z^{ε}_t exists and is unique for $t \in [0, \tau_R]$.

Now, let η be a smooth function such that $\eta(x) \equiv 1$ for $|x| \le R$ and $\eta \equiv 0$ for $|x| > R + 1$. Define $\hat{\sigma}_R(z) = \eta(z)\tilde{\sigma}^{\varepsilon}_{t,\upsilon}(z)$ and $\hat{b}_R(z) = \eta(z)\tilde{b}^{\varepsilon}_{t,\upsilon}(z)$. Now consider the equation

$$
\hat{Z}^{\varepsilon}_t = Z_0
$$

Bibliography