

A generalization of Itô calculus and large deviations theory

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SECTION 1

INTRODUCTION AND MOTIVATION

Quick revision and notations

- ▷ Let $T \in (0, \infty)$, and denote $t \in [0, T]$.
- ▷ Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$ be a filtered probability space.
- ▷ B_\bullet is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbb{P})$.
- ▷ Properties of B_\bullet
 - starts at 0
 - has independent increments
 - $B_t - B_s \sim \mathcal{N}(0, t - s)$
 - continuous paths (a.s.)
 - is a.s. nowhere differentiable
 - has **unbounded linear variation** ☹️
 - has **bounded quadratic variation** 😊
 - $\mathbb{E}(B_t B_s) = s \wedge t$
 - martingale
- ▷ **Naive integration w.r.t. B_t : not possible.**
- ▷ A stochastic process X_\bullet is called **adapted** to \mathcal{F}_\bullet if X_t is measurable w.r.t. $\mathcal{F}_t \forall t$.

Martingales and Markov processes

Definition Let X_\bullet is a L^1 -bounded \mathcal{F}_\bullet -adapted stochastic process, and let $0 \leq s \leq t \leq T$. Then X_\bullet is called a

- martingale if $\mathbb{E}(X_t | \mathcal{F}_s) = X_s$,
- submartingale if $\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s$, and
- supermartingale if $\mathbb{E}(X_t | \mathcal{F}_s) \leq X_s$.

Definition A stochastic process X_\bullet is called **Markov** if for any $0 \leq s \leq t \leq T$, we have

$$\mathbb{P}(X_t \in E | \mathcal{F}_s) = \mathbb{P}(X_t \in E | X_s).$$

Remark If (X_n) is a discrete-time martingale and (H_n) is an adapted process, then the process $Y_n = \sum_{j=0}^{n-1} H_j(X_{j+1} - X_j) =: (H \bullet X)_n$ is itself a martingale and called a **martingale transform** of (X_n) .

Wiener integral for $f \in L^2[0, T]$

▷ Definition of the integral:

1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t) \, dB_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
2. $f \in L^2[0, T]$: Use step functions approximating f to extend the integral a.s.

▷ Properties of the integral:

- Linear.
- **Gaussian distribution** with mean 0 and variance $\|f\|_{L^2[0, T]}^2$ (Itô isometry).
- Agrees with the Riemann–Stieltjes integral for continuous functions of bounded variation.

▷ Properties of the associated process $I_\bullet = \int_0^\bullet f(t) \, dB_t$:

- continuity.
- martingale.

▷ Problem: Cannot integrate stochastic processes.

Trying to integrate stochastic processes

▷ $\int_0^T B_t \, dB_t \stackrel{?}{=}$

Since B_t is continuous, let us try Riemann–Stieltjes integral. Consider a sequence of partitions Δ_n such that $\|\Delta_n\| \rightarrow 0$. Then

$$\int_0^T B_t \, dB_t = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} B_{t_j^*} \Delta B_j.$$

▷ Choosing different endpoints for t_j^* gives us different results.

t_j^*	$\int_0^t B_s \, dB_s$	Intuitive?	\mathbb{E}	Martingale?	Theory
left	$\frac{1}{2} (B_t^2 - t)$	☹	0	☺	Itô
mid	$\frac{1}{2} (B_t^2)$	☺	$\frac{1}{2}t$	☹	Stratonovich
right	$\frac{1}{2} (B_t^2 + t)$	☹	t	☹	

▷ Which one do we choose?

Itô integral [Itô44] for $X_{\bullet} \in L^2_{\text{ad}}([0, T] \times \Omega)$

▷ Definition of the integral:

1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \tilde{\zeta}_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, dB_t = \sum_{j=0}^{n-1} \tilde{\zeta}_j \Delta B_j$.
2. $X \in L^2_{\text{ad}}([0, T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

▷ Properties of the integral:

- Linear.
- Mean 0 and variance $\|f\|_{L^2[0, T]}^2$ (Itô isometry).
- For X_{\bullet} continuous, $\int_0^T X_t \, dB_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, dB_t = \lim \sum_{j=0}^{n-1} X_{t_j} \Delta B_j$.

▷ Properties of the associated process $I_{\bullet} = \int_0^{\bullet} X_t \, dB_t$:

- continuity.
- martingale.

▷ Example: $\int_0^t B_u \, dB_u = \frac{1}{2}(B_t^2 - t) \quad \forall t$.

Itô integral for X_\cdot such that $\int_0^T X_t^2 dt < \infty$ a.s.

- ▷ Definition: Use sequences of processes in $L_{\text{ad}}^2([0, T] \times \Omega)$ approximating X in probability to extend the integral in probability.
- ▷ Properties of the integral:
 - Linear.
 - Mean and variance? ☹
- ▷ Properties of the associated process $I_\cdot = \int_0^\cdot X_t dB_t$:
 - continuity.
 - local martingale.

The Itô formula

- ▷ An Itô process is a process of the form $X_\bullet = X_0 + \int_0^\bullet m_t \, dt + \int_0^\bullet \sigma_t \, dB_t$.
Equivalently expressed as $dX_t = m_t \, dt + \sigma_t \, dB_t$.

Theorem ([Itô44]) Let X_t be a d -dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d -dimensional Itô process, and

$$df(X_t) = \langle (Df)(X_t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D^2 f)(X_t) dX_t \rangle,$$

where we use the rule $dB_t \otimes dB_t = I_d \, dt$.

- ▷ Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, $d\mathcal{E}_t = -\frac{1}{2}\sigma^2 \mathcal{E}_t \, dt + \sigma \mathcal{E}_t \, dB_t + \frac{1}{2}\sigma^2 \mathcal{E}_t (dB_t)^2$.

Exponential processes and the Girsanov theorem

- ▷ Let h_\bullet be a stochastic process. The associated exponential process is defined as

$$\mathcal{E}_\bullet^{(h)} = \exp \left(\int_0^\bullet h_t \, dB_t - \frac{1}{2} \int_0^\bullet h_t^2 \, dt \right).$$

- ▷ The exponential process is a martingale if and only if $\mathbb{E} \mathcal{E}_t^{(h)} = 1 \, \forall t$.
- ▷ The Novikov condition: The exponential process is a martingale if $\mathbb{E} \exp \left(\frac{1}{2} \int_0^T h_t^2 \, dt \right) < \infty$.
- ▷ The Girsanov theorem [Gir60]: The translated stochastic process $W_\bullet = B_\bullet + \int_0^\bullet h_t \, dt$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(h)}$.

Stochastic differential equations

- ▷ Let $\zeta \in L^2(\Omega)$ be independent of B_\cdot , and $m, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ have ‘nice’ measurability. Then a \mathcal{F}_t -adapted stochastic process X_t is called a solution of the **stochastic *integral* equation** $X_t = \zeta + \int_0^t m(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$ if for each t , the X_t satisfies the integral equation a.s.
- ▷ Stochastic *differential* equation $dX_t = m(t, X_t) dt + \sigma(t, X_t) dB_t$, $X_0 = \zeta$ is a *formal representation*.

Theorem (Existence and uniqueness, Markov property) The SDE above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

- (Lipschitz condition) $|m(t, x) - m(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq M|x - y|^2$ a.s.
- (growth condition) $|m(t, x)|^2 + |\sigma(t, y)|^2 \leq M(1 + |x|^2)$ a.s.

The solution is a Markov process.

Moreover if $\zeta \in \mathbb{R}$ and m, σ are function of only x , then the solution is also stationary.

- Example: For σ constant, $d\mathcal{E}_t = \sigma \mathcal{E}_t dB_t$, $\mathcal{E}_0 = 1$ is solved by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

- ▷ How do we define the double integral?
- ▷ Naive idea: $\int_0^t \int_0^t dB_u dB_v = \int_0^t dB_u \int_0^t dB_v = B_t^2$.
But $\mathbb{E}B_t^2 = t \neq 0$, so **no martingale property**. ☹
- ▷ Itô's idea: remove the diagonal to get

$$\int_0^t \int_0^t dB_u dB_v = 2 \int_0^t \int_0^v dB_u dB_v = 2 \int_0^t B_v dB_v = B_t^2 - t.$$

Theorem ([Itô51]) Let $f \in L^2([0, T]^n)$ and \hat{f} be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) dB_{t_1} \cdots dB_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) dB_{t_n} \right) dB_{t_{n-1}} \cdots dB_{t_1}.$$

SECTION 2

A GENERALIZATION OF ITÔ CALCULUS

Motivation

- ▷ Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T dB_s dB_t = \int_0^T B_T dB_t \stackrel{?}{=} B_T^2$.
- ▷ Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so **no martingale property** ☹.
- ▷ Stochastic differential equations with anticipation:

$$dX_t = X_t dB_t$$

$$X_0 = B_1$$

$$dY_t = B_T dB_t$$

$$Y_0 = 1$$

- ▷ Problem: We want to define $\int_0^T Z(\cdot) dB_t$, where $Z(\cdot)$ is not (necessarily) adapted.
- ▷ Some approaches:
 - Enlargement of filtration $\mathcal{G}_\cdot = \mathcal{F}_\cdot \vee B_1$, with Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} ds \right) + \int_0^t \frac{B_T - B_s}{T-s} ds$.
 - White noise theory
 - Malliavin calculus

The new integral [AK08; AK10]: Idea

- A process Y^\bullet and filtration \mathcal{F}_\bullet are called **instantly independent** if Y^t and \mathcal{F}_t are independent $\forall t$.
Example: The process $(B_T - B_\bullet)$ is instantly independent of the filtration generated by B_\bullet .
- Ideas
 1. Decompose the integrand into **adapted** and **instantly independent** parts.
 2. Evaluate the **adapted** and the **instantly independent** parts at the **left** and **right** endpoints.
- Consider two continuous stochastic processes, X_t **adapted** and Y^t **instantly independent** w.r.t. \mathcal{F}_\bullet . Then the integral $\int_0^T X_t Y^t dB_t$ is **defined** as

$$\int_0^T X_t Y^t dB_t \triangleq \lim_{\|\Delta_n\| \rightarrow 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

provided that the limit exists in probability.

- Now, for any stochastic process $Z(t) = \sum_{k=1}^n X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.
- This is well-defined [HKS+16].

A simple example

▷ In the following, \lim is the limit in L^2 .

$$\begin{aligned}\int_0^t B_T \, dB_t &= \int_0^t (B_t + (B_T - B_t)) \, dB_t = \int_0^t B_t \, dB_t + \int_0^t (B_T - B_t) \, dB_t \\ &= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\ &= \lim \sum_{j=0}^{n-1} (B_T - \Delta B_j) \Delta B_j \\ &= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t\end{aligned}$$

▷ Note that $\mathbb{E}(B_T B_t - t) = 0$.

▷ In general, $\mathbb{E} \int_0^t Z(s) \, dB_s = 0$. 😊

The near-martingale property

- ▷ Question: What are the analogues of the martingale property and the Markov property?
- ▷ Answer for martingales: “near-martingales” [KSS12b].
- ▷ Example: $\mathbb{E}(B_T B_t - t \mid \mathcal{F}_s) = B_s^2 - s \neq B_T B_s - s$, but $\mathbb{E}(B_T B_s - s \mid \mathcal{F}_s) = B_s^2 - s$. ☺
- ▷ Let $Z(t)$ be a process such that $\mathbb{E} |Z(t)| < \infty \forall t$, and $0 \leq s \leq t \leq T$. Then $Z(t)$ is called a
 - near-martingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) = \mathbb{E}(Z(s) \mid \mathcal{F}_s)$,
 - near-submartingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \geq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$, and
 - near-supermartingale if $\mathbb{E}(Z(t) \mid \mathcal{F}_s) \leq \mathbb{E}(Z(s) \mid \mathcal{F}_s)$.

Theorem ([HKS+17]) Let $Z(\cdot)$ be a stochastic process bounded in L^1 , and $X_\cdot = \mathbb{E}(Z(\cdot) \mid \mathcal{F}_\cdot)$. Then X_\cdot is a (sub/super)martingale if and only if $Z(\cdot)$ is a near-(sub/super)martingale.

Theorem ([KSS12b]) Let f, ϕ be continuous function on \mathbb{R} . Under integrability conditions, the processes $X_\cdot = \int_0^\cdot f(B_t) \phi(B_T - B_t) dB_t$ and $Y^\bullet = \int_\cdot^T f(B_t) \phi(B_T - B_t) dB_t$ are near-martingales.

A generalized Itô formula [HKS+16]

Process	Definition	Representation
Itô	$X_t = X_0 + \int_0^t m_s \, ds + \int_0^t \sigma_s \, dB_s$	$dX_t = m_t \, dt + \sigma_t \, dB_t$
instantly independent	$Y^t = Y^0 + \int_t^T \eta^s \, ds + \int_t^T \zeta^s \, dB_s$	$dY^t = -\eta^t \, dt - \zeta^t \, dB_t$

Here η^t and ζ^t are instantly independent such that Y^t is also instantly independent.

Theorem ([HKS+16]) Let $dX_t = m_t \, dt + \sigma_t \, dB_t$ be an d -dimensional Itô process, and $dY^t = -\eta^t \, dt - \zeta^t \, dB_t$ be a \tilde{d} -dimensional instantly independent process. If $f(x, y) \in C^2(\mathbb{R}^2)$, then

$$\begin{aligned}
 df(X_t, Y^t) = & \langle (D_x f)(X_t, Y^t), dX_t \rangle + \frac{1}{2} \langle dX_t, (D_x^2 f)(X_t, Y^t) dX_t \rangle \\
 & + \langle (D_y f)(X_t, Y^t), dY^t \rangle - \frac{1}{2} \langle dY^t, (D_y^2 f)(X_t, Y^t) dY^t \rangle,
 \end{aligned}$$

where we use the rules $dB_t \otimes dB_t = I_d \, dt$.

Iterated integrals

Theorem ([Itô51]) Let $f \in L^2([0, T]^n)$ and \hat{f} be its symmetrization. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = n! \int_0^T \dots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, \dots, t_n) \, dB_{t_n} \right) \, dB_{t_{n-1}} \dots dB_{t_1}.$$

Theorem ([AK10]) Let $f \in L^2([0, T]^n)$. Then

$$\int_{[0, T]^n} f(t_1, \dots, t_n) \, dB_{t_1} \dots dB_{t_n} = \int_0^T \dots \int_0^T f(t_1, \dots, t_n) \, dB_{t_n} \dots dB_{t_1}.$$

Example[HKS+16]: For the new integral, $\int_0^T \left(\int_0^T B_u \, du \right) \, dB_v = \int_0^T \left(\int_0^T B_u \, dB_v \right) \, du$.

Generalization of Itô isometry

Theorem ([KSS12b]) Let ϕ be an analytic function on \mathbb{R} . Then under integrability conditions and for each t ,

$$\mathbb{E} \left[\left(\int_0^t \phi(B_T - B_s) \, dB_s \right)^2 \right] = \int_0^t \mathbb{E} \left[(\phi(B_T - B_s))^2 \right] \, ds$$

Theorem ([KSS13]) Let f, ϕ be C^1 function on \mathbb{R} . Then

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T f(B_t) \phi(B_T - B_t) \, dB_t \right)^2 \right] &= \int_0^T \mathbb{E} \left[(f(B_t) \phi(B_T - B_t))^2 \right] \, dt \\ &\quad + 2 \int_0^T \int_0^t \mathbb{E} \left[f(B_s) \phi'(B_T - B_s) f'(B_s) \phi(B_T - B_s) \right] \, ds \, dt. \end{aligned}$$

A generalization of Girsanov theorem

Theorem ([KPS13]) Let X_\bullet and Y^\bullet be continuous square-integrable stochastic processes such that X_\bullet is adapted and Y^\bullet is instantly independent.

Then translated stochastic process $W_\bullet = B_\bullet + \int_0^\bullet (X_t + Y^t) dt$ is a near-martingale under the probability measure $\widetilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(X+Y)}$.

SECTION 3

LARGE DEVIATIONS THEORY

What is it about?

- ▷ A theory to find probabilities of rare events that decay exponentially.
- ▷ Started by Swedish actuarials Fredrik Esscher, Harald Cramér, Filip Lundberg.
- ▷ Unified by Varadhan in his 1966 paper [[Var66](#)].
- ▷ Example: A problem faced by the insurance industry.
 - Value of claims received on the n th day: X_n \$.
 - Steady income from premium: x \$/day.
 - Planning period: n days.
 - Average expenditure: $\bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j$ \$/day.
 - *Question*: How should the company decide on the premium?
 - *Idea*: Determine x such that $\mathbb{P} \{ \bar{X}_n > x \} < \varepsilon$ (specified).

Insurance problem: setup

1. Let the following hold:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- (X_n) is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moment generating function M .
- $\mathbb{E}X_1 = m$, $\mathbb{V}X_1 = \sigma^2$, and $X_1 \sim \mu$.
- $\bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j$.

2. Asymptotic behavior of \bar{X}_n :

- Weak law of large numbers: $\bar{X}_n \xrightarrow{\mathbb{P}} m$.
- Central limit theorem: $\sqrt{n}(\bar{X}_n - m) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$.

3. What is the rate for LLN?

Insurance problem: large deviation bounds

1. For $x > m$ and an arbitrary $\theta > 0$, we get

$$\mathbb{P} \left\{ \bar{X}_n \geq x \right\} = \mathbb{P} \left\{ e^{\theta n \bar{X}_n} \geq e^{\theta n x} \right\} \leq e^{-\theta n x} \mathbb{E} \left(e^{\theta n \bar{X}_n} \right) = e^{-\theta n x} M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}.$$

2. Since θ was arbitrary, we have

$$\mathbb{P} \left\{ \bar{X}_n \geq x \right\} \leq \inf_{\theta} e^{-n(\theta x - \log M_X(\theta))} = e^{-n \sup_{\theta} (\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim} \frac{1}{n} \log \mathbb{P} \left\{ \bar{X}_n \in F \right\} \leq - \inf_F I \quad \forall F \text{ closed.}$$

4. We can also obtain a large deviation lower bound using an exponential change of measure

$$\underline{\lim} \frac{1}{n} \log \mathbb{P} \left\{ \bar{X}_n \in G \right\} \geq - \inf_G I \quad \forall G \text{ open.}$$

5. We informally write $\mathbb{P} \left\{ \bar{X}_n \in dx \right\} \asymp e^{-nI(x)} dx$ for $x \in \mathbb{R}$.

Definition of large deviation principle

- ▷ The setup: (X_n) is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space (\mathcal{X}, d) .
- ▷ A function $I : \mathcal{X} \rightarrow [0, \infty]$ is called a **rate function** if it has compact level sets.

Definition (X_n) is said to satisfy the **large deviation principle** on \mathcal{X} with rate function I if the large deviation **upper** and **lower** bounds hold.

Example

Theorem ([Cra38]) Let (X_n) be a sequence of i.i.d. real random variables with finite moment generating function M . Then (X_n) follows large deviation principle with rate function $I(x) = \sup_{\theta} (\theta x - \log M(\theta))$.

Applications of the Cramér theorem

Rate functions for some common distributions

Distribution	$M(\theta)$	$I(x)$
Bernoulli(p)	$1 - p + pe^\theta$	$\left(x \log x + (1 - x) \log(1 - x) - \left(x \log \frac{1 - p}{p} + \log p \right) \right) \mathbb{1}_{[0,1]}(x) + \infty \mathbb{1}_{[0,1]^c}(x)$
Poisson(λ)	$e^{\lambda(e^\theta - 1)}$	$\left(\lambda - x + x \log \frac{x}{\lambda} \right) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
Exp(λ)	$\left(1 - \frac{\theta}{\lambda} \right)^{-1}$	$(\lambda x - 1 + x \log(\lambda x)) \mathbb{1}_{[0,\infty)}(x) + \infty \mathbb{1}_{(-\infty,0)}(x)$
$\mathcal{N}(m, \sigma^2)$	$e^{m\theta + \frac{1}{2}\sigma^2\theta^2}$	$\frac{(x - m)^2}{2\sigma^2}$
$\chi^2(k)$	$(1 - 2\theta)^{-\frac{k}{2}}$	$\frac{1}{2} \left(x - k + k \log \frac{k}{x} \right)$

The Schilder theorem

- ▷ Aim: Estimate the probability that a scaled-down sample path of a **Brownian motion** will stray far from the mean path.
- ▷ Let C_x denote the set of continuous functions from $[0, T]$ to \mathbb{R}^d starting at x , and let $\text{CM}_x = \{\omega \in C_x : \omega \text{ is absolutely continuous and } \omega'_t \in L^2[0, T]\}$.
- ▷ Let B_\cdot be a d -dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0, T]; \mathbb{R}^d)$
- ▷ Let $\frac{1}{\sqrt{n}}B_t \sim W^{(n)}$. Then $W^{(n)} = \mathcal{N}\left(0, \frac{t}{n}\right) \xrightarrow{\mathcal{D}} \delta_0$ as $n \rightarrow \infty$.

Theorem ([Sch66]) On the Banach space $(C_0, \|\cdot\|_\infty)$, the sequence of probability measures $(W^{(n)})$ satisfies LDP with the rate function $I : C_0 \rightarrow \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2} \int_0^T |\omega'_t|^2 dt \right) \mathbb{1}_{\text{CM}_0}(\omega) + \infty \mathbb{1}_{\text{CM}_0^c}(\omega)$$

The Freidlin–Wentzell theorem

- ▷ Aim: Estimate the probability that a scaled-down sample path of an **Itô diffusion** will stray far from the mean path.
- ▷ Let $X_{\bullet}^{(n)}$ be the solution of the d -dimensional stochastic differential equation $\mathrm{d}X_t^{(n)} = m(X_t^{(n)}) \mathrm{d}t + \frac{1}{\sqrt{n}} \sigma(X_t^{(n)}) \mathrm{d}B_t$, $X_0^{(n)} = x$, where m, σ are sufficiently nice.
- ▷ Let $W_x^{(n)}$ denote the law of $X_{\bullet}^{(n)}$ starting at x .
- ▷ As $n \rightarrow \infty$, $W_x^{(n)} \xrightarrow{\mathcal{D}} \delta_{\tilde{\zeta}}$, where $\tilde{\zeta}$ solves the ODE $\dot{\tilde{\zeta}}(t) = m(\tilde{\zeta}(t))$, $\tilde{\zeta}(0) = x$.

Theorem ([FW12]) For any fixed x , the sequence of probability measures $(W_x^{(n)})$ satisfies LDP with the rate function $I_x : C_0 \rightarrow \overline{\mathbb{R}}$ given by

$$I_x(\omega) = \left(\frac{1}{2} \int_0^T \langle \omega'_t - b(\omega_t), A^{-1}(\omega_t) (\omega'_t - b(\omega_t)) \rangle \mathrm{d}t \right) \mathbb{1}_{\mathrm{CM}_x}(\omega) + \infty \mathbb{1}_{\mathrm{CM}_x^c}(\omega),$$

where $A = \sigma \sigma^*$.

SECTION 4

THE WAY FORWARD

Possible research directions

- ▷ Identify the class of integrable processes under the new integral.
- ▷ Give a broader generalization of the Itô isometry for the new integral.
- ▷ Provide a broader generalization of the Girsanov theorem.
- ▷ Formulate the extension to stochastic differential equations with anticipating coefficients.
- ▷ Develop the near-Markov property for the new integral.
- ▷ Prove Freidlin–Wentzell type results for stochastic differential equations with anticipating initial conditions.
- ▷ Study LDP results for SDEs with anticipating coefficients.
- ▷ Analyze LDP for linear SPDEs with anticipating initial conditions.

Thank you!

APPENDIX

Laplace principle and equivalence to LDP

Definition (Laplace principle) (X_n) is said to satisfy the Laplace principle on \mathcal{X} with rate function I if for all bounded continuous functions h , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp(-nh(X_n)) = \inf_{\mathcal{X}} (h + I)$$

Theorem (X_n) satisfies LP on \mathcal{X} with rate function I if and only if (X_n) satisfies LDP on \mathcal{X} with the same rate function I .

Some important results

- Uniqueness of the rate function.
- Continuity principle.
- Superexponential approximation preserves Laplace principle.

Thank you!

Bibliography

- Ayed, W. & Kuo, H. H. (2008). An extension of the Itô integral. *Communications on Stochastic Analysis*, 2(3). doi:10.31390/cosa.2.3.05
- (2010). An extension of the Itô integral: Toward a general theory of stochastic integration. *Theory of Stochastic Processes*, 16(32), 17–28. Retrieved from <http://mi.mathnet.ru/thsp56>
- Cramér, H. (1938). Sur un nouveau théorème-limite de la théorie des probabilités. *Actualités Scientifiques et Industrielles*, 736, 5–23.
- Freidlin, M. I. & Wentzell, A. D. (2012). *Random Perturbations of Dynamical Systems*. (3 ed.). Springer-Verlag. doi:10.1007/978-3-642-25847-3
- Girsanov, I. V. (1960). On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures. *Theory of Probability & Its Applications*, 5, 285–301. doi:10.1137/1105027
- Hwang, C. R., Kuo, H. H., Saitô, K., & Zhai, J. (2016). A general Itô formula for adapted and instantly independent stochastic processes. *Communications on Stochastic Analysis*, 10(3). doi:10.31390/cosa.10.3.05

- _____ (2017). Near-martingale Property of Anticipating Stochastic Integration. *Communications on Stochastic Analysis*, 11(4). doi:10.31390/cosa.11.4.06
- Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad.*, 20(8), 519–524. doi:10.3792/pia/1195572786
- _____ (1951). Multiple Wiener Integral. *J. Math. Soc. Japan*, 3(1), 157–169. doi:10.2969/jmsj/00310157
- Kuo, H. H., Peng, Y., & Szozda, B. (2013). Generalization of the anticipative Girsanov theorem. *Communications on Stochastic Analysis*, 7(4). doi:10.31390/cosa.7.4.06
- Kuo, H. H., Sae-Tang, A., & Szozda, B. (2012b). A stochastic integral for adapted and instantly independent stochastic processes. In A stochastic integral for adapted and instantly independent stochastic processes., *Stochastic Processes, Finance and Control*. (Vol. 29, pp. 53–71). Author. doi:10.1142/9789814383318_0003
- _____ (2013). And isometry formula for a new stochastic integral. In And isometry formula for a new stochastic integral., *Quantum Probability and Related Topics*. (Vol. 29, pp. 222–232). Author. doi:10.1142/9789814447546_0014
- Schilder, M. (1966). Some Asymptotic Formulas for Wiener Integrals. *Transactions of the American Mathematical Society*, 125(1), 63–85. doi:10.2307/1994588

Varadhan, S. R. S. (1966). Asymptotic probabilities and differential equations. *Communications on Pure and Applied Mathematics*, 19, 261–286. doi:<https://doi.org/10.1002/cpa.3160190303>