Generalization of stochastic calculus and its applications in large deviations theory

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§ 1
Introduction and motivation

Quick revision and notations

- Let $T \in (0, \infty)$, and denote $\mathbb{T} = [0, T]$ as the index set for t.
- Let $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$ be a filtered probability space.
- $B_\cdot$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$.
- Properties of $B_\cdot$
  - starts at 0
  - has independent increments
  - $B_t - B_s \sim \mathcal{N}(0, t-s)$
  - continuous paths
  - has unbounded linear variation ☹
  - has bounded quadratic variation ☺
  - $\mathbb{E}(B_t B_s) = s \wedge t$
  - martingale
- Naive stochastic integration w.r.t. B_t: not possible.
- A stochastic process X_t is called $(\mathcal{F}_t)$-adapted if $\forall t$, X_t is measurable w.r.t. $\mathcal{F}_t$.

Wiener integral ($f \in L^2[0,T]$)

- Definition

  1. Step functions $f = \sum_{j=0}^{n-1} c_j \mathbb{1}_{[t_j, t_{j+1})}(t)$: Define $\int_0^T f(t)\,\mathrm{d}B_t = \sum_{j=0}^{n-1} c_j \Delta B_j$, where $\Delta B_j = B_{t_{j+1}} - B_{t_j}$.
  2. $f \in L^2[0,T]$: Use step functions approximating f to extend the integral a.s.

- Properties

  ⋆ Linear
  ⋆ Gaussian distribution with mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry)
  ⋆ Corresponds to the Riemann–Stieltjes integral for $f \in C[0,T]$

- The associated process: $I_t = \int_0^t X_t\,\mathrm{d}B_t$

  ⋆ continuous
  ⋆ martingale

- Problem: Cannot integrate stochastic processes.

Itô integral ($X \in L^2_{\text{ad}}([0,T] \times \Omega)$)

- Definition

  1. Adapted step processes $X_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$: define $\int_0^T X_t \, \mathrm{d}B_t = \sum_{j=0}^{n-1} \xi_j \Delta B_j$.
  2. $X \in L^2_{\text{ad}}([0,T] \times \Omega)$: use step processes approximating X to extend the integral in $L^2(\Omega)$.

- Properties

  ⋆ Linear
  ⋆ Mean 0 and variance $\|f\|^2_{L^2[0,T]}$ (Itô isometry)
  ⋆ For $X_\cdot$ continuous, $\int_0^T X_t \, \mathrm{d}B_t = \lim \int_0^T X_{\lfloor \frac{tn}{n} \rfloor} \, \mathrm{d}B_t$, for example $\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}\left(B_t^2 - t\right)$

- The associated process: $I_t = \int_0^t X_t \, \mathrm{d}B_t$

  ⋆ continuous
  ⋆ martingale

- Example: $\int_0^T B_t \, \mathrm{d}B_t = \frac{1}{2}(B_T - T)$.

Itô integral ($\int_0^T X_t^2 \, \mathrm{d}t < \infty$ a.s.)

- Definition: Use sequences of processes in $L^2_{\mathrm{ad}}([0,T] \times \Omega)$ approximating X in probability to extend the integral in probability.

- Properties

  ⋆ Linear
  ⋆ Mean 0, but variance? 😐

- The associated process: $I_t = \int_0^t X_t \, \mathrm{d}B_t$

  ⋆ continuous
  ⋆ local martingale

- Example: $\int_0^T e^{B_t^2} \, \mathrm{d}B_t = \int_0^{B_1} e^{t^2} \, \mathrm{d}t - \int_0^T B_t e^{B_t^2} \, \mathrm{d}t.$

Itô formula

- An Itô process is a process of the form $X_t = X_0 + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$, equivalently expressed as $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t$.
  [Only makes sense when $\int_0^T \left(|m(s, X_s)| + |\sigma(s, X_s)|^2\right)\mathrm{d}s < \infty$ a.s.]

Theorem ([Itô44]) Let X_t be a d-dimensional Itô process, and let $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R})$. Then $f(X_t)$ is also a d-dimensional Itô process, and

$$\mathrm{d}f(X_t) = \langle (\mathrm{D}f)(X_t), \mathrm{d}X_t\rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D^2 f)(X_t)\,\mathrm{d}X_t\right\rangle,$$

where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: For σ constant, $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$, we have $\mathrm{d}\mathcal{E}_t = -\frac{1}{2}\sigma^2\mathcal{E}_t\,\mathrm{d}t + \sigma\mathcal{E}_t\,\mathrm{d}B_t + \frac{1}{2}\sigma^2\mathcal{E}_t(\mathrm{d}B_t)^2$.

Exponential processes and Girsanov theorem

TODO

Let $h \in L^2[0,T]$. Then the translated stochastic process $W_t = B_t - \int_0^t h(s)\,\mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(\sigma B_T - \frac{1}{2}\sigma^2 T\right) =: \mathcal{E}_T^h$.

Then $\tilde{\mathbb{P}} \sim \mathbb{P}$ and the process $Z_t := \mathbb{E}\left(\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \mid \mathcal{F}_t\right)$ is a martingale.

Stochastic differential equations

- Let $\xi \in L^2(\Omega)$ be independent of $B_\cdot$, and $m, \sigma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}[0,T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}$ measurable such that $m(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable $\forall t$.
  Then a $\mathcal{F}_t$-adapted stochastic process X_t is called a solution of the stochastic *integral* equation $X_t = \xi + \int_0^t m(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$ if for each t, the X_t satisfies the integral equation a.s.
- The stochastic differential equation $\mathrm{d}X_t = m(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t,\; X_0 = \xi$ is a symbolic representation of the stochastic integral equation.

Theorem (Existence and uniqueness, Markov property) The stochastic differential equation above has a unique solution if there exists an $M > 0$ such that the following two conditions are satisfied:

⋆ (Lipschitz condition) $|m(t,x) - m(t,y)| + |\sigma(t,x) - \sigma(t,y)|^2 \leq M(1 + |x|^2)$ a.s.

⋆ (growth condition) $|m(t,x)| + |\sigma(t,y)|^2 \leq M(1 + |x|^2)$ a.s.

The solution is a Markov process. Moreover if $\xi \in \mathbb{R}$, then the solution is also stationary.

- Example: For σ constant, the solution of $\mathrm{d}\mathcal{E}_t = \sigma \mathcal{E}_t\,\mathrm{d}B_t, \mathcal{E}_0 = 1$ is given by $\mathcal{E}_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$.

Multiple Wiener–Itô integrals

§ 2

Generalization of Itô calculus

Motivation

- Iterated integrals: Consider the iterated integral $\int_0^T \int_0^T \mathrm{d}B_s \, \mathrm{d}B_t = \int_0^T B_T \, \mathrm{d}B_t \overset{?}{=} B_T B_t$.
- Note that $\mathbb{E}(B_T B_t) = T \wedge t = t \neq 0$, so no martingale property ☹.
- Stochastic differential equations with anticipation

$$\begin{aligned} \mathrm{d}X_t &= X_t \, \mathrm{d}B_t & \qquad \mathrm{d}Y_t &= B_T \, \mathrm{d}B_t \\ X_0 &= B_1 & \qquad Y_0 &= 1 \end{aligned}$$

- Problem: We want to define $\int_0^T X_t \, \mathrm{d}B_t$, where $X_\cdot$ is not adapted (anticipating).
- Some approaches
  - ⋆ Itô's decomposition of integrand $B_t = \left(B_t - \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s\right) + \int_0^t \frac{B_T - B_s}{T-s} \, \mathrm{d}s$
  - ⋆ Enlargement of filtration
  - ⋆ White noise theory
  - ⋆ ...

The new integral [AK08; AK10]: Idea

- A process $Y^{\cdot}$ and filtration $\mathcal{F}_{\cdot}$ are called instantly independent if Y^t and $\mathcal{F}_t$ are independent $\forall t$.
- Ideas
  1. Decompose the integrand into adapted and instantly independent parts.
  2. Evaluate the adapted and the instantly independent parts at the left and right endpoints.
- Consider two continuous stochastic processes, X_t adapted and Y^t instantly independent w.r.t. $\mathcal{F}_{\cdot}$. Then the integral $\int_0^T X_t Y^t \, dB_t$ is defined as

$$\int_0^T X_t Y^t \, dB_t \triangleq \lim_{\|\Delta_n\| \to 0} \sum_{j=0}^{n-1} X_{t_j} Y^{t_{j+1}} \Delta B_j,$$

  provided that the limit exists in probability.
- Now, for any stochastic process $Z(t) = \sum_{k=1}^{n} X_t^{(k)} Y_{(k)}^t$ we extend the definition by linearity.
- This is well-defined [HKS+16].

A simple example

- In the following, denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and lim is the limit in L^2.

$$\begin{aligned}\int_0^t B_T \, \mathrm{d}B_t &= \int_0^t (B_t + (B_T - B_t)) \, \mathrm{d}B_t = \int_0^t B_t \, \mathrm{d}B_t + \int_0^t (B_T - B_t) \, \mathrm{d}B_t \\ &= \lim \sum_{j=0}^{n-1} B_{t_j} \Delta B_j + \lim \sum_{j=0}^{n-1} (B_T - B_{t_{j+1}}) \Delta B_j \\ &= \lim \sum_{j=0}^{n-1} \left(B_T - \Delta B_j\right) \Delta B_j \\ &= B_T \lim \sum_{j=0}^{n-1} \Delta B_j - \lim \sum_{j=0}^{n-1} (\Delta B_j)^2 = B_T B_t - t\end{aligned}$$

- Note that $\mathbb{E}(B_T B_t - t) = 0$.
- In general, $\mathbb{E} \int_0^t Z(t) \, \mathrm{d}B_t = 0$. ☺

Generalized Itô formula [HKS+16]

- Let $\mathrm{d}X_t = m(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$ be an d-dimensional Itô process, $Y^t = \tilde{m}(t)\,\mathrm{d}t + \tilde{\sigma}(t)\,\mathrm{d}B_t$ be a $\tilde{d}$-dimensional instantly independent process, $f(x,y) \in C^2(\mathbb{R}^2)$. Then

$$\mathrm{d}f(X_t, Y^t) = \langle (\mathrm{D}_x f)(X_t, Y^t), \mathrm{d}X_t \rangle + \frac{1}{2}\left\langle \mathrm{d}X_t, (D_x^2 f)(X_t, Y^t)\ \mathrm{d}X_t \right\rangle$$
$$+ \left\langle (\mathrm{D}_y f)(X_t, Y^t), \mathrm{d}Y^t \right\rangle - \frac{1}{2}\left\langle \mathrm{d}Y^t, (D_y^2 f)(X_t, Y^t)\ \mathrm{d}Y^t \right\rangle,$$

  where we use the rule $\mathrm{d}B_t \otimes \mathrm{d}B_t = I_d\,\mathrm{d}t$.

- Example: TODO

Iterated integrals

Theorem ([Itô51]) Let $f \in L^2([0,T]^n)$ and $\hat{f}$ be its symmetrization. Then

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\,\mathrm{d}B_{t_1}\dots\mathrm{d}B_{t_n} = n! \int_0^T \cdots \int_0^{t_{n-1}} \hat{f}(t_1,\dots,t_n)\,\mathrm{d}B_{t_n}\dots\mathrm{d}B_{t_1},$$

Theorem ([AK10]) Let $f \in L^2([0,T]^n)$. Then

$$\int_{[0,T]^n} f(t_1,\dots,t_n)\,\mathrm{d}B_{t_1}\dots\mathrm{d}B_{t_n} = \int_0^T \cdots \int_0^T f(t_1,\dots,t_n)\,\mathrm{d}B_{t_n}\dots\mathrm{d}B_{t_1}.$$

Near-martingale property [HKS+17]

- Question: What are the analogues of the martingale property and the Markov property?
- Partial answer: near-martingales
- Let $Z(t)$ be a stochastic process such that $\mathbb{E}\,|Z(t)| < \infty \; \forall t$, and $0 \le s \le t \le T$. Then, with respect to $\mathcal{F}_\cdot$, the process $Z(t)$ is called a
  - ⋆ near-martingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) = 0$,
  - ⋆ near-submartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \ge 0$, and
  - ⋆ near-supermartingale if $\mathbb{E}(Z(t) - Z(s) \mid \mathcal{F}_s) \le 0$.

§ 3

Large deviations theory

Motivation: an example

1. Setup. Let the following hold:
   - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
   - (X_n) is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moment generating function M.
   - $\mathbb{E}X_1 = m$, $\mathbb{V}X_1 = \sigma^2$, and $X_1 \sim \mu$.
   - $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$.
2. Asymptotic behavior of $\bar{X}_n$:
   - Weak law of large numbers: $\bar{X}_n \overset{\mathbb{P}}{\to} m$.
   - Central limit theorem: $\sqrt{n}\bar{X}_n \overset{w^*}{\to} \sqrt{n}m + \mathcal{N}(0, \sigma^2)$.
3. But at what speed?
4. We want to *control large deviations from the mean*.

Example: large deviation bounds

1. Fixing $x > m$ and forcing the exponential with a free parameter $\theta > 0$, we get

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} = \mathbb{P}\left\{e^{\theta n \bar{X}_n} \geq e^{\theta n x}\right\} \leq e^{-\theta n x}\mathbb{E}\left(e^{\theta n \bar{X}_n}\right) = e^{-\theta n x} M_X(\theta)^n = e^{-n(\theta x - \log M_X(\theta))}$$

2. Since θ was arbitrary, we have

$$\mathbb{P}\left\{\bar{X}_n \geq x\right\} \leq \inf_\theta e^{-n(\theta x - \log M_X(\theta))} = e^{-n \sup_\theta(\theta x - \log M_X(\theta))} =: e^{-nI(x)}.$$

3. Generalizing, we get the large deviation upper bound

$$\overline{\lim}\, \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \leq -\inf_{\bar{E}} I \qquad \forall E \in \mathscr{B}.$$

4. We can also obtain a lower bound too using an exponential change of measure

$$\underline{\lim}\, \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -\inf_{\mathring{E}} I \qquad \forall E \in \mathscr{B}.$$

5. So informally, we get $\mathbb{P}\left\{\bar{X}_n = x\right\} \asymp e^{-nI(x)}$ for $x \in \mathbb{R}$.

Definitions

- The setup: (X_n) is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space $(\mathcal{X}, d)$.
- A function $I : \mathcal{X} \to [0, \infty]$ is called a rate function if it has compact level sets.
- I is lower semicontinuous and attains its infimum on a nonempty closed set.
- For any Borel set E, denote $I(E) = \inf_{x \in E} I(x)$.

Definition (X_n) is said to satisfy the large deviation principle on $\mathcal{X}$ with rate function I if the following two conditions hold.

$$\text{(upper bound)} \qquad \overline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in F\right\} \leq -I(F) \qquad \forall F \text{ closed}$$

$$\text{(lower bound)} \qquad \underline{\lim} \frac{1}{n} \log \mathbb{P}\left\{\bar{X}_n \in E\right\} \geq -I(G) \qquad \forall G \text{ open}$$

Cramér theorem

Theorem ([Cra38]) Let (X_n) be a sequence of i.i.d. real random variables with finite moment generating function M. Then (X_n) follows large deviation principle with rate function $I(x) = \sup_\theta (\theta x - \log M(\theta))$.

Rate function for some common distributions for X.

Distribution	$M(\theta)$	$I(x)$
$Bern(p)$	$1 - p + pe^\theta$	$x \log \frac{x}{1-p} + (x-1)\log \frac{p}{x-1}$
$Pois(\lambda)$	$e^{\lambda(e^\theta - 1)}$	$\lambda - x + x \log \frac{x}{\lambda}$
$Exp(\lambda)$	$(1 - \theta\lambda^{-1})^{-1}$	$\lambda x - 1 + x \log(\lambda x)$
$\mathcal{N}(m, \sigma^2)$	$e^{m\theta + \frac{1}{2}\sigma^2\theta^2}$	$\frac{(x-m)^2}{2\sigma^2}$
$\chi^2(k)$	$(1 - 2\theta)^{-\frac{k}{2}}$	$\frac{1}{2}\left(x - k + k \log \frac{k}{x}\right)$

Sanov theorem

LD in ∞-dimensions — Schilder theorem

Aim: Estimate the probability that a scaled-down sample path of a Brownian motion will stray far from the mean path (the 0 function).

Setup

- Let $B_\cdot$ be a d-dimensional Brownian motion, so $B_\cdot \in C_0 = C_0([0,T];\mathbb{R}^d)$
- $\forall \varepsilon > 0$, let W_ε denote the law of $\sqrt{\varepsilon} B_\cdot$
- Let $\mathrm{CM} = \left\{\omega \in C_0 : \omega \in \mathrm{AC}, \text{ and } \dot{\omega}_t \in L^2[0,T]\right\}]$

Theorem On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 \,\mathrm{d}t\right) \mathbb{1}_{\mathrm{AC}}(\omega) + \infty \mathbb{1}_{\mathrm{AC}^\complement}(\omega)$$

Freidlin–Wentzell theorem

Aim: Estimate the probability that a scaled-down sample path of an Itô diffusion will stray far from the mean path.

Setup

- Let $B_{\cdot}$ be a d-dimensional Brownian motion, so $B_{\cdot} \in C_0 = C_0([0,T];\mathbb{R}^d)$
- $\forall \varepsilon > 0$, let $X^{(\varepsilon)}$ be a $\mathbb{R}^d$-valued Itô diffusion solving an Itô SDE of the form

$$\mathrm{d}X_t^{(\varepsilon)} = b(X_t^{(\varepsilon)})\,\mathrm{d}t + \sigma(X_t^{(\varepsilon)})\sqrt{\varepsilon}\,\mathrm{d}B_t, \qquad X_0^{(\varepsilon)} = 0.$$

- $\forall \varepsilon > 0$, let W_ε denote the law of $X_{\cdot}^{(\varepsilon)}$.

Theorem (Freidlin, Wentzell (year?)) On the Banach space $(C_0, \|\cdot\|_\infty)$, the family of probability measures $\{W_\varepsilon : \varepsilon > 0\}$ satisfy the large deviations principle with the rate function $I : C_0 \to \overline{\mathbb{R}}$ given by

$$I(\omega) = \left(\frac{1}{2}\int_0^T |\dot{\omega}_t - b(\omega_t)|^2\,\mathrm{d}t\right)\mathbb{1}_{H^1([0,T];\mathbb{R}^d)}(\omega) + \infty\mathbb{1}_{H^1([0,T];\mathbb{R}^d)^\complement}(\omega)$$

§ 4
Conclusion

Open areas for research

- ⋆ Extension to SDEs with anticipating coefficients
- ⋆ Near-Markov property
- ⋆ Girsanov theorem for anticipating integrals
- ⋆ Freidlin-Wintzell type result for SDEs with anticipation

> The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

§ 5
Sample slides

Possible areas of interest

- Extension to SDEs with anticipating coefficients
- Near-Markov property
- Girsanov theorem for generalized integration
- Freidlin-Wintzell type result for SDEs with anticipating initial conditions

Theorem (Cramér, 1938) Let $X_1, X_2, \ldots$be a series of i.i.d. real random variables with finite logarithmic moment generating function, for example $\Lambda(t) < \infty \ \forall t \in \mathbb{R}$.
Then the Legendre transform of Λ, $\Lambda^* = \sup_{t\in\mathbb{R}}(tx - \Lambda(t))$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq nx\right) = -\Lambda^*(x) \quad \forall x > \mathbb{E}(X_1)$$

Freidlin–Wentzell theorem

Column 1

The Earth, as a habitat for animal life, is in old age and has a fatal illness. Several, in fact. It would be happening whether humans had ever evolved or not. But our presence is like the effect of an old-age patient who smokes many packs of cigarettes per day—and we humans are the cigarettes.

Itô table

$\times$	$\mathrm{d}t$	$\mathrm{d}B_t$
$\mathrm{d}t$	0	0
$\mathrm{d}B_t$	0	$\mathrm{d}t$

Something

- One
- Two
- Three
- Four

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